Chance Constrained Linear Quadratic Optimal Control

$$
\begin{aligned}
J^*(x) = \min_{\{\pi_k\}} \quad & \mathbb{E}\left(\|x(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 \right) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k) + w(k), \\
& u(k) = \pi_k(x(k)), \\
& w(k) \sim \mathcal{N}(0, \Sigma_w), \ \text{i.i.d.}, \\
& \Pr(h_{x,j}^\mathsf{T} x(k) \leq b_{x,j}) \geq p_{x,j} \ \forall j = 1, \dots, N_{c,x}, \\
& \Pr(h_{u,j}^\mathsf{T} u(k) \leq b_{u,j}) \geq p_{u,j} \ \forall j = 1, \dots, N_{c,u}, \\
& x(0) = x
\end{aligned}
$$

Approach

1. Restrict policy class: $\pi_k(x) = K_k x + v_k$
2. Derive mean-variance dynamics
3. Reformulate cost function
4. Reformulate constraints

Mean-Variance Dynamics under Affine Policy

$$x(k+1) = Ax(k) + Bu(k) + w(k), \qquad \mathbb{E}(w(k)) = 0,\ \mathrm{var}(w(k)) = \Sigma_w \text{ i.i.d.}$$

To simplify, we introduce some notation:

$$\begin{aligned} \bar{x}(k) &:= \mathbb{E}(x(k)) \\ \bar{u}(k) &:= \mathbb{E}(u(k)) \\ \Sigma^x(k) &:= \mathrm{var}(x(k)) = \mathrm{var}(d(k)) \end{aligned} \qquad \begin{aligned} d(k) &:= x(k) - \bar{x}(k) \\ \bar{d}(k) &:= \mathbb{E}(d(k)) = 0 \end{aligned}$$

where the expectations are understood conditioned on the initial state $x(0)$

Choosing an affine tube policy class $\pi_k(x) = K_k(x - \bar{x}(k)) + v_k$ we have

$$\begin{aligned} u(k) &= K_k d(k) + v_k \\ \bar{u}(k) &= v_k \end{aligned} \qquad \text{resulting in } \rightarrow \qquad \begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) + B\bar{u}(k) \\ \Sigma^x(k+1) &= (A + BK_k)\Sigma^x(k)(A + BK_k)^\mathsf{T} + \Sigma_w \end{aligned}$$

Reformulation of Cost Function

$$\begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) + B\bar{u}(k) \\ \Sigma^x(k+1) &= (A + BK_k)\Sigma^x(k)(A + BK_k)^\mathsf{T} + \Sigma_w \end{aligned}$$

Expected value of quadratic form:

$$\mathbb{E}_z\left(\|z\|_Q^2\right) = \|\mathbb{E}_z(z)\|_Q^2 + \mathrm{tr}(Q\,\mathrm{var}_z(z))$$

Using this allows us to reformulate cost function in term of mean and variance!

$$\mathbb{E}\left(\|x(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1}\|x(k)\|_Q^2 + \|u(k)\|_R^2\right)$$

$$= \|\bar{x}(\bar{N})\|_P^2 + \mathrm{tr}(P\Sigma^x(\bar{N})) + \sum_{k=0}^{\bar{N}-1}\|\bar{x}(k)\|_Q^2 + \mathrm{tr}(Q\Sigma^x(k)) + \|\bar{u}(k)\|_R^2 + \mathrm{tr}(RK_k\Sigma^x(k)K_k^\mathsf{T})$$

Gaussian Half-Space Chance Constraint I

Given affine policy, state distributions remain Gaussian. For a half-space chance constraint

$$x(k) \sim \mathcal{N}(\bar{x}(k), \Sigma^x(k))$$
$$\mathcal{X} = \{x \,|\, h^\top x \leq b\}$$

we can construct the marginal distribution in direction of the constraint

$$h^\top x(k) \sim \mathcal{N}(h^\top \bar{x}(k), h^\top \Sigma^x(k) h) \text{ } \textbf{(scalar!)}$$

$$\Pr(x(k) \in \mathcal{X}) = \Pr(h^\top x(k) \leq b) = \phi\left(\frac{b - h^\top \bar{x}(k)}{\sqrt{h^\top \Sigma^x(k) h}}\right)$$

- ϕ is the cumulative distribution function of the standard normal distribution (available)

$$\phi(\tilde{x}) := \Pr(x \leq \tilde{x}) \text{ with } x \sim \mathcal{N}(0, 1)$$

- depends only on $\bar{x}(k)$ and $\Sigma^x(k)$ (available)

Gaussian Half-Space Chance Constraint II

$$\begin{aligned}
\Pr(x(k) \in \mathcal{X}) \geq p &\Leftrightarrow \phi\left(\frac{b - h^\top \bar{x}(k)}{\sqrt{h^\top \Sigma^x(k) h}}\right) \geq p \\
&\Leftrightarrow \frac{b - h^\top \bar{x}(k)}{\sqrt{h^\top \Sigma^x(k) h}} \geq \phi^{-1}(p) \\
&\Leftrightarrow -h^\top \bar{x}(k) \geq -b + \sqrt{h^\top \Sigma^x(k) h}\, \phi^{-1}(p) \\
&\Leftrightarrow h^\top \bar{x}(k) \leq b \underbrace{- \sqrt{h^\top \Sigma^x(k) h}\, \phi^{-1}(p)}_{\text{tightening/backoff term}}
\end{aligned}$$

with ϕ^{-1} the inverse cumulative distribution function of the standard normal distribution (available)

Tightened half-space constraint when optimizing over $\bar{x}(k)$

Gaussian Linear Quadratic Control with Half-Space Constraint

$$
\begin{aligned}
J^*(x) = \min_{\{\pi_k\}} \quad & \mathbb{E}\left(\|x(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 \right) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k) + w(k), \\
& u(k) = \pi_k(x(k)), \\
& w(k) \sim \mathcal{N}(0, \Sigma_w), \ \text{i.i.d.}, \\
& \Pr(h_{x,j}^\top x(k) \le b_{x,j}) \ge p_{x,j} \ \forall j = 1, \dots, N_{c,x}, \\
& \Pr(h_{u,j}^\top u(k) \le b_{u,j}) \ge p_{u,j} \ \forall j = 1, \dots, N_{c,u}, \\
& x(0) = x
\end{aligned}
$$

Gaussian Linear Quadratic Control with Half-Space Constraint

$$
\begin{aligned}
J^*_{\mathrm{det}}(x) = \min_{\{v_k, K_k\}} \quad & \|\bar{x}(\bar{N})\|_P^2 + \mathrm{tr}(P\Sigma^x(\bar{N})) + \sum_{k=0}^{\bar{N}-1} \|\bar{x}(k)\|_Q^2 + \|\bar{u}(k)\|_R^2 + \mathrm{tr}((Q + K_k^\top R K_k)\Sigma^x(k)) \\
\text{s.t.} \quad & \bar{x}(k+1) = A\bar{x}(k) + Bv_k, \\
& \Sigma^x(k+1) = (A + BK_k)\Sigma^x(k)(A + BK_k)^\top + \Sigma_w, \\
& h_{x,j}^\top \bar{x}(k) \leq b_{x,j} - \sqrt{h_{x,j}^\top \Sigma^x(k) h_{x,j}}\, \phi^{-1}(p) \ \forall j = 1, \dots, N_{c,x}, \\
& h_{u,j}^\top v_k \leq b_{u,j} - \sqrt{h_{u,j}^\top K_k \Sigma^x(k) K_k^\top h_{u,j}}\, \phi^{-1}(p) \ \forall j = 1, \dots, N_{c,u}, \\
& \bar{x}(0)) = x,\ \Sigma^x(0) = 0
\end{aligned}
$$

- With affine policy: deterministic optimization problem over mean and variances (non-convex)

Gaussian Linear Quadratic Control with Half-Space Constraint

$$
\begin{aligned}
\tilde{J}^*_{\text{pre}}(x) = \min_{\{v_k\}} \quad & \|\bar{x}(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|\bar{x}(k)\|_Q^2 + \|\bar{u}(k)\|_R^2 \\
\text{s.t.} \quad & \bar{x}(k+1) = A\bar{x}(k) + Bv_k, \\
& h_{x,j}^\top \bar{x}(k) \le b_{x,j} - \sqrt{h_{x,j}^\top \Sigma^x(k) h_{x,j}}\, \phi^{-1}(p) \;\forall j = 1, \dots, N_{c,x}, \\
& h_{u,j}^\top v_k \le b_{u,j} - \sqrt{h_{u,j}^\top K_k \Sigma^x(k) K_k^\top h_{u,j}}\, \phi^{-1}(p) \;\forall j = 1, \dots, N_{c,u}, \\
& \bar{x}(0) = x
\end{aligned}
$$

- With affine policy: deterministic optimization problem over mean and variances (non-convex)
- Fixing K_k and only optimizing over v_k reduces problem to 'standard' MPC QP
- Variance dynamics independent of v_k $\to$ tightening can be precomputed