Topic 8 —- rotations in 3D —

Use the vector angular velocity, torque and angular momentum and the cross product in 3D to study the rotations of symmetrical rigid bodies such as spheres and symmetric tops. Calculate the velocity of any point $\vec{r}$ on a rigid body in terms of the velocity of its center of mass and angular velocity about the center of mass using $\dot{\vec{r}} = \vec{v}_{CM} + \vec{\omega} \times \vec{r}$. Find the angular momentum of a symmetric top in terms of its angular velocity, the direction of its symmetry axis and its moments of inertia.

For a rigid body rotating around a fixed pivot point or its CM, the angular velocity vector is a vector in the direction of the axis of rotation (with the sign determined as usual by the right-hand rule) and with magnitude equal to the rate of rotation about the axis, ω.

$$\vec{\omega} = \omega\,\hat{n} \qquad \omega > 0$$

The miracle is that $\vec{\omega}$ vectors are vectors — they rotate like vectors and they can be added like vectors — even if they have different directions — this may not sound like much, but it is a little amazing because rotations don't work that way —

Amazing vector equation — if at some time t any vector $\vec{r}(t)$ if rotating about a fixed origin with $\vec{\omega}$ then

$$\dot{\vec{r}}(t) = \vec{\omega} \times \vec{r}(t)$$

This works because the magnitude of $\vec{\omega} \times \vec{r}(t)$ is the angular speed $\omega = |\vec{\omega}|$ times the perpendicular distance from the axis of rotation, $r \sin \theta$ where θ is the angle between $\vec{r}$ and $\vec{\omega}$. The direction is given properly by the right hand rule.

You should be able to see all this in the animation in **sl-8**.

This could only have worked if $\vec{\omega}$ is a vector, because the left-hand side is a vector.

Some tricks with vectors and dot products -

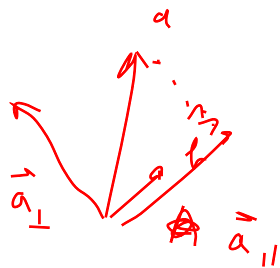
unit vector in direction of a vector $\vec{a}$ — $\hat{a} = \vec{a}/|\vec{a}|$

component of a vector $\vec{a}$ along ~~is~~ a unit vector $\hat{b}$ is just the dot product $\vec{a} \cdot \hat{b}$ — you can see this easily by going to a coordinate system in which $\hat{b}$ is $(1, 0, 0)$ — typical vector argument

break up a vector $\vec{a}$ into a vector $\vec{a}_{\|}$ and $\vec{a}_{\perp}$ parallel and perpendicular to $\hat{b}$

$$\vec{a}_{\|} = \hat{b}\,(\hat{b} \cdot \vec{a})$$

$$\vec{a}_{\perp} = \vec{a} - \hat{b}\,(\hat{b} \cdot \vec{a})$$



$$\hat{b} \cdot \vec{a}_{\perp} = \hat{b} \cdot \vec{a} - \hat{b} \cdot \hat{b}\,(\hat{b} \cdot \vec{a}) = 0$$

$$\vec{a}_\perp + \vec{a}_\parallel = \vec{a}$$

Angular momentum of a rigid body about its CM for arbitrary $\vec{\omega}$

$$\vec{L} = \sum_j \vec{r}_j \times (m_j\, \vec{v}_j) = \sum_j m_j\, \vec{r}_j \times (\vec{\omega} \times \vec{r}_j)$$

proportional to the components of $\vec{\omega}$ — and it is a vector

Uniform sphere rotating about its CM — the vector angular momentum must be in the direction of the angular velocity vector because there is no other vector around, and so it must just be proportional to $\vec{\omega}$

$$\vec{L}_{\text{sphere}} = I\, \vec{\omega}$$

Symmetrical top rotating about it CM — now there is a special direction associated with the top — $\hat{e}_s$ — the symmetry axis

now the vector angular momentum can have a piece proportional to the angular velocity vector but it can also have another piece proportional to the component of the angular momentum along the symmetry axis —

$$\boxed{\vec{L} = I_\perp \vec{\omega} + (I_s - I_\perp)\, \hat{e}_s\, (\hat{e}_s \cdot \vec{\omega})}$$

If $\vec{\omega} \cdot \hat{e}_s = 0$ then $\vec{L} = I_\perp \vec{\omega}$ so $I_\perp$ is the moment of inertia about any axis perpendicular to $\hat{e}_s$.

If $\vec{\omega}$ is $\parallel$ to $\hat{e}_s$ $\cancel{= 0}$ then $\hat{e}_s\, (\hat{e}_s \cdot \vec{\omega}) = \vec{\omega}$ and therefore

$$\vec{L} = I_\perp \vec{\omega} + (I_s - I_\perp)\, \hat{e}_s\, (\hat{e}_s \cdot \vec{\omega}) = I_\perp \vec{\omega} + (I_s - I_\perp)\, \vec{\omega} = I_s \vec{\omega}$$

so I_s is the moment of inertia about the symmetry axis $\hat{e}_s$.

$\vec{\omega} = \omega \hat{n}$

$$\hat{n} \cdot \vec{L} = \hat{n} \cdot \left(I_\perp \omega \hat{n} + (I_s - I_\perp)\, \hat{e}_s (\hat{e}_s \cdot \vec{\omega}) \right)$$

$$I_\perp \omega + (I_s - I_\perp)\, \omega\, (\hat{n} \cdot \hat{e}_s)^2$$

If you know $\vec{L}$ and $\hat{e}_s$ you can also find $\vec{\omega}$ — the simplest way is to write

$$\vec{L} = I_\perp \vec{\omega} + (I_s - I_\perp)\, \hat{e}_s\, (\hat{e}_s \cdot \vec{\omega})$$

and take dot product with $\hat{e}_s$

$$\hat{e}_s \cdot \vec{L} = I_\perp\, (\hat{e}_s \cdot \vec{\omega}) + (I_s - I_\perp)\, (\hat{e}_s \cdot \vec{\omega})\, (\hat{e}_s \cdot \hat{e}_s) = I_s\, (\hat{e}_s \cdot \vec{\omega})$$

This tells us the component of $\vec{\omega} \parallel$ to $\hat{e}_s$, at least if $I_s \neq 0$.

$$(\hat{e}_s \cdot \vec{\omega}) = (\hat{e}_s \cdot \vec{L})/I_s$$

Then putting this back in, we can solve for $\vec{\omega}$

$$\vec{L} = I_\perp \vec{\omega} + (I_s - I_\perp)\, \hat{e}_s\, (\hat{e}_s \cdot \vec{\omega}) = I_\perp \vec{\omega} + (I_s - I_\perp)\, \hat{e}_s\, (\hat{e}_s \cdot \vec{L})/I_s$$

$$\vec{\omega} = \vec{L}/I_\perp - (I_s - I_\perp)\, \hat{e}_s\, (\hat{e}_s \cdot \vec{L})/(I_s\, I_\perp)$$

$$\boxed{\vec{L} = \frac{1}{I_\perp}\Big(\vec{L} - \hat{e}_s\, (\hat{e}_s \cdot \vec{L})\Big) + \frac{1}{I_s}\, \hat{e}_s\, \Big((\hat{e}_s \cdot \vec{L})\Big)}$$

In words this says that $\vec{\omega}$ is the piece of $\vec{L}$ perpendicular to $\hat{e}_s$ divided by $I_\perp$ plus the piece of $\vec{L}$ parallel to $\hat{e}_s$ divided by I_s.

So now, we can go back and forth between $\vec{\omega}$ and $\vec{L}$ for symmetric tops. This, along with the amazing vector equation, is the key understanding their rotational motion.

Rotation about a fixed axis produces a torque on the axis unless the axis is perpendicular or parallel to the symmetry axis.

You should be able to feel this in your bones staring at the second animation in **sl-8.nb**.

For example, if the top is prolate, there is more mass out at the ends of the symmetry axis and so to keep $\vec{\omega}$ along the z axis there must the a torque constantly twisting the symmetry axis of the top towards the z axis.

What if the top is floating in space and (hence “free” — we call this the **free symmetric top**)

This means that for a free symmetric top, $\vec{\omega}$ cannot be constant unless it is exactly parallel or perpendicular to the symmetry axis — so in general both the symmetry axis and $\vec{\omega}$ are changing.

in general how does the symmetry axis move? (this is what you actually see)

$$\vec{L} = I_\perp\,\vec{\omega} + (I_s - I_\perp)\,\hat{e}_s\,(\hat{e}_s \cdot \vec{\omega})$$

$$\vec{\omega} = \frac{\vec{L}}{I_\perp} - \hat{e}_s\,\frac{I_s - I_\perp}{I_\perp}\,(\hat{e}_s \cdot \vec{\omega})$$

$$\frac{d}{dt}\hat{e}_s = \vec{\omega} \times \hat{e}_s$$

This is says that $\hat{e}_s$ is instantaneously rotating about $\vec{\omega}$, but that doesn't help much because $\vec{\omega}$ is changing, as we will see. But

$$\frac{d}{dt}\hat{e}_s = \left(\frac{\vec{L}}{I_\perp} - \hat{e}_s\,\frac{I_s - I_\perp}{I_\perp}\,(\hat{e}_s \cdot \vec{\omega})\right) \times \hat{e}_s = \frac{\vec{L}}{I_\perp} \times \hat{e}_s \qquad \checkmark$$

This is very simple but it tells us a lot about what the system looks like — the unit vector $\hat{e}_s$ is just undergoing uniform circular motion about the fixed vector $\vec{L}$ with angular speed $|\vec{L}_{\text{top}}|/I_\perp$! That means that the top is precessing around its fixed angular momentum even though there is no torque. Look at the third animation.

Notice in the animation that the angle between $\vec{L}$ remains constant as the symmetry axis precesses. You can see this analyically

$$\frac{d}{dt}\left(\vec{L}\cdot\hat{e}_s\right) = \vec{L}\cdot\frac{d}{dt}\hat{e}_s = \vec{L}\cdot\frac{\vec{L}}{I_\perp}\times\hat{e}_s = 0$$

but we know that $\vec{L}\cdot\hat{e}_s = I_s\,\vec{\omega}\cdot\hat{e}_s$ and so this means that $\vec{\omega}\cdot\hat{e}_s$ is also constant which in turn means that our fundamental relation can be written

$$\vec{\omega} = \frac{\vec{L}}{I_\perp} - \hat{e}_s\,\frac{I_s - I_\perp}{I_\perp}\,(\hat{e}_s\cdot\vec{\omega}) = \frac{\vec{L}}{I_\perp} - \Omega\,\hat{e}_s$$

where

$$\Omega \equiv \hat{e}_s\,\frac{I_s - I_\perp}{I_\perp}\,(\hat{e}_s\cdot\vec{\omega})$$

and this implies that the three vectors $\vec{L}$, $\vec{\omega}$ and $\hat{e}_s$ lie in a plane

What this actually looks like depends on the sign of Ω, and we will look at it in the animation **sl-8.nb** and the large animation **free-symmetric-top.nb**.