Functional Analysis - MATH7320

Joel Sleeba
joelsleeba1@gmail.com

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Contents

Chapter 1

Banach Spaces

Textbook : A Course in Functional Analysis, John Conway

Functional analysis is the study of Topological Vector Spaces.

Definition 1.0.1. Let X be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A seminorm on X is a map $\|\cdot\| : X \to [0, \infty)$ such that

- $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{F}, \forall x \in X$
- $\|x + y\| \le \|x\| + \|y\|$

In addition if $\forall x \neq 0, \|x\| \neq 0$, we say $\|\cdot\|$ is a norm on X

Norm induces a metric $d(x, y) = \|x - y\|$

Note. Let X be a normed space. Then the maps

- $+ : X \times X \to X : (x, y) \to x + y$
- $\cdot : \mathbb{F} \times X \to X : (\alpha, x) \to \alpha x$

are continuous.

Hence every normed space is a topological vector space.

Example 1.0.1. $\mathbb{F}^n$ with ℓ_p-norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}}$$

Example 1.0.2. $\mathbb{F}^n$ with ℓ_∞-norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\infty = \max\{|a_i|\}$$

Example 1.0.3. Consider $C_{00} = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n = 0$ except for finitely many $n \in \mathbb{N}\}$ which can be identified by collection of functions $f : \mathbb{N} \to \mathbb{F}$ with finite support.

Then

$$\|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}$$

is a norm on C_{00}

Proposition 1.0.1. *Let X, Y be normed space, and let $T : X \to Y$ be linear. Then the following are equivalent.*

- *T is continuous*
- *T is continuous on 0*
- *T is continuous on any point $x \in X$*
- *$\exists M > 0$ such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$*

Proof. $(1 \implies 2)$ It is clear that if T is continuous, then it is continuous at 0 from the definition of continuity.

$(2 \implies 3)$ Let $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in X that converge to x. Then the sequence $\{y_n = x_n - x\}$ converge to zero by the algebra of limits. By the continuity of T at zero, $\{T(y_n) = T(x_n) - T(x)\}$ converge to 0. Therefore $T(x_n) \to T(x)$. And this shows T is sequentially continuous at $x \in X$. Since the space is a metric space, sequential continuity is equivalent to continuity.

$(4 \implies 2)$ Let $x \in X$. Then $\|T(0) - T(x)\| = \|T(x)\| \leq M\|x\| = M\|0 - x\|$. Hence T is continuous at 0.

$(3 \implies 1)$

$(2 \implies 4)$

□

Example 1.0.4. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be defined as $T(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1, 0, \ldots, 0)$. Is T convergent for any norm $\|\cdot\|_1, \|\cdot\|_2$ in the domain and range?

Proof. verify □

Example 1.0.5. Consider identity function $I : C_{00} \to C_{00}$. Let the norm in domain be $\|\cdot\|_\infty$ and that in range be $\|\cdot\|_1$. Is the function continuous? What if the norms in domain and range are switched?

Proof. verify □

Note. Let X be a space with two norms $\|\cdot\|_1, \|\cdot\|_2$. When is the two norms topologically equivalent?

When $\exists M, M'$ such that $\|x\|_1 \leq M\|x\|_2$ and $\|x\|_2 \leq M'\|x\|_1$ Equivalently, when the identity map between the two spaces with their respective norms are bi-continuous. (See 4th equivalent statement of previous proposition)

Theorem 1.0.1. *Let X and Y be normed spaces, and $T : X \to Y$ be linear. Assume X is finite dimensional. Then T is continuous.*

Proof. Since $T(X) \leq Y$ is finite dimensional, we may assume without loss of generality that Y is also finite dimensional and T is onto. Let $\{x_1, x_2, \dots x_n\}$ be a basis for X. Define another norm on X as follows. For every $x = \sum_{i=1}^n \alpha_i x_i \in X$,

$$\|x\|' = \sum_{i=1}^n |\alpha_i|(\|T(x_i)\| + \|x_i\|)$$

verify that this is a norm. Then for every $x \in X$, we have

$$\|T(x)\| \leq \sum_{i=1}^n |\alpha_i|\|T(x_i)\| \leq \|x\|'$$

Hence T is bound with respect to the norm $\|\cdot\|'$ on X, since all norms are equivalent on X. Therefore T is continuous w.r.t to the original norm on X. $\square$

Corollary 1.0.1.1. *Let X be a finite dimensional vector space. Then any two norms in X are equivalent.*

Proof. Let $\{e_1, e_2, \dots e_n\}$ be a basis for X. For each $x = \sum_{i=1}^n \alpha_i e_i \in X$, define

$$\|x\|_\infty = \max\{|\alpha_i|\}$$

Then $\|\cdot\|_\infty$ is a norm and we'll show every norm on X is equivalent to this norm. Let $\|\cdot\|$ be an arbitrary norm on X. For each $x = \sum_{i=1}^n \alpha_i e_i \in X$, we have

$$\begin{aligned}
\|x\| &= \|\sum_{i=1}^n \alpha_i e_i\| \\
&\leq \sum_{i=1}^n |\alpha_i|\|e_i\| \\
&\leq \max\{|\alpha_i|\} \sum_{i=1}^n \|e_i\| \\
&\leq \|x\|_\infty \sum_{i=1}^n \|e_i\|
\end{aligned}$$

Therefore the identity map $I : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|)$ is continuous. Since the set $K = \{x \in X \ : \ \|x\|_\infty \leq 1\}$ is compact, K is also compact in $(X, \|\cdot\|)$ and the restriction $\mathrm{Id}|_K$ is also a homeomorphism. verify In particular, the set $\{x \in X \ : \ \|x\|_\infty < 1\}$ is an open neighborhood of $0 \in (X, \|\cdot\|)$ By the Heine-Borel theorem, the unit ball $B = \{x \in X \ : \ \|x\|_2 \leq 1\}$ is compact. Hence B is compact in $(X, \|\cdot\|)$. verify the last line. □

Corollary 1.0.1.2.

- *Every finite dimensional normed space is complete*
- *If X is a normed space and Z is a finite dimensional subspace of X, then Z is the closed in X*

Proof.

- Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Let $\|\cdot\|_2$ be the euclidean norm on X. Then by the theorem above there exists $M \in \mathbb{R}$ such that $\frac{1}{M}\|x\|_2 \leq \|x\| \leq M\|x\|_2$ for all $x \in X$.

  Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|)$. Then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy in $(X, \|\cdot\|_2)$. Since the latter space is complete, so is $(X, \|\cdot\|)$.
- verify

□

Note. If $T, S : X \to Y$ are continuous linear maps between normed space, then $T + S$ is also continuous. Also, $\forall \alpha \in \mathbb{F}$, αT is continuous.

Thus the space $B(X, Y)$ of all continuous linear maps from X to Y is a subspace of all linear maps between X and Y.

Definition 1.0.2. Let X, Y be normed spaces. and $T \in B(X, Y)$. We define the operator norm of T as

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} \ : \ x \in X, x \neq 0\right\}$$

Lemma 1.0.1. *Let $T \in B(X, Y)$. Then the following are equivalent*

- $\|T\|$
- $\sup\{\|T(x)\| \ : \ x \in X, \|x\| \leq 1\}$
- $\sup\{\|T(x)\| \ : \ x \in X, \|x\| < 1\}$
- $\inf\{M \geq 0 \ : \ \|T(x)\| \leq M\|x\|, \forall x \in X\}$

Proof. verify □

Proposition 1.0.2. *The operator norm is a norm in $B(X, Y)$*

Proof. verify □

1.1 Hahn Banach Theorem

Lemma 1.1.1. *Let X be a complex normed space. Let $f : X \to \mathbb{R}$ be an $\mathbb{R}$-linear functional. Then $g : X \to \mathbb{C}$ defined as $g(x) = f(x) - if(ix)$ is $\mathbb{C}$-linear*

Conversely if $g : X \to \mathbb{C}$ is a $\mathbb{C}$-linear map, then $f := \Re \circ g : X \to \mathbb{R}$ is $\mathbb{R}$-linear.

Moreover $\|f\| = \|g\|$.

Proof. We'll prove that $\|f\| = \|g\|$ and leave the rest for the reader (verify).

Since $|f(x)| \leq |g(x)|$, for all $x \in X$, it is easy to see that $\|f\| \leq \|g\|$. Conversely, $\forall \epsilon > 0, \exists x_o \in X$ with $\|x_o\| = 1$ such that $|g(x_o)| > \|g\| - \epsilon$. If $g(x_o) = re^{i\theta}$, take $\alpha = e^{-i\theta}$. Then $f(\alpha x_o) = \Re(re^{-i\theta}e^{i\theta}) = r = g(\alpha x_o)$. Then $\|f\| \geq |f(\alpha x_o)| = |g(\alpha x_o)| = |\alpha||g(x_o)| = |g(x_o)| > \|g\| - \epsilon$. Since ϵ is arbitrary, this gives $\|f\| \geq \|g\|$ □

Theorem 1.1.1 (Hahn-Banach Extension Theorem)**.** *Let X be a normed space over $\mathbb{R}$, Z be a subspace of X and let $\phi : Z \to \mathbb{R}$ be a continuous linear functional. Then there exists a linear functional $\psi : X \to \mathbb{R}$ such that $\psi|_Z = \phi$ and $\|\phi\| = \|\psi\|$.*

Proof. Assume $\|\phi\| = 1$ (If this is not the case, we can always scale the functional down to norm 1). Now we'll extend ϕ from Z to a subspace with one dimension higher than Z, preserving the norm. Let $x_o \in (X \setminus Z)$ and $Y = \text{Span}\{\{x_o\} \cup Z\}$ be the set one dimension higher than Z. Assume ψ is the extension of ϕ to Y. Then ψ will be completely characterized, if we know the value of $\psi(x_o)$. We look to see what real values we can assign $\psi(x_o)$ satisfying our conditions. Let $y = z + x_o \in Y$ where $z \in Z$ (We must be taking an arbitrary element $y = z + \alpha x_o \in Y$, but if we know the image of $y = z + x_o$ for all $z \in Z$ under ψ, then we can get the image of $y = z + \alpha x_o$ for any $\alpha \in \mathbb{R}$ by scaling). Norm preserveness demands that for all $z \in Z$, we must have

$$-\|z + x_o\| \leq \psi(y) = \psi(z) + \psi(x_o) \leq \|z + x_o\|$$

Since ψ agrees with ϕ on Z, this is equivalent to

$$-\phi(z) - \|z + x_o\| \leq \psi(x_o) \leq \|z + x_o\| - \phi(z) \tag{1.1}$$

Moreover since we normalized ϕ to have norm 1, we know ψ must also have norm 1. Then by triangle inequality, we get that for all $a, b \in Y$

$$\psi(a) - \psi(b) = \psi(a - b) \leq \|a - b\| = \|(a + x_o) - (b + x_o)\| \leq \|a + x_o\| + \|b + x_o\|$$

which gives

$$-\psi(b) - \|b + x_o\| \leq \|a + x_o\| - \psi(a)$$

Since this inequality is true for all $a, b \in Y$, taking supremum and infimum over all the possible $a, b \in Y$ preserves the inequality. Hence we get

$$\sup_{b\in Y} \Big\{ -\psi(b) - \|b + x_o\| \Big\} \leq \inf_{a\in Y} \Big\{ \|a + x_o\| - \psi(a) \Big\} \tag{1.2}$$

Substituting $a = b = z$ in Equation 1.2 guarantees the existence of $\psi(x_o)$ satisfying Equation 1.1. Hence we get an extension (namely ψ) of ϕ to Y preserving the norm. Since Z was an arbitrary subspace of X, this is true for all such subspaces of X.

Now we will employ Zorn's lemma to get an extension of ϕ from Z to the whole of X. Consider the collection of all linear extensions of ϕ, i.e

$$\mathcal{S} = \Big\{ (\psi_Y, Y) \;:\; Z \subset Y,\; \psi_Y|_Y = \phi,\; \|\psi_Y\| = \|\phi\| \Big\}$$

Then we define a partial order in the collection $\mathcal{S}$ as $(\psi_X, X) \leq (\psi_Y, Y)$ if and only if $X \subset Y$ and $\psi_Y|_X = \psi_X$. Now let $\mathscr{C}$ be a chain in $\mathcal{S}$. Consider the set

$$\tilde{Y}_{\mathscr{C}} = \bigcup_{(\psi_Y, Y)\in\mathscr{C}} Y$$

and the map $\psi_{\tilde{Y}_{\mathscr{C}}} : \tilde{Y}_{\mathscr{C}} \to \mathbb{R}$ defined as

$$\psi_{\tilde{Y}_{\mathscr{C}}}(x) = \psi_Y(x), \quad \text{where } x \in Y,\; \text{for } (\psi_Y, Y) \in \mathscr{C}$$

To see this map is well defined, assume $x \in X$ and $x \in Y$ for $(\psi_X, X), (\psi_Y, Y) \in \mathscr{C}$. Then either $(\psi_X, X) \leq (\psi_Y, Y)$ or $(\psi_Y, Y) \leq (\psi_X, X)$ since $\mathscr{C}$ is totally ordered. WLOG assume $(\psi_X, X) \leq (\psi_Y, Y)$, then by definition we get that $\psi_Y|X = \psi_X$. This gives that $\psi_Y(x) = \psi_X(x)$. Hence we get that $\psi_{\tilde{Y}_{\mathscr{C}}}$ is well defined. In a similar fashion we can verify that $\psi_{\tilde{Y}_{\mathscr{C}}}$ is a linear functional.

Now we claim that $(\tilde{Y}_{\mathscr{C}}, \psi_{\tilde{Y}_{\mathscr{C}}})$ is the upper bound of the chain $\mathscr{C}$. By the definition of $\tilde{Y}$, we see that there cannot be an element (ψ_Y, Y) in the chain $\mathscr{C}$, with $\tilde{Y} \subset Y$. Hence the only remaining thing to show is that for all $(\psi_X, X) \in \mathscr{C}$, we have $\psi_{\tilde{Y}_{\mathscr{C}}}|_X = \psi_X$. But this also follows from the definition of the map $\psi_{\tilde{Y}_{\mathscr{C}}}$.

Since $\mathscr{C}$ was taken to be an arbitrary chain in the collection $\mathcal{S}$, we get that every chain in $\mathcal{S}$ has an upper bound. Then by Zorn's lemma, the collection $\mathcal{S}$ has a maximal element (ψ, Y). We claim that in this maximal element, $Y = X$. If not, we can extend ψ to a space one dimension above Y like we did in the beginning contradicting the maximality of (ψ, Y). Hence the maximal element is (ψ, X). This by definition of the collection S, is the required extension for (ϕ, Z). $\square$

Remark 1.1.1. Note that in the proof above, we only used the scaling property and triangle inequality of the norm, hence we can relax the condition for norm and replace it with a seminorm, without messing up the proof.

Theorem 1.1.2 (Hahn-Banach Extension Theorem for $\mathbb{C}$)**.** *Same statement of Theorem 1.1.1 with only the field changed to* $\mathbb{C}$*.*

Proof. Consider X as a normed linear space over $\mathbb{R}$. Let $f = \Re \circ \phi : Z \to \mathbb{R}$ and apply Theorem 1.1.1 on f to get a real linear functional $\tilde{f} : X \to \mathbb{R}$ with the required properties. Now we claim that $\tilde{\phi}$ defined as $\tilde{\phi}(x) = \tilde{f}(x) - i\tilde{f}(ix)$ is the required extension.

First we show $\tilde{\phi}_Z = \phi$. To see this first we notice that if ϕ can be written as $\phi(x) = f(x) + ig(x)$ where f, g are real valued functionals, then since $-\phi(x) = i\phi(ix) = if(ix) - g(ix)$. Hence $0 = \phi(x) - \phi(x) = (f(x) - g(ix)) + i(g(x) + f(ix))$. Since real part and imaginary part must be equal to 0, we get that $g(x) = -f(ix)$. Therefore we get $\phi(x) = f(x) - if(ix)$. Now we get $\tilde{\phi}|_Z = \phi$ immediately since $\tilde{f}|_Z = f$. To finish the proof, we also have to show that $\|\phi\| = \|\tilde{\phi}\|$. But this follows easily from Lemma 1.1.1 as $\|\phi\| = \|f\| = \|\tilde{f}\| = \|\tilde{\phi}\|$. □

Remark 1.1.2. It is quite natural to be confused about the well defineness of the expression $f(ix)$ when we are considering X as a normed linear space over $\mathbb{R}$ in the beginning of the proof. But note that since X initially was a complex normed linear space, viewing it as a space over $\mathbb{R}$ doesn't change or remove any elements from the space. Hence $ix \in X$ even though X is viewed as a real normed linear space.

Definition 1.1.1. A sublinear map is a functoin $\rho : X \to \mathbb{R}$ with the properites

- $\rho(rx) = r\rho(x), \forall r \in \mathbb{R}$
- $\rho(x + y) \leq \rho(x) + \rho(y)$

Definition 1.1.2. Let X be a normed space. Then the dual of X, denoted by X^*, is the space $B(X, \mathbb{F})$

Lemma 1.1.2. *Let X be a normed space and $x \in X$. Then $\exists f \in X^*$ such that*

$$\|f\| = 1 \text{ and } f(x) = \|x\|$$

Proof. Let $Z = \text{Span}\{x\}$. Define $g : Z \to \mathbb{F}$ as $g(\alpha x) = \alpha\|x\|$. Then $\|g\| = 1$. By the Hahn Banach theorem, g has an extension f which preserve the norm and extends g to X. □

Corollary 1.1.2.1. *Let X be a normed space and $x \in X$, then we have*

$$\|x\| = \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\}$$

Proof. If f is any linear functional with $\|f\| \leq 1$, then $|f(x)| \leq \|f\|\|x\| = \|x\|$. Hence $\|x\| \leq \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\}$. Now let f_x be the functional we get from Lemma 1.1.2. Then $f_x \in X^*$ and $\|f_x\| = 1$ with $f_x(x) = |f(x)| = \|x\|$. Hence we get that the inequality is actually an equality, and this proves the corollary. □

Definition 1.1.3. For every $x \in X$, define a linear map $\hat{x} : X^* \to \mathbb{F}$ by $\hat{x}(f) = f(x)$

Theorem 1.1.3. *For every $x \in X$, $\hat{x} \in (X^*)^*$. The map $\rho : x \to \hat{x}$ is an isometric linear map.*

Proof. The fact that $\hat{x}$ is linear and bounded and the map $X \ni x \to \hat{x} \in X^{**}$ is linear follows from the definition of $f + g$ and λf.

By definition and Corollary 1.1.2.1

$$\begin{aligned}\|\hat{x}\| &= \sup\{|\hat{x}(f) \ : \ f \in X^*, \|f\| \leq 1\} \\ &= \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\} \\ &= \|f\|\end{aligned}$$

□

Definition 1.1.4. A normed space X is said to be reflexive if the map $\rho : X \to X^{**} := x \to \hat{x}$ is surjective. (This is a stronger condition than $X \equiv X^{**}$)

Theorem 1.1.4. *There are isometric isomorphisms between*

- $(\mathbf{c}_0)^*$ *and* ℓ^1
-
- $(\ell^1)^*$ *and* ℓ^∞

Proof.

- Let $(x_n) \in \ell^1$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \to \mathbb{F}$ defined as

$$\phi_{(x_n)} : (y_n) \to \sum_{n \in \mathbb{N}} x_n y_n$$

We claim that $\phi_{(x_n)}$ is a continuous linear functional. But first we should see that the sum is well defined. Since $y_n \to 0$, there is an $N \in \mathbb{N}$ such that $|y_n| < 1$ for all $n \geq N$. Since

$$\Big|\sum_{i=N}^{\infty} x_n y_n\Big| \leq \sum_{i=N}^{\infty} |x_n||y_n| \leq \|(x_n)\|_1$$

we see that the sum is well defined and the map makes sense. Also since $(y_n) + (z_n) = (y_n + z_n) \in \mathbf{c}_0$ whenever $(y_n), (z_n) \in \mathbf{c}_0$, we get that

$$\sum_{n\in\mathbb{N}} x_n(y_n + z_n) = \sum_{n\in\mathbb{N}} x_n y_n + \sum_{n\in\mathbb{N}} x_n z_n$$

which shows the linearity of the map $\phi_{(x_n)}$.

Now we show that $\|\phi_{(x_n)}\| = \|(x_n)\|_1$. We immediately see that for $(y_n) \in c_0$ with $\|(y_n)\|_{\sup} = \sup_{n\in\mathbb{N}} y_n = 1$,

$$|\phi_{(x_n)}((y_n))| = \Big|\sum_{n\in\mathbb{N}} x_n y_n\Big| \leq \|(y_n)\|_{\sup}\Big(\sum_{n\in\mathbb{N}} |x_n|\Big) \leq \|(x_n)\|_1$$

which gives $\|\phi_{(x_n)}\| \leq \|(x_n)\|_1$. Now let $\theta_j \in [0, 2\pi)$ such that $|x_j| = e^{i\theta_j} x_j$. Now consider the sequence $s_m \in \mathbf{c}_0$ defined as $s_m = \sum_{j=1}^{m} e^{i\theta_j} e_j$, where e_j is the sequence with jth entry 1 and the rest of the entries 0. Since $(x_n) \in \ell_1$, for all $\epsilon \geq 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{i=N_\epsilon+1}^{\infty} |x_i| < \epsilon$$

Then since

$$|\phi_{(x_n)}(s_{N_\epsilon})| = \Big|\sum_{n=1}^{N_\epsilon} e^{i\theta_j} x_n\Big| = \sum_{i=1}^{N_\epsilon} |x_n| = \|(x_n)\| - \sum_{i=N_\epsilon+1}^{\infty} |x_n| \geq \|(x_n)\| - \epsilon$$

and $\epsilon > 0$ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|$

Hence we see that the map $(x_n) \to \phi_{(x_n)}$ is an isometric linear map. Now for surjectivity, let $\phi \in \mathbf{c}_0^*$. We claim that the sequence $(y_n) = (\phi(e_n)) \in \ell^1$ and $\phi = \phi_{(y_n)}$. Let $\theta_j \in [0, 2\pi)$ such that $e^{i\theta_j} y_j = |y_j|$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{j=1}^{N} |\phi(e_j)| &= \sum_{j=1}^{N} e^{i\theta_j}\phi(e_j) \\
&= \phi\Big(\sum_{j=1}^{N} e^{i\theta_j} e_j\Big) \\
&\leq \|\phi\|\Big\|\sum_{j=1}^{N} e^{i\theta_j} e_j\Big\| \\
&= \|\phi\|
\end{aligned}$$

Since this is true for all $N \in \mathbb{N}$, taking the limits as $N \to \infty$, the inequality is preserved and we get that $(y_n) \in \ell^1$. Moreover $\phi = \phi_{(y_n)}$ follows from the definition of $\phi_{(x_n)}$. Hence we get that $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$.

-
- The proof of this will be extremely similar to what we attempted before when we proved $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$. Let $(x_n) \in \ell^\infty$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \to \mathbb{C}$ defined as
$$\phi_{(x_n)} : (y_n) \to \sum_{n \in \mathbb{N}} x_n y_n$$
By a similar way as we did in the above equivalence we see that $\phi_{(x_n)}$ is linear. Moreover since
$$\Big|\sum_{n \in \mathbb{N}} x_n y_n\Big| \leq \|(x_n)\|_\infty \Big|\sum_{n \in \mathbb{N}} y_n\Big| = \|(x_n)\|_\infty \|(y_n)\|_1$$
we see that $\|\phi_{(x_n)}\| \leq \|(x_n)\|_\infty$. To get the reverse inequality, Let $\|(x_n)\|_\infty = M$, then for any $\epsilon > 0$, there exist some x_k in the sequence (x_n) such that $|x_k - M| < \epsilon$. Now consider the sequence $e_k \in \ell^1$ with kth entry 1 and all the rest of them 0. We get that
$$|\phi_{(x_n)}(e_k)| = |x_k| \geq \|(x_n)\|_\infty - \epsilon$$
Since ϵ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|_\infty$. Hence the map $(x_n) \to \phi_{(x_n)}$ is an isometry. To show that it is indeed a bijection, assume $\phi \in (\ell^1)^*$, then consider the sequence $y_n = \phi(e_n)$. Since ϕ is continuous, it is bounded above by $\|\phi\|$ and we get that $y_n \leq \|\phi\|$. Therefore $(y_n) \in \ell^\infty$. Moreover we can verify like above that $\phi = \phi_{(y_n)}$ from the definition of $\phi_{(y_n)}$. Hence we get $(\ell^1)^* \cong^{\text{iso}} \ell^\infty$.

□

Theorem 1.1.5. *Let $1 < p < \infty$, and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$*

Proof. Let $(a_n) \in \ell^p, (b_n) \in \ell^q$, then $\sum_{n \in \mathbb{N}} a_n \bar{b}_n$ is the map to check for isometric isomorphism. Use Holder's inequality as needed. verify □

1.2 Banach Limits

Theorem 1.2.1. *There exists $\phi \in (\ell^\infty)^*$ satisfying the following*

1 $\forall (a_n) \in \ell^\infty$ *with* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, $\phi((a_n)) \geq 0$

2 If (a_n) is convergent, then $\phi((a_n)) = \lim_{n\to\infty} a_n$

3 If $(a_n) \in \ell^\infty$ and $b_n = a_{n+1}$, then $\phi((b_n)) = \phi((a_n))$

Moreover such ϕ is called a Banach limit.

Proof. Let $S : \ell^\infty(\mathbb{R} \to \ell^\infty(\mathbb{R})$ and $T = I - S$ where I is the identity map. Also let $V = \text{Range}(T) + c$ where $c \in \mathbf{c}$, the set of convergent sequences.

Define $\phi : V \to \mathbb{R}$, $\phi(a_n - a_{n+1} + x_n) = \lim_{n\to\infty} x_n$.

- Claim 1: ϕ is well defined
- Claim 2: $\|\phi\| = 1$

Assuming the claims, by Hahn Banach, ϕ extends to $\tilde{\phi} \in \ell^\infty(\mathbb{R})$ with $\|\tilde{\phi}\| = 1$. Then by the last lemma we get $\tilde{\phi}((y_n)) \geq 0$ for all $(y_n) \in ell^\infty(\mathbb{R})$ with $y_n \geq 0$ $\square$

Proof of Claim 1. Suppose that $(a_n) \in \ell^\infty$ is a sequence such that $a_n - a_{n+1}$ converges, say $a_n \to a_{n+1} \to \alpha$. If $\alpha > 0$, then $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n - a_{n+1} > \frac{\alpha}{2}$. So $a_N > \frac{\alpha}{2} + a_{N+1} > \ldots > k\frac{\alpha}{2} + a_{N+k}$. So for all $k \in \mathbb{N}$, $a_N - a_{N+k} = k\frac{\alpha}{2} \to \infty$ contradicting our assumption that $a_n - a_{n+1}$ converges.

Now assume that $(a_n), (b_n) \in \ell^\infty(\mathbb{R})$ with $(x_n), (y_n) \in \mathbf{c}$ such that $a_n - a_{n+1} + x_n = b_n - b_{n+1} + y_n$. Then $(a_n - b_n) - (a_{n+1} - b_{n+1}) = y_n - x_n$. Then since RHS is a convergent limit, LHS must be convergent, which we get from above that it must converge to zero. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ $\square$

Proof Claim 2. To complete the proof, define $\Psi : \ell^\infty \to \mathbb{C}$ by $\Psi((a_n + ib_n)) = \tilde{\phi}(a_n) + i\tilde{\phi}(b_n)$ verify $\square$

Corollary 1.2.1.1. *ℓ^1 is not reflexive*

Proof. Let $\phi \in (\ell^\infty)^*$ be a Banach limit. FTOC, assume $\exists f = (\alpha_n) \in \ell^1$ such that

$$\phi((a_n)) = \sum_{i=1}^{\infty} a_n \overline{\alpha_n}$$

Then for all $m \in \mathbb{N}$, $\overline{\alpha_m} = \phi(\delta_m) = 0$, where $\delta_m = (0, 0, \ldots, 1, 0, 0, \ldots)$. But this contradicts since we assumed $\phi \neq 0$ by the Hahn Banach extension from c_0 $\square$

Lemma 1.2.1. *Let $\psi \in (\ell^\infty)^*$. then the following are equivalent.*

1 $\|\psi\| = \psi((1, 1, 1, \ldots))$

2 If $(a_n) \in \ell^\infty$ with $a_n \geq 0, \forall n \in \mathbb{N}$. Then $\psi((a_n)) \geq 0$

Proof. (1 $\implies$ 2) FTSOC assume $\exists(a_n) \in \ell^\infty$, $\psi((a_n)) < 0$. WLOG, assume $|a_n| \leq 1$ for all $n \in \mathbb{N}$. let $b_n = 1 - a_n$. Then $0 \leq b_n \leq 1$ and

$$\psi((b_n)) > \psi((1,1,1,\ldots)) - \psi((a_n)) > \psi((1,1,1,\ldots))$$

So

$$\|\psi\| \geq |\psi((b_n))| \geq \psi((1,1,\ldots))$$

(2 $\implies$ 1) Let $(a_n) \in \ell^\infty$ with $|a_n| \leq 1$, then $0 \leq 1-a_n$. So $\psi((1-a_n)) \geq 0$ and therefore $\psi((1,1,1,\ldots)) \geq \psi((a_n))$. Similarly $\psi((-a_n)) \leq \psi((1,1,1,\ldots))$ which gives $|\psi((a_n))| \leq \psi((1,1,1,\ldots))$ $\square$

1.3 Quotient Spaces

Definition 1.3.1. Let X be a normed space and $Y \leqslant X$ be a closed subspace. For every $x \in X$, define

$$\|x + Y\| = \inf\{\|x + y\| \;:\; y \in Y\}$$

Lemma 1.3.1. *This defines as norm on $\frac{X}{Y}$. If X is complete, then $\frac{X}{Y}$ is complete.*

Proof. Obviously, $\|x+Y\| \geq 0$ for all $x \in X$, and $\|x+z+Y\| \leq \|x+Y\|+\|y+Y\|$. Similarly, we can also show homogeneity.

Now assume $x \in X$ is such that $\|x+Y\| = 0$. Then there is a sequence $(y_n) \in Y$ such that $\|x - y_n\| \to 0$, that is $y_n \to x$. Since Y is closed, we get $x \in Y$.

To show the second part of the lemma, consider the sequence $(x_n + Y) \in X/Y$ such that $\sum_{n\in\mathbb{N}} \|x_n - Y\| < \infty$. For each $n \in \mathbb{N}$, choose $y_n \in Y$ such that

$$\|x_n + y_n\| \leq \|x_n + Y\| + \frac{1}{2^n}$$

Then $\sum_{n\in\mathbb{N}} \|x_n + y_n\| \leq \infty$. Since X is complete, the sequence $\sum_{n\in\mathbb{N}} x_n + y_n$ converges to say $z \in X$. Then

$$\begin{aligned}
\|(z+Y) - \sum_{n=k}^{n}(x_n + Y)\| &= \|\Big(z - \sum_{n=k}^{n} x_n\Big) + Y\| \\
&= \|\Big(z - \sum_{i=1}^{k}(x_n + y_n)\Big) + Y\| \\
&\leq \|\Big(z - \sum_{i=1}^{k}(x_n + y_n)\Big)\|
\end{aligned}$$

which converges to 0 as $k \to \infty$ $\square$

Lemma 1.3.2. *The canocial map, $q : X \to \frac{X}{Y}$ is a continuous open map. A subset $E \subset X/Y$ is open iff $q^{-1}(E) \subset X$ is open.*

Proof. Since $\|x+Y\| \le \|x\|$, for all $x \in X$, we see that the map q is a contraction. Thus for all open $E \subset X/Y$, we get $q^{-1}(E)$ is open.

Conersely, assume that $A \subset X$ is open. Let $x \in A$ and $r > 0$ such that $B_r(a) \subset A$. Let $z \in X$ such that $\|q(a) - q(z)\| < r$. So, $\|(a-z)+Y\| < r$. Then $\exists y \in Y$ such that $\|a - z - y\| < r$. So $z + y \in B_r(a), q(z+y) = q(z) \in q(B_r(a))$. So $B_r(q(a)) \subset q(B_r(a)) \subset q(A)$. Thus $q(A)$ is open. □

1.4 Open Mapping Theorem and Closed Graph Theorem

Theorem 1.4.1 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a surjective bounded linear map. Then T is an open map i.e $T(E)$ is open in Y i.e if $E \subset X$, then $T(E)$ is open in Y.*

Steps of proof. See Prof. Blecher's Notes on Functional Analysis

Use Baire category theorem to show that $\overline{T(B_r(0))}$ has a non-empty interior.

- Then use linearity of T to show that $0 \in \overline{T(B_{2r}(0))}$.

□

Proof. Since Y is complete, by the Baire catergory theorem it is of the second category. Let $r \ge 0$, then

$$\begin{aligned} Y &= T(X) \\ &= T\Big(\bigcup_{n=1}^{\infty} B_{nr}(0)\Big) \\ &= T\Big(\bigcup_{n=1}^{\infty} nB_r(0)\Big) \\ &= \bigcup_{n=1}^{\infty} n\overline{T(B_r(0))} \end{aligned}$$

Then by BCT, there exist some $n \in \mathbb{N}$ such that $\mathrm{int}(\overline{T(B_r(0))}) \neq \emptyset$. Let $y_0 \in \mathrm{int}(\overline{T(B_r(0))})$. So there exists $\epsilon > 0$, such that $B_\epsilon(y_0) \subset \overline{T(B_r(0))}$. Let $w \in B_Y(0, \epsilon)$. Then $y_0 + w \in B_Y(y_0, \epsilon)$, and $\exists (x_n) \subset B_X(0, r)$ such that $T(x_n) \to$

$y_0 + w$. Also $\exists (z_n) \in B_X(0, r)$ such that $T(z_n) \to y_0$. Then $T(x_n - z_n) \to w$, so $w \in \overline{T(B_X(0, 2r))}$. Since w was an arbitrary element in $B_Y(0, \epsilon)$ we see that $B_Y(0, \epsilon) \subset \overline{T(B_X(0, 2r))}$. So $0 \in \text{int}(\overline{T(B_X(0, s))})$ for $s > 0$.

Now fix $t > 0$ and let $y_0 \in \overline{T(B_X(0, t))}$. By the above there exists $\epsilon > 0$ such that $B_Y(0, \epsilon) \subset \overline{T(B_X(0, \frac{t}{2}))}$. Then $(y_0 + B_Y(0, \epsilon)) \cap T(B_X(0, \frac{t}{2})) \neq \emptyset$. So $\exists x \in B_t(0)$ such that $T(x_1) = y_0 - y_1$ where $y_1 \in B_Y(0, \epsilon) \subset \overline{T(B_X(0, \frac{t}{2}))}$.

Similarly $\exists y_2 \in \overline{T(B_{\frac{t}{4}}(0))}$ and $x_2 \in B_{\frac{t}{2}}(0)$ such that $T(x_2) = y_1 - y_2$. Thus inductively we can choose $y_{n+1} \in \overline{T(B_{\frac{t}{2^{n+1}}}(0))}$ and $x_{n+1} \in B_{\frac{t}{2^n}}(0)$ such that $T(x_{n+1}) = y_n - y_{n+1}$. Now since we constructed nicely, $\sum_{i=0}^{\infty} x_n$ converge. verify. Moreover for all $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} T(x_n) = y_0 - y_N$$

Also notice that $y_n \to 0$. Hence $y_0 = \lim_{N\to\infty}(y_0 - y_N) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n T(x_n) = \lim_{N\to\infty} T(\sum_{n=1}^{N} x_n) = T(x) \in T(B_{2t}(0))$. So

$$\overline{T(B_t(0))} \subset T(B_{2t}(0))$$

Now to complete the proof, let E be an open subset of X. Let $x_0 \in E$ be such that $y_0 = T(x_0)$. Let $\epsilon > 0$ be such that $x_0 + B_\epsilon(0) = B_\epsilon(x_0) \subset E$. So $y_0 + T(B_\epsilon(0)) = T(B_\epsilon(x_0)) \subset T(E)$. By the above $\exists \delta > 0$ such that $B_\delta(0) \subset T(B_\epsilon(0))$ ☐

Find examples where this fails if we slack the conditions

Corollary 1.4.1.1. *Let X and Y be Banach spaces and $T : X \to Y$ be a bijective bounded linear map. Then $T^{-1} : Y \to X$ is bounded.*

Proof. verify ☐

Theorem 1.4.2 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a linear map. Then T is bounded if and only if graph of T, defined as $g(T) = \{(x, T(x)) \ : \ x \in X\}$ is closed in the product topology of $X \times Y$.*

Proof. Define the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$ on $X \times Y$. Then $X \times Y$ with this norm is a Banach space. verify.

Assume T is continuous. Then if $(x_n, T(x_n))$ is Cauchy in $g(T)$ then x_n and $T(x_n)$ must be cauchy in X and Y respectively. By the completeness of the spaces X and Y, we get $x_n \to x \in X$ and $T(x_n) \to y \in Y$. Moreover by continuity of T, we get $T(x_n) \to x$. Since the Banach space is Hausdorff, we get $y = x$ and that $(x, T(x)) \in g(T)$ making it closed.

Conversely, define $S : X \to g(T)$ as $S(x) = (x, T(x))$. S is linear and bijective. Assume $g(T)$ is closed, hence a Banach space. Observe that $S^{-1} : g(T) \to X$ is bounded(contractive). By the open mapping theorem, S is bounded. Assume $x_n \to z$. So $S(x_n) \to S(z)$. Then $(x_n, T(x_n)) \to (z, T(z))$, which gives $T(x_n) \to T(z)$. $\square$

Chapter 2

Topological Vector Spaces

Example 2.0.1. Let X be a vector space and let $f : X \to \mathbb{C}$ be a linear map. Define $\phi : X \to R^+ := \phi(x) = |f(x)|$. Then ϕ is a seminorm.

2.1 Weak and Weak * Topologies

Remark 2.1.1. Let X be a TVS and $A \subset X^*$. We denote by $\sigma(X, A)$, the topology on X defined by A. (initial topology). Recall that $\sigma(X, A)$ is Hausdorff if and only if A separate points of X.

$\sigma(X, X^*)$ is called the weak topology on X.

Also recall that $X \hookrightarrow X^{**}$ by the evaluation maps. Hence we can view X as a subset X^{**}. And with this identification, we call $\sigma(X^*, X)$ the weak $*$ topology on X^*

Definition 2.1.1. Let S be any set. Let I be a directed set. A net in S indexed by Λ is a function $f : \Lambda \to S$. We denote the net by $(x_\lambda)_{\lambda\in\Lambda}$.

In addition if S is a topological space, we say a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in S$ if for all open set U in S with $x \in U$, there exists an $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$ we have $x_\lambda \in U$.

See how nets become the topological equivalent of sequences in metric spaces. Find what property of the metric space makes it enough to be indexed by a countable totally ordered set for openness.

Remark 2.1.2. By definition, a basis of open neighborhoods of a point $x_0 \in X$ in $\sigma(X, A)$ is given

$$\bigcup_{\substack{p_1,p_2,\ldots,p_n\in A\\ \epsilon_1,\epsilon_2,\ldots,\epsilon_n>0}} \bigcap_{i=1}^{n} \{z \in Z | p_k(z - x_0) < \epsilon_k\}$$

So the basis in a weak topology is

$$\bigcup_{f_1,f_2,\ldots,f_n\in X^*} \quad_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n>0} (x_0) = \left\{z \in Z \Big| |f_k(z - x_0)| < \epsilon_k, \text{ for all } k = 1, 2, \ldots n\right\}$$

Example 2.1.1. Let S be a topological space, $E \subset S$, $x_0 \in \overline{E}$. Then there is a net $(x_\alpha) \subset E$ such that $x_\alpha \to x_0$.

Proof. Consider the collection $\mathscr{T}$ of all open sets of S that contain x_0. Order $\mathscr{T}$ by the reserve inclusion. That is $A \le B$ if $B \subset A$. This makes $\mathscr{T}$, a directed set. Now each $\lambda \in \mathscr{T}$ has a nonempty intersection with E being an open set containing the limit point x_0 of E. For each $\lambda \in \mathscr{T}$ choose an $x_\lambda \in \lambda \cap E$. Then we claim that the net $(x_\lambda) \to x_0$. □

Example 2.1.2. Let $X = \ell^1$. Then $X^* = \ell^\infty$. Then the weak * topology on ℓ^∞ is given by the pointwise convergence of a net $(f_\alpha) \subset \ell^\infty$ converges to $f \in \ell^\infty$ if and only if $f_\alpha(n) \to f(n)$ for all $n \in \mathbb{N}$

Theorem 2.1.1 (Tychonoff). *The product of compact sets is compact.*

Corollary 2.1.1.1. *Let X be a compact Hausdorff space. Then for any set S, the set $\{\phi : S \to X\} = X^S$ is compact wrt to pointwise convergence.*

Theorem 2.1.2 (Banach-Alaoglu Theorem). *Let X be a normed space. Then the closed unit ball $\overline{B_{X^*}(0,1)} = \{f \in X^* \ : \ \|f\| \le 1\} = E$ is weak * compact.*

Proof. Let $\bar{B}$ be the closed unit ball of X. Then by Tychonoff theorem, $\bar{D}^{\bar{B}}$ is compact. Define $\phi : E \to \bar{D}^{\bar{B}}$ as $\phi(f)(x) := f(x)$. Observe that ϕ is injective. Also observe that ϕ is continuous (weak * in LHS, and pointwise in RHS).

Next we show that the image of ϕ a closed subset of $\bar{D}^{\bar{B}}$, hence compact. Let f_i be a net in E such that $\phi(f_i) \to \psi$ pointwise for some $\psi \in \bar{D}^{\bar{B}}$.

Define $g : X \to C$ as $g(x) = \alpha\psi(\frac{x}{\alpha})$ where $\|x\| \le \alpha$. For this to be well defined we must have $\|x\|\psi(\frac{x}{\|x\|}) = \alpha\psi(\frac{x}{\alpha})$ for any $\alpha > \|x\|$. But we get this since ψ is a pointwise limit of linear functionals. Moreover we get that g is linear for the same reason. Thus $\psi = \phi(g)$ and so $\phi(E)$ is closed.

It only remains to show that the inverse of ϕ is continuous. verify. □

Remark 2.1.3. The closed unit ball of a normed space Y is compact w.r.t the norm topology if and only if Y is finite dimensional.

Proof. verify □

Theorem 2.1.3. *Let X be a normed space. Then E is weak * metrizable iff X is separable.*

Proof. Assume X is separable. Let $\{x_n \ : \ n \in \mathbb{N}\}$ be a dense subset of X. For every $f, g \in E$, define $d(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n}|f(\frac{x_n}{\|x_n\|} - g(\frac{x_n}{\|x_n\|})|$. Check that d is a metric.

Now asssume $f_i \to f$ weakly in E. Then $f_i(x_n) \to f(x_n)$ for all $n \in \mathbb{N}$ and $d(f_i, f) \to 0$ (verify).

Assume E is metrizable. Then $\exists\{U_n : n \in \mathbb{N}\}$ of weak * open neighborhoods of 0 such that $\cap_{n=1}^{\infty} U_n = \{0\}$. So, for each $n \in \mathbb{N}$, there exists a finite set $A_n \in X$ and $\epsilon > 0$ such that the (subbasis sets) $\{f \in E : |f| \leq \epsilon \forall x \in A_n \subset U_n$. Now let $A = \cup_{n=1}^{\infty} A_n$. Let $\phi \in E$ such that $\phi(x) = 0$ for all $x \in A$. □

Definition 2.1.2. Let X and Y be normed spaces and $T \in B(X,Y)$. The adjoint of T, denoted by $T^* \in B(Y^*, X^*)$, is the map $T^* : f \to f \circ T$

Proposition 2.1.1. $\|T\| = \|T^*\|$

Proof. $|T^*(f)| \leq \|f \circ T\| \leq \|T\|\|f\|$ implies $\|T^*\| \leq \|T\|$

$$\begin{aligned}\|T^*\| &= \sup\{\|T^*(\phi)\| : \phi \in Y^*, \|\phi\| \leq 1\} \\ &= \sup\{|\phi(T(x))| : \phi \in Y^*, x \in X, \|\phi\| \leq 1, \|x\| \leq 1\} \\ &= \|T\|\end{aligned}$$

Note that the last equality is a consequence of HBT since it guarantees the existence of $\phi_y \in Y^*$ with $\|\phi_y\| \leq 1$ and $\phi_y(y) = |y|$. □

Lemma 2.1.1. *For any $T \in B(X,Y)$, $T^* : Y^* \to X^*$ is weak * continuous (in both spaces)*

Proof. Let $\phi_i \to \phi$ weakly in Y^*. Then by definition for all $y \in Y$, $\phi_i(y) \to \phi(y)$. Then for $x \in X$, $T^*(\phi_i)(x) = \phi_i(T(x)) \to \phi(T(x)) = (T^*(\phi))(x)$ which shows the continuity of T^*. □

Lemma 2.1.2. *For any normed space X, $i_X(X)$ is weak * dense in X^{**}.*

Proof. verify □

Example 2.1.3. Is i_{X^*} weak * - weak * continuous.

2.2 Locally Convex Topological Vector Spaces

Lemma 2.2.1. *Let X be a normed space and $x_1, x_2, \ldots, x_n \in X$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \geq 0$. Then the set*

$$\bigcup_{x_1,x_2,\ldots,x_n,\epsilon_1,\epsilon_2,\ldots,\epsilon_n} (\phi)$$

is convex. Moreover any topological vector spaces with the topology induced by a family of seminorms is locally convex.

Review convexity arguments in Arveson's 'Subalgebras of C Algebras'*

Proof. verify □

Definition 2.2.1. Let X be a vector space and $E \subset X$ be a convex subset. An element $a \in E$ is called an extreme point of E if whenever $x, y \in E$, $0 \le t \le 1$ with $a = tx + (1-t)y$, then $x = y = a$.

Example 2.2.1. Let $\bar{D} = \{\alpha \in \mathbb{C} \ : \ |\alpha| \le 1\}$. Then $\bar{D}$ is convex with $\mathrm{Ext}(\bar{D}) = S^1$

Theorem 2.2.1 (Krein-Milman Theorem)**.** *Let X be a locally convex space, and let K be a compact convex subset of X. Then the $Ext(K) \neq \emptyset$ and indeed $K = \overline{co}(Ext(K))$*

Definition 2.2.2. Let V be a vector space and $S \subset V$. The convex hull of S is defined as

$$\mathrm{co}(S) = \Big\{ \sum_{i=1}^{n} t_i x_i \ \Big| \ 0 \le t \le 1, \sum t_i = 1, x_i \in S \Big\}$$

Lemma 2.2.2. *Let K be a convex set. $x_0 \in Ext(K)$ if and only if $K \setminus \{x_0\}$ is convex.*

Proof. If $K \setminus \{x_0\}$ is not convex, since K is convex, x_0 can be written as the convex combination of elements in $K \setminus \{x_0\}$ which makes $x_0 \notin \mathrm{Ext}(K_0)$. Conversely if $x_0 \in \mathrm{Ext}(K)$, then x_0 cannnot be written as the convex combination of elements of K. Hence $K \setminus \{x_0\}$ is closed under convex combinations, making in convex. □

Theorem 2.2.2 (Krein-Milman Theorem)**.** *Let X be a locally convex space, and let K be a compact convex subset of X. Then the $Ext(K) \neq \emptyset$ and indeed $K = \overline{co}(Ext(K))$*

Proof. We first prove that the $\mathrm{Ext}(K) \neq \emptyset$. Note that $K \setminus \{x_0\}$ is a relatively open subset of K since $\{x_0\}$ is closed and $K \setminus \{x_0\} = \{x_0\}^c$ relative to K.

Now let $\mathcal{A}$ be the collection of all relatively open convex proper subsets of K. Note that $\emptyset \in \mathcal{A}$, therefore $\mathcal{A}$ is nonempty. Equip $\mathcal{A}$ with the partial order defined by the set inclusion. Let $\mathscr{C}$ be a chain in $\mathcal{A}$ and $F_{\mathscr{C}} = \cup_{C \in \mathscr{C}} C$. $F_{\mathscr{C}}$ is relatively open being the union of relatively open subsets of K. To see that $F_{\mathscr{C}}$ is convex, let $x, y \in F_{\mathscr{C}}$. Then since $\mathscr{C}$ is a chain, there exist a $C \in \mathscr{C}$ such that $x, y \in \mathscr{C}$. Then by the convexity of C, $tx + (1-t)y \in C \subset F_{\mathscr{C}}$ for all $t \in [0,1]$.

Zorn's Lemma to find a maximal proper open convex subset of K

We claim that $F_{\mathscr{C}}$ is a proper subset of K. For the sake of contradiction, assume $F_{\mathscr{C}} = K$. Since K is compact and C is open in K for all $C \in \mathscr{C}$, there are finitely many $C_1 \subset C_2 \subset \ldots \subset C_k \in \mathscr{C}$ which cover K (i.e $K = \cup_{n=1}^{k} C_n$). Hence we get $C_k = K$, which is absurd since C_k must be a proper subset of K. Hence $F_{\mathscr{C}} \in \mathcal{A}$ and thus every chain must have an upper bound in $\mathcal{A}$. Now by Zorn's lemma, $\mathcal{A}$ has a maximal element K_0.

Constructing and open convex subset containing K_0

Since K is a connected space (path connected by a straight line, being convex), we know that the only clopen subsets are $\emptyset$ and K. Since we know that K_0 is open being in $\mathcal{A}$, we see that $K_0 \neq K$ and $K_0 \neq \emptyset$. Therefore $\overline{K_0} \neq K_0$. Let $x_0 \in \overline{K_0} \backslash K_0$, $y_o \in K_0$ and $0 < t < 1$. Define $\varphi_{t,y_0} : K \to K$ such that $\varphi_{t,y_0}(z) = ty_0 + (1-t)z$. Then φ_{t,y_0} is $(1-t$ Lipschitz) continuous relative to K and thus $\varphi_{t,y_0}^{-1}(K_0)$ is open in K. By the convexity of K_0, we get $K_0 \subset \phi_{t,y_0}^{-1}(K_0)$.

Also $\varphi_{t,y_0}^{-1}(K_0)$ is convex. Let $a, b \in \phi_{t,y_0}^{-1}(K)$. Then $ty_0+(1-t)a, ty_0+(1-t)b \in K_0$. By the convexity of K_0 we get $r(ty_0 + (1-t)a) + (1-r)(ty_0 + (1-t)b) = ty_0 + (1-t)(ra + (1-r)b) = \phi_{t,y_0}(ra + (1-r)b) \in K_0$ for all $r \in [0, 1]$. Thus $ra + (1-r)b \in \phi_{t,y_0}^{-1}(K_0)$ for all $r \in [0, 1]$. Hence we get $\phi_{t,y_0}^{-1}(K_0)$ is convex.

We claim, $x_0 \in \varphi_{t,y_0}^{-1}(K_0)$, then the maximality of K_0 will force $\phi_{t,y_0}^{-1}(K_0) = K$. Let U be a convex neighborhood of $0 \in X$ containing $-x$ for all $x \in U$ (just take $-U$ and intersect with U) such that $y_o + E \subset K_0$, where $E = K \cap U$. Let $w = \varphi_{t,y_0}(x_0)$. Since $x_0 \in \overline{K_0}$, for any $r > 0$, there exists $x_r \in K_0$ such that $x_r \in (x_0 + rE) \cap K_0 \neq \emptyset$. In particular, let $r = \frac{t}{1-t}$. Then by linearity, we get $\big(x_0 + \frac{t}{1-t}E\big) \cap K_0 = (\frac{t}{1-t})E \cap (K_0 - x_0) \neq \emptyset$. Choose z in the above set. Then

I can't picturize the choice of z

$$y_0 - \Big(\frac{1-t}{t}\Big)z \;\in\; y_0 + E \subset K_0$$

and $x_0 + z \in K_0$. Since K_0 is convex,

$$t\Big(y_0 - \frac{(1-t)}{t}z\Big) + (1-t)(x_0 + z) = \phi_{t,y_0}(x_0) \in K_0$$

Thus $\phi_{t,y_0}^{-1}(K_0) = K$.

Now we claim that $K = K_0 \cup \{x_0\}$. For the sake of contradiction assume $\exists p \in K$ such that $p \notin K_0 \cup \{x_0\}$. Since the space is Hausdorff and locally convex, x_0 has an open convex neighborhood E in X such that $p \notin E$. Let $E' = E \cap K$, $a \in K_0, b \in E'$ and $0 < r < 1$. Then since $\phi_{t,y_0}(K) = K_0$ for all $t \in [0, 1], y_0 \in K_0$, we get $\phi_{r,a}(b) = ra + (1-r)b \in K_0$. So $K_0 \cup E'$ is convex (Sine we know that K_0, E' are convex, we only need to worry about $rx + (1-r)y$ for $x \in K_0, y \in E'$. But $\phi_{r,x}$ takes care of that). $K_0 \cup E'$ is also open in K. Hence by maximality, we get $K_0 \cup E' = K$. But this is a contradiction since $p \notin K_0 \cup E'$. Thus by Lemma 2.2.2, we see that $x_0 \in \mathrm{Ext}(K)$.

Next we prove $K = \overline{co}(\mathrm{Ext}(K))$. Let $P = \overline{co}(\mathrm{Ext}(K))$ and for the sake of contradiction assume $P \neq K$. Let $x_0 \in K \setminus P$. Now by the geometric Hahn-Banach separation theorem, we get that there is a continuous linear functional $\phi : X \to \mathbb{R}$ and a number $\alpha, \epsilon \in \mathbb{R}$ such that

$$\Re\phi(x_0) \leq \alpha < \alpha + \epsilon \leq \Re\phi(p), \quad \forall p \in P$$

□

Lemma 2.2.3. *Let K_1, K_2 be compact convex subsets of a locally compact TVS X. Then*

$$\overline{co}(K_1 \cup K_2) = (co)(K_1 \cup K_2)$$

Proof. verify. We'll show that $\text{co}(K_1 \cup K_2)$ is compact and hence closed. Let $x = \alpha_1 a_1, \alpha_2 a_2, \ldots, \alpha_n a_n + \beta_1 b_1, \beta_2 b_2, \ldots, \beta_m b_n \in \text{co}(K_1 \cup K_2)$, where $\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{m} \beta_i = 1$. Then

$$x = (\sum_{i=1}^{n} \alpha_i) \underbrace{\left(\sum_{i=1}^{n} \Big(\frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} \Big) a_i \right)}_{\in\, K_1} + (\sum_{i=1}^{m} \beta_i) \underbrace{\left(\sum_{i=1}^{m} \Big(\frac{\beta_i}{\sum_{i=1}^{m} \beta_i} \Big) b_i \right)}_{\in\, K_2}$$

Hence every element $x \in \text{co}(K_1 \cup K_2)$, can be written as $x = ta + (1-t)b$ where $a \in K_1, b \in K_2$.

Now let $x_\lambda = t_\lambda a_\lambda + (1-t_\lambda) b_\lambda$ be a net in $\text{co}(K_1 \cup K_2)$, for $\lambda \in \Lambda$, $a_\lambda \in K_1, b_\lambda \in K_2$. Since (a_λ) is a net in the compact set K_1, there is a subnet a_σ for $\sigma \in \Sigma \subseteq \Lambda$, such that $a_\sigma \to a \in K_1$. By similar reasoning b_σ has a convergent subnet b_π for $\pi \in \Pi \subseteq \Sigma$, such that $b_\pi \to b \in K_2$. Again t_π is a net in the compact space $[0,1]$, hence is has a convergent subnet t_ω for $\omega \in \Omega \subseteq \Pi$ such that $t_\omega \to t$ in $[0,1]$.

Now consider the subnet $x_\omega = t_\omega a_\omega + (1-t_\omega)\beta_\omega$ of x_λ. Since $\Omega \subseteq \Pi \subseteq \Sigma$, $t_\omega \to t, \beta_\omega \to b$ and $a_\omega \to a$. Therefore by the continuity of the scalar product and addition in the TVS, we get $x_\omega \to t\alpha + (1-t)\beta \in \text{co}(K_1 \cup K_2)$. Hence we get $\text{co}(K_1 \cup K_2)$ is compact. □

Theorem 2.2.3 (Inverse Krein-Milman)**.** *Let K be a compact convex subset of a locally convex topological vector space X. Let $A \subset K$ be a closed subset of K. If $K = \overline{co}(A)$, then $Ext(K) \subset A$.*

Note that $\text{Prob}[0,1]$, the collection of probability measures identified as a subspace of a $C([0,1])^*$ is convex, weak * compact with $\text{Ext}(K) = \{\delta_x \ : \ x \in [0,1]\}$

Proof. FSTOC, assume $\exists x_0 \in \text{Ext}(K)$, $x_0 \notin A$. Since A is compact, $\exists y_1, y_2, \ldots, y_n \in A$ and an open convex neighborhood B of 0 such that

$$A \subset \cup_{i=1}^{n} (y_i + B)$$

and $x_0 \notin y_i + \overline{B}$ for all $i = 1, 2, \ldots, n$. Let $B_i = (y_i + \overline{B}) \cap K$. Then B_i is a compact convex subset of K for each i. Hence by the previous lemma, we get

$$\text{co}(B_1 \cup B_2 \cup \ldots \cup B_n) = \overline{\text{co}}(B_1 \cup B_2 \cup \ldots \cup B_n) \supset \overline{\text{co}}(A) = K$$

Thus $\exists b_i \in B_i$ and $0 \le t_i \le 1$, $\sum_{i=1}^n t_i = 1$ such that

I struggle at finding the contradiction

$$x_0 = t_n b_1 + t_n b_2 + \ldots + t_n b_n$$

Since $x_0 \in \text{Ext}(K)$, this forces $x_0 = b_j$ for some $1 \le j \le n$. This contradicts the assumption that $x_0 \notin y_i + \overline{B}$. Hence $x_0 \in A$. $\square$

Note that in the following attempt to prove the theorem, it is not obvious why U is convex. If we try to argue using arguments to the proof of separating a compact set and a point using open sets in a Hausdorff space, we will eventually need to show that the finite open subcover of A sits inside a closed convex set that does not contain x_0, which again is not obvious.

Proof. FTSOC, assume that $\exists x_0 \in \text{Ext}(K) \setminus A$. Since the TVS is Hausdorff, there exist convex open sets U, V such that $A \subset U, x_0 \in V$, and $U \cap V = \emptyset$. Moreover we claim that $\overline{U} \cap V = \emptyset$. Otherwise if $x \in \overline{V} \cap U$, then for any net $(x_\lambda) \in V$ that converge to x, by the definition of convergence $x_{\lambda_n} \in U$ for all λ_n greater that some λ_N. This would contradict the assumption that $U \cap V = \emptyset$. Hence we see that $A \subset \overline{V}$, and therefore $\overline{\text{co}}(A) \subset \overline{V}$. But this would again contradict the fact that $\overline{\text{co}}(A) = K$ since $x_0 \notin \overline{V}$. $\square$

Why is U convex?

Example 2.2.2. Let X be an infinite dimensional normed space. Then the set $A = \{x \in X \ : \ \|x\| = 1\}$ is norm closed. But the weak closure of A is the set

$$\overline{A}^w = \overline{\{x \in X \ : \ \|x\| = 1\}}^w = \{y \in X \ : \ \|y\| \le 1\}$$

Hence A is an example of a norm closed set, which is not weak closed.

Theorem 2.2.4. *Let X be a normed space and let $K \subset X$ be convex subset of X. Then the norm and the weak closure of K coincide.*

Proof. Since norm topology is stronger than weak topology, we get $\overline{K}^{\|\cdot\|} \subset \overline{K}^w$. Let $x \in X$ such that $x \notin \overline{K}^{\|\cdot\|}$. Now since $\{x\}, K$ are convex and compact, by Hahn-Banach separation theorem, there is a $f \in X^*, s \in \mathbb{R}, \epsilon > 0$ such that

$$|f(x)| \le s < s + \epsilon \le |f(y)|, \quad \forall y \in \overline{K}^{\|\cdot\|}$$

Since the set $\{z \in X \ : \ |f(z)| \ge s + \epsilon\}$ is weakly closed, and contains K, it must contain $\overline{K}^w$. Hence $x \notin \overline{K}^w$ $\square$

Corollary 2.2.4.1. *Let X be a normed space, and $(x_i)_{i \in I}$ be a net in X such that $x_i \to x$ weakly in X. Then there exists a net $(y_j)_{j \in J}$ of finite convex combinations of $\{x_i \ : \ i \in I\}$ such that $y_i \to x$ in norm.*

Proof. verify □

Proposition 2.2.1. *If K is a convex subset of a LCTVS. Then $\overline{K}$ is also convex.*

Proof. verify □

Proposition 2.2.2. *Show that $\boldsymbol{c}_0$ is weakly closed and weak * dense in ℓ_∞.*

Proof. verify □

Theorem 2.2.5 (Krein-Smulian Theorem)**.** *Let X be a Banach space, and let C be a convex subset of X^*. Then C is weak * closed if and only if $C \cap \{f \in X^* : \|f\| \leq r\}$ is weak * closed for all $r \in \mathbb{R}^+$.*

This should even work if we just take $n \in \mathbb{N}$. verify.

Corollary 2.2.5.1. *Let Z be a subspace of X^*. Then Z is weak * closed if and only if $\{\phi \in Z : \|\phi\| \leq 1\}$ is closed.*

Corollary 2.2.5.2. *Let X be a separable Banach space. Then a convex subset Z of X^* is weak * closed if and only if it is weak * sequentially closed.*

Proof. Since X is separable, for every $r > 0$, the set $\{f \in X^* : \|f\| \leq r\}$ is weak * metrizable. Thus $Z \cap \{f \in X^* : \|f\| \leq 1\}$ is weak * closed iff it is weak * sequentially closed. □

Corollary 2.2.5.3. *Let X be a separable Banach space, and $\phi \in X^{**}$. Then ϕ is weak * continuous if and only if ϕ is sequentially continuous i.e $f_n \to f$ weak * in X^* implies $\phi(f_n) \to \phi(f)$.*

Proof. Assume ϕ is sequentially weak * continuous. Let $C = \mathrm{Ker}(\phi)$ be a subspace of X^*. Let $g_n \in C$ be a sequence such that $g_n \to g \in X^*$. Then by assumption, $0 = \phi(g_n) \to \phi(g)$ implies $\phi(g) = 0$ and therefore $g \in C$. This shows that C is weak * sequentially closed, hence weak * closed by the separability of X. Hence ϕ is weak * continuous. □

Example 2.2.3. Let $S = \{n\delta_n : n \in \mathbb{N}\} \subset \ell^\infty$. We show that $0 \in \overline{S}^{w^*}$

Proof. Let $f \in \ell^1$. Then the set $\{n \in \mathbb{N} : |f(n)| < \epsilon/n\}$ is infinite. (Otherwise this would contradict $f \in \ell^1$). Thus $\exists N \in \mathbb{N}$ such that $N|f_i(N)| < \epsilon$ for all $i = 1, 2, \ldots m$. And therefore

$$N\delta_n \in \bigcup_{f_1, f_2, \ldots, f_N, \epsilon} (0)$$

□

Definition 2.2.3. A subset S of a vector space V is called balanced if $\forall s \in S, \alpha \in \mathbb{F}$ with $|\alpha| \leq 1, \alpha s \in S$.

Lemma 2.2.4. *Let X be a topological vector space, then every open neighborhood of* 0 *contains a balanced open neighborhood of O.*

Proof. verify $\square$

Lemma 2.2.5. *All n-dimensional topological vector spaces are isomorphic as topological vector spaces.*

Proof. For the case $n = 1$, and $\mathbb{F} = \mathbb{C}$.

Assume τ is a topology on $\mathbb{C}$ that turns it into a topological vector space. Now think of $i : \mathbb{C} \to (\mathbb{C}, \tau) := x \to x$ as the composition of $\mathbb{C} \to \mathbb{C} \times (\mathbb{C}, \tau) := x \to (x, 1)$ and $\mathbb{C} \times (\mathbb{C}, \tau) \to (\mathbb{C}, \tau) := (x, y) \to xy$. Then we see that i is the composition of these maps which are continuous by the definition of the product topology and the TVS. Hence, i is continuous.

To show that i^{-1} is continuous, consider the annulus $A = \{\alpha \in \mathbb{C} \ : \ 1 \leq |\alpha| \leq 2\}$. Then since A is compact in the usual topology and $i(A) = A$ is a continuous image, we get that A is open in τ. Hence $A^c \ni 0$ is open and by the lemma above has a balanced open neighborhood of 0 in it. (Show that this is actually an open disk). $\square$

Theorem 2.2.6. *Let X be a normed space. Then the closed unit ball of X is compact in norm topology if and only if X is finite dimensional.*

Proof. Suppose X is infinite dimensional normed space and let $\bar{B}$ be the closed unit ball. Let $x_1 \in \bar{B}$ and let $Y_1 = \text{span}\{x_1\}$. Then Y_1 is a closed subspace of X. Since X is a non-zero normed space, let $x_2 \in X$ such that $\|x_2 + Y_1\| = \frac{1}{2}$. Repeat the construction in the proof of Reisz lemma. $\square$

Lemma 2.2.6. *Let X be a normed space. Then $i(X)$ is weak * dense in X^{**}*

Proof. Let $C = \overline{B}^{w*}$, where B is the closed unit ball. Then C is compact convex. FSTOC, assume $\exists \phi \in X^{**}$ such that $\|\phi\| \leq 1$, $\phi \notin C$. Then by HBT, there is a $f \in X^{**}$ and $r \in \mathbb{R}$, $\epsilon > 0$ such that $\Re f(y) \leq r < r + \epsilon \leq \Re f(\phi)$ for all $y \in i(X)$. This implies $\|f\| \leq r$, hence $|f(\phi)| = |\phi(f)| \leq \|\phi\| \|f\| < r$ which gives a contradiction. $\square$

Show that $\Re f(x) \leq r\|x\|$ imply $\|f\| \leq \alpha$

Theorem 2.2.7. *Let X be a Banach space. Then the closed unit ball $\bar{B}$ is weakly compact if and only if X is reflexive.*

Proof. If X is reflexive, the weak and weak * topology coincides and the Banach Alaouglu gives the proof. verify

Assume $\bar{B}$ is weakly compact. Observe that then the map $i : X \to X^{**}$ is continuous when we equip X with weak topology and X^{**} with weak topology. Thus $i(\bar{B})$ is weak * compact. Moreover $i(\bar{B})$ is weak * dense in the closed unit ball of X^{**}. Hence the result. $\square$

Chapter 3

Hilbert Spaces

Definition 3.0.1. Recall that a complex inner product o a complex vector space is a map

$$\langle\ ,\ \rangle : X \times X \to \mathbb{C}$$

such that

(1) $\langle x, x\rangle \geq 0$ for all $x \in X$

(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(3) $\langle \alpha x + z, y\rangle = \alpha\langle x, y\rangle + \langle z, y\rangle$

Recall the norms induced by the inner product and the Cauchy-Schwarz inequality.

Definition 3.0.2. Complete inner product spaces are called Hilbert spaces

Proposition 3.0.1. *Let X be an inner product space. Then the inner product of X extends to an inner product on the completion (unique metric space completion) of X, turns it into a Hilbert space.*

Definition 3.0.3. If $x, y \in H$, the Hilbert space, we say $x \perp y$ if $\langle x, y\rangle = 0$

Definition 3.0.4. Given a set $S \subset H$, $S^{\perp} = \{y \in H\ :\ \langle x, y\rangle = 0\}$

Proposition 3.0.2. *Let H, K be Hilbert spaces and $T : H \to K$ be linear. Then the following are equivalent.*

(1) T is isometry

(2) $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$

Proof. See Homework-5 □

Proposition 3.0.3. *For all $x, y \in H$, a Hilbert space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Example 3.0.1. Show that c_{00} under the usual inner product is not complete and its completion is $\ell^2(\mathbb{N})$.

Proof. Consider the sequence $x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots)$. Then clearly $x_n \in \mathbf{c}_0$. Moreover $x_n \to x = (1, \frac{1}{2}, \dots \frac{1}{n}, \frac{1}{n+1}, \dots)$ in the norm by the inner product. But $x \notin \mathbf{c}_{00}$. But $x \in \ell^2(\mathbb{N})$. Moreover, the same process can be used to approximate any element in $\ell^2(\mathbb{N})$ by elements in $\mathbf{c}_{00}$. Thus we see that $\mathbf{c}_{00}$ is dense in $\ell^2(\mathbb{N})$ $\square$

Example 3.0.2. $L^2(\mathbb{R}, \mu)$ with

$$\langle f, g \rangle = \int_{\mathbb{R}} f\overline{g} \, d\mu$$

is a Hilbert space.

Example 3.0.3. Let J be any set $\ell^2(J) = \{f : J \to \mathbb{C} \ : \ \sum_{j \in J} |f(j)|^2 < \infty\}$ with the usual inner product is a Hilbert space.

Definition 3.0.5. An orthonormal basis for H is a maximal orthonormal set.

Theorem 3.0.1. *Let H be a Hilbert space and J be an orthonormal basis for H. Then there exists a bijective linear isometry $T : H \to \ell^2(J)$.*

Theorem 3.0.2. *Let $\mathcal{H}$ be a Hilbert space and let C be a non-empty closed convex subset of $\mathcal{H}$. Then there exist a unique vector $x \in C$ such that $\|x\| \leq \|\eta\|$ for all $\eta \in C$.*

construction of the proof is a bit tricky

Proof. Let $d = \inf\{\|\eta\| \ : \ \eta \in C\}$ and choose a sequence $\eta_n \in C$ such that $\|\eta_n\| \to d$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|\eta_n\|^2 < d^2 + \epsilon$ for all $n \geq N$. Then for all $m, n \geq N$, we have

$$\|\eta_n - \eta_m\|^2 = 2(\|\eta_n\|^2 + \|\eta_m\|^2) - 4\|\frac{1}{2}(\eta_n + \eta_m)\|^2 \leq 4(d^2 + \epsilon) - 4d^2 = 4\epsilon$$

(Note that since C is compact, $\frac{1}{2}\eta_n + \frac{1}{2}\eta_m \in C$ and therefore, by the definition of d, we get $\|\frac{1}{2}(\eta_n + \eta_m)\| \geq d$.) Hence the sequence η_n is Cauchy and hence convergent since the space is complete. Let $\eta = \lim_{n\to\infty} \eta_n$. Since C is closed $\eta \in C$ and clearly $\|\eta\| = d$.

To see the uniqueness, assume $\alpha \in C$, and $\|\alpha\| = d$. Then

$$\begin{aligned} \|\eta - \alpha\|^2 &= 2(\|\eta\|^2 + \|\alpha\|^2) - 4\|\frac{1}{2}(\eta + \alpha)\|^2 \\ &\leq 4d^2 - 4d^2 = 0 \end{aligned}$$

Verify the second inequality. $\square$

Corollary 3.0.2.1. *Let $\eta \in \mathcal{H}$ and C be as before. Then there exist a unique vector $x \in C$ such that $d(\eta, C) = \|x - \eta\|$*

Proof. Apply above theorem to $C' = C - \eta$. Since C is closed and convex, so will be its translation $C - \eta$. $\square$

Proposition 3.0.4 (Pythagoras Theorem)**.** *Let $x, y \in \mathcal{H}$ an inner product space, and $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.*

Lemma 3.0.1. *Let $\mathcal{H}$ be a Hilbert space and K be a nontrivial closed subspace. Let $\eta \in \mathcal{H}$. Then $\xi \in K$ satisfy $\|\xi - \eta\| = d(\eta, K)$ iff $\xi - \eta \perp K$.*

Proof. Let $\xi - \eta \perp K$ and $k \in K$. Then $\eta - k = (\eta - \xi) + (\xi - k)$ and by Pythagoras theorem, we get

$$\|\eta - k\|^2 = \|\eta - \xi\|^2 + \|\xi - k\|^2 \geq \|\eta - \xi\|^2$$

Thus we see that $\|\eta - \xi\| = d(\eta, K)$.

Conversely, let $\|\eta - \xi\| = d(\eta, K)$. Then for all $\rho \in K$ and $t > 0$, we have

cool proof technique

$$\begin{aligned}\|\eta - \xi\|^2 &\leq \|\eta - (\xi + t\rho)\|^2 \\ &= \|\eta - \xi - t\rho\|^2 \\ &= \|\eta - \xi\|^2 + t^2\|\rho\|^2 - 2t\Re\langle \eta - \xi, \rho\rangle\end{aligned}$$

Hence we see that $|2\Re\langle \eta - \xi, \rho\rangle| \leq t\|\rho\|^2$ Since $t > 0$ was arbitrary, limiting it to zero, we get $\Re\langle \eta - \xi, \rho\rangle = 0$. Replacing ρ with $-i\rho$ will give the imaginary part is also zero. $\square$

Remark 3.0.1. If K is a closed subspace of the Hilbert space $\mathcal{H}$, then $\mathcal{H} = K \oplus K^\perp$. To see this notice that $K \cap K^\perp = \{0\}$. Now if $x \in \mathcal{H}$, then there is a $k_x \in K$ such that $d(x, K) = \|x - k_x\|$. Moreover, from Lemma 3.0.1, we see that $x - k_x \in K^\perp$. Hence $x = k_x + (x - k_x) \in K \oplus K^\perp$.

Theorem 3.0.3 (Reisz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and $f \in \mathcal{H}^*$. Then there exists a unique $\eta_f \in \mathcal{H}$ such that $f(x) = \langle x, \eta_f\rangle$ for all $x \in \mathcal{H}$. The map $\phi : \mathcal{H}^* \to \mathcal{H} := f \to \eta_f$ is conjugate linear isometric bijection.*

Proof. If f is the zero linear functional, $n_f = 0$ and we're done. If not let $K = \text{Ker}(f)$. Then K has co-dimension 1. Consider $f|_{K^\perp}$. Let $v \in K^\perp$ such that $f(v) = 1$. Clearly $K^\perp = \text{span}(v)$. And therefore for any $\alpha v \in K^\perp$, $f(\alpha v) = \alpha = \langle \alpha v, \frac{v}{\|v\|^2}\rangle$. Choose $\eta_f = \frac{v}{\|v\|^2}$. Now verify. $\square$

3.1 Orthogonal Projections

Theorem 3.1.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear map. Then there exists a unique bounded linear map $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ satisfying $\langle Tx, y\rangle_{\mathcal{H}_2} = \langle x, T^*y\rangle_{\mathcal{H}_1}$ for all $x \in \mathcal{H}_1, y \in \mathcal{H}_2$.*

Proof. For every given $y \in \mathcal{H}_2$ define a linear functional $f^y : H_1 \to \mathbb{C}$ as $f^y(x) = \langle Tx, y\rangle$. Since f^y is bounded, $f^y \in \mathcal{H}_1^*$. Hence by Reisz representation, there is a unique $T^*(y) \in \mathcal{H}_1$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

Uniqueness follows from the fact that in any inner product space X, if $x, y \in X$ such that $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in X$, then $\langle x-y, z\rangle = 0$ for all $z \in X$, in particular $z = x-y$. Then by the positive definiteness of the inner product, we get $x = y$. $\square$

Theorem 3.1.2. *Let $\mathcal{H}$ be a Hilbert space and K a closed subspace. For every $\eta \in \mathcal{H}$, denote by $P_K(\eta)$, the unique closest vector in K, closest to η. Then*

(1) $P_K : \mathcal{H} \to \mathcal{H}$ is linear, bounded with $\|P_K\| = 1$ and idempotent.

(2) $P_K^* = P_K$ (self-adjoint)

Proof. (1) Let $\eta_1, \eta_2 \in \mathcal{H}$ and $\alpha \in \mathbb{C}$. Then using Lemma 3.0.1 for all $\xi \in K$, we have

$$\begin{aligned}\langle \alpha\eta_1 + \eta_2 - \alpha P_K(\eta_1) - P_K(\eta_2), \xi\rangle &= \alpha\langle \eta_1 - P_K(\eta_1), \xi\rangle + \langle \eta_2 - P_K(\eta_2), \xi\rangle \\ &= 0\end{aligned}$$

Thus we get $P_K(\alpha\eta_1+\eta_2) = \alpha P_K(\eta_1)+P_K(\eta_2)$ again by the same Lemma 3.0.1. Moreover since $P_K(x)$ is a vector in K it the vector in K closest to $P_K(x)$ is $P_K(x)$ itself. Thus we see that $P_K^2(x) = P_K(x)$. Hence idempotent. Moreover since K is closed, we have $X = K \oplus K^\perp$ (Remark 3.0.1). Thus for all $x \in X$, we can write $x = k_x + k_x'$, where $k_x \in K$ and $k_x' \in K^\perp$. Then $P_K(x) = k_x$ and triangle inequality shows $\|P_K(x)\| = \|k_x\| \leq \|x\|$. Thus $\|P_K\| \leq 1$.

(2) Since $X = K \oplus K^\perp$, let $x = k_x + k_x'$ and $y = k_y + k_y'$. Then

$$\langle P(x), y\rangle = \langle k_x, k_y + k_y'\rangle = \langle k_x, k_y\rangle = \langle k_x + k_x', k_y\rangle = \langle x, P_K(y)\rangle$$

and the uniqueness of the adjoint proves that $P_K^* = P_K$

$\square$

Proposition 3.1.1. $I - P_K = P_{K^\perp}$

Proof. Let $x \in X$ and $k \in K$. Then

$$\begin{aligned}
\langle (I - P_k)(x), k \rangle &= \langle x - P_k(x), k \rangle \\
&= \langle x, k \rangle - \langle P_k(x), k \rangle \\
&= \langle x, k \rangle - \langle x, P_k(k) \rangle \\
&= \langle x, k \rangle - \langle x, k \rangle \\
&= 0
\end{aligned}$$

Shows that □

Proposition 3.1.2. *Let K be a closed subspace of H. Let $E \subset K$ be an orthonormal basis for K. Extend E to an orthonormal basis $\tilde{E}$ for H. Then*

$$P_K|_E = I_K, \quad P_K|_{\tilde{E}\setminus E} = 0$$

Remark 3.1.1 (Parserval's Inequality). Let H be a Hilbert space. Let E be an orthonormal set. Then for every vector $\eta \in H$,

$$\|\eta\|^2 \geq \sum_{e \in E} |\langle \eta, e \rangle|^2$$

Lemma 3.1.1. *Let S be a nonempty subset of H. Then*

$$(S^{\perp})^{\perp} = \overline{span(S)}$$

Proof. Notice from the above proposition that $\mathrm{Ker}(P_K) = K^{\perp}$. Thus $(S^{\perp})^{\perp} = \mathrm{Ker}(P_{S^{\perp}})$. Since $P_{S^{\perp}}(s) = 0$ for all $s \in S$, we see that $S \subset \mathrm{Ker}(P_{S^{\perp}})$. Moreover, since $\mathrm{Ker}(P_{S^{\perp}})$ is a closed subspace, we see that $\overline{\mathrm{span}}(S) \subset \mathrm{Ker}(P_{S^{\perp}}) = (S^{\perp})^{\perp}$. □

Corollary 3.1.2.1. *Let E be an orthonormal subset of H. Then the following are equivalent.*

(1) E is an orthonormal basis

(2) $(E^{\perp})^{\perp} = H$

(3) $\overline{Span(S)} = H$

(4) $\|\eta\|^2 = \sum_{e \in E} |\langle \eta, e \rangle|^2, \ \forall \eta \in H$

Proof. verify □

Proposition 3.1.3. *Let K be a closed subspace of a Hilbert space H. Then $P_K = P_K^*$*

Proof. Let $k \in K$ and $x \in H$. Then

$$\langle x, P_K^*(k)\rangle = \langle P_K(x), k\rangle = \langle P_K(x) - x, k\rangle + \langle x, k\rangle = \langle x, k\rangle$$

for all $x \in H$. Hence $P_K^*(k) = k$. Conversely, verify $\square$

Theorem 3.1.3. *Let $P \in B(H)$ be an idempotent. Then the following are equivalent.*

(1) $P = P_K$ for some closed subspace $K \leqslant H$

(2) $P = P^*$

(3) $\|P\| = 1$

Proof. $1 \implies 2, 1 \implies 3$ is easily known from above. To see $2 \implies 1$. Let $K = \mathrm{Im}(P)$. Let $\rho \in K^\perp$. Then for all $\eta \in H$,

$$\langle P(\rho), \eta\rangle = \langle \rho, P^*(\eta)\rangle = \langle \rho, P(\eta)\rangle = 0$$

So $P|_{K^\perp} = 0$. Hence $P = P_k$.

Conversely, assume $P \neq P_K$. Then $\exists \rho \in K^\perp$ such that $P(\rho) = 0$. For each $n \in \mathbb{N}$, we have $\|\rho + nP(\rho)\|^2 = \|\rho\|^2 + n^2\|P(\rho)\|^2$. And

$$\|P(\rho + nP(\rho))\|^2 = (n+1)^2\|P(\rho)\|^2$$

So for large n, we have

$$\|P(\rho + nP(\rho))\| > \|\rho + P(\rho)\|$$

so $\|P\| > 1$. $\square$

3.2 Compact Operators

Definition 3.2.1. let X, Y be Banach Spaces. A linear map $T : X \to Y$ is called a compact operator if $\overline{T(B_1^X)}$ is compact. We denote by $K(X, Y)$, the set of all compact operators.

Example 3.2.1. If either X or Y is finite, then every linear map $T : X \to Y$ is compact. If X be any infinite dimensional Banach space. Then $T = \mathrm{Id} : X \to X$ is not compact.

Definition 3.2.2. $T : X \to Y$ is called a finite rank if the dimension of T is finite. Then the dimension of the image is called the rank of the operator. Let $F(X, Y)$ denote finite rank operators.

Lemma 3.2.1. *Every compact operator is bounded.*

Proof. Every compact set is bounded in any metric space. $\square$

Theorem 3.2.1. *Let H be a Hilbert space. Then $K(H) = \overline{F(H)}^{\|\cdot\|}$*

Proof. It is evident that $K(H) \subset K(H)$. We'll now show that $\overline{F(H)} \subset K(H)$ Let T_n be a sequence in $F(H)$ and $T_n \to T \in B(H)$ (in norm). We'll show that the image of $T(B_1(H))$ is closed and totally bounded. Then since the space is complete, this would give a convergent subsequence and hence would be complete.

Let $\epsilon > 0$ be given. Then there exist some $N \in \mathbb{H}$ such that $\|T_n - T\| < \epsilon$. Since T_N is compact, $\exists \eta_1, \eta_2, \ldots \eta_k \in B_1^H$

$$\overline{T_N(B_1^H)} \subset \bigcup_{\eta \in B_1^H} B_\epsilon(T_N(\eta))$$

Hence by the compactness of $\overline{T_N(B_1^H)}$ it has a finite open cover. Then

$$\overline{T_N(B_1^H)} \subset \bigcup_{i=1}^{n} B_\epsilon(T_N(\eta))$$

Let $\eta \in B_1^H$ be arbitrary. Then $\exists i \leq j \leq k$ such that $\|T_N(\eta) - T_N(\eta_i)\| < \epsilon$. Then

$$\begin{aligned} \|T(\eta) - T(\eta_i)\| &\leq \|T(\eta) - T_N(\eta)\| + \|T_N(\eta) - T_N(\eta_i)\| + \|T_N(\eta_i) - T(\eta_i)\| \\ &< 3\epsilon \end{aligned}$$

Conversely, let $T \in B(H)$ be compact and $\epsilon > 0$ be given. So $\exists \eta_1, \eta_2, \ldots, \eta_m \in B_1(H)$ such that

$$T(B_1(H)) \subset \bigcup_{i=1}^{n} B_\epsilon(T(\eta_i))$$

Assume that H is separable. Let $\{e_n \ : \ n \in \mathbb{N}\}$ be an orthonormal basis for H and let $P_n = P_{\text{Span}\{e_1, e_2, \ldots, e_n\}}$. Then choose $N > 0$ such that for all $n > N$, $\forall i = 1, 2, \ldots m$,

$$\|(P_n T - T)(\eta_i)\| < \epsilon$$

Then $\forall \xi \in H$ with $\|\xi\| = 1$. Choose $1 \leq j \leq m$ such that $\|T(\xi) - T(\eta_j)\| < \epsilon$. Then

$$\begin{aligned} \|P_n T(\xi) - T(\xi)\| &\leq \|P_n T(\xi) - P_n T(\eta_j)\| + \|P_n T(\eta_j) - T(\eta_j)\| + \|T(\eta_j) - T(\xi)\| \\ &< 3\epsilon \end{aligned}$$

Which gives that $\|P_n T - T\| < 3\epsilon$ for all $n \in \mathbb{N}$.

The generalization of the case when H is non-separable, will use the fact that $T(H)$ is separable and using an orthonormal basis for the pre-image of $T(H)$, which will again be separable. $\square$

Remark 3.2.1. Let K be a separable Hilbert space. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for K. For each $n \in \mathbb{N}$, let P_n be the projection to the $\mathrm{Span}\{e_1, e_2, \ldots, e_n\}$. Then for all $x \in K$

$$P_n(x) \to x$$

pointwise

Lemma 3.2.2. *If $T \in B(H)$ is compact, then $T(H)$ is separable.*

Proof. verify □

Corollary 3.2.1.1. *The set $K(H)$ of all compact operators on H is a closed two-sided ideal in $B(H)$.*

Proof. verify. Use the fact that compact operators are the closure of finite rank operators. □

Corollary 3.2.1.2. *$T \in K(H)$ implies $T^* \in K(H)$.*

Proof. verify. Use the fact that T is finite rank implies T^* is finite rank. □

Example 3.2.2. Let

$$T : L^2([0,1], m) \to L^2([0,1], m)$$

be defined as $T(f)(x) = xf(x)$. Prove that $T = T^*$ and that T has no eigenvectors.

Only for joseph: If $\xi = \eta$ almost everywhere, then show that $f\xi = f\eta$ almost everywhere if $f \in C([0,1])$

Proof. Homework □

Example 3.2.3. Let α_n be a bounded sequence in $\mathbb{C}$. Consider $T : \ell^2 \to \ell^2$, such that

$$T(\delta_n) = \alpha_n \delta_n$$

for all $n \in \mathbb{N}$. Then T is bounded with $\|T\| = \|(\alpha_n)\|_\infty$.

Example 3.2.4. Prove that the T above is compact if and only if $(\alpha_n) \in c_{\mathbf{0}}$ Prove that $T^*(\delta_n) = \overline{\alpha_n}\delta_n$

Proof. Homework □

Theorem 3.2.2. *Let H be separable Hilbert space and $T \in K(H)$ be normal. i.e $TT^* = T^*T$. Then there exist an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of H, and a sequence $(\alpha_n) \in c_{\mathbf{0}}$, such that $T(e_n) = \alpha_n e_n$.*

Lemma 3.2.3. *Let $T \in K(H)$ be self-adjoint. Then $\exists 0 \neq \eta \in \mathbb{C}$ such that $Ker(T - \lambda I) \neq \{0\}$.*

Lemma 3.2.4. *If T is a compact operator, and $0 \neq \lambda \in \mathbb{C}$, Then $Ker(T - \lambda I)$ if finite dimensional.*

Definition 3.2.3. Let $T \in B(H)$. A subspace $W \leqslant H$ is said to be invariant under T if $T(W) \subset W$. We say W reduces T, if $T(W) \subset W$ and $T(W^{\perp}) \subset W^{\perp}$

Example 3.2.5. Let $T \in B(H)$. Prove that $W \subset H$ is invariant under T if and only if $P_W T = T P_W$ Prove that W reduces T if and only if $P_W T = T P_W$ and $P_{W^{\perp}} T = T P_{W^{\perp}}$.

Proof. Homework □

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Lemma 3.2.5. *If $T \in B(H)$ is self adjoint. Then every eigenvalue of T is real.*

Proof. verify □

Example 3.2.6. Let $\alpha_n \in c_{\mathbf{0}}$. Define $T : \ell^2 \to \ell^2$, $T(\delta_n) = \delta_n \delta_n$ for all $n \in \mathbb{N}$. Then the set of eigenvalues of T is precisely the set of α_ns.

Lemma 3.2.6. *If $T \in B(\mathcal{H})$ is self adjoint, then $\|T\| = \sup\{|\langle T\xi, \xi\rangle| \ : \ \xi \in \mathcal{H}, \|\xi\| \leq 1\}$*

Proof. The $\geq$ inequality is clear from Cauchy-Schwarz and the definition of operator norm. To see the converse, let $\alpha, \beta \in \mathcal{H}$ with $\|\alpha\| = \|\beta\| = 1$. Then by the self-adjointness of T, we get

$$\begin{aligned}\langle T(\alpha+\beta), \alpha+\beta\rangle - \langle T(\alpha-\beta), \alpha-\beta\rangle &= 2(\langle T\alpha, \beta\rangle + \langle T\beta, \alpha\rangle) \\ &= 4\Re\langle T\alpha, \beta\rangle\end{aligned}$$

Now

$$\begin{aligned}4|\Re\langle T\alpha, \beta\rangle| &\leq \|T\|\|\alpha+\beta\|^2 + \|T\|\|\alpha-\beta\|^2 \\ &= 2\|T\|(\|\alpha\|^2 + \|\beta\|^2) \\ &= 4\|T\|\end{aligned}$$

Now choosing α, β appropriately gives $\langle T\alpha, \beta\rangle \leq \|T\|$. □

Lemma 3.2.7. *Let $T \in K(\mathcal{H})$ and $0 \neq \lambda \in \mathbb{C}$. Then the space*

$$Ker(T - \lambda I)$$

is finite dimensional.

Proof. verify □

Theorem 3.2.3 (Spectral theorem for compact self-adjoint operators)**.** *Let $\mathcal{H}$ be a separable Hilbert space and $T \in K(\mathcal{H})$ be self-adjoint. Then there is an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and a sequence $\{\alpha_n\} \in c_0$ such that*

$$Te_n = \alpha_n e_n$$

for all $n \in \mathbb{N}$.

Proof. We claim that T has an eigenvalue λ such that $|\lambda| = \|T\|$. By Lemma 3.2.6, $\exists(\xi_n)$ of unit vectors in $\mathcal{H}$ such that

$$|\langle T\xi_n, \xi_n\rangle| \to \|T\|$$

Hence by taking a subsequence ξ_{n_k}, we get

$$\langle T\xi_{n_k}, \xi_{n_k}\rangle \to \lambda$$

where $\lambda = \pm\|T\|$. Since T is compact, by passing to a subsequence, we may also assume $T\xi_{n_{k_j}} \to \eta \in H$. Then we have

$$\begin{aligned}\|(T-\lambda I)\xi_{n_{k_j}}\|^2 &= \|T\xi_{n_{k_j}}\|^2 + \lambda^2\|\xi_{n_{k_j}}\|^2 - 2\Re\langle T\xi_{n_{k_j}}, \lambda\xi_{n_{k_j}}\rangle \\ &\leq \|T\|^2 + \lambda^2\end{aligned}$$

Now in the first equality above, since $2\Re\langle T\xi_{n_{k_j}}, \xi_{n_{k_j}}\rangle \to 2\lambda^2$ as $n \to \infty$, we get that $T\xi_{n_{k_j}} - \lambda\xi_n \to 0$ and $T\xi_{n_{k_j}} \to \eta$. Thus $\lambda\xi_{n_{k_j}} \to \eta$ and thus applying T to both sides, we get

$$T\xi_{n_{k_j}} \to \frac{1}{\lambda}T\eta$$

and hence we get $T\eta = \lambda\eta$. If $\eta = 0$, this will contradict our precious assumption that T is nonzero since $|\langle T\xi_n, \xi_n\rangle| \to \|T\|$.

Note that the kernel of $T - \lambda I$, H_1 is invariant under T and has finite dimension (Otherwise T will break compactness unless $T = 0$ or the space $\mathcal{H}$ is finite dimensional.) Moreover, $T(H_1^\perp) \subset H_1^\perp$ and $T_1 := T|_{H_1^\perp}$ is self adjoint. Since T is compact, its restriction T_1 is compact. Hence $T_1 \in K(H_1^\perp)$. Applying the claim again it follows that T_1 has an eigenvalue λ_1 such that $|\lambda_1| = \|T_1\|$.

Continuing this construction, we get $|\lambda| \geq |\lambda_1| \geq |\lambda_2| \geq \ldots$ such that the $\mathrm{Ker}(T - \lambda_i I) \perp \mathrm{Ker}(T - \lambda_j I)$ for all $i \neq j$.

For each k (where λ_k exist), choose an orthonormal basis E_k for $\mathrm{Ker}(T - \lambda_k I)$ and let $E = \cup E_k$. If $\mathcal{H}$ is finite dimensional, there is nothing new to prove. Hence assume $\mathcal{H}$ is infinite dimensional. If $\text{Ker}(T - \lambda_{n+1}I) = \big(\text{Ker}(T - \lambda I) \oplus \text{Ker}(T - \lambda_1 I) \oplus \ldots \oplus \text{Ker}(T - \lambda_n I)\big)^{\perp}$, then the above lemma gives $\lambda_{n+1} = 0$. In this case E conatins finitely many eigenvectors corresponding to nonzero eigenvalues and infinitely many vectors corresponding to the 0 eigenvalue. Hence we are done in this case. This happens only when T is finite rank.

Otherwise, T has infinitely man eigenvalues and therefore E is an infinite set, hence by a homework problem, the sequence $\lambda_n \to 0$. verify.

It remains to show that E is an orthonormal basis for $\mathcal{H}$. Let F be the collection of all eigenvectors of T and let $\mathcal{M} = \overline{\text{Span}}(F)$. Then $T(\mathcal{M}) \subset \mathcal{M}$, and so $T(\mathcal{M}^{\perp}) \subset \mathcal{M}^{\perp}$. Thus $T|_{\mathcal{M}^{\perp}}$ has no non-zero eigenvectors. Thus from what we have proved in the first part $\mathcal{M}^{\perp}$ must be $\{0\}$. □

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Lemma 3.2.8. *Every $T \in B(\mathcal{H})$ can be decomposed as a linear combination of two self-adjoint operators.*

Proof.

$$T = \frac{T + T^*}{2} + i\frac{(T - T^*)}{2i}$$

□

Lemma 3.2.9. *$T \in B(\mathcal{H})$ is normal if and only if $T + T^*$ and $T - T^*$ commutes*

Proof.

$$(T \mp T^*)(T \pm T^*) = T^2 \pm T^*T \mp TT^* + T^{*2}$$

□

Lemma 3.2.10. *$T \in B(\mathcal{H})$ is normal if and only if $T = T_1 + iT_2$ where $T_1, T_2 \in S(\mathcal{H})$ and $T_1T_2 = T_2T_1$*

Proof. Use above lemma □

Lemma 3.2.11. *Let $T \in B(\mathcal{H}), \lambda \in \mathbb{C}$ be an eigenvalue of T, and let $S \in B(\mathcal{H})$ with $ST = TS$. Then Ker$(T - \lambda I)$ is invariant under S*

Proof. verify □

Lemma 3.2.12. *Let $T \in B(\mathcal{H})$, be self-adjoint. Let $\lambda_1 \neq \lambda_2 \in \sigma(T)$ and $\xi_1, \xi_2 \in \mathcal{H}$ their corresponding eigenvectors, then $\langle \xi_1, \xi_2 \rangle = 0$*

Theorem 3.2.4 (Spectral theorem for compact normal operators)**.** *Let $\mathcal{H}$ be a separable Hilbert space and $T \in K(\mathcal{H})$ be normal. Then there is an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and a sequence $\{\alpha_n\} \in c_{\boldsymbol{0}}$ such that*

$$Te_n = \alpha_n e_n$$

for all $n \in \mathbb{N}$.

Proof. Let $T \in \mathcal{K}(\mathcal{H})$ be normal. So $\exists S_1, S_2 \in \mathcal{K}(\mathcal{H}) \cap S(\mathcal{H})$ such that $T = S_1 + iS_2$ and $S_1S_2 = S_1S_2$. Then by spectral theorem for self-adjoint operators, we get $\sigma(S_1)$, the set of eigenvalues of S_1 such that

$$H = \bigoplus_{\lambda\in\sigma(S_1)} \mathrm{Ker}(S_1 - \lambda I)$$

where each $\mathrm{Ker}(S_1 - \lambda I)$ is finite dimensional.

Since S_2 commutes with S_1, $\mathrm{Ker}(S_1 - \lambda I)$ is invariant under S_2 and $S_2|_{\mathrm{Ker}(S_1-\lambda I)}$ is self-adjoint and compact. Thus, by the first part of the proof, for each $\lambda \in \sigma(S_1)$, we can choose and orthonormal basis E_λ for $\mathrm{Ker}(S_1 - \lambda I)$ consisting of eigenvectors of S_2.

Observe that if $\xi \in \mathcal{H}$ is such that $S_1\xi = \lambda\xi$ and $S_2\xi = \beta\xi$, then $S_2\xi = \beta\xi$. Then

$$T\xi = (\lambda + i\beta)\xi$$

Now let $E = \cup_{\lambda\in\sigma(S_1)} E_\lambda$. Then E is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of T. □

Lemma 3.2.13. *Let V be a vector space and $f_1, f_2, \ldots, f_n : V \to \mathbb{C}$ linear. Then $f \in span\{f_1, f_2, \ldots, f_n\}$ iff*

$$\bigcap_{k=1}^{n} Ker(f_k) \subset Ker(f)$$

Proof. One part is easy □

Lemma 3.2.14.

$$(X, weak)^* = X^*$$

19/11/2024 Given T as above, denote by $\sigma(T)$, the set of eigenvalues of T and for each eigenvalue $\lambda \in \sigma(T)$, denote $P_\lambda := P_{\mathrm{Ker}(T-\lambda I)}$. Then spectral theorem gives

$$T = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda$$

Exercise 3.2.1. Show $\sum_{\lambda\in\Lambda} f(\lambda)P_\lambda$ convergences pointwise and in norm. i.e

$$\left(\sum_{i=1}^{N} f(\lambda_i)P_{\lambda_i}\right)$$

Proof. verify □

Definition 3.2.4. Let Ξ be a non-empty subset of $B(\mathcal{H})$. The commutant of Ξ, is the set $\Xi' = \{S \in B(\mathcal{H}) \ : \ ST = TS, \forall T \in \Xi\}$

Exercise 3.2.2. 1. Prove that for any set Ξ' is a closed subspace of $B(\mathcal{H})$.

2. $\Xi' = \operatorname{span}(\Xi)'$

Proof. verify □

Theorem 3.2.5 (Functional calculus for compact normal operators)**.** *Let $T \in B(H)$ be compact normal with spectral decomposition*

$$T = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda.$$

For each $f \in L^\infty(\mathbb{C})$, define $f(T) := \sum_{\lambda\in\sigma(T)} f(\lambda)P_\lambda$. Then

(1) $\forall f \in \ell^\infty(\mathbb{C}), \|f(T)\| = \sup\{|f(\lambda)| \ : \ \lambda \in \sigma(T)\}$

(2) The map $\ell^\infty(\mathbb{C}) \to B(\mathcal{H})$ defines as $f \to f(T)$ is linear.

(3) $\forall f, g \in \ell^\infty(\mathbb{C}), (fg)(T) = f(T)g(T)$

(4) $\overline{f}(T) = f(T)^*$

(5) $\{f(T) \ : \ f \in \ell^\infty(\mathbb{C})\} = \{T\}''$

Proof. (1) exercise

(2) exercise

(3) exercise

(4) exercise

(5) Let $f \in \ell^\infty(\mathbb{C})$, and let $S \in \{T\}'$. Since $ST = TS$, for every $\lambda \in \sigma(T)$, the eigenspace $\mathrm{Ker}(T - \lambda I)$ is invariant under S. Hence they are all reducing for S. Then $SP_\lambda = P_\lambda S$. Thus,

$$\begin{aligned} Sf(T) &= S \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda \\ &= \sum_{\lambda \in \sigma(T)} f(\lambda) S P_\lambda \\ &= \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda S \\ &= f(T) S \end{aligned}$$

So we get $f(T) \in \{T\}''$

Note that $\forall \lambda \neq \lambda' \in \sigma(T)$, $P_\lambda P_{\lambda'} = 0$. Then for all $\gamma \in \sigma(T)$, we get $P_\gamma T = P_\gamma \sum_{\lambda \in \sigma(T)} \lambda P_\lambda = \gamma P_\gamma$. So $P_\gamma \in \{T\}'$. Thus $RP_\gamma = P_\gamma R$ for all $\gamma \in \sigma(T)$.

Fix $\gamma \in \sigma(T)$. Given $A \in B(P_\gamma(\mathcal{H}))$. Let $S := P_\gamma A P_\gamma : \mathcal{H} \to \mathcal{H}$. Then

$$\begin{aligned} ST &= P_\gamma A P_\gamma \sum_{\lambda \in \sigma(T)} \lambda P_\lambda \\ &= \gamma P_\gamma A P_\gamma \\ &= \gamma S \\ &= TS \end{aligned}$$

Thus $RS = SR$. This shows that $R|_{P_\gamma(\mathcal{H})}$ commutes with every $A \in B(P_\gamma(\mathcal{H}))$. Hence there exist such a scalar $f(\gamma) \in \mathbb{C}$ such that $R|_{P_\gamma(\mathcal{H})} = f(\gamma) I_{P_\gamma(\mathcal{H})}$. Thus $R = \sum_{\lambda \in \sigma(T)} f(\gamma) P_\gamma$ and $f \in \ell^\infty(\mathbb{C})$.

Corollary 3.2.5.1. *The mapping above restricted to $\sigma(T)$ is an isometric bijective linear multiplicative $*$-preserving map.*

Definition 3.2.5. Let $T \in B(\mathcal{H})$. Define $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$

$\square$

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Definition 3.2.6. A Banach algebra $\mathcal{A}$ is a Banach space with a multiplicatit a ring with addition satisfying $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. We say $\mathcal{A}$ is unital if the ring above is unital with multiplicative identity $1_{\mathcal{A}}$. Units (invertible elements) may also exist similarly.

Definition 3.2.7. Given a Banach algebra $\mathcal{A}$, and $a \in \mathcal{A}$, the spectrum of a, denoted by

$$\sigma(a) = \{\lambda \in \mathbb{C} \ : \ \lambda 1_{\mathcal{A}} - a \text{ is not invertible in } \mathcal{A}\}$$

Example 3.2.7. Let $(\alpha_n) \in \mathbf{c}_0$ and $T : \ell^2 \to \ell^2$ such that $T((x_n)) = (\alpha_n x_n)$. We claim that

$$\sigma(T) = \{\alpha_n \ : \ n \in \mathbb{N}\} \cup \{0\}$$

Proof. Since T is a compact operator (see Homework-5), we must have $0 \in \sigma(T)$. Otherwise if T is invertible, we'll get $I = T \circ T^{-1}$ be also compact, which is a contradiction since ℓ^2 is infinite dimensional. Moreover since $(T - \alpha_n I)(e_n) = 0$, we have $\alpha_n \in \sigma(T)$.

Now assume that $\beta \notin \{\alpha_n \ : \ n \in \mathbb{N}\} \cup \{0\}$. Then let $S \in B(\ell^2)$ defined by

$$S(e_n) = \frac{1}{\alpha - \beta} e_n$$

Then show that indeed $S \in B(\ell^2)$ and $(T - \beta I)S = S(T - \beta I) = I$ □

Example 3.2.8. Let $T \in B(L^2([0,1]))$ such that $T(f)(x) = xf(x)$ for all $f \in L^2([0,1])$. Then $\sigma(T) = [0,1]$.

Proof. First let us see that $0 \in \sigma(T)$. Suppose $S = T^{-1}$ exists. Then $\forall n \in \mathbb{N}$ if $f = S\chi_{[0,1]}$, then $Tf = \chi_{[0,1]}$. So $xf(x) = \chi_{[0,1]}$. But this is absurd, since $\frac{1}{x}\chi_{[0,1]} \notin L^2([0,1])$. □

Lemma 3.2.15. *Let $S \in B(\mathcal{H})$ such that $\|S - T\| \leq 1$. Then S is invertible.*

Proof. Since $\|S - I\| \leq 1$,

$$\sum_{n\in\mathbb{N}} \|(S - I)^n\| \leq \sum_{n\in\mathbb{N}} \|S - I\|^n$$

Moreover since $B(\mathcal{H})$ is a Banach space, absolutely convergent sequences converge and this gives that $\sum_{n\in\mathbb{N}} (I - S)^n$ converges. Thus

$$R = \sum_{n=1}^{\infty} (I - S)^n$$

exists. Now for each $N \in \mathbb{N}$, we have

$$\begin{aligned} S\Big(\sum_{n=0}^{N} (I - S)^n\Big) = (I - (I - S))\Big(\sum_{n=1}^{N} (I - S)^n\Big) &= \sum_{n=0}^{n} (I - S)^n - \sum_{n=0}^{N} (I - S)^{n+1} \\ &= I - (I - S)^{N+1} \end{aligned}$$

which converges to 0 as $n \to \infty$. □

Corollary 3.2.5.2. *The set of invertible operators is invertible in $B(\mathcal{H})$.*

Proof. Let S be invertible. Then

$$\|T - S\| = \|S(S^{-1}T - I)\| \leq \|S\|\|S^{-1}T - I\|$$

$\square$

Theorem 3.2.6. *For any $T \in B(\mathcal{H})$, $\sigma(T)$ is a non-empty, compact subspace of $\mathbb{C}$.*

Proof. Observe that the function $f : \lambda \to T - \lambda I$ is continuous. Then $f^{-1}(G(\mathcal{A}))$ is open, where $G(\mathcal{A})$ is the collection of all invertible elements of $B(\mathcal{H})$. But $\sigma(T)^c = f^{-1}(G(\mathcal{A}))$. So we see $\sigma(T)$ is closed. Moreover let $\lambda \in \mathbb{C}$ with $\|T\| < |\lambda|$. Then,

$$-T + \lambda I = \lambda(\frac{-T}{\lambda} + I)$$

Then $\|S - I\| = \|\frac{T}{\lambda}\| = \frac{\|T\|}{|\lambda|} < 1$ Hence show that $\sigma(T)$ is bounded. $\square$