MATH6320 - Theory of Functions of a Real Variable
Assignment 8

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1. **Solution:** Let x_n be a sequence in R_f that converge to $x \in \mathbb{C}$. We'll be done if we prove that $x \in R_f$. Let $\varepsilon > 0$ be given. Then there is a $N_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n > N_{\frac{\varepsilon}{2}}$. Hence $B_\varepsilon(x) \supseteq B_{\frac{\varepsilon}{2}}(x_n)$ for all $n > N_{\frac{\varepsilon}{2}}$. Therefore
$$f^{-1}(B_\varepsilon(x)) \supseteq f^{-1}(B_{\frac{\varepsilon}{2}}(x_n))$$
But $f^{-1}(B_{\frac{\varepsilon}{2}}(x_n)) = A_{x_n,\frac{\varepsilon}{2}}$ and $f^{-1}(B_\varepsilon(x)) = A_{x,\varepsilon}$. Since $x_n \in R_f$ by assumption, we see that $\mu(A_{x_n,\frac{\varepsilon}{2}}) > 0$. Then by the monotonicity of the measure, we see that for all $n > N_{\frac{\varepsilon}{2}}$
$$\mu(A_{x,\varepsilon}) = \mu(f^{-1}(B_\varepsilon(x))) \geq \mu(f^{-1}(B_{\frac{\varepsilon}{2}}(x_n))) = \mu(A_{x_n,\frac{\varepsilon}{2}}) > 0$$
Since $\varepsilon > 0$ was chosen arbitrarily, we see that $\mu(A_{x,\varepsilon}) > 0$ for all $\varepsilon > 0$. Hence $x \in R_f$, by the definition of R_f.

2. **Solution:** Let $f \in L^1(m)$ be bounded ($|f(x)| < M$) such that $A = \{x \in \mathbb{R} : f(x) \neq 0\}$ has finite measure $m(A) < \infty$. Note that the Lebesgue measure is a regular, Borel measure and the space $\mathbb{R}$ is locally compact and Hausdorff. Then by Luzin's theorem, for any given $\varepsilon > 0$, there is a $g_\varepsilon \in C_c(\mathbb{R})$ such that for $E_\varepsilon = \{x \in \mathbb{R} \ : \ f(x) \neq g_\varepsilon(x)\}$, we have $\mu(E_\varepsilon) < \frac{\varepsilon}{4M}$ and $|g_\varepsilon(x)| < M$ for

all $x \in \mathbb{R}$. Then

$$\begin{aligned}
\int |f - g_\varepsilon| \; dm &= \int_{E_\varepsilon} |f - g_\varepsilon| \; dm + \int_{E_\varepsilon^c} |f - g_\varepsilon| \; dm \\
&= \int_{E_\varepsilon} |f - g_\varepsilon| \; dm + 0 \\
&\leq 2Mm(E_\varepsilon) \\
&< 2M\frac{\varepsilon}{4M} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

Again, since $g_\varepsilon \in C_c(X)$, it is Riemann integrable and there is a partition $P_\varepsilon = \{p_1 < p_2 < \cdots < p_n\}$ of the compact support $K = \text{supp}(g_\varepsilon)$ (Without loss of generality, we can assume that this K is an interval $[p_1, p_n]$. In case it is not, Extend K to its convex closure) such that

$$\int g_\varepsilon(x) \; dx < m_{P_\varepsilon}(g_\varepsilon) + \frac{\varepsilon}{2}$$

where the integral above is the Reimann integral and $m_{P_\varepsilon}(\varepsilon)$ is the lower Reimann sum of g_ε on the partition P_ε.

Then consider the step function

$$h = \sum_{i=1}^{n-1} \chi_{[p_i, p_{i+1})} \inf_{x \in [p_i, p_{i+1}]} g_\varepsilon(x)$$

By definition, we see that $g_\varepsilon \geq h$. Hence $g_\varepsilon - h = |g_\varepsilon - h|$. Moreover,

$$\int h(x) \; dx = m_{P_\varepsilon}(g_\varepsilon)$$

Therefore,

$$\int |g_\varepsilon - h| \; dx = \int (g_\varepsilon - h) \; dx = \int g_\varepsilon \; dx - \int h \; dx = \int g_\varepsilon \; dx - m_{P_\varepsilon}(f) < \frac{\varepsilon}{2}$$

Since Riemann integral and Lebesgue integral agree on Reimann integrable functions, we get

$$\int |g_\varepsilon - h| \; dm = \int |g_\varepsilon - h| \; dx < \frac{\varepsilon}{2}$$

By triangle inequality, we know that $|f-h| \leq |f-g_\varepsilon| + |g_\varepsilon - h|$. Then by the linearity and monotonicity of the integral on positive functions, we see that

$$\int |f-h|\ dm \leq \int |f-g_\varepsilon|\ dm + \int |g_\varepsilon - h|\ dm < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Now let $f \in L^1(m)$ and $\varepsilon > 0$ be given. Consider the set $B_n = \{x \in \mathbb{R} \ :\ \frac{1}{n} \leq |f(x)| \leq n\}$. Clearly $f_n = f\chi_{B_n}$ converge pointwise to f. To see this let $x \in \mathbb{R}$. If $f(x) = 0$, then each $f_n(x) = 0$ and we've nothing to prove. Otherwise there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < |f(x)|$. Then $f_n(x) = f(x)$ for all $n > N$, and we're done. Hence we see that $|f - f_n|$ converge pointwise to 0.

Also, notice that $|f_n| < |f|$. Therefore by triangle inequality, $|f - f_n| \leq 2|f|$ which is again in $L^1(m)$. Therefore by dominated convergence theorem,

$$\lim_{n\to\infty} \int |f - f_n|\ dm = 0$$

Thus there is an N_ε such that

$$\int |f - f_{N_\varepsilon}|\ dm < \frac{\varepsilon}{2}$$

Moreover for every $n \in \mathbb{N}$, $\frac{1}{n}\chi_{B_n} \leq f\chi_{B_n}$ and therefore

$$\frac{1}{n} m(B_n) = \int \frac{1}{n}\chi_{B_n}\ dm \leq \int f\chi_{B_N}\ dm \leq \int f\ dm < \infty$$

Shows that $m(B_{N_\epsilon}) < \infty$. Then $f\chi_{B_n}$ is a bounded function ($|f\chi_{B_n}| < n$) with $\{x \in \mathbb{R}\ :\ f\chi_{B_n}(x) \neq 0\} = B_n$. Thus by the first part of the proof there is a step function h_n such that

$$\int |f_n - h_n|\ dm < \frac{\varepsilon}{2}$$

Then specifically for $n = N_\varepsilon$, by the triangle inequality and the linearity and monotonicity of the integral, we get

$$\begin{aligned}\int |f - h_n|\ dm &\leq \int |f - f_{N_\varepsilon}|\ dm + \int |f_{N_\varepsilon} - h_{N_\varepsilon}|\ dm \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

3. **Solution:** We'll first show that this holds for step functions. Let

$$s = \sum_{i=1}^{n} a_i \chi_{[a_i, b_i)}$$

where $a_i < b_i \leq a_{i+1}$ for each i. Then

$$s_t(x) := s(x-t) = \sum_{i=1}^{n} a_i \chi_{[a_i, b_i)}(x-t) = \sum_{i=1}^{n} a_i \chi_{[a_i+t, b_i+t)}(x)$$

Then

$$s_t - s = \sum_{i=1}^{n} a_i \chi_{[a_i+t, b_i+t)} - \sum_{i=1}^{n} a_i \chi_{[a_i, b_i)} = \sum_{i=1}^{n} a_n \left(\chi_{[a_i+t, b_i+t)} - \chi_{[a_i, b_i)}\right)$$

Now when $0 < t < \min\{b_i - a_i\}$ (such t must exist, since $a_i < b_i$ for each i) and $M = \max\{|a_i|\}$, we see that

$$|s_t - s| = \left| \sum_{i=1}^{n} a_n \left(\chi_{[b_i, b_i+t)} - \chi_{[a_i, a_i+t)}\right) \right| \leq M \sum_{i=1}^{n} \left(\chi_{[b_i, b_i+t)} + \chi_{[a_i, a_i+t)}\right)$$

Then

$$\int |s_t - s| \; dm \leq M \sum_{i=1}^{n} 2t = 2Mnt$$

Since M, n does not depend on t, taking limits as $t \to 0$, we see that

$$0 \leq \lim_{t \to 0} \int |s_t - s| \; dm \leq \lim_{t \to 0} \; 2Mnt = 0$$

Now for the general case, let $f \in L^1(\mu)$ and $\epsilon > 0$ be given. Then by the previous answer there is a step function s such that

$$\int |f - s| \; dm < \frac{\varepsilon}{3}$$

Moreover, by the first part of this proof, there is a $t_\varepsilon > 0$ such that for all $t \in [0, t_\varepsilon]$

$$\int |s_t - s| \; dm < \frac{\varepsilon}{3}$$

Also notice that $f_t - s_t = (f - s)_t$. Since Lebesgue measure is translation invariant, we get that

$$\int |f_t - s_t| \; dm = \int |(f - s)_t| \; dm = \int |f - s| \; dm < \frac{\varepsilon}{3}$$

Thus we see that for all $t \in [0, t_\varepsilon]$,

$$\int |f - f_t| \; dm \leq \int |f - s| \; dm + \int |s - s_t| \; dm + \int |s_t - f_t| \; dm < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Since ε was arbitrary, we have proved the statement for general $f \in L^1(m)$.