

# MATH6320 - Functions of a Real Variable

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# Chapter 1

## 1.1 Course Info

Bernhard Bodmann  
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Tue 10-11AM, Wed 1-2PM

Email for organizational stuff and meet for a course related conceptual stuff

- Canvas
- MS Teams

**Textbook :** Walter Rudin, Real & Complex Analysis, Chapters 1-9  
Midterm test, October 10, in class  
Grading: 30% HW, 30% Midterm, 40% Final

## 1.2 Notations and Basic Definitions

**Definition 1.2.1.** Let  $X$  be a set and  $P(X)$  be its power set. A subset  $\tau \subset P(X)$  is called a topology on  $X$  provided

- $\emptyset, X \in \tau$
- If  $E_1, E_2, \dots, E_n \in \tau$ , then  $\cap_{j=1}^n E_j \in \tau$
- If  $J$  is any index set and for each  $j \in J$ ,  $E_j \in \tau$  then  $\cup_{j \in J} E_j \in \tau$

**Example 1.2.1.** Given a set  $X$ ,  $\{\emptyset, X\}$  is a topology known as in-discrete topology.

**Definition 1.2.2.** Let  $(X, d)$  be a metric space with  $d : X \times X \rightarrow \mathbb{R}^+$  satisfying positive definiteness, symmetry, and triangle inequality.

**Definition 1.2.3.** We say  $E \subset X$  is open if for each  $x \in E$ , there is an  $\epsilon \geq 0$  such that  $\{y \in X : d(x, y) \leq \epsilon\} \subset E$

**Example 1.2.2.** Let  $\tau$  be the set of all open subsets of  $X$ , where  $(X, d)$  is a metric space, then  $\tau$  forms a topology. verify this

**Definition 1.2.4.** Let  $X$  be a set and  $\tau$  a topology on  $X$ , then we call  $(X, \tau)$  a topological space. Elements of  $\tau$  are called open sets.

**Definition 1.2.5.** Let  $X$  be a set,  $\beta \subset P(X)$  such that

- $\forall x \in X, \exists B \in \beta$  such that  $x \in B$
- If  $x \in X, B_1, B_2 \in \beta$  and if  $x \in B_1 \cap B_2$ , then there is  $B_3 \in \beta$  such that  $x \in B_3 \subset B_1 \cap B_2$

Then  $\beta$  is called a basis

**Theorem 1.2.1.** *If  $\beta$  is a basis then,  $\tau$ , the collection of all (empty or non-empty) unions of elements of  $\beta$  form a topology on  $X$ .*

*Proof.* It is clear from the definition of  $\tau$  that arbitrary unions of sets in  $\tau$  is again in  $\tau$ . Also the first property guarantees that  $X \in \tau$ . Since empty unions are also considered,  $\emptyset \in \tau$ . Hence all that remains is to show that finite intersections of sets in  $\tau$  is again in  $\tau$ .

Let  $U_1, U_2 \in \tau$ , once we show that  $U_1 \cap U_2 \in \tau$ , we can use induction to show  $\cap_{i=1}^n U_i \in \tau$  when  $U_1, U_2, \dots, U_n \in \tau$ . Let  $x \in U_1 \cap U_2$ . Since  $U_1, U_2$  are unions of elements from  $\beta$ , there exists  $B_1, B_2 \in \beta$  such that  $x \in B_1 \subset U_1$  and  $x \in B_2 \subset U_2$ . Then by the second property of the basis, there exists  $B_x \in \beta$  with  $x \in B_x \subset B_1 \cap B_2 \subset U_1 \cap U_2$ . Since  $x \in U_1 \cap U_2$  was arbitrary, we get

$$U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B_x$$

Thus  $U_1 \cap U_2 \in \tau$  and hence  $\tau$  is a topology. □

**Example 1.2.3.** Let  $\beta = \{(p, q) : p, q \in \mathbb{Q}, p < q\} \subset P(\mathbb{R})$ . Then  $\beta$  is a basis and the topology generated by  $\beta$  is the usual euclidean topology on  $\mathbb{R}$  obtained from the metric  $d(x, y) = |x - y|$ .

**Example 1.2.4.** Let  $X = [-\infty, \infty]$  and  $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty] : a \in \mathbb{R}\}$  Then  $\beta$  is a basis.

**Example 1.2.5.** Let  $J$  be a set and  $\mathbb{R}^J = \{f : J \rightarrow \mathbb{R}\}$ . Let  $\beta$  contain all the sets of the form  $\{f : J \rightarrow \mathbb{R} : f(j_1) \in U_1, f(j_2) \in U_2, \dots, f(j_n) \in U_n\}$  where  $n \in \mathbb{N}, j_1, j_2, \dots, j_n \in J$  and  $U_1, U_2, \dots, U_n$  are open sets in  $\mathbb{R}$ .

Then  $\beta$  is a basis and the topology generated by  $\beta$  is called the product topology in  $\mathbb{R}^J$ .

If  $J$  is uncountable, then this topology  $\mathbb{R}^J$  is not metrizable. **verify.**

**Definition 1.2.6.** Let  $X$  be a set  $\mathcal{M} \subset P(X)$  is a  $\sigma$ -algebra, if

- $X \in \mathcal{M}$
- If  $A \in \mathcal{M}$ , then  $A^c \in \mathcal{M}$
- If  $A_1, A_2, \dots, A_j, \dots \in \mathcal{M}$ , then  $\cup_{j=1}^{\infty} A_j \in \mathcal{M}$

Then we call  $(X, \mathcal{M})$  a measurable space, and  $\mathcal{M}$  contains measurable sets.

**Theorem 1.2.2.** Let  $X$  be a set, and  $F \subset P(X)$ , then there exists a unique  $\sigma$ -algebra  $\mathcal{M}$  such that,

- $F \subset \mathcal{M}$
- If  $\mathcal{N}$  is a  $\sigma$ -algebra on  $X$ , and  $F \subset \mathcal{N}$ , then  $\mathcal{M} \subset \mathcal{N}$

Then  $\mathcal{M}$  is called a  $\sigma$ -algebra generated by  $F$

# Chapter 2

Assignment 1 is posted. Submissions due Aug 29.

## 2.1 Warm up

**Example 2.1.1.** Let  $X = \{1, 2, 3\}$ ,  $F = \{\{1, 2\}, \{1, 3\}\}$ . Then the smallest topology containing  $F$  is  $\{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$ , and the  $\sigma$ -algebra generated by  $F$  is the power set,  $P(X)$ .

## 2.2 continues

*Proof.* Proof of [Theorem 1.2.2](#).

Consider all  $\sigma$ -algebras containing  $F$ , let  $\Omega = \{\mathcal{N} \subset P(X) : \mathcal{N} \supset F, \mathcal{N} \text{ is a } \sigma\text{-algebra}\}$ .  $\Omega$  is non-empty since  $P(X) \in \Omega$ . Let

$$\mathcal{M} = \bigcap_{\mathcal{N} \in \Omega} \mathcal{N}$$

Then we claim  $\mathcal{M}$  is a  $\sigma$ -algebra. To see this

- $X \in \mathcal{M}$ , because  $X \in \mathcal{N}$ , for each  $\mathcal{N} \in \Omega$ .
- If  $E \in \mathcal{M}$ , then  $E \in \mathcal{N}$  for each  $\mathcal{N} \in \Omega$ . Then  $E^c \in \mathcal{N}$  for each  $\mathcal{N} \in \Omega$  and thus  $E^c \in \mathcal{M}$ .
- If  $A_1, A_2, \dots \in \mathcal{M}$ , then  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$  because since each  $A_i \in \mathcal{N}$  and  $\mathcal{N}$  is a  $\sigma$ -algebra,  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{N}$  for each  $\mathcal{N} \in \Omega$ .

Moreover,  $F \subset \mathcal{M}$  since  $F \subset \mathcal{N}$  for each  $\mathcal{N} \in \Omega$ . Finally, if  $\mathcal{N}$  is a  $\sigma$ -algebra with  $\mathcal{N} \supset F$ , then  $\mathcal{N} \in \Omega$ . Then  $\mathcal{M} \subset \mathcal{N}$ . To prove uniqueness, let  $\mathcal{M}_0$  be a  $\sigma$ -algebra which satisfies the required properties defining  $\Omega$ . By intersection operation giving  $\mathcal{M}$ , and  $\mathcal{M}_0 \in \Omega$ ,  $\mathcal{M} \subset \mathcal{M}_0$ . Additionally, if  $\mathcal{M}_0$  satisfies that  $\mathcal{M}_0 \subset \mathcal{N}$  for each  $\mathcal{N} \in \Omega$ , then  $\mathcal{M}_0 \subset \mathcal{M}$ . Thus  $\mathcal{M}_0 = \mathcal{M}$ .  $\square$

We combine concepts of topologies and  $\sigma$ -algebras.

**Definition 2.2.1.** Let  $(X, \tau)$  be any topological space. The  $\sigma$ -algebra,  $\mathcal{B}$  generated by the topology  $\tau$  is called the Borel  $\sigma$ -algebra. Elements of  $\mathcal{B}$  are called Borel sets.

**Definition 2.2.2.** Let  $X, Y$  be topological spaces. A map  $f : X \rightarrow Y$  is continuous if the inverse image of any open set is open. The map  $f$  is continuous at  $x \in X$  if every open set  $V \subset Y$  with  $f(x) \in V$ , there is an open set  $W \subset X$  with  $f(W) \subset V$ .

**Theorem 2.2.1.** A map  $f : X \rightarrow Y$  is continuous if and only if it is continuous at each  $x \in X$ .

*Proof.* ( $\implies$ ) If  $f$  is continuous and  $x \in X$ ,  $V \subset Y$  is open and  $f(x) \in V$ , then by continuity,  $f^{-1}(V)$  is open and  $x \in f^{-1}(V)$ . This holds for any such  $x$  and  $V$ , thus  $f$  is continuous at  $x \in X$ . Since  $x$  was arbitrarily chosen,  $f$  is continuous at each  $x \in X$ .

( $\impliedby$ ) Suppose  $f$  is continuous at each  $x \in X$ . Let  $V$  be an open subset of  $Y$ . Need to show that  $W = f^{-1}(V)$  is open. For each  $x \in W$ , there is a  $W_x \subset X$  which is open with  $x \in W_x$  and  $f(W_x) \subset V$  by the continuity of  $f$  at  $x$ . Now take

$$Y = \bigcup_{x \in W} W_x$$

Then  $Y$  is open being a union of open sets. Also it contains each  $x \in W$ . Hence  $W \subset Y$ . But again,  $W_x \subset W = f^{-1}(V)$  for each  $x \in W$  and taking the unions preserve the inclusion. Hence we get  $W = Y$ . Since we already know  $Y$  is open, this gives us  $W = f^{-1}(V)$  is open.  $\square$

**Proposition 2.2.1.** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then so is  $g \circ f : X \rightarrow Z$ .

*Proof.* Let  $V \subset Z$  be an open set. Then  $f^{-1}(V)$  is open in  $Y$  by the continuity of  $f$ . Similarly,  $g^{-1}(f^{-1}(V))$  is open in  $X$  by the continuity of  $g$ . But  $g^{-1}(f^{-1}(V)) = (g \circ f)^{-1}(V)$ . Since  $V$  was arbitrarily open, we get that  $g \circ f$  is continuous.  $\square$

**Definition 2.2.3.** Let  $X$  be a measurable space and  $Y$  a topological space. Then a map  $f : X \rightarrow Y$  is called measurable, if all inverse images of open sets are measurable.

**Proposition 2.2.2.** Let  $X$  be a measurable space,  $Y$  be a topological space, then  $f : X \rightarrow Y$  is measurable if and only if  $f^{-1}(B)$  is measurable for each Borel set  $B$ .



*Proof.* (  $\implies$  ) Every open set is a Borel set. So this is true by inclusion.

(  $\impliedby$  ) Suppose  $f$  is measurable. Let  $M = \{E \subset Y : f^{-1}(E) \text{ is measurable}\}$ . We know  $M$  contains all open sets (Since we assume  $f$  is measurable). Moreover since  $f^{-1}(\cup_{j \in J} U_j) = \cup_{j \in J} f^{-1}(U_j)$  for any open sets  $U_j \subset Y$  with index set  $J$ , and  $f^{-1}(\cap_{i=1}^n U_i) = \cap_{i=1}^n f^{-1}(U_i)$ , we get that  $M$  is a  $\sigma$ -algebra.

Since  $M$  contains all open sets,  $M$  contains the Borel  $\sigma$ -algebra in  $Y$ . Hence  $f^{-1}(B)$  is measurable for every Borel set  $B$ .  $\square$

# Chapter 3

## 3.1 Warm up

**Example 3.1.1.** Let  $\mathcal{M}$  be a  $\sigma$ -algebra on a set  $X$  and  $\mathcal{B}$  be the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . For any given set  $A \subset X$ , consider the function  $\chi_A : X \rightarrow \mathbb{R}$  defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

The function  $\chi_A$  is measurable if and only if  $A \in \mathcal{M}$ .

To see this if  $\chi_A$  is measurable, then inverse image of every Borel set is measurable. Consider the Borel set  $(\frac{1}{2}, \frac{3}{2})$ , then  $\chi_A^{-1}(\frac{1}{2}, \frac{3}{2}) = A \in \mathcal{M}$ .

Conversely, assume  $A \in \mathcal{M}$ , Take  $B \in \mathcal{B}$ , the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . Consider  $\chi_A^{-1}(B)$ . We get

$$\chi_A^{-1}(B) = \begin{cases} X, & \{0, 1\} \in B \\ A, & 0 \notin B, 1 \in B \\ A^c, & 0 \in B, 1 \notin B \\ \emptyset, & 0, 1 \notin B \end{cases}$$

In all these cases, we get  $\chi_A^{-1}(B)$  to be an element of  $\mathcal{M}$ , since  $\emptyset, X \in \mathcal{M}$ . and if  $A \in \mathcal{M}$ , then  $A^c \in \mathcal{M}$ . This implies  $\chi_A$  is measurable.

## 3.2 Main Course

**Definition 3.2.1.** Let  $X, Y$  be topological spaces. We say that a function  $f : X \rightarrow Y$  is Borel measurable if  $f^{-1}(V)$  is a Borel set whenever  $V$  is an open set (or equivalently a Borel set because of [Proposition 2.2.2](#))

**Proposition 3.2.1.** *If  $f : X \rightarrow Y$  is a continuous function, then it is Borel measurable.*

*Proof.* For every open set  $E \subset Y$ , by assumption  $f^{-1}(E)$  is open. So it is in the Borel  $\sigma$ -algebra on  $X$ .  $\square$

### 3.3 Algebra of measurable functions

**Theorem 3.3.1.** *Let  $X$  be a measurable space,  $Y, Z$  be topological spaces. If  $f : X \rightarrow Y$  is measurable and  $g : Y \rightarrow Z$  is Borel measurable, then  $g \circ f : X \rightarrow Z$  is measurable.*

*Proof.* Let  $V \subset Z$  be an open set. We have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ . Now since  $g$  is Borel measurable, we get  $g^{-1}(V)$  is Borel measurable in  $Y$ . Again since  $f$  is measurable and  $g^{-1}(V)$  is a Borel measurable, we get  $f^{-1}(g^{-1}(V))$  is measurable in  $X$ .  $\square$

Next we consider forming ordered pairs of measurable functions.

**Lemma 3.3.1.** *If  $V \subset \mathbb{R}^2$  is open, then there are open rectangles  $\{R_j\}_{j \in \mathbb{N}}$ , such that  $R_j = (a_j, b_j) \times (c_j, d_j)$  and  $V = \bigcup_{j=1}^{\infty} R_j$*

*Proof.* Since rational  $(a, b) \times (c, d)$ ,  $a, b, c, d \in \mathbb{Q}$  generate the euclidean topology on  $\mathbb{R}^2$  (product topology on  $\mathbb{R} \times \mathbb{R}$  is the euclidean topology in  $\mathbb{R}^2$ ), we obtain a countable union of all such rectangles contained in  $V$ .  $\square$

**Theorem 3.3.2.** *Let  $X$  be a measurable space. If  $u, v : X \rightarrow \mathbb{R}$  are measurable, then  $f : X \rightarrow \mathbb{R}^2$  defined as  $f(x) = (u(x), v(x))$  is measurable.*

*Proof.* Let  $R = (a, b) \times (c, d) \subset \mathbb{R}^2$ . Then

$$\begin{aligned} f^{-1}(R) &= \{x \in X : u(x) \in (a, b), v(x) \in (c, d)\} \\ &= \{x \in X : u(x) \in (a, b)\} \cap \{x \in X : v(x) \in (c, d)\} \end{aligned}$$

Hence  $f^{-1}(R)$  is measurable.

Given any open set  $V \in \mathbb{R}^2$ , consider appropriate  $\{R_j\}_{j \in \mathbb{N}}$  such that  $V = \bigcup_{j=1}^{\infty} R_j$ . Then  $f^{-1}(V) = f^{-1}(\bigcup_{j=1}^{\infty} R_j) = \bigcup_{j=1}^{\infty} f^{-1}(R_j)$ . Thus  $f^{-1}(V)$  is measurable.  $\square$

Next we establish that measurability is preserved under algebraic operations.

**Proposition 3.3.1.** *Let  $f : X \rightarrow \mathbb{C}$  be such that  $f = u + iv$  with real valued  $u, v : X \rightarrow \mathbb{R}$ . If  $u, v$  are measurable, then  $f$  is measurable. And conversely, if  $f$  is measurable, then so are  $u, v$ , and  $|f| = \sqrt{u^2 + v^2}$ .*

*Proof.* Let  $u, v$  be measurable, then  $h : X \rightarrow \mathbb{R}^2 := x \rightarrow (u(x), v(x))$  is measurable by [Theorem 3.3.2](#). Also  $g : \mathbb{R}^2 \rightarrow \mathbb{C} : (x, y) \rightarrow x + iy$  is continuous. Hence we get that  $f = g \circ h$  is measurable.

For converse use that  $\Re : \mathbb{C} \rightarrow \mathbb{R}$  is a continuous function. So is  $\Im : \mathbb{C} \rightarrow \mathbb{R}$ , and  $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}$ . Then use that  $u = \Re \circ f$ ,  $v = \Im \circ f$ ,  $|f| = |\cdot| \circ f$ .  $\square$

**Proposition 3.3.2.** *If  $f, g : X \rightarrow \mathbb{C}$  are measurable, then  $f + g$  and  $fg$  are measurable.*

*Proof.* Suppose  $f, g$  are measurable. Then  $F(x) = (f(x), g(x))$  defines a measurable function. Next consider  $\phi : \mathbb{C}^2 \rightarrow \mathbb{C} := (a, b) \mapsto a + b$ . By continuity of  $\phi$ ,  $\phi \circ F$  is measurable, and we obtain  $(\phi \circ F)(x) = f(x) + g(x)$

To show  $fg$  is measurable use the continuity of  $\psi : \mathbb{C}^2 \rightarrow \mathbb{C} := (a, b) \mapsto ab$  and compose it with  $F$ .  $\square$

Can we find a simple test for measurability of a real-valued function?

# Chapter 4

## 4.1 Warm up

Let  $\mathcal{M}$  be a  $\sigma$ -algebra on  $X$  and  $A_1, A_2, \dots, A_n \in \mathcal{M}$ . Why does

$$f(x) = \sum_{i=1}^n c_i \chi_{A_i}$$

define a measurable function?

*Proof.* Use [Proposition 3.3.2](#). Interpreting  $c_i \chi_{A_i}$  as product of  $\chi_{A_i}$  with a constant function, we observe  $c_i \chi_{A_i}$  is measurable. Then using that the sum of two measurable functions is measurable in an inductive fashion, we get that the finite sum defining  $f$  also measurable.  $\square$

## 4.2 Continues

**Lemma 4.2.1.** *Let  $f : X \rightarrow [-\infty, \infty]$ . Then  $f$  is measurable if and only if  $f^{-1}((a, \infty])$  is measurable for each  $a \in \mathbb{R}$*

*Proof.* (  $\implies$  ) If  $f$  is measurable, then by  $(a, \infty]$  being open, we get that  $f^{-1}((a, \infty])$  is measurable. This is true for all  $a \in \mathbb{R}$ . So the claimed property holds.

(  $\impliedby$  ) Suppose for each  $a \in \mathbb{R}$ ,  $f^{-1}((a, \infty])$  is measurable. Then since we also have that  $(f^{-1}((a, \infty])^c = f^{-1}((a, \infty]^c) = f^{-1}([-\infty, a])$ , Now therefore  $f^{-1}([-\infty, a])$  is measurable for all  $a \in \mathbb{R}$ .

Now

$$[-\infty, b) = \bigcup_{n=1}^{\infty} \left[ -\infty, b - \frac{1}{n} \right]$$

so,

$$\begin{aligned} f^{-1}([-\infty, b)) &= f^{-1}\left(\bigcup_{n=1}^{\infty} [-\infty, b - \frac{1}{n}]\right) \\ &= \bigcup_{n=1}^{\infty} f^{-1}\left([-\infty, b - \frac{1}{n}]\right) \in \mathcal{M} \end{aligned}$$

Next we use  $(a, b) = [-\infty, b) \cap (a, \infty]$  so we get  $f^{-1}(a, b)$  to be measurable. Thus we have shown measurability for inverse images of a basis. Now let  $V \subset [-\infty, \infty]$  be an open set. Then there are four cases.

1.  $V$  is a countable union of rational open intervals. i.e  $-\infty, \infty \notin V$
2.  $-\infty \in V, \infty \notin V$ . Then  $V = [-\infty, b) \cup V_o$ , where  $V_o$  is of case 1, and  $[-\infty, b)$  is the union of countable sequence of rational half-infinite intervals. ( Let  $b_n$  be a rational sequence monotonically increasing to  $b$ , then  $\bigcup_{n=1}^{\infty} [-\infty, b_n] = [-\infty, b)$ ).
3.  $-\infty \notin V, \infty \in V$ . Then  $V = V_o \cup (a, \infty]$ , where  $V_o$  is a countable union of open intervals in  $\mathbb{R}$ .
4.  $-\infty, \infty \in V$ . Then  $V = [-\infty, b) \cup V_o \cup (a, \infty]$ , where  $V_o$  is a countable union of open intervals in  $\mathbb{R}$ .

In all these cases, we get  $f^{-1}(V)$  to be measurable. □

*Remark 4.2.1.* Given a sequence  $(a_n)$  in  $[-\infty, \infty]$ , let  $b_j = \sup_{n \leq j} a_n$ . Then for each  $j$ ,  $b_{j+1} \leq b_j$ . So  $\beta = \lim_{n \rightarrow \infty} b_j$  exists in  $[-\infty, \infty]$ .

**Definition 4.2.1.** Let  $(a_n)$  be a sequence in  $[-\infty, \infty]$  and  $(b_j)$  be as above, then  $\beta = \inf_{j \in \mathbb{N}} b_j$  is known as the  $\lim_{j \rightarrow \infty} \sup a_j$  or  $\overline{\lim}_{n \rightarrow \infty} a_j$

Similarly defining  $c_j = \inf_{n \geq j} a_n$  gives  $\lim_{j \rightarrow \infty} \inf a_j = \sup c_j$

**Definition 4.2.2.** Let  $f_n : X \rightarrow [-\infty, \infty]$  be a sequence of functions, define the limit supremum of the sequence of functions as

$$(\limsup_{n \rightarrow \infty} f_n)(x) = \lim_{n \rightarrow \infty} \sup f_n(x)$$

*Remark 4.2.2.* If  $(f_n(x))$  converges for each  $x$ , then we say the sequence of functions converges pointwise.

**Proposition 4.2.1.** Let  $(f_n)$  be a sequence of  $[-\infty, \infty]$  value functions, then

$$g(x) = \sup_{n \geq n_0} f_n(x), \quad h(x) = \lim_{n \rightarrow \infty} \sup f_n(x)$$

are measurable functions.

*Proof.* We only need to show that  $g^{-1}(a, \infty]$  is measurable for each  $a \in \mathbb{R}$ . We consider

$$g^{-1}((a, \infty]) = \{x \in X : g(x) > a\}$$

Now  $g(x) > a$ , then  $f_n(x) \geq a$  for all  $n \geq n_0$ . Thus we get

$$\begin{aligned} g^{-1}((a, \infty]) &= \bigcup_{n=n_0}^{\infty} \{x \in X : f_n(x) > a\} \\ &= \bigcup_{n=n_0}^{\infty} f^{-1}((a, \infty]) \end{aligned}$$

Thus we see  $g$  is measurable. Similarly we can show this holds true if we replace sup with inf in the definition of  $g$

Now since we know that composition of measurable functions are measurable, we get that  $\inf \sup f_n(x) = h(x)$  is measurable.

Similarly we can also show that  $\sup \inf f_n$  is also measurable.  $\square$

**Definition 4.2.3.** Let  $X$  be a set, a function  $s : X \rightarrow \mathbb{C}$  is called a simple function if the range of  $s$  is finite.

**Proposition 4.2.2.** A function  $s : X \rightarrow \mathbb{C}$  is simple if and only if there exists mutually disjoint sets  $A_1, A_2, \dots, A_n \subset X$ , and  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$  with

$$s = \sum_{j=1}^n \alpha_j \chi_{A_j}$$

*Proof.* ( $\implies$ ) by definition.

( $\impliedby$ ) Let  $s$  be a simple function with range  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . Then take  $A_j = s^{-1}(\alpha_j)$ . Then  $A_j$ s partition  $X$  and

$$s(x) = \sum_{j=1}^n \alpha_j \chi_{A_j}(x)$$

$\square$

# Chapter 5

**Theorem 5.0.1.** *If  $f : X \rightarrow [0, \infty]$  is measurable, then there exists a sequence  $(s_n)_{n \in \mathbb{N}}$  of simple non-negative real valued functions such that*

- i each  $s_n$  is measurable*
- ii sequence  $(s_n)$  is non-decreasing*
- iii  $(s_n)$  converge pointwise to  $f$*

*Proof.* Define a 'staircase to plateau' functions, (defined in the homework-2, question 3) defined as

$$\phi_n(x) = \begin{cases} 0, & x < 0 \\ k2^{-n}, & k2^{-n} \leq x < (k+1)2^{-n}, \quad k \in \{0, 1, 2, \dots, \} \\ n, & x \geq n \end{cases}$$

and then let  $s_n = \phi_n \circ f$ . We first prove the theorem for the special case  $f = \phi : [0, \infty) \rightarrow [0, \infty) : \phi(t) = t$ .

We have  $0 \leq \phi_1(t) \leq \phi_2(t) \leq \dots$  for each  $t \in \mathbb{R}$  and for  $t \leq n$ ,

$$|\phi_n(t) - \phi(t)| \leq \frac{1}{2^n}$$

so since  $\phi(t) < \infty$ ,  $\phi_n(t) \rightarrow \phi(t)$  for each fixed  $t \in \mathbb{R}$ . We also known from the homework that each  $\phi_n$  are Borel measurable.

For the general case, we take  $s_n = \phi_n \circ f$ . Then similar to what we got above, we get  $0 \leq s_1 \leq s_2 \leq \dots$  while each  $s_n$  is simple. Also for each  $t \in \mathbb{R}$ ,  $s_n(t) \rightarrow f(t)$ .  $\square$

**Definition 5.0.1.** Let  $(X, \mathcal{M})$  be a measurable space, and  $Z = [0, \infty]$  or  $Z = \mathbb{C}$ . A function  $\mu : \mathcal{M} \rightarrow Z$  is called countably additive (or  $\sigma$ -additive) if given  $A_1, A_2, \dots \in \mathcal{M}$  such that  $A_i \cap A_j = \emptyset$  if  $i \neq j$ , we have

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$



If  $Z = [0, \infty]$  and if there is a  $A \in \mathcal{M}$  such that  $\mu(A) \leq \infty$ , then we say that  $\mu$  is a measure (or a positive measure). And we call  $(X, \mathcal{M}, \mu)$  a measure space.

If  $Z = \mathbb{C}$ , then we call  $\mu$  a complex measure.

**Example 5.0.1.** We give examples of different measures.

- $X = \mathbb{N}, \mathcal{M} = P(\mathbb{N}), \mu(S) = |S|$ . This is called the counting measure.
- $X = \mathbb{N}, \mathcal{M} = P(\mathbb{N}), \mu(S) = \sum_{j \in S} \frac{1}{2^j}$

## 5.1 Properties of Measures

**Proposition 5.1.1.** *Let  $\mu$  be a (positive) measure on a  $\sigma$ -algebra  $\mathcal{M}$ . Then*

(1)  $\mu(\emptyset) = 0$

(2)  $A_1, A_2, \dots, A_n$  with  $A_i \cap A_j = \emptyset$  for each  $i \neq j$ , then

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j)$$

(3) If  $A, B \in \mathcal{M}$  with  $A \subset B$ , then  $\mu(A) \leq \mu(B)$ . And if  $\mu(B) \leq \infty$ , then

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

(4) If  $A_1 \subset A_2 \subset \dots$  with all  $A_j \in \mathcal{M}$ , then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j \rightarrow \infty} \mu(A_j)$$

(5) If  $A_1 \supset A_2 \supset \dots$  with all  $A_j \in \mathcal{M}$ , and there is  $j_0 \in \mathbb{N}$  with  $\mu(A_{j_0}) \leq \infty$ , then

$$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j \rightarrow \infty} \mu(A_j)$$

*Proof.* 1 Let  $A \in \mathcal{M}$  with  $\mu(A) \leq \infty$ .

2

3

4 WLOG assume  $j_o = 1$ . Consider the sets  $B_j = A_1 \setminus A_j$ . Then we apply the above property to get

$$\mu\left(\bigcup_{j=1}^{\infty}(A_1 \setminus A_j)\right) = \mu(A_1) - \lim_{j \rightarrow \infty} \mu(A_j)$$

But we see that  $\cup_{j=1}^{\infty}(A_1 \setminus A_j) = \cup_{j=1}^{\infty}(A_1 \cap A_j^c)$ . Now since each  $A_j \subset A_1$ , we get this to be equal to  $A_1 \setminus \cup_{j=1}^{\infty} A_j^c = A_1 \cap$

□

# Chapter 6

## 6.1 Integrals

**Definition 6.1.1.** Define the integral of a measurable simple function  $s : X \rightarrow [0, \infty]$  defined in the standard form as

$$s = \sum_{j=1}^n \alpha_j \chi_{A_j}$$

with  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  as the range of  $S$  and  $A_j = s^{-1}(\{\alpha_j\})$  by

$$\int s \, d\mu = \sum_{j=1}^n \alpha_j \mu(A_j)$$

We adopt the convention  $0 \times \infty = 0$  from now onwards.

**Lemma 6.1.1.** *Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $A_1, A_2, \dots, A_n \in \mathcal{M}$  and  $B_1, B_2, \dots, B_{n'} \in \mathcal{M}$  with the  $A_j$ s are mutually disjoint, as well as  $B_j$ s, and*

$$\bigcup_{j=1}^n A_j = X = \bigcup_{j=1}^{n'} B_j$$

*Let  $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, \infty]$  and  $\beta_1, \beta_2, \dots, \beta_{n'} \in [0, \infty]$  such that*

$$t = \sum_{j=1}^{n'} \beta_j \chi_{B_j} \leq s = \sum_{j=1}^n \alpha_j \chi_{A_j}$$

*then*

$$\sum_{j=1}^{n'} \beta_j \mu(B_j) \leq \sum_{j=1}^n \alpha_j \mu(A_j)$$

*Proof.*

$$\begin{aligned}
\sum_{j=1}^{n'} \beta_j \mu(B_j) &= \sum_{j=1}^n \beta_j \mu\left(B_j \cap \left(\bigcup_{l=1}^n A_l\right)\right) \\
&= \sum_{j=1}^{n'} \beta_j \mu\left(\bigcup_{l=1}^n B_j \cap A_l\right) \\
&= \sum_{j=1}^{n'} \sum_{l=1}^n \beta_j \mu(B_j \cap A_l)
\end{aligned}$$

By a similar deduction, we get that

$$\sum_{l=1}^n \alpha_l \mu(A_l) = \sum_{l=1}^n \sum_{j=1}^{n'} \alpha_l \mu(A_l \cap B_j)$$

Since we know that  $t \leq s$ , comparing the values of the function at  $A_l \cap B_j$ , we get that  $\beta_j \leq \alpha_l$ . This immediately gives us our needed result.  $\square$

**Corollary 6.1.0.1.** *If a measurable simple function has two representations*

$$s = \sum_{j=1}^n \alpha_j \chi_{A_j} = \sum_{j=1}^{n'} \beta_j \chi_{B_j}$$

*with disjoint measurable sets as before, then*

$$\int s \, d\mu = \sum_{j=1}^n \alpha_j \mu(A_j) = \sum_{j=1}^{n'} \beta_j \mu(B_j)$$

*Proof.* Use the fact that  $a = b$  is equivalent to  $a \leq b$  and  $b \leq a$  and use above lemma.  $\square$

**Definition 6.1.2.** Let  $(X, \mathcal{M}, \mu)$  be a measurable space,  $s : X \rightarrow [0, \infty]$  a measurable simple function,

$$s = \sum_{j=1}^n \alpha_j \chi_{A_j}$$

with  $\{A_j\}_{j=1}^n$  disjoint, measurable, then we define for  $E \in \mathcal{M}$

$$\int_E s \, d\mu = \sum_{j=1}^n \alpha_j \mu(A_j \cap E)$$

**Lemma 6.1.2.** If  $s, t$  are non-negative measurable, simple functions and  $t \leq s$  and  $E \in \mathcal{M}$ , then

$$\int_E t \, d\mu \leq \int_E s \, d\mu$$

*Proof.* Proof is exactly like before lemma, just replacing  $\mu(A_j)$  with  $\mu(A_j \cap E)$ .  $\square$

*Remark 6.1.1.* If  $s : X \rightarrow [0, \infty]$  is simple and measurable, then

$$\int s \, dx = \sup \left\{ \int_E t \, d\mu : 0 \leq t \leq s \text{ is measurable and simple.} \right\}$$

**Definition 6.1.3.** For  $f : X \rightarrow [0, \infty]$  measurable, we define

$$\int_E f \, d\mu = \sup_{\substack{0 \leq t \leq f \\ t \text{ is simple}}} \int_E t \, d\mu$$

**Example 6.1.1.** We will give some examples of measurable functions.

- $X = \mathbb{N}, \mathcal{M} = P(\mathbb{N}), \mu$  is the counting measure.  $f : \mathbb{N} \rightarrow [0, \infty]$ . Then let

$$s_N(n) = \begin{cases} f(n), & n \leq N \\ 0, & \text{otherwise} \end{cases}$$

Now if  $\sum_{j=1}^{\infty} f(j) \leq \infty$ , then  $f(j) \rightarrow 0$  as  $j \rightarrow \infty$ . Thus if  $t \leq f$  and  $t$  is simple, then there is  $N \in \mathbb{N}$  such that  $t(j) = 0$  for each  $j \geq N$ . Then by comparison,  $0 \leq t \leq s_N \leq f$  and finally, we have

$$\sum_{j=1}^{\infty} t(j) \leq \sum_{j=1}^{\infty} s_N(j) \leq \sum_{j=1}^{\infty} f(j)$$

so taking supremums, we get

$$\sup_{\substack{0 \leq t \leq f \\ t \text{ is simple}}} \sum_{j=1}^{\infty} t(j) = \sup_{N \in \mathbb{N}} \sum_{j \in \mathbb{N}} s_N(j) = \sum_{j=1}^{\infty} f(j)$$

# Chapter 7

*Remark 7.0.1.* Let  $(X, \mathcal{M}, \mu)$  be a measure space, a simple function  $s : X \rightarrow [0, \infty]$ , then  $\phi : \mathcal{M} \rightarrow [0, \infty]$  defined as

$$\phi(E) = \int_E s \, d\mu$$

is a measure.

*Proof.* Since our definition demands that measure of some set should be finite, we verify this first. We see that

$$\phi(\emptyset) = \int_{\emptyset} s \, d\mu = 0$$

Now to prove countable disjoint additivity, consider the disjoint collection  $\{E_l\}_{l \in \mathbb{N}}$ . And assume that  $s = \sum_{j=1}^n \alpha_j \chi_{A_j}$  with  $\alpha_j \in [0, \infty]$ , with  $A_j$ s disjoint. Then for  $E = \cup_{l=1}^{\infty} E_l$ , we have

$$\begin{aligned} \phi(E) &= \sum_{j=1}^n \alpha_j \mu(A_j \cap E) \\ &= \sum_{j=1}^n \sum_{l \in \mathbb{N}} \alpha_j \mu(A_j \cap E_l) \\ &= \sum_{l \in \mathbb{N}} \sum_{j=1}^n \alpha_j \mu(A_j \cap E_l) \\ &= \sum_{l \in \mathbb{N}} \int_{E_l} s \, d\mu \end{aligned}$$

□

## 7.1 Properties of Integrals

**Theorem 7.1.1.** *The integral of a non-negative measurable function from a measure space  $(X, \mathcal{M}, \mu)$  has the following properties*

- (1) If  $0 \leq f \leq g$ , then  $\int_E f(x) \, dx \leq \int_E g \, d\mu$
- (2) If  $A \subset B$ ,  $A, B \in \mathcal{M}$ , then  $\int_A f \, d\mu \leq \int_B f \, d\mu$
- (3) If  $c \in [0, \infty)$ ,  $E \in \mathcal{M}$ , then  $\int_E cf \, d\mu = c \int_E f \, d\mu$
- (4) If  $f = 0$ , or  $\mu(E) = 0$ , then  $\int_E f \, d\mu = 0$
- (5) For all  $E \in \mathcal{M}$ ,

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu$$

*Proof.* (1) By definition

$$\int f \, d\mu = \sup_{\substack{t \text{ is simple} \\ t \text{ is measurable} \\ 0 \leq t \leq f}} \int_E t \, d\mu$$

then the simple function  $t \leq f$  is also  $t \leq g$ . Hence suping over simple functions under  $g$ , every simple function under  $f$  is included.

- (2) Let  $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$  be a simple function  $0 \leq s \leq f$  with  $\int s \, dx + \epsilon > \int f \, d\mu$ . Using the inclusion  $A \subset B$ , we get

$$\int_A s \, d\mu = \sum_{n \in \mathbb{N}} \alpha_n$$

- (3) Suppose  $s = \sum_{j=1}^n \alpha_j \chi_{A_j}$  is a simple function with disjoint  $A_j$ s. Then  $s \chi_E = \sum_{j=1}^n \alpha_j \chi_{A_j \cap E}$  is also simple (and measurable), and

$$\int_E s \, dx = \sum_{j=1}^n \alpha_j \mu(A_j \cap E) = \int s \chi_E \, dx$$

Hence the statement is true for simple measurable functions. Next, consider  $f$  non-negative measurable, then for  $\epsilon \geq 0$ , we have a simple measurable function  $s$  with  $\int_E s \, d\mu + \epsilon > \int_E f \, d\mu$ . Then by preceding part,

$$\int s \chi_E \, d\mu + \epsilon > \int_E f \, d\mu$$

Also  $s\chi_E \leq f\chi_E$ . So

$$\int f\chi_E d\mu + \epsilon \geq \sup_{t \text{ is simple}} \int s\chi_E d\mu + \epsilon > \int f d\mu$$

Taking  $\epsilon \rightarrow 0$  gives

$$\int f\chi_E d\mu \geq \int_E f d\mu$$

For the reverse inequality, note that  $f\chi_E \leq f$ , and use similar circus.

□

**Theorem 7.1.2** (Monotone convergence theorem). *Let  $(X, \mathcal{M}, \mu)$  be a measure space, given a sequence  $f_n : X \rightarrow [0, \infty]$  of measurable functions and they are monotone increasing, i.e for each  $x \in X$ ,  $0 \leq f_1(x) \leq f_2(x) \leq \dots$ , then*

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int \lim_{n \rightarrow \infty} f_n d\mu$$

*Proof.* Let  $f = \lim_{n \rightarrow \infty} f_n$  be the pointwise limit. Then  $f$  is measurable. From  $f_n \leq f_{n+1}$ , we get that

$$\int f_n d\mu \leq \int f_{n+1} d\mu$$

so both sides of the claimed identity exist, and from  $f_n \leq f$ , we also know that

$$\int f_n d\mu \leq \int f d\mu$$

which taking the limits give us,

$$\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu$$

Now let  $s : X \rightarrow [0, \infty]$  be a simple measurable function  $s \leq f$ . Choose  $0 \leq c < 1$ , and define  $E_n = \{x \in X : f_n(x) \geq cs(x)\} = (f_n - s)^{-1}([0, \infty])$ . **Verify that difference between an extended real valued function and a real valued function is measurable, then  $E_n$  is measurable.** This gives a nested sequence  $E_1 \subset E_2 \subset \dots$ . If  $f(x) > 0$ , then by  $f(x) > cs(x)$  and  $f_n(x) \rightarrow f(x)$ , there is  $n \in \mathbb{N}$  such that  $x \in E_n$ . On the other hand if  $f(x) = 0$ , then  $cs(x) = 0 = f(x)$ , so  $x \in E_n$  for all  $n \in \mathbb{N}$ . We see that each  $x \in X$  is in the union  $\cup_{n=1}^{\infty} E_n$ . Hence  $X = \cup_{n=1}^{\infty} E_n$ . Now we define  $\phi : \mathcal{M} \rightarrow [0, \infty]$  by

$$\phi(E) = \int_E s d\mu$$



which is a measure and  $\phi(X) = \phi(\cup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \phi(E_n)$  by Theorem 7.1.1. We rewrite this as

$$\begin{aligned} \int_X s \, d\mu &= \lim_{n \rightarrow \infty} \int_{E_n} s \, d\mu \\ &= \lim_{n \rightarrow \infty} \int_X s \chi_{E_n} \, d\mu \\ &\leq \lim_{n \rightarrow \infty} \int_X \frac{1}{c} f_n \, d\mu \end{aligned}$$

Now take sup over all such simple (bounded) functions  $s \leq f$  and let  $c \rightarrow 1$ . **Finish this proof.**  $\square$

# Chapter 8

*Remark 8.0.1.* Suppose  $A_1, A_2, \dots$ . Consider their characteristic functions  $\chi_{A_n}$  and let  $\limsup_{k \geq n} = \chi_A$ . What is  $A$ ?

$$\begin{aligned} \limsup \chi_{A_n} &= \lim_{n \rightarrow \infty} \sup_{k \geq n} \chi_{A_k} \\ &= \lim_{n \rightarrow \infty} \chi_{\cup_{k \geq n} A_k} \end{aligned}$$

**Theorem 8.0.1.** Let  $(X, \mathcal{M}, \mu)$  be a measurable space,  $f, g : X \rightarrow [0, \infty]$  be measurable, then

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu$$

*Proof.* For  $s, t : X \rightarrow [0, \infty]$  simple and measurable, by definition

$$\int (s + t) d\mu = \int s d\mu + \int t d\mu$$

Considering sequences of simple measurable functions  $(s_n)_{n=1}^\infty, (t_n)_{n=1}^\infty$  such that  $s_n(x) \nearrow f(x), t_n(x) \nearrow g(x)$  for each  $x \in X$ . Then by monotone convergence theorem

$$\int s_n d\mu \rightarrow \int f d\mu \quad \int t_n d\mu \rightarrow \int g d\mu$$

and since  $s_n(x) + t_n(x) \nearrow f(x) + g(x)$  for each  $x \in X$  then again by MCT we get

$$\int (s_n + t_n) d\mu \rightarrow \int (f + g) d\mu$$

□

**Corollary 8.0.1.1.** If  $(f_n)_{n=1}^\infty$  is a sequence of functions  $f_n : X \rightarrow [0, \infty]$ , then

$$\int \sum_{i=1}^\infty f_n d\mu = \sum_{i=1}^\infty \int f_n d\mu$$

*Proof.* Let  $g_m = \sum_{n=1}^m f_n$ . Then  $(g_m)$  forms an increasing sequence, so

$$\begin{aligned} \int \sum_{n \in \mathbb{N}} f_n d\mu &= \int \lim_{n \rightarrow \infty} g_n d\mu \\ &= \lim_{m \rightarrow \infty} \int \sum_{i=1}^m f_i d\mu \end{aligned}$$

□

**Theorem 8.0.2.** *If  $f : [0, \infty]$  is measurable on  $(x, \mathcal{M}, \mu)$ , then  $\phi : \mathcal{M} \rightarrow [0, \infty]$ ,*

$$\phi(E) = \int_E f d\mu$$

*defines a measure  $\phi$  and for any  $g : X \rightarrow [0, \infty]$ , and for any measurable  $g : X \rightarrow [0, \infty]$*

$$\int g d\phi = \int gf d\mu$$

*Proof.*  $\phi(\emptyset) = 0$  since the integral of every simple measurable function  $s \leq f$  over  $\emptyset$  is 0.

Let  $(E_n)_{n=1}^\infty$  be a disjoint sequence of sets  $E = \cup_{j=1}^\infty E_j$ , then

$$\phi(E) = \int f d\mu = \int f \chi_{X_E} dx = \int f \chi_{\cup_{n=1}^\infty E_n} d\mu = \int f \left( \sum_{n \in \mathbb{N}} \chi_{E_n} \right) d\mu = \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu$$

which is exactly  $\sum_{n \in \mathbb{N}} \phi(E_n)$ . This gives that  $\phi$  is a measure.

To see the claimed identity, we first show that

$$\int s d\phi = \int sf d\mu$$

for  $s : X \rightarrow [0, \infty)$  simple measurable, with

$$s(x) = \sum_{j=1}^n \alpha_j \chi_{A_j}(x)$$

Then we see that

$$\begin{aligned}
\int s \, d\mu &= \sum_{j=1}^n \alpha_j \phi(A_j) \\
&= \sum_{j=1}^n \alpha_j \int_{A_j} f \, d\mu \\
&= \int \left( \sum_{j=1}^n \alpha_j \chi_{A_j} \right) f \, d\mu \\
&= \int s f \, d\mu
\end{aligned}$$

Now for any given  $g : X \rightarrow [0, \infty]$ , we approximate  $g$  with a simple measurable sequence  $s_n \nearrow g$ . Then by monotone functions, we get

$$\begin{aligned}
\int g \, d\phi &= \lim_{n \rightarrow \infty} \int s_n \, d\phi \\
&= \lim_{n \rightarrow \infty} \int s_n f \, d\mu \\
&= \int \lim_{n \rightarrow \infty} s_n f \, d\mu \\
&= \int \phi \, d\mu
\end{aligned}$$

□

**Definition 8.0.1.** We define the space  $L^1(\mu)$  of integrable functions on a measurable functions  $(X, \mathcal{M}, \mu)$  to consist of all measurable  $f : X \rightarrow \mathbb{C}$  such that

$$\int |f| \, d\mu \leq \infty$$

*Remark 8.0.2.* If  $f$  is measurable,  $\mathbb{C}$  valued, such that  $f = u + iv$  where  $u, v$  are real valued measurable functions. Then let  $u^+ = \max\{0, u\}, u^- = \max\{0, -u\}$ . Then  $u^+, u^-$  are measurable functions. Similarly, we get  $v^+, v^-$  also to be measurable functions. Then we get  $f = u^+ - u^- + i(v^+ - v^-)$  and we define the integral as

$$\int f \, d\mu = \int u^+ \, d\mu - \int u^- \, d\mu + i \int v^+ \, d\mu - i \int v^- \, d\mu$$

# Chapter 9

*Remark 9.0.1* (Warm up). Assume there is a measure  $\mu$  on  $\mathbb{R}^+$ , for all Borel-measurable functions, and  $\mu([a, b]) = b - a$  for each  $a \leq b$  and for continuous function  $f$ ,

$$\int_{[a,b]} f \, d\mu = \int_a^b f \, dx$$

Is the function

$$f(x) = \begin{cases} 1, & x = 0 \\ \frac{\sin(x)}{x}, & x > 0 \end{cases}$$

**Theorem 9.0.1.**  $L^1(\mu)$  is a vector space for  $f, g \in L^1(\mu)$ . Moreover

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$$

*Proof.* We know that for  $\alpha, \beta \in \mathbb{C}$ ,

$$|\alpha f + \beta g| \leq |\alpha||f| + |\beta||g|$$

Then using the properties of integration, we get that

$$\int |\alpha f + \beta g| d\mu \leq \int |\alpha||f| d\mu + \int |\beta||g| d\mu = |\alpha|\|f\|_1 + |\beta|\|g\|_1 < \infty$$

Now to prove the rest, we'll assume  $f, g$  are  $\mathbb{R}$ -valued functions and let  $h = f + g$ . Then we have  $h^+ - h^- = f^+ - f^- + g^+ - g^- = f^+ + g^+ - (f^- + g^-)$ , which gives

$$\begin{aligned} \int h^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu &= \int h^+ + f^- + g^- \, d\mu \\ &= \int h^- + f^+ + g^+ \, d\mu \\ &= \int h^- \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu \end{aligned}$$

Now rearranging things up, we get what we need for reals. verify similarly for Complex case. □

*Note.* What can we say about  $f$ ?

**Theorem 9.0.2.** *If  $f \in L^1(\mu)$ , then*

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

*Proof.* If  $f$  was  $\mathbb{R}$ -valued, then

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu + \int f^- \, d\mu \right| \leq \left| \int f^+ \, d\mu \right| + \left| \int f^- \, d\mu \right| = \int |f| \, d\mu$$

Now in general, if  $f$  is a  $\mathbb{C}$ -valued function, then let the integral be equal to  $z$ . Now if  $z = 0$ , we have nothing to prove. If  $z \neq 0$ , then multiply  $f$  with  $\alpha = \frac{\bar{z}}{|z|}$ . Then integral of  $\alpha f$  will be real and we'll be good.  $\square$

# Chapter 10

**Theorem 10.0.1** (Fatou's Lemma). *If  $(f_n)$  is a sequence of measurable functions  $f_n : X \rightarrow [0, \infty]$ , then*

$$\int \liminf_{n \rightarrow \infty} f_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu$$

*Proof.* Let  $g_m(x) = \inf_{n \geq m} f_n(x)$ . Then  $0 \leq g_1(x) \leq g_2(x) \leq \dots$ . Then by MCT, we get

$$\int \lim_{m \rightarrow \infty} g_m \, d\mu = \lim_{m \rightarrow \infty} \int g_m \, d\mu(x)$$

Also see that if  $n \geq m$ , then  $f_n \geq g_m$  and therefore, we get

$$\int f_n \, d\mu \geq \int g_m \, d\mu$$

So

$$\inf_{n \geq m} \int f_n \, d\mu \geq \int g_m \, d\mu$$

Now taking  $m \rightarrow \infty$  on both sides, we get

$$\liminf_{n \rightarrow \infty} \int f_n \, d\mu \geq \int \liminf_{n \rightarrow \infty} f_n \, d\mu$$

which proves the theorem. □

**Example 10.0.1.** Let  $\mu$  be the counting measure on  $X = \{0, 1\}$ . Let

$$f_{2n}(x) = \begin{cases} 0, & x = 0 \\ 1, & x = 1 \end{cases} \quad f_{2n+1}(x) = \begin{cases} 1, & x = 0 \\ 0, & x = 1 \end{cases}$$

Then  $\int \liminf_{n \rightarrow \infty} f_n \, d\mu = 0 \leq 1 = \liminf_{n \rightarrow \infty} \int f_n \, d\mu$

**Theorem 10.0.2** (Lebesgue dominated convergence theorem). *Let  $(X, \mathcal{M}, \mu)$  be a measurable space. If  $f_n : X \rightarrow \mathbb{C}$  defines a sequence of measurable functions pointwise converging to  $f$ , and there is a  $g \in L^1(\mu)$  such that*

$$|f_n| \leq g, \quad \forall n \in \mathbb{N}$$

*Then  $f \in L^1(\mu)$  and*

$$\int |f_n - f| \, d\mu \rightarrow 0$$

*So we exchange limits and integral and write*

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$$

*Proof.* We have  $|f| \leq g$  since  $|f_n| \leq g$  for all  $n \in \mathbb{N}$  and  $f_n \rightarrow f$  pointwise. Consider  $h_n = 2g - |f_n - f| \geq 0$  (Use triangle inequality to show that  $h_n \geq 0$ ). Fatou's lemma gives

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \, d\mu &\geq \int \lim_{n \rightarrow \infty} (2g - |f_n - f|) \, d\mu \\ &= 2 \int g \, d\mu + \int \liminf_{n \rightarrow \infty} (-|f_n - f|) \, d\mu \\ &= 2 \int g \, d\mu - \int \limsup_{n \rightarrow \infty} (|f_n - f|) \, d\mu \end{aligned}$$

But we also have

$$\liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \, dx \leq 2 \int g \, d\mu + \liminf_{n \rightarrow \infty} \int |f_n - f| \, d\mu$$

Hairy logic. Verify with Rudin. □

## 10.1 Measure Zero

**Definition 10.1.1.** We say that a property  $P$  holds almost everywhere if

$$\mu(\{x \in X : P \text{ does not hold at } x\}) = 0$$

**Theorem 10.1.1.** *If  $f : X \rightarrow [0, \infty]$  and  $\int f \, d\mu = 0$ , then  $f = 0$  almost everywhere. Conversely, if  $f = 0$  almost everywhere then  $\int f \, d\mu = 0$ .*



*Proof.* Let  $E_n = \{s \in X : f(x) \geq \frac{1}{n}\}$  and  $E = \cup_{n=1}^{\infty} E_n = \{x \in X : f(x) > 0\}$ . Note that  $E$  is measurable since each of  $E_i$  is measurable. So

$$\begin{aligned} 0 &= \int f \, d\mu \geq \int f \chi_{E_n} \, d\mu \\ &\geq \int \frac{1}{n} \chi_{E_n} \, dx \\ &= \frac{1}{n} \mu(E_n) \geq 0 \end{aligned}$$

Hence  $\mu(E_n) = 0$  for each  $n \in \mathbb{N}$ . Hence  $E$  is a measure zero set. Therefore  $f$  is zero almost everywhere.

Conversely if  $f = 0$  almost everywhere, then let

$$g(x) = \begin{cases} 0, & f(x) = 0 \\ \infty, & \text{otherwise} \end{cases}$$

Then  $g$  is a measurable simple function with  $g > f$  and  $\int g \, d\mu = \infty$ . Hence  $\int f \, d\mu = 0$ .  $\square$

**Theorem 10.1.2.** *If  $f_n : X \rightarrow \mathbb{C}$  defines a sequence of measurable functions and if*

$$\sum_{n \in \mathbb{N}} |f_n| \in L^1(\mu).$$

*Then*

$$\sum_{n \in \mathbb{N}} f_n \in L^1(\mu)$$

*and the series  $\sum_{n \in \mathbb{N}} f_n$  converges almost everywhere. See theorem*

*Proof.* We assume each  $f_n$  is defined on  $X \setminus S_n$  with  $\mu(S_n) = 0$ . We have to show that there exist a set  $S$  with  $\mu(S) = 0$  and  $\forall x \notin S$ ,  $\sum_{n \in \mathbb{N}} f_n(x)$  converges. Let

$$f(x) = \sum_{n \in \mathbb{N}} |f_n(x)|$$

By MCT

$$\sum_{n \in \mathbb{N}} \int |f_n| \, d\mu = \int f \, d\mu \leq \infty$$

This implies  $\{x : f(x) = \infty\}$  has measure zero. Hence if  $x \notin S_n$  and  $x \notin \{x : f(x) = \infty\}$ , then  $\sum_{n \in \mathbb{N}} f_n(x)$  converges absolutely. Thus  $S = \cup_{n=1}^{\infty} S_n \cup \{x : f(x) = \infty\}$  is measure zero and  $x \in S^c$   $\square$

**Definition 10.1.2.** Let  $(X, \mathcal{M}, \mu)$  be a measure space. If for any  $E \in \mathcal{M}$  and  $F \subset E$ ,  $\mu(E) = 0$  implies  $F \subset \mathcal{M}$ , then  $\mu$  is called complete.

# Chapter 11

*Note* (Warm up). Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $f : X \rightarrow [0, \infty]$ , with  $f \in L^1(\mu)$ . Let  $E = \{x \in X : f(x) \geq 1\}$ . Then show  $\mu(E) < \infty$ .

This is Chebyshev's inequality for general measures.

*Remark 11.0.1.* Consider the distance (semi-metric) between sets in  $\mathcal{M}$ , defined as  $\mu(A \Delta B)$ . Let  $f : X \rightarrow [0, \infty]$  be a function  $f \in L^1(\mu)$ . Now let  $\phi$  be a measure defined as  $d\phi = f d\mu$ . Then define  $\tilde{d}(A, B) = \phi(A \Delta B) = \int_{A \Delta B} f d\mu$ . Then if  $d(A_n, B) \rightarrow 0$  will imply  $\tilde{d}(A_n, B) \rightarrow 0$ .

**Theorem 11.0.1.** Any measure space  $(X, \mathcal{M}, \mu)$  can be equipped with a complete extension of  $\mu$  on the collection of sets,  $\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, \mu(B \setminus A) = 0\}$  in which case we define  $\mu^*(E) = \mu(A)$ , which gives a complete measure on  $\mathcal{M}^*$ .

*Proof.* First, we establish  $\mu^*$  is well defined, that is it does not depend on the particular choice of the subset  $A \subset E$ . To see this, let  $A' \subset E \subset B'$  such that  $\mu(B' \setminus A') = 0$ . By the inclusions,  $A \subset E \subset B'$ . So we get

$$A \setminus A' \subset E \setminus A' \subset B' \setminus A'$$

Thus by monotonicity of  $\mu$ , we get  $\mu(A \setminus A') = 0$ . Moreover by symmetry of  $A$  and  $A'$ , we get  $\mu(A' \setminus A) = 0$ . Thus we get  $\mu(A) = \mu(A \setminus A') + \mu(A \cap A') = \mu(A' \setminus A) + \mu(A' \cap A) = \mu(A')$ . Hence we see that the definition of  $\mu^*$  is well defined.

Now we show that  $\mathcal{M}^*$  is actually a  $\sigma$ -algebra. We immediately see that  $\mu^*(\emptyset) = 0$ .

- $\mathcal{M} \subset \mathcal{M}^*$  implies  $X \in \mathcal{M}^*$
- Let  $E \in \mathcal{M}^*$ , then there are  $A, B \in \mathcal{M}$  with  $A \subset E \subset B$  and  $\mu(B \setminus A) = 0$ . Thus  $B^c \subset E^c \subset A^c$ . Then  $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \cap A) = 0$  shows  $E^c \in \mathcal{M}^*$ .

- Let  $(E_j)$  be a countable collection of disjoint sets in  $\mathcal{M}^*$ . Then there are subsets  $A_j, B_j \in \mathcal{M}$  with  $A_j \subset E_j \subset B_j$ , with  $\mu(B_j \setminus A_j) = 0$ . Then let

$$A = \bigcup_{j=1}^{\infty} A_j \quad E = \bigcup_{j=1}^{\infty} E_j \quad B = \bigcup_{j=1}^{\infty} B_j$$

Then we have  $A \subset E \subset B$ . Moreover since each  $E_j$  are disjoint, we get  $A_j$  are disjoint.

Now show  $\mu^*$  is countably additive and then show  $\mu^*$  is complete. verify  $\square$

*Remark 11.0.2.* Consider  $C([0, 1])$  equipped with the sup norm. Recall that this is a Banach space. Let  $\lambda : C([0, 1]) \rightarrow \mathbb{C}$  be defined as

$$\lambda(f) = \int_0^1 f(x) \, dx$$

Recall also that  $|\lambda(f)| \leq \lambda(|f|) \leq \|f\|_{\infty}$ . Hence we see  $\lambda$  is a bounded linear functional. Therefore we see that we can associate the Riemann integral with a linear functional. We ask if we can go back i.e if we have a linear functional on  $C([0, 1])$ , can we get a measure to integrate functions on  $C([0, 1])$

# Chapter 12

## 12.1 Recap on topology

**Definition 12.1.1.** Let  $(X, \tau)$  be a topological space. A set  $E$  is called closed if its complement is open. The closure of  $E$  is the smallest closed subset containing  $E$ .

$$\overline{E} = \bigcap_{\substack{F^c \in \tau \\ E \subset F}} F$$

We can check  $\overline{E}$  is closed by looking at  $\overline{E}^c$ .

**Definition 12.1.2.** A set  $K \subset X$  is called compact if every open cover of  $K$  has a finite subcover.

**Definition 12.1.3.**  $(X, \tau)$  is Hausdorff ( $T_2$ ) if for any  $p \neq q \in X$  there are open sets  $U, V \in \tau$  such that  $p \in U, q \in V$  and  $U \cap V = \emptyset$ .

**Definition 12.1.4.** A neighborhood of  $p \in X$  is an open set  $U \in \tau$  containing  $p$ .

**Definition 12.1.5.**  $X$  is called locally compact if any point  $p \in X$  has a neighborhood  $V$  with compact  $\overline{V}$ .

**Theorem 12.1.1.** *Let  $X$  be a topological space. If  $K \subset X$  is compact and  $F \subset K$  is closed, then  $F$  is compact.*

*Proof.* Make any covering of  $F$  into a covering of  $K$ , by adding  $F^c$ , then get a finite subcover for  $K$ , then remove  $F^c$  from this subcover if it's there. Now you got a finite subcover for  $F$ .  $\square$

**Theorem 12.1.2.** *Let  $X$  be a topological Hausdorff space. Then if  $K \subset X$  is compact,  $p \notin K$ , then there are open sets  $U, V$  such that  $K \subset V, p \in U, U \cap V = \emptyset$ . (not that we are not claiming regularity).*

*Proof.* For each  $q \in K$ , there is an open set  $U_q, V_q$  with  $q \in V_q, p \in V_q, V_q \cap U_q = \emptyset$ . Then  $K \subset \bigcap_{q \in K} V_q$ . Then since  $K$  is compact, there is a finite subcover  $V_{q_1}, V_{q_2}, \dots, V_{q_n}$  of  $K$ . Now let  $V = \bigcup_{i=1}^n V_{q_i}$  and  $U = \bigcap_{i=1}^n U_{q_i}$  both of which are open. Then  $K \subset V, p \in U$  and  $U \cap V = \emptyset$ .  $\square$

**Theorem 12.1.3.** *If  $K_\alpha$  is a collection of nonempty compact subsets of a topological Hausdorff space  $X$  indexed by  $A$ , and if for each finite subset  $B \subset A$ ,  $\bigcap_{\beta \in B} K_\beta \neq \emptyset$  then*

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$$

*Proof.* If  $\bigcap_{\alpha \in A} K_\alpha = \emptyset$ , then  $K_\alpha^c$  forms an open cover for  $K_{\alpha_0}$ . Now use the compactness property. verify  $\square$

**Theorem 12.1.4.** *If  $X, Y$  are topological spaces, if  $f : X \rightarrow Y$  is continuous, and  $K$  is compact, then  $f(K)$  is compact.*

*Proof.* Let  $U_\alpha$  be an open cover for  $f(K)$ , then  $f^{-1}(U_\alpha)$  forms an open cover for  $K$ . Now by the compactness there is a finite cover  $f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), \dots, f^{-1}(U_{\alpha_n})$ . Therefore  $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$  is a finite subcover of  $f(K)$ .  $\square$

**Definition 12.1.6.** Let  $X$  be a topological space,  $f : X \rightarrow \mathbb{C}$ . Then the support of  $f$  is defined as  $\text{supp } f = \overline{\{x \in X : f(x) \neq 0\}}$ . See that  $\text{supp}(f+g) \subset \text{supp}(f) \cup \text{supp}(g)$

We denote  $C_c(X)$  to be the set of continuous functions which have compact support.  $C_c(X)$  is a subspace of the vector space  $C(X)$ .

**Theorem 12.1.5** (Urysohn Lemma). *Let  $X$  be a locally compact Hausdorff space. If  $X$  is compact,  $V$  is open and  $K \subset V$ , then there is a function  $f \in C_c(X)$  with*

$$\chi_K \leq f \leq \chi_V$$

# Chapter 13

**Theorem 13.0.1** (Urysohn Lemma). *Let  $X$  be a locally compact Hausdorff space. If  $X$  is compact,  $V$  is open and  $K \subset V$ , then there is a function  $f \in C_c(X)$  with*

$$\chi_K \leq f \leq \chi_V$$

*Proof.* Get a finite cover for  $K$  whose closure is contained in  $V$  □

**Definition 13.0.1.** Let  $X$  be locally Hausdorff. A linear functional  $\lambda : X \rightarrow \mathbb{C}$  is positive, if  $\lambda(x) \geq 0$  for each  $x \in X$ .

*Remark 13.0.1.* Suppose  $X$  is locally compact,  $\mu$  a measure on a  $\sigma$ -algebra  $\mathcal{M}$ ,  $\mathcal{M}$  containing Borel sets. If  $f \in C(X)$  and  $f(x) \geq 0$  for each  $x \in X$ , then  $\int f d\mu \geq 0$ .

If every compact set has finite measure, then each  $f \in C_c(X)$  is in  $L^1(\mu)$ . And  $\lambda(f) = \int f d\mu$  defines a positive linear functional on  $C_c(X)$ . Conversely, if each  $f \in C_c(X)$  is in  $L^1(\mu)$ , then we know for each compact  $K$ , we have  $\mu(K) < \infty$ . To see this, take  $V$  open with  $K \subset V$ ,  $\bar{V}$  compact and use Urysohn's Lemma to construct  $f \in C_c(X)$ ,  $\chi_K \leq f \leq \chi_V$ . Then by monotonicity,

$$0 \leq \int \chi_K d\mu \leq \int f d\mu < \infty$$

**Theorem 13.0.2** (Riesz Representation Theorem). *Let  $X$  be a locally compact Hausdorff space. If  $\lambda$  is a positive linear functional on  $C_c(X)$ , then there exists a  $\sigma$ -algebra  $\mathcal{M}$  and a complete (positive) measure  $\mu$ , uniquely determined by  $\lambda$  such that*

- (1)  $\mathcal{M} \supset B(X)$ , the Borel sigma algebra.
- (2)  $\lambda(f) = \int f d\mu$  for each  $f \in C_c(X)$ .
- (3)  $\mu(K) < \infty$  for each compact  $K$ .

(4) for  $E \in \mathcal{M}$ ,

$$\mu(E) = \inf_{\substack{V \text{ is open} \\ E \subset V}} \mu(V)$$

(5) If  $E$  is open or  $E \in \mathcal{M}$  and  $\mu(E) < \infty$ , then

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}$$

*Proof.* We will only prove the uniqueness and refer Rudin for the proof. Assume  $\mu_1, \mu_2$  satisfy these properties. Take  $K$  compact,  $\epsilon > 0$ , then from iv) we know that there exist open sets  $V_1, V_2$  containing  $K$  and  $\mu_i(V_i) - \epsilon < \mu_i(K)$ . Take  $V = V_1 \cap V_2 \cap V_3$  with  $V$ . **prove the rest.**  $\square$

# Chapter 14

**Theorem 14.0.1.** *Let  $X$  be a locally compact Hausdorff space. If  $X$  is  $\sigma$ -compact and a Borel measure  $\nu$ , that assigns each compact set  $K$  the measure  $\nu(K) < \infty$  then the  $\mu$  given by Reisz representation theorem satisfies*

1. *If  $E \in \mathcal{M}$ ,  $\epsilon > 0$ , there is an open set  $V$  and a closed set  $C$  with  $C \subset E \subset V$  and  $\mu(V \setminus C) < \epsilon$ .*
2. *If  $E \in \mathcal{M}$ , then there is an  $F_\sigma$  set  $F$  (countable union of closed sets) and an  $G_\delta$  set  $G$  (countable intersection of open sets) with  $F \subset E \subset G$  and  $\mu(G \setminus F) = 0$ .*
3.  *$\mu$  is regular*

*Proof.* 1. If  $\mu(E) < \infty$ , then it holds by Reisz representation theorem. Next consider  $E \in \mathcal{M}$  with  $\mu(E) = \infty$ . Recall that  $X = \cup_{j=1}^{\infty} K_j$ , where each  $K_j$  is compact. Let  $\epsilon > 0$ . Take intersection with  $K_j$ , then we have  $\mu(E \cap K_j) < \infty$ . So we have open sets  $V_j$  such that  $K_j \cap E \subset V_j$  and  $\mu(V_j \setminus (K_j \cap E)) < \frac{\epsilon}{2^{j+1}}$ .  $V_j$ s are guaranteed by the (4) in the Reisz representation theorem. Take  $V = \cup_{j=1}^{\infty} V_j$ . We have  $V \setminus E \subset \cup_{j=1}^{\infty} (V_j \setminus (K_j \cap E))$ . So we get  $\mu(V \setminus E) < \frac{\epsilon}{2}$ .

Again consider  $E^c$  and using the same analysis, we get an open set  $W$  such that  $E^c \subset W$  and  $\mu(W \setminus E^c) < \epsilon/2$ . Now let  $C = W^c$ , this gives  $\mu(E \setminus C) = \mu(W \setminus E^c) = \frac{\epsilon}{2}$ . **Now show that  $\mu(W \setminus C) < \epsilon$ .** Then we're done.

2. Repeat i) for a sequence of  $\epsilon_n = \frac{1}{n}$ . Then we get a corresponding  $C_n \subset E \subset V_n$ . Take  $V = \cap_{n=1}^{\infty} V_n$ ,  $C = \cup_{n=1}^{\infty} C_n$ . Then we're done.
3. (4), (5) of Reisz representation theorem gives the outer regularity, and outer regularity when  $\mu(E) < \infty$ . We only need to show inner regularity when  $\mu(E) = \infty$ . Therefore, we need a sequence  $A_n$  of compact sets such that  $A_n \subset E$  for each  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \mu(A_n) = \infty$ . From (1), taking  $\epsilon = 1$ , we have  $C \subset E$ , where  $\mu(E \setminus C) < 1$ . Hence we see  $\mu(C) = \infty$ .



Now from the  $\sigma$ -compactness, we get  $X = \cup_{n=1}^{\infty} K_n$  for  $K_n$  compact. We can further demand  $K_n$ s are increasing since if not we can take finite unions of everything below. Now let  $C_n = K_n \cap C$  and we have

$$\infty = \mu(C) = \lim_{n \rightarrow \infty} \mu(C_n)$$

□

## 14.1 Lebesgue Measure

**Definition 14.1.1.** A  $k$ -cell in  $\mathbb{R}^n$  is a set of the form

$$A = \{x = (x_1, x_2, \dots, x_k) : a_j \leq^\circ x_j \leq^\circ b_j, \leq^\circ \in \{\leq, <\}\}$$

We define  $\text{vol}(A) = \prod_{j=1}^k (b_j - a_j)$

**Theorem 14.1.1.** *There is a  $\sigma$ -algebra  $\mathcal{M}$  including Borel sets on  $\mathbb{R}^n$  and measure  $m$  on  $\mathcal{M}$  such that*

- (1)  $m(V) = \text{vol}(V)$  if  $V$  is a  $k$ -cell
- (2)  $m$  restricted to Borel sets is a regular measure
- (3)  $m$  is translation invariant

*Proof.* For any  $f \in C_c(\mathbb{R}^k)$ . Let  $\Lambda(f) = \int f dV$  be the Riemann integral. Then  $\Lambda$  is a positive linear functional on  $C_c(\mathbb{R}^k)$ . Reisz representation theorem gives a measure  $m$  out of  $\Lambda$  which has regularity and defined on a  $\sigma$ -algebra  $\mathcal{M}$  which contains the Borel sets.

- (1) Let  $V$  be an open  $k$ -cell. Pick compact  $k$ -cells nested increasing with with union  $V = \cup_{j=1}^{\infty} V_j$ . By Urysohn's lemma, there are  $f_n \in C_c(\mathbb{R}^n)$  such that  $\chi_{V_n} \leq f_n \leq \chi_V$  where  $V_n$  is compact and  $V$  is open. Then

$$m(V_n) = \int \chi_{V_n} dm \leq \int f_n dm \leq \int \chi_V dm = m(V)$$

Now taking  $n \rightarrow \infty$ , by monotone convergence theorem, we get  $m(V_n) \rightarrow m(V)$ . Hence by sandwich, we get  $\int f_n dm \rightarrow m(V)$ .

Similarly

$$\text{vol}(V_n) \leq \int f_n dV \leq \text{vol}(V)$$

Then we can choose  $V_k$  such that  $\text{vol}(V_k) \rightarrow \text{vol}(V)$ , then we get

- (2) Property of Reisz representation measure
- (3) Fix  $a \in \mathbb{R}^k$  and define  $\lambda : \mathcal{M} \rightarrow [0, \infty] := \lambda(E) = m(a + E)$ . **Verify that  $\lambda$  is a measure on  $\mathcal{M}$ .**

Also define translation of functions  $f \in C_c(\mathbb{R}^k)$  as  $f \rightarrow f_a$ , where  $f_a(x) = f(x - a)$ . We have seen for Riemann integrals that

$$\int_{\mathbb{R}^k} f \, dV = \int_{\mathbb{R}^k} f_a \, dV$$

By the extension (Reisz, i guess),

$$\int f \, dm = \int f_a \, dm$$

Moreover if  $K$  is compact, and  $V$  open with  $K \subset V$ , we have  $f \in C_c(\mathbb{R}^k)$  with  $\chi_K \leq f \leq \chi_V$ . Then  $\chi_{K+a} \leq f_a \leq \chi_{V+a}$ .

Next choose any compact set  $K$  in  $\mathbb{R}^k$ . Define a distance from  $K$  as  $\phi_K(x) = \inf_{y \in K} |x - y|$ . Then  $\phi_K$  is uniformly continuous on  $\mathbb{R}^k$ . Pick  $V_k = \phi_K^{-1}((\frac{-1}{n}, \frac{1}{n}))$ . Then  $V_n \supset V_{n+1} \supset \dots$  and  $K = \cap_{n=1}^{\infty} V_n$ .

Now choose a sequence  $(f_n) \in C_c(\mathbb{R}^k)$  such that  $\chi_K \leq f_n \leq \chi_{V_n}$  and  $f_1 \geq f_2 \geq \dots$  (By choosing minima among the first few functions).

Then we get

$$\begin{aligned} m(K) &= \inf_{n \in \mathbb{N}} \int f_n \, dm \\ &= \inf_{n \in \mathbb{N}} \int (f_n)_a \, dm \\ &= \lambda(K) \end{aligned}$$

Now we have showed that  $\lambda = \mu$  for compact sets in  $\mathbb{R}^k$ . Now we should prove the same for the open sets of  $\mathbb{R}^k$ . Now by the  $\sigma$ -compactness of  $\mathbb{R}^k$ , we get our desired translation invariance.

□

# Chapter 15

## 15.1 Vitali Sets

**Theorem 15.1.1.** *If  $\mathcal{M}$  is a  $\sigma$ -algebra on  $\mathbb{R}$  and  $\lambda : \mathcal{M} \rightarrow [0, \infty]$  is a translation invariant measure with  $0 < \lambda([0, 1)) < \infty$ , then there is  $E \subset [0, 1)$  such that  $E \notin \mathcal{M}$ .*

*Proof.* Endow  $[0, 1)$  with an equivalence relation  $a \sim b \iff a - b \in \mathbb{Q}$ . This gives a partition of  $[0, 1)$  by the equivalence classes. Now from each of these classes pick (by AOC) one representative element and build the set  $E$ . Observe that for  $r, s \in \mathbb{Q}$ ,  $(E + s) \cap (E + r) = \emptyset$  if and only if  $r = s$ .

Also note that

$$[0, 1) \subset \bigcup_{r \in \mathbb{Q} \cap [-1, 1]} (E + r)$$

Therefore

$$E \subset [0, 1) \subset \bigcup_{r \in \mathbb{Q} \cap [-1, 1]} (E + r) \subset [-1, 2)$$

verify the rest, its easy. □

**Theorem 15.1.2** (Luzin's theorem). *Let  $X$  be a locally compact Hausdorff space.*

- (1)  $\mu$  is a regular measure on a  $\sigma$ -algebra  $\mathcal{M}$  containing  $B(X)$
- (2)  $f : X \rightarrow \mathbb{C}$  is measurable
- (3) there is a  $A \in \mathcal{M}$  such that  $\mu(A) < \infty$  and  $f = 0$  on  $A^c$

*Given  $\epsilon > 0$  there is a  $g \in C_c(X)$  such that  $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$*

# Chapter 16

**Theorem 16.0.1** (Luzin's theorem). *Let  $X$  be a locally compact Hausdorff space.*

- (1)  $\mu$  is a regular measure on a  $\sigma$ -algebra  $\mathcal{M}$  containing  $B(X)$
- (2)  $f : X \rightarrow \mathbb{C}$  is measurable
- (3) there is a  $A \in \mathcal{M}$  such that  $\mu(A) < \infty$  and  $f = 0$  on  $A^c$

*Given  $\epsilon > 0$  there is a  $g \in C_c(X)$  such that  $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$  and  $\sup\{|g(x)| : x \in X\} \leq \sup\{|f(x)| : x \in X\}$ .*

*Proof.* Suppose for now  $A$  is compact. (We can assume this since the measure is regular and we can find a compact set  $K \subset A$  such that  $f = 0$  almost everywhere in  $K^c$ .) We'll do the  $A$  not compact case later.

Choose  $V$  open such that  $A \subset V$  and  $\bar{V}$  is compact. We'll first prove the existence of the desired  $g$  if  $f$  is simple. Let

$$f = \sum_{j=1}^n \alpha_j \chi_{A_j}$$

where each  $A_j$  is disjoint and  $\cup_{j=1}^n A_j = A$ . Again each of the  $\mu(A_j) \leq \mu(A) < \infty$ . Hence by the regularity of the measure there are compact sets  $K_j \subset A_j$  such that  $\mu(A_j \setminus K_j) < \frac{\epsilon}{2^{j+1}}$ .

Since  $K_j$  are compact and disjoint, we can find collection of disjoint open sets  $V_j$  such that  $K_j \subset V_j$ . **verify this, I am not sure.**

Moreover by replacing  $V_j$  with  $V_j \cap V$ , we can assume  $V_j \subset V$ . Now by the outer regularity of the measure, we can assume  $\mu(V_j \setminus K_j) < \frac{\epsilon}{2^{j+1}}$ . Now by Urysohn, there is a  $g_j \in C_c(X)$  such that  $\chi_{K_j} \leq g_j \leq \chi_{V_j}$ . Let

$$g = \sum_{j=1}^n \alpha_j g_j$$

Then  $g$  is continuous being the finite sum of continuous function. Moreover since  $\cup_{j=1}^n V_j \subset V$ , we get  $\text{supp}(g) \subset \overline{V}$ . Also

$$|g(x)| \leq \max\{|\alpha_j|\} \max_{x \in A} |f(x)|$$

Now we see that  $f(x) = g(x)$  for all  $x \in K_j$  and  $x \in (A_j \cup V_j)^c$ . Since  $K_j \subset V_j$ , the set where they possibly disagree is

$$D = \bigcup_{j=1}^n (V_j \setminus K_j) \cup \bigcup_{j=1}^n (A_j \setminus K_j)$$

Add a diagram for ease of reasoning

Now by the subadditivity of  $\mu$ , we get  $\mu(D) < \epsilon$  and we have proved the result for  $A$  compact and  $f$  simple.

Now for the case when  $0 \leq f < 1$ , let  $s_n$  be the sequence of simple functions  $0 \leq s_1 \leq s_2 \leq \dots \leq$  with  $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ . Let  $t_n = s_n - s_{n-1}$ , where  $s_0 = 0$ . Each  $t_n$  is simple and  $t_n = 0$  on  $A^c$  and by construction, we get

$$t_n \leq \frac{1}{2^{n-1}} \chi_{B_n}$$

for some set  $B_n$ .

Now we use the first part of the proof on  $t_n$ s to get a corresponding  $g_n \in C_c(X)$ . Then  $g_n$  satisfy

(1)

(2)

(3)

Let  $g = \sum_{n \in \mathbb{N}} g_n$ , which converges uniformly as  $|g_n| \leq \frac{1}{2^{n-1}}$  by Wierestrass. Hence  $g \in C_c(X)$  and  $\text{supp}(g) \subset \overline{V}$ .

We know that  $f = \sum_{n=1}^{\infty} t_n$  from the definition of  $t_n$ . So the set  $D = \{x \in X : f(x) \neq g(x)\}$  is a subset of  $\cup_{n=1}^{\infty} \{x \in X : t_n(x) \neq g_n(X)\}$ . Now the subadditivity of  $\mu$  gives that  $\mu(D) < \epsilon$ .

Next, if  $f$  is non-negative, bounded, the result follows from scaling  $f$ . Again if  $f \geq 0$  is measurable and possibly unbounded, we have  $\cap_{n=1}^{\infty} \{x \in X : f(x) \geq n\} = \emptyset$ . Moreover  $\mu(\{f \geq 1\}) \leq \mu(A) < \infty$ . Hence by the continuity of the measure from above, we get  $\mu(\{f \geq n\}) \rightarrow 0$ . Hence we can replace  $f$  with  $f \chi_{f < n}$  for some appropriate  $n$ .

Now if the function is general complex, we can split it as the sum and difference of four non-negative measurable functions and continue the analysis. Finally if  $A$  is not compact, we can find a  $K \subset A$  such that  $K$  is compact and  $\mu(A \setminus K)$  is arbitrarily small by the inner regularity of the measure  $\mu$  for finite sets.  $\square$

# Chapter 17

**Definition 17.0.1.** A function  $f$  of a topological space  $X$  is called lower semi-continuous if for all  $\alpha \in \mathbb{R}$ ,  $\{x \in X : f(x) > \alpha\}$  is open.

**Example 17.0.1.** If  $V$  is open, then  $\chi_V$  is lower semi-continuous because the  $\{x \in X : f(x) > \alpha\}$  has choices  $\phi, V, X$ , all of them are open.

**Definition 17.0.2.** A function is called upper semi-continuous if for all  $\alpha \in \mathbb{R}$ , the set  $\{x \in X : f(x) < \alpha\}$  is open.

*Remark 17.0.1.* If  $f : X \rightarrow \mathbb{R}$  is lower semi-continuous, then  $-f$  is upper semi-continuous.

**Example 17.0.2.** If  $V$  is open, then  $\chi_{V^c} = 1 - \chi_V$  is upper semi-continuous.

**Proposition 17.0.1.** If  $f, g$  are lower semi-continuous, so is  $f + g$ .

*Proof.*

$$\{x \in X : f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{R}} (\{x : f(x) > r\} \cap \{x : g(x) < \alpha - r\})$$

□

**Proposition 17.0.2.** If  $u_1 \leq u_2 \leq \dots$  are all lower semi-continuous, then so is  $\lim_{n \rightarrow \infty} u_n = u$ .

*Proof.*

$$\{u > \alpha\} = \bigcup_{n \in \mathbb{N}} \{u_n > \alpha\}$$

□

**Corollary 17.0.0.1.** A monotone increasing sequence of continuous functions converges to a lower semi-continuous function.

**Theorem 17.0.1** (Vitali-Caratheodory Theorem). *Let  $X$  be locally compact and Hausdorff,  $\mu$  be a regular Borel measure. If  $f : X \rightarrow \mathbb{R}$  in  $L^1(\mu)$ , then there is an upper semi-continuous function  $u$  and a lower semi-continuous function  $v$  such that  $u \leq f \leq v$  and  $\int (v - u) d\mu < \epsilon$ .*

*Proof.* Assume  $f \geq 0$ . There exists an increasing sequence of simple functions  $(s_n)$  converging (pointwise) to  $f$ . Considering as before,  $t_n = s_n - s_{n-1}$  with  $s_0 = 0$ , we see that each  $t_n$  is simple and  $f = \sum_{n \in \mathbb{N}} t_n$ .

Then since the  $t_n$  are simple, expanding them out into the standard simple function form and re-indexing them, we get

$$f = \sum_{j=1}^{\infty} c_j \chi_{E_j}$$

Note that we're not claiming  $E_j$ s are disjoint. Since  $f \in L^1(\mu)$ , we can apply monotone convergence theorem. Thus

$$\sum_{j=1}^{\infty} \underbrace{\int c_j \chi_{E_j} d\mu}_{c_j \mu(E_j)} = \int f d\mu < \infty$$

If  $c_j = 0$ , discard. Otherwise we see that  $\mu(E_j) < \infty$  for each  $j \in \mathbb{N}$ . By regularity,  $\exists K_j$  compact and  $V_j$  open such that  $K_j \subset E_j \subset V_j$  and  $\mu(V_j \setminus K_j) < \frac{\epsilon}{2^j c_j}$ . As a consequence of convergence of  $\sum_{j=1}^{\infty} c_j \mu(E_j)$ , we have  $N \in \mathbb{N}$  such that  $\sum_{j=N+1}^{\infty} c_j \mu(E_j) < \epsilon$ . Let

$$u = \sum_{j=1}^N c_j \chi_{K_j} \quad \text{and} \quad v = \sum_{j=1}^{\infty} c_j \chi_{V_j}$$

Then we see that  $u$  is upper semi-continuous and  $v$  is lower semi-continuous and

$$v - u = \sum_{j=1}^N c_j \chi_{V_j \setminus K_j} + \sum_{j=N+1}^{\infty} c_j \chi_{V_j}$$

Thus,

$$\begin{aligned}
\int (v - u) \, d\mu &= \int \left( \sum_{j=1}^N c_j \chi_{V_j \setminus K_j} + \sum_{j=N+1}^{\infty} c_j \chi_{V_j} \right) \, d\mu \\
&= \sum_{j=1}^N c_j \mu(V_j \setminus K_j) + \sum_{j=N+1}^{\infty} c_j \mu(V_j) \\
&\leq \sum_{j=1}^N c_j \frac{\epsilon}{2^j c_j} + \\
&< \epsilon +
\end{aligned}$$

Now to complete the proof, apply this result to  $f^+$  and  $f^-$ . Then since  $f = f^+ - f^-$  and we get upper and lower semi-continuous functions  $u_+, v_+$  for  $f^+$  and  $u_-, v_-$  for  $f^-$ . Let  $u = u_+ - v_-, v = v_+ - u_-$  gives  $u \leq f \leq v$  and satisfy the properties.  $\square$



# Chapter 18

## $L^p$ Spaces

**Definition 18.0.1.** A function  $\phi : (a, b) \rightarrow \mathbb{R}$  is called convex if

$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y)$$

for all  $x, y \in (a, b)$  and  $0 \leq t \leq 1$ .

**Proposition 18.0.1.** A function  $\phi : (a, b) \rightarrow \mathbb{R}$  is convex if and only if for  $u, s, t$  with  $a < u \leq t \leq s < b$ , we have

$$\phi(t) \leq \phi(s) \frac{u - t}{u - s} + \phi(u) \frac{t - s}{u - s}$$

or equivalently using

$$\phi(t) - \phi(s) = \frac{t - s}{u - s}(\phi(u) - \phi(s))$$

satisfies

$$\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(s)}{u - s}$$

**Theorem 18.0.1.** A function  $\phi : (a, b) \rightarrow \mathbb{R}$  that is convex is continuous.

*Proof.* Let  $S = (s, \phi(s))$ ,  $X = (x, \phi(x))$ ,  $Y = (y, \phi(y))$ , with  $a < s \leq x \leq y < b$ .

Draw secants and refer Rudin. □

**Theorem 18.0.2** (Jensen's Inequality). Let  $(X, \mathcal{M}, \mu)$  be a measure space with  $\mu(X) = 1$ . If  $f \in L^1(\mu)$  and for each  $x \in X$ ,  $a < f(x) < b$  and  $\phi$  is convex on  $(a, b)$ , then

$$\phi\left(\int f \, d\mu\right) \leq \int (\phi \circ f) \, d\mu$$

*Proof.* We know by convexity that for  $u \leq s \leq t$ ,

$$\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(s)}{u - s}$$

Then there is  $\beta$  such that

$$\frac{\phi(t) - \phi(s)}{t - s} \leq \beta \leq \frac{\phi(u) - \phi(s)}{u - s}$$

Consider LHS Inequality to get

$$\begin{aligned} \phi(t) - \phi(s) &\leq \beta(t - s) \\ \phi(s) &\geq \phi(t) + \beta(s - t) \end{aligned}$$

for  $s < t$ , and similarly by the RHS we get

$$\phi(u) - \phi(s) \geq \beta(u - s)$$

Hence in both the cases ( $t = f(x)$ ,  $u = f(x)$ )

$$\phi(f(x)) - \phi(s) - \beta(f(x) - s) \geq 0$$

Now integrating this gives

$$\int \phi \circ f \, d\mu - \phi(t) - \beta \left( \int f \, d\mu - s \right) \geq 0$$

Choosing  $s = \int f \, d\mu$  gives out inequality.  $\square$

**Example 18.0.1.** Take  $\mu$  to be the probability measure on  $X = \{1, 2, 3, \dots, n\}$ , assume  $\mu(\{j\}) = \alpha_j > 0$ . Then for  $b_1, b_2, \dots, b_n > 0$ , we have

$$b_1^{\alpha_1} b_2^{\alpha_2} \dots b_n^{\alpha_n} \leq \sum_{j=1}^n \alpha_j b_j$$

*Proof.* Use the convexity of  $x \rightarrow e^x$ , and let  $b_j = e^{c_j}$ .  $\square$

**Theorem 18.0.3** (Holder's Inequality). *Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $f, g : X \rightarrow [0, \infty]$  be measurable. Then for  $1 < p < \infty$ , with  $1/p + 1/q = 1$ , then*

$$\int f g \, d\mu \leq \left( \int f^p \, d\mu \right)^{\frac{1}{p}} \left( \int g^q \, d\mu \right)^{\frac{1}{q}} \equiv \|f\|_p \|g\|_q$$

and

$$\left( \int (f + g)^p \, d\mu \right)^{\frac{1}{p}} \leq \|f\|_p + \|g\|_p$$

# Chapter 19

**Theorem 19.0.1** (Holder's & Minkowski Inequality). *Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $f, g : X \rightarrow [0, \infty]$  be measurable. Then for  $1 \leq p < \infty$ , with  $1/p + 1/q = 1$ , then*

$$\int fg \, d\mu \leq \left( \int f^p \, d\mu \right)^{\frac{1}{p}} \left( \int g^q \, d\mu \right)^{\frac{1}{q}} = \|f\|_p \|g\|_q$$

and

$$\left( \int (f + g)^p \, d\mu \right)^{\frac{1}{p}} \leq \|f\|_p + \|g\|_p$$

*Proof.* Let  $A = \|f\|_p, B = \|g\|_p$ . If  $A = 0$  or  $A = \infty$ , or  $B = 0$ , or  $B = \infty$ , we have nothing to show. Hence assume that  $0 < A, B < \infty$ . Let  $F(x) = \frac{f(x)}{A}, G(x) = \frac{g(x)}{B}$ . We also define  $s, t : X \rightarrow \mathbb{R}$  as

$$F(x) = e^{\frac{s(x)}{p}}, \quad G(x) = e^{\frac{t(x)}{q}}$$

By convexity of the exponential function, we have

$$e^{s/p+t/q} \leq \frac{1}{p}e^s + \frac{1}{q}e^t$$

In terms of  $F, G$ , this is

$$F(x)G(x) \leq \frac{1}{p}(F(x))^p + \frac{1}{q}(G(x))^p$$

Hence integrating both sides, we get

$$\int F(x)G(x) \, d\mu \leq \frac{1}{p} \int (F(x))^p \, d\mu + \frac{1}{q} \int (G(x))^p \, d\mu$$

Now writing this in terms of  $f, g$  gives us

$$\begin{aligned}\frac{1}{AB} \int fg \, d\mu &\leq \frac{1}{p} \frac{1}{A^p} \int f^p \, d\mu + \frac{1}{q} \frac{1}{B^q} \int g^q \, d\mu \\ &= \frac{1}{p} \frac{1}{A^p} \|f\|_p^p + \frac{1}{q} \frac{1}{B^q} \|g\|_q^q \\ &= 1/p + 1/q = 1\end{aligned}$$

Thus we get Holder inequality.

For Minkowski, consider

$$\begin{aligned}(f + g)^p &= (f + g)(f + g)^{p-1} \\ &= f(f + g)^{p-1} + g(f + g)^{p-1}\end{aligned}$$

Now integrating both sides and carefully applying Holder's inequality, we get

$$\begin{aligned}\int (f + g)^p \, d\mu &= \int f(f + g)^{p-1} \, d\mu + \int g(f + g)^{p-1} \, d\mu \\ &= \left( \int f^p \, d\mu \right)^p \left( \int (f + g)^{(p-1)q} \, d\mu \right)^q + \left( \int g^q \, d\mu \right)^q \left( \int (f + g)^{(p-1)p} \, d\mu \right)^p \\ &= \end{aligned}$$

verify

□

**Definition 19.0.1.** Let  $0 < p < \infty$ .  $f : X \rightarrow \mathbb{C}$  measurable on  $(X, \mathcal{M}, \mu)$ . We define

$$\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p}$$

We also write  $L^p(\mu) = \{f : X \rightarrow \mathbb{C} : \|f\|_p < \infty\}$

**Definition 19.0.2.** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f : X \rightarrow [0, \infty]$  be measurable. The essential supremum of  $f$  is

$$\text{ess sup } f = \inf \{ \alpha : \mu(\{f > \alpha\}) = 0 \}$$

**Proposition 19.0.1.** With  $(X, \mathcal{M}, \mu)$ ,  $f$  be as above.  $\beta = \text{ess sup } f$ . Then

$$\mu(\{f > \beta\}) = 0$$

**Definition 19.0.3.** For  $(X, \mathcal{M}, \mu)$ ,  $f$  as above,

$$\|f\|_\infty = \text{ess sup } \|f\|$$

and  $L^\infty(\mu)$  be the set of all  $f$  with  $\|f\|_\infty < \infty$

We add a case of Holder's inequality for  $\|\cdot\|_\infty$ .

**Theorem 19.0.2.** *If  $(X, \mathcal{M}, \mu)$  is as usual  $f, g$  measurable,  $f \in L^1(\mu), g \in L^\infty(\mu)$ , then  $fg \in L^1(\mu)$  and*

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$$

*Proof.* Take  $E = \{x \in X : |g(x)| > \|g\|_\infty\}$ . Then  $E$  has measure zero, and

$$\begin{aligned} \int |fg| d\mu &= \int_{X \setminus E} |fg| d\mu + \int_E |fg| d\mu \\ &\leq \|g\|_\infty \int_{X \setminus E} |f| d\mu \\ &\leq \|g\|_\infty \|f\|_1 \end{aligned}$$

□

**Theorem 19.0.3.** *let  $(X, \mathcal{M}, \mu)$  be as usual,  $f, g$  measurable  $f, g \in L^\infty(\mu)$ . Then*

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

*Proof.* Notice that

$$\begin{aligned} \{x : |f(x) + g(x)| > \|f\|_\infty + \|g\|_\infty\} &\subset \{x : |f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty\} \\ &\subset \{x : |f(x)| > \|f\|_\infty\} \cup \{x : |g(x)| > \|g\|_\infty\} \end{aligned}$$

Since both the sets at the end is of measure zero. Hence we get the inequality. □

**Theorem 19.0.4.** *For each  $1 \leq p \leq \infty$ ,  $L^p(\mu)$  is a normed vector space over  $\mathbb{C}$  provided we identify functions that are equal almost everywhere.*

*Proof.* Positive definiteness follows from the identification of functions in the space. Homogeneity follows from the definition of  $\|\cdot\|_p$ . And triangle inequality is the Minkowski inequality. We have shown that for the cases  $1 \leq p < \infty$ , that  $\|\cdot\|_p$  is a norm. □

**Lemma 19.0.1.** *Let  $(f_n) \in L^p(\mu)$  be a Cauchy sequence in  $1 \leq p \leq \infty$ . Then there exists a subsequence  $(f_{n_j})$  which is convergent pointwise almost everywhere.*

# Chapter 20

*Remark 20.0.1.* Consider the counting measure  $\mu$ , on  $\mathbb{N}$ . Find a sequence of functions  $f_n : \mathbb{N} \rightarrow [0, \infty)$ , such that  $\|f_n\|_1 \rightarrow 0$  and  $g = \sup_n f_n \notin L^1(\mu)$ .

**Lemma 20.0.1.** *Let  $(f_n) \in L^p(\mu)$  be a Cauchy sequence in  $1 \leq p \leq \infty$ . Then there exists a subsequence  $(f_{n_j})$  which is convergent pointwise almost everywhere.*

*Proof.* First suppose,  $p < \infty$ . Starting from a Cauchy sequence, choose a subsequence  $n_1 < n_2 < \dots$  such that for each  $k \in \mathbb{N}$

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k}$$

Let

$$g_l = \sum_{k=1}^l |f_{n_{k+1}} - f_{n_k}| \quad g = \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

Then  $g_n^p \leq g_{n+1}^p \leq \dots$  and  $g_n^p \rightarrow g^p$ . Then by monotone convergence theorem,

$$\int g_n^p d\mu \rightarrow \int g^p d\mu$$

Moreover, using Minkowski's inequality, we get

$$\begin{aligned} \|g_l\|_p &\leq \sum_{k=1}^l \|f_{n_{k+1}} - f_{n_k}\| \\ &\leq \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\| \\ &\leq 1 \end{aligned}$$

By monotone convergence, we get  $\|g\|_p \leq 1$ . In particular  $g$  is finite almost everywhere. Hence

$$f = \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$$

is absolutely convergent almost everywhere. So by telescoping series for almost every  $x \in X$

$$\begin{aligned} f(x) &= \lim_{l \rightarrow \infty} \sum_{k=1}^l (f_{n_{k+1}} - f_{n_k})(x) \\ &= \lim_{l \rightarrow \infty} (f_{n_{l+1}}(x) - f_{n_1}(x)) \end{aligned}$$

So  $f_{n_l}$  converges for almost every  $x \in X$ .

Next, we consider  $p = \infty$ . For  $n, k \in \mathbb{N}$ , let

$$E_{n,k} = \{x \in X : |f_n(x) - f_k(x)| > \|f_n - f_k\|_\infty\}$$

Then  $\mu(E_{n,k}) = 0$ , by the definition of essential supremum. Moreover  $E = \cup_{n,k=1}^\infty E_{n,k}$  also has measure 0. On  $E^c$ , for each  $k, n \in \mathbb{N}$ , we have

$$|f_n(x) - f_k(x)| \leq \|f_n - f_k\|$$

This means  $f_n|_{E^c}$  converges uniformly. □

**Theorem 20.0.1.** *For  $1 \leq p \leq \infty$ ,  $L^p(\mu)$  is a complete metric space. (After identifying functions that are equal almost everywhere.)*

*Proof.* (1) For  $p = \infty$ , the proof in the above lemma is the proof

(2) For the rest of the  $p$ , consider the Cauchy sequence  $f_n$  in  $L^p(\mu)$ ,  $p < \infty$ . It has a pointwise almost everywhere converging subsequence converging to  $f$ . We need to show that  $f \in L^p(\mu)$  and convergence is in norm. That is  $\|f_n - f\|_p \rightarrow 0$ .

We apply Fatou's lemma to the function  $g_k = |f_n - f_{n_k}|^p$  to get

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int |f_n - f_{n_k}|^p d\mu &\geq \int \liminf_{k \rightarrow \infty} |f_n - f_{n_k}|^p d\mu \\ &= \|f_n - f\|_p^p \end{aligned}$$

Given  $\epsilon > 0$ , since  $f_n$  is Cauchy in  $L^p(\mu)$ , there is a  $N$  such that for  $n, m \geq N$ , we have

$$\epsilon^p > \|f_n - f_m\|_p^p = \int |f_n - f_m|^p d\mu$$

By taking  $m = n_k \rightarrow \infty$ , we then get

$$\epsilon^p \geq \|f_n - f\|_p^p$$

This implies  $f \in L^p(\mu)$ , by

$$\|f\|_p \leq \|f - f_n\|_p + \|f_n\|_p$$

Now that fact that  $\|f - f_n\|_p \rightarrow 0$ , we get  $f \in L^1(\mu)$ . □

# Chapter 21

## 21.1 Approximations by simple or continuous functions

**Theorem 21.1.1.** *Let  $(X, \mathcal{M}, \mu)$  be a measure space, denote by  $S$ , the collection of simple measurable functions with finite measurable support. Then for  $1 \leq p < \infty$ ,  $S \subset L^p(\mu)$  and  $S$  is dense in  $L^p(\mu)$ .*

*Proof.* Given  $f \in L^p(\mu)$ , we need to find a sequence  $s_n$  in  $S$  such that  $s_n \rightarrow f$  in  $L^1(\mu)$ . First suppose that  $f : X \rightarrow [0, \infty)$ . We know a sequence of simple measurable functions  $s_n$  such that  $0 \leq s_1 \leq s_2 \leq \dots$  and

$$\lim_{n \rightarrow \infty} s_n(x) = f(x)$$

for each  $x \in X$ . Applying dominated convergence theorem, since  $|s_n - f| \leq f$ , for  $f \in L^p(\mu)$  gives

$$\|f - s_n\|_p^p = \int |f - s_n|^p d\mu \leq \int |f|^p d\mu < \infty$$

we get  $\|f - s_n\|_p \rightarrow 0$

Now taking a general  $f \in L^p(\mu)$ , writing  $f = u_+ - u_- + i(v_+ - v_-)$  and repeating the process for these gives  $s = s_+ - s_- + i(t_+ - t_-)$  where  $s_{\pm}, t_{\pm} \in S$  and

$$\|s_{\pm} - u_{\pm}\|_p, \|t_{\pm} - v_{\pm}\|_p < \varepsilon$$

hence by triangle inequality, we get

$$\|s - f\|_p < 4\varepsilon$$

We can make RHS arbitrarily small, so  $S$  is dense in  $L^p(\mu)$ . □

**Theorem 21.1.2.** *Let  $X$  be a locally compact Hausdorff space with  $1 \leq p < \infty$ , then  $C_c(X)$  is dense in  $L^p(\mu)$ .*



*Proof.* It is enough to show  $\overline{C_c(X)}$  includes  $S$ . Given  $s \in S$ , let  $A = \{s \neq 0\}$  with  $\mu(A) < \infty$ . Then by Luzin's theorem, there is a  $g \in C_c(X)$  such that

$$\|g\|_\infty \leq \|s\|_\infty \quad \text{and} \quad \mu(E_\varepsilon) < \varepsilon$$

where  $E_\varepsilon = \{x \in X \mid g(x) \neq s(x)\}$ . Since  $|g(x) - s(x)| \leq 2\|s\|_\infty$ , we get

$$\begin{aligned} \|g - s\|_p &= \left( \int |g - s|^p d\mu \right)^{\frac{1}{p}} \\ &= \left( \int_{E_\varepsilon} |g - s|^p d\mu \right)^{\frac{1}{p}} \end{aligned}$$

On this set,  $|g - s| \leq 2\|s\|_\infty$  gives

$$\begin{aligned} \|g - s\|_\infty &\leq \left( \int_{E_\varepsilon} (2\|s\|_\infty)^p d\mu \right)^{\frac{1}{p}} \\ &< 2\|s\|_\infty \varepsilon^{1/p} \end{aligned}$$

Since we can make  $\varepsilon$  arbitrarily small, we get the density.  $\square$

*Remark 21.1.1.* This theorem proves that  $L^p(\mu)$  is the completion of  $(C_c(\mathbb{R}^k), d_p)$  where for  $f, g \in C_c(\mathbb{R}^k)$ ,  $d_p(f, g) = \|f - g\|_p$ . The limit of a Cauchy sequence in  $C_c(\mathbb{R}^k)$  is determined almost everywhere.

If  $p = \infty$ , then the completion of  $C_c(\mathbb{R}^k)$  is not  $L^\infty(m)$ , but  $C_o(\mathbb{R}^k)$ .

**Definition 21.1.1.** Let  $X$  be locally compact Hausdorff, we say a continuous function  $f$  vanishes at infinity and write  $f \in C_o(X)$  if for  $\varepsilon > 0$ , we can find a compact set  $K$  such that  $|f(x)| < \varepsilon$  for all  $x \notin K$ .

**Theorem 21.1.3.** Let  $X$  be locally compact Hausdorff, then  $C_o(X)$  is the completion of  $C_c(X)$  with  $\|\cdot\|_\infty$ .

*Proof.* Let  $f \in C_o(X)$ ,  $\varepsilon > 0$ , we can choose  $K$  such that  $K$  is compact and  $|f(x)| < \varepsilon$  for all  $x \in K^c$ . Using Urysohn's lemma, there is a  $g \in C_c(X)$  such that  $\chi_K \leq g \leq 1$ , then  $h = fg \in C_c(X)$  and

$$\begin{aligned} \|h - f\|_\infty &= \|f(1 - g)\|_\infty \\ &= \|f(1 - g)\chi_{K^c}\|_\infty \\ &\leq \varepsilon \|1 - g\|_\infty \\ &\leq \varepsilon \end{aligned}$$

$\square$

**Proposition 21.1.1.** *Show that if  $\mu(X) < \infty$ , with  $p \leq r \leq \infty$ , then*

$$L^r(\mu) \subset L^p(\mu)$$

*Given  $f \in L^r(\mu)$ .*

*Proof.* Let  $f \in L^r(\mu)$ . Then,

$$\begin{aligned} \|f\|_p^p &= \int |f|^p \, dm \\ &\leq \left( \int |f|^r \, dm \right)^{p/r} \left( \int 1 \, dm \right)^{1-p/r} \\ &\leq \|f\|_r^p \mu(X)^{1-\frac{p}{r}} \\ &< \infty \end{aligned}$$

□

# Chapter 22

## Inner Product Spaces

**Definition 22.0.1.** Let  $\mathcal{H}$  be a vector space over  $\mathbb{C}$ . A sesquilinear form is a function

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$$

satisfying

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle x + \alpha z, y \rangle = \langle x, y \rangle + \alpha \langle z, y \rangle$

for all  $x, y, z \in \mathcal{H}, \alpha \in \mathbb{C}$ . It is said to be positive semidefinite (positive definite) if  $\langle x, x \rangle \geq 0$  ( $\langle x, x \rangle > 0$  for all  $x \in \mathcal{H} \setminus \{0\}$ ) for all  $x \in \mathcal{H}$ .

A positive definite sesquilinear form makes  $\mathcal{H}$  an inner product space.

**Example 22.0.1.** Take  $L^2(\mu)$  (functions identified almost everywhere) with the natural inner product is an inner product.

**Proposition 22.0.1.** If  $\mathcal{H}$  is a complex vector space with a positive semidefinite sesquilinear form and  $\langle x, x \rangle = 0$ , then  $\langle x, y \rangle = 0$  for all  $y \in \mathcal{H}$ .

*Proof.* Take  $\alpha \in \mathbb{C}$  and consider

$$\begin{aligned} \langle x + \alpha y, x + \alpha y \rangle &= \langle x, x \rangle + \alpha \langle y, x \rangle + \overline{\alpha} \langle y, y \rangle \\ &= 2\Re(\overline{\alpha} \langle x, y \rangle) + |\alpha|^2 \langle y, y \rangle \end{aligned}$$

Now if  $\langle x, y \rangle \neq 0$ , then either  $\langle y, y \rangle = 0$  or nonzero. If  $\langle y, y \rangle = 0$ , take  $\alpha = -\langle x, y \rangle$  to get

$$\langle x + \alpha y, x + \alpha y \rangle = \underbrace{2\Re(-\overline{\langle x, y \rangle} \langle x, y \rangle)}_{<0}$$

which is a contradiction.

Now if  $\langle y, y \rangle \neq 0$ , take  $\alpha = i\langle x, y \rangle$  to get a similar contradiction, which makes  $\Re(\overline{\alpha}\langle x, y \rangle) = 0$   $\square$

**Definition 22.0.2.** If  $\langle \cdot, \cdot \rangle$  is a positive semidefinite sesquilinear form, then

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

is a seminorm.

If  $\langle \cdot, \cdot \rangle$  is positive definite, then  $x \rightarrow \|x\|$  is a norm.

**Theorem 22.0.1** (Cauchy-Schwarz). *If  $\langle \cdot, \cdot \rangle$  is a positive semidefinite sesquilinear form on  $\mathcal{H}$ , then for  $x, y \in \mathcal{H}$*

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

*Proof.* If  $\|y\| = 0$ , then previous proposition takes care of the proof. If not, choose  $\alpha = \frac{\langle x, y \rangle}{\langle y, y \rangle}$  and consider

$$\begin{aligned} 0 &\leq \langle x - \alpha y, x - \alpha y \rangle \\ &= \|x\|^2 - 2\Re(\alpha \langle y, x \rangle) + |\alpha|^2 \langle y, y \rangle \\ &= \|x\|^2 - 2\frac{\langle x, y \rangle^2}{\|y\|^2} + \frac{\langle x, y \rangle^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \end{aligned}$$

which gives our inequality.  $\square$

**Theorem 22.0.2.** Let  $\mathcal{H}$  be a vector space over  $\mathbb{C}$  with a positive semidefinite sesquilinear  $\langle \cdot, \cdot \rangle$  form and the associated seminorm  $\| \cdot \|$ , then for all  $x, y \in \mathcal{H}$ ,

$$\|x + y\| \leq \|x\| + \|y\|$$

*Proof.* verify □

*Remark 22.0.1.* If  $\langle \cdot, \cdot \rangle$  is an inner product space, then  $\| \cdot \|$  defines a norm in  $\mathcal{H}$ .

**Definition 22.0.3.** If  $\mathcal{H}$  be an inner product. If  $\mathcal{H}$  is complete with respect to the topology induced by the inner product, then it is called a Hilbert space.

**Example 22.0.2.**  $L^2(\mu)$ , with functions identified that agrees almost everywhere is a Hilbert space when endowed with the inner product

$$\langle f, g \rangle = \int f \bar{g} \, d\mu$$

**Proposition 22.0.2.** Let  $\mathcal{H}$  be a Hilbert space. Then for  $g \in \mathcal{H}$

$$\lambda_g : \mathcal{H} \rightarrow \mathbb{C} := f \mapsto \langle f, g \rangle$$

is a linear, uniformly continuous functional.

*Proof.* Use Cauchy-Schwarz inequality. □

**Definition 22.0.4.** Let  $H$  be a Hilbert space. We say  $x, y \in \mathcal{H}$  are orthogonal if  $\langle x, y \rangle = 0$ . We also write  $x \perp y$ .

If  $S \subset \mathcal{H}$ , define

$$S^\perp = \{x \in \mathcal{H} : x \perp s, \forall s \in S\}$$

**Theorem 22.0.3.** If  $S \subset H$ , then  $S^\perp$  is a closed subspace of  $\mathcal{H}$ .

*Proof.* Let  $z \in \mathcal{H}$ . Then  $K_z = z^\perp = \text{Ker}(\lambda_z)$  is closed. Observe that

$$S^\perp = \bigcap_{s \in S} K_s$$

is closed as well. □

**Lemma 22.0.1.** Let  $\mathcal{M}$  be a closed subspace of a Hilbert space  $\mathcal{H}$ , and  $h \in \mathcal{H}$ , then there is a unique  $m \in \mathcal{M}$  that minimizes the distance to  $h$

*Proof.* We recall the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

and write for  $x, y \in \mathcal{H}$ ,

$$\|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x + y\|^2$$

Let  $\delta = \inf\{\|m - h\| : m \in \mathcal{M}\}$ . There there is a sequence of  $m_j \in \mathcal{M}$  such that  $\|m_j - h\| \rightarrow \delta$ . To show that  $m_j$  is a Cauchy sequence, consider  $x = m_j - h, y = m_i - h$ . Then

$$\frac{x + y}{2} = \frac{m_i + m_j}{2} - h$$

and we see that

$$\left\| \frac{x + y}{2} \right\| = \left\| \frac{m_i + m_j}{2} - h \right\|$$

Then by prarllelogram law,

$$\begin{aligned} \|m_j - m_i\|^2 &= 2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 \\ &= 2(\|m_j - h\|^2 + \|m_i - h\|^2 - \|m_i + m_j - 2h\|^2) \end{aligned}$$

verify

This shows that, we can make  $\|m_i - m_j\|$  arbitrarily small by requiring  $i, j \in \mathbb{N}$  for a similarly large  $\mathbb{N}$ , meaning  $m_j$  is Cauchy. Since  $\mathcal{M}$  is closed and a closed and a closed subset of a complete metric space,  $\mathcal{M}$  is complete, so there is a point  $m \in \mathcal{M}$ , where  $m_j$  converges to. We'll prove the uniqueness in the next lecture.  $\square$

**Theorem 22.0.4** (Orthogonal Projections). *If  $M$  is a closed subspace of a Hilbert space  $\mathcal{H}$ , then for each  $h \in \mathcal{H}$ , there is a unique pair  $m \in M$  and  $n \in M^\perp$  such that  $h = m + n$  and  $\|h\|^2 = \|m\|^2 + \|n\|^2$ . Moreover, the maps  $P(h) = m$ ,  $Q(h) = n$  are linear and write  $m = Ph$ ,  $n = Qh$*

*Proof.* Fix  $h \in \mathcal{H}$ . Pick  $m \in M$  that is nearest to  $h$ , using precedent lemma. Let  $n = h - m$ . We will show  $n \in M^\perp$ . For any  $x \in M$ ,  $\alpha \in \mathbb{C}$ , consider

$$\|n - \alpha x\|^2 = \|n\|^2 - 2\Re(\alpha \langle x, n \rangle) + |\alpha|^2 \|x\|^2$$

and  $\|h\|^2 = \|m\|^2 + \|n\|^2$ . Suppose  $\langle x, n \rangle \neq 0$ . Choose  $\alpha = \frac{t}{\langle x, n \rangle}$  for  $t \in \mathbb{R}$ . Then

$$\|n - \alpha x\|^2 = \|n\|^2 - 2t + \frac{t^2 \|x\|^2}{|\langle x, n \rangle|^2}$$

For sufficiently small  $t$ , we have  $2t > \frac{t^2 \|x\|^2}{|\langle x, n \rangle|^2}$ . Then we'd get

$$\|n - \alpha x\|^2 < \|n\|^2$$

Replacing  $n$  with  $h - m$ , we get

$$\|h - (m + \alpha x)\|^2 < \|h - m\|^2$$

which contradicts the optimality of  $m$  for distance to  $h$ . We conclude  $\langle x, n \rangle = 0$ . This is true for each  $x \in M$ . Thus, we get  $n \in M^\perp$ .

Now to see that the choice of  $m$  (and  $n$ ) is unique, let  $h = m' + n'$ , with  $m' \in M, n' \in M^\perp$ . Then  $m + n = m' + n'$ , which implies

$$\underbrace{m - m'}_{\in M} = \underbrace{n' - n}_{\in M^\perp}$$

which forces  $m = m', n = n'$ , since  $M \cap M^\perp = \{0\}$

Now for the linearity of  $P, Q$ , let  $h = h_1 + \alpha h_2$ , where  $h_1 = m_1 + n_1, h_2 = m_2 + n_2$  for  $m_i \in M, n_i \in M^\perp$ . Then

$$h = \underbrace{m_1 + \alpha m_2}_{\in M} + \underbrace{n_1 + \alpha n_2}_{\in M^\perp}$$

This shows  $P : h \rightarrow m, Q : h \rightarrow n$  are linear maps. □

**Definition 22.0.5.** The maps  $P, Q$  above are called orthogonal projections onto  $M$  and  $M^\perp$ , respectively.

**Corollary 22.0.4.1.** *Let  $M$  be a proper closed subspace in a Hilbert space  $\mathcal{H}$ . Then  $M^\perp \neq \{0\}$ .*

**Exercise 22.0.1.** Let  $M \subset L^2(\mathbb{R})$  such that

$$M = \{f \in L^2(\mathbb{R}) : f(x) = \alpha_n \text{ for almost every } x \in [n, n+1)\}$$

## 22.1 Reisz Representation Theorem

**Theorem 22.1.1** (Reisz Representation Theorem). *Let  $\Lambda : \mathcal{H} \rightarrow \mathbb{C}$  be a continuous linear functional on a Hilbert space. Then there is a unique  $y \in \mathcal{H}$  such that  $\Lambda(x) = \langle x, y \rangle$*

*Proof.* Assume that  $\Lambda \neq 0$ . Then  $M = \text{Ker}(\Lambda)$  is a proper closed linear subspace of  $\mathcal{H}$ . Then so is  $M^\perp$ . Let  $0 \neq v, w \in M^\perp$ . Then  $\Lambda(v) \neq 0 \neq \Lambda(w)$ . Then since

$$\Lambda\left(\frac{v}{\Lambda(v)} - \frac{w}{\Lambda(w)}\right) = 0$$

forces  $\frac{v}{\Lambda(v)} - \frac{w}{\Lambda(w)} \in M \cap M^\perp = \{0\}$ . Hence  $v \in \text{span}(w)$ . Thus we see that  $M$  has co-dimension 1. i.e  $M^\perp$  has dimension 1.  $\square$