Hw_4

Local Lattice Planner:

(1) Build an ego-graph of the linear modeled robot
(2) Select the best trajectory closest to the planning target

The modelling and how to select the best trajectory by using OBVP have been given here, please follow the annotation in the code, finish the homework step by step. Enjoy it~

1. Modelling
   a) Objective, minimize the integral of squared accelerate

$$J = \int_0^T g(x,u)dt = \int_0^T (1 + u^T R u)dt = \int_0^T (1 + a_x^2 + a_y^2 + a_z^2)dt$$

   b) State, Input and System equation

$$x = \begin{pmatrix} p_x \\ p_y \\ p_z \\ v_x \\ v_y \\ v_z \end{pmatrix}, u = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \dot{x} = f(x,u) = \begin{pmatrix} v_x \\ v_y \\ v_z \\ a_x \\ a_y \\ a_z \end{pmatrix}$$

2. Solving
   a) Costate $\lambda = (\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \quad \lambda_6)^T$
   b) Define the Hamiltonian function

$$H(x,u,\lambda) = g(x,u) + \lambda^T f(x,u),$$
$$H(x,u,\lambda) = \left(1 + a_x^2 + a_y^2 + a_z^2\right) + \lambda^T f(x,u)$$
$$\dot{\lambda} = -\nabla H(x^*,u^*,\lambda) = (0 \quad 0 \quad 0 \quad -\lambda_1 \quad -\lambda_2 \quad -\lambda_3)^T$$

   c) The costate is solved as

$$\lambda = \begin{pmatrix} 2\alpha_1 \\ 2\alpha_2 \\ 2\alpha_3 \\ -2\alpha_1 t - 2\beta_1 \\ -2\alpha_2 t - 2\beta_2 \\ -2\alpha_3 t - 2\beta_3 \end{pmatrix}$$

   d) The optimal input is solved as

$$u^* = \arg\min_{a(t)} H(x^*(t), u(t), \lambda(t)) = \begin{pmatrix} \alpha_1 t + \beta_1 \\ \alpha_2 t + \beta_2 \\ \alpha_3 t + \beta_3 \end{pmatrix}$$

   e) The optimal state trajectory is solved as

$$x^* = \begin{pmatrix} \frac{1}{6}\alpha_1 t^3 + \frac{1}{2}\beta_1 t^2 + v_{x0} t + p_{x0} \\ \frac{1}{6}\alpha_2 t^3 + \frac{1}{2}\beta_2 t^2 + v_{y0} t + p_{y0} \\ \frac{1}{6}\alpha_3 t^3 + \frac{1}{2}\beta_3 t^2 + v_{z0} t + p_{z0} \\ \frac{1}{2}\alpha_1 t^2 + \beta_1 t + v_{x0} \\ \frac{1}{2}\alpha_2 t^2 + \beta_2 t + v_{y0} \\ \frac{1}{2}\alpha_3 t^2 + \beta_3 t + v_{z0} \end{pmatrix}, initial\ state: x(0) = \begin{pmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \\ v_{x0} \\ v_{y0} \\ v_{z0} \end{pmatrix}$$

   f) α, β are solved as

$$\begin{pmatrix} \frac{1}{6}T^3 & 0 & 0 & \frac{1}{2}T^2 & 0 & 0 \\ 0 & \frac{1}{6}T^3 & 0 & 0 & \frac{1}{2}T^2 & 0 \\ 0 & 0 & \frac{1}{6}T^3 & 0 & 0 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & 0 & 0 & T & 0 & 0 \\ 0 & \frac{1}{2}T^2 & 0 & 0 & T & 0 \\ 0 & 0 & \frac{1}{2}T^2 & 0 & 0 & T \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

$$\begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix} = \begin{pmatrix} p_{xf} - v_{x0}T - p_{x0} \\ p_{yf} - v_{y0}T - p_{y0} \\ p_{zf} - v_{x0}T - p_{z0} \\ v_{xf} - v_{x0} \\ v_{yf} - v_{y0} \\ v_{zf} - v_{z0} \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -\frac{12}{T^3} & 0 & 0 & \frac{6}{T^2} & 0 & 0 \\ 0 & -\frac{12}{T^3} & 0 & 0 & \frac{6}{T^2} & 0 \\ 0 & 0 & -\frac{12}{T^3} & 0 & 0 & \frac{6}{T^2} \\ \frac{6}{T^2} & 0 & 0 & -\frac{2}{T} & 0 & 0 \\ 0 & \frac{6}{T^2} & 0 & 0 & -\frac{2}{T} & 0 \\ 0 & 0 & \frac{6}{T^2} & 0 & 0 & -\frac{2}{T} \end{pmatrix} \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

g) The cost

$$J = \int_0^T (1 + a_x^2 + a_y^2 + a_z^2)dt$$

$$J = T + \left(\frac{1}{3}{\alpha_1}^2T^3 + \alpha_1\beta_1T^2 + {\beta_1}^2T\right) + \left(\frac{1}{3}{\alpha_2}^2T^3 + \alpha_2\beta_2T^2 + {\beta_2}^2T\right)$$

$$+ \left(\frac{1}{3}{\alpha_3}^2T^3 + \alpha_3\beta_3T^2 + {\beta_3}^2T\right)$$

h) J only depends on T, and the boundary states (known), so we can even get an optimal T!

   i. You can use the Mathematica obtain the optimal analytic expression of T
   ii. You can use Numerical calculation method obtain the optimal approximate solution of T