MATH7320 Functional Analysis
Homework 3

Joel Sleeba

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1. **Solution:** We easily see that if $f : X \to \mathbb{C}$ is a bounded linear functional, then since $\{0\}$ is closed in $\mathbb{C}$, then $\text{Ker}(f) = f^{-1}(0)$ is closed. Conversely, if $K = \text{Ker}(f)$, then the canonical map $q : X \to X/K$ and the map $\tilde{f} : X/K \to \mathbb{C}$ defined as $\tilde{f}([x]) = f(x)$ is continuous since X/K is finite dimensional and all linear maps between finite dimensional spaces are continuous. Now $f = \tilde{f} \circ q$ gives that f is continuous.

2. **Solution:** Let x_n be a Cauchy sequence in X. Then $[x_n]$ is Cauchy in X/Y since $\|[x_n] - [x_m]\| = \|[x_n - x_m]\| \leq \|x_n - x_m\|$. Therefore $[x_n] \to [x]$ in X/Y. Let x_{n_k} be a subsequence of x_n such that $\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^{k+1}}$. Then $\|[x_{n_k} - x_{n_{k+1}}]\| < \frac{1}{2^{k+1}}$ and for all k, there exists $y_k \in Y$ such that $\|x_{n_k} - x - y_k\| < \frac{1}{2^k}$. We claim that y_k is a Cauchy in Y. Let $0 < n \leq m$, then

$$\begin{aligned}
\|y_m - y_n\| &= \|(x_{n_n} - x - y_n) - (x_{n_m} - x - y_m) - (x_{n_n} - x_{n_m})\| \\
&\leq \|(x_{n_n} - x - y_n)\| + \|(x_{n_m} - x - y_m)\| + \|x_{n_n} - x_{n_m}\| \\
&< \frac{1}{2^n} + \frac{1}{2^m} + \frac{1}{2^n} \\
&\leq \frac{3}{2^n}
\end{aligned}$$

Therefore y_n is Cauchy and converges to $y \in Y$. Hence $x_{n_k} \to x + y$. Since the space is Hausdorff we get that $x_n \to x + y$.

3. **Solution:** Let $S = \{y_n \in Y\}$ be dense in Y and $T = \{[x_n] \in X/Y\}$ be dense in X/Y. Construct a new collection U of elements of X using the axiom of choice by selecting an element $x \in [x_n]$ for each $[x_n] \in T$. Now we claim that the set $Z = S + U = \{y + x \ : \ y \in S, x \in U\}$ is dense in X. Clearly we see that Z is countable since the cardinality of Z is the cardinality of $S \times U$.

   Let $x \in X$ and $\epsilon > 0$. Then by density of T in X/Y, there exist an $[x_n]$ such that $\|[x] - [x_n]\| = \|[x - x_n]\| < \frac{\epsilon}{3}$. Then there is a $y \in Y$ such that $\|x - x_n - y\| < \frac{2\epsilon}{3}$. Now again by the density of S in Y, there is a $y_n \in S$ such that $\|y - y_n\| < \frac{\epsilon}{3}$. Then

   $$\|x-(x_n+y_n)\| = \|x-x_n-y_n+(y_n-y)\| \leq \|x-x_n-y\|+\|y-y_n\| < \frac{2\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$$

   Hence we see that Z is dense in X.