# MATH7320 - Functional Analysis
# Homework 5

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1. not finished

   **Solution:**

2. not finished

   **Solution:** Since $S \perp S^{\perp}$, clearly $S \subset (S^{\perp})^{\perp}$. Moreover, we know that $(S^{\perp})^{\perp} = \mathrm{Ker}(P_{S^{\perp}})$. Therefore $(S^{\perp})^{\perp}$ is a closed subspace. Hence $\overline{\mathrm{span}}(S) \subset (S^{\perp})^{\perp}$. Conversely if $x \in (S^{\perp})^{\perp}$, then $x \perp S^{\perp}$

3. not finished

   **Solution:** Let $T_n \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ be a Cauchy sequence.

4. **Solution:** Since we know that $I = P_{\mathcal{M}} + P_{\mathcal{M}^{\perp}}$, we see that $X = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then for all $x \in X$, $x = m + m'$ for unique $m \in \mathcal{M}, m' \in \mathcal{M}^{\perp}$. Then $\pi(x) = m' + \mathcal{M}$ for $\pi : X \to X/\mathcal{M}$. Moreover

   $$\|x\| = \|m\| + \|m'\| \quad \text{and} \quad \|\pi(x)\| = \|m'\|$$

   Thus we see that $\pi|_{\mathcal{M}^{\perp}}$ is isometric.

5. **Solution:**

6. **Solution:**

7. **Solution:**

   (a) ($1 \implies 2$) If $T$ is an isometry, then expanding $\langle Tx, Ty\rangle = \langle x, y\rangle$ for some $x, y \in \mathcal{H}$, we get

   $$\langle Tx, Ty\rangle + \langle Ty, Tx\rangle = \langle x, y\rangle + \langle y, x\rangle$$

   which gives $\Re\langle x, y\rangle = \Re\langle Tx, Ty\rangle$. Now replace $x$ with $ix$ to get $\Im\langle Tx, Ty\rangle = \Im\langle x, y\rangle$. Since real and imaginary parts are equal, we see that $\langle Tx, Ty\rangle = \langle x, y\rangle$

   (b) ($2 \implies 3$) If $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$, then $\langle T^*Tx, y\rangle = \langle x, y\rangle$, which implies $\langle (T^*T - I)x, y\rangle = 0$ for all $x, y \in \mathcal{H}$. Then Reisz Representation theorem shows that if $y \neq 0$, we must have $T^*T - I = 0$.

   (c) ($3 \implies 2$) If $T^*T = I$, then

   $$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &= \langle x, x\rangle \\ &= \|x\|^2 \end{aligned}$$

   which shows that $T$ is an isometry.

8. **Solution:**

   (a) ($1 \implies 3$) If $T$ is normal isometry, we see that $TT^* = T^*T$, and previous question proves that $TT^* = T^*T = I$.

   (b) ($3 \implies 2$) If $TT^* = T^*T = I$, then it is clear from the previous question that $T$ is an isometry. To see that it is a bijection, let $x \in H$, since $T(T^*(x)) = x$, we see that $x \in T(\mathcal{H})$. Hence $T$ is a bijection.

   (c) ($2 \implies 1$). We just need to show normality of $T$. Since $T$ is given to be an isometric bijection, $T$ has an inverse, $P$. Since $T$ is bijective $P$ is also isometric and linear. To see linearity, notice that

   $$P(x + y) = P(T(Px) + T(Py)) = P(T(Px + Py)) = Px + Py$$

We claim $P = T^*$. To see this, note that

$$\langle PTx, y\rangle = \langle x, y\rangle = \langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$$

Hence we see that $\langle (PT - T^*T)x, y\rangle = 0$ for al $x, y \in \mathcal{H}$. Therefore by Reisz representation, we have $PT - T^*T = (P - T^*)T = 0$. Since $T$ is bijective, this forces $P = T^*$ and we get the normality.

9. **Solution:** Let $x \in \mathrm{Ker}(T)$. Then $0 = \langle Tx, y\rangle = \langle x, T^*y\rangle$ for any $y \in \mathcal{H}$, shows that $y \in T(\mathcal{H})^\perp$.

   Conversely if $x \in T^*(H)^\perp$. Then $0 = \langle x, T^*y\rangle = \langle Tx, y\rangle =$ for any $y \in \mathcal{H}$ shows that $x \in \mathrm{Ker}(T)$ by Reisz representation theorem.

10. not finished

    **Solution:**

11. not finished

    **Solution:**

12. not finished

    **Solution:**

13. not finished

    **Solution:**

    (a) If $M$ is invariant under $T$, then $T(M) \subset M$, This shows $T(m) = P_M T(m)$ for all $m \in M$. Thus $TP_M = P_M T P_M$. Conversely if $TP_M = P_M T P_M$, then $T(m) = TP_M(m) = P_M T P_M(m) = P_M T(m)$ for all $m \in M$, shows that $T(m) \subset M$ for all $m \in M$. Thus $M$ is invariant under $T$.

    (b) If $M$ reduces $T$, then $M$ and $M^\perp$ is invariant under $T$. Thus we see that for $x = m + m'$ for $m \in M, m' \in M^\perp$,

    $$P_M T(x) = P_M(T(m) + T(m')) = P_M(T(m)) = T(m) = T(P_M(x))$$

    Thus $P_M T = TP_M$.

Conversely, if $P_M T = T P_M$, then for $m \in M$, we get

$$P_M T(m) = T P_M(m) = T(m)$$

which shows $T(m) \in M$ and for $m' \in M^{\perp}$, we get

$$P_M T(m') = T P_M(m') = T(0) = 0$$

Hence $T(m') \perp M$ which implies $T(m') \in M^{\perp}$. Thus we see that $M$ reduces $T$.

(c) If $M$ reduces $T$, then it is clear that $M$ is invariant under $T$. Moreover from before, we see that $T P_M = P_M T$, then

$$\langle P_M T(x), y \rangle = \langle T P_M(x), y \rangle = \langle P_M(x), T^*(y) \rangle$$

(d)

(e)