MATH 7320, Functional Analysis I

HOMEWORK 1

Notation: For each $1 \le p < \infty$ we denote by ℓ^p the set of all sequences $(c_n)_{n=1}^{\infty}$ of complex numbers such that $\sum_{n=1}^{\infty} |c_n|^p < \infty$. We denote by ℓ^∞ the set of all bounded sequences of complex numbers, and by $\mathbf{c}_0$ the of all convergent sequences of complex numbers. We equip ℓ^∞ and $\mathbf{c}_0$ with the sup-norm $\|\cdot\|_\infty$.

1. Read/recall the Minkowski's inequality from your previous courses or from the literature. It implies that the formula $\|(c_n)\|_p := \left(\sum_{n=1}^{\infty} |c_n|^p\right)^{\frac{1}{p}}$ defines a norm on ℓ_p.
2. Determine, and prove which of the following normed spaces is a Banach space (i.e. a complete normed space): $(\ell^1, \|\cdot\|_1)$, $(\ell^\infty, \|\cdot\|_\infty)$, $(\mathbf{c}_0, \|\cdot\|_\infty)$, $(\mathbf{c}_{00}, \|\cdot\|_\infty)$.
3. Define the linear map $f : \mathbf{c}_0 \to \mathbb{C}$ by
$$f\big((c_n)\big) = \sum_{n=1}^{\infty} \frac{c_n}{2^n}$$
for all $(c_n) \in \mathbf{c}_0$. Show that f is continuous, and find its norm.
4. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces.
   a) Show that $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a normed space.
   b) Show that if $\mathcal{Y}$ is a Banach space, then $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a Banach space.
   c) Assume that $\mathcal{B}(\mathcal{X}, \mathbb{C}) \neq \{0\}$ (this follows from the Hahn-Banach Theorem as we discussed in the last lecture). Show that if $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a Banach space then $\mathcal{Y}$ is a Banach space.
5. Prove that a normed space $\mathcal{X}$ is a Banach space if and only if whenever (x_n) is a sequence in $\mathcal{X}$ such that $\sum \|x_n\| < \infty$, then $\sum x_n$ is convergent in $\mathcal{X}$.

   (**Definition:** A normed space is said to be separable if it has a countable dense subset.) Prove $\mathbf{c}_0$ and ℓ^1 are separable, but l^∞ is not.
6. Prove that every Banach space has a Hamel basis (i.e. a basis in the sense of linear algebra). (Hint: use Zorn's lemma).
7. Show that if $\mathcal{X}$ is an infinite dimensional Banach space, then any Hamel basis of $\mathcal{X}$ is uncountable. (Hint: use the Baire category theorem).
8. Let $\mathcal{X}$ is an inner product space. Show that the formula $\|x\| := \sqrt{\langle x, x, \rangle}$ defines a norm on $\mathcal{X}$.

(**Definition:** A complete inner product space is called a Hilbert space.
Given an inner product space $\mathcal{X}$, a subset $\mathcal{E} \subset \mathcal{X}$ is said to be an orthonormal basis for $\mathcal{X}$, if $\|e\| = 1$ for all $e \in \mathcal{E}$, $\langle e, e' \rangle = 0$ for all $e \neq e' \in \mathcal{E}$, and for every $x \in X$ there is a sequence (e_n) in $\mathcal{E}$ and a sequence (c_n) in $\mathbb{F}$ such that $x = \sum c_n e_n$.

9. Show that if $\mathcal{H}$ is an infinite dimensional Hilbert space, then an orthonormal basis cannot be a Hamel basis.