Quantum Computing

Problem sheet 1

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Exercise 1.1 (Complex vector spaces)

a)

First we simplify:

$$
\begin{aligned}
z_1 &= 3 - i + (2 - i)(-1 + i) \\
&= 3 - i + (-2 + 2i) + (i + 1) \\
&= 3 - i + (-1 + 3i) \\
&= 3 + (-1 + 2i) \\
&= 2 + 2i
\end{aligned}
$$

This gives us the real and imaginary part $\Re(z_1) = \Im(z_1) = 2$, as well as the complex conjugate $z_1^* = 2 - 2i$ and by pythagoras theorem the absolute value $\sqrt{2^2 + 2^2} = \sqrt{8}$.

b)

$$\prod_{k=-4}^{4} \sqrt{36-k^2} + ik$$

$$= \left(\sqrt{36} + 0k\right) \cdot \prod_{\substack{k=-4 \\ k \neq 0}}^{k=4} \sqrt{36-k^2} + ik$$

$$= 6 \cdot \prod_{k=1}^{4} \underbrace{\left(\sqrt{36-k^2} + ik\right)\left(\sqrt{36-k^2} - ik\right)}$$

$$(36-k^2) \cancel{- ik(36-k^2) + ik(36-k^2)} \underbrace{- i^2k^2}_{+k^2}$$

$$= 6 \cdot \quad (36-1)(36-2)(36-3)(36-4)$$

c)

Given that $\{|0\rangle, |1\rangle, ..., |d-1\rangle\}$ is an ONB, we also know that $\{\langle 0|, \langle 1|, ..., \langle d-1|\}$ is an ONB of the dual-space $\mathbb{C}^{d^*}$ with $\langle i \,|\, j\rangle = \delta_{ij}$.

Therefore given a ket $|\psi\rangle$ which has a decomposition into basis vectors $\sum_{k=0}^{d-1} \psi_k |k\rangle$, because $\mathbb{C}^{d^*}$ is a vector space, we can find the ψ_i by

$$
\begin{aligned}
\langle i \,|\, \psi\rangle &= \langle i| \sum_{k=0}^{d-1} \phi_k |k\rangle \\
&= \delta_{0i}\psi_0 + \cdots + \delta_{ii}\phi_i + \cdots + \delta_{i(d-1)}\phi_{d_1} \\
&= \psi_i
\end{aligned}
$$

d)

A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

We already know that $\mathbb{C}^d$ is a complete vector space under the euclidean norm. Moreover it is also a complete vector space under the induced distance function of the inner product, defined by:

$$\langle x, y\rangle = y^\dagger x = \bar{y}^t x$$

we therefore only need to check that $\langle \cdot, \cdot\rangle$ is an inner product.

1. We have *conjugate symmetry*:

$$
\begin{aligned}
\overline{\langle y, x\rangle} &= \overline{\bar{x}^t y} \\
&= x^t \bar{y} \\
&= \bar{y}^t x \\
&= y^\dagger x \\
&= \langle x, y\rangle
\end{aligned}
$$

2. We have *linearity in the first argument*:

$$
\begin{aligned}
\langle \lambda x_1 + \mu x_2, y\rangle &= \bar{y}^t(\lambda x_1 + \mu x_2) \\
&= \begin{pmatrix} \bar{y}_1 & \dots & \bar{y}_d \end{pmatrix} \begin{pmatrix} \lambda x_{1_1} + \mu x_{2_1} \\ \vdots \\ \lambda x_{1_d} + \mu x_{2_d} \end{pmatrix} \\
&= \begin{pmatrix} \bar{y}_1 & \dots & \bar{y}_d \end{pmatrix} \left(\begin{pmatrix} \lambda x_{1_1} \\ \vdots \\ \lambda x_{1_d} \end{pmatrix} + \begin{pmatrix} \mu x_{2_1} \\ \vdots \\ \mu x_{2_d} \end{pmatrix} \right) \\
&= \begin{pmatrix} \bar{y}_1 & \dots & \bar{y}_d \end{pmatrix} \begin{pmatrix} \lambda x_{1_1} \\ \vdots \\ \lambda x_{1_d} \end{pmatrix} + \begin{pmatrix} \bar{y}_1 & \dots & \bar{y}_d \end{pmatrix} \begin{pmatrix} \mu x_{2_1} \\ \vdots \\ \mu x_{2_d} \end{pmatrix} \\
&= \lambda \begin{pmatrix} \bar{y}_1 & \dots & \bar{y}_d \end{pmatrix} \begin{pmatrix} x_{1_1} \\ \vdots \\ x_{1_d} \end{pmatrix} + \mu \begin{pmatrix} \bar{y}_1 & \dots & \bar{y}_d \end{pmatrix} \begin{pmatrix} x_{2_1} \\ \vdots \\ x_{2_d} \end{pmatrix} \\
&= \lambda\langle x_1, y\rangle + \mu\langle x_2, y\rangle
\end{aligned}
$$

3. We have that the inner product of an element with itself is *positive-definite*:

$$
\begin{aligned}
\langle x, x\rangle &= \bar{x}^t x \\
&= \begin{pmatrix} \bar{x}_1 & \dots & \bar{x}_d \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \\
&= \underbrace{\underbrace{\bar{x}_1 x_1}_{\substack{=0 \Leftrightarrow x_1 = 0 \\ >0\,\forall x_1}} + \dots + \underbrace{\bar{x}_d x_d}_{\substack{=0 \Leftrightarrow x_d = 0 \\ >0\,\forall x_d}}}_{\substack{=0 \Leftrightarrow x = 0 \\ >0\,\forall x}}
\end{aligned}
$$

Exercise 1.2 (Postulates: Quantum states and their transformations)

a)

$$
Y = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}
$$

$$
Y^\dagger = \begin{pmatrix} \bar{1} & \overline{-i} \\ \bar{i} & \bar{1} \end{pmatrix}^t = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = Y
$$

b)

$M \in U(\mathbb{C}^{3\times 3}) \Leftrightarrow M^\dagger M = \mathbb{1}_{\mathbb{C}^3}$

$$
\begin{aligned}
\det(M) &= \cos(\theta)^2 e^{i\phi} - i\sin(\theta) i \sin(\theta) e^{i\phi} \\
&= e^{i\phi}(\cos(\theta)^2 + \sin(\theta)\sin(\theta)) \\
&= e^{i\phi}(\cos(\theta)^2 + \sin(\theta)^2) \\
&= e^{i\phi}
\end{aligned}
$$

Therefore $|det(M)| \neq 1$ for all $\phi, \theta \in \mathbb{R}$

$$
\begin{pmatrix} \cos(\theta) & 0 & -i\sin(\theta) \\ 0 & e^{i\phi} & 0 \\ -i\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}^{\dagger} = \begin{pmatrix} \cos(\theta) & 0 & i\sin(\theta) \\ 0 & e^{-i\phi} & 0 \\ i\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}
$$

$$
\begin{aligned}
&\begin{pmatrix} \cos(\theta) & 0 & -i\sin(\theta) \\ 0 & e^{i\phi} & 0 \\ -i\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & i\sin(\theta) \\ 0 & e^{-i\phi} & 0 \\ i\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\
= &\begin{pmatrix} \cos(\theta)^2 + \sin(\theta)^2 & 0 & i\cos(\theta)\sin(\theta) - i\sin(\theta)\cos(\theta) \\ 0 & e^{i\phi}e^{-i\phi} & 0 \\ -i\sin(\theta)\cos(\theta) + \cos(\theta) i \sin(\theta) & 0 & -i^2\sin(\theta)^2 + \cos(\theta)^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi - i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Finally we can conclude that for all $\phi, \theta \in \mathbb{C}$ the matrix M is unitary.

c)

$$
H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

$$
H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
$$

There cannot be another matrix that describes the same linear transformation. I am unsure what is meant with: > any other unitary matrices that describe the same transformation of states? Does this mean the same linear transformation?

1.3 (Observables)

a)

We need $M^\dagger = M$ or equivalently $a_{ij} = \bar{a}_{ji}$ for a matrix to be hermitian. For the special case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have $a = \bar{a}$, $b = \bar{c}$ and $d = \bar{d}$

b)

Assume M is self-adjoint and given a choice of orthonormal basis vectors $(e_i)_{1 \le i \le d}$ we choose the following representation

$$|\phi\rangle = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_d \end{bmatrix} \quad \langle\phi| = \begin{bmatrix} \bar{\psi}_1 \dots \bar{\psi}_d \end{bmatrix} \quad M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1d} \\ \overline{m_{12}} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \overline{m_{1d}} & \dots & \dots & m_{dd} \end{bmatrix}$$

We can now calculate the following using the fact that M is a linear operator[1] :

$$\begin{aligned}
\langle \psi \mid M \mid \psi \rangle &= \langle\psi| \left(M |\psi\rangle\right) = \left(\langle\psi| M\right) |\psi\rangle \\
&= \psi^\dagger (M\psi) \\
&= \begin{bmatrix} \bar{\psi}_1 \dots \bar{\psi}_d \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1d} \\ \overline{m_{12}} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \overline{m_{1d}} & \dots & \dots & m_{dd} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_d \end{bmatrix} \\
&= \begin{bmatrix} \bar{\psi}_1 \dots \bar{\psi}_d \end{bmatrix} \begin{bmatrix} m_{11}\psi_1 + \dots + m_{1d}\psi_d \\ \vdots \\ \overline{m_{1d}}\psi_1 + \dots + m_{dd}\psi_d \end{bmatrix} \\
&= \sum_{k=1}^{d} \bar{\psi}_k \left(\sum_{l=1}^{d} m_{kl}\psi_l \right) \\
&= \underbrace{\sum_{k=1}^{d} m_{kk} \bar{\psi}_k \psi_k}_{\in \mathbb{R}} + \underbrace{\sum_{k=2}^{d} \sum_{l=1}^{k-1} m_{kl} \bar{\psi}_k \psi_l + \overline{m_{kl}} \bar{\psi}_l \psi_k}_{\in \mathbb{R}}
\end{aligned}$$

The right underbrace can be checked by writing $m_{kl} = (a + bi), \psi_k = (c + di), \dots$ and then multiplying out.

[1] https://en.wikipedia.org/wiki/Bra%E2%80%93ket_notation#Linear_operators

c)

$$\begin{aligned}\langle 1 \,|\, \sigma_x \,|\, 1 \rangle &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle 1 \,|\, \sigma_y \,|\, 1 \rangle &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -i \\ 0 \end{bmatrix} \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle 1 \,|\, \sigma_z \,|\, 1 \rangle &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= -1\end{aligned}$$

$$\begin{aligned}\langle 0 \,|\, \sigma_x \,|\, 0 \rangle &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle 0 \,|\, \sigma_y \,|\, 0 \rangle &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ i \end{bmatrix} \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle 0 \,|\, \sigma_z \,|\, 0 \rangle &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 1\end{aligned}$$

d)

Due to the associativity of matrix-multiplication, we can calculate $\langle v \,|\, (XY) \,|\, v\rangle$ instead of $\langle v|\, X(Y\,|v\rangle)$. Therefore applying the Pauli observables to $H\,|0\rangle$ and $H\,|1\rangle$ yields:

$$\begin{aligned}\langle 1 \,|\, (\sigma_x H) \,|\, 1\rangle &= \begin{bmatrix}0 & 1\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right)\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle 1 \,|\, (\sigma_y H) \,|\, 1\rangle &= \begin{bmatrix}0 & 1\end{bmatrix}\left(\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right)\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-i & i\\ i & i\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{i}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle 1 \,|\, (\sigma_z H) \,|\, 1\rangle &= \begin{bmatrix}0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right)\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle 0 \,|\, (\sigma_x H) \,|\, 0\rangle &= \begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right)\begin{bmatrix}1\\ 0\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle 0 \,|\, (\sigma_y H) \,|\, 0\rangle &= \begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right)\begin{bmatrix}1\\ 0\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}-i & i\\ i & i\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\\ &= \frac{-i}{\sqrt{2}}\end{aligned}$$

$$
\begin{aligned}
\langle 0 \,|\, (\sigma_z H) \,|\, 0 \rangle &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$