14.1 The Linear Extended Kalman Filter (EKF)

Consider the following nonlinear dynamical system model with continuous nonlinear time measurement model as:

$$\dot{x}(t) = f(x, u, t) + G(t)\mathrm{w}(t) \tag{14.1a}$$

$$\tilde{y}(t) = h(x, t) + \mathrm{v}(t) \tag{14.1b}$$

where $f(x, u, t)$ and $h(x, t)$ are assumed to be continuously differentiable and $\mathrm{v}(t)$, $\mathrm{w}(t)$ are uncorrelated Gaussian noises. The Feedback formulation based on the estimate $\hat{x}$ is given by:

$$\dot{\hat{x}}(t) = F(\hat{x}, u, t) + K(t)[\tilde{y}(t) - h(\hat{x}, t)] \tag{14.2a}$$

$$\hat{y}(t) = h(\hat{x}, t) \tag{14.2b}$$

To find $K(t)$, the error is defined as:

$$\tilde{x}(t) = \hat{x}(t) - x(t) \tag{14.3a}$$

$$\Delta y = \tilde{y} - \hat{y} \tag{14.3b}$$

$$\dot{\tilde{x}}(t) = \dot{\hat{x}}(t) - \dot{x}(t) \tag{14.3c}$$

Plugging Equation (14.1a) and Equation (14.2a) into Equation (14.3c) results in:

$$\dot{\tilde{x}}(t) = f(\hat{x}, u, t) + K(t)[\tilde{y}(t) - h(\hat{x}, t)] - [f(x, u, t) + G(t)\mathrm{w}(t)] \tag{14.4}$$

Expanding $f(\hat{x}, u, t)$ and $h(x, t)$ about $\hat{x}$ yields:

$$f(x, u, t) \approx f(\hat{x}, u, t) + \frac{\partial f}{\partial x}|_{\hat{x}} [x - \hat{x}], \quad F(\hat{x}, t) = \frac{\partial f}{\partial x}|_{\hat{x}} \tag{14.5}$$

$$h(x, t) \approx h(\hat{x}, t) + \frac{\partial h}{\partial x}|_{\hat{x}}[x - \hat{x}], \quad H(\hat{x}, t) = \frac{\partial h}{\partial x}|_{\hat{x}} \tag{14.6}$$

Plugging Equation (14.5) and Equation (14.6) into $\dot{\tilde{x}}$

$$\begin{aligned}\dot{\tilde{x}}(t) = {} & f(\hat{x}, u, t)\hat{x}(t) + K[h(\hat{x}, t) + H(x - \hat{x}) + \mathrm{v}(t) - h(\hat{x}, t)] \\ & - [f(x, u, t) + G(t)\mathrm{w}(t)]\end{aligned} \tag{14.7}$$

$$\dot{\tilde{x}}(t) = f(\hat{x}, u, t)\hat{x}(t) + K[H(x - \hat{x}) + \mathrm{v}(t)] - [f(x, u, t) + G(t)\mathrm{w}(t)] \tag{14.8}$$

$$\dot{\tilde{x}}(t) = [F(\hat{x}, t) - KH(\hat{x}, t)]\tilde{x} + K\mathrm{v} - G\mathrm{w} \tag{14.9}$$

The linearized system has the same form as the continuous Kalman filter:

$$K(t) = P(t)H^T(t)R^{-1}(t) \tag{14.10}$$

To simplify the equations, just let $P(t) = P, F(t) = F, G(t) = G$ and so on.

$$\dot{P}(t) = FP + PF^T - KRK^T + GQG^T \tag{14.11}$$

Plugging Equation (14.10) into Equation (14.11) yields:

$$\dot{P}(t) = FP + PF^T - PH^T R^{-1} HP^T + GQG^T \tag{14.12}$$

Summary of Linear Kalman Filter (Table 3.8)

Model:

$$\dot{x}(t) = f(x, u, t) + G(t)\mathrm{w}(t), \qquad \mathrm{w}(t) \sim N\big(0, Q(t)\big) \tag{14.13a}$$

$$\tilde{y}(t) = h(x, t) + \mathrm{v}(t), \qquad \mathrm{v}(t) \sim N\big(0, Q(t)\big) \tag{14.13b}$$

Initialize:

$$\hat{x}(t_0) = \hat{x}_0 \tag{14.14a}$$

$$P_0 = E\{\hat{x}(t_0)\hat{x}^T(t_0)\} \tag{14.14b}$$

Gain:

$$K(t) = P(t)H^T(t)(t)R^{-1}(t), \tag{14.15}$$

Covariance:

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) - P^T(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t) \tag{14.16a}$$

$$F(t) = \frac{\partial f}{\partial x}|_{\hat{x}}, \quad H(t) = \frac{\partial h}{\partial x}|_{\hat{x}} \tag{14.16b}$$

Estimate:

$$\dot{\hat{x}}(t) = f(\hat{x}, u, t) + K(t)[\tilde{y}(t) - h(\hat{x}, t)] \tag{14.17}$$

14.2 The Continuous Discrete Extended Kalman Filter (EKF)

The continuous-discrete (EKF) handles nonlinear systems by linearizing around the current estimate, offering greater robustness to initial errors than the standard (LKF). Its iterative updates refine the state estimate, yielding higher accuracy and smoother convergence. Below is the summary of (EKF)

Model:

$$\dot{x}(t) = f(x,u,t) + G(t)\mathrm{w}(t) \quad \mathrm{w}(t) \sim N\big(0, Q(t)\big) \quad \text{Continuous} \tag{14.18a}$$

$$\tilde{y}_k = h(x_k) + \mathrm{v}_k \quad \mathrm{v}_k \sim N\big(0, Q(t)\big) \quad \text{Discrete} \tag{14.18b}$$

Initialize:

$$\hat{x}(t_0) = \hat{x}_0 \tag{14.19a}$$

$$P_0 = E\{\hat{x}(t_0)\hat{x}^T(t_0)\} \tag{14.19b}$$

Gain:

$$K_k = P_k^{-1} H_k^T \left[H_k P_k^- H_k^T + R_k\right]^{-1} \tag{14.20a}$$

$$H_k(\hat{x}_k^-) = \frac{\partial h}{\partial x}|\hat{x}_k^- \tag{14.21b}$$

Update:

$$\hat{x}_k^+ = \hat{x}_k^- + K_k[\tilde{y}_k - h(\hat{x}_k^-)] \tag{14.22a}$$

$$P_k^+ = [I - K_k H_k] P_k^- \tag{14.22b}$$

Propagation:

$$\dot{\hat{x}}(t) = f(\hat{x}, u, t) \tag{14.23a}$$

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) \tag{14.23b}$$

$$F(t) = \frac{\partial f}{\partial x}|_{\hat{x}(t), u(t)} \tag{14.23c}$$

14.3 Unscented Transformation (UT) Filtering

UT Filtering is used because it provides more accurate and robust state estimates for nonlinear systems without requiring Jacobians or differentiable models. It captures nonlinear behavior better than the EKF by using a higher-order approximation of the mean and covariance. The UT is represented by the model:

$$x_{k+1} = f(x_k, \mathrm{w}_k, u_k, k) \tag{14.24a}$$

$$\tilde{y}_k = h(x_k, u_k, \mathrm{v}_k, k) \tag{14.24b}$$

$$\hat{x}_k^+ = \hat{x}_k^- + K_k \mathrm{e}_k^- \tag{14.24c}$$

$$P_k^+ = P_k^- - K_k P_k^{e_y e_x} K_k^T \tag{14.24d}$$

The innovations process given by:

$$\mathrm{e}_k^- \equiv \tilde{y}_k - \hat{y}_k^- \tag{14.25}$$

The covariance of e_k^- is defined by $P_k^{e_y e_y}$.The gain K_k is obtained by:

$$K_k = P_k^{e_y e_x} \left(P_k^{e_y e_y}\right)^{-1} \tag{14.26}$$

where $P_k^{e_y e_y}$ is the cross-correlation matrix between $\hat{x}_k^-$ and $\hat{y}_k^-$.
The final augmented covariance matrix is given by:

$$P_k^a = \begin{bmatrix} P_k^+ & P_k^{xw} & P_k^{xv} \\ (P_k^{xw})^T & Q_k & P_k^{wv} \\ (P_k^{xv})^T & (P_k^{wv})^T & R_k \end{bmatrix} \tag{14.27}$$

The general formulation for the propagation equation is given by

$$\sigma_k, (\, 2L \text{ cloumns}) \pm \gamma\sqrt{P_k^a} \tag{14.28a}$$

$$\chi_k^{a(0)} = \hat{x}_k^a \tag{14.28b}$$

$$\chi_k^{a(i)} = \sigma_k^{(i)} + \hat{x}_k^a \tag{14.28c}$$

Where $\hat{x}_k^a$ is an augmented state defined by

$$\mathrm{x}_k^a = \begin{bmatrix} \mathrm{x}_k \\ \mathrm{w}_k \\ \mathrm{v}_k \end{bmatrix}, \hat{\mathrm{x}}_k^a = \begin{bmatrix} \hat{x}_k \\ 0_{q\times 1} \\ 0_{m\times 1} \end{bmatrix} \tag{14.29}$$

L is the size of the vector x_k^a. The parameter γ is given by

$$\gamma = \sqrt{L + \lambda} \tag{14.30a}$$

Where

- λ is a scaling parameter , $\lambda = \alpha^2(L + \kappa) - L$ (14.30b)
- α is the spread of sigma points (10^{-3})
- κ is a scaling parameter, $\kappa = 0$
- β is used to incorporate prior knowledge of x $(Guassion \to \beta = 2)$

The transformed set of sigma points is evaluated for each of the points by

$$\chi_{k+1}^{x(i)} = f\left(\chi_k^{x(i)}, \chi_k^{\mathrm{w}(i)}, u_k, k\right) \tag{14.31}$$

Where $\chi_k^{x(i)}$ is a vector of the first n elements of $\chi_k^{\mathrm{w}(i)}$is a vector of the next q elements of $\chi_k^{a(i)}$

$$\chi_k^{a(i)} = \begin{bmatrix} \chi_k^{x(i)} \\ \chi_k^{\mathrm{w}(i)} \\ \chi_k^{\mathrm{v}(i)} \end{bmatrix} \tag{14.32}$$

where $\chi_k^{\mathrm{v}(i)}$ is a vector of the last l elements of $\chi_k^{a(i)}$, which will be used to compute the output covariance. We now define the following weights:

$$W_0^{\text{mean}} = \frac{\lambda}{L+\lambda}\,, \tag{14.33a}$$

$$W_0^{\text{cov}} = \frac{\lambda}{L+\lambda} + (1 - \alpha^2 + \beta) \tag{14.33b}$$

$$W_0^{\text{mean}} = W_0^{\text{cov}} = \frac{1}{2(L+\lambda)}, \qquad i = 1,2,\ldots,2L \tag{14.33c}$$

The predicted mean for the state estimate is calculated using a weighted sum of the points $\chi_k^{x(i)}$

$$\hat{x}_k^- = \sum_{i=0}^{2L} W_i^{mean} \chi_k^{x(i)} \tag{14.34}$$

The predicated covariance is given by

$$P_k^- = \sum_{i=0}^{2L} W_0^{\text{cov}} \left[\chi_k^{x(i)} - \hat{x}_k^-\right]\left[\chi_k^{x(i)} - \hat{x}_k^-\right]^T \tag{14.35}$$

The mean observation is given by

$$\hat{y}_k^- = \sum_{i=0}^{2L} W_i^{mean} \gamma_k^{(i)} \tag{14.36}$$

Where

$$\gamma_k^{(i)} = h\left(\chi_k^{x(i)}, u_k, \chi_k^{\text{v}(i)}, k\right) \tag{14.37}$$

The output covariance is given by

$$P_k^{yy} = \sum_{i=0}^{2L} W_i^{\text{cov}} \left[\gamma_k^{(i)} - \hat{y}_k^-\right]\left[\gamma_k^{(i)} - \hat{y}_k^-\right]^T \tag{14.38}$$

Then, the innovation covariance is simply given by

$$P_k^{e_y e_y} = P_k^{yy} \tag{14.39}$$

Finally, the cross-correlation matrix is determined using:

$$P_k^{e_y e_x} = \sum_{i=0}^{2L} W_i^{\text{cov}} \left[\chi_k^{x(i)} - \hat{x}_k^-\right]\left[\gamma_k^{(i)} - \hat{y}_k^-\right]^T \tag{14.40}$$