MATH 6320
Homework 7

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1. not finished

   Solution:

2. **Solution:** Since $r \in D_8$, has order 4, if $\phi : D_8 \to D_8$ is any automorphism, then $\phi(r)$ must also have the same order. Hence the possible $\phi(r)$ are $r, r^{-1} \in D_8$. Similarly since $|s| = 2$, $\phi(s)$ also must have order 2, which gives $\phi(s) \in \{s, r^2, sr, sr^2, sr^3\}$. But since $\phi(r) \in \{r, r^3\}$, if $\phi(s) = r^2$, $\phi(D_8) = \langle r \rangle$, and ϕ will not be an automorphism. Hence $\phi(s) \in \{s, sr, sr^2, sr^3\}$. Since s, r generate D_8, and each of them have 4 and 2 possible options, by the counting argument, $\mathrm{Aut}(D_8)$ can have at most 8 elements.

3. not finished

   Solution: Since $D_8 \trianglelefteq D_{16}$, we see that $\phi : D_{16} \to \mathrm{Aut}(D_8) : g \to \phi_g$, where $\phi_g : h \to ghg^{-1}$ is a well defined map. Since

$$\begin{aligned}\phi_g \phi_{g'}(h) &= \phi_g(g'h(g')^{-1}) \\ &= gg'h(g')^{-1}g^{-1} \\ &= (gg')h(gg')^{-1} \\ &= \phi_{gg'}(h)\end{aligned}$$

   we see that ϕ is a group homomorphism. Moreover, we know that $\mathrm{Ker}(\phi) = C_{D_{16}}(D_8) = \langle r \rangle = \{r, e\}$. Hence by the first isomorphism theorem, we see that $\phi(D_{16}) = \frac{D_{16}}{\langle r^4 \rangle} \cong D_8$. Moreover,

4. not finished

   Solution: From what we proved in the class, we know that if $H \leqslant G$, then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$. Hence in the question, we know that $N_{S_p}(P)/C_{S_p}(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$.

   Since P is a cyclic group of order p, $P \cong \mathbb{Z}/p\mathbb{Z}$ and hence the number of automorphisms of P are precisely $p-1$.

   Also $C_{S_p}(P) = P$. To show this, we first notice that every non-identity element in P must be a p-cycle. If $p = 2$, $P = S_p$ and we've noting to prove. Hence assume $p > 2$, then without loss of generality, assume $(1\ 2 \ldots\ p) \in P$. Then $(1\ 2)(1\ 2 \ldots\ p) = (2\ 3 \ldots\ p) \neq (1\ 3\ 4 \ldots p) = (1\ 2 \ldots\ p)(1\ 2)$.

5. **Solution:** Let $(1, k) \in C_K(H)$. Then for any $(h, 1) \in G$,

$$(h, k) = (h\varphi(1)(1), k) = (h, 1)(1, k) = (1, k)(h, 1) = (1\varphi(k)(h), k)$$

   forces $\varphi(k)(h) = h$. Since this is true for all $h \in H$, we see that $\phi(k)$ is the trivial automorphism of H. Hence $k \in \mathrm{Ker}(\phi)$.

   Conversely, if $k \in \mathrm{Ker}(\phi)$, then $\phi(k)(h) = h$ for all $h \in H$. Then for any $(h, 1) \in H$(identified as a subgroup of G)

$$(h, 1)(1, k) = (h\varphi(1)(1), k) = (h, k) = (\phi(k)(h), k) = (1, k)(h, 1)$$

   shows that $(1, k) \in C_K(H)$. Hence $C_K(H) = \mathrm{Ker}(\varphi)$.

6. not finished

   Solution: We know that $\mathrm{Hol}(H) = H \rtimes_\phi \mathrm{Aut}(H)$, where $\phi : \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ is the identity map.

   (a) We notice that $H = Z_2 \times Z_2 \cong V_4$, the Klein 4 group. Therefore, let $H = V_4 = \{1, a, b, c\}$. Since we know that any two of a, b, c generate the group V_4 we see that any permutation of a, b, c will be a group automorphism. Hence we see that $\mathrm{Aut}(H) \cong S_3$. Hence we see that $\mathrm{Hol}(Z_2 \times Z_2) \cong H \rtimes K$, where $H = Z_2 \times Z_2$ and $K \cong S_3$. Also, $|H \rtimes K| = |H \times K| = |H| \times |K| = 4 \times 6 = 24$

   (b)

7. not finished

Solution: We know that since $75 = 3 \times 5^2$, the fundamental theorem for Abelian groups immediately gives two groups $Z_3 \times Z_{5^2} \cong Z_{75}$ and $Z_3 \times Z_5 \times Z_5$.