Foundations of Robotics

Lec 3: Rigid-Body Motions (Rotation)

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Outline

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Rigid-Body Motions

In the previous lecture, we studied two fundamental properties of a robot's C-space:

- its dimension, or the number of degrees of freedom;
- the shape of the space, or the topology of the space.

In order to describe a rigid body's position and orientation systematically, we would need to attach a reference frame to the body and study the mathematical descriptions for rotations (this lecture) and transformations (next lecture) respectively.

We explore the answers to the following (example) questions:

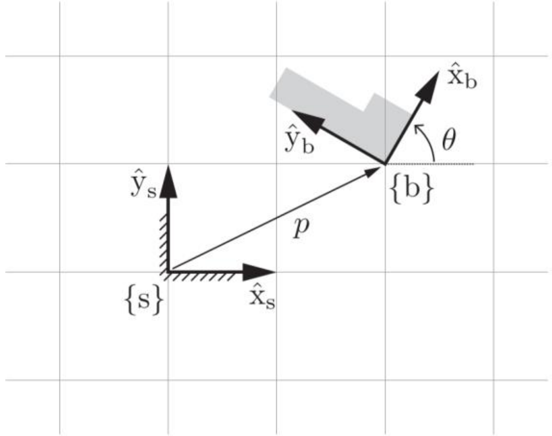
- How can we describe the pose* of the tip of a manipulator with respect to its base?
- How can we describe the displacement* of the tip of a manipulator with respect to its base?
- If viewing from the tip of the manipulator, how to describe the base?

* pose = position + orientation (位姿 = 位置 + 姿态/朝向)
displacement = translation + rotation (位移 = 平移 + 旋转)

Rigid-Body Motions

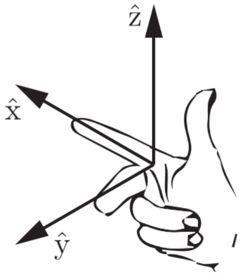
Notations

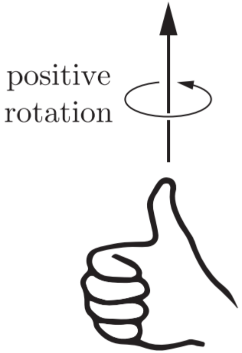
- v – free vector
- v – vector in a coordinate system
- $\hat{\mathrm{v}}$ – unit vector (hat notation)
- {b} – body frame
- {s} – space frame or fixed frame



Conventions

- Always assume that a length scale for physical space has been chosen.
- All frames in this book/lecture are <u>stationary</u>, <u>inertial</u> frames!
- All frames are right-handed!



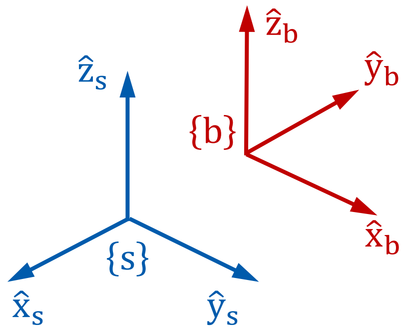


Rigid-Body Motions

Example: how to describe the position and orientation of {b} relative to {s}?

Let p denote the vector from {s} origin to {b} origin.
In terms of the {s} coordinates, p can be expressed as

$$p = p_1\hat{\mathrm{x}}_\mathrm{s} + p_2\hat{\mathrm{y}}_\mathrm{s} + p_3\hat{\mathrm{z}}_\mathrm{s}$$



The axes of frame {b} can also be expressed as

$$\begin{aligned}
\hat{\mathrm{x}}_\mathrm{b} &= r_{11}\hat{\mathrm{x}}_\mathrm{s} + r_{21}\hat{\mathrm{y}}_\mathrm{s} + r_{31}\hat{\mathrm{z}}_\mathrm{s}\\
\hat{\mathrm{y}}_\mathrm{b} &= r_{12}\hat{\mathrm{x}}_\mathrm{s} + r_{22}\hat{\mathrm{y}}_\mathrm{s} + r_{32}\hat{\mathrm{z}}_\mathrm{s}\\
\hat{\mathrm{z}}_\mathrm{b} &= r_{13}\hat{\mathrm{x}}_\mathrm{s} + r_{23}\hat{\mathrm{y}}_\mathrm{s} + r_{33}\hat{\mathrm{z}}_\mathrm{s}
\end{aligned}$$

Define $p \in \mathbb{R}^3$ and $R \in \mathbb{R}^{3\times 3}$ as

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad R = [\hat{\mathrm{x}}_\mathrm{b} \;\; \hat{\mathrm{y}}_\mathrm{b} \;\; \hat{\mathrm{z}}_\mathrm{b}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

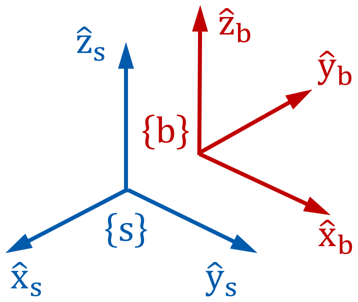
Example values:

$$p = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Outline

Rotation Matrices

Recall that the three columns of R correspond to the body frame unit axes $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$. The following constraints must be satisfied.



(1) The unit norm condition: $\hat{\mathrm{x}}_\mathrm{b}$, $\hat{\mathrm{y}}_\mathrm{b}$ and $\hat{\mathrm{z}}_\mathrm{b}$ are all unit vectors

$$
\begin{aligned}
r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\
r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\
r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1
\end{aligned}
$$

$$
R = [\hat{\mathrm{x}}_\mathrm{b} \ \ \hat{\mathrm{y}}_\mathrm{b} \ \ \hat{\mathrm{z}}_\mathrm{b}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

(2) The orthogonality condition: $\hat{\mathrm{x}}_\mathrm{b} \cdot \hat{\mathrm{y}}_\mathrm{b} = \hat{\mathrm{x}}_\mathrm{b} \cdot \hat{\mathrm{z}}_\mathrm{b} = \hat{\mathrm{y}}_\mathrm{b} \cdot \hat{\mathrm{z}}_\mathrm{b} = 0$

$$
\begin{aligned}
r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0 \\
r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} &= 0 \\
r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} &= 0
\end{aligned}
$$

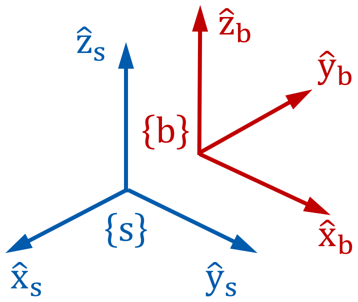
These six constraints can be expressed more compactly as

$$R^\top R = I$$ → Orthogonal Matrices!

Rotation Matrices

Recall that frames are right-handed ($\hat{\mathrm{x}}_\mathrm{b} \times \hat{\mathrm{y}}_\mathrm{b} = \hat{\mathrm{z}}_\mathrm{b}$) rather then left-handed ($\hat{\mathrm{x}}_\mathrm{b} \times \hat{\mathrm{y}}_\mathrm{b} = -\hat{\mathrm{z}}_\mathrm{b}$). This implies an additional constraint.

$$\det R = 1$$



It comes from the formula for evaluating the determinant of a 3×3 matrix M, where we substitute the columns for R into the formula.

$$\det M = a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$$

$$R = [\hat{\mathrm{x}}_\mathrm{b} \ \hat{\mathrm{y}}_\mathrm{b} \ \hat{\mathrm{z}}_\mathrm{b}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the constraint $\det R = 1$ does not change the number of <u>independent</u> continuous variables needed to parameterize R. (In other words, the number of independent constraints for R is still six.)

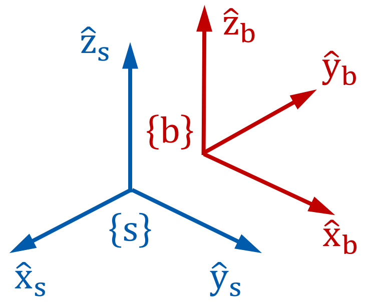
Now we are ready to introduce the definitions of Special Orthogonal Groups $SO(3)$ and $SO(2)$.

Rotation Matrices

Definition: The Special Orthogonal Group $SO(3)$, a.k.a. the group of rotation matrices, is the set of all 3×3 real matrices R that satisfy

(1) $R^\top R = I$ → Orthogonal

(2) $\det R = 1$ → Special



Definition: The Special Orthogonal Group $SO(2)$ is the set of all 2×2 real matrices R that satisfy

(1) $R^\top R = I$

(2) $\det R = 1$

$$R = [\hat{x}_b \ \hat{y}_b \ \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the definition it follows that every $R \in SO(2)$ can be written as

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta \in [0, 2\pi)$.

Rotation Matrices

Examples of $R \in SO(3)$ about coordinate frame axes are

$$\mathrm{Rot}(\hat{\mathrm{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \mathrm{Rot}(\hat{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

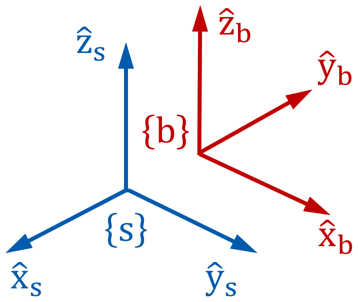
More generally, as we will see later in the lecture, for $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$,

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Note that $\mathrm{Rot}(\hat{\omega}, \theta) = \mathrm{Rot}(-\hat{\omega}, -\theta)$.

Properties of Rotation Matrices

Group Axioms
- Closure: $R_1 R_2 \in SO(n)$
- Associativity: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
- Identity element: exists I such that $RI = IR = R$
- Inverse element: exists R^{-1} such that $RR^{-1} = R^{-1}R = I$



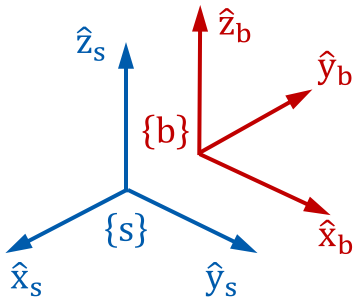
Propositions
- $R^{-1} = R^\top$ (from orthogonal matrices)
- $R_1 R_2 \neq R_2 R_1$ (generally not commutative, except for $SO(2)$)
- $\|Rx\| = \|x\|$ (length preserving)

$$R = [\hat{\mathrm{x}}_\mathrm{b} \;\; \hat{\mathrm{y}}_\mathrm{b} \;\; \hat{\mathrm{z}}_\mathrm{b}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The $SO(n)$ groups are also called <u>matrix Lie groups</u> because the elements of the group form a differentiable manifold.

Uses of Rotation Matrices

There are a few uses for a rotation matrix R. In summary, it can be thought of as a representation, or as an operator.



As a representation, it can

(1) represent an orientation.

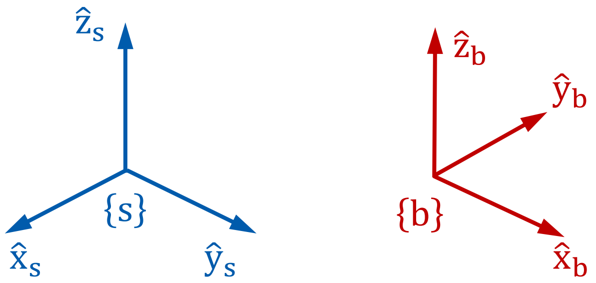
As an operator, it can

(2) change the reference frame (of a vector or a frame),
(3) rotate a vector or a frame (in its current frame).

$$R = [\hat{x}_b \ \hat{y}_b \ \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Uses of Rotation Matrices

(1) Represent an orientation



$$\hat{x}_b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{y}_b = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{z}_b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_{sb} = [\hat{x}_b \ \hat{y}_b \ \hat{z}_b] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

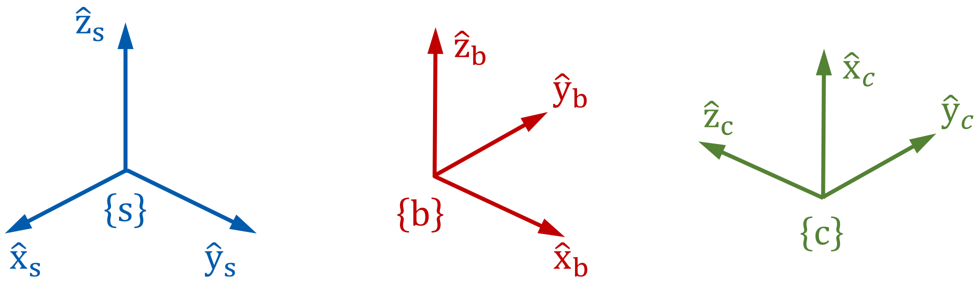
$$R_{bs} = R_{sb}^{-1} = R_{sb}^{\top} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Subscript convention: we refer to the orientation of frame $\{b\}$ relative to frame $\{s\}$ as R_b (implicitly) or R_{sb} (explicitly).

- Recall one of the properties of rotation matrices $R^{-1} = R^{\top}$.

Uses of Rotation Matrices

(2) Change the reference frame (of ~~a vector or~~ a frame)



$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{bc} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Subscript cancellation: for matrix multiplication

$$R_{ab}R_{bc}R_{cd}R_{de} = R_{a\cancel{b}}R_{\cancel{bc}}R_{\cancel{cd}}R_{\cancel{d}e} = R_{ae}$$

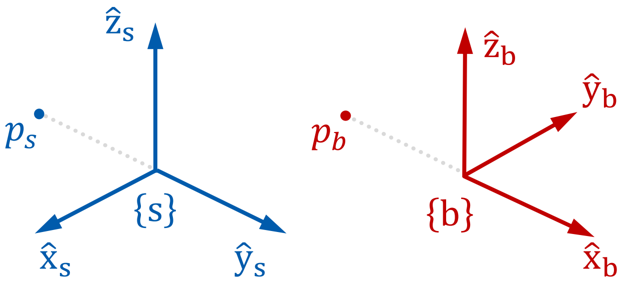
$$R_{sc} = R_{sb}R_{bc} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Known $R_{sb}, R_{bc} \Longrightarrow R_{sc} = R_{sb}R_{bc}$

Known $R_{sb}, R_{cb} \Longrightarrow R_{sc} = R_{sb}R_{bc} = R_{sb}\, R_{cb}^{-1} = R_{sb}\, R_{cb}^{\top}$

Uses of Rotation Matrices

(2) Change the reference frame (of a vector ~~or a frame~~)



$$p_b = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Subscript cancellation: for a vector multiplied by a matrix

$$R_{ab}p_b = R_{a\cancel{b}}p_{\cancel{b}} = p_a$$

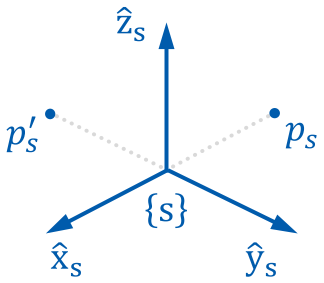
Known $R_{sb}, p_b \Longrightarrow p_s = R_{sb}p_b$

Known $R_{bs}, p_b \Longrightarrow p_s = R_{sb}p_b = R_{bs}^{-1}p_b = R_{bs}^{\top}p_b$

$$p_s = R_{sb}p_b = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Uses of Rotation Matrices

(3) Rotate a vector ~~or a frame~~ (in its current frame)



$$p_s = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$R = \mathrm{Rot}(\hat{z}, 90°) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

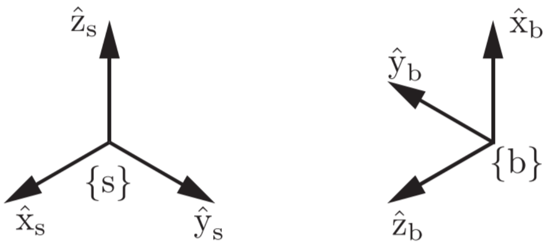
- $\mathrm{Rot}(\hat{\omega}, \theta)$: a rotation of angle θ about unit axis $\hat{\omega}$
- Subscript cancellation does not apply.

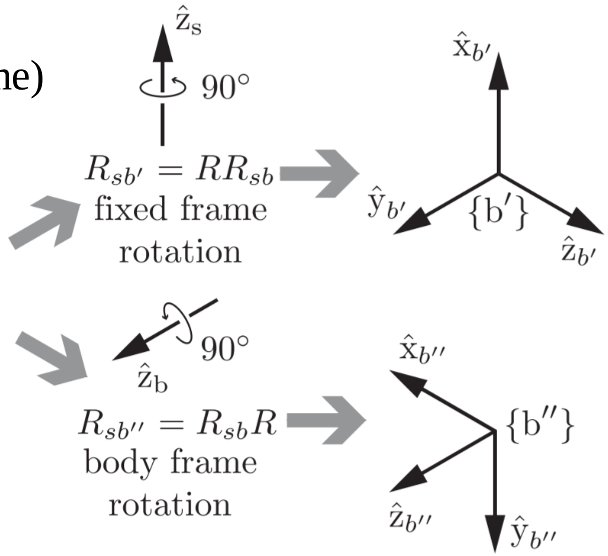
$$p'_s = Rp_s = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Known $R, p_s \Longrightarrow p'_s = Rp_s$

Uses of Rotation Matrices

(3) Rotate ~~a vector or~~ a frame (in its current frame)





$$R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R = \mathrm{Rot}(\hat{z}, 90°) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

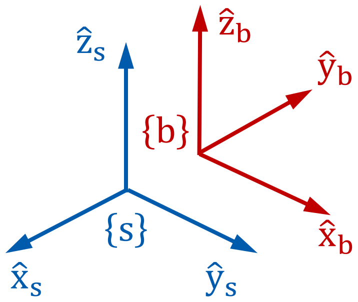
- Pre-multiply by R: rotate R_{sb} about $\hat{z}$ in frame {s}
- Post-multiply by R: rotate R_{sb} about $\hat{z}$ in frame {b}*

* The rotation axis $\hat{z}$ is considered to be in frame {b}, but the rotated frame (the result) is still represented/referenced in frame {s}. It also follows our convention of subscript: both $R_{sb'}$ and $R_{sb''}$ have s as the first subscript.

Uses of Rotation Matrices

Summary of the uses of rotation matrices

(1) represent an orientation
(2) change the reference frame (of a vector or a frame)
(3) rotate a vector or a frame (in its current frame)



Be careful about your wording!

- The orientation of {b} relative to {s}
- The rotation from {s} to {b}*
  - ➢ Representation or change its frame?

$$R = [\hat{x}_b \ \hat{y}_b \ \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Be very clear in your mind about two questions below:

- Is this matrix R_{sb} or R_{bs}?
- Is it pre-multiplying or post-multiplying?

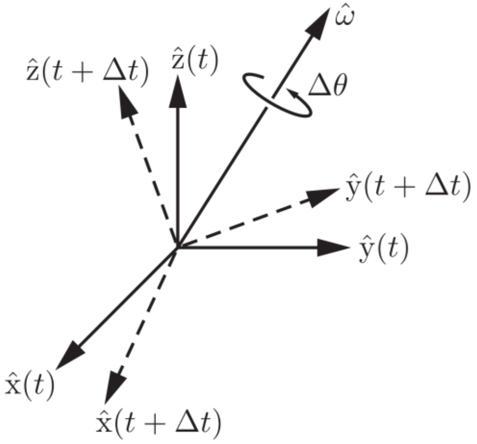
* In the literature, you may have seen many expressions like this, which can be ambiguous.

Outline

Angular Velocities

Suppose that a frame with unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ is attached to a rotating body.

If we examine the body frame at times t and $t + \Delta t$, the change in frame orientation can be described as a rotation of angle θ about some unit axis $\widehat{\mathrm{w}}$ passing through the origin.



Define $\dot{\theta} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$ as the rate of rotation, and $\widehat{\mathrm{w}}$ the instantaneous axis of rotation.

Define **angular velocity** w* as

$$\mathrm{w} = \widehat{\mathrm{w}}\dot{\theta}$$

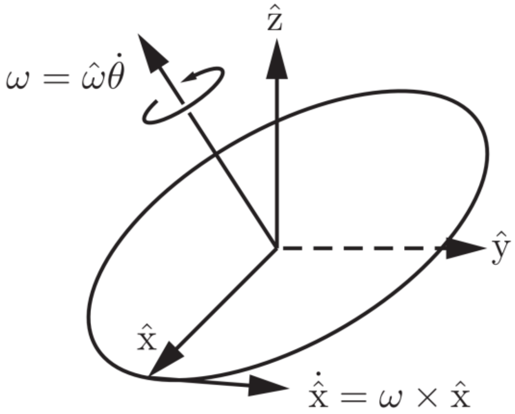
→ Only three parameters!

* Note that the axis w is coordinate-free; it is not yet represented in any particular reference frame.

Angular Velocities

To express in coordinates, we have to choose a reference frame.*

Let us start with {s} and let $\omega_s \in \mathbb{R}^3$ be the angular velocity w expressed in {s}.



The time rate of change in each axis is

$$\dot{\hat{x}} = \omega_s \times \hat{x}$$
$$\dot{\hat{y}} = \omega_s \times \hat{y}$$
$$\dot{\hat{z}} = \omega_s \times \hat{z}$$

Denote x, y, z as r_1, r_2, r_3 respectively, then

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3$$

Rearrange into the following matrix form

$$\dot{R} = [\omega_s \times r_1 \quad \omega_s \times r_2 \quad \omega_s \times r_3] = \omega_s \times R$$

We derived $\dot{R}$, the time rate of change of R!

* We can choose any reference frame, but two natural choices are {s} and {b}.

Angular Velocities

To eliminate the cross product, we introduce a new notation to rewrite $\omega_s \times R$ as $[\omega_s]R$.

Given a vector $x = [x_1\ x_2\ x_3]^\top \in \mathbb{R}^3$, define

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The matrix $[x]$ is a 3×3 skew-symmetric matrix representation of x; that is

$$[x] = -[x]^\top$$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$; it is the Lie algebra of the Lie group $SO(3)$. It consists of all possible $\dot{R}$ when $R = I$.

With this bracket notation*, we can rewrite $\dot{R} = \omega_s \times R$ as $\dot{R} = [\omega_s]R$.

* In some literature, it is also referred to as hat operator (i.e. a × b = âb).

Properties of Angular Velocities

Proposition: For any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, we have $R[\omega]R^\top = [R\omega]$.

Proof:

$$R[\omega]R^\top = \begin{bmatrix} r_1^\top(\omega \times r_1) & r_1^\top(\omega \times r_2) & r_1^\top(\omega \times r_3) \\ r_2^\top(\omega \times r_1) & r_2^\top(\omega \times r_2) & r_2^\top(\omega \times r_3) \\ r_3^\top(\omega \times r_1) & r_3^\top(\omega \times r_2) & r_3^\top(\omega \times r_3) \end{bmatrix} = \begin{bmatrix} 0 & -r_3^\top\omega & r_2^\top\omega \\ r_3^\top\omega & 0 & -r_1^\top\omega \\ -r_2^\top\omega & r_1^\top\omega & 0 \end{bmatrix} = [R\omega]$$

Now let ω_b be the angular velocity w expressed in {b}, and we write R explicitly as R_{sb}. Let us find the relations between {s} and {b}.

- $\omega_s = R_{sb}\omega_b$
- $\omega_b = R_{bs}\omega_s = R_{sb}^{-1}\omega_s = R_{sb}^\top\omega_s$
- $\dot{R}_{sb} = [\omega_s]R_{sb} \quad \Longrightarrow \quad [\omega_s] = \dot{R}_{sb}R_{sb}^{-1} = \dot{R}_{sb}R_{sb}^\top$
- $[\omega_b] = \left[R_{sb}^\top\omega_s\right] = R_{sb}^\top[\omega_s]R_{sb} = R_{sb}^\top\left(\dot{R}_{sb}R_{sb}^\top\right)R_{sb} = R_{sb}^\top\dot{R}_{sb}$

We refer to ω as the vector representation of angular velocity w, and refer to $[\omega]$ as the matrix representation of angular velocity w.

Outline

1. Rigid-Body Motions
2. Rotation Matrices
3. Angular Velocities
4. **Exponential Coordinate Representation of Rotation**
5. Homework

Exponential Coordinate Representation of Rotation

We now introduce a three-parameter representation for rotations. (Finally!!)

The **exponential coordinates** parameterize $R \in SO(3)$ in terms of a rotation axis $\hat{\omega}$ (unit vector) and an angle of rotation θ.

- The vector $\hat{\omega}\theta \in \mathbb{R}^3$ serves as the **exponential coordinate** representation of the rotation.
- Writing $\hat{\omega}$ and θ individually is the **axis-angle** representation of a rotation.

four parameters!

Three interpretations for $\hat{\omega}\theta$

1)	the axis $\hat{\omega}$ and rotation angle θ		were rotated by θ about $\hat{\omega}$,	
2)	the angular velocity $\hat{\omega}\theta$ expressed in {s}	such that, if a frame initially coincident with {s}	followed $\hat{\omega}\theta$ for one unit of time,	its final orientation relative to {s} would be expressed by R.
3)	the angular velocity $\hat{\omega}$ expressed in {s}		followed $\hat{\omega}$ for θ unit of time,	

Exponential Coordinate Representation of Rotation

Let us further investigate the relation between exponential coordinates and rotation matrices.

Recall results from linear differential equations.

Form	Equation	Solution	Series Expansion
scalar	$\dot{x}(t) = ax(t)$	$x(t) = e^{at}x_0$	$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$
matrix	$\dot{x}(t) = Ax(t)$	$x(t) = e^{At}x_0$	$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$

matrix exponential

It can be shown that if A is constant and finite, then the series is guaranteed to converge.

Exponential Coordinate Representation of Rotation

Suppose that a vector $p(0)$ is rotated by θ about $\hat{\omega}$ to $p(\theta)$.

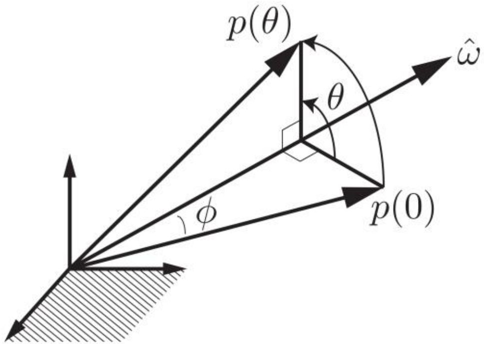
The velocity $\dot{p}$ is then given by

$$\dot{p} = \hat{\omega} \times p$$

Alternatively,

$$\dot{p} = [\hat{\omega}]p$$

which is a linear differential equation of the form $\dot{x} = Ax$.



The solution is given by

$$p(\theta) = e^{[\hat{\omega}]\theta} p(0)$$

Expand the matrix exponential $e^{[\hat{\omega}]\theta}$ in series form

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots$$

Exponential Coordinate Representation of Rotation

We can show that

$$[\hat{\omega}]^3 = -[\hat{\omega}]$$
$$[\hat{\omega}]^4 = -[\hat{\omega}]^2$$
$$[\hat{\omega}]^5 = -[\hat{\omega}]^3 = [\hat{\omega}]$$

$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2\frac{\theta^2}{2!} + [\hat{\omega}]^3\frac{\theta^3}{3!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2 \\ &= I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \end{aligned}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

<u>Rodrigues' formula</u>

- The matrix exponential of $[\hat{\omega}]\theta = [\hat{\omega}\theta] \in so(3)$ is

$$\mathrm{Rot}(\hat{\omega},\theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3)$$

- We have shown how to use matrix exponential to construct a rotation matrix from a rotation axis $\hat{\omega}$ and an angle θ. The other way around?

Exponential Coordinate Representation of Rotation

The matrix logarithm is the inverse of the matrix exponential.

Lie group

$$\exp: \ [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R \in SO(3)$$
$$\log: \ R \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3)$$

Lie algebra

To derive matrix logarithm, let us expand each entry for $e^{[\hat{\omega}]\theta}$.

$$e^{[\hat{\omega}]\theta} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

Setting the above matrix equal to $R \in SO(3)$ and playing with rearrangements can lead to the solutions. We skip them for now and please refer to the textbook for details if interested.

Exponential Coordinate Representation of Rotation

Matrix Logarithm

- Given $R \in SO(3)$, find a $\theta \in [0, \pi]$ and a unit rotation axis $\hat{\omega} \in \mathbb{R}^3$, such that $e^{[\hat{\omega}]\theta} = R$.
  - The vector $\hat{\omega}\theta \in \mathbb{R}^3$ comprises the exponential coordinates for R.
  - The skew-symmetric matrix $[\hat{\omega}]\theta \in so(3)$ is the matrix logarithm of R.

(a) If $R = I$ then $\theta = 0$ and $\hat{\omega}$ is undefined.

(b) If $\mathrm{tr}(R) = -1$ then $\theta = \pi$ and $\hat{\omega}$ can be set equal to any of the following three vectors.

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix}, \quad \hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix}, \quad \hat{\omega} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

Note that if $\hat{\omega}$ is a solution, then so is $-\hat{\omega}$.

(c) Otherwise, $\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}(R) - 1)\right) \in [0, \pi)$ and $[\hat{\omega}] = \frac{1}{2 \sin \theta}\left(R - R^{\top}\right)$.

Outline

Homework

(1) Which of the following matrices are rotation matrices? Explain why (not).

$$R_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad R_3 = \begin{bmatrix} 0.75 & 0.25 & 0.6124 \\ 0.25 & 0.75 & -0.6124 \\ -0.6124 & 0.6124 & 0.5 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_5 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad R_6 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

(2) Textbook Exercises: 3.1*, 3.9, 3.15, 3.28 (Bonus: 3.10)

(3) Lab Assignments: Closed-loop control of the Turtlebot robot

- In Gazebo simulation, write a script using ROS (in Python or C++) to make the robot run along the sides of a square (higher accuracy requirement this time).

* Hint about 3.1 (e), (g): https://blog.csdn.net/myan/article/details/647511 （推荐阅读：孟岩理解矩阵系列，共三篇）
Hint about 3.1 (i), (j): You may use a calculator or any programming language to facilitate computation.

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Thanks for Listening！