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Quantum Computing – Problem sheet 1

due on: 12.04.2022 at 12:30

Return: either in the lecture, online [1] or in the mailbox next to room 25.32.03.33

Disclaimer: Every participant has to submit their own solution.
Collaborations are only allowed if they are disclosed on the first page.

Exercise 1.1 *(Complex vector spaces):*

The fields of real and complex numbers are denoted as $\mathbb{R}$ and $\mathbb{C}$ respectively. By $\mathbb{C}^d$ we denote a d-dimensional vector space over $\mathbb{C}$, i.e., a space of column vectors of length d with complex entries. To denote (column) vectors in $\mathbb{C}^d$ we use the Dirac (also called "braket") notation $|\psi\rangle \in \mathbb{C}^d$ (read as "ket" ψ). To every "ket" vector $|\psi\rangle$ we associate a "bra" vector $\langle\psi|$, which is a row vector with entries equal to the complex conjugate of the corresponding entries of $|\psi\rangle$. The imaginary unit we denote as i, i.e., $\mathrm{i}^2 = -1$.

a) Find a simplified expression for the complex number

$$z_1 := 3 - \mathrm{i} + (2 - \mathrm{i})(-1 + \mathrm{i})\,. \tag{1}$$

What are its real and imaginay part, its complex conjugate and its absolute value?

b) Find the following product of complex numbers

$$z_2 := \prod_{k=-4}^{4} \left(\sqrt{36 - k^2} + \mathrm{i}\,k\right). \tag{2}$$

Hint: *This can be done without technical calculations.*

c) Let $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ be an orthonormal basis in $\mathbb{C}^d$. Every vector $|\psi\rangle \in \mathbb{C}^d$ can be written as

$$|\psi\rangle = \sum_{k=0}^{d-1} \psi_k\, |k\rangle\,, \tag{3}$$

with $\psi_k \in \mathbb{C}$. Verify that for any $j \in \{0, 1, \dots, d-1\}$ the coefficient ψ_j in Eq. (3) can be calculated as $\psi_j = \langle j|\psi\rangle$.

d) Verify that $\mathbb{C}^d$ with its inherit inner product is a Hilbert space.

Hint: *Only verify the three required properties of the inner product.*

△

Exercise 1.2 *(Postulates: Quantum states and their transformations):*

Normalized vectors such as $|\psi\rangle \in \mathbb{C}^d$ are used to describe (pure) quantum states in quantum mechanics. Linear operators on $\mathbb{C}^d$ are used to describe operations on the states.
For an $n \times m$ matrix $X \in \mathbb{C}^{n\times m}$, define its *conjugate transpose* $X^\dagger \in \mathbb{C}^{m\times n}$ as an $m \times n$ matrix componentwise by $X^\dagger_{i,j} := \overline{X}_{j,i}$. The map $(\,\cdot\,)^\dagger : \mathbb{C}^{n\times m} \to \mathbb{C}^{m\times n}$ can be applied to vectors too, namely $|\psi\rangle^\dagger = \langle\psi|$.
A square matrix $U \in \mathbb{C}^{d\times d}$ is said to be *unitary* if $UU^\dagger = U^\dagger U = \mathbb{1}_d$, where the product of U and $U^\dagger$ is the usual "dot" product of matrices and $\mathbb{1}_d \in \mathbb{C}^{d\times d}$ is the identity matrix.
For a qubit system, we define the two *computational basis* state vectors in $\mathbb{C}^2$ to be

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4}$$

a) Find the conjugate transpose of the following matrix

$$Y := \begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}. \tag{5}$$

b) For which values of the parameters $\phi, \theta \in \mathbb{R}$ is the following matrix unitary?

$$M := \begin{bmatrix} \cos(\theta) & 0 & -\mathrm{i}\sin(\theta) \\ 0 & e^{\mathrm{i}\phi} & 0 \\ -\mathrm{i}\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \tag{6}$$

c) Consider the following so-called *Hadamard matrix*

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{7}$$

Find the action of this unitary matrix on the state vectors $|0\rangle$ and $|1\rangle$. Are there any other unitary matrices that describe the same transformation of states?

△

Exercise 1.3 *(Observables):*

Measurements in quantum mechanics are described by self-adjoint operators. An operator on $\mathbb{C}^d$ is said to be *self-adjoint*, or *Hermitian*, if $M^\dagger = M$. The space of such self-adjoint operators is denoted by $\mathrm{Herm}(\mathbb{C}^d)$. The *expectation value* of $M \in \mathrm{Herm}(\mathbb{C}^d)$ with respect to state vector $|\psi\rangle \in \mathbb{C}^d$ is defined as $\langle\psi| M |\psi\rangle \in \mathbb{R}$ and can be calculated by the rules of matrix multiplication. Often, the notation $\langle M\rangle := \langle\psi| M |\psi\rangle$ is used when the state can be inferred from the context.

a) Consider a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{8}$$

with $a, b, c, d \in \mathbb{C}$. What are the conditions on a, b, c, d that make the above matrix Hermitian?

b) Show that $\langle\psi| M |\psi\rangle$ is indeed always real whenever M is self-adjoint.

c) Calculate the expectation values of the following three observables, called Pauli X, Y and Z, with respect to the two state vectors $|0\rangle$ and $|1\rangle$:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{9}$$

d) Apply the Hadamard transform H in Eq. (7) to the states $|0\rangle$ and $|1\rangle$ and calculate the expectation values of the Pauli obseravles with respect to those two states.

Is there another way to do these calculations?

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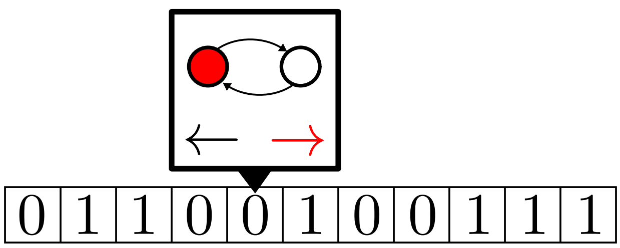
Exercise 1.4 *(Turing machines):*

The Turing machine is a simple mathematical model of computation. It consists of a tape and a machine with a head which moves on this tape according to pre-defined instructions and what is written on the tape. Any Turing machine has a finite number of internal states, and during the computation the machine changes from one state to another. See Wikipedia for a proper definition.



a) Consider a problem of finding a sum modulo 2 of eight binary numbers. Let these numbers be written on a tape from left to right consecutively and the head of the Turing machine be places on the first number. Write a program, i.e., provide a set of instructions for a Turing machine that finds the sum of the given eight numbers, writes it as a ninth number on the tape, and halts. Try to find the optimal program with respect to the number of steps needed and the size of the internal memory of the machine (number of the internal states).

△

[1] M. Kliesch, *Sciebo web repository for uploads*, https://uni-duesseldorf.sciebo.de/s/78Zg8wHqU29yv3h (2022), [Online; accessed 31-March-2022].