

# Functional Analysis - MATH7320

Joel Sleeba  
joelsleeba1@gmail.com

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# Chapter 1

## Banach Spaces

**Textbook :** A Course in Functional Analysis, John Conway

Functional analysis is the study of Topological Vector Spaces.

**Definition 1.0.1.** Let  $X$  be a vector space (over  $\mathbb{R}$  or  $\mathbb{C}$ ). A seminorm on  $X$  is a map  $\|\cdot\| : X \rightarrow [0, \infty)$  such that

- $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{F}, \forall x \in X$
- $\|x + y\| \leq \|x\| + \|y\|$

In addition if  $\forall x \neq 0, \|x\| \neq 0$ , we say  $\|\cdot\|$  is a norm on  $X$

Norm induces a metric  $d(x, y) = \|x - y\|$

*Note.* Let  $X$  be a normed space. Then the maps

- $+: X \times X \rightarrow X : (x, y) \rightarrow x + y$
- $\cdot : \mathbb{F} \times X \rightarrow X : (\alpha, x) \rightarrow \alpha x$

are continuous.

Hence every normed space is a topological vector space.

**Example 1.0.1.**  $\mathbb{F}^n$  with  $\ell_p$ -norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_p = \left( \sum_{i=1}^n |\alpha_i|^p \right)^{\frac{1}{p}}$$

**Example 1.0.2.**  $\mathbb{F}^n$  with  $\ell_\infty$ -norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\infty = \max\{|\alpha_i|\}$$

**Example 1.0.3.** Consider  $C_{00} = \{(a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n = 0 \text{ except for finitely many } n \in \mathbb{N}\}$  which can be identified by collection of functions  $f : \mathbb{N} \rightarrow \mathbb{F}$  with finite support.

Then

$$\|(a_n)\|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}$$

is a norm on  $C_{00}$

**Proposition 1.0.1.** Let  $X, Y$  be normed space, and let  $T : X \rightarrow Y$  be linear. Then the following are equivalent.

- $T$  is continuous
- $T$  is continuous on 0
- $T$  is continuous on any point  $x \in X$
- $\exists M > 0$  such that  $\|T(x)\|_Y \leq M\|x\|_X$  for all  $x \in X$

*Proof.* (1  $\implies$  2) It is clear that if  $T$  is continuous, then it is continuous at 0 from the definition of continuity.

(2  $\implies$  3) Let  $x \in X$  and  $\{x_n\}_{n \in \mathbb{N}}$  be any sequence in  $X$  that converge to  $x$ . Then the sequence  $\{y_n = x_n - x\}$  converge to zero by the algebra of limits. By the continuity of  $T$  at zero,  $\{T(y_n) = T(x_n) - T(x)\}$  converge to 0. Therefore  $T(x_n) \rightarrow T(x)$ . And this shows  $T$  is sequentially continuous at  $x \in X$ . Since the space is a metric space, sequential continuity is equivalent to continuity.

(4  $\implies$  2) Let  $x \in X$ . Then  $\|T(0) - T(x)\| = \|T(x)\| \leq M\|x\| = M\|0 - x\|$ . Hence  $T$  is continuous at 0.

(3  $\implies$  1)

(2  $\implies$  4)

□

**Example 1.0.4.** Let  $T : \mathbb{F}^n \rightarrow \mathbb{F}^n$  be defined as  $T(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, 0, \dots, 0)$ . Is  $T$  convergent for any norm  $\|\cdot\|_1, \|\cdot\|_2$  in the domain and range?

*Proof.* verify

□

**Example 1.0.5.** Consider identity function  $I : C_{00} \rightarrow C_{00}$ . Let the norm in domain be  $\|\cdot\|_{\infty}$  and that in range be  $\|\cdot\|_1$ . Is the function continuous? What if the norms in domain and range are switched?

*Proof.* verify

□

*Note.* Let  $X$  be a space with two norms  $\|\cdot\|_1, \|\cdot\|_2$ . When are the two norms topologically equivalent?

When  $\exists M, M'$  such that  $\|x\|_1 \leq M\|x\|_2$  and  $\|x\|_2 \leq M'\|x\|_1$ . Equivalently, when the identity map between the two spaces with their respective norms are bi-continuous. (See 4th equivalent statement of previous proposition)

**Theorem 1.0.1.** *Let  $X$  and  $Y$  be normed spaces, and  $T : X \rightarrow Y$  be linear. Assume  $X$  is finite dimensional. Then  $T$  is continuous.*

*Proof.* Since  $T(X) \leq Y$  is finite dimensional, we may assume without loss of generality that  $Y$  is also finite dimensional and  $T$  is onto. Let  $\{x_1, x_2, \dots, x_n\}$  be a basis for  $X$ . Define another norm on  $X$  as follows. For every  $x = \sum_{i=1}^n \alpha_i x_i \in X$ ,

$$\|x\|' = \sum_{i=1}^n |\alpha_i| (\|T(x_i)\| + \|x_i\|)$$

verify that this is a norm. Then for every  $x \in X$ , we have

$$\|T(x)\| \leq \sum_{i=1}^n |\alpha_i| \|T(x_i)\| \leq \|x\|'$$

Hence  $T$  is bound with respect to the norm  $\|\cdot\|'$  on  $X$ , since all norms are equivalent on  $X$ . Therefore  $T$  is continuous w.r.t to the original norm on  $X$ .  $\square$

**Corollary 1.0.1.1.** *Let  $X$  be a finite dimensional vector space. Then any two norms in  $X$  are equivalent.*

*Proof.* Let  $\{e_1, e_2, \dots, e_n\}$  be a basis for  $X$ . For each  $x = \sum_{i=1}^n \alpha_i e_i \in X$ , define

$$\|x\|_\infty = \max\{|\alpha_i|\}$$

Then  $\|\cdot\|_\infty$  is a norm and we'll show every norm on  $X$  is equivalent to this norm. Let  $\|\cdot\|$  be an arbitrary norm on  $X$ . For each  $x = \sum_{i=1}^n \alpha_i e_i \in X$ , we have

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^n \alpha_i e_i \right\| \\ &\leq \sum_{i=1}^n |\alpha_i| \|e_i\| \\ &\leq \max\{|\alpha_i|\} \sum_{i=1}^n \|e_i\| \\ &\leq \|x\|_\infty \sum_{i=1}^n \|e_i\| \end{aligned}$$

Therefore the identity map  $I : (X, \|\cdot\|_\infty) \rightarrow (X, \|\cdot\|)$  is continuous. Since the set  $K = \{x \in X : \|x\|_\infty \leq 1\}$  is compact,  $K$  is also compact in  $(X, \|\cdot\|)$  and the restriction  $\text{Id}|_K$  is also a homeomorphism. **verify** In particular, the set  $\{x \in X : \|x\|_\infty < 1\}$  is an open neighborhood of  $0 \in (X, \|\cdot\|)$ . By the Heine-Borel theorem, the unit ball  $B = \{x \in X : \|x\|_2 \leq 1\}$  is compact. Hence  $B$  is compact in  $(X, \|\cdot\|)$ . **verify the last line.**  $\square$

**Corollary 1.0.1.2.** • *Every finite dimensional normed space is complete*

- *If  $X$  is a normed space and  $Z$  is a finite dimensional subspace of  $X$ , then  $Z$  is the closed in  $X$*

*Proof.* • Let  $(X, \|\cdot\|)$  be a finite dimensional normed space. Let  $\|\cdot\|_2$  be the euclidean norm on  $X$ . Then by the theorem above there exists  $M \in \mathbb{R}$  such that  $\frac{1}{M}\|x\|_2 \leq \|x\| \leq M\|x\|_2$  for all  $x \in X$ .

Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $(X, \|\cdot\|)$ . Then  $(x_n)_{n \in \mathbb{N}}$  is also Cauchy in  $(X, \|\cdot\|_2)$ . Since the latter space is complete, so is  $(X, \|\cdot\|)$ .

- **verify**

$\square$

*Note.* If  $T, S : X \rightarrow Y$  are continuous linear maps between normed space, then  $T + S$  is also continuous. Also,  $\forall \alpha \in \mathbb{F}$ ,  $\alpha T$  is continuous.

Thus the space  $B(X, Y)$  of all continuous linear maps from  $X$  to  $Y$  is a subspace of all linear maps between  $X$  and  $Y$ .

**Definition 1.0.2.** Let  $X, Y$  be normed spaces. and  $T \in B(X, Y)$ . We define the operator norm of  $T$  as

$$\|T\| = \sup \left\{ \frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0 \right\}$$

**Lemma 1.0.1.** *Let  $T \in B(X, Y)$ . Then the following are equivalent*

- $\|T\|$
- $\sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$
- $\sup\{\|T(x)\| : x \in X, \|x\| < 1\}$
- $\inf\{M \geq 0 : \|T(x)\| \leq M\|x\|, \forall x \in X\}$

*Proof.* **verify**

$\square$

**Proposition 1.0.2.** *The operator norm is a norm in  $B(X, Y)$*

*Proof.* **verify**

$\square$

## 1.1 Hahn Banach Theorem

**Lemma 1.1.1.** *Let  $X$  be a complex normed space. Let  $f : X \rightarrow \mathbb{R}$  be an  $\mathbb{R}$ -linear functional. Then  $g : X \rightarrow \mathbb{C}$  defined as  $g(x) = f(x) - if(ix)$  is  $\mathbb{C}$ -linear*

*Conversely if  $g : X \rightarrow \mathbb{C}$  is a  $\mathbb{C}$ -linear map, then  $f := \Re \circ g : X \rightarrow \mathbb{R}$  is  $\mathbb{R}$ -linear.*

*Moreover  $\|f\| = \|g\|$ .*

*Proof.* We'll prove that  $\|f\| = \|g\|$  and leave the rest for the reader (verify).

Since  $|f(x)| \leq |g(x)|$ , for all  $x \in X$ , it is easy to see that  $\|f\| \leq \|g\|$ . Conversely,  $\forall \epsilon > 0, \exists x_o \in X$  with  $\|x_o\| = 1$  such that  $|g(x_o)| > \|g\| - \epsilon$ . If  $g(x_o) = re^{i\theta}$ , take  $\alpha = e^{-i\theta}$ . Then  $f(\alpha x_o) = \Re(re^{-i\theta}e^{i\theta}) = r = g(\alpha x_o)$ . Then  $\|f\| \geq |f(\alpha x_o)| = |g(\alpha x_o)| = |\alpha| |g(x_o)| = |g(x_o)| > \|g\| - \epsilon$ . Since  $\epsilon$  is arbitrary, this gives  $\|f\| \geq \|g\|$   $\square$

**Theorem 1.1.1** (Hahn-Banach Extension Theorem). *Let  $X$  be a normed space over  $\mathbb{R}$ ,  $Z$  be a subspace of  $X$  and let  $\phi : Z \rightarrow \mathbb{R}$  be a continuous linear functional. Then there exists a linear functional  $\psi : X \rightarrow \mathbb{R}$  such that  $\psi|_Z = \phi$  and  $\|\phi\| = \|\psi\|$ .*

*Proof.* Assume  $\|\phi\| = 1$  (If this is not the case, we can always scale the functional down to norm 1). Now we'll extend  $\phi$  from  $Z$  to a subspace with one dimension higher than  $Z$ , preserving the norm. Let  $x_o \in (X \setminus Z)$  and  $Y = \text{Span}\{\{x_o\} \cup Z\}$  be the set one dimension higher than  $Z$ . Assume  $\psi$  is the extension of  $\phi$  to  $Y$ . Then  $\psi$  will be completely characterized, if we know the value of  $\psi(x_o)$ . We look to see what real values we can assign  $\psi(x_o)$  satisfying our conditions. Let  $y = z + x_o \in Y$  where  $z \in Z$  (We must be taking an arbitrary element  $y = z + \alpha x_o \in Y$ , but if we know the image of  $y = z + x_o$  for all  $z \in Z$  under  $\psi$ , then we can get the image of  $y = z + \alpha x_o$  for any  $\alpha \in \mathbb{R}$  by scaling). Norm preserveness demands that for all  $z \in Z$ , we must have

$$-\|z + x_o\| \leq \psi(y) = \psi(z) + \psi(x_o) \leq \|z + x_o\|$$

Since  $\psi$  agrees with  $\phi$  on  $Z$ , this is equivalent to

$$-\phi(z) - \|z + x_o\| \leq \psi(x_o) \leq \|z + x_o\| - \phi(z) \quad (1.1)$$

Moreover since we normalized  $\phi$  to have norm 1, we know  $\psi$  must also have norm 1. Then by triangle inequality, we get that for all  $a, b \in Y$

$$\psi(a) - \psi(b) = \psi(a - b) \leq \|a - b\| = \|(a + x_o) - (b + x_o)\| \leq \|a + x_o\| + \|b + x_o\|$$

which gives

$$-\psi(b) - \|b + x_o\| \leq \|a + x_o\| - \psi(a)$$

Since this inequality is true for all  $a, b \in Y$ , taking supremum and infimum over all the possible  $a, b \in Y$  preserves the inequality. Hence we get

$$\sup_{b \in Y} \left\{ -\psi(b) - \|b + x_o\| \right\} \leq \inf_{a \in Y} \left\{ \|a + x_o\| - \psi(a) \right\} \quad (1.2)$$

Substituting  $a = b = z$  in Equation 1.2 guarantees the existence of  $\psi(x_o)$  satisfying Equation 1.1. Hence we get an extension (namely  $\psi$ ) of  $\phi$  to  $Y$  preserving the norm. Since  $Z$  was an arbitrary subspace of  $X$ , this is true for all such subspaces of  $X$ .

Now we will employ Zorn's lemma to get an extension of  $\phi$  from  $Z$  to the whole of  $X$ . Consider the collection of all linear extensions of  $\phi$ , i.e

$$\mathcal{S} = \left\{ (\psi_Y, Y) : Z \subset Y, \psi_Y|_Z = \phi, \|\psi_Y\| = \|\phi\| \right\}$$

Then we define a partial order in the collection  $\mathcal{S}$  as  $(\psi_X, X) \leq (\psi_Y, Y)$  if and only if  $X \subset Y$  and  $\psi_Y|_X = \psi_X$ . Now let  $\mathcal{C}$  be a chain in  $\mathcal{S}$ . Consider the set

$$\tilde{Y}_{\mathcal{C}} = \bigcup_{(\psi_Y, Y) \in \mathcal{C}} Y$$

and the map  $\psi_{\tilde{Y}_{\mathcal{C}}} : \tilde{Y}_{\mathcal{C}} \rightarrow \mathbb{R}$  defined as

$$\psi_{\tilde{Y}_{\mathcal{C}}}(x) = \psi_Y(x), \text{ where } x \in Y, \text{ for } (\psi_Y, Y) \in \mathcal{C}$$

To see this map is well defined, assume  $x \in X$  and  $x \in Y$  for  $(\psi_X, X), (\psi_Y, Y) \in \mathcal{C}$ . Then either  $(\psi_X, X) \leq (\psi_Y, Y)$  or  $(\psi_Y, Y) \leq (\psi_X, X)$  since  $\mathcal{C}$  is totally ordered. WLOG assume  $(\psi_X, X) \leq (\psi_Y, Y)$ , then by definition we get that  $\psi_Y|_X = \psi_X$ . This gives that  $\psi_Y(x) = \psi_X(x)$ . Hence we get that  $\psi_{\tilde{Y}_{\mathcal{C}}}$  is well defined. In a similar fashion we can verify that  $\psi_{\tilde{Y}_{\mathcal{C}}}$  is a linear functional.

Now we claim that  $(\tilde{Y}_{\mathcal{C}}, \psi_{\tilde{Y}_{\mathcal{C}}})$  is the upper bound of the chain  $\mathcal{C}$ . By the definition of  $\tilde{Y}$ , we see that there cannot be an element  $(\psi_Y, Y)$  in the chain  $\mathcal{C}$ , with  $\tilde{Y} \subset Y$ . Hence the only remaining thing to show is that for all  $(\psi_X, X) \in \mathcal{C}$ , we have  $\psi_{\tilde{Y}_{\mathcal{C}}}|_X = \psi_X$ . But this also follows from the definition of the map  $\psi_{\tilde{Y}_{\mathcal{C}}}$ .

Since  $\mathcal{C}$  was taken to be an arbitrary chain in the collection  $\mathcal{S}$ , we get that every chain in  $\mathcal{S}$  has an upper bound. Then by Zorn's lemma, the collection  $\mathcal{S}$  has a maximal element  $(\psi, Y)$ . We claim that in this maximal element,  $Y = X$ . If not, we can extend  $\psi$  to a space one dimension above  $Y$  like we did in the beginning contradicting the maximality of  $(\psi, Y)$ . Hence the maximal element is  $(\psi, X)$ . This by definition of the collection  $\mathcal{S}$ , is the required extension for  $(\phi, Z)$ .  $\square$

*Remark 1.1.1.* Note that in the proof above, we only used the scaling property and triangle inequality of the norm, hence we can relax the condition for norm and replace it with a seminorm, without messing up the proof.



**Theorem 1.1.2** (Hahn-Banach Extension Theorem for  $\mathbb{C}$ ). *Same statement of Theorem 1.1.1 with only the field changed to  $\mathbb{C}$ .*

*Proof.* Consider  $X$  as a normed linear space over  $\mathbb{R}$ . Let  $f = \Re \circ \phi : Z \rightarrow \mathbb{R}$  and apply Theorem 1.1.1 on  $f$  to get a real linear functional  $\tilde{f} : X \rightarrow \mathbb{R}$  with the required properties. Now we claim that  $\tilde{\phi}$  defined as  $\tilde{\phi}(x) = \tilde{f}(x) - i\tilde{f}(ix)$  is the required extension.

First we show  $\tilde{\phi}|_Z = \phi$ . To see this first we notice that if  $\phi$  can be written as  $\phi(x) = f(x) + ig(x)$  where  $f, g$  are real valued functionals, then since  $-\phi(x) = i\phi(ix) = if(ix) - g(ix)$ . Hence  $0 = \phi(x) - \phi(x) = (f(x) - g(ix)) + i(g(x) + f(ix))$ . Since real part and imaginary part must be equal to 0, we get that  $g(x) = -f(ix)$ . Therefore we get  $\phi(x) = f(x) - if(ix)$ . Now we get  $\tilde{\phi}|_Z = \phi$  immediately since  $\tilde{f}|_Z = f$ . To finish the proof, we also have to show that  $\|\phi\| = \|\tilde{\phi}\|$ . But this follows easily from Lemma 1.1.1 as  $\|\phi\| = \|f\| = \|\tilde{f}\| = \|\tilde{\phi}\|$ .  $\square$

*Remark 1.1.2.* It is quite natural to be confused about the well defineness of the expression  $f(ix)$  when we are considering  $X$  as a normed linear space over  $\mathbb{R}$  in the beginning of the proof. But note that since  $X$  initially was a complex normed linear space, viewing it as a space over  $\mathbb{R}$  doesn't change or remove any elements from the space. Hence  $ix \in X$  even though  $X$  is viewed as a real normed linear space.

**Definition 1.1.1.** A sublinear map is a function  $\rho : X \rightarrow \mathbb{R}$  with the properties

- $\rho(rx) = r\rho(x), \forall r \in \mathbb{R}$
- $\rho(x + y) \leq \rho(x) + \rho(y)$

**Definition 1.1.2.** Let  $X$  be a normed space. Then the dual of  $X$ , denoted by  $X^*$ , is the space  $B(X, \mathbb{F})$

**Lemma 1.1.2.** Let  $X$  be a normed space and  $x \in X$ . Then  $\exists f \in X^*$  such that

$$\|f\| = 1 \text{ and } f(x) = \|x\|$$

*Proof.* Let  $Z = \text{Span}\{x\}$ . Define  $g : Z \rightarrow \mathbb{F}$  as  $g(\alpha x) = \alpha\|x\|$ . Then  $\|g\| = 1$ . By the Hahn Banach theorem,  $g$  has an extension  $f$  which preserve the norm and extends  $g$  to  $X$ .  $\square$

**Corollary 1.1.2.1.** Let  $X$  be a normed space and  $x \in X$ , then we have

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\}$$

*Proof.* If  $f$  is any linear functional with  $\|f\| \leq 1$ , then  $|f(x)| \leq \|f\|\|x\| = \|x\|$ . Hence  $\|x\| \leq \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\}$ . Now let  $f_x$  be the functional we get from [Lemma 1.1.2](#). Then  $f_x \in X^*$  and  $\|f_x\| = 1$  with  $f_x(x) = |f(x)| = \|x\|$ . Hence we get that the inequality is actually an equality, and this proves the corollary.  $\square$

**Definition 1.1.3.** For every  $x \in X$ , define a linear map  $\hat{x} : X^* \rightarrow \mathbb{F}$  by  $\hat{x}(f) = f(x)$

**Theorem 1.1.3.** For every  $x \in X$ ,  $\hat{x} \in (X^*)^*$ . The map  $\rho : x \rightarrow \hat{x}$  is an isometric linear map.

*Proof.* The fact that  $\hat{x}$  is linear and bounded and the map  $X \ni x \rightarrow \hat{x} \in X^{**}$  is linear follows from the definition of  $f + g$  and  $\lambda f$ .

By definition and [Corollary 1.1.2.1](#)

$$\begin{aligned}\|\hat{x}\| &= \sup\{|\hat{x}(f)| : f \in X^*, \|f\| \leq 1\} \\ &= \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\} \\ &= \|x\|\end{aligned}$$

$\square$

**Definition 1.1.4.** A normed space  $X$  is said to be reflexive if the map  $\rho : X \rightarrow X^{**} : x \rightarrow \hat{x}$  is surjective. (This is a stronger condition than  $X \equiv X^{**}$ )

**Theorem 1.1.4.** There are isometric isomorphisms between

- $(c_0)^*$  and  $\ell^1$
- 
- $(\ell^1)^*$  and  $\ell^\infty$

*Proof.* • Let  $(x_n) \in \ell^1$ . Then consider the map  $\phi_{(x_n)} : c_0 \rightarrow \mathbb{F}$  defined as

$$\phi_{(x_n)} : (y_n) \rightarrow \sum_{n \in \mathbb{N}} x_n y_n$$

We claim that  $\phi_{(x_n)}$  is a continuous linear functional. But first we should see that the sum is well defined. Since  $y_n \rightarrow 0$ , there is an  $N \in \mathbb{N}$  such that  $|y_n| < 1$  for all  $n \geq N$ . Since

$$\left| \sum_{i=N}^{\infty} x_i y_i \right| \leq \sum_{i=N}^{\infty} |x_i| |y_i| \leq \|(x_n)\|_1$$

we see that the sum is well defined and the map makes sense. Also since  $(y_n) + (z_n) = (y_n + z_n) \in \mathbf{c}_0$  whenever  $(y_n), (z_n) \in \mathbf{c}_0$ , we get that

$$\sum_{n \in \mathbb{N}} x_n(y_n + z_n) = \sum_{n \in \mathbb{N}} x_n y_n + \sum_{n \in \mathbb{N}} x_n z_n$$

which shows the linearity of the map  $\phi_{(x_n)}$ .

Now we show that  $\|\phi_{(x_n)}\| = \|(x_n)\|_1$ . We immediately see that for  $(y_n) \in c_0$  with  $\|(y_n)\|_{\sup} = \sup_{n \in \mathbb{N}} y_n = 1$ ,

$$|\phi_{(x_n)}((y_n))| = \left| \sum_{n \in \mathbb{N}} x_n y_n \right| \leq \|(y_n)\|_{\sup} \left( \sum_{n \in \mathbb{N}} |x_n| \right) \leq \|(x_n)\|_1$$

which gives  $\|\phi_{(x_n)}\| \leq \|(x_n)\|_1$ . Now let  $\theta_j \in [0, 2\pi)$  such that  $|x_j| = e^{i\theta_j} x_j$ . Now consider the sequence  $s_m \in \mathbf{c}_0$  defined as  $s_m = \sum_{j=1}^m e^{i\theta_j} e_j$ , where  $e_j$  is the sequence with  $j$ th entry 1 and the rest of the entries 0. Since  $(x_n) \in \ell_1$ , for all  $\epsilon \geq 0$  there exists an  $N_\epsilon \in \mathbb{N}$  such that

$$\sum_{i=N_\epsilon+1}^{\infty} |x_i| < \epsilon$$

Then since

$$|\phi_{(x_n)}(s_{N_\epsilon})| = \left| \sum_{n=1}^{N_\epsilon} e^{i\theta_j} x_n \right| = \sum_{i=1}^{N_\epsilon} |x_n| = \|(x_n)\| - \sum_{i=N_\epsilon+1}^{\infty} |x_n| \geq \|(x_n)\| - \epsilon$$

and  $\epsilon > 0$  was arbitrary, we get that  $\|\phi_{(x_n)}\| = \|(x_n)\|$

Hence we see that the map  $(x_n) \rightarrow \phi_{(x_n)}$  is an isometric linear map. Now for surjectivity, let  $\phi \in \mathbf{c}_0^*$ . We claim that the sequence  $(y_n) = (\phi(e_n)) \in \ell^1$  and  $\phi = \phi_{(y_n)}$ . Let  $\theta_j \in [0, 2\pi)$  such that  $e^{i\theta_j} y_j = |y_j|$ . Then for any  $N \in \mathbb{N}$ , we have

$$\begin{aligned} \sum_{j=1}^N |\phi(e_j)| &= \sum_{j=1}^N e^{i\theta_j} \phi(e_j) \\ &= \phi \left( \sum_{j=1}^N e^{i\theta_j} e_j \right) \\ &\leq \|\phi\| \left\| \sum_{j=1}^N e^{i\theta_j} e_j \right\| \\ &= \|\phi\| \end{aligned}$$

Since this is true for all  $N \in \mathbb{N}$ , taking the limits as  $N \rightarrow \infty$ , the inequality is preserved and we get that  $(y_n) \in \ell^1$ . Moreover  $\phi = \phi_{(y_n)}$  follows from the definition of  $\phi_{(x_n)}$ . Hence we get that  $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$ .

•

- The proof of this will be extremely similar to what we attempted before when we proved  $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$ . Let  $(x_n) \in \ell^\infty$ . Then consider the map  $\phi_{(x_n)} : \mathbf{c}_0 \rightarrow \mathbb{C}$  defined as

$$\phi_{(x_n)} : (y_n) \rightarrow \sum_{n \in \mathbb{N}} x_n y_n$$

By a similar way as we did in the above equivalence we see that  $\phi_{(x_n)}$  is linear. Moreover since

$$\left| \sum_{n \in \mathbb{N}} x_n y_n \right| \leq \|(x_n)\|_\infty \left| \sum_{n \in \mathbb{N}} y_n \right| = \|(x_n)\|_\infty \|(y_n)\|_1$$

we see that  $\|\phi_{(x_n)}\| \leq \|(x_n)\|_\infty$ . To get the reverse inequality, Let  $\|(x_n)\|_\infty = M$ , then for any  $\epsilon > 0$ , there exist some  $x_k$  in the sequence  $(x_n)$  such that  $|x_k - M| < \epsilon$ . Now consider the sequence  $e_k \in \ell^1$  with  $k$ th entry 1 and all the rest of them 0. We get that

$$|\phi_{(x_n)}(e_k)| = |x_k| \geq \|(x_n)\|_\infty - \epsilon$$

Since  $\epsilon$  was arbitrary, we get that  $\|\phi_{(x_n)}\| = \|(x_n)\|_\infty$ . Hence the map  $(x_n) \rightarrow \phi_{(x_n)}$  is an isometry. To show that it is indeed a bijection, assume  $\phi \in (\ell^1)^*$ , then consider the sequence  $y_n = \phi(e_n)$ . Since  $\phi$  is continuous, it is bounded above by  $\|\phi\|$  and we get that  $y_n \leq \|\phi\|$ . Therefore  $(y_n) \in \ell^\infty$ . Moreover we can verify like above that  $\phi = \phi_{(y_n)}$  from the definition of  $\phi_{(y_n)}$ . Hence we get  $(\ell^1)^* \cong^{\text{iso}} \ell^\infty$ . □

**Theorem 1.1.5.** Let  $1 < p < \infty$ , and  $q \in \mathbb{R}$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $(\ell^p)^* \cong \ell^q$

*Proof.* Let  $(a_n) \in \ell^p, (b_n) \in \ell^q$ , then  $\sum_{n \in \mathbb{N}} a_n \bar{b}_n$  is the map to check for isometric isomorphism. Use Holder's inequality as needed. verify □

## 1.2 Banach Limits

**Theorem 1.2.1.** There exists  $\phi \in (\ell^\infty)^*$  satisfying the following

$$1 \ \forall (a_n) \in \ell^\infty \text{ with } a_n \geq 0 \text{ for all } n \in \mathbb{N}, \phi((a_n)) \geq 0$$

2 If  $(a_n)$  is convergent, then  $\phi((a_n)) = \lim_{n \rightarrow \infty} a_n$

3 If  $(a_n) \in \ell^\infty$  and  $b_n = a_{n+1}$ , then  $\phi((b_n)) = \phi((a_n))$

Moreover such  $\phi$  is called a Banach limit.

*Proof.* Let  $S : \ell^\infty(\mathbb{R}) \rightarrow \ell^\infty(\mathbb{R})$  and  $T = I - S$  where  $I$  is the identity map. Also let  $V = \text{Range}(T) + c$  where  $c \in \mathbf{c}$ , the set of convergent sequences.

Define  $\phi : V \rightarrow \mathbb{R}$ ,  $\phi(a_n - a_{n+1} + x_n) = \lim_{n \rightarrow \infty} x_n$ .

- Claim 1:  $\phi$  is well defined

- Claim 2:  $\|\phi\| = 1$

Assuming the claims, by Hahn Banach,  $\phi$  extends to  $\tilde{\phi} \in \ell^\infty(\mathbb{R})$  with  $\|\tilde{\phi}\| = 1$ . Then by the last lemma we get  $\tilde{\phi}((y_n)) \geq 0$  for all  $(y_n) \in \ell^\infty(\mathbb{R})$  with  $y_n \geq 0$   $\square$

*Proof of Claim 1.* Suppose that  $(a_n) \in \ell^\infty$  is a sequence such that  $a_n - a_{n+1}$  converges, say  $a_n \rightarrow a_{n+1} \rightarrow \alpha$ . If  $\alpha > 0$ , then  $\exists N \in \mathbb{N}$  such that for all  $n > N$ ,  $a_n - a_{n+1} > \frac{\alpha}{2}$ . So  $a_N > \frac{\alpha}{2} + a_{N+1} > \dots > k\frac{\alpha}{2} + a_{N+k}$ . So for all  $k \in \mathbb{N}$ ,  $a_N - a_{N+k} = k\frac{\alpha}{2} \rightarrow \infty$  contradicting our assumption that  $a_n - a_{n+1}$  converges.

Now assume that  $(a_n), (b_n) \in \ell^\infty(\mathbb{R})$  with  $(x_n), (y_n) \in \mathbf{c}$  such that  $a_n - a_{n+1} + x_n = b_n - b_{n+1} + y_n$ . Then  $(a_n - b_n) - (a_{n+1} - b_{n+1}) = y_n - x_n$ . Then since RHS is a convergent limit, LHS must be convergent, which we get from above that it must converge to zero. Then  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$   $\square$

*Proof Claim 2.* To complete the proof, define  $\Psi : \ell^\infty \rightarrow \mathbb{C}$  by  $\Psi((a_n + ib_n)) = \tilde{\phi}(a_n) + i\tilde{\phi}(b_n)$  **verify**  $\square$

**Corollary 1.2.1.1.**  $\ell^1$  is not reflexive

*Proof.* Let  $\phi \in (\ell^\infty)^*$  be a Banach limit. FTOC, assume  $\exists f = (\alpha_n) \in \ell^1$  such that

$$\phi((a_n)) = \sum_{i=1}^{\infty} a_n \overline{\alpha_n}$$

Then for all  $m \in \mathbb{N}$ ,  $\overline{\alpha_m} = \phi(\delta_m) = 0$ , where  $\delta_m = (0, 0, \dots, 1, 0, 0, \dots)$ . But this contradicts since we assumed  $\phi \neq 0$  by the Hahn Banach extension from  $c_0$   $\square$

**Lemma 1.2.1.** Let  $\psi \in (\ell^\infty)^*$ . then the following are equivalent.

1  $\|\psi\| = \psi((1, 1, 1, \dots))$

2 If  $(a_n) \in \ell^\infty$  with  $a_n \geq 0, \forall n \in \mathbb{N}$ . Then  $\psi((a_n)) \geq 0$

*Proof.* (1  $\implies$  2) FTSOC assume  $\exists(a_n) \in \ell^\infty$ ,  $\psi((a_n)) < 0$ . WLOG, assume  $|a_n| \leq 1$  for all  $n \in \mathbb{N}$ . let  $b_n = 1 - a_n$ . Then  $0 \leq b_n \leq 1$  and

$$\psi((b_n)) > \psi((1, 1, 1, \dots)) - \psi((a_n)) > \psi((1, 1, 1, \dots))$$

So

$$\|\psi\| \geq |\psi((b_n))| \geq \psi((1, 1, \dots))$$

(2  $\implies$  1) Let  $(a_n) \in \ell^\infty$  with  $|a_n| \leq 1$ , then  $0 \leq 1 - a_n$ . So  $\psi((1 - a_n)) \geq 0$  and therefore  $\psi((1, 1, 1, \dots)) \geq \psi((a_n))$ . Similarly  $\psi((-a_n)) \leq \psi((1, 1, 1, \dots))$  which gives  $|\psi((a_n))| \leq \psi((1, 1, 1, \dots))$   $\square$

## 1.3 Quotient Spaces

**Definition 1.3.1.** Let  $X$  be a normed space and  $Y \leq X$  be a closed subspace. For every  $x \in X$ , define

$$\|x + Y\| = \inf\{\|x + y\| : y \in Y\}$$

**Lemma 1.3.1.** This defines a norm on  $\frac{X}{Y}$ . If  $X$  is complete, then  $\frac{X}{Y}$  is complete.

*Proof.* Obviously,  $\|x + Y\| \geq 0$  for all  $x \in X$ , and  $\|x + z + Y\| \leq \|x + Y\| + \|y + Y\|$ . Similarly, we can also show homogeneity.

Now assume  $x \in X$  is such that  $\|x + Y\| = 0$ . Then there is a sequence  $(y_n) \in Y$  such that  $\|x - y_n\| \rightarrow 0$ , that is  $y_n \rightarrow x$ . Since  $Y$  is closed, we get  $x \in Y$ .

To show the second part of the lemma, consider the sequence  $(x_n + Y) \in X/Y$  such that  $\sum_{n \in \mathbb{N}} \|x_n - Y\| < \infty$ . For each  $n \in \mathbb{N}$ , choose  $y_n \in Y$  such that

$$\|x_n + y_n\| \leq \|x_n + Y\| + \frac{1}{2^n}$$

Then  $\sum_{n \in \mathbb{N}} \|x_n + y_n\| \leq \infty$ . Since  $X$  is complete, the sequence  $\sum_{n \in \mathbb{N}} x_n + y_n$  converges to say  $z \in X$ . Then

$$\begin{aligned} \|(z + Y) - \sum_{n=k}^n (x_n + Y)\| &= \left\| \left( z - \sum_{n=k}^n x_n \right) + Y \right\| \\ &= \left\| \left( z - \sum_{i=1}^k (x_n + y_n) \right) + Y \right\| \\ &\leq \left\| \left( z - \sum_{i=1}^k (x_n + y_n) \right) \right\| \end{aligned}$$

which converges to 0 as  $k \rightarrow \infty$   $\square$

**Lemma 1.3.2.** *The canonical map,  $q : X \rightarrow \frac{X}{Y}$  is a continuous open map. A subset  $E \subset X/Y$  is open iff  $q^{-1}(E) \subset X$  is open.*

*Proof.* Since  $\|x + Y\| \leq \|x\|$ , for all  $x \in X$ , we see that the map  $q$  is a contraction. Thus for all open  $E \subset X/Y$ , we get  $q^{-1}(E)$  is open.

Conersely, assume that  $A \subset X$  is open. Let  $x \in A$  and  $r > 0$  such that  $B_r(x) \subset A$ . Let  $z \in X$  such that  $\|q(x) - q(z)\| < r$ . So,  $\|(x - z) + Y\| < r$ . Then  $\exists y \in Y$  such that  $\|x - z - y\| < r$ . So  $z + y \in B_r(x)$ ,  $q(z + y) = q(z) \in q(B_r(x))$ . So  $B_r(q(x)) \subset q(B_r(x)) \subset q(A)$ . Thus  $q(A)$  is open.  $\square$

## 1.4 Open Mapping Theorem and Closed Graph Theorem

**Theorem 1.4.1** (Open Mapping Theorem). *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  be a surjective bounded linear map. Then  $T$  is an open map i.e  $T(E)$  is open in  $Y$  i.e if  $E \subset X$ , then  $T(E)$  is open in  $Y$ .*

*Steps of proof.* See Prof. Blecher's Notes on Functional Analysis

Use Baire category theorem to show that  $\overline{T(B_r(0))}$  has a non-empty interior.

- Then use linearity of  $T$  to show that  $0 \in \overline{T(B_{2r}(0))}$ .

$\square$

*Proof.* Since  $Y$  is complete, by the Baire category theorem it is of the second category. Let  $r \geq 0$ , then

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B_{nr}(0)\right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB_r(0)\right) \\ &= \bigcup_{n=1}^{\infty} n\overline{T(B_r(0))} \end{aligned}$$

Then by BCT, there exist some  $n \in \mathbb{N}$  such that  $\text{int}(\overline{T(B_r(0))}) \neq \emptyset$ . Let  $y_0 \in \text{int}(\overline{T(B_r(0))})$ . So there exists  $\epsilon > 0$ , such that  $B_\epsilon(y_0) \subset \overline{T(B_r(0))}$ . Let  $w \in B_Y(0, \epsilon)$ . Then  $y_0 + w \in B_Y(y_0, \epsilon)$ , and  $\exists(x_n) \subset B_X(0, r)$  such that  $T(x_n) \rightarrow$

$y_0 + w$ . Also  $\exists(z_n) \in B_X(0, r)$  such that  $T(z_n) \rightarrow y_0$ . Then  $T(x_n - z_n) \rightarrow w$ , so  $w \in \overline{T(B_X(0, 2r))}$ . Since  $w$  was an arbitrary element in  $B_Y(0, \epsilon)$  we see that  $B_Y(0, \epsilon) \subset \overline{T(B_X(0, 2r))}$ . So  $0 \in \text{int}(\overline{T(B_X(0, s))})$  for  $s > 0$ .

Now fix  $t > 0$  and let  $y_0 \in \overline{T(B_X(0, t))}$ . By the above there exists  $\epsilon > 0$  such that  $B_Y(0, \epsilon) \subset \overline{T(B_X(0, \frac{t}{2}))}$ . Then  $(y_0 + B_Y(0, \epsilon)) \cap \overline{T(B_X(0, \frac{t}{2}))} \neq \emptyset$ . So  $\exists x \in B_t(0)$  such that  $T(x_1) = y_0 - y_1$  where  $y_1 \in B_Y(0, \epsilon) \subset \overline{T(B_X(0, \frac{t}{2}))}$ .

Similarly  $\exists y_2 \in \overline{T(B_{\frac{t}{4}}(0))}$  and  $x_2 \in B_{\frac{t}{2}}(0)$  such that  $T(x_2) = y_1 - y_2$ . Thus inductively we can choose  $y_{n+1} \in \overline{T(B_{\frac{t}{2^{n+1}}}(0))}$  and  $x_{n+1} \in B_{\frac{t}{2^n}}(0)$  such that  $T(x_{n+1}) = y_n - y_{n+1}$ . Now since we constructed nicely,  $\sum_{i=0}^{\infty} x_n$  converge. **verify**. Moreover for all  $N \in \mathbb{N}$ , we have

$$\sum_{n=1}^N T(x_n) = y_0 - y_N$$

Also notice that  $y_n \rightarrow 0$ . Hence  $y_0 = \lim_{N \rightarrow \infty} (y_0 - y_N) = \lim_{N \rightarrow \infty} \sum_{n=1}^N T(x_n) = \lim_{N \rightarrow \infty} T(\sum_{n=1}^N x_n) = T(x) \in \overline{T(B_{2t}(0))}$ . So

$$\overline{T(B_t(0))} \subset T(B_{2t}(0))$$

Now to complete the proof, let  $E$  be an open subset of  $X$ . Let  $x_0 \in E$  be such that  $y_0 = T(x_0)$ . Let  $\epsilon > 0$  be such that  $x_0 + B_\epsilon(0) = B_\epsilon(x_0) \subset E$ . So  $y_0 + T(B_\epsilon(0)) = T(B_\epsilon(x_0)) \subset T(E)$ . By the above  $\exists \delta > 0$  such that  $B_\delta(0) \subset \overline{T(B_\epsilon(0))}$ .  $\square$

**Find examples where this fails if we slack the conditions**

**Corollary 1.4.1.1.** *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  be a bijective bounded linear map. Then  $T^{-1} : Y \rightarrow X$  is bounded.*

*Proof.* **verify**  $\square$

**Theorem 1.4.2** (Closed Graph Theorem). *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  be a linear map. Then  $T$  is bounded if and only if graph of  $T$ , defined as  $g(T) = \{(x, T(x)) : x \in X\}$  is closed in the product topology of  $X \times Y$ .*

*Proof.* Define the norm  $\|(x, y)\| = \|x\|_X + \|y\|_Y$  on  $X \times Y$ . Then  $X \times Y$  with this norm is a Banach space. **verify**.

Assume  $T$  is continuous. Then if  $(x_n, T(x_n))$  is Cauchy in  $g(T)$  then  $x_n$  and  $T(x_n)$  must be cauchy in  $X$  and  $Y$  respectively. By the completeness of the spaces  $X$  and  $Y$ , we get  $x_n \rightarrow x \in X$  and  $T(x_n) \rightarrow y \in Y$ . Moreover by continuity of  $T$ , we get  $T(x_n) \rightarrow T(x)$ . Since the Banach space is Hausdorff, we get  $y = T(x)$  and that  $(x, T(x)) \in g(T)$  making it closed.



Conversely, define  $S : X \rightarrow g(T)$  as  $S(x) = (x, T(x))$ .  $S$  is linear and bijective. Assume  $g(T)$  is closed, hence a Banach space. Observe that  $S^{-1} : g(T) \rightarrow X$  is bounded (contractive). By the open mapping theorem,  $S$  is bounded. Assume  $x_n \rightarrow z$ . So  $S(x_n) \rightarrow S(z)$ . Then  $(x_n, T(x_n)) \rightarrow (z, T(z))$ , which gives  $T(x_n) \rightarrow T(z)$ .  $\square$

# Chapter 2

## Topological Vector Spaces

**Example 2.0.1.** Let  $X$  be a vector space and let  $f : X \rightarrow \mathbb{C}$  be a linear map. Define  $\phi : X \rightarrow \mathbb{R}^+ := \phi(x) = |f(x)|$ . Then  $\phi$  is a seminorm.

### 2.1 Weak and Weak \* Topologies

*Remark 2.1.1.* Let  $X$  be a TVS and  $A \subset X^*$ . We denote by  $\sigma(X, A)$ , the topology on  $X$  defined by  $A$ . (initial topology). Recall that  $\sigma(X, A)$  is Hausdorff if and only if  $A$  separate points of  $X$ .

$\sigma(X, X^*)$  is called the weak topology on  $X$ .

Also recall that  $X \hookrightarrow X^{**}$  by the evaluation maps. Hence we can view  $X$  as a subset  $X^{**}$ . And with this identification, we call  $\sigma(X^*, X)$  the weak \* topology on  $X^*$ .

**Definition 2.1.1.** Let  $S$  be any set. Let  $I$  be a directed set. A net in  $S$  indexed by  $\Lambda$  is a function  $f : \Lambda \rightarrow S$ . We denote the net by  $(x_\lambda)_{\lambda \in \Lambda}$ .

In addition if  $S$  is a topological space, we say a net  $(x_\lambda)_{\lambda \in \Lambda}$  converges to a point  $x \in S$  if for all open set  $U$  in  $S$  with  $x \in U$ , there exists an  $\lambda_0 \in \Lambda$  such that for all  $\lambda \geq \lambda_0$  we have  $x_\lambda \in U$ .

*Remark 2.1.2.* By definition, a basis of open neighborhoods of a point  $x_0 \in X$  in  $\sigma(X, A)$  is given

$$\bigcup_{\substack{p_1, p_2, \dots, p_n \in A \\ \epsilon_1, \epsilon_2, \dots, \epsilon_n > 0}} \bigcap_{i=1}^n \{z \in X \mid p_i(z - x_0) < \epsilon_i\}$$

So the basis in a weak topology is

$$\bigcup_{f_1, f_2, \dots, f_n \in X^*} \bigcap_{\epsilon_1, \epsilon_2, \dots, \epsilon_n > 0} (x_0) = \{z \in X \mid |f_k(z - x_0)| < \epsilon_k, \text{ for all } k = 1, 2, \dots, n\}$$

See how nets become the topological equivalent of sequences in metric spaces. Find what property of the metric space makes it enough to be indexed by a countable totally ordered set for openness.

**Example 2.1.1.** Let  $S$  be a topological space,  $E \subset S$ ,  $x_0 \in \overline{E}$ . Then there is a net  $(x_\alpha) \subset E$  such that  $x_\alpha \rightarrow x_0$ .

*Proof.* Consider the collection  $\mathcal{T}$  of all open sets of  $S$  that contain  $x_0$ . Order  $\mathcal{T}$  by the reverse inclusion. That is  $A \leq B$  if  $B \subset A$ . This makes  $\mathcal{T}$ , a directed set. Now each  $\lambda \in \mathcal{T}$  has a nonempty intersection with  $E$  being an open set containing the limit point  $x_0$  of  $E$ . For each  $\lambda \in \mathcal{T}$  choose an  $x_\lambda \in \lambda \cap E$ . Then we claim that the net  $(x_\lambda) \rightarrow x_0$ .  $\square$

**Example 2.1.2.** Let  $X = \ell^1$ . Then  $X^* = \ell^\infty$ . Then the weak \* topology on  $\ell^\infty$  is given by the pointwise convergence of a net  $(f_\alpha) \subset \ell^\infty$  converges to  $f \in \ell^\infty$  if and only if  $f_\alpha(n) \rightarrow f(n)$  for all  $n \in \mathbb{N}$

**Theorem 2.1.1** (Tychonoff). *The product of compact sets is compact.*

**Corollary 2.1.1.1.** *Let  $X$  be a compact Hausdorff space. Then for any set  $S$ , the set  $\{\phi : S \rightarrow X\} = X^S$  is compact wrt to pointwise convergence.*

**Theorem 2.1.2** (Banach-Alaoglu Theorem). *Let  $X$  be a normed space. Then the closed unit ball  $\overline{B_{X^*}}(0, 1) = \{f \in X^* : \|f\| \leq 1\} = E$  is weak \* compact.*

*Proof.* Let  $\bar{B}$  be the closed unit ball of  $X$ . Then by Tychonoff theorem,  $\bar{D}^{\bar{B}}$  is compact. Define  $\phi : E \rightarrow \bar{D}^{\bar{B}}$  as  $\phi(f)(x) := f(x)$ . Observe that  $\phi$  is injective. Also observe that  $\phi$  is continuous (weak \* in LHS, and pointwise in RHS).

Next we show that the image of  $\phi$  a closed subset of  $\bar{D}^{\bar{B}}$ , hence compact. Let  $f_i$  be a net in  $E$  such that  $\phi(f_i) \rightarrow \psi$  pointwise for some  $\psi \in \bar{D}^{\bar{B}}$ .

Define  $g : X \rightarrow C$  as  $g(x) = \alpha\psi(\frac{x}{\alpha})$  where  $\|x\| \leq \alpha$ . For this to be well defined we must have  $\|x\|\psi(\frac{x}{\|x\|}) = \alpha\psi(\frac{x}{\alpha})$  for any  $\alpha > \|x\|$ . But we get this since  $\psi$  is a pointwise limit of linear functionals. Moreover we get that  $g$  is linear for the same reason. Thus  $\psi = \phi(g)$  and so  $\phi(E)$  is closed.

It only remains to show that the inverse of  $\phi$  is continuous. **verify**.  $\square$

**Proposition 2.1.1.** *The closed unit ball of a normed space  $Y$  is compact w.r.t the norm topology if and only if  $Y$  is finite dimensional.*

*Proof.* **verify**  $\square$

**Theorem 2.1.3.** *Let  $X$  be a normed space. Then  $E$  is weak \* metrizable iff  $X$  is separable.*

*Proof.* Assume  $X$  is separable. Let  $\{x_n : n \in \mathbb{N}\}$  be a dense subset of  $X$ . For every  $f, g \in E$ , define  $d(f, g) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} |f(\frac{x_n}{\|x_n\|}) - g(\frac{x_n}{\|x_n\|})|$ . Check that  $d$  is a metric.

Now assume  $f_i \rightarrow f$  weakly in  $E$ . Then  $f_i(x_n) \rightarrow f(x_n)$  for all  $n \in \mathbb{N}$  and  $d(f_i, f) \rightarrow 0$  (**verify**).

Assume  $E$  is metrizable. Then  $\exists \{U_n : n \in \mathbb{N}\}$  of weak  $*$  open neighborhoods of 0 such that  $\cap_{n=1}^{\infty} U_n = \{0\}$ . So, for each  $n \in \mathbb{N}$ , there exists a finite set  $A_n \in X$  and  $\epsilon > 0$  such that the (subbasis sets)  $\{f \in E : |f| \leq \epsilon \forall x \in A_n \subset U_n\}$ . Now let  $A = \cup_{n=1}^{\infty} A_n$ . Let  $\phi \in E$  such that  $\phi(x) = 0$  for all  $x \in A$ .  $\square$

**Definition 2.1.2.** Let  $X$  and  $Y$  be normed spaces and  $T \in B(X, Y)$ . The adjoint of  $T$ , denoted by  $T^* \in B(Y^*, X^*)$ , is the map  $T^* : f \rightarrow f \circ T$

**Proposition 2.1.2.**  $\|T\| = \|T^*\|$

*Proof.*  $|T^*(f)| \leq \|f \circ T\| \leq \|T\| \|f\|$  implies  $\|T^*\| \leq \|T\|$

$$\begin{aligned} \|T^*\| &= \sup\{\|T^*(\phi)\| : \phi \in Y^*, \|\phi\| \leq 1\} \\ &= \sup\{|\phi(T(x))| : \phi \in Y^*, x \in X, \|\phi\| \leq 1, \|x\| \leq 1\} \\ &= \|T\| \end{aligned}$$

Note that the last equality is a consequence of HBT since it guarantees the existence of  $\phi_y \in Y^*$  with  $\|\phi_y\| \leq 1$  and  $\phi_y(y) = |y|$ .  $\square$

**Lemma 2.1.1.** For any  $T \in B(X, Y)$ ,  $T^* : Y^* \rightarrow X^*$  is weak  $*$  continuous (in both spaces)

*Proof.* Let  $\phi_i \rightarrow \phi$  weakly in  $Y^*$ . Then by definition for all  $y \in Y$ ,  $\phi_i(y) \rightarrow \phi(y)$ . Then for  $x \in X$ ,  $T^*(\phi_i)(x) = \phi_i(T(x)) \rightarrow \phi(T(x)) = (T^*(\phi))(x)$  which shows the continuity of  $T^*$ .  $\square$

**Lemma 2.1.2.** For any normed space  $X$ ,  $i_X(X)$  is weak  $*$  dense in  $X^{**}$ .

*Proof.* **verify**  $\square$

**Example 2.1.3.** Is  $i_{X^*}$  weak  $*$  - weak  $*$  continuous.

## 2.2 Locally Convex Topological Vector Spaces

**Lemma 2.2.1.** Let  $X$  be a normed space and  $x_1, x_2, \dots, x_n \in X$  and  $\epsilon_1, \epsilon_2, \dots, \epsilon_n \geq 0$ . Then the set

$$\bigcup_{x_1, x_2, \dots, x_n, \epsilon_1, \epsilon_2, \dots, \epsilon_n} (\phi)$$

is convex. Moreover any topological vector spaces with the topology induced by a family of seminorms is locally convex.

*Proof.* **verify**  $\square$

Review  
convexity  
arguments  
in  
Arveson's  
'Subalgebras of  $C^*$   
Algebras'

**Definition 2.2.1.** Let  $X$  be a vector space and  $E \subset X$  be a convex subset. An element  $a \in E$  is called an extreme point of  $E$  if whenever  $x, y \in E$ ,  $0 \leq t \leq 1$  with  $a = tx + (1 - t)y$ , then  $x = y = a$ .

**Example 2.2.1.** Let  $\bar{D} = \{\alpha \in \mathbb{C} : |\alpha| \leq 1\}$ . Then  $\bar{D}$  is convex with  $\text{Ext}(\bar{D}) = S^1$

**Theorem 2.2.1** (Krein-Milman Theorem). *Let  $X$  be a locally convex space, and let  $K$  be a compact convex subset of  $X$ . Then the  $\text{Ext}(K) \neq \emptyset$  and indeed  $K = \overline{\text{co}}(\text{Ext}(K))$*

**Definition 2.2.2.** Let  $V$  be a vector space and  $S \subset V$ . The convex hull of  $S$  is defined as

$$\text{co}(S) = \left\{ \sum_{i=1}^n t_i x_i \mid 0 \leq t \leq 1, \sum t_i = 1, x_i \in S \right\}$$

**Lemma 2.2.2.** *Let  $K$  be a convex set.  $x_0 \in \text{Ext}(K)$  if and only if  $K \setminus \{x_0\}$  is convex.*

*Proof.* If  $K \setminus \{x_0\}$  is not convex, since  $K$  is convex,  $x_0$  can be written as the convex combination of elements in  $K \setminus \{x_0\}$  which makes  $x_0 \notin \text{Ext}(K_0)$ . Conversely if  $x_0 \in \text{Ext}(K)$ , then  $x_0$  cannot be written as the convex combination of elements of  $K$ . Hence  $K \setminus \{x_0\}$  is closed under convex combinations, making in convex.  $\square$

**Theorem 2.2.2** (Krein-Milman Theorem). *Let  $X$  be a locally convex space, and let  $K$  be a compact convex subset of  $X$ . Then the  $\text{Ext}(K) \neq \emptyset$  and indeed  $K = \overline{\text{co}}(\text{Ext}(K))$*

*Proof.* We first prove that the  $\text{Ext}(K) \neq \emptyset$ . Note that  $K \setminus \{x_0\}$  is a relatively open subset of  $K$  since  $\{x_0\}$  is closed and  $K \setminus \{x_0\} = \{x_0\}^c$  relative to  $K$ .

Now let  $\mathcal{A}$  be the collection of all relatively open convex proper subsets of  $K$ . Note that  $\emptyset \in \mathcal{A}$ , therefore  $\mathcal{A}$  is nonempty. Equip  $\mathcal{A}$  with the partial order defined by the set inclusion. Let  $\mathcal{C}$  be a chain in  $\mathcal{A}$  and  $F_{\mathcal{C}} = \cup_{C \in \mathcal{C}} C$ .  $F_{\mathcal{C}}$  is relatively open being the union of relatively open subsets of  $K$ . To see that  $F_{\mathcal{C}}$  is convex, let  $x, y \in F_{\mathcal{C}}$ . Then since  $\mathcal{C}$  is a chain, there exist a  $C \in \mathcal{C}$  such that  $x, y \in C$ . Then by the convexity of  $C$ ,  $tx + (1 - t)y \in C \subset F_{\mathcal{C}}$  for all  $t \in [0, 1]$ .

We claim that  $F_{\mathcal{C}}$  is a proper subset of  $K$ . For the sake of contradiction, assume  $F_{\mathcal{C}} = K$ . Since  $K$  is compact and  $C$  is open in  $K$  for all  $C \in \mathcal{C}$ , there are finitely many  $C_1 \subset C_2 \subset \dots \subset C_k \in \mathcal{C}$  which cover  $K$  (i.e  $K = \cup_{n=1}^k C_n$ ). Hence we get  $C_k = K$ , which is absurd since  $C_k$  must be a proper subset of  $K$ . Hence  $F_{\mathcal{C}} \in \mathcal{A}$  and thus every chain must have an upper bound in  $\mathcal{A}$ . Now by Zorn's lemma,  $\mathcal{A}$  has a maximal element  $K_0$ .

Since  $K$  is a connected space (path connected by a straight line, being convex), we know that the only clopen subsets are  $\emptyset$  and  $K$ . Since we know that  $K_0$  is open

Zorn's  
Lemma to  
find a  
maximal  
proper  
open  
convex  
subset of  $K$

Construct-  
ing and  
open  
convex  
subset  
containing  
 $K_0$

being in  $\mathcal{A}$ , we see that  $K_0 \neq K$  and  $K_0 \neq \emptyset$ . Therefore  $\overline{K_0} \neq K_0$ . Let  $x_0 \in \overline{K_0} \setminus K_0$ ,  $y_0 \in K_0$  and  $0 < t < 1$ . Define  $\varphi_{t,y_0} : K \rightarrow K$  such that  $\varphi_{t,y_0}(z) = ty_0 + (1-t)z$ . Then  $\varphi_{t,y_0}$  is  $(1-t)$  Lipschitz continuous relative to  $K$  and thus  $\varphi_{t,y_0}^{-1}(K_0)$  is open in  $K$ . By the convexity of  $K_0$ , we get  $K_0 \subset \phi_{t,y_0}^{-1}(K_0)$ .

Also  $\varphi_{t,y_0}^{-1}(K_0)$  is convex. Let  $a, b \in \phi_{t,y_0}^{-1}(K)$ . Then  $ty_0 + (1-t)a, ty_0 + (1-t)b \in K_0$ . By the convexity of  $K_0$  we get  $r(ty_0 + (1-t)a) + (1-r)(ty_0 + (1-t)b) = ty_0 + (1-t)(ra + (1-r)b) = \phi_{t,y_0}(ra + (1-r)b) \in K_0$  for all  $r \in [0, 1]$ . Thus  $ra + (1-r)b \in \phi_{t,y_0}^{-1}(K_0)$  for all  $r \in [0, 1]$ . Hence we get  $\phi_{t,y_0}^{-1}(K_0)$  is convex.

We claim,  $x_0 \in \varphi_{t,y_0}^{-1}(K_0)$ , then the maximality of  $K_0$  will force  $\phi_{t,y_0}^{-1}(K_0) = K$ . Let  $U$  be a convex neighborhood of  $0 \in X$  containing  $-x$  for all  $x \in U$  (just take  $-U$  and intersect with  $U$ ) such that  $y_0 + E \subset K_0$ , where  $E = K \cap U$ . Let  $w = \varphi_{t,y_0}(x_0)$ . Since  $x_0 \in \overline{K_0}$ , for any  $r > 0$ , there exists  $x_r \in K_0$  such that  $x_r \in (x_0 + rE) \cap K_0 \neq \emptyset$ . In particular, let  $r = \frac{t}{1-t}$ . Then by linearity, we get  $(x_0 + \frac{t}{1-t}E) \cap K_0 = (\frac{t}{1-t})E \cap (K_0 - x_0) \neq \emptyset$ . Choose  $z$  in the above set. Then

$$y_0 - \left(\frac{1-t}{t}\right)z \in y_0 + E \subset K_0$$

and  $x_0 + z \in K_0$ . Since  $K_0$  is convex,

$$t\left(y_0 - \frac{(1-t)}{t}z\right) + (1-t)(x_0 + z) = \phi_{t,y_0}(x_0) \in K_0$$

Thus  $\phi_{t,y_0}^{-1}(K_0) = K$ .

Now we claim that  $K = K_0 \cup \{x_0\}$ . For the sake of contradiction assume  $\exists p \in K$  such that  $p \notin K_0 \cup \{x_0\}$ . Since the space is Hausdorff and locally convex,  $x_0$  has an open convex neighborhood  $E$  in  $X$  such that  $p \notin E$ . Let  $E' = E \cap K$ ,  $a \in K_0, b \in E'$  and  $0 < r < 1$ . Then since  $\phi_{t,y_0}(K) = K_0$  for all  $t \in [0, 1]$ ,  $y_0 \in K_0$ , we get  $\phi_{r,a}(b) = ra + (1-r)b \in K_0$ . So  $K_0 \cup E'$  is convex (Sine we know that  $K_0, E'$  are convex, we only need to worry about  $rx + (1-r)y$  for  $x \in K_0, y \in E'$ . But  $\phi_{r,x}$  takes care of that).  $K_0 \cup E'$  is also open in  $K$ . Hence by maximality, we get  $K_0 \cup E' = K$ . But this is a contradiction since  $p \notin K_0 \cup E'$ . Thus by [Lemma 2.2.2](#), we see that  $x_0 \in \text{Ext}(K)$ .

Next we prove  $K = \overline{\text{co}}(\text{Ext}(K))$ . Let  $P = \overline{\text{co}}(\text{Ext}(K))$  and for the sake of contradiction assume  $P \neq K$ . Let  $x_0 \in K \setminus P$ . Now by the geometric Hahn-Banach separation theorem, we get that there is a continuous linear functional  $\phi : X \rightarrow \mathbb{R}$  and a number  $\alpha, \epsilon \in \mathbb{R}$  such that

$$\Re \phi(x_0) \leq \alpha < \alpha + \epsilon \leq \Re \phi(p), \quad \forall p \in P$$

□

I can't  
picturize  
the choice  
of  $z$

**Lemma 2.2.3.** *Let  $K_1, K_2$  be compact convex subsets of a locally compact TVS  $X$ . Then*

$$\overline{\text{co}}(K_1 \cup K_2) = (\text{co})(K_1 \cup K_2)$$

*Proof.* **verify.** We'll show that  $\text{co}(K_1 \cup K_2)$  is compact and hence closed. Let  $x = \alpha_1 a_1, \alpha_2 a_2, \dots, \alpha_n a_n + \beta_1 b_1, \beta_2 b_2, \dots, \beta_m b_m \in \text{co}(K_1 \cup K_2)$ , where  $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i = 1$ . Then

$$x = \left( \sum_{i=1}^n \alpha_i \right) \underbrace{\left( \sum_{i=1}^n \left( \frac{\alpha_i}{\sum_{i=1}^n \alpha_i} \right) a_i \right)}_{\in K_1} + \left( \sum_{i=1}^m \beta_i \right) \underbrace{\left( \sum_{i=1}^m \left( \frac{\beta_i}{\sum_{i=1}^m \beta_i} \right) b_i \right)}_{\in K_2}$$

Hence every element  $x \in \text{co}(K_1 \cup K_2)$ , can be written as  $x = ta + (1-t)b$  where  $a \in K_1, b \in K_2$ .

Now let  $x_\lambda = t_\lambda a_\lambda + (1-t_\lambda)b_\lambda$  be a net in  $\text{co}(K_1 \cup K_2)$ , for  $\lambda \in \Lambda$ ,  $a_\lambda \in K_1, b_\lambda \in K_2$ . Since  $(a_\lambda)$  is a net in the compact set  $K_1$ , there is a subnet  $a_\sigma$  for  $\sigma \in \Sigma \subseteq \Lambda$ , such that  $a_\sigma \rightarrow a \in K_1$ . By similar reasoning  $b_\sigma$  has a convergent subnet  $b_\pi$  for  $\pi \in \Pi \subseteq \Sigma$ , such that  $b_\pi \rightarrow b \in K_2$ . Again  $t_\pi$  is a net in the compact space  $[0, 1]$ , hence it has a convergent subnet  $t_\omega$  for  $\omega \in \Omega \subseteq \Pi$  such that  $t_\omega \rightarrow t$  in  $[0, 1]$ .

Now consider the subnet  $x_\omega = t_\omega a_\omega + (1-t_\omega)b_\omega$  of  $x_\lambda$ . Since  $\Omega \subseteq \Pi \subseteq \Sigma$ ,  $t_\omega \rightarrow t, b_\omega \rightarrow b$  and  $a_\omega \rightarrow a$ . Therefore by the continuity of the scalar product and addition in the TVS, we get  $x_\omega \rightarrow ta + (1-t)b \in \text{co}(K_1 \cup K_2)$ . Hence we get  $\text{co}(K_1 \cup K_2)$  is compact.  $\square$

**Theorem 2.2.3** (Inverse Krein-Milman). *Let  $K$  be a compact convex subset of a locally convex topological vector space  $X$ . Let  $A \subset K$  be a closed subset of  $K$ . If  $K = \overline{\text{co}}(A)$ , then  $\text{Ext}(K) \subset A$ .*

Note that  $\text{Prob}[0, 1]$ , the collection of probability measures identified as a subspace of a  $C([0, 1])^*$  is convex, weak \* compact with  $\text{Ext}(K) = \{\delta_x : x \in [0, 1]\}$

*Proof.* FSTOC, assume  $\exists x_0 \in \text{Ext}(K), x_0 \notin A$ . Since  $A$  is compact,  $\exists y_1, y_2, \dots, y_n \in A$  and an open convex neighborhood  $B$  of 0 such that

$$A \subset \cup_{i=1}^n (y_i + B)$$

and  $x_0 \notin y_i + \overline{B}$  for all  $i = 1, 2, \dots, n$ . Let  $B_i = (y_i + \overline{B}) \cap K$ . Then  $B_i$  is a compact convex subset of  $K$  for each  $i$ . Hence by the previous lemma, we get

$$\text{co}(B_1 \cup B_2 \cup \dots \cup B_n) = \overline{\text{co}}(B_1 \cup B_2 \cup \dots \cup B_n) \supset \overline{\text{co}}(A) = K$$

Thus  $\exists b_i \in B_i$  and  $0 \leq t_i \leq 1$ ,  $\sum_{i=1}^n t_i = 1$  such that

$$x_0 = t_n b_1 + t_n b_2 + \dots + t_n b_n$$

I struggle  
at finding  
the contra-  
diction

Since  $x_0 \in \text{Ext}(K)$ , this forces  $x_0 = b_j$  for some  $1 \leq j \leq n$ . This contradicts the assumption that  $x_0 \notin y_i + \overline{B}$ . Hence  $x_0 \in A$ .  $\square$

Note that in the following attempt to prove the theorem, it is not obvious why  $U$  is convex. If we try to argue using arguments to the proof of separating a compact set and a point using open sets in a Hausdorff space, we will eventually need to show that the finite open subcover of  $A$  sits inside a closed convex set that does not contain  $x_0$ , which again is not obvious.

*Proof.* FTSOC, assume that  $\exists x_0 \in \text{Ext}(K) \setminus A$ . Since the TVS is Hausdorff, there exist convex open sets  $U, V$  such that  $A \subset U, x_0 \in V$ , and  $U \cap V = \emptyset$ . Moreover we claim that  $\overline{U} \cap V = \emptyset$ . Otherwise if  $x \in \overline{U} \cap V$ , then for any net  $(x_\lambda) \in V$  that converge to  $x$ , by the definition of convergence  $x_{\lambda_n} \in U$  for all  $\lambda_n$  greater than some  $\lambda_N$ . This would contradict the assumption that  $U \cap V = \emptyset$ . Hence we see that  $A \subset \overline{U}$ , and therefore  $\overline{\text{co}}(A) \subset \overline{U}$ . But this would again contradict the fact that  $\overline{\text{co}}(A) = K$  since  $x_0 \notin \overline{U}$ .  $\square$

Why is  $U$   
convex?

**Example 2.2.2.** Let  $X$  be an infinite dimensional normed space. Then the set  $A = \{x \in X : \|x\| = 1\}$  is norm closed. But the weak closure of  $A$  is the set

$$\overline{A}^w = \overline{\{x \in X : \|x\| = 1\}}^w = \{y \in X : \|y\| \leq 1\}$$

Hence  $A$  is an example of a norm closed set, which is not weak closed.

**Theorem 2.2.4.** Let  $X$  be a normed space and let  $K \subset X$  be convex subset of  $X$ . Then the norm and the weak closure of  $K$  coincide.

*Proof.* Since norm topology is stronger than weak topology, we get  $\overline{K}^{\|\cdot\|} \subset \overline{K}^w$ . Let  $x \in X$  such that  $x \notin \overline{K}^{\|\cdot\|}$ . Now since  $\{x\}, K$  are convex and compact, by Hahn-Banach separation theorem, there is a  $f \in X^*, s \in \mathbb{R}, \epsilon > 0$  such that

$$|f(x)| \leq s < s + \epsilon \leq |f(y)|, \quad \forall y \in \overline{K}^{\|\cdot\|}$$

Since the set  $\{z \in X : |f(z)| \geq s + \epsilon\}$  is weakly closed, and contains  $K$ , it must contain  $\overline{K}^w$ . Hence  $x \notin \overline{K}^w$ .  $\square$

**Corollary 2.2.4.1.** Let  $X$  be a normed space, and  $(x_i)_{i \in I}$  be a net in  $X$  such that  $x_i \rightarrow x$  weakly in  $X$ . Then there exists a net  $(y_j)_{j \in J}$  of finite convex combinations of  $\{x_i : i \in I\}$  such that  $y_j \rightarrow x$  in norm.



*Proof.* **verify** □

**Proposition 2.2.1.** *If  $K$  is a convex subset of a LCTVS. Then  $\overline{K}$  is also convex.*

*Proof.* **verify** □

**Proposition 2.2.2.** *Show that  $c_0$  is weakly closed and weak  $*$  dense in  $\ell_\infty$ .*

*Proof.* **verify** □

**Theorem 2.2.5** (Krein-Smulian Theorem). *Let  $X$  be a Banach space, and let  $C$  be a convex subset of  $X^*$ . Then  $C$  is weak  $*$  closed if and only if  $C \cap \{f \in X^* : \|f\| \leq r\}$  is weak  $*$  closed for all  $r \in \mathbb{R}^+$ .*

*This should even work if we just take  $n \in \mathbb{N}$ . **verify**.*

**Corollary 2.2.5.1.** *Let  $Z$  be a subspace of  $X^*$ . Then  $Z$  is weak  $*$  closed if and only if  $\{\phi \in Z : \|\phi\| \leq 1\}$  is closed.*

**Corollary 2.2.5.2.** *Let  $X$  be a separable Banach space. Then a convex subset  $Z$  of  $X^*$  is weak  $*$  closed if and only if it is weak  $*$  sequentially closed.*

*Proof.* Since  $X$  is separable, for every  $r > 0$ , the set  $\{f \in X^* : \|f\| \leq r\}$  is weak  $*$  metrizable. Thus  $Z \cap \{f \in X^* : \|f\| \leq 1\}$  is weak  $*$  closed iff it is weak  $*$  sequentially closed. □

**Corollary 2.2.5.3.** *Let  $X$  be a separable Banach space, and  $\phi \in X^{**}$ . Then  $\phi$  is weak  $*$  continuous if and only if  $\phi$  is sequentially continuous i.e  $f_n \rightarrow f$  weak  $*$  in  $X^*$  implies  $\phi(f_n) \rightarrow \phi(f)$ .*

*Proof.* Assume  $\phi$  is sequentially weak  $*$  continuous. Let  $C = \text{Ker}(\phi)$  be a subspace of  $X^*$ . Let  $g_n \in C$  be a sequence such that  $g_n \rightarrow g \in X^*$ . Then by assumption,  $0 = \phi(g_n) \rightarrow \phi(g)$  implies  $\phi(g) = 0$  and therefore  $g \in C$ . This shows that  $C$  is weak  $*$  sequentially closed, hence weak  $*$  closed by the separability of  $X$ . Hence  $\phi$  is weak  $*$  continuous. □

**Example 2.2.3.** Let  $S = \{n\delta_n : n \in \mathbb{N}\} \subset \ell^\infty$ . We show that  $0 \in \overline{S}^{w*}$

*Proof.* Let  $f \in \ell^1$ . Then the set  $\{n \in \mathbb{N} : |f(n)| < \epsilon/n\}$  is infinite. (Otherwise this would contradict  $f \in \ell^1$ ). Thus  $\exists N \in \mathbb{N}$  such that  $N|f_i(N)| < \epsilon$  for all  $i = 1, 2, \dots, m$ . And therefore

$$N\delta_n \in \bigcup_{f_1, f_2, \dots, f_N, \epsilon} (0)$$

□

**Definition 2.2.3.** A subset  $S$  of a vector space  $V$  is called balanced if  $\forall s \in S, \alpha \in \mathbb{F}$  with  $|\alpha| \leq 1, \alpha s \in S$ .

**Lemma 2.2.4.** Let  $X$  be a topological vector space, then every open neighborhood of 0 contains a balanced open neighborhood of 0.

*Proof.* **verify** □

**Lemma 2.2.5.** All  $n$ -dimensional topological vector spaces are isomorphic as topological vector spaces.

*Proof.* For the case  $n = 1$ , and  $\mathbb{F} = \mathbb{C}$ .

Assume  $\tau$  is a topology on  $\mathbb{C}$  that turns it into a topological vector space. Now think of  $i : \mathbb{C} \rightarrow (\mathbb{C}, \tau) := x \rightarrow x$  as the composition of  $\mathbb{C} \rightarrow \mathbb{C} \times (\mathbb{C}, \tau) := x \rightarrow (x, 1)$  and  $\mathbb{C} \times (\mathbb{C}, \tau) \rightarrow (\mathbb{C}, \tau) := (x, y) \rightarrow xy$ . Then we see that  $i$  is the composition of these maps which are continuous by the definition of the product topology and the TVS. Hence,  $i$  is continuous.

To show that  $i^{-1}$  is continuous, consider the annulus  $A = \{\alpha \in \mathbb{C} : 1 \leq |\alpha| \leq 2\}$ . Then since  $A$  is compact in the usual topology and  $i(A) = A$  is a continuous image, we get that  $A$  is open in  $\tau$ . Hence  $A^c \ni 0$  is open and by the lemma above has a balanced open neighborhood of 0 in it. **(Show that this is actually an open disk)**. □

**Theorem 2.2.6.** Let  $X$  be a normed space. Then the closed unit ball of  $X$  is compact in norm topology if and only if  $X$  is finite dimensional.

*Proof.* Suppose  $X$  is infinite dimensional normed space and let  $\bar{B}$  be the closed unit ball. Let  $x_1 \in \bar{B}$  and let  $Y_1 = \text{span}\{x_1\}$ . Then  $Y_1$  is a closed subspace of  $X$ . Since  $X$  is a non-zero normed space, let  $x_2 \in X$  such that  $\|x_2 + Y_1\| = \frac{1}{2}$ . Repeat the construction in the proof of Reisz lemma. □

**Lemma 2.2.6.** Let  $X$  be a normed space. Then  $i(X)$  is weak \* dense in  $X^{**}$

*Proof.* Let  $C = \bar{B}^{w*}$ , where  $B$  is the closed unit ball. Then  $C$  is compact convex. FSTOC, assume  $\exists \phi \in X^{**}$  such that  $\|\phi\| \leq 1, \phi \notin C$ . Then by HBT, there is a  $f \in X^{**}$  and  $r \in \mathbb{R}, \epsilon > 0$  such that  $\Re f(y) \leq r < r + \epsilon \leq \Re f(\phi)$  for all  $y \in i(X)$ . This implies  $\|f\| \leq r$ , hence  $|f(\phi)| = |\phi(f)| \leq \|\phi\| \|f\| < r$  which gives a contradiction. □

Show that  
 $\Re f(x) \leq r\|x\|$  imply  
 $\|f\| \leq r$

**Theorem 2.2.7.** Let  $X$  be a Banach space. Then the closed unit ball  $\bar{B}$  is weakly compact if and only if  $X$  is reflexive.

*Proof.* If  $X$  is reflexive, the weak and weak \* topology coincides and the Banach Alaouglu gives the proof. **verify**

Assume  $\bar{B}$  is weakly compact. Observe that then the map  $i : X \rightarrow X^{**}$  is continuous when we equip  $X$  with weak topology and  $X^{**}$  with weak topology. Thus  $i(\bar{B})$  is weak \* compact. Moreover  $i(\bar{B})$  is weak \* dense in the closed unit ball of  $X^{**}$ . Hence the result.  $\square$

# Chapter 3

## Hilbert Spaces

**Definition 3.0.1.** Recall that a complex inner product on a complex vector space is a map

$$\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$$

such that

- (1)  $\langle x, x \rangle \geq 0$  for all  $x \in X$
- (2)  $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- (3)  $\langle \alpha x + z, y \rangle = \alpha \langle x, y \rangle + \langle z, y \rangle$

Recall the norms induced by the inner product and the Cauchy-Schwarz inequality.

**Definition 3.0.2.** Complete inner product spaces are called Hilbert spaces

**Proposition 3.0.1.** *Let  $X$  be an inner product space. Then the inner product of  $X$  extends to an inner product on the completion (unique metric space completion) of  $X$ , turns it into a Hilbert space.*

**Definition 3.0.3.** If  $x, y \in H$ , the Hilbert space, we say  $x \perp y$  if  $\langle x, y \rangle = 0$

**Definition 3.0.4.** Given a set  $S \subset H$ ,  $S^\perp = \{y \in H : \langle x, y \rangle = 0\}$

**Proposition 3.0.2.** *Let  $H, K$  be Hilbert spaces and  $T : H \rightarrow K$  be linear. Then the following are equivalent.*

- (1)  $T$  is isometry
- (2)  $\langle Tx, Ty \rangle = \langle x, y \rangle$  for all  $x, y \in H$

*Proof.* See Homework-5

□

**Proposition 3.0.3.** For all  $x, y \in H$ , a Hilbert space, then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

**Example 3.0.1.** Show that  $c_{00}$  under the usual inner product is not complete and its completion is  $\ell^2(\mathbb{N})$ .

*Proof.* Consider the sequence  $x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots)$ . Then clearly  $x_n \in c_{00}$ . Moreover  $x_n \rightarrow x = (1, \frac{1}{2}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots)$  in the norm by the inner product. But  $x \notin c_{00}$ . But  $x \in \ell^2(\mathbb{N})$ . Moreover, the same process can be used to approximate any element in  $\ell^2(\mathbb{N})$  by elements in  $c_{00}$ . Thus we see that  $c_{00}$  is dense in  $\ell^2(\mathbb{N})$   $\square$

**Example 3.0.2.**  $L^2(\mathbb{R}, \mu)$  with

$$\langle f, g \rangle = \int_{\mathbb{R}} f \bar{g} \, d\mu$$

is a Hilbert space.

**Example 3.0.3.** Let  $J$  be any set  $\ell^2(J) = \{f : J \rightarrow \mathbb{C} : \sum_{j \in J} |f(j)|^2 < \infty\}$  with the usual inner product is a Hilbert space.

**Definition 3.0.5.** An orthonormal basis for  $H$  is a maximal orthonormal set.

**Theorem 3.0.1.** Let  $H$  be a Hilbert space and  $J$  be an orthonormal basis for  $H$ . Then there exists a bijective linear isometry  $T : H \rightarrow \ell^2(J)$ .

**Theorem 3.0.2.** Let  $\mathcal{H}$  be a Hilbert space and let  $C$  be a non-empty closed convex subset of  $\mathcal{H}$ . Then there exist a unique vector  $x \in C$  such that  $\|x\| \leq \|\eta\|$  for all  $\eta \in C$ .

*construc-  
tion of the  
proof is a  
bit tricky* *Proof.* Let  $d = \inf\{\|\eta\| : \eta \in C\}$  and choose a sequence  $\eta_n \in C$  such that  $\|\eta_n\| \rightarrow d$ . Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that  $\|\eta_n\|^2 < d^2 + \epsilon$  for all  $n \geq N$ . Then for all  $m, n \geq N$ , we have

$$\|\eta_n - \eta_m\|^2 = 2(\|\eta_n\|^2 + \|\eta_m\|^2) - 4\|\frac{1}{2}(\eta_n + \eta_m)\|^2 \leq 4(d^2 + \epsilon) - 4d^2 = 4\epsilon$$

(Note that since  $C$  is compact,  $\frac{1}{2}\eta_n + \frac{1}{2}\eta_m \in C$  and therefore, by the definition of  $d$ , we get  $\|\frac{1}{2}(\eta_n + \eta_m)\| \geq d$ .) Hence the sequence  $\eta_n$  is Cauchy and hence convergent since the space is complete. Let  $\eta = \lim_{n \rightarrow \infty} \eta_n$ . Since  $C$  is closed  $\eta \in C$  and clearly  $\|\eta\| = d$ .

To see the uniqueness, assume  $\alpha \in C$ , and  $\|\alpha\| = d$ . Then

$$\begin{aligned} \|\eta - \alpha\|^2 &= 2(\|\eta\|^2 + \|\alpha\|^2) - 4\|\frac{1}{2}(\eta + \alpha)\|^2 \\ &\leq 4d^2 - 4d^2 = 0 \end{aligned}$$

Verify the second inequality.  $\square$

**Corollary 3.0.2.1.** *Let  $\eta \in \mathcal{H}$  and  $C$  be as before. Then there exist a unique vector  $x \in C$  such that  $d(\eta, C) = \|x - \eta\|$*

*Proof.* Apply above theorem to  $C' = C - \eta$ . Since  $C$  is closed and convex, so will be its translation  $C - \eta$ .  $\square$

**Proposition 3.0.4** (Pythagoras Theorem). *Let  $x, y \in \mathcal{H}$  an inner product space, and  $x \perp y$ , then  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .*

**Lemma 3.0.1.** *Let  $\mathcal{H}$  be a Hilbert space and  $K$  be a nontrivial closed subspace. Let  $\eta \in \mathcal{H}$ . Then  $\xi \in K$  satisfy  $\|\xi - \eta\| = d(\eta, K)$  iff  $\xi - \eta \perp K$ .*

*Proof.* Let  $\xi - \eta \perp K$  and  $k \in K$ . Then  $\eta - k = (\eta - \xi) + (\xi - k)$  and by Pythagoras theorem, we get

$$\|\eta - k\|^2 = \|\eta - \xi\|^2 + \|\xi - k\|^2 \geq \|\eta - \xi\|^2$$

Thus we see that  $\|\eta - \xi\| = d(\eta, K)$ .

Conversely, let  $\|\eta - \xi\| = d(\eta, K)$ . Then for all  $\rho \in K$  and  $t > 0$ , we have

$$\begin{aligned} \|\eta - \xi\|^2 &\leq \|\eta - (\xi + t\rho)\|^2 \\ &= \|\eta - \xi - t\rho\|^2 \\ &= \|\eta - \xi\|^2 + t^2\|\rho\|^2 - 2t\Re\langle\eta - \xi, \rho\rangle \end{aligned}$$

cool proof  
technique

Hence we see that  $|2\Re\langle\eta - \xi, \rho\rangle| \leq t\|\rho\|^2$ . Since  $t > 0$  was arbitrary, limiting it to zero, we get  $\Re\langle\eta - \xi, \rho\rangle = 0$ . Replacing  $\rho$  with  $-i\rho$  will give the imaginary part is also zero.  $\square$

*Remark 3.0.1.* If  $K$  is a closed subspace of the Hilbert space  $\mathcal{H}$ , then  $\mathcal{H} = K \oplus K^\perp$ . To see this notice that  $K \cap K^\perp = \{0\}$ . Now if  $x \in \mathcal{H}$ , then there is a  $k_x \in K$  such that  $d(x, K) = \|x - k_x\|$ . Moreover, from **Lemma 3.0.1**, we see that  $x - k_x \in K^\perp$ . Hence  $x = k_x + (x - k_x) \in K \oplus K^\perp$ . Now replacing  $K$  with  $K^\perp$ , we see that  $(K^\perp)^\perp = K$ .

**Theorem 3.0.3** (Reisz Representation Theorem). *Let  $\mathcal{H}$  be a Hilbert space and  $f \in \mathcal{H}^*$ . Then there exists a unique  $\eta_f \in \mathcal{H}$  such that  $f(x) = \langle x, \eta_f \rangle$  for all  $x \in \mathcal{H}$ . The map  $\phi : \mathcal{H}^* \rightarrow \mathcal{H} := f \rightarrow \eta_f$  is conjugate linear isometric bijection.*

*Proof.* If  $f$  is the zero linear functional,  $\eta_f = 0$  and we're done. If not let  $K = \text{Ker}(f)$ . Then  $K$  has co-dimension 1. Consider  $f|_{K^\perp}$ . Let  $v \in K^\perp$  such that  $f(v) = 1$ . Clearly  $K^\perp = \text{span}(v)$ . And therefore for any  $\alpha v \in K^\perp$ ,  $f(\alpha v) = \alpha = \langle \alpha v, \frac{v}{\|v\|^2} \rangle$ . Choose  $\eta_f = \frac{v}{\|v\|^2}$ . Now verify.  $\square$

### 3.1 Orthogonal Projections

**Theorem 3.1.1.** *Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces, and  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear map. Then there exists a unique bounded linear map  $T^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  satisfying  $\langle Tx, y \rangle_{\mathcal{H}_2} = \langle x, T^*y \rangle_{\mathcal{H}_1}$  for all  $x \in \mathcal{H}_1, y \in \mathcal{H}_2$ .*

*Proof.* For every given  $y \in \mathcal{H}_2$  define a linear functional  $f^y : \mathcal{H}_1 \rightarrow \mathbb{C}$  as  $f^y(x) = \langle Tx, y \rangle$ . Since  $f^y$  is bounded,  $f^y \in \mathcal{H}_1^*$ . Hence by Reisz representation, there is a unique  $T^*(y) \in \mathcal{H}_1$  such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$ .

Uniqueness follows from the fact that in any inner product space  $X$ , if  $x, y \in X$  such that  $\langle x, z \rangle = \langle y, z \rangle$  for all  $z \in X$ , then  $\langle x - y, z \rangle = 0$  for all  $z \in X$ , in particular  $z = x - y$ . Then by the positive definiteness of the inner product, we get  $x = y$ .  $\square$

**Theorem 3.1.2.** *Let  $\mathcal{H}$  be a Hilbert space and  $K$  a closed subspace. For every  $\eta \in \mathcal{H}$ , denote by  $P_K(\eta)$ , the unique closest vector in  $K$ , closest to  $\eta$ . Then*

(1)  $P_K : \mathcal{H} \rightarrow \mathcal{H}$  is linear, bounded with  $\|P_K\| = 1$  and idempotent.

(2)  $P_K^* = P_K$  (self-adjoint)

*Proof.* (1) Let  $\eta_1, \eta_2 \in \mathcal{H}$  and  $\alpha \in \mathbb{C}$ . Then using [Lemma 3.0.1](#) for all  $\xi \in K$ , we have

$$\begin{aligned} \langle \alpha\eta_1 + \eta_2 - \alpha P_K(\eta_1) - P_K(\eta_2), \xi \rangle &= \alpha \langle \eta_1 - P_K(\eta_1), \xi \rangle + \langle \eta_2 - P_K(\eta_2), \xi \rangle \\ &= 0 \end{aligned}$$

Thus we get  $P_K(\alpha\eta_1 + \eta_2) = \alpha P_K(\eta_1) + P_K(\eta_2)$  again by the same [Lemma 3.0.1](#). Moreover since  $P_K(x)$  is a vector in  $K$  it the vector in  $K$  closest to  $P_K(x)$  is  $P_K(x)$  itself. Thus we see that  $P_K^2(x) = P_K(x)$ . Hence idempotent. Moreover since  $K$  is closed, we have  $X = K \oplus K^\perp$  ([Remark 3.0.1](#)). Thus for all  $x \in X$ , we can write  $x = k_x + k'_x$ , where  $k_x \in K$  and  $k'_x \in K^\perp$ . Then  $P_K(x) = k_x$  and triangle inequality shows  $\|P_K(x)\| = \|k_x\| \leq \|x\|$ . Thus  $\|P_K\| \leq 1$ .

(2) Since  $X = K \oplus K^\perp$ , let  $x = k_x + k'_x$  and  $y = k_y + k'_y$ . Then

$$\langle P(x), y \rangle = \langle k_x, k_y + k'_y \rangle = \langle k_x, k_y \rangle = \langle k_x + k'_x, k_y \rangle = \langle x, P_K(y) \rangle$$

and the uniqueness of the adjoint proves that  $P_K^* = P_K$

$\square$

**Proposition 3.1.1.**  $I - P_K = P_{K^\perp}$

*Proof.* Let  $x \in X$  and  $k \in K$ . Then

$$\begin{aligned}
\langle (I - P_k)(x), k \rangle &= \langle x - P_k(x), k \rangle \\
&= \langle x, k \rangle - \langle P_k(x), k \rangle \\
&= \langle x, k \rangle - \langle x, P_k(k) \rangle \\
&= \langle x, k \rangle - \langle x, k \rangle \\
&= 0
\end{aligned}$$

Shows that □

**Proposition 3.1.2.** *Let  $K$  be a closed subspace of  $\mathcal{H}$ . Let  $E \subset K$  be an orthonormal basis for  $K$ . Extend  $E$  to an orthonormal basis  $\tilde{E}$  for  $\mathcal{H}$ . Then*

$$P_K|_E = I_K, \quad P_K|_{\tilde{E} \setminus E} = 0$$

*Remark 3.1.1* (Parseval's Inequality). Let  $\mathcal{H}$  be a Hilbert space. Let  $E$  be an orthonormal set. Then for every vector  $\eta \in \mathcal{H}$ ,

$$\|\eta\|^2 \geq \sum_{e \in E} |\langle \eta, e \rangle|^2$$

**Lemma 3.1.1.** *Let  $S$  be a nonempty subset of  $\mathcal{H}$ . Then*

$$(S^\perp)^\perp = \overline{\text{span}}(S)$$

*Proof.* Notice from the above proposition that  $\text{Ker}(P_K) = K^\perp$ . Thus  $(S^\perp)^\perp = \text{Ker}(P_{S^\perp})$ . Since  $P_{S^\perp}(s) = 0$  for all  $s \in S$ , we see that  $S \subset \text{Ker}(P_{S^\perp})$ . Moreover, since  $\text{Ker}(P_{S^\perp})$  is a closed subspace, we see that  $\overline{\text{span}}(S) \subset \text{Ker}(P_{S^\perp}) = (S^\perp)^\perp$ .

Now see that if  $A \subset B$ , then  $B^\perp \subset A^\perp$ . Note that if  $\langle x, b \rangle = 0$  for all  $b \in B$ , then  $\langle x, a \rangle = 0$  for all  $a \in A$  since  $A \subset B$ . Thus we see that  $B^\perp \subset A^\perp$ . Then since  $S \subset \overline{\text{span}}(S)$ , we see that  $\overline{\text{span}}(S)^\perp \subset S^\perp$  and  $(S^\perp)^\perp \subset (\overline{\text{span}}(S)^\perp)^\perp$ . Thus by [Remark 3.0.1](#), we see that  $(S^\perp)^\perp \subset \overline{\text{span}}(S)$ . □

**Corollary 3.1.2.1.** *Let  $E$  be an orthonormal subset of  $\mathcal{H}$ . Then the following are equivalent.*

- (1)  $E$  is an orthonormal basis
- (2)  $(E^\perp)^\perp = \mathcal{H}$
- (3)  $\overline{\text{span}}(E) = \mathcal{H}$
- (4)  $\|\eta\|^2 = \sum_{e \in E} |\langle \eta, e \rangle|^2, \quad \forall \eta \in \mathcal{H}$



*Proof.*  $1 \implies 2 \implies 3$  follows easily from [Lemma 3.1.1](#). To see  $3 \implies 4$ , since  $E$  is orthonormal and  $\overline{\text{span}}(E) = \mathcal{H}$ , we claim that it will be an orthonormal basis. Because if  $y \perp E$ , then  $y \perp \text{span}(E)$ . If  $x_n \in \text{span}(E)$  such that  $x_n \rightarrow x$ , then since  $\langle y, x_n \rangle = 0$  for each  $n \in \mathbb{N}$ , we get

$$|\langle x, y \rangle| = |\langle x - x_n, y \rangle + \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq \|x - x_n\| \|y\|$$

Since  $x_n \rightarrow x$ , we see that  $\langle x, y \rangle = 0$ . Thus we see that  $E$  is an orthonormal basis. Thus we see that for all  $\eta \in \mathcal{H}$ ,

$$\eta = \sum_{e \in E} \langle \eta, e \rangle e$$

and Parseval's equality follows. □

**Theorem 3.1.3.** *Let  $P \in B(\mathcal{H})$  be an idempotent. Then the following are equivalent.*

- (1)  $P = P_K$  for some closed subspace  $K \leq \mathcal{H}$
- (2)  $P = P^*$
- (3)  $\|P\| = 1$

*Proof.*  $1 \implies 2, 3$  is proved in [Theorem 3.1.2](#). To see  $2 \implies 1$ . Let  $K = \text{Im}(P)$ . Let  $\rho \in K^\perp$ . Then for all  $\eta \in H$ ,

$$\langle P(\rho), \eta \rangle = \langle \rho, P^*(\eta) \rangle = \langle \rho, P(\eta) \rangle = 0$$

So  $P|_{K^\perp} = 0$ . Since  $P^2 = P$ ,  $P|_K = I$ . Hence  $P = P_K$ .

To see  $3 \implies 1$ , assume  $P \neq P_K$ . Then  $\exists \rho \in K^\perp$  such that  $P(\rho) \neq 0$ . For each  $n \in \mathbb{N}$ , we have  $\|\rho + nP(\rho)\|^2 = \|\rho\|^2 + n^2\|P(\rho)\|^2$  by [Proposition 3.0.4](#). And

$$\|P(\rho + nP(\rho))\|^2 = (n+1)^2\|P(\rho)\|^2$$

nice Hence we have

technique

$$\|P(\rho + nP(\rho))\|^2 - \|\rho + nP(\rho)\|^2 = 2n\|P(\rho)\|^2 + \|P(\rho)\|^2 - \|\rho\|^2$$

So for large  $n$ , we have

$$\|P(\rho + nP(\rho))\| > \|\rho + P(\rho)\|$$

so  $\|P\| > 1$ . □

## 3.2 Compact Operators

**Definition 3.2.1.** let  $\mathcal{X}, \mathcal{Y}$  be Banach Spaces. A linear map  $T : \mathcal{X} \rightarrow \mathcal{Y}$  is called a compact operator if  $\overline{T(B_1^{\mathcal{X}})}$  is compact, where  $B_1^{\mathcal{X}}$  is the unit ball of  $\mathcal{X}$ . We denote by  $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ , the set of all compact operators.

**Example 3.2.1.** If either  $\mathcal{X}$  or  $\mathcal{Y}$  is finite, then every linear map  $T : \mathcal{X} \rightarrow \mathcal{Y}$  is compact. If  $\mathcal{X}$  be any infinite dimensional Banach space. Then  $T = \text{Id} : \mathcal{X} \rightarrow \mathcal{X}$  is not compact. This follows from [Theorem 2.2.6](#).

**Definition 3.2.2.**  $T : \mathcal{X} \rightarrow \mathcal{Y}$  is called a finite rank if the dimension of  $T$  is finite. Then the dimension of the image is called the rank of the operator. Let  $\mathcal{F}(\mathcal{X}, \mathcal{Y})$  denote finite rank operators.

**Lemma 3.2.1.** *Every compact operator is bounded.*

*Proof.* Every compact set is bounded in any metric space. □

**Theorem 3.2.1.** *Let  $\mathcal{H}$  be a Hilbert space. Then  $\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}^{\|\cdot\|}$*

*Proof.* It is evident that  $\mathcal{F}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ . We'll now show that  $\overline{\mathcal{F}(\mathcal{H})} \subset \mathcal{K}(\mathcal{H})$ . Let  $T_n$  be a sequence in  $\mathcal{F}(\mathcal{H})$  and  $T_n \rightarrow T \in B(\mathcal{H})$  (in norm). We'll show that  $\overline{T(B_1^{\mathcal{H}})}$  is closed and totally bounded, which is equivalent to compactness in metric spaces.

Let  $\epsilon > 0$  be given. Then there exist some  $N \in \mathbb{N}$  such that  $\|T_N - T\| < \epsilon$ . Moreover

$$\overline{T_N(B_1^{\mathcal{H}})} \subset \bigcup_{\eta \in B_1^{\mathcal{H}}} B_{\epsilon}(T_N(\eta))$$

By the compactness of  $\overline{T_N(B_1^{\mathcal{H}})}$  there exist  $\eta_1, \eta_2, \dots, \eta_k \in B_1^{\mathcal{H}}$  such that

$$\overline{T_N(B_1^{\mathcal{H}})} \subset \bigcup_{i=1}^k B_{\epsilon}(T_N(\eta_i))$$

Thus for an arbitrary  $\eta \in B_1^{\mathcal{H}}$ ,  $\exists i \leq k$  such that  $\|T_N(\eta) - T_N(\eta_i)\| < \epsilon$ . Then

$$\begin{aligned} \|T(\eta) - T(\eta_i)\| &\leq \|T(\eta) - T_N(\eta)\| + \|T_N(\eta) - T_N(\eta_i)\| + \|T_N(\eta_i) - T(\eta_i)\| \\ &< 3\epsilon \end{aligned}$$

Thus we see that

$$T(B_1^{\mathcal{H}}) \subset \bigcup_{i=1}^k B_{3\epsilon}(T(\eta_i))$$

and hence

$$\overline{T(B_1^{\mathcal{H}})} \subset \bigcup_{i=1}^k \overline{B_{3\epsilon}(T(\eta_i))}$$

which gives total boundedness. Closure of  $\overline{T(B_1^{\mathcal{H}})}$  is obvious. Hence we see that  $\overline{T(B_1^{\mathcal{H}})}$ . Thus  $T$  is compact and we see  $\overline{\mathcal{F}(\mathcal{H})} \subset \mathcal{K}(\mathcal{H})$ .

Conversely, let  $T \in B(\mathcal{H})$  be compact and  $\epsilon > 0$  be given. So  $\exists \eta_1, \eta_2, \dots, \eta_m \in B_1(\mathcal{H})$  such that

$$\overline{T(B_1^{\mathcal{H}})} \subset \bigcup_{i=1}^n B_{\epsilon}(T(\eta_i))$$

Assume that  $\mathcal{H}$  is separable. Let  $\{e_n : n \in \mathbb{N}\}$  be an orthonormal basis for  $\mathcal{H}$  and let  $P_n = P_{\text{Span}\{e_1, e_2, \dots, e_n\}}$ . Then choose  $N > 0$  such that for all  $n > N$ ,  $\forall i = 1, 2, \dots, m$ ,

$$\|(P_n T - T)(\eta_i)\| < \epsilon$$

Then  $\forall \xi \in \mathcal{H}$  with  $\|\xi\| = 1$ . Choose  $1 \leq j \leq m$  such that  $\|T(\xi) - T(\eta_j)\| < \epsilon$ . Then

$$\begin{aligned} \|P_n T(\xi) - T(\xi)\| &\leq \|P_n T(\xi) - P_n T(\eta_j)\| + \|P_n T(\eta_j) - T(\eta_j)\| + \|T(\eta_j) - T(\xi)\| \\ &< 3\epsilon \end{aligned}$$

Which gives that  $\|P_n T - T\| < 3\epsilon$  for all  $n \in \mathbb{N}$ .

The generalization of the case when  $\mathcal{H}$  is non-separable, will use the fact that  $T(\mathcal{H})$  is separable and using an orthonormal basis for the pre-image of  $T(\mathcal{H})$ , which will again be separable.  $\square$

**Proposition 3.2.1.** *Let  $\mathcal{K}$  be a separable Hilbert space. Let  $\{e_n : n \in \mathbb{N}\}$  be an orthonormal basis for  $\mathcal{K}$ . For each  $n \in \mathbb{N}$ , let  $P_n$  be the projection to the  $\text{Span}\{e_1, e_2, \dots, e_n\}$ . Then for all  $x \in \mathcal{K}$*

$$P_n(x) \rightarrow x$$

*pointwise*

*Proof.* Direct application of [Corollary 3.1.2.1\(iv\)](#)  $\square$

**Lemma 3.2.2.** *If  $T \in B(\mathcal{H})$  is compact, then  $T(\mathcal{H})$  is separable.*

*Proof.* For the sake of contradiction, assume that  $T(\mathcal{H})$  is not separable. Then  $T(\mathcal{H})$  must have an uncountable orthonormal set  $E$ . This is because if  $E$  is a countable basis, then  $\mathbb{Q}E + i\mathbb{Q}E$  will be a countable dense subset of  $T(\mathcal{H})$  making it separable. [verify](#)  $\square$

**Corollary 3.2.1.1.** *The set  $\mathcal{K}(\mathcal{H})$  of all compact operators on  $\mathcal{H}$  is a closed two-sided ideal in  $B(\mathcal{H})$ .*

*Proof.* Use the fact that compact operators are the closure of finite rank operators. □

**Corollary 3.2.1.2.**  *$T \in \mathcal{K}(\mathcal{H})$  implies  $T^* \in \mathcal{K}(\mathcal{H})$ .*

*Proof.* Use the fact that  $T$  is finite rank implies  $T^*$  is finite rank. □

**Example 3.2.2.** Let

$$T : L^2([0, 1], m) \rightarrow L^2([0, 1], m)$$

be defined as  $T(f)(x) = xf(x)$ . Prove that  $T = T^*$  and that  $T$  has no eigenvectors.

Only for Joseph: If  $\xi = \eta$  almost everywhere, then show that  $f\xi = f\eta$  almost everywhere if  $f \in C([0, 1])$

*Proof.* See homework-5 □

**Example 3.2.3.** Let  $\alpha_n$  be a bounded sequence in  $\mathbb{C}$ . Consider  $T : \ell^2 \rightarrow \ell^2$ , such that

$$T(\delta_n) = \alpha_n \delta_n$$

for all  $n \in \mathbb{N}$ . Then  $T$  is bounded with  $\|T\| = \|(\alpha_n)\|_\infty$ .

**Example 3.2.4.** Prove that the  $T$  above is compact if and only if  $(\alpha_n) \in c_0$ . Prove that  $T^*(\delta_n) = \overline{\alpha_n} \delta_n$

*Proof.* See homework-5 □

### 3.3 Spectral Theorem for Compact Normal Operators

We'll be working to prove the following theorem.

**Theorem 3.3.1.** *Let  $\mathcal{H}$  be separable Hilbert space and  $T \in \mathcal{K}(\mathcal{H})$  be normal. i.e.  $TT^* = T^*T$ . Then there exist an orthonormal basis  $\{e_n : n \in \mathbb{N}\}$  of  $\mathcal{H}$ , and a sequence  $(\alpha_n) \in c_0$ , such that  $T(e_n) = \alpha_n e_n$ .*

**Lemma 3.3.1.** *Let  $T \in \mathcal{K}(\mathcal{H})$  be self-adjoint. Then there exist a non-zero  $\lambda \in \mathbb{C}$  such that  $\text{Ker}(T - \lambda I) \neq \{0\}$ .*

*Proof.* See the first part of the proof of [Theorem 3.3.2](#). □

**Definition 3.3.1.** Let  $T \in B(\mathcal{H})$ . A subspace  $W \leq H$  is said to be invariant under  $T$  if  $T(W) \subset W$ . We say  $W$  reduces  $T$ , if  $T(W) \subset W$  and  $T(W^\perp) \subset W^\perp$

**Example 3.3.1.** Let  $T \in B(\mathcal{H})$ . Prove that  $W \subset \mathcal{H}$  is invariant under  $T$  if and only if  $P_W T = T P_W$ . Prove that  $W$  reduces  $T$  if and only if  $P_W T = T P_W$  and  $P_{W^\perp} T = T P_{W^\perp}$ .

*Proof.* See homework-5 □

**Proposition 3.3.1.** Let  $T \in B(\mathcal{H})$  be self adjoint with  $T(\mathcal{M}) \subset \mathcal{M}$ , where  $\mathcal{M}$  is a closed subspace of  $\mathcal{H}$ . Then  $\mathcal{M}$  reduces  $T$ .

*Proof.* Let  $m \in \mathcal{M}$  and  $x \in \mathcal{M}^\perp$ . Then

$$\langle T(x), m \rangle = \langle x, T(m) \rangle = 0$$

shows that  $T(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$ . □

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**Lemma 3.3.2.** If  $T \in B(\mathcal{H})$  is self adjoint. Then every eigenvalue of  $T$  is real.

*Proof.* Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $T$  with eigenvector  $v_\lambda$ . Then

$$\lambda \|v\|^2 = \langle \lambda v_\lambda, v_\lambda \rangle = \langle T(v_\lambda), v_\lambda \rangle = \langle v_\lambda, T(v_\lambda) \rangle = \langle v_\lambda, \lambda v_\lambda \rangle = \bar{\lambda} \|v_\lambda\|^2$$

forces  $\lambda = \bar{\lambda}$ . □

**Example 3.3.2.** Let  $\alpha_n \in c_0$ . Define  $T : \ell^2 \rightarrow \ell^2$ ,  $T(\delta_n) = \delta_n \delta_n$  for all  $n \in \mathbb{N}$ . Then the set of eigenvalues of  $T$  is precisely the set of  $\alpha_n$ s.

**Lemma 3.3.3.** If  $T \in B(\mathcal{H})$  is self adjoint, then  $\|T\| = \sup\{|\langle T\xi, \xi \rangle| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}$

*Proof.*  $\langle T\xi, \xi \rangle \leq \|T\|$  inequality is clear from Cauchy-Schwarz and definition of operator norm. To see the converse, let  $\alpha, \beta \in \mathcal{H}$  with  $\|\alpha\| = \|\beta\| = 1$ . Then by the self-adjointness of  $T$ , we get

$$\begin{aligned} \langle T(\alpha + \beta), \alpha + \beta \rangle - \langle T(\alpha - \beta), \alpha - \beta \rangle &= 2(\langle T\alpha, \beta \rangle + \langle T\beta, \alpha \rangle) \\ &= 4\Re\langle T\alpha, \beta \rangle \end{aligned}$$

Now with enough rotation on  $\beta$ , we get  $\langle T\alpha, \beta \rangle = \Re\langle T\alpha, \beta \rangle$ . Moreover, if  $A = \sup\{|\langle T\xi, \xi \rangle| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}$ , notice that  $|\langle Tx, x \rangle| \leq A\|x\|^2$ . Then we get

$$\begin{aligned} 4|\langle T\alpha, \beta \rangle| &\leq A\|\alpha + \beta\|^2 + A\|\alpha - \beta\|^2 \\ &= 2A(\|\alpha\|^2 + \|\beta\|^2) \\ &= 4A \end{aligned}$$

Thus we see that  $\|T\| \leq A$ . □

**Lemma 3.3.4.** *Let  $T \in \mathcal{K}(\mathcal{H})$  and  $0 \neq \lambda \in \mathbb{C}$ . Then the space*

$$\text{Ker}(T - \lambda I)$$

*is finite dimensional.*

*Proof.* For the sake of contradiction, assume that  $K = \text{Ker}(T - \lambda I)$  is not finite dimensional. Then  $\overline{T(K \cap B_1^{\mathcal{H}})} = \overline{\lambda K \cap B_1^{\mathcal{H}}}$ , is a closed non-compact (Theorem 2.2.6) subspace of  $\overline{T(B_1^{\mathcal{H}})}$ . This contradicts the compactness of  $T$ .  $\square$

**Theorem 3.3.2** (Spectral theorem for compact self-adjoint operators). *Let  $\mathcal{H}$  be a separable Hilbert space and  $T \in \mathcal{K}(\mathcal{H})$  be self-adjoint. Then there is an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$  and a sequence  $\{\alpha_n\} \in c_0$  such that*

$$Te_n = \alpha_n e_n$$

*for all  $n \in \mathbb{N}$ .*

*Proof.* We claim that  $T$  has an eigenvalue  $\lambda \in \mathbb{R}$  (Lemma 3.3.2) such that  $|\lambda| = \|T\|$ . By Lemma 3.3.3,  $\exists(\xi_n)$  of unit vectors in  $\mathcal{H}$  such that

$$|\langle T\xi_n, \xi_n \rangle| \rightarrow \|T\|$$

Hence by taking a subsequence  $\xi_{n_k}$ , we get

$$\langle T\xi_{n_k}, \xi_{n_k} \rangle \rightarrow \lambda$$

where  $\lambda = \pm\|T\|$ . Since  $T$  is compact, by passing to a subsequence, we may also assume  $T\xi_{n_{k_j}} \rightarrow \eta \in \mathcal{H}$ . Then we have

$$\begin{aligned} \|(T - \lambda I)\xi_{n_{k_j}}\|^2 &= \|T\xi_{n_{k_j}}\|^2 + \lambda^2\|\xi_{n_{k_j}}\|^2 - 2\Re\langle T\xi_{n_{k_j}}, \lambda\xi_{n_{k_j}} \rangle \\ &\leq \|T\|^2 + \lambda^2 \end{aligned}$$

Now in the first equality above, since  $2\Re\langle T\xi_{n_{k_j}}, \lambda\xi_{n_{k_j}} \rangle \rightarrow 2\lambda^2$  as  $n \rightarrow \infty$ , we get that  $T\xi_{n_{k_j}} - \lambda\xi_{n_{k_j}} \rightarrow 0$  and since  $T\xi_{n_{k_j}} \rightarrow \eta$ , by the uniqueness of the limit, we get  $\lambda\xi_{n_{k_j}} \rightarrow \eta$  and thus applying  $T$  to both sides, we get

$$\lambda T\xi_{n_{k_j}} \rightarrow T\eta$$

and hence we get  $T\eta = \lambda\eta$ . If  $\eta = 0$ , this will contradict our previous assumption that  $T$  is nonzero since  $|\langle T\xi_n, \xi_n \rangle| \rightarrow \|T\|$ .

Note that the kernel of  $T - \lambda I$ ,  $H_1$  is invariant under  $T$  being the eigenspace, and has finite dimension (Lemma 3.3.4). Moreover,  $T(H_1^\perp) \subset H_1^\perp$  by Proposition 3.3.1.

Therefore  $T_1 := T|_{H_1^\perp}$  is self adjoint. Since  $T$  is compact, its restriction  $T_1$  is compact. Hence  $T_1 \in \mathcal{K}(H_1^\perp)$ . Applying the claim again it follows that  $T_1$  has an eigenvalue  $\lambda_1$  such that  $|\lambda_1| = \|T_1\|$ .

Continuing this construction, we get  $|\lambda| \geq |\lambda_1| \geq |\lambda_2| \geq \dots$  such that the  $\text{Ker}(T - \lambda_i I) \perp \text{Ker}(T - \lambda_j I)$  for all  $i \neq j$ .

For each  $k$  (where  $\lambda_k$  exist), choose an orthonormal basis  $E_k$  for  $\text{Ker}(T - \lambda_k I)$  and let  $E = \cup E_k$ . If  $\mathcal{H}$  is finite dimensional, there is nothing new to prove. Hence assume  $\mathcal{H}$  is infinite dimensional. If  $\text{Ker}(T - \lambda_{n+1} I) = (\text{Ker}(T - \lambda_1 I) \oplus \text{Ker}(T - \lambda_2 I) \oplus \dots \oplus \text{Ker}(T - \lambda_n I))^\perp$ , then the above lemma gives  $\lambda_{n+1} = 0$ . In this case  $E$  contains finitely many eigenvectors corresponding to nonzero eigenvalues and infinitely many vectors corresponding to the 0 eigenvalue. Hence we are done in this case. This happens only when  $T$  is finite rank.

Otherwise,  $T$  has infinitely many eigenvalues and therefore  $E$  is an infinite set, hence by [Example 3.2.4](#), the sequence  $\lambda_n \rightarrow 0$ .

It remains to show that  $E$  is an orthonormal basis for  $\mathcal{H}$ . Let  $F$  be the collection of all eigenvectors of  $T$  and let  $\mathcal{M} = \overline{\text{span}}(F)$ . Then  $T(\mathcal{M}) \subset \mathcal{M}$ , and so  $T(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$  by [Proposition 3.3.1](#). Thus  $T|_{\mathcal{M}^\perp}$  has no non-zero eigenvectors. Thus from what we have proved in the first part,  $\mathcal{M}^\perp$  must be  $\{0\}$ .  $\square$

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**Lemma 3.3.5.** *Every  $T \in B(\mathcal{H})$  can be decomposed as a linear combination of two self-adjoint operators.*

*Proof.*

$$T = \frac{T + T^*}{2} + i \frac{(T - T^*)}{2i}$$

Verify that  $\frac{T+T^*}{2}$  and  $\frac{T-T^*}{2i}$  are self adjoint.  $\square$

**Lemma 3.3.6.**  *$T \in B(\mathcal{H})$  is normal if and only if  $T + T^*$  and  $T - T^*$  commutes*

*Proof.*

$$\begin{aligned} (T + T^*)(T - T^*) &= T^2 + T^*T - TT^* - T^{*2} \\ (T - T^*)(T + T^*) &= T^2 - T^*T + TT^* - T^{*2} \end{aligned}$$

$\square$

**Lemma 3.3.7.**  *$T \in B(\mathcal{H})$  is normal if and only if  $T = T_1 + iT_2$  where  $T_1, T_2 \in S(\mathcal{H})$  and  $T_1T_2 = T_2T_1$*

*Proof.* Directly apply above lemma  $\square$

**Lemma 3.3.8.** *Let  $T \in B(\mathcal{H})$ ,  $\lambda \in \mathbb{C}$  be an eigenvalue of  $T$ , and let  $S \in B(\mathcal{H})$  with  $ST = TS$ . Then  $\text{Ker}(T - \lambda I)$  is invariant under  $S$*

*Proof.* Let  $M = \text{Ker}(T - \lambda I)$  and  $m \in M$ . Then

$$\begin{aligned} (T - \lambda I)(S(m)) &= TS(m) - \lambda S(m) \\ &= ST(m) - \lambda S(m) \\ &= S(\lambda m) - \lambda S(m) \\ &= \lambda S(m) - \lambda S(m) = 0 \end{aligned}$$

□

**Lemma 3.3.9.** *Let  $T \in B(\mathcal{H})$ , be self-adjoint. Let  $\lambda_1 \neq \lambda_2 \in \sigma(T)$  and  $\xi_1, \xi_2 \in \mathcal{H}$  their corresponding eigenvectors, then  $\langle \xi_1, \xi_2 \rangle = 0$*

*Proof.*

$$0 = \langle T\xi_1, \xi_2 \rangle - \langle \xi_1, T\xi_2 \rangle = \lambda_1 \langle \xi_1, \xi_2 \rangle - \lambda_2 \langle \xi_1, \xi_2 \rangle = (\lambda_1 - \lambda_2) \langle \xi_1, \xi_2 \rangle$$

□

**Theorem 3.3.3** (Spectral theorem for compact normal operators). *Let  $\mathcal{H}$  be a separable Hilbert space and  $T \in \mathcal{K}(\mathcal{H})$  normal. Then there is an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$  and a sequence  $(\alpha_n) \in \mathbf{c}_0$  such that*

$$Te_n = \alpha_n e_n$$

for all  $n \in \mathbb{N}$ .

*Proof.* Let  $T \in \mathcal{K}(\mathcal{H})$  be normal. So  $\exists S_1, S_2 \in \mathcal{K}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$  such that  $T = S_1 + iS_2$  and  $S_1 S_2 = S_2 S_1$ . Then by spectral theorem for self-adjoint operators, we get  $\sigma(S_1)$ , the set of eigenvalues of  $S_1$  such that

$$H = \bigoplus_{\lambda \in \sigma(S_1)} \text{Ker}(S_1 - \lambda I)$$

where each  $\text{Ker}(S_1 - \lambda I)$  is finite dimensional.

Since  $S_2$  commutes with  $S_1$ ,  $\text{Ker}(S_1 - \lambda I)$  is invariant under  $S_2$  by [Lemma 3.3.8](#), and  $S_2|_{\text{Ker}(S_1 - \lambda I)}$  is self-adjoint by [Proposition 3.3.1](#) and compact. Thus, by the first part of the proof, for each  $\lambda \in \sigma(S_1)$ , we can choose an orthonormal basis  $E_\lambda$  for  $\text{Ker}(S_1 - \lambda I)$  consisting of eigenvectors of  $S_2$ .

Observe that if  $\xi \in \mathcal{H}$  is such that  $S_1 \xi = \lambda \xi$  and  $S_2 \xi = \beta \xi$ , then

$$T\xi = (\lambda + i\beta)\xi$$

Now let  $E = \bigcup_{\lambda \in \sigma(S_1)} E_\lambda$ . Then  $E$  is an orthonormal basis for  $\mathcal{H}$  consisting of eigenvectors of  $T$ . □



**Lemma 3.3.10.** Let  $V$  be a vector space and  $f_1, f_2, \dots, f_n : V \rightarrow \mathbb{C}$  linear. Then  $f \in \text{span}\{f_1, f_2, \dots, f_n\}$  iff

$$\bigcap_{k=1}^n \text{Ker}(f_k) \subset \text{Ker}(f)$$

*Proof.* If  $f \in \text{span}\{f_1, f_2, \dots, f_n\}$  and if  $x \in \text{Ker}(f_i)$  for each  $1 \leq i \leq n$ , then clearly  $f(x) = 0$ . Conversely **verify**.  $\square$

**Lemma 3.3.11.**

$$(X, \text{weak})^* = X^*$$

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Given  $T$  as above, denote by  $\sigma(T)$ , the set of eigenvalues of  $T$  and for each eigenvalue  $\lambda \in \sigma(T)$ , denote  $P_\lambda := P_{\text{Ker}(T - \lambda I)}$ . Then spectral theorem gives

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda$$

**Definition 3.3.2.** Let  $\Xi$  be a non-empty subset of  $B(\mathcal{H})$ . The commutant of  $\Xi$ , is the set  $\Xi' = \{S \in B(\mathcal{H}) : ST = TS, \forall T \in \Xi\}$

**Exercise 3.3.1.** 1. Prove that for any set  $\Xi'$  is a closed subspace of  $B(\mathcal{H})$ .

$$2. \Xi' = \text{span}(\Xi)'$$

*Proof.* (1) Let  $(S_n)$  be a sequence in  $\Xi'$  such that  $S_n \rightarrow S$  in  $B(\mathcal{H})$ . Then by the continuity of the map  $m_T : B(\mathcal{H}) \rightarrow B(\mathcal{H}) : R \rightarrow TR$  as  $\|m_T(R)\| \leq \|T\| \|R\|$ , we see that  $S_n T \rightarrow ST$  and similarly by the continuity of  $m'_T : S \rightarrow TS$ , we see that  $TS_n \rightarrow TS$ . Then by the algebra of limits, for  $T \in \Xi$ , we see that

$$0 = S_n T - TS_n \rightarrow ST - TS$$

which forces  $S \in \Xi'$ .

(2)

$\square$

**Theorem 3.3.4** (Functional calculus for compact normal operators). Let  $T \in B(\mathcal{H})$  be compact normal with spectral decomposition

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda.$$

For each  $f \in L^\infty(\mathbb{C})$ , define  $f(T) := \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda$ . Then

- (1)  $\forall f \in \ell^\infty(\mathbb{C}), \|f(T)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}$
- (2) The map  $\ell^\infty(\mathbb{C}) \rightarrow B(\mathcal{H})$  defines as  $f \rightarrow f(T)$  is linear.
- (3)  $\forall f, g \in \ell^\infty(\mathbb{C}), (fg)(T) = f(T)g(T)$
- (4)  $\overline{f}(T) = f(T)^*$
- (5)  $\{f(T) : f \in \ell^\infty(\mathbb{C})\} = \{T\}''$

*Proof.* (1) **exercise**

(2) **exercise**

(3) **exercise**

(4) **exercise**

(5) Let  $f \in \ell^\infty(\mathbb{C})$ , and let  $S \in \{T\}'$ . Since  $ST = TS$ , by **Lemma 3.3.8** for every  $\lambda \in \sigma(T)$ , the eigenspace  $\text{Ker}(T - \lambda I)$  is invariant under  $S$ . Hence they are all reducing for  $S$ . i.e  $SP_\lambda = P_\lambda S$ . Thus,

$$\begin{aligned}
 Sf(T) &= S \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda \\
 &= \sum_{\lambda \in \sigma(T)} f(\lambda) SP_\lambda \\
 &= \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda S \\
 &= f(T)S
 \end{aligned}$$

So we get  $f(T) \in \{T\}''$

Let  $R \in \{T\}''$ . Note that  $\forall \lambda \neq \lambda' \in \sigma(T)$ ,  $P_\lambda P_{\lambda'} = 0$  by **Lemma 3.3.9**. Then for all  $\gamma \in \sigma(T)$ , we get

$$P_\gamma T = P_\gamma \sum_{\lambda \in \sigma(T)} \lambda P_\lambda = \gamma P_\gamma$$

So  $P_\gamma \in \{T\}'$ . Thus  $RP_\gamma = P_\gamma R$  for all  $\gamma \in \sigma(T)$ .

Fix  $\gamma \in \sigma(T)$ . Given  $A \in B(P_\gamma(\mathcal{H}))$ . Let  $S := P_\gamma A P_\gamma : \mathcal{H} \rightarrow \mathcal{H}$ . Then

$$\begin{aligned}
 ST &= P_\gamma A P_\gamma \sum_{\lambda \in \sigma(T)} \lambda P_\lambda \\
 &= \gamma P_\gamma A P_\gamma \\
 &= \gamma S \\
 &= TS
 \end{aligned}$$

Thus  $RS = SR$ . This shows that  $R|_{P_\gamma(\mathcal{H})}$  commutes with every  $A \in B(P_\gamma(\mathcal{H}))$ . Hence there exist such a scalar  $f(\gamma) \in \mathbb{C}$  such that  $R|_{P_\gamma(\mathcal{H})} = f(\gamma)I_{P_\gamma(\mathcal{H})}$ . Thus  $R = \sum_{\lambda \in \sigma(T)} f(\lambda)P_\lambda$  and  $f \in \ell^\infty(\mathbb{C})$ .

**Corollary 3.3.4.1.** *The mapping above restricted to  $\sigma(T)$  is an isometric bijective linear multiplicative  $*$ -preserving map.*

**Definition 3.3.3.** Let  $T \in B(\mathcal{H})$ . Define  $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$

□

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**Definition 3.3.4.** A Banach algebra  $\mathcal{A}$  is a Banach space with a multiplication ring with addition satisfying  $\|ab\| \leq \|a\|\|b\|$  for all  $a, b \in \mathcal{A}$ . We say  $\mathcal{A}$  is unital if the ring above is unital with multiplicative identity  $1_{\mathcal{A}}$ . Units (invertible elements) may also exist similarly.

**Definition 3.3.5.** Given a Banach algebra  $\mathcal{A}$ , and  $a \in \mathcal{A}$ , the spectrum of  $a$ , denoted by

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \text{ is not invertible in } \mathcal{A}\}$$

**Example 3.3.3.** Let  $(\alpha_n) \in \mathbf{c}_0$  and  $T : \ell^2 \rightarrow \ell^2$  such that  $T((x_n)) = (\alpha_n x_n)$ . We claim that

$$\sigma(T) = \{\alpha_n : n \in \mathbb{N}\} \cup \{0\}$$

*Proof.* Since  $T$  is a compact operator (see Homework-5), we must have  $0 \in \sigma(T)$ . Otherwise if  $T$  is invertible, we'll get  $I = T \circ T^{-1}$  be also compact, which is a contradiction since  $\ell^2$  is infinite dimensional. Moreover since  $(T - \alpha_n I)(e_n) = 0$ , we have  $\alpha_n \in \sigma(T)$ .

Now assume that  $\beta \notin \{\alpha_n : n \in \mathbb{N}\} \cup \{0\}$ . Then let  $S \in B(\ell^2)$  defined by

$$S(e_n) = \frac{1}{\alpha - \beta} e_n$$

Then **show that indeed  $S \in B(\ell^2)$**  and  $(T - \beta I)S = S(T - \beta I) = I$  □

**Example 3.3.4.** Let  $T \in B(L^2([0, 1]))$  such that  $T(f)(x) = xf(x)$  for all  $f \in L^2([0, 1])$ . Then  $\sigma(T) = [0, 1]$ .

*Proof.* First let us see that  $0 \in \sigma(T)$ . Suppose  $S = T^{-1}$  exists. Then  $\forall n \in \mathbb{N}$  if  $f = S\chi_{[0, 1]}$ , then  $Tf = \chi_{[0, 1]}$ . So  $xf(x) = \chi_{[0, 1]}$ . But this is absurd, since  $\frac{1}{x}\chi_{[0, 1]} \notin L^2([0, 1])$ . □

**Lemma 3.3.12.** *Let  $S \in B(\mathcal{H})$  such that  $\|S - I\| \leq 1$ . Then  $S$  is invertible.*

*Proof.* Since  $\|S - I\| \leq 1$ ,

$$\sum_{n \in \mathbb{N}} \|(S - I)^n\| \leq \sum_{n \in \mathbb{N}} \|S - I\|^n$$

Moreover since  $B(\mathcal{H})$  is a Banach space, absolutely convergent sequences converge and this gives that  $\sum_{n \in \mathbb{N}} (I - S)^n$  converges. Thus

$$R = \sum_{n=1}^{\infty} (I - S)^n$$

exists. Now for each  $N \in \mathbb{N}$ , we have

$$\begin{aligned} S \left( \sum_{n=0}^N (I - S)^n \right) &= (I - (I - S)) \left( \sum_{n=1}^N (I - S)^n \right) = \sum_{n=0}^n (I - S)^n - \sum_{n=0}^N (I - S)^{n+1} \\ &= I - (I - S)^{N+1} \end{aligned}$$

which converges to 0 as  $n \rightarrow \infty$ . □

**Corollary 3.3.4.2.** *The set of invertible operators is invertible in  $B(\mathcal{H})$ .*

*Proof.* Let  $S$  be invertible. Then

$$\|T - S\| = \|S(S^{-1}T - I)\| \leq \|S\| \|S^{-1}T - I\|$$

□

**Theorem 3.3.5.** *For any  $T \in B(\mathcal{H})$ ,  $\sigma(T)$  is a non-empty, compact subspace of  $\mathbb{C}$ .*

*Proof.* Observe that the function  $f : \lambda \rightarrow T - \lambda I$  is continuous. Then  $f^{-1}(G(\mathcal{A}))$  is open, where  $G(\mathcal{A})$  is the collection of all invertible elements of  $B(\mathcal{H})$ . But  $\sigma(T)^c = f^{-1}(G(\mathcal{A}))$ . So we see  $\sigma(T)$  is closed. Moreover let  $\lambda \in \mathbb{C}$  with  $\|T\| < |\lambda|$ . Then,

$$-T + \lambda I = \lambda \left( \frac{-T}{\lambda} + I \right)$$

Then  $\|S - I\| = \left\| \frac{T}{\lambda} \right\| = \frac{\|T\|}{|\lambda|} < 1$  Hence show that  $\sigma(T)$  is bounded. □