Numerical Algorithms SS 2024 Homework 2

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## Convergence behavior of Jacobi and Gauss-Seidel (5 points)

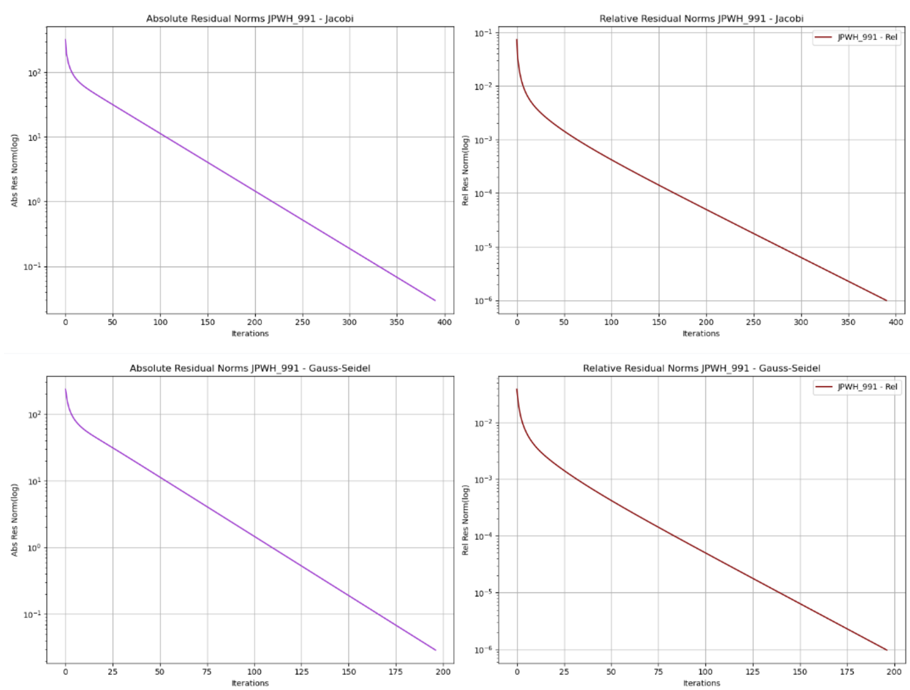


Figure 1: JPWH_991 Residual Plots

For matrix "***JPWH_991***" the graphs for the absolute and relative residual look really similar in their behaviour for both the Jacobi as well as the Gauss-Seidel method. As the absolute residual gets rather low($10^{-1}$) and the relative residual reaching a magnitude of $10^{-6}$, which we defined as our tolerance of our both methods - suggests that we **reach convergance with this matrix** with both algorithms.

One difference that we can observe is that with **Jacobi method** we reach the specified relative residual in 400 iterations whereas Gauss-Seidel only takes half as many iterations(200). So for "JPWH_991" Gauss-Seidel seems to perform better.

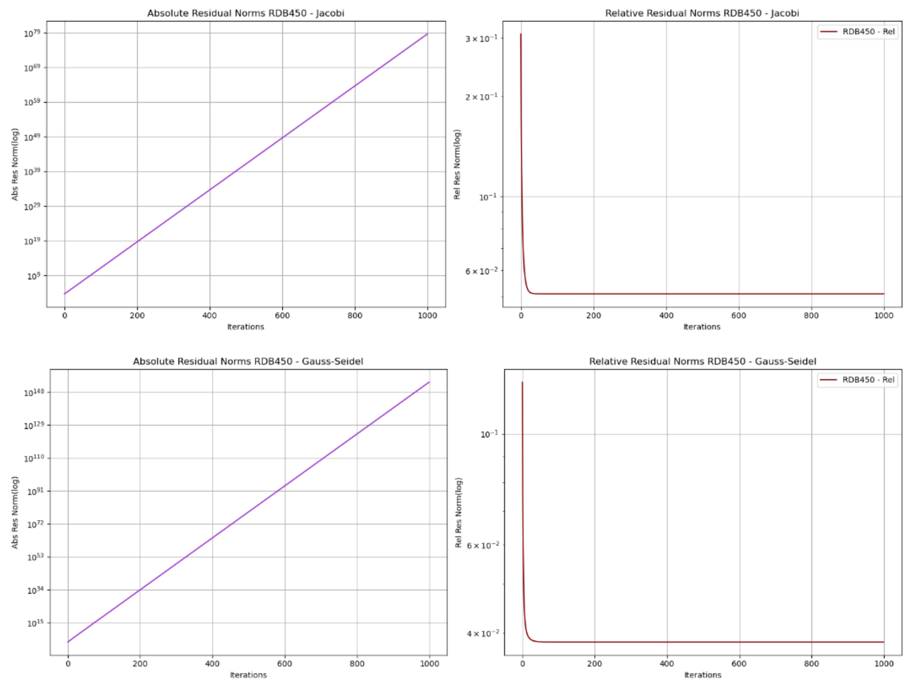


Figure 2: RDB450 Residual Plots

Again for matrix "***RDB450***" the graphs for the absolute and relative residual look really similar in their behaviour for both the Jacobi as well as the Gauss-Seidel method. But a key difference to the results with the last matrix is that the absolute residual starts rather high and further increases by the number of iterations until we reach our iteration limit of 1000, which is a **strong characteristic of divergence**($10^{79}$ and $10^{148}$). The graph of the relative residual shows that for one we never reach our goal of $10^{-6}$ relative residual, but also it seems that it gets a tiny bit better at first after a few iterations before our relative residual begins to stagnate at 4 or 6 * $10^{-2}$ respectively and never gets better onward. Both methods do not converge, and although the relative residual of the **Gauss-Seidel** shows a maginally lower residual at iteration 1000(4 * $10^{-2}$), I am hesident to call it "better" because both methods fail to converge and get us a satisfying result.

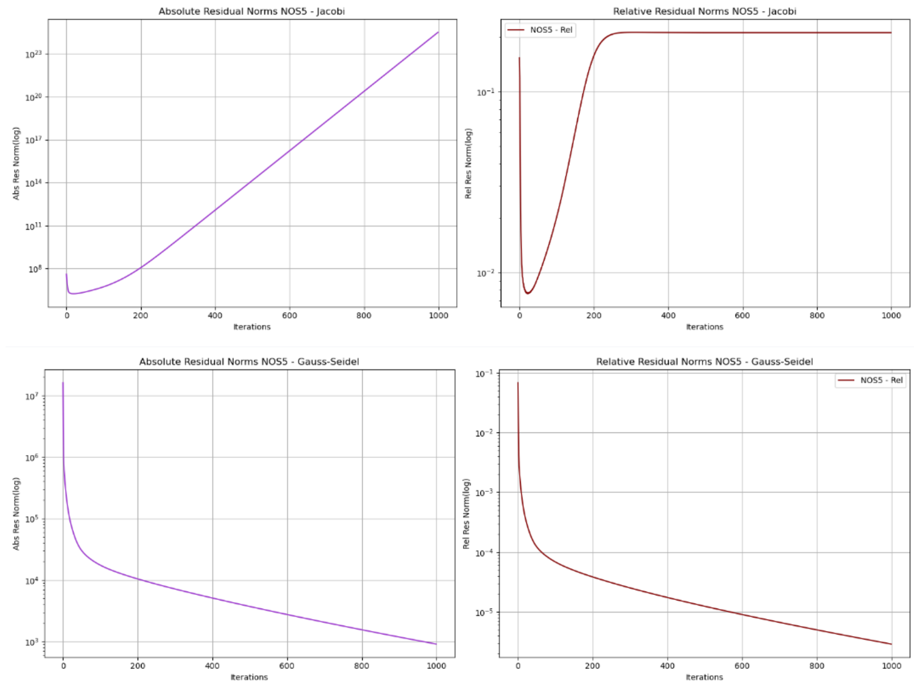


Figure 3: NOS5 Residual Plots

The plots for matrix "***NOS5***" get interesting, because this time we can observe completely different behaviours for our iterative methods. When applying the Jacobi Methode on the given matrix, we can see that the absolute residual starts high but decreases at the beginning, before changing its trend drastically and increases exponentially after certain iterations(absolute residual intervall $[10^6,\ 10^{23}]$). If we look at the relative residual we can see a similar behaviour, where our accuracy improves at the beginning before skyrocketing again after some iterations before it starts to stagnate at about slightly above $10^{-1}$. Both metrics suggest **divergence**. When we look at the results of Gauss-Seidel, we can see a completely different picture. Looking at the absolute residual, we first start at a high absolute residual($10^7$) before steadily decreasing significantly. Same trend can be observed with the relative residual which starts at $10^{-1}$ and improves steadily till $10^{-5}$. Both **trends suggest convergance**, but we can see that we will not reach our desired tolerance of $10^{-6}$ in under 1000 iterations, therefore also the **Gauss-Seidel** approach fails to achieve that - but it suggest convergance after certain alternative conditions(increase iterations, preprocessing the input matrix etc), therefore a significant improvement to the Jacobi Methode in this case.

Throughout the 2 later matrixes the relative residuals showed different trends to the absolute residual variant. I would explain it as such, that that phenomena occurs, because of the normalization of the discrepancy of our approximation to the true solution in the relative residual (among others). While the former shows the absolute error magnitude without normalization it can also be rather sensitive of certain scales. But the relative residual can also like descriped before can show initial improvements before growing exponentially again, in case of convergence(while the absolute residual steadily increases in a linear manner).

## Reason for convergence respectively divergence (5 points)

A rather important metric to signify if a specific iteration matrix G leads to convergence or not seems to be the spectral radius. We get this property by retrieving the largest absolut eigenvalue form our iteration matrix G. We interpret the results as following if:

$$\rho(G) < 1 \; convergance$$

$$\rho(G) \geq 1 \; divergence$$

if we now compute the iteration matrix G for every Jacobi or Gauss-Seidel application and retrieve there respective eigenvalues in correlation to the test matrices we get the following result, which perfectly mimics the behavior of each descript above.

| | JPWH_991 | RDB450 | NOS5 |
|---|---|---|---|
| Jacobi | 0.979722 | 1.18848 | 1.048947 |
| Gauss-Seidel | 0.959915 | 1.412484 | 0.997904 |

Figure 4: spectral radii for the test matrices

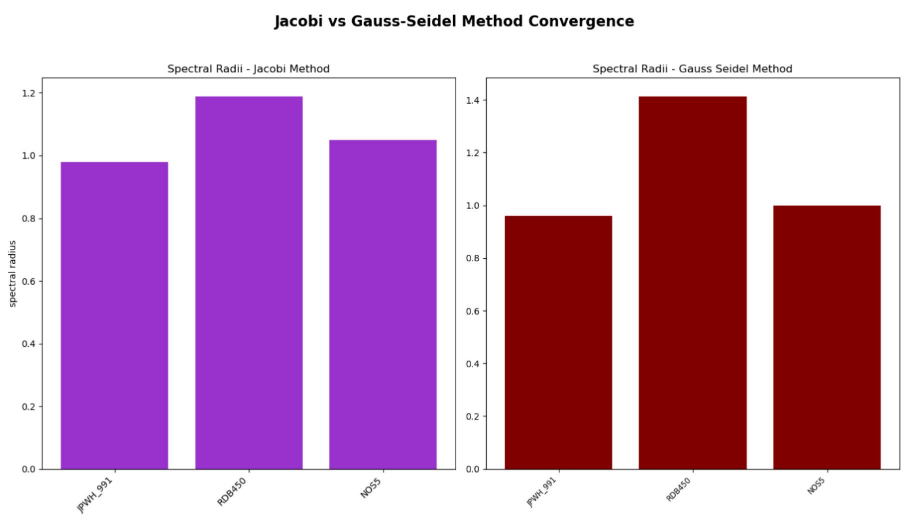


Figure 5: spectral radii bar plot