Department of Mathematics University of Houston

MATH 6320

Theory of Functions of a Real Variable
Fall 2024

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 7, due Thursday, October 31, 11:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Find an example of a sequence of Riemann integrable functions $(f_n)_{n=1}^\infty$, $f_n : [0,1] \to [0,\infty)$ so that $\lim_{n\to\infty} \int_{[0,1]} f_n dm = 0$ and $g = \sup_{n\in\mathbb{N}} f_n$ is not in $L^1(m)$, where m is the Lebesgue measure on $[0,1]$. Next, explain how you can modify your example to obtain the same property for a sequence of continuous functions $(f_n)_{n=1}^\infty$. Hint: Start with g and let $f_n = \chi_{I_n} g$, where each I_n is a closed interval.

Problem 2

Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f : X \to [0,\infty)$ be bounded, measurable. Prove that

$$\int f d\mu = \inf\Big\{\sum_{j=1}^n \mu(A_j) \sup_{x\in A_j} f(x) : A_j's \text{ partition } X\Big\}$$

where it is assumed that each partition contains only measurable $A_j \in \mathcal{M}$. Also show that this identity can fail if the assumption that f bounded is replaced by $f \in L^1(\mu)$.

Problem 3

Let V be an open set in $\mathbb{R}$ and μ a Borel measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) < \infty$. Show that $f(x) = \mu(x+V)$ defines a lower semicontinuous function f on $\mathbb{R}$.