深蓝学院
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Foundations of Robotics

Lec 4: Rigid-Body Motions (Transformation)

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Outline

1. Homogeneous Transformation Matrices
2. Twists
3. Exponential Coordinate Representation of Rigid-Body Motions
4. Wrenches
5. Homework

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1. Homogeneous Transformation Matrices
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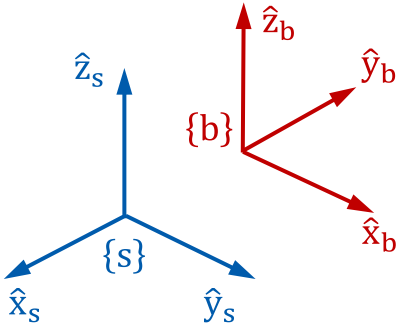
Homogeneous Transformation Matrices

In the previous lecture, we discussed how to describe rotations in rigid-body motions.
In this lecture, we discuss general rigid-body motions that include translations.

Example: how to describe the configuration of {b} relative to {s}?



Define $p \in \mathbb{R}^3$ and $R \in SO(3)$ as

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad R = [\hat{x}_b \;\; \hat{y}_b \;\; \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Alternatively, define $T \in SE(3)$ as

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example values:

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

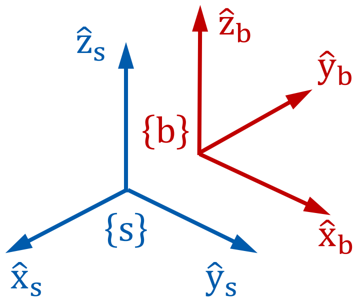
Definition: The Special Euclidean Group $SE(3)$, a.k.a. the group of rigid-body motions or homogeneous transformation matrices, is the set of all 4×4 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$ is a column vector.

Recall that the rotation matrix $R \in SO(3)$ has to satisfy

(1) $R^\top R = I$
(2) $\det R = 1$



$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

Definition: The Special Euclidean Group $SE(2)$ is the set of all 3×3 real matrices T of the form

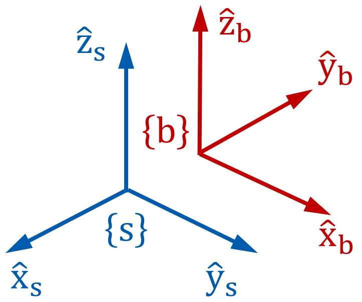
$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

where $R \in SO(2)$ and $p \in \mathbb{R}^2$ is a column vector.

A matrix $T \in SE(2)$ is always of the form

$$T = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta \in [0, 2\pi)$.



$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of Transformation Matrices

Group Axioms

- Closure: $T_1 T_2 \in SE(n)$
- Associativity: $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- Identity element: exists I such that $TI = IT = T$
- Inverse element: exists T^{-1} such that $TT^{-1} = T^{-1}T = I$

Propositions

- $T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix}$
- $T_1 T_2 \neq T_2 T_1$ (generally not commutative)
- $\|Tx - Ty\| = \|x - y\|$ (length preserving)
- $\langle Tx - Tz, Ty - Tz \rangle = \langle x - z, y - z \rangle$ (angle preserving)

* Recall in linear algebra that how to compute the inverse of a 2×2 matrix:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

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* $\|\cdot\|$ denotes Euclidean norm.

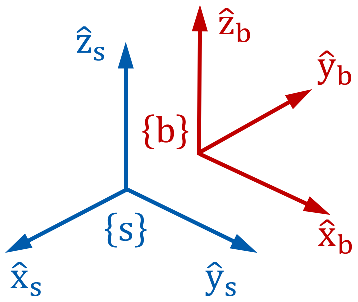
* The notation of Tx implies the use of homogeneous coordinates, where a ‘1’ is implicitly appended to x, making it a four-dimensional vector.

* This operation can be written explicitly as $T\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$. Therefore, when we write Tx, we mean $Rx + p$.

* $\langle \cdot,\cdot \rangle$ denotes Euclidean inner product.

Uses of Transformation Matrices

As was the case for rotation matrices, there are three major uses for a transformation matrix T. In summary, it can be thought of as a representation, or as an operator.



As a representation, it can

(1) represent the configuration (position and orientation).

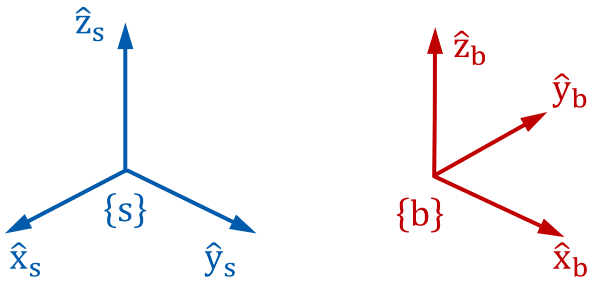
$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As an operator, it can

(2) change the reference frame (of a vector or a frame),
(3) displace a vector or a frame (in its current frame).

Uses of Transformation Matrices

(1) Represent a configuration (position and orientation)



$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad p_{sb} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

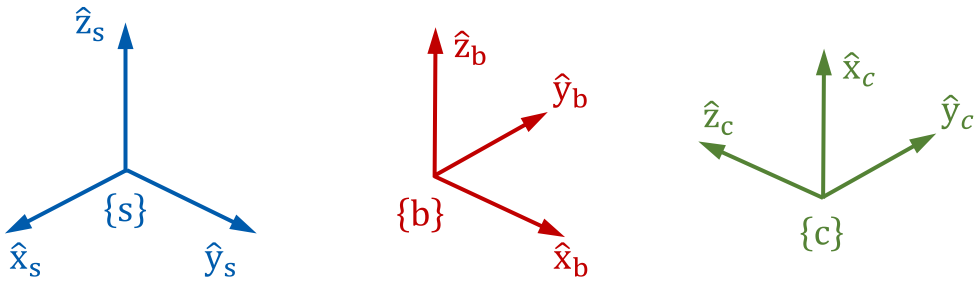
- Subscript convention: T_{sb} represents the configuration of frame {b} relative to frame {s}.
- The inverse of a transformation matrix

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix}$$

$$T_{bs} = \begin{bmatrix} R_{sb}^\top & -R_{sb}^\top p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Uses of Transformation Matrices

(2) Change the reference frame (of ~~a vector or~~ a frame)



$$T_{sb} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bc} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- <u>Subscript cancellation</u>: for matrix multiplication

$$T_{ab}T_{bc}T_{cd}T_{de} = T_{a\cancel{b}}T_{\cancel{bc}}T_{\cancel{cd}}T_{\cancel{d}e} = T_{ae}$$

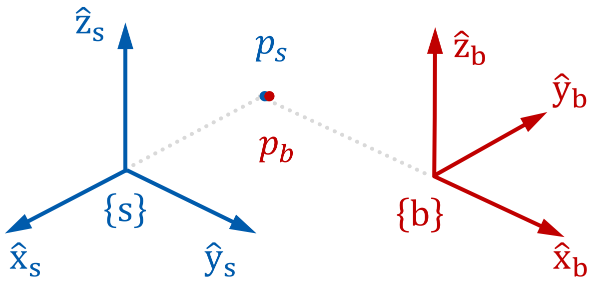
$$\text{Known } T_{sb}, T_{bc} \Longrightarrow T_{sc} = T_{sb}T_{bc}$$

$$\text{Known } T_{sb}, T_{cb} \Longrightarrow T_{sc} = T_{sb}T_{bc} = T_{sb}T_{cb}^{-1}$$

$$T_{sc} = T_{sb}T_{bc} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Uses of Transformation Matrices

(2) Change the reference frame (of a vector ~~or a frame~~)



$$p_b = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$T_{sb} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Subscript cancellation: for a vector multiplied by a matrix

$$T_{ab}p_b = T_{a\cancel{b}}p_{\cancel{b}} = p_a$$

Known $T_{sb}, p_b \Longrightarrow p_s = T_{sb}p_b$

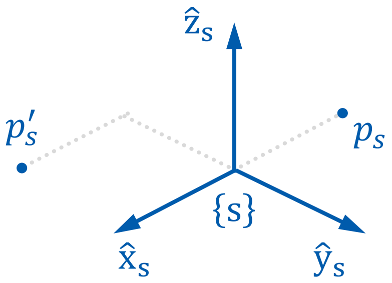
Known $T_{bs}, p_b \Longrightarrow p_s = T_{sb}p_b = T_{bs}^{-1}p_b$

$$\begin{bmatrix} p_s \\ 1 \end{bmatrix} = T_{sb}\begin{bmatrix} p_b \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Longrightarrow p_s = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Uses of Transformation Matrices

(3) Displace a vector ~~or a frame~~ (in its current frame)



$$\mathrm{Rot}(\hat{\omega},\theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Subscript cancellation does not apply.
- $\mathrm{Rot}(\hat{\omega},\theta)$: a rotation of angle θ about unit axis $\hat{\omega}$.
- $\mathrm{Trans}(p)$: a translation by vector p.

$$\text{Known } T, p_s \Longrightarrow p'_s = Tp_s = \mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega},\theta)p_s$$

$$p_s = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T = \mathrm{Trans}(p)\mathrm{Rot}(\hat{z}, 90°)^*$$

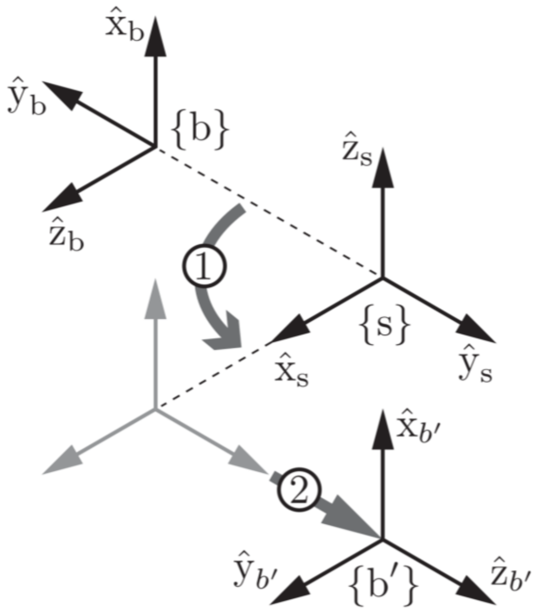
$$= \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p'_s = Tp_s = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

* $T = \mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega},\theta)$ follows our definition $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, whereas $T = \mathrm{Rot}(\hat{\omega},\theta)\mathrm{Trans}(p)$ leads to $T = \begin{bmatrix} R & Rp \\ 0 & 1 \end{bmatrix}$.

Uses of Transformation Matrices

(3) Displace ~~a vector or~~ a frame (in its current frame)



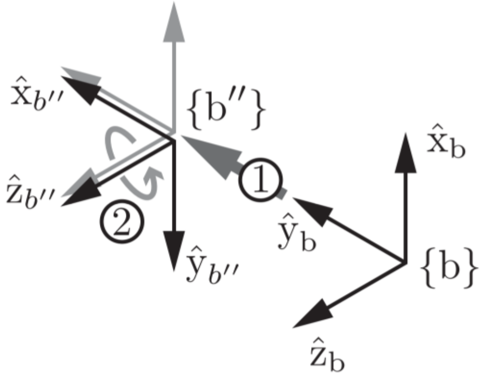
$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

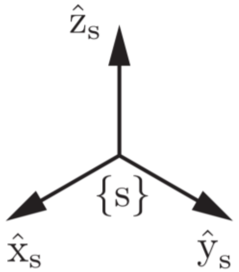
$$\begin{aligned} T_{sb'} &= TT_{sb} \\ &= \mathrm{Trans}(p)\mathrm{Rot}(\hat{z}, 90°)T_{sb} \\ &= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Pre-multiplication!
(interpreted in fixed frame)

Uses of Transformation Matrices

(3) Displace ~~a vector or~~ a frame (in its current frame)





$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} T_{sb''} &= T_{sb}T \\ &= T_{sb}\mathrm{Trans}(p)\mathrm{Rot}(\hat{\mathrm{z}}, 90°) \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Post-multiplication!
(interpreted in body frame)

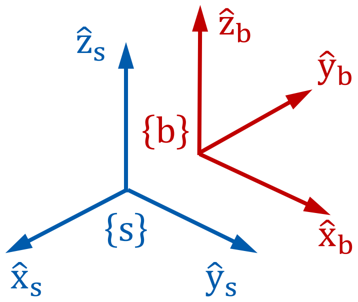
Uses of Transformation Matrices

Summary of the uses of transformation matrices

(1) represent a configuration (position and orientation)

(2) change the reference frame (of a vector or a frame)

(3) displace a vector or a frame (in its current frame)



$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outline

1. Homogeneous Transformation Matrices

2. Twists

3. Exponential Coordinate Representation of Rigid-Body Motions

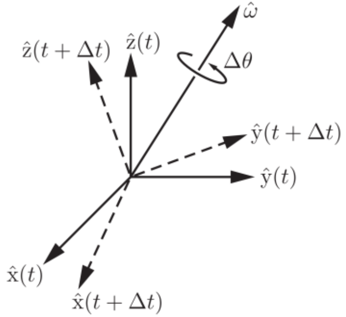
4. Wrenches

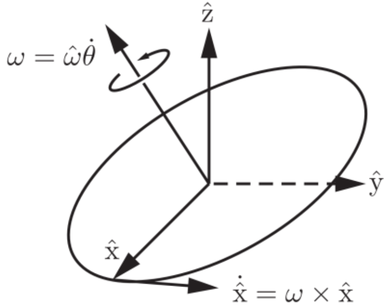
5. Homework

Twists

Recall what we have discussed about angular velocity in the previous lecture.

	Angular velocity	Relations between {s} and {b}
Vector Representation	$\omega = \hat{\omega}\dot{\theta}$	$\omega_s = R_{sb}\omega_b$ $\omega_b = R_{sb}^{\top}\omega_s$
Matrix Representation	$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ $\dot{R} = \omega \times R = [\omega]R$	$[\omega_s] = \dot{R}_{sb}R_{sb}^{\top}$ $[\omega_b] = R_{sb}^{\top}\dot{R}_{sb}$





Twists

We now consider both the linear and angular velocities of a moving frame!

Let's start with the counterpart of $[\omega_b] = R_{sb}^{\top}\dot{R}_{sb}$ for transformation matrices.

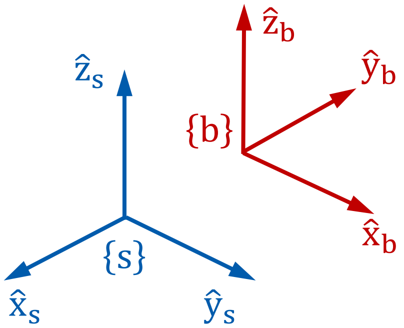
$$T_{sb}^{-1}\dot{T}_{sb} = \begin{bmatrix} R_{sb}^{\top} & -R_{sb}^{\top}p_{sb} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{R}_{sb} & \dot{p}_{sb} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} R_{sb}^{\top}\dot{R}_{sb} & R_{sb}^{\top}\dot{p}_{sb} \\ 0 & 0 \end{bmatrix}$$

linear velocity of {b} in {s}

$$= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}$$

linear velocity of {b} in {b}



Define the spatial velocity in the body frame, or simply the **body twist**, to be $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$.

Let $[\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb} = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$

- We have overloaded the bracket notation!
- This is the matrix representation of a twist!

Twists

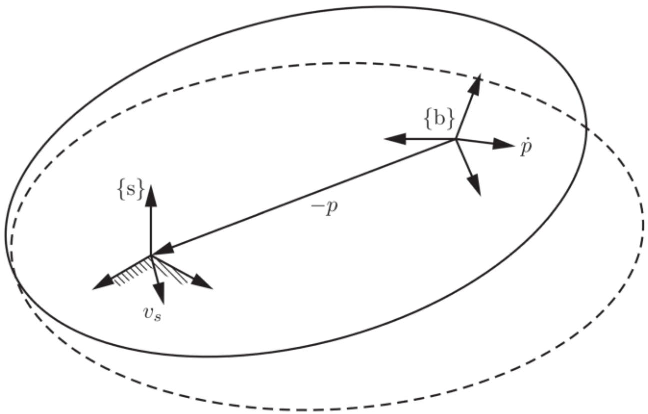
Now that we have a physical interpretation for $T_{sb}^{-1}\dot{T}_{sb}$, let us evaluate $\dot{T}_{sb}T_{sb}^{-1}$:

$$\dot{T}_{sb}T_{sb}^{-1} = \begin{bmatrix} \dot{R}_{sb} & \dot{p}_{sb} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R_{sb}^{\top} & -R_{sb}^{\top}p_{sb} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}_{sb}R_{sb}^{\top} & \dot{p}_{sb} - \dot{R}_{sb}R_{sb}^{\top}p_{sb} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}$$

NOT the linear velocity of {b} in {s}



Define the <u>spatial velocity in the space frame</u>, or simply the **<u>spatial twist</u>**, to be $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$.

Let $[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1} = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$

* The interpretation of v_s: we can rewrite $v_s = \dot{p}_{sb} - \omega_s \times p_{sb} = \dot{p}_{sb} + \omega_s \times (-p_{sb})$. Imagining the moving body to be infinitely large, v_s is the instantaneous velocity of the point on this body currently at the fixed-frame origin, expressed in the fixed frame.

v_b is the linear velocity of a point at the origin of {b} expressed in {b};
v_s is the linear velocity of a point at the origin of {s} expressed in {s}.

Twists

Let's have a brief summary about angular velocity and twist.

	Vector Representation	Matrix Representation	Relations between {s} and {b}
Angular Velocity	$\omega = \hat{\omega}\dot{\theta} \in \mathbb{R}^3$	$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{3\times 3} \in so(3)$	$\omega_s = R_{sb}\omega_b$ $\omega_b = R_{sb}^{\top}\omega_s$ $[\omega_s] = \dot{R}_{sb}R_{sb}^{\top}$ $[\omega_b] = R_{sb}^{\top}\dot{R}_{sb}$
Twist	$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$	$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}_{4\times 4} \in se(3)$	$[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1}$ $[\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb}$ $\mathcal{V}_s \overset{?}{\leftrightarrow} \mathcal{V}_b$

Twists

In matrix representation, the relations between $\mathcal{V}_s$ and $\mathcal{V}_b$ are as follows.

$$[\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb} = T_{sb}^{-1}[\mathcal{V}_s]T_{sb}$$
$$[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1} = T_{sb}[\mathcal{V}_b]T_{sb}^{-1}$$

Writing out the products explicitly, we have

$$[\mathcal{V}_s] = \begin{bmatrix} R_{sb}[\omega_b]R_{sb}^\top & -R_{sb}[\omega_b]R_{sb}^\top p_{sb} + R_{sb}v_b \\ 0 & 0 \end{bmatrix}$$

Applying $R[\omega]R^\top = [R\omega]$ and $-[\omega]p = [p]\omega$, and recalling $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}_{4\times 4}$, we have

$$\begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [R_{sb}\omega_b] & [p_{sb}]R_{sb}\omega_b + R_{sb}v_b \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R_{sb} & 0 \\ [p_{sb}]R_{sb} & R_{sb} \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

It is worth assigning a name to this matrix that relates $\mathcal{V}_s$ and $\mathcal{V}_b$!

Twists

Given $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$, its **adjoint representation** $[\mathrm{Ad}_T]$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

With this, if we are provided T_{sb} and $\mathcal{V}_b$, then $\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = \begin{bmatrix} R_{sb} & 0 \\ [p_{sb}]R_{sb} & R_{sb} \end{bmatrix}\mathcal{V}_b$.

Properities

- $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$
- $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}] = \begin{bmatrix} R^\top & 0 \\ -R^\top[p] & R^\top \end{bmatrix}$

The second property follows from the first on choosing $T_1 = T^{-1}$ and $T_2 = T$.

It also indicates that, if we are provided T_{sb} and $\mathcal{V}_s$, then $\mathcal{V}_b = [\mathrm{Ad}_{T_{sb}}]^{-1}\mathcal{V}_s = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$.

Twists

Let's have a brief summary about angular velocity and twist.

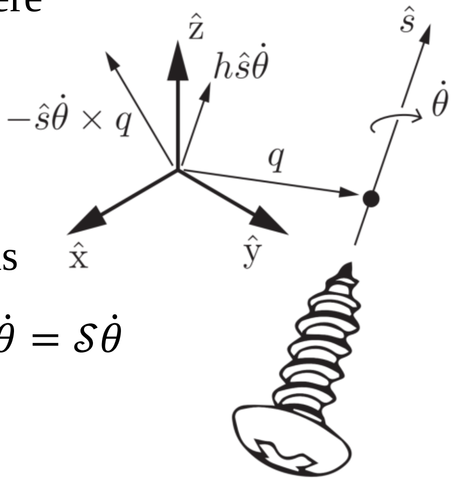
	Vector Representation	Matrix Representation	Relations between {s} and {b}
Angular Velocity	$\omega = \hat{\omega}\dot{\theta} \in \mathbb{R}^3$	$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{3\times 3} \in so(3)$	$\omega_s = R_{sb}\omega_b$ $\omega_b = R_{sb}^\top \omega_s$ $[\omega_s] = \dot{R}_{sb}R_{sb}^\top$ $[\omega_b] = R_{sb}^\top \dot{R}_{sb}$
Twist	$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$	$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}_{4\times 4} \in se(3)$	$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$ $\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$ $[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1}$ $[\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb}$

Screw Interpretation of a Twist

Recall that an angular velocity ω can be viewed as $\omega = \hat{\omega}\dot{\theta}$, which rotates about a unit axis. A twist $\mathcal{V}$ can also be interpreted as a <u>**screw axis**</u> $\mathcal{S}$ and a velocity $\dot{\theta}$ about the screw axis.

One representation of a screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$, where

- $q \in \mathbb{R}^3$ is any point on the axis,
- $\hat{s}$ is a unit vector in the direction of the axis, and
- h is the screw pitch (pitch = linear speed / angular speed).



We can express twist $\mathcal{V}$ in terms of screw $\{q, \hat{s}, h\}$ and velocity $\dot{\theta}$ as

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \omega \\ -\omega \times q + \dot{q} \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix} \dot{\theta} = \mathcal{S}\dot{\theta}$$

linear motion at the origin
induced by rotation about the axis
(in the plane orthogonal to $\hat{s}$)

translation along
the screw axis

In this representation, q is not unique (any point on the axis), and h can be infinite if $\omega = 0$.

Screw Interpretation of a Twist

The other representation of a screw axis $\mathcal{S}$ is a <u>normalized version</u> of twist $\mathcal{V} = (\omega, v)$.

Define the unit (normalized) screw axis $\mathcal{S} = (\mathcal{S}_\omega, \mathcal{S}_v) \in \mathbb{R}^6$, where*

(1) if $\omega \neq 0$, then $\mathcal{S}_\omega = \frac{\omega}{\|\omega\|}$, $\mathcal{S}_v = \frac{v}{\|\omega\|}$, $\dot{\theta} = \|\omega\|$

(2) if $\omega = 0$, then $\mathcal{S}_\omega = 0$, $\mathcal{S}_v = \frac{v}{\|v\|}$, $\dot{\theta} = \|v\|$

In summary, there are three types of motion.

Motion	ω	h	$\mathcal{S}_\omega$	$\mathcal{S}_v$	$\dot{\theta}$	Example
pure linear	$\omega = 0$	$h = \infty$	$\mathcal{S}_\omega = 0$	$\|\mathcal{S}_v\| = 1$	$\dot{\theta} = \|v\|$	prismatic joint
pure rotation	$\omega \neq 0$	$h = 0$	$\|\mathcal{S}_\omega\| = 1$	$\mathcal{S}_v = \frac{v}{\|\omega\|}$	$\dot{\theta} = \|\omega\|$	revolute joint
general rigid-body	$\omega \neq 0$	others	$\|\mathcal{S}_\omega\| = 1$	$\mathcal{S}_v = \frac{v}{\|\omega\|}$	$\dot{\theta} = \|\omega\|$	/

* To make it clear during the derivation, we use a different notation $(\mathcal{S}_\omega, \mathcal{S}_v)$ to define the unit screw axis $\mathcal{S}$. In the textbook (Definition 3.24) and all other places when the meaning is clear from the context, we can define $\mathcal{S} = (\omega, v)$ using the same notation as a general twist $\mathcal{V} = (\omega, v)$.

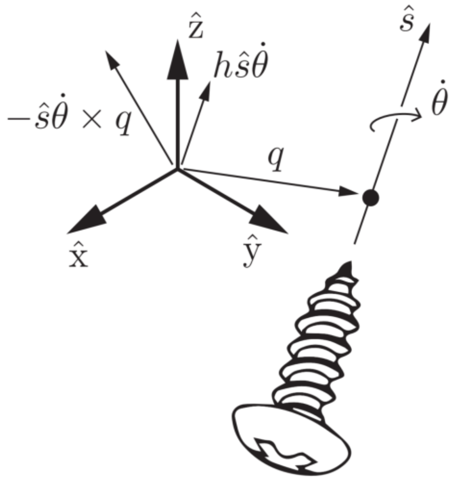
Screw Interpretation of a Twist

Example: find the screw axis $\mathcal{S}$ provided that

- $q = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \hat{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $h = 1$

$$\mathcal{V} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix} \dot{\theta} = \mathcal{S}\dot{\theta}$$

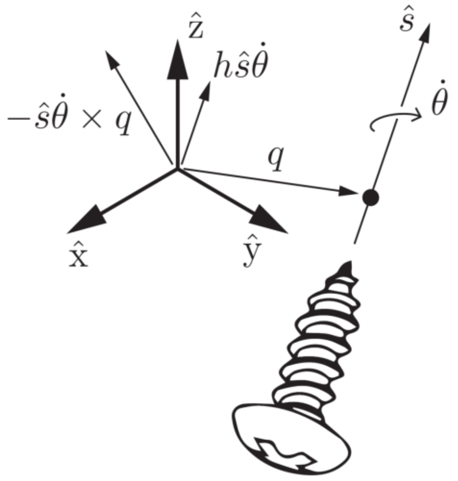
$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



Screw Interpretation of a Twist

Example: find velocity $\dot{\theta}$ and the screw axis $\mathcal{S}$ in terms of $\{q, \hat{s}, h\}$, provided that

- $\mathcal{V} = (0, 0, 1, 2, 1, 0)^\top$



Extract $\{q, \hat{s}, h\}$ from $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix} \dot{\theta} = \begin{bmatrix} \mathcal{S}_\omega \\ \mathcal{S}_v \end{bmatrix} \dot{\theta}$

- If $\omega \neq 0$, then $\dot{\theta} = \|\omega\|$, $\hat{s} = \mathcal{S}_\omega = \frac{\omega}{\|\omega\|}$, $\mathcal{S}_v = \frac{v}{\|\omega\|}$, $h = \frac{\mathcal{S}_\omega^\top \mathcal{S}_v}{\dot{\theta}}$
- If $\omega = 0$, then $\dot{\theta} = \|v\|$, $\hat{s} = \mathcal{S}_v = \frac{v}{\|v\|}$, $h = \infty$

Outline

1. Homogeneous Transformation Matrices

2. Twists

3. Exponential Coordinate Representation of Rigid-Body Motions

4. Wrenches

5. Homework

Exponential Coordinate Representation of Rigid-body Motions

By analogy to the rotation case, we can define a matrix exponential and a matrix logarithm for general rigid-body motions.

Chasles-Mozzi theorem: every rigid-body displacement can be expressed as a displacement along a fixed screw axis $\mathcal{S}$ in space.

Exponential Coordinate Representation	Lie Group	Lie Algebra
$\exp : [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$ $\log : R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$	R	$[\hat{\omega}]\theta = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} \theta$
$\exp : [\mathcal{S}]\theta \in se(3) \rightarrow T \in SE(3)$ $\log : T \in SE(3) \rightarrow [\mathcal{S}]\theta \in se(3)$	$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$	$[\mathcal{S}]\theta = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \theta$

Exponential Coordinate Representation of Rigid-body Motions

We can derive a closed-form expression for the matrix exponential $e^{[\mathcal{S}]\theta}$.

$$e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots$$

$$= I + \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}\theta + \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}^2\frac{\theta^2}{2!} + \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}^3\frac{\theta^3}{3!} + \cdots$$

$$= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots$$

We can apply the identity $[\omega]^3 = -[\omega]$ to simplify $G(\theta)$.

$$G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots$$

$$= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots\right)[\omega]^2$$

$$= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$$

Exponential Coordinate Representation of Rigid-body Motions

Matrix Exponential of Rigid-Body Motions

Let $\mathcal{S} = (\omega, v)$ be a (unit) screw axis.

- If $\|\omega\| = 1$, then for any distance $\theta \in \mathbb{R}$ traveled along the axis,

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right)v \\ 0 & 1 \end{bmatrix} \in SE(3)$$

- If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

Recall that the matrix exponential of rotations (a.k.a. Rodrigues' formula) is

$$e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3)$$

Exponential Coordinate Representation of Rigid-body Motions

<u>Matrix Logarithm of Rigid-Body Motions</u>

Given $T = (R, p) \in SE(3)$, find a $\theta \in [0, \pi]$ and a (unit) screw axis $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$ (where at least one of $\|\omega\|$ and $\|v\|$ is unity) such that $e^{[\mathcal{S}]\theta} = T$.

- The vector $\mathcal{S}\theta \in \mathbb{R}^6$ comprises the exponential coordinates for T.
- The matrix $[\mathcal{S}]\theta \in se(3)$ is the matrix logarithm of T.

(a) If $R = I$ then set $\omega = 0$, $v = \frac{p}{\|p\|}$, and $\theta = \|p\|$.

(b) Otherwise, use the matrix logarithm on $SO(3)$ to determine ω (written as $\hat{\omega}$ in the $SO(3)$ algorithm) and θ for R. Then v is calculated as

$$v = G^{-1}(\theta)p,$$

where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2.$$

Outline

1. Homogeneous Transformation Matrices
2. Twists
3. Exponential Coordinate Representation of Rigid-Body Motions
4. Wrenches
5. Homework

Wrenches

Recall in physics that a force f can create a torque or moment m at a point r:

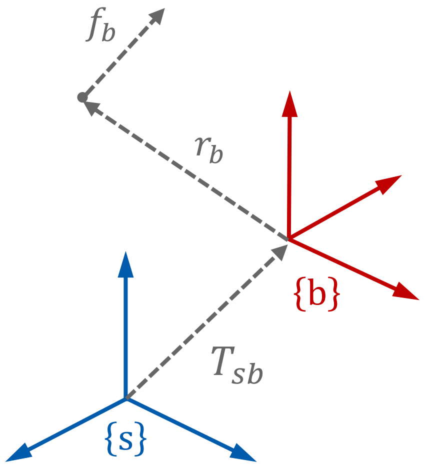
$$m = r \times f$$

As with twists, we can merge the moment and force into a single six-dimensional spatial force, or wrench, expressed in the body frame, $\mathcal{F}_b$:

$$\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} \in \mathbb{R}^6$$

Recall that the dot product of a force and a velocity is a power, and power is a coordinate-independent quantity. From this, we know that

$$\mathcal{V}_b^\top \mathcal{F}_b = \mathcal{V}_s^\top \mathcal{F}_s$$



Wrenches

From the previous lecture we know that $\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$.

With this, we can derive

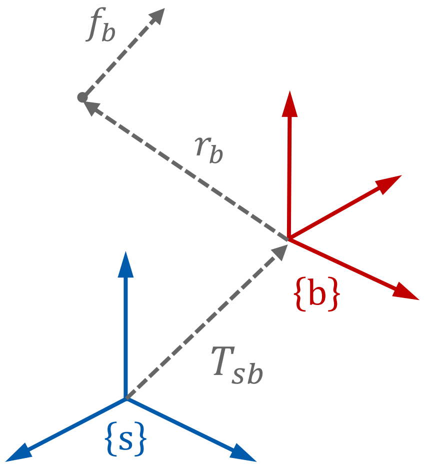
$$\begin{aligned}\mathcal{V}_b^\top \mathcal{F}_b &= \mathcal{V}_s^\top \mathcal{F}_s \\ &= \left([\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b\right)^\top \mathcal{F}_s \\ &= \mathcal{V}_b^\top [\mathrm{Ad}_{T_{sb}}]^\top \mathcal{F}_s\end{aligned}$$

Since this must hold for all $\mathcal{V}_b$, this simplifies to

$$\mathcal{F}_b = [\mathrm{Ad}_{T_{sb}}]^\top \mathcal{F}_s$$

Going the other way,

$$\mathcal{F}_s = [\mathrm{Ad}_{T_{bs}}]^\top \mathcal{F}_b$$



Outline

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Homework

(1) Which of the following matrices are transformation matrices? Explain why (not).

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad T_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad T_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) Textbook Exercises: 3.15(c), 3.16*, 3.17, 3.26, 3.27

(3) A supplementary video from 3Blue1Brown (recommended to watch): the geometric interpretation of matrix exponential

- https://www.bilibili.com/video/BV11y4y1b7c5/
- https://www.youtube.com/watch?v=O85OWBJ2ayo

* Hint about 3.16 (i), (j): You may use a calculator or any programming language/software to facilitate computation.

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Thanks for Listening！