MATH 6320 - Functions of One Real Variable
Homework 7

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1. **Solution:** Consider the sequence of functions

$$g_{n,k}(x) = n\chi_{[\frac{k}{2^n},\frac{k+1}{2^n}]}(x), \quad \text{where } k \in \{0,1,2,\dots,2^n-1\}, n \in \mathbb{N}$$

Order then with the lexicographic ordering to get $f_1, f_2, f_3, \dots$. Let $f_r = g_{n,k}$. Since each f_n is simple, they are Reimann integrable and

$$\int f_r\ dm = \int n\chi_{[\frac{k}{2^n},\frac{k+1}{2^n}]}\ dm = \frac{n}{2^n}$$

Since each n has only finitely many elements $k \in \{1,2,3,\dots 2^n-1\}$, we see that as $r \to \infty$, $n \to \infty$. Thus

$$\lim_{r\to\infty} \int f_r\ dm = \lim_{n\to\infty} \frac{n}{2^n} = 0$$

But then we see that for any $x \in [0,1]$ and $M \in \mathbb{N}$, there exist $k_0 < 2^{M+1} \in \mathbb{N}\cup\{0\}$ such that $\frac{k_0}{2^{M+1}} \le x \le \frac{k_0+1}{2^{M+1}}$. Thus we see that $g_{(M+1),k_0}(x) = M+1 > M$. Then for all $x \in [0,1]$,

$$\sup_{r\in\mathbb{N}} f_r(x) = \infty$$

Hence if $g = \sup_{n\in\mathbb{N}} f_n$, then $g = \infty\chi_X$, which clearly is not in $L^1(m)$ as $\int g\ dm = \infty$.

Next, to get a sequence of continuous functions f_r which satisfy with the same property as above, let

$$K_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], \quad \text{where } k \in \{0,1,\dots 2^n-1\}, n \in \mathbb{N}$$

and

$$U_{n,k} = \begin{cases} [0, \frac{2}{2^n}), & k = 0 \\ (\frac{k-1}{2^n}, \frac{k+2}{2^n}), & 1 \le k < 2^n - 1 \\ (\frac{2^n-2}{2^n}, 1], & k = 2^n - 1 \end{cases}$$

Then we notice that each $K_{n,k}$ is compact and $U_{n,k}$ is open, with $K_{n,k} \subset U_{n,k}$. Since $[0,1]$ is compact, it is also locally compact and by Urysohn's lemma, there are continuous functions $h_{n,k}$ such that

$$\chi_{K_{n,k}} \prec h_{n,k} \prec \chi_{U_{n,k}}$$

Let $g_{n,k} = nh_{n,k}$. Then we see that $g_{n,k}$ are continuous with

$$\int g_{n,k}\ dm = n \int h_{n,k}\ dm \le n \int \chi_{U_{n,k}}\ dm = \frac{3n}{2^n}$$

Now index $g_{n,k}$ by the lexicographic ordering on (n,k) to get a sequence $f_1, f_2, \ldots$. Then by the same arguments as in the previous choice for f_r, we see that

$$\sup_{r \in \mathbb{N}} f_r(x) = \infty$$

and

$$\lim_{r \to \infty} \int f_r\ dm = \lim_{n \to \infty} \frac{3n}{2^n} = 0$$

2. **Solution:** Let $A_1, A_2, \ldots, A_n$ be any partition of X where each A_j is measurable. Define a simple function

$$s = \sum_{j=1}^{n} \chi_{A_j} \sup_{x \in A_j} f(x)$$

Since each A_j is measurable, we see that s is measurable. Moreover $f < s$ since $f < s$ in each of the set A_j. Then,

$$\int f\ d\mu \le \int s\ d\mu = \sum_{j=1}^{n} \mu(A_j) \sup_{x \in A_j} f(x)$$

Since $A_1, A_2, \ldots, A_n$ was an arbitrary partition of X into measurable sets, the above inequality holds for all such finite partitions. Then taking the infimum among all such partitions preserve the inequality. Thus we get

$$\int f \; d\mu \leq \inf \left\{ \sum_{j=1}^{n} \mu(A_j) \sup_{x \in A_j} f(x) \;\; : \;\; A_j\text{'s partition } X \right\}$$

Now consider the sequence of functions

$$s_n(x) = \begin{cases} 0, & x < 0 \\ (k+1)2^{-n}, & k2^{-n} < x \leq (k+1)2^{-n}, \quad k \in \mathbb{N} \end{cases}$$

Then $s_1 \geq s_2 \geq \ldots$. We notice that s_n is a slight variation of the familiar 'staircase-to-plateau' function. We also observe that each s_n is measurable and s_n converge pointwise to the identity function in the positive part of the real numbers.

Then consider the sequence $\phi_n = s_n \circ f$. Since s_n, f are measurable functions, ϕ_n is also a measurable function. Since f is bounded, there is an $M \in \mathbb{N}$ such that $f(x) \in [0, M]$ for all $x \in X$. Hence ϕ_n can take at most $2^n M$ values. Therefore ϕ_n are simple measurable functions. Hence

$$\phi_n = \sum_{i=1}^{m} a_i \chi_{A_i}$$

where A_is are measurable sets partition X. Moreover by virtue of the definition, we see that $a_i = \sup_{x \in A_i} f(x)$. Hence

$$\phi_n = \sum_{i=1}^{m} \chi_{A_i} \sup_{x \in A_i} f(x)$$

Again, since $s_n(r) \geq s_{n+1}(r)$ for all $r \in \mathbb{R}$, we see that $s_n(f(x)) \geq s_{n+1}(f(x))$ for all $x \in X$. Hence $\phi_1 \geq \phi_2 \geq \phi_3 \geq \ldots \geq f$. Since s_n converge pointwise to to the identity on $[0, \infty)$, and f is bounded, ϕ_n converge pointwise to f.

Moreover note that since f is bounded above by M, each ϕ_n is bounded above by M. Hence $|\phi_n| \leq M\chi_X$ and

$$\int M\chi_X \; d\mu = M\mu(X) < \infty$$

Then by dominated convergence theorem, we get

$$\lim_{n \to \infty} \int \phi_n \; d\mu = \int f \; d\mu$$

Thus we see that

$$\inf\left\{\sum_{i=1}^{m}\chi_{A_i}\sup_{x\in A_i} f(x)\ \ : A_j\text{'s partition } X\right\}\leq\int f\ d\mu$$

which gives our equality.

To see that the above equality need not hold if f is not bounded, consider $X=(0,1)$ with the restricted Lebesgue measure. Consider the function $f : (0,1)\to\mathbb{R} := x\to\frac{1}{\sqrt{x}}$. Then by the fact that Lebesgue and Riemann integral agrees for continuous function, we see that

$$\int f\ dm=2$$

If A is any non-null measurable set containing a neighborhood of 0, we see that $\mu(A)\sup_{x\in A} f(x)=\infty$. Since null sets cannot finitely cover any neighborhood, we see that any finite partition of X must contain a non-null set that intersect all neighborhoods of 0. Thus we see that for any partition $\{A_1,A_2,\ldots,A_n\}$ of X

$$\sum_{j=1}^{n}\mu(A_j)\sup_{x\in A_j} f(x)=\infty$$

which gives us

$$\inf\left\{\sum_{j=1}^{n}\mu(A_j)\sup_{x\in A_j} f(x)\ :\ A_j\text{'s partition } X\right\}=\infty\neq\int f\ dm$$

3. **Solution:** We need to show that the set $f^{-1}((y_0,\infty))$ is open for all $y_0\in\mathbb{R}$. But

$$\begin{aligned} f^{-1}((y_0,\infty))\text{ is open} &\iff f^{-1}((-\infty,y_0])\text{ is closed}\\ &\iff \{x\in\mathbb{R}\ :\ \mu(x+V)\leq y_0\}\text{ is closed}\end{aligned}$$

Let $(x_n)_{n=1}^{\infty}\subset\{x\in\mathbb{R}\ :\ \mu(x+B)\leq y\}$ be a sequence such that $x_n\to x_0\in\mathbb{R}$. We need to show that $\mu(x_0+V)\leq y$.

Since V is open and addition is a continuous function, we see that $x_0 + V$ is also open. Now let

$$V_n = \{y \in x_0 + V \ : \ B_{\frac{1}{n}}(y) \subset x_0 + V\} = \bigcup_{\substack{y \,\in\, x_0+V \\ B_{\frac{1}{n}}(y) \,\subset\, x_0+V}} B_{\frac{1}{n}}(y)$$

Then it is clear that V_n is open for each V_n and $V_1 \subset V_2 \dots V_n \subset V_{n+1} \dots$ since $B_{\frac{1}{n+1}}(y) \subset B_{\frac{1}{n}}(y)$. Since $x_0 + V$ is open, each $y \in x_0 + V$ is contained in an open ball $B_{1/n}(y) \subset x_0 + V$ for some $n \in \mathbb{N}$. Thus we see that

$$\bigcup_{n=1}^{\infty} V_n = x_0 + V$$

Then by the continuity of the measure from below, we see that

$$\mu(V_n) \nearrow \mu(x_0 + V)$$

Now consider the set

$$\begin{aligned} D_n = (x_n + V) \cap (x_0 + V) &= \{y \in x_0 + V \ : \ y \in x_n + V\} \\ &= \{y \in x_0 + V \ : \ (y - x_n) + x_0 \in x_0 + V\} \\ &= \{y \in x_0 + V \ : \ y + (x_0 - x_n) \in x_0 + V\} \\ &\supseteq \{y \in x_0 + V \ : \ B_{2|x_0 - x_n|}(y) \subset x_0 + V\} \end{aligned}$$

Since $x_n \to x_0$, for each N, there is an $N_N > N$ (we can demand $N_N > N$) such that for all $n > N_N$, we have $2|x_n - x_0| < \frac{1}{N}$. Then for all $n > N_N$,

$$\begin{aligned} \{y \in x_0 + V \ : \ B_{2|x_0 - x_n|}(y) \subset x_0 + V\} &\supseteq \{y \in x_0 + V \ : \ B_{\frac{1}{N}}(y) \subset x_0 + V\} \\ &= V_N \end{aligned}$$

Therefore for all $n > N_N$, we get $V_N \subset D_n \subset x_0 + V$ and

$$\mu(V_N) \leq \mu(D_n) \leq \mu(x_0 + V)$$

Since we know that $\mu(V_N) \to \mu(x_0 + V)$ as $N \to \infty$, we get

$$\mu(D_n) \to \mu(x_0 + V)$$

being sandwiched between $\mu(V_N)$ and $\mu(x_0+V)$. Again, since $V_N \subset D_n \subset x_n+V$ for all $n > N_N$, we get

$$\mu(V_N) \leq \mu(D_n) \leq \mu(x_n + V)$$

for all $n > N_N$. Then, taking the limits must preserve the inequality and we see that

$$\mu(x_0 + V) = \lim_{n\to\infty} \mu(V_n) \leq \lim_{n\to\infty} \mu(x_n + V)$$

Now since $\mu(x_n + V) \leq y$ for each x_n, we get

$$\mu(x_0 + V) \leq \lim_{n\to\infty} \mu(x_n + V) \leq y$$

which proves our assertion.