Abstract Algebra (Math 6302), Fall 2024
Homework Assignment 4

Due at 9:59pm on Monday, September 30. Turn in your completed assignment (single .pdf file) on Canvas. For full credit, show your work and justify your answers.

(1) Show that if G is an Abelian group with order greater than 1 which is not isomorphic to C_p for a prime number p, then G is not a simple group.

(2) Suppose that G is a group and that for all i in some indexing set I, the group H_i is normal in G. Prove that

$$\bigcap_{i \in I} H_i \trianglelefteq G.$$

(3) Prove that for all groups G, $\mathrm{Z}(G) \trianglelefteq G$.

(4) Find a proper, non-trivial, normal subgroup in D_{14}, and prove that it is normal.

(5) Let $G = Q_8$ and let $S = \{i, -i, j\} \subseteq G$. Compute $C_G(S)$ and $N_G(S)$.

(6) Prove that if $H \trianglelefteq G$ and if $\phi : G \to K$ is a surjective homomorphism, then $\phi(H) \trianglelefteq K$.

(7) Prove that if $H \trianglelefteq G$ then for all $g \in G$ with finite order, the order of gH in G/H divides the order of g in G.

(8) Let $G = \mathbb{Q}$ and $H = \mathbb{Z} \leqslant G$.
   (a) Prove that G/H is an infinite group.
   (b) Prove that every element of G/H has finite order.
   (c) Prove that for every $n \in \mathbb{N}$, there is an element of G/H with order n.

(9) Write

$$G = D_{16} = \langle r, s \mid r^8 = s^2 = e,\ rs = sr^{-1} \rangle \quad \text{and}$$
$$K = V_4 = \langle a, b \mid a^2 = b^2 = e,\ ab = ba \rangle.$$

   (a) Prove that there is a homomorphism $\phi : G \to K$ with $\phi(r) = a$ and $\phi(s) = b$.
   (b) Use the first isomorphism theorem to prove that $D_{16}/\langle r^2 \rangle \cong V_4$.

(10) Prove that if G is a group and $G/\mathrm{Z}(G)$ is cyclic, then G is Abelian.

(11) Let $G = \mathbb{Z}^2$ and $H = \langle (2,1), (2,5) \rangle \leqslant G$.
   (a) Find a complete set of distinct coset representatives for G/H.
   (b) Prove that $G/H \cong C_8$.

(12) Let $G = \mathbb{R}^2$ and suppose that $v_1, v_2 \in G$ are $\mathbb{R}$-linearly independent vectors.

(a) Find a complete set of distinct coset representatives for $G/\langle v_1, v_2\rangle$.

(b) Prove that, for all pairs of $\mathbb{R}$-linearly independent vectors $u_1, u_2 \in G$,

$$G/\langle u_1, u_2\rangle \cong G/\langle v_1, v_2\rangle.$$