MATH6302 - Modern Algebra
Homework 6

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1. Assume G acts transitively on a finite set A and H be a normal subgroup of G. Let $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}$ be distinct orbits of H on A.
   (a) Prove that G permutes the set $\mathcal{O}$ and that the action of G is transitive on $\mathcal{O}$. Deduce that all the orbits $\mathcal{O}_k$ have the same cardinality.
   (b) Prove that if $a \in \mathcal{O}_1$, then $|\mathcal{O}_1| = |H : H \cap G_a|$ and show that $r = |G : HG_a|$

Solution: For notational convenience we'll use $\mathcal{O}^a$ for the orbit of $a \in A$ under the action of H and $\mathcal{O}_k$ for the corresponding order in $\mathcal{O}$ as given in the question. Moreover we note that $\{\mathcal{O}^a \ : \ a \in A\} = \mathcal{O}$ by definition.

(a) Let $g \in G, a \in A$, and $ga = b$. Since $a \in \mathcal{O}^a$, we get $b \in g\mathcal{O}^a$. We claim that $g\mathcal{O}^a = \mathcal{O}^b$. Let $gx \in g\mathcal{O}^a$. By the definition of the orbit, $x = ha$ for some $h \in H$. Since $H \trianglelefteq G$, $gH = Hg$, and $gh = \tilde{h}g$ for some $\tilde{h} \in H$. Then

$$gx = g(ha) = (gh)a = (\tilde{h}g)a = \tilde{h}(ga) = \tilde{h}b$$

shows $gx \in \mathcal{O}_b$. Conversely, if $hb \in \mathcal{O}^b$, again by the normality of H in G, there is some $\tilde{h} \in H$ such that $hg = g\tilde{h}$. Then

$$hb = h(ga) = (hg)a = (g\tilde{h})a = g(\tilde{h}a) \in g\mathcal{O}^a$$

Hence we see that $g\mathcal{O}^a = \mathcal{O}^b$. In essence, we have verified that the map $\phi_g : \mathcal{O} \to \mathcal{O} := \mathcal{O}^a \to \mathcal{O}^{ga}$ is well defined for all $g \in G$.

Since $\mathcal{O}$ is a finite set, to show that ϕ_g is a permutation (bijection), we just need to verify surjectivity. Let $\mathcal{O}^b \in \mathcal{O}$. Then $\phi_g(\mathcal{O}^{g^{-1}b}) = \mathcal{O}^{gg^{-1}b} = \mathcal{O}^b$ shows that ϕ_g is a permutation. Moreover if $g' \in G$, then $\phi_{gg'}(\mathcal{O}^a) = \mathcal{O}^{gg'a} = \phi_g(\mathcal{O}^{ga}) = \phi_g(\phi_{g'}(\mathcal{O}^a))$ shows that $\phi_{gg'} = \phi_g \circ \phi_{g'}$.

Therefore, $\phi : G \to S_{\mathcal{O}} := g \to \phi_g$ is a well defined action (permutation representation) of G on $\mathcal{O}$. Moreover if $\mathcal{O}^a, \mathcal{O}^b \in \mathcal{O}$, since G acts transitively on A, there is a $g \in G$ such that $ga = b$. Then

$$\phi_g(\mathcal{O}^a) = \mathcal{O}^{ga} = \mathcal{O}^b$$

shows that G acts transitively on $\mathcal{O}$.

Now from the orbit-stabilizer theorem, we know that $|\mathcal{O}^a| = |H : H_a|$ where H_a is the stabilizer of $a \in A$ under the action by H. Since the cosets of H_a partition H sets of equal cardinality, we'll be proving $|\mathcal{O}^a| = |\mathcal{O}^b|$, if we show $|H_a| = |H_b|$ for any $a, b \in A$. Since G acts transitively on A there is a $g \in G$ such that $ga = b$. Then we claim $H_b = gH_ag^{-1}$. This follows from the following equivalences.

$$\begin{aligned} h \in H_a &\iff ha = a \\ &\iff hg^{-1}b = g^{-1}b \\ &\iff ghg^{-1}b = b \\ &\iff ghg^{-1} \in H_b \end{aligned}$$

Moreover gH_ag^{-1} and H_a has the same cardinality since the map $H_a \to gH_ag^{-1} := h \to ghg^{-1}$ has an inverse $gH_ag^{-1} \to H_a := x \to g^{-1}xg$. Then we're done.

(b) To prove $|\mathcal{O}^a| = |H : H \cap G_a|$, it is enough to show that $H_a = H \cap G_a$, then the relation will easily follow from the first theorem we proved in class. Since H borrows the action of G on A, $H_a = H \cap G_a$ follows from the definition of the stabilizer of a.

Now we'll go on to show that $r = |G : HG_a|$. Since G acts transitively on $\mathcal{O}$, the orbit of $\mathcal{O}^a$ under this action is the whole $\mathcal{O}$. Also, since H is normal, we notice that $HG_a = G_aH$. Hence by a similar reasoning as above, to show $r = |\mathcal{O}| = |G : HG_a|$, it is enough to show that HG_a is the stabilizer of $\mathcal{O}^a$. That is $G_{\mathcal{O}^a} = HG_a$.

If $g \in G_{\mathcal{O}^a}$, then $g\mathcal{O}^a = \mathcal{O}^a$. This implies for all $h \in H$, there is a $\tilde{h} \in H$ such that $gha = \tilde{h}a$. Hence $ga = gh^{-1}g^{-1}gha = gh^{-1}g^{-1}\tilde{h}a$. By the normality of H, $gh^{-1}g^{-1} = h' \in H$ and gives $ga = h'\tilde{h}a$ and hence $(h'\tilde{h})^{-1}ga = a$. This gives $(h'\tilde{h})^{-1}g \in G_a$ and therefore $g \in HG_a$.

Conversely, by the normality of H, for all $h \in H$, there exist a $\tilde{h} \in H$ such that $gh = \tilde{h}g$. If $h'g \in HG_a$ then $h'g\mathcal{O}^a = h'gHa = \{h'gha \ : \ h \in H\} = \{h'\tilde{h}ga \ : \ \tilde{h} \in H\} = \{h'\tilde{h}a \ : \ \tilde{h} \in H\} = Ha = \mathcal{O}^a$. Hence $h'g \in G_{\mathcal{O}^a}$

Thus we've shown that $G_{\mathcal{O}^a} = HG_a$ and the result follows.

2. Prove that if H has finite index n, then there is a normal subgroup K of G with $K \leqslant H$ such that $|G : K| \leq n!$

   Solution: Consider G/H, the collection of left cosets of H in G. Let G act on G/H by left multiplication. We see that if gH is any coset of H, its stabilizer

$$\begin{aligned} G_{gH} &= \{x \in G \ : \ xgH = gH\} \\ &= \{x \in G \ : \ x \in gHg^{-1}\} \\ &= gHg^{-1} \end{aligned}$$

   Therefore the kernel of the action is $K = \cap_{g \in G} G_{gH} = \cap_{g \in G} gHg^{-1}$. We claim that this is our required subgroup K.

   Since K is the kernel of the left multiplication action on G/H, it is the kernel of the corresponding permutation representation $\phi : G \to S_{G/H}$. Therefore we see that K is normal. Moreover by definition $K = \cap_{g \in G} gHg^{-1} \subset H$, shows that $K \leqslant H$.

   Now by the first isomorphism theorem, G/K is isomorphic to a subgroup of $S_{G/H}$. Hence $|G : K| = |G/K| \leq |S_{G/H}|$. Since $|G/H| = n$, $S_{G/H}$ is isomorphic to S_n, which have $n!$ elements. Thus, $|G : K| \leq n!$

3. Let G be a group and $\pi : G \to S_G$ be the left regular representation. Prove that if x is an element of order n and $|G| = mn$, then $\pi(x)$ is a product of m n-cycles. Also prove that if $\pi(x)$ is an odd permutation, then m is odd and n is even.

   Solution: Since $|x| = n$, and the map π is an injective homomorphism, we see that $\pi(x)$ is also of order n. Now if $\pi(x)$ has a cycle of order less than n, then we get that there is a $g \in G$ such that $\pi(x)^k g = x^k g = g$ for some $k < n$. But this forces $x^k = e$ for $k < n$, contradicting the order assumption on x. Therefore we see that $\pi(x)$ is a product of n-cycles.

   Moreover, if $\pi(x)$ is not a product of m n-cycles, then there is some element $g \in G$ that if fixed by $\pi(x)$. That is $\pi(x)g = xg = g$. But this forces $x = e$ again contradicting the order assumptions on x. Hence we see that $\pi(x)$ is precisely a product of m n-cycles.

   Since we have shown that $\pi(x)$ is a product of m n-cycles, we get that the sign of $\pi(x)$ is the parity of $m \times (n - 1)$. Now if $\pi(x)$ is an odd permutation, we get that both m and $n - 1$ has to be odd which forces $|x| = n$ to be even and $\frac{|G|}{|x|} = m$ to be odd.

4. Let G, π as in the previous exercise. Prove that if G contains an odd permutation, then G has a subgroup of index 2.

   Solution: Since $\pi(G)$ contains an odd permutation, we see that $\pi(G) \nleqslant A_G$, the alternating subgroup of S_G. Note that since G is a finite group, elements of G can be indexed and therefore S_G can be identified with S_n where $n = |G|$. Similarly we can identify A_G with A_n.

   Now exercise 3 from section 3.3 of the textbook shows that if $H \trianglelefteq G$ is a subgroup of prime index p, then for all $K \leqslant G$ either

   (1) $K \leqslant H$ or

   (2) $G = HK$ and $|K : K \cap H| = p$

   Replace $G = S_n, H = A_n, K = \pi(G)$ to the above statement. Since $|S_n : A_n| = 2$ and $\pi(G) \nleqslant A_n$, we get that $|\pi(G) : \pi(G) \cap A_n| = 2$. Since the left regular representation is faithful (injective), the preimage of $\pi(G) \cap A_n$ under π will have index 2 in G.

5. Prove that if $|G| = 2k$ where k is odd then G has a subgroup of index 2.

   Solution: Since G is a finite group and $2 \big| |G|$, G has an element x with $|x| = 2$ by Cauchy's theorem. Since $|x| = 2$ is even and $\frac{|G|}{|x|} = k$ is odd, by question 3, for the regular representation $\pi : G \to S_G$, we get that $\pi(x)$ is a odd permutation. Now by question 4, we get that G has a subgroup of index 2.

6. Prove that if M is a maximal subgroup of G, either $N_G(M) = M$ or $N_G(M) = G$. Deduce that if M is a maximal subgroup of G that is not normal in G, then the number of non-identity elements of G that are contained in the conjugates of M is at most $(|M| - 1)|G : M|$.

   Solution: Assume M is a maximal subgroup of G. Then $N_G(M)$ is a subgroup of G which contains M since $mMm^{-1} = M$ for all $m \in M$. Hence the maximality of M forces $N_G(M)$ to be either G or M.

   Now if M is not normal, we get that $N_G(M) = M$. Moreover $gMg^{-1} = hMh^{-1}$ if and only if $M = (g^{-1}h)M(g^{-1}h)^{-1}$ if and only if $g^{-1}h \in N_G(M) = M$.

   Consider the conjugate action of G on the subsets of G. We get that the number of conjugate classes (number of elements in the orbit) of M is equal to $|G : N_G(M)| = |G : M|$. Moreover each conjugate classes of M have $|M| - 1$

non-identity elements. Then it is evident that the number of elements of G which are in the conjugate classes of G is atmost $(|M|-1)|G:M|$.

7. Assume that H is a proper subgroup of the finite group G, Prove $G \neq \cup_{g \in G} gHg^{-1}$.

   Solution: Let H be a proper subgroup of G. Then $H \leqslant M$ for some maximal subgroup M of G. Existence of such a maximal subgroup is guaranteed because the group is a finite, and have only a finite number of subgroups). Then $gHg^{-1} \subset gMg^{-1}$ for all $g \in G$.

   If M is normal, then $gMg^{-1} = M$ for all $g \in G$ and thus $gHg^{-1} \subset M$ for all all $g \in G$ proves our statement. If M is not normal, from the above problem, we get that

$$|\bigcup_{g \in G} gMg^{-1}| \leq (|M|-1)|G:M|+1 = \frac{|M|-1}{|M|}|G|+\frac{|M|}{|M|} = |G|+(1-\frac{|G|}{|M|}) < |G|$$

   where the 1 is added above to include the identity element in the conjugates and the last inequality is because $|M| < |G|$.

   Hence $|\cup_{g \in G} gHg^{-1}| \leq |\cup_{g \in G} gMg^{-1}| < |G|$ which proves our assertion.

8. Let p, q be primes with $p < q$. Prove that a non-abelian group G of order pq has a non-normal subgroup of index q, so that there exists an injective homomorphism into S_q. Deduce that G is isomorphic to a subgroup of the normalizer in S_q of the cyclic group generated by the q-cycle $(1, 2, \ldots q)$.

   Solution: Since $p < q$, we know that every subgroup G of index p is normal. If every subgroup of index q is also normal, this would force every subgroup of G to be normal. Then if H_p, H_q are any subgroups of order p, q respectively, since they are both normal with $H_p \cap H_q = \{e\}$ and $H_q H_p = H_p H_q = G$ (this is because $H_p H_q$ have H_p and H_q as subgroups and Lagrange's theorem forces $H_p H_q = G$), we get that $G \cong H_p \times H_q \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Hence we see that G is Abelian. Hence if our group is non-Abelian, we must have a non-normal subgroup H of index q (order p).

   Now consider the left regular action of G on G/H, the left cosets of G. We know from question 2 that the kernel of this action is $K = \cap_{g \in G} gHg^{-1} \leqslant H$. Since $K \leqslant H$ and $|H| = p$(prime), by Lagrange's theorem, either $K = H$ or $K = \{e\}$. If $K = H$, then this would force $gHg^{-1} = H$ for all $g \in H$, making H normal and contradicting our assumption. Therefore, $K = \{e\}$.

This shows that the action of G on G/H is faithful. Hence the corresponding homomorphism $\phi : G \to S_{G/H}$ is injective. Since $|G/H| = q$, indexing elements of G/H by the numbers $1, 2, \ldots q$ gives an injective homomorphism from $G \to S_q$.

Since q is also a prime that divides the order of $|G|$, Cauchy's theorem guarantees the existence of a subgroup $K \leqslant G$ of order q. Moreover any subgroup of order q must have index p, and since p is the smallest prime dividing the order of the group, we see that K must be normal in G. Then $HK = KH = G$ since it contains elements of order p, q. Now let k be a generator of K. We can choose to index the elements in $S_{G/H}$ with $1, 2, ...q$ such that $\phi(k) = (1, 2, \ldots q)$. Then clearly $\phi(k) \in N_{S_q}\langle(1, 2, \ldots q)\rangle$. If h is a generator of our above subgroup H, then for any element $(1, 2, \ldots q)^n \in \langle(1, 2, \ldots q)\rangle$, we have

$$\begin{aligned}
\phi(h)(1, 2, \ldots q)^n \phi(h)^{-1} &= \phi(h)\phi(k)\phi(h^{-1}) \\
&= \phi(hk^n h^{-1}) \\
&= \phi(k^m) \qquad \text{by the normality of } K \\
&= (1, 2, \ldots q)^m \quad \in \quad \langle(1, 2, \ldots q)\rangle
\end{aligned}$$

This shows that $\phi(h) \in N_{S_q}\langle(1, 2, \ldots q)\rangle$.

Now since h, k are generators for H and K respectively, $\langle h, k\rangle = G$ (since $HK = G$). Hence $\phi(h), \phi(k) \in N_{S_q}\langle(1, 2, \ldots q)\rangle$ gives $\phi(G) \leqslant N_{S_q}\langle(1, 2, \ldots q)\rangle$.