Department of Mathematics University of Houston

MATH 6320

Theory of Functions of a Real Variable
Fall 2024

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 1, due Thursday, August 29, 11:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let X be an uncountable set. Let M denote the collection of subsets containing each $E \subset X$ such that E is at most countable (including $E = \emptyset$) or $E^c \equiv X \setminus E$ is at most countable. Is M a σ-algebra?

Problem 2

Let X be a measurable space, Y a set and consider a map $f : X \to Y$. Show that

$$\mathcal{M} := \{E \subset Y : f^{-1}(E) \text{ is a measurable subset of } X\}$$

defines a σ-algebra of subsets of Y.

Problem 3

Let $\mathcal{M}$ be a σ-algebra on a set X. Show that if $\mathcal{M}$ is not finite, then it is not countably infinite.

Hint: Assuming $\mathcal{M}$ is at most countable, then for each $x \in X$, consider $B_x = \cap_{E \in \mathcal{M}: x \in E} E$, the "smallest" element in $\mathcal{M}$ containing x. Show that for any $\{x, y\} \subset X$, either $B_x = B_y$ or $B_x \cap B_y = \emptyset$. Next, observe that for each $E \in \mathcal{M}$, $E = \cup_{x \in E} B_x$. Use this to argue that if $\mathcal{M}$ is infinite, then so is $\beta = \{B_x : x \in X\}$. Choose $\{x_1, x_2, \dots\} \subset X$ such that $A_j \equiv B_{x_j}$ forms a mutually disjoint sequence $\{A_1, A_2, \dots\}$ of non-empty sets in $\mathcal{M}$. Next, consider the map $i : \{0, 1\}^{\mathbb{N}} \to \mathcal{M}$, $i(a) = \cup_{j: a_j = 1} A_j$ that associates with a binary sequence the corresponding union from the sequence of sets. Why is i one-to-one? You may then quote that the set of binary sequences is uncountable.