SSY281 Model Predictive Control

Assignment A2

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Question 1

(a) In the question a, it is a "Equal number of inputs and outputs" system which follows the equation shown below:

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix} \tag{1}$$

Through taking the value of matrix A, B, C and y_s into calculation, we can get the value of X_s and u_s:

$$X_s = \begin{bmatrix} 0.1745 \\ 0.0722 \\ -3.1416 \\ 0.0008 \end{bmatrix}, u_s = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix} \tag{2}$$

(b) In the question b, it is a "less inputs than outputs" system, which follows the equation shown below:

$$\begin{gathered} \min_{x_s,u_s} \left(|Cx_s - y_{sp}|_Q^2 \right), \quad Q \succeq 0 \\ \begin{bmatrix} I-A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0 \\ Eu_s \leq e \\ FCx_s \leq f. \end{gathered} \tag{3}$$

Since we only have equality constraints, taking the equality constraints and values of y_s into Matlab function **quadprog**,then we can get the steady states and input of the system:

$$X_s = \begin{bmatrix} -0.0000 \\ -0.0000 \\ -3.1416 \\ 0.0000 \end{bmatrix}, u_s = 0 \tag{4}$$

(c) In the question c, where p¡m, it is a "more inputs than controlled outputs" system. Such a system can be solved by finding the feasible steady-state targets:

$$
\begin{gathered}
\min_{x_s,u_s}\left(|u_s-u_{sp}|^2_{R_s}+|C_yx_s-y_{sp}|^2_{Q_s}\right),\quad R_s\succ 0\\
\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix}\begin{bmatrix} x_s \\ u_s \end{bmatrix}=\begin{bmatrix} 0 \\ z_{sp} \end{bmatrix}\\
Eu_s\le e\\
FC_zx_s\le f
\end{gathered}
\tag{5}
$$

Since we only have equality constraints, taking A, B, C_z and Z_s into the **Quadprog**, we can get the value of steady states and outputs:

$$
X_s=\begin{bmatrix} 0.1745 \\ 0.0722 \\ 0 \\ 0.0008 \end{bmatrix}, u_s=\begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix}
\tag{6}
$$

In the case b, more output than input, we can not expect all states reach to a steady states. If we take theX_s back to the equation (1), the y_s does not match the set-point before. Thus, system (case b) output could not settle at y_s. However, for the case a and case c, system output can settle at y_s, because they can match.

Question 2

For the system contains noise, the system can be constructed as augmented system which is shown below:

$$
\begin{gathered}
\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix}=\begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}\begin{bmatrix} x(k) \\ d(k) \end{bmatrix}+\begin{bmatrix} B \\ 0 \end{bmatrix}u(k)\\
y(k)=\begin{bmatrix} C & C_d \end{bmatrix}\begin{bmatrix} x(k) \\ d(k) \end{bmatrix}
\end{gathered}
\tag{7}
$$

(a) Before designing the kalman filter, it is necessary to check if the system is detectable. According to the knowledge of linear control, if the system is observable, then it must be detectable. If the system is not observable and the eigenvalue outside the unit disk, then the system is not detectable.

$$
\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
\tag{8}
$$

Through calculating the rank of observe matrix, **case one** and **case three** are full rank, which means they are observable(detectable). However, for the **case two**, which is not observable, we may have to calculate its eigenvalues. Through checking the eigenvalues of **case two**(which is shown below), unobservable modes are not stable(outside the unit circle). Thus, **case two** is not detectable.

$$eigenvalue = \begin{bmatrix} 1.0000 \\ 1.0949 \\ 0.9126 \\ 0.9636 \\ 1.0000 \\ 1.0000 \end{bmatrix} \tag{9}$$

(b) For kalman filter, which can be calculated through discrete-time raccati equation, the equation is shown below:

$$\begin{aligned} L &= PC^\top \left[CPC^\top + R\right]^{-1} \\ P &= APA^\top - APC^\top \left[CPC^\top + R\right]^{-1} CPA^\top + Q. \end{aligned} \tag{10}$$

The P of Raccati equation could be solved by the idare function, with A, C, Q, R matrix, where Q, R are the noise matrix of the system. Since only **case one** and **case three** are detectable, then the kalman gain of them are shown below:

$$L_1 = \begin{bmatrix} 1.0912 & -0.4025 \\ 8.7809 & -3.1649 \\ 0.4246 & 0.2332 \\ 0.3672 & -0.0835 \\ -0.3508 & 0.4763 \end{bmatrix}, L_3 = \begin{bmatrix} 1.2496 & 0.0099 \\ 10.4751 & 0.2567 \\ 0.0111 & 0.2279 \\ 0.2721 & 1.8818 \\ -0.4815 & -0.0090 \\ -0.0102 & 0.5109 \end{bmatrix} \tag{11}$$

(c) If we only want to focus on regulating the second output, we can use H matrix to multiply the y_s. Additionally, the matrix should follow the equation shown below.

$$\begin{gathered} \begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d} \\ z_{sp} - HC_d\hat{d} \end{bmatrix} \\ Eu_s \le e \\ FHCx_s \le f - FHC_d\hat{d} \end{gathered} \tag{12}$$

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_d\hat{d} \\ z_{sp} - HC_d\hat{d} \end{bmatrix} \tag{13}$$

Through the equation shown above, we can get the Mss matrix of **case one** and **case three**:

$$Mss1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, Mss3 = \begin{bmatrix} 0 & 0.0000 \\ 0 & 0.0000 \\ 0 & -1.0000 \\ 0 & 0.0000 \\ 0 & -1.0000 \end{bmatrix} \tag{14}$$

(d) Since the system have N and M control horizon, then the equality constraints of our system should be as equation (15), which means that after M steps, u_M should keep same until N steps. Additionally, in the URHC, the matlab code should be expressed like figure 1:

$$\begin{bmatrix} I & 0 & \cdots & \cdots & 0 & -B & 0s & \cdots & 0 \\ -A & I & \cdots & \cdots & 0 & 0 & -B & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & -B & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & -B \\ \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & I & 0 & \cdots & \cdots & -B \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_M \\ \vdots \\ u_M \end{bmatrix} = \begin{bmatrix} Ax_0 \\ \vdots \\ 0 \end{bmatrix} \tag{15}$$

```
I     = eye(n);
Aeq1 = kron(eye(N),I)+kron(diag(ones(N-1,1),-1),-A);
Aeqb1= kron(eye(40),-B);
Aeqb2=zeros(1*n,40*p);
Aeqb2(:,end)=-B;
Aeq2=[Aeqb1;repmat(Aeqb2,10,1)];
Aeq  = [Aeq1 Aeq2];
beq  = [A*x0;zeros(n*(N-1),1)];
```

Figure 1: code of equality constraints

To construct off-set augmented system, what we should do is to follow the MPC block diagram, which is shown in the figure 2. Additionally, the off-set free control is to use the $M_s s$ matrix we calculated above and the augmented system (equation (7)).

Through calculating kalman gain and taking value of states into the equation

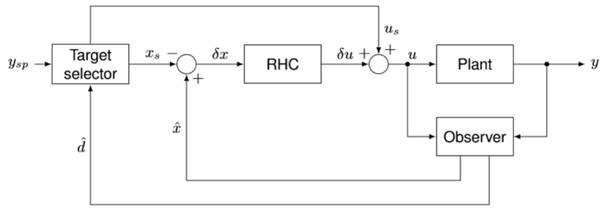


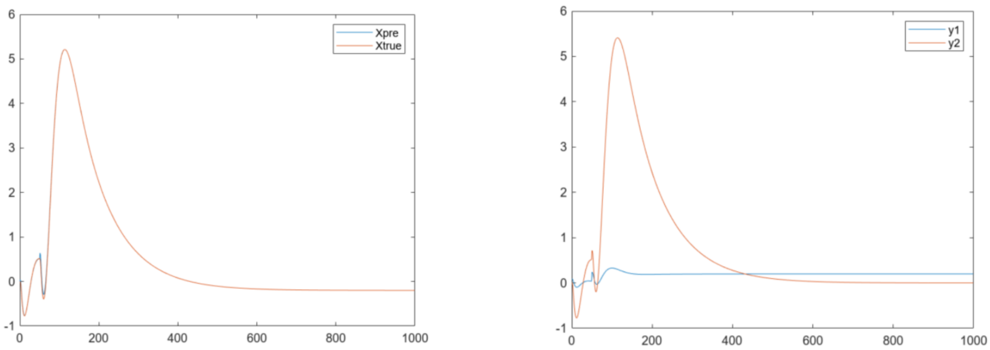
Figure 2: MPC block diagram

shown below, we can get $\hat{x}$ and $\hat{d}$. Additionally, taking the $\hat{x}$ and $\hat{d}$ into the Target selector and RHC controller, we can get the δu nad take it to calculate next step.

$$\hat{x}(k \mid k) = \hat{x}(k \mid k-1) + L(k)[y(k) - C\hat{x}(k \mid k-1)], \quad \hat{x}(0 \mid 0) = x_0 \tag{16}$$

$$\hat{x}(k+1 \mid k) = A\hat{x}(k \mid k) + Bu(k). \tag{17}$$

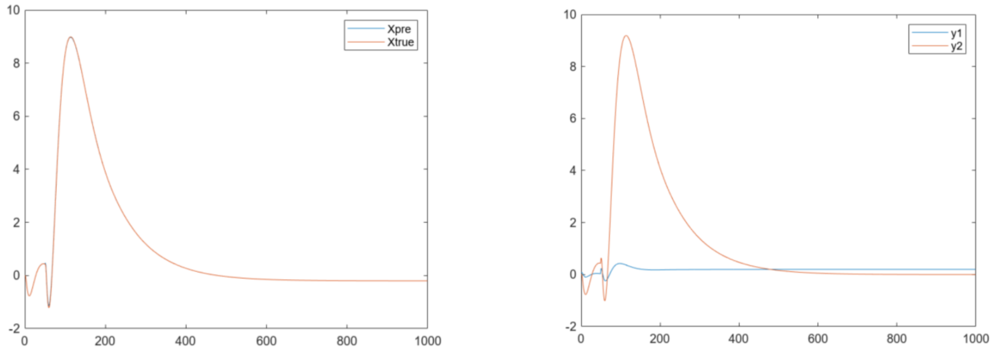
After 1000 time steps calculation, we get the prediction and true value of x_3 and value of y/



(a) prediction and true value of x_3 *(b) plot of y*

Figure 3: A figure with case one

Additionally, the plot of case three is shown below:

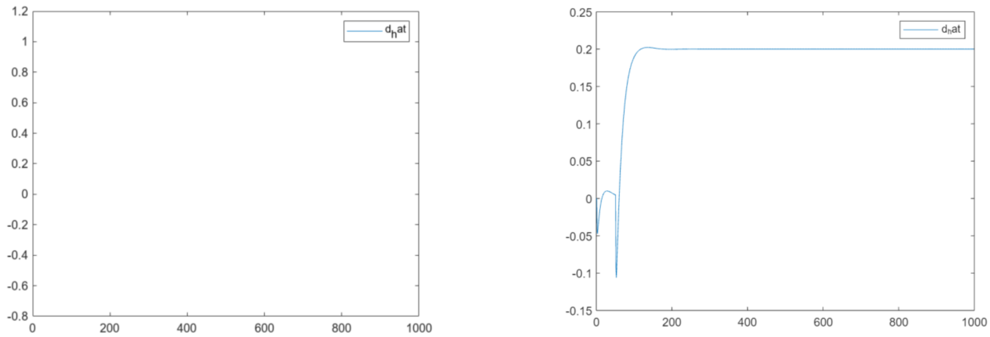


(a) prediction and true value of x_3

(b) plot of y

Figure 4: A figure with case three

As is shown in the figure 5, **case one** can not track the $\hat{d}$, but the $\hat{d}$ can be tracked very well in the **case three**, Thus, the controller of **case three** can remove the offset.



(a) $\hat{d}$ of case one

(b) $\hat{d}$ of case three

Figure 5: A figure with disturbance