Modern Algebra 1 - MATH 6302

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Chapter 1

1.1 Internal Direct products

Suppose G is a group and $H, K \leqslant G$. Define

$$HK = \{hk \ : \ h \in H, k \in K\}$$

Theorem 1.1.1. *Suppose $|H|, |K| < \infty$, then $|HK| = \frac{|H||K|}{|H \cap K|}$.*

Proof. We write $HK = \cup_{h \in H} hK$. Then $h_1K = h_2K$ if and only if $h_2^{-1}h_1 \in K$. This is equivalent to saying $h_2^{-1}h_2 \in H \cap K$. verify □

Theorem 1.1.2. *If $H, K \leqslant G$ then $HK \leqslant G$ if and only if $HK = KH$.*

Corollary 1.1.2.1. *If H is a normal subgroup of G, then $HK \leqslant G$*

Theorem 1.1.3. *If $H, K \leqslant G$, $|H|, |K| < \infty$, H, K normal in G, and $H \cap K = \{e\}$, then $HK \cong H \times K$. In this case HK is called the internal direct product of H and K.*

Proof. By the corollary, $HK \leqslant G$. Therefore consider the map $\phi : HK \to H \times K := hk \to (h, k)$.

- ϕ is well defined. We know that $|HK| = \frac{|H||K|}{|H \cap K|} = |H||K|$. Hence every element in HK has a unique representation as hk. Thus we see that ϕ is well defined.

- ϕ is a homomorphism. Let $h_1k_1, h_2k_2 \in HK$. Want to show that $\phi(h_1k_1h_2k_2) = \phi(h_1k_1)\phi(h_2k_2) = (h_1, k_1)(h_2, k_2) = (h_1h_1, k_1k_2)$. Equilvalently we want to show that $h_1k_1h_2k_2 = h_1h_2k_1k_2$, which is again equivalent to showing $hk = kh$ for all $h \in H, k \in K$. This is again equivalent to showing $[h, k] = h^{-1}k^{-1}hk = e$ for all $h \in H, k \in K$.

  To see why $[h, k] = \{e\}$. see that $[h, k] = h^{-1}(k^{-1}hk) = h^{-1}h' \in H$ for some $h' \in H$. Similarly $[h, k] = (h^{-1}k^{-1}h)k = k'k \in K$ for some $k' \in K$.

Both of these are by the normality of H and K. But since we know that $H \cap K = \{e\}$, we see that $[h, k] = e$ for all h, k. Hence we are done.

$\square$

Chapter 2

Example 2.0.1. Let $G = D_{12}$, $H = \langle r \rangle, K = \langle s \rangle$. Then $H \cap K = \{e\}$, so $|HK| = 12$ and $HK \cong D_{12}$, but $HK \ncong H \times K$ since $H \times K$ is Abelian and D_{12} is not.

Theorem 2.0.1 (Diamond Isomorphism Theorem)**.** *Suppose* $A, B \leqslant G$ *and* $A \subseteq N_G(B)$*, then* $AB \leqslant G, B \lhd AB, A \cap B \trianglelefteq$ *and* $AB/B \cong A/A \cap B$

$$\begin{tikzcd}
 & G & \\
 & AB \arrow[u] & \\
A \arrow[ur] & & B \arrow[ul, Rightarrow] \\
 & A \cap B \arrow[ul, Rightarrow] \arrow[ur] &
\end{tikzcd}$$

Proof. verify $\square$

Theorem 2.0.2. *Let* $H, K \trianglelefteq G$ *and* $H \leqslant K$*, then* $H \trianglelefteq K$*,* $K/H \trianglelefteq G/H$*,*

$$(G/H)/(K/H) \cong G/K$$