MATH6320 - Functions of a Real Variable

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November 21, 2024

Contents

Chapter 1

1.1 Course Info

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PGH 641A
Tue 10-11AM, Wed 1-2PM

Email for organizational stuff and meet for a course related conceptual stuff

- Canvas
- MS Teams

Textbook : Walter Rudin, Real & Complex Analysis, Chapters 1-9
Midterm test, October 10, in class
Grading: 30% HW, 30% Midterm, 40% Final

1.2 Notations and Basic Definitions

Definition 1.2.1. Let X be s set and $P(X)$ be its power set. A subset $\tau \subset P(X)$ is called a topology on X provided

- $\emptyset, X \in \tau$
- If $E_1, E_2, \ldots E_n \in \tau$, then $\cap_{j=1}^{n} E_j \in \tau$
- If J is any index set and for each $j \in J$, $E_j \in \tau$ then $\cup_{j \in J} E_j \in \tau$

Example 1.2.1. Given a set X, $\{\emptyset, X\}$ is a topology known as in-discrete topology.

Definition 1.2.2. Let (X, d) be a metric space with $d : X \times X \to \mathbb{R}^+$ satisfying positive definiteness, symmetry, and triangle inequality.

Definition 1.2.3. We say $E \subset X$ is open if for each $x \in E$, there is an $\epsilon \geq 0$ such that $\{y \in X \ : d(x, y) \leq \epsilon\} \subset E$

Example 1.2.2. Let τ be the set of all open subsets of X, where (X, d) is a metric space, then τ forms a topology. verify this

Definition 1.2.4. Let X be a set and τ a topology on X, then we call (X, τ) a topological space. Elements of τ are called open sets.

Definition 1.2.5. Let X be a set, $\beta \subset P(X)$ such that

- $\forall x \in X, \exists B \in \beta$ such that $x \in B$
- If $x \in X, B_1, B_2 \in \beta$ and if $x \in B_1 \cap B_2$, then there is $B_3 \in \beta$ such that $x \in B_3 \subset B_1 \cap B_2$

Then β is called a basis

Theorem 1.2.1. *If β is a basis then, τ, the collection of all (empty or non-empty) unions of elements of β form a topology on X.*

Proof. It is clear from the definition of τ that arbitrary unions of sets in τ is again in τ. Also the first property guarantees that $X \in \tau$. Since empty unions are also considered, $\emptyset \in \tau$. Hence all that remains is to show that finite intersections of sets in τ is again in τ.

Let $U_1, U_2 \in \tau$, once we show that $U_1 \cap U_2 \in \tau$, we can use induction to show $\cap_{i=1}^n U_i \in \tau$ when $U_1, U_2, \ldots, U_n \in \tau$. Let $x \in U_1 \cap U_2$. Since U_1, U_2 are unions of elements from β, there exists $B_1, B_2 \in \beta$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then by the second property of the basis, there exists $B_x \in \beta$ with $x \in B_x \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Since $x \in U_1 \cap U_2$ was arbitrary, we get

$$U_1 \cap U_2 = \bigcap_{x \in U_1 \cap U_2} B_x$$

Thus $U_1 \cap U_2 \in \tau$ and hence τ is a topology. $\square$

Example 1.2.3. Let $\beta = \{(p, q) \ : p, q \in \mathbb{Q}, p < q\} \subset P(\mathbb{R})$. Then β is a basis and the topology generated by β is the usual euclidean topology on $\mathbb{R}$ obtained from the metric $d(x, y) = |x - y|$.

Example 1.2.4. Let $X = [-\infty, \infty]$ and $\beta = \{(a, b) \ : \ a, b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) \ : \ b \in \mathbb{R}\} \cup \{(a, \infty] \ : \ a \in \mathbb{R}\}$ Then β is a basis.

Example 1.2.5. Let J be a set and $\mathbb{R}^J = \{f : J \to \mathbb{R}\}$. Let β contain all the sets of the form $\{f : J \to \mathbb{R} \ : \ f(j_1) \in U_1, f(j_2) \in U_2, \dots, f(j_n) \in U_n\}$ where $n \in \mathbb{N}, j_1, j_2, \dots, j_n \in J$ and $U_1, U_2, \dots U_n$ are open sets in $\mathbb{R}$.

Then β is a basis and the topology generated by β is called the product topology in $\mathbb{R}^J$

If J is uncountable, then this topology $\mathbb{R}^J$ is not metrizable. verify.

Definition 1.2.6. Let X be a set $\mathscr{M} \subset P(X)$ is a σ-algebra, if

- $X \in \mathscr{M}$
- If $A \in \mathscr{M}$, then $A^c \in \mathscr{M}$
- If $A_1, A_2, \dots, A_j, \dots \in \mathscr{M}$, then $\cup_{j=1}^{\infty} A_j \in \mathscr{M}$

Then we call $(X, \mathscr{M})$ a measurable space, and $\mathscr{M}$ contains measurable sets.

Theorem 1.2.2. *Let X be a set, and $F \subset P(X)$, then there exists a unique σ-algebra $\mathscr{M}$ such that,*

- *$F \subset \mathscr{M}$*
- *If $\mathscr{N}$ is a σ-algebra on X, and $F \subset \mathscr{N}$, then $\mathscr{M} \subset \mathscr{N}$*

Then $\mathscr{M}$ is called a σ-algebra generated by F

Chapter 2

Assignment 1 is posted. Submissions due Aug 29.

2.1 Warm up

Example 2.1.1. Let $X = \{1, 2, 3\}, F = \{\{1, 2\}, \{1, 3\}\}$. Then the smallest topology containing F is $\{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$, and the σ-algebra generated by F is the power set, $P(X)$.

2.2 continues

Proof. Proof of Theorem 1.2.2.

Consider all σ-algebras containing F, let $\Omega = \{\mathscr{N} \subset P(X) : \mathscr{N} \supset F, \mathscr{N}$ is a $\sigma-$ algebra$\}$. Ω is non-empty since $P(X) \subset \Omega$. Let

$$\mathscr{M} = \cap_{\mathscr{N} \in \Omega} \mathscr{N}$$

Then we claim $\mathscr{M}$ is a σ-algebra. To see this

- $X \in \mathscr{M}$, because $X \in \mathscr{N}$, for each $\mathscr{N} \in \Omega$.
- If $E \in \mathscr{M}$, then $E \in \mathscr{N}$ for each $\mathscr{N} \in \Omega$. Then $E^c \in \mathscr{N}$ for each $\mathscr{N} \in \Omega$ and thus $E^c \in \mathscr{M}$.
- If $A_1, A_2, \ldots \in \mathscr{M}$, then $\cup_{j=1}^{\infty} A_j \in \mathscr{M}$ because since each $A_i \in \mathscr{N}$ and $\mathscr{N}$ is a σ-algebra, $\cup_{j=1}^{\infty} A_j \in \mathscr{N}$ for each $\mathscr{N} \in \Omega$.

Moreover, $F \subset \mathscr{M}$ since $F \subset \mathscr{N}$ for each $\mathscr{N} \in \Omega$. Finally, if $\mathscr{N}$ is a σ-algebra with $\mathscr{N} \supset F$, then $\mathscr{N} \in \Omega$. Then $\mathscr{M} \subset \mathscr{N}$. To prove uniqueness, let $\mathscr{M}_0$ be a σ-algebra which satisfies the required properties defining Ω. By intersection operation giving $\mathscr{M}$, and $\mathscr{M}_0 \in \Omega$, $M \subset M_0$. Additionally, if $\mathscr{M}_0$ satisfies that $\mathscr{M}_0 \subset \mathscr{N}$ for each $\mathscr{N} \in \Omega$, then $\mathscr{M}_0 \subset \mathscr{M}$. Thus $\mathscr{M}_0 = \mathscr{M}$. $\square$

We combine concepts of topologies and σ-algebras.

Definition 2.2.1. Let (X, τ) be any topological space. The σ-algebra, $\mathscr{B}$ generated by the topology τ is called the Borel σ-algebra. Elements of $\mathscr{B}$ are called Borel sets.

Definition 2.2.2. Let X, Y be topological spaces. A map $f : X \to Y$ is continuous if the inverse image of any open set is open. The map f is continuous at $x \in X$ if every open set $V \subset Y$ with $f(x) \in V$, there is an open set $W \subset X$ with $f(W) \subset V$.

Theorem 2.2.1. *A map $f : X \to Y$ is continuous if and only if it is continuous at each $x \in X$.*

Proof. ($\Longrightarrow$) If f is continuous and $x \in X$, $V \subset Y$ is open and $f(x) \in V$, then by continuity, $f^{-1}(V)$ is open and $x \in f^{-1}(V)$. This holds for any such x and V, thus f is continuous at $x \in X$. Since x was arbitrarily chosen, f is continuous at each $x \in X$.

($\Longleftarrow$) Suppose f is continuous at each $x \in X$. Let V be an open subset of Y. Need to show that $W = f^{-1}(V)$ is open. For each $x \in W$, there is a $W_x \subset X$ which is open with $x \in W_x$ and $f(W_x) \subset V$ by the continuity of f at x. Now take

$$Y = \bigcup_{x \in W} W_x$$

Then Y is open being a union of open sets. Also it contains each $x \in W$. Hence $W \subset Y$. But again, $W_x \subset W = f^{-1}(V)$ for each $x \in W$ and taking the unions preserve the inclusion. Hence we get $W = Y$. Since we already know Y is open, this gives us $W = f^{-1}(V)$ is open. $\square$

Proposition 2.2.1. *If $f : X \to Y$ and $f : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*

Proof. Let $V \subset Z$ be an open set. Then $f^{-1}(V)$ is open in Y by the continuity of f. Similarly, $g^{-1}(f^{-1}(V))$ is open in X by the continuity of g. But $g^{-1}(f^{-1}(V)) = (g \circ f)^{-1}(V)$. Since V was arbitrarily open, we get that $g \circ f$ is continuous. $\square$

Definition 2.2.3. Let X be a measurable space and Y a topological space. Then a map $f : X \to Y$ is called measurable, if all inverse images of open sets are measurable.

Proposition 2.2.2. *Let X be a measurable space, Y be a topological space, then $f : X \to Y$ is measurable if and only if $f^{-1}(B)$ is measurable for each Borel set B.*

Proof. ($\Longrightarrow$) Every open set is a Borel set. So this is true by inclusion.

($\Longleftarrow$) Suppose f is measurable. Let $M = \{E \subset Y \ : \ f^{-1}(E) \text{ is measurable } \}$. We know M contains all open sets (Since we assume f is measurable). Moreover since $f^{-1}(\cup_{j\in J} U_j) = \cup_{j\in J} f^{-1}(U_j)$ for any open sets $U_j \subset Y$ with index set J, and $f^{-1}(\cap_{i=1}^n U_i) = \cap_{i=1}^n f^{-1}(U_i)$, we get that M is a σ-algebra.

Since M contains all open sets, M contains the Borel σ-algebra in Y. Hence $f^{-1}(B)$ is measurable for every Borel set B. $\square$

Chapter 3

3.1 Warm up

Example 3.1.1. Let $\mathcal{M}$ be a σ-algebra on a set X and B be the Borel σ-algebra on $\mathbb{R}$. For any given set $A \subset X$, consider the function $\chi_A : X \to \mathbb{R}$ defined as

$$\chi_A(x) = \begin{cases} 1, x & \in A \\ 0, x & \notin A \end{cases}$$

The function χ_A is measurable if and only if $A \in \mathcal{M}$.

To see this if χ_A is measurable, then inverse image of every Borel set is measurable. Consider the Borel set $(\frac{1}{2}, \frac{3}{2})$, then $\chi_A^{-1}(\frac{1}{2}, \frac{3}{2}) = A \in \mathcal{M}$.

Conversely, assume $A \in \mathcal{M}$, Take $B \in \mathcal{B}$, the Borel σ-algebra of $\mathbb{R}$. Consider $\chi_A^{-1}(B)$. We get

$$\chi_A^{-1}(B) = \begin{cases} X, & \{0,1\} \in B \\ A, & 0 \notin B, 1 \in B \\ A^c, & 0 \in B, 1 \notin B \\ \emptyset, & 0, 1 \notin B \end{cases}$$

In all these cases, we get $\chi_A^{-1}(B)$ to be an element of $\mathcal{M}$, since $\emptyset, X \in \mathcal{M}$. and if $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$. This implies χ_A is measurable.

3.2 Main Course

Definition 3.2.1. Let X, Y be topological spaces. We say that a function $f : X \to Y$ is Borel measurable if $f^{-1}(V)$ is a Borel set whenever V is an open set (or equivalently a Borel set because of Proposition 2.2.2)

Proposition 3.2.1. *If $f : X \to Y$ is a continuous function, then it is Borel measurable.*

Proof. For every open set $E \subset Y$, by assumption $f^{-1}(E)$ is open. So it is in the Borel σ-algebra on X. $\square$

3.3 Algebra of measurable functions

Theorem 3.3.1. *Let X be a measurable space, Y, Z be topological spaces. If $f : X \to Y$ is measurable and $g : Y \to Z$ is Borel measurable, then $g \circ f : X \to Z$ is measurable.*

Proof. Let $V \subset Z$ be an open set. We have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. Now since g is Borel measurable, we get $g^{-1}(V)$ is Borel measurable in Y. Again since f is measurable and $g^{-1}(V)$ is a Borel measurable, we get $f^{-1}(g^{-1}(V))$ is measurable in X. $\square$

Next we consider forming ordered pairs of measurable functions.

Lemma 3.3.1. *If $V \subset \mathbb{R}^2$ is open, then there are open rectangles $\{R_j\}_{j\in\mathbb{N}}$, such that $R_j = (a_j, b_j) \times (c_j, d_j)$ and $V = \cup_{j=1}^{\infty} R_j$*

Proof. Since rational $(a, b) \times (c, d)$, $a, b, c, d \in \mathbb{Q}$ generate the euclidean topology on $\mathbb{R}^2$ (product topology on $\mathbb{R} \times \mathbb{R}$ is the euclidean topology in $\mathbb{R}^2$), we obtain a countable union of all such rectangles contained in V. $\square$

Theorem 3.3.2. *Let X be a measurable space. If $u, v : X \to \mathbb{R}$ are measurable, then $f : X \to \mathbb{R}^2$ defined as $f(x) = (u(x), v(x))$ is measurable.*

Proof. Let $R = (a, b) \times (c, d) \subset \mathbb{R}^2$. Then

$$\begin{aligned} f^{-1}(R) &= \{x \in X \ : \ u(x) \in (a, b), v(x) \in (c, d)\} \\ &= \{x \in X \ : \ u(x) \in (a, b)\} \cap \{x \in X \ : \ v(x) \in (c, d)\} \end{aligned}$$

Hence $f^{-1}(R)$ is measurable.

Given any open set $V \in \mathbb{R}^2$, consider appropriate $\{R_j\}_{j\in\mathbb{N}}$ such that $V = \cup_{j=1}^{\infty} R_j$. Then $f^{-1}(V) = f^{-1}(\cup_{j=1}^{\infty} R_j) = \cup_{j=1}^{\infty} f^{-1}(R_j)$. Thus $f^{-1}(V)$ is measurable. $\square$

Next we establish that measurabilty is preserved under algebraic operations.

Proposition 3.3.1. *Let $f : X \to \mathbb{C}$ be such that $f = u + iv$ with real valued $u, v : X \to R$. If u, v are measurable, then f is measurable. And conversely, if f is measurable, then so are u, v, and $|f| = \sqrt{u^2 + v^2}$.*

Proof. Let u, v be measurable, then $h : X \to \mathbb{R}^2 := x \to (u(x), v(x))$ is measurable by Theorem 3.3.2. Also $g : \mathbb{R}^2 \to \mathbb{C} : (x, y) \to x + iy$ is continuous. Hence we get that $f = g \circ h$ is measurable.

For converse use that $\Re : \mathbb{C} \to \mathbb{R}$ is a continuous function. So is $\Im : \mathbb{C} \to \mathbb{R}$, and $|\cdot| : \mathbb{C} \to \mathbb{R}$. Then use that $u = \Re \circ f,\ v = \Im \circ f, |f| = |\cdot| \circ f$. □

Proposition 3.3.2. *If $f, g : X \to \mathbb{C}$ are measurable, then $f + g$ and fg are measurable.*

Proof. Suppose f, g are measurable. Then $F(x) = (f(x), g(x))$ defines a measurable function. Next consider $\phi : \mathbb{C}^2 \to \mathbb{C} := (a, b) = a + b$. By continuity of ϕ, $\phi \circ F$ is measurable, and we obtain $(\phi \circ F)(x) = f(x) + g(x)$

To show fg is measurable use the continuity of $\psi : \mathbb{C}^2 \to \mathbb{C} := (a, b) \to ab$ and compose it with F. □

Can we find a simple test for measurability of a real-valued function?

Chapter 4

4.1 Warm up

Let $\mathcal{M}$ be a σ-algebra on X and $A_1, A_2, \ldots, A_n \in \mathcal{M}$. Why does

$$f(x) = \sum_{i=1}^{n} c_j \chi_{A_j}$$

define a measurable function?

Proof. Use Proposition 3.3.2. Interpreting $c_j \chi_{A_j}$ as product of χ_{A_j} with a constant function, we observe $c_j \chi_{A_j}$ is measurable. Then using that the sum of two measurable functions is measurable in an inductive fashion, we get that the finite sum defining f also measurable. □

4.2 Continues

Lemma 4.2.1. *Let $f : X \to [-\infty, \infty]$. Then f is measurable if and only if $f^{-1}((a, \infty])$ is measurable for each $a \in \mathbb{R}$*

Proof. ($\implies$) If f is measurable, then by $(a, \infty]$ being open, we get that $f^{-1}((a, \infty])$ is measurable. This is true for all $a \in \mathbb{R}$. So the claimed property holds.

($\impliedby$) Suppose for each $a \in \mathbb{R}$, $f^{-1}((a, \infty])$ is measurable. Then since we also have that $(f^{-1}(a, \infty])^c = f^{-1}((a, \infty]^c) = f^{-1}([-\infty, a])$, Now therefore $f^{-1}([-\infty, a])$ is measurable for all $a \in \mathbb{R}$.

Now

$$[-\infty, b) = \bigcup_{n=1}^{\infty} \left[-\infty, b - \frac{1}{n} \right]$$

so,

$$
\begin{aligned}
f^{-1}([-\infty, b)) &= f^{-1}\Big(\bigcup_{n=1}^{\infty}[-\infty, b-\frac{1}{n}]\Big) \\
&= \bigcup_{n=1}^{\infty} f^{-1}\Big([-\infty, b-\frac{1}{n}]\Big) \in \mathcal{M}
\end{aligned}
$$

Next we use $(a,b) = [-\infty, b) \cap (a, \infty]$ so we get $f^{-1}(a,b)$ to be measurable. Thus we have shown measurability for inverse images of a basis. Now let $V \subset [-\infty, \infty]$ be an open set. Then there are four cases.

1. V is a countable union of rational open intervals. i.e $-\infty, \infty \notin V$
2. $-\infty \in V, \infty \notin V$. Then $V = [-\infty, b) \cup V_o$, where V_o is of case 1, and $[-\infty, b)$ is the union of countable sequence of rational half-infinite intervals. (Let b_n be a rational sequence monotonically increasing to b, then $\cup_{n=1}^{\infty}[-\infty, b_n] = [-\infty, b)$).
3. $-\infty \notin V, \infty \in V$. Then $V = V_o \cup (a, \infty]$, where V_o is a countable union of open intervals in $\mathbb{R}$.
4. $-\infty, \infty \in V$. Then $V = [-\infty, b) \cup V_o \cup (a, \infty]$, where V_o is a countable union of open intervals in $\mathbb{R}$.

In all these cases, we get $f^{-1}(V)$ to be measurable. □

Remark 4.2.1. Given a sequence (a_n) in $[-\infty, \infty]$, let $b_j = \sup_{n \leq j} a_n$. Then for each j, $b_{j+1} \leq b_j$. So $\beta = \lim_{n\to\infty} b_j$ exists in $[-\infty, \infty]$.

Definition 4.2.1. Let (a_n) be a sequence in $[-\infty, \infty]$ and (b_j) be as above, then $\beta = \inf_{j \in \mathbb{N}} b_j$ is known as the $\lim_{j\to\infty} \sup a_j$ or $\overline{\lim}_{n\to\infty} a_j$

Similarly defining $c_j = \inf_{n \geq j} a_n$ gives $\lim_{j\to\infty} \inf a_j = \sup c_j$

Definition 4.2.2. Let $f_n : X \to [-\infty, \infty]$ be a sequence of functions, define the limit supremum of the sequence of functions as

$$
(\lim_{n\to\infty} \sup f_n)(x) = \lim_{n\to\infty} \sup f_n(x)
$$

Remark 4.2.2. If $(f_n(x))$ converges for each x, then we say the sequence of functions converges pointwise.

Proposition 4.2.1. *Let (f_n) be a sequence of $[-\infty, \infty]$ value functions, then*

$$
g(x) = \sup_{n \geq n_0} f_n(x), \quad h(x) = \lim_{n\to\infty} \sup f_n(x)
$$

are measurable functions.

Proof. We only need to show that $g^{-1}(a, \infty]$ is measurable for each $a \in \mathbb{R}$. We consider

$$g^{-1}((a, \infty]) = \{x \in X \;:\; g(x) > a)\}$$

Now $g(x) > a$, then $f_n(x) \geq a$ for all $n \geq n_0$. Thus we get

$$\begin{aligned} g^{-1}((a, \infty]) &= \bigcup_{n=n_0}^{\infty} \{x \in X \;:\; f_n(x) > a\} \\ &= \bigcup_{n=n_0}^{\infty} f^{-1}((a, \infty]) \end{aligned}$$

Thus we see g is measurable. Similarly we can show this holds true if we replace sup with inf in the definition of g

Now since we know that composition of measurable functions are measurable, we get that $\inf \sup f_n(x) = h(x)$ is measurable.

Similarly we can also show that $\sup \inf f_n$ is also measurable. □

Definition 4.2.3. Let X be a set, a function $s : X \to \mathbb{C}$ is called a simple function if the range of s is finite.

Proposition 4.2.2. *A function $s : X \to \mathbb{C}$ is simple if and only if there exists mutually disjoint sets $A_1, A_2, \ldots, A_n \subset X$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ with*

$$s = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

Proof. ($\Longrightarrow$) by definition.

($\Longleftarrow$) Let s be a simple function with range $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then take $A_j = s^{-1}(\alpha_j)$. Then A_js partition X and

$$s(x) = \sum_{j=1}^{n} \alpha_j \chi_{A_j}(x)$$

□

Chapter 5

Theorem 5.0.1. *If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple non-negative real valued functions such that*

i each s_n is measurable

ii sequence (s_n) is non-decreasing

iii (s_n) converge pointwise to f

Proof. Define a 'staircase to plateau' functions, (defined in the homework-2, question 3) defined as

$$\phi_n(x) = \begin{cases} 0, & x < 0 \\ k2^{-n}, & k2^{-n} \le x < (k+1)2^{-n},\ k \in \{0, 1, 2, \ldots, \} \\ n, & x \ge n \end{cases}$$

and then let $s_n = \phi_n \circ f$. We first prove the theorem for the special case $f = \phi : [0, \infty) \to [0, \infty) := \phi(t) = t$.

We have $0 \le \phi_1(t) \le \phi_2(t) \le \ldots$ for each $t \in \mathbb{R}$ and for $t \le n$,

$$|\phi_n(t) - \phi(t)| \le \frac{1}{2^n}$$

so since $\phi(t) < \infty$, $\phi_n(t) \to \phi(t)$ for each fixed $t \in \mathbb{R}$. We also known from he homework that each ϕ_n are Borel measurable.

For the general case, we take $s_n = \phi_n \circ f$. Then similar to what we got above, we get $0 \le s_1 \le s_2 \le \ldots$ while each s_n is simple. Also for each $t \in \mathbb{R}$, $s_n(t) \to f(t)$. $\square$

Definition 5.0.1. Let $(X, \mathcal{M})$ be a measurable space, and $Z = [0, \infty]$ or $Z = \mathbb{C}$. A function $\mu : \mathcal{M} \to Z$ is called countably additive (or σ-additive) if given $A_1, A_2, \ldots \in \mathcal{M}$ such that $A_i \cap A_j = \emptyset$ if $i \ne j$, we have

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{n} \mu(A_j)$$

If $Z = [0, \infty]$ and if there is a $A \in \mathcal{M}$ such that $\mu(A) \leq \infty$, then we say that μ is a measure (or a positive measure). And we call $(X, \mathcal{M}, \mu)$ a measure space.

If $Z = \mathbb{C}$, then we call μ a complex measure.

Example 5.0.1. We give examples of different measures.

- $X = \mathbb{N}, \mathcal{M} = P(\mathbb{N}), \mu(S) = |S|$. This is called the counting measure.
- $X = \mathbb{N}, \mathcal{M} = P(\mathbb{N}), \mu(S) = \sum_{j \in S} \frac{1}{2^j}$

5.1 Properties of Measures

Proposition 5.1.1. *Let μ be a (positive) measure on a σ-algebra $\mathcal{M}$. Then*

(1) $\mu(\emptyset) = 0$

(2) $A_1, A_2, \ldots, A_n$ *with* $A_i \cap A_j = \emptyset$ *for each* $i \neq j$*, then*

$$\mu\Big(\cup_{j=1}^n A_j \Big) = \sum_{j=1}^n \mu(A_j)$$

(3) If $A, B \in \mathcal{M}$ *with* $A \subset B$*, then* $\mu(A) \leq \mu(B)$*. And if* $\mu(B) \leq \infty$*, then*

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

(4) If $A_1 \subset A_2 \subset \ldots$ *with all* $A_j \in \mathcal{M}$*, then*

$$\mu\Big(\cup_{j=1}^\infty A_j \Big) = \lim_{j \to \infty} \mu(A_j)$$

(5) If $A_1 \supset A_2 \supset \ldots$ *with all* $A_j \in \mathcal{M}$*, and ther is* $j_o \in \mathbb{N}$ *with* $\mu(A_{j_o}) \leq \infty$*, then*

$$\mu\Big(\cap_{j=1}^\infty A_j \Big) = \lim_{j \to \infty} \mu(A_j)$$

Proof. 1 Let $A \in \mathcal{M}$ with $\mu(A) \leq \infty$.

2

3

4 WLOG assume $j_o = 1$. Consider the sets $B_j = A_1 \setminus A_j$. Then we apply the above property to get

$$\mu\Big(\bigcup_{j=1}^{\infty}(A_1 \setminus A_j)\Big) = \mu(A_1) - \lim_{j\to\infty} \mu(A_j)$$

But we see that $\cup_{j=1}^{\infty}(A_1 \setminus A_j) = \cup_{j=1}^{\infty}(A_1 \cap A_j^c)$. Now since each $A_j \subset A_1$, we get this to be equal to $A_1 \setminus \cup_{j=1}^{\infty} A_j^c = A_1 \cap$

$\square$

Chapter 6

6.1 Integrals

Definition 6.1.1. Define the integral of a measurable simple function $s : X \to [0, \infty]$ defined in the standard form as

$$s = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

with $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ as the range of S and $A_j = s^{-1}(\{\alpha_j\})$ by

$$\int s \, d\mu = \sum_{j=1}^{n} \alpha_j \mu(A_j)$$

We adopt the convention $0 \times \infty = 0$ from now onwards.

Lemma 6.1.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $A_1, A_2, \ldots, A_n \in \mathcal{M}$ and $B_1, B_2, \ldots, B_{n'} \in \mathcal{M}$ with the A_js are mutually disjoint, as well as B_js, and*

$$\bigcup_{j=1}^{n} A_j = X = \bigcup_{j=1}^{n'} B_j$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, \infty]$ and $\beta_1, \beta_2, \ldots, \beta_n' \in [0, \infty]$ such that

$$t = \sum_{j=1}^{n'} \beta_j \chi_{B_j} \leq s = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

then

$$\sum_{j=1}^{n'} \beta_j \mu(B_j) \leq \sum_{j=1}^{n} \alpha_j \mu(A_j)$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{n'} \beta_j \mu(B_j) &= \sum_{j=1}^{n} \beta_j \mu\Big(B_j \bigcap (\bigcup_{l=1}^{n} A_l)\Big) \\
&= \sum_{j=1}^{n'} \beta_j \mu\Big(\bigcup_{l=1}^{n} B_j \cap A_l\Big) \\
&= \sum_{j=1}^{n'} \sum_{l=1}^{n} \beta_j \mu\Big(B_j \cap A_l\Big)
\end{aligned}$$

By a similar deduction, we get that

$$\sum_{l=1}^{n} \alpha_j \mu(A_j) = \sum_{l=1}^{n} \sum_{j=1}^{n'} \alpha_l \mu(A_l \cap B_j)$$

Since we know that $t \leq s$, comparing the values of the function at $A_l \cap B_j$, we get that $\beta_j \leq \alpha_l$. This immediately gives us our needed result. □

Corollary 6.1.0.1. *If a measurable simple function has two representations*

$$s = \sum_{j=1}^{n} \alpha_j \chi_{A_j} = \sum_{j=1}^{n'} \beta_j \chi_{B_j}$$

with disjoint measurable sets as before, then

$$\int s \; d\mu = \sum_{j=1}^{n} \alpha_j \mu(A_j) = \sum_{j=1}^{n'} \beta_j \mu(B_j)$$

Proof. Use the fact that $a = b$ is equivalent to $a \leq b$ and $b \leq a$ and use above lemma. □

Definition 6.1.2. Let $(X, \mathcal{M}, \mu)$ be a mesurable space, $s : X \to [0, \infty]$ a measurable simple function,

$$s = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

with $\{A_j\}_{j=1}^{n}$ disjoint, measurable, then we define for $E \in \mathcal{M}$

$$\int_E s \; d\mu = \sum_{j=1}^{n} \alpha_j \mu(A_j \cap E)$$

Lemma 6.1.2. *If s, t are non-negative measurable, simple fucntions and $t \leq s$ and $E \in \mathcal{M}$, then*
$$\int_E t \; d\mu \leq \int_E s \; d\mu$$

Proof. Proof is exactly like before lemma, just replacing $\mu(A_j)$ with $\mu(A_j \cap E)$. $\square$

Remark 6.1.1. If $s : X \to [0, \infty]$ is simple and measurable, then

$$\int s \; dx = \sup\{\int_E t d\mu \; : \; 0 \leq t \leq s \text{ is measurable and simple.}\}$$

Definition 6.1.3. For $f : X \to [0, \infty]$ measurable, we define

$$\int_E f d\mu = \sup_{\substack{0 \leq t \leq f \\ t \text{ is simple}}} \int_E t \; d\mu$$

Example 6.1.1. We will give some examples of measurable functions.

- $X = \mathbb{N}, \mathcal{M} = P(\mathbb{N}), \mu$ is the counting measure. $f : \mathbb{N} \to [0, \infty]$. Then let

  $$s_N(n) = \begin{cases} f(n), & n \leq N \\ 0, & \text{otherwise} \end{cases}$$

  Now if $\sum_{j=1}^{\infty} f(j) \leq \infty$, then $f(j) \to \infty$ as $j \to \infty$. Thus if $t \leq f$ and t is simple, then there is $N \in \mathbb{N}$ such that $t(j) = 0$ for each $j \geq N$. Then by comparison, $0 \leq t \leq s_n \leq f$ and finally, we have

  $$\sum_{j=1}^{\infty} t(j) \leq \sum_{j=1}^{\infty} s_N(j) \leq \sum_{j=1}^{\infty} f(j)$$

  so taking supremums, we get

  $$\sup_{\substack{0 \leq t \leq f \\ t \text{ is simple}}} \sum_{j=1}^{\infty} t(j) = \sup_{N \in \mathbb{N}} \sum_{j \in \mathbb{N}} s_N(j) = \sum_{j=1}^{\infty} f(j)$$

Chapter 7

Remark 7.0.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, a simple function $s : X \to [0, \infty]$, then $\phi : \mathcal{M} \to [0, \infty]$ defined as

$$\phi(E) = \int_E s \; d\mu$$

is a measure.

Proof. Since our definiton demands that measure of some set should be finite, we verify this first. We see that

$$\phi(\emptyset) = \int_{\emptyset} s \; d\mu = 0$$

Now to prove countable disjoint additivity, consider the disjoint collection $\{E_l\}_{l \in \mathbb{N}}$. And assume that $s = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$ with $\alpha_j \in [0, \infty]$, with A_js disjoint. Then for $E = \cup_{l=1}^{\infty} E_l$, we have

$$\begin{aligned}
\phi(E) &= \sum_{j=1}^{n} \alpha_j \mu(A_j \cap E) \\
&= \sum_{j=1}^{n} \sum_{l \in \mathbb{N}} \alpha_j \mu(A_j \cap E_l) \\
&= \sum_{l \in \mathbb{N}} \sum_{j=1}^{n} \alpha_j \mu(A_j \cap E_l) \\
&= \sum_{l \in \mathbb{N}} \int_{E_l} s \; d\mu
\end{aligned}$$

□

7.1 Properties of Integrals

Theorem 7.1.1. *The interal of a non-negative measurable function from a measure space $(X, \mathcal{M}, \mu)$ has the following properties*

(1) If $0 \le f \le g$, then $\int_E f(x)\ dx \le \int_E g\ d\mu$

(2) If $A \subset B$, $A, B \in \mathcal{M}$, then $\int_A f\ d\mu \le \int_B f\ d\mu$

(3) If $c \in [0, \infty)$, $E \in \mathcal{M}$, then $\int_E cf\ d\mu = c \int_E f\ d\mu$

(4) If $f = 0$, or $\mu(E) = 0$, then $\int_E f\ d\mu = 0$

(5) For all $E \in \mathcal{M}$,

$$\int_E f\ d\mu = \int_X f\chi_E\ d\mu$$

Proof. (1) By definition

$$\int f\ d\mu = \sup_{\substack{t \text{ is simple} \\ t \text{ is measurable} \\ 0 \le t \le f}} \int_E t\ d\mu$$

then the simple function $t \le f$ is also $t \le g$. Hence suping over simple functions under g, every simple function under f is included.

(2) Let $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$ be a simple function $0 \le s \le f$ with $\int s\ dx + \epsilon > \int f\ d\mu$. Using the inclusion $A \subset B$, we get

$$\int_A s\ d\mu = \sum_{n \in \mathbb{N}} \alpha_n$$

(3) Suppose $s = \sum_{j=1}^n \alpha_j \chi_{A_j}$ is a simple function with disjoint A_js. Then $s\chi_E = \sum_{j=1}^n \alpha_j \chi_{A_j \cap E}$ is also simple (and measurable), and

$$\int_E s\ dx = \sum_{j=1}^n \alpha_j \mu(A_j \cap E) = \int s\chi_E\ dx$$

Hence the statement is true for simple measurable functions. Next, consider f non-negative measurable, then for $\epsilon \ge 0$, we have a simple measurable function s with $\int_E s\ d\mu + \epsilon > \int_E f\ d\mu$. Then by preceding part,

$$\int s\chi_E\ d\mu + \epsilon > \int_E f\ d\mu$$

Also $s\chi_E \leq f\chi_E$. So

$$\int f\chi_E \, d\mu + \epsilon \geq \sup_{t \text{ is simple}} \int s\chi_E \, d\mu + \epsilon > \int f \, d\mu$$

Taking $\epsilon \to 0$ gives

$$\int f\chi_E \, d\mu \geq \int_E f \, d\mu$$

For the reverse inequaltiy, note that $f\chi_E \leq f$, and use similar circus.

$\square$

Theorem 7.1.2 (Monotone convergence theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, given a sequence $f_n : X \to [0, \infty]$ of measurable functions and they are monotone increasing, i.e for each $x \in X$, $0 \leq f_1(x) \leq f_2(x) \leq \dots$, then*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

Proof. Let $f = \lim_{n\to\infty} f_n$ be the pointwise limit. Then f is measurable. From $f_n \leq f_{n+1}$, we get that

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu$$

so both sides of the claimed identity exist, and from $f_n \leq f$, we also know that

$$\int f_n \, d\mu \leq \int f \, d\mu$$

which taking the limits give us,

$$\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$$

Now let $s : X \to [0, \infty]$ be a simple measurable function $s \leq f$. Choose $0 \leq c < 1$, and define $E_n = \{x \in X \ : \ f_n(x) \geq cs(x)\} = (f_n - s)^{-1}([0, \infty])$. Verify that difference between an extended real valued function and a real valued function is measurable, then E_n is measurable. This gives a nested sequence $E_1 \subset E_2 \subset \dots$. If $f(x) > 0$, then by $f(x) > cs(x)$ and $f_n(x) \to f(x)$, there is $n \in \mathbb{N}$ such that $x \in E_n$. On the other hand if $f(x) = 0$, then $cs(x) = 0 = f(x)$, so $x \in E_n$ for all $n \in \mathbb{N}$. We see that each $x \in X$ is in the union $\cup_{n=1}^{\infty} E_n$. Hence $X = \cup_{n=1}^{\infty} E_n$. Now we define $\phi : \mathcal{M} \to [0, \infty]$ by

$$\phi(E) = \int_E s \, d\mu$$

which is a measure and $\phi(X) = \phi(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \phi(E_n)$ by Theorem 7.1.1. We rewrite this as

$$\begin{aligned}\int_X s\ d\mu &= \lim_{n\to\infty} \int_{E_n} s\ d\mu \\ &= \lim_{n\to\infty} \int_X s\chi_{E_n}\ d\mu \\ &\leq \lim_{n\to\infty} \int_X \frac{1}{c} f_n\ d\mu\end{aligned}$$

Now take sup over all such simple (bounded) functions $s \leq f$ and let $c \to 1$. Finish this proof. $\square$

Chapter 8

Remark 8.0.1. Suppose $A_1, A_2, \ldots$. Consider their characteristic functions χ_{A_n} and let $\limsup_{k \geq n} = \chi_A$. What is A?

$$\begin{aligned}\limsup \chi_{A_n} &= \lim_{n\to\infty} \sup_{k\geq N} \chi_{A_k} \\ &= \lim_{n\to\infty} \chi_{\cup_{k\geq n} A_k}\end{aligned}$$

Theorem 8.0.1. *Let $(X, \mathcal{M}, \mu)$ be a measurable space, $f, g : X \to [0, \infty]$ be measurable, then*

$$\int (f+g)\ d\mu = \int f\ d\mu + \int g\ d\mu$$

Proof. For $s, t : X \to [0, \infty]$ simple and measurable, by definition

$$\int (s+t)\ d\mu = \int s\ d\mu + \int t\ d\mu$$

Considering sequences of simple measurable functions $(s_n)_{n=1}^{\infty}, (t_n)_{n=1}^{\infty}$ such that $s_n(x) \nearrow f(x), t_n(x) \nearrow g(x)$ for each $x \in X$. Then by monotone convergence theorem

$$\int s_n\ d\mu \to \int f\ d\mu \quad \int t_n\ d\mu \to \int g\ d\mu$$

and since $s_n(x) + t_n(x) \nearrow f(x) + g(x)$ for each $x \in X$ then again by MCT we get

$$\int (s_n + t_n)\ d\mu \to \int (f+g)\ d\mu$$

□

Corollary 8.0.1.1. *If $(f_n)_{n=1}^{\infty}$ is a sequence of functions $f_n : X \to [0, \infty]$, then*

$$\int \sum_{i=1}^{\infty} f_n\ d\mu = \sum_{i=1}^{\infty} \int f\ d\mu$$

Proof. Let $g_m = \sum_{n=1}^m f_n$. Then (g_m) forms an incrasing sequence, so

$$\begin{aligned}\int \sum_{n\in\mathbb{N}} f_n \; d\mu &= \int \lim_{n\to\infty} g_m d\mu \\ &= \lim_{m\to\infty} \int \sum_{i=1}^{m} f_i \; d\mu\end{aligned}$$

$\square$

Theorem 8.0.2. *If $f : [0,\infty]$ is maeasurable on $(x, \mathcal{M}, \mu)$, then $\phi : \mathcal{M} \to [0,\infty]$,*

$$\phi(E) = \int_E f d\mu$$

defines a measure ϕ and for any $g : X \to [0,\infty]$, and for any measurable $g : X \to [0,\infty]$

$$\int g \; d\phi = \int gf \; d\mu$$

Proof. $\phi(\emptyset) = 0$ since the integral of every simple measurable function $s \leq f$ over $\emptyset$ is 0.

Let $(E_n)_{n=1}^\infty$ be a disjoint sequne of sets $E = \cup_{j=1}^\infty E_j$, then

$$\phi(E) = \int f \; d\mu = \int f\chi_{X_E} \; dx = \int f\chi_{\cup_{n=1}^\infty E_n} \; d\mu = \int f(\sum_{n\in\mathbb{N}} \chi_{E_n}) \; d\mu = \sum_{n\in\mathbb{N}} \int_{E_n} f \; d\mu$$

which is exactly $\sum_{n\in\mathbb{N}} \phi(E_n)$. This gives that ϕ is a measure.

To see the claimed identity, we first show that

$$\int s \; d\phi = \int sf \; d\mu$$

for $s : X \to [0,\infty)$ simple measurable, with

$$s(x) = \sum_{j=1}^{n} \alpha_j \chi_{A_j}(x)$$

Then we see that

$$\begin{aligned}
\int s \; d\mu &= \sum_{j=1}^{n} \alpha_i \phi(A_j) \\
&= \sum_{j=1}^{n} \alpha_j \int_{A_j} f \; d\mu \\
&= \int \Big(\sum_{j=1}^{n} \alpha_j \chi_{A_j} \Big) f \; d\mu \\
&= \int sf \; d\mu
\end{aligned}$$

Now for any given $g : X \to [0, \infty]$, we approximate g with a simple measurable seqeunce $s_n \nearrow g$. Then by monotone functoins, we get

$$\begin{aligned}
\int g \; d\phi &= \lim_{n\to\infty} \int s_n \; d\phi \\
&= \lim_{n\to\infty} \int s_n f \; d\mu \\
&= \int \lim_{n\to\infty} s_n f \; d\mu \\
&= \int \phi \; d\mu
\end{aligned}$$

□

Definition 8.0.1. We define the space $L^1(\mu)$ of integrable functions on a measurable functions $(X, \mathcal{M}, \mu)$ to consist of all measurable $f : X \to \mathbb{C}$ such that

$$\int |f| \; d\mu \leq \infty$$

Remark 8.0.2. If f is measurable, $\mathbb{C}$ valued, such that $f = u + iv$ where u, v are real valued measurable functions. Then let $u^+ = \max\{0, u\}, u^- = \max\{0, -u\}$. Then u^+, u^- are measurable functions. Similarly, we get v^+, v^- also to be measurable functions. Then we get $f = u^+ - u^- + i(v^+ - v^-)$ and we define the integral as

$$\int f \; d\mu = \int u^+ \; d\mu - \int u^- \; d\mu + i \int v^+ \; d\mu - i \int v^- \; d\mu$$

Chapter 9

Remark 9.0.1 (Warm up). Assume there is a measure μ on $\mathbb{R}^+$, for all Borel-measurable functions, and $\mu([a,b]) = b - a$ for each $a \leq b$ and for continuous function f,

$$\int_{[a,b]} f \, d\mu = \int_a^b f \, dx$$

Is the function

$$f(x) = \begin{cases} 1, & x = 0 \\ \frac{\sin(x)}{x}, & x > 0 \end{cases}$$

Theorem 9.0.1. *$L^1(\mu)$ is a vector space for $f, g \in L^1(\mu)$. Moreover*

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$$

Proof. We know that for $\alpha, \beta \in \mathbb{C}$,

$$|\alpha f + \beta g| \leq |\alpha||f| + |\beta||g|$$

Then using the properties of integration, we get that

$$\int |\alpha f + \beta g| d\mu \leq \int |\alpha||f| d\mu + \int |\beta||g| \, d\mu = |\alpha| \|f\|_1 + \beta \|g\|_1 \leq \infty$$

Now to prove the rest, we'll assume f, g are $\mathbb{R}$-valued functions and let $h = f + g$. Then we have $h^+ - h^- = f^+ - f^- + g^+ - g^- = f^+ + g^+ - (f^- + g^-)$, which gives

$$\begin{aligned} \int h^+ \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu &= \int h^+ + f^- + g^- \, d\mu \\ &= \int h^- + f^+ + g^+ \, d\mu \\ &= \int h^- \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu \end{aligned}$$

Now rearranging things up, we get what we need for reals. verify similarly for Complex case. ∎

Note. What can we say about f?

Theorem 9.0.2. *If $f \in L^1(\mu)$, then*

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

Proof. If f was $\mathbb{R}$-valued, then

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu + \int f^- \, d\mu \right| \leq \left| \int f^+ \, d\mu \right| + \left| \int f^- \, d\mu \right| = \int |f| \, d\mu$$

Now in general, if f is a $\mathbb{C}$-valued function, then let the integral be equal to z. Now if $z = 0$, we have nothing to prove. If $z \neq 0$, then multiply f with $\alpha = \frac{\bar{z}}{|z|}$. Then integral of αf will be real and we'll be good. $\square$

Chapter 10

Theorem 10.0.1 (Fatou's Lemma)**.** *If (f_n) is a sequence of measurable functions $f_n : X \to [0, \infty]$, then*

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Proof. Let $g_m(x) = \inf_{n\geq m} f_n(x)$. Then $0 \leq g_1(x) \leq g_2(x) \leq \ldots$. Then by MCT, we get

$$\int \lim_{m\to\infty} g_m \, d\mu = \lim_{n\to\infty} \int g_m \, d\mu(x)$$

Also see that if $n \geq m$, then $f_n \geq g_m$ and therefore, we get

$$\int f_n \, d\mu \geq \int g_m \, d\mu$$

So

$$\inf_{n\geq m} \int f_n \, d\mu \geq \int g_m \, d\mu$$

Now taking $m \to \infty$ on both sides, we get

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \liminf_{n\to\infty} f_n \, d\mu$$

which proves the theorem. □

Example 10.0.1. Let μ be the counting measure on $X = \{0, 1\}$. Let

$$f_{2n}(x) = \begin{cases} 0, & x = 0 \\ 1, & x = 1 \end{cases} \qquad f_{2n+1} = \begin{cases} 1, & x = 0 \\ 0, & x = 1 \end{cases}$$

Then $\int \lim_{n\to\infty} \inf f_n \, d\mu = 0 \leq 1 = \lim_{n\to\infty} \inf \int f_n \, d\mu$

Theorem 10.0.2 (Lebesgue dominated convergence theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measurable space. If $f_n : X \to \mathbb{C}$ defines a sequence of measurable functions pointwise converging to f, and there is a $g \in L^1(\mu)$ such that*

$$|f_n| \leq g, \quad \forall n \in \mathbb{N}$$

Then $f \in L^1(\mu)$ and

$$\int |f_n - f| \; d\mu \to 0$$

So we exchange limits and integral and write

$$\lim_{n\to\infty} \int f_n \; d\mu = \int f \; d\mu$$

Proof. We have $|f| \leq g$ since $|f_n| \leq g$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise. Consider $h_n = 2g - |f_n - f| \geq 0$ (Use triangle inequality to show that $h_n \geq 0$). Fatou's lemma gives

$$\begin{aligned}
\liminf_{n\to\infty} \int (2g - |f_n - f|) \; d\mu &\geq \int \lim_{n\to\infty} (2g - |f_n - f|) \; d\mu \\
&= 2\int g \; d\mu + \int \liminf_{n\to\infty} (-|f_n - f|) \; d\mu \\
&= 2\int g \; d\mu - \int \limsup_{n\to\infty} (|f_n - f|) \; d\mu
\end{aligned}$$

But we also have

$$\liminf_{n\to\infty} \int (2g - |f_n - f|) \; dx \leq 2\int g \; d\mu + \liminf_{n\to\infty} \int |f_n - f| \; d\mu$$

Hairy logic. Verify with Rudin. □

10.1 Measure Zero

Definition 10.1.1. We say that a property P holds almost everywhere if

$$\mu\big(\{x \in X \; : \; \text{P does not hold at} x\}\big) = 0$$

Theorem 10.1.1. *If $f : X \to [0, \infty]$ and $\int f \; d\mu = 0$, then $f = 0$ almost everywhere. Conversely, if $f = 0$ almost everywhere then $\int f \; d\mu = 0$.*

Proof. Let $E_n = \{s \in X \ : \ f(x) \geq \frac{1}{n}\}$ and $E = \cup_{n=1}^{\infty} E_n = \{x \in X \ : \ f(x) > 0\}$. Note that E is measurable since each of E_i is measurable. So

$$
\begin{aligned}
0 = \int f \, d\mu &\geq \int f \chi_{E_n} \, d\mu \\
&\geq \int \frac{1}{n} \chi_{E_n} \, dx \\
&= \frac{1}{n} \mu(E_n) \geq 0
\end{aligned}
$$

Hence $\mu(E_n) = 0$ for each $n \in \mathbb{N}$. Hence E is a measure zero set. Therefore f is zero almost everywhere.

Conversely if $f = 0$ almost everywhere, then let

$$
g(x) = \begin{cases} 0, & f(x) = 0 \\ \infty, & \text{otherwise} \end{cases}
$$

Then g is a measurable simple function with $g > f$ and $\int g \, d\mu =$. Hence $\int f \, d\mu = 0$. $\square$

Theorem 10.1.2. *If $f_n : X \to \mathbb{C}$ defines a sequence of measurable functions and if*

$$
\sum_{n \in \mathbb{N}} |f_n| \in L^1(\mu).
$$

Then

$$
\sum_{n \in \mathbb{N}} f_n \in L^1(\mu)
$$

and the series $\sum_{n \in \mathbb{N}} f_n$ converges almost everywhere. See theorem

Proof. We assume each f_n is defined on $X \setminus S_n$ with $\mu(S_n) = 0$. We have to show that there exist a set S with $\mu(S) = 0$ and $\forall x \notin S$, $\sum_{n \in \mathbb{N}} f_n(x)$ converges. Let

$$
f(x) = \sum_{n \in \mathbb{N}} |f_n(x)|
$$

By MCT

$$
\sum_{n \in \mathbb{N}} \int |f_n| \, d\mu = \int f \, d\mu \leq \infty
$$

This implies $\{x \ : \ f(x) = \infty\}$ has measure zero. Hence if $x \notin S_n$ nad $x \notin \{x \ : \ f(x) = \infty\}$, then $\sum_{n \in \mathbb{N}} f_n(x)$ converges absolutely. Thus $S = \cup_{n=1}^{\infty} S_n \cup \{x \ : \ f(x) = \infty\}$ is measure zero and $x \in S^c$ $\square$

Definition 10.1.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. If for any $E \in \mathcal{M}$ and $F \subset E$, $\mu(E) = 0$ implies $F \subset \mathcal{M}$, then μ is called complete.

Chapter 11

Note (Warm up). Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, \infty]$, with $f \in L^1(\mu)$. Let $E = \{x \in X \ : \ f(x) \geq 1\}$. Then show $\mu(E) < \infty$.

This is Chebyshev's inequality for general measures.

Remark 11.0.1. Consider the distance (semi-metric) between sets in $\mathcal{M}$, defined as $\mu(A \Delta B)$. Let $f : X \to [0, \infty]$ be a function $f \in L^1(\mu)$. Now let ϕ be a measure defined as $d\phi = f d\mu$. Then define $\tilde{d}(A, B) = \phi(A \Delta B) = \int_{A \Delta B} f \ d\mu$. Then if $d(A_n, B) \to 0$ will imply $\tilde{d}(A_n, B) \to 0$.

Theorem 11.0.1. *Any measure space $(X, \mathcal{M}, \mu)$ can be equipped with a complete extension of μ on the collection of sets, $\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, \mu(B \backslash A) = 0\}$ in which case we define $\mu^*(E) = \mu(A)$, which gives a complete measure on $\mathcal{M}^*$.*

Proof. First, we establish μ^* is well defined, that is it does not depend on the particular choice of the subset $A \subset E$. To see this, let $A' \subset E \subset B'$ such that $\mu(B' \setminus A') = 0$. By the inclusions, $A \subset E \subset B'$. So we get

$$A \setminus A' \ \subset \ E \setminus A' \ \subset \ B' \setminus A'$$

Thus by monotonicity of μ, we get $\mu(A \setminus A') = 0$. Moreover by symmetry of A and A', we get $\mu(A' \setminus A) = 0$. Thus we get $\mu(A) = \mu(A \setminus A') + \mu(A \cap A') = \mu(A' \setminus A) + \mu(A' \cap A) = \mu(A')$. Hence we see that the definition of μ^* is well defined.

Now we show that $\mathcal{M}^*$ is actually a σ-algebra. We immediately see that $\mu^*(\emptyset) = 0$.

- $\mathcal{M} \subset \mathcal{M}^*$ implies $X \in \mathcal{M}^*$
- Let $E \in \mathcal{M}^*$, then there are $A, B \in \mathcal{M}$ with $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Thus $B^c \subset E^c \subset A^c$. Then $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \cap A) = 0$ shows $E^c \in \mathcal{M}^*$.

- Let (E_j) be a countable collection of disjoint sets in $\mathcal{M}^*$. Then there are subsets $A_j, B_j \in \mathcal{M}$ with $A_j \subset E_j \subset B_j$, with $\mu(B_j \setminus A_j) = 0$. Then let

$$A = \bigcup_{j=1}^{\infty} A_j \quad E = \bigcup_{j=1}^{\infty} E_j \quad B = \bigcup_{j=1}^{\infty} B_j$$

  Then we have $A \subset E \subset B$. Moreover since each E_j are disjoint, we get A_j are disjoint.

Now show μ^* is countably additive and then show μ^* is complete. verify $\square$

Remark 11.0.2. Consider $C([0,1])$ equipped with the sup norm. Recall that this is a Banach space. Let $\lambda : C([0,1]) \to \mathbb{C}$ be defined as

$$\lambda(f) = \int_0^1 f(x)\ dx$$

Recall also that $|\lambda(f)| \leq \lambda(|f|) \leq \|f\|_\infty$. Hence we see λ is a bounded linear functional. Therefore we see that we can associate the Riemann integral with a linear functional. We ask if we can go back i.e if we have a linear functional on $C([0,1])$, can we get a measure to integrate functions on $C([0,1])$

Chapter 12

12.1 Recap on topology

Definition 12.1.1. Let (X, τ) be a topological space. A set E is called closed if its complement is open. The closure of E is the smallest closed subset containing E.

$$\overline{E} = \bigcap_{\substack{F^c \in \tau \\ E \subset F}} F$$

We can check $\overline{E}$ is closed by looking at $\overline{E}^c$.

Definition 12.1.2. A set $K \subset X$ is called compact if every open cover of K has a finite subcover.

Definition 12.1.3. (X, τ) is Hausdorff (T_2) if for any $p \neq q \in X$ there are open sets $U, V \in \tau$ such that $p \in U, q \in V$ and $U \cap V = \emptyset$.

Definition 12.1.4. A neighborhood of $p \in X$ is an open set $U \in \tau$ containing p.

Definition 12.1.5. X is called locally compact if any point $p \in X$ has a neighborhood V with compact $\overline{V}$.

Theorem 12.1.1. *Let X be a topological space. If $K \subset X$ is compact and $F \subset K$ is closed, then F is compact.*

Proof. Make any covering of F into a covering of K, by adding F^c, the get a finite subcover for K, then remove F^c from this subcover if its there. Now you got a finite subcover for F. $\square$

Theorem 12.1.2. *let X be a topological Hausdorff space. Then if $K \subset X$ is compact, $p \notin K$, then there are open set U, V such that $K \subset V$, $p \in U$, $U \cap V = \emptyset$. (not that we are not claiming regularity).*

Proof. For each $q \in K$, there is an open set U_q, V_q with $q \in V_q, p \in V_q, V_q \cap U_q = \emptyset$. Then $K \subset \cap_{q\in K} V_q$. Then since K is compact, there is a finite subcover $V_{q_1}, V_{q_2}, \ldots V_{q_n}$ of K. Now let $V = \cup_{i=1}^n V_{q_i}$ and $U = \cap_{i=1}^n U_{q_i}$ both of which are open. Then $K \subset V, p \in U$ and $U \cap V = \emptyset$. □

Theorem 12.1.3. *If K_α is a collection of nonempty compact subsets of a topological Hausdorff space X indexed by A, and if for each finite subset $B \subset A$, $\cap_{\beta\in B} K_\beta \neq \emptyset$ then*

$$\cap_{\alpha\in A} K_\alpha \neq \emptyset$$

Proof. If $\cap_{\alpha\in A} K_\alpha = \emptyset$, then K_α^c forms an open cover for K_{α_0}. Now use the compactness property. verify □

Theorem 12.1.4. *If X, Y are topological spaces, if $f : X \to Y$ is continuous, and K is compact, then $f(K)$ is compact.*

Proof. Let U_α be an open cover for $f(K)$, then $f^{-1}(U_\alpha)$ forms an open cover for K. Now by the compactness there is a finite cover $f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), \ldots, f^{-1}(U_{\alpha_n})$. Therefore $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ is a finite subcover of $f(K)$. □

Definition 12.1.6. Let X be a topological space, $f : X \to \mathbb{C}$. Then the support of f is defined as $\mathrm{supp} f = \overline{\{x \in X \ : \ f(x) \neq 0\}}$. See that $\mathrm{supp}(f+g) \subset \mathrm{supp}(f) \cup \mathrm{supp}(g)$

We denote $C_c(X)$ to be the set of continuous functions which have compact support. $C_c(X)$ is a subspace of the vector space $C(X)$.

Theorem 12.1.5 (Urysohn Lemma)**.** *Let X be a locally compact Hausdorff space. If X is compact, V is open and $K \subset V$, then there is a function $f \in C_c(X)$ with*

$$\chi_K \leq f \leq \chi_V$$

.

Chapter 13

Theorem 13.0.1 (Urysohn Lemma)**.** *Let X be a locally compact Hausdorff space. If X is compact, V is open and $K \subset V$, then there is a function $f \in C_c(X)$ with*

$$\chi_K \leq f \leq \chi_V$$

.

Proof. Get a finite cover for K whose closure is contained in V $\square$

Definition 13.0.1. Let X be locally Hausdorff. A linear functional $\lambda : X \to \mathbb{C}$ is positive, if $\lambda(x) \geq 0$ for each $x \in X$.

Remark 13.0.1. Suppose X is locally compact, μ a measure on a σ-algebra $\mathcal{M}$, $\mathcal{M}$ containing Borel sets. If $f \in C(X)$ and $f(x) \geq 0$ for each $x \in X$, then $\int f \, d\mu \geq 0$.

If every compact set has finite measure, then each $f \in C_c(X)$ is in $L^1(\mu)$. And $\lambda(f) = \int f \, d\mu$ defines a positive linear functional on $C_c(X)$. Conversely, if each $f \in C_c(X)$ is in $L^1(\mu)$, then we know for each compact K, we have $\mu(K) < \infty$. To see this, take V open with $K \subset V$, $\overline{V}$ compact and use Urysohn's Lemma to construct $f \in C_c(X)$, $\chi_K \leq f \leq \chi_V$. Then by monotonicity,

$$0 \leq \int X_k \, d\mu \leq \int f \, d\mu < \infty$$

Theorem 13.0.2 (Reisz Representation Theorem)**.** *Let X be a locally compact Hausdorff space. If λ is a positive linear functional on $C_c(X)$, then there exists a σ-algebra $\mathcal{M}$ and a complete (positive) measure μ, uniquely determined by λ such that*

(1) $\mathcal{M} \supset B(X)$, the Borel sigma algebra.

(2) $\lambda(f) = \int f \, d\mu$ for each $f \in C_c(X)$.

(3) $\mu(K) < \infty$ for each compact K.

(4) for $E \in \mathcal{M}$,

$$\mu(E) = \inf_{\substack{V \text{ is open} \\ E \subset V}} \mu(V)$$

(5) If E is open or $E \in \mathcal{M}$ and $\mu(E) < \infty$, then

$$\mu(E) = \sup\{\mu(K) \ : \ K \subset E, \ K \textit{ is compact}\}$$

Proof. We will only prove the uniqueness and refer Rudin for the proof. Assume μ_1, μ_2 satisfy these properties. Take K compact, $\epsilon > 0$, then from iv) we know that there exist open sets V_1, V_2 containing K and $\mu_i(V_i) - \epsilon < \mu_i(K)$. Take $V = V_1 \cap V_2 \cap V_3$ with V. prove the rest. $\square$

Chapter 14

Theorem 14.0.1. *Let X be a locally compact Hausdorff space. If X is σ-compact and a Borel measure ν, that assigns each compact set K the measure $\nu(K) < \infty$ then the μ given by Reisz representation theorem satisfies*

1. *If $E \in \mathcal{M}$, $\epsilon > 0$, there is an open set V and a closed set C with $C \subset E \subset V$ and $\mu(V \setminus C) < \epsilon$.*
2. *If $E \in \mathcal{M}$, then there is an F_σ set F (countable union of closed sets) and an G_δ set G (countable intersection of open sets) with $F \subset E \subset G$ and $\mu(G \setminus F) = 0$.*
3. *μ is regular*

Proof.

1. If $\mu(E) < \infty$, then it holds by Reisz representation theorem. Next consider $E \in \mathcal{M}$ with $\mu(E) = \infty$. Recall that $X = \cup_{j=1}^{\infty} K_j$, where each K_j is compact. Let $\epsilon > 0$. Take intersection with K_j, then we have $\mu(E \cap K_j) < \infty$. So we have open sets V_j such that $K_j \cap E \subset V_j$ and $\mu(V_j \setminus (K_j \cap E)) < \frac{\epsilon}{2^{j+1}}$. V_js are guaranteed by the (4) in the Reisz representation theorem. Take $V = \cup_{j=1}^{\infty} V_j$. We have $V \setminus E \subset \cup_{j=1}^{\infty} (V_j \setminus (K_j \cap E))$. So we get $\mu(V \setminus E) < \frac{\epsilon}{2}$.

   Again consider E^c and using the same analysis, we get an open set W such that $E^c \subset W$ and $\mu(W \setminus E^c) < \epsilon/2$. Now let $C = W^c$, this gives $\mu(E \setminus C) = \mu(W \setminus E^c) = \frac{\epsilon}{2}$. Now show that $\mu(W \setminus C) < \epsilon$. Then we're done.

2. Repeat i) for a sequence of $\epsilon_n = \frac{1}{n}$. Then we get a corresponding $C_n \subset E \subset V_n$. Take $V = \cap_{n=1}^{\infty} V_n, C = \cup_{n=1}^{\infty} C_n$. Then we're done.

3. (4), (5) of Reisz representation theorem gives the outer regualarity, and outer regularity when $\mu(E) < \infty$. We only need to show inner regularity when $\mu(E) = \infty$. Therefore, we need a sequence A_n of compact sets such that $A_n \subset E$ for each $n \in \mathbb{N}$ and $\lim_{n \to \infty} \mu(A_n) = \infty$. From (1), taking $\epsilon = 1$, we have $C \subset E$, where $\mu(E \setminus C) < 1$. Hence we see $\mu(C) = \infty$.

Now from the σ-compactness, we get $X = \cup_{n=1}^{\infty} K_n$ for K_n compact. We can further demand K_ns are increasing since if not we can take finite unions of everything below. Now let $C_n = K_n \cap C$ and we have

$$\infty = \mu(C) = \lim_{n\to\infty} \mu(C_n)$$

$\square$

14.1 Lebesgue Measure

Definition 14.1.1. A k-cell in $\mathbb{R}^n$ is a set of the form

$$A = \{\lambda = (x_1, x_2, \ldots, x_k) \ : \ a_j \leq^\circ x_j \leq^\circ b_j, \ \ \leq^\circ \in \{\leq, <\}\}$$

We define $\text{vol}(A) = \Pi_{j=1}^k (b_j - a_j)$

Theorem 14.1.1. *There is a σ-algebra $\mathcal{M}$ including Borel sets on $\mathbb{R}^n$ and measure m on $\mathcal{M}$ such that*

(1) $m(V) = vol(V)$ if V is a k-cell

(2) m restricted to Borel sets is a regular measure

(3) m is translation invariant

Proof. For any $f \in C_c(\mathbb{R}^k)$. Let $\Lambda(f) = \int f \ dV$ be the Riemann integral. Then Λ is a positive linear functional on $C_c(\mathbb{R}^k)$. Reisz representation theorem gives a measure m out of Λ which has regularity and defined on a σ-algebra $\mathcal{M}$ which contains the Borel sets.

(1) Let V be an open k-cell. Pick compact k-cells nested increasing with with union $V = \cup_{j=1}^{\infty} V_j$. By Urysohn's lemma, there are $f_n \in C_c(\mathbb{R}^n)$ such that $\chi_{V_n} \leq f_n \leq \chi_V$ where V_n is compact and V is open. Then

$$m(V_n) = \int \chi_{V_n} \ dm \leq \int f_n \ dm \leq \int \chi_V \ dm = m(V)$$

Now taking $n \to \infty$, by monotone convergence theorem, we get $m(V_n) \to m(V)$. Hence by sandwich, we get $\int f_n \ dm \to m(V)$.

Similarly

$$\text{vol}(V_n) \leq \int f_n \ dV \leq \text{vol}(V)$$

Then we can choose V_k such that $\text{vol}(V_k) \to \text{vol}(V)$, then we get

(2) Property of Reisz representation measure

(3) Fix $a \in \mathbb{R}^k$ and define $\lambda : \mathcal{M} \to [0, \infty] := \lambda(E) = m(a + E)$. Verify that λ is a measure on $\mathcal{M}$.

Also define translation of functions $f \in C_c(\mathbb{R}^k)$ as $f \to f_a$, where $f_a(x) = f(x - a)$. We have seen for Riemann integrals that

$$\int_{\mathbb{R}^k} f \; dV = \int_{\mathbb{R}^k} f_a \; dV$$

By the extension (Reisz, i guess),

$$\int f \; dm = \int f_a \; dm$$

Moreover if K is compact, and V open with $K \subset V$, we have $f \in C_c(\mathbb{R}^k)$ with $\chi_K \leq f \leq \chi_V$. Then $\chi_{K+a} \leq f_a \leq \chi_{V+a}$.

Next choose any compact set K in $\mathbb{R}^k$. Define a distance from K as $\phi_k(x) = \inf_{y \in K} |x - y|$. Then ϕ_K is uniformly continuous on $\mathbb{R}^k$. Pick $V_k = \phi_K^{-1}((\frac{-1}{n}, \frac{1}{n}))$. Then $V_n \supset V_{n+1} \supset \dots$ and $K = \cap_{n=1}^{\infty} V_n$.

Now choose a sequence $(f_n) \in C_c(\mathbb{R}^k)$ such that $\chi_k \leq f_n \leq \chi_{V_n}$ and $f_1 \geq f_2 \geq \dots$ (By choosing minima among the first few functions).

Then we get

$$\begin{aligned} m(K) &= \inf_{n \in \mathbb{N}} \int f_n \; dm \\ &= \inf_{n \in \mathbb{N}} \int (f_n)_a \; dm \\ &= \qquad\qquad\qquad\qquad\qquad = \lambda(K) \end{aligned}$$

Now we have showed that $\lambda = \mu$ for compact sets in $\mathbb{R}^k$. Now we should prove the same for the open sets of $\mathbb{R}^k$. Now by the σ-compactness of $\mathbb{R}^k$, we get our desired translation invariance.

$\square$

Chapter 15

15.1 Vitali Sets

Theorem 15.1.1. *If $\mathcal{M}$ is a σ-algebra on $\mathbb{R}$ and $\lambda : \mathcal{M} \to [0, \infty]$ is a translation invariant measure with $0 < \lambda([0, 1)) < \infty$, then there is $E \subset [0, 1)$ such that $E \notin \mathcal{M}$.*

Proof. Endow $[0, 1)$ with an equivalence relation $a \sim b \iff a - b \in \mathbb{Q}$. This gives a partition of $[0, 1)$ by the equivalence classes. Now from each of these classes pick (by AOC) one representative element and build the set E. Observe that for $r, s \in \mathbb{Q}$, $(E + s) \cap (E + r) = \emptyset$ if and only if $r = s$.

Also note that

$$[0, 1) \ \subset \ \cup_{r \in \mathbb{Q} \cap [-1,1]} (E + r)$$

Therefore

$$E \ \subset [0, 1) \ \subset \ \cup_{r \in \mathbb{Q} \cap [-1,1]} (E + r) \ \subset \ [-1, 2)$$

verify the rest, its easy. □

Theorem 15.1.2 (Luzin's theorem)**.** *Let X be a locally compact Hausdorff space.*

(1) μ is a regular measure on a σ-algebra $\mathcal{M}$ containing $B(X)$

(2) $f : X \to \mathbb{C}$ is measurable

(3) there is a $A \in \mathcal{M}$ such that $\mu(A) < \infty$ and $f = 0$ on A^c

Given $\epsilon > 0$ there is a $g \in C_c(X)$ such that $\mu(\{x \in X \ : \ f(x) \neq g(x)\}) < \epsilon$

Chapter 16

Theorem 16.0.1 (Luzin's theorem)**.** *Let X be a locally compact Hausdorff space.*

(1) μ is a regular measure on a σ-algebra $\mathcal{M}$ containing $B(X)$

(2) $f : X \to \mathbb{C}$ is measurable

(3) there is a $A \in \mathcal{M}$ such that $\mu(A) < \infty$ and $f = 0$ on A^c

Given $\epsilon > 0$ there is a $g \in C_c(X)$ such that $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$ and $\sup\{|g(x)| : x \in X\} \leq \sup\{|f(x)| : x \in X\}$.

Proof. Suppose for now A is compact. (We can assume this since the measure is regular and we can find a compact set $K \subset A$ such that $f = 0$ almost everywhere in K^c.) We'll do the A not compact case later.

Choose V open such that $A \subset V$ and $\overline{V}$ is compact. We'll first prove the existence of the desired g if f is simple. Let

$$f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

where each A_j is disjoint and $\cup_{j=1}^{n} A_j = A$. Again each of the $\mu(A_j) \leq \mu(A) < \infty$. Hence by the regularity of the measure there are compact sets $K_j \subset A_j$ such that $\mu(A_j \setminus K_j) < \frac{\epsilon}{2^{j+1}}$.

Since K_j are compact and disjoint, we can find collection of disjoint open sets V_j such that $K_j \subset V_j$. verify this, I am not sure.

Moreover by replacing V_j with $V_j \cap V$, we can assume $V_j \subset V$. Now by the outer regularity of the measure, we can assume $\mu(V_j \setminus K_j) < \frac{\epsilon}{2^{j+1}}$. Now by Urysohn, there is a $g_i \in C_c(X)$ such that $\chi_{K_j} \leq g_j \leq \chi_{V_j}$. Let

$$g = \sum_{j=1}^{n} \alpha_j g_j$$

Then g is continuous being the finite sum of continuous function. Moreover since $\cup_{j=1}^{n} V_j \subset V$, we get $\text{supp}(g) \subset \overline{V}$. Also

$$|g(x)| \le \max\{|\alpha_j|\}$$
$$\max_{x \in A} |f(x)|$$

Now we see that $f(x) = g(x)$ for all $x \in K_j$ and $x \in (A_j \cup V_j)^c$. Since $K_j \subset V_j$, the set where they possibly disagree is

Add a diagram for ease of reasoning

$$D = \bigcup_{j=1}^{n} (V_j \setminus K_j) \quad \cup \quad \bigcup_{j=1}^{n} (A_j \setminus K_j)$$

Now by the subadditivity of μ, we get $\mu(D) < \epsilon$ and we have proved the result for A compact and f simple.

Now for the case when $0 \le f < 1$, let s_n be the sequence of simple functions $0 \le s_1 \le s_2 \le \ldots \le$ with $\lim_{n\to\infty} s_n(x) = f(x)$. Let $t_n = s_n - s_{n-1}$, where $s_0 = 0$. Each t_n is simple and $t_n = 0$ on A^c and by construction, we get

$$t_n \le \frac{1}{2^{n-1}} \chi_{B_n}$$

for some set B_n.

Now we use the first part of the proof on t_ns to get a corresponding $g_n \in C_c(X)$. Then g_n satisfy

(1)

(2)

(3)

Let $g = \sum_{n \in \mathbb{N}} g_n$, which converges uniformly as $|g_n| \le \frac{1}{2^{n-1}}$ by Wierestrass. Hence $g \in C_c(X)$ and $\text{supp}(g) \subset \overline{V}$.

We know that $f = \sum_{n=1}^{\infty} t_n$ from the definition of t_n. So the set $D = \{x \in X \ : \ f(x) \neq g(x)\}$ is a subset of $\cup_{n=1}^{\infty}\{x \in X \ : \ t_n(x) \neq g_n(X)\}$. Now the subadditivity of μ gives that $\mu(D) < \epsilon$.

Next, if f is non-negative, bounded, the result follows from scaling f. Again if $f \ge 0$ is measurable and possibly unbounded, we have $\cap_{n=1}^{\infty}\{x \in X \ : \ f(x) \ge n\} = \emptyset$. Moreover $\mu(\{f \ge 1\}) \le \mu(A) < \infty$. Hence by the continuity of the measure from above, we get $\mu(\{f \ge n\}) \to 0$. Hence we can replace f with $f\chi_{f<n}$ for some appropriate n.

Now if the function is general complex, we can split it as the sum and difference of four non-negative measurable functions and continue the analysis. Finally if A is not compact, we can find a $K \subset A$ such that K is compact and $\mu(A \setminus K)$ is arbitrarily small by the inner regularity of the measure μ for finite sets. $\square$

Chapter 17

Definition 17.0.1. A function f of a topological space X is called lower semi-continuous if for all $\alpha \in \mathbb{R}$, $\{x \in X \;:\; f(x) > \alpha\}$ is open.

Example 17.0.1. If V is open, then χ_V is lower semi-continuous because the $\{x \in X \;:\; f(x) > \alpha\}$ has choices ϕ, V, X, all of them are open.

Definition 17.0.2. A function is called upper semi-continuous if for all $\alpha \in \mathbb{R}$, the set $\{x \in X \;:\; f(x) < \alpha\}$ is open.

Remark 17.0.1. If $f : X \to \mathbb{R}$ is lower semi-continuous, then $-f$ is upper semi-continuous.

Example 17.0.2. If V is open, then $\chi_{V^c} = 1 - \chi_V$ is upper semi-continuous.

Proposition 17.0.1. *If f, g are lower semi-continuous, so is $f + g$.*

Proof.

$$\{x \in X \;:\; f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{R}} \big(\{x \;:\; f(x) > r\} \cap \{x \;:\; g(x) < \alpha - r\}\big)$$

□

Proposition 17.0.2. *If $u_1 \leq u_2 \leq \dots$ are all lower semi-continuous, then so is $\lim_{n\to\infty} u_n = u$.*

Proof.

$$\{u > \alpha\} = \bigcup_{n \in \mathbb{N}} \{u_\alpha > \alpha\}$$

□

Corollary 17.0.0.1. *A monotone increasing sequence of continuous functions converges to a lower semi-continuous function.*

Theorem 17.0.1 (Vitali-Caratheodory Theorem)**.** *Let X be locally compact and Hausdorff, μ be a regular Borel measure. If $f : X \to \mathbb{R}$ in $L^1(\mu)$, then there is an upper semi-continuous function u and a lower semi-continuous function v such that $u \leq f \leq v$ and $\int (v-u)\ d\mu < \epsilon$.*

Proof. Assume $f \geq 0$. There exists an increasing sequence of simple functions (s_n) converging (pointwise) to f. Considering as before, $t_n = s_n - s_{n-1}$ with $s_0 = 0$, we see that each t_n is simple and $f = \sum_{n\in\mathbb{N}} t_n$.

Then since of the t_n are simple, expanding them out into the standard simple function form and re-indexing them, we get

$$f = \sum_{j=1}^{\infty} c_j \chi_{E_j}$$

Note that we're not claiming E_js are disjoint. Since $f \in L^1(\mu)$, we can apply monotone convergence theorem. Thus

$$\sum_{j=1}^{\infty} \underbrace{\int c_j \chi_{E_j}\ d\mu}_{c_j\mu(E_j)} = \int f\ d\mu < \infty$$

If $c_j = 0$, discard. Otherwise we see that $\mu(E_j) < \infty$ for each $j \in \mathbb{N}$. By regularity, $\exists K_j$ compact and V_j open such that $K_j \subset E_j \subset V_j$ and $\mu(V_j \setminus K_j) < \frac{\epsilon}{2^j c_j}$. As a consequence of convergence of $\sum_{j=1}^{\infty} c_j\mu(E_j)$, we have $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} c_j\mu(E_j) < \epsilon$. Let

$$u = \sum_{j=1}^{N} c_j \chi_{K_j} \quad \text{and} \quad v = \sum_{j=1}^{\infty} c_j \chi_{V_j}$$

Then we see that u is upper semi-continuous and v is lower semi-continuous and

$$v - u = \sum_{j=1}^{N} c_j \chi_{V_j \setminus K_j} + \sum_{j=N+1}^{\infty} c_j \chi_{V_j}$$

Thus,

$$\begin{aligned}
\int (v-u)\, d\mu &= \int \big(\sum_{j=1}^{N} c_j \chi_{V_j \setminus K_j} + \sum_{j=N+1}^{\infty} c_j \chi_{V_j}\big)\big)\, d\mu \\
&= \sum_{j=1}^{N} c_j \mu(V_j \setminus K_j) + \sum_{j=N+1}^{\infty} c_j \mu(V_j) \\
&\leq \sum_{j=1}^{N} c_j \frac{\epsilon}{2^j c_j} + \\
&< \epsilon +
\end{aligned}$$

Now to complete the proof, apply this result to f^+ and f^-. Then since $f = f^+ - f^-$ and we get upper and lower semi-continuous functions u_+, v_+ for f^+ and u_-, v_- for f^-. Let $u = u_+ - v_-, v = v_+ - u_-$ gives $u \leq f \leq v$ and satisfy the properties. $\square$

Chapter 18

L^p Spaces

Definition 18.0.1. A function $\phi : (a, b) \to \mathbb{R}$ is called convex if

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

for all $x, y \in (a, b)$ and $0 \leq t \leq 1$.

Proposition 18.0.1. *A function $\phi : (a, b) \to \mathbb{R}$ is convex if and only if for u, s, t with $a < u \leq t \leq s < b$, we have*

$$\phi(t) \leq \phi(s)\frac{u-t}{u-s} + \phi(u)\frac{t-s}{u-s}$$

or equivalently using

$$\phi(t) - \phi(s) = \frac{t-s}{u-s}(\phi(u) - \phi(s))$$

satisfies

$$\frac{\phi(t) - \phi(s)}{t-s} \leq \frac{\phi(u) - \phi(s)}{u-s}$$

Theorem 18.0.1. *A function $\phi : (a, b) \to \mathbb{R}$ that is convex is continuous.*

Proof. Let $S = (s, \phi(s)), X = (x, \phi(x)), Y = (y, \phi(y))$, with $a < s \leq x \leq y < b$.
Draw secands and refer Rudin. $\square$

Theorem 18.0.2 (Jensen's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. If $f \in L^1(\mu)$ and for each $x \in X$, $a < f(x) < b$ and ϕ is convex on (a, b), then*

$$\phi\left(\int f \, d\mu\right) \leq \int (\phi \circ f) \, d\mu$$

Proof. We know by convexity that for $u \leq s \leq t$,

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(s)}{u-s}$$

Then there is β such that

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \beta \leq \frac{\phi(u)-\phi(s)}{u-s}$$

Consider LHS Inequality to get

$$\begin{aligned} \phi(t)-\phi(s) &\leq \beta(t-s) \\ \phi(s) &\geq \phi(t)+\beta(s-t) \end{aligned}$$

for $s < t$, and similarly by the RHS we get

$$\phi(u)-\phi(s) \geq \beta(u-s)$$

Hence in both the cases $(t = f(x),\ u = f(x))$

$$\phi(f(x)) - \phi(s) - \beta(f(x)-s) \geq 0$$

Now integrating this gives

$$\int \phi \circ f \ d\mu - \phi(t) - \beta\Big(\int f \ d\mu - s\Big) \geq 0$$

Choosing $s = \int f \ d\mu$ gives out inequality. □

Example 18.0.1. Take μ to be the probablity measure on $X = \{1, 2, 3, \ldots n\}$, assume $\mu(\{j\}) = \alpha_j > 0$. Then for $b_1, b_2, \ldots, b_n > 0$, we have

$$b_1^{\alpha_1} b_2^{\alpha_2} \ldots b_n^{\alpha_n} \leq \sum_{j=1}^{n} \alpha_j b_j$$

Proof. Use the convexity of $x \to e^x$, and let $b_j = e^{c_j}$. □

Theorem 18.0.3 (Holder's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $f, g : X \to [0, \infty]$ be measurable. Then for $1 < p < \infty$, with $1/p + 1/q = 1$, then*

$$\int fg \ d\mu \leq \left(\int f^p \ d\mu\right)^{\frac{1}{p}} \left(\int g^q \ d\mu\right)^{\frac{1}{q}} \equiv \|f\|_p \|g\|_q$$

and

$$\left(\int (f+g)^p \ d\mu\right)^{\frac{1}{p}} \leq \|f\|_p + \|g\|_p$$

Chapter 19

Theorem 19.0.1 (Holder's & Minkowski Inequality)**.** *Let* $(X, \mathcal{M}, \mu)$ *be a measure space,* $f, g : X \to [0, \infty]$ *be measurable. Then for* $1 \leq p < \infty$*, with* $1/p + 1/q = 1$*, then*

$$\int fg \, d\mu \leq \left(\int f^p \, d\mu \right)^{\frac{1}{p}} \left(\int g^q \, d\mu \right)^{\frac{1}{q}} = \|f\|_p \|g\|_q$$

and

$$\left(\int (f+g)^p \, d\mu \right)^{\frac{1}{p}} \leq \|f\|_p + \|g\|_p$$

Proof. Let $A = \|f\|_p, B = \|g\|_p$. If $A = 0$ or $A = \infty$, or $B = 0$, or $B = \infty$, we have nothing to show. Hence assume that $0 < A, B < \infty$. Let $F(x) = \frac{f(x)}{A}, G(x) = \frac{g(x)}{B}$. We also define $s, t : X \to \mathbb{R}$ as

$$F(x) = e^{\frac{s(x)}{p}}, \quad G(x) = e^{\frac{t(x)}{q}}$$

By convexity of the exponential function, we have

$$e^{s/p + t/q} \leq \frac{1}{p} e^s + \frac{1}{q} e^t$$

In terms of F, G, this is

$$F(x)G(x) \leq \frac{1}{p}(F(x))^p + \frac{1}{q}(G(x))^p$$

Hence integrating both sides, we get

$$\int F(x)G(x) \, d\mu \;\leq\; \frac{1}{p} \int (F(x))^p \, d\mu + \frac{1}{q} \int (G(x))^p \, d\mu$$

Now writing this in terms of f, g gives us

$$\begin{aligned}\frac{1}{AB}\int fg\,d\mu &\leq \frac{1}{p}\frac{1}{A^p}\int f^p\,d\mu + \frac{1}{q}\frac{1}{B^q}\int g^q\,d\mu\\ &= \frac{1}{p}\frac{1}{A^p}\|f\|_p^p + \frac{1}{q}\frac{1}{B^q}\|g\|_q^q\\ &= 1/p + 1/q = 1\end{aligned}$$

Thus we get Holder inequality.

For Minkowski, consider

$$\begin{aligned}(f+g)^p &= (f+g)(f+g)^{p-1}\\ &= f(f+g)^{p-1} + g(f+g)^{p-1}\end{aligned}$$

Now integrating both sides and carefully applying Holder's inequality, we get

$$\begin{aligned}\int (f+g)^p\,dm &= \int f(f+g)^{p-1}\,d\mu + \int g(f+g)^{p-1}\,d\mu\\ &= \left(\int f^p\,d\mu\right)^p\left(\int (f+g)^{(p-1)q}\,d\mu\right)^q + \left(\int g^q\,d\mu\right)^q\left(\int (f+g)^{(p-1)p}\,d\mu\right)^p\\ &=\end{aligned}$$

verify $\square$

Definition 19.0.1. Let $0 < p < \infty$. $f : X \to \mathbb{C}$ measurable on $(X, \mathcal{M}, \mu)$. We define

$$\|f\|_p = \left(\int |f|^p\,d\mu\right)^p$$

We also write $L^p(\mu) = \{f : X \to \mathbb{C} \;:\; \|f\|_p < \infty\}$

Definition 19.0.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be measurable. The essential supremeum of f is

$$\operatorname{ess\,sup} f = \inf\{\alpha \;:\; \mu(\{f > \alpha\}) = 0\}$$

Proposition 19.0.1. *With $(X, \mathcal{M}, \mu), f$ be as above. $\beta = \operatorname{ess\,sup} f$. Then*

$$\mu(\{f > \beta\}) = 0$$

Definition 19.0.3. For $(X, \mathcal{M}, \mu), f$ as above,

$$\|f\|_\infty = \operatorname{ess\,sup}\|f\|$$

and $L^\infty(\mu)$ be the set of all f with $\|f\|_\infty < \infty$

We add a case of Holder's inequality for $\|\cdot\|_\infty$.

Theorem 19.0.2. *If $(X, \mathcal{M}, \mu)$ is as usual f, g measurable, $f \in L^1(\mu), g \in L^\infty(\mu)$, then $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$$

Proof. Take $E = \{x \in X \ : \ |g(x)| > \|g\|_\infty\}$. Then E has measure zero, and

$$\begin{aligned}\int |fg| \ d\mu &= \int_{X\setminus E} |fg| \ d\mu + \int_E |fg| \ d\mu \\ &\leq \|g\|_\infty \int_{X\setminus E} |f| \ d\mu \\ &\leq \|g\|_\infty \|f\|_1\end{aligned}$$

□

Theorem 19.0.3. *let $(X, \mathcal{M}, \mu)$ be as usual, f, g measurable $f, g \in L^\infty(\mu)$. Then*

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

Proof. Notice that

$$\begin{aligned}\{x : |f(x)+g(x)| > \|f\|_\infty + \|g\|_\infty\} &\subset \{x : |f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty\} \\ &\subset \{x : |f(x)| > \|f\|_\infty\} \ \cup \ \{x : |g(x)| > \|g\|_\infty\}\end{aligned}$$

Since both the sets at the end is of measure zero. Hence we get the inequality. □

Theorem 19.0.4. *For each $1 \leq p \leq \infty$, $L^p(\mu)$ is a normed vector space over $\mathbb{C}$ provided we identify functions that are equal almost everywhere.*

Proof. Positive definiteness follows from the identification of functions in the space. Homogeneity follows from the definition of $\|\cdot\|_p$. And triangle inequality is the Minkowski inequality. We have shown that for the cases $1 \leq p < \infty$, that $\|\cdot\|_p$ is a norm. □

Lemma 19.0.1. *Let $(f_n) \in L^p(\mu)$ be a Cauchy sequence in $1 \leq p \leq \infty$. Then there exists a subsequence (f_{n_j}) which is convergent pointwise almost everywhere.*

Chapter 20

Remark 20.0.1. Consider the counting measure μ, on $\mathbb{N}$. Find a sequence of functions $f_n : \mathbb{N} \to [0, \infty)$, such that $\|f_n\|_1 \to 0$ and $g = \sup_n f_n \notin L^1(\mu)$.

Lemma 20.0.1. *Let $(f_n) \in L^p(\mu)$ be a Cauchy sequence in $1 \leq p \leq \infty$. Then there exists a subsequence (f_{n_j}) which is convergent pointwise almost everywhere.*

Proof. First suppose, $p < \infty$. Starting from a Cauchy sequence, choose a subsequence $n_1 < n_2 < \ldots$ such that for each $k \in \mathbb{N}$

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k}$$

Let

$$g_l = \sum_{k=1}^{l} |f_{n_{k+1}} - f_{n_k}| \quad g = \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

Then $g_n^p \leq g_{n+1}^p \leq \ldots$ and $g_n^p \to g^p$. Then by monotone convergence theorem,

$$\int g_n^p \, d\mu \to \int g^p \, d\mu$$

Moreover, using Minkowski's inequality, we get

$$\begin{aligned}\|g_l\|_p &\leq \sum_{k=1}^{l} \|f_{n_{k+1}} - f_{n_k}\| \\ &\leq \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\| \\ &\leq 1\end{aligned}$$

By monotone convergence, we get $\|g\|_p \leq 1$. In particular g is finite almost everywhere. Hence

$$f = \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$$

is absolutely convergent almost everywhere. So by telescoping series for almost every $x \in X$

$$f(x) = \lim_{l\to\infty} \sum_{k=1}^{l} (f_{n_{k+1}} - f_{n_k})(x)$$
$$= \lim_{l\to\infty} (f_{n_{l+1}}(x) - f_{n_1}(x))$$

So f_{n_l} converges for almost every $x \in X$.

Next, we consider $p = \infty$. For $n, k \in \mathbb{N}$, let

$$E_{n,k} = \{x \in X \ : \ |f_n(x) - f_k(x)| > \|f_n - f_k\|_\infty\}$$

Then $\mu(E_{n,k}) = 0$, by the definition of essential supremum. Moreover $E = \cup_{n,k=1}^{\infty} E_{n,k}$ also has measure 0. On E^c, for each $k, n \in \mathbb{N}$, we have

$$|f_n(x) - f_k(x)| \leq \|f_n - f_k\|$$

This means $f_n|_E^c$ converges uniformly. □

Theorem 20.0.1. *For $1 \leq p \leq \infty$, $L^p(\mu)$ is a complete metric space. (After identifying functions that are equal almost everywhere.)*

Proof. (1) For $p = \infty$, the proof in the above lemma is the proof

(2) For the rest of the p, consider the Cauchy sequence f_n in $L^p(\mu)$, $p < \infty$. It has c pointwise almost everywhere converging subseqence converging to f. We need to show that $f \in L^p(\mu)$ and convergence is in norm. That is $\|f_n - f\|_p \to 0$.

We apply Fatou's lemma to the function $g_k = |f_n - f_{n_k}|^p$ to get

$$\liminf_{k\to\infty} \int |f_n - f_{n_k}|^p \, d\mu \geq \int \liminf_{k\to\infty} |f_n - f_{n_k}|^p \, d\mu$$
$$= \|f_n - f\|^p$$

Given $\epsilon > 0$, since f_n is Cauchy in $L^p(\mu)$, there is a N such that for $n, m \geq N$, we have

$$\epsilon^p > \|f_n - f_m\|_p^p = \int |f_n - f_m|^p \, d\mu$$

By taking $m = n_k \to \infty$, we then get

$$\epsilon^p \geq \|f_n - f\|_p^p$$

This implies $f \in L^p(\mu)$, by

$$\|f\|_p \leq \|f - f_n\|_p + \|f_n\|_p$$

Now that fact that $\|f - f_n\|_p \to 0$, we get $f \in L^1(\mu)$.

□

Chapter 21

21.1 Approximations by simple or continuous functions

Theorem 21.1.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, denote by S, the collection of simple measurable functions with finite measurable support. Then for $1 \leq p < \infty$, $S \subset L^p(\mu)$ and S is dense in $L^p(\mu)$.*

Proof. Given $f \in L^p(\mu)$, we need to find a sequence s_n in S such that $s_n \to f$ in $L^1(\mu)$. First suppose that $f : X \to [0, \infty)$. We know a sequence of simple measurable functions s_n such that $0 \leq s_1 \leq s_2 \leq \ldots$ and

$$\lim_{n \to \infty} s_n(x) = f(x)$$

for each $x \in X$. Applying dominated convergence theorem, since $|s_n - f| \leq f$, for $f \in L^p(\mu)$ gives

$$\|f - s_n\|_p^p = \int |f - s_n|^p \, d\mu \leq \int |f|^p \, d\mu < \infty$$

we get $\|f - s_n\|_p \to 0$

Now taking a general $f \in L^p(\mu)$, writing $f = u_+ - u_- + i(v_+ - v_-)$ and repeating the process for these gives $s = s_+ - s_- + i(t_+ - t_-)$ where $s_\pm, t_\pm \in S$ and

$$\|s_\pm - u_\pm\|_p, \|t_\pm - v_\pm\|_p < \varepsilon$$

hence by triangle inequality, we get

$$\|s - f\|_p < 4\varepsilon$$

We can make RHS arbitarily small, so S is dense in $L^p(\mu)$. □

Theorem 21.1.2. *Let X be a locally compact Hausdorff space with $1 \leq p < \infty$, then $C_c(X)$ is dense in $L^p(\mu)$.*

Proof. It is enough to show $\overline{C_c(X)}$ includes S. Given $s \in S$, let $A = \{s \neq 0\}$ with $\mu(A) < \infty$. Then by Luzin's theorem, there is a $g \in C_c(X)$ such that

$$\|g\|_\infty \leq \|s\|_\infty \quad \text{and} \quad \mu(E_\varepsilon) < \varepsilon$$

where $E_\varepsilon = \{x \in X \ \ g(x) \neq s(x)\}$. Since $|g(x) - s(x)| \leq 2\|s\|_\infty$, we get

$$\begin{aligned}\|g - s\|_p &= \Big(\int |g - s|^p \, d\mu\Big)^{\frac{1}{p}} \\ &= \Big(\int_{E_\varepsilon} |g - s|^p \, d\mu\Big)^{\frac{1}{p}}\end{aligned}$$

On this set, $|g - s| \leq 2\|s\|_\infty$ gives

$$\begin{aligned}\|g - s\|_\infty &\leq \Big(\int_{E_\varepsilon} (2\|s\|_\infty)^p \, d\mu\Big)^{\frac{1}{p}} \\ &< 2\|s\|_\infty \varepsilon^{1/p}\end{aligned}$$

Since we can make ε arbitrarily small, we get the density. $\square$

Remark 21.1.1. This theorem proves that $L^p(\mu)$ is the completion of $(C_c(\mathbb{R}^k), d_p)$ where for $f, g \in C_c(\mathbb{R}^k)$, $d_p(f, g) = \|f - g\|_p$. The limit of a Cauchy sequence in $C_c(\mathbb{R}^k)$ is determined almost everywhere.

If $p = \infty$, then the completion of $C_c(\mathbb{R}^k)$ is not $L^\infty(m)$, but $C_o(\mathbb{R}^k)$.

Definition 21.1.1. Let X be locally compact Hausdorff, we say a continuous function f vanishes at infinity and write $f \in C_o(X)$ if for $\varepsilon > 0$, we can find a compact set K such that $|f(x)| < \varepsilon$ for all $x \notin K$.

Theorem 21.1.3. *Let X be locally compact Hausdorff, then $C_o(X)$ is the completion of $C_c(X)$ with $\|\cdot\|_\infty$.*

Proof. Let $f \in C_o(X), \varepsilon > 0$, we can choose K such that K is compact and $|f(x)| < \varepsilon$ for all $x \in K^c$. Using Urysohn's lemma, threre is a $g \in C_c(X)$ such that $\chi_K \leq g \leq 1$, then $h = fg \in C_c(X)$ and

$$\begin{aligned}\|h - f\|_\infty &= \|f(1 - g)\|_\infty \\ &= \|f(1 - g)\chi_{K^c}\|_\infty \\ &\leq \varepsilon\|1 - g\|_\infty \\ &\leq \varepsilon\end{aligned}$$

$\square$

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Proposition 21.1.1. *Show that if $\mu(X) < \infty$, with $p \leq r \leq \infty$, then*

$$L^r(\mu) \subset L^p(\mu)$$

Given $f \in L^r(\mu)$.

Proof. Let $f \in L^r(\mu)$. Then,

$$\begin{aligned}
\|f\|_p^p &= \int |f|^p \; dm \\
&\leq \left(\int |f|^r \; dm \right)^{p/r} \left(\int 1 \; dm \right)^{1-p/r} \\
&\leq \|f\|_r^p \mu(X)^{1-\frac{p}{r}} \\
&< \infty
\end{aligned}$$

□

Chapter 22

Inner Product Spaces

Definition 22.0.1. Let $\mathcal{H}$ be a vector space over $\mathbb{C}$. A sesquilinear form is a function

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

satisfying

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle x + \alpha z, y \rangle = \langle x, y \rangle + \alpha \langle x, z \rangle$

for all $x, y, z \in \mathcal{H}, \alpha \in \mathbb{C}$. It is said to be positive semidefinite (positive definite) if $\langle x, x \rangle \geq 0$ ($\langle x, x \rangle > 0$ for all $x \in \mathcal{H} \setminus \{0\}$) for all $x \in \mathcal{H}$.

A positive definite sesquilinear forms makes $\mathcal{H}$ an inner product space.

Example 22.0.1. Take $L^2(\mu)$ (functions identified almost everywhere) with the natural inner product is an inner product.

Proposition 22.0.1. *If $\mathcal{H}$ is a complex vector space with a positive semidefinite sesquilinear form and $\langle x, x \rangle = 0$, then $\langle x, y \rangle = 0$ for all $y \in \mathcal{H}$.*

Proof. Take $\alpha \in \mathbb{C}$ and consider

$$\begin{aligned} \langle x + \alpha y, x + \alpha y \rangle &= \langle x, x \rangle + \alpha \langle y, x \rangle + \overline{\alpha} \langle y, y \rangle \\ &= 2\Re\big(\overline{\alpha}\langle x, y \rangle\big) + |\alpha|^2 \langle y, y \rangle \end{aligned}$$

Now if $\langle x, y \rangle \neq 0$, then either $\langle y, y \rangle = 0$ or nonzero. If $\langle y, y \rangle = 0$, take $\alpha = -\langle x, y \rangle$ to get

$$\langle x + \alpha y, x + \alpha y \rangle = \underbrace{2\Re\big(- \overline{\langle x, y \rangle} \langle x, y \rangle \big)}_{<0}$$

which is a contradiction.

Now if $\langle y, y\rangle \neq 0$, take $\alpha = i\langle x, y\rangle$ to get a similiar contradiction, which makes $\Re(\overline{\alpha}\langle x, y\rangle) = 0$ □

Definition 22.0.2. If $\langle\cdot,\cdot\rangle$ is a positive semidefinite sesquilinear form, then

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}$$

is a seminorm.

If $\langle\cdot,\cdot\rangle$ is positive definite, then $x \to \|x\|$ is a norm.

Theorem 22.0.1 (Cauchy-Schwarz)**.** *If $\langle\cdot,\cdot\rangle$ is a positive semidefinite sesquilinear form on $\mathcal{H}$, then for $x, y \in \mathcal{H}$*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Proof. If $\|y\| = 0$, then previous proposition takes care of the proof. If not, choose $\alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$ and consider

$$\begin{aligned}
0 &\leq \langle x - \lambda y, x - \lambda y\rangle \\
&= \|x\|^2 - 2\Re\big(\lambda\langle y, x\rangle\big) + |\lambda|^2\langle y, y\rangle \\
&= \|x\| - 2\frac{\langle x, y\rangle^2}{\|y\|^2} + \frac{\langle x, y\rangle^2}{\|y\|^2} \\
&= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}
\end{aligned}$$

which gives our inequality. □

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Theorem 22.0.2. *Let $\mathcal{H}$ be a vector space over $\mathbb{C}$ with a positive semidefinite sesquilinear $\langle\cdot,\cdot\rangle$ form and the associated seminorm $\|\cdot\|$, then for all $x, y \in \mathcal{H}$,*

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof. verify □

Remark 22.0.1. If $\langle\cdot,\cdot\rangle$ is an inner product space, then $\|\cdot\|$ defines a norm in $\mathcal{H}$.

Definition 22.0.3. If $\mathcal{H}$ be an inner product. If $\mathcal{H}$ is complete with respect to the topology induced by the inner product, then it is called a Hilbert space.

Example 22.0.2. $L^2(\mu)$, with functions identified that agrees almost everywhere is a Hilbert space when endowed with the inner product

$$\langle f, g\rangle = \int f\overline{g}\, d\mu$$

Proposition 22.0.2. *Let $\mathcal{H}$ be a Hilbert space. Then for $g \in \mathcal{H}$*

$$\lambda_g : \mathcal{H} \to \mathbb{C} := f \to \langle f, g \rangle$$

is a linear, uniformly continuous functional.

Proof. Use Cauchy-Schwarz inequality. □

Definition 22.0.4. Let H be a Hilbert space. We say $x, y \in \mathcal{H}$ are orthogonal if $\langle x, y \rangle = 0$. We also write $x \perp y$.

If $S \subset \mathcal{H}$, define

$$S^{\perp} = \{x \in \mathcal{H} \ : \ x \perp s, \forall s \in S\}$$

Theorem 22.0.3. *If $S \subset H$, then $S^{\perp}$ is a closed subspace of $\mathcal{H}$.*

Proof. Let $z \in \mathcal{H}$. Then $K_z = z^{\perp} = \mathrm{Ker}(\lambda_z)$ is closed. Observe that

$$S^{\perp} = \bigcap_{s \in S} K_s$$

is closed as well. □

Lemma 22.0.1. *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$, and $h \in \mathcal{H}$, then there is a unique $m \in \mathcal{M}$ that minimizes the distance to h*

Proof. We recall the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

and write for $x, y \in \mathcal{H}$,

$$\|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x + y\|^2$$

Let $\delta = \inf\{\|m - h\| \ : \ m \in \mathcal{M}\}$. There there is a sequence of $m_j \in \mathcal{M}$ such that $\|m_j - h\| \to \delta$. To show that m_j is a Cauchy seqeunce, consider $x = m_j - h, y = m_i - h$. Then

$$\frac{x + y}{2} = \frac{m_i + m_j}{2} - h$$

and we see that

$$\left\|\frac{x + y}{2}\right\| = \left|\frac{m_i + m_j}{2} - h\right|$$

Then by prarllelogram law,

$$\begin{aligned}\|m_j - m_i\|^2 &= 2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 \\ &= 2(\|m_j - h\|^2 + \|m_i - h\|^2 - \|m_i + m_j - 2h\|^2)\end{aligned}$$

verify

This shows that, we can make $\|m_i - m_j\|$ arbitrarily small by requiring $i, j \in \mathbb{N}$ for a similarly large $\mathbb{N}$, meaning m_j is Cauchy. Since $\mathcal{M}$ is closed and a closed and a closed subset of a complete metric space, $\mathcal{M}$ is complete, so there is a point $m \in \mathcal{M}$, where m_j converges to. We'll prove the uniqueness in the next lecture. □

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Theorem 22.0.4 (Orthogonal Projections)**.** *If M is a closed subspace of a Hilbert space $\mathcal{H}$, then for each $h \in \mathcal{H}$, there is a unique pair $m \in M$ and $n \in M^\perp$ such that $h = m + n$ and $\|h\|^2 = \|m\|^+\|n\|^2$. Moreover, the maps $P(h) = m$, $Q(h) = n$ are linear and write $m = Ph, n = Qh$*

Proof. Fix $h \in \mathcal{H}$. Pick $m \in M$ that is nearest to h, using precedent lemma. Let $n = h - m$. We will show $n \in M^\perp$. For any $x \in M$, $\alpha \in \mathbb{C}$, consider

$$\|n - \alpha x\|^2 = \|n\|^2 - 2\Re(\alpha\langle x, n\rangle) + |\alpha|^2\|x\|^2$$

and $\|h\|^2 = \|m\|^2 + \|n\|^2$. Suppose $\langle x, n\rangle \neq 0$. Choose $\alpha = \frac{t}{\langle x,n\rangle}$ for $t \in \mathbb{R}$. Then

$$\|n - \alpha x\|^2 = \|n\|^2 - 2t + \frac{t^2\|x\|^2}{|\langle x, n\rangle|^2}$$

For sufficiently small t, we have $2t > \frac{t^2\|x\|^2}{|\langle x,n\rangle|^2}$. Then we'd get

$$\|n - \alpha x\|^2 < \|n\|^2$$

Replacing n with $h - m$, we get

$$\|h - (m + \alpha x)\|^2 < \|h - m\|^2$$

which contradicts the optimality of m for distance to h. We conclude $\langle x, n\rangle = 0$. This is true for each $x \in M$. Thus, we get $n \in M^\perp$.

Now to see that the choice of m(and n) is unique, let $h = m' + n'$, with $m' \in M, n' \in M^\perp$. Then $m + n = m' + n'$, which implies

$$\underbrace{m - m'}_{\in M} = \underbrace{n' - n}_{\in M^\perp}$$

which forces $m = m', n = n'$, since $M \cap M^\perp = \{0\}$

Now for the linearity of P, Q, let $h = h_1 + \alpha h_2$, where $h_1 = m_1 + n_1, h_2 = m_2 + n_2$ for $m_i \in M, n_i \in M^\perp$. Then

$$h = \underbrace{m_1 + \alpha m_2}_{\in M} + \underbrace{n_1 + \alpha n_2}_{\in M^\perp}$$

This shows $P : h \to m, Q : h \to n$ are linear maps. $\square$

Definition 22.0.5. The maps P, Q above are called orthogonal projections onto M and $M^\perp$, respectively.

Corollary 22.0.4.1. *Let M be a proper closed subspace in a Hilbert space $\mathcal{H}$. Then $M^\perp \neq \{0\}$.*

Exercise 22.0.1. Let $M \subset L^2(\mathbb{R})$ such that

$$M = \{f \in L^2(\mathbb{R}) \ : \ f(x) = \alpha_n \text{ for almost every } x \in [n, n+1)\}$$

21/11/2024

22.1 Reisz Representation Theorem

Theorem 22.1.1 (Reisz Representation Theorem)**.** *Let $\Lambda : \mathcal{H} \to \mathbb{C}$ be a continuous linear functional on a Hilbert space. Then there is a unique $y \in \mathcal{H}$ such that $\Lambda(x) = \langle x, y \rangle$*

Proof. Assume that $\Lambda \neq 0$. Then $M = \mathrm{Ker}(\Lambda)$ is a proper closed linear subspace of $\mathcal{H}$. Then so is $M^\perp$. Let $0 \neq v, w \in M^\perp$. Then $\Lambda(v) \neq 0 \neq \Lambda(w)$. Then since

$$\Lambda\Big(\frac{v}{\Lambda(v)} - \frac{w}{\Lambda(w)}\Big) = 0$$

forces $\frac{v}{\Lambda(v)} - \frac{w}{\Lambda(w)} \in M \cap M^\perp = \{0\}$. Hence $v \in \mathrm{span}(w)$. Thus we see that M has co-dimension 1. i.e $M^\perp$ has dimension 1.

Now consider P, the orthogonal projection to M and $Q = 1 - P$. Then $x = Px + Qx$ for all $x \in \mathcal{H}$. Hence

$$\Lambda(x) = \Lambda(Px + Qx) = \Lambda(Qx)$$

Since $M^\perp$ is a one dimensional subspace $Qx = \alpha v$ for $\alpha \in \mathbb{C}$ and $0 \neq v \in M^\perp$ with $\Lambda(v) = 1$. This gives that

$$\begin{aligned}
\Lambda(x) &= \alpha\Lambda(v) \\
&= \alpha \\
&= \alpha\langle v, \frac{v}{\|v\|^2}\rangle \\
&= \langle \alpha v, \frac{v}{\|v\|^2}\rangle \\
&= \langle Qx, \frac{v}{\|v\|^2}\rangle + 0 \\
&= \langle Qx, \frac{v}{\|v\|^2}\rangle + \langle Px, \frac{v}{\|v\|^2}\rangle \quad \text{(Since } Px \perp M^\perp\text{)} \\
&= \langle Qx + Px, \frac{v}{\|v\|^2}\rangle \\
&= \langle x, \frac{v}{\|v\|^2}\rangle
\end{aligned}$$

□

22.2 Orthonormal Sets

Definition 22.2.1. A family $\{u_\alpha\}_{\alpha \in A}$ is called orthonormal if $\langle u_\alpha, u_\beta\rangle = \delta_{a,b}$ for each $\alpha, \beta \in A$.

If $x \in \mathcal{H}$, then $\langle x, u_\alpha\rangle$ is called a Fourier coefficient of x relative to u_α.

We consider finite orthonormal sets first.

Proposition 22.2.1. *Let $\{u_\alpha\}_{\alpha \in A}$ be an orthonormal set and $F \subset A$ be finite. Let $M_F = span\{u_\alpha\}_{\alpha \in F}$.*

(a) If $\phi : A \to \mathbb{C}$, $\phi|_{A \setminus F} = 0$, then there is $y \in M_F$ such that

$$y = \sum_{\alpha \in F} \phi(\alpha) u_\alpha$$

and $\phi(\alpha) = \langle y, u_\alpha\rangle$ for each $a \in A$. Also

$$\|y\|^2 = \sum_{\alpha \in F} |\phi(\alpha)|^2$$

This says that the orthogonal projection is the best approximation to the subspace

(b) If $x \in \mathcal{H}$, then

$$\left\| x = \sum_{\alpha \in F} \langle x, u_\alpha\rangle u_\alpha \right\| \leq \|x - s\| \quad \text{for any } s \in M_F$$

Proof. (a) Straightforward.

(b) Let $S(x) = \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha$. Note that $\langle S(x), u_\alpha \rangle = \langle x, u_\alpha \rangle$ for each $\alpha \in F$. Thus we see that $\langle x - S(x), u_\alpha \rangle = 0$ for each $\alpha \in F$. Because $M_F = \operatorname{span}\{u_\alpha : \alpha \in F\}$, we see that $(x - S(x)) \perp v$ for any $v \in M_F$. Thus $(x - S(x)) \perp (S(x) - v)$. Thus be Pythagoras theorem, we get

$$\|x - v\|^2 = \|x - S(x)\|^2 + \|v - S(x)\|^2 \geq \|x - S(x)\|^2$$

Thus we see that $S(x)$ is the best approximation of x on to the space M_F. Thus we see that S is the orthogonal projection to M.

$\square$