MATH 7320, Functional Analysis I

HOMEWORK 4 – Due Friday October 25

1. Determine the extreme points of the closed unit ball of the following complex Banach spaces:

   i) $\ell^p$ for $1 \le p \le \infty$

   ii) $L^p([0,1],\mu)$ for $1 \le p \le \infty$, where $\mu$ is the Lebesgue measure.

   iii) $C([0,1])$.

   iv) $C_0(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \mid f \text{ is continuous and } \lim_{|x|\to\infty} f(x) = 0\}$.

   v) $\mathcal{B}(\ell^2)$.

2. Denote $C_b(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and bounded}\}$ and $C_0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } \lim_{x\to\pm\infty} f(x) = 0\}$, and equip both with the sup-norm. For each $f \in C_0(\mathbb{R})$ define the semi-norm $p_f$ on $C_b(\mathbb{R})$ by $p_f(g) = \|fg\|$ and let $\tau$ be the topology on $C_b(\mathbb{R})$ defined by the family of semi-norms $\{p_f : f \in C_0(\mathbb{R})\}$.

   i) Prove that both normed spaces are complete, and $C_b(\mathbb{R})$ is also complete with respect to $\tau$.

   ii) Prove that the identity map $\big(C_b(\mathbb{R}), \|\cdot\|\big) \to \big(C_b(\mathbb{R}), \tau\big)$ is continuous but not open.