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Foundations of Robotics

Lec 5: Forward Kinematics

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Outline

1. Forward Kinematics
2. Product of Exponentials (Space Frame)
3. Product of Exponentials (Body Frame)
4. Denavit–Hartenberg Parameters
5. Homework

Outline

Forward Kinematics

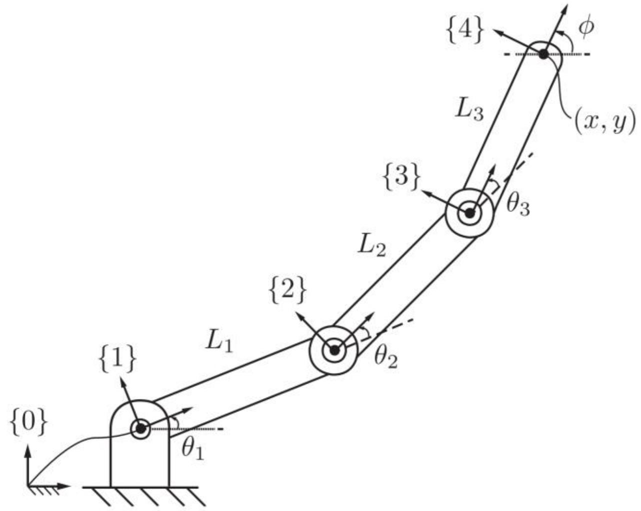
Consider a general n degree-of-freedom open chain.

Forward Kinematics

- $\theta \in \mathbb{R}^n \quad \rightarrow \quad T(\theta) \in SE(3)$
- Given joint coordinates θ, calculate the configuration (position and orientation) $T(\theta)$ of the end-effector frame.

Inverse Kinematics

- $T(\theta) \in SE(3) \quad \rightarrow \quad \theta \in \mathbb{R}^n$
- Given a homogeneous transform $X \in SE(3)$ and forward kinematics $T(\theta)$, find solutions θ that satisfy $T(\theta) = X$.



Forward Kinematics

Example: a 3R planar open-chain manipulator

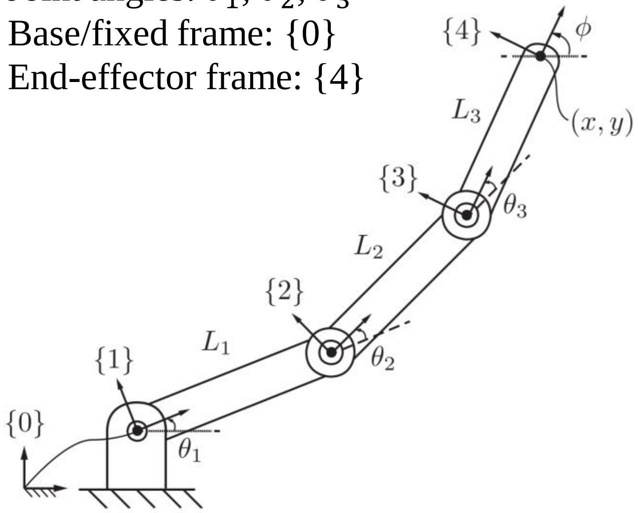
$$\theta \in \mathbb{R}^3 \quad \rightarrow \quad T(\theta) \in SE(2)$$

The position (x, y) and orientation ϕ of the end-effector frame as functions of the joint angles $(\theta_1, \theta_2, \theta_3)$ are given by

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3),$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3),$$
$$\phi = \theta_1 + \theta_2 + \theta_3.$$

Though this analysis can be done using only basic trigonometry, it can become considerably more complicated for general spatial chains.

Link lengths: L_1, L_2, L_3
Joint angles: θ_1, θ_2, θ_3
Base/fixed frame: {0}
End-effector frame: {4}



$$T(\theta) = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \in SE(2)$$

Forward Kinematics

A more systematic method? Attaching reference frames!

- Attach three link reference frames {1}, {2}, and {3}.*

The forward kinematics can then be written as a product of four homogeneous transformation matrices:

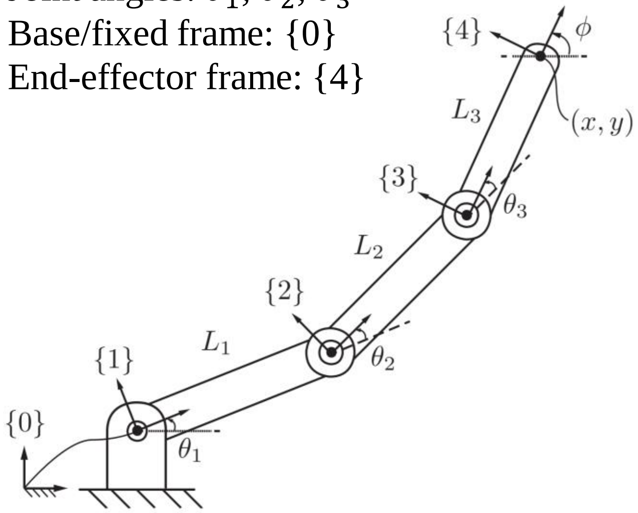
$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$

where

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{34} = \begin{bmatrix} 1 & 0 & L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Link lengths: L_1, L_2, L_3
Joint angles: θ_1, θ_2, θ_3
Base/fixed frame: {0}
End-effector frame: {4}



* Reference frames are attached to links (not joints!), though they can be located at joints where two links intersect. Each link should have its own frames, and joint angles can describe the change between two link frames.

Forward Kinematics

How about a general spatial open chain?

$$\theta \in \mathbb{R}^3 \quad \rightarrow \quad T(\theta) \in SE(3)$$

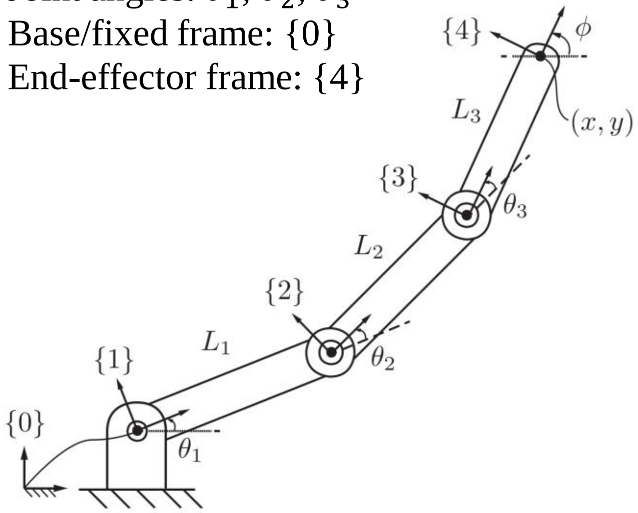
We need a more powerful, general and systematic method to compute T_{01}, T_{12}, T_{23}, and T_{34}.

Fortunately, we have learned the exponential coordinate representation in previous lectures ☺

Formulate the motion at each joint using screw axis $\mathcal{S}$, and the corresponding transformation matrix T can be obtained by $T = e^{[\mathcal{S}]\theta}$.

This approach is named Product of Exponentials (PoE).

Link lengths: L_1, L_2, L_3
Joint angles: θ_1, θ_2, θ_3
Base/fixed frame: {0}
End-effector frame: {4}



Outline

Product of Exponentials (Space Frame)

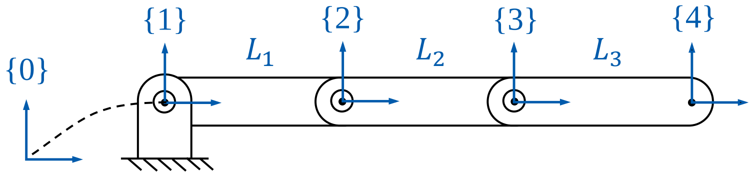
We start with the initial configuration where all joint angles are set to zero.

$$\theta_1 = \theta_2 = \theta_3 = 0$$

$$T_{04} = M$$

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(define M as the “home” transformation matrix)



Product of Exponentials (Space Frame)

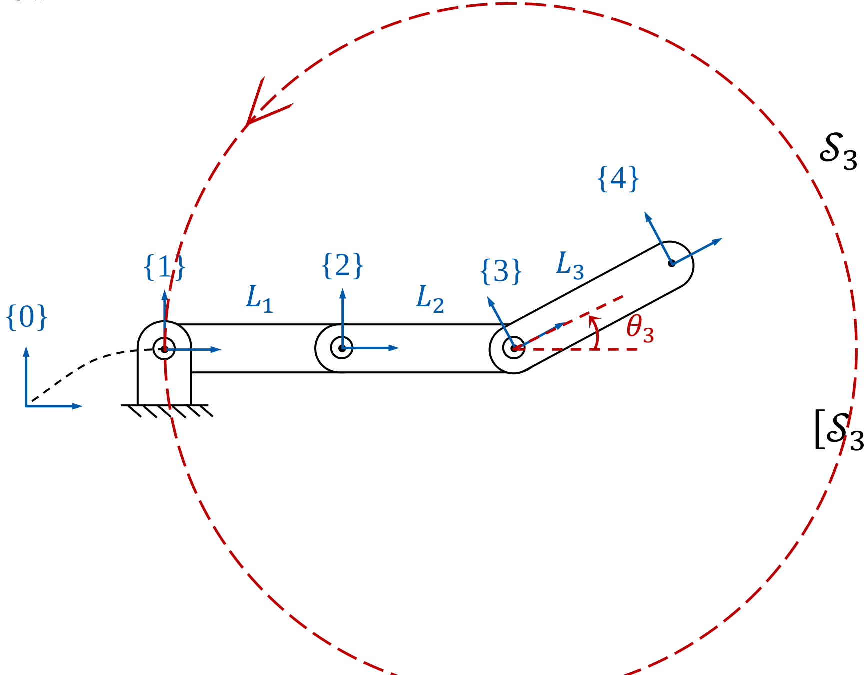
$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now consider the motion at each joint axis.

$$\theta_1 = \theta_2 = 0,\ \theta_3 \in S^1$$

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M$$

$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$ (recall this in $\{q, \hat{s}, h\}$ representation)



$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_3 &= -\omega_3 \times q_3 \\ &= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L_1 + L_2 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix} \end{aligned}$$

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega_3] & v_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

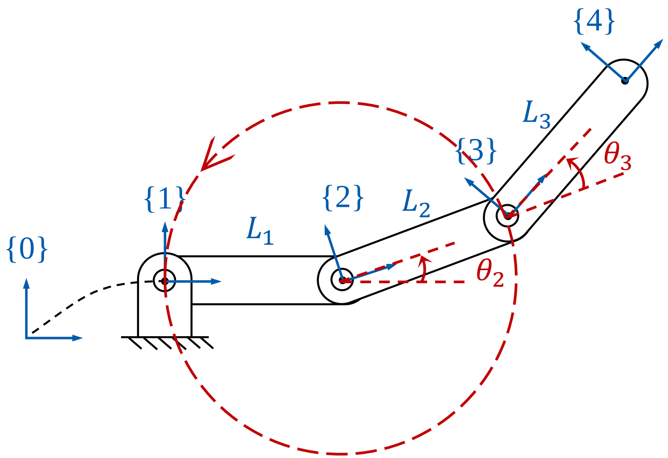
Product of Exponentials (Space Frame)

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now consider the motion at each joint axis.

$\theta_1 = 0,\ \theta_2 \in S^1,\ \theta_3 = \text{fixed}$

$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$



$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$ (recall this in $\{q, \hat{s}, h\}$ representation)

$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$$

$$v_2 = -\omega_2 \times q_2 = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

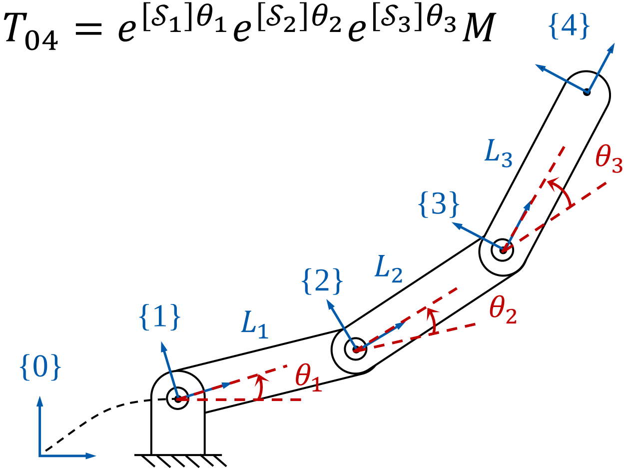
Product of Exponentials (Space Frame)

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now consider the motion at each joint axis.

$\theta_1 \in S^1$, $\theta_2 =$ fixed, $\theta_3 =$ fixed

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$



We found $T_{04}(\theta) \in SE(3)$ as the forward kinematics of this manipulator!

$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$ (recall this in $\{q, \hat{s}, h\}$ representation)

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = -\omega_1 \times q_1 = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

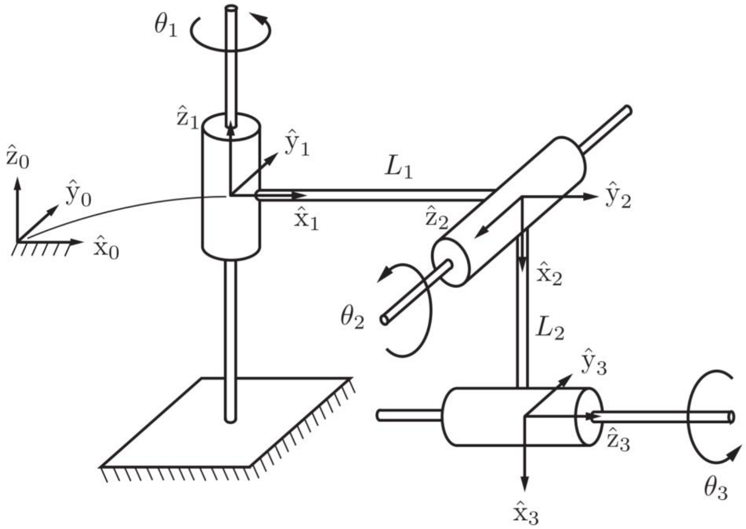
$$[\mathcal{S}_1] = \begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Product of Exponentials (Space Frame)

$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: A 3R spatial open chain

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$



$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, -1, 0)$	$(0, 0, -L_1)$
3	$(1, 0, 0)$	$(0, -L_2, 0)$

Outline

Product of Exponentials (Body Frame)

Mathematical equivalence of Product of Exponentials in space frame and in body frame.

$$
\begin{aligned}
T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} M M^{-1} e^{[\mathcal{S}_n]\theta_n} M \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} M e^{M^{-1}[\mathcal{S}_n]\theta_n M} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} M e^{[\mathcal{B}_n]\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n} \\
&= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}
\end{aligned}
$$

Add $I = MM^{-1}$

Recall property $e^{M^{-1}PM} = M^{-1}e^{P}M$

θ_n is a scalar variable $\Rightarrow$ $\theta_n M = M\theta_n$

Denote $\mathcal{B}$ as the screw axis in body frame, then $\mathcal{B}_i = \left[\mathrm{Ad}_{M^{-1}}\right]\mathcal{S}_i$ or $[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M$

Repeat for all screw axes

Product of Exponentials (Body Frame)

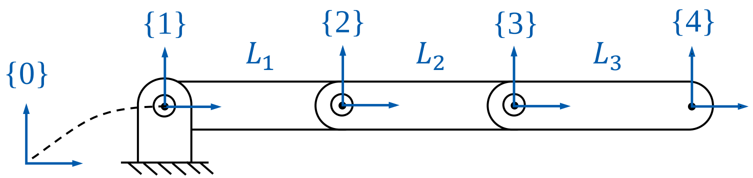
We start with the initial configuration where all joint angles are set to zero.

$$\theta_1 = \theta_2 = \theta_3 = 0$$

$$T_{04} = M$$

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(define M as the “home” transformation matrix)



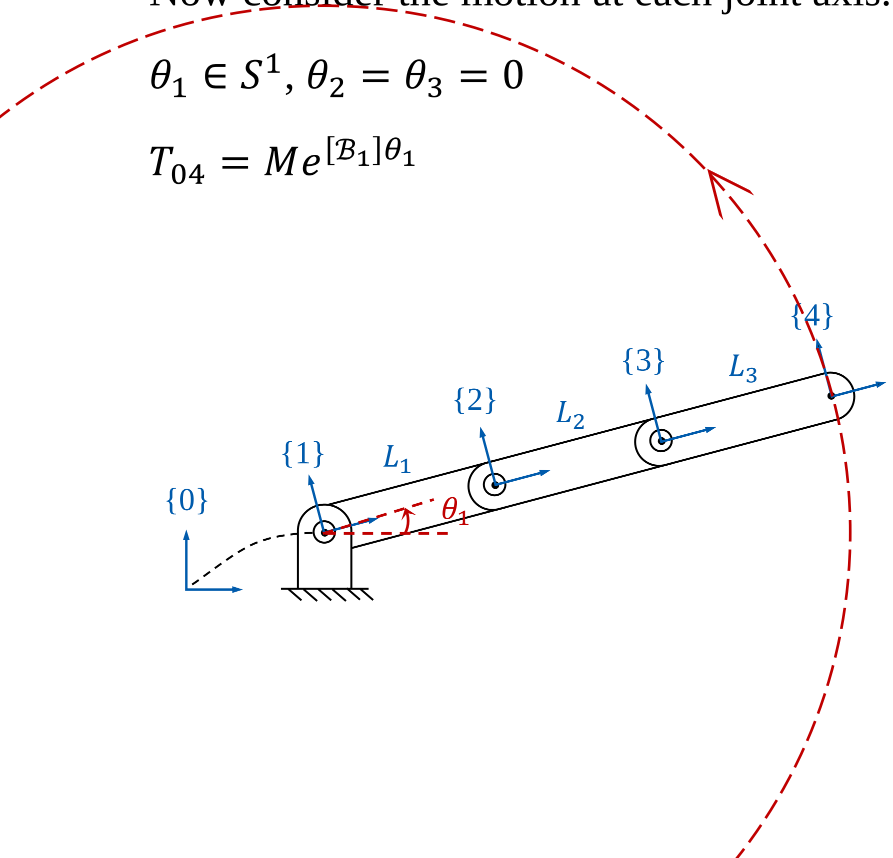
Product of Exponentials (Body Frame)

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now consider the motion at each joint axis.

$\theta_1 \in S^1, \theta_2 = \theta_3 = 0$

$T_{04} = Me^{[\mathcal{B}_1]\theta_1}$



$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$ (recall this in $\{q, \hat{s}, h\}$ representation)

$$\mathcal{B}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix}$$

$$v_1 = -\omega_1 \times q_1$$
$$= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -(L_1 + L_2 + L_3) \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix}$$

$$[\mathcal{B}_1] = \begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

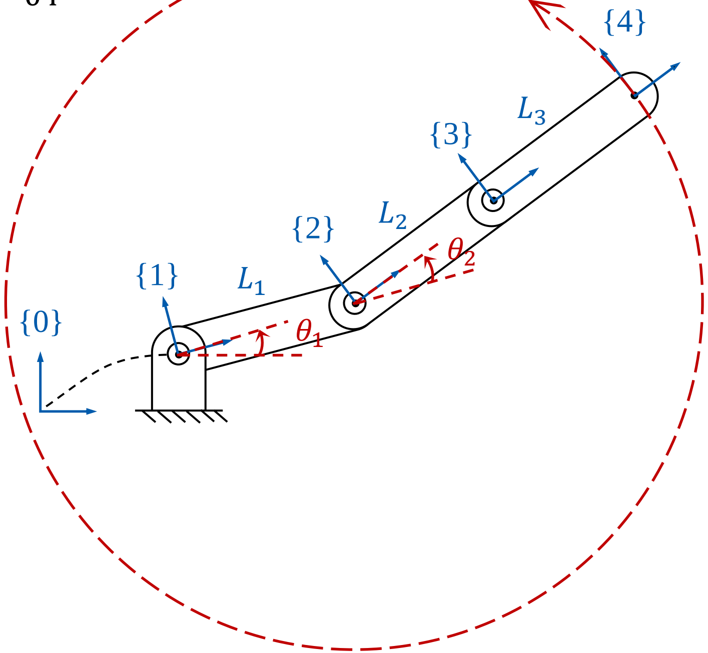
Product of Exponentials (Body Frame)

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now consider the motion at each joint axis.

$\theta_1 = \text{fixed}, \theta_2 \in S^1, \theta_3 = 0$

$T_{04} = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}$



$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$ (recall this in $\{q, \hat{s}, h\}$ representation)

$$\mathcal{B}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_2 + L_3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_2 &= -\omega_2 \times q_2 \\ &= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -(L_2 + L_3) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ L_2 + L_3 \\ 0 \end{bmatrix} \end{aligned}$$

$$[\mathcal{B}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & L_2 + L_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

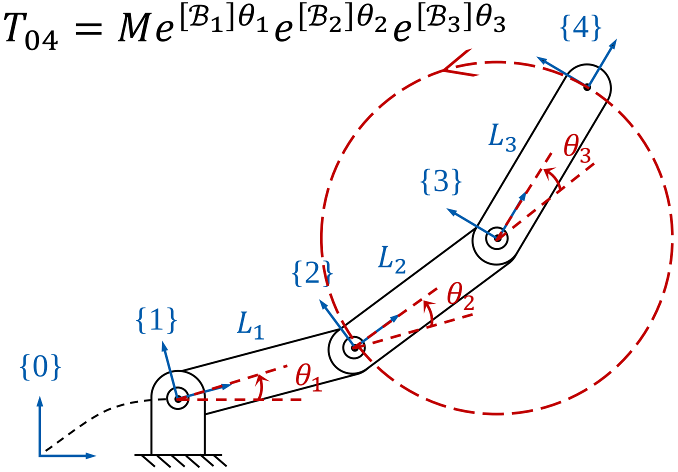
Product of Exponentials (Body Frame)

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now consider the motion at each joint axis.

$\theta_1 = \text{fixed}, \theta_2 = \text{fixed}, \theta_3 \in S^1$

$T_{04} = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}e^{[\mathcal{B}_3]\theta_3}$



$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$ (recall this in $\{q, \hat{s}, h\}$ representation)

$$\mathcal{B}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_3 &= -\omega_3 \times q_3 \\ &= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -L_3 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ L_3 \\ 0 \end{bmatrix} \end{aligned}$$

$$[\mathcal{B}_3] = \begin{bmatrix} [\omega_3] & v_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & L_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

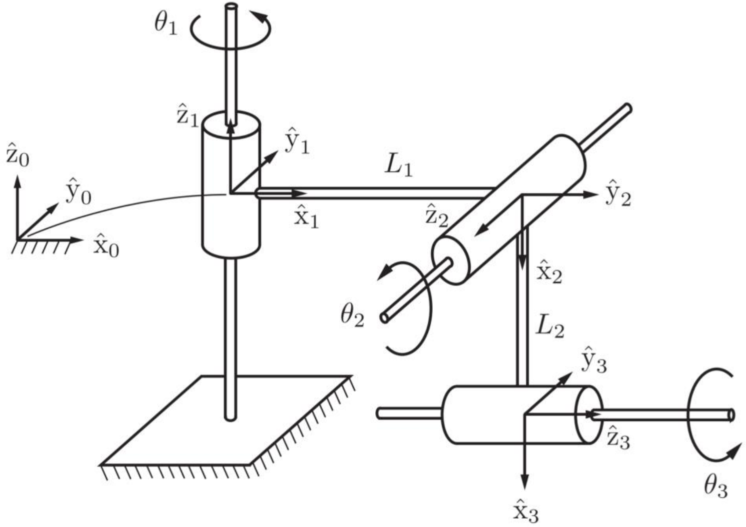
Product of Exponentials (Body Frame)

$$\mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: A 3R spatial open chain

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \ldots e^{[\mathcal{B}_n]\theta_n}$$



$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(-1, 0, 0)$	$(0, L_1, 0)$
2	$(0, -1, 0)$	$(0, 0, L_2)$
3	$(0, 0, 1)$	$(0, 0, 0)$

Outline

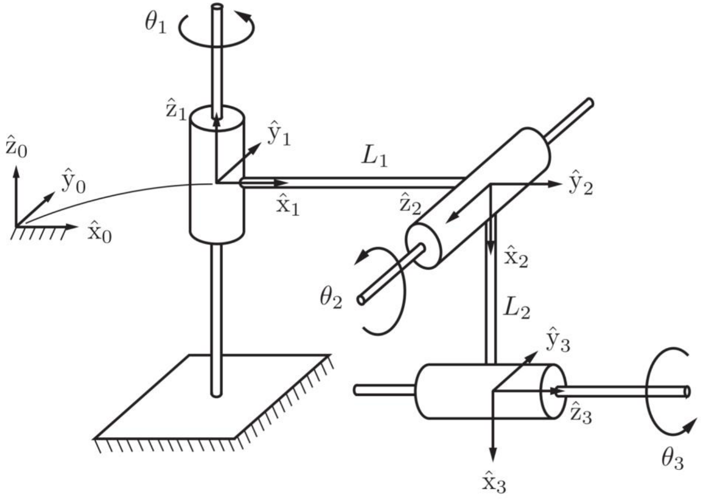
Denavit–Hartenberg Parameters

Recall the problem formulation of forward kinematics:

$$\theta \in \mathbb{R}^n \quad \rightarrow \quad T(\theta) \in SE(3)$$

Written as the product of transformation matrices:

$$T(\theta) = T_{0n} = T_{01}T_{12} \cdots T_{n-1,n}$$



Product of Exponentials

- $T_{0n} = e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_n]\theta_n} M$
- No need to assign frames (no dependence)
- The product of exponentials of each screw axis

D-H Parameters

- $T_{0n} = T_{01}(\theta_1)T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n)$
- Assign frames according to <u>specific rules</u>
- Use <u>four</u> parameters to represent each $T_{i-1,i}(\theta_i)$

Denavit–Hartenberg Parameters

D-H Parameters

- Four parameters associated with a particular convention for attaching reference frames.
- Introduced by Jacques Denavit and Richard Hartenberg in 1955.
- Four parameters are proven to be sufficient and minimal.

Why sufficient?

- By taking advantage of specific rules used in frame assignment (reduced unknowns about the robot's structure).
- D-H parameters cannot describe any transformation between two arbitrary frames (instead, only a subset that follows the D-H convention to assign frames).

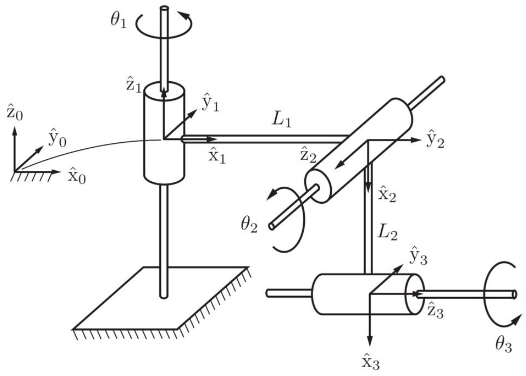
How to use? Two basic steps.

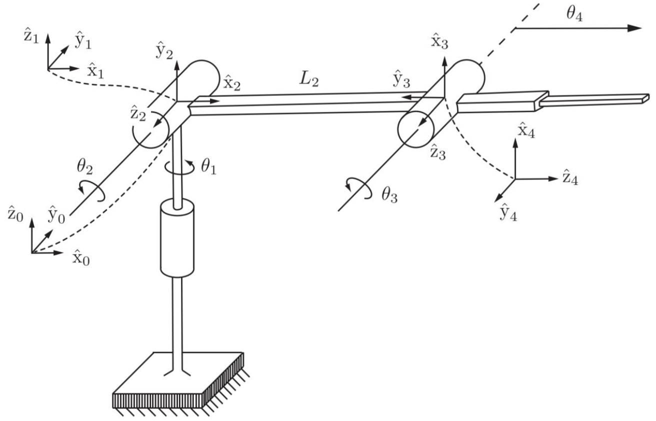
1) Assign frames according to specific rules.
2) Find four parameters between every two frames.

Denavit–Hartenberg Parameters

Step one: assign frames according to specific rules

1) Assign $\hat{z}$ axes for all frames
   - Align with each screw axis
   - Revolute: positive direction by right-hand rule
   - Prismatic: positive direction along translation
2) Determine the origin of each frame
   - At the intersection of screw $i-1$ and mutual perpendicular line (if exists)
   - At the intersection of screw $i-1$ and screw i
3) Assign $\hat{x}$ axes for all frames
   - Along the mutual perpendicular line (if exists)
   - Always perpendicular to screw $i-1$ and screw i (may have two options to pick $\hat{x}_{i-1}$, both valid)
4) Determine $\hat{y}$ axes by right-hand rule





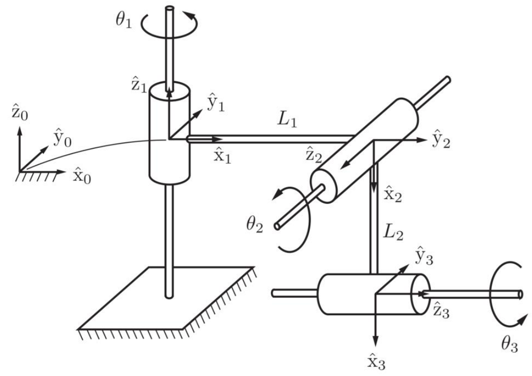
Denavit–Hartenberg Parameters

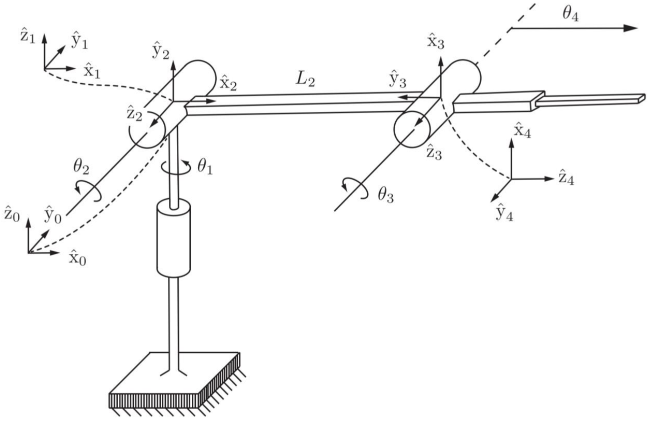
Step two: find four parameters between every two frames

- Once the frame is assigned, there is only one set of parameters that could be correct.
- If there is no way to move from frame $\{i-1\}$ to frame $\{i\}$ by the four tentative parameters, go back and double check frame assignment.
- Keep in mind the physical meaning of four parameters.

$$T_{i-1,i} = \mathrm{Rot}(\hat{x}, \alpha_{i-1})\ \mathrm{Trans}(\hat{x}, a_{i-1})\ \mathrm{Trans}(\hat{z}, d_i)\ \mathrm{Rot}(\hat{z}, \phi_i)$$

$$= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i \cos\alpha_{i-1} & \cos\phi_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i \sin\alpha_{i-1} \\ \sin\phi_i \sin\alpha_{i-1} & \cos\phi_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Denavit–Hartenberg Parameters

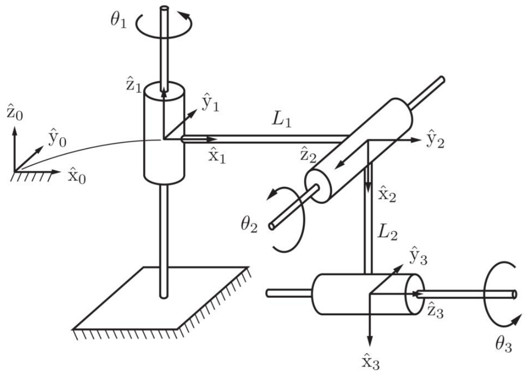
$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\ \mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\ \mathrm{Trans}(\hat{\mathrm{z}}, d_i)\ \mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$$

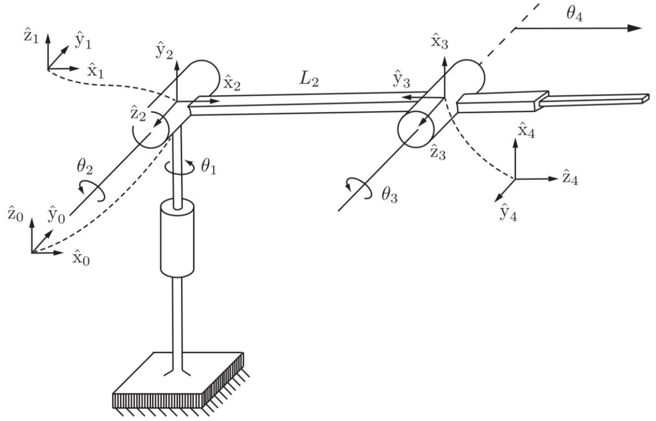
$$\mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

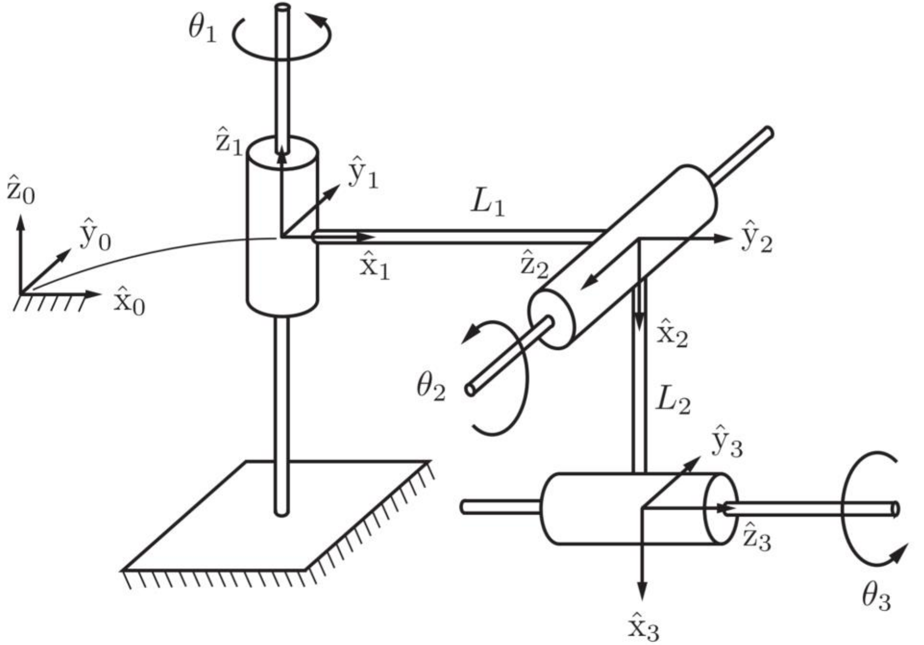
$$\mathrm{Trans}(\hat{\mathrm{z}}, d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i) = \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & 0 \\ \sin\phi_i & \cos\phi_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



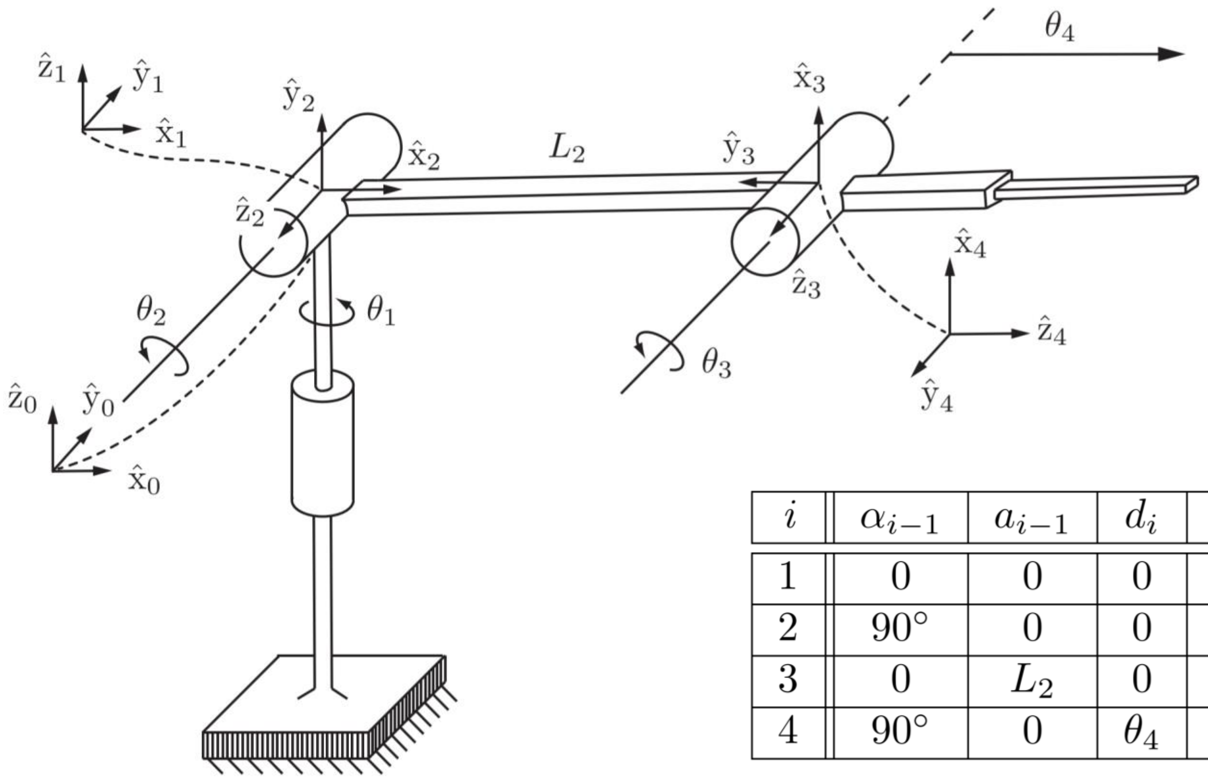


Denavit–Hartenberg Parameters



i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	L_1	0	$\theta_2 - 90°$
3	−90°	L_2	0	θ_3

Denavit–Hartenberg Parameters



i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	0	0	θ_2
3	0	L_2	0	$\theta_3 + 90°$
4	$90°$	0	θ_4	0

Summary

Forward Kinematics: $\theta \in \mathbb{R}^n \rightarrow T(\theta) \in SE(3)$

Product of Exponentials

- No dependence on frame assignment; find exponentials of screw axes only
- In space frame: $T(\theta) = e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$
- In body frame: $T(\theta) = M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \ldots e^{[\mathcal{B}_n]\theta_n}$

D-H Parameters

- A minimal number of parameters associated with a convention for attaching frames
- $T_{0n} = T_{01}(\theta_1) T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n)$
- $T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\ \mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\ \mathrm{Trans}(\hat{\mathrm{z}}, d_i)\ \mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$

For an open chain with n one-dof joints:

- PoE approach requires $7n$ parameters (n for joint variables, $6n$ for screw axes)
- D-H approach requires $4n$ parameters (n for joint variables, $3n$ for robot's structure)

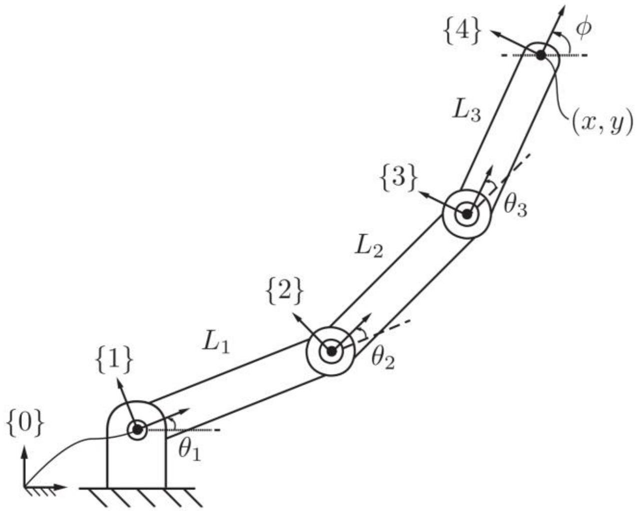
Outline

1. Forward Kinematics
2. Product of Exponentials (Space Frame)
3. Product of Exponentials (Body Frame)
4. Denavit–Hartenberg Parameters
5. Homework

Homework

(1) Textbook Exercises: 4.7 and 4.9 (for PoE)

(2) Derive D-H parameters for this 3R open chain and the mechanisms in exercises 4.7 and 4.9



(3) Lab Assignments: Forward kinematics

- Provided the schematic of a 3R manipulator, write two scripts (in C++ or Python) to implement PoE and D-H approaches respectively to compute the forward kinematics.
- The scripts will be tested using a few test cases (input: joint variables; expected output: the position of the end effector)

Thanks for Listening！