Functional Analysis - MATH7320

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Contents

Chapter 1

Textbook : A Course in Functional Analysis, John Conway

Functional analysis is the study of Topological Vector Spaces.

Definition 1.0.1. Let X be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A seminorm on X is a map $\|\cdot\| : X \to [0, \infty)$ such that

- $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{F}, \forall x \in X$
- $\|x + y\| \leq \|x\| + \|y\|$

In addition if $\forall x \neq 0, \|x\| \neq 0$, we say $\|\cdot\|$ is a norm on X

Norm induces a metric $d(x, y) = \|x - y\|$

Note. Let X be a normed space. Then the maps

- $+ : X \times X \to X : (x, y) \to x + y$
- $\cdot : \mathbb{F} \times X \to X : (\alpha, x) \to \alpha x$

are continuous.

Hence every normed space is a topological vector space.

Example 1.0.1. $\mathbb{F}^n$ with ℓ_p-norm defined as

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}}$$

Example 1.0.2. $\mathbb{F}^n$ with ℓ_∞-norm defined as

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\infty = \max\{|a_i|\}$$

Example 1.0.3. Consider $C_{00} = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n = 0$ except for finitely many $n \in \mathbb{N}\}$ which can be identified by collection of functions $f : \mathbb{N} \to \mathbb{F}$ with finite support.

Then

$$\|(a_n)\|_p = \Big(\sum_{n=1}^{\infty} |a_n|^p\Big)^{\frac{1}{p}}$$

is a norm on C_{00}

Proposition 1.0.1. *Let X, Y be normed space, and let $T : X \to Y$ be linear. Then the following are equivalent.*

- *T is continuous*
- *T is continuous on 0*
- *T is continuous on any point $x \in X$*
- *$\exists M > 0$ such that $\|T(x)\|_Y \le M\|x\|_X$ for all $x \in X$*

Proof. ($1 \implies 2$) It is clear that if T is continuous, then it is continuous at 0 from the definition of continuity.

($2 \implies 3$) Let $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in X that converge to x. Then the sequence $\{y_n = x_n - x\}$ converge to zero by the algebra of limits. By the continuity of T at zero, $\{T(y_n) = T(x_n) - T(x)\}$ converge to 0. Therefore $T(x_n) \to T(x)$. And this shows T is sequentially continuous at $x \in X$. Since the space is a metric space, sequential continuity is equivalent to continuity.

($4 \implies 2$) Let $x \in X$. Then $\|T(0) - T(x)\| = \|T(x)\| \le M\|x\| = M\|0 - x\|$. Hence T is continuous at 0.

($3 \implies 1$)

($2 \implies 4$)

□

Example 1.0.4. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be defined as $T(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, 0, \dots, 0)$. Is T convergent for any norm $\|\cdot\|_1, \|\cdot\|_2$ in the domain and range?

Proof. verify □

Example 1.0.5. Consider identity function $I : C_{00} \to C_{00}$. Let the norm in domain be $\|\cdot\|_\infty$ and that in range be $\|\cdot\|_1$. Is the function continuous? What if the norms in domain and range are switched?

Proof. verify □

Note. Let X be a space with two norms $\|\cdot\|_1, \|\cdot\|_2$. When is the two norms topologically equivalent?

When $\exists M, M'$ such that $\|x\|_1 \leq M\|x\|_2$ and $\|x\|_2 \leq M'\|x\|_1$ Equivalently, when the identity map between the two spaces with their respective norms are bi-continuous. (See 4th equivalent statement of previous proposition)

Theorem 1.0.1. *Let X and Y be normed spaces, and $T : X \to Y$ be linear. Assume X is finite dimensional. Then T is continuous.*

Proof. Since $T(X) \leq Y$ is finite dimensional, we may assume without loss of generality that Y is also finite dimensional and T is onto. Let $\{x_1, x_2, \dots x_n\}$ be a basis for X. Define another norm on X as follows. For every $x = \sum_{i=1}^n \alpha_i x_i \in X$,

$$\|x\|' = \sum_{i=1}^n |\alpha_i|(\|T(x_i)\| + \|x_i\|)$$

verify that this is a norm. Then for every $x \in X$, we have

$$\|T(x)\| \leq \sum_{i=1}^n |\alpha_i|\|T(x_i)\| \leq \|x\|'$$

Hence T is bound with respect to the norm $\|\cdot\|'$ on X, since all norms are equivalent on X. Therefore T is continuous w.r.t to the original norm on X. $\square$

Corollary 1.0.1.1. *Let X be a finite dimensional vector space. Then any two norms in X are equivalent.*

Proof. Let $\{e_1, e_2, \dots e_n\}$ be a basis for X. For each $x = \sum_{i=1}^n \alpha_i e_i \in X$, define

$$\|x\|_\infty = \max\{|\alpha_i|\}$$

Then $\|\cdot\|_1$ is a norm and we'll show every norm on X is equivalent to this norm. Let $\|\cdot\|$ be an arbitrary norm on X. For each $x = \sum_{i=1}^n \alpha_i e_i \in X$, we have

$$\begin{aligned}
\|x\| &= \|\sum_{i=1}^n \alpha_i e_i\| \\
&\leq \sum_{i=1}^n |\alpha_i|\|e_i\| \\
&\leq \max\{|\alpha_i|\} \sum_{i=1}^n \|e_i\| \\
&\leq \|x\|_\infty \sum_{i=1}^n \|e_i\|
\end{aligned}$$

Therefore the identity map $I : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|)$ is continuous. Since the set $K = \{x \in X \ : \ \|x\|_\infty \leq 1\}$ is compact, K is also compact in $(X, \|\cdot\|)$ and the restriction $\mathrm{Id}|_K$ is also a homeomorphism. verify In particular, the set $\{x \in X \ : \ \|x\|_\infty < 1\}$ is an open neighborhood of $0 \in (X, \|\cdot\|)$ By the Heine-Borel theorem, the unit ball $B = \{x \in X \ : \ \|x\|_2 \leq 1\}$ is compact. Hence B is compact in $(X, \|\cdot\|)$. verify the last line. $\square$

Chapter 2

2.1 continues

Corollary 2.1.0.1.

- *Every finite dimensional normed space is complete*
- *If X is a normed space and Z is a finite dimensional subspace of X, then Z is the closed in X*

Proof.

- Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Let $\|\cdot\|_2$ be the euclidean norm on X. Then by the theorem above there exists $M \in \mathbb{R}$ such that $\frac{1}{M}\|x\|_2 \leq \|x\| \leq M\|x\|_2$ for all $x \in X$.

  Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|)$. Then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy in $(X, \|\cdot\|_2)$. Since the latter space is complete, so is $(X, \|\cdot\|)$.
- verify

□

Note. If $T, S : X \to Y$ are continuous linear maps between normed space, then $T + S$ is also continuous. Also, $\forall \alpha \in \mathbb{F}$, αT is continuous.

Thus the space $B(X, Y)$ of all continuous linear maps from X to Y is a subspace of all linear maps between X and Y.

Definition 2.1.1. Let X, Y be normed spaces. and $T \in B(X, Y)$. We define the operator norm of T as

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} \;:\; x \in X, x \neq 0\right\}$$

Lemma 2.1.1. *Let $T \in B(X, Y)$. Then the following are equivalent*

- $\|T\|$
- $\sup\{\|T(x)\| \;:\; x \in X, \|x\| \leq 1\}$

- $\sup\{\|T(x)\| \ : \ x \in X, \|x\| < 1\}$
- $\inf\{M \geq 0 \ : \ \|T(x)\| \leq M\|x\|, \forall x \in X\}$

Proof. verify $\square$

Proposition 2.1.1. *The operator norm is a norm in* $B(X, Y)$

Proof. verify $\square$

Chapter 3

Hahn Banach Theorem

Lemma 3.0.1. *Let X be a complex normed space. Let $f : X \to \mathbb{R}$ be an $\mathbb{R}$-linear functional. Then $g : X \to \mathbb{C}$ defined as $g(x) = f(x) - if(ix)$ is $\mathbb{C}$-linear*

Conversely if $g : X \to \mathbb{C}$ is a $\mathbb{C}$-linear map, then $f := \Re \circ g : X \to \mathbb{R}$ is $\mathbb{R}$-linear.

Moreover $\|f\| = \|g\|$.

Proof. We'll prove that $\|f\| = \|g\|$ and leave the rest for the reader (verify).

Since $|f(x)| \leq |g(x)|$, for all $x \in X$, it is easy to see that $\|f\| \leq \|g\|$. Conversely, $\forall \epsilon > 0, \exists x_o \in X$ with $\|x_o\| = 1$ such that $|g(x_o)| > \|g\| - \epsilon$. If $g(x_o) = re^{i\theta}$, take $\alpha = e^{-i\theta}$. Then $f(\alpha x_o) = \Re(re^{-i\theta}e^{i\theta}) = r = g(\alpha x_o)$. Then $\|f\| \geq |f(\alpha x_o)| = |g(\alpha x_o)| = |\alpha||g(x_o)| = |g(x_o)| > \|g\| - \epsilon$. Since ϵ is arbitrary, this gives $\|f\| \geq \|g\|$ □

Theorem 3.0.1 (Hahn-Banach Extension Theorem)**.** *Let X be a normed space over $\mathbb{R}$, Z be a subspace of X and let $\phi : Z \to \mathbb{R}$ be a continuous linear functional. Then there exists a linear functional $\psi : X \to \mathbb{R}$ such that $\psi|_Z = \phi$ and $\|\phi\| = \|\psi\|$.*

Proof. Assume $\|\phi\| = 1$ (If this is not the case, we can always scale the functional down to norm 1). Now we'll extend ϕ from Z to a subspace with one dimension higher than Z, preserving the norm. Let $x_o \in (X \setminus Z)$ and $Y = \text{Span}\{\{x_o\} \cup Z\}$ be the set one dimension higher than Z. Assume ψ is the extension of ϕ to Y. Then ψ will be completely characterized, if we know the value of $\psi(x_o)$. We look to see what real values we can assign $\psi(x_o)$ satisfying our conditions. Let $y = z + x_o \in Y$ where $z \in Z$ (We must be taking an arbitrary element $y = z + \alpha x_o \in Y$, but if we know the image of $y = z + x_o$ for all $z \in Z$ under ψ, then we can get the image of $y = z + \alpha x_o$ for any $\alpha \in \mathbb{R}$ by scaling). Norm preserveness demands that for all $z \in Z$, we must have

$$-\|z + x_o\| \leq \psi(y) = \psi(z) + \psi(x_o) \leq \|z + x_o\|$$

Since ψ agrees with ϕ on Z, this is equivalent to

$$-\phi(z) - \|z + x_o\| \leq \psi(x_o) \leq \|z + x_o\| - \phi(z) \tag{3.1}$$

Moreover since we normalized ϕ to have norm 1, we know ψ must also have norm 1. Then by triangle inequality, we get that for all $a, b \in Y$

$$\psi(a) - \psi(b) = \psi(a - b) \leq \|a - b\| = \|(a + x_o) - (b + x_o)\| \leq \|a + x_o\| + \|b + x_o\|$$

which gives

$$-\psi(b) - \|b + x_o\| \leq \|a + x_o\| - \psi(a)$$

Since this inequality is true for all $a, b \in Y$, taking supremum and infimum over all the possible $a, b \in Y$ preserves the inequality. Hence we get

$$\sup_{b \in Y} \Big\{ -\psi(b) - \|b + x_o\| \Big\} \leq \inf_{a \in Y} \Big\{ \|a + x_o\| - \psi(a) \Big\} \tag{3.2}$$

Substituting $a = b = z$ in Equation 3.2 guarantees the existence of $\psi(x_o)$ satisfying Equation 3.1. Hence we get an extension (namely ψ) of ϕ to Y preserving the norm. Since Z was an arbitrary subspace of X, this is true for all such subspaces of X.

Now we will employ Zorn's lemma to get an extension of ϕ from Z to the whole of X. Consider the collection of all linear extensions of ϕ, i.e

$$\mathcal{S} = \Big\{ (\psi_Y, Y) \ : \ Z \subset Y, \ \psi_Y|_Y = \phi, \ \|\psi_Y\| = \|\phi\| \Big\}$$

Then we define a partial order in the collection $\mathcal{S}$ as $(\psi_X, X) \leq (\psi_Y, Y)$ if and only if $X \subset Y$ and $\psi_Y|_X = \psi_X$. Now let $\mathscr{C}$ be a chain in $\mathcal{S}$. Consider the set

$$\tilde{Y}_{\mathscr{C}} = \bigcup_{(\psi_Y, Y) \in \mathscr{C}} Y$$

and the map $\psi_{\tilde{Y}_{\mathscr{C}}} : \tilde{Y}_{\mathscr{C}} \to \mathbb{R}$ defined as

$$\psi_{\tilde{Y}_{\mathscr{C}}}(x) = \psi_Y(x), \quad \text{where } x \in Y, \ \text{for } (\psi_Y, Y) \in \mathscr{C}$$

To see this map is well defined, assume $x \in X$ and $x \in Y$ for $(\psi_X, X), (\psi_Y, Y) \in \mathscr{C}$. Then either $(\psi_X, X) \leq (\psi_Y, Y)$ or $(\psi_Y, Y) \leq (\psi_X, X)$ since $\mathscr{C}$ is totally ordered. WLOG assume $(\psi_X, X) \leq (\psi_Y, Y)$, then by definition we get that $\psi_Y|X = \psi_X$. This gives that $\psi_Y(x) = \psi_X(x)$. Hence we get that $\psi_{\tilde{Y}_{\mathscr{C}}}$ is well defined. In a similar fashion we can verify that $\psi_{\tilde{Y}_{\mathscr{C}}}$ is a linear functional.

Now we claim that $(\tilde{Y}_{\mathscr{C}}, \psi_{\tilde{Y}_{\mathscr{C}}})$ is the upper bound of the chain $\mathscr{C}$. By the definition of $\tilde{Y}$, we see that there cannot be an element (ψ_Y, Y) in the chain $\mathscr{C}$, with $\tilde{Y} \subset Y$. Hence the only remaining thing to show is that for all $(\psi_X, X) \in \mathscr{C}$, we have $\psi_{\tilde{Y}_{\mathscr{C}}}|_X = \psi_X$. But this also follows from the definition of the map $\psi_{\tilde{Y}_{\mathscr{C}}}$.

Since $\mathscr{C}$ was taken to be an arbitrary chain in the collection $\mathcal{S}$, we get that every chain in $\mathcal{S}$ has an upper bound. Then by Zorn's lemma, the collection $\mathcal{S}$ has a maximal element (ψ, Y). We claim that in this maximal element, $Y = X$. If not, we can extend ψ to a space one dimension above Y like we did in the beginning contradicting the maximality of (ψ, Y). Hence the maximal element is (ψ, X). This by definition of the collection S, is the required extension for (ϕ, Z). □

Remark 3.0.1. Note that in the proof above, we only used the scaling property and triangle inequality of the norm, hence we can relax the condition for norm and replace it with a seminorm, without messing up the proof.

Theorem 3.0.2 (Hahn-Banach Extension Theorem for $\mathbb{C}$)**.** *Same statement of Theorem 3.0.1 with only the field changed to* $\mathbb{C}$.

Proof. Consider X as a normed linear space over $\mathbb{R}$. Let $f = \Re \circ \phi : Z \to \mathbb{R}$ and apply Theorem 3.0.1 on f to get a real linear functional $\tilde{f} : X \to \mathbb{R}$ with the required properties. Now we claim that $\tilde{\phi}$ defined as $\tilde{\phi}(x) = \tilde{f}(x) - i\tilde{f}(ix)$ is the required extension.

First we show $\tilde{\phi}_Z = \phi$. To see this first we notice that if ϕ can be written as $\phi(x) = f(x) + ig(x)$ where f, g are real valued functionals, then since $-\phi(x) = i\phi(ix) = if(ix) - g(ix)$. Hence $0 = \phi(x) - \phi(x) = (f(x) - g(ix)) + i(g(x) + f(ix))$. Since real part and imaginary part must be equal to 0, we get that $g(x) = -f(ix)$. Therefore we get $\phi(x) = f(x) - if(ix)$. Now we get $\tilde{\phi}|_Z = \phi$ immediately since $\tilde{f}|_Z = f$. To finish the proof, we also have to show that $\|\phi\| = \|\tilde{\phi}\|$. But this follows easily from Lemma 3.0.1 as $\|\phi\| = \|f\| = \|\tilde{f}\| = \|\tilde{\phi}\|$. □

Remark 3.0.2. It is quite natural to be confused about the well defineness of the expression $f(ix)$ when we are considering X as a normed linear space over $\mathbb{R}$ in the beginning of the proof. But note that since X initially was a complex normed linear space, viewing it as a space over $\mathbb{R}$ doesn't change or remove any elements from the space. Hence $ix \in X$ even though X is viewed as a real normed linear space.

Chapter 4

Definition 4.0.1. A sublinear map is a functoin $\rho : X \to \mathbb{R}$ with the properites

- $\rho(rx) = r\rho(x), \forall r \in \mathbb{R}$
- $\rho(x + y) \leq \rho(x) + \rho(y)$

Definition 4.0.2. Let X be a normed space. Then the dual of X, denoted by X^*, is the space $B(X, \mathbb{F})$

Lemma 4.0.1. *Let X be a normed space and $x \in X$. Then $\exists f \in X^*$ such that*

$$\|f\| = 1 \text{ and } f(x) = \|x\|$$

Proof. Let $Z = \text{Span}\{x\}$. Define $g : Z \to \mathbb{F}$ as $g(\alpha x) = \alpha\|x\|$. Then $\|g\| = 1$. By the Hahn Banach theorem, g has an extension f which preserve the norm and extends g to X. $\square$

Corollary 4.0.0.1. *Let X be a normed space and $x \in X$, then we have*

$$\|x\| = \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\}$$

Proof. If f is any linear functional with $\|f\| \leq 1$, then $|f(x)| \leq \|f\|\|x\| = \|x\|$. Hence $\|x\| \leq \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\}$. Now let f_x be the functional we get from Lemma 4.0.1. Then $f_x \in X^*$ and $\|f_x\| = 1$ with $f_x(x) = |f(x)| = \|x\|$. Hence we get that the inequality is actually an equality, and this proves the corollary. $\square$

Definition 4.0.3. For every $x \in X$, define a linear map $\hat{x} : X^* \to \mathbb{F}$ by $\hat{x}(f) = f(x)$

Theorem 4.0.1. *For every $x \in X$, $\hat{x} \in (X^*)^*$. The map $\rho : x \to \hat{x}$ is an isometric linear map.*

Proof. The fact that $\hat{x}$ is linear and bounded and the map $X \ni x \to \hat{x} \in X^{**}$ is linear follows from the definition of $f+g$ and λf.

By definition and Corollary 4.0.0.1

$$\begin{aligned} \|\hat{x}\| &= \sup\{|\hat{x}(f) \ : \ f \in X^*, \|f\| \leq 1\} \\ &= \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\} \\ &= \|f\| \end{aligned}$$

□

Definition 4.0.4. A normed space X is said to be reflexive if the map $\rho : X \to X^{**} := x \to \hat{x}$ is surjective. (This is a stronger condition than $X \equiv X^{**}$)

Theorem 4.0.2. *There are isometric isomorphisms between*

- $(\mathbf{c}_0)^*$ *and* ℓ^1
-
- $(\ell^1)^*$ *and* ℓ^∞

Proof.

- Let $(x_n) \in \ell^1$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \to \mathbb{F}$ defined as

$$\phi_{(x_n)} : (y_n) \to \sum_{n \in \mathbb{N}} x_n y_n$$

We claim that $\phi_{(x_n)}$ is a continuous linear functional. But first we should see that the sum is well defined. Since $y_n \to 0$, there is an $N \in \mathbb{N}$ such that $|y_n| < 1$ for all $n \geq N$. Since

$$\Big|\sum_{i=N}^{\infty} x_n y_n\Big| \leq \sum_{i=N}^{\infty} |x_n||y_n| \leq \|(x_n)\|_1$$

we see that the sum is well defined and the map makes sense. Also since $(y_n) + (z_n) = (y_n + z_n) \in \mathbf{c}_0$ whenever $(y_n), (z_n) \in \mathbf{c}_0$, we get that

$$\sum_{n \in \mathbb{N}} x_n (y_n + z_n) = \sum_{n \in \mathbb{N}} x_n y_n + \sum_{n \in \mathbb{N}} x_n z_n$$

which shows the linearity of the map $\phi_{(x_n)}$.

Now we show that $\|\phi_{(x_n)}\| = \|(x_n)\|_1$. We immediately see that for $(y_n) \in c_0$ with $\|(y_n)\|_{\sup} = \sup_{n \in \mathbb{N}} y_n = 1$,

$$|\phi_{(x_n)}((y_n))| = \Big|\sum_{n \in \mathbb{N}} x_n y_n\Big| \leq \|(y_n)\|_{\sup}\Big(\sum_{n \in \mathbb{N}} |x_n|\Big) \leq \|(x_n)\|_1$$

which gives $\|\phi_{(x_n)}\| \leq \|(x_n)\|_1$. Now let $\theta_j \in [0, 2\pi)$ such that $|x_j| = e^{i\theta_j} x_j$. Now consider the sequence $s_m \in \mathbf{c}_0$ defined as $s_m = \sum_{j=1}^{m} e^{i\theta_j} e_j$, where e_j is the sequence with jth entry 1 and the rest of the entries 0. Since $(x_n) \in \ell_1$, for all $\epsilon \geq 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{i=N_\epsilon+1}^{\infty} |x_i| < \epsilon$$

Then since

$$|\phi_{(x_n)}(s_{N_\epsilon})| = \Big|\sum_{n=1}^{N_\epsilon} e^{i\theta_j} x_n\Big| = \sum_{i=1}^{N_\epsilon} |x_n| = \|(x_n)\| - \sum_{i=N_\epsilon+1}^{\infty} |x_n| \geq \|(x_n)\| - \epsilon$$

and $\epsilon > 0$ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|$

Hence we see that the map $(x_n) \to \phi_{(x_n)}$ is an isometric linear map. Now for surjectivity, let $\phi \in \mathbf{c}_0^*$. We claim that the sequence $(y_n) = (\phi(e_n)) \in \ell^1$ and $\phi = \phi_{(y_n)}$. Let $\theta_j \in [0, 2\pi)$ such that $e^{i\theta_j} y_j = |y_j|$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{j=1}^{N} |\phi(e_j)| &= \sum_{j=1}^{N} e^{i\theta_j} \phi(e_j) \\
&= \phi\Big(\sum_{j=1}^{N} e^{i\theta_j} e_j\Big) \\
&\leq \|\phi\| \Big\| \sum_{j=1}^{N} e^{i\theta_j} e_j \Big\| \\
&= \|\phi\|
\end{aligned}$$

Since this is true for all $N \in \mathbb{N}$, taking the limits as $N \to \infty$, the inequality is preserved and we get that $(y_n) \in \ell^1$. Moreover $\phi = \phi_{(y_n)}$ follows from the definition of $\phi_{(x_n)}$. Hence we get that $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$.

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- The proof of this will be extremely similar to what we attempted before when we proved $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$. Let $(x_n) \in \ell^\infty$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \to \mathbb{C}$ defined as

$$\phi_{(x_n)} : (y_n) \to \sum_{n \in \mathbb{N}} x_n y_n$$

By a similar way as we did in the above equivalence we see that $\phi_{(x_n)}$ is linear. Moreover since

$$\Big|\sum_{n\in\mathbb{N}} x_n y_n\Big| \leq \|(x_n)\|_\infty \Big|\sum_{n\in\mathbb{N}} y_n\Big| = \|(x_n)\|_\infty \|(y_n)\|_1$$

we see that $\|\phi_{(x_n)}\| \leq \|(x_n)\|_\infty$. To get the reverse inequality, Let $\|(x_n)\|_\infty = M$, then for any $\epsilon > 0$, there exist some x_k in the sequence (x_n) such that $|x_k - M| < \epsilon$. Now consider the sequence $e_k \in \ell^1$ with kth entry 1 and all the rest of them 0. We get that

$$|\phi_{(x_n)}(e_k)| = |x_k| \geq \|(x_n)\|_\infty - \epsilon$$

Since ϵ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|_\infty$. Hence the map $(x_n) \to \phi_{(x_n)}$ is an isometry. To show that it is indeed a bijection, assume $\phi \in (\ell^1)^*$, then consider the sequence $y_n = \phi(e_n)$. Since ϕ is continuous, it is bounded above by $\|\phi\|$ and we get that $y_n \leq \|\phi\|$. Therefore $(y_n) \in \ell^\infty$. Moreover we can verify like above that $\phi = \phi_{(y_n)}$ from the definition of $\phi_{(y_n)}$. Hence we get $(\ell^1)^* \ \cong^{\text{iso}} \ell^\infty$.

□

Theorem 4.0.3. *Let $1 < p < \infty$, and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$*

Proof. Let $(a_n) \in \ell^p, (b_n) \in \ell^q$, then $\sum_{n\in\mathbb{N}} a_n \bar{b_n}$ is the map to check for isometric isomorphism. Use Holder's inequality as needed. verify □

Theorem 4.0.4. *There exists $\phi \in (\ell^\infty)^*$ satisfying the following*

1 $\forall (a_n) \in \ell^\infty$ with $a_n \geq 0$ for all $n \in \mathbb{N}$, $\phi((a_n)) \geq 0$

2 If (a_n) is convergent, then $\phi((a_n)) = \lim_{n\to\infty} a_n$

3 If $(a_n) \in \ell^\infty$ and $b_n = a_{n+1}$, then $\phi((b_n)) = \phi((a_n))$

Moreover such ϕ is called a Banach limit.

Proof. We'll prove this later. □

Corollary 4.0.4.1. *ℓ^1 is not reflexive*

Proof. Let $\phi \in (\ell^\infty)^*$ be a Banach limit. FTOC, assume $\exists f = (\alpha_n) \in \ell^1$ such that

$$\phi((a_n)) = \sum_{i=1}^{\infty} a_n \overline{\alpha_n}$$

Then for all $m \in \mathbb{N}$, $\overline{\alpha_m} = \phi(\delta_m) = 0$, where $\delta_m = (0, 0, \ldots, 1, 0, 0, \ldots)$. But this contradicts since we assumed $\phi \neq 0$ by the Hahn Banach rextension from c_0 □

Lemma 4.0.2. *Let $\psi \in (\ell^\infty)^*$. then the following are equivalent.*

1 $\|\psi\| = \psi((1, 1, 1, \ldots))$

2 If $(a_n) \in \ell^\infty$ *with* $a_n \geq 0, \forall n \in \mathbb{N}$. *Then* $\psi((a_n)) \geq 0$

Proof. ($1 \implies 2$) FTSOC assume $\exists (a_n) \in \ell^\infty$, $\psi((a_n)) < 0$. WLOG, assume $|a_n| \leq 1$ for all $n \in \mathbb{N}$. let $b_n = 1 - a_n$. Then $0 \leq b_n \leq 1$ and

$$\psi((b_n)) > \psi((1, 1, 1, \ldots)) - \psi((a_n)) > \psi((1, 1, 1, \ldots))$$

So

$$\|\psi\| \geq |\psi((b_n))| \geq \psi((1, 1, \ldots))$$

($2 \implies 1$) Let $(a_n) \in \ell^\infty$ with $|a_n| \leq 1$, then $0 \leq 1 - a_n$. So $\psi((1 - a_n)) \geq 0$ and therefore $\psi((1, 1, 1, \ldots)) \geq \psi((a_n))$. Similarly $\psi((-a_n)) \leq \psi((1, 1, 1, \ldots))$ which gives $|\psi((a_n))| \leq \psi((1, 1, 1, \ldots))$ $\square$

Chapter 5

Theorem 5.0.1. *There exists $\phi \in (\ell^\infty)^*$ satisfying the following*

1 $\forall (a_n) \in \ell^\infty$ *with* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, $\phi((a_n)) \geq 0$

2 *If* (a_n) *is convergent, then* $\phi((a_n)) = \lim_{n\to\infty} a_n$

3 *If* $(a_n) \in \ell^\infty$ *and* $b_n = a_{n+1}$, *then* $\phi((b_n)) = \phi((a_n))$

Moreover such ϕ is called a Banach limit.

Proof. Let $S : \ell^\infty(\mathbb{R} \to \ell^\infty(\mathbb{R})$ and $T = I - S$ where I is the identity map. Also let $V = \text{Range}(T) + c$ where $c \in \mathbf{c}$, the set of convergent sequences.

Define $\phi : V \to \mathbb{R}$, $\phi(a_n - a_{n+1} + x_n) = \lim_{n\to\infty} x_n$.

- Claim 1: ϕ is well defined
- Claim 2: $\|\phi\| = 1$

Assuming the claims, by Hahn Banach, ϕ extends to $\tilde{\phi} \in \ell^\infty(\mathbb{R})$ with $\|\tilde{\phi}\| = 1$. Then by the last lemma we get $\tilde{\phi}((y_n)) \geq 0$ for all $(y_n) \in ell^\infty(\mathbb{R})$ with $y_n \geq 0$ □

Proof of Claim 1. Suppose that $(a_n) \in \ell^\infty$ is a sequence such that $a_n - a_{n+1}$ converges, say $a_n \to a_{n+1} \to \alpha$. If $\alpha > 0$, then $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n - a_{n+1} > \frac{\alpha}{2}$. So $a_N > \frac{\alpha}{2} + a_{N+1} > \ldots > k\frac{\alpha}{2} + a_{N+k}$. So for all $k \in \mathbb{N}$, $a_N - a_{N+k} = k\frac{\alpha}{2} \to \infty$ contradicting our assumption that $a_n - a_{n+1}$ converges.

Now assume that $(a_n), (b_n) \in \ell^\infty(\mathbb{R})$ with $(x_n), (y_n) \in \mathbf{c}$ such that $a_n - a_{n+1} + x_n = b_n - b_{n+1} + y_n$. Then $(a_n - b_n) - (a_{n+1} - b_{n+1}) = y_n - x_n$. Then since RHS is a convergent limit, LHS must be convergent, which we get from above that it must converge to zero. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ □

Proof Claim 2. verify □

To complete the proof, define $\Psi : \ell^\infty \to \mathbb{C}$ by $\Psi((a_n + ib_n)) = \tilde{\phi}(a_n) + i\tilde{\phi}(b_n)$ verify

5.1 Quotient Spaces

Definition 5.1.1. Let X be a normed space and $Y \leqslant X$ be a closed subspace. For every $x \in X$, define

$$\|x + Y\| = \inf\{\|x + y\| \ : \ y \in Y\}$$

Lemma 5.1.1. *This defines as norm on $\frac{X}{Y}$. If X is complete, then $\frac{X}{Y}$ is complete.*

Proof. Obviously, $\|x+Y\| \geq 0$ for all $x \in X$, and $\|x+z+Y\| \leq \|x+Y\|+\|y+Y\|$. Similarly, we can also show homogeneity.

Now assume $x \in X$ is such that $\|x+Y\| = 0$. Then there is a sequence $(y_n) \in Y$ such that $\|x - y_n\| \to 0$, that is $y_n \to x$. Since Y is closed, we get $x \in Y$.

To show the second part of the lemma, consider the sequence $(x_n + Y) \in X/Y$ such that $\sum_{n\in\mathbb{N}} \|x_n - Y\| < \infty$. For each $n \in \mathbb{N}$, choose $y_n \in Y$ such that

$$\|x_n + y_n\| \leq \|x_n + Y\| + \frac{1}{2^n}$$

Then $\sum_{n\in\mathbb{N}} \|x_n + y_n\| \leq \infty$. Since X is complete, the sequence $\sum_{n\in\mathbb{N}} x_n + y_n$ converges to say $z \in X$. Then

$$\begin{aligned}
\|(z + Y) - \sum_{n=k}^{n}(x_n + Y)\| &= \|\Big(z - \sum_{n=k}^{n} x_n\Big) + Y\| \\
&= \|\Big(z - \sum_{i=1}^{k}(x_n + y_n)\Big) + Y\| \\
&\leq \|\Big(z - \sum_{i=1}^{k}(x_n + y_n)\Big)\|
\end{aligned}$$

which converges to 0 as $k \to \infty$ ☐

Lemma 5.1.2. *The canocial map, $q : X \to \frac{X}{Y}$ is a continuous open map. A subset $E \subset X/Y$ is open iff $q^{-1}(E) \subset X$ is open.*

Proof. Since $\|x+Y\| \leq \|x\|$, for all $x \in X$, we see that the map q is a contraction. Thus for all open $E \subset X/Y$, we get $q^{-1}(E)$ is open.

Conersely, assume that $A \subset X$ is open. Let $x \in A$ and $r > 0$ such that $B_r(a) \subset A$. Let $z \in X$ such that $\|q(a) - q(z)\| < r$. So, $\|(a - z) + Y\| < r$. Then $\exists y \in Y$ such that $\|a - z - y\| < r$. So $z + y \in B_r(a), q(z + y) = q(z) \in q(B_r(a))$. So $B_r(q(a)) \subset q(B_r(a)) \subset q(A)$. Thus $q(A)$ is open. ☐

Chapter 6

Theorem 6.0.1 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a surjective bounded linear map. Then T is an open map i.e $T(E)$ is open in Y i.e if $E \subset X$, then $T(E)$ is open in Y.*

Steps of proof. See Prof. Blecher's Notes on Functional Analysis

Use Baire category theorem to show that $\overline{T(B_r(0))}$ has a non-empty interior.

- Then use linearity of T to show that $0 \in \overline{T(B_{2r}(0))}$.

$\square$

Proof. Since Y is complete, by the Baire catergory theorem it is of the second category. Let $r \geq 0$, then

$$\begin{aligned}
Y &= T(X) \\
&= T\Big(\bigcup_{n=1}^{\infty} B_{nr}(0)\Big) \\
&= T\Big(\bigcup_{n=1}^{\infty} nB_r(0)\Big) \\
&= \bigcup_{n=1}^{\infty} n\overline{T(B_r(0))}
\end{aligned}$$

Then by BCT, there exist some $n \in \mathbb{N}$ such that $\text{int}(\overline{T(B_r(0))}) \neq \emptyset$. Let $y_0 \in \text{int}(\overline{T(B_r(0))})$. So there exists $\epsilon > 0$, such that $B_\epsilon(y_0) \subset \overline{T(B_r(0))}$. Let $w \in B_Y(0, \epsilon)$. Then $y_0 + w \in B_Y(y_0, \epsilon)$, and $\exists(x_n) \subset B_X(0, r)$ such that $T(x_n) \to y_0 + w$. Also $\exists(z_n) \in B_X(0, r)$ such that $T(z_n) \to y_0$. Then $T(x_n - z_n) \to w$, so $w \in \overline{T(B_X(0, 2r))}$. Since w was an arbitrary element in $B_Y(0, \epsilon)$ we see that $B_Y(0, \epsilon) \subset \overline{T(B_X(0, 2r))}$. So $0 \in \text{int}(\overline{T(B_X(0, s))})$ for $s > 0$.

Now fix $t > 0$ and let $y_0 \in \overline{T(B_X(0,t))}$. By the above there exists $\epsilon > 0$ such that $B_Y(0,\epsilon) \subset \overline{T(B_X(0,\frac{t}{2}))}$. Then $(y_0 + B_Y(0,\epsilon)) \cap T(B_X(0,\frac{t}{2})) \neq \emptyset$. So $\exists x \in B_t(0)$ such that $T(x_1) = y_0 - y_1$ where $y_1 \in B_Y(0,\epsilon) \subset \overline{T(B_X(0,\frac{t}{2}))}$.

Similarly $\exists y_2 \in \overline{T(B_{\frac{t}{4}}(0))}$ and $x_2 \in B_{\frac{t}{2}}(0)$ such that $T(x_2) = y_1 - y_2$. Thus inductively we can choose $y_{n+1} \in \overline{T(B_{\frac{t}{2^{n+1}}}(0))}$ and $x_{n+1} \in B_{\frac{t}{2^n}}(0)$ such that $T(x_{n+1}) = y_n - y_{n+1}$. Now since we constructed nicely, $\sum_{i=0}^{\infty} x_n$ converge. verify. Moreover for all $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} T(x_n) = y_0 - y_N$$

Also notice that $y_n \to 0$. Hence $y_0 = \lim_{N\to\infty}(y_0 - y_N) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n T(x_n) = \lim_{N\to\infty} T(\sum_{n=1}^{N} x_n) = T(x) \in T(B_{2t}(0))$. So

$$\overline{T(B_t(0))} \subset T(B_{2t}(0))$$

Now to complete the proof, let E be an open subset of X. Let $x_0 \in E$ be such that $y_0 = T(x_0)$. Let $\epsilon > 0$ be such that $x_0 + B_\epsilon(0) = B_\epsilon(x_0) \subset E$. So $y_0 + T(B_\epsilon(0)) = T(B_\epsilon(x_0)) \subset T(E)$. By the above $\exists \delta > 0$ such that $B_\delta(0) \subset T(B_\epsilon(0))$ □

Find examples where this fails if we slack the conditions

Corollary 6.0.1.1. *Let X and Y be Banach spaces and $T : X \to Y$ be a bijective bounded linear map. Then $T^{-1} : Y \to X$ is bounded.*

Proof. verify □

Theorem 6.0.2 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a linear map. Then T is bounded if and only if graph of T, defined as $g(T) = \{(x, T(x)) \ : \ x \in X\}$ is closed in the product topology of $X \times Y$.*

Proof. Define the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ on $X \times Y$. Then $X \times Y$ with this norm is a Banach space. verify.

Assume T is continuous. Then if $(x_n, T(x_n))$ is Cauchy in $g(T)$ then x_n and $T(x_n)$ must be cauchy in X and Y respectively. By the completeness of the spaces X and Y, we get $x_n \to x \in X$ and $T(x_n) \to y \in Y$. Moreover by continuity of T, we get $T(x_n) \to x$. Since the Banach space is Hausdorff, we get $y = x$ and that $(x, T(x)) \in g(T)$ making it closed.

Conversely, define $S : X \to g(T)$ as $S(x) = (x, T(x))$. S is linear and bijective. Assume $g(T)$ is closed, hence a Banach space. Observe that $S^{-1} : g(T) \to X$ is bounded(contractive). By the open mapping theorem, S is bounded. Assume $x_n \to z$. So $S(x_n) \to S(z)$. Then $(x_n, T(x_n)) \to (z, T(z))$, which gives $T(x_n) \to T(z)$. □

Chapter 7

Example 7.0.1. Let X be a vector space and let $f : X \to \mathbb{C}$ be a linear map. Define $\phi : X \to R^+ := \phi(x) = |f(x)|$. Then ϕ is a seminorm.

Remark 7.0.1. Let X be a TVS and $A \subset X^*$. We denote by $\sigma(X, A)$, the topology on X defined by A. (initial topology). Recall that $\sigma(X, A)$ is Hausdorff if and only if A separate points of X.

$\sigma(X, X^*)$ is called the weak topology on X.

Also recall that $X \hookrightarrow X^{**}$ by the evaluation maps. Hence we can view X as a subset X^{**}. And with this identification, we call $\sigma(X^*, X)$ the weak $*$ topology on X^*

Definition 7.0.1. Let S be any set. Let I be a directed set. A net in S indexed by Λ is a function $f : \Lambda \to S$. We denote the net by $(x_\lambda)_{\lambda\in\Lambda}$.

In addition if S is a topological space, we say a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in S$ if for all open set U in S with $x \in U$, there exists an $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$ we have $x_\lambda \in U$.

See how nets generalize the sequences to topological spaces from the metric space. For example, consider the definition of closedness in a metric space and a topological space. Find what exact property of the metric space makes it enough to be indexed by a countable totally ordered set for openess.

Remark 7.0.2. By definition, a basis of open neighborhoods of a point $x_0 \in X$ in $\sigma(X, A)$ is given

$$\bigcup_{\substack{p_1,p_2,\ldots,p_n\in A\\ \epsilon_1,\epsilon_2,\ldots,\epsilon_n>0}} \bigcap_{i=1}^{n} \{z \in Z | p_k(z - x_0) < \epsilon_k\}$$

So the basis in a weak topology is

$$\bigcup_{f_1,f_2,\ldots,f_n\in X^*\ \epsilon_1,\epsilon_2,\ldots,\epsilon_n>0} (x_0) = \{z \in Z \Big| |f_k(z - x_0)| < \epsilon_k, \text{ for all } k = 1, 2, \ldots n\}$$

Example 7.0.2. Let S be a topologicla space, $E \subset S$, $x_0 \in E'$. Then there is a net $(x_\alpha) \subset E$ such that $x_\alpha \to x_0$.

Proof. □

Example 7.0.3. Let $X = \ell^1$. Then $X^* = \ell^\infty$. Then the weak * topology on ℓ^∞ is given by the pointwise convergence of a net $(f_\alpha) \subset \ell^\infty$ converges to $f \in \ell^\infty$ if and only if $f_\alpha(n) \to f(n)$ for all $n \in \mathbb{N}$

Chapter 8

Theorem 8.0.1 (Tychonoff). *The product of compact sets is compact.*

Corollary 8.0.1.1. *Let X be a compact Hausdorff space. Then for any set S, the set $\{\phi : S \to X\} = X^S$ is compact wrt to pointwise convergence.*

Theorem 8.0.2 (Banach-Alaoglu Theorem). *Let X be a normed space. Then the closed unit ball $\overline{B_{X^*}(0,1)} = \{f \in X^* \ : \ \|f\| \leq 1\} = E$ is weak * compact.*

Proof. Let $\bar{B}$ be the closed unit ball of X. Then by Tychonoff theorem, $\bar{D}^{\bar{B}}$ is compact. Define $\phi : E \to \bar{D}^{\bar{B}}$ as $\phi(f)(x) := f(x)$. Observe that ϕ is injective. Also observe that ϕ is continuous (weak * in LHS, and pointwise in RHS).

Next we show that the image of ϕ a closed subset of $\bar{D}^{\bar{B}}$, hence compact. Let f_i be a net in E such that $\phi(f_i) \to \psi$ pointwise for some $\psi \in \bar{D}^{\bar{B}}$.

Define $g : X \to C$ as $g(x) = \alpha\psi(\frac{x}{\alpha})$ where $\|x\| \leq \alpha$. For this to be well defined we must have $\|x\|\psi(\frac{x}{\|x\|}) = \alpha\psi(\frac{x}{\alpha})$ for any $\alpha > \|x\|$. But we get this since ψ is a pointwise limit of linear functionals. Moreover we get that g is linear for the same reason. Thus $\psi = \phi(g)$ and so $\phi(E)$ is closed.

It only remains to show that the inverse of ϕ is continuous. verify. $\square$

Remark 8.0.1. The closed unit ball of a normed space Y is compact w.r.t the norm topology if and only if Y is finite dimensional.

Proof. verify $\square$

Theorem 8.0.3. *Let X be a normed space. Then E is weak * metrizable iff X is separable.*

Proof. Assume X is separable. Let $\{x_n \ : \ n \in \mathbb{N}\}$ be a dense subset of X. For every $f, g \in E$, define $d(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n}|f(\frac{x_n}{\|x_n\|} - g(\frac{x_n}{\|x_n\|})|$. Check that d is a metric.

Now asssume $f_i \to f$ weakly in E. Then $f_i(x_n) \to f(x_n)$ for all $n \in \mathbb{N}$ and $d(f_i, f) \to 0$ (verify).

Assume E is metrizable. Then $\exists\{U_n : n \in \mathbb{N}\}$ of weak * open neighborhoods of 0 such that $\cap_{n=1}^{\infty} U_n = \{0\}$. So, for each $n \in \mathbb{N}$, there exists a finite set $A_n \in X$ and $\epsilon > 0$ such that the (subbasis sets) $\{f \in E : |f| \leq \epsilon \forall x \in A_n \subset U_n$. Now let $A = \cup_{n=1}^{\infty} A_n$. Let $\phi \in E$ such that $\phi(x) = 0$ for all $x \in A$. $\square$

Chapter 9

Definition 9.0.1. Let X and Y be normed spaces and $T \in B(X, Y)$. The adjoint of T, denoted by $T^* \in B(Y^*, X^*)$, is the map $T^* : f \to f \circ T$

Proposition 9.0.1. $\|T\| = \|T^*\|$

Proof. $|T^*(f)| \leq \|f \circ T\| \leq \|T\|\|f\|$ implies $\|T^*\| \leq \|T\|$

$$\begin{aligned}\|T^*\| &= \sup\{\|T^*(\phi)\| \;:\; \phi \in Y^*, \|\phi\| \leq 1\} \\ &= \sup\{\|\phi(T(x))\| \;:\; \phi \in Y^*, x \in X, \|\phi\| \leq 1, \|x\| \leq 1\} \qquad = \|T\|\end{aligned}$$

□

Lemma 9.0.1. *For any $T \in B(X, Y)$, $T^* : Y^* \to X^*$ is weak * continuous (in both spaces)*

Proof. Let $\phi_i \to \phi$ weakly in Y^*. Then by definition for all $y \in Y$, $\phi_i(y) \to \phi(y)$. Then for $x \in X$, $T^*(\phi_i)(x) = \phi_i(T(x)) \to \phi(T(x)) = (T^*(\phi))(x)$ which shows the continuity of T^*. □

Lemma 9.0.2. *For any normed space X, $i_X(X)$ is weak * dense in X^{**}.*

Proof. verify □

Example 9.0.1. Is i_{X^*} weak * - weak * continuous.

Lemma 9.0.3. *Let X be a normed space and $x_1, x_2, \ldots, x_n \in X$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \geq 0$. Then the set*

$$\bigcup_{x_1, x_2, \ldots, x_n, \epsilon_1, \epsilon_2, \ldots, \epsilon_n} (\phi)$$

is convex. Moreover any topological vector spaces with the topology induced by a family of seminorms is locally convex. Refer back to the proofs of Arveson.

Proof. verify □

Definition 9.0.2. Let X be a vector space and $E \subset X$ be a convex subset. An element $a \in E$ is called an extreme point of E if whenever $x, y \in E$, $0 \leq t \leq 1$ with $a = tx + (1-t)y$, then $x = y = a$.

Example 9.0.2. Let $\bar{D} = \{\alpha \in \mathbb{C} \ : \ |\alpha| \leq 1\}$. Then $\bar{D}$ is convex with $\mathrm{Ext}(\bar{D}) = S^1$

Theorem 9.0.1 (Krein-Milman Theorem)**.** *Let X be a locally convex space, and let K be a compact convex subset of X. Then the $Ext(K) \neq \emptyset$ and indeed $K = \overline{co}(Ext(K))$*

Definition 9.0.3. Let V be a vector space and $S \subset V$. The convex hull of S is defined as

$$\mathrm{co}(S) = \left\{ \sum_{i=1}^{n} t_i x_i \,\middle|\, 0 \leq t \leq 1, \sum t_i = 1, x_i \in S \right\}$$

Chapter 10

Theorem 10.0.1 (Krein-Milman Theorem)**.** *Let X be a locally convex space, and let K be a compact convex subset of X. Then the Ext$(K) \neq \emptyset$ and indeed $K = \overline{co}(Ext(K))$*

Proof. We first prove that the $\mathrm{Ext}(K) \neq \emptyset$. Observe that a point $x_0 \in \mathrm{Ext}(K)$ if and only if $K \setminus \{x_0\}$ is convex. Also note that $K \setminus \{x_0\}$ is a relatively open subset of K since $\{x_0\}$ is closed and $K \setminus \{x_0\} = \{x_0\}^c$ relative to K.

Now let $\mathcal{A}$ be the collection of all relatively open convex proper subsets of K. Note that $\emptyset \in \mathcal{A}$, therefore $\mathcal{A}$ is nonempty. Equip $\mathcal{A}$ with the partial order defined by the set inclusion. Let $\mathscr{C}$ be a chain in $\mathcal{A}$ and $F_{\mathscr{C}} = \cup_{C \in \mathscr{C}} C$. $F_{\mathscr{C}}$ is relatively open being the union of relatively open subsets of K. To see that $F_{\mathscr{C}}$ is convex, let $x, y \in F_{\mathscr{C}}$. Then since $\mathscr{C}$ is a chain, there exist a $C \in \mathscr{C}$ such that $x, y \in \mathscr{C}$. Then clearly $tx + (1-t)y \in C \subset F_{\mathscr{C}}$.

We claim that $F_{\mathscr{C}}$ is a proper subset of K. For the sake of contradiction, assume $F_{\mathscr{C}} = K$. Since K is compact and C is open in K for all $C \in \mathscr{C}$, there are finitely many $C_1 \subset C_2 \subset \ldots \subset C_k \in \mathscr{C}$ which cover K (i.e $K = \cup_{n=1}^{k} C_n$). Hence we get $C_k = K$, which is absurd since C_k must be a proper subset of K. Hence $F_{\mathscr{C}} \in \mathcal{A}$ and thus every chain must have an upper bound in $\mathcal{A}$. Now by Zorn's lemma, $\mathcal{A}$ has a maximal element K_0.

Since K is a connected space (being a TVS), we know that the only clopen subsets are $\emptyset$ and K. Since we know that K_0 is open being in $\mathcal{A}$, we see that $K_0 \neq K$ and $K_0 \neq \emptyset$. Therefore $\overline{K_0} \neq K_0$. Let $x_0 \in \overline{K_0} \setminus K_0$, $y_o \in K_0$ and $0 < t < 1$. Define $\varphi_{t,y_0} : K \to K$ such that $\varphi_{t,y_0}(z) = ty_0 + (1-t)z$. Then φ_{t,y_0} is (Lipschitz) continuous relative to K and thus $\varphi_{t,y_0}^{-1}(K_0)$ is open in K. Also $\varphi_{t,y_0}^{-1}(K_0)$ is convex.

We claim, $x_0 \in \varphi_{t,y_0}^{-1}(K_0)$. Let U be a convex neighborhood of $0 \in X$ containing $-x$ for all $x \in U$ (just take $-U$ and intersect with U) such that $(y_o + U) \cap K \subset K_0$. Let $w = \varphi_{t,y_0}(x_0)$. Since $x_0 \in \overline{K_0}$, there exists $x_1 \in K_0$ such that $(x_0 + (1-t)U) \cap K_0 \neq \emptyset$. Then $(\frac{t}{1-t}U) \cap (K_0 - x_0) \neq \emptyset$. Choose z in the above set. Then

$$y_0 - (\frac{1-t}{t} z) \in y_0 + E \subset K_0$$

and $x_0 + z \in K_0$. Since K_0 is convex,

$$t(y_0 - \frac{(1-t)}{t}z) + (1-t)(x_0 + z) = \phi_{t,y_0}(x_0) \in K_0$$

Then by the maximality of K_0, we get that $\varphi_{t,y_0}^{-1}(K_0) = K$.

Now we claim that $K = K_0 \cup \{x_0\}$. For the sake of contradiction assume $\exists p \in K$ such that $p \notin K_0 \cup \{x_0\}$. Since the space is Hausdorff and compact, x_0 has an open convex neighborhood E in X such that $p \notin E$. Let $E' = E \cap K$, $a \in K_0, b \in E'$ and $0 < r < 1$. Then since $\varphi_{t,y_0}(K) = K_0$ for all $t, y_0 \in K_0$, we get $\varphi_{r,b}(b) = ra + (1-r)b \in K_0$. So $K_0 \cup E'$ is convex (Sine we know that K_0, E' are compact, we only need to worry about $rx + (1-r)y$ for $x \in K_0, y \in E'$, but $\varphi_{r,b}$ takes care of that). $K_0 \cup E'$ is also open in K. Hence by maximality, we get $K_0 \cup E' = K$. But this is a contradiction since $p \notin K_0 \cup E'$. Thus by the equivalence at the beginning, we see that $x_0 \in \text{Ext}(K)$.

Next we prove $K = \overline{co}(\text{Ext}(K))$. Let $P = \overline{co}(\text{Ext}(K))$ and for the sake of contradiction assume $P \neq K$. Let $x_0 \in K \setminus P$. Let E be an open convex neighborhood of $0 \in X$ such that $(x_0 + E') \cap P = \emptyset$ for $E' = E \cap K$. Define $\phi : X \to \mathbb{R}$ such that

$$\phi(x) = \inf\{0 \leq t \big| x \in tE\}$$

Observe that $E = \{x \in X \ \phi(x) < 1\}$. For every $r \geq 0$ and $x \in X$, $\phi(rx) = r\phi(x)$, and for all $x, y \in X$, $\phi(x+y) \leq \phi(x) + \phi(y)$. Define $f : \mathbb{R}\{x_0\} \to \mathbb{R}$, $f(rx_0) = r\phi(x_0)$, for all $r \in \mathbb{R}$. For every $r \geq 0$, we have $f(rx_0) = r\phi(x_0) = \phi(rx_0)$. For $r < 0$, we have $f(rx_0) = r\phi(x_0) = -\phi(-rx_0) \leq 0 \leq \phi(rx_0)$. So $f \leq \phi$ on $\mathbb{R}x_0$. Then by HBT theorem, there is an extension $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(x) \leq \phi(x)$ for all $x \in X$.

For Fun we'll use something else now. Let P be as before. $\square$

Chapter 11

Lemma 11.0.1. *Let K_1, K_2 be compact convex subsets of a locally compact TVS X. Then*

$$\overline{co}(K_1 \cup K_2) = (co)(K_1 \cup K_2)$$

Proof. verify. We'll show that $\mathrm{co}(K_1 \cup K_2)$ is compact and hence closed. Let $x = \alpha_1 a_1, \alpha_2 a_2, \ldots, \alpha_n a_n + \beta_1 b_1, \beta_2 b_2, \ldots, \beta_m b_n \in \mathrm{co}(K_1 \cup K_2)$, where $\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{m} \beta_i = 1$. Then

$$x = (\sum_{i=1}^{n} \alpha_i) \underbrace{\left(\sum_{i=1}^{n} \Big(\frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} \Big) a_i \right)}_{\in \, K_1} + (\sum_{i=1}^{m} \beta_i) \underbrace{\left(\sum_{i=1}^{m} \Big(\frac{\beta_i}{\sum_{i=1}^{m} \beta_i} \Big) b_i \right)}_{\in \, K_2}$$

Hence every element $x \in \mathrm{co}(K_1 \cup K_2)$, can be written as $x = ta + (1-t)b$ where $a \in K_1, b \in K_2$.

Now let $x_\lambda = t_\lambda a_\lambda + (1-t_\lambda) b_\lambda$ be a net in $\mathrm{co}(K_1 \cup K_2)$, for $\lambda \in \Lambda$, $a_\lambda \in K_1, b_\lambda \in K_2$. Since (a_λ) is a net in the compact set K_1, there is a subnet a_σ for $\sigma \in \Sigma \subseteq \Lambda$, such that $a_\sigma \to a \in K_1$. By similar reasoning b_σ has a convergent subnet b_π for $\pi \in \Pi \subseteq \Sigma$, such that $b_\pi \to b \in K_2$. Again t_π is a net in the compact space $[0,1]$, hence is has a convergent subnet t_ω for $\omega \in \Omega \subseteq \Pi$ such that $t_\omega \to t$ in $[0,1]$.

Now consider the subnet $x_\omega = t_\omega a_\omega + (1-t_\omega)\beta_\omega$ of x_λ. Since $\Omega \subseteq \Pi \subseteq \Sigma$, $t_\omega \to t, \beta_\omega \to b$ and $a_\omega \to a$. Therefore by the continuity of the scalar product and addition in the TVS, we get $x_\omega \to t\alpha + (1-t)\beta \in \mathrm{co}(K_1 \cup K_2)$. Hence we get $\mathrm{co}(K_1 \cup K_2)$ is compact. $\square$

Theorem 11.0.1 (Inverse Krein-Milman)**.** *Let K be a compact convex subset of a locally convex topological vector space X. Let $A \subset K$ be a closed subset of K. If $K = \overline{co}(A)$, then Ext$(K) \subset A$.*

Note that Prob$[0,1]$, the collection of probability measures identified as a subspace of a $C([0,1])^*$ is convex, weak * compact with $\mathrm{Ext}(K) = \{\delta_x \ : \ x \in [0,1]\}$

Proof. FSTOC, assume $\exists x_0 \in \text{Ext}(K)$, $x_0 \notin A$. Since A is compact, $\exists y_1, y_2, \ldots, y_n \in A$ and an open convex neighborhood B of 0 such that

$$A \subset \cup_{i=1}^{n}(y_i + B)$$

and $x_0 \notin y_i + \overline{B}$ for all $i = 1, 2, \ldots, n$. Let $B_i = (y_i + \overline{B}) \cap K$. Then B_i is a compact convex subset of K for each i. Hence by the previous lemma, we get

$$\text{co}(B_1 \cup B_2 \cup \ldots \cup B_n) = \overline{\text{co}}(B_1 \cup B_2 \cup \ldots \cup B_n) \supset \overline{\text{co}}(A) = K$$

Thus $\exists b_i \in B_i$ and $0 \leq t_i \leq 1$, $\sum_{i=1}^{n} t_i = 1$ such that

$$x_0 = t_n b_1 + t_n b_2 + \ldots + t_n b_n$$

Since $x_0 \in \text{Ext}(K)$, this forces $x_0 = b_j$ for some $1 \leq j \leq n$. This contradicts the assumption that $x_0 \notin A$. $\square$