MATH 7320, Functional Analysis I

HOMEWORK 3

1. Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{F}$ a linear functional. Prove that f is continuous iff $\ker(f)$ is closed.

2. Let $\mathcal{X}$ be a normed space and $\mathcal{Y} \leq \mathcal{X}$ be a subspace. Assume that both $\mathcal{Y}$ and the quotient space $\mathcal{X}/\mathcal{Y}$ are complete. Prove that $\mathcal{X}$ is complete.

3. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y} \leq \mathcal{X}$ a closed subspace. Prove that if $\mathcal{Y}$ and $\mathcal{X}/\mathcal{Y}$ are both separable, then $\mathcal{X}$ is separable.

4. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y} \leq \mathcal{X}$ a closed subspace. Show that $(\mathcal{X}/\mathcal{Y})^*$ is isometrically isomorphic to
$$\mathcal{Y}^{\perp} := \{f \in \mathcal{X}^* : f|_{\mathcal{Y}} \equiv 0\}.$$

5. Let $\mathcal{Y} = \{(\alpha_n) \in \ell^1 : \alpha_{2n} = 0, \ \forall n \in \mathbb{N}\}$. Show that $\mathcal{Y}$ is a closed subspace of ℓ^1, and that the quotient space $\ell^1/\mathcal{Y}$ is isometrically isomorphic to ℓ^1.

6. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Show that the following are equivalent:
   i) there is $c > 0$ such that $\|x\| \leq c\|T(x)\|$ for all $x \in \mathcal{X}$;
   ii) T is injective and $T(\mathcal{X})$ is closed.

7. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Prove that if $T(\mathcal{X}) \subseteq \mathcal{Y}$ is not closed, then $T(\mathcal{X})$ is of first category.