MATH 7320, Functional Analysis
Homework I

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1. **Solution:** Minkowski's inequality states that if $a = (a_n)_{n\in\mathbb{N}}, b = (b_n)_{n\in\mathbb{N}}$ are elements of ℓ_p, then $\|a+b\|_p \leq \|a\|_p + \|b\|_p$. To prove this, we'll use the generalized Young's inequality which states that

2. **Solution:**

3. **Solution:**

4. **Solution:**

5. Show that a normed space χ is a Banach space if and only if whenever (x_n) is a sequence with $\sum_{n\in\mathbb{N}} \|x_n\| \leq \infty$ implies $\sum_{n\in\mathbb{N}} x_n$ converges.

   Solution: ($\implies$). Assume that χ is a Banach space and (x_n) is a sequence with $\sum_{n\in\mathbb{N}} \|x_n\| \leq \infty$.

   Consider the sequence $s_n = \sum_{i=1}^{n} x_i$. Then

   $$\|s_n - s_m\| = \left\| \sum_{i=n}^{m} x_i \right\| \leq \sum_{i=n}^{m} \|x_i\|$$

   Since we know that $\sum_{n\in\mathbb{N}} \|x_i\| \leq \infty$, for any given $\epsilon \geq 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that for all $m, n \geq N_\epsilon$, $\|s_n - s_m\| < \epsilon$. This implies s_n is a Cauchy sequence in χ. Now since by assumption we know that the space is complete, hence s_n converges. This implies $\sum_{n\in\mathbb{N}} x_i$ converges.

($\Longleftarrow$) Assume that y_n is a Cauchy sequence in χ. Then consider a subsequence y_{n_k} with $\|y_{n_k} - y_{n_{k-1}}\| < \frac{\epsilon}{2^k}$. (This choice can be made by choosing $y_{n_k} = y_j$, where $j \geq N_{\frac{\epsilon}{2^k}}$). Now we construct another sequence x_k from this such that $x_1 = y_{n_1}$ and $x_k = y_{n_k} - y_{n_{k-1}}$. Then

$$\Big\| \sum_{n\in\mathbb{N}} x_n \Big\| \leq \sum_{n\in\mathbb{N}} \|x_n\| = \sum_{n\in\mathbb{N}} \|y_{n_k} - y_{n_{k-1}}\| \leq \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^k} = \epsilon$$

and our assumption gives that $\sum_{k\in\mathbb{N}} x_k = \lim_{k\to\infty} y_{n_k}$ converges. Therefore since a subsequence of a Cauchy sequence converge, the original sequence must also converge to the same limit. Hence we get that y_n also converges. Since y_n was an arbitrary Cauchy sequence in χ, this gives that every Cauchy sequence in χ converges, completing the space.

6. Show that c_0 with the sup norm is separable, while ℓ_∞ is not.

   Solution: Let e_i be the sequence with i-th entry 1, with the rest of the entries 0. Now we claim the set $A = \{\sum_{i\in\mathbb{N}} r_i e_i \ : \ r_i \in \mathbb{Q}, \text{ finitely many } r_i\text{s are non-zero } \}$ is a countable dense set for c_0. A is countable since it is indexed by $\otimes_{i\in\mathbb{N}} Q_i$, where $Q_i = \mathbb{Q}$. Clearly we see that each $e_i \in c_0$. Hence $A \subset c_0$.

   Now let $x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ be a sequence in c_0. Then for any $\epsilon > 0$, consider $y_i \in \mathbb{Q}$ such that $|x_i - y_i| \leq \frac{\epsilon}{2^i}$. Density of $\mathbb{Q}$ in $\mathbb{R}$ guarantees the existence of such y_is. Then $y = (y_1, y_2, \ldots, y_n, 0, 0, \ldots) \in A$ and $\|x - y\|_\infty \leq \epsilon$, which guarantees A is dense in c_0. Hence c_0 is separable.

   Now for $I \subset \mathbb{N}$, let e_I be defined as

$$e_I(x) = \begin{cases} 1, & x \in I \\ 0, & x \notin I \end{cases}$$

   we see that $\|e_I - e_J\|_\infty = 1$ for all $I \neq J$. Since the number of subsets of $\mathbb{N}$ is uncountable, the balls $B_I = B(e_I, \frac{1}{2})$ is an uncountable disjoint collection of open balls in ℓ_∞. Hence ℓ_∞ cannot have a countable dense subset.

7. **Solution:**

8. Show that any infinite dimensional Banach Space cannot have a countable Hamel basis (Hint: Use Baire's category theorem)

Solution: Baire's category theorem states that in a complete metric space no open set can be constructed as a countable union of nowhere dense sets.

Now assume B is a Banach space (over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) with a countable Hamel basis $\beta = \{b_1, b_2, \ldots\}$ with $|b_i| = 1$ (Even if the initial choice is not of norm 1, we can always normalize it). Consider the subsets $B_i = \{rb_i \ : \ r \in \mathbb{F}\}$. Since every open ball $B(0, r)$ around $\mathbf{0}$ must contain scalar multiples $\frac{r}{2}b_i$ for all $i \in \mathbb{N}$, we get that none of B_is contains an open set. Therefore each of B_i are nowhere dense with $B = \cup_{n=1}^{\infty} B_i$. This contradicts the Baire's category theorem. Hence a Banach space cannot have a countable Hamel basis. Verify where are we using the completion of the space.

9. **Solution:**

10. Show that an orthonormal basis for an infinite dimensional Hilbert space cannot be a Hamel basis.

Solution: Let E be an orthonormal basis for the Hilbert space H. We will find an $x \in H$ which cannot be written as a finite linear combination of elements in E, hence showing that E is not a Hamel basis.

Let $(e_i)_{i\in\mathbb{N}} \subset E$ be a sequence of elements in E, and $a_n = \frac{1}{2^n}$. Consider the element

$$x = \sum_{n\in\mathbb{N}} a_n e_n = \sum_{n\in\mathbb{N}} \frac{e_n}{2^n}$$

$x \in H$, since H is a complete space and x is the limit of the sequence $s_n = \sum_{i=1}^{n} a_i e_i$. Moreover we see that

$$\|x\|^2 = |\langle x, x\rangle| = \sum_{n\in\mathbb{N}} a_n^2 = \sum_{n\in\mathbb{N}} \frac{1}{2^{2n}} = \sum_{n\in\mathbb{N}} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

We claim that x cannot be written as a finite linear combination of elements in E. On contrary assume that

$$x = \sum_{j=1}^{m} b_j e'_j$$

where $e'_j \in E$. Then by orthogonality we get that $\langle e_i, e'_j\rangle = 1$ if and only if $e_i = e'_j$ and otherwise 0. This implies $a_n = b_j$ if and only if $e_n = e'_j$. Then we will get

$$\mathbf{0} = x - x = \sum_{n\in\mathbb{N}} a_n e_n - \sum_{j=1}^{n} b_j e'_j = \sum_{\substack{n\in\mathbb{N} \\ a_n \notin \{b_1, b_2, \ldots, b_m\}}} a_n e_n$$

which implies

$$0 = \|\mathbf{0}\|^2 = \sum_{\substack{n \in \mathbb{N} \\ a_n \notin \{b_1, b_2, \ldots, b_m\}}} |a_n|^2$$

This happens if and only if $a_n = 0$ for all $a_n \notin \{b_1, b_2, \ldots, b_m\}$ which contradicts our choice for a_n.