2019 18th European Control Conference (ECC)
Napoli, Italy, June 25-28, 2019

Collision avoidance for uncertain nonlinear systems with moving obstacles using robust Model Predictive Control

Raffaele Soloperto[1], Johannes Köhler[1], Matthias A. Müller[2], Frank Allgöwer[1]

Abstract— **In this paper, we provide a novel robust collision avoidance approach that is based on a general tube-based MPC framework. We consider collision avoidance for general nonlinear uncertain systems with moving obstacles. The resulting optimization problem can be handled by standard nonlinear programming solvers. Moreover, we provide formal guarantees, such as recursive feasibility, constraint satisfaction, as well as robust collision avoidance. We demonstrate the efficacy of the proposed method through a simulation of an autonomous car during realistic manoeuvres.**

I. INTRODUCTION

Motivation

In several practical applications, an autonomous system evolves in an environment with obstacles, which may be represented by humans, objects, as well as other systems. In such scenarios, a fundamental task of the controller is the ability to plan a collision-free trajectory in order to guarantee the safety of the controlled object and of the obstacles. Facilitated by the increase of computational power, real-time optimization-based trajectory planning has become more common in different areas, spanning from autonomous cars to robots [1], [2]. These methods rely on the online solution of an optimal control problem, based on the knowledge of the system and of the surrounding environment. In particular, Model Predictive Control (MPC) is one of the most promising approaches to handle such multivariable constrained control problems [3], [4], [5]. MPC uses the model of the process to control for predicting the evolution of the system over a finite horizon while optimizing a user-defined cost function. This procedure is then repeated at each time instant in a receding horizon fashion. In general, the performance of an optimization-based trajectory planning approach is influenced by the uncertainty in the model and in the environment. Guaranteeing hard constraint satisfaction (e.g. collision avoidance) in the presence of uncertainty is of crucial importance, and can be tackled within a robust MPC framework. In the literature, different such robust MPC approaches have been proposed. Within this work, we exploit so-called tube-based methods such as, e.g., [6], [7], [8], [9], [10]. These methods typically construct a pre-stabilizing controller together with a tube that contains the real system state. Robust constraint satisfaction is then ensured by tightening the nominal constraints according to the constructed tube.

In this paper, we make use of the ability of robust MPC to predict both the tube containing the actual trajectory of the system, as well as the uncertain movements of the obstacles, in order to guarantee safe collision avoidance in presence of uncertainty.

[1]Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany, e-mail: {raffaele.soloperto, johannes.koehler, frank.allgower}@ist.uni-stuttgart.de.

[2] Matthias A. Müller is with the Institute of Automatic Control, Leibniz University Hannover, 30167 Hannover, Germany, e-mail: mueller@irt.unihannover.de.

This work was supported by the International Max Planck Research School for Intelligent Systems (IMPRS-IS), and by the German Research Foundation under Grant GRK 2198/1.

Related work

There exists a vast amount of literature on collision avoidance, spanning from optimization-based collision avoidance methods, to dynamic programming [11], graph search [12] or reachability analysis [13]. However, since collision avoidance problems are, in general, NP-hard [14], the majority of the practical approaches are often problem specific. For a comprehensive review on existing obstacles avoidance algorithms, we refer the reader to [15], [16], [17]. Optimization-based collision avoidance methods express the collision avoidance problem as an optimal control problem, which is then solved through numerical optimization techniques. Common approaches in obstacle avoidance study the simplified case of point-mass models, and then consider the shape of the object by inflating the obstacle. The case of full-dimensional objects has been studied in [1], where the authors assume that all the involved objects are rectangular, and then formulate collision avoidance by keeping all the vertices of the controlled object outside the obstacles. In [18], the authors consider a general full-dimensional object, introduce the notion of signed distance, and propose a sequential linearization algorithm in order to overcome the non-convexity of the signed distance problem.

Collision avoidance can be incorporated into standard MPC approaches as general nonlinear constraints. However, this results in a non-convex and, in general, non-smooth optimization problem, which might not be suitable for standard solvers, and thus real time application. In [19], a smooth reformulation of the collision avoidance problem for point-mass controlled objects and polyhedral obstacles is proposed, while in [20] this procedure is generalized to full-dimensional objects, where exact knowledge of the controlled object and of the obstacles is considered.

Contribution

In this paper, we provide a novel robust collision avoidance approach that is based on a general tube-based robust MPC framework. In order to obtain a smooth reformulation of the collision avoidance constraints, we utilize the methods

978-3-907144-00-8 ©2019 EUCA

in [19], [20]. The novelty of the proposed method can be summarized as follows

- General uncertain nonlinear systems can be handled. We consider general classes of uncertainty, such as model mismatch and external disturbances.
- We consider the movement of obstacles to be subject to uncertainty, and to change their position and shape in time.
- Formal theoretical guarantees are provided, such as recursive feasibility, constraint satisfaction and robust collision avoidance.

The proposed framework is formulated such that various existing tube-based MPC approaches can be employed. As a particular example, we show the efficacy of the proposed approach by using a novel nonlinear robust MPC framework, proposed in [10]. As a simulation example, we demonstrate how a nonlinear system, representing a car, behaves during an over-taking manoeuvre in a realistic scenario. A video animation of the shown example can be found at `https://www.youtube.com/watch?v=YeftO1QYJk8`.

Notation

The quadratic norm with respect to a positive definite matrix $Q = Q^\top$ is denoted by $\|x\|_Q^2 = x^\top Q x$. The Minkowski sum of two sets is defined by $\mathbb{U} \oplus \mathbb{V} := \{u+v : u \in \mathbb{U}, v \in \mathbb{V}\}$. The positive real numbers are $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$. The power set of a set $\mathbb{S}$ is denoted by $2^{\mathbb{S}}$.

II. Standard Robust MPC

This section introduces the nonlinear system dynamics, and summarizes the general tube-based MPC approach.

A. Nonlinear System

We consider the following nonlinear perturbed discrete-time system

$$x_{t+1} = f_w(x_t, u_t, d_t). \tag{1}$$

with state $x_t \in \mathbb{R}^n$, control input $u_t \in \mathbb{R}^m$, and disturbance $d_t \in \mathbb{D} \subset \mathbb{R}^q$. We impose state and input constraints as follows

$$(x_t, u_t) \in \mathcal{Z}, \quad t \geq 0, \tag{2}$$

for some $\mathcal{Z} \subseteq \mathbb{R}^{n+m}$.

B. Robust MPC scheme

In the following, we present the general formulation of a robust MPC scheme, together with the related assumptions.

Assumption 1. *There exists a function $\Phi : 2^{\mathbb{R}^n} \times \mathbb{R}^m \times 2^{\mathbb{R}^q} \to 2^{\mathbb{R}^n}$ that bounds all the possible open-loop states, for a given input u_t, such that*

$$f_w(x_t, u_t, d_t) \in \mathbb{X}_{t+1} := \Phi(\mathbb{X}_t, u_t, \mathbb{D}),$$

for all $x_t \in \mathbb{X}_t$, $(x_t, u_t) \in \mathbb{Z}$ and all $d \in \mathbb{D}$. Furthermore, for all $u \in \mathbb{U}$, the function Φ satisfies the following monotonicity property

$$\mathbb{X}_1 \subseteq \mathbb{X}_2 \Rightarrow \Phi(\mathbb{X}_1, u, \mathbb{D}) \subseteq \Phi(\mathbb{X}_2, u, \mathbb{D}). \tag{3}$$

Remark 1. *In the case of unstable systems, the function Φ can grow unbounded, hence, a pre-stabilizing control law is added and then the reformulated stabilized system is considered, compare [6], [10].*

We define the open-loop cost as follows

$$J_N(\mathbb{X}_{\cdot|t}, u_{\cdot|t}) = \sum_{k=0}^{N-1} \ell(\mathbb{X}_{k|t}, u_{k|t}) + V_f(\mathbb{X}_{N|t}), \tag{4}$$

with some stage cost ℓ and terminal cost V_f.

Assumption 2. *There exists a terminal region $\mathcal{X}_f \subseteq \mathbb{R}^n$, a terminal cost $V_f : 2^{\mathbb{R}^n} \to \mathbb{R}$, and a terminal controller $\kappa_f : 2^{\mathbb{R}^n} \to \mathbb{R}^m$ such that for all $\mathbb{X} \subseteq \mathcal{X}_f$ and for all $x \in \mathbb{X}$*

$$\begin{aligned}
&\Phi(\mathbb{X}, \kappa_f(\mathbb{X}), \mathbb{D}) \subseteq \mathcal{X}_f, \\
&V_f(\Phi(\mathbb{X}, \kappa_f(\mathbb{X}))) \leq V_f(\mathbb{X}) - l(\mathbb{X}, \kappa_f(\mathbb{X})), \\
&(x, \kappa_f(\mathbb{X})) \in \mathcal{Z}.
\end{aligned}$$

Finally, the optimization problem is defined as follows

$$V_N(x_t) := \min_{u_{\cdot|t}, \mathbb{X}_{\cdot|t}} J_N(\mathbb{X}_{0|t}, u_{\cdot|t}) \tag{5a}$$

$$\text{s.t. } \mathbb{X}_{k+1|t} = \Phi(\mathbb{X}_{k|t}, u_{k|t}, \mathbb{D}), \tag{5b}$$

$$\{x_t\} \in \mathbb{X}_{0|t}, \tag{5c}$$

$$\mathbb{X}_{N|t} \subseteq \mathcal{X}_f, \tag{5d}$$

$$(x_{t+k}, u_{t+k}) \in \mathcal{Z}, \ \forall x_{t+k} \in \mathbb{X}_{k|t}, \tag{5e}$$

$$k = 0, \ldots, N-1.$$

The solution of (5) are the optimal sets $\mathbb{X}^*_{\cdot|t}$ and the input trajectory $u^*_{\cdot|t}$. The resulting closed-loop system is given by

$$x_{t+1} = f_w(x_t, u_t, d_t), \quad u_t = u^*_{0|t}. \tag{6}$$

Remark 2. *Assumption 1 is a general condition used in tube-based MPC. In this paper, we do not consider a specific choice of Φ, but we rather focus on a more general and conceptual level. This gives the user the freedom to choose any tube-based robust MPC formulation satisfying Assumption 1. In the following, we briefly discuss how Assumption 1 is exemplarily satisfied in [6], [7], [8], [9], [10].*

1) Linear systems subject to bounded additive disturbance.:

- *In [6] the sets $\mathbb{X}_{\cdot|t}$ represent the forward reachable sets of the stabilized disturbed system, starting from $\mathbb{X}_{0|t} = \{x_{0|t}\}$, and centered around some nominal predicted trajectory $x_{\cdot|t}$.*
- *In [7] the sets $\mathbb{X}_{\cdot|t}$ have a constant size and are given by the so-called minimal disturbance invariant set (compare [21]) centered around some nominal predicted trajectory $x_{\cdot|t}$. The initial state $x_{0|t}$ of such a trajectory is optimized at each time instant.*

2) Nonlinear system subject to bounded disturbance:

- *In [8] only additive disturbances are considered. The sets $\mathbb{X}_{\cdot|t}$ are hyperboxes constructed to over-approximate the forward reachable sets of the system. Since the system is not assumed to be stable (nor pre-stabilized), the sets $\mathbb{X}_{\cdot|t}$ might be subject to a fast* *and unbounded increase. The initial set is defined as $\mathbb{X}_{0|t} := \{x_t\}$.*

- *In [9] nonlinear input-affine uncertain systems are considered. The sets $\mathbb{X}_{k|t}$ are ellipsoidal robust forward invariant tubes, parametrized through online optimized matrices $Q_{k|t} \in \mathbb{R}^{n\times n}$, and centered around some nominal trajectory, starting from $\mathbb{X}_{0|t} := \{x_t\}$.*
- *In [10] uncertain nonlinear systems are considered, which are assumed to be locally incrementally stabilizable. The initial set is defined as $\mathbb{X}_{0|t} := \{x_t\}$. The tube $\mathbb{X}_{\cdot|t}$ is then implicitly defined by an additional scalar optimization variable that characterizes the sublevel sets of an incremental Lyapunov function, centered around some nominal trajectory (compare Sec. IV).*
  In both [9] and [10], the tubes $\mathbb{X}_{k|t}$ are bounded for stable and unstable systems, for any $k \geq 0$. Moreover, the optimal control problem (5) *can also aim at minimizing their size. In [9], the predicted tubes might be more accurate, i.e. smaller, than the tubes predicted as proposed in [10], on the other hand, the approach in [9] strongly increases the computational demand for large n, while in [10] the increase in computational demand is moderate.*

Theorem 1. *Let Assumptions 1 and 2 hold, and suppose that Problem* (5) *is feasible at time $t = 0$. Then* (5) *is recursively feasible and the constraints* (2) *are satisfied for the closed-loop system* (6).

Proof. This is a standard result in robust MPC, see, e.g., [6], [7], [8], [9], [10]. □

III. Robust Collision Avoidance

This section shows how to incorporate robust collision avoidance constraints into a general robust MPC framework as shown in Section II-B. In particular, Section III-A and Section III-B mathematically describe the property of the object and of the obstacles, respectively. Section III-D provides additional conditions on the tube and on the so-called *safe terminal region*. In Section III-E, we show how the general collision avoidance constraints can be converted into a smooth reformulation, based on [19], [20]. Finally, in Section III-F we present the proposed robust MPC formulation for collision avoidance, and the theoretical analysis is detailed in Section III-G.

Remark 3. *For simplicity, we will consider the controlled object, the obstacles and their over-approximations, to be polytopes. However, any compact convex sets, represented by a conic relation, can be used without loss of theoretical guarantees.*

A. Object description

In this paper, we consider the controlled object $\mathbb{E}(x_t)$ to be modelled as a *point-mass*, i.e. with no shape and dimension, as well as a *full-dimensional* object.

1) Point-mass Object: The object has no shape nor dimension, hence, it can be described by its position, defined by a general nonlinear function of the state, $p_x : \mathbb{R}^n \to \mathbb{R}^{n_p}$, as follows

$$\mathbb{E}(x_t) := p_x(x_t). \tag{7}$$

2) Full-Dimensional Controlled Object: The object occupies a certain space $\mathbb{E}(x_t) \subseteq \mathbb{R}^{n_p}$, defined as

$$\mathbb{E}(x_t) := p_x(x_t) \oplus \mathbb{B}(x_t), \tag{8}$$

where $\mathbb{B}(x_t)$ is a convex polytope

$$\mathbb{B}(x_t) := \{y \in \mathbb{R}^{n_p} : E(x_t)y \leq e(x_t)\}, \tag{9}$$

for some $E : \mathbb{R}^n \to \mathbb{R}^{n_h \times n_p}$, $e : \mathbb{R}^n \to \mathbb{R}^{n_h}$, where $n_h \in \mathbb{N}$ is the number of half-spaces defining the polytope $\mathbb{B}$.

Remark 4. *Note that* (8) *and* (9) *do not impose a fixed shape nor a fixed size to the controlled object.*

A common formulation of $\mathbb{E}(x_t)$ is the following

$$\mathbb{E}(x_t) = p_x(x_t) \oplus R(x_t)\mathbb{B},$$

where $R(x_t)$ is the rotation matrix and $\mathbb{B}$ is the physical shape of the object.

B. Obstacle Description

In this paper, we consider $M \in \mathbb{N}$, $M \geq 0$ obstacles that the system must avoid. We study the case where each obstacle may move its center and/or vary its shape with time, i.e., can be defined as a general set $\mathbb{O}^m_t \subset \mathbb{R}^{n_p}$, $m = 1, \ldots, M$. We do not assume to exactly know the current space occupied by the obstacle, nor its evolution in time $\mathbb{O}^m_{t+k}$, $k \geq 0$, however, we assume to know an over-approximation of $\mathbb{O}^m_{t+k}$.

Assumption 3. *There exist known prediction sets $\mathbb{O}^m_{k|t}$ defined as*

$$\mathbb{O}^m_{k|t} = \{y \in \mathbb{R}^{n_p} : A^m_{k|t} y \leq b^m_{k|t}\}, \quad k = 0, \ldots, N, \tag{10}$$

such that

$$\mathbb{O}^m_{t+k} \subseteq \mathbb{O}^m_{k|t},\ k = 0, \ldots, N,\ m = 1, \ldots, M \tag{11a}$$

$$\mathbb{O}^m_{k|t+1} \subseteq \mathbb{O}^m_{k+1|t},\ k = 0, \ldots, N-1,\ m = 1, \ldots, M. \tag{11b}$$

Condition (11a) implies that the obstacle at time $t + k$ is contained in the k-steps ahead prediction at time t of the obstacle, while (11b) ensures that the prediction of the obstacle gets more accurate with time, see Fig. 1. This concept is similar to the tube constructed in a general tube-based MPC, where the evolution of the system predicted at time $t+1$ for k steps ahead is contained inside the predicted evolution at time t for $k+1$ steps ahead.

C. Obstacle avoidance formulation

The mathematical formulation of collision avoidance can be written as follows

$$\mathbb{E}(x_t) \cap \mathbb{O}_t^m = \emptyset, \; m = 1, \dots, M, \; t \geq 0. \tag{12}$$

However, (12) is, in general, non-convex and non-differentiable. A common way of smoothly formulating collision avoidance is based on the notion of *distance* [19]:

$$\text{dist}(\mathbb{E}(x), \mathbb{O}) := \min_r \{\|r\| : (\mathbb{E}(x) \oplus r) \cap \mathbb{O} \neq \emptyset\}.$$

Collision avoidance can then be ensured by requiring

$$\text{dist}(\mathbb{E}(x), \mathbb{O}) > d_{\min} \geq 0. \tag{13}$$

Remark 5. *An alternative solution to the distance function is the so-called signed distance function. The usage of the signed distance function can modify the strong collision avoidance constraint in* (12)*, into soft constraints by allowing the object to penetrate the obstacle, compare [19], [20].*

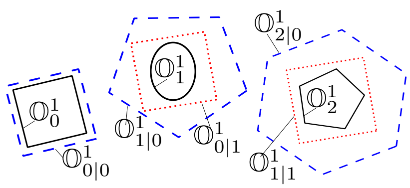


Fig. 1: Exemplary illustration of the real and predicted obstacle number 1 at time $t = 0, \; 1, \; 2$.

D. Tube for robust collision avoidance

Assumption 4. *There exists a region* $\mathbb{X}_{\mathit{safe}} \subseteq \mathbb{R}^{n_p}$*, such that*

$$\mathit{dist}(\mathbb{X}_{\mathit{safe}}, \mathbb{O}^m_{N|t}) > d_{\min}, \quad \forall t \geq 0, \; m = 1, \dots, M. \tag{14}$$

Assumption 4 guarantees the existence of a region such that if the object is contained in such a region then it does not collide with any obstacle.

Assumption 5. *There exists a function* $\Theta : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^{n_p}}$ *defined as*

$$\Theta(\mathbb{X}_{k|t}) := \mathbb{S}_{k|t} := \{y \in \mathbb{R}^{n_p} : G_{k|t} y \leq g_{k|t}\} \tag{15}$$

such that the following conditions are satisfied

$$\mathbb{E}(x) \subseteq \Theta(\mathbb{X}_{k|t}), \quad \forall x \in \mathbb{X}_{k|t}, \tag{16a}$$
$$\mathbb{X}_{k|t+1} \subseteq \mathbb{X}_{k+1|t} \Rightarrow \Theta(\mathbb{X}_{k|t+1}) \subseteq \Theta(\mathbb{X}_{k+1|t}), \tag{16b}$$
$$\Theta(\mathcal{X}_f) \subseteq \mathbb{X}_{\mathit{safe}}. \tag{16c}$$

The function Θ overapproximates the space occupied by the object knowing that the state is contained inside the set $\mathbb{X}_{k|t}$. In Section IV, we show an example of how $\mathbb{S}_{k|t}$ can be defined.

Remark 6. *The tightest set* $\Theta(\mathbb{X}_{k|t})$ *that satisfies* (16a), (16b) *is given by*

$$\Theta(\mathbb{X}_{k|t}) := \cup_{x \in \mathbb{X}_{k|t}} \mathbb{E}(x). \tag{17}$$

However, this set is not necessarily a polytope. Moreover, an analytical description of this set might be difficult to derive, hence, an over-approximation is typically used.

E. Smooth Obstacle Avoidance formulation

In the following, we exploit the subsequent proposition which has been established in [19], [20].

Proposition 1. *Let* $d_{\min} \geq 0$*. Then the following equivalence holds*

$$\begin{aligned}
&\mathit{dist}(\Theta(\mathbb{X}_{k|t}), \mathbb{O}^m_{k|t}) > d_{\min} \Leftrightarrow \exists \lambda^m_{k|t} \geq 0, \; \mu^m_{k|t} \geq 0 : \\
&- g^{m\top}_{k|t} \mu^m_{k|t} - b^{m\top}_{k|t} \lambda^m_{k|t} > d_{\min} \\
&G^{m\top}_{k|t} \mu^m_{k|t} + A^{m\top}_{k|t} \lambda^m_{k|t} = 0 \\
&\|A^{m\top}_{k|t} \lambda^m_{k|t}\| \leq 1
\end{aligned} \tag{18}$$

where $\lambda^m_{k|t}, \; \mu^m_{k|t} \in \mathbb{R}^{n_p}$.

Proposition 1 yields a smooth reformulation (18) of the posed collision avoidance constraints (13), which can be handled by standard gradient-based optimization algorithms.

F. Collision avoidance with robust MPC

In the following, we present the proposed robust MPC scheme. The basic idea is to employ the knowledge of the tubes $\mathbb{X}_{k|t}$ and sets $\Theta(\mathbb{X}_{k|t})$ to ensure robust constraint satisfaction and robust collision avoidance. The proposed nonlinear robust MPC scheme is based on the following optimization problem:

$$V_N(x_t, t) = \min_{u_{\cdot|t}, \mathbb{X}_{\cdot|t}, \lambda^m_{\cdot|t}, \mu^m_{\cdot|t}} J_N(\mathbb{X}_{\cdot|t}, u_{\cdot|t}) \tag{19a}$$
$$\text{s.t. } \mathbb{X}_{k+1|t} = \Phi(\mathbb{X}_{k|t}, u_{k|t}, \mathbb{D}), \tag{19b}$$
$$\{x_t\} \in \mathbb{X}_{0|t}, \tag{19c}$$
$$\mathbb{X}_{N|t} \subseteq \mathcal{X}_f, \tag{19d}$$
$$(x_{t+k}, u_{t+k}) \in \mathcal{Z}, \; \forall x_{t+k} \in \mathbb{X}_{k|t}, \tag{19e}$$
$$- g^\top_{k|t} \mu^m_{k|t} - b^{m\top}_{k|t} \lambda^m_{k|t} > d_{\min}, \tag{19f}$$
$$G^\top_{k|t} \mu^m_{k|t} + A^{m\top}_{k|t} \lambda^m_{k|t} = 0, \tag{19g}$$
$$\|A^{m\top}_{k|t} \lambda^m_{k|t}\| \leq 1, \; \lambda^m_{k|t} \geq 0, \; \mu^m_{k|t} \geq 0, \tag{19h}$$
$$k = 0, \dots, N-1, \quad m = 1, \dots, M,$$

with $A^m_{k|t}$, $b^m_{k|t}$ defined as in (10) and $G_{k|t}$, $g_{k|t}$ as in (15).

The solution of (19) are the optimal sets $\mathbb{X}^*_{\cdot|t}$, the input trajectory $u^*_{\cdot|t}$, and the dual variables $\lambda^{m*}_{\cdot|t}$, $\mu^{m*}_{\cdot|t}$.

The resulting closed-loop system is given by

$$x_{t+1} = f_w(x_t, u_t, d_t), \quad u_t = u^*_{0|t}. \tag{20}$$

G. Theoretical Analysis

We are now in a position to state the main result of this work.

Theorem 2. *Let Assumptions 1, 2, 3, 4 and 5 hold, and suppose that* (19) *is feasible at time* $t = 0$*. Then* (19) *is recursively feasible, the constraints* (2) *are satisfied and the object avoids the obstacles, satisfying* (13)*, for the closed-loop* (20).

Proof. As shown in [22, Prop. 3.31], (11b) in Assumption 3 implies that there exists a matrix $H^m_{k|t+1}$, with non-negative entries, such that, for $k=0,\dots,N-1$

$$H^m_{k|t+1}A^m_{k|t+1}=A^m_{k+1|t} \tag{21a}$$
$$H^m_{k|t+1}b^m_{k|t+1}\le b^m_{k+1|t} \tag{21b}$$

1) Candidate solution for robust MPC: Recursive satisfaction of (19b), (19c), (19d), (19e) follows as in Theorem 1. In particular, as is standard in robust MPC, to this end candidate sequences $\mathbb{X}_{\cdot|t+1}$ and $u_{\cdot|t+1}$ are defined satisfying

$$\mathbb{X}_{k|t+1}\subseteq\mathbb{X}^*_{k+1|t},\quad k=0,\dots,N-1, \tag{22a}$$
$$u_{k|t+1}:=u^*_{k+1|t},\quad k=0,\dots,N-2, \tag{22b}$$
$$u_{N-1|t+1}:=\kappa_f(\mathbb{X}_{N-1|t+1}), \tag{22c}$$
$$\mathbb{X}_{N|t+1}:=\Phi(\mathbb{X}_{N-1|t+1},u_{N-1|t+1},\mathbb{D}). \tag{22d}$$

Recursive satisfaction of (19d) is ensured by Assumption 2 and (22d), while recursive satisfaction of (19e) follows from the definition of the candidate solution in (22a), (22b), (22c), (22d) and Assumption 2.

2) Candidate Solution for collision avoidance: Assumptions 5 and (22a) imply that, for $k=0,\dots,N-1$, $\Theta(\mathbb{X}_{k|t+1})\subseteq\Theta(\mathbb{X}^*_{k+1|t})$, hence, again by [23, Prop. 3.31], there exists a matrix $F_{k|t+1}$ with non-negative entries, such that

$$F_{k|t+1}G_{k|t+1}=G_{k+1|t}, \tag{23a}$$
$$F_{k|t+1}g_{k|t+1}\le g_{k+1|t}. \tag{23b}$$

For $k=0,\dots,N-2$, we define

$$\lambda^m_{k|t+1}:=H^{m\top}_{k|t+1}\lambda^{m*}_{k+1|t}, \tag{24a}$$
$$\mu^m_{k|t+1}:=F^\top_{k|t+1}\mu^{m*}_{k+1|t}. \tag{24b}$$

A suitable choice for $\lambda^m_{N-1|t+1}$ and $\mu^m_{N-1|t+1}$ will be discussed below.

3) Recursive feasibility of (19g) *for* $k=0,\dots,N-2$*:*

$$\begin{aligned}&G^\top_{k|t+1}\mu^m_{k|t+1}+A^{m\top}_{k|t+1}\lambda^m_{k|t+1}\\ \overset{(24a),(24b)}{=}&G^\top_{k|t+1}F^\top_{k|t+1}\mu^{m*}_{k+1|t}+A^{m\top}_{k|t+1}H^{m\top}_{k|t+1}\lambda^{m*}_{k+1|t}\\ \overset{(21a),(23a)}{=}&G^\top_{k+1|t}\mu^{m*}_{k+1|t}+A^{m\top}_{k+1|t}\lambda^{m*}_{k+1|t}\\ =&0.\end{aligned} \tag{25}$$

4) Recursive feasibility of (19f) *for* $k=0,\dots,N-2$*:*

$$\begin{aligned}&-g^\top_{k|t+1}\mu^m_{k|t+1}-b^{m\top}_{k|t+1}\lambda^m_{k|t+1}\\ \overset{(24a),(24b)}{=}&-g^\top_{k|t+1}F^\top_{k|t+1}\mu^{m*}_{k+1|t}-b^{m\top}_{k|t+1}H^{m\top}_{k|t+1}\lambda^{m*}_{k+1|t}\\ \overset{(23b),(21b)}{\ge}&-g^\top_{k+1|t}\mu^{m*}_{k+1|t}-b^{m\top}_{k+1|t}\lambda^{m*}_{k+1|t}>d_{\min}.\end{aligned}$$

5) Recursive feasibility of (19h) *for* $k=0,\dots,N-2$*:*

$$\begin{aligned}&\|A^{m\top}_{k|t+1}\lambda^m_{k|t+1}\|\overset{(24a)}{=}\|A^{m\top}_{k|t+1}H^{m\top}_{k|t+1}\lambda^{m*}_{k+1|t}\|\\ \overset{(21a)}{=}&\|A^{m\top}_{k+1|t}\lambda^{m*}_{k+1|t}\|\le 1.\end{aligned}$$

Finally, the non-negativity of $H^{m\top}_{k|t+1}$ and $F^\top_{k|t+1}$ imply that the following holds

$$\lambda^m_{k|t+1}\overset{(24a)}{=}H^{m\top}_{k|t+1}\lambda^{m*}_{k+1|t}\ge 0,\quad \mu^m_{k|t+1}\overset{(24b)}{=}F^\top_{k|t+1}\mu^{m*}_{k+1|t}\ge 0.$$

The existence of suitable $\lambda^m_{N-1|t+1}$ and $\mu^m_{N-1|t+1}$ is guaranteed by Assumption 4, (16c) and (22d), knowing that (19d) is satisfied for $t+1$. In particular, Proposition 1 implies that $\lambda^m_{N-1|t+1}$ and $\mu^m_{N-1|t+1}$ satisfy (19f), (19g) and (19h) for $k=N-1$. $\square$

IV. Application to Safe Autonomous Driving

This section illustrates an application of the proposed framework combined with the nonlinear robust MPC approach proposed in [10]. We consider two cars, travelling in the same direction. The car in front, considered as an obstacle, travels at time $t=0s$ at a constant speed of 90km/h (known with a 5% accuracy), and it is subject to a random bounded acceleration. The rear car, the object, tries to drive with a constant velocity of 110km/h. The resulting MPC scheme generates a trajectory which allows the controlled car to safely over-take the obstacle. After this manoeuvre, the controlled car continues to follow the reference trajectory with no visible tracking error. A complete video of the example can be found at `https://www.youtube.com/watch?v=Yeft01QYJk8`.

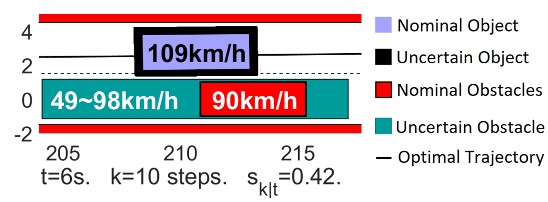


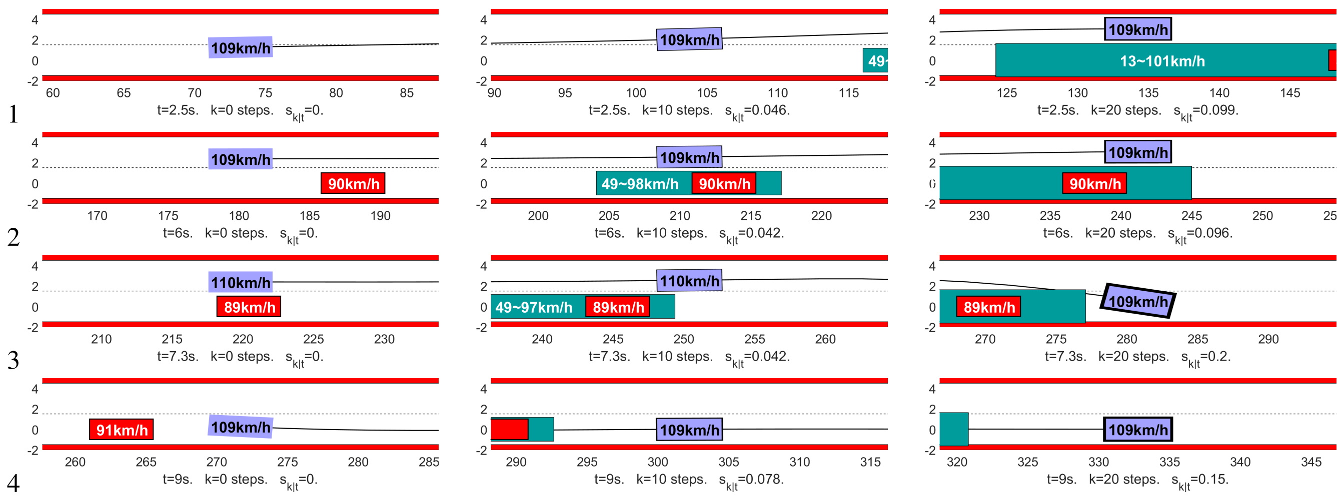
Fig. 2: Description of the object and obstacles. The predicted velocity of the uncertain object at time t is shown inside the nominal object. The range of velocity that the obstacle can have at time $t+k$ is shown inside the uncertain obstacle, while, for comparison, we also show the nominal obstacle, representing where the obstacle would be if it travels with a constant velocity of $v_{obs,t}$. Note that the nominal object is over-approximated by the uncertain object.

A. Considered model

We consider a nonlinear system representing an autonomous car, defined as follows

$$\dot{x}=\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{\psi}\\ \dot{v}\\ \dot{\beta}\end{bmatrix}=\begin{bmatrix}v\cdot\cos(\psi+\beta)\\ v\cdot\sin(\psi+\beta)\\ \frac{v}{w_1+l_r}\cdot\sin(\beta)\\ u_1+w_2|u_1|\\ u_2+w_3|u_2|\end{bmatrix}, \tag{26}$$

where $x_1\in[-\infty,\infty]$ and $x_2\in[-\infty,\infty]$ represent the coordinates of the center of mass of the car, $\psi\in[-\infty,\infty]$ indicates its rotation, $v\in[14m/s,36m/s]$ is the longitudinal velocity, and $\beta\in[-37°,37°]$ is the angle of the current velocity of the center of mass with respect to the longitudinal axis of the car, compare [24]. The inputs $u_1\in[-10m/s^2,1m/s^2]$, $u_2\in[-10°/s,10°/s]$ control the acceleration of the car and the velocity of the steering angle, respectively. The parameter $l_r=1.7m$ represents the



(a) - closed-loop trajectory (b) - 10 steps ahead prediction (c) - 20 steps ahead prediction

Fig. 3: Over-taking manoeuvre as a result of safe obstacle avoidance. Each row shows on the left the closed-loop scenario at the considered time t, in the middle the $k = 10$ steps ahead prediction, while on the right the $k = 20$ steps ahead prediction.

distance from the center of the mass of the car to the rear axle. The uncertainty acting on the system are represented by the values $w_1 \in [-0.05, 0.05]$, $w_2 \in [-0.1, 0.1]$ and $w_3 \in [-0.1, 0.1]$. The nominal prediction model is given by (26), with $w_1 = w_2 = w_3 = 0$. Similarly to the example shown in [25], the system is incrementally stabilizable with an incremental Lyapunov function $V_\delta(x, z)$

$$V_\delta(x, z) := \|x - z\|^2_{P(z)}, \tag{27}$$

with $P(z) := (X_0 + \sum_{i=1}^{n} \theta_i(z) X_i)^{-1}$, based on a Quasi-LPV parametrization. For the implementation we use a Euler discretization with a sampling time of $h = 100$ ms, as done in [24].

B. Object and obstacles description

Object: The physical space occupied by the controlled car is defined as $\mathbb{E}(x_t) = R(\psi_t)\mathbb{B} + p(x_t)$, where $R(\psi_t) \in \mathbb{R}^{2\times 2}$ is the rotation matrix, $p(x) := [x_1\ x_2]^\top$, and $\mathbb{B}$ is a hyper-box with a length of 4.5 m and a width of 2.0 m.

Obstacle: Firstly, we consider the border of the street as non-moving obstacles. In addition, we consider one moving obstacle $\mathbb{O}_t$, represented by a vehicle with the same shape as the object, $\mathbb{B}$. At time $t = 0$s, the obstacle travels at a velocity of 90km/h, while for $t \geq 0$ it is subject to a random acceleration a_t, such that $a_t \in [-10, 1] m/s^2$.

C. Predicted Object and Obstacles

1) Sets $\mathbb{O}_{\cdot|t}$, (10): The predicted over-approximation of the moving obstacle, $\mathbb{O}_{k|t}$, satisfying Assumption 3, is done by employing the bounds on the acceleration a_t that the obstacle can be subject to, combined with a noisy knowledge of the velocity of the obstacle at time t, with an error of $\pm 5\%$, and by considering slight movements on the transverse direction.

2) Tube $\mathbb{X}_{\cdot|t}$: The predicted tube $\mathbb{X}_{\cdot|t}$, defined as

$$\mathbb{X}_{k|t} := \{\tilde{x} \in \mathbb{R}^n | V_\delta(\tilde{x}, x_{k|t}) \leq s^2_{k|t}\},$$

is parametrized with an online computed scalar $s_{k|t} \geq 0$ and the incremental Lyapunov function (27), compare [10]. The tube propagation (Assumption 1) is given by

$$s_{k+1|t} = \rho s_{k|t} + \tilde{w}_\delta(x_{k|t}, u_{k|t}, s_{k|t}), \tag{28}$$

where $\rho \in (0, 1)$ is a contraction constant of the incremental Lyapunov function, compare [10, Ass. 2]. The function $\tilde{w}_\delta : \mathbb{R}^{n+m+1} \to \mathbb{R}_{\geq 0}$ represents a scalar bound of the mismatch between the real and the nominal system, [10, Ass. 5]. For the considered example we have $\rho = 0.3679$, and

$$\tilde{w}_\delta(x, u, s) = c_{w_1} v \sin(|\beta|) + c_{w_2}|u_1| + c_{w_3}|u_2| + L_w s \tag{29}$$

with $c_{w_1} = 0.0278$, $c_{w_2} = 0.0197$, $c_{w_3} = 0.0826$ and $L_w = 0.3384$.

3) Sets $\Theta(\mathbb{X}_{\cdot|t})$, (15): We over-approximate the uncertain prediction of the object by suitably scaling the shape of the object $\mathbb{B}$ around its nominal prediction $x_{k|t}$. Hence, $\Theta(\mathbb{X}_{k|t})$ is defined as

$$\Theta(\mathbb{X}_{k|t}) := (1 + L_{\mathbb{B}} s_{k|t})\mathbb{B} + p(x_{k|t}), \quad L_{\mathbb{B}} = 1.35.$$

D. MPC Optimization problem

We consider a nominal reference trajectory $[x_{r,t},\ u_{r,t}]$, where $x_{r,t} = [0,\ 0,\ 0,\ 30.5m/s,\ 0]^\top$, and $u_{r,t} = [0,\ 0]^\top$ that the car is supposed to follow. We define the open-loop cost J_N (4) as

$$J_N(\mathbb{X}_{\cdot|t}, u_{\cdot,t}, t) = \sum_{k=0}^{N-1} \|x_{k|t} - x_{r,t+k}\|^2_Q + \|u_{k|t} - u_{r,t+k}\|^2_R$$

where[1] $Q = \mathrm{diag}(0, 1, 0, 100, 0)$ and $R = \mathrm{diag}(0.001, 100)$, and $N = 20$. The optimization problem is an instantiation

[1] An appropriate tuning of the weighting matrices Q and R is fundamental for having a desired, e.g. a smooth human-like, behaviour of the car.

of Problem (19) with the addition of (28) and (29) as constraints needed for the tube dynamics and for the uncertainty propagation, based on [10].

Note that in the practical implementation we considered an incremental Lyapunov function computed on a continuous time system, and no terminal region. This simplifies the implementation and is common practice in several applications.

E. Discussion

Here we explain and discuss the simulation results shown in Fig. 3. We consider a prediction horizon of 2.0s. At time $t = 2.5$s, in (a-1), the controlled car is ca. $20m$ behind the the moving obstacle, while the 20 steps ahead prediction (c-1) shows that the predicted car is over-taking the over-approximation of the obstacle. We see that due to the uncertainties present in the system and in the movement of the obstacle, their predictions over-approximate their real size. At time $t = 6.0$s, (a-2) the car gets closer to the obstacle, while the object in the 20 steps ahead prediction (c-2) is close to end the manoeuvre. At time $t = 7.3s$, in (c-3), the 20 steps ahead predicted object safely returns to the reference lane, however, this dynamic manoeuvre increases the size of the tube $s_{k|t}$, i.e., the uncertainty in the predicted object. Lastly, at time $t = 9.0s$, in (a-4), the controlled object safely concludes the manoeuvre, and continues travelling at the desired velocity, similarly to its predictions in (b-4) and (c-4). It is important to note that the reference trajectory does not consider any over-taking manoeuvre, but this naturally arises from the solution of the optimal control problem (19) including the collision avoidance constraints (19f)-(19h). Moreover, it is also important to see that during the whole manoeuvre, the MPC scheme provides a safety distance between the object and the moving obstacle, which guarantees collision avoidance even in case of a sudden brake from the moving obstacle. This is an important feature that an autonomous car must have in order to ensure the safety of the people and objects involved in the scenario. This would not necessarily be guaranteed in other standard collision avoidance approaches which do not explicitly consider uncertainty, such as, e.g., [20]. The algorithm explicitly employs the uncertainty in the obstacle movements, and captures the effect of limit dynamics on the uncertainty of the object movements. Such effects would be hard, if at all possible, to replicate with (a time-varying) d_{min} in a nominal MPC formulation. This represents a significant and fundamental improvement toward the application of collision avoidance approaches in real scenarios.

V. Conclusion

We have presented a robust collision avoidance approach based on a general nonlinear robust MPC framework. The scheme is applicable to *uncertain nonlinear systems* with uncertain obstacle movements. We have demonstrated the applicability of the proposed framework with a realistic example, showing an autonomous car during a safe over-taking manoeuvre.

References

[1] J. Schulman, Y. Duan, J. Ho, A. Lee, I. Awwal, H. Bradlow, J. Pan, S. Patil, K. Goldberg, and P. Abbeel, "Motion planning with sequential convex optimization and convex collision checking," *The International Journal of Robotics Research*, vol. 33, no. 9, pp. 1251–1270, 2014.

[2] A. Liniger, A. Domahidi, and M. Morari, "Optimization-based autonomous racing of 1: 43 scale RC cars," *Optimal Control Applications and Methods*, vol. 36, no. 5, pp. 628–647, 2015.

[3] J. B. Rawlings and D. Q. Mayne, *Model predictive control: Theory and design*. Nob Hill Pub., 2009.

[4] B. Kouvaritakis and M. Cannon, *Model predictive control: Classical, Robust and Stochastic*. Springer, 2016.

[5] L. Grüne and J. Pannek, *Nonlinear Model Predictive Control*. Springer, 2017.

[6] L. Chisci, J. A. Rossiter, and G. Zappa, "Systems with persistent disturbances: predictive control with restricted constraints," *Automatica*, vol. 37, pp. 1019–1028, 2001.

[7] D. Q. Mayne, M. M. Seron, and S. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, pp. 219–224, 2005.

[8] D. Limon, J. Bravo, T. Alamo, and E. Camacho, "Robust MPC of constrained nonlinear systems based on interval arithmetic," *IEE Proceedings-Control Theory and Applications*, vol. 152, no. 3, pp. 325–332, 2005.

[9] M. E. Villanueva, R. Quirynen, M. Diehl, B. Chachuat, and B. Houska, "Robust MPC via min–max differential inequalities," *Automatica*, vol. 77, pp. 311–321, 2017.

[10] J. Köhler, R. Soloperto, M. A. Müller, and F. Allgöwer, "A computationally efficient robust model predictive control framework for uncertain nonlinear systems," *IEEE Transactions on Automatic Control*, 2019, submitted. Online available: https://www.ist.uni-stuttgart.de/de/institut/team/PDFs_MA-Seiten/JK/Robust_Nonlin.pdf.

[11] G. Schildbach and F. Borrelli, "A dynamic programming approach for nonholonomic vehicle maneuvering in tight environments," in *Intelligent Vehicles Symposium (IV), 2016*. IEEE, 2016, pp. 151–156.

[12] M. Likhachev and D. Ferguson, "Planning long dynamically feasible maneuvers for autonomous vehicles," *The International Journal of Robotics Research*, vol. 28, no. 8, pp. 933–945, 2009.

[13] K. Margellos and J. Lygeros, "Hamilton–Jacobi formulation for reach–avoid differential games," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1849–1861, 2011.

[14] J. Canny, *The complexity of robot motion planning*. MIT press, 1988.

[15] S. M. LaValle, *Planning algorithms*. Cambridge university press, 2006.

[16] M. Campbell, M. Egerstedt, J. P. How, and R. M. Murray, "Autonomous driving in urban environments: approaches, lessons and challenges," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 368, no. 1928, pp. 4649–4672, 2010.

[17] B. Paden, M. Čáp, S. Z. Yong, D. Yershov, and E. Frazzoli, "A survey of motion planning and control techniques for self-driving urban vehicles," *IEEE Transactions on intelligent vehicles*, vol. 1, no. 1, pp. 33–55, 2016.

[18] B. Li and Z. Shao, "A unified motion planning method for parking an autonomous vehicle in the presence of irregularly placed obstacles," *Knowledge-Based Systems*, vol. 86, pp. 11–20, 2015.

[19] R. B. Patel and P. J. Goulart, "Trajectory generation for aircraft avoidance maneuvers using online optimization," *Journal of guidance, control, and dynamics*, vol. 34, no. 1, pp. 218–230, 2011.

[20] X. Zhang, A. Liniger, and F. Borrelli, "Optimization-based collision avoidance," *arXiv preprint arXiv:1711.03449*, 2017.

[21] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical problems in engineering*, vol. 4, no. 4, pp. 317–367, 1998.

[22] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Springer, 2008.

[23] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[24] J. Kong, M. Pfeiffer, G. Schildbach, and F. Borrelli, "Kinematic and dynamic vehicle models for autonomous driving control design." in *Intelligent Vehicles Symposium*, 2015, pp. 1094–1099.

[25] J. Köhler, M. A. Müller, and F. Allgöwer, "A nonlinear model predictive control framework using reference generic terminal ingredients," *IEEE Transactions on Automatic Control*, 2018, submitted.