MATH730 Functional Analysis
Homework 4

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1. **Solution:**

   (a) We approach this problem in cases.

   i. Let $1 < p < \infty$. We claim that the unit circle $\mathbb{T}_p = \{x \in \ell_p \ : \ \|x\|_p = 1\}$ is the collection of all extreme points of the unit ball of ℓ_p, which we'll denote by B_p.

   To see that the elements of $\mathbb{T}_p$ are indeed extreme points of the unit ball, assume $x = ta + (1-t)b$, where $x \in \mathbb{T}_p, a, b \in B_p$. Then Minkowski inequality $1 = \|x\| \leq t\|a\| + (1-t)\|b\|$ forces $\|a\| = \|b\| = 1$. Thus we see that $1 = \|x\| = \|ta\| + \|(1-t)y\|$. Thus the Minkowski inequality is an equality here. We know that in ℓ_p spaces where $1 < p < \infty$, the Minkwoski inequality is an equality if and only if $(1-t)b = kta$ for some $k > 0$. Thus we get $x = ta + kta = (k+1)ta$. Now using the fact that a, x and $(k+1)ta$ all must have norm 1, gives us that $(k+1)t = 1$ and thus $a = x$. Replacing a with x in $x = ta + (1-t)b$ gives us that $b = x$. Thus we see that $\mathbb{T}_p \subset \text{Ext}(B_p)$.

   Now to prove that $\mathbb{T}_p$ are precisely the extreme points, we'll show that $\overline{\text{co}}(\mathbb{T}_p) = B$. Then inverse Krein-Milman would show that $\text{Ext}(B_p) \subset \mathbb{T}_p$. Let $x \in B$. Then $\frac{x}{\|x\|_p}, \frac{-x}{\|x\|_p} \in \mathbb{T}_p$ and

$$x = \left(\frac{1 + \|x\|_p}{2} \right) \frac{x}{\|x\|_p} - \left(\frac{1 - \|x\|_p}{2} \right) \frac{x}{\|x\|_p}$$

   shows that $\overline{\text{co}}(\mathbb{T}_p) = B_p$. Hence we're done.

   ii. Let $p = 1$. Then we claim that $S = \{re_j \ : \ e_j(n) = \delta_j(n), |r| = 1\}$ are all the extreme points of the unit ball of ℓ_1, which we'll denote by B_1.

To see that re_j is an extreme point, assume that $re_j = tx + (1-t)y$, where $x, y \in B_1$. Then if $x_j = x(j), y_j = y(j)$, we see that $r = tx_j + (1-t)y_j$ fails if either $|x_j| < 1$ or $|y_j| < 1$. Thus $|x_j|, |y_j| \geq 1$ Moreover since $|x_j| \leq \|x\|_1 = 1 = \|y\|_1 \geq |y_j|$, we see that $|x_j| = |y_j| = 1$. Now if $i \neq j$ and $x_i \neq 0$. Then $|x_j| + |x_i| = 1 + |x_i| > \|x\|_1$ is a contradiction. Thus we see that $x = z_1 e_j$ for some $|z_1| = 1$. By the same reasoning, we get that $y = z_2 e_j$ for some $|z_2| = 1$. Then we see that $r = tz_1 + (1-t)z_2$ for the above z_1, z_2. But the strict convexity of $\mathbb{C}$ forces $r = z_1 = z_2$ which gives $x = y = re_j$ Hence we get $S \subset \mathrm{Ext}(B_1)$.

Let $x = (x_1, x_2, \ldots) \in B_1$. We'll show that if $0 < |x_j| < 1$ for any $j \in \mathbb{N}$, then $x \notin \mathrm{Ext}(B_1)$. Without loss of generality, assume that $0 < |x_1| < 1$. Then there exist a $\epsilon > 0$ such that $B_\epsilon(x_1) \subset \mathbb{D}$, the closed unit ball of $\mathbb{C}$. Let $y \in B_\epsilon(0)$. Then $|x_1 + y|, |x_1 - y| < |x_1| + \epsilon$. Since ϵ was arbitrary, we can find y such that $|x_1 + y|, |x_1 - y| < |x_1|$. Then for $a = (x_1 + y, x_2, \ldots), b = (x_1 - y, x_2 - y, \ldots)$, we see that $a, b \in B_1$ and

$$x = \frac{1}{2}a + \frac{1}{2}b$$

Thus the only extreme points of B_1 are those sequences $x = (x_1, x_2, \ldots)$ with $|x_i| = 1, 0$. But the fact that $\|x\| = 1$ forces $x \in S$. Thus we see that $\mathrm{Ext}(B_1) = S$

iii. Let $p = \infty$. We claim that $S = \{x = (x_1, x_2, \ldots) : |x_i| = 1\}$ are all the extreme points of the unit ball of ℓ_∞, which we'll denote by B_∞.

To see that elements of S are extreme points of B_∞, let $x \in S$ and assume that $x = ta + (1-t)b$ for $a, b \in B_\infty$. Then $x_j = ta_j + (1-t)b_j$ for all $j \in \mathbb{N}$ with $-1 \leq |a_j|, |b_j| \leq 1$. Since $|x_j| = 1$, by the same reasoning, we used for ℓ_1 case, we get $|a_j| = |b_j| = 1$. Then again the strict convexity of $\mathbb{C}$ gives us that $x_j = a_j = b_j$. Since j was arbitrary, we see $x = a = b$. Thus $S \subset \mathrm{Ext}(B_\infty)$.

Assume $x = (x_1, x_2, \ldots) \in B_\infty$ such that $x \notin S$. Without loss of generality assume that $|x_1| < 1$. Note that $|x_1| \not> 1$ since $1 = \|x\| \geq |x_1|$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x_1) \subset \mathbb{D}$, the closed unit ball in $\mathbb{C}$. Let $y \in B_\epsilon(0)$, then $|x_1 + y|, |x_1 - y| < 1$. Thus $a = (x_1 + y, x_2, \ldots), b = (x_1 - y, x_2, \ldots) \in B_\infty$. Then

$$\frac{a+b}{2} = \frac{1}{2}(x_1 + y, x_2, \ldots) + \frac{1}{2}(x_1 - y, x_2, \ldots, x_n) = (x_1, x_2, \ldots) = x$$

shows that x is not an extreme point. Thus, we get that $\mathrm{Ext}(B_\infty) = S$.

(b) Here also we approach using subparts.

i. Let $1 < p < \infty$. We claim that $\mathbb{T}_p = \{f \in L^p([0,1]) \ : \ \|f\|_p = 1\}$ is the collection of all extreme points of the closed unit ball of $L^1([0,1])$. We notice that the same proof for ℓ_p also works for $L^p([0,1])$.

ii. If $p = 1$, we claim that there are no extreme points for the closed unit ball B. To see this let $f \in B$, the unit ball of L^1 with $\int |f| \ d\mu = 1$. Then there is a non-null set E, where $|f(x)| < \frac{1}{2}$ for all $x \in E$. Then $\int |f|\chi_E \ d\mu \leq \frac{\mu(E)}{2}$. And since

$$\int |f| \ d\mu = 1 = \int |f|\chi_E \ d\mu + \int |f|\chi_{E^c} \ d\mu$$

we get

$$\int |f|\chi_{E^c} d \ \mu = 1 - \int |f|\chi_E \ d\mu \leq 1 - \frac{\mu(E)}{2}$$

Therefore $\frac{2f\chi_E}{\mu(E)}, \frac{f\chi_{E^c}}{2-\mu(E)} \in B$ and

$$\frac{\mu(E)}{2}\left(\frac{2f\chi_E}{\mu(E)}\right) + \left(1 - \frac{\mu(E)}{2}\right)\frac{f\chi_{E^c}}{1 - \frac{\mu(E)}{2}} = f\chi_E + f\chi_{E^c} = f$$

shows that f is not an extreme point of B.

iii. I claim that the collection $S = \{\chi_E - \chi_{E^c} \ : \ E \subset [0,1], E \in M_\sigma\}$ are the extreme points of unit ball of $L^\infty([0,1])$.

(c) I claim that the only extreme points of the unit ball of $C([0,1])_{\mathbb{R}} = \{f : [0,1] \to \mathbb{R} \ : \ f \text{ is continuous}\}$ are the $\mathbf{1}, -\mathbf{1}$ functions. If $f \notin \{\mathbf{1}, -\mathbf{1}\}$ is any function in the closed unit ball of $C([0,1])$,

$$g = \frac{f+1}{2}, \quad h = f + \frac{(f-1)}{2}$$

are two functions in the unit ball of $C([0,1])$ with

$$f = \frac{g}{2} + \frac{h}{2}$$

Hence there can be no other extreme points than $\{\mathbf{1}, -\mathbf{1}\}$.

Now for the case of complex valued functions we claim that the corresponding extreme points are the set $S = \{f \in C([0,1])_{\mathbb{C}} \ : \ |f(x)| = 1, \text{ for all } x \in [0,1]\}$. If $f \notin S$, then $\exists x_0 \in [0,1]$ such that $|f(x)| < 1$. Then we can find a function $g \in C([0,1])_{\mathbb{C}}$ with $\|g\| \leq 1, \|f - g\| \leq \frac{1}{2}$

that vary from f only on a neighborhood of x_0. Now chose $h = 2f - g$. Then $\|h\| \leq 1$ with

$$f = \frac{g+h}{2}$$

shows that all the extreme points are in S.

Conversely, if $f \in S$ and $f = tg + (1-t)h$, then $f(x) = tg(x) + (1-t)g(x)$ for all $x \in [0,1]$ and thus taking absolute values on both sides, the strict convexity of $\mathbb{C}$ forces $f(x) = g(x) = h(x)$ for all $x \in [0,1]$. Thus we see that f is an extreme point of the unit ball of $C([0,1])$.

(d) I claim that there are no extreme points for the closed unit ball in $C_0(\mathbb{C})$, denoted by B. Let $g \in B$ with $\|g\| = 1$. Then there is a compact set K such that $|g(x)| < \frac{1}{2}$ whenever $x \notin K$. Let $h \in C_0(\mathbb{C})$ such that $h(x) = 0$ on K but $\|h\| = \frac{1}{4}$. Then $\|h + g\| = 1$ and

$$g = \frac{g+h}{2} + \frac{g-h}{2}$$

Shows g is not an extreme points. Now if $\|g\| \neq 1$, then we can rescale it to have norm 1 and proceed as above.

2. **Solution:**

(a) i. $(C_b(\mathbb{R}), \|\cdot\|_\infty)$ is complete.

Let f_n be a Cauchy sequence in $C_b(\mathbb{R})$. Then $f_n(x)$ is Cauchy for all $x \in \mathbb{R}$. Since $\mathbb{C}$ is complete $f_n(x)$ converge for each $x \in \mathbb{R}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. We'll show that $f_n \to f$ in the sup norm, and that $f \in C_b(\mathbb{R})$. This will show that $C_b(\mathbb{R})$ is complete under the sup norm.

Let $\epsilon > 0$. Then there is a N_ϵ such that for all $n, m \geq N_\epsilon$, we have

$$|f_n(x) - f_m(x)| < \epsilon, \qquad \text{for all } x \in \mathbb{R}$$

Now taking limit as $m \to \infty$, we get that $\|f_n - f\| < \epsilon$. Now for $f \in C_b(\mathbb{R})$, we notice that the convergence is uniform which guaranteed the boundedness and continuity of the f.

ii. $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is complete.

Let f_n be Cauchy sequence in $C_0(\mathbb{R})$ and f be the function as before. Since $C_0(\mathbb{R}) \subset C_b(\mathbb{R})$, most of the proof follows similarly as before. We just need to show that f vanishes at infinty. Let $\epsilon > 0$. Let

f_n be the function in the sequence such that $\|f_n - f\| \leq \frac{\epsilon}{2}$. Since $f_n \in C_0(\mathbb{R})$, there is a compact set $K \subset \mathbb{R}$ such that $\|f_n(x)\| < \frac{\epsilon}{2}$ for all $x \in K^c$. Then we claim that $|f(x)| < \epsilon$ for all $x \in K^c$. Let $x \notin K^c$, then

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq \|f - f_n\| + |f_n(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

shows that $f \in C_0(\mathbb{R})$

iii. $(C_b(\mathbb{R}), \tau)$ is complete.

We just need to show any Cauchy net f_λ converges in τ. Let f_λ be a Cauchy net in the unit ball of $C_b(\mathbb{R}), \tau$. Then for $\epsilon > 0$, there is a λ_ϵ such that for all $\lambda_1, \lambda_2 > \lambda_\epsilon$, we get $\rho_g(f_{\lambda_1} - f_{\lambda_2}) < \epsilon$ for all $g \in C_0(\mathbb{R})$. This is equivalent to $\|gf_{\lambda_1} - gf_{\lambda_2}\|_\infty < \epsilon$ for all $C_0(\mathbb{R})$. Since $g \in C_0(\mathbb{R})$ and f_λ is bounded, we see that gf_λ is a Cauchy net in $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ and hence converges to some ϕ_g for each $g \in C_0(\mathbb{R})$. Let $g \in C_0(\mathbb{R})$ such that $0 < g(x) < 1$ for all $x \in \mathbb{R}$. Since f_λ is a net in the unit ball of $C_b(\mathbb{R})$, we get $gf_\lambda < g$ for all f_λ. Hence taking limits preserve the inequality and we see that $\phi_g < g$. Hence $\frac{\phi_g}{g} < 1$ and $\frac{\phi_g}{g} \in C_b(\mathbb{R})$.

Now tracing back our construction of ϕ_g, we see that $f_\lambda \to \frac{\phi_g}{g}$. Since any Cauchy net can be rescaled to be inside the unit ball, we see that $C_b(\mathbb{R})$ is complete in τ.

(b) Let $f_n \to f$ in $(C_b(\mathbb{R}), \|\cdot\|_\infty)$. We have to show that $f_n \to f$ in τ, which is equivalent to show that $\|g(f_n - f)\| \to 0$ for all $g \in C_b(\mathbb{R})$. But since $\|g(f_n - f)\| < \|g\|\|f_n - f\|$, by the algebra of limits, we get that $f_n \to f$ in τ. Hence open sets of τ are open in $\|\cdot\|_\infty$.

To show that the converse is not true, consider the sequence of functions $f_n \in C_b(\mathbb{R})$ such that $\chi_{[-n,n]} < f_n < \chi_{[-n-1,n+1]}$. Existence of such functions are guaranteed by the Urysohn's lemma, since $(-n-1, n+1) \subset [-n-1, n+1]$ (We don't even need Urysohn if I hand draw). Then $\|f_n - f_{n+1}\| = 1$ and thus f_n is not Cauchy in $\|\cdot\|_\infty$. But we claim that f_n is Cauchy in τ.

Let $\epsilon > 0$ and $g \in C_0(\mathbb{R})$ be given. Then there is a compact set $K \subset \mathbb{R}$ such that $g(x) < \epsilon$ for all $x \in K^c$. Moreover there is an $N \in \mathbb{N}$ such that

$K \subset [-N, N]$. Then for $m > n > N$, we have

$$|g(x)f_n(x) - g(x)f_m(x)| = 0, \qquad \text{when } x \in K$$

since $f_n(x) = f_m(x) = 1$ when $x \in K$. And

$$|g(x)f_n(x) - g(x)f_m(x)| \geq |g(x)| < \epsilon, \qquad \text{when } x \in K^c$$

since $f_m(x) - f_n(x) < 1$ everywhere. Thus we see that $\rho_g(f_n - f_m) < \epsilon$. Since $g \in C_0(\mathbb{R})$ was arbitrary, this gives that f_n is Cauchy in τ. Hence we see that the topology of τ and $\|\cdot\|_\infty$ in $C_b(\mathbb{R})$ are not the same.