MATH 7320, Functional Analysis I

HOMEWORK 2

1. Let $\mathcal{X}$ be a normed space and $\mathcal{M}$ be a subspace of $\mathcal{X}$. Prove that $\mathcal{M}$ is dense in $\mathcal{X}$ iff $f \in \mathcal{X}^*$ and $f|_{\mathcal{M}} = 0$ implies $f = 0$.

2. Let $\mathcal{X}$ be a normed space.

    i) Show that if $\mathcal{X}$ is reflexive, then it is complete.

    ii) Show that $\mathcal{X}$ is reflexive iff $\mathcal{X}^{**}$ is reflexive.

    iii) Show that if $\mathcal{X}$ is reflexive, then every closed subspace of $\mathcal{X}$ is reflexive.

    iv) Show that if $\mathcal{X}$ is reflexive, then for every $f \in \mathcal{X}^*$ there exists $\mathbf{x} \in \mathcal{X}$ such that $\|\mathbf{x}\| \leq 1$ and $f(x) = \|f\|$.

    v) Show that $\mathbf{c}_0$ is not reflexive.

3. Let $f \in (\ell^1)^*$. Show that there exists a unique sequence $(a_n) \in \ell^\infty$ such that for every sequence $(x_n) \in \ell^1$ we have
$$f\big((x_n)\big) = \sum_{n \in \mathbb{N}} x_n \overline{a_n}\,.$$
Moreover, the map $(\ell^1)^* \to \ell^\infty$ defined by $f \mapsto (a_n)$ is an isometric isomorphism (aka a surjective isometric linear vegetarian map).

4. Read/recall the Hölder's inequality from your previous courses or from the literature.

5. Let $1 < p < \infty$ and $q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(a_n) \in \ell^q$. Define $f : \ell^p \to \mathbb{C}$ by
$$f\big((x_n)\big) = \sum_{n \in \mathbb{N}} x_n \overline{a_n} \quad \forall (x_n) \in \ell^p\,.$$
Show that $f \in (\ell^p)^*$ and moreover, the map $\ell^q \to (\ell^p)^*$ defined by $(a_n) \mapsto f$ is a continuous linear map.