Probability Homework 1

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1 Computational Project

1.1 Task 1

1.2 Task 2

1.2.1 2.1

1.2.2 2.2

1.3 Task 3

1.3.1 3.1

1.3.2 3.2

1.4 Task 4

1.4.1 4.1

1.4.2 4.2

1.4.3 4.3

1.4.4 4.4

2 Written Problem Solving

The random lifetime T (in hours) of the light bulb in an overhead projector follows an exponential distribution with mean $= M$ hours. During a normal workweek, the projector is used for a random number N of lectures lasting exactly one hour each. The random variable N has a Poisson distribution with mean $= K$.
We recall:

$$T \sim \text{Exp}\left[\lambda = \frac{1}{M}\right] \text{ has PDF } f^T(t) = P(T = t) = \frac{1}{M} e^{-\frac{t}{M}}, \text{ for } t > 0$$

$$N \sim \text{Poisson}\left[\lambda = K\right] \text{ has PDF } f^N(n) = P(N = n) = \frac{k^n}{n!} e^{-k}, \text{ for } n = 0, 1, \ldots$$

2.1 Problem 1

Compute the conditional probability $P(T > N|N = n) = P(T > n|N = n)$ for any integer $n = 0, 1, 2...$ and use the result to compute the probability $P(T > N)$ as an explicit function $u(M, K)$.

$$\begin{aligned} P(T > N \mid N = n) &= P(T > n) = 1 - P(T \leq n) \\ &= 1 - \int_0^n \frac{1}{M} e^{-\frac{t}{M}} dt = 1 - \int_{-\frac{n}{M}}^0 e^x dx \\ &= 1 - \left[e^x\right]_{-\frac{n}{M}}^0 \\ &\boxed{= e^{-\frac{n}{M}}} \end{aligned}$$

2.2 Problem 2

use (1) to compute the probability $p(M,K)$ that a projector with a newly installed lightbulb will actually last for a whole normal workweek without changing the light bulb

$$\begin{aligned}
p(M,K) &= P(T \geq n) \\
&= \sum_{n=0}^{\infty} P(T > N \mid N = n) \cdot P(N = n) \\
&= \sum_{n=0}^{\infty} e^{\frac{-n}{M}} \cdot \frac{k^n}{n!} e^{-k} = e^{-k} \sum_{n=0}^{\infty} \frac{\left[e^{-\frac{1}{m}} k\right]^n}{n!} \\
&= e^{-k} \cdot e^{\left[ke^{-1/n}\right]} = e^{-k+ke^{-1/n}} \\
&\boxed{= e^{k\left(e^{-1/n}-1\right)}}
\end{aligned}$$

2.3 Problem 3

Consider that M is imposed by the light bulb commercial brand , and indicate how to compute $g(M)$ (either by a formula or by numerical computation) such that $K \leq g(M)$ will force $p(M,K) \geq 0.95$.
$K \leq g(M) \Rightarrow p(M,K) \geq 0.95$

$$\begin{aligned}
e^{k\left(e^{-1/M}-1\right)} &= 0.95 \\
k\left(e^{-1/M} - 1\right) &= \ln(0.95) \\
k &= \boxed{\frac{\ln(0.95)}{e^{-1/M} - 1} = g(M)}
\end{aligned}$$

2.4 Problem 4

Restrictions on lectures scheduling have succeeded to impose $K = g(M)$. Compute the probability that a projector with a newly installed light bulb will actually last for 4 successive workweeks without changing the light bulb.

Let X be the number of lectures in 4 successive workweeks $X = N_1 + N_2 + N_3 + N_4$, where each N_i are identically and independently distributed $\sim$ Poisson$[\lambda = k]$

Then, $X \sim$ Poisson$[\lambda = 4k]$ with the PDF $f^X(x) = P(X = x) = \frac{(4k)^x}{x!} e^{-(4k)}$, for $n = 0, 1, \ldots$

$$\begin{aligned}
P(T > X) &= \sum_{x=0}^{\infty} P(T > X \mid X = x) \cdot P(X = x) \\
&= \sum_{x=0}^{\infty} e^{-\frac{x}{M}} \cdot \frac{(4k)^x}{x!} e^{-4k} \\
&= e^{-4k} \sum_{x=0}^{\infty} \frac{\left(4ke^{-1/n}\right)^x}{x!} \\
&= e^{-4k} e^{4ke^{-1/n}} = e^{4k\left(e^{-1/m}-1\right)}
\end{aligned}$$

Now, we substitute in the value for k we found in Problem 3

$$\begin{aligned}
P(T > X) &= e^{4k\left(e^{-1/m}-1\right)} \\
&= \exp\left[4\left(\frac{\ln(0.95)}{e^{-1/M} - 1}\right)\left(e^{-1/M} - 1\right)\right] \\
&= e^{4\ln(0.95)} = (0.95)^4 \\
&= \boxed{0.81450625}
\end{aligned}$$