MATH 7320 Functional Analysis
Homework 2

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1. **Solution:** One side is easy using the conitnuity of the functional, for the other way, use the functional $d(x, \mathcal{M})$

2. **Solution:**

   (a) If X is reflexive, the map $X \to X^{**} := x \to \text{ev}_x$ is an isometric isomorphism. Moreover we know that $X^{**} = B(X^*, \mathbb{C})$ is complete since $\mathbb{C}$ is complete. Therefore by the isometric isomorphism, we get that X is complete.

   (b) Let $i_X : X \to X^{**} : x \to \text{ev}_x$ be the canonical injection map. Then first we show that $i_{X^{**}} = (i_X)^{**}$, that is the canonical injection of the double dual is the double dual of the canonical injection.

   First we notice that since $i_X : X \to X^{**}$, the dual of it $i_X^* : X^{***} \to X^*$ and $i_X^{**} : X^{**} \to X^{****}$ as the usual dual of linear transformations. Moreover $i_{X^{**}} : X^{**} \to X^{****}$ shows that the domain and codomain of the maps are same, therefore considering the equality of the maps makes sense. (Notice that actually $i_{X^{**}}$ is a map from $X^{**} \to (X^{**})^{**}$, but by definition it follows that $(X^{**})^{**} := (X^{***})^* = X^{****}$).

   (c) Let $M \subset X$ be a closed subset of a reflexive space X. Consider the subset $F \subset X^*$ such that $F = \{f \in X^* \ : \ f|_M = \mathbf{0}\}$. Show that $X^*/F \cong^{\text{iso}} M^*$ (For this it is enough to show that the restriction map $f \to f|_M$ has its kernel F. And for isometry, play with the quotient norm).