Abstract Algebra (Math 6302), Fall 2024
Homework Assignment 2

Due at 9:59pm on Wednesday, September 11. Turn in your completed assignment (single .pdf file) on Canvas. For full credit, show your work and justify your answers.

(1) Give an example of a binary relation on $\mathbb{Z}$ which is reflexive and symmetric, but not transitive.

(2) Define a relation $\sim$ on the set $\{(a,b) \in \mathbb{Z}^2 : b \neq 0\}$ by the rule that $(a,b) \sim (c,d)$ if and only if $ad - bc = 0$. Prove that $\sim$ is an equivalence relation.

(3) Determine which of the following groups are cyclic (justify your answers). For the ones that are cyclic, find a generator.

(a) $C_2 \times C_5$

(b) $C_4 \times C_4$

(c) $\mathbb{Z}$

(d) $\mathbb{Q}$

(e) $\mathbb{Z} \times \mathbb{Z}$

(f) $(\mathbb{Z}/18\mathbb{Z})^\times$

(g) $(\mathbb{Z}/36\mathbb{Z})^\times$

(h) $(\mathcal{P}(S), \triangle)$, where S is a set with $n \geq 2$ elements

(4) Prove that $(\mathcal{P}(\{1,2\}), \triangle)$ is isomorphic to V_4.

(5) Suppose that $G_1, \ldots, G_n$ are groups. Prove that the direct product $G_1 \times \cdots \times G_n$ is Abelian if and only if G_i is Abelian, for all $1 \leq i \leq n$.

(6) Let $d, n \in \mathbb{N}$ and suppose that $d | n$. Write $C_n = \langle x \rangle$ and let $H \subseteq C_n$ be defined by

$$H = \{x^d, x^{2d}, x^{3d}, \ldots, x^n\}.$$

Define a relation $\sim$ on C_n by $g \sim h$ if and only if $g^{-1}h \in H$. Prove that $\sim$ is an equivalence relation.

(7) Let $G = (\mathbb{Z}/900\mathbb{Z})^\times$.

(a) Compute $|G|$.

(b) Find a representative $0 \leq a < 900$ with $a = 11^{-1}$ in G.

(8) Find the set of all integer solutions x to the equation $5457x = 3317 \bmod 5885$.

(9) Find all integers x satisfying the system of equations

$$\begin{aligned} x &= 43 \bmod 101, \\ x &= 10 \bmod 103, \text{ and} \\ x &= 96 \bmod 107. \end{aligned}$$

(10) Given that 10 is a primitive root modulo 313, find all residue classes x mod 313 which satisfy $x^3 = 1$ mod 313.

(11) What is the ones digit of the number $7^{7^{7^7}}$?

(12) Find the smallest positive integer which is equivalent modulo 211 to 8^{1074}.

(13) Let $n \in \mathbb{N}$. Prove that for all $a \in \mathbb{Z}$, the residue class $\overline{a}$ is a generator for $\mathbb{Z}/n\mathbb{Z}$ if and only if $(a, n) = 1$.

(14) How many elements of the group $(\mathbb{Z}/103\mathbb{Z})^\times$ are generators for the group?

(15) Let g be Graham's number (see lecture 13). Find the smallest positive integer x satisfying $x = g$ mod 121.

(16) Given that 5 is a primitive root modulo 97, find all residue classes x mod 9797 which satisfy $x^3 = 1$ mod 9797.