First Post

Vorlesung 2

Definition 0.1. $F \in L(V, W)$ can be represented by a matrix $A \in C^{n \times m}$ with $n = \dim V$ and $m = \dim W$ when basises are chosen.

Definition 0.2 (Complex Hilbert space)**.** A complete (every cauchy sequence converges) complex vector space $\mathcal{H}$ with a sequilinear inner product

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

Definition 0.3. A Basis $(e_1, \ldots, e_n) \subset \mathcal{H}$ with

$$\langle e_i, e_j \rangle = \delta_{ij}$$

Definition 0.4. We define the dual space

$$\mathcal{H}^* = \{f : \mathcal{H} \overset{linear}{\to} \mathbb{C}\}$$

Definition 0.5.

$$\begin{aligned} \mathcal{H}^* &= \{f : \mathcal{H} \overset{linear}{\to} \mathbb{C}\} \\ &= \{\langle x, \cdot \rangle | x \in \mathcal{H}\} \end{aligned}$$

We will use the bra-ket notation:

Definition 0.6 (bra-ket Notation)**.** $\varphi, \psi \in \mathcal{H}$:

$$\begin{aligned} \varphi = |\varphi\rangle &\in \mathcal{H} && \text{bra} \\ \langle \varphi, \cdot \rangle = \langle \varphi| &\in L(\mathcal{H}) && \text{ket} \\ \langle \varphi, \psi \rangle &= \langle \varphi|\psi\rangle \end{aligned}$$

Example of BraKet

Definition 0.7 (Rank-1 operators)**.** Let $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$ The "*Outer product*" acts on $|\eta\rangle \in \mathcal{H}$ as

$$(|\psi\rangle\langle\varphi|)(|\eta\rangle) = |\psi\rangle\langle\varphi|\eta\rangle$$

Definition 0.8 (self-adjoint operators, aka. Hermitian operators)**.**

$$\mathrm{Herm}(\mathcal{H}) = \{A \in L(\mathcal{H}) | A^\dagger = A\}$$

Definition 0.9 (unitary operators)**.**

$$U(\mathcal{H}) = \{A \in L(\mathcal{H}) | A^\dagger = A^{-1}\}$$

Definition 0.10 (normal operators)**.**

$$\{A \in L(\mathcal{H}) | A^\dagger A = AA^\dagger\}$$

is the set of normal operators

Proposition 0.11. *$A \in L(\mathcal{H})$ ist diagonalizable if there is a basis of eigenvectors*

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Proposition 0.12 (diagonalizable (continued))**.** *$A \in L(\mathcal{H}$ is unitarily diagonalizable if there is an orthonormalbasisoof eigenvectors of A*

Theorem 0.13. *Let $A \in L(\mathcal{H}$. A is uniformly diagonalizable $\Leftrightarrow$ A is invertible*

Theorem 0.14 (spectral decomposition)**.** *$A \in L(\mathcal{H}$ normal, $n = \dim(\mathcal{H}$. then*

$$A = \sum_{j=1}^{n} a_j \left|a_j\right\rangle \left\langle a_j\right|$$

for eigenvalues $a_i, \ldots, a_n$ of A and corresponding eigenvectors $\left|a_1\right\rangle, \ldots, \left|a_n\right\rangle$.

Why do we call a matrix which is normal (ie $AA^\dagger = A^\dagger A$) *unitarily diagonalizable*?

Consider $\mathcal{H} = \mathbb{C}^d$ and $A \in L(\mathbb{C}^n)$ normal. We set $U = [\left|a_1\right\rangle, \ldots, \left|a_n\right\rangle]$ which is a matrix that has as its collumns the eigenvectors of A. We can now calculate $U^\dagger = \begin{bmatrix} \langle a_1| \\ \vdots \\ \langle a_n| \end{bmatrix}$ which leads to

$$U^\dagger U = \begin{bmatrix} \langle a_1 \,|\, a_1\rangle & \langle a_1 \,|\, a_2\rangle & \ldots & \langle a_1 \,|\, a_n\rangle \\ \langle a_2 \,|\, a_1\rangle & \langle a_2 \,|\, a_2\rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle a_n \,|\, a_1\rangle & \ldots & & \langle a_n \,|\, a_n\rangle \end{bmatrix} = \mathbf{1}_{\mathbb{C}^{n\times n}}$$

Therefore every normal matrix has a is uniformly diagonalizable.

Definition 0.15 (positive semi-definite operators)**.** $A \in \text{Herm}(\mathcal{H}$ is positive semi-definite (PSD) if

$$\langle \psi \,|\, A \,|\, \psi \rangle \geq 0, \quad \forall \,|\psi\rangle \in \mathcal{H}$$

Notation

$$A \geq 0$$

Definition 0.16.

$$\text{spec}(A) = \{a_j \in \mathbb{C} | a_j \text{ is an eigenvector of } A\}$$

Let A be normal. Then $\text{spec}(A^\dagger) = \overline{\text{spec}(A)} = \{\overline{a_j}\}_j$

A is ...	relation to spec(A)
self-adjoint	$\text{spec}(A) \subset \mathbb{R}$
unitary	$\text{spec}(A) \subset \{e^{i\phi} \| \phi \in \mathbb{R}\}$
positive semi-definite	$\text{spec}(A) \subset \mathbb{R}_0^+$
a projector ($A^2 = A$)	$\text{spec}(A) \subset \{0, 1\}$

Theorem 0.17.

$$A = \sum_{j=1}^{n} \underbrace{a_j}_{degenerate\ in\ general (\exists i \neq j : a_i = a_j)} \underbrace{|a_j\rangle \langle a_j|}_{rank 1}$$

Chapter 2: Quantum mechanics

Preparation: probability theory (finite sample spaces)

goal::include amplitudes of our vectors $|0\rangle, |1\rangle, \ldots$

Definition 0.18 (probability vector)**.** A vector $p \in \mathbb{R}1d$ is a probability vector if

$$p_i \geq 0, \quad \sum_{i=1}^{d} p_i = 1$$

Definition 0.19 (random variables)**.** A with d outcomes from $(a_1, \ldots, a_d) \subset \mathbb{R}$. A is a function associated with a probability vector p

$$\mathbb{P}[i] = \mathbb{P}\left[A \underset{\wedge}{=} a_i\right] = p_i$$

Quantum mechanical notation

Replace

$$p \to \rho = \mathrm{diag}(p)$$
$$A \to A = \mathrm{diag}(a_1, \dots, a_d)$$

Implications

- $\mathrm{Tr}[\rho] = \sum_i p_i = 1$
- $\langle A \rangle = \mathrm{Tr}(\rho A)$
- $A = \sum_{a \in \mathrm{spec}(A)} a P_a$
  - $A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_d \end{bmatrix} \overset{a_1=a_2=a}{=} \begin{bmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a_d \end{bmatrix} \Rightarrow P_a = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$
- $\mathbb{P}[a] = \mathbb{P}[A = a] = \sum_{j : a_j = a} p_j = \mathrm{Tr}[P_{a\rho}]$

interpretations of probabilities: lack of knowledge

conditioned on observing a:

$$\rho \to \rho_a = |a\rangle\langle a| \qquad \text{non-degenerate case}$$
$$\rho \to \rho_a = \frac{P_a \rho P_a}{\mathrm{Tr}[P_a \rho]} \qquad \text{general case}$$

Check: Say $a_1 = a_2 = \dots = a_r = a, \quad a_r + 1, \dots, a_d$ conditional probability of $j \in [r]$

$$\mathbb{P}[j] = \frac{\mathbb{P}[j \text{ AND } A = a]}{\mathbb{P}[A = a]} = \frac{p_j}{\sum_{j \in [r]} p_j} = \frac{p_{jj}}{\mathrm{Tr}(P_a \rho)}$$

Definition 0.20 (density operator)**.** The set of density operators ρ is :

$$S(\mathcal{H} := \{\rho \in \mathrm{Herm}(\mathcal{H}) | \rho \geq 0,\ \mathrm{Tr}[\rho] = 1\}$$

Proposition 0.21.

$$\rho_1, \rho_2 \in S(\mathcal{H}),\ \lambda \in [0,1] \lambda \rho_1 + (1-\lambda)\rho_2 \in S(\mathcal{H})$$