MATH6320 - Theory of Functions of a Real Variable
Assignment 9

Joel Sleeba

November 14, 2024

1. not finished

Solution:

(a) Let $r < p < s$, where $r, s \in E$. Then by the convexity of $[r, s] \subset \mathbb{R}$, there is a $t \in [0, 1]$ such that $p = tr + (1-t)s$. Then Holder's inequality on $\frac{1}{t}$ and $\frac{1}{(1-t)}$ gives,

$$\begin{aligned}\int |f|^p \, d\mu &= \int |f|^{tr} |f|^{(1-t)s} \, d\mu \\ &\leq \left(\int |f|^{\frac{tr}{t}} \, dm \right)^t \left(\int |f|^{\frac{(1-t)s}{(1-t)}} \, dm \right)^{1-t} \\ &= \left(\int |f|^r \, dm \right)^t \left(\int |f|^s \, dm \right)^{1-t} \\ &= \|f\|_r^{rt} \|f\|_s^{s(1-t)}\end{aligned}$$

Thus we get $\|f\|_p \leq \|f\|_r^{\frac{rt}{p}} \|f\|_s^{\frac{s(1-t)}{p}}$

For the sake of contradiction, assume that $\|f\|_p > \max\{\|f\|_r, \|f\|_s\}$. Then by the monotonicity of the function $x \to x^k$, where $k > 0$, we get

$$\|f\|_p^{\frac{rt}{p}} > \|f\|_r^{\frac{rt}{p}} \quad \text{and} \quad \|f\|_p^{\frac{s(1-t)}{p}} > \|f\|_s^{\frac{s(1-t)}{p}}$$

Then we'll get

$$\|f\|_p = \|f\|_p^{\frac{rt}{p}} \|f\|_p^{\frac{s(1-t)}{p}} > \|f\|_r^{\frac{rt}{p}} \|f\|_s^{\frac{s(1-t)}{p}}$$

contradicting our previous result. Hence we see that $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$

(b) Let $0 < \epsilon$. Consider the set $A_\epsilon = \{x \in X \ : \ \|f\|_\infty < |f(x)| + \epsilon\}$. Then

$$\begin{aligned}\int_X |f|^p \ d\mu &\geq \int_{A_\epsilon} |f|^p \ d\mu \\ &\geq \int_{A_\epsilon} (\|f\|_\infty - \epsilon)^p \ d\mu \\ &= (\|f\|_\infty - \epsilon)^p \mu(A_\epsilon)\end{aligned}$$

Since we are given that $\|f\|_\infty \in (0, \infty]$, there is an $\varepsilon > 0$ such that $\|f\|_\infty > \varepsilon$. Moreover since $\|f\|_r < \infty$, the above inequality forces $\mu(A_\varepsilon) < \infty$. Then taking power $\frac{1}{p}$ to the above inequality, we get

$$\|f\|_p \geq (\|f\|_\infty - \epsilon)\mu(A_\varepsilon)^{\frac{1}{p}}$$

Now taking limits, we get

$$\lim_{p\to\infty} \inf \|f\|_p \geq (\|f\|_\infty - \varepsilon)$$

since $\mu(A_\varepsilon)^{\frac{1}{p}} \to 1$ as $p \to \infty$. Again since $\varepsilon > 0$ was arbitrary, we get

$$\lim_{p\to\infty} \inf \|f\|_p \geq \|f\|_\infty$$

Now to get the other inequality, observe that

$$\begin{aligned}\int |f|^p \ d\mu &= \int |f|^r \ d\mu \int |f|^{p-r} \ d\mu \\ &\leq \|f\|_\infty^{p-r} \int |f|^r \ d\mu\end{aligned}$$

Hence we get

$$\|f\|_p = \left(\int |f|^p \ d\mu\right)^{1/p} \leq \|f\|_\infty^{\frac{p-r}{p}} \left(\int |f|^r \ d\mu\right)^{\frac{1}{p}} = \|f\|_\infty \|f\|_r^{\frac{r}{p}}$$

Thus taking limits, we see that

$$\lim_{p\to\infty} \sup \|f\|_p \leq \|f\|_\infty$$

as $\|f\|_r^{\frac{r}{p}} \to 0$ as $p \to \infty$ since $\|f\|_r < \infty$

Combining both the inequalities, we see

$$\lim_{p\to\infty} \sup \|f\|_p \leq \|f\|_\infty \leq \lim_{p\to\infty} \inf \|f\|_p$$

Thus

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$