Abstract Algebra (Math 6302), Fall 2024
Homework Assignment 3

Due at 9:59pm on Friday, September 20. Turn in your completed assignment (single .pdf file) on Canvas. For full credit, show your work and justify your answers.

(1) Draw the lattice of subgroups of Q_8.

(2) Draw the lattice of subgroups of $\mathbb{Z}/36\mathbb{Z}$.

(3) Give an example of a group G and a set $H \subseteq G$ which is closed under multiplication, but which is not a subgroup of G.

(4) Suppose that $G_1, \dots, G_n$ are finitely generated groups. Prove that the direct product $G_1 \times \cdots \times G_n$ is finitely generated.

(5) Let $G = \mathrm{GL}_2(\mathbb{Q})$ be the multiplicative group of invertible 2×2 matrices with coefficients in $\mathbb{Q}$. Prove that G is not finitely generated.

(6) Suppose that $G = \langle g_1, \dots, g_n \rangle$ is an *Abelian* group and that $|g_i| < \infty$ for each $1 \leq i \leq n$. Prove that
$$|G| \leq |g_1||g_2| \cdots |g_n|.$$

(7) Let $\sigma, \tau \in S_{12}$ be defined by
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 11 & 12 & 9 & 2 & 6 & 8 & 7 & 5 & 1 & 10 & 3 & 4 \end{pmatrix}$$
and
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 10 & 5 & 12 & 1 & 8 & 11 & 7 & 4 & 2 & 9 & 6 \end{pmatrix}.$$

(a) Find the cycle decompositions of σ and τ.

(b) Write σ and τ as products of transpositions.

(c) Compute the cycle decomposition of $\sigma^2\tau$.

(d) Compute the cycle decomposition of $(\sigma\tau)^{-1}$.

(8) How many 5-cycles are there in S_{10}?

(9) Suppose that $n \geq 36$ and that $\sigma \in S_n$ is a 36-cycle.

(a) List all possible values of $|\sigma^k|$, for $k \in \mathbb{Z}$.

(b) How many integers $0 \leq k < 36$ have the property that $\langle \sigma \rangle = \langle \sigma^k \rangle$?

(10) Suppose that $n \geq 4$. Give an example of two elements $\sigma_1, \sigma_2 \in S_n$ with the property that
$$|\langle \sigma_1, \sigma_2 \rangle| > |\sigma_1| \cdot |\sigma_2|.$$

(Comment: By Problem 6, this can't happen in an *Abelian* group.)

(11) Let $n \in \mathbb{N}$ and suppose that $\sigma_1, \ldots, \sigma_\ell$ are *disjoint* cycles in S_n. Prove that

$$|\sigma_1\sigma_2 \cdots \sigma_\ell| = \operatorname{lcm}(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_\ell|).$$

(12) Use the result of the previous problem to solve the following two sub-problems:

(a) List all possible orders of elements of S_{10}.

(b) What is the smallest value of $n \in \mathbb{N}$ with the property that S_n contains a subgroup isomorphic to C_{30}?

(13) Prove that $A_4 = \langle (12)(34), (123) \rangle$.

(14) Determine whether or not the following pairs of groups are isomorphic. For the ones that are, find an explicit isomorphism between them.

(a) $8\mathbb{Z}$ and $\mathbb{Z}$

(b) $\mathbb{Z}$ and $\mathbb{Q}$

(c) $\mathbb{Q}$ and $\mathbb{R}$

(d) $\mathbb{R}$ and $\mathrm{SL}_2(\mathbb{R})$

(e) $(\mathbb{Z}/49\mathbb{Z})^\times$ and $\mathbb{Z}/42\mathbb{Z}$

(f) $C_2 \times C_6$ and $C_2 \times C_2 \times C_3$

(g) $(\mathcal{P}(\{1,2\}), \triangle)$ and C_4

(h) $(\mathcal{P}(\{1,2,3\}), \triangle)$ and $C_2 \times C_2 \times C_2$

(15) Prove that for all $n \in \mathbb{N}$, there are exactly $\varphi(n)$ distinct automorphisms of the group $\mathbb{Z}/n\mathbb{Z}$.

(16) Find all automorphisms of the group $C_2 \times C_2$.

(17) Given that 5 is a primitive root modulo 23, determine all possible integers $0 \le b < 23$ with the property that there is an automorphism ϕ of $(\mathbb{Z}/23\mathbb{Z})^\times$ satisfying $\phi(5) = b$.