MATH7320 - Functional Analysis
Homework 5

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1. **Solution:** Since ϕ is distance preserving, we get

$$\begin{aligned}
\|x\|^2 + \|y\|^2 - 2\Re\langle x, y\rangle &= \langle x - y, x - y\rangle \\
&= \|x - y\|^2 \\
&= \|\phi(x) - \phi(y)\|^2 \\
&= \langle \phi(x) - \phi(y), \phi(x) - \phi(y)\rangle \\
&= \|\phi(x)\|^2 + \|\phi(y)\|^2 - 2\Re\langle \phi(x), \phi(y)\rangle \\
&= \|x\|^2 + \|y\|^2 - 2\Re\langle \phi(x), \phi(y)\rangle
\end{aligned}$$

Thus we get $\Re\langle x, y\rangle = \Re\langle \phi(x), \phi(y)\rangle$ for all $x, y \in \mathcal{H}$.

Now let $x, y \in \mathcal{H}$ and $p, q \in \mathcal{H}$ such that $\phi(p) = r\phi(x), \phi(q) = s\phi(y)$, for $r, s \in \mathbb{R}$. Surjectivity of ϕ allows us to find p, q. Then

$$\begin{aligned}
\|\phi(rx + sy) - \phi(p) - \phi(q)\|^2 &= \langle \phi(rx + sy) - \phi(p) - \phi(q), \phi(rx + sy) - \phi(p) - \phi(q)\rangle \\
&= \|\phi(rx + sy)\|^2 + \|\phi(p)\|^2 + \|\phi(q)\|^2 - 2\Re\langle \phi(p), \phi(q)\rangle \\
&\quad - 2\Re\langle \phi(rx + sy), \phi(p)\rangle - 2\Re\langle \phi(rx + sy), \phi(q)\rangle \\
&= \|\phi(rx + sy)\|^2 + r^2\|\phi(x)\|^2 + s^2\|\phi(y)\|^2 - 2\Re\langle r\phi(x), s\phi(y)\rangle \\
&\quad - 2\Re\langle \phi(rx + sy), r\phi(x)\rangle - 2\Re\langle \phi(rx + sy), s\phi(y)\rangle \\
&= \|\phi(rx + sy)\|^2 + r^2\|\phi(x)\|^2 + s^2\|\phi(y)\|^2 - 2rs\Re\langle \phi(x), \phi(y)\rangle \\
&\quad - 2r\Re\langle \phi(rx + sy), \phi(x)\rangle - 2s\Re\langle \phi(rx + sy), \phi(y)\rangle \\
&= \|rx + sy\|^2 + r^2\|x\|^2 + s^2\|y\|^2 - 2\Re\langle rx, sy\rangle \\
&\quad - 2\Re\langle rx + sy, rx\rangle - 2\Re\langle rx + sy, sy\rangle \\
&= \|rx + sy - rp - sq\|^2 \\
&= 0
\end{aligned}$$

2. not finished

   Solution: Since $S \perp S^{\perp}$, clearly $S \subset (S^{\perp})^{\perp}$. Moreover, we know that $(S^{\perp})^{\perp} = \mathrm{Ker}(P_{S^{\perp}})$. Therefore $(S^{\perp})^{\perp}$ is a closed subspace. Hence $\overline{\mathrm{span}}(S) \subset (S^{\perp})^{\perp}$. Conversely if $x \in (S^{\perp})^{\perp}$, then $x \perp S^{\perp}$

3. not finished

   Solution: Let $T_n \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ be a Cauchy sequence.

4. **Solution:** Since we know that $I = P_{\mathcal{M}} + P_{\mathcal{M}^{\perp}}$, we see that $X = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then for all $x \in X$, $x = m + m'$ for unique $m \in \mathcal{M}, m' \in \mathcal{M}^{\perp}$. Then $\pi(x) = m' + \mathcal{M}$ for $\pi : X \to X/\mathcal{M}$. Moreover

   $$\|x\| = \|m\| + \|m'\| \quad \text{and} \quad \|\pi(x)\| = \|m'\|$$

   Thus we see that $\pi|_{\mathcal{M}^{\perp}}$ is isometric.

5. verify

   Solution: From what's given, it is evident that whenever $f_i \to 0$ weak $*$, $Tf_i \to 0$. Since the spaces X, Y are linear this gives us that T is weak $*$ continuous. Since the closed unit ball, $\overline{B}$ is weak $*$ compact by Banach-Alaoglu, and the continuity of T gives that $T(\overline{B})$ is compact.

6. **Solution:** Without loss of generality, assume that $\|T\| \leq 1$. Notice that $\overline{T(B_1)}$, the closure of the image of the unit ball is compact since T is a compact operator. Let $T(e_{i_n})$ be an arbitrary subsequence of $T(e_i)$. Since $T(e_{i_n})$ is a sequence in a compact space, it has a convergent subsequence $T(e_{i_{n_k}})$. We claim $T(e_{i_{n_k}})$ converge to zero. Let $x = \lim_{k\to\infty} T(e_{i_{n_k}})$. Since the convergence is in norm, we see that $T(e_{i_{n_k}}) \to x$ weakly.

   Now, for any $x \in \mathcal{H}$,

   $$\|x\|^2 \geq \sum_{n\in\mathbb{N}} \langle x, e_n \rangle$$

   Hence $\langle e_n, x \rangle \to 0$ for any $x \in \mathcal{H}$. Thus we see that $e_n \to 0$, weakly. Specifically, we see that

   $$\langle Te_n, y \rangle = \langle e_n, T^* y \rangle \to 0$$

for any $y \in \mathcal{H}$. Shows that $T(e_n) \to 0$ weakly. Since weak topology is Hausdorff, we see that $x = 0$.

Since we have shown that any arbitrary subsequence of $T(e_n)$ has a subsequence that converge to 0, we get that $T(e_j) \to 0$. Hence we are done.

7. **Solution:**

   (a) ($1 \implies 2$) If T is an isometry, then expanding $\langle Tx, Ty \rangle = \langle x, y \rangle$ for some $x, y \in \mathcal{H}$, we get

   $$\langle Tx, Ty \rangle + \langle Ty, Tx \rangle = \langle x, y \rangle + \langle y, x \rangle$$

   which gives $\Re\langle x, y \rangle = \Re\langle Tx, Ty \rangle$. Now replace x with ix to get $\Im\langle Tx, Ty \rangle = \Im\langle x, y \rangle$. Since real and imaginary parts are equal, we see that $\langle Tx, Ty \rangle = \langle x, y \rangle$

   (b) ($2 \implies 3$) If $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$, then $\langle T^*Tx, y \rangle = \langle x, y \rangle$, which implies $\langle (T^*T - I)x, y \rangle = 0$ for all $x, y \in \mathcal{H}$. Then Reisz Representation theorem shows that if $y \neq 0$, we must have $T^*T - I = 0$.

   (c) ($3 \implies 2$) If $T^*T = I$, then

   $$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \langle T^*Tx, x \rangle \\ &= \langle x, x \rangle \\ &= \|x\|^2 \end{aligned}$$

   which shows that T is an isometry.

8. **Solution:**

   (a) ($1 \implies 3$) If T is normal isometry, we see that $TT^* = T^*T$, and previous question proves that $TT^* = T^*T = I$.

   (b) ($3 \implies 2$) If $TT^* = T^*T = I$, then it is clear from the previous question that T is an isometry. To see that it is a bijection, let $x \in H$, since $T(T^*(x)) = x$, we see that $x \in T(\mathcal{H})$. Hence T is a bijection.

   (c) ($2 \implies 1$). We just need to show normality of T. Since T is given to be an isometric bijection, T has an inverse, P. Since T is bijective P is also isometric and linear. To see linearity, notice that

$$P(x+y) = P(T(Px)+T(Py)) = P(T(Px+Py)) = Px+Py$$

We claim $P = T^*$. To see this, note that

$$\langle PTx, y\rangle = \langle x, y\rangle = \langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$$

Hence we see that $\langle (PT - T^*T)x, y\rangle = 0$ for al $x, y \in \mathcal{H}$. Therefore by Reisz representation, we have $PT - T^*T = (P - T^*)T = 0$. Since T is bijective, this forces $P = T^*$ and we get the normality.

9. **Solution:** Let $x \in \mathrm{Ker}(T)$. Then $0 = \langle Tx, y\rangle = \langle x, T^*y\rangle$ for any $y \in \mathcal{H}$, shows that $y \in T(\mathcal{H})^\perp$.

   Conversely if $x \in T^*(H)^\perp$. Then $0 = \langle x, T^*y\rangle = \langle Tx, y\rangle =$ for any $y \in \mathcal{H}$ shows that $x \in \mathrm{Ker}(T)$ by Reisz representation theorem.

10. **Solution:** Consider the map $T|_{\mathrm{Ker}(T)^\perp} : \mathrm{Ker}(T)^\perp \to T(H)$. Clearly the map is surjective and linear. Hence $\mathrm{Ker}(T)^\perp \cong T(H)$, which is finite dimensional. From the previous question, we know that $T^*(\mathcal{H}) = \mathrm{Ker}(T)^\perp$. Hence we see that T^* is finite rank.

11. **Solution:**

$$\begin{aligned}\|Tf\|^2 &= \int |xf(x)|^2 \, d\mu \\ &\le \int |f|^2 \, d\mu \\ &= \|f\|^2\end{aligned}$$

shows that T is a contraction. Hence $T \in B(L^2[0,1])$.

Moreover for any $f, g \in L^2([0,1])$, we get

$$\langle Tf, g\rangle = \int |xf(x)\bar{g}(x)|^2 \, d\mu = \int |f(x)\bar{x}g(x)|^2 \, d\mu = \langle f, Tg\rangle$$

Hence $T^* = T$. To prove injectivity of T, assume $T(f) = T(g)$, then

$$0 = (T(f) - T(g))(x) = x(f(x) - g(x))$$

forces $f = g$. (Note that the equalities above is almost everywhere). To see that T is not surjective, we claim that there $T(f) \neq \chi_{[0,1]} \in L^2([0,1])$ for any $f \in L^2([0,1])$. If such f exist, then $f(x) = \frac{\chi_{[0,1]}(x)}{x} \notin L^2([0,1])$.
For the sake of contradiction, assume that $\lambda \in \mathbb{C}$ such that $Tf = \lambda f$ for some $f \in L^2([0,1])$. Then we must have

$$xf(x) = T(f)(x) = \lambda f(x)$$

almost everywhere. This forces $x = \lambda$ or $f = 0$ almost everywhere. Since x cannot be equal to λ except possibly only at $x = \lambda$ (measure zero set), we see that $f = 0$ almost everywhere. Thus λ cannot be an eigenvector of T.

12. **Solution:**

(a) Let $x = (x_n) \in \ell^2$. Then

$$\begin{aligned}\|T(x)\|_2^2 &= \sum_{n\in\mathbb{N}} |\alpha_n x_n|^2 \\ &\leq \|(\alpha_n)\|_\infty^2 \sum_{n\in\mathbb{N}} |x_n|^2 \\ &= \|(\alpha_n)\|_\infty \|(x_n)\|_2^2\end{aligned}$$

shows that $\|T\| \leq \|(\alpha_n)\|_\infty$. Moreover since each $\|\delta_n\| = 1$, and

$$\|T(\delta_n)\| = |\alpha_n|$$

taking supremum over n, we see T attains the norm $\|(\alpha_n)\|_\infty$.

(b) One direction is the direct application of question 6. Conversely if $(\alpha_n) \in c_\mathbf{0}$, then for any open cover U which cover $T(B)$, we can find a finite subcover by first taking an element which cover 0, then there can only by at most finite $T(\delta_n)$ outside that open ball. Now by the fact that finite dimensional closed unit balls are compact, we get compactness of T.

(c) Notice that if $x = (x_n), y = (y_n) \in \ell^2$, then

$$x = \sum_{n\in\mathbb{N}} x_n\delta_n, \quad y = \sum_{n\in\mathbb{N}} y_n\delta_n$$

and

$$\langle Tx, y\rangle = \sum_{n\in\mathbb{N}} \alpha_n x_n \overline{y_n} = \sum_{n\in\mathbb{N}} x_n \overline{\overline{\alpha_n} y_n}$$

Since this is true for all $x, y \in \ell^2$, by the uniqueness of the adjoint, we get our assertion.

13. **Solution:**

(a) If M is invariant under T, then $T(M) \subset M$, This shows $T(m) = P_M T(m)$ for all $m \in M$. Thus $TP_M = P_M TP_M$. Conversely if $TP_M = P_M TP_M$, then $T(m) = TP_M(m) = P_M TP_M(m) = P_M T(m)$ for all $m \in M$, shows that $T(m) \subset M$ for all $m \in M$. Thus M is invariant under T.

(b) If M reduces T, then M and $M^\perp$ is invariant under T. Thus we see that for $x = m + m'$ for $m \in M, m' \in M^\perp$,

$$P_M T(x) = P_M(T(m) + T(m')) = P_M(T(m)) = T(m) = T(P_M(x))$$

Thus $P_M T = TP_M$.

Conversely, if $P_M T = TP_M$, then for $m \in M$, we get

$$P_M T(m) = TP_M(m) = T(m)$$

which shows $T(m) \in M$ and for $m' \in M^\perp$, we get

$$P_M T(m') = TP_M(m') = T(0) = 0$$

Hence $T(m') \perp M$ which implies $T(m') \in M^\perp$. Thus we see that M reduces T.

(c) If M reduces T, then it is clear that M is invariant under T. Let $m \in M$. Then for $x \in M^\perp$, we get

$$\langle T^* m, x\rangle = \langle m, Tx\rangle = 0$$

since $Tx \in M^\perp$. Thus $T^*(m) \in M$.

(d) If M reduces T Then $\mathcal{H} = M \oplus M^\perp$ and $TP_M = P_M T$. Notice also that $T|_M = TP_M$. Then

$$P_M T^* = P_M^* T^* = (TP_M)^*$$

Moreover, we know M reduces T^* also. Hence we get $P_M T^* = T^* P_M$. Thus we get

$$(T|_M)^* = (TP_M)^* = P_M T^* = T^* P_M = T^*|_M$$

(e) No. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be represented by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Now take $M = \text{span}(e_1)$. The fact is evident.

14. not finished

Solution:

(a) Given P, Q are projections, if $P+Q$ is a projection, then

$$P+Q = (P+Q)^2 = P^2 + PQ + QP + Q^2 = P + QP + PQ + Q$$

shows that $PQ = -QP$. Then for any $x, y \in \mathcal{H}$, verify

Conversely if $P(\mathcal{H}) \perp Q(\mathcal{H})$, then for all $x, y \in \mathcal{H}$, we get

$$\langle x, PQy \rangle = \langle Px, Qy \rangle = \langle QPx, y \rangle = 0$$

Then Reisz representation theorem forces $PQ = QP = 0$, thus we see

$$(P+Q)^2 = P^2 + PQ + QP + Q^2 = P + QP + PQ + Q = P + Q$$

Also $(P+Q)^* = P^* + Q^* = P + Q$. Hence we see that $P+Q$ is a projection.

If this happens, then it is clear that $\mathrm{Ker}(P) \cap \mathrm{Ker}(Q) \subset \mathrm{Ker}(P+Q)$. Conversely let $(P+Q)(y) = 0$, then $P(y) = -Q(y)$. But since $P(\mathcal{H}) \perp Q(\mathcal{H})$, this forces $P(y) = Q(y) = 0$. Hence $\mathrm{Ker}(P+Q) = \mathrm{Ker}(P) \cap \mathrm{Ker}(Q)$.

Moreover since $P(\mathcal{H}) \perp Q(\mathcal{H})$, we immediately see that $(P+Q)(\mathcal{H}) = P(\mathcal{H}) \oplus Q(\mathcal{H})$

(b)
- If PQ is a projection, we must have $(PQ)^* = PQ$. Then for all $x, y \in \mathcal{H}$,

$$\langle PQ(x), y \rangle = \langle Qx, Py \rangle = \langle x, QPy \rangle$$

  Then uniqueness of the adjoint forces $PQ = QP$.
- If $PQ = QP$, then

$$\begin{aligned}
(P+Q-QP)^2 &= P^2 + Q^2 + QPQP + PQ + QP - PQP - QQP - QPP - QPQ \\
&= P + Q + QQPP + QP + QP - QPP - QQP - QPP - QQP \\
&= P + Q + QP + QP + QP - QP - QP - QP - QP \\
&= P + Q - QP
\end{aligned}$$

  and $(P+Q-QP)^* = P^* + Q^* - P^*Q^* = P + Q - PQ = P + Q - QP$. shows that $P+Q-QP$ is a projection.

- If $P+Q-QP$ is a projection, then

$$P+Q-QP=(P+Q-QP)^*=P^*+Q^*-P^*Q^*=P+Q-PQ$$

  forces $QP=PQ$

If the above happens, then for any $x\in\mathcal{H}$, $PQ(x)=QP(x)$ forces that $PQ(x)\in P(\mathcal{H})$ and $QP(x)\in Q(\mathcal{H})$. Hence we see that $PQ(\mathcal{H})\subset P(\mathcal{H})\cap Q(\mathcal{H})$. Conversely, if $y\in P(\mathcal{H})\cap Q(\mathcal{H})$, then for some $a,b\in\mathcal{H}$, $P(a)=y=Q(b)$. Then $Q(P(a))=Q(y)=Q^2(b)=Q(b)=y$ and $P(Q(b))=P(y)=P^2(a)=P(a)=y$ shows that $y\in PQ(\mathcal{H})$. Hence we see that $P(\mathcal{H})\cap Q(\mathcal{H})=PQ(\mathcal{H})$.

Similarly, if $y\in\mathrm{Ker}(P)$, $z\in\mathrm{Ker}(Q)$, then since $QP=PQ$, we get

$$\begin{aligned}(P+Q)(y+z)&=(P+Q)^2(y+z)\\&=(P+Q)(P(z)+Q(y))\\&=P(z)+Q(y)+QP(z)+PQ(y)\\&=P(z)+Q(y)\end{aligned}$$