MATH6320 - Modern Algebra
Homework 5

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1. Prove that if H is a normal subgroup of prime index p, then for all $K \leqslant G$ either $K \leqslant H$ or $G = HK$ with $|K : K \cap H| = p$

   Solution: Since H is normal in G, we see that $N_G(H) = G$, and therefore $K \leqslant N_G(H)$. Now using the second isomorphism theorem we get

   (i) HK is a subgroup of G

   (ii) $H \trianglelefteq HK$

   (iii) $K \cap H \trianglelefteq K$

   (iv) $HK/H \cong K/K \cap H$

   Assume that K is not a subgroup of H. Then H is a proper normal subgroup of HK. Since the cosets of H in HK are also cosets in G we have $HK/K \subset G/K$. By the normality of H in HK, we get that HK/H is algebraically closed under the group operation. Since G/H is a finite group, by the subgroup criterion for finite groups HK/H must be a subgroup of G/H. Moreover $|G/H| = |G : H| = p$ is prime, and $|HK/H| > 1$ (since H is a proper subgroup of HK). Then Lagrange's theorem forces $HK/H = G/H$.

   Now if HK is a proper subgroup of G, then there is $g \in G$ such that $g \notin HK$. This forces $gH \in G/H$ but $gH \notin HK/H$, which is a contradiction. Hence $HK = G$.

   By (iv) from above, we get that $G/H \cong K/K \cap H$. Then $p = |G : H| = |G/H| = |K/K \cap H| = |K : K \cap H|$

2. Let C be a normal subgroup of the group A and let D be a normal subgroup

of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Solution: First we'll show that $(C \times D) \trianglelefteq (A \times B)$. For this let $(a, b) \in (A \times B)$ and $(c, d) \in C \times D$. Then $(a, b)(c, d)(a, b)^{-1} = (a, b)(c, d)(a^{-1}, b^{-1}) = (aca^{-1}, bdb^{-1}) \in C \times D$ since $C \trianglelefteq A, D \trianglelefteq B$. Hence we get, $(C \times D) \trianglelefteq (A \times B)$.

Now consider the map $\phi : A \times B \to (A/C) \times (B/D) := (a, b) \to (aC, bD)$. Clearly ϕ is surjective, since if $(aC, bD) \in (A/C) \times (B/D)$, then $\phi(a, b) = (aC, bD)$. Also, it is a group homomorphism. If $(p, q), (r, s) \in A \times B$, then $\phi(pr, qs) = (prC, qsD) = (pC, qD)(rC, sD) = \phi(p, q)\phi(r, s)$.

Moreover, the $\text{Ker}(\phi) = C \times D$, since $\phi(a, b) = (aC, bD) = \mathbf{0} \in (A/C) \times (B/D)$ if and only if $aC = C$ and $bD = D$, which is equivalent to $a \in C$ and $d \in D$. Hence by the first isomorphism theorem, we get $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

3. Let M, N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

Solution: Consider the map $\phi : G \to (G/M) \times (G/N) := g \to (gM, gN)$. We'll show that this map is a surjective group homomorphism with $\text{Ker}(\phi) = M \cap N$. Then first isomorphism theorem will give us our required result.

First to show that it is a group homomorphism, let $g, h \in G$. Then $\phi(gh) = (ghM, ghN) = (gM, gN)(hM, hN) = \phi(g)\phi(h)$. Hence ϕ is a group homomorphism.

Now to show the surjectivity of ϕ, let $(gM, hN) \in (G/M) \times (G/N)$. Since $MN = G$ is a group, we see that $MN = G = NM$. Hence there exist $m_g, m'_g, m_h, m'_h \in M$ and $n_g, n'_g, n_h, n'_h \in N$ such that $g = m_g n_g = n'_g m'_g$ and $h = m_h n_h = n'_h m'_h$. Then

$$gM = n'_g m'_g M = n'_g M = n'_g m_h M$$

and

$$hN = m_h n_h N = m_h N = n'_g (m_h N) n'^{-1}_g = n'_g m_h N$$

where $m_h N = n'_g (m_h N) n'^{-1}_g$ follows from the fact that cosets of a normal subgroup are stable under the conjugation action. Therefore we see that $\phi(n'_g m_h) = (n'_g m_h M, n'_g m_h N) = (gM, hN)$ and hence ϕ is surjective.

Now it only remains to show that $\text{Ker}(\phi) = M \cap N$. $\phi(g) = (gM, gN) = \mathbf{0} = (M, N)$ if and only if $g \in M$ and $g \in N$. Hence we see that $\phi(g) = \mathbf{0}$ if and only if $g \in M \cap N$. Hence $\text{Ker}(\phi) = M \cap N$.

4. Let p be a prime and let G be a group of p-power roots of 1 in $\mathbb{C}$. Prove that the map $z \to z^p$ is a surjective homomorphism. Deduce that G is isomorphic to a proper quotient of itself.

   Solution: Note that the group $G = \{z \in \mathbb{C} \ : \ z^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$ and $\phi : G \to G$ is the map with $\phi(z) = z^p$. First we'll show that the map is well defined. Let $z \in G$ with $z^{p^n} = 1$. Then $\phi(z)^{p^{n-1}} = (z^p)^{p^{n-1}} = z^{p^n} = 1$. Hence $\phi(z) \in G$.

   Now to see that ϕ is a homomorphism, let $z, w \in G$. Then $\phi(zw) = (zw)^p = z^p w^p = \phi(z)\phi(w)$. Hence ϕ is a homomorphism.

   To prove the surjectivity of ϕ, let $z = e^{i\theta} \in G$ with $z^{p^n} = 1$. This is equivalent to $p^n\theta = 2m\pi$ for some $m \in \mathbb{N}$. Let $w = e^{i\theta/p}$. Since $w^{p^{n+1}} = (e^{i\theta/p})^{p^{n+1}} = e^{i\theta p^n} = e^{2m\pi} = 1$, we see that $w \in G$. Moreover $\phi(w) = z$. Hence ϕ is a surjective homomorphism.

   Now by the first isomorphism theorem, we see that $G \cong G/\mathrm{Ker}(\phi)$.

5. Let p be a prime and let G be a group of order $p^\alpha m$ where $p \not| m$. Assume P is a subgroup of G of order p^α and N be a normal subgroup of order $p^\beta n$, where $p \not| n$. Prove that $|P \cap N| = p^\beta$ and $|PN/N| = p^{\alpha-\beta}$.

   Solution: Note that since $P \cap N \leqslant P, N$, we have $|P \cap N|\big|p^\alpha$ and $|P \cap N|\big|p^\beta n$. Since $\beta \le \alpha$ and $(p^\alpha, p^\beta n) = p^\beta$, we get $|P \cap N|\big|p^\beta$. Let $|P \cap N| = p^k$ where $k \le \beta$. Since N is a normal subgroup of G, we know that PN is a subgroup of G with $|PN| = \frac{|P||N|}{|P \cap N|} = p^{\alpha+\beta-k}n$. Again since PN is a subgroup of G, by Lagrange's theorem, we get $p^{\alpha+\beta-k}n\big|p^\alpha n$. Therefore,

$$\begin{aligned} \alpha + \beta - k &\le \alpha \\ \beta - k &\le 0 \\ \beta &\le k \end{aligned}$$

   Hence $k = \beta$, and we get $|P \cap N| = p^\beta$.

   Now since $N \trianglelefteq G$ and $P \leqslant N_G(N) = G$, by the second isomorphism theorem $PN/N \cong P/(P \cap N)$. Therefore $|PN/N| = |P/(P \cap N)| = \frac{|P|}{|P \cap N|} = \frac{p^\alpha}{p^\beta} = p^{\alpha-\beta}$.

6. Give the list of invariant factors for all Abelian groups of order 105, 44100.

Solution:

(a) $105 = 21 \times 5 = 3 \times 5 \times 7$. If $Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_k}$ is an invariant factor decomposition of the group of order 105, we know that $n_{i+1}|n_i$. But since the powers of each of $2, 3, 5$ in the prime factorization of 105, is just 1, there is no way other than Z_{105} to represent an Abelian group of order 105.

(b) $44100 = 2^2 \times 3^3 \times 5^2 \times 7^2$. Therefore the possible invariant factor decompositions of a group of order 44100 are as

- Z_{44100}
- $Z_{22050} \times Z_2$
- $Z_{14700} \times Z_3$
- $Z_{8820} \times Z_5$
- $Z_{6300} \times Z_7$
- $Z_{7350} \times Z_6$
- $Z_{4410} \times Z_{10}$
- $Z_{3150} \times Z_{14}$
- $Z_{2920} \times Z_{15}$
- $Z_{2100} \times Z_{21}$
- $Z_{1050} \times Z_{42}$
- $Z_{1260} \times Z_{35}$
- $Z_{1470} \times Z_{30}$
- $Z_{630} \times Z_{70}$
- $Z_{420} \times Z_{105}$
- $Z_{210} \times Z_{210}$

7. Give the list of elementary divisors for all Abelian groups of order 105, 44100

Solution:

(a) $105 = 3 \times 5 \times 7$. If $A_1 \times A_2 \times \ldots \times A_k$ is the elementary divisor decomposition of the group of order 105, then each $|A_1| = 3, |A_2| = 5$ and $|A_3| = 7$. Since every group of prime power is cyclic, we get that the only possible decomposition is $Z_3 \times Z_5 \times Z_7$.

(b) $44100 = 2^2 3^2 5^2 7^2$. Hence if $A_2 \times A_3 \times A_5 \times A_7$ is the elementary divisor decomposition, then $|A_1| = 2^2, |A_2| = 3^2, |A_3| = 5^2$ and $|A_4| = 7^2$. Again each A_p can be isomorphic to either Z_{p^2} or $Z_p \times Z_p$. So the complete list of elementary divisors for all Abelian groups of order 44100 is

- $Z_4 \times Z_9 \times Z_{25} \times Z_{49}$
- $(Z_2 \times Z_2) \times Z_9 \times Z_{25} \times Z_{49}$
- $Z_4 \times (Z_3 \times Z_3) \times Z_{25} \times Z_{49}$
- $Z_4 \times Z_9 \times (Z_5 \times Z_5) \times Z_{49}$
- $Z_4 \times Z_9 \times Z_{25} \times (Z_7 \times Z_7)$
- $Z_4 \times Z_9 \times (Z_5 \times Z_5) \times (Z_7 \times Z_7)$
- $Z_4 \times (Z_3 \times Z_3) \times Z_{25} \times (Z_7 \times Z_7)$
- $(Z_2 \times Z_2) \times Z_9 \times Z_{25} \times (Z_7 \times Z_7)$
- $(Z_2 \times Z_2) \times Z_9 \times (Z_5 \times Z_5) \times Z_{49}$

- $(Z_2 \times Z_2) \times (Z_3 \times Z_3) \times Z_{25} \times Z_{49}$
- $Z_4 \times (Z_3 \times Z_3) \times (Z_5 \times Z_5) \times Z_{49}$
- $(Z_2 \times Z_2) \times (Z_3 \times Z_3) \times (Z_5 \times Z_5) \times Z_{49}$
- $(Z_2 \times Z_2) \times (Z_3 \times Z_3) \times Z_{25} \times (Z_7 \times Z_7)$
- $(Z_2 \times Z_2) \times Z_9 \times (Z_5 \times Z_5) \times (Z_7 \times Z_7)$
- $Z_4 \times (Z_3 \times Z_3) \times (Z_5 \times Z_5) \times (Z_7 \times Z_7)$
- $(Z_2 \times Z_2) \times (Z_3 \times Z_3) \times (Z_5 \times Z_5) \times (Z_7 \times Z_7)$

8. (a) Prove that any finite group has a finite exponent
   (b) Give an example of an infinite group with finite exponent
   (c) Does a finite group of exponent m always contain an element of order m?

   Solution:

   (a) We know that for all finite group G with order $|G|$, $g^{|G|} = e$ for all $g \in G$. Therefore the exponent of the group can be at most $|G|$.

   (b) Let $G = \{z \in \mathbb{C} \ : \ z^n = 1 \text{ for some } n \in \mathbb{Z}\}$. Then

9. Let A be a finite Abelian group and let p be a prime. Let

$$A^p = \{a^p \ : \ a \in A\} \quad \text{and} \quad A_p = \{x \ : \ x^p = 1\}$$

   (a) Prove that $A/A^p \cong A_p$
   (b) Prove that the number of subgroups of A of order p equals the number of subgroups of A of index p.

   Solution:

   (a) We'll first show that A^p, A_p are subgroups of A. Let $a^p, b^p \in A^p$. Then $(a^p)^{-1}b^p = a^{-p}b^p = (a^{-1})^p b^p = (a^{-1}b)^p$. Since $a^{-1}b \in A$, we get that $(a^p)^{-1}b^p = (a^{-1}b)^p \in A^p$. Hence A^p is a subgroup. Now let $a, b \in A_p$. Then $a^p = b^p = 1$. Then $(a^{-1}b)^p = (a^{-1})^p b^p = (a^p)^{-1}b^p = 1$. Hence $a^{-1}b \in A_p$. Hence A_p is also is a subgroup. Since A is Abelian, both of these are normal in A. Now let $A \cong A_1 \times A_2 \times \ldots \times A_n$ with $|A_i| = p_i^{\alpha_i}$ and $p_i > p_{i+1}$ where each p_i are distinct primes.

   We'll show that both the groups A/A^p and A_p have exponent p. Then both their orders must be a power of p. Then if we show that the order of both these groups are the same, by the fundamental theorem of finitely generated Abelian groups, we'll get that they are isomorphic.

To see that A_p has exponent p, let $a \in A_p$, then by the definition of A_p, we see that $a^p = 1$. Since p is a prime, we see that the exponent of the group should be either p or 1.

Now for A/A^p, let $aA^p \in A/A^p$, then $(aA^p)^p = a^pA^p = A^p$ implies the exponent of A/A^p should be either p or 1 by the same reasoning.

If the exponent of $A_p = 1$, A_p must be the trivial group, which forces the homomorphism $\phi : A \to A : a \to a^p$ to be isomorphic, since $\text{Ker}(\phi) = A_p$. This gives $\phi(A) = A^p$ and that $A/A^p = A/A = \{0\} = A_p$.

If the exponent of the group $A_p = p$, then there is an element in $a \in A$ with order p. Then aA^p must also have order p forcing the exponent of A/A^p to be p. Hence we see that the groups A_p and A/A^p has the same exponent.

Now by the same map ϕ above, we see that $A/A_p \cong A^p$. Therefore $\frac{|A|}{|A_p|} = |A^p|$ and we get $\frac{|A|}{|A^p|} = |A_p|$, which gives us that the order of the groups under consideration are the same. Hence we get $A/A^p \cong A_p$.

(b) Let G be a subgroup of A of order p. Then $g^p = 1$ for all $g \in G$. Hence $G \subset A_p$, which gives $G \leqslant A_p$, since A_p itself is a subgroup. Conversely if H is a subgroup of A_p of order p, then trivially it is a subgroup of A of the same order. Hence there's a correspondence between subgroups of A of order p and the subgroups of A_p of order p. Since $A_p \cong A/A^p$, we get that the number of subgroups of A_p of order p is equal to the number of subgroups of A/A^p of order p.

Now let G be a subgroup A of index p. Then $A/G = \{[h_1], [h_2], \ldots, [h_p]\}$. Then $h_i^p \in G$. Since h_i can be chosen to be any $a \in A$, we see that $A^p \leqslant G$. Therefore, by the third isomorphism theorem, we see that $p = |A/G| = |\frac{A/A^p}{G/A^p}|$. Hence G/A^p is a subgroup of A/A^p of index p. Conversely if H/A^p is a subgroup of A/A^p of index p, let $\frac{A/A^p}{H/A^p} = \{[a_1A^p], [a_2A^p], \ldots, [a_pA^p]\}$ then

$$\tilde{H} = \bigcup_{i=1}^{p} a_iA^p$$

10. Let n and k be positive integers and let A be the free Abelian group of rank n. Prove that A/kA is isomorphic to the direct product of n copies of $\mathbb{Z}/k\mathbb{Z}$.

Solution: Since we know that $A \cong \mathbb{Z}^n$, in essence, we just need to show that $\mathbb{Z}^n/k\mathbb{Z}^n \cong (\mathbb{Z}/k\mathbb{Z})^n$. Consider the map $\phi : \mathbb{Z}^n \to (\mathbb{Z}/k\mathbb{Z})^n$ as $\phi(a_1, a_2, \ldots, a_n) = ([a_1], [a_2], \ldots, [a_n])$, where $[a_i] = a_i + k\mathbb{Z}$.

Clearly ϕ is a homomorphism. Let $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$. Then $\phi(a^{-1}b) = \phi(-a_1 + b_1, -a_2 + b_2, \ldots, -a_n + b_n) = ([b_1 - a_1], [b_2 - a_2], \ldots, [b_n - a_n]) = ([b_1], [b_2], \ldots, [b_n]) - ([a_1], [a_2], \ldots, [a_n]) = \phi(b) - \phi(a)$.

Moreover, ϕ is a surjection. If $([a_1], [a_2], \ldots, [a_n]) \in (\mathbb{Z}/k\mathbb{Z})^n$, then $\phi(a_1, a_2, \ldots, a_n) = ([a_1], [a_2], \ldots, [a_n])$, shows that ϕ is surjective.

Now we'll show that $\mathrm{Ker}(\phi) = k\mathbb{Z}^n$. $\phi(a_1, a_2, \ldots, a_n) = ([a_1], [a_2], \ldots, [a_n]) = 0$ if and only if $[a_i] \in k\mathbb{Z}$ for each $1 \leq i \leq n$. This is equivalent to $(a_1, a_2, \ldots, a_n) \in (k\mathbb{Z})^n = k\mathbb{Z}^n$. Hence we get that $\mathrm{Ker}(\phi) = k\mathbb{Z}^n$.

Now by the first isomorphism theorem, we get that $\mathbb{Z}^n/k\mathbb{Z}^n \cong (\mathbb{Z}/k\mathbb{Z})^n$.