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Systems with persistent disturbances: predictive control with restricted constraints☆

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Predictive regulation of linear discrete-time systems subject to unknown but bounded disturbances and to state/control constraints is addressed. An algorithm based on constraint restrictions is presented and its stability properties are analysed.

Abstract

This paper addresses predictive state regulation of linear discrete-time systems subject to persistent bounded disturbances and to state and/or control constraints. It is well known that the joint presence of constraints and disturbances can drive a predictive controller to infeasibility and instability. Here it is shown how robustness against persistent bounded disturbances can be enforced by inserting in the predictive controller suitable constraint restrictions. The robust predictive controller obtained in this way guarantees, for all admissible disturbances, constraint fulfillment and asymptotic state regulation, i.e. convergence of the state to a minimal robust invariant set, provided that the initial state is feasible. Simulation results demonstrate the effectiveness of the proposed approach. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Predictive control represents an effective control design methodology for handling both hard constraints and performance issues. The predictive control law is, in fact, obtained by minimizing a moving horizon performance index which explicitly takes into account input and/or state constraints. Stability under nominal conditions, i.e. when there are neither uncertainties in the model nor disturbances acting on the plant, has been thoroughly investigated (Keerthi & Gilbert, 1988) and is now well established. On the other hand, the issue of robustness still deserves further attention although recently there have been several novel contributions (see, e.g., Bemporad & Morari (1999)) for a survey. In particular, some papers have addressed robustness with respect to model uncertainty (Kothare, Balakrishnan, & Morari, 1996; Lee & Yu, 1997; Badgwell, 1997; Bemporad & Mosca, 1998). Some references have also considered robustness with respect to disturbances (Casavola & Mosca, 1996; Gossner, Kouvaritakis, & Rossiter, 1997; Mayne & Schroeder, 1997; Scokaert & Mayne, 1998). The present work concentrates on the latter source of uncertainty, i.e. the presence of bounded disturbances.

It is well known that the action of a bounded disturbance, even non-persistent, can destabilize a predictive controller which is stabilizing for the nominal case. The key point is that to be able to guarantee stability, the algorithm must first ensure feasibility, that is the existence of an admissible input sequence satisfying constraints and meeting terminal conditions. However, feasibility is not ensured by the nominal cost constrained minimization for future state vectors under any disturbance realization.

A possible approach is to carry out a min–max optimization as proposed by Lee and Yu (1997) and Scokaert

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and Mayne (1998). This approach, however, suffers from two major drawbacks, as also remarked by Badgwell (1997): (i) the resulting optimization problem is computationally more expensive even if it is possible to design min–max controllers with a finite-dimensional parameterization; (ii) the min–max paradigm of optimizing performance for the "worst-case" disturbance represents an unrealistic scenario and may yield poor performance whenever the disturbance realization gets close to zero. For the above reasons, a more sensible approach seems to minimize the nominal performance index while imposing constraint fulfillment for all admissible disturbances. This can, for instance, be accomplished by resorting to suitable restrictions of the constraints which take care of the effects of disturbances. This idea has been pursued in Casavola and Mosca (1996) for *robust command governors* and in Gossner et al. (1997) for *robust predictive controllers*. This paper proposes a robust predictive controller based on constraint restriction, but improves considerably on the results obtained in Gossner et al. (1997): (i) a state-space framework is used hence allowing consideration of multivariable systems and general state constraints (Gossner et al. (1997) considered only SISO systems modelled by transfer function and subject to bound input constraints only); (ii) allowance is made for an arbitrary stabilizing terminal control law in the predictive control law; the work of Gossner et al. (1997) was restricted to dead-beat terminal conditions of Kouvaritakis, Rossiter, and Chang (1992); (iii) it is shown that the proposed robust predictive controller not only guarantees asymptotic stability in the disturbance-free case and boundedness of the state response in presence of disturbances, but in the latter case also ensures convergence of the state to the smallest possible robust invariant set, i.e. the set of all states reachable from the origin under the disturbance input and the terminal linear constant feedback.

2. Problem formulation

2.1. Set notation

For any subsets $\mathscr{A}$, $\mathscr{B}$ of $\mathbb{R}^n$, for any matrix M mapping $\mathbb{R}^n$ onto $\mathbb{R}^l$ and for any subset $\mathscr{C}$ of $\mathbb{R}^l$, the following sets are defined: $\mathscr{A}+\mathscr{B}=\{a+b|a\in\mathscr{A},b\in\mathscr{B}\}$; $\mathscr{A}-\mathscr{B}=\{a-b|a\in\mathscr{A},b\in\mathscr{B}\}$; $M(\mathscr{A})=\{Ma|a\in\mathscr{A}\}$; $M^{-1}(\mathscr{C})=\{a|Ma\in\mathscr{C}\}$; $\mathscr{A}\sim\mathscr{B}=\{a|a+\mathscr{B}\subseteq\mathscr{A}\}$; $int(\mathscr{A})$ denotes the interior of $\mathscr{A}$.

2.2. System notation

The paper addresses state feedback control for the linear, time-invariant, discrete-time system

$$x_{t+1}=Ax_t+Bu_t+Dw_t \tag{1}$$

subject to the *disturbance input*

$$w_t\in\mathscr{W}\subset\mathbb{R}^q \tag{2}$$

and to the *state and control constraints*

$$x_t\in\mathscr{X}\subset\mathbb{R}^n,\qquad u_t\in\mathscr{U}\subset\mathbb{R}^m. \tag{3}$$

Hereafter a disturbance sequence w_t satisfying (2) for all $t\geqslant 0$ will be called *admissible*. It will be assumed throughout the paper that

(A1) the pair (A,B) is stabilizable;
(A2) the sets $\mathscr{U}$, $\mathscr{X}$, $\mathscr{W}$ contain the origin as an interior point;
(A3) $\mathscr{U}$ and $\mathscr{W}$ are compact sets.

2.3. Control objective

The objective is to design, via predictive control, a nonlinear state feedback

$$u_t=g(x_t) \tag{4}$$

which regulates the state of system (1) "*as close as possible*" *to the origin while satisfying the state and control constraints* (3) *for all admissible disturbances* (2). *A further objective is that the nonlinear feedback* (4) *reduce to a linear well-tuned feedback whenever this is compatible with the constraints* (3).

Clearly, the presence of a persistent disturbance (2) acting on the system (1) means it is not possible to guarantee asymptotic regulation, i.e. that $\lim_{t\to\infty}x_t=0$. Then an achievable goal, the best that can be hoped for, is to steer the initial state x_0 to a neighborhood of the origin Ω. Of course the requirement for optimal regulation performance then translates to the target set Ω being as small as possible since this amounts to minimizing the state sensitivity to disturbances. To meet the desire for linear control when $x_t\in\Omega$, consider a stabilizing LTI state feedback

$$u_t=Fx_t, \tag{5}$$

designed so as to provide a satisfactory, or optimal in some sense (e.g. LQ, $\mathscr{H}_\infty$, ℓ_1), control performance to system (1) in the absence of constraints. The control law (5) will be regarded as a-priori fixed throughout the paper and referred to as the *nominal feedback*. The corresponding asymptotically stable nominal closed-loop system is

$$x_{t+1}=\Phi x_t+Dw_t,\qquad w_t\in\mathscr{W}. \tag{6}$$

where $\Phi\triangleq A+BF$ is the closed-loop state transition matrix for which it is assumed that

(A4) Φ has all eigenvalues strictly inside the unit circle.

2.4. Invariant sets

In order to characterize the target set Ω, it is convenient to introduce the notion of invariant set (Kolmanovsky & Gilbert, 1995).

Definition 1. A set $\Sigma \subset \mathbb{R}^n$ is disturbance (d)-invariant for the uncontrolled system (6) if for any $x \in \Sigma$, $Ax + Dw \in \Sigma$ for all $w \in \mathscr{W}$.

In particular, if in the above definition $\mathscr{W} = \{0\}$, the set Σ is simply *invariant*. Let $\mathscr{X}_c \triangleq \mathscr{X} \cap F^{-1}(\mathscr{U})$ be the set of the states which satisfy both the state and the control constraints in (3) under the nominal feedback (5). The control objectives can therefore be accomplished by taking the target set Ω as the minimal d-invariant subset of $\mathscr{X}_c$ for the closed-loop system (6). Such an Ω can be identified using some results presented in Kolmanovsky and Gilbert (1995). Let

$$\mathscr{R}_j \triangleq \sum_{i=0}^{j-1} \Phi^i D \mathscr{W} \tag{7}$$

denote the set of states of the nominal linear closed-loop system (6) which are reachable in j steps from the origin (under the disturbance input w). Then:

(i) The sequence of sets $\mathscr{R}_j$ has a limit $\mathscr{R}_\infty$ as $j \to \infty$, and $\mathscr{R}_\infty$ is a compact d-invariant set.
(ii) $\mathscr{R}_\infty$ is the minimal d-invariant set, i.e. if Σ is d-invariant then $\mathscr{R}_\infty \subset \Sigma$.

Therefore, there exists a d-invariant subset of $\mathscr{X}_c$ if and only if the following assumption holds:

(A5) $\mathscr{R}_\infty \subset \mathscr{X}_c$.

2.5. Restated control objective

Motivated by the above discussion, the control objective is restated in more convenient form as follows.

Design a nonlinear state feedback (4) *such that, for all initial states* x_0 *in a suitable domain of attraction, the following two requirements are met*:

(R1) *the closed-loop system*

$$x_{t+1} = Ax_t + Bg(x_t) + Dw_t, \quad w_t \in \mathscr{W}, \tag{8}$$

satisfies the constraints (3) *for all* $t \geqslant 0$;

(R2) *the nonlinear feedback* (4) *asymptotically approaches the nominal feedback* (5), *i.e.*

$$\lim_{t\to\infty} [g(x_t) - Fx_t] = 0. \tag{9}$$

Notice that (9), in turn, implies that

$$x_t \to \mathscr{R}_\infty \quad \text{as} \quad t \to \infty. \tag{10}$$

To see this, note that (8) can be rewritten as

$$x_{t+1} = \Phi x_t + B[g(x_t) - Fx_t] + Dw_t. \tag{11}$$

Now, by (9), the bracketed term vanishes for $t \to \infty$. Hence (11) can be thought of as the state equation of a linear asymptotically stable system subject to a vanishing input $c_t = g(x_t) - Fx_t$ and to a persistent input w_t. For such a system clearly $\lim_{t\to\infty} [x_t - \sum_{k=0}^{t-1} \Phi^k D w_{t-k-1}] = 0$, from which (10) follows.

It is convenient (Rossiter et al., 1997) to introduce the control variable $c_t \triangleq u_t - Fx_t$ as the difference between the control input u_t and the nominal feedback Fx_t. System (1) is accordingly rewritten as

$$\begin{aligned} x_{t+1} &= \Phi x_t + Bc_t + Dw_t, \\ u_t &= Fx_t + c_t \end{aligned} \tag{12}$$

and requirement (9) becomes more simply

(R2′) $\lim_{t\to\infty} c_t = 0$.

3. Robust predictive control via constraint restriction

In this section a method is proposed for designing, via predictive control, a nonlinear state feedback (4) which meets the control objectives (R1) and (R2′) stated in the previous section. For expository reasons, we present first a predictive control algorithm, referred to as *Nominal Predictive Control* (NPC) which synthesizes the control law neglecting disturbances. Next we shall introduce a modified controller, referred to as *Robust Predictive Control* (RPC) (Rossiter, Kouvaritakis, & Rice, 1998), to account for disturbances. In order to do that, some further notation has to be introduced: $c_{t+k|t}$, $k \geqslant 0$, will denote the future control moves, with respect to the nominal feedback, planned at time t; $x_{t+k|t}$ and $u_{t+k|t}$ the disturbance-free predictions of x_{t+k} and u_{t+k}, respectively, given the state x_t and $c_{t+j|t}$, $0 \leqslant j < k$.

3.1. NPC

The NPC algorithm can be summarized as follows:
NPC: At sample time t, given the state x_t:

(1) Minimize, with respect to the control sequence $C_t \triangleq [c_{t|t}^{\mathrm{T}}, c_{t+1|t}^{\mathrm{T}}, \ldots, c_{t+N-1|t}^{\mathrm{T}}]^{\mathrm{T}}$, the quadratic cost

$$J(C_t) \triangleq \sum_{k=0}^{N-1} c_{t+k|t}^{\mathrm{T}} \Psi c_{t+k|t}, \quad \Psi = \Psi^{\mathrm{T}} > 0, \tag{13}$$

subject to the constraints

$$x_{t|t} = x_t, \tag{14}$$

$$x_{t+k+1|t} = \Phi x_{t+k|t} + Bc_{t+k|t} \quad k \geqslant 0, \tag{15}$$

$$u_{t+k|t} = Fx_{t+k|t} + c_{t+k|t} \quad k \geqslant 0, \tag{16}$$

$$c_{t+k|t} = 0 \quad k \geqslant N, \tag{17}$$

$$x_{t+k|t} \in \mathscr{X}, \quad u_{t+k|t} \in \mathscr{U} \quad k \geqslant 0 \tag{18}$$

and let $\hat{C}_t \triangleq [\hat{c}_{t|t}^{\mathrm{T}}, \hat{c}_{t+1|t}^{\mathrm{T}}, \ldots, \hat{c}_{t+N-1|t}^{\mathrm{T}}]^{\mathrm{T}}$ be the optimal solution;

(2) Set $c_t = \hat{c}_{t|t}$ and apply $u_t = Fx_t + c_t$ to the plant.

Remark 2. The integer $N > 0$ in (13) and (17) is called the control horizon and represents the number of free control moves $c_{t+k|t}$ considered in the optimization problem. Clearly, a larger N implies better performance and a larger feasibility region at the price of a higher computational load: in practice, a suitable tradeoff is required.

Remark 3. The cost (13) suitably penalizes, via the weight matrix $\Psi > 0$, the deviation from the nominal control (5) and may be related by appropriate choice of Ψ to an infinite-horizon LQ criterion. If F is the unconstrained LQ feedback gain minimizing a given quadratic cost

$$\sum_{k=0}^{\infty} x_{t+k|t}^{\mathrm{T}} Q x_{t+k|t} + u_{t+k|t}^{\mathrm{T}} R u_{t+k|t}, \quad Q = Q^{\mathrm{T}} \geqslant 0,$$
$$R = R^{\mathrm{T}} > 0 \tag{19}$$

and the pair $(A, Q^{1/2})$ is detectable, it is easy to see that there is an appropriate choice of Ψ, namely $\Psi = R + B^{\mathrm{T}}PB$ where P is the unique symmetric nonnegative definite solution of the algebraic Riccati equation $P = A^{\mathrm{T}}PA - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA + Q$, for which minimization of (13) subject to (14)–(18) yields the optimal constrained LQ control corresponding to the cost (19) under the terminal condition $u_{t+k|t} = Fx_{t+k|t}$, $k \geqslant N$. In fact, this follows from the well known LQ formula

$$\sum_{k=0}^{\infty} x_{t+k|t}^{\mathrm{T}} Q x_{t+k|t} + u_{t+k|t}^{\mathrm{T}} R u_{t+k|t}$$
$$= x_t^{\mathrm{T}} P x_t + \sum_{k=0}^{\infty} (u_{t+k|t} - Fx_{t+k|t})^{\mathrm{T}} \times (R + B^{\mathrm{T}}PB)(u_{t+k|t} - Fx_{t+k|t}).$$

Remark 4. The predictive controller implicitly defines a nonlinear feedback (4) from the solution of the constrained optimization (13)–(18) by means of the so-called receding horizon control strategy; that is the first element $c_t \triangleq \hat{c}_{t|t} = c(x_t)$ of the optimal control sequence, clearly depending on x_t, is used to define $g(x_t) = Fx_t + c(x_t)$.

Remark 5. The infinitely many constraints (18) can actually be reduced to a finite number. In fact, a finite integer $i^* \geqslant 0$ can be determined off-line (Gilbert & Tan, 1991) such that (18) needs only to be imposed for $k = 0, 1, \ldots, N + i^*$.

3.2. RPC

The constraints (18) involve only the nominal (disturbance-free) predictions. Therefore, they do not provide any guarantee that the true system state and input will satisfy the constraints (3) due to the presence of the disturbance (2). Hence NPC, as well as many standard predictive control algorithms, can lose feasibility and hence their guarantee of stability.

The RPC algorithm replaces the original constraints (18) with more stringent ones which preserve feasibility despite the presence of disturbances. RPC improves on the work of Gossner et al. (1997) and derives from the following observation. The true state and input responses are given by

$$x_{t+k} = x_{t+k|t} + \sum_{i=1}^{k} \Phi^{i-1} D w_{t+k-i},$$
$$u_{t+k} = u_{t+k|t} + \sum_{i=1}^{k} F\Phi^{i-1} D w_{t+k-i}, \tag{20}$$

where the first terms represent the disturbance-free predictions and the latter terms represent the forced responses due to the disturbance. Hence, with $\mathscr{R}_k$ defined as in (7), a sufficient condition for the constraints $x_{t+k} \in \mathscr{X}$ and $u_{t+k} \in \mathscr{U}$ to be satisfied, is to impose restricted constraints on the nominal predictions, i.e.

$$x_{t+k|t} \in \mathscr{X}_k, \quad u_{t+k|t} \in \mathscr{U}_k, \quad k \geqslant 0 \tag{21}$$

with

$$\mathscr{X}_k \triangleq \mathscr{X} \sim \mathscr{R}_k, \quad \mathscr{U}_k \triangleq \mathscr{U} \sim F\mathscr{R}_k. \tag{22}$$

In fact, $x_{t+k|t} \in \mathscr{X} \sim \mathscr{R}_k \Rightarrow x_{t+k|t} + \mathscr{R}_k \subset \mathscr{X}$ and consequently $x_{t+k} \in \mathscr{X}$ for all admissible disturbances. The same arguments can be used to show that $u_{t+k|t} \in \mathscr{U}_k$ implies $u_{t+k} \in \mathscr{U}$. Notice that $\mathscr{X}_k$ and $\mathscr{U}_k$ are non-empty due to assumption (A5).

Based on the above arguments, the following RPC algorithm is proposed.

RPC: At sample time t, given the state x_t, compute the control u_t as in the NPC algorithm by only replacing the constraints (18) with (21)–(22).

3.3. Implementation issues

To make the RPC practically implementable, it is necessary to reduce the constraints (21) to a finite number. Since $c_{t+k|t} = 0$ for $k \geqslant N$, these constraints are related to the *maximal output admissible set* (Kolmanovsky & Gilbert, 1995)

$$\Sigma_0 \triangleq \{x\colon \Phi^i x \in \mathscr{X}_i, F\Phi^i x \in \mathscr{U}_i \text{ for all } i \geqslant 0\}, \tag{23}$$

viz. the largest d-invariant subset of $\mathscr{X}_c$. In fact, the constraints $x_{t+k|t} \in \mathscr{X}_k$ and $u_{t+k|t} \in \mathscr{U}_k$ $\forall k \geqslant N$ are equivalent to the terminal constraint $x_{t+N|t} \in \Sigma_0 \sim \mathscr{R}_N$. Under weak conditions, e.g. observability of the pair (Φ, F),

Σ_0 turns out to be finitely determined (Kolmanovsky & Gilbert, 1995), i.e. there exists i^* such that

$$\Sigma_0 \triangleq \{x\colon \Phi^i x \in \mathscr{X}_i, F\Phi^i x \in \mathscr{U}_i \text{ for } i = 0,1,\ldots,i^*\}. \tag{24}$$

Provided that the sets $\mathscr{X},\mathscr{U},\mathscr{W}$ are convex polyhedra, the finite determination index i^* can be determined off-line by *Linear Programming* (LP) tools. Hence, constraints (21) can be expressed as

$$x_{t+k|t} \in \mathscr{X}_k,\ u_{t+k|t} \in \mathscr{U}_k, \quad k = 0,1,\ldots,N-1,$$

$$x_{t+N|t} \in \Sigma_0 \sim \mathscr{R}_N \tag{25}$$

and the RPC optimization (13)–(17), (21) becomes a *Quadratic Programming* (QP) problem

$$\hat{C}_t = \arg\min_{C_t} C_t^{\mathrm{T}} \boldsymbol{\Psi} C_t \quad \text{subject to} \quad M_N C_t + L_N x_t \leqslant v_N$$

where $\boldsymbol{\Psi} \triangleq diag\{\Psi,\Psi,\ldots,\Psi\}$; "$\leqslant$" denotes "componentwise inequality"; and L_N, M_N, v_N are suitable matrices, representing (25), which can be calculated off-line.

The RPC algorithm proposed in this paper extends the algorithm in Gossner et al. (1997) in the following directions:

(1) A state instead of input–output framework is adopted. This allows to consider also general state constraints as well as generic sets $\mathscr{U}$ and $\mathscr{W}$ instead of balls in the infinity norm.
(2) The nominal feedback is not limited to deadbeat. Notice that a deadbeat feedback makes $\mathscr{R}_\infty = \mathscr{R}_n$ and reduces the computation of the sets $\mathscr{X}_k$ and $\mathscr{U}_k$ to a finite number. However, as shown above, as far as Σ_0 is finitely determined, this restriction is unnecessary. Conversely, a deadbeat nominal control could reduce significantly the feasibility region of the algorithm.

4. Main results

In this section, the properties of RPC will be analysed. First introduce the control sequence $\tilde{C}_{t+1} \triangleq [\hat{c}_{t+1|t}^{\mathrm{T}},\ldots,\hat{c}_{t+N-1|t}^{\mathrm{T}},0]^{\mathrm{T}}$ obtained by the concatenation of the optimal "tail" at sample time t, i.e. $[\hat{c}_{t+1|t}^{\mathrm{T}},\ldots,\hat{c}_{t+N-1|t}^{\mathrm{T}}]^{\mathrm{T}}$, with a terminal zero element. The following terminology will also be used.

Definition 6. A control sequence $C_t \triangleq [c_{t|t}^{\mathrm{T}}, c_{t+1|t}^{\mathrm{T}},\ldots,c_{t+N-1|t}^{\mathrm{T}}]^{\mathrm{T}}$ is said to be admissible for state x_t if the constraints (14)–(17) and (21) are satisfied. A state x_t is said to be feasible if there exists a sequence C_t admissible for x_t.

4.1. Feasibility and stability

The lemma and theorem below give sufficient conditions under which RPC guarantees the required objectives (R1) and (R2′).

Lemma 7. *For system* (1) *under RPC feedback* $u_t = Fx_t + c_t$, *with* $c_t = \hat{c}_{t|t}$, *the following implication holds*

$$\hat{C}_t \text{ is admissible for } x_t \;\Rightarrow\; \tilde{C}_{t+1} \text{ is admissible for } x_{t+1}. \tag{26}$$

Proof. Denote by $(\hat{x}_{t+k|t}, \hat{u}_{t+k|t})$ the predictions associated to the control sequence $\hat{C}_t$ and state x_t, and by $(\tilde{x}_{t+1+k|t+1}, \tilde{u}_{t+1+k|t+1})$ the predictions for control $\tilde{C}_{t+1}$ and state x_{t+1}. It is easy to check that

$$\tilde{x}_{t+1+k-1|t+1} = \hat{x}_{t+k|t} + \Phi^{k-1} D w_t, \quad k \geqslant 1,$$

$$\tilde{u}_{t+1+k-1|t+1} = \hat{u}_{t+k|t} + F\Phi^{k-1} D w_t, \quad k \geqslant 1.$$

Since, by assumption, $\hat{C}_t$ is admissible for x_t, we have $\hat{x}_{t+k|t} \in \mathscr{X}_k$ and $\hat{u}_{t+k|t} \in \mathscr{U}_k$, hence

$$\tilde{x}_{t+1+k-1|t+1} \in \mathscr{X}_k + \Phi^{k-1} D\mathscr{W},$$

$$\tilde{u}_{t+1+k-1|t+1} \in \mathscr{U}_k + F\Phi^{k-1} D\mathscr{W}.$$

Using (22) and the properties of the set difference $\sim$ (Kolmanovsky & Gilbert, 1995)

$$\begin{aligned}
&\mathscr{X}_k + \Phi^{k-1} D\mathscr{W} \\
&\quad = (\mathscr{X} \sim \mathscr{R}_k) + \Phi^{k-1} D\mathscr{W} \\
&\quad = [(\mathscr{X} \sim \mathscr{R}_{k-1}) \sim \Phi^{k-1} D\mathscr{W}] + \Phi^{k-1} D\mathscr{W} \\
&\quad = (\mathscr{X}_{k-1} \sim \Phi^{k-1} D\mathscr{W}) + \Phi^{k-1} D\mathscr{W} \subset \mathscr{X}_{k-1}
\end{aligned}$$

and similarly $\mathscr{U}_k + F\Phi^{k-1} D\mathscr{W} \subset \mathscr{U}_{k-1}$. Hence, $\tilde{x}_{t+1+k-1|t+1} \in \mathscr{X}_{k-1}, \tilde{u}_{t+1+k-1|t+1} \in \mathscr{U}_{k-1}$ from which $\tilde{C}_{t+1}$ is admissible for x_{t+1}.

Theorem 8. *Provided that the initial state* x_0 *is feasible, system* (1) *under RPC feedback* $u_t = Fx_t + c_t$, *with* $c_t = \hat{c}_{t|t}$, *satisfies the following properties*: (i) $x_t \in \mathscr{X}$ *and* $u_t \in \mathscr{U}$ *for all* $t \geqslant 0$; (ii) $\lim_{t\to\infty} c_t = 0$; (iii) $x_t \to \mathscr{R}_\infty$ *as* $t \to \infty$.

Proof. The hypothesis that x_0 is feasible along with (26) imply that x_t is feasible for all $t \geqslant 0$ and hence (i) holds. To show convergence of c_t, introduce the Bellman function $V_t = V(x_t) \triangleq \sum_{k=0}^{N-1} \hat{c}_{t+k|t}^{\mathrm{T}} \Psi \hat{c}_{t+k|t}$, $\Psi = \Psi^{\mathrm{T}} > 0$. Define the cost $\tilde{V}_{t+1}$ associated to the control sequence $\tilde{C}_{t+1}$, i.e. $\tilde{V}_{t+1} \triangleq \sum_{k=1}^{N-1} \hat{c}_{t+k|t}^{\mathrm{T}} \Psi \hat{c}_{t+k|t} = V_t - c_t^{\mathrm{T}} \Psi c_t$. However, at sample instant $t+1$, an admissible $\hat{C}_{t+1}$ is selected to minimize V_{t+1} and therefore can be different from $\tilde{C}_{t+1}$. Hence it is clear that

$$V_{t+1} \leqslant \tilde{V}_{t+1} \Rightarrow V_t - V_{t+1} \geqslant c_t^{\mathrm{T}} \Psi c_t \geqslant 0. \tag{27}$$

Hence $\{V_t\}_{t \geqslant 0}$ is a nonnegative monotonic nonincreasing scalar sequence and, as $t \to \infty$, must converge to $V_\infty < \infty$. Summing the $V_t - V_{t+1}$ of (27), for t from 0 to ∞, we have

$$\infty > V_0 - V_\infty \geqslant \sum_{t=0}^{\infty} c_t^{\mathrm{T}} \Psi c_t \geqslant 0 \Rightarrow \lim_{t \to \infty} c_t^{\mathrm{T}} \Psi c_t = 0$$

which, as $\Psi > 0$, proves (ii). Finally, thanks to assumption (A4) and (ii),

$$\lim_{t \to \infty} x_t = \lim_{t \to \infty} \left[\Phi^t x_0 + \sum_{k=1}^{t} \Phi^{k-1} B c_{t-k} + \sum_{k=1}^{t} \Phi^{k-1} D w_{t-k} \right] = \lim_{t \to \infty} \left[\sum_{k=1}^{t} \Phi^{k-1} D w_{t-k} \right]$$

which, in turn, proves (iii). □

Hence, by virtue of Theorem 8, under initial feasibility RPC guarantees convergence of the state to the terminal region $\Omega = \mathscr{R}_\infty$. Notice that (18) does not ensure (26); in fact, even if x_t is feasible from which $x_{t+1|t} \in \mathscr{X}$ and $u_{t+1|t} \in \mathscr{U}$, x_{t+1} may not be feasible in that $x_{t+1} = x_{t+1|t} + D w_t$ and/or $u_{t+1} = u_{t+1|t} + F D w_t$ may not belong to the respective sets $\mathscr{X}$ and $\mathscr{U}$.

4.2. The domain of attraction

From an analysis point of view, it is important to characterize the set of feasible initial conditions for the RPC algorithm. Since there may not exist admissible control sequences for all $x \in \mathscr{X}$, feasible initial conditions will form a subset of $\mathscr{X}$; this set will be denoted by $\underline{\Sigma}_N$ to stress its dependence on the control horizon N. By the previous theorem, $\underline{\Sigma}_N$ clearly coincides with the *domain of attraction* for the closed-loop system under RPC feedback in that each state $x_0 \in \underline{\Sigma}_N$ is asymptotically steered to the terminal region $\mathscr{R}_\infty$. Consider system (12) whose response clearly depends on x_0, $\{w_t\}_{t \geqslant 0}$ and $\{c_t\}_{t \geqslant 0}$. Then, $\underline{\Sigma}_N$ is the set of all initial states x_0 for which there exists a control sequence $\{c_0, c_1, \ldots, c_{N-1}, 0, 0, \ldots\}$ such that the disturbance-free response of (12), i.e. the response to $w_t = 0$ for all $t \geqslant 0$, satisfies the restricted constraints $x_t \in \mathscr{X}_t$ and $u_t \in \mathscr{U}_t$ for all $t \geqslant 0$, where $\mathscr{X}_t$ and $\mathscr{U}_t$ are defined in (22). $\underline{\Sigma}_N$ is also the set of initial conditions x_0 for which there exists a sequence $C(x_0) = \{c_0(x_0), c_1(x_0), \ldots, c_{N-1}(x_0), 0, 0, \ldots\}$ which drives the plant state into Σ_0 in N steps, for all admissible disturbances. Since $\underline{\Sigma}_0 = \Sigma_0$ is d-invariant, $\{\underline{\Sigma}_N\}$ is a nested sequence, i.e. $\underline{\Sigma}_N \subseteq \underline{\Sigma}_{N+1}$. Whenever $\mathscr{W}, \mathscr{X}, \mathscr{U}$ are convex polyhedra, $\underline{\Sigma}_0$ and, consequently, $\underline{\Sigma}_N$ are also convex polyhedra. In particular,

$$\underline{\Sigma}_N = \{x \mid \exists c \colon M_N c + L_N x \leqslant v_n\}$$

and can be easily computed by LP techniques.

5. Comparisons

The aim of this section is to assess the performance of the RPC algorithm and to compare it with existing predictive control techniques. In particular, the following issues will be specifically addressed: (a) comparison with NPC; (b) comparison with the "deadbeat" RPC of Gossner et al. (1997); (c) connection with "command governors". The arguments will also be illustrated with the aid of the running example described hereafter.

Running example. Consider systems (1)–(4) and nominal feedback (5) with

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F = [-0.7434, -1.0922], \tag{28}$$

$$\mathscr{X} = \mathbb{R}^2, \quad \mathscr{U} = [-1, 1], \quad \mathscr{W} = \{w \colon \|w\|_\infty \leqslant 0.12\},$$

where F is the LQ gain corresponding to $Q = I_2$ and $R = 0.01$ in (19). Notice that the system is *antistable*, i.e. has both eigenvalues strictly outside the unit circle.

All the figures referred to in the sequel, will concern the above running example.

5.1. Comparison with NPC

Let us denote by $\bar{\Sigma}_N$ the domain of attraction of NPC when $w_t = 0$, that is the disturbance free case. $\bar{\Sigma}_N$ represents the set of states which would be steered asymptotically to the origin by NPC in the absence of disturbances. In particular, $\bar{\Sigma}_0$ is the largest invariant subset of $\mathscr{X}_c$.

For fixed N and F, it is interesting to compare the domains of attraction $\underline{\Sigma}_N$ and $\bar{\Sigma}_N$ to quantify the performance limitations due to the presence of disturbances; of course, $\underline{\Sigma}_N \subset \bar{\Sigma}_N$. Fig. 1 shows the three sets $\mathscr{R}_\infty \subset$

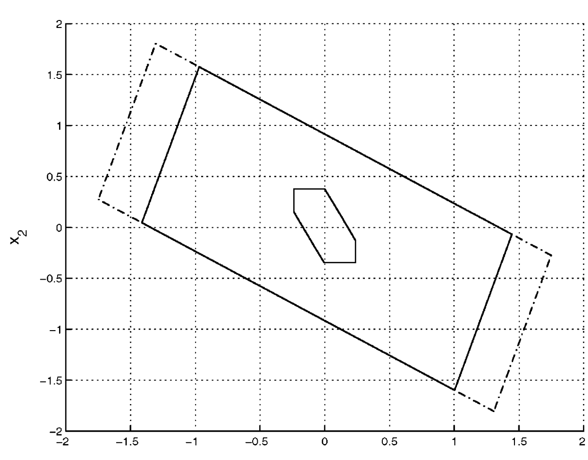

Fig. 1. Sets $\mathscr{R}_\infty \subset \underline{\Sigma}_0 = \Sigma_0 \subset \bar{\Sigma}_0$.

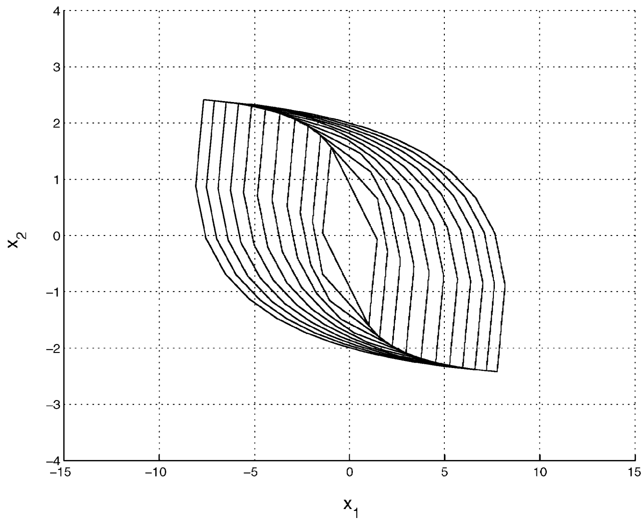

Fig. 2. Sets $\underline{\Sigma}_N$, $1 \leqslant N \leqslant 10$.

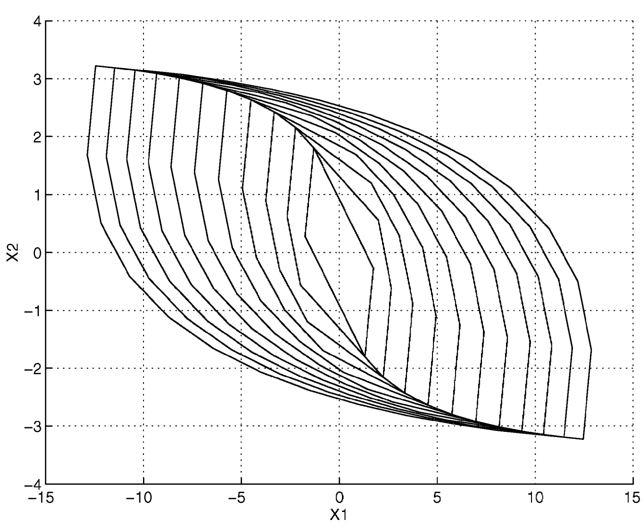

Fig. 3. Sets $\bar{\Sigma}_N$, $1 \leqslant N \leqslant 10$.

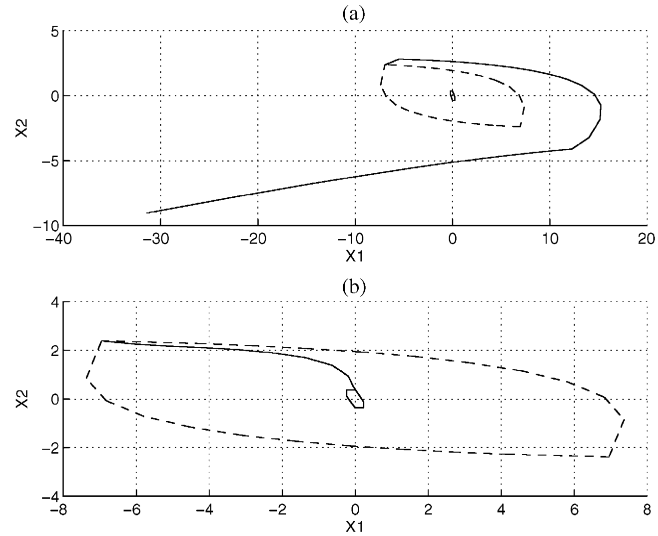

Fig. 4. State response x_t: (a) NPC; (b) RPC.

$\underline{\Sigma}_0 \subset \bar{\Sigma}_0$ (for both $\underline{\Sigma}_0$ and $\bar{\Sigma}_0$ the finite determination index is $i^* = 2$). Figs. 2 and 3 show $\underline{\Sigma}_N$ and, respectively, $\bar{\Sigma}_N$ for $N = 1,2,\dots,10$. Due to antistability of the system, both sequences of sets converge to compact sets.

Simulations compare RPC using restricted constraints (21) and (22) with NPC which imposes the original unrestricted constraints (18). A control horizon $N = 10$ has been selected. In all simulations the possibly infeasible control variable $Fx_t + c_t$ provided by the predictive controller has been clipped in $[-1,1]$ so as to satisfy the hard bounds imposed on the actuator input. Fig. 4 shows the state responses of both RPC and NPC for an initial state $x_0 = [-6.9, 2.38]^{\mathrm{T}} \in \underline{\Sigma}_{10}$ and an extremal (worst-case) disturbance realization. It can be seen that NPC loses feasibility and gives an unstable response of x_t (Fig. 4a). RPC on the other hand, as expected, guarantees state convergence to the robust control invariant set $\mathscr{R}_\infty$, see Fig. 4b. This figure also depicts the sets $\underline{\Sigma}_{10}$ (the domain of attraction of RPC for $N = 10$) and $\mathscr{R}_\infty$.

It is clear that RPC has coped well with the unknown disturbance and still was able to drive the state close to the origin. NPC, without knowledge and use of the restricted constraints was unstable.

5.2. Comparison with "deadbeat" RPC

One of the extensions of the proposed algorithm with respect to the work of Gossner et al. (1997) is the possibility to use any stabilizing, not necessarily deadbeat, terminal feedback F. To this end, notice that the size of the region $\underline{\Sigma}_N$, besides getting larger for an increasing control horizon N, also depends critically on the choice of F.

Fig. 5 compares the regions $\underline{\Sigma}_0$, $\underline{\Sigma}_5$ and $\underline{\Sigma}_{10}$ of Fig. 1 with the ones relative to the deadbeat regulator. It is evident that the choice of F in (28) outperforms the deadbeat regulator in terms of size of the feasibility regions. Notice, in particular, that F in (28) with $N = 5$ provides a feasibility region which is approximately the same of the deadbeat feedback with $N = 10$. That is, in general, a dead-beat terminal condition implies a higher computational load for the same set sizes.

5.3. Connection with the Command Governor (CG)

The framework adopted in this paper also allows us to consider the so-called *Command Governor* (CG) approach to the control of constrained systems (see Casavola & Mosca (1996); Gilbert & Kolmanovsky (1999)) for CGs in presence of persistent disturbances. Although CGs have been introduced for set-point tracking, they can be analysed, without almost any loss of generality, in a

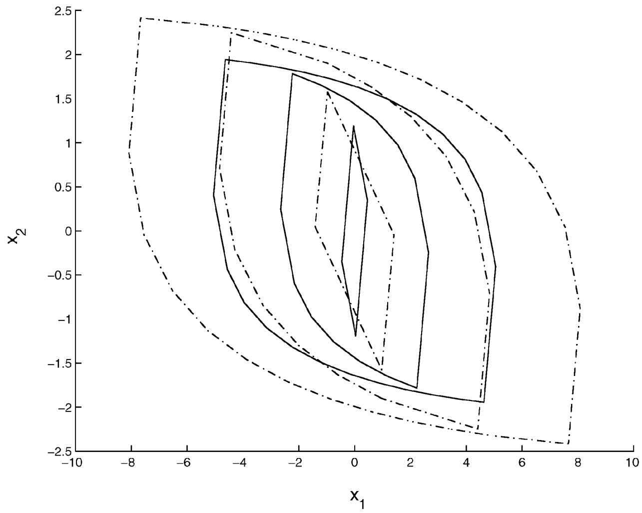

Fig. 5. Sets $\underline{\Sigma}_0$,$\underline{\Sigma}_5$,$\underline{\Sigma}_{10}$ relative to the feedback F in (28) (dash–dotted lines) and to the deadbeat feedback (solid lines).

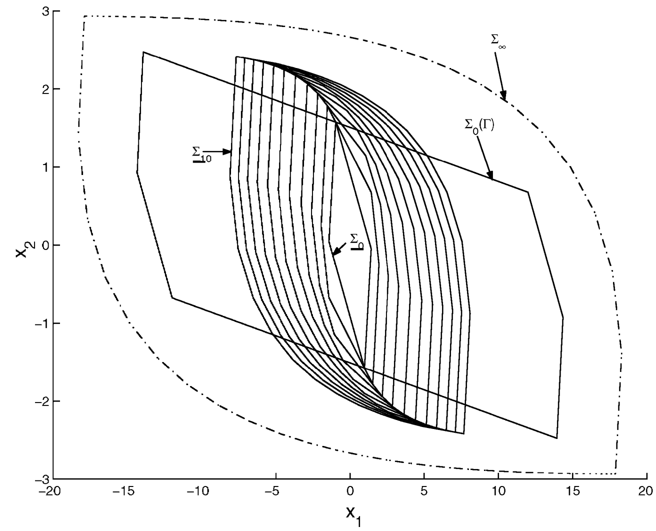

Fig. 6. Sets $\Sigma_0(\Gamma)$, Σ_∞ and $\underline{\Sigma}_N$, $1 \leqslant N \leqslant 10$.

regulation context. Essentially the CG can be recast in the formulation adopted in this paper by

(1) setting $N = 0$;
(2) considering, in place of (16), a control sequence

$$u_{t+k|t} = Fx_{t+k|t} + \gamma, \quad k \geqslant 0,$$

parametrized by a constant command $\gamma \in \mathbb{R}^m$;
(3) minimizing, with respect to γ, the cost

$$J(\gamma) \triangleq \gamma^{\mathrm{T}} \Psi_\gamma \gamma \triangleq \|\gamma\|^2_{\Psi_\gamma}$$

where $\Psi_\gamma = \Psi_\gamma^{\mathrm{T}} > 0$.

The CG uses, therefore, the "constant command" γ as a degree of freedom for the satisfaction of constraints (3) instead of the N control moves of predictive control. This different parametrization of the control sequence implies, in turn, different properties (e.g. domain of attraction and performance) of the CG and RPC, respectively.

In order to highlight the connection between the CG and RPC, some new sets have to be introduced. Let $x(\gamma) = (I - \Phi)^{-1} B\gamma$ and $u(\gamma) = [F(I - \Phi)^{-1}B + I]\gamma$ denote the equilibrium state and, respectively, input corresponding to γ. Then,

$$\Gamma \triangleq \{\gamma : \ x(\gamma) \in int(\mathscr{X}_\infty), u(\gamma) \in int(\mathscr{U}_\infty)\}$$

is the set of *statically admissible* commands, i.e. the set of γ which satisfy the constraints on x and u in steady state. For each $\gamma \in \Gamma$,

$$\Sigma_0(\gamma) \triangleq \{x : \Phi^k(x - x(\gamma)) + x(\gamma) \in \mathscr{X}_k \text{ and } F\Phi^k(x - x(\gamma)) + u(\gamma) \in \mathscr{U}_k, \ \forall k \geqslant 0\}$$

is the largest d-invariant subset of $\mathscr{X}_c$ under the control law $u_t = Fx_t + \gamma$. Clearly $\Sigma_0 = \Sigma_0(0)$. Hence, the CG can also be interpreted as follows: at time t, given the state x_t, it provides

$$\gamma_t = \gamma(x_t) = \arg\min_{\gamma \in \Gamma} \|\gamma\|^2_{\Psi_\gamma} \quad \text{subject to } x_t \in \Sigma_0(\gamma)$$

and applies the feedback $g(x_t) = Fx_t + \gamma(x_t)$ to the plant. Let

$$\Sigma_0(\Gamma) \triangleq \bigcup_{\gamma \in \Gamma} \Sigma_0(\gamma). \tag{29}$$

In the CG literature it has been proved that if $x_0 \in \Sigma_0(\Gamma)$ there exists a finite time $\bar{t}$ such that $\gamma_t = 0$ for $t \geqslant \bar{t}$. Hence the CG-regulator ensures the desired regulation properties (R1) and (R2) with a domain of attraction $\Sigma_0(\Gamma)$ to be compared with $\underline{\Sigma}_N$ of RPC. Such a comparison is illustrated in Fig. 6 relatively to the considered example for which $\Gamma = [-8.64, 8.64]$. Notice that the CG and RPC have complementary effects on the domain of attraction: the CG enlarges the domain Σ_0 of the linear regulator (5) to $\Sigma_0(\Gamma)$ by (29), i.e. varying γ over Γ; conversely RPC enlarges Σ_0 to $\underline{\Sigma}_N$ via N free control moves. Clearly, the increase of N improves performance; recall in fact that if F and Ψ are chosen according to some infinite-horizon quadratic criterion, then RPC approaches the infinite-horizon constrained LQ regulator as $N \to \infty$.

From the above interpretation it is evident that not only the CG can be exploited for regulation, as described above, but also RPC can be used to some extent for set-point tracking. For a given admissible command value γ, it suffices to apply RPC in the coordinates $\tilde{x} = x - x(\gamma)$ and $\tilde{u} = u - u(\gamma)$. In fact, the feedback control law $\tilde{u} = F\tilde{x}$ automatically provides the feedforward command $\gamma = u(\gamma) - Fx(\gamma)$. It turns out that x_t is feasible

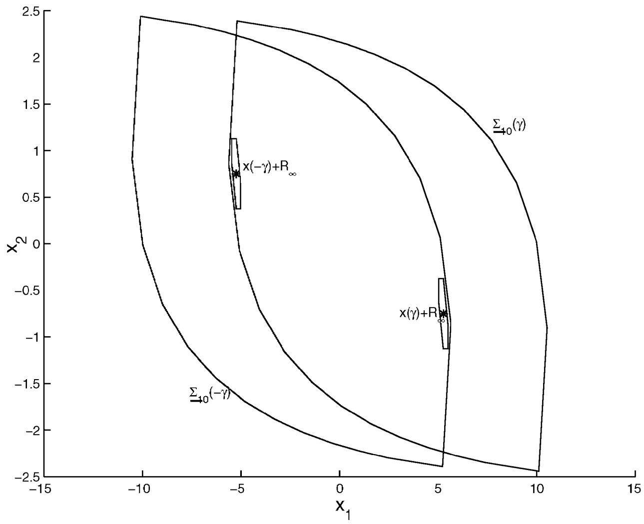

Fig. 7. Sets $x(\gamma) + \mathscr{R}_\infty \subset \underline{\Sigma}_{10}(-\gamma)$ and $x(-\gamma) + \mathscr{R}_\infty \subset \underline{\Sigma}_{10}(\gamma)$ for $\gamma = 3.3$.

for a given $\gamma \in \Gamma$ if $x_t \in \underline{\Sigma}_N(\gamma)$ where $\underline{\Sigma}_N(\gamma)$ is obtained from $\Sigma_0(\gamma)$ adding N free control moves. Assuming that after each setpoint change there is enough time to reach ultimate boundedness in the neighborhood $\mathscr{R}_\infty$ of the desired equilibrium, the transition from γ_1 to γ_2 is possible if $x(\gamma_1) + \mathscr{R}_\infty \subset \underline{\Sigma}_N(\gamma_2)$. Fig. 7 illustrates the capabilities of the RPC regulator as a set-point tracker: with a control horizon $N = 10$ it is possible to move the set-point in the interval $[-3.3, 3.3]$ without losing feasibility. For a sufficiently large N it is clearly possible to achieve the full set-point range $\Gamma = [-8.64, 8.64]$.

The complementary benefits of the CG (degree of freedom γ) and of predictive control ($c_{t+k|t}$, $0 \leqslant k \leqslant N-1$) can, of course, be combined (Rossiter, Gossner, & Kouvaritakis, 1996) by parametrizing the control sequence as

$$u_{t+k|t} = Fx_{t+k|t} + c_{t+k|t} + \gamma, \quad k \geqslant 0,$$

$$c_{t+k|t} = 0, \quad k \geqslant N.$$

The invariant set analysis tools (Kolmanovsky & Gilbert, 1995; Blanchini, 1999) allow the computation of the set $\underline{\Sigma}_N(\Gamma)$ and, hence, can be used to tune off-line the control horizon N according to the desired set-point range, state-space operating region and performance level.

6. Conclusion

The problem of regulating, via state feedback, a system subject to both hard constraints and persistent bounded disturbances by means of predictive control techniques has been addressed. It has been shown that suitable constraint restrictions can be adopted in order to robustify the predictive controller with respect to the unknown but bounded disturbance in such a way that, if the initial state is feasible, feasibility is guaranteed in the future system evolution for all admissible disturbance realizations. Since the constraint restrictions can be computed off-line, this method has the advantage of requiring no significant on-line extra computational burden.

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