

Introduction to Discrete-Time Signals and Systems

Welcome to Discrete Time Signals and Systems

- This is an introductory course on **signal processing** that studies **signals** and **systems**

DEFINITION

Signal (n): A detectable physical quantity . . . by which messages or information can be transmitted (Merriam-Webster)

- Signals carry **information**
- Examples:
 - Speech signals transmit language via acoustic waves
 - Radar signals transmit the position and velocity of targets via electromagnetic waves
 - Electrophysiology signals transmit information about processes inside the body
 - Financial signals transmit information about events in the economy

Welcome to Discrete Time Signals and Systems

- **Systems** manipulate the information carried by signals

DEFINITION

Signal processing involves the theory and application of

- filtering, coding, transmitting, estimating, detecting, analyzing, recognizing, synthesizing, recording, and reproducing signals by digital or analog devices or techniques
- where signal includes audio, video, speech, image, communication, geophysical, sonar, radar, medical, musical, and other signals

(IEEE Signal Processing Society Constitutional Amendment, 1994)



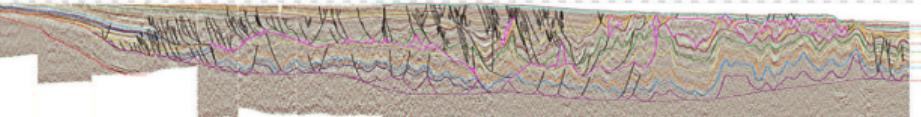
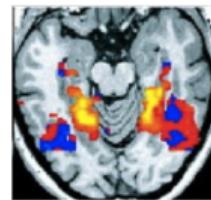
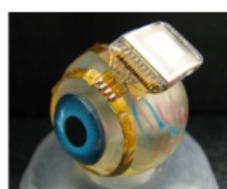
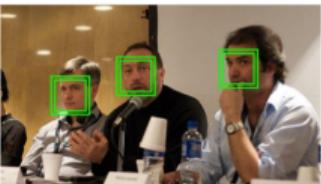
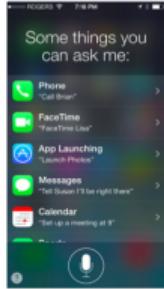
Signal Processing

- Signal processing has traditionally been a part of electrical and computer engineering
- But now expands into applied mathematics, statistics, computer science, geophysics, and host of application disciplines
- Initially **analog** signals and systems implemented using resistors, capacitors, inductors, and transistors



- Since the 1940s increasingly **digital** signals and systems implemented using computers and computer code (Matlab, Python, C, ...)
 - Advantages of digital include stability and programmability
 - As computers have shrunk, digital signal processing has become ubiquitous

Digital Signal Processing Applications



Discrete Time Signals and Systems

- This edX course consists of one-half of the core Electrical and Computer Engineering course entitled "Signals and Systems" taught at Rice University in Houston, Texas, USA (see www.dsp.rice.edu)
- Goals: Develop intuition into and learn how to reason analytically about signal processing problems
- Video lectures, primary sources, supplemental materials, practice exercises, homework, programming case studies, final exam
- Integrated Matlab!
- The course comes in two halves
 - Part 1: Time Domain
 - Part 2: Frequency Domain



Course Outline



■ Part 1: Time Domain

- Week 1: Types of Signals
- Week 2: Signals Are Vectors
- Week 3: Systems
- Week 4: Convolution
- Week 5: Study Week, Practice Exam, Final Exam

■ Part 2: Frequency Domain

- Week 1: Discrete Fourier Transform (DFT)
- Week 2: Discrete-Time Fourier Transform (DTFT)
- Week 3: z Transform
- Week 4: Analysis and Design of Discrete-Time Filters
- Week 5: Study Week, Practice Exam, Final Exam

What You Should Do Each Week

- Watch the Lecture videos
- Do the Exercises (on the page to the right of the videos)
- As necessary, refer to the lesson's Supplemental Resources (the page to the right of the exercises)
- Do the homework problems
- Some weeks will also have graded MATLAB case study homework problems

Logistics and Grading

- How to get help: **Course Discussion** page
 - Use a thread set up for a particular topic, or
 - Start a new thread
- Rules for discussion
 - Be respectful and helpful
 - Do not reveal answers to any problem that will be graded
- Grading

Quick Questions	15%
Homework	30%
Homework Free Response Questions	15%
Pre-Exam Survey	5%
Final exam	30%
Post-Exam Survey	5%

- Passing grade: 60%

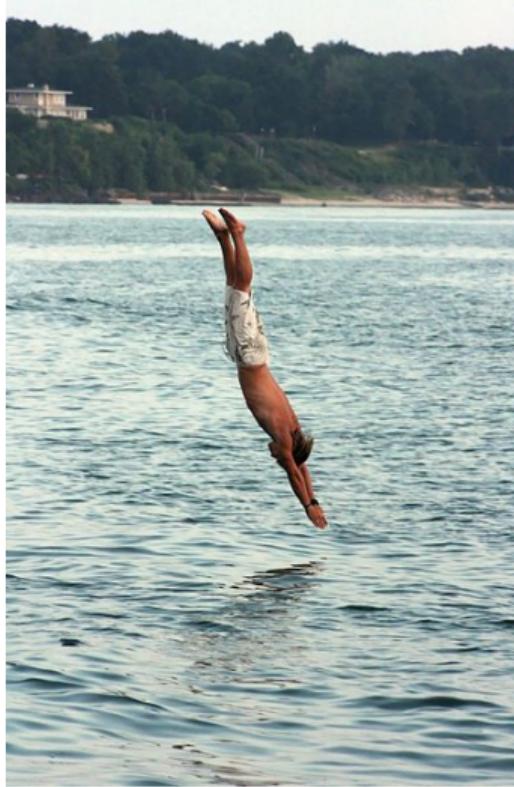
Supplemental Resources

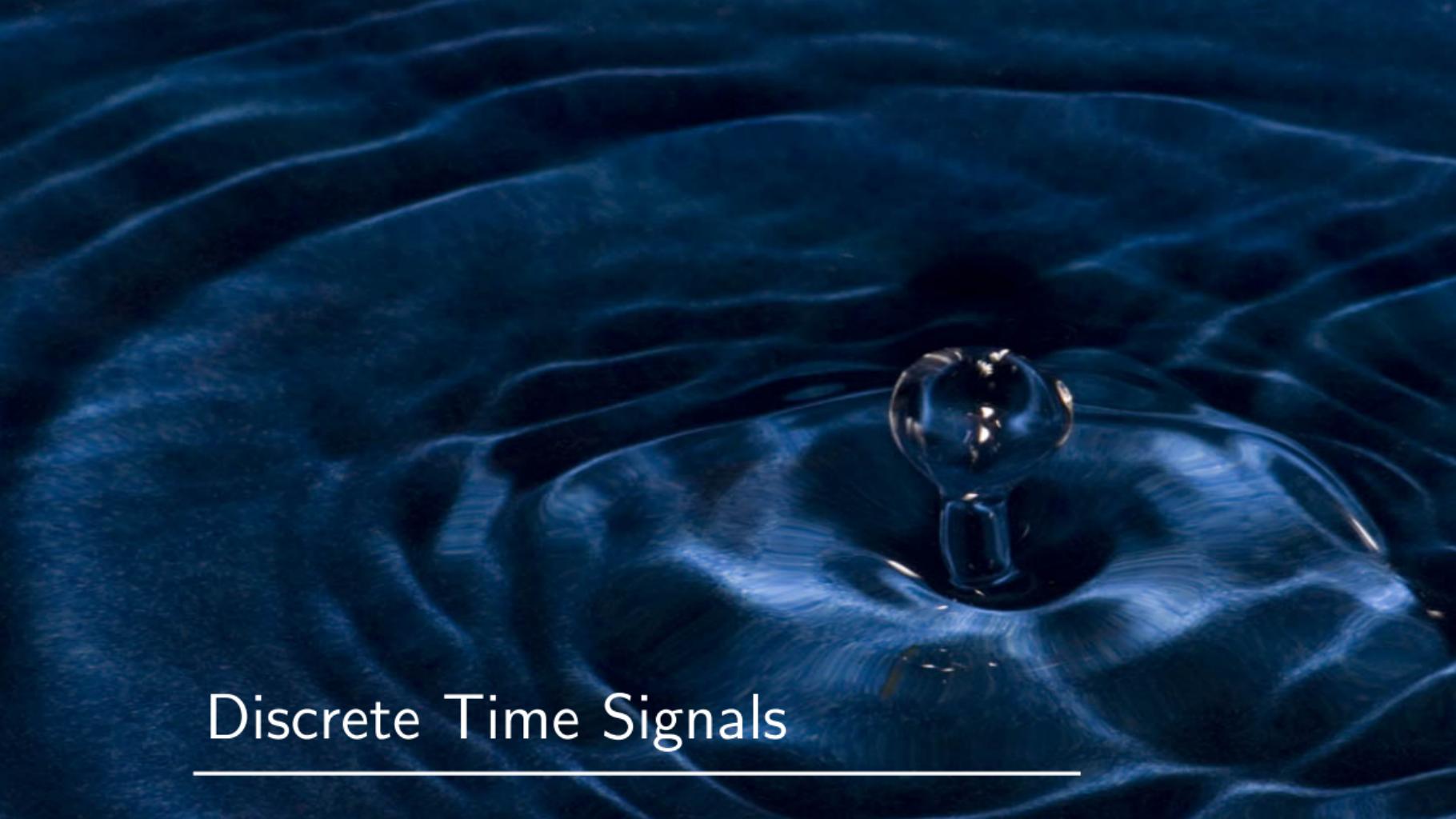
- After the video lecture and a practice exercise or two, you will often see additional **Supplemental Resources**
- Sometimes these will contain background material to provide motivation for the topic
- Sometimes these will provide a refresher of pre-requisite concepts
- Sometimes these will provide deeper explanations of the content (more rigorous proofs, etc.)
- Sometimes a particular signal processing application will be showcased
- **Important:** Though the content in these resources will not be assessed in the homework or exam, you may find that they help you to understand a concept better or increase your interest in it

Before You Start

- **Important:** This is a mathematical treatment of signals and systems (no pain, no gain!)
- Please make sure you have a solid understanding of
 - Complex numbers and arithmetic
 - Linear algebra (vectors, matrices, dot products, eigenvectors, bases . . .)
 - Series (finite and infinite)
 - Calculus of a single variable (derivatives and integrals)
 - Matlab
- To test your readiness or refresh your knowledge, visit the “Pre-class Mathematics Refresher” section of the course

Discrete Time Signals and Systems





Discrete Time Signals

Signals

DEFINITION

Signal (n): A detectable physical quantity . . . by which messages or information can be transmitted (Merriam-Webster)

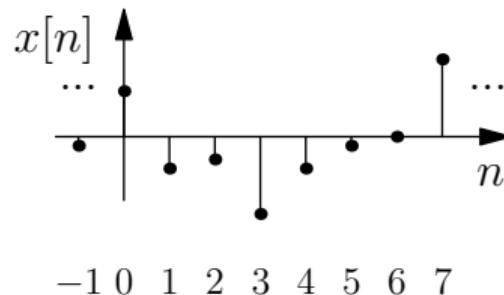
- Signals carry **information**
- Examples:
 - Speech signals transmit language via acoustic waves
 - Radar signals transmit the position and velocity of targets via electromagnetic waves
 - Electrophysiology signals transmit information about processes inside the body
 - Financial signals transmit information about events in the economy
- **Signal processing systems** manipulate the information carried by signals
- This is a course about signals and systems

Signals are Functions

DEFINITION

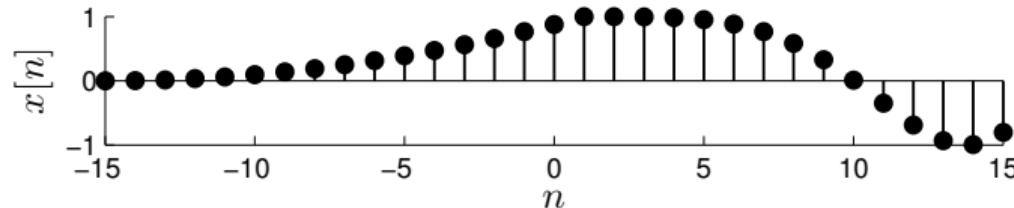
A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
 - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
 - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or \mathbb{C}

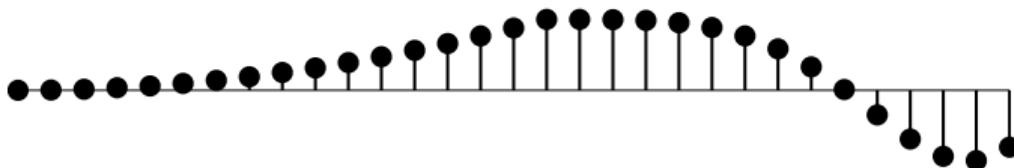


Plotting Real Signals

- When $x[n] \in \mathbb{R}$ (ex: temperature in a room at noon on Monday), we use one signal plot

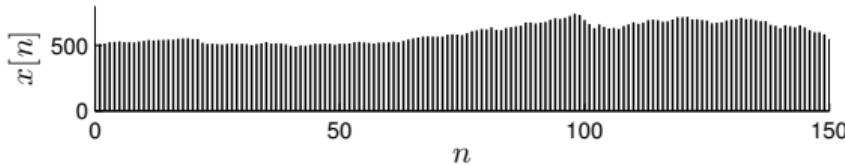


- When it is clear from context, we will often suppress the labels on one or both axes, like this

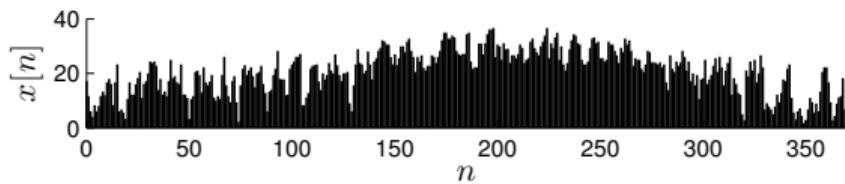


A Menagerie of Signals

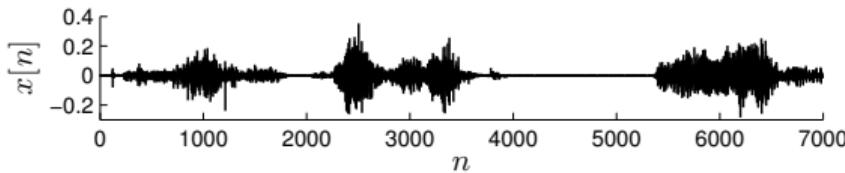
- Google Share daily share price for 5 months



- Temperature at Houston Intercontinental Airport in 2013 (Celcius)

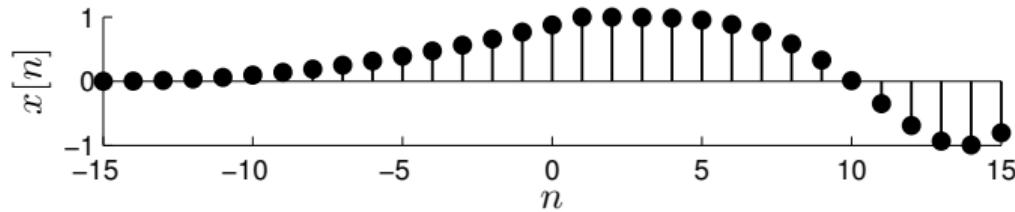


- Excerpt from Shakespeare's *Hamlet*

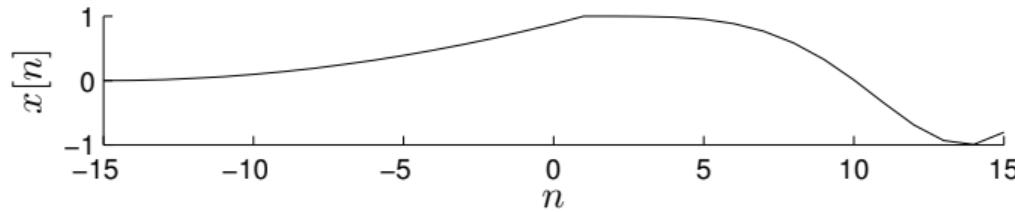


Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable n is discrete (integer)
- To plot a discrete-time signal in a program like Matlab, you should use the stem or similar command and not the plot command
- Correct:



- Incorrect:

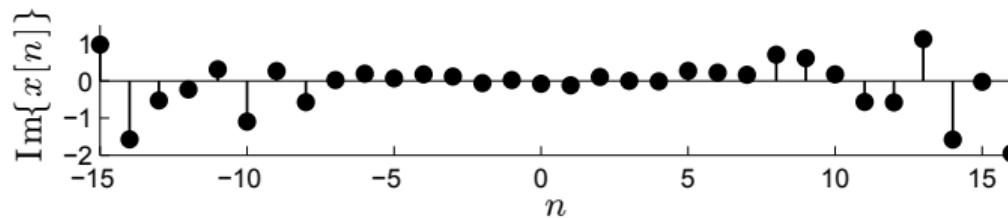
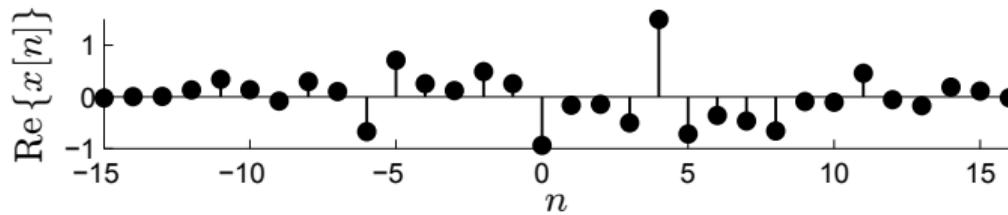


Plotting Complex Signals

- Recall that a complex number $a \in \mathbb{C}$ can be equivalently represented two ways:
 - Polar form: $a = |a| e^{j\angle a}$
 - Rectangular form: $a = \text{Re}\{a\} + j \text{Im}\{a\}$
- Here $j = \sqrt{-1}$ (engineering notation; mathematicians use $i = \sqrt{-1}$)
- When $x[n] \in \mathbb{C}$ (ex: magnitude and phase of an electromagnetic wave), we use two signal plots
 - Rectangular form: $x[n] = \text{Re}\{x[n]\} + j \text{Im}\{x[n]\}$
 - Polar form: $x[n] = |x[n]| e^{j\angle x[n]}$

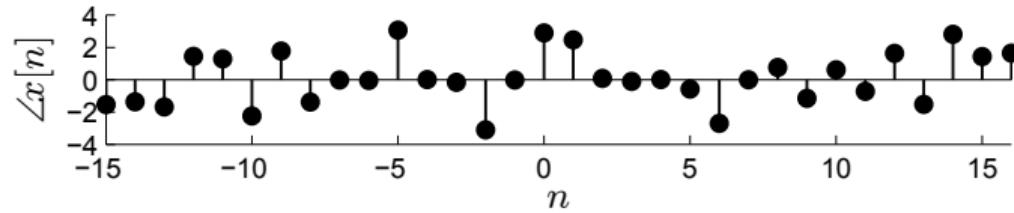
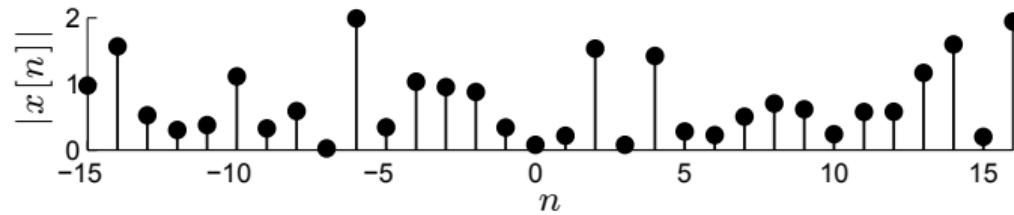
Plotting Complex Signals (Rectangular Form)

- Rectangular form: $x[n] = \operatorname{Re}\{x[n]\} + j \operatorname{Im}\{x[n]\} \in \mathbb{C}$



Plotting Complex Signals (Polar Form)

- Polar form: $x[n] = |x[n]| e^{j\angle(x[n])} \in \mathbb{C}$

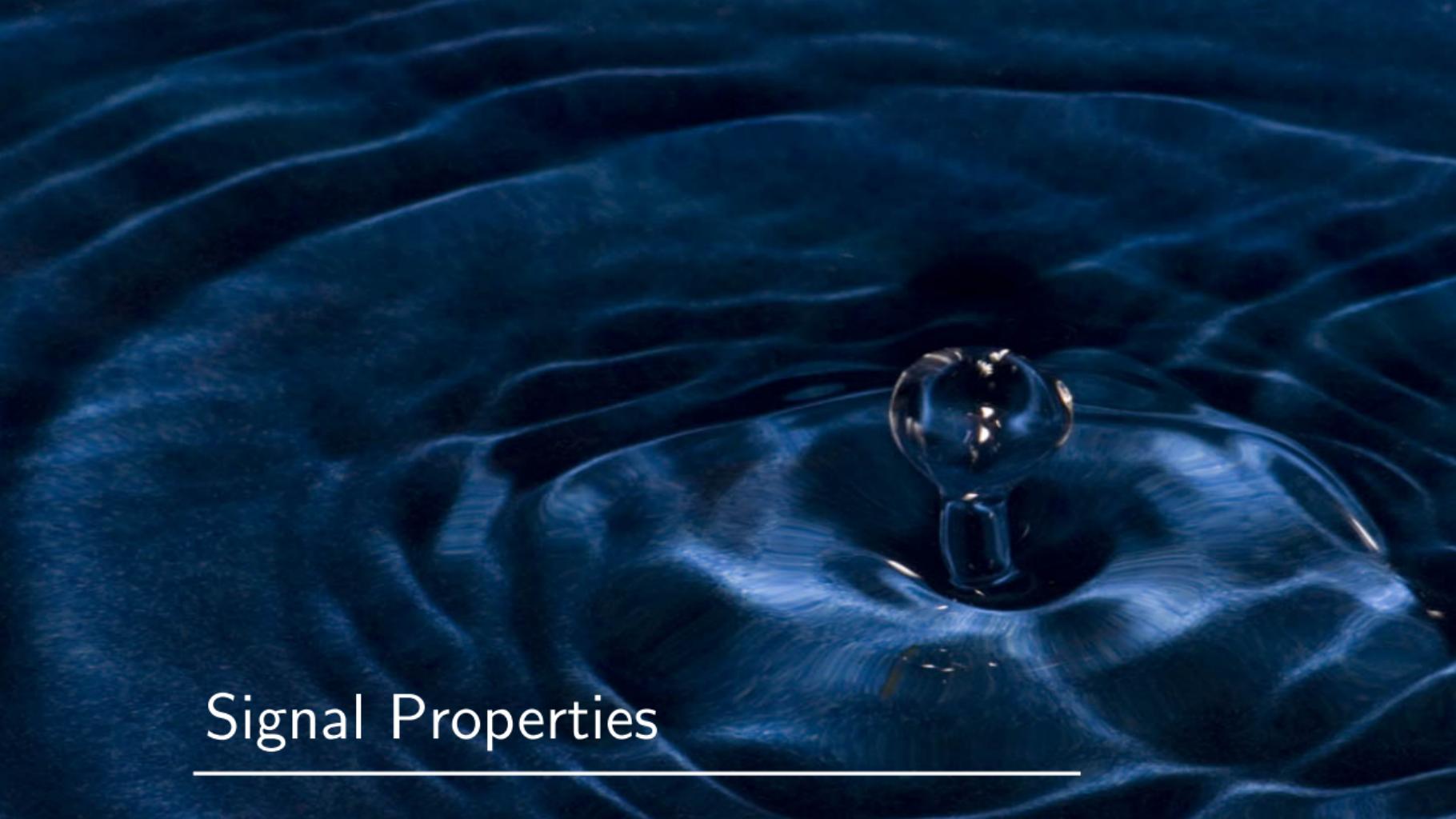


Summary

- Discrete-time signals

- Independent variable is an integer: $n \in \mathbb{Z}$ (will refer to as time)
- Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or \mathbb{C}

- Plot signals correctly!



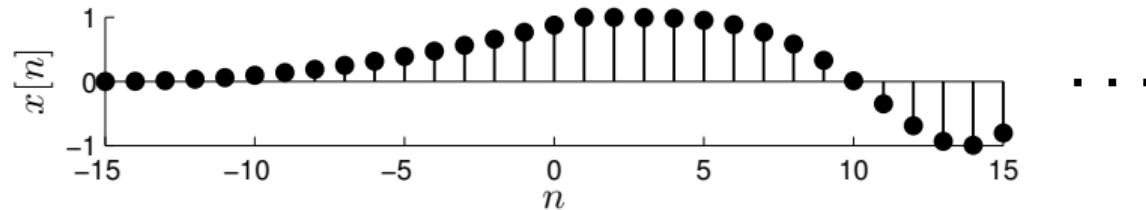
Signal Properties

Signal Properties

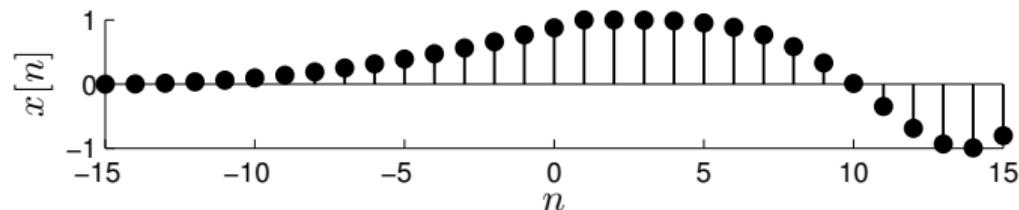
- Infinite/finite-length signals
- Periodic signals
- Causal signals
- Even/odd signals
- Digital signals

Finite/Infinite-Length Signals

- An **infinite-length** discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



- A **finite-length** discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$



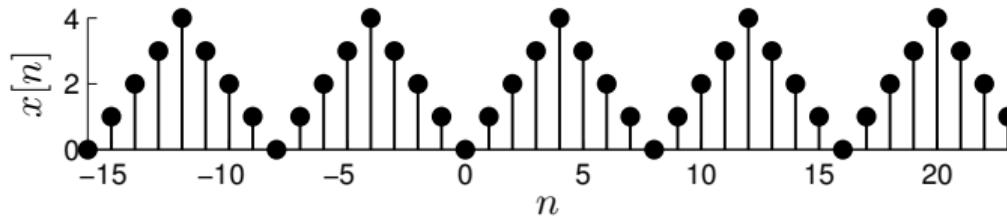
- Important: a finite-length signal is undefined for $n < N_1$ and $n > N_2$

Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

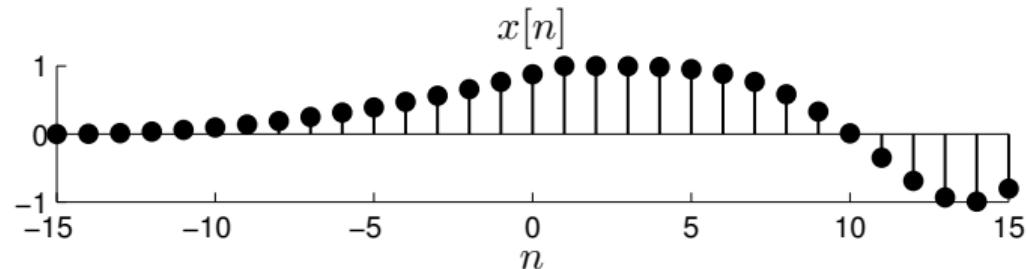
Converting between Finite and Infinite-Length Signals

- Convert an infinite-length signal into a finite-length signal by **windowing**
- Convert a finite-length signal into an infinite-length signal by either
 - (infinite) **zero padding**, or
 - **periodization**

Windowing

- Converts a longer signal into a shorter one

$$y[n] = \begin{cases} x[n] & N_1 \leq n \leq N_2 \\ 0 & \text{otherwise} \end{cases}$$

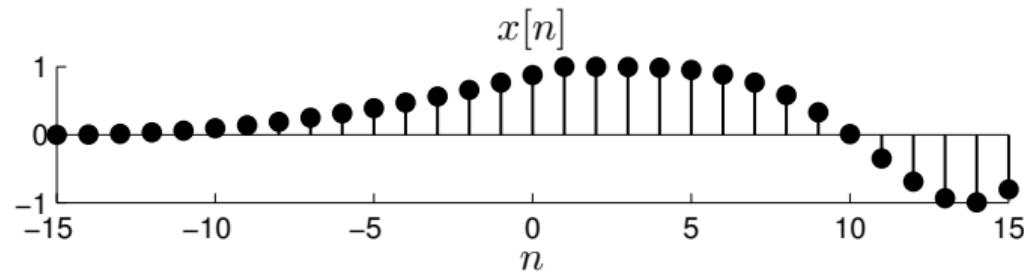


Zero Padding

- Converts a shorter signal into a longer one
- Say $x[n]$ is defined for $N_1 \leq n \leq N_2$

- Given $N_0 \leq N_1 \leq N_2 \leq N_3$

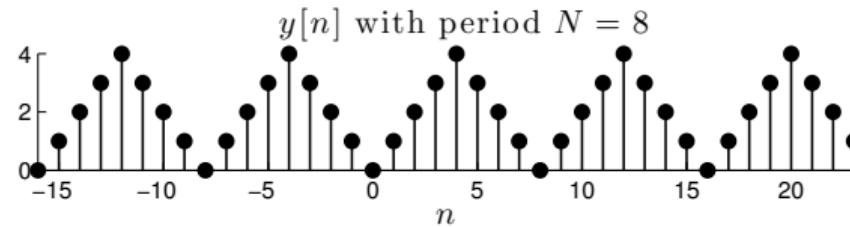
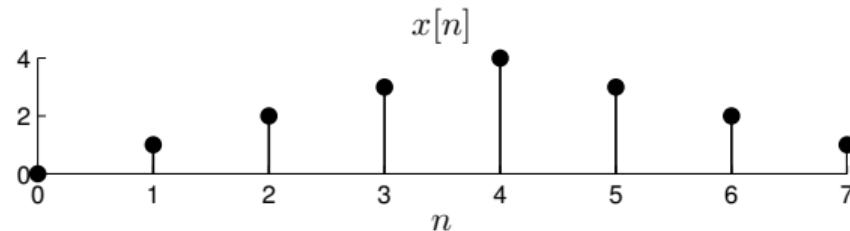
$$y[n] = \begin{cases} 0 & N_0 \leq n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \leq N_3 \end{cases}$$



Periodization

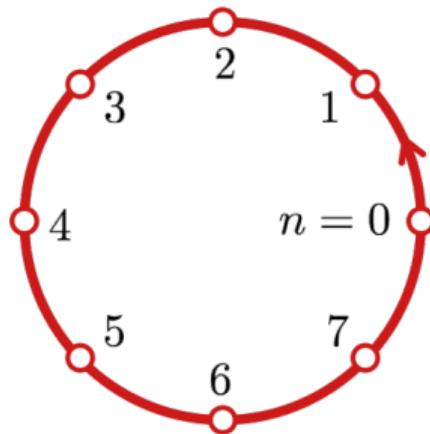
- Converts a finite-length signal into an infinite-length, periodic signal
- Given finite-length $x[n]$, replicate $x[n]$ periodically with period N

$$\begin{aligned}y[n] &= \sum_{m=-\infty}^{\infty} x[n - mN], \quad n \in \mathbb{Z} \\&= \cdots + x[n + 2N] + x[n + N] + x[n] + x[n - N] + x[n - 2N] + \cdots\end{aligned}$$



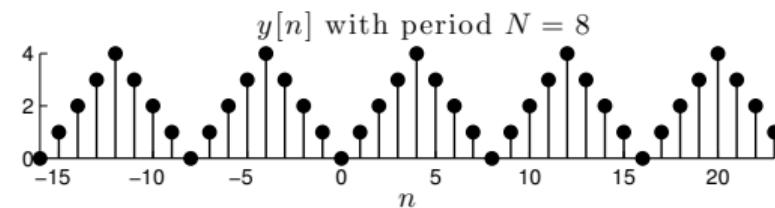
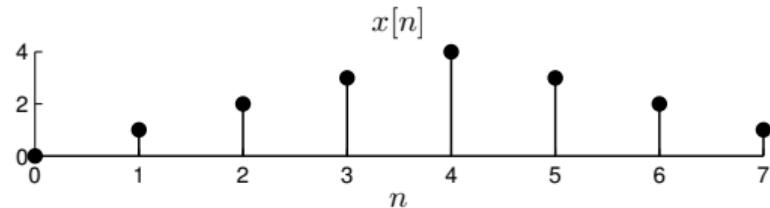
Useful Aside – Modular Arithmetic

- Modular arithmetic with **modulus** N ($\text{mod-}N$) takes place on a **clock** with N “hours”
 - Ex: $(12)_8$ (“twelve mod eight”)
- Modulo arithmetic is inherently **periodic**
 - Ex: $\dots (-12)_8 = (-4)_8 = (4)_8 = (12)_8 = (20)_8 \dots$

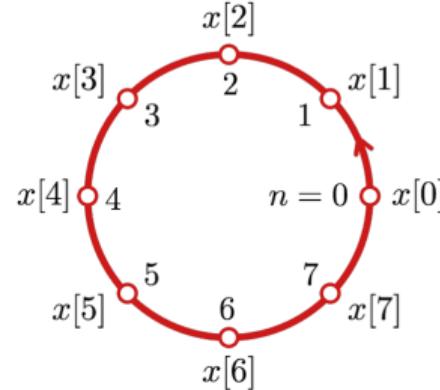


Periodization via Modular Arithmetic

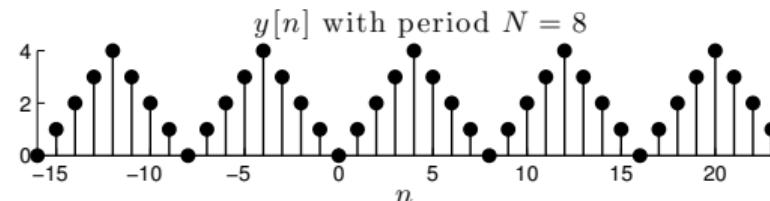
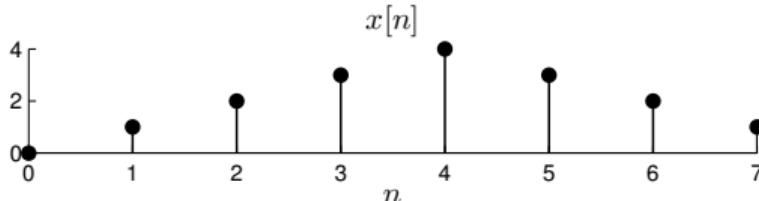
- Consider a length- N signal $x[n]$ defined for $0 \leq n \leq N - 1$
- A convenient way to express periodization with period N is $y[n] = x[(n)_N]$, $n \in \mathbb{Z}$



- Important interpretation
 - Infinite-length signals live on the (infinite) number **line**
 - Periodic signals live on a **circle** – a clock with N “hours”



Finite-Length and Periodic Signals are Equivalent

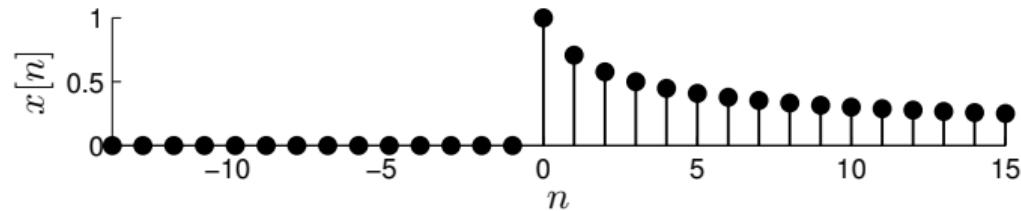


- All of the information in a periodic signal is contained in **one period** (of finite length)
- Any finite-length signal can be periodized
- Conclusion: We can and will think of finite-length signals and periodic signals interchangeably
- We can choose the most convenient viewpoint for solving any given problem
 - Application: Shifting finite length signals

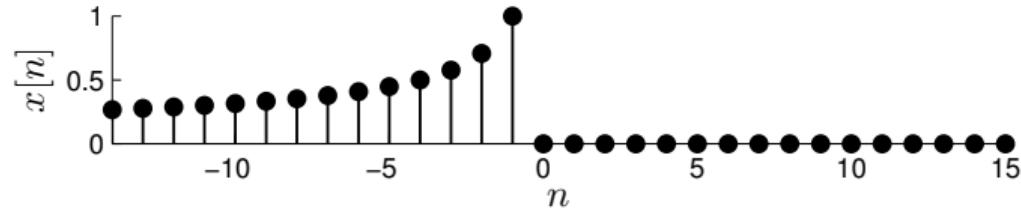
Causal Signals

DEFINITION

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.



- A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$

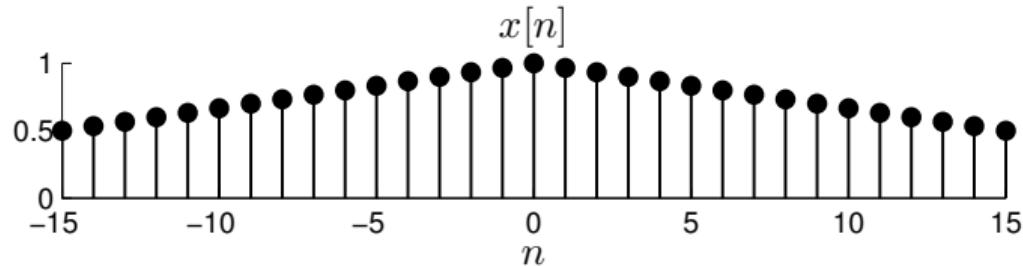


- A signal $x[n]$ is **acausal** if it is not causal

Even Signals

DEFINITION

A real signal $x[n]$ is **even** if $x[-n] = x[n]$

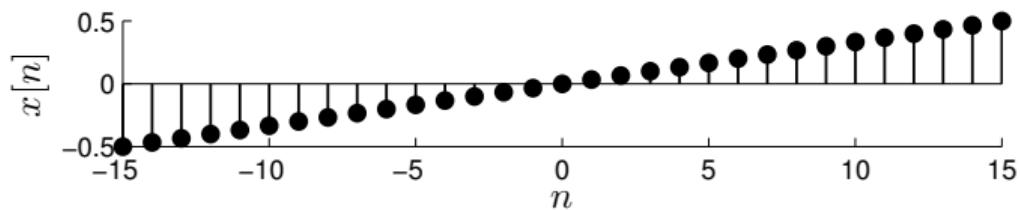


- Even signals are symmetrical around the point $n = 0$

Odd Signals

DEFINITION

A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$



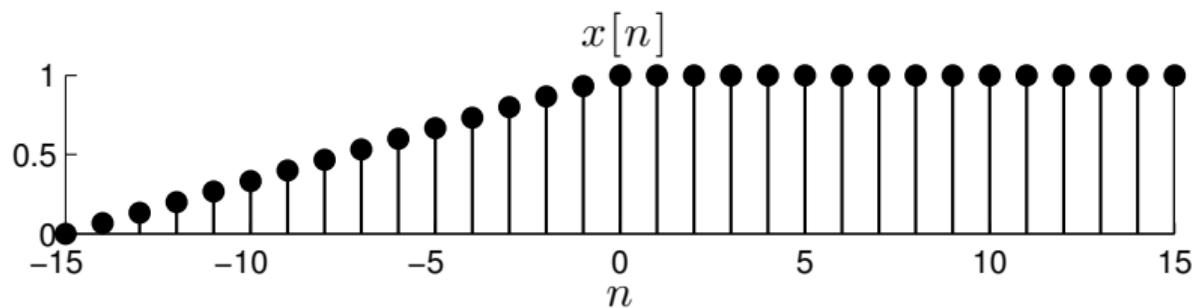
- Odd signals are anti-symmetrical around the point $n = 0$

Even+Odd Signal Decomposition

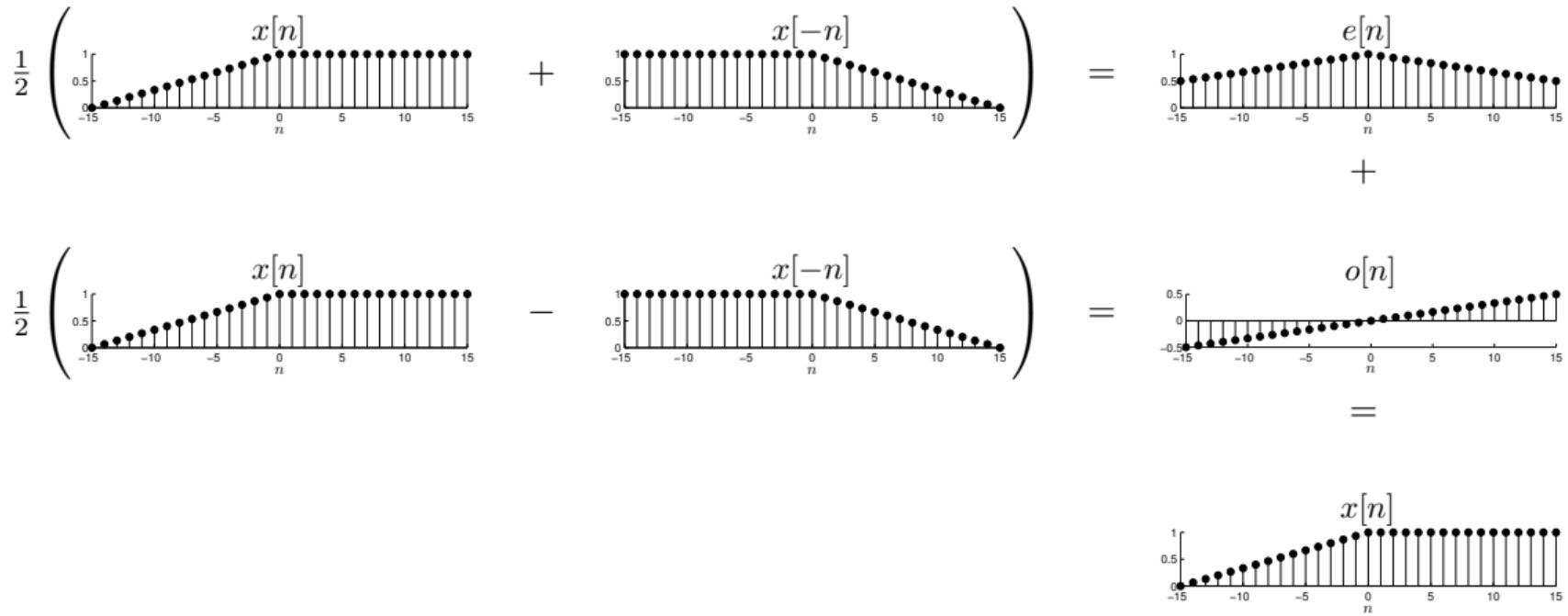
- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part: $e[n] = \frac{1}{2} (x[n] + x[-n])$ (easy to verify that $e[n]$ is even)
- Odd part: $o[n] = \frac{1}{2} (x[n] - x[-n])$ (easy to verify that $o[n]$ is odd)
- **Decomposition** $x[n] = e[n] + o[n]$
- Verify the decomposition:

$$\begin{aligned} e[n] + o[n] &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= \frac{1}{2}(x[n] + x[-n] + x[n] - x[-n]) \\ &= \frac{1}{2}(2x[n]) = x[n] \quad \checkmark \end{aligned}$$

Even+Odd Signal Decomposition in Pictures



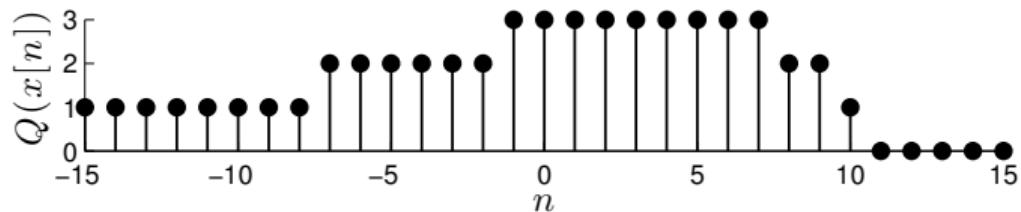
Even+Odd Signal Decomposition in Pictures



Digital Signals

- **Digital signals** are a special sub-class of discrete-time signals

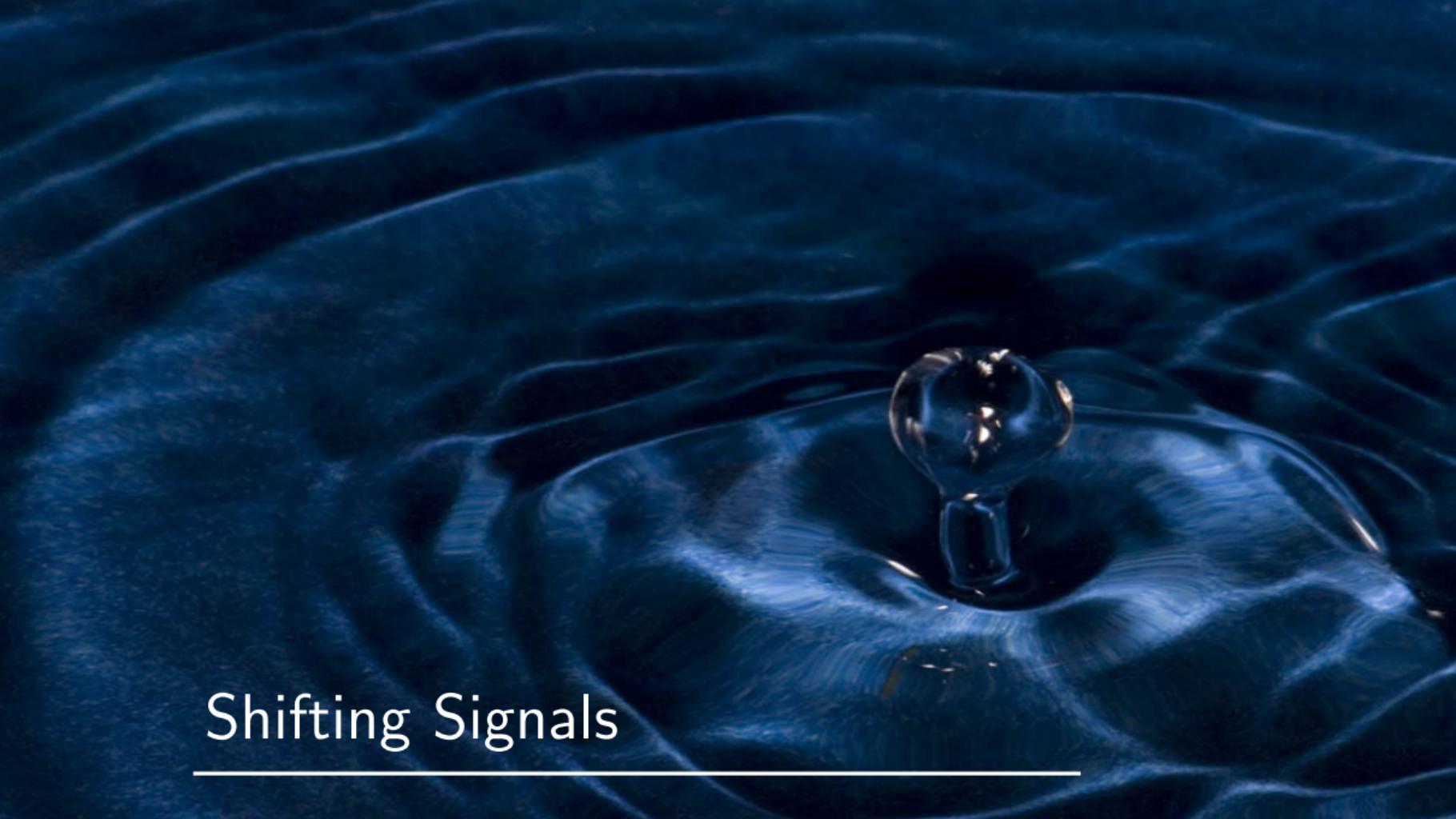
- Independent variable is still an integer: $n \in \mathbb{Z}$
- Dependent variable is from a **finite set of integers**: $x[n] \in \{0, 1, \dots, D - 1\}$
- Typically, choose $D = 2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
- Ex: Digital signal with $q = 2$ bits $\Rightarrow D = 2^2 = 4$ levels



- Ex: Compact discs use $q = 16$ bits $\Rightarrow D = 2^{16} = 65536$ levels

Summary

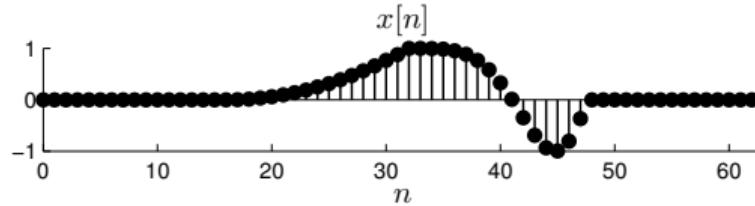
- Signals can be classified many different ways (real/complex, infinite/finite-length, periodic/aperiodic, causal/acausal, even/odd, ...)
- Finite-length signals are equivalent to periodic signals; modulo arithmetic useful



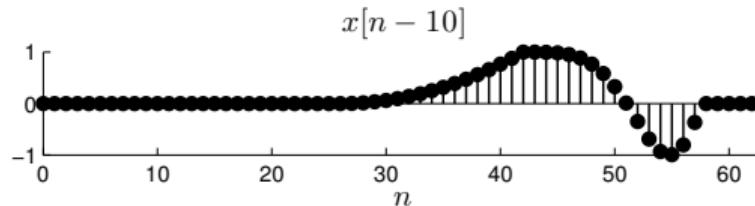
Shifting Signals

Shifting Infinite-Length Signals

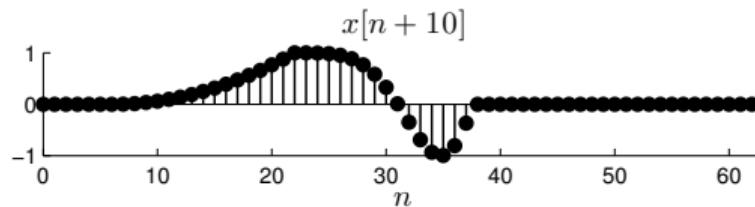
- Given an infinite-length signal $x[n]$, we can **shift** it back and forth in time via $x[n - m]$, $m \in \mathbb{Z}$



- When $m > 0$, $x[n - m]$ shifts to the **right** (forward in time, delay)



- When $m < 0$, $x[n - m]$ shifts to the **left** (back in time, advance)

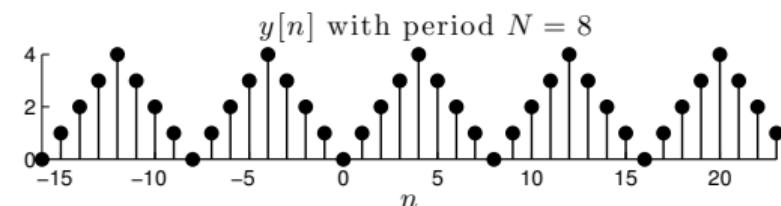
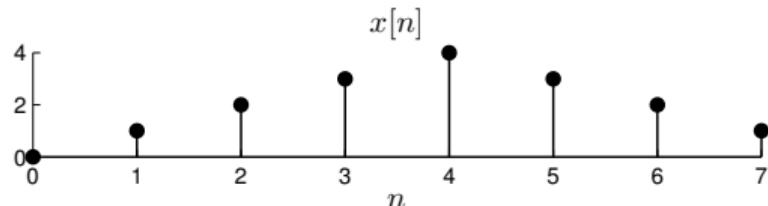


Periodic Signals and Modular Arithmetic

- A convenient way to express a signal $y[n]$ that is periodic with period N is

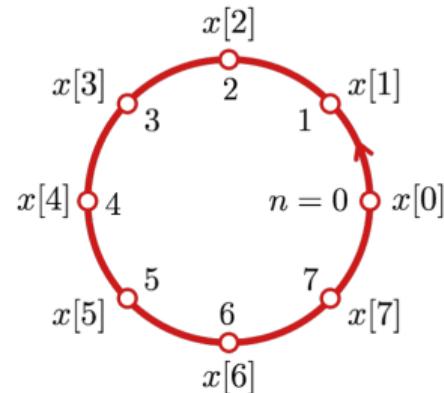
$$y[n] = x[(n)_N], \quad n \in \mathbb{Z}$$

where $x[n]$, defined for $0 \leq n \leq N - 1$, comprises one period



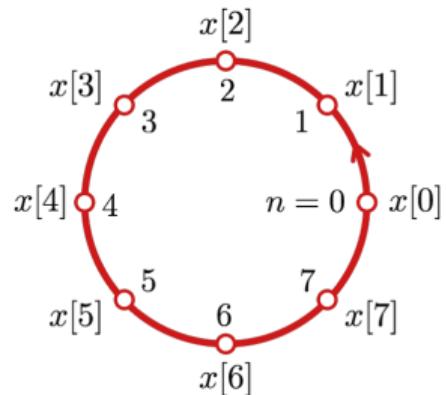
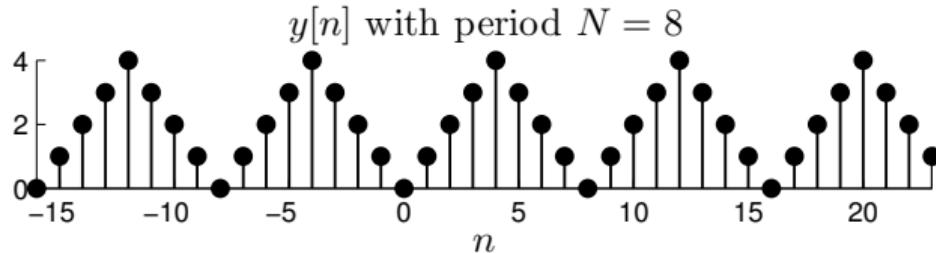
- Important interpretation

- Infinite-length signals live on the (infinite) number line
- Periodic signals live on a circle
– a clock with N “hours”

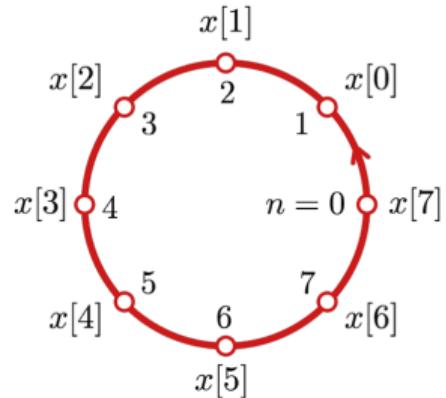
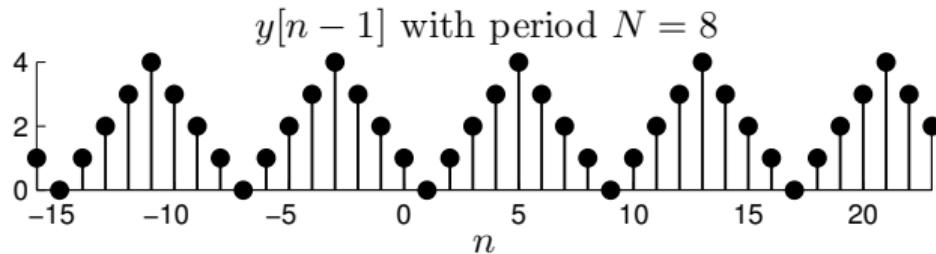


Shifting Periodic Signals

- Periodic signals can also be shifted; consider $y[n] = x[(n)_N]$



- Shift one sample into the future: $y[n - 1] = x[(n - 1)_N]$



Shifting Finite-Length Signals

- Consider finite-length signals x and v defined for $0 \leq n \leq N - 1$ and suppose " $v[n] = x[n - 1]$ "

$$v[0] = ??$$

$$v[1] = x[0]$$

$$v[2] = x[1]$$

$$v[3] = x[2]$$

⋮

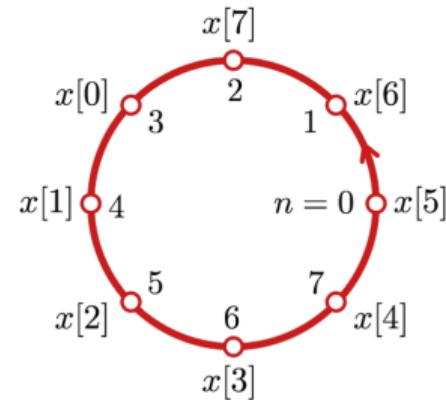
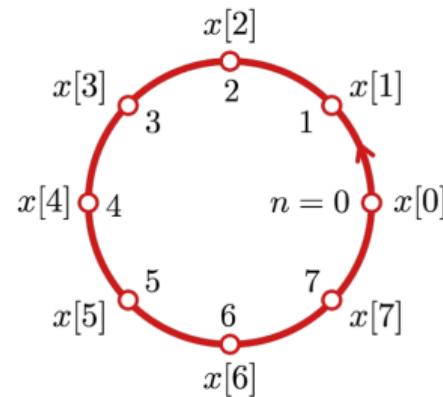
$$v[N - 1] = x[N - 2]$$

$$?? = x[N - 1]$$

- What to put in $v[0]$? What to do with $x[N - 1]$? We don't want to invent/lose information
- Elegant solution: Assume x and v are both periodic with period N ; then $v[n] = x[(n - 1)_N]$
- This is called a **periodic** or **circular shift** (see `circshift` and `mod` in Matlab)

Circular Shift Example

- Elegant formula for circular shift of $x[n]$ by m time steps: $x[(n - m)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(n - 3)_8]$



Circular Shift Example

- Elegant formula for circular shift of $x[n]$ by m time steps: $x[(n - m)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(n - m)_N]$

$$v[0] = x[5]$$

$$v[1] = x[6]$$

$$v[2] = x[7]$$

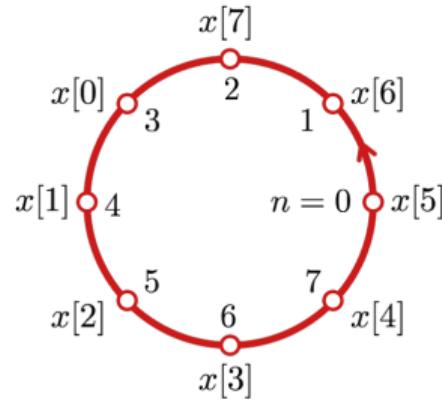
$$v[3] = x[0]$$

$$v[4] = x[1]$$

$$v[5] = x[2]$$

$$v[6] = x[3]$$

$$v[7] = x[4]$$



Circular Time Reversal

- For infinite length signals, the transformation of reversing the time axis $x[-n]$ is obvious
- Not so obvious for periodic/finite-length signals
- Elegant formula for reversing the time axis of a periodic/finite-length signal: $x[(-n)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(-n)_N]$

$$v[0] = x[0]$$

$$v[1] = x[7]$$

$$v[2] = x[6]$$

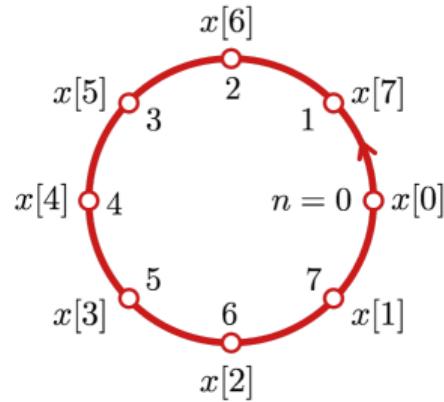
$$v[3] = x[5]$$

$$v[4] = x[4]$$

$$v[5] = x[3]$$

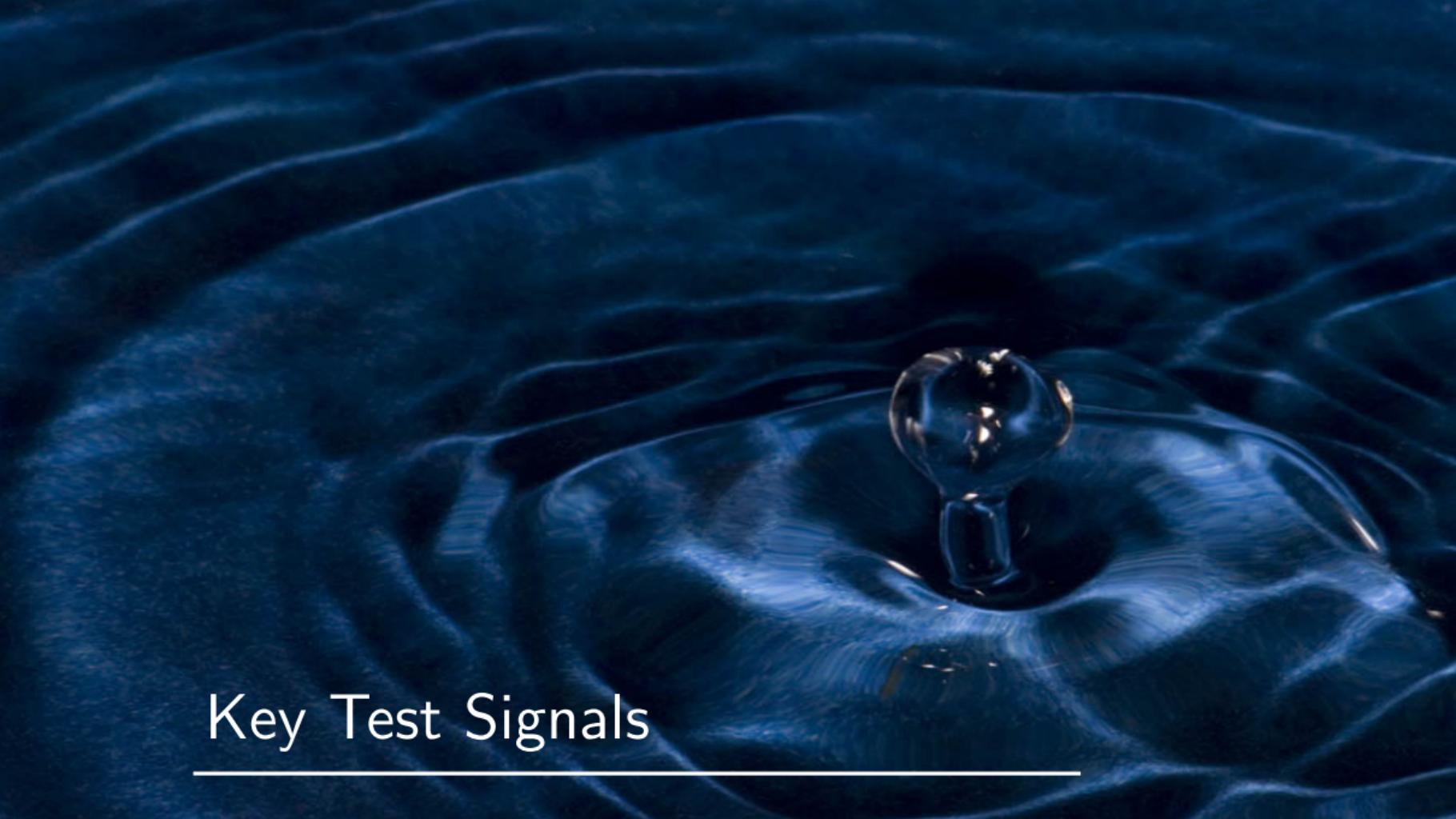
$$v[6] = x[2]$$

$$v[7] = x[1]$$



Summary

- Shifting a signal moves it forward or backward in time
- Modulo arithmetic provides and easy way to shift periodic signals



Key Test Signals

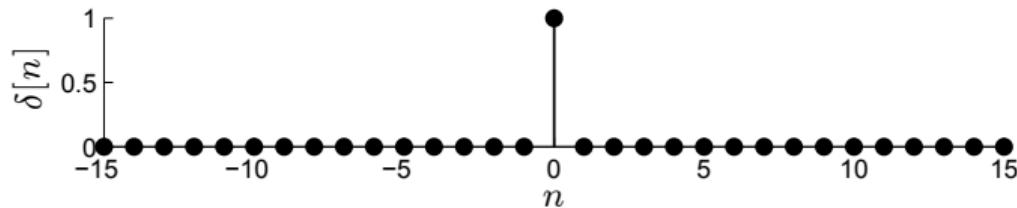
A Toolbox of Test Signals

- Delta function
- Unit step
- Unit pulse
- Real exponential
- Still to come: sinusoids, complex exponentials
- **Note:** We will introduce the test signals as infinite-length signals, but each has a finite-length equivalent

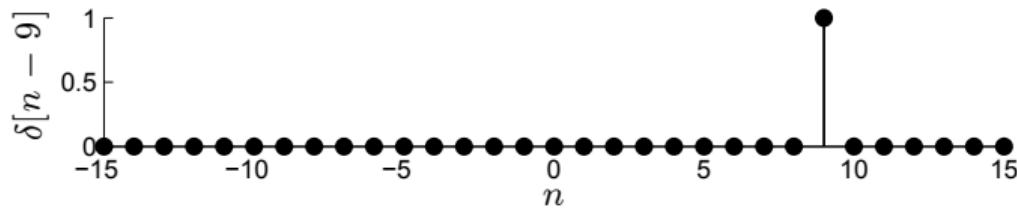
Delta Function

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



- The shifted delta function $\delta[n - m]$ peaks up at $n = m$; here $m = 9$

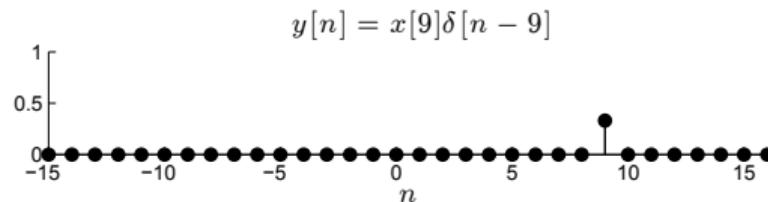
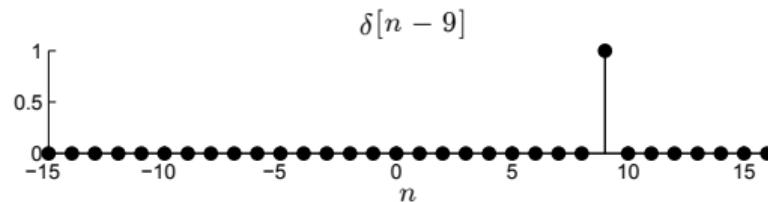
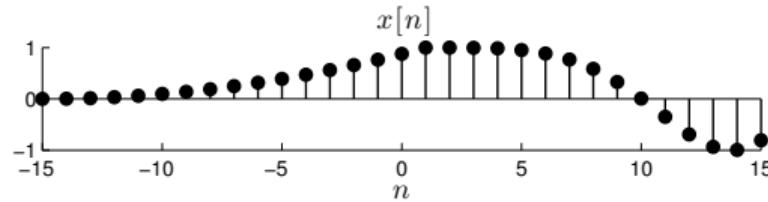


Delta Functions Sample

- Multiplying a signal by a shifted delta function picks out one sample of the signal and sets all other samples to zero

$$y[n] = x[n] \delta[n - m] = x[m] \delta[n - m]$$

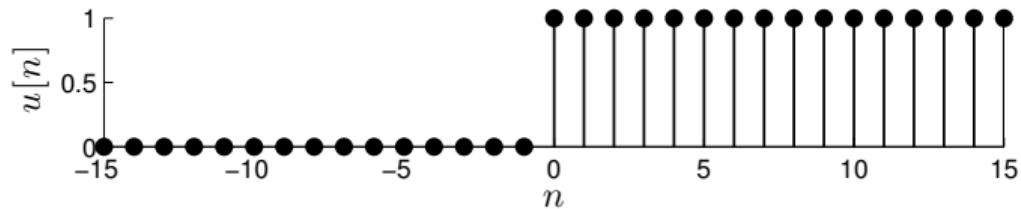
- Important: m is a fixed constant, and so $x[m]$ is a constant (and not a signal)



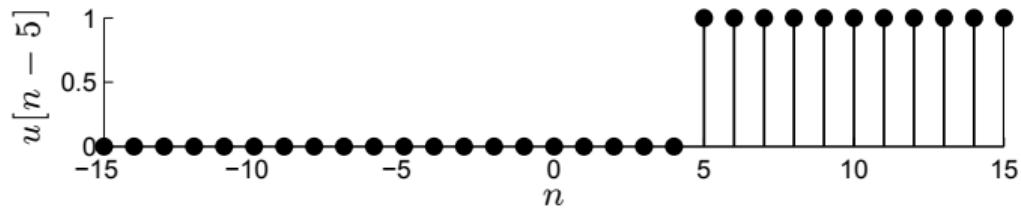
Unit Step

DEFINITION

The **unit step** $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n - m]$ jumps from 0 to 1 at $n = m$; here $m = 5$

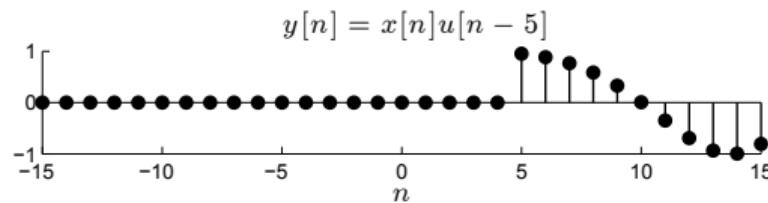
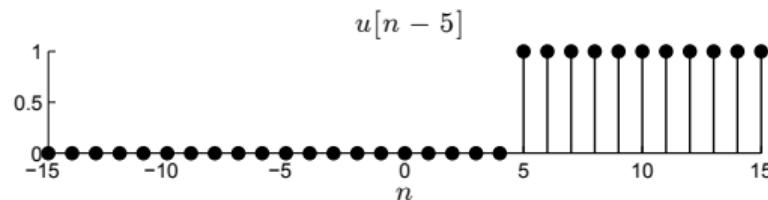
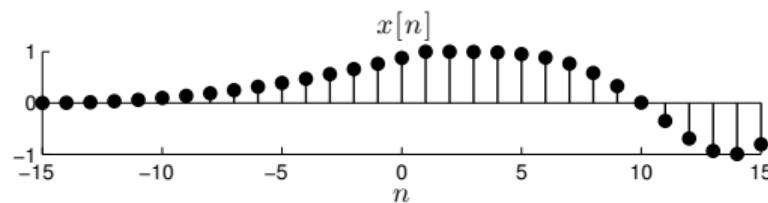


Unit Step Selects Part of a Signal

- Multiplying a signal by a shifted unit step function zeros out its entries for $n < m$

$$y[n] = x[n] u[n - m]$$

(Note: For $m = 0$, this makes $y[n]$ causal)

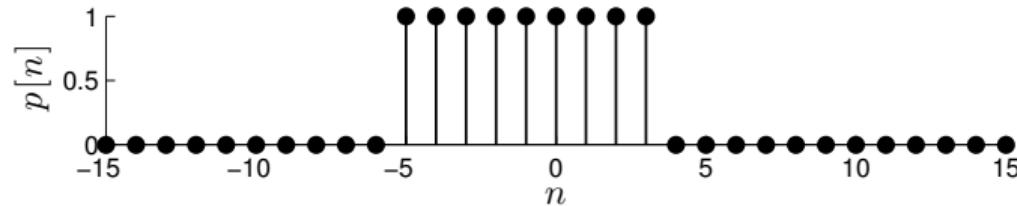


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \leq n \leq N_2 \\ 0 & n > N_2 \end{cases}$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

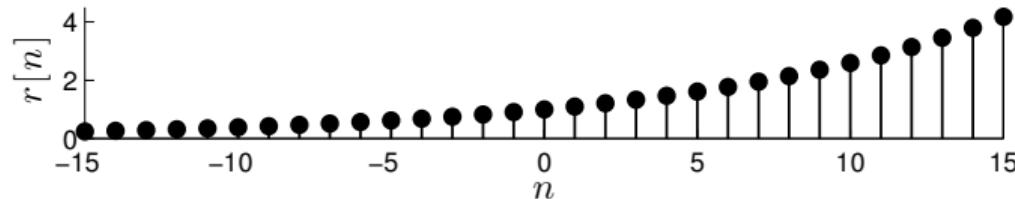
$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

Real Exponential

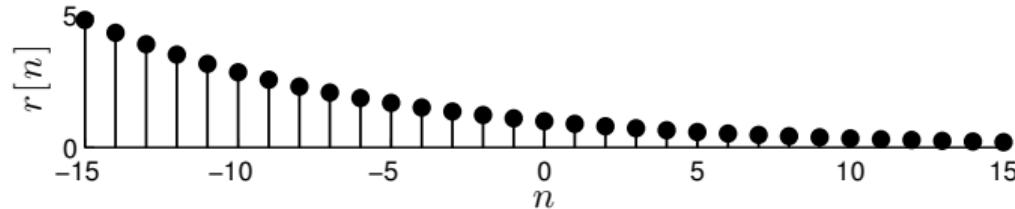
DEFINITION

The **real exponential** $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$



- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$



Summary

- We will use our test signals often, especially the delta function and unit step



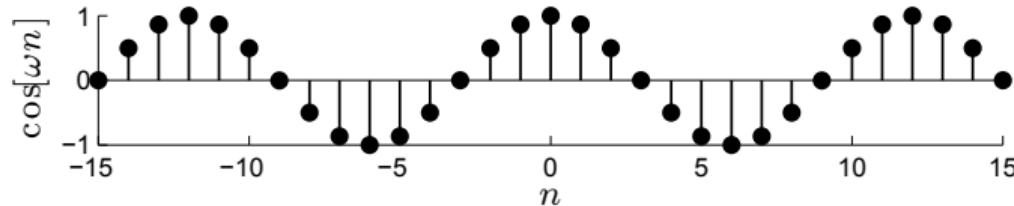
Sinusoids

Sinusoids

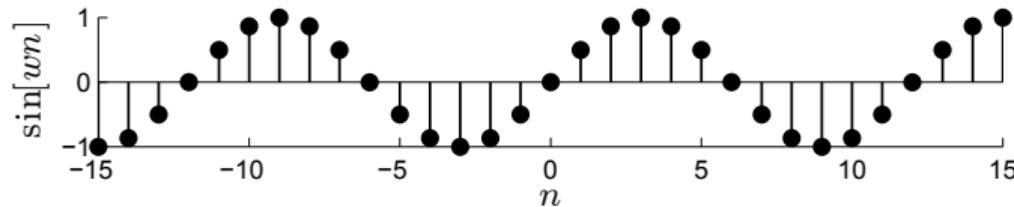
- Sinusoids appear in myriad disciplines, in particular signal processing
- They are the basis (literally) of Fourier analysis (DFT, DTFT)
- We will introduce
 - Real-valued sinusoids
 - (Complex) sinusoid
 - Complex exponential

Sinusoids

- There are two natural real-valued sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)
- $\cos(\omega n)$ (even)

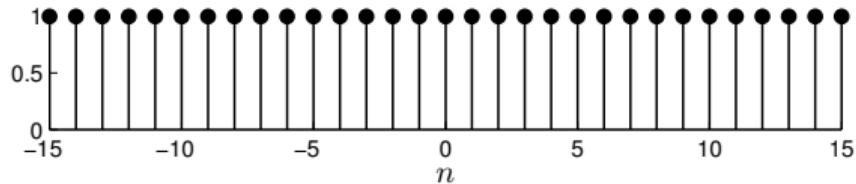


- $\sin(\omega n)$ (odd)

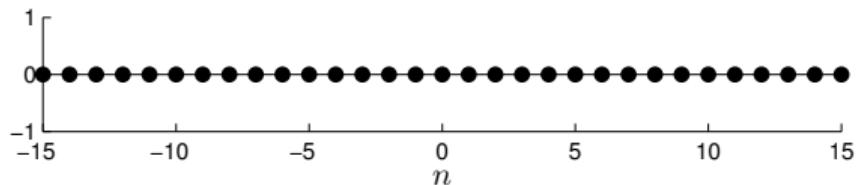


Sinusoid Examples

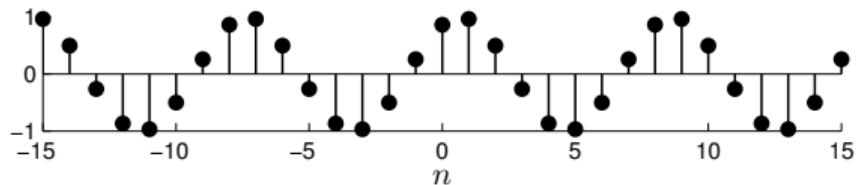
- $\cos(0n)$



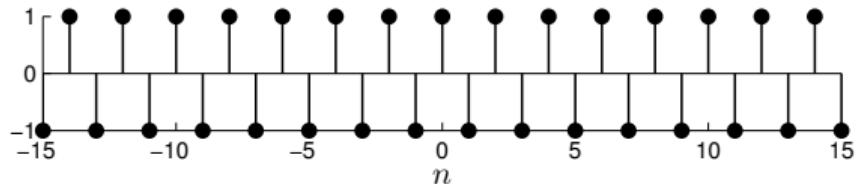
- $\sin(0n)$



- $\sin(\frac{\pi}{4}n + \frac{2\pi}{6})$



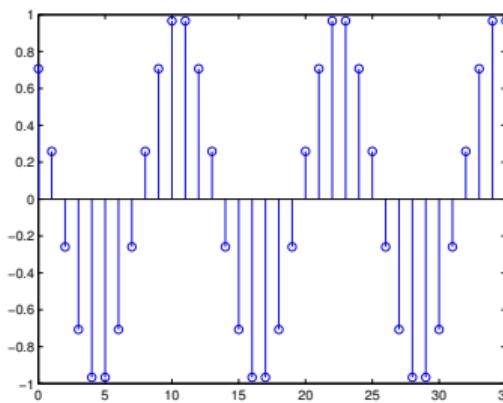
- $\cos(\pi n)$



Get Comfortable with Sinusoids!

- It's easy to play around in Matlab to get comfortable with the properties of sinusoids

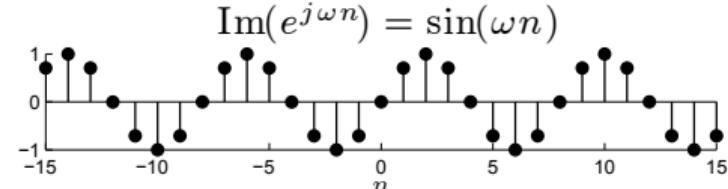
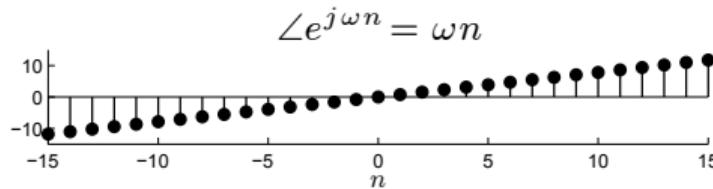
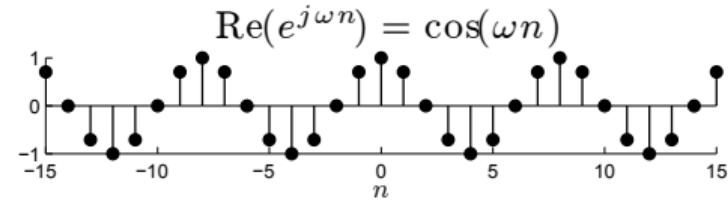
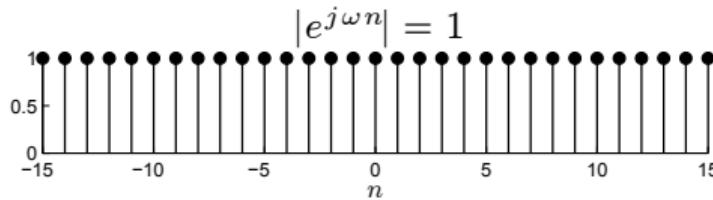
```
N=36;  
n=0:N-1;  
omega=pi/6;  
phi=pi/4;  
x=cos(omega*n+phi);  
stem(n,x)
```



Complex Sinusoid

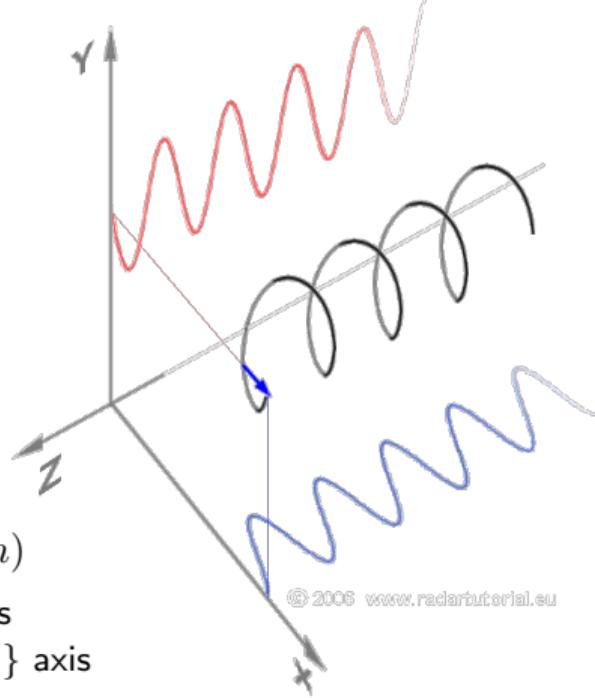
- The complex-valued sinusoid combines both the cos and sin terms (via Euler's identity)

$$e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)$$



A Complex Sinusoid is a Helix

$$e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)$$



- A complex sinusoid is a **helix** in 3D space ($\text{Re}\{\cdot\}$, $\text{Im}\{\cdot\}$, n)
 - **Real part** (cos term) is the projection onto the $\text{Re}\{\cdot\}$ axis
 - **Imaginary part** (sin term) is the projection onto the $\text{Im}\{\cdot\}$ axis
- Frequency ω determines rotation speed and direction of helix
 - $\omega > 0 \Rightarrow$ anticlockwise rotation
 - $\omega < 0 \Rightarrow$ clockwise rotation

Complex Sinusoid is a Helix (Animation)

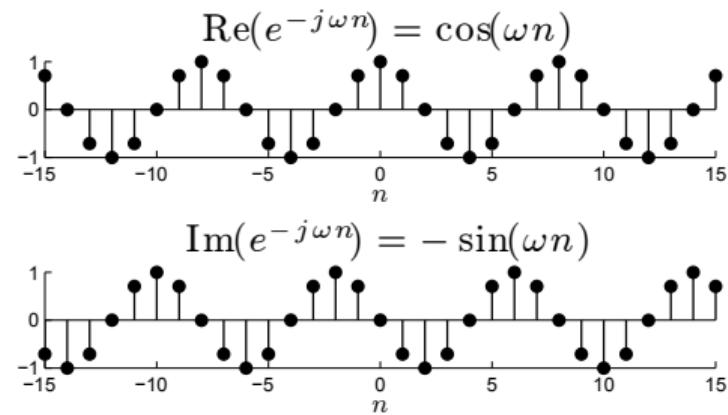
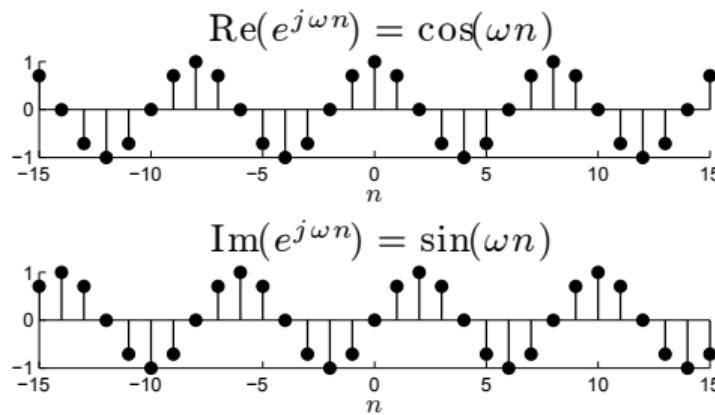
- Complex sinusoid animation

Negative Frequency

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency $-\omega$

$$e^{j(-\omega)n} = e^{-j\omega n} = \cos(-\omega n) + j \sin(-\omega n) = \cos(\omega n) - j \sin(\omega n)$$

- Also note: $e^{j(-\omega)n} = e^{-j\omega n} = (e^{j\omega n})^*$
- Bottom line: negating the frequency is equivalent to complex conjugating a complex sinusoid, which flips the sign of the imaginary, sin term

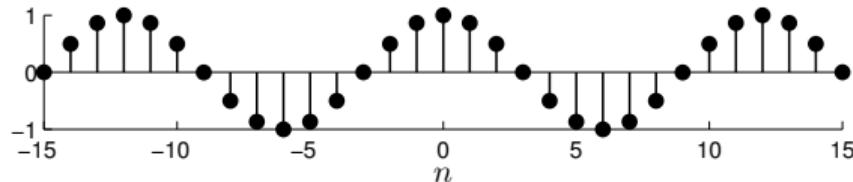


Phase of a Sinusoid

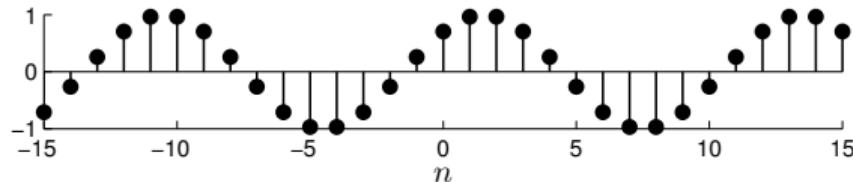
$$e^{j(\omega n + \phi)}$$

- ϕ is a (frequency independent) shift that is referenced to one period of oscillation

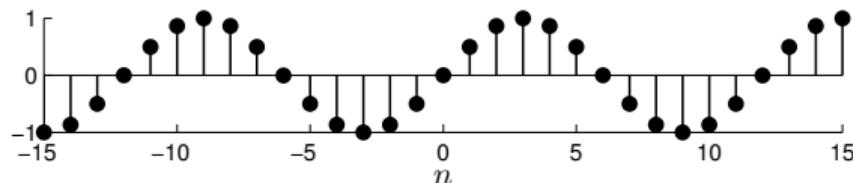
- $\cos\left(\frac{\pi}{6}n - 0\right)$



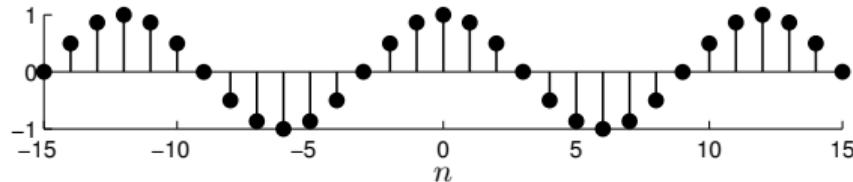
- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right)$



- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}n\right)$

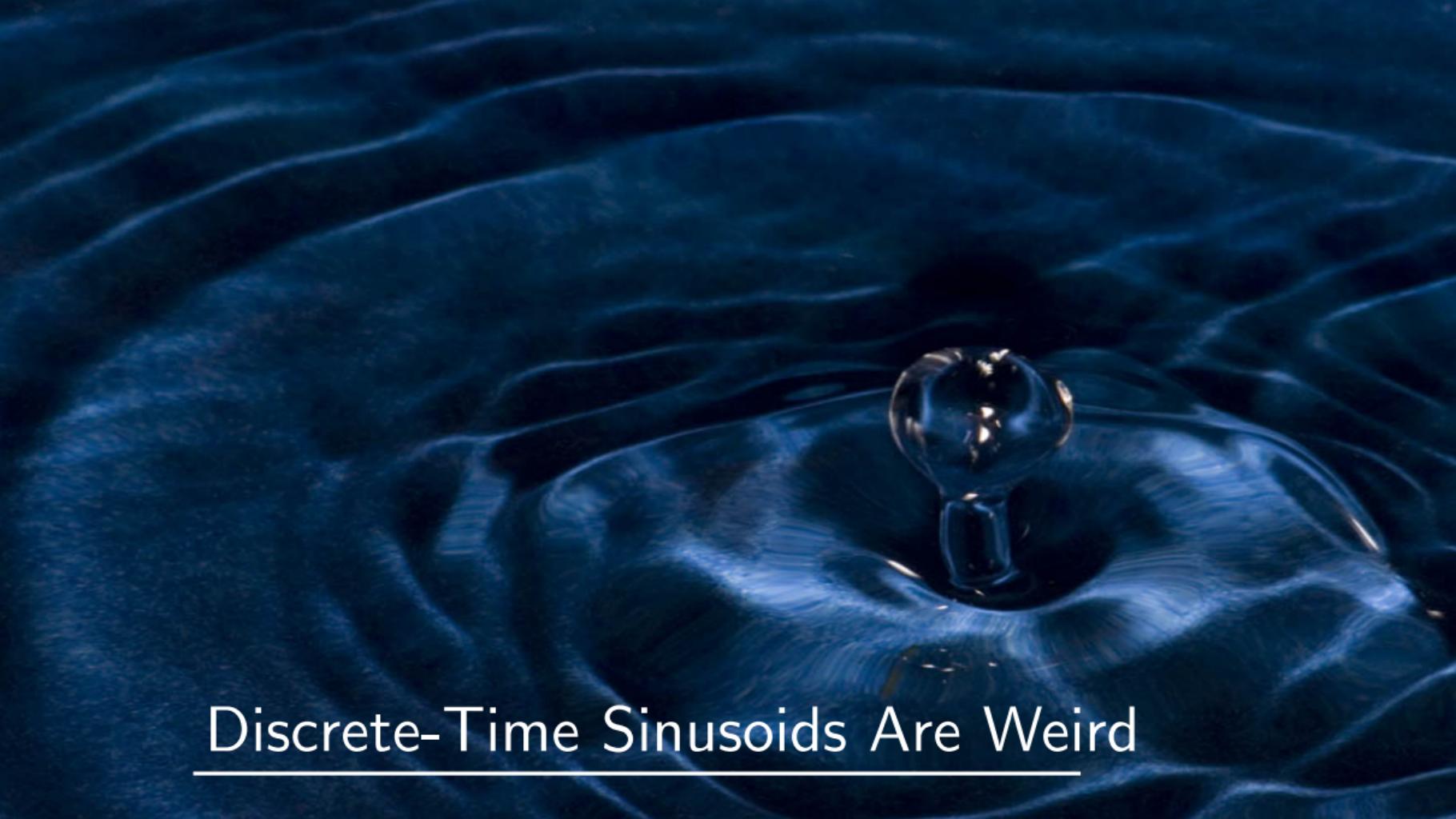


- $\cos\left(\frac{\pi}{6}n - 2\pi\right) = \cos\left(\frac{\pi}{6}n\right)$



Summary

- Sinusoids play a starring role in both the theory and applications of signals and systems
- A sinusoid has a **frequency** and a **phase**
- A complex sinusoid is a helix in three-dimensional space and naturally induces the sine and cosine
- Negative frequency is nothing to be scared by; it just means that the helix spins backwards



Discrete-Time Sinusoids Are Weird

Discrete-Time Sinusoids are Weird!

- Discrete-time sinusoids $e^{j(\omega n + \phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperiodicity

Weird Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies

- $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
- $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$

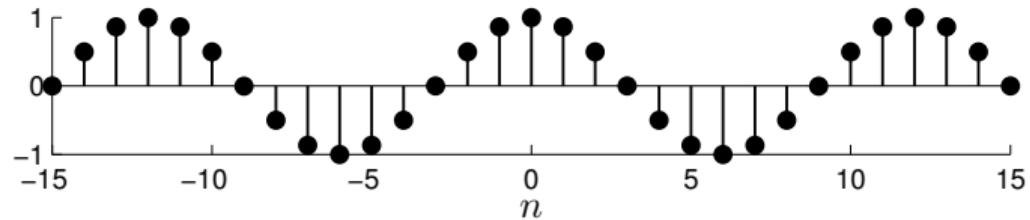
- But note that

$$x_2[n] = e^{j((\omega + 2\pi)n + \phi)} = e^{j(\omega n + \phi) + j2\pi n} = e^{j(\omega n + \phi)} e^{j2\pi n} = e^{j(\omega n + \phi)} = x_1[n]$$

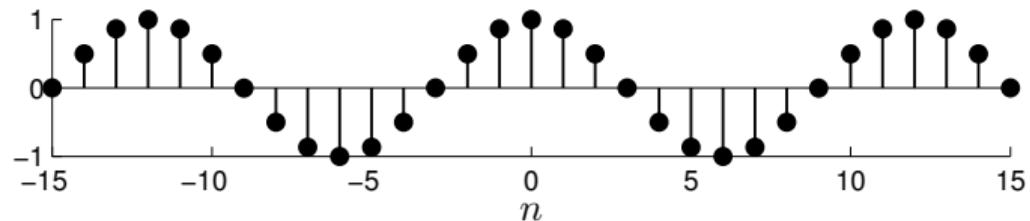
- The signals x_1 and x_2 have different frequencies but are **identical!**
- We say that x_1 and x_2 are aliases; this phenomenon is called **aliasing**
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega + 2\pi m)n + \phi)}$, $m \in \mathbb{Z}$

Aliasing of Sinusoids – Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



- $x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$



Alias-Free Frequencies

- Since

$$x_3[n] = e^{j(\omega + 2\pi m)n + \phi} = e^{j(\omega n + \phi)} = x_1[n] \quad \forall m \in \mathbb{Z}$$

the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length 2π

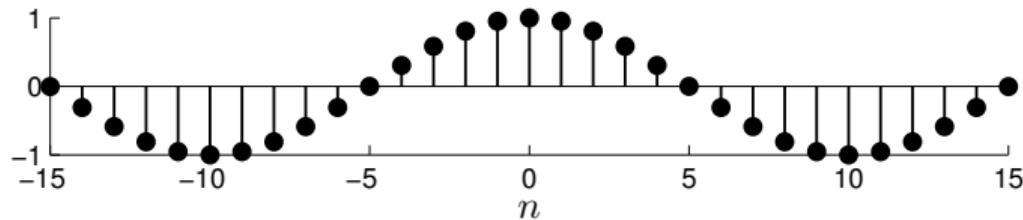
- Convenient to interpret the frequency ω as an **angle**
(then aliasing is handled automatically; more on this later)
- Two intervals are typically used in the signal processing literature (and in this course)
 - $0 \leq \omega < 2\pi$
 - $-\pi < \omega \leq \pi$

Low and High Frequencies

$$e^{j(\omega n + \phi)}$$

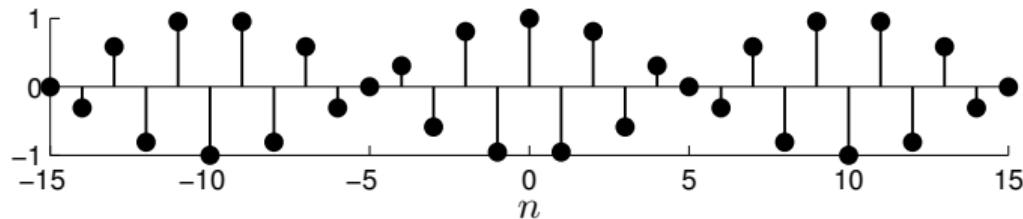
- **Low frequencies:** ω close to 0 or 2π rad

Ex: $\cos\left(\frac{\pi}{10}n\right)$



- **High frequencies:** ω close to π or $-\pi$ rad

Ex: $\cos\left(\frac{9\pi}{10}n\right)$



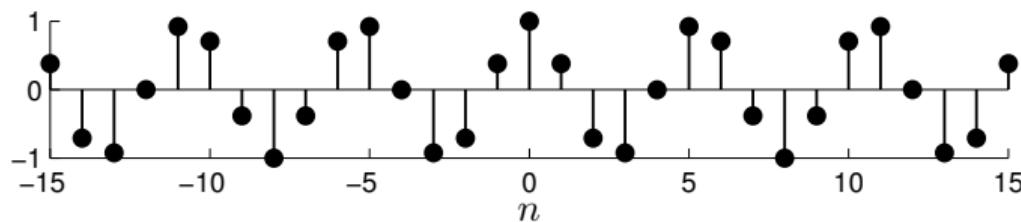
Weird Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n + \phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that x_1 is periodic with period N , since

$$x_1[n+N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} = e^{j(\omega n + \phi)} e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16}n)$, $N = 16$



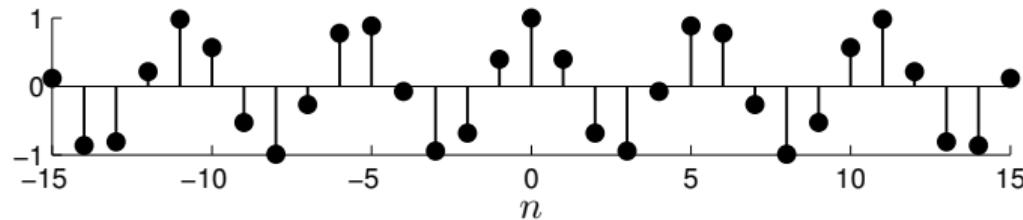
- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)
- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_2[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n + \phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

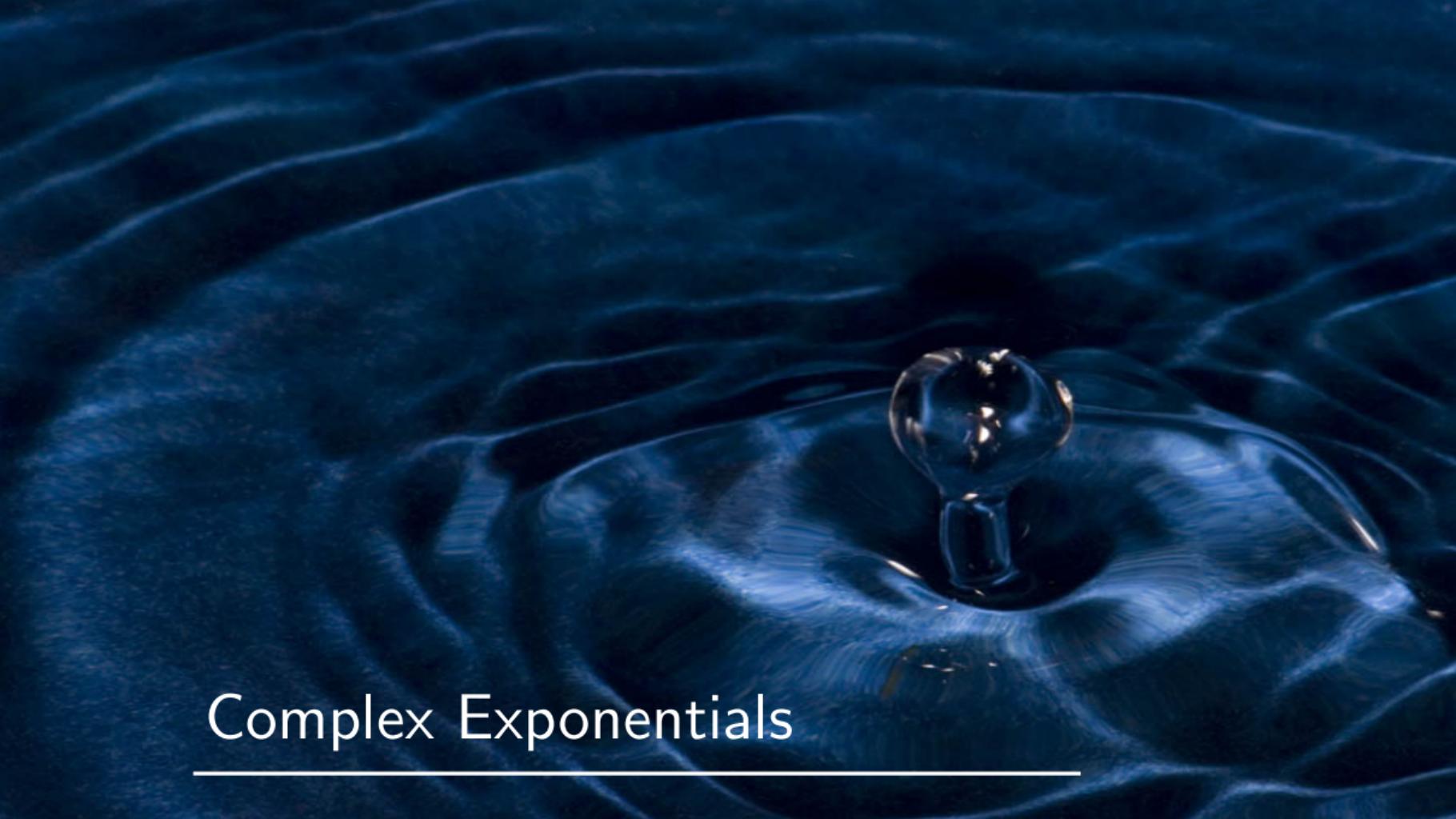
- Which means that
 - **Most** discrete-time sinusoids are **not** periodic!
 - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Harmonic Sinusoids (Matlab)

- [Click here](#) to view a MATLAB demo that visualizes harmonic sinusoids.

Summary

- Discrete-time sinusoids $e^{j(\omega n + \phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperiodicity
- The only sinusoids that are periodic: Harmonic sinusoids $e^{j(\frac{2\pi k}{N}n + \phi)}$, $n, k, N \in \mathbb{Z}$



Complex Exponentials

Complex Exponential

- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Generalize to $e^{\text{General Complex Numbers}}$
- Consider the general complex number $z = |z| e^{j\omega}$, $z \in \mathbb{C}$
 - $|z| = \text{magnitude of } z$
 - $\omega = \angle(z)$, phase angle of z
 - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**
- Now we have
$$z^n = (|z|e^{j\omega})^n = |z|^n(e^{j\omega})^n = |z|^n e^{j\omega n}$$
 - $|z|^n$ is a **real exponential** (a^n with $a = |z|$)
 - $e^{j\omega n}$ is a **complex sinusoid**

Complex Exponential is a Spiral

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

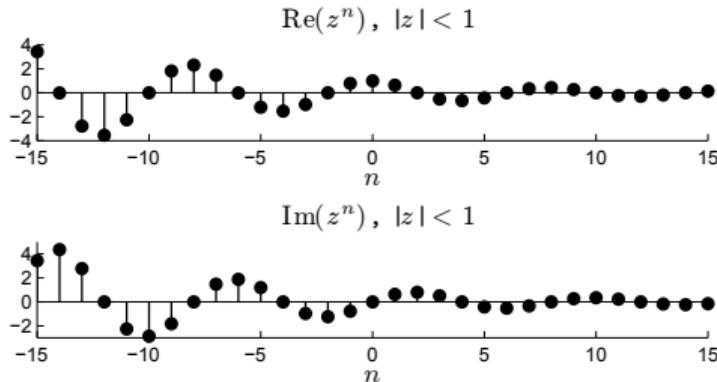
- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**
- z^n is a helix with expanding radius (spiral)

Complex Exponential is a Spiral

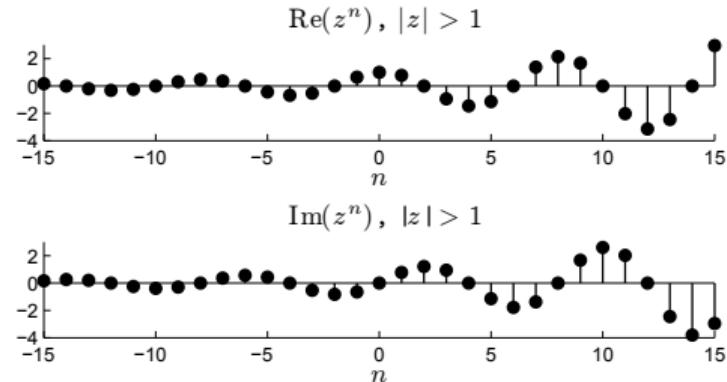
$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**

$$|z| < 1$$



$$|z| > 1$$



Complex Exponentials and z Plane (Matlab)

- [Click here](#) to view a MATLAB demo plotting the signals z^n .

Summary

- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Complex exponential: Generalize $e^{j(\omega n + \phi)}$ to $e^{\text{General Complex Numbers}}$
- A complex exponential is the product of a real exponential and a complex sinusoid
- A complex exponential is a spiral in three-dimensional space