TERMINAL FANO FOURFOLDS AND ANTICANONICAL LINEAR SECTIONS

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Abstract. We study terminal $\mathbb{Q}$-Fano 4-folds of index 1 realized as low-codimension complete intersections in weighted projective space and their anticanonical linear sections. We first construct families of terminal $\mathbb{Q}$-Fano 4-folds of index 1 in low-codimension weighted complete intersection models. Then we compute if the linear system contains an anticanonical divisor that is a quasismooth isolated canonical Calabi–Yau 3-fold. Building on [Qur25], which treated the cases of $h^0(-K_X) = 0$ and $h^0(-K_X) \geq 2$, we complete the analysis by investigating the borderline case $h^0(-K_X) = 1$. We also enlarge the list of examples in the previously studied cases by performing the computations with larger search bounds.

1. Introduction

A *Fano* variety is a normal projective variety X if some positive multiple $-iK_X$, of its anticanonical Weil divisor $-K_X$ is Cartier and ample. If X contains only terminal $\mathbb{Q}$-factorial singularities then it is called a terminal $\mathbb{Q}$-Fano variety. The *Fano index* of X is the largest integer $i \geq 1$ such that $-K_X \sim iA$ for some ample Weil $\mathbb{Q}$-Cartier divisor A. If X is a terminal $\mathbb{Q}$-Fano 4-fold and $Y \subset X$ is an effective Weil divisor with $K_Y \sim 0$, We call Y an *isolated canonical Calabi–Yau* 3*-fold* if Y is normal, has isolated canonical singularities.

Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_N)$ be a weighted projective space. A *weighted complete intersection* (WCI) is a subvariety $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}$ cut out by c weighted homogeneous equations $f_1 = \cdots = f_c = 0$ of degrees $\deg(f_\ell) = d_\ell$, where the weighted homogeneous coordinates satisfy $\deg(x_i) = w_i$. We exclude the case that X is an *intersection with a linear cone*, meaning that $d_\ell = w_j$ for some ℓ, j.

An n-dimensional subvariety $X \subset \mathbb{P}$ is *wellformed* if $\mathbb{P}$ is wellformed and X does not contain any $n-1$-dimensional singular stratum of $\mathbb{P}$. A weighted complete intersection $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}$ is *quasismooth* if its affine cone is smooth away from the origin. A *quotient singularity* of type $\frac{1}{r}(a_1, \ldots, a_n)$ is a quotient of $\mathbb{A}^n$ by a cyclic group μ_r of order r given

$$\epsilon : x_i \mapsto e^{a_i} x_i, \ i = 1, \ldots, n,$$

where r is a positive integer and $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$. It is called *isolated* if all a_is are coprime to r and it is called *terminal (canonical)* if and only if

$$\frac{1}{r} \sum_{i=1}^{n} \overline{ka_i} > (\geq) 1 \text{ for all } k = 1, \ldots, r-1.$$

Key words and phrases. Fano varieties, 4-folds, Weighted complete intersections.

If X contains at worst terminal singularities then it is called a terminal Fano variety. In this article, we will construct terminal Fano 4-folds with isolated singularities, ITF4, if no confusion can arise.

The classification of Fano varieties is a central theme in algebraic geometry. It is well known that, in each fixed dimension, there are only finitely many deformation families of smooth Fano varieties [KMM92]. In dimension at least three, terminal singularities [Rei83] are essential in the minimal model program, and consequently the classification of terminal Fano varieties have attracted considerable attention. For threefolds, the graded ring database [BK, BK22] records all possible Hilbert series together with the expected numerical and geometric data, such as terminal quotient singularities, ambient weighted projective space, equation degrees, and Hilbert numerator, that may arise for a terminal $\mathbb{Q}$-Fano 3-fold and a significant progress on the classification of such varieties have been achieved in [Rei80, IF00, ABR02, Tak02, BKR12, BKQ18, CD20]. In dimension four, terminal Fano 4-fold weighted projective spaces were classified by Kasprzyk [Kas13] and Terminal index-one Fano 4-fold hypersurfaces were classified by Brown and Kasprzyk [BK16]. In particular, the graded ring database [BK] contains 490 families of isolated terminal Fano 4-fold hypersurfaces (ITF4s) of index 1.

In our index-1 Fano fourfold setting, adjunction gives $K_Y \sim 0$ for $Y \in |-K_X|$, so anticanonical divisors provide a natural source of Calabi–Yau threefolds. We also focus on deciding when $|-K_X|$ contains a quasismooth canonical Calabi–Yau 3-fold section.

1.1. **Results.** In this article we describe the construction and classification of isolated terminal Fano 4-folds (ITF4s) of index 1 that admit an anticanonical embedding as a wellformed, quasismooth weighted complete intersection (WCI) in a weighted projective space. We divide our families into three cases, denoted by type-K_1, type-K_0, and type-K_2, following the terminology for the latter two introduced in [Qur25].

We also record, for type-K_1 and type-K_2, whether the anticanonical system admits a quasismooth isolated canonical Calabi–Yau 3-fold section. For the comparison we use the list of canonical Calabi–Yau 3-folds compiled by Brown–Kasprzyk–Zhou [BKZ22], available on the graded ring database [BK].

1.1.1. ***Type-K_1.*** We call an index-1 ITF4 X to be of *type-K_1* if $h^0(-K_X) = 1$, but the (unique up to scaling) anticanonical divisor $Y \in |-K_X|$ is *not* a quasismooth Calabi–Yau threefold with isolated canonical orbifold points. Such cases typically occur when X contains at least one orbifold point whose polarization has no coordinate of weight 1, i.e., a point of type

$$\frac{1}{r}(a_1, a_2, a_3, a_4), \qquad a_i \geq 2.$$

However, we also find two examples in which every orbifold point has a weight 1 coordinate, but the anticanonical section is still not a quasismooth Calabi–Yau 3-fold; see Section 2. In particular, these families do not appear among the canonical Calabi–Yau 3-fold sections listed in [BKZ22].

Theorem 1.1 (The case $h^0(-K_X) = 1$). *Let $X \subset \mathbb{P}^{4+c}(w_0, \ldots, w_{4+c})$ be a wellformed, quasismooth, isolated terminal Fano 4-fold of index 1, realized as a weighted complete* *intersection of codimension $c \in \{2,3\}$. Assume $h^0(-K_X) = 1$, and write $W := \sum_{i=0}^{4+c} w_i$. If $W \leq 120$ for $c = 2$ and $W \leq 80$ for $c = 3$, then there are m such families, and exactly $\#K_1$ of them are of type-K_1 (that is, the unique anticanonical member $Y \in |-K_X|$ is not a quasismooth Calabi–Yau 3-fold with isolated canonical orbifold points). The numerical summary is given in Table 1.*

TABLE 1. ITF4s of Type-K_1

Attributes	Codimension $c = 2$	Codimension $c = 3$
m	415	51
$\#K_1$	276	42
$\#\text{CY3} \in \lvert -K_X \rvert$	139	9
W	120	80

1.1.2. ***Type-K_2***. We call an index-1 ITF4 X *type-K_2* if $h^0(-K_X) \geq 2$, yet a general member $Y \in |-K_X|$ fails to be a quasismooth Calabi–Yau threefold with isolated canonical orbifold points. In our setting, this only occurs whenever X contains at least one orbifold point with no weight 1 polarization, i.e., a point of type

$$\frac{1}{r}(a_1, a_2, a_3, a_4), \qquad a_i \geq 2.$$

Theorem 1.2 (The case $h^0(-K_X) \geq 2$)**.** *Let $X \subset \mathbb{P}^{4+c}(w_0, \ldots, w_{4+c})$ be a wellformed, quasismooth, isolated terminal Fano 4-fold of index 1, realized as a weighted complete intersection of codimension $c \in \{2,3\}$. Assume $h^0(-K_X) \geq 2$, and write $W := \sum_{i=0}^{4+c} w_i$. If $W \leq 120$ for $c = 2$ and $W \leq 80$ for $c = 3$, then there are m such families, and exactly $\#K_2$ of them are of type-K_2 (that is, a general member of $|-K_X|$ is not a quasismooth Calabi–Yau 3-fold with isolated canonical orbifold points). Equivalently, the remaining $m-\#K_2$ families admit an anticanonical Calabi–Yau 3-fold section. The numerical summary is given in Table 2.*

1.1.3. ***Type-K_0***. We call an index-1 ITF4 X *type-K_0* if its anticanonical linear system is empty, that is, $h^0(-K_X) = 0$. In particular, X has no anticanonical divisor, and hence no Calabi–Yau 3-fold arises as a hyperplane section via $|-K_X|$.

Theorem 1.3 (The case $h^0(-K_X) = 0$)**.** *Let $X \subset \mathbb{P}^{4+c}(w_0, \ldots, w_{4+c})$ be a wellformed, quasismooth, isolated terminal Fano 4-fold of index 1, realized as a weighted complete intersection of codimension $c \in \{2,3\}$. Assume $h^0(-K_X) = 0$, and write $W := \sum_{i=0}^{4+c} w_i$. If $W \leq 120$ for $c = 2$ and $W \leq 80$ for $c = 3$, then there are $\#K_0$ such families. In this*

TABLE 2. ITF4s of Type-K_2

Attributes	Codimension $c = 2$	Codimension $c = 3$
m	273	21
$\#K_2$	11	8
$\#\text{CY3} \in \lvert -K_X\rvert$	262	13
W	120	80

case $|-K_X| = \emptyset$ by definition, so X admits no anticanonical Calabi–Yau 3-fold section. The numerical summary is given in Table 3.

TABLE 3. ITF4s of Type-K_0

Attributes	Codimension $c = 2$	Codimension $c = 3$
$m = \#K_0$	94	15
W	120	80

Remark 1.4. In other words, let $X \subset \mathbb{P}^{4+c}(w_0, \ldots, w_{4+c})$, with $\sum_i w_i \leq W$, be an index-1 isolated terminal Fano 4-fold satisfying the hypotheses of one of Theorems 1.1, 1.2, or 1.3. Then X occurs in our lists, and it occurs uniquely up to isomorphism. Concretely, within these bounds we obtain 782 families in codimension 2 and 70 families in codimension 3, recorded in Tables 1, 2, and 3. The full datasets are available on GitHub[1].

Remark 1.5. We also searched for ITF4s realized as codimension-4 weighted complete intersections, but found no quasismooth examples. In codimension 4 we searched up to $\sum_i w_i \leq 64$ and obtained 13 candidate isolated terminal Fano 4-folds. Among these, seven satisfy $h^0(-K_X) \geq 2$, and none has empty anticanonical system. Apart from the smooth complete intersection $X_{2,2,2,2} \subset \mathbb{P}^8$, every candidate has at least one non-terminal orbifold point.

From the Tables 1 and 2, we see that there are 401 codimension 2 and 22 codimension 3 cases of weighted complete intersection quasismooth isolated canonical Calabi–Yau 3-folds that appear as linear sections of these ITF4s, which exactly matches the number of such cases listed on [BK] for the given range of weights.

Theorem 1.6 (canonical Calabi–Yau 3-fold sections)**.** *Let $Y \subset \mathbb{P}(w_1, \ldots, w_6)$ be a well-formed, quasismooth codimension 2 weighted complete intersection Calabi–Yau threefold*

[1] `https://github.com/QureshiMI/TerminalFano4folds`

with isolated canonical singularities, as recorded on the graded ring database [BK]*. Then there exists an index-*1 *terminal Fano* 4*-fold*

$$X \subset \mathbb{P}(1, w_1, \dots, w_6)$$

such that

(i) X *has only isolated terminal quotient singularities of type* $\frac{1}{r}(1, a, b, c)$*, and*
(ii) Y *occurs as a special anticanonical divisor on* X*, namely the coordinate hyperplane section* $Y = X \cap (x_0 = 0)$*.*

Remark 1.7. The converse fails in general: if $X \subset \mathbb{P}(1, w_1, \dots, w_6)$ is an index-1 terminal Fano 4-fold with quotient points of type $\frac{1}{r}(1, a, b, c)$, the special section $Y = X \cap (x_0 = 0)$ need not be a quasismooth canonical Calabi–Yau threefold (hence need not appear in [BK]); see Example 2.2.

We perform our analysis in the following stages. First, we run the format-search procedure of [Qur17, BKZ22] (see Section 3) to produce a list of numerical candidates that matches a Hilbert series of a terminal Fano 4-fold as a weighted complete intersection in codimension 2 and 3. Second, for each candidate we confirm that it is realized by an actual family with desired properties (quasismooth, wellformed, isolated terminal) by combining theoretical checks with explicit computations as outlined in Section 3. We do not claim completeness beyond the search region: there is no known *apriori* bound on the total weight $W = \sum_i w_i$ for these formats. Accordingly, for each format we have to choose a weight bound compatible with the available computational resources, and the lists reported here are complete within those stated bounds.

1.2. **Context and Motivation.** This paper continues the explicit classification program for terminal $\mathbb{Q}$-Fano varieties via graded rings and weighted projective models. On the one hand, our constructions extend the established threefold story in low codimension. On the other hand, dimension four exhibits new behaviour which is not visible in the threefold geography, and understanding this discrepancy is a main motivation for our study.

Our starting point is to construct families of isolated terminal Fano 4-folds of index 1 (ITF4s) in low codimension, primarily as weighted complete intersections and, more generally, within Gorenstein formats as in [Qur25]. The classification problem is naturally bounded in fixed dimension (beginning with the smooth case [KMM92] and culminating in Birkar's proof of the Borisov–Alexeev–Borisov conjecture [Bir21]). The graded ring viewpoint is particularly well suited to symbolic computation: one enumerates candidate Hilbert series and baskets, matches these to a format, and then verifies the output by wellformedness, stratum analysis and Jacobian computations.

A second motivation, and the main theme of this paper, concerns the behavior of the anticanonical linear system. For Fano threefolds of index 1, the general elephant philosophy (Reid, Shokurov) predicts that under suitable hypotheses a general member of $|-K|$ is a $K3$ surface with at worst Du Val singularities [Rei87, Ale94]. In higher dimension Höring and Śmiech show that for a canonical Gorenstein Fano fivefold the system $|-K|$ is nonempty, and if a general element is reduced then it has canonical singularities [HŚ20].

For our index-1 Fano fourfolds, adjunction gives $K_Y \sim 0$ for $Y \in |-K_X|$, so anticanonical divisors are a natural source of Calabi–Yau threefolds. The basic question is therefore concrete and testable: when does $|-K_X|$ contain an anticanonical divisor which is an isolated canonical Calabi–Yau threefold or otherwise?

In codimension ≤ 4 there are only a handful of terminal $\mathbb{Q}$-Fano threefolds with empty anticanonical system, whereas in dimension four we find many such families, across each format; see Table 5. Also, when $|-K_X| \neq \emptyset$, the generic anticanonical section need not be an isolated canonical Calabi–Yau threefold, and this phenomenon occurs widely in such families; see Table 4. In this paper we also isolate the borderline case $h^0(-K_X) = 1$, see Table 1 where $|-K_X|$ contains a distinguished divisor Y unique up to scaling, and the problem becomes a deterministic singularity test for that specific section. We carry out this analysis for codimension 2 and 3 complete intersection models, and we enlarge the previously studied ranges [Qur25] by repeating the computations with larger search bounds.

2. Terminal $\mathbb{Q}$-Fano 4-folds and linear sections

In this section we relate index-1 isolated terminal $\mathbb{Q}$-Fano 4-folds $X \subset \mathbb{P}(1, w_1, \ldots)$ with $|-K_X| \neq \emptyset$ to their anticanonical sections $Y \in |-K_X|$, which satisfy $K_Y \sim 0$ by adjunction and hence are natural Calabi–Yau 3-fold candidates. Lemma 2.1 records how cyclic quotient singularities restrict to Y, while the examples show that quasismoothness can still fail globally. Conversely, Theorem 1.6 realises the GRDB Calabi–Yau threefolds as special hyperplane sections of suitable terminal Fano 4-folds. These points are the input for our separation into type-K_1 and type-K_2.

Lemma 2.1. *Let $n \geq 2$ and let $X \subset \mathbb{P}(1, w_1, \ldots, w_N)$ be a $\mathbb{Q}$-Fano variety of index 1 and dimension n, with at worst isolated cyclic quotient singularities. Let $H = (x_0 = 0)$ be the quasismooth divisor cut by the weight 1 coordinate and $Y := X \cap H$ is quasismooth.*

Suppose that $P \in X \cap H$ is a singular point of type

$$(X, P) \;\cong\; \frac{1}{r}(1, a_1, \ldots, a_{n-1}), \qquad \sum_{i=1}^{n-1} a_i \equiv 1 \pmod{r}.$$

Then the induced singularity of Y at P is

$$(Y, P) \;\cong\; \frac{1}{r}(a_1, \ldots, a_{n-1}).$$

Proof. Since $(X, P) \cong \frac{1}{r}(1, a_1, \ldots, a_{n-1})$, there exists an analytic neighborhood of P isomorphic to

$$(\mathbb{A}^n/\mu_r, 0),$$

where $\mu_r = \langle \epsilon \rangle$ acts on coordinates $(u, v_1, \ldots, v_{n-1})$ by

$$\epsilon \cdot (u, v_1, \ldots, v_{n-1}) \mapsto (\epsilon^1 u, \epsilon^{a_1} v_1, \ldots, \epsilon^{a_{n-1}} v_{n-1}).$$

The congruence $\sum a_i \equiv 1 \pmod{r}$ is the local Gorenstein condition.

The divisor $H = (x_0 = 0)$ is a form of weight 1; after choosing the above coordinates we may take its local equation to be $u = 0$. Therefore (Y, P) is the quotient of the hyperplane $\{u = 0\} \cong \mathbb{A}^{n-1}$ by the induced μ_r-action on $(v_1, \ldots, v_{n-1})$. Restricting the action gives

$$\epsilon \cdot (v_1, \ldots, v_{n-1}) \mapsto (\epsilon^{a_1} v_1, \ldots, \epsilon^{a_{n-1}} v_{n-1}),$$

hence

$$(Y, P) \cong \frac{1}{r}(a_1, \ldots, a_{n-1}),$$

as required. $\square$

Lemma 2.1 describes the local behavior of the coordinate hyperplane section: if X has a terminal orbifold point of type $\frac{1}{r}(1, a_1, \ldots, a_{n-1})$ lying on $H = (x_0 = 0)$, then the induced singularity on $Y = X \cap H$ is the canonical quotient $\frac{1}{r}(a_1, \ldots, a_{n-1})$. In particular, for an index-1 ITF4 $X \subset \mathbb{P}(1, w_1, \ldots)$ whose orbifold points all have a weight 1 polarization, every anticanonical section $Y \in |-K_X|$ has $K_Y \sim 0$ and only cyclic quotient singularities of the expected canonical type.

Conversely, our construction goes in the other direction: by Theorem 1.6, every wellformed, quasismooth codimension 2 and codimension 3 weighted complete intersection Calabi–Yau threefold with isolated canonical singularities recorded in [BK, BKZ22] occurs as a special hyperplane section $Y = X \cap (x_0 = 0)$ of some index-1 terminal Fano 4-fold $X \subset \mathbb{P}(1, w_1, \ldots, w_6)$.

However, quasismoothness of Y is a global condition and can fail even when X is quasismooth, because after setting $x_0 = 0$ one of the defining equations may vanish identically on a coordinate stratum, as depicted by Example below.

Example 2.2. Consider the wellformed quasismooth terminal Fano 4-fold of index 1

$$X = (f_{52} = f_{66} = 0) \subset \mathbb{P}(1, 2, 6, 11, 25, 33, 41) = \mathbb{P}(x_0, y, z, u, v, w, t),$$

with basket of quotient singularities

$$\frac{1}{3}(1, 2, 2, 2), \quad 2 \times \frac{1}{11}(1, 2, 3, 6), \quad \frac{1}{25}(1, 6, 8, 11), \quad \frac{1}{41}(1, 2, 6, 33).$$

By adjunction, the hyperplane section $Y := X \cap (x_0 = 0)$ satisfies $K_Y \sim 0$, hence Y is a Calabi–Yau 3-fold.

After setting $x_0 = 0$ the section is a codimension 2 complete intersection

$$Y = Y_{52,66} \subset \mathbb{P}(2, 6, 11, 25, 33, 41).$$

On the coordinate stratum $\Sigma = \mathbb{P}(6, 25, 33) \subset \mathbb{P}(2, 6, 11, 25, 33, 41)$ there is no monomial of weighted degree 52 in the variables of weights $6, 25, 33$, so

$$f_{52}\big|_{\Sigma} \equiv 0.$$

On the other hand, there are monomials of degree 66 on Σ (for instance w^2 with $\deg(w) = 33$), so for general f_{66} the restriction $f_{66}|_{\Sigma}$ is not identically zero. Consequently,

$$Y \cap \Sigma = \Sigma \cap (f_{66} = 0)$$

is a curve, and along this curve the Jacobian rank condition for quasismoothness drops. Hence Y is not quasismooth.

This illustrates how a fixed anticanonical section can fail quasismoothness even when the ambient fourfold X is quasismooth: the equation(s) that ensure transversality on X may vanish identically on a coordinate stratum after imposing $x_0 = 0$. $\square$

The following example describes the behavior when we have singular points with no polarization of weight 1.

Example 2.3. Let

$$X = X_{6,6,6} \subset \mathbb{P}(1,1,1,2,3,3,3,5)$$

be a codimension 3 complete intersection terminal Fano 4-fold. The basket contains the orbifold point

$$P = P_t \in X, \qquad (X,P) \cong \frac{1}{5}(2,3,3,3).$$

Indeed, in the chart $t \neq 0$ (t is the weight 5 variable) the ambient space is $\mathbb{A}^7/\mu_5(1,1,1,2,3,3,3)$, and since $6 = 5+1$, all the general defining equations contain terms tx_i (with x_i of weight 1), so locally we eliminate the three weight 1 variables and obtain the quotient type $\frac{1}{5}(2,3,3,3)$.

Now take the "linear" section $Y = X \cap (x_1 = 0)$. At P the coordinate x_1 is not among the local parameters on the index-one cover of X (it is eliminated by the equations), so $(x_1 = 0)$ does not cut out a coordinate hyperplane in the local quotient chart. $\square$

3. Algorithm and computations

In this section, we describe all the ingredients and steps required to prove the Theorem 1.1, Theorem 1.1 1.2 and Theorem 1.3. We use the format search algorithm of [Qur17, BKZ22] to search for the candidate ITF4s. Then we prove the existence, quasismoothness and terminality of each 4-folds by using the following algorithmic and computational approach.

3.1. **Algorithm for candidate isolated orbifolds.** We recall the candidate-generation algorithm of [Qur17] used in this paper to produce numerical candidates for terminal Fano 4-folds. The main input is the orbifold Riemann–Roch framework of Buckley–Reid–Zhou [BRZ13], which expresses the Hilbert series $P_X(t)$ as a sum of a "smooth" contribution and terms coming from isolated orbifold points. If X has a basket of isolated orbifold points

$$\mathcal{B} = \{\, k_i \times Q_i : k_i \in \mathbb{Z}_{>0} \,\},$$

then

$$P_X(t) = P_{\mathrm{smooth}}(t) + \sum_{Q_i \in \mathcal{B}} k_i \, P_{Q_i}(t), \tag{1}$$

where $P_{\mathrm{smooth}}(t)$ is the smooth part and each $P_{Q_i}(t)$ is the local contribution determined by the singularity type at Q_i.

Fix the dimension, the Fano index i, and a polarizing divisor A such that

$$K_X = \mathcal{O}(-iA).$$

For a fixed adjunction number (read off from $P_X(t)$), the algorithm searches for all possible baskets $\mathcal{B}$ compatible with (1). For the reader's convenience we outline the steps in the case of Fano 4-folds.

(i) **Choose the ambient space.** Enumerate weighted projective spaces $\mathbb{P}(w_0, \dots, w_N)$ and for each record

$$P_{\mathbb{P}}(t) = \frac{1}{\prod_{j=0}^{N}(1 - t^{w_j})}, \qquad K_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}\left(-\sum_{j=0}^{N} w_j\right).$$

(ii) **List complete intersection types.** For each $\mathbb{P}(w_0, \dots, w_N)$, list codimension-$c$ degree vectors $\mathbf{d} = (d_1, \dots, d_c)$ satisfying

$$i = \sum_{j=0}^{N} w_j - \sum_{\ell=1}^{c} d_\ell, \qquad d_\ell \neq w_j \quad \forall j, \ell,$$

so that for $X = X_{d_1,\dots,d_c} \subset \mathbb{P}(w_0, \dots, w_N)$ we have $K_X \cong \mathcal{O}_X(-iA)$ and X is not a linear-cone intersection.

(iii) **Compute Hilbert series.** For each admissible $\mathbf{d}$, compute the WCI Hilbert series

$$P_X(t) = \frac{\prod_{\ell=1}^{c}(1 - t^{d_\ell})}{\prod_{j=0}^{N}(1 - t^{w_j})}.$$

Extract the smooth contribution $P_{\mathrm{smooth}}(t)$ from the initial coefficients of $P_X(t)$.

(iv) **Match the orbifold part.** Form the difference $P_X(t) - P_{\mathrm{smooth}}(t)$ and test whether it can be written as

$$P_X(t) - P_{\mathrm{smooth}}(t) = \sum_{Q \in \mathcal{B}} k_Q \, P_Q(t), \qquad k_Q \in \mathbb{Z}_{\geq 0},$$

for some basket $\mathcal{B}$ of terminal cyclic quotient types compatible with the weights w_i.

(v) **Record candidates.** Retain $(\mathbb{P}(w_0, \dots, w_N), \mathbf{d}, \mathcal{B})$ whenever such a decomposition exists, and repeat over all ambient choices and all admissible degree vectors.

3.2. **Existence, consistency and wellformedness.** We briefly describe the verification step; for further details see [Qur19]. In the present paper all candidates are weighted complete intersections, so once the ambient weights and degrees are fixed we obtain an explicit family by choosing general weighted homogeneous equations of the prescribed degrees. We then check that the resulting 4-fold X is consistent with the Hilbert series decomposition produced by Algorithm 3.1, namely that X contains exactly the basket of isolated terminal quotient singularities predicted by the orbifold Riemann–Roch analysis.

The check is carried out by intersecting X with each singular toric stratum of the ambient weighted projective space $\mathbb{P}^n(w_i)$, using Magma. Since we allow only isolated orbifold points on X, each such intersection must be finite (equivalently, of dimension at most 0), which in particular enforces wellformedness. A candidate is rejected if X meets a positive-dimensional singular stratum of $\mathbb{P}^n(w_i)$, or if any isolated orbifold point occurring on X is non-terminal, or if the observed basket does not match the predicted multiplicities.

3.3. **Quasismoothness.** The main geometric verification step is to prove that a candidate weighted complete intersection $X \subset \mathbb{P}^n(w_i)$ is quasismooth. Even for complete intersections, this is not automatic: the equations have prescribed degrees, and the corresponding linear systems may have nontrivial base loci dictated by the combinatorics of the weights. A Bertini-type statement implies that a general choice of equations is quasismooth away from the reduced base locus, so the problem reduces to controlling what happens along the base locus and, in particular, on the singular strata of the ambient weighted projective space.

In this paper we certify quasismoothness using explicit computation. For each candidate we choose generic equations of the prescribed degrees (with random coefficients in a suitable finite field), and we apply the Jacobian criterion on the affine cone, stratum by stratum. These computations are implemented in MAGMA [BCP97]. In practice this is effective for the codimension 2 and 3 complete intersections considered here, but the cost grows quickly with the size of the weights.

For codimension 2 candidates we sometimes complement the Jacobian computations with Fletcher's quasismoothness criteria [IF00, Theorem 8.7], which provides a fast sufficient check in many cases.

4. Analysis of results

In this section we analyze the output for wellformed, quasismooth index-1 isolated terminal Fano 4-folds in codimension 2 and 3, following the steps in (Section 3). We split the discussion according to the anticanonical linear system: when $|-K_X| \neq \emptyset$ we test whether the anticanonical divisor(s) give quasismooth canonical Calabi–Yau 3-folds (distinguishing the deterministic case $h^0(-K_X) = 1$ from the generic case $h^0(-K_X) \geq 2$), and compare with the list in [BKZ22]. When $|-K_X| = \emptyset$ we record the vanishing of $|-\ell K_X|$ for small ℓ, and exhibit extremal examples. The tables in each subsection summarize the distribution of families and highlight the exceptional type-$K_\bullet$ behavior.

4.1. **Isolated terminal Fano 4-folds with $|-K_X| \neq \emptyset$.** In this section, we record the output of our format search results for wellformed, quasismooth codimension 2 and 3 weighted complete intersection models of index-1 isolated terminal Fano 4-folds X with $|-K_X| \neq \emptyset$. When $h^0(-K_X) = 1$ the anticanonical system has a single member, so the Calabi–Yau section test is deterministic; when $h^0(-K_X) \geq 2$ we test whether a general member of $|-K_X|$ is a quasismooth canonical Calabi–Yau threefold. In both cases we compare the resulting quasismooth Calabi–Yau sections with the list of Brown–Kasprzyk–Zhou [BKZ22].

4.1.1. *Summary.* Table 4 summarizes the quasismooth families found (up to the weight bound W) by codimension and by the value of $h^0(-K_X)$. The column "#QS CY3 $\in |-K_X|$" counts those families whose anticanonical section occurs among the quasismooth canonical Calabi–Yau 3-folds listed in [BKZ22]. The final column "#QS-$K_\bullet$" records the remaining quasismooth families: it counts type-K_1 cases in the rows with $h^0(-K_X) = 1$, and type-K_2 cases in the rows with $h^0(-K_X) \geq 2$.

Table 4. Combined summary of quasismooth ITF4 families with $|-K_X| \neq \emptyset$

Format	W	$h^0(-K_X)$	# QS Fano 4-folds	#QS CY3 $\in |-K_X|$	#QS-$K_\bullet$
C.I cod. 2	120	1	415	139	274+2
		2	202	191	11
		$3\dots7$	71	71	0
C.I cod. 3	80	1	51	9	42
		2	16	9	7
		3	3	2	1
		$4\dots8$	2	2	0

4.1.2. *Sample Examples.* The following are some examples representing extreme and interesting phenomena in this case.

Example 4.1. In Table 4, we list the last entry in the first row as 274+2 to show that there are two examples of ITF4s with all orbifold points of type $\frac{1}{r}(1,a,b,c)$ but $|-K_X|$ has no quasismooth canonical Calabi–Yau 3-fold, unlike 139 other cases. We described one case in detail in Example 2.2. The other one is the ITF4

$$X = X_{40,54} \subset \mathbb{P}(1,2,5,6,19,27,35)$$

with basket

$$\mathcal{B} = \left\{\tfrac{1}{3}(1,2,2,2),\ \tfrac{1}{5}(1,1,2,2),\ \tfrac{1}{19}(1,5,6,8),\ \tfrac{1}{35}(1,2,6,27)\right\}.$$

However, on the Calabi–Yau 3-fold section

$$Y := X \cap (x_0 = 0) = Y_{40,54} \subset \mathbb{P}(2,5,6,19,27,35)$$

there is no monomial of weighted degree 40 in the variables of weights $19, 27, 35$, so $f_{40}|_\Sigma \equiv 0$, whereas there are monomials of degree 54 on Σ (for instance u^2, with $\deg(u) = 27$). Hence $Y \cap \Sigma = \Sigma \cap (f_{54} = 0)$ is a curve and the Jacobian rank drops along it; therefore Y is not quasismooth. $\square$

Example 4.2 (An extreme $h^0(-K_X) = 2$ case with no canonical CY 3-fold section)**.** Among the 11 quasismooth ITF4 families of type-K_2 with $h^0(-K_X) = 2$, the following has the largest number of distinct orbifold types. Let

$$X \subset \mathbb{P}(1,1,3,7,11,11,11,22,22)$$

be the quasismooth index-1 terminal Fano 4-fold in our list with basket

$$\mathcal{B} = \left\{\tfrac{1}{3}(1,2,2,2),\ \tfrac{1}{7}(3,4,4,4),\ 4 \times \tfrac{1}{11}(1,1,3,7)\right\}.$$

Since $\mathcal{B}$ contains the point $\frac{1}{7}(3,4,4,4)$ with *no weight* 1 *polarization* the linear section contains worse than canonical singularities. $\square$

Example 4.3. A codimension 3 example of type-K_2, with largest value of $h^0(-K_X)$ is the weighted complete intersection

$$X = X_{6,6,6} \subset \mathbb{P}(1,1,1,2,3,3,3,5) = \mathbb{P}(x_0, x_1, x_2, y, z_1, z_2, z_3, t),$$

which is a wellformed, quasismooth isolated terminal Fano 4-fold of index 1 with basket

$$\mathcal{B} = \left\{ \frac{1}{5}(2,3,3,3) \right\}.$$

X has an orbifold point with no weight 1 polarization and linear section contains worse than canonical singularities.

4.2. **Isolated terminal Fano 4-folds with $|-K_X|$ empty.** This section describes the results of Theorem 1.3 in the complete intersection cases $c = 2, 3$. We focus on those candidates for which the anticanonical system is empty, and we highlight the extreme behaviour, namely examples with the largest m such that $h^0(-\ell K_X) = 0$ for every $1 \leq \ell \leq m$. Table 5 summarizes the outcome of our computer search and the subsequent vanishing and quasismoothness tests.

4.2.1. *Table summary.* Table 5 is organized by codimension. The first row #QS Fanos records the total number of quasismooth ITF4 families in our search window (up to the weight bound W in the second row). The rows $h^0(-\ell K_X) = 0$, $\ell \leq n$ count, among these quasismooth families, those for which the linear systems $|-\ell K_X|$ are empty for all $\ell \leq n$. Finally, the row $\#K_0$ records the total number of ITF4s in given codimension with $|-K_X| = \emptyset$ and hence has no Calabi–Yau 3-fold linear section.

Table 5. Summary of isolated terminal Fano 4-folds with $|-K_X| = \emptyset$

Attribute	Codimension 2	Codimension 3
#QS Fanos	782	87
W	120	80
$h^0(-K_X) = 0$	72	8
$h^0(-\ell K_X) = 0,\ \ell \leq 2$	19	3
$h^0(-\ell K_X) = 0,\ \ell \leq 3$	2	3
$h^0(-\ell K_X) = 0,\ \ell \leq 4$	1	1
$\#K_0$	94	15

4.2.2. *Sample Examples.* In each codimension, we list an example that represents the extreme case, i.e., the examples with largest m with $h^0(-lK_X) = 0, 1 \leq l < m$.

Example 4.4. Similarly, in codimension 2 consider the weighted complete intersection

$$X = X_{36,40} \subset \mathbb{P}(5^2, 7, 8, 9, 12, 31) = \mathbb{P}(5, 5, 7, 8, 9, 12, 31),$$

which is index 1 since $\sum w_i - \sum d_j = 77 - 76 = 1$. The smallest ambient weight is 5, hence

$$h^0(-\ell K_X) = h^0(X, \mathcal{O}_X(\ell)) = 0 \qquad \text{for } 1 \leq \ell \leq 4.$$

The basket is

$$\mathcal{B} = \left\{ \tfrac{1}{3}(1,2,2,2),\ 8 \times \tfrac{1}{5}(2,2,3,4),\ \tfrac{1}{7}(2,3,5,5),\ \tfrac{1}{31}(5,7,8,12) \right\}.$$

Example 4.5. In the codimension 3 complete intersection format, an extreme case for the emptiness of anticanonical multiples occurs with $m = 4$. Consider the weighted complete intersection

$$X = X_{32,24,20} \subset \mathbb{P}(5, 6, 7, 9, 10, 11, 13, 16).$$

Since

$$\sum w_i = 5 + 6 + 7 + 9 + 10 + 11 + 13 + 16 = 77, \qquad 32 + 24 + 20 = 76,$$

adjunction gives $K_X \sim \mathcal{O}_X(-1)$, so X is an index-1 $\mathbb{Q}$-Fano 4-fold. Moreover, because the smallest ambient weight is 5, there are no weighted forms of degree ℓ for $\ell \leq 4$, hence

$$h^0(-\ell K_X) = h^0\big(\mathcal{O}_X(\ell)\big) = 0 \qquad \text{for } 1 \leq \ell \leq 4.$$

One quasismooth example produced by our search has basket

$$\mathcal{B} = \left\{ \tfrac{1}{3}(1,1,1,1),\ 2 \times \tfrac{1}{5}(1,1,1,3),\ \tfrac{1}{7}(2,2,5,6),\ \tfrac{1}{9}(1,4,7,7),\ \tfrac{1}{11}(5,5,6,7),\ \tfrac{1}{13}(3,5,9,10) \right\}.$$

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