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Foundations of Robotics

Lec 6: Inverse Kinematics

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Outline

1. Inverse Kinematics
2. Analytic Approach (Trigonometry)
3. Numerical Approach (Newton’s Method)
4. Homework

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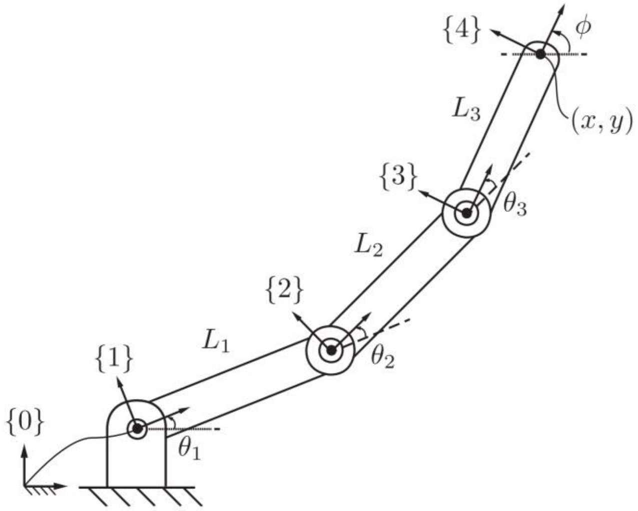
Inverse Kinematics

Consider a general n degree-of-freedom open chain.

Forward Kinematics
- $\theta \in \mathbb{R}^n \quad \rightarrow \quad T(\theta) \in SE(3)$
- Given joint coordinates θ, calculate the configuration (position and orientation) $T(\theta)$ of the end-effector frame.

Inverse Kinematics
- $T(\theta) \in SE(3) \quad \rightarrow \quad \theta \in \mathbb{R}^n$
- Given a homogeneous transform $X \in SE(3)$ and forward kinematics $T(\theta)$, find solutions θ that satisfy $T(\theta) = X$.



Inverse Kinematics

Example: a 2R planar open-chain manipulator

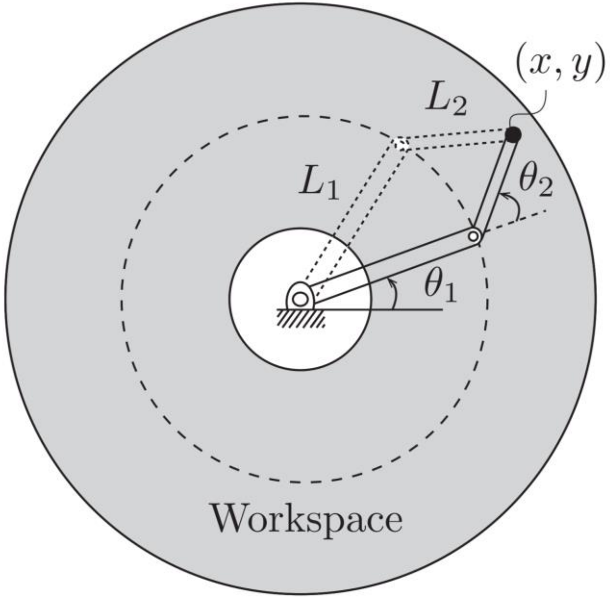
$$T(\theta) \in SE(2) \quad \rightarrow \quad \theta \in \mathbb{R}^2$$

Considering only the position of end-effector and ignoring its orientation, the forward kinematics can be expressed as

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

Provided a desired end-effector position (x, y), there will be either zero, one or two solutions depending on where (x, y) lies.

Assuming $L_1 > L_2$

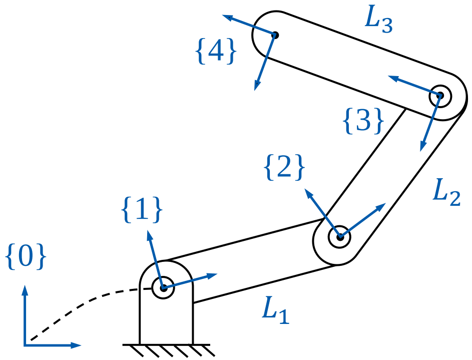


Inverse Kinematics

Example: a 3R planar open-chain manipulator

$$T(\theta) \in SE(2) \quad \rightarrow \quad \theta \in \mathbb{R}^3$$

Considering only the position of end-effector and ignoring its orientation, the forward kinematics can be expressed as



$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

Provided a desired end-effector position (x, y), there will be either zero, one or multiple solutions depending on where (x, y) lies.

Inverse Kinematics

Two ways to solve inverse kinematics:

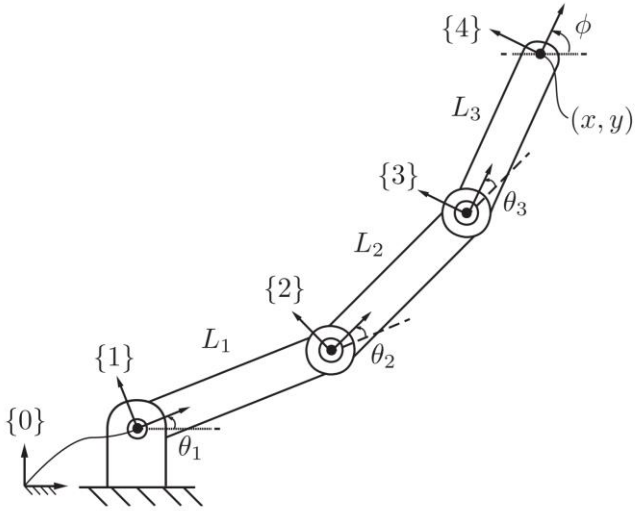
Analytic Approach

- Derive equations based on the geometric structure (e.g., trigonometry) of the mechanism
- Provide zero, one or multiple solutions in closed-form

Numerical Approach

- Iteratively solve a non-linear optimization problem (e.g., by Newton-Raphson method)
- Require an initial guess; always produce one solution (best approximation, may not be optimal)

In practice, they both can be applied: analytic solution can serve as the initial guess for numerical methods.



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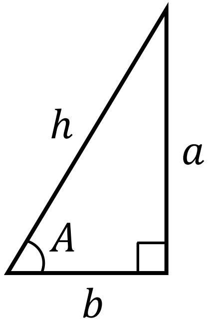
Analytic Approach (Trigonometry)

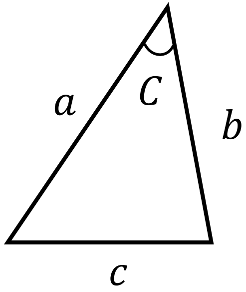
Math tools from trigonometry:



$\sin A$, $\cos A$, $\tan A$

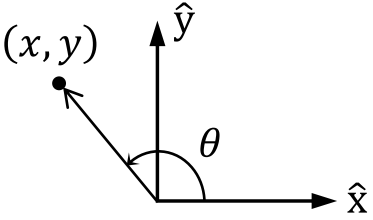
$\operatorname{asin}\frac{a}{h}, \operatorname{acos}\frac{b}{h}, \operatorname{atan}\frac{a}{b}$

Trigonometric ratios



$c^2 = a^2 + b^2 - 2ab\cos C$

Law of cosines



$\theta = \operatorname{atan2}(y, x) \in (-\pi, \pi]$

Two-argument arctangent

Analytic Approach (Trigonometry)

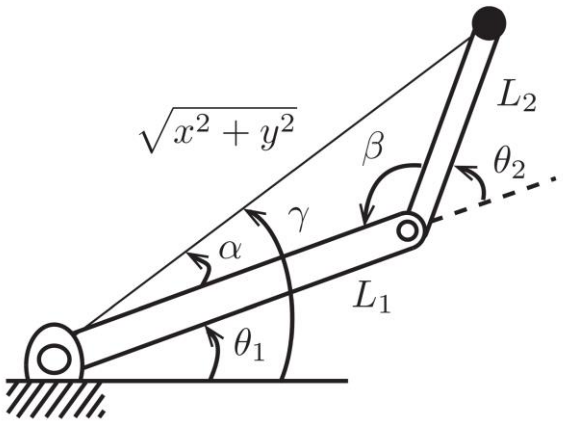
Example: A 2R planar open-chain manipulator

In the current "elbow-down" setup, as shown in the figure, joint angles can be obtained as follows.

- Compute $\gamma = \text{atan2}(y, x)$
- Compute α from law of cosines
- Obtain $\theta_1 = \gamma - \alpha$
- Compute β from law of cosines
- Obtain $\theta_2 = \pi - \beta$

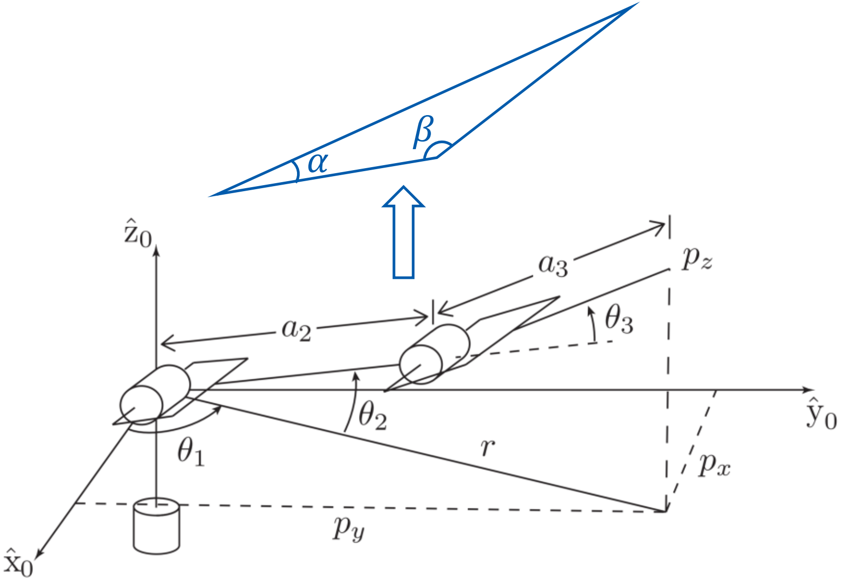
The other solution in the "elbow-up" setup can be obtained similarly: $\theta_1 = \gamma + \alpha$, $\theta_2 = \beta - \pi$

Again, no solution exists if (x, y) is not reachable, and exactly one solution exists if (x, y) lies on the boundary.



Analytic Approach (Trigonometry)

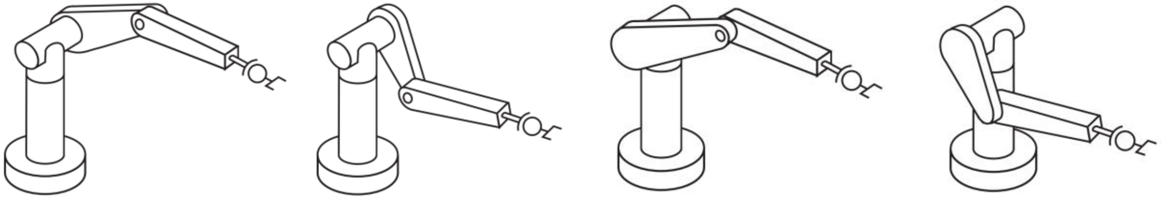
Example: A 3R spatial open-chain manipulator



One possible solution:

- Obtain $\theta_1 = \mathrm{atan2}(p_y, p_x)$
- Compute $\gamma = \mathrm{atan}(p_z/r)$
- Compute α from law of cosines
- Obtain $\theta_2 = \gamma - \alpha$
- Compute β from law of cosines
- Obtain $\theta_3 = \pi - \beta$

A more challenging case: what if joint axes are not aligned with coordinate axes (i.e. with an offset)?



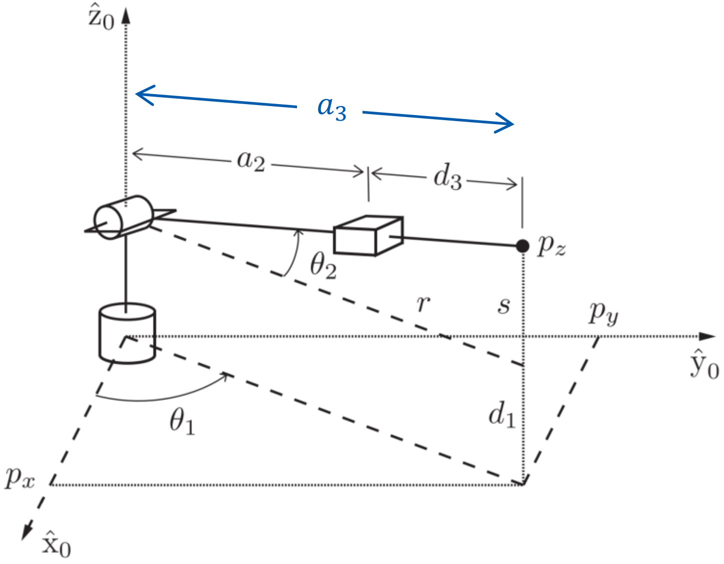
Analytic Approach (Trigonometry)

Example: An RRP spatial open-chain manipulator

One possible solution:

- Obtain $\theta_1 = \text{atan2}(p_y, p_x)$
- Compute $s = p_z - d_1$
- Compute $r = \sqrt{p_x^2 + p_y^2}$
- Obtain $\theta_2 = \text{atan}\, s/r$
- Compute $a_3 = \sqrt{r^2 + s^2}$
- Obtain $d_3 = a_3 - a_2$

Again, the solution can be more complicated if the shoulder joint is placed with an offset.



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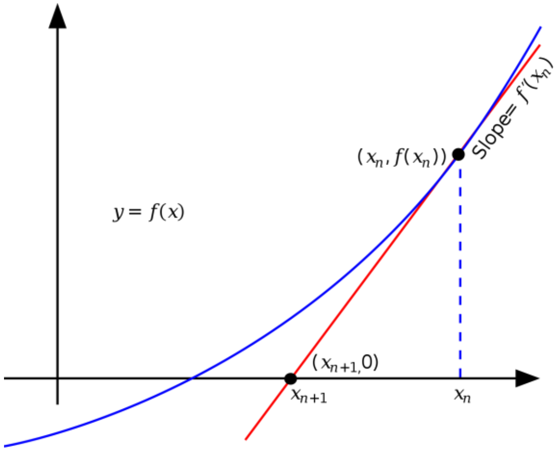
Numerical Approach (Newton's Method)

Newton-Raphson Method (a.k.a. Newton's Method)

- An iterative approach for finding the roots of a nonlinear equation.

Example of a scalar function

- Problem to solve: $f(x) = 0$
- Initial guess: $x = x_0$
- At the nth iteration: $x = x_n$
- Taylor expansion of $f(x)$ at x_n:
  $f(x) = f(x_n) + f'(x_n)(x - x_n) + h.o.t.$
- Ignore higher-order terms, set $f(x) = 0$ and solve for x to obtain $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Repeat until some stopping criterion is satisfied



* Figure taken from wikipedia: https://en.wikipedia.org/wiki/Newton's_method

Numerical Approach (Newton's Method)

Newton's Method in Optimization

- Find the roots of the derivative of a twice-differentiable function (solution to $f'(x) = 0$).

Example of a scalar function

- Problem to solve

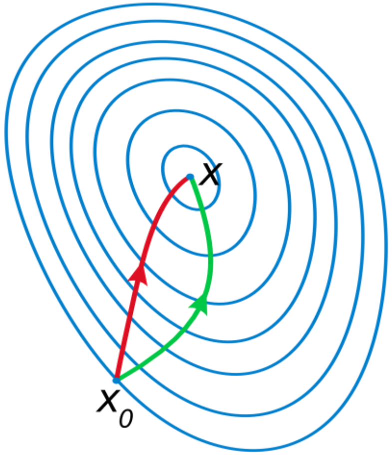
$$\min f(x)$$

- In each iteration, Taylor expansion of $f(x)$ at x_n up to second order terms

$$f'(x) = f'(x_n) + f''(x_n)(x - x_n) = 0$$

- Update equation

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$



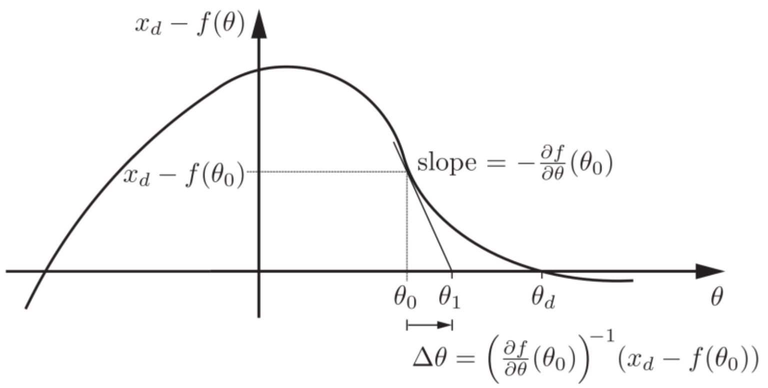
Green: Gradient Descent; Red: Newton's Method

* Figure taken from wikipedia: https://en.wikipedia.org/wiki/Newton's_method_in_optimization

Numerical Approach (Newton's Method)

Newton's Method in Inverse Kinematics

- Inverse kinematics: find solution to $T(\theta) = X$
- Define $g(\theta) = x_d - f(\theta) = X - T(\theta)$
- Solve $g(\theta) = 0$ using Newton's method



Example of a vector function

- Problem to solve

$$g(\theta) = x_d - f(\theta) = 0$$

- Taylor expansion

$$x_d = f(\theta) = f(\theta^k) + J(\theta^k)(\theta - \theta^k) + h.o.t.$$

- Update equation

$$\theta^{k+1} = \theta^k + J^{-1}(\theta^k)(x_d - f(\theta^k))$$

$$J(\theta^k) = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1^k} & \cdots & \frac{\partial f_1}{\partial \theta_n^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \theta_1^k} & \cdots & \frac{\partial f_m}{\partial \theta_n^k} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Note: J^{-1} exists only when J is square and nonsingular!

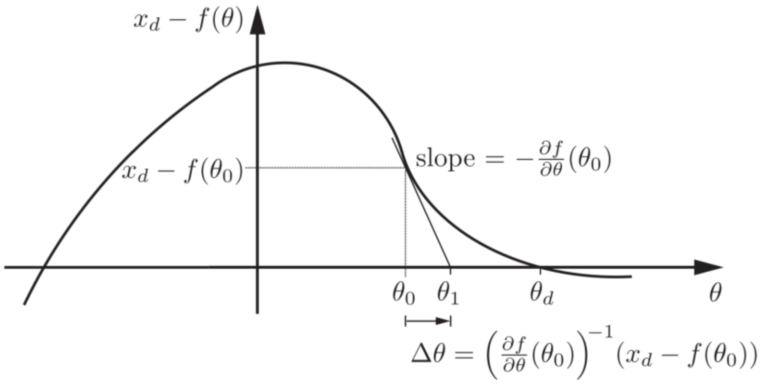
Numerical Approach (Newton's Method)

In cases where J is not invertible, we can apply pseudoinverse $J^{\dagger}$ instead.

1. The robot has more joints n than the end-effector coordinates m ("fat" Jacobian)
   - $J^{\dagger} = J^{\top}\left(JJ^{\top}\right)^{-1}$
   - Called a right inverse since $JJ^{\dagger} = I$

2. The robot has fewer joints n than the end-effector coordinates m ("tall" Jacobian)
   - $J^{\dagger} = \left(J^{\top}J\right)^{-1}J^{\top}$
   - Called a left inverse since $J^{\dagger}J = I$

In practice, we often use linear solvers to avoid the computation of inverse matrices.



$$J(\theta^k) = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1^k} & \cdots & \frac{\partial f_1}{\partial \theta_n^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \theta_1^k} & \cdots & \frac{\partial f_m}{\partial \theta_n^k} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Numerical Approach (Newton's Method)

Algorithm to reach a provided 3D point

1. Initialization
   - Given an end-effector position $x_d \in \mathbb{R}^m$
   - Given an initial guess $\theta^0 \in \mathbb{R}^n$
   - Set $k = 0$
2. While $\left\|x_d - f\left(\theta^k\right)\right\| > \epsilon$ for some small ϵ:
   - Compute $\Delta\theta = J^\dagger\left(\theta^k\right)\left(x_d - f\left(\theta^k\right)\right)$
   - Update $\theta^{k+1} = \theta^k + \Delta\theta$
   - Increment k

Right inverse: $J^\dagger = J^\top\left(JJ^\top\right)^{-1}$

Left inverse: $J^\dagger = \left(J^\top J\right)^{-1}J^\top$

$$J\left(\theta^k\right) = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1^k} & \cdots & \frac{\partial f_1}{\partial \theta_n^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \theta_1^k} & \cdots & \frac{\partial f_m}{\partial \theta_n^k} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Numerical Approach (Newton's Method)

Algorithm to reach a provided 6D pose

1. Initialization
   - Given an end-effector pose $T_{sd} \in SE(3)$
   - Given an initial guess $\theta^0 \in \mathbb{R}^n$
   - Set $k = 0$
2. While $\|\omega_b\| > \epsilon_\omega$ or $\|v_b\| > \epsilon_v$ for some small ϵ_ω and ϵ_v :
   - Compute $[\mathcal{V}_b] = \log\left(T_{sb}^{-1}(\theta^k)T_{sd}\right)$
   - Compute $\Delta\theta = J^\dagger(\theta^k)\mathcal{V}_b$
   - Update $\theta^{k+1} = \theta^k + \Delta\theta$
   - Increment k

Jacobian should act on body twist

desired state: T_{sd}
current state: $T_{sb}(\theta^k)$
difference: $T_{bd}(\theta^k) = T_{bs}(\theta^k)T_{sd} = T_{sb}^{-1}(\theta^k)T_{sd}$

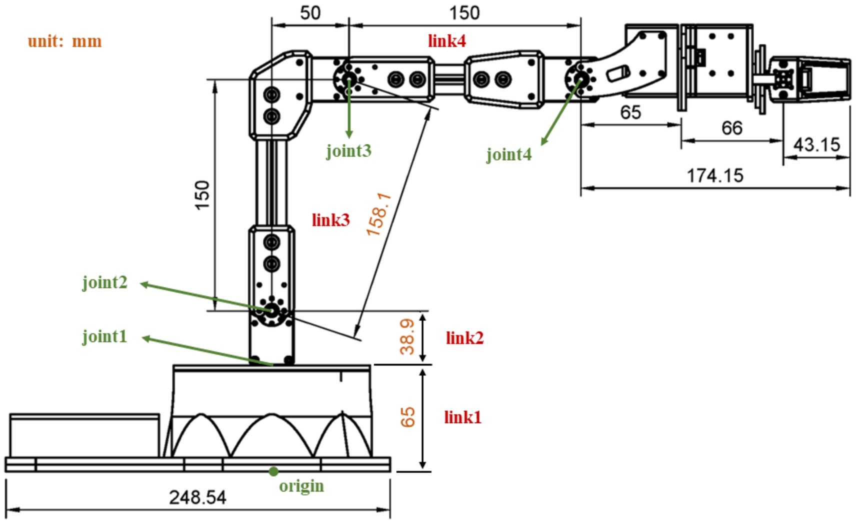
Right inverse: $J^\dagger = J^\top(JJ^\top)^{-1}$

Left inverse: $J^\dagger = (J^\top J)^{-1}J^\top$

$$J(\theta^k) = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1^k} & \cdots & \frac{\partial f_1}{\partial \theta_n^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \theta_1^k} & \cdots & \frac{\partial f_m}{\partial \theta_n^k} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Outline

Homework



(1) Lab Assignments: Inverse Kinematics

- Provided the schematic of a 3R manipulator, write two scripts (in C++ or Python) to implement analytic and numerical methods to solve the inverse kinematics.
- The scripts will be tested using a few test cases (input: the position of the end effector; expected output: joint variables)

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Thanks for Listening！