Project # 5

Due: Wednesday 11/26/2025

Consider the orbit determination problem as shown in Fig. 1 and detailed in [1]. In this problem range (ρ), azimuth (az), and elevation (el) measurements are obtained from a ground station to determine the location of a spacecraft. The observation equations are given by

$$\begin{aligned} \|\boldsymbol{\rho}\| &= \rho = \sqrt{\rho_u^2 + \rho_e^2 + \rho_n^2} \\ az &= \tan^{-1}\left(\frac{\rho_e}{\rho_n}\right) \\ el &= \sin^{-1}\left(\frac{\rho_u}{\|\boldsymbol{\rho}\|}\right) \end{aligned} \tag{1}$$

where, $\{u, e, n\}$ defines the up, east, north coordinate frame (see appendix for definitions).

The objective of this project is to estimate the states (position, $\boldsymbol{r}_(t)$, and velocity $\dot{\boldsymbol{r}}_(t)$) of the spacecraft from range, azimuth, and elevation measurements using sequential state estimation.

Project Steps

Generating Measurements

Starting from the true initial conditions,

$$\begin{aligned} \boldsymbol{r}_0 &= [7000, 1000, 200]^{\mathrm{T}} \qquad \text{km} \\ \dot{\boldsymbol{r}}_0 &= [4, 7, 2]^{\mathrm{T}} \qquad \text{km/s} \end{aligned}$$

1. Numerically integrate the equations of motion for the two-body problem, Eq. (5), to generate the true inertial states of the target spacecraft, $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$. The simulation time will be for 3000 seconds with measurements given every 10 seconds.

2. Use Eq. (3) to generate the inertial range vector $\boldsymbol{\rho}$. The observer latitude is $\phi = 5^\circ$, its inertial sidereal time is $\Theta_0 = 10^\circ$, $\|\boldsymbol{R}\| = R = 6371$ km, and Earth angular velocity is $\omega_\oplus = 7.2921159 \times 10^{-5}$ rad/s.

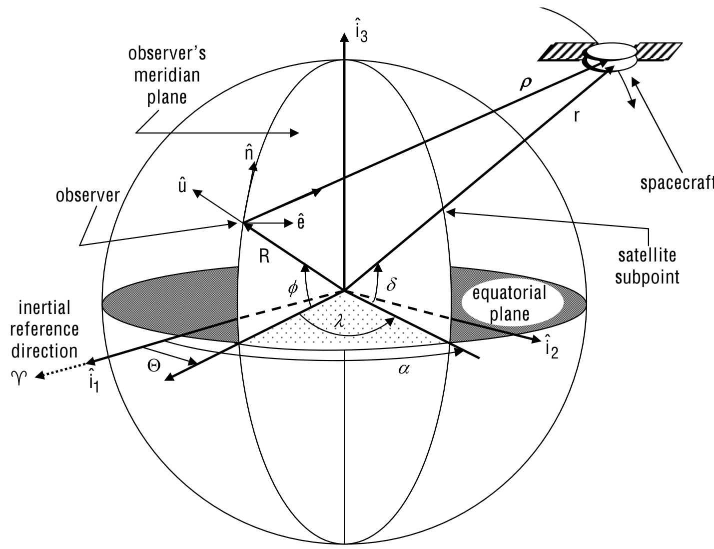

Figure 1: The Orbit Determination Problem [1]

3. Use the coordinate transformation, Eq. (4) to obtain the range vector in the observer frame (up, east, north).
4. Using Eq. (1), generate the range magnitude, azimuth, and elevation true values.
5. Generate the noisy measurements by adding zero-mean Gaussian noise to the true measurements with $\sigma_\rho = 1$ km, and $\sigma_{az} = \sigma_{el} = 0.01^\circ$.

Initial Guess

Start with the initial guess

$$\begin{aligned} \hat{\boldsymbol{r}}_0 &= [6990, 1, 1]^{\mathrm{T}} \qquad \text{km} \\ \dot{\hat{\boldsymbol{r}}}_0 &= [1, 1, 1]^{\mathrm{T}} \qquad \text{km/s} \end{aligned} \tag{2}$$

where, $\hat{\boldsymbol{x}}_0 = \left[\hat{\boldsymbol{r}}_0^{\mathrm{T}}, \dot{\hat{\boldsymbol{r}}}_0^{\mathrm{T}}\right]^{\mathrm{T}}$

Extended Kalman Filter

The EKF flowchart is shown in Fig. 2 [1].

Model	$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k,\ \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Initialize	$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ $P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\,\tilde{\mathbf{x}}^T(t_0)\right\}$
Gain	$K_k = P_k^- H_k^T(\hat{\mathbf{x}}_k^-)[H_k(\hat{\mathbf{x}}_k^-)P_k^- H_k^T(\hat{\mathbf{x}}_k^-) + R_k]^{-1}$ $H_k(\hat{\mathbf{x}}_k^-) \equiv \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right\vert_{\hat{\mathbf{x}}_k^-}$
Update	$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)]$ $P_k^+ = [I - K_k H_k(\hat{\mathbf{x}}_k^-)]P_k^-$
Propagation	$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t)$ $\dot{P}(t) = F(t)\,P(t) + P(t)\,F^T(t) + G(t)\,Q(t)\,G^T(t)$ $F(t) \equiv \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right\vert_{\hat{\mathbf{x}}(t),\,\mathbf{u}(t)}$

Figure 2: Continuous Discrete Extended Kalman Filter [1]

The Unscented Kalman Filter

The implementation steps for the UKF are shown in Fig. 3. Where, $\gamma = \sqrt{L+\lambda}$, and $\lambda = \alpha^2\,(L+k) - L$. Set $\alpha = 10^{-3}$, and $k = 0$.

Results and Analysis

1. Implement the EKF for the orbit estimation problem. Show your results by plotting the estimation error along with the 3σ bounds of the error covariance.

2. Implement the UKF for the orbit estimation problem. As in the EKF show the errors and the errors covariance.

3. **Bonus 25%** Increase the Δt (time space between measurements) to $20, 50$, and 100 seconds and compare the results for both the EKF and UKF.

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{w}_k, \mathbf{u}_k, k)$$
$$\tilde{\mathbf{y}}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k, k)$$

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k \mathbf{e}_k^-$$
$$P_k^+ = P_k^- - K_k P_k^{e_y e_y} K_k^T$$
$$\mathbf{e}_k^- \equiv \tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k^-$$
$$K_k = P_k^{e_x e_y} (P_k^{e_y e_y})^{-1}$$

$$\boldsymbol{\sigma}_k \leftarrow 2L \text{ columns from } \pm\gamma\sqrt{P_k^a}$$
$$\chi_k^{a(0)} = \hat{\mathbf{x}}_k^a$$
$$\chi_k^{a(i)} = \boldsymbol{\sigma}_k^{(i)} + \hat{\mathbf{x}}_k^a$$
$$\mathbf{x}_k^a = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{w}_k \\ \mathbf{v}_k \end{bmatrix}, \quad \hat{\mathbf{x}}_k^a = \begin{bmatrix} \hat{\mathbf{x}}_k \\ \mathbf{0}_{q\times 1} \\ \mathbf{0}_{m\times 1} \end{bmatrix}$$

$$W_0^{\text{mean}} = \frac{\lambda}{L+\lambda}$$
$$W_0^{\text{cov}} = \frac{\lambda}{L+\lambda} + (1 - \alpha^2 + \beta)$$
$$W_i^{\text{mean}} = W_i^{\text{cov}} = \frac{1}{2(L+\lambda)}, \quad i = 1, 2, \ldots, 2L$$

$$\chi_{k+1}^{x(i)} = \mathbf{f}(\chi_k^{x(i)}, \chi_k^{w(i)}, \mathbf{u}_k, k)$$

$$\hat{\mathbf{x}}_k^- = \sum_{i=0}^{2L} W_i^{\text{mean}} \chi_k^{x(i)}$$

$$P_k^- = \sum_{i=0}^{2L} W_i^{\text{cov}} [\chi_k^{x(i)} - \hat{\mathbf{x}}_k^-][\chi_k^{x(i)} - \hat{\mathbf{x}}_k^-]^T$$

$$\gamma_k^{(i)} = \mathbf{h}(\chi_k^{x(i)}, \mathbf{u}_k, \chi_k^{v(i)}, k)$$

$$\hat{\mathbf{y}}_k^- = \sum_{i=0}^{2L} W_i^{\text{mean}} \gamma_k^{(i)}$$

$$P_k^{yy} = \sum_{i=0}^{2L} W_i^{\text{cov}} [\gamma_k^{(i)} - \hat{\mathbf{y}}_k^-][\gamma_k^{(i)} - \hat{\mathbf{y}}_k^-]^T$$

$$P_k^{e_y e_y} = P_k^{yy}$$

$$P_k^{e_x e_y} = \sum_{i=0}^{2L} W_i^{\text{cov}} [\chi_k^{x(i)} - \hat{\mathbf{x}}_k^-][\gamma_k^{(i)} - \hat{\mathbf{y}}_k^-]^T$$

Figure 3: The Unscented Kalman Filter [1]

Appendix

As shown in Fig. 1, in an inertial frame, the slant range vector $\boldsymbol{\rho}$ is given by,

$$\boldsymbol{\rho} = \begin{bmatrix} x - \|\boldsymbol{R}\| \cos\phi \cos\Theta \\ y - \|\boldsymbol{R}\| \cos\phi \sin\Theta \\ z - \|\boldsymbol{R}\| \sin\phi \end{bmatrix} \tag{3}$$

where, x, y, z define the position of the spacecraft measured from the center of the Earth, $\boldsymbol{r} = [x, y, z]^{\mathrm{T}}$, ϕ is the latitude of the observer, and Θ is the local sidereal time of the observer.

The coordinate transformation from the inertial frame to the observer's frame is then given by

$$\begin{bmatrix} \rho_u \\ \rho_e \\ \rho_n \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\rho} \tag{4}$$

The dynamics of the orbit problem is given by

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} \tag{5}$$

where, $\mu = 398600.4415$ km^3/s^2 is the Earth gravitational parameter, and $r = \sqrt{x^2 + y^2 + z^2}$.

The STM is given by,

$$\dot{\Phi} = F\Phi = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ F_{21} & 0_{3\times3} \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi 21 & \Phi_{22} \end{bmatrix} \tag{6}$$

where,

$$F_{21} = \begin{bmatrix} \frac{3\mu x^2}{r^5} - \frac{\mu}{r^3} & \frac{3\mu xy}{r^5} & \frac{3\mu xz}{r^5} \\ \frac{3\mu xy}{r^5} & \frac{3\mu y^2}{r^5} - \frac{\mu}{r^3} & \frac{3\mu yz}{r^5} \\ \frac{3\mu xz}{r^5} & \frac{3\mu yz}{r^5} & \frac{3\mu z^2}{r^5} - \frac{\mu}{r^3} \end{bmatrix}$$

For this problem we will not be estimating system parameters $\boldsymbol{p}$ or biases $\boldsymbol{b}$. Hence, The H matrix is defined as

$$H = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \Phi \tag{7}$$

where, $\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} = \left[\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{r}}, \frac{\partial \boldsymbol{h}}{\partial \dot{\boldsymbol{r}}}\right] = [H_{11}, 0_{3\times3}]$. Where,

$$H_{11} = \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \\ \frac{\partial az}{\partial x} & \frac{\partial az}{\partial y} & \frac{\partial az}{\partial z} \\ \frac{\partial el}{\partial x} & \frac{\partial el}{\partial y} & \frac{\partial el}{\partial z} \end{bmatrix} \tag{8}$$

$$\frac{\partial ||\boldsymbol{\rho}||}{\partial x} = (\rho_u \cos\phi \cos\Theta - \rho_e \sin\Theta - \rho_n \sin\phi \cos\Theta)/||\boldsymbol{\rho}||$$
$$\frac{\partial ||\boldsymbol{\rho}||}{\partial y} = (\rho_u \cos\phi \sin\Theta + \rho_e \cos\Theta - \rho_n \sin\phi \sin\Theta)/||\boldsymbol{\rho}||$$
$$\frac{\partial ||\boldsymbol{\rho}||}{\partial z} = (\rho_u \sin\phi + \rho_n \cos\phi)/||\boldsymbol{\rho}||$$

$$\frac{\partial \mathrm{az}}{\partial x} = \frac{1}{(\rho_n^2 + \rho_e^2)}(\rho_e \sin\phi \cos\Theta - \rho_n \sin\Theta)$$
$$\frac{\partial \mathrm{az}}{\partial y} = \frac{1}{(\rho_n^2 + \rho_e^2)}(\rho_e \sin\phi \sin\Theta + \rho_n \cos\Theta)$$
$$\frac{\partial \mathrm{az}}{\partial z} = -\frac{1}{(\rho_n^2 + \rho_e^2)}\rho_e \cos\phi$$

$$\frac{\partial \mathrm{el}}{\partial x} = \frac{1}{||\boldsymbol{\rho}||(||\boldsymbol{\rho}||^2 - \rho_u^2)^{1/2}}\left(||\boldsymbol{\rho}|| \cos\phi \cos\Theta - \rho_u \frac{\partial ||\boldsymbol{\rho}||}{\partial x}\right)$$
$$\frac{\partial \mathrm{el}}{\partial y} = \frac{1}{||\boldsymbol{\rho}||(||\boldsymbol{\rho}||^2 - \rho_u^2)^{1/2}}\left(||\boldsymbol{\rho}|| \cos\phi \sin\Theta - \rho_u \frac{\partial ||\boldsymbol{\rho}||}{\partial y}\right)$$
$$\frac{\partial \mathrm{el}}{\partial z} = \frac{1}{||\boldsymbol{\rho}||(||\boldsymbol{\rho}||^2 - \rho_u^2)^{1/2}}\left(||\boldsymbol{\rho}|| \sin\phi - \rho_u \frac{\partial ||\boldsymbol{\rho}||}{\partial z}\right)$$

Figure 4: Partials of the Measurement Model

The elements of the matrix H_{11} are shown in Fig. 4 as detailed in [1]. Please note that $\|\boldsymbol{\rho}\| = \rho$.

References

[1] Crassidis, J. L. and Junkins, J. L., *Optimal estimation of dynamic systems*, CRC press, 2011.