# Project # 4

Due: Monday 11/10/2025

Consider the orbit determination problem as shown in Fig. 1 and detailed in [1]. In this problem range ($\rho$), azimuth ($az$), and elevation ($el$) measurements are obtained from a ground station to determine the location of a spacecraft. The observation equations are given by

$$\begin{aligned} \|\boldsymbol{\rho}\| &= \rho = \sqrt{\rho_u^2 + \rho_e^2 + \rho_n^2} \\ az &= \tan^{-1}\left(\frac{\rho_e}{\rho_n}\right) \\ el &= \sin^{-1}\left(\frac{\rho_u}{\|\boldsymbol{\rho}\|}\right) \end{aligned} \tag{1}$$

where, $\{u, e, n\}$ defines the up, east, north coordinate frame (see appendix for definitions).

The objective of this project is to determine the initial conditions (position, $\boldsymbol{r}_0$, and velocity $\dot{\boldsymbol{r}}_0$) of the spacecraft from range, azimuth, and elevation measurements using GLSDC.

## Project Steps

### Generating Measurements

Starting from the true initial conditions,

$$\begin{aligned} \boldsymbol{r}_0 &= [7000, 1000, 200]^{\mathrm{T}} \qquad \text{km} \\ \dot{\boldsymbol{r}}_0 &= [4, 7, 2]^{\mathrm{T}} \qquad \text{km/s} \end{aligned}$$

1. Numerically integrate the equations of motion for the two-body problem, Eq. (5), to generate the true inertial states of the target spacecraft, $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$. The simulation time will be for 100 seconds with measurements given every 10 seconds.

2. Use Eq. (3) to generate the inertial range vector $\boldsymbol{\rho}$. The observer latitude is $\phi = 5^\circ$, its inertial sidereal time is $\Theta_0 = 10^\circ$, $\|\boldsymbol{R}\| = R = 6371$ km, and Earth angular velocity is $\omega_\oplus = 7.2921159 \times 10^{-5}$ rad/s.

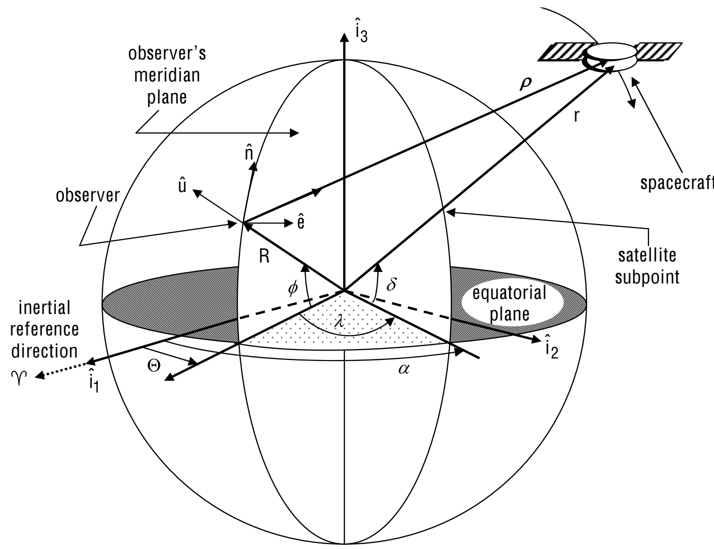


Figure 1: The Orbit Determination Problem [1]

3. Use the coordinate transformation, Eq. (4) to obtain the range vector in the observer frame (up, east, north).
4. Using Eq. (1), generate the range magnitude, azimuth, and elevation true values.
5. Generate the noisy measurements by adding zero-mean Gaussian noise to the true measurements with $\sigma_\rho = 1$ km, and $\sigma_{az} = \sigma_{el} = 0.01^\circ$.

## Initial Guess

to initialize the GLSDC algorithm, start with the initial guess

$$\begin{aligned} \hat{\boldsymbol{r}}_0 &= [6990, 1, 1]^{\mathrm{T}} \qquad \text{km} \\ \dot{\hat{\boldsymbol{r}}}_0 &= [1, 1, 1]^{\mathrm{T}} \qquad \text{km/s} \end{aligned} \tag{2}$$

where, $\hat{\boldsymbol{x}}_0 = \left[\hat{\boldsymbol{r}}_0^{\mathrm{T}}, \dot{\hat{\boldsymbol{r}}}_0^{\mathrm{T}}\right]^{\mathrm{T}}$

## GLSDC

The GLSDC flowchart is shown in Fig. 2 [1]. To implement, follow the steps:

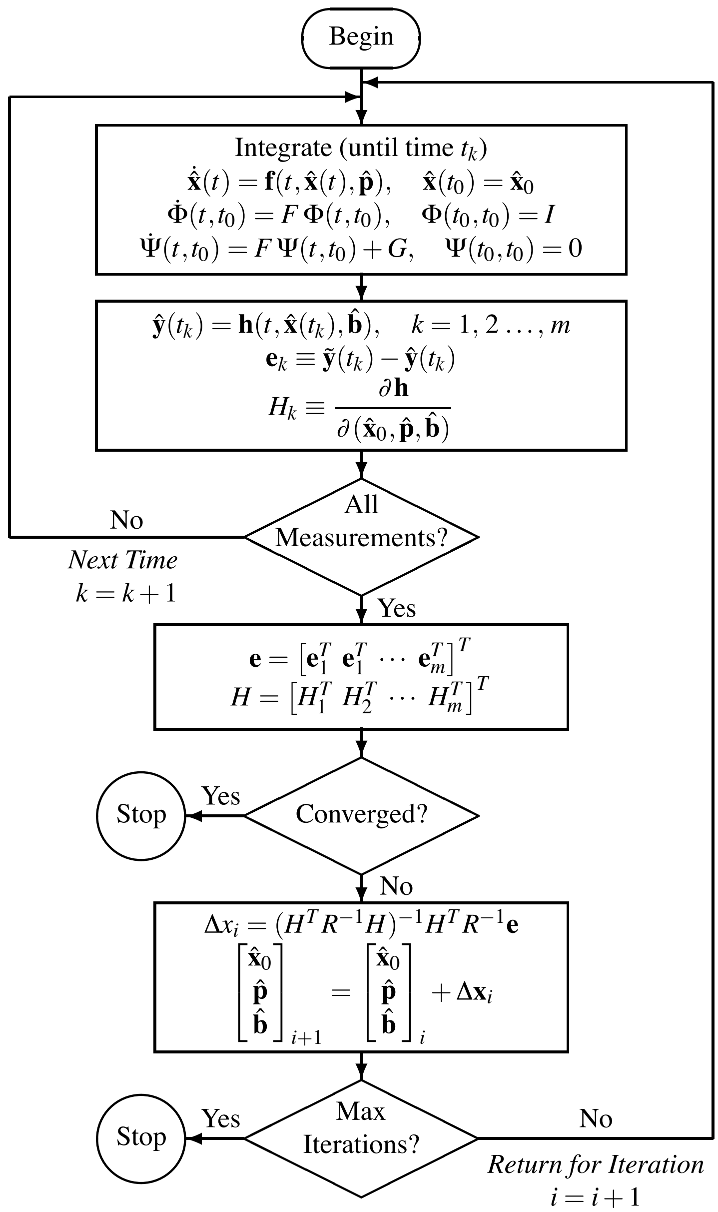


Figure 2: GLSDC for Orbit Determination [1]

1. Numerically integrate the orbit problem, Eq. (5), and the state transition matrix, Eq. (6), simultaneously. Use the initial guess for the initial position and velocity in Eq. (2). Here the numerical integration will be performed for $t \in [0, 100]$ seconds, generating the states every 10 seconds.

2. Obtain the estimated measurements $\hat{\boldsymbol{y}}(t, \hat{\boldsymbol{x}}(t))$ by substituting the obtained estimated states $\hat{\boldsymbol{x}}$ into Eq. (3) then Eq. (4).

3. Compute the error vector $\boldsymbol{e} = \tilde{\boldsymbol{y}} - \hat{\boldsymbol{y}}$ and check for convergence.

4. If not converged, compute the $H$ matrix as detailed in Eq. (7), Eq. (8), and Fig. 3.

5. Update the estimate using the least squares solution and check for maximum iterations.

6. If not at maximum iterations, repeat steps 1 through 5 using the updated initial conditions.

### Results and Analysis

First as in [1], show a table with the iterations and the final solution to indicate convergence. Do not be alarmed if your solution does not converge in 7 iterations as in [1] or does not converge at all. You probably need to change the seed of the random variable. Next, show the $1\sigma - 3\sigma$ bounds of the obtained solution. Finally, as was done in project 3, run 1000 Monte-Carlo simulations and show the $1\sigma - 3\sigma$ bounds of the obtained solutions.

## Appendix

As shown in Fig. 1, in an inertial frame, the slant range vector $\boldsymbol{\rho}$ is given by,

$$\boldsymbol{\rho} = \begin{bmatrix} x - \|\boldsymbol{R}\| \cos\phi \cos\Theta \\ y - \|\boldsymbol{R}\| \cos\phi \sin\Theta \\ z - \|\boldsymbol{R}\| \sin\phi \end{bmatrix} \tag{3}$$

where, $x, y, z$ define the position of the spacecraft measured from the center of the Earth, $\boldsymbol{r} = [x, y, z]^{\mathrm{T}}$, $\phi$ is the latitude of the observer, and $\Theta$ is the local sidereal time of the observer.

The coordinate transformation from the inertial frame to the observer's frame is then given by

$$\begin{bmatrix} \rho_u \\ \rho_e \\ \rho_n \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\rho} \tag{4}$$

The dynamics of the orbit problem is given by

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} \tag{5}$$

where, $\mu = 398600.4415$ km$^3$/s$^2$ is the Earth gravitational parameter, and $r = \sqrt{x^2 + y^2 + z^2}$.

The STM is given by,

$$\dot{\Phi} = F\Phi = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ F_{21} & 0_{3\times3} \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi 21 & \Phi_{22} \end{bmatrix} \tag{6}$$

where,

$$F_{21} = \begin{bmatrix} \frac{3\mu x^2}{r^5} - \frac{\mu}{r^3} & \frac{3\mu xy}{r^5} & \frac{3\mu xz}{r^5} \\ \frac{3\mu xy}{r^5} & \frac{3\mu y^2}{r^5} - \frac{\mu}{r^3} & \frac{3\mu yz}{r^5} \\ \frac{3\mu xz}{r^5} & \frac{3\mu yz}{r^5} & \frac{3\mu z^2}{r^5} - \frac{\mu}{r^3} \end{bmatrix}$$

For this problem we will not be estimating system parameters $\boldsymbol{p}$ or biases $\boldsymbol{b}$. Hence, The $H$ matrix is defined as

$$H = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \Phi \tag{7}$$

where, $\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} = \left[\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{r}}, \frac{\partial \boldsymbol{h}}{\partial \dot{\boldsymbol{r}}}\right] = [H_{11}, 0_{3\times3}]$. Where,

$$H_{11} = \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \\ \frac{\partial az}{\partial x} & \frac{\partial az}{\partial y} & \frac{\partial az}{\partial z} \\ \frac{\partial el}{\partial x} & \frac{\partial el}{\partial y} & \frac{\partial el}{\partial z} \end{bmatrix} \tag{8}$$

The elements of the matrix $H_{11}$ are shown in Fig. 3 as detailed in [1]. Please note that $\|\boldsymbol{\rho}\| = \rho$.

# References

[1] Crassidis, J. L. and Junkins, J. L., *Optimal estimation of dynamic systems*, CRC press, 2011.

$$\frac{\partial ||\boldsymbol{\rho}||}{\partial x} = (\rho_u \cos\phi \cos\Theta - \rho_e \sin\Theta - \rho_n \sin\phi \cos\Theta)/||\boldsymbol{\rho}||$$
$$\frac{\partial ||\boldsymbol{\rho}||}{\partial y} = (\rho_u \cos\phi \sin\Theta + \rho_e \cos\Theta - \rho_n \sin\phi \sin\Theta)/||\boldsymbol{\rho}||$$
$$\frac{\partial ||\boldsymbol{\rho}||}{\partial z} = (\rho_u \sin\phi + \rho_n \cos\phi)/||\boldsymbol{\rho}||$$

$$\frac{\partial \mathrm{az}}{\partial x} = \frac{1}{(\rho_n^2 + \rho_e^2)}(\rho_e \sin\phi \cos\Theta - \rho_n \sin\Theta)$$
$$\frac{\partial \mathrm{az}}{\partial y} = \frac{1}{(\rho_n^2 + \rho_e^2)}(\rho_e \sin\phi \sin\Theta + \rho_n \cos\Theta)$$
$$\frac{\partial \mathrm{az}}{\partial z} = -\frac{1}{(\rho_n^2 + \rho_e^2)}\rho_e \cos\phi$$

$$\frac{\partial \mathrm{el}}{\partial x} = \frac{1}{||\boldsymbol{\rho}||(||\boldsymbol{\rho}||^2 - \rho_u^2)^{1/2}}\left(||\boldsymbol{\rho}|| \cos\phi \cos\Theta - \rho_u \frac{\partial ||\boldsymbol{\rho}||}{\partial x}\right)$$
$$\frac{\partial \mathrm{el}}{\partial y} = \frac{1}{||\boldsymbol{\rho}||(||\boldsymbol{\rho}||^2 - \rho_u^2)^{1/2}}\left(||\boldsymbol{\rho}|| \cos\phi \sin\Theta - \rho_u \frac{\partial ||\boldsymbol{\rho}||}{\partial y}\right)$$
$$\frac{\partial \mathrm{el}}{\partial z} = \frac{1}{||\boldsymbol{\rho}||(||\boldsymbol{\rho}||^2 - \rho_u^2)^{1/2}}\left(||\boldsymbol{\rho}|| \sin\phi - \rho_u \frac{\partial ||\boldsymbol{\rho}||}{\partial z}\right)$$

Figure 3: Partials of the Measurement Model