MATH 7320, Functional Analysis I

HOMEWORK 5 – Due Friday November 15

1. Let $\mathcal{H}$ be a Hilbert space and $\phi : \mathcal{H} \to \mathcal{H}$ be a surjective distance-preserving function with $\phi(0) = 0$. Prove that $\phi(r\xi + s\eta) = r\phi(\xi) + s\phi(\eta)$ for all $r, s \in \mathbb{R}$ and $\xi, \eta \in \mathcal{H}$.

2. Let $\mathcal{H}$ be a Hilbert space and $S \subset \mathcal{H}$ a non-empty subset. Prove that $(S^\perp)^\perp = \overline{\text{span}}(S)$.

3. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Show that the space $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ of all linear compact operators from $\mathcal{X}$ to $\mathcal{Y}$ is a closed subspace of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. Moreover, if $\mathcal{Z}$ is a normed space, $T \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{B}(\mathcal{X}, \mathcal{Z})$, then $S \circ T \in \mathcal{K}(\mathcal{X}, \mathcal{Z})$.

4. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \leq \mathcal{H}$ a closed subspace. Show that the map $\pi : \mathcal{M}^\perp \to \mathcal{H}/\mathcal{M}$ defined by $\pi(\xi) = \xi + \mathcal{M}$, is isometric.

5. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Suppose that $T \in \mathcal{B}(\mathcal{X}^*, \mathcal{Y})$ has the property that $\|Tf_i\| \to 0$ whenever $(f_i)_{i \in I}$ is a net in $\mathcal{X}^*$ such that $f_i(x) \to 0$ for all $x \in \mathcal{X}$. Prove that T is a compact operator.

6. Show that if $T \in \mathcal{K}(\mathcal{H})$ and $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$, then $T(e_n) \to 0$.

7. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Prove that the following are equivalent:

   i) T is an isometry;

   ii) $\langle T(\xi), T(\eta) \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$;

   iii) $T^*T = I$.

8. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Prove that the following are equivalent:

   i) T is a normal isometry;

   ii) T is an isometric bijection;

   iii) $T^*T = TT^* = I$.

9. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Show that $\ker(T) = T^*(\mathcal{H})^\perp$.

10. Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{F}(\mathcal{H})$. Show that $T^* \in \mathcal{F}(\mathcal{H})$.

11. Let $T : L^2([0,1]) \to L^2([0,1])$ be defined by $T(f)(x) = xf(x)$. Prove that $T \in \mathcal{B}(L^2([0,1]))$, $T = T^*$, T is injective but not bijective, and that T has no eigenvectors.

12. Let $(\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty$, and let $T : \ell^2 \to \ell^2$ be defined by $T(\delta_n) = \alpha_n \delta_n$ for all $n \in \mathbb{N}$.

    i) Show that T is bounded with $\|T\| = \|(\alpha_n)\|_\infty$.

    ii) Prove that T is compact if and only if $(\alpha_n) \in \mathbf{c_0}$

    iii) Prove that $T^*(\delta_n) = \overline{\alpha_n}\delta_n$

13. Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{M} \leq \mathcal{H}$ a closed subspace.

    i) Prove that $\mathcal{M}$ is invariant under T if and only if $TP_\mathcal{M} = P_\mathcal{M}TP_\mathcal{M}$.

    ii) Prove that $\mathcal{M}$ reduces T if and only if $TP_\mathcal{M} = P_\mathcal{M}T$.

    iii) Prove that $\mathcal{M}$ reduces T if and only if $\mathcal{M}$ is invariant under both T and T^*.

    iv) Show that if $\mathcal{M}$ reduces T, then $(T|_\mathcal{M})^* = T^*|_\mathcal{M}$

    v) Is part iii) true if $\mathcal{M}$ is only invariant under T?

14. Let $\mathcal{H}$ be a Hilbert space, and $P, Q \in \mathcal{B}(\mathcal{H})$ be projections, i.e. $P^2 = P = P^*$ and similarly for Q.

    i) Prove that $P + Q$ is a projection if and only if $P(\mathcal{H}) \perp Q(\mathcal{H})$, and in this case $(P+Q)(\mathcal{H}) = P(\mathcal{H}) + Q(\mathcal{H})$ and $\ker(P+Q) = \ker(P) \cap \ker(Q)$.

    ii) Prove that PQ is a projection if and only if $PQ = QP$ if and only if $P + Q - PQ$ is a projection, and in this case $(PQ)(\mathcal{H}) = P(\mathcal{H}) \cap Q(\mathcal{H})$ and $\ker(P+Q) = \ker(P) + \ker(Q)$.