# SSY281 Model Predictive Control

## Assignment A6

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# Question 1

(a) According to the equation of Lyapunov equation $A^T SA - S = -Q$, taking A,B and $S = I_4$ into calculation, we can get the matrix of Q:

$$\begin{bmatrix} -0.6964 & -0.8415 & -0.0002 & -0.0396 \\ -0.8415 & -0.0083 & 0 & 0.0091 \\ -0.0002 & 0 & 0 & -0.0098 \\ -0.0396 & 0.0091 & -0.0098 & 0.0726 \end{bmatrix} \tag{1}$$

To check if the system fits the Lyapunov function, we have to use eigenvalue to check if $Q \succ 0$, and the eigenvalue of Q is shown below, which means that it does not fit the the Lyapunov function:

$$eig(Q) = \begin{bmatrix} -1.2621 & -0.0013 & 0.0727 & 0.5586 \end{bmatrix} \tag{2}$$

Then, we have to check the $eig(S - Q) = [0.4414 \quad 0.9273 \quad 1.0013 \quad 2.2621]$, which means that the the system is unstable- small perturbations in the initial state of the system will grow over time. Then, we have to check the eigenvalue of the A. As is shown below, some eigenvalues of A are outside the unit circle, which means that the system is unstable.

$$eig(A) = \begin{bmatrix} 1.0000 & 1.0949 & 0.9126 & 0.9636 \end{bmatrix} \tag{3}$$

(b) Through given K, we can Q matrix through the equation shown below:

$$Q = S - (A - BK)^T * S * (A - BK) \tag{4}$$

$$Q = 1.0e + 03 * \begin{bmatrix} -4.3342 & -0.4685 & 0.0489 & 0.1222 \\ -0.4685 & -0.0504 & 0.0053 & 0.0134 \\ 0.0489 & 0.0053 & -0.0006 & -0.0014 \\ 0.1222 & 0.0134 & -0.0014 & -0.0025 \end{bmatrix} \tag{5}$$

Through check the eigenvalues of the Q, which means Q can not fit the Lyapunov function:

$$eig(Q) = 1.0e + 03 * \begin{bmatrix} -4.3889 & -0.0000 & 0.0002 & 0.0010 \end{bmatrix} \tag{6}$$

However, through checking the close-loop eigenvalue $eig(A - BK)$, we found that the eigenvalues are in the unit circle, which means that the close-loop system is stable.

$$eig(A - BK) = \begin{bmatrix} 0.1894 + 0.0000i \\ 0.9156 + 0.0101i \\ 0.9156 - 0.0101i \\ 0.9901 + 0.0000i \end{bmatrix} \tag{7}$$

If a system cannot fit the Lyapunov function, then we cannot directly apply the Lyapunov stability theorem to determine the stability of the system. However, if the eigenvalues of the closed-loop system A - BK are inside the unit circle, then we can say that the closed-loop system is stable in the sense of the Lyapunov.

(c) Through taking the value of Q into dlypa function, we can get the matrix of S:

$$S = 1.0e + 04 * \begin{bmatrix} 3.9766 & 0.4327 & -0.0937 & -0.1030 \\ 0.4327 & 0.0477 & -0.0103 & -0.0113 \\ -0.0937 & -0.0103 & 0.0124 & 0.0025 \\ -0.1030 & -0.0113 & 0.0025 & 0.0028 \end{bmatrix} \tag{8}$$

Through checking the eigenvalue of S and Q, S and Q are all positive define matrix, which satisfy the Lyapunov function, and we can say that the close-loop system is stable.

# Question 2

(a) Constantly, we can use $eig(A + BK)$ to check if a system is stable or not. However, according to the definition of K, it always depends on A,B,P and R. So the K is independent with Q, which means that the selection of Q will not affect the stability when N =1.

(b) According to the definition of dp programming, until the $eig(A + Bk)$ inside the unit circle.

$$\begin{aligned} &u^*(k) = K(k)x(k), \quad k = 0, \ldots, N-1 \\ &K(k) = -\left(R + B^\top P(k+1)B\right)^{-1} B^\top P(k+1)A \end{aligned} \tag{9}$$

$$P(k-1) = Q + A^\top P(k)A - A^\top P(k)B\left(R + B^\top P(k)B\right)^{-1} B^\top P(k)A, \quad P(N) = P_f \tag{10}$$

Then, we can get the shortest N=38 and the corresponding feedback gain is shown below:

$$K = \begin{bmatrix} -45.6628 & -5.1672 & 0.0068 & 0.3898 \end{bmatrix} \tag{11}$$

(c) Through the equation shown below, we can calculate the Q matrix:

$$Q = P - (A + BK)^T P(A + BK) \tag{12}$$

$$Q = \begin{bmatrix} 592.7558 & 76.9147 & 14.5515 & 16.2516 \\ 76.9147 & 10.7621 & 1.5806 & 1.6131 \\ 14.5515 & 1.5806 & 0.0006 & -0.3597 \\ 16.2516 & 1.6131 & -0.3597 & 0.3749 \end{bmatrix} \tag{13}$$

To check if $Q \succ 0$, we calculate the $eig(Q)$ shown below, which means that Q is not positive define and can not satisfy Lyapunov function.

**Thus, $V_N(x)$ is not a a Lyapunov function.**

$$eig(Q) = \begin{bmatrix} 603.5452 & -1.1403 & 0.9999 & 0.4886 \end{bmatrix} \tag{14}$$

However, through calculating the $eig(A+BK)$, eigenvalues are still inside the unit circle, which means the close loop system is stable:

$$eig(A + BK) = \begin{bmatrix} 0.5584 & 0.9226 & 0.9544 & 0.9998 \end{bmatrix} \tag{15}$$

If a system cannot fit the Lyapunov function, then we cannot directly apply the Lyapunov stability theorem to determine the stability of the system. However, if the eigenvalues of the closed-loop system A + BK are inside the unit circle, then we can say that the closed-loop system is stable in the sense of the Lyapunov.

(d) Using idare command to calculate the $P_f$ and taking it into MPC controller(N=1) into calculation, then we can get the $P_f$ ($eig > 0$, postive define)and $K$ shown below:

$$P_f = 1.0e + 04 * \begin{bmatrix} 4.8730 & 0.5323 & -0.1038 & -0.1154 \\ 0.5323 & 0.0587 & -0.0114 & -0.0127 \\ -0.1038 & -0.0114 & 0.0125 & 0.0027 \\ -0.1154 & -0.0127 & 0.0027 & 0.0030 \end{bmatrix} \tag{16}$$

$$K = \begin{bmatrix} -67.7456 & -7.5229 & 0.6939 & 0.9025 \end{bmatrix} \tag{17}$$

Using the equation(12) to calculate the Q, we can get the Q matrix shown below:

$$Q = 1.0e + 03 * \begin{bmatrix} 4.5905 & 0.5096 & -0.0470 & -0.0611 \\ 0.5096 & 0.0576 & -0.0052 & -0.0068 \\ -0.0470 & -0.0052 & 0.0015 & 0.0006 \\ -0.0611 & -0.0068 & 0.0006 & 0.0018 \end{bmatrix} \tag{18}$$

Through checking the eigenvalues of Q, we can get that:

$$eig(Q) = 1.0e + 03 * \begin{bmatrix} 4.6484 & 0.0010 & 0.0010 & 0.0010 \end{bmatrix} \tag{19}$$

Then we can get that $Q \succ 0$ and $P_f \succ 0$, and $V_N(x)$ **is a Lyapunov function for the close-loop system.**

# Question 3

(a) According to the definition of the closed-loop system, we can get that the equation of feedback gain of closed-loop system:

$$K = -(B^T PB + R)^{-1} B^T PA \tag{20}$$

In the question, we know that:

$$A = \begin{bmatrix} \bar{A}_{(n-1)\times n} \\ 0_{1\times n} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{21}$$

$$P_f = \text{diag}\left(\begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix}\right), p_1, \dots, p_n > 0, \tag{22}$$

Then, through calculating we can get that $B^T PA$ is a zero matrix, which means that $K = -(B^T PB + R)^{-1} B^T PA = 0$. To check the eigenvalue of the close-loop system, we can find that the stability totally depends on the matrix A.

$$eig(A + BK) = eig(A) \tag{23}$$

Since the open loop system is unstable, we can say that we can not stabilize the system regardless Q and R.

# Question 4

As is shown in the figure 1 and 2, which are plot of $X_1$, $X_2$ and trajectory of the system when $P_f = I_2$(choosing $X_f = C_{inf} \cap X$). The system failed to converge and can not reach to the origin, finally out of X0, which means that the system is infeasible and unstable.

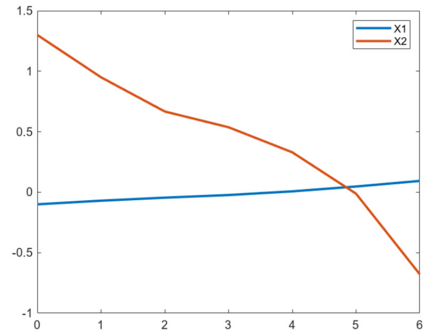


*Figure 1: plot of $X_1$ and $X_2$ when pf=$I_2$*

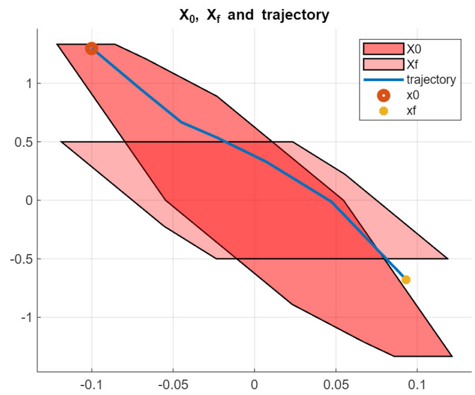


*Figure 2: $X_0, X_f$ and trajectory*

However, taking the $P_f$ calculated by Riccati equation, then we can get the plot shown below. As we can see that $X_1$, $X_2$ converge and finally reach to the origin in the$X_f$, which means that they are stable and feasible.

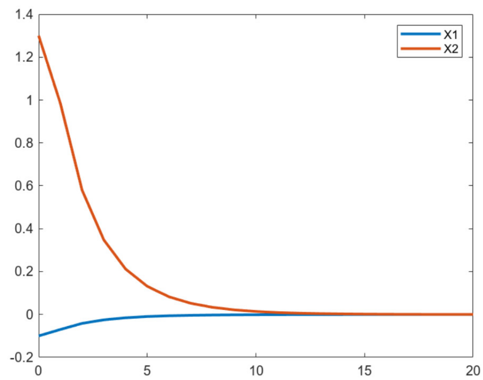


*Figure 3: plot of $X_1$ and $X_2$ when pf=raccatisolution*

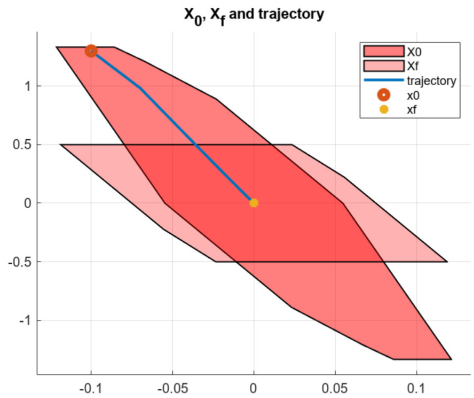


*Figure 4: $X_0, X_f$ and trajectory*