Functional Analysis - MATH7320

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Contents

Chapter 1

Textbook : A Course in Functional Analysis, John Conway
Functional analysis is the study of Topological Vector Spaces.

Definition 1.0.1. Let X be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A seminorm on X is a map $\|\cdot\| : X \to [0, \infty)$ such that

- $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{F}, \forall x \in X$
- $\|x + y\| \leq \|x\| + \|y\|$

In addition if $\forall x \neq 0, \|x\| \neq 0$, we say $\|\cdot\|$ is a norm on X

Norm induces a metric $d(x, y) = \|x - y\|$

Note. Let X be a normed space. Then the maps

- $+ : X \times X \to X : (x, y) \to x + y$
- $\cdot : \mathbb{F} \times X \to X : (\alpha, x) \to \alpha x$

are continuous.

Hence every normed space is a topological vector space.

Example 1.0.1. $\mathbb{F}^n$ with ℓ_p-norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_p = \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}}$$

Example 1.0.2. $\mathbb{F}^n$ with ℓ_∞-norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\infty = \max\{|a_i|\}$$

Example 1.0.3. Consider $C_{00} = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n = 0$ except for finitely many $n \in \mathbb{N}\}$ which can be identified by collection of functions $f : \mathbb{N} \to \mathbb{F}$ with finite support.

Then

$$\|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}$$

is a norm on C_{00}

Proposition 1.0.1. *Let X, Y be normed space, and let $T : X \to Y$ be linear. Then the following are equivalent.*

- *T is continuous*
- *T is continuous on 0*
- *T is continuous on any point $x \in X$*
- *$\exists M > 0$ such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$*

Proof. ($1 \implies 2$) It is clear that if T is continuous, then it is continuous at 0 from the definition of continuity.

($2 \implies 3$) Let $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in X that converge to x. Then the sequence $\{y_n = x_n - x\}$ converge to zero by the algebra of limits. By the continuity of T at zero, $\{T(y_n) = T(x_n) - T(x)\}$ converge to 0. Therefore $T(x_n) \to T(x)$. And this shows T is sequentially continuous at $x \in X$. Since the space is a metric space, sequential continuity is equivalent to continuity.

($4 \implies 2$) Let $x \in X$. Then $\|T(0) - T(x)\| = \|T(x)\| \leq M\|x\| = M\|0 - x\|$. Hence T is continuous at 0.

($3 \implies 1$)

($2 \implies 4$)

□

Example 1.0.4. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be defined as $T(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, 0, \dots, 0)$. Is T convergent for any norm $\|\cdot\|_1, \|\cdot\|_2$ in the domain and range?

Proof. verify □

Example 1.0.5. Consider identity function $I : C_{00} \to C_{00}$. Let the norm in domain be $\|\cdot\|_\infty$ and that in range be $\|\cdot\|_1$. Is the function continuous? What if the norms in domain and range are switched?

Proof. verify □

Note. Let X be a space with two norms $\|\cdot\|_1, \|\cdot\|_2$. When is the two norms topologically equivalent?

When $\exists M, M'$ such that $\|x\|_1 \leq M\|x\|_2$ and $\|x\|_2 \leq M'\|x\|_1$ Equivalently, when the identity map between the two spaces with their respective norms are bi-continuous. (See 4th equivalent statement of previous proposition)

Theorem 1.0.1. *Let X and Y be normed spaces, and $T : X \to Y$ be linear. Assume X is finite dimensional. Then T is continuous.*

Proof. Since $T(X) \leq Y$ is finite dimensional, we may assume without loss of generality that Y is also finite dimensional and T is onto. Let $\{x_1, x_2, \dots x_n\}$ be a basis for X. Define another norm on X as follows. For every $x = \sum_{i=1}^{n} \alpha_i x_i \in X$,

$$\|x\|' = \sum_{i=1}^{n} |\alpha_i|(\|T(x_i)\| + \|x_i\|)$$

verify that this is a norm. Then for every $x \in X$, we have

$$\|T(x)\| \leq \sum_{i=1}^{n} |\alpha_i|\|T(x_i)\| \leq \|x\|'$$

Hence T is bound with respect to the norm $\|\cdot\|'$ on X, since all norms are equivalent on X. Therefore T is continuous w.r.t to the original norm on X. $\square$

Corollary 1.0.1.1. *Let X be a finite dimensional vector space. Then any two norms in X are equivalent.*

Proof. Let $\{e_1, e_2, \dots e_n\}$ be a basis for X. For each $x = \sum_{i=1}^{n} \alpha_i e_i \in X$, define

$$\|x\|_\infty = \max\{|\alpha_i|\}$$

Then $\|\cdot\|_\infty$ is a norm and we'll show every norm on X is equivalent to this norm. Let $\|\cdot\|$ be an arbitrary norm on X. For each $x = \sum_{i=1}^{n} \alpha_i e_i \in X$, we have

$$\begin{aligned}
\|x\| &= \|\sum_{i=1}^{n} \alpha_i e_i\| \\
&\leq \sum_{i=1}^{n} |\alpha_i|\|e_i\| \\
&\leq \max\{|\alpha_i|\} \sum_{i=1}^{n} \|e_i\| \\
&\leq \|x\|_\infty \sum_{i=1}^{n} \|e_i\|
\end{aligned}$$

Therefore the identity map $I : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|)$ is continuous. Since the set $K = \{x \in X \ : \ \|x\|_\infty \leq 1\}$ is compact, K is also compact in $(X, \|\cdot\|)$ and the restriction $\mathrm{Id}|_K$ is also a homeomorphism. verify In particular, the set $\{x \in X \ : \ \|x\|_\infty < 1\}$ is an open neighborhood of $0 \in (X, \|\cdot\|)$ By the Heine-Borel theorem, the unit ball $B = \{x \in X \ : \ \|x\|_2 \leq 1\}$ is compact. Hence B is compact in $(X, \|\cdot\|)$. verify the last line. $\square$

Chapter 2

2.1 continues

Corollary 2.1.0.1.

- *Every finite dimensional normed space is complete*
- *If X is a normed space and Z is a finite dimensional subspace of X, then Z is the closed in X*

Proof.

- Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Let $\|\cdot\|_2$ be the euclidean norm on X. Then by the theorem above there exists $M \in \mathbb{R}$ such that $\frac{1}{M}\|x\|_2 \leq \|x\| \leq M\|x\|_2$ for all $x \in X$.

  Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|)$. Then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy in $(X, \|\cdot\|_2)$. Since the latter space is complete, so is $(X, \|\cdot\|)$.
- verify

□

Note. If $T, S : X \to Y$ are continuous linear maps between normed space, then $T + S$ is also continuous. Also, $\forall \alpha \in \mathbb{F}$, αT is continuous.

Thus the space $B(X, Y)$ of all continuous linear maps from X to Y is a subspace of all linear maps between X and Y.

Definition 2.1.1. Let X, Y be normed spaces. and $T \in B(X, Y)$. We define the operator norm of T as

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} \;:\; x \in X, x \neq 0\right\}$$

Lemma 2.1.1. *Let $T \in B(X, Y)$. Then the following are equivalent*

- $\|T\|$
- $\sup\{\|T(x)\| \;:\; x \in X, \|x\| \leq 1\}$

- $\sup\{\|T(x)\| \; : \; x \in X, \|x\| < 1\}$
- $\inf\{M \geq 0 \; : \; \|T(x)\| \leq M\|x\|, \forall x \in X\}$

Proof. verify $\square$

Proposition 2.1.1. *The operator norm is a norm in* $B(X, Y)$

Proof. verify $\square$

Chapter 3

Hahn Banach Theorem

Lemma 3.0.1. *Let X be a complex normed space. Let $f : X \to \mathbb{R}$ be an $\mathbb{R}$-linear functional. Then $g : X \to \mathbb{C}$ defined as $g(x) = f(x) - if(ix)$ is $\mathbb{C}$-linear*

Conversely if $g : X \to \mathbb{C}$ is a $\mathbb{C}$-linear map, then $f := \Re \circ g : X \to \mathbb{R}$ is $\mathbb{R}$-linear.

Moreover $\|f\| = \|g\|$.

Proof. We'll prove that $\|f\| = \|g\|$ and leave the rest for the reader (verify).

Since $|f(x)| \leq |g(x)|$, for all $x \in X$, it is easy to see that $\|f\| \leq \|g\|$. Conversely, $\forall \epsilon > 0, \exists x_o \in X$ with $\|x_o\| = 1$ such that $|g(x_o)| > \|g\| - \epsilon$. If $g(x_o) = re^{i\theta}$, take $\alpha = e^{-i\theta}$. Then $f(\alpha x_o) = \Re(re^{-i\theta}e^{i\theta}) = r = g(\alpha x_o)$. Then $\|f\| \geq |f(\alpha x_o)| = |g(\alpha x_o)| = |\alpha||g(x_o)| = |g(x_o)| > \|g\| - \epsilon$. Since ϵ is arbitrary, this gives $\|f\| \geq \|g\|$ □

Theorem 3.0.1 (Hahn-Banach Extension Theorem)**.** *Let X be a normed space over $\mathbb{R}$, Z be a subspace of X and let $\phi : Z \to \mathbb{R}$ be a continuous linear functional. Then there exists a linear functional $\psi : X \to \mathbb{R}$ such that $\psi|_Z = \phi$ and $\|\phi\| = \|\psi\|$.*

Proof. Assume $\|\phi\| = 1$ (If this is not the case, we can always scale the functional down to norm 1). Now we'll extend ϕ from Z to a subspace with one dimension higher than Z, preserving the norm. Let $x_o \in (X \setminus Z)$ and $Y = \text{Span}\{\{x_o\} \cup Z\}$ be the set one dimension higher than Z. Assume ψ is the extension of ϕ to Y. Then ψ will be completely characterized, if we know the value of $\psi(x_o)$. We look to see what real values we can assign $\psi(x_o)$ satisfying our conditions. Let $y = z + x_o \in Y$ where $z \in Z$ (We must be taking an arbitrary element $y = z + \alpha x_o \in Y$, but if we know the image of $y = z + x_o$ for all $z \in Z$ under ψ, then we can get the image of $y = z + \alpha x_o$ for any $\alpha \in \mathbb{R}$ by scaling). Norm preserveness demands that for all $z \in Z$, we must have

$$-\|z + x_o\| \leq \psi(y) = \psi(z) + \psi(x_o) \leq \|z + x_o\|$$

Since ψ agrees with ϕ on Z, this is equivalent to

$$-\phi(z) - \|z + x_o\| \leq \psi(x_o) \leq \|z + x_o\| - \phi(z) \tag{3.1}$$

Moreover since we normalized ϕ to have norm 1, we know ψ must also have norm 1. Then by triangle inequality, we get that for all $a, b \in Y$

$$\psi(a) - \psi(b) = \psi(a - b) \leq \|a - b\| = \|(a + x_o) - (b + x_o)\| \leq \|a + x_o\| + \|b + x_o\|$$

which gives

$$-\psi(b) - \|b + x_o\| \leq \|a + x_o\| - \psi(a)$$

Since this inequality is true for all $a, b \in Y$, taking supremum and infimum over all the possible $a, b \in Y$ preserves the inequality. Hence we get

$$\sup_{b \in Y} \Big\{ -\psi(b) - \|b + x_o\| \Big\} \leq \inf_{a \in Y} \Big\{ \|a + x_o\| - \psi(a) \Big\} \tag{3.2}$$

Substituting $a = b = z$ in Equation 3.2 guarantees the existence of $\psi(x_o)$ satisfying Equation 3.1. Hence we get an extension (namely ψ) of ϕ to Y preserving the norm. Since Z was an arbitrary subspace of X, this is true for all such subspaces of X.

Now we will employ Zorn's lemma to get an extension of ϕ from Z to the whole of X. Consider the collection of all linear extensions of ϕ, i.e

$$\mathcal{S} = \Big\{ (\psi_Y, Y) \ : \ Z \subset Y, \ \psi_Y|_Y = \phi, \ \|\psi_Y\| = \|\phi\| \Big\}$$

Then we define a partial order in the collection $\mathcal{S}$ as $(\psi_X, X) \leq (\psi_Y, Y)$ if and only if $X \subset Y$ and $\psi_Y|_X = \psi_X$. Now let $\mathscr{C}$ be a chain in $\mathcal{S}$. Consider the set

$$\tilde{Y}_{\mathscr{C}} = \bigcup_{(\psi_Y, Y) \in \mathscr{C}} Y$$

and the map $\psi_{\tilde{Y}_{\mathscr{C}}} : \tilde{Y}_{\mathscr{C}} \to \mathbb{R}$ defined as

$$\psi_{\tilde{Y}_{\mathscr{C}}}(x) = \psi_Y(x), \quad \text{where } x \in Y, \text{ for } (\psi_Y, Y) \in \mathscr{C}$$

To see this map is well defined, assume $x \in X$ and $x \in Y$ for $(\psi_X, X), (\psi_Y, Y) \in \mathscr{C}$. Then either $(\psi_X, X) \leq (\psi_Y, Y)$ or $(\psi_Y, Y) \leq (\psi_X, X)$ since $\mathscr{C}$ is totally ordered. WLOG assume $(\psi_X, X) \leq (\psi_Y, Y)$, then by definition we get that $\psi_Y|X = \psi_X$. This gives that $\psi_Y(x) = \psi_X(x)$. Hence we get that $\psi_{\tilde{Y}_{\mathscr{C}}}$ is well defined. In a similar fashion we can verify that $\psi_{\tilde{Y}_{\mathscr{C}}}$ is a linear functional.

Now we claim that $(\tilde{Y}_{\mathscr{C}}, \psi_{\tilde{Y}_{\mathscr{C}}})$ is the upper bound of the chain $\mathscr{C}$. By the definition of $\tilde{Y}$, we see that there cannot be an element (ψ_Y, Y) in the chain $\mathscr{C}$, with $\tilde{Y} \subset Y$. Hence the only remaining thing to show is that for all $(\psi_X, X) \in \mathscr{C}$, we have $\psi_{\tilde{Y}_{\mathscr{C}}}|_X = \psi_X$. But this also follows from the definition of the map $\psi_{\tilde{Y}_{\mathscr{C}}}$.

Since $\mathscr{C}$ was taken to be an arbitrary chain in the collection $\mathcal{S}$, we get that every chain in $\mathcal{S}$ has an upper bound. Then by Zorn's lemma, the collection $\mathcal{S}$ has a maximal element (ψ, Y). We claim that in this maximal element, $Y = X$. If not, we can extend ψ to a space one dimension above Y like we did in the beginning contradicting the maximality of (ψ, Y). Hence the maximal element is (ψ, X). This by definition of the collection S, is the required extension for (ϕ, Z). □

Remark 3.0.1. Note that in the proof above, we only used the scaling property and triangle inequality of the norm, hence we can relax the condition for norm and replace it with a seminorm, without messing up the proof.

Theorem 3.0.2 (Hahn-Banach Extension Theorem for $\mathbb{C}$)**.** *Same statement of Theorem 3.0.1 with only the field changed to* $\mathbb{C}$*.*

Proof. Consider X as a normed linear space over $\mathbb{R}$. Let $f = \Re \circ \phi : Z \to \mathbb{R}$ and apply Theorem 3.0.1 on f to get a real linear functional $\tilde{f} : X \to \mathbb{R}$ with the required properties. Now we claim that $\tilde{\phi}$ defined as $\tilde{\phi}(x) = \tilde{f}(x) - i\tilde{f}(ix)$ is the required extension.

First we show $\tilde{\phi}_Z = \phi$. To see this first we notice that if ϕ can be written as $\phi(x) = f(x) + ig(x)$ where f, g are real valued functionals, then since $-\phi(x) = i\phi(ix) = if(ix) - g(ix)$. Hence $0 = \phi(x) - \phi(x) = (f(x) - g(ix)) + i(g(x) + f(ix))$. Since real part and imaginary part must be equal to 0, we get that $g(x) = -f(ix)$. Therefore we get $\phi(x) = f(x) - if(ix)$. Now we get $\tilde{\phi}|_Z = \phi$ immediately since $\tilde{f}|_Z = f$. To finish the proof, we also have to show that $\|\phi\| = \|\tilde{\phi}\|$. But this follows easily from Lemma 3.0.1 as $\|\phi\| = \|f\| = \|\tilde{f}\| = \|\tilde{\phi}\|$. □

Remark 3.0.2. It is quite natural to be confused about the well defineness of the expression $f(ix)$ when we are considering X as a normed linear space over $\mathbb{R}$ in the beginning of the proof. But note that since X initially was a complex normed linear space, viewing it as a space over $\mathbb{R}$ doesn't change or remove any elements from the space. Hence $ix \in X$ even though X is viewed as a real normed linear space.

Chapter 4

Definition 4.0.1. A sublinear map is a functoin $\rho : X \to \mathbb{R}$ with the properites

- $\rho(rx) = r\rho(x), \forall r \in \mathbb{R}$
- $\rho(x + y) \leq \rho(x) + \rho(y)$

Definition 4.0.2. Let X be a normed space. Then the dual of X, denoted by X^*, is the space $B(X, \mathbb{F})$

Lemma 4.0.1. *Let X be a normed space and $x \in X$. Then $\exists f \in X^*$ such that*

$$\|f\| = 1 \text{ and } f(x) = \|x\|$$

Proof. Let $Z = \mathrm{Span}\{x\}$. Define $g : Z \to \mathbb{F}$ as $g(\alpha x) = \alpha\|x\|$. Then $\|g\| = 1$. By the Hahn Banach theorem, g has an extension f which preserve the norm and extends g to X. □

Corollary 4.0.0.1. *Let X be a normed space and $x \in X$, then we have*

$$\|x\| = \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\}$$

Proof. If f is any linear functional with $\|f\| \leq 1$, then $|f(x)| \leq \|f\|\|x\| = \|x\|$. Hence $\|x\| \leq \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\}$. Now let f_x be the functional we get from Lemma 4.0.1. Then $f_x \in X^*$ and $\|f_x\| = 1$ with $f_x(x) = |f(x)| = \|x\|$. Hence we get that the inequality is actually an equality, and this proves the corollary. □

Definition 4.0.3. For every $x \in X$, define a linear map $\hat{x} : X^* \to \mathbb{F}$ by $\hat{x}(f) = f(x)$

Theorem 4.0.1. *For every $x \in X$, $\hat{x} \in (X^*)^*$. The map $\rho : x \to \hat{x}$ is an isometric linear map.*

Proof. The fact that $\hat{x}$ is linear and bounded and the map $X \ni x \to \hat{x} \in X^{**}$ is linear follows from the definition of $f + g$ and λf.

By definition and Corollary 4.0.0.1

$$\begin{aligned}\|\hat{x}\| &= \sup\{|\hat{x}(f) \ : \ f \in X^*, \|f\| \leq 1\} \\ &= \sup\{|f(x)| \ : \ f \in X^*, \|f\| \leq 1\} \\ &= \|f\|\end{aligned}$$

□

Definition 4.0.4. A normed space X is said to be reflexive if the map $\rho : X \to X^{**} := x \to \hat{x}$ is surjective. (This is a stronger condition than $X \equiv X^{**}$)

Theorem 4.0.2. *There are isometric isomorphisms between*

- $(\mathbf{c}_0)^*$ *and* ℓ^1
-
- $(\ell^1)^*$ *and* ℓ^∞

Proof.

- Let $(x_n) \in \ell^1$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \to \mathbb{F}$ defined as

$$\phi_{(x_n)} : (y_n) \to \sum_{n \in \mathbb{N}} x_n y_n$$

We claim that $\phi_{(x_n)}$ is a continuous linear functional. But first we should see that the sum is well defined. Since $y_n \to 0$, there is an $N \in \mathbb{N}$ such that $|y_n| < 1$ for all $n \geq N$. Since

$$\Big|\sum_{i=N}^{\infty} x_n y_n\Big| \leq \sum_{i=N}^{\infty} |x_n||y_n| \leq \|(x_n)\|_1$$

we see that the sum is well defined and the map makes sense. Also since $(y_n) + (z_n) = (y_n + z_n) \in \mathbf{c}_0$ whenever $(y_n), (z_n) \in \mathbf{c}_0$, we get that

$$\sum_{n \in \mathbb{N}} x_n(y_n + z_n) = \sum_{n \in \mathbb{N}} x_n y_n + \sum_{n \in \mathbb{N}} x_n z_n$$

which shows the linearity of the map $\phi_{(x_n)}$.

Now we show that $\|\phi_{(x_n)}\| = \|(x_n)\|_1$. We immediately see that for $(y_n) \in c_0$ with $\|(y_n)\|_{\sup} = \sup_{n \in \mathbb{N}} y_n = 1$,

$$|\phi_{(x_n)}((y_n))| = \Big|\sum_{n \in \mathbb{N}} x_n y_n\Big| \leq \|(y_n)\|_{\sup}\Big(\sum_{n \in \mathbb{N}} |x_n|\Big) \leq \|(x_n)\|_1$$

which gives $\|\phi_{(x_n)}\| \leq \|(x_n)\|_1$. Now let $\theta_j \in [0, 2\pi)$ such that $|x_j| = e^{i\theta_j} x_j$. Now consider the sequence $s_m \in \mathbf{c}_0$ defined as $s_m = \sum_{j=1}^{m} e^{i\theta_j} e_j$, where e_j is the sequence with jth entry 1 and the rest of the entries 0. Since $(x_n) \in \ell_1$, for all $\epsilon \geq 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{i=N_\epsilon+1}^{\infty} |x_i| < \epsilon$$

Then since

$$|\phi_{(x_n)}(s_{N_\epsilon})| = \Big|\sum_{n=1}^{N_\epsilon} e^{i\theta_j} x_n\Big| = \sum_{i=1}^{N_\epsilon} |x_n| = \|(x_n)\| - \sum_{i=N_\epsilon+1}^{\infty} |x_n| \geq \|(x_n)\| - \epsilon$$

and $\epsilon > 0$ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|$

Hence we see that the map $(x_n) \to \phi_{(x_n)}$ is an isometric linear map. Now for surjectivity, let $\phi \in \mathbf{c}_0^*$. We claim that the sequence $(y_n) = (\phi(e_n)) \in \ell^1$ and $\phi = \phi_{(y_n)}$. Let $\theta_j \in [0, 2\pi)$ such that $e^{i\theta_j} y_j = |y_j|$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{j=1}^{N} |\phi(e_j)| &= \sum_{j=1}^{N} e^{i\theta_j} \phi(e_j) \\
&= \phi\Big(\sum_{j=1}^{N} e^{i\theta_j} e_j\Big) \\
&\leq \|\phi\| \Big\| \sum_{j=1}^{N} e^{i\theta_j} e_j \Big\| \\
&= \|\phi\|
\end{aligned}$$

Since this is true for all $N \in \mathbb{N}$, taking the limits as $N \to \infty$, the inequality is preserved and we get that $(y_n) \in \ell^1$. Moreover $\phi = \phi_{(y_n)}$ follows from the definition of $\phi_{(x_n)}$. Hence we get that $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$.

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- The proof of this will be extremely similar to what we attempted before when we proved $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$. Let $(x_n) \in \ell^\infty$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \to \mathbb{C}$ defined as
$$\phi_{(x_n)} : (y_n) \to \sum_{n \in \mathbb{N}} x_n y_n$$

By a similar way as we did in the above equivalence we see that $\phi_{(x_n)}$ is linear. Moreover since

$$\Big|\sum_{n\in\mathbb{N}} x_n y_n\Big| \leq \|(x_n)\|_\infty \Big|\sum_{n\in\mathbb{N}} y_n\Big| = \|(x_n)\|_\infty \|(y_n)\|_1$$

we see that $\|\phi_{(x_n)}\| \leq \|(x_n)\|_\infty$. To get the reverse inequality, Let $\|(x_n)\|_\infty = M$, then for any $\epsilon > 0$, there exist some x_k in the sequence (x_n) such that $|x_k - M| < \epsilon$. Now consider the sequence $e_k \in \ell^1$ with kth entry 1 and all the rest of them 0. We get that

$$|\phi_{(x_n)}(e_k)| = |x_k| \geq \|(x_n)\|_\infty - \epsilon$$

Since ϵ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|_\infty$. Hence the map $(x_n) \to \phi_{(x_n)}$ is an isometry. To show that it is indeed a bijection, assume $\phi \in (\ell^1)^*$, then consider the sequence $y_n = \phi(e_n)$. Since ϕ is continuous, it is bounded above by $\|\phi\|$ and we get that $y_n \leq \|\phi\|$. Therefore $(y_n) \in \ell^\infty$. Moreover we can verify like above that $\phi = \phi_{(y_n)}$ from the definition of $\phi_{(y_n)}$. Hence we get $(\ell^1)^* \cong^{\text{iso}} \ell^\infty$.

□

Theorem 4.0.3. *Let $1 < p < \infty$, and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$*

Proof. Let $(a_n) \in \ell^p, (b_n) \in \ell^q$, then $\sum_{n\in\mathbb{N}} a_n \bar{b_n}$ is the map to check for isometric isomorphism. Use Holder's inequality as needed. verify □

Theorem 4.0.4. *There exists $\phi \in (\ell^\infty)^*$ satisfying the following*

1 $\forall (a_n) \in \ell^\infty$ *with* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, $\phi((a_n)) \geq 0$

2 If (a_n) *is convergent, then* $\phi((a_n)) = \lim_{n\to\infty} a_n$

3 If $(a_n) \in \ell^\infty$ *and* $b_n = a_{n+1}$, *then* $\phi((b_n)) = \phi((a_n))$

Moreover such ϕ is called a Banach limit.

Proof. We'll prove this later. □

Corollary 4.0.4.1. *ℓ^1 is not reflexive*

Proof. Let $\phi \in (\ell^\infty)^*$ be a Banach limit. FTOC, assume $\exists f = (\alpha_n) \in \ell^1$ such that

$$\phi((a_n)) = \sum_{i=1}^{\infty} a_n \overline{\alpha_n}$$

Then for all $m \in \mathbb{N}$, $\overline{\alpha_m} = \phi(\delta_m) = 0$, where $\delta_m = (0, 0, \ldots, 1, 0, 0, \ldots)$. But this contradicts since we assumed $\phi \neq 0$ by the Hahn Banach rextension from c_0 □

Lemma 4.0.2. *Let $\psi \in (\ell^\infty)^*$. then the following are equivalent.*

1 $\|\psi\| = \psi((1, 1, 1, \ldots))$

2 If $(a_n) \in \ell^\infty$ *with* $a_n \geq 0, \forall n \in \mathbb{N}$. *Then* $\psi((a_n)) \geq 0$

Proof. ($1 \implies 2$) FTSOC assume $\exists (a_n) \in \ell^\infty$, $\psi((a_n)) < 0$. WLOG, assume $|a_n| \leq 1$ for all $n \in \mathbb{N}$. let $b_n = 1 - a_n$. Then $0 \leq b_n \leq 1$ and

$$\psi((b_n)) > \psi((1, 1, 1, \ldots)) - \psi((a_n)) > \psi((1, 1, 1, \ldots))$$

So

$$\|\psi\| \geq |\psi((b_n))| \geq \psi((1, 1, \ldots))$$

($2 \implies 1$) Let $(a_n) \in \ell^\infty$ with $|a_n| \leq 1$, then $0 \leq 1 - a_n$. So $\psi((1 - a_n)) \geq 0$ and therefore $\psi((1, 1, 1, \ldots)) \geq \psi((a_n))$. Similarly $\psi((-a_n)) \leq \psi((1, 1, 1, \ldots))$ which gives $|\psi((a_n))| \leq \psi((1, 1, 1, \ldots))$ □

Chapter 5

Theorem 5.0.1. *There exists $\phi \in (\ell^\infty)^*$ satisfying the following*

1 $\forall (a_n) \in \ell^\infty$ *with* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, $\phi((a_n)) \geq 0$

2 If (a_n) *is convergent, then* $\phi((a_n)) = \lim_{n\to\infty} a_n$

3 If $(a_n) \in \ell^\infty$ *and* $b_n = a_{n+1}$, *then* $\phi((b_n)) = \phi((a_n))$

Moreover such ϕ *is called a Banach limit.*

Proof. Let $S : \ell^\infty(\mathbb{R} \to \ell^\infty(\mathbb{R})$ and $T = I - S$ where I is the identity map. Also let $V = \text{Range}(T) + c$ where $c \in \mathbf{c}$, the set of convergent sequences.

Define $\phi : V \to \mathbb{R}$, $\phi(a_n - a_{n+1} + x_n) = \lim_{n\to\infty} x_n$.

- Claim 1: ϕ is well defined
- Claim 2: $\|\phi\| = 1$

Assuming the claims, by Hahn Banach, ϕ extends to $\tilde{\phi} \in \ell^\infty(\mathbb{R})$ with $\|\tilde{\phi}\| = 1$. Then by the last lemma we get $\tilde{\phi}((y_n)) \geq 0$ for all $(y_n) \in ell^\infty(\mathbb{R})$ with $y_n \geq 0$ □

Proof of Claim 1. Suppose that $(a_n) \in \ell^\infty$ is a sequence such that $a_n - a_{n+1}$ converges, say $a_n \to a_{n+1} \to \alpha$. If $\alpha > 0$, then $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n - a_{n+1} > \frac{\alpha}{2}$. So $a_N > \frac{\alpha}{2} + a_{N+1} > \ldots > k\frac{\alpha}{2} + a_{N+k}$. So for all $k \in \mathbb{N}$, $a_N - a_{N+k} = k\frac{\alpha}{2} \to \infty$ contradicting our assumption that $a_n - a_{n+1}$ converges.

Now assume that $(a_n), (b_n) \in \ell^\infty(\mathbb{R})$ with $(x_n), (y_n) \in \mathbf{c}$ such that $a_n - a_{n+1} + x_n = b_n - b_{n+1} + y_n$. Then $(a_n - b_n) - (a_{n+1} - b_{n+1}) = y_n - x_n$. Then since RHS is a convergent limit, LHS must be convergent, which we get from above that it must converge to zero. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ □

Proof Claim 2. verify □

To complete the proof, define $\Psi : \ell^\infty \to \mathbb{C}$ by $\Psi((a_n + ib_n)) = \tilde{\phi}(a_n) + i\tilde{\phi}(b_n)$ verify

5.1 Quotient Spaces

Definition 5.1.1. Let X be a normed space and $Y \leqslant X$ be a closed subspace. For every $x \in X$, define

$$\|x + Y\| = \inf\{\|x + y\| \ : \ y \in Y\}$$

Lemma 5.1.1. *This defines as norm on $\frac{X}{Y}$. If X is complete, then $\frac{X}{Y}$ is complete.*

Proof. Obviously, $\|x+Y\| \geq 0$ for all $x \in X$, and $\|x+z+Y\| \leq \|x+Y\|+\|y+Y\|$. Similarly, we can also show homogeneity.

Now assume $x \in X$ is such that $\|x+Y\| = 0$. Then there is a sequence $(y_n) \in Y$ such that $\|x - y_n\| \to 0$, that is $y_n \to x$. Since Y is closed, we get $x \in Y$.

To show the second part of the lemma, consider the sequence $(x_n + Y) \in X/Y$ such that $\sum_{n\in\mathbb{N}} \|x_n - Y\| < \infty$. For each $n \in \mathbb{N}$, choose $y_n \in Y$ such that

$$\|x_n + y_n\| \leq \|x_n + Y\| + \frac{1}{2^n}$$

Then $\sum_{n\in\mathbb{N}} \|x_n + y_n\| \leq \infty$. Since X is complete, the sequence $\sum_{n\in\mathbb{N}} x_n + y_n$ converges to say $z \in X$. Then

$$\begin{aligned}
\|(z + Y) - \sum_{n=k}^{n}(x_n + Y)\| &= \|\Big(z - \sum_{n=k}^{n} x_n\Big) + Y\| \\
&= \|\Big(z - \sum_{i=1}^{k}(x_n + y_n)\Big) + Y\| \\
&\leq \|\Big(z - \sum_{i=1}^{k}(x_n + y_n)\Big)\|
\end{aligned}$$

which converges to 0 as $k \to \infty$ $\square$

Lemma 5.1.2. *The canocial map, $q : X \to \frac{X}{Y}$ is a continuous open map. A subset $E \subset X/Y$ is open iff $q^{-1}(E) \subset X$ is open.*

Proof. Since $\|x+Y\| \leq \|x\|$, for all $x \in X$, we see that the map q is a contraction. Thus for all open $E \subset X/Y$, we get $q^{-1}(E)$ is open.

Conersely, assume that $A \subset X$ is open. Let $x \in A$ and $r > 0$ such that $B_r(a) \subset A$. Let $z \in X$ such that $\|q(a) - q(z)\| < r$. So, $\|(a - z) + Y\| < r$. Then $\exists y \in Y$ such that $\|a - z - y\| < r$. So $z + y \in B_r(a), q(z + y) = q(z) \in q(B_r(a))$. So $B_r(q(a)) \subset q(B_r(a)) \subset q(A)$. Thus $q(A)$ is open. $\square$

Chapter 6

Theorem 6.0.1 (Open Mapping Theorem). *Let X and Y be Banach spaces and $T : X \to Y$ be a surjective bounded linear map. Then T is an open map i.e $T(E)$ is open in Y i.e if $E \subset X$, then $T(E)$ is open in Y.*

Steps of proof. See Prof. Blecher's Notes on Functional Analysis

Use Baire category theorem to show that $\overline{T(B_r(0))}$ has a non-empty interior.

- Then use linearity of T to show that $0 \in \overline{T(B_{2r}(0))}$.

$\square$

Proof. Since Y is complete, by the Baire catergory theorem it is of the second category. Let $r \geq 0$, then

$$\begin{aligned} Y &= T(X) \\ &= T\Big(\bigcup_{n=1}^{\infty} B_{nr}(0)\Big) \\ &= T\Big(\bigcup_{n=1}^{\infty} nB_r(0)\Big) \\ &= \bigcup_{n=1}^{\infty} n\overline{T(B_r(0))} \end{aligned}$$

Then by BCT, there exist some $n \in \mathbb{N}$ such that $\text{int}(\overline{T(B_r(0))}) \neq \emptyset$. Let $y_0 \in \text{int}(\overline{T(B_r(0))})$. So there exists $\epsilon > 0$, such that $B_\epsilon(y_0) \subset \overline{T(B_r(0))}$. Let $w \in B_Y(0, \epsilon)$. Then $y_0 + w \in B_Y(y_0, \epsilon)$, and $\exists (x_n) \subset B_X(0, r)$ such that $T(x_n) \to y_0 + w$. Also $\exists (z_n) \in B_X(0, r)$ such that $T(z_n) \to y_0$. Then $T(x_n - z_n) \to w$, so $w \in \overline{T(B_X(0, 2r))}$. Since w was an arbitrary element in $B_Y(0, \epsilon)$ we see that $B_Y(0, \epsilon) \subset \overline{T(B_X(0, 2r))}$. So $0 \in \text{int}(\overline{T(B_X(0, s))})$ for $s > 0$.

Now fix $t > 0$ and let $y_0 \in \overline{T(B_X(0,t))}$. By the above there exists $\epsilon > 0$ such that $B_Y(0,\epsilon) \subset \overline{T(B_X(0,\frac{t}{2}))}$. Then $(y_0 + B_Y(0,\epsilon)) \cap T(B_X(0,\frac{t}{2})) \neq \emptyset$. So $\exists x \in B_t(0)$ such that $T(x_1) = y_0 - y_1$ where $y_1 \in B_Y(0,\epsilon) \subset \overline{T(B_X(0,\frac{t}{2}))}$.

Similarly $\exists y_2 \in \overline{T(B_{\frac{t}{4}}(0))}$ and $x_2 \in B_{\frac{t}{2}}(0)$ such that $T(x_2) = y_1 - y_2$. Thus inductively we can choose $y_{n+1} \in \overline{T(B_{\frac{t}{2^{n+1}}}(0))}$ and $x_{n+1} \in B_{\frac{t}{2^n}}(0)$ such that $T(x_{n+1}) = y_n - y_{n+1}$. Now since we constructed nicely, $\sum_{i=0}^{\infty} x_n$ converge. verify. Moreover for all $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} T(x_n) = y_0 - y_N$$

Also notice that $y_n \to 0$. Hence $y_0 = \lim_{N\to\infty}(y_0 - y_N) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n T(x_n) = \lim_{N\to\infty} T(\sum_{n=1}^{N} x_n) = T(x) \in T(B_{2t}(0))$. So

$$\overline{T(B_t(0))} \subset T(B_{2t}(0))$$

Now to complete the proof, let E be an open subset of X. Let $x_0 \in E$ be such that $y_0 = T(x_0)$. Let $\epsilon > 0$ be such that $x_0 + B_\epsilon(0) = B_\epsilon(x_0) \subset E$. So $y_0 + T(B_\epsilon(0)) = T(B_\epsilon(x_0)) \subset T(E)$. By the above $\exists \delta > 0$ such that $B_\delta(0) \subset T(B_\epsilon(0))$ □

Find examples where this fails if we slack the conditions

Corollary 6.0.1.1. *Let X and Y be Banach spaces and $T : X \to Y$ be a bijective bounded linear map. Then $T^{-1} : Y \to X$ is bounded.*

Proof. verify □

Theorem 6.0.2 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a linear map. Then T is bounded if and only if graph of T, defined as $g(T) = \{(x, T(x)) : x \in X\}$ is closed in the product topology of $X \times Y$.*

Proof. Define the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ on $X \times Y$. Then $X \times Y$ with this norm is a Banach space. verify.

Assume T is continuous. Then if $(x_n, T(x_n))$ is Cauchy in $g(T)$ then x_n and $T(x_n)$ must be cauchy in X and Y respectively. By the completeness of the spaces X and Y, we get $x_n \to x \in X$ and $T(x_n) \to y \in Y$. Moreover by continuity of T, we get $T(x_n) \to x$. Since the Banach space is Hausdorff, we get $y = x$ and that $(x, T(x)) \in g(T)$ making it closed.

Conversely, define $S : X \to g(T)$ as $S(x) = (x, T(x))$. S is linear and bijective. Assume $g(T)$ is closed, hence a Banach space. Observe that $S^{-1} : g(T) \to X$ is bounded(contractive). By the open mapping theorem, S is bounded. Assume $x_n \to z$. So $S(x_n) \to S(z)$. Then $(x_n, T(x_n)) \to (z, T(z))$, which gives $T(x_n) \to T(z)$. □

Chapter 7

Example 7.0.1. Let X be a vector space and let $f : X \to \mathbb{C}$ be a linear map. Define $\phi : X \to R^+ := \phi(x) = |f(x)|$. Then ϕ is a seminorm.

Remark 7.0.1. Let X be a TVS and $A \subset X^*$. We denote by $\sigma(X, A)$, the topology on X defined by A. (initial topology). Recall that $\sigma(X, A)$ is Hausdorff if and only if A separate points of X.

$\sigma(X, X^*)$ is called the weak topology on X.

Also recall that $X \hookrightarrow X^{**}$ by the evaluation maps. Hence we can view X as a subset X^{**}. And with this identification, we call $\sigma(X^*, X)$ the weak $*$ topology on X^*

Definition 7.0.1. Let S be any set. Let I be a directed set. A net in S indexed by Λ is a function $f : \Lambda \to S$. We denote the net by $(x_\lambda)_{\lambda\in\Lambda}$.

In addition if S is a topological space, we say a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in S$ if for all open set U in S with $x \in U$, there exists an $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$ we have $x_\lambda \in U$.

See how nets generalize the sequences to topological spaces from the metric space. For example, consider the definition of closedness in a metric space and a topological space. Find what exact property of the metric space makes it enough to be indexed by a countable totally ordered set for openess.

Remark 7.0.2. By definition, a basis of open neighborhoods of a point $x_0 \in X$ in $\sigma(X, A)$ is given

$$\bigcup_{\substack{p_1,p_2,\dots,p_n\in A \\ \epsilon_1,\epsilon_2,\dots,\epsilon_n>0}} \bigcap_{i=1}^{n} \{z \in Z | p_k(z - x_0) < \epsilon_k\}$$

So the basis in a weak topology is

$$\bigcup_{f_1,f_2,\dots,f_n\in X^* \ \epsilon_1,\epsilon_2,\dots,\epsilon_n>0} (x_0) = \{z \in Z \Big| |f_k(z - x_0)| < \epsilon_k, \text{ for all } k = 1, 2, \dots n\}$$

Example 7.0.2. Let S be a topological space, $E \subset S$, $x_0 \in \overline{E}$. Then there is a net $(x_\alpha) \subset E$ such that $x_\alpha \to x_0$.

Proof. Consider the collection $\mathscr{T}$ of all open sets of S that contain x_0. Order $\mathscr{T}$ by the reserve inclusion. That is $A \leq B$ if $B \subset A$. This makes $\mathscr{T}$, a directed set. Now each $\lambda \in \mathscr{T}$ has a nonempty intersection with E being an open set containing the limit point x_0 of E. For each $\lambda \in \mathscr{T}$ choose an $x_\lambda \in \lambda \cap E$. Then we claim that the net $(x_\lambda) \to x_0$. $\square$

Example 7.0.3. Let $X = \ell^1$. Then $X^* = \ell^\infty$. Then the weak * topology on ℓ^∞ is given by the pointwise convergence of a net $(f_\alpha) \subset \ell^\infty$ converges to $f \in \ell^\infty$ if and only if $f_\alpha(n) \to f(n)$ for all $n \in \mathbb{N}$

Chapter 8

Theorem 8.0.1 (Tychonoff)**.** *The product of compact sets is compact.*

Corollary 8.0.1.1. *Let X be a compact Hausdorff space. Then for any set S, the set $\{\phi : S \to X\} = X^S$ is compact wrt to pointwise convergence.*

Theorem 8.0.2 (Banach-Alaoglu Theorem)**.** *Let X be a normed space. Then the closed unit ball $\overline{B_{X^*}(0,1)} = \{f \in X^* \ : \ \|f\| \leq 1\} = E$ is weak * compact.*

Proof. Let $\bar{B}$ be the closed unit ball of X. Then by Tychonoff theorem, $\bar{D}^{\bar{B}}$ is compact. Define $\phi : E \to \bar{D}^{\bar{B}}$ as $\phi(f)(x) := f(x)$. Observe that ϕ is injective. Also observe that ϕ is continuous (weak * in LHS, and pointwise in RHS).

Next we show that the image of ϕ a closed subset of $\bar{D}^{\bar{B}}$, hence compact. Let f_i be a net in E such that $\phi(f_i) \to \psi$ pointwise for some $\psi \in \bar{D}^{\bar{B}}$.

Define $g : X \to C$ as $g(x) = \alpha\psi(\frac{x}{\alpha})$ where $\|x\| \leq \alpha$. For this to be well defined we must have $\|x\|\psi(\frac{x}{\|x\|}) = \alpha\psi(\frac{x}{\alpha})$ for any $\alpha > \|x\|$. But we get this since ψ is a pointwise limit of linear functionals. Moreover we get that g is linear for the same reason. Thus $\psi = \phi(g)$ and so $\phi(E)$ is closed.

It only remains to show that the inverse of ϕ is continuous. verify. □

Remark 8.0.1. The closed unit ball of a normed space Y is compact w.r.t the norm topology if and only if Y is finite dimensional.

Proof. verify □

Theorem 8.0.3. *Let X be a normed space. Then E is weak * metrizable iff X is separable.*

Proof. Assume X is separable. Let $\{x_n \ : \ n \in \mathbb{N}\}$ be a dense subset of X. For every $f, g \in E$, define $d(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n}|f(\frac{x_n}{\|x_n\|} - g(\frac{x_n}{\|x_n\|})|$. Check that d is a metric.

Now asssume $f_i \to f$ weakly in E. Then $f_i(x_n) \to f(x_n)$ for all $n \in \mathbb{N}$ and $d(f_i, f) \to 0$ (verify).

Assume E is metrizable. Then $\exists\{U_n : n \in \mathbb{N}\}$ of weak * open neighborhoods of 0 such that $\cap_{n=1}^{\infty} U_n = \{0\}$. So, for each $n \in \mathbb{N}$, there exists a finite set $A_n \in X$ and $\epsilon > 0$ such that the (subbasis sets) $\{f \in E : |f| \leq \epsilon \forall x \in A_n \subset U_n$. Now let $A = \cup_{n=1}^{\infty} A_n$. Let $\phi \in E$ such that $\phi(x) = 0$ for all $x \in A$. $\square$

Chapter 9

Definition 9.0.1. Let X and Y be normed spaces and $T \in B(X, Y)$. The adjoint of T, denoted by $T^* \in B(Y^*, X^*)$, is the map $T^* : f \to f \circ T$

Proposition 9.0.1. $\|T\| = \|T^*\|$

Proof. $|T^*(f)| \leq \|f \circ T\| \leq \|T\|\|f\|$ implies $\|T^*\| \leq \|T\|$

$$\begin{aligned}\|T^*\| &= \sup\{\|T^*(\phi)\| \ : \ \phi \in Y^*, \|\phi\| \leq 1\} \\ &= \sup\{|\phi(T(x))| \ : \ \phi \in Y^*, x \in X, \|\phi\| \leq 1, \|x\| \leq 1\} \\ &= \|T\|\end{aligned}$$

Note that the last equality is a consequence of HBT since it guarantees the existence of $\phi_y \in Y^*$ with $\|\phi_y\| \leq 1$ and $\phi_y(y) = |y|$. □

Lemma 9.0.1. *For any $T \in B(X, Y)$, $T^* : Y^* \to X^*$ is weak * continuous (in both spaces)*

Proof. Let $\phi_i \to \phi$ weakly in Y^*. Then by definition for all $y \in Y$, $\phi_i(y) \to \phi(y)$. Then for $x \in X$, $T^*(\phi_i)(x) = \phi_i(T(x)) \to \phi(T(x)) = (T^*(\phi))(x)$ which shows the continuity of T^*. □

Lemma 9.0.2. *For any normed space X, $i_X(X)$ is weak * dense in X^{**}.*

Proof. verify □

Example 9.0.1. Is i_{X^*} weak * - weak * continuous.

Lemma 9.0.3. *Let X be a normed space and $x_1, x_2, \ldots, x_n \in X$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \geq 0$. Then the set*

$$\bigcup_{x_1, x_2, \ldots, x_n, \epsilon_1, \epsilon_2, \ldots, \epsilon_n} (\phi)$$

is convex. Moreover any topological vector spaces with the topology induced by a family of seminorms is locally convex. Refer back to the proofs of Arveson.

Proof. verify $\square$

Definition 9.0.2. Let X be a vector space and $E \subset X$ be a convex subset. An element $a \in E$ is called an extreme point of E if whenever $x, y \in E$, $0 \leq t \leq 1$ with $a = tx + (1-t)y$, then $x = y = a$.

Example 9.0.2. Let $\bar{D} = \{\alpha \in \mathbb{C} \ : \ |\alpha| \leq 1\}$. Then $\bar{D}$ is convex with $\mathrm{Ext}(\bar{D}) = S^1$

Theorem 9.0.1 (Krein-Milman Theorem)**.** *Let X be a locally convex space, and let K be a compact convex subset of X. Then the $Ext(K) \neq \emptyset$ and indeed $K = \overline{co}(Ext(K))$*

Definition 9.0.3. Let V be a vector space and $S \subset V$. The convex hull of S is defined as

$$\mathrm{co}(S) = \left\{ \sum_{i=1}^{n} t_i x_i \;\middle|\; 0 \leq t \leq 1, \sum t_i = 1, x_i \in S \right\}$$

Chapter 10

Lemma 10.0.1. *Let K be a convex set. $x_0 \in Ext(K)$ if and only if $K \setminus \{x_0\}$ is convex.*

Proof. If $K \setminus \{x_0\}$ is not convex, since K is convex, x_0 can be written as the convex combination of elements in $K \setminus \{x_0\}$ which makes $x_0 \notin \text{Ext}(K_0)$. Conversely if $x_0 \in \text{Ext}(K)$, then x_0 cannnot be written as the convex combination of elements of K. Hence $K \setminus \{x_0\}$ is closed under convex combinations, making in convex. $\square$

Theorem 10.0.1 (Krein-Milman Theorem)**.** *Let X be a locally convex space, and let K be a compact convex subset of X. Then the $Ext(K) \neq \emptyset$ and indeed $K = \overline{co}(Ext(K))$*

Proof. We first prove that the $\text{Ext}(K) \neq \emptyset$. Note that $K \setminus \{x_0\}$ is a relatively open subset of K since $\{x_0\}$ is closed and $K \setminus \{x_0\} = \{x_0\}^c$ relative to K.

Now let $\mathcal{A}$ be the collection of all relatively open convex proper subsets of K. Note that $\emptyset \in \mathcal{A}$, therefore $\mathcal{A}$ is nonempty. Equip $\mathcal{A}$ with the partial order defined by the set inclusion. Let $\mathscr{C}$ be a chain in $\mathcal{A}$ and $F_{\mathscr{C}} = \cup_{C \in \mathscr{C}} C$. $F_{\mathscr{C}}$ is relatively open being the union of relatively open subsets of K. To see that $F_{\mathscr{C}}$ is convex, let $x, y \in F_{\mathscr{C}}$. Then since $\mathscr{C}$ is a chain, there exist a $C \in \mathscr{C}$ such that $x, y \in \mathscr{C}$. Then by the convexity of C, $tx + (1-t)y \in C \subset F_{\mathscr{C}}$ for all $t \in [0, 1]$.

Zorn's Lemma to find a maximal proper open convex subset of K

We claim that $F_{\mathscr{C}}$ is a proper subset of K. For the sake of contradiction, assume $F_{\mathscr{C}} = K$. Since K is compact and C is open in K for all $C \in \mathscr{C}$, there are finitely many $C_1 \subset C_2 \subset \ldots \subset C_k \in \mathscr{C}$ which cover K (i.e $K = \cup_{n=1}^{k} C_n$). Hence we get $C_k = K$, which is absurd since C_k must be a proper subset of K. Hence $F_{\mathscr{C}} \in \mathcal{A}$ and thus every chain must have an upper bound in $\mathcal{A}$. Now by Zorn's lemma, $\mathcal{A}$ has a maximal element K_0.

Constructing and open convex subset containing K_0

Since K is a connected space (path connected by a straight line, being convex), we know that the only clopen subsets are $\emptyset$ and K. Since we know that K_0 is open being in $\mathcal{A}$, we see that $K_0 \neq K$ and $K_0 \neq \emptyset$. Therefore $\overline{K_0} \neq K_0$. Let $x_0 \in \overline{K_0} \setminus K_0$, $y_o \in K_0$ and $0 < t < 1$. Define $\varphi_{t,y_0} : K \to K$ such that $\varphi_{t,y_0}(z) = ty_0 + (1-t)z$. Then φ_{t,y_0} is $(1 - t$ Lipschitz) continuous relative to K and thus $\varphi_{t,y_0}^{-1}(K_0)$ is open in K. By the convexity of K_0, we get $K_0 \subset \phi_{t,y_0}^{-1}(K_0)$.

Also $\varphi_{t,y_0}^{-1}(K_0)$ is convex. Let $a, b \in \phi_{t,y_0}^{-1}(K)$. Then $ty_0+(1-t)a, ty_0+(1-t)b \in K_0$. By the convexity of K_0 we get $r(ty_0 + (1-t)a) + (1-r)(ty_0 + (1-t)b) = ty_0 + (1-t)(ra + (1-r)b) = \phi t, y_0(ra + (1-r)b) \in K_0$ for all $r \in [0,1]$. Thus $ra + (1-r)b \in \phi_{t,y_0}^{-1}(K_0)$ for all $r \in [0,1]$. Hence we get $\phi_{t,y_0}^{-1}(K_0)$ is convex.

We claim, $x_0 \in \varphi_{t,y_0}^{-1}(K_0)$, then the maximality of K_0 will force $\phi_{t,y_0}^{-1}(K_0) = K$. Let U be a convex neighborhood of $0 \in X$ containing $-x$ for all $x \in U$ (just take $-U$ and intersect with U) such that $y_o + E \subset K_0$, where $E = K \cap U$. Let $w = \varphi_{t,y_0}(x_0)$. Since $x_0 \in \overline{K_0}$, for any $r > 0$, there exists $x_r \in K_0$ such that $x_r \in (x_0 + rE) \cap K_0 \neq \emptyset$. In particular, let $r = \frac{t}{1-t}$. Then by linearity, we get $\left(x_0 + \frac{t}{1-t}E\right) \cap K_0 = (\frac{t}{1-t})E \cap (K_0 - x_0) \neq \emptyset$. Choose z in the above set. Then

I can't picturize the choice of z

$$y_0 - \Big(\frac{1-t}{t}\Big)z \;\in\; y_0 + E \subset K_0$$

and $x_0 + z \in K_0$. Since K_0 is convex,

$$t\Big(y_0 - \frac{(1-t)}{t}z\Big) + (1-t)(x_0 + z) = \phi_{t,y_0}(x_0) \in K_0$$

Thus $\phi_{t,y_0}^{-1}(K_0) = K$.

Now we claim that $K = K_0 \cup \{x_0\}$. For the sake of contradiction assume $\exists p \in K$ such that $p \notin K_0 \cup \{x_0\}$. Since the space is Hausdorff and locally convex, x_0 has an open convex neighborhood E in X such that $p \notin E$. Let $E' = E \cap K$, $a \in K_0, b \in E'$ and $0 < r < 1$. Then since $\phi_{t,y_0}(K) = K_0$ for all $t \in [0,1], y_0 \in K_0$, we get $\phi_{r,a}(b) = ra + (1-r)b \in K_0$. So $K_0 \cup E'$ is convex (Sine we know that K_0, E' are convex, we only need to worry about $rx + (1-r)y$ for $x \in K_0, y \in E'$. But $\phi_{r,x}$ takes care of that). $K_0 \cup E'$ is also open in K. Hence by maximality, we get $K_0 \cup E' = K$. But this is a contradiction since $p \notin K_0 \cup E'$. Thus by Lemma 10.0.1, we see that $x_0 \in \text{Ext}(K)$.

Next we prove $K = \overline{co}(\text{Ext}(K))$. Let $P = \overline{co}(\text{Ext}(K))$ and for the sake of contradiction assume $P \neq K$. Let $x_0 \in K \setminus P$. Now by the geometric Hahn-Banach separation theorem, we get that there is a continuous linear functional $\phi : X \to \mathbb{R}$ and a number $\alpha, \epsilon \in \mathbb{R}$ such that

$$\Re\phi(x_0) \leq \alpha < \alpha + \epsilon \leq \Re\phi(p), \quad \forall p \in P$$

$\square$

Chapter 11

Lemma 11.0.1. *Let K_1, K_2 be compact convex subsets of a locally compact TVS X. Then*

$$\overline{co}(K_1 \cup K_2) = (co)(K_1 \cup K_2)$$

Proof. verify. We'll show that $\text{co}(K_1 \cup K_2)$ is compact and hence closed. Let $x = \alpha_1 a_1, \alpha_2 a_2, \ldots, \alpha_n a_n + \beta_1 b_1, \beta_2 b_2, \ldots, \beta_m b_n \in \text{co}(K_1 \cup K_2)$, where $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i = 1$. Then

$$x = (\sum_{i=1}^n \alpha_i) \underbrace{\left(\sum_{i=1}^n \Big(\frac{\alpha_i}{\sum_{i=1}^n \alpha_i} \Big) a_i \right)}_{\in K_1} + (\sum_{i=1}^m \beta_i) \underbrace{\left(\sum_{i=1}^m \Big(\frac{\beta_i}{\sum_{i=1}^m \beta_i} \Big) b_i \right)}_{\in K_2}$$

Hence every element $x \in \text{co}(K_1 \cup K_2)$, can be written as $x = ta + (1-t)b$ where $a \in K_1, b \in K_2$.

Now let $x_\lambda = t_\lambda a_\lambda + (1-t_\lambda) b_\lambda$ be a net in $\text{co}(K_1 \cup K_2)$, for $\lambda \in \Lambda$, $a_\lambda \in K_1, b_\lambda \in K_2$. Since (a_λ) is a net in the compact set K_1, there is a subnet a_σ for $\sigma \in \Sigma \subseteq \Lambda$, such that $a_\sigma \to a \in K_1$. By similar reasoning b_σ has a convergent subnet b_π for $\pi \in \Pi \subseteq \Sigma$, such that $b_\pi \to b \in K_2$. Again t_π is a net in the compact space $[0,1]$, hence is has a convergent subnet t_ω for $\omega \in \Omega \subseteq \Pi$ such that $t_\omega \to t$ in $[0,1]$.

Now consider the subnet $x_\omega = t_\omega a_\omega + (1-t_\omega)\beta_\omega$ of x_λ. Since $\Omega \subseteq \Pi \subseteq \Sigma$, $t_\omega \to t, \beta_\omega \to b$ and $a_\omega \to a$. Therefore by the continuity of the scalar product and addition in the TVS, we get $x_\omega \to t\alpha + (1-t)\beta \in \text{co}(K_1 \cup K_2)$. Hence we get $\text{co}(K_1 \cup K_2)$ is compact. □

Theorem 11.0.1 (Inverse Krein-Milman)**.** *Let K be a compact convex subset of a locally convex topological vector space X. Let $A \subset K$ be a closed subset of K. If $K = \overline{co}(A)$, then $Ext(K) \subset A$.*

Note that $\text{Prob}[0,1]$, the collection of probability measures identified as a subspace of a $C([0,1])^*$ is convex, weak * compact with $\text{Ext}(K) = \{\delta_x \ : \ x \in [0,1]\}$

Proof. FSTOC, assume $\exists x_0 \in \mathrm{Ext}(K)$, $x_0 \notin A$. Since A is compact, $\exists y_1, y_2, \ldots, y_n \in A$ and an open convex neighborhood B of 0 such that

$$A \subset \cup_{i=1}^{n}(y_i + B)$$

and $x_0 \notin y_i + \overline{B}$ for all $i = 1, 2, \ldots, n$. Let $B_i = (y_i + \overline{B}) \cap K$. Then B_i is a compact convex subset of K for each i. Hence by the previous lemma, we get

$$\mathrm{co}(B_1 \cup B_2 \cup \ldots \cup B_n) = \overline{\mathrm{co}}(B_1 \cup B_2 \cup \ldots \cup B_n) \supset \overline{\mathrm{co}}(A) = K$$

Thus $\exists b_i \in B_i$ and $0 \leq t_i \leq 1$, $\sum_{i=1}^{n} t_i = 1$ such that

I struggle at finding the contradiction

$$x_0 = t_n b_1 + t_n b_2 + \ldots + t_n b_n$$

Since $x_0 \in \mathrm{Ext}(K)$, this forces $x_0 = b_j$ for some $1 \leq j \leq n$. This contradicts the assumption that $x_0 \notin y_i + \overline{B}$. Hence $x_0 \in A$. $\square$

Note that in the following attempt to prove the theorem, it is not obvious why U is convex. If we try to argue using arguments to the proof of separating a compact set and a point using open sets in a Hausdorff space, we will eventually need to show that the finite open subcover of A sits inside a closed convex set that does not contain x_0, which again is not obvious.

Proof. FTSOC, assume that $\exists x_0 \in \mathrm{Ext}(K) \setminus A$. Since the TVS is Hausdorff, there exist convex open sets U, V such that $A \subset U, x_0 \in V$, and $U \cap V = \emptyset$. Moreover we claim that $\overline{U} \cap V = \emptyset$. Otherwise if $x \in \overline{V} \cap U$, then for any net $(x_\lambda) \in V$ that converge to x, by the definition of convergence $x_{\lambda_n} \in U$ for all λ_n greater that some λ_N. This would contradict the assumption that $U \cap V = \emptyset$. Hence we see that $A \subset \overline{V}$, and therefore $\overline{\mathrm{co}}(A) \subset \overline{V}$. But this would again contradict the fact that $\overline{\mathrm{co}}(A) = K$ since $x_0 \notin \overline{V}$. $\square$

Why is U convex?

Chapter 12

Example 12.0.1. Let X be an infinite dimensional normed space. Then the set $A = \{x \in X \ : \ \|x\| = 1\}$ is norm closed. But the weak closure of A is the set

$$\overline{A}^w = \overline{\{x \in X \ : \ \|x\| = 1\}}^w = \{y \in X \ : \ \|y\| \leq 1\}$$

Hence A is an example of a norm closed set, which is not weak closed.

Theorem 12.0.1. *Let X be a normed space and let $K \subset X$ be convex subset of X. Then the norm and the weak closure of K coincide.*

Proof. Since norm topology is stronger than weak topology, we get $\overline{K}^{\|\cdot\|} \subset \overline{K}^w$. Let $x \in X$ such that $x \notin \overline{K}^{\|\cdot\|}$. Now since $\{x\}, K$ are convex and compact, by Hahn-Banach separation theorem, there is a $f \in X^*, s \in \mathbb{R}, \epsilon > 0$ such that

$$|f(x)| \leq s < s + \epsilon \leq |f(y)|, \quad \forall y \in \overline{K}^{\|\cdot\|}$$

Since the set $\{z \in X \ : \ |f(z)| \geq s + \epsilon\}$ is weakly closed, and contains K, it must contain $\overline{K}^w$. Hence $x \notin \overline{K}^w$ □

Corollary 12.0.1.1. *Let X be a normed space, and $(x_i)_{i \in I}$ be a net in X such that $x_i \to x$ weakly in X. Then there exists a net $(y_j)_{j \in J}$ of finite convex combinations of $\{x_i \ : \ i \in I\}$ such that $y_i \to x$ in norm.*

Proof. verify □

Proposition 12.0.1. *If K is a convex subset of a LCTVS. Then $\overline{K}$ is also convex.*

Proof. verify □

Proposition 12.0.2. *Show that $\mathbf{c}_0$ is weakly closed and weak * dense in ℓ_∞.*

Proof. verify □

Theorem 12.0.2 (Krein-Smulian Theorem)**.** *Let X be a Banach space, and let C be a convex subset of X^*. Then C is weak * closed if and only if $C \cap \{f \in X^* : \|f\| \leq r\}$ is weak * closed for all $r \in \mathbb{R}^+$.*

This should even work if we just take $n \in \mathbb{N}$. verify.

Corollary 12.0.2.1. *Let Z be a subspace of X^*. Then Z is weak * closed if and only if $\{\phi \in Z : \|\phi\| \leq 1\}$ is closed.*

Corollary 12.0.2.2. *Let X be a separable Banach space. Then a convex subset Z of X^* is weak * closed if and only if it is weak * sequentially closed.*

Proof. Since X is separable, for every $r > 0$, the set $\{f \in X^* : \|f\| \leq r\}$ is weak * metrizable. Thus $Z \cap \{f \in X^* : \|f\| \leq 1\}$ is weak * closed iff it is weak * sequentially closed. □

Corollary 12.0.2.3. *Let X be a separable Banach space, and $\phi \in X^{**}$. Then ϕ is weak * continuous if and only if ϕ is sequentially continuous i.e $f_n \to f$ weak * in X^* implies $\phi(f_n) \to \phi(f)$.*

Proof. Assume ϕ is sequentially weak * continuous. Let $C = \mathrm{Ker}(\phi)$ be a subspace of X^*. Let $g_n \in C$ be a sequence such that $g_n \to g \in X^*$. Then by assumption, $0 = \phi(g_n) \to \phi(g)$ implies $\phi(g) = 0$ and therefore $g \in C$. This shows that C is weak * sequentially closed, hence weak * closed by the separability of X. Hence ϕ is weak * continuous. □

Chapter 13

Example 13.0.1. Let $S = \{n\delta_n \ : \ n \in \mathbb{N}\} \subset \ell^\infty$. We show that $0 \in \overline{S}^{w^*}$

Proof. Let $f \in \ell^1$. Then the set $\{n \in \mathbb{N} \ : \ |f(n)| < \epsilon/n\}$ is infinite. (Otherwise this would contradict $f \in \ell^1$). Thus $\exists N \in \mathbb{N}$ such that $N|f_i(N)| < \epsilon$ for all $i = 1, 2, \dots m$. And therefore

$$N\delta_n \in \bigcup_{f_1, f_2, \dots, f_N, \epsilon} (0)$$

□

Definition 13.0.1. A subset S of a vector space V is called balanced if $\forall s \in S, \alpha \in \mathbb{F}$ with $|\alpha| \leq 1, \alpha s \in S$.

Lemma 13.0.1. *Let X be a topological vector space, then every open neighborhood of 0 contains a balanced open neighborhood of O.*

Proof. verify □

Lemma 13.0.2. *All n-dimensional topological vector spaces are isomorphic as topological vector spaces.*

Proof. For the case $n = 1$, and $\mathbb{F} = \mathbb{C}$.

Assume τ is a topology on $\mathbb{C}$ that turns it into a topological vector space. Now think of $i : \mathbb{C} \to (\mathbb{C}, \tau) := x \to x$ as the composition of $\mathbb{C} \to \mathbb{C} \times (\mathbb{C}, \tau) := x \to (x, 1)$ and $\mathbb{C} \times (\mathbb{C}, \tau) \to (\mathbb{C}, \tau) := (x, y) \to xy$. Then we see that i is the composition of these maps which are continuous by the definition of the product topology and the TVS. Hence, i is continuous.

To show that i^{-1} is continuous, consider the annulus $A = \{\alpha \in \mathbb{C} \ : \ 1 \leq |\alpha| \leq 2\}$. Then since A is compact in the usual topology and $i(A) = A$ is a continuous image, we get that A is open in τ. Hence $A^c \ni 0$ is open and by the lemma above has a balanced open neighborhood of 0 in it. (Show that this is actually an open disk). □

Theorem 13.0.1. *Let X be a normed space. Then the closed unit ball of X is compact in norm topology if and only if X is finite dimensional.*

Proof. Suppose X is infinite dimensional normed space and let $\bar{B}$ be the closed unit ball. Let $x_1 \in \bar{B}$ and let $Y_1 = \text{span}\{x_1\}$. Then Y_1 is a closed subspace of X. Since X is a non-zero normed space, let $x_2 \in X$ such that $\|x_2 + Y_1\| = \frac{1}{2}$. Repeat the construction in the proof of Reisz lemma. □

Lemma 13.0.3. *Let X be a normed space. Then $i(X)$ is weak * dense in X^{**}*

Proof. Let $C = \overline{B}^{w*}$, where B is the closed unit ball. Then C is compact convex. FSTOC, assume $\exists \phi \in X^{**}$ such that $\|\phi\| \leq 1$, $\phi \notin C$. Then by HBT, there is a $f \in X^{**}$ and $r \in \mathbb{R}$, $\epsilon > 0$ such that $\Re f(y) \leq r < r + \epsilon \leq \Re f(\phi)$ for all $y \in i(X)$. This implies $\|f\| \leq r$, hence $|f(\phi)| = |\phi(f)| \leq \|\phi\|\|f\| < r$ which gives a contradiction. □

Show that $\Re f(x) \leq r\|x\|$ imply $\|f\| \leq \alpha$

Theorem 13.0.2. *Let X be a Banach space. Then the closed unit ball $\bar{B}$ is weakly compact if and only if X is reflexive.*

Proof. If X is reflexive, the weak and weak * topology coincides and the Banach Alaouglu gives the proof. verify

Assume $\bar{B}$ is weakly compact. Observe that then the map $i : X \to X^{**}$ is continuous when we equip X with weak topology and X^{**} with weak topology. Thus $i(\bar{B})$ is weak * compact. Moreover $i(\bar{B})$ is weak * dense in the closed unit ball of X^{**}. Hence the result. □

Chapter 14

Definition 14.0.1. Recall that a complex inner product o a complex vector space is a map

$$\langle\ ,\ \rangle : X \times X \to \mathbb{C}$$

such that

(1) $\langle x, x\rangle \geq 0$ for all $x \in X$

(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(3) $\langle \alpha x + z, y\rangle = \alpha\langle x, y\rangle + \langle z, y\rangle$

Recall the norms induced by the inner product and the Cauchy-Schwarz inequality.

Definition 14.0.2. Complete inner product spaces are called Hilbert spaces

Proposition 14.0.1. *Let X be an inner product space. Then the inner product of X extends to an inner product on the completion (unique metric space completion) of X, turns it into a Hilbert space.*

Definition 14.0.3. If $x, y \in H$, the Hilbert space, we say $x \perp y$ if $\langle x, y\rangle = 0$

Definition 14.0.4. Given a set $S \subset H$, $S^{\perp} = \{y \in H \ : \ \langle x, y\rangle = 0\}$

Proposition 14.0.2. *Let H, K be Hilbert spaces and $T : H \to K$ be linear. Then the following are equivalent.*

(1) T is isometry

(2) $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$

Proof. verify □

Proposition 14.0.3. *For all $x, y \in H$, a Hilbert space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Example 14.0.1. Show that c_{00} under the usual inner product is not complete and its completion is $\ell^2(\mathbb{N})$.

Proof. verify □

Example 14.0.2. $L^2(\mathbb{R}, \mu)$ with

$$\langle f, g \rangle = \int_{\mathbb{R}} f\overline{g} \; d\mu$$

is a Hilbert space.

Example 14.0.3. Let J be any set $\ell^2(J) = \{f : J \to \mathbb{C} \; : \; \sum_{j \in J} |f(j)|^2 < \infty\}$ with the usual inner product is a Hilbert space.

Definition 14.0.5. An orthonormal basis for H is a maximal orthonormal set.

Theorem 14.0.1. *Let H be a Hilbert space and J be an orthonormal basis for H. Then there exists a bijective linear isometry $T : H \to \ell^2(J)$.*

Chapter 15

Theorem 15.0.1. *Let H be a Hilbert space and let C be a non-empty closed convex subset of H. Then there exist a unique vector $x \in C$ such that $\|x\| \leq \|\eta\|$ for all $\eta \in C$.*

Proof. Let $d = \inf\{\|\eta\| : \eta \in C\}$ and choose a sequence $\eta_n \in C$ such that $\|\eta_n\| \to d$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|\eta_n\|^2 < d^2 + \epsilon$ for all $n \geq N$. Then for all $m, n \geq N$, we have

$$\|\eta_n - \eta_m\|^2 = 2(\|\eta_n\|^2 + \|\eta_m\|^2) - 4\|\frac{1}{2}(\eta_n + \eta_m)\|^2 \leq 4(d^2 + \epsilon) - 4d^2 = 4\epsilon$$

Hence the sequence η_n is Cauchy and hence convergent since the space is complete. Let $\eta = \lim_{n\to\infty} \eta_n$. Since C is closed $\eta \in C$ and clearly $\|\eta\| = d$.

To see the uniqueness, assume $\alpha \in C$, and $\|\alpha\| = d$. Then

$$\begin{aligned}\|\eta - \alpha\|^2 &= 2(\|\eta\|^2 + \|\alpha\|^2) - 4\|\frac{1}{2}(\eta + \alpha)\|^2 \\ &\leq 4d^2 - 4d^2 = 0\end{aligned}$$

Verify the second inequality. □

Corollary 15.0.1.1. *Let $\eta \in H$. Then there exist a unique vector $x \in C$ such that $d(\eta, C) = \|x - \eta\|$*

Proof. Apply above theorem to $C' = C \setminus \{\eta\}$. □

Proposition 15.0.1 (Pythagoras Theorem)**.** *Let $x, y \in H$ an inner product space, and $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.*

Lemma 15.0.1. *Let H be a Hilbert space and K be a nontrivial closed subspace. Let $\eta \in H$. Then $\xi \in K$ satisfy $\|\xi - \eta\| = d(\eta, K)$ iff $\xi - \eta \perp K$.*

Theorem 15.0.2 (Reisz Representation Theorem)**.** *Let H be a Hilbert space and $f \in H^*$. Then there exists a unique $\eta_f \in H$ such that $f(x) = \langle x, \eta_f \rangle$ for all $x \in H$. The map $\phi : H^* \to H := f \to \eta_f$ is conjugate linear isometric bijection.*

Proof. verify ☐

Theorem 15.0.3. *Let H_1, H_2 be Hilbert spaces, and $T : H_1 \to H_2$ be a bounded linear map. Then there exists a unique bounded linear map $T^* : H_2 \to H_1$ satisfying $\langle Tx, y \rangle_{H_2} = \langle x, T^*y \rangle_{H_1}$ for all $x \in H_1, y \in H_2$.*

Proof. For every given $y \in H_2$ define a linear functional $f^y : H_1 \to \mathbb{C}$ as $f^y(x) = \langle Tx, y \rangle$. Since f^y is bounded, $f^y \in H_1^*$. Hence by Reisz representation, there is a unique $T^*(y) \in H_1$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

Uniquness follows from the fact that in any inner product space X, if $x, y \in X$ such that $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in X$, then $x = y$ Verify the linearity. ☐

Theorem 15.0.4. *Let H be a Hilbert space and K a closed subspace. For every $\eta \in H$, denote by $P_K(\eta)$, the unique closest vector in K, closest to η. Then*

(1) $P_K : H \to H$ is linear, bounded with $\|P_K\| = 1$ and idempotent.

(2) $P_K^* = P_K$ (self-adjoint)

Proof. (1) Let $\eta_1, \eta_2 \in H$ and $\alpha \in \mathbb{C}$. Then for all $\xi \in K$, we have

$$\begin{aligned}\langle \alpha\eta_1 + \eta_2 - \alpha P_K(\eta_1) - P_K(\eta_2), \xi \rangle &= \alpha\langle \eta_1 - P_K(\eta_1), \xi \rangle + \langle \eta_2 - P_K(\eta_2), \xi \rangle \\ &= 0\end{aligned}$$

If $K \neq \{0\}$ and $0 \neq$

(2)

☐

Chapter 16

Lemma 16.0.1. *Let H be a Hilbert space and K be a nontrivial closed subspace. Let $\eta \in H$. Then $\xi \in K$ satisfy $\|\xi - \eta\| = d(\eta, K)$ iff $\xi - \eta \perp K$.*

Proof. Let K be a closed subspace of H and let $\eta \in H$. Let $\xi \in K$ such that $\|\eta - \xi\| = d(\eta, K)$. Then for all $\rho \in K$ and $t > 0$, we have

$$\begin{aligned}
\|\eta - \xi\|^2 &\leq \|\eta - (\xi + t\rho)\|^2 \\
&= \|\eta - \xi - t\rho\|^2 \\
&= \|\eta - \xi\|^2 + t^2\|\rho\|^2 - 2t\Re\langle \eta - \xi, \rho\rangle
\end{aligned}$$

Hence we see that $|2\Re\langle \eta - \xi, \rho\rangle| \leq t\|\rho\|^2$ Since $t > 0$ was arbitrary, limiting it to zero, we get $\Re\langle \eta - \xi, \rho\rangle = 0$. Replacing ρ with $-i\rho$ will give the imaginary part is also zero.

Conversely, assume that $(\eta - \xi) \perp K$. Then for all $\rho \in K$, we have

$$\begin{aligned}
\|\eta - \rho\|^2 &= \|(\eta - \xi) + (\xi - \rho)\|^2 \\
&= \|\eta - \xi\|^2 + \|\xi - \rho\|^2 \\
&\geq \|\eta - \xi\|^2
\end{aligned}$$

$\square$

Proposition 16.0.1. $I - P_K = P_{K^\perp}$

Proof. Let $x \in X$ and $k \in K$. Then

$$\begin{aligned}
\langle (I - P_k)(x), k\rangle &= \langle x - P_k(x), k\rangle \\
&= \langle x, k\rangle - \langle P_k(x), k\rangle \\
&= \langle x, k\rangle - \langle x, P_k(k)\rangle \\
&= \langle x, k\rangle - \langle x, k\rangle \\
&= 0
\end{aligned}$$

Shows that $\square$

Proposition 16.0.2. *Let K be a closed subspace of H. Let $E \subset K$ be an o.n.b for K. Extend E to an o.n.b $\tilde{E}$ for H. Then*

$$P_K|_E = I_K, \quad P_K|_{\tilde{E}\setminus E} = 0$$

Remark 16.0.1 (Parserval's Inequality). Let H be a Hilbert space. Let E be an orthonormal set. Then for every vector $\eta \in H$,

$$\|\eta\|^2 \geq \sum_{e\in E} |\langle \eta, e\rangle|^2$$

Lemma 16.0.2. *Let S be a nonempty subset of H. Then*

$$(S^\perp)^\perp = \overline{Span(S)}$$

Corollary 16.0.0.1. *Let E be an orthonormal subset of H. Then the following are equivalent.*

(1) E *is an o.n.b*

(2) $(E^\perp)^\perp = H$

(3) $\overline{Span(S)} = H$

(4) $\|\eta\|^2 = \sum_{e\in E} |\langle \eta, e\rangle|^2, \ \forall \eta \in H$

Proposition 16.0.3. *Let K be a closed subspace of a Hilbert space H. Then $P_K = P_K^*$*

Proof. Let $k \in K$ and $x \in H$. Then

$$\langle x, P_K^*(k)\rangle = \langle P_K(x), k\rangle = \langle P_K(x) - x, k\rangle + \langle x, k\rangle = \langle x, k\rangle$$

for all $x \in H$. Hence $P_K^*(k) = k$. Conversely, verify □

Theorem 16.0.1. *Let $P \in B(H)$ be an idempotent. Then the following are equivalent.*

(1) $P = P_K$ *for some closed subspace* $K \leqslant H$

(2) $P = P^*$

(3) $\|P\| = 1$

Proof. $1 \implies 2, 1 \implies 3$ is easily known from above. To see $2 \implies 1$. Let $K = \mathrm{Im}(P)$. Let $\rho \in K^{\perp}$. Then for all $\eta \in H$,

$$\langle P(\rho), \eta \rangle = \langle \rho, P^*(\eta) \rangle = \langle \rho, P(\eta) \rangle = 0$$

So $P|_{K^{\perp}} = 0$. Hence $P = P_k$.

Conversely, assume $P \neq P_K$. Then $\exists \rho \in K^{\perp}$ such that $P(\rho) = 0$. For each $n \in \mathbb{N}$, we have $\|\rho + nP(\rho)\|^2 = \|\rho\|^2 + n^2\|P(\rho)\|^2$. And

$$\|P(\rho + nP(\rho))\|^2 = (n+1)^2\|P(\rho)\|^2$$

So for large n, we have

$$\|P(\rho + nP(\rho))\| > \|\rho + P(\rho)\|$$

so $\|P\| > 1$. $\square$

Chapter 17

Definition 17.0.1. let X, Y be Banach Spaces. A linear map $T : X \to Y$ is called a compact operator if $\overline{T(B_1^X)}$ is compact. We denote by $K(X, Y)$, the set of all compact operators.

Example 17.0.1. If either X or Y is finite, then every linear map $T : X \to Y$ is compact. If X be any infinite dimensional Banach space. Then $T = \mathrm{Id} : X \to X$ is not compact.

Definition 17.0.2. $T : X \to Y$ is called a finite rank if the dimension of T is finite. Then the dimension of the image is called the rank of the operator. Let $F(X, Y)$ denote finite rank operators.

Lemma 17.0.1. *Every compact operator is bounded.*

Proof. Every compact set is bounded in any metric space. □

Theorem 17.0.1. *Let H be a Hilbert space. Then $K(H) = \overline{F(H)}^{\|\cdot\|}$*

Proof. It is evident that $K(H) \subset K(H)$. We'll now show that $\overline{F(H)} \subset K(H)$ Let T_n be a sequence in $F(H)$ and $T_n \to T \in B(H)$ (in norm). We'll show that the image of $T(B_1(H))$ is closed and totally bounded. Then since the space is complete, this would give a convergent subsequence and hence would be complete.

Let $\epsilon > 0$ be given. Then there exist some $N \in \mathbb{H}$ such that $\|T_n - T\| < \epsilon$. Since T_N is compact, $\exists \eta_1, \eta_2, \ldots \eta_k \in B_1^H$

$$\overline{T_N(B_1^H)} \subset \bigcup_{\eta \in B_1^H} B_\epsilon(T_N(\eta))$$

Hence by the compactness of $\overline{T_N(B_1^H)}$ it has a finite open cover. Then

$$\overline{T_N(B_1^H)} \subset \bigcup_{i=1}^{n} B_\epsilon(T_N(\eta))$$

Let $\eta \in B_i^H$ be arbitrary. □