Department of Mathematics University of Houston

MATH 6320

**Theory of Functions of a Real Variable**
**Fall 2024**

**First name:** ____________ **Last name:** ____________ **Points:**

# Assignment 9, due Thursday, November 14, 11:30am

**Please staple this cover page to your homework.** When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

## Problem 1

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f$ be a complex-valued measurable function on $X$. Let

$$\phi(p) = \int |f|^p d\mu$$

and $E = \{p > 0 : \phi(p) < \infty\}$. Assume $\|f\|_\infty \in (0, \infty]$.

(a) Show that if $r < p < s$, and $r, s \in E$, then

$$\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\},$$

so $p \in E$. Hint: Hölder's inequality.

(b) Assume that $r \in E$ for some $r > 0$ and prove

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty .$$

## Problem 2

Suppose $1 \leq p \leq \infty$, $(f_n)_{n=1}^\infty$ is a sequence in $L^p(\mu)$, $\|f_n - f\|_p \to 0$ and $f_n \to g$ pointwise almost everywhere. What is the relationship between $f$ and $g$ and why?

## Problem 3

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^p(\mu)$, $(f_n)_{n=1}^\infty$ a sequence in $L^p(\mu)$ with $f_n \to f$ pointwise almost everywhere and $\|f_n\|_p \to \|f\|_p$. Show that $\|f - f_n\|_p \to 0$ using Egoroff's theorem, splitting $X = A \cup B$ such that $\int_A |f|^p d\mu < \epsilon$ and $f_n \to f$ uniformly on $B$ with $\mu(B) < \infty$, in combination with Fatou's lemma applied to $\int_B |f_n|^p d\mu$ which shows

$$\limsup_n \int_A |f_n|^p d\mu \leq \epsilon.$$