CH.5 Transform Analysis of LTI systems

Introduction :

→ In Ch. 2 and Ch. 3, the emphasis was on the transforms (DTFT and Z-transform) and their properties.

→ In this chapter, we develop in more detail the representation and analysis of LTI system using the Fourier and z-transform

→ This chapter is essential background for our discussion in Ch. 6 of the implementation of LTI systems and in Ch. 7 of the design of such systems.

$$x[n] \longrightarrow \overset{h[n]}{\boxed{\text{LTI}}} \longrightarrow y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$X(z) \longrightarrow \underset{\text{system Function}}{\boxed{H(z)}} \longrightarrow Y(z) = H(z)X(z)$$

$$y[n] = Z^{-1}\{Y(z)\}$$

$$X(e^{jw}) = e^{jwn} \longrightarrow \underset{\text{Frequency response.}}{\boxed{H(e^{jw})}} \longrightarrow H(e^{jw})\, e^{jwn}.$$

Both $H(e^{jw})$ and $H(Z)$ are useful in the analysis and representation of LTI systems. That is because we can infer many properties of the system response from them.

Frequency Response of LTI systems

$$Y(e^{jw}) = H(e^{jw})\, X(e^{jw})$$

$$|Y(e^{jw})| = |H(e^{jw})|\,|X(e^{jw})|$$

$$\angle Y(e^{jw}) = \angle H(e^{jw}) + \angle X(e^{jw})$$

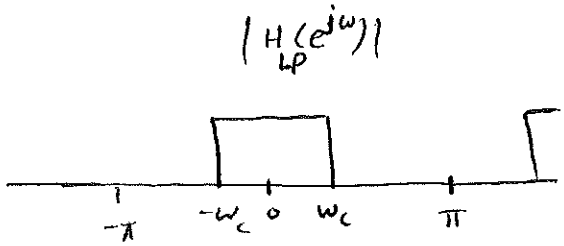
$|H(e^{jw})|$: magnitude Response (Gain)

$\angle H(e^{jw})$: phase Response (phase shift).

Note that : Undesirable effect of the system on the input signal is called magnitude or phase distortions or Both.

example : Ideal LPF

$$H_{LP}(e^{jw}) = \begin{cases} 1 & |w| < w_c \\ 0 & w_c < |w| \leq \pi \end{cases}$$



$H_{LP}(e^{jw})$ is periodic with period of 2π.

$|H_{LP}(e^{jw})| = 1$ LPF selects low frequency components and Rejects high " " .

$\angle H(e^{jw}) = 0 \Rightarrow$ No delay or phase distortion

remember $h_{LP}[n] = \dfrac{\sin w_c n}{\pi n} \quad -\infty < n < \infty$

Group delay (phase distortion and delay)

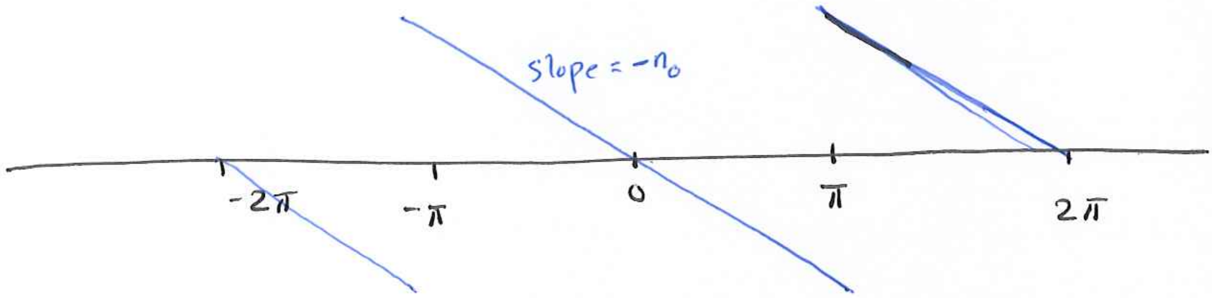
→ Another useful representation of the phase response is through defining the group delay as :-

$$\tau(w) = grd[H(e^{jw})] = -\frac{d}{dw}\{\angle H(e^{jw})\}$$

Example :- consider the ideal delay system

$h[n] = \delta[n-n_0]$ n_0 : integer.

$$H(e^{jw}) = 1\, e^{-jn_0 w}$$
$$|H(e^{jw})| = 1 \quad \text{(unity Gain)}$$
$$\angle H(e^{jw}) = -n_0 w \quad \text{(Linear phase response)}$$



* note that the time delay (or advance if $n_0 < 0$) is associated with phase that is linear with frequency.
* note also $\tau(w) = -\frac{d}{dw}\{-n_0 w\} = n_0$ samples.

In many applications, delay distortion would be tolerated and is considered as a simple form of phase distortion.

* for example, in designing ideal LPF and other LTI systems, we are willing to accept linear-phase response rather than zero phase response.

Example:

$$H_{LP}(e^{jw}) = \begin{cases} e^{-jwn_0} & |w| < w_c \\ 0 & w_c < |w| \leq \pi \end{cases}$$

$$h_{LP}[n] = \frac{\sin(w_c(n-n_0))}{\pi(n-n_0)}, \quad -\infty < n < \infty$$

$$\angle H_{LP}(e^{jw}) = -wn_0$$

(In general, it can be $= -wn_0 + \phi$ still linear)

$\Rightarrow$ Group delay: $\tau(w) = n_0$ samples.

$\Rightarrow$ Group delay measures the linearity of the phase. It can be thought as $\tau(w)$: average delay for all frequencies.

Example:

$$h[n] = \delta[n-4]$$

$$H(z) = z^{-4}$$

$$H(e^{jw}) = H(z)\Big|_{z=e^{jw}} = e^{-j4w}$$

$$\angle H(e^{jw}) = -4w$$

$$\tau(w) = 4 \text{ samples.}$$

Example: $h[n] = \frac{1}{3}\{\underset{\uparrow}{1}, 1, 1\}$ 3 point Moving Average system.

$$= \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

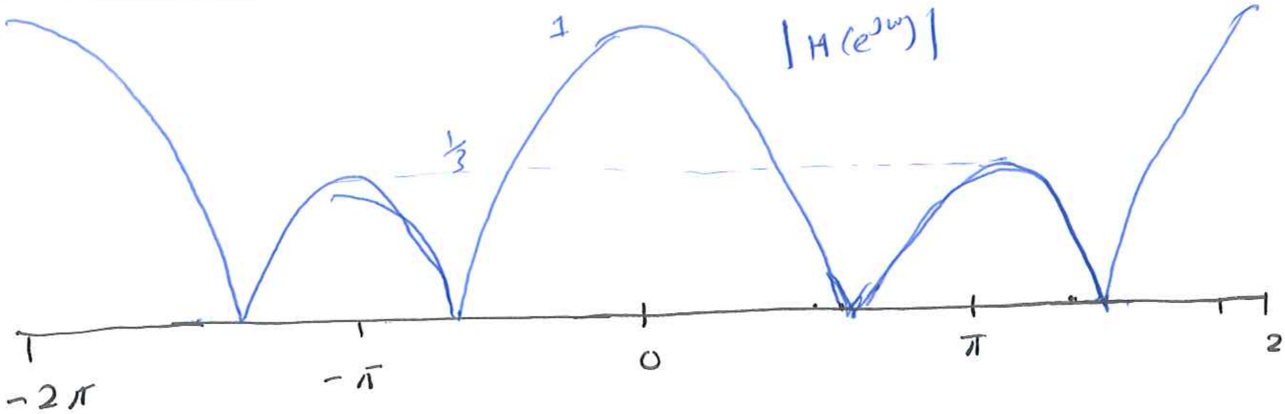
$$H(z) = \frac{1}{3}(1 + z^{-1} + z^{-2})$$

$$H(e^{jw}) = \frac{1}{3}(1 + e^{-jw} + e^{-j2w})$$

$$= \frac{1}{3}e^{-jw}(e^{jw} + 1 + e^{-jw}) = \frac{1}{3}e^{-jw}(1 + 2\cos w)$$

$$|H(e^{jw})| = \frac{1}{3}(1 + 2\cos w)$$

$$\angle H(e^{jw}) = -w \Rightarrow \tau(w) = 1 \text{ sample}$$

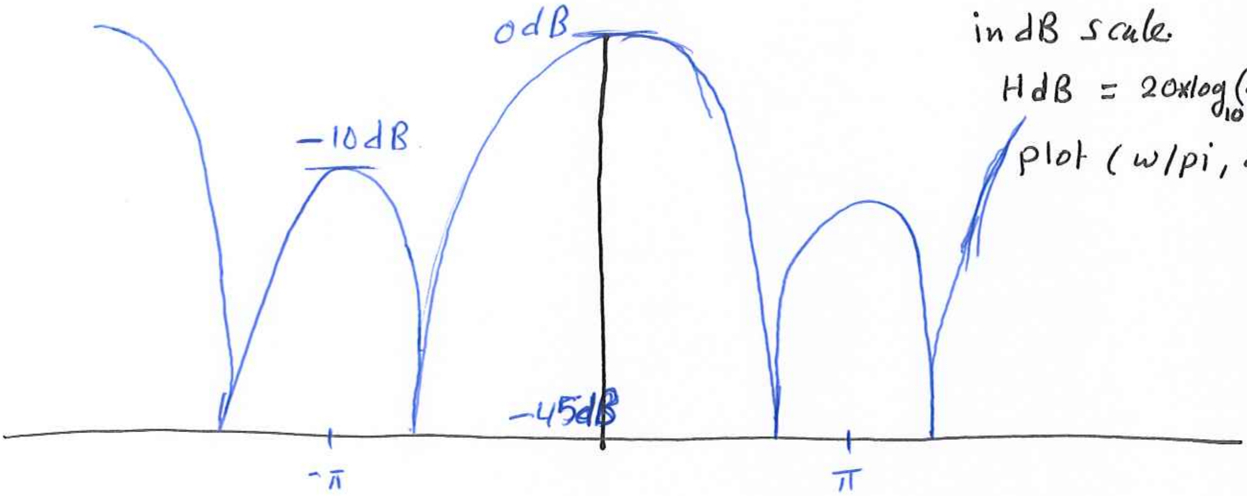


Matlab:

```
w = -2:0.01:2
w = pi*w
H = (1/3)*(1+2*cos(w))
plot (w/pi , abs(H))
```

in dB scale.

```
HdB = 20*log10(abs(H))
plot (w/pi, abs(HdB))
```



Exercise : let $h[n] = \frac{1}{5}\{1, 1, 1, 1, 1\}$, 5 point moving Averager

① Find $H(z)$ and the ROC

② Find $H(e^{j\omega})$

③ Find and plot the Amplitude response $|H(e^{j\omega})|$

④ Find the group delay.

⑤ Is the system LPF, HPF, BPF, BRF ?

Note: X_{dB} scale $= 20 \log_{10}(X_{linear\ scale})$

or $Gain_{dB} = 10 \log \underbrace{|H(e^{j\omega})|^2}_{Energy} = 20 \log |H(e^{j\omega})|$

Illustration of Effects of Group Delay and Attenuatio

As an illustration of the effects of phase, group delay, and attenuation, consider the specific system having system function

$$H(z) = \underbrace{\left(\frac{(1-.98e^{j.8\pi}z^{-1})(1-.98e^{-j.8\pi}z^{-1})}{(1-.8e^{j.4\pi}z^{-1})(1-.8e^{-j.4\pi}z^{-1})}\right)}_{H_1(z)} \underbrace{\prod_{k=1}^{4}\left(\frac{(c_k^* - z^{-1})(c_k - z^{-1})}{(1-c_k z^{-1})(1-c_k^* z^{-1})}\right)^2}_{H_2(z)} \quad (15)$$

with $c_k = 0.95e^{j(.15\pi+.02\pi k)}$ for $k = 1, 2, 3, 4$ and $H_1(z)$ and $H_2(z)$ defined as indicated. The pole-zero plot for the overall system function $H(z)$ is shown in Figure 2, where the factor $H_1(z)$ in Eq. (15) contributes the complex conjugate pair of poles at $z = 0.8e^{\pm j.4\pi}$ as well as the pair of zeros close to the unit circle at $z = .98e^{\pm j.8\pi}$.

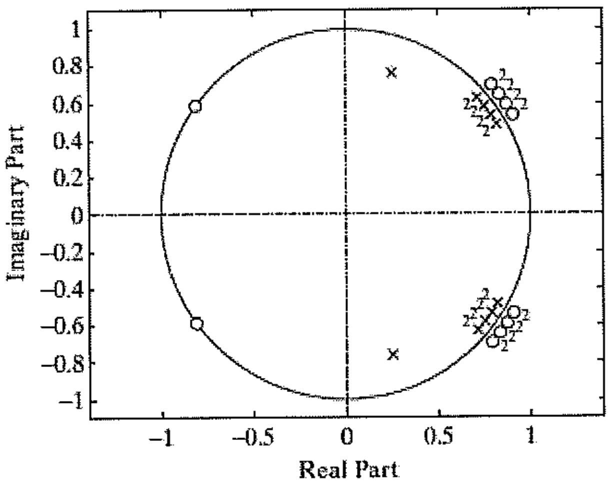


Figure 2 Pole-zero plot for the filter in the example of Section 1.2. (The number 2 indicates double-order poles and zeroes.)

The factor $H_2(z)$ in Eq. (15) contributes the groups of double-order poles at $z = c_k = 0.95e^{\pm j(.15\pi+.02\pi k)}$ and double-order zeros at $z = 1/c_k = 1/0.95e^{\mp j(.15\pi+.02\pi k)}$ for $k = 1, 2, 3, 4$. By itself, $H_2(z)$ represents an allpass system (see Section 5), i.e., $|H_2(e^{j\omega})| = 1$ for all ω. As we will see, $H_2(z)$ introduces a large amount of group delay over a narrow band of frequencies.

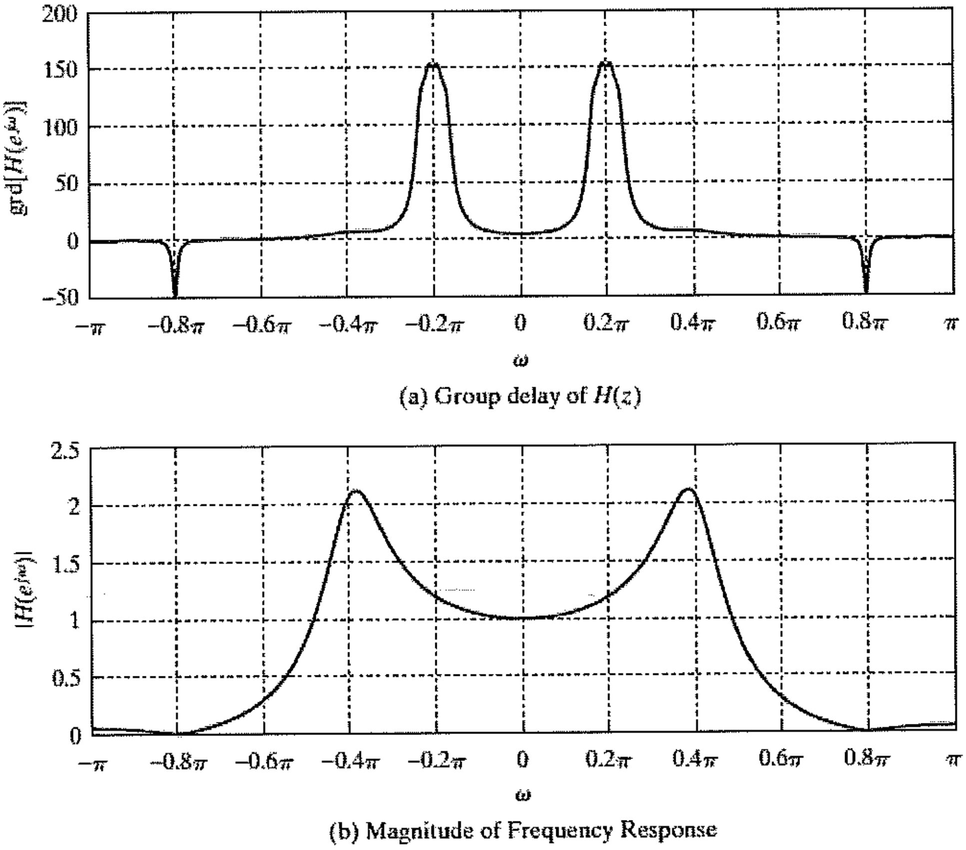


Figure 4 Frequency response of system in the example of Section 1.2;
(a) Group delay function, grd[$H(e^{j\omega})$], (b) Magnitude of frequency response, $|H(e^{j\omega})|$.

In Figure 5(a) we show an input signal $x[n]$ consisting of three narrowband pulses separated in time. Figure 5(b) shows the corresponding DTFT magnitude $|X(e^{j\omega})|$. The pulses are given by

$$x_1[n] = w[n]\cos(0.2\pi n), \tag{16a}$$

$$x_2[n] = w[n]\cos(0.4\pi n - \pi/2), \tag{16b}$$

$$x_3[n] = w[n]\cos(0.8\pi n + \pi/5). \tag{16c}$$

Each sinusoid is shaped into finite-duration pulse using hamming widow (it will be discussed in ch7).

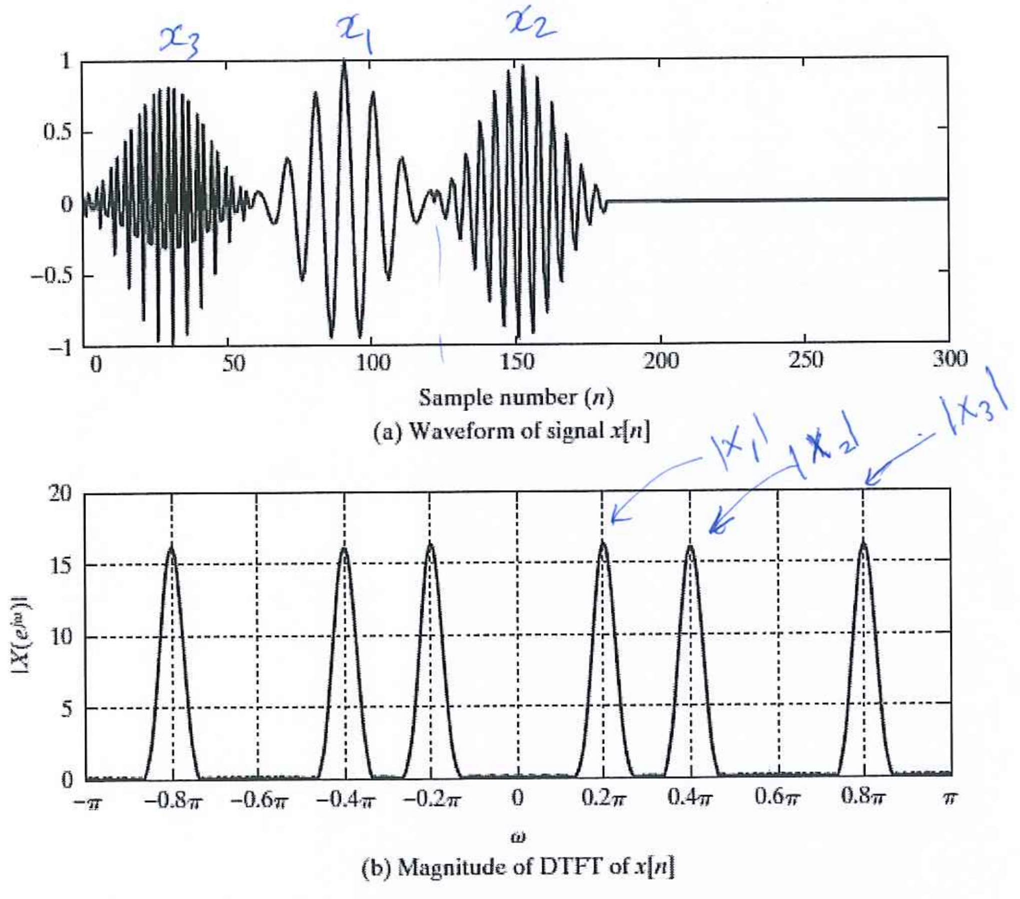


Figure 5 Input signal for example of Section 1.2; (a) Input signal $x[n]$, (b) Corresponding DTFT magnitude $|X(e^{j\omega})|$.

Each of the frequency packets or groups associated with each of the narrowband pulses will be affected by the filter response magnitude and group delay over the frequency band of that group.

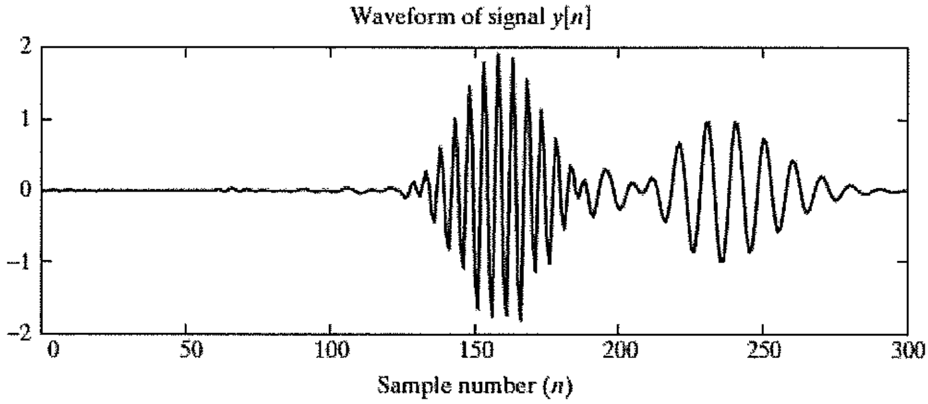


Figure 6 Output signal for the example of Section 1.2.

5.2 Systems Characterized by Linear Constant-Coefficient Difference Equation (LCCDE)

Systems and filters are most typically characterized and realized through LCCDE

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

* In ch. 6, we discuss various computational structures for realizing such systems.
* In ch. 7, we discuss various procedures for obtaining the parameters of the difference equation (a_k's, b_k's) to approximate a desired frequency response.

* In this section, we examine the properties and characteristics of LTI systems represented by LCCDE.

* The system function has the following form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{k=M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

each factor $(1-c_k z^{-1})$ contributes a zero at $z=c_k$ and a pole at $z=0$

" " $(1-d_k z^{-1})$ " a zero at $z=0$ and a pole at $z=d_k$

Example 5.1 2^{nd} order System

let $H(z) = \dfrac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}$

Find the difference equation ?

Solution :

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z)+\frac{1}{4}Y(z)z^{-1}-\frac{3}{8}Y(z)z^{-2} = X(z)+2X(z)z^{-1}+X(z)z^{-2}$$

apply $\mathcal{Z}^{-1}\{\ \}$

$$y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2] = x[n]+2x[n-1]+x[n-2]$$

* Stability and causality :-
  * From the Difference equation, we can obtain $H(z)$ but not the ROC
  * For $H(z)$ there is a number of choices for the ROC
    $\Rightarrow$ The Difference equation does't uniquely specify the impulse response $h[n]$ of the LTI system.
    Each choice of ROC Leads to different $h[n]$.

  For example, if we assume that the system is causal,
  $\Rightarrow h[n]$ is right-sided $\Rightarrow$ The ROC is outside the outermost pole.

  if we assume that it is stable $\Rightarrow h[n]$ is absolutely summable, i.e.,

$$\sum |h[n]z^{-n}| < \infty \Rightarrow \text{when } |z|=1 \Rightarrow \sum |h[n]| < \infty$$

* Stability $\equiv$ ROC of $H(z)$ includes the unit circle.

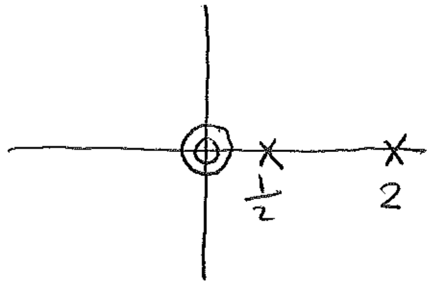
Example 5.2 Determine the ROC for the following LTI system:

$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$

Solution :-

$$H(z) = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}}$$

$$= \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - 2 z^{-1})}$$



Three possible ROC's :-

① $|z| > 2$ $\Rightarrow$ System is causal and Not stable (Right sided sequence)

② $|z| < \frac{1}{2}$ $\Rightarrow$ System is not causal, Not stable (left sided)

③ $\frac{1}{2} < |z| < 2$ $\Rightarrow$ Not causal stable. (Two sided)

Remember, All poles of $H(z)$ must be inside the unit circle for the system to be both stable and causal.

Inverse Systems

For LTI system with system Function $H(z)$,

$$X(z) \longrightarrow \boxed{H(z)} \xrightarrow{Y(z)} \boxed{H_i(z)} \longrightarrow X(z)$$

System Function   Inverse system Function

the cascade connection is equivalent to

$$X(z) \longrightarrow \boxed{G(z)} \longrightarrow X(z)$$

$G(z)$: overall effective system Function

$$G(z) = H(z)\, H_i(z) = 1 \Rightarrow \boxed{H_i(z) = \frac{1}{H(z)}}$$

in time-domain :-

$$g[n] = h[n] * h_i[n] = \delta[n]$$

in frequency domain, the Freq. response of the inverse system

$$H_i(e^{jw}) = \frac{1}{H(e^{jw})}$$

Notes: $\boxed{1}$ Not all systems have an inverse !

For example, the ideal low pass filter (LPF) does't have.

(there is no way to recover the frequency components above the cutoff frequency that are set to zero)

[2] Many systems do have inverse,

For example, systems with rational functions

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

with zeros at $z = c_k$

poles at $z = d_k$

possible zeros and/or poles at $z = 0$ and $z = \infty$

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N} (1 - d_k z^{-1})}{\prod_{k=1}^{M} (1 - c_k z^{-1})}$$

the poles of $H_i(z)$ are zeros of $H(z)$ and vice versa.

Question: What ROC to associate with $H_i(z)$?

Answer: Using convolution theorem, the ROC of $H(z)$ and $H_i(z)$ must overlap.

$\Rightarrow$ Thus, any appropriate ROC for $H_i(z)$ that overlaps with the ROC of $H(z)$ is a valid ROC for $H_i(z)$.

Example 5.3 Find $h_i[n]$?

$$H(z) = \frac{1 - 0.5 z^{-1}}{1 - 0.9 z^{-1}} \quad \text{with ROC } |z| > 0.9$$

solution: $H_i(z) = \dfrac{1 - 0.9 z^{-1}}{1 - 0.5 z^{-1}}$ with one pole at $z = 0.5$

there is two possible ROCs for $H_i(z)$

Choice 1: $|z| > 0.5$ overlaps with the ROC of $H(z)$: $|z| > 0.9$

Choice 2: $|z| < 0.5$ doesn't not overlaps.

$\Rightarrow$ we choose choice number 1:

$\Rightarrow h_i[n] = (0.5)^n u[n] - 0.9 (0.5)^{n-1} u[n-1]$

$\Rightarrow$ the inverse is both causal and stable.

Example 5.4 Determine $h_i[n]$ for

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9 z^{-1}} \qquad |z| > 0.9$$

solution :

$$H_i(z) = \frac{1 - 0.9 z^{-1}}{z^{-1} - 0.5} * \frac{-2}{-2} = \frac{-2(1 - 0.9 z^{-1})}{(1 - 2 z^{-1})}$$

multiply by $\frac{z}{z}$

$$\Rightarrow H_i(z) = \frac{-2(z - 0.9)}{(z - 2)}$$

pole at $z = 2$
zero at $z = 0.9$

two possible ROC's :-

① $|z| < 2$ it overlaps with ROC of $H(z)$ $|z| > 0.9$

$$\Rightarrow H_i(z) = \frac{-2}{1 - 2 z^{-1}} + \frac{1.8}{1 - 2 z^{-1}} z^{-1}$$

left sided $\Rightarrow h_{i,1}[n] = -2 \cdot -(2)^n u[-n-1] - 1.8\,(2)^{n-1} u[-(n-1)-1]$

$$= 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-n]$$

it is stable and non causal inverse system.

② $|z| > 2$ also, it overlaps with the ROC of $H(z)$: $|z| > 0.9$

right sided $\Rightarrow h_{i,2}[n] = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1]$

it is causal and not stable inverse system

Conclusion : An LTI system is stable and causal and also has a stable and causal inverse if and only if the poles and zeros of $H(z)$ are inside the unit circle.

Such systems are referred to as "minimum-phase systems" (will be discussed later)

minimum phase system $\equiv$ system with negligable phase effect or distortion

5.5 All-Pass Systems

A stable function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} = \frac{-a^*\left(1 - \frac{1}{a^*} z^{-1}\right)}{\left(1 - a z^{-1}\right)}$$

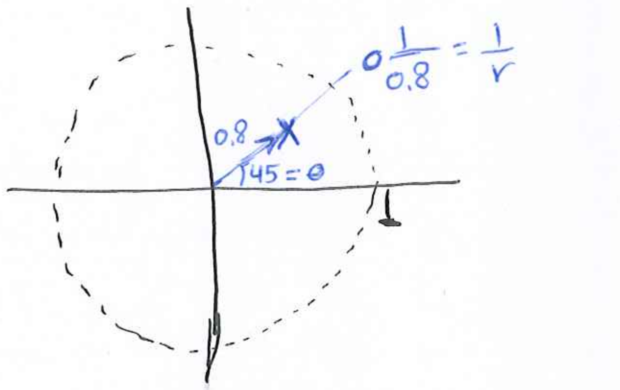
has a frequency-response magnitude independent of ω

All-Pass $\equiv$ Poles and zeros are conjugate reciprocal

pole $\quad z = re^{j\theta} \xrightarrow{\text{conjugate}} z^* = re^{-j\theta} \xrightarrow{\text{reciprocal}} \frac{1}{z^*} = \frac{1}{re^{-j\theta}}$

$\Rightarrow$ zero $\quad \frac{1}{z^*} = \frac{1}{r} e^{j\theta}$

note that, they have the same θ.



To show that $H_{ap}(z)$ has a frequency-response magnitude independent of ω:-

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - a e^{-j\omega}} = e^{-j\omega} \cdot \frac{1 - a^* e^{+j\omega}}{1 - a e^{-j\omega}}$$

note that,

$$|e^{-j\omega}| = 1$$

numerator and denominator factors are complex conjugate of each other $\Rightarrow$ they have the same magnitude.

$$\Rightarrow |H_{ap}(e^{j\omega})| = 1 \quad \text{or constant over } 0 \leq \omega \leq 2\pi$$

The most general form for the system function of an all-pass system with a real valued impulse response is given by (complex poles being paired)

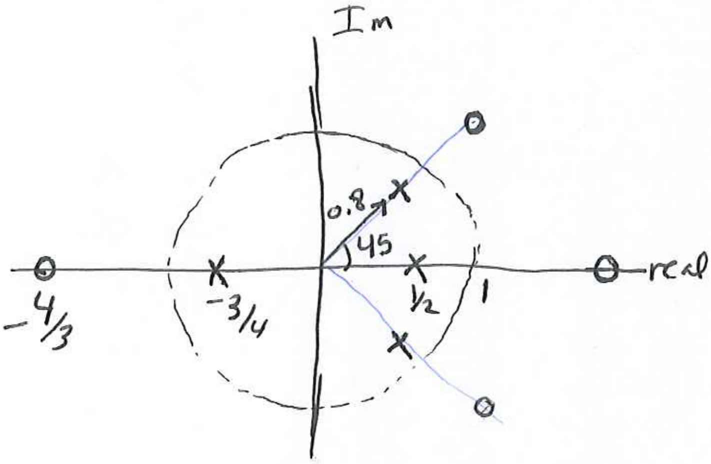
$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k^* z^{-1})(1 - e_k z^{-1})}$$

A : constant

d_k : real poles

e_k : Complex poles

example: Typical pole-zero plot for an all-pass system



Notes :-

* Min phase system (all zeros and poles are inside unit circle) the main effect on the magnitude

* All-pass system (poles and zeros are conjugate reciprocal) the main effect is on the phase

Example : All pass system

let $H(z) = \dfrac{z^{-1} - 0.9}{1 - 0.9 z^{-1}}$

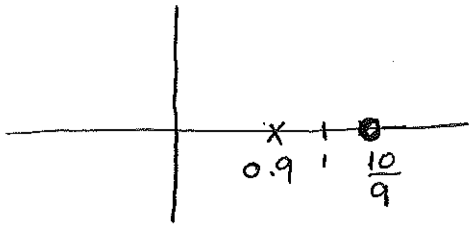
then $H(e^{jw}) = e^{-jw} \dfrac{1 - 0.9 e^{+jw}}{1 - 0.9 e^{-jw}} = e^{-jw} \dfrac{1 - 0.9(\cos w + j \sin w)}{1 - 0.9(\cos w - j \sin w)}$

$|H(e^{jw})| = 1, \quad -\pi < w < \pi$

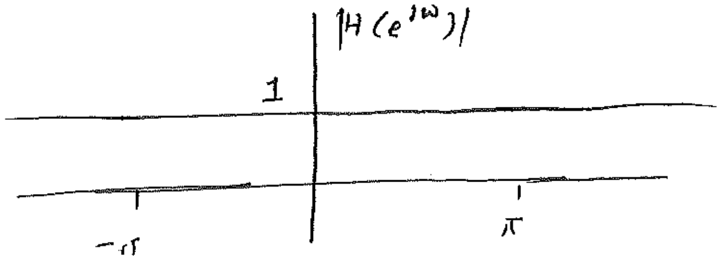
$$\angle H(e^{jw}) = -w + \tan^{-1}\frac{-0.9 \sin w}{1 - 0.9\cos w} - \tan^{-1}\frac{0.9 \sin w}{1 - 0.9 \cos w}$$

$$= -w - 2\tan^{-1}\frac{0.9 \sin w}{1 - 0.9\cos w}$$

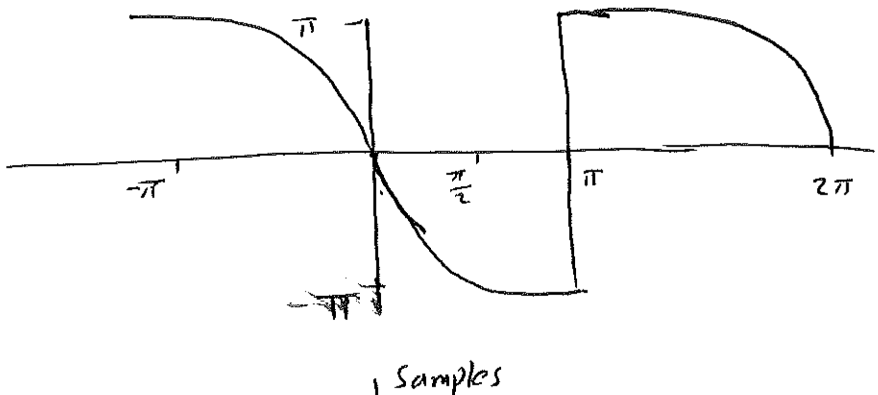
zero-pole plot



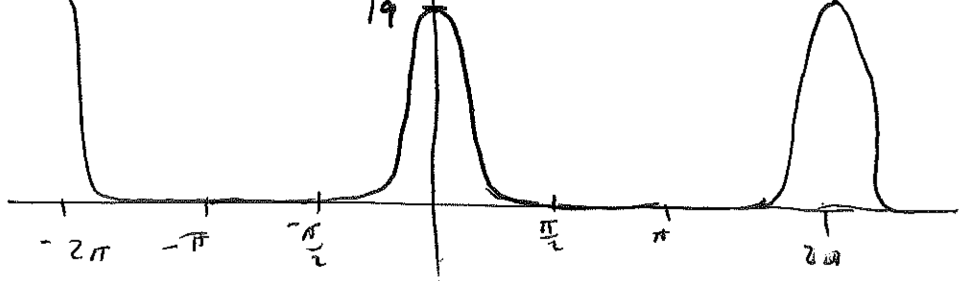
Amp. freq. Response



phase response



Group delay



5.6 Minimum-phase and all-pass System Decomposition

Any rational system function can be factorized as a cascade of minimum phase system and all-pass systems

$$\boxed{H(z) = H_{min}(z)\, H_{ap}(z)}$$

$H_{min}(z)$ contains all the poles and zeros of $H(z)$ that lie inside the unit circle, together with zeros that are the conjugate reciprocals of the zeros of $H(z)$ that lie outside the unit circle.

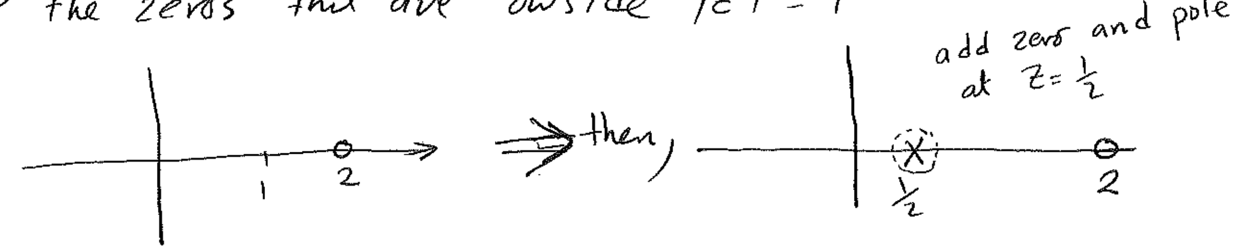
$H_{ap}(z)$ contains all zeros of $H(z)$ that lie outside the unit circle, together with poles to cancel the reflected conjugate reciprocal zeros in $H_{min}(z)$.

Steps to Decompose $H(z)$:

① Take zeros that lie outside $|z| = 1$ and move them to $H_{ap}(z)$

② Add poles to $H_{ap}(z)$ in conjugate reciprocal locations of the zeros that are outside $|z| = 1$

If,



③ put zeros inside $|z| < 1$ to cancel poles added to $H_{ap}(z)$

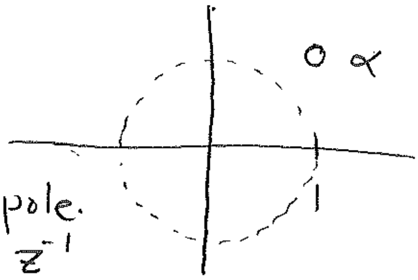
Example: Suppose that $H(z)$ has one zero outside $|z|=1$ such that

$$H(z) = H_1(z)\,(1-\alpha z^{-1}) \qquad |\alpha| > 1$$

$H_1(z)$ is min. phase

solution:

$$H(z) = -\alpha\, H_1(z)\left(z^{-1} - \frac{1}{\alpha}\right)$$



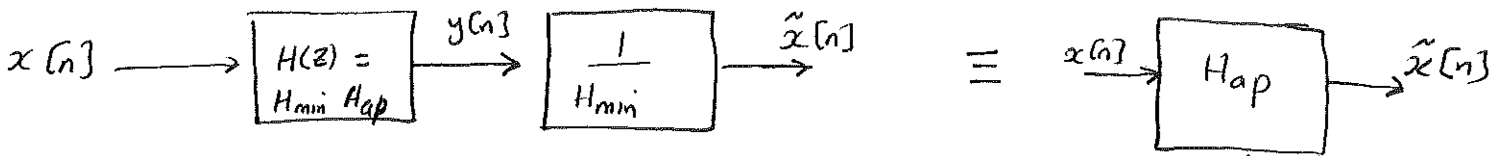
add pole inside $|z| = 1$
add zero inside $|z| = 1$ to cancel the pole.

$$\Rightarrow H(z) = -\alpha\, H_1(z)\left(z^{-1} - \frac{1}{\alpha}\right) \cdot \frac{1 - \frac{1}{\alpha^*} z^{-1}}{1 - \frac{1}{\alpha^*} z^{-1}}$$

$$= \underbrace{-\alpha\, H_1(z) \cdot \left(1 - \frac{1}{\alpha^*} z^{-1}\right)}_{H_{min}(z)} \cdot \underbrace{\frac{\left(z^{-1} - \frac{1}{\alpha}\right)}{\left(1 - \frac{1}{\alpha^*} z^{-1}\right)}}_{H_{ap}(z)}$$

note that, the min phase portion of any system has a stable and causal inverse system.

* If we are interested to remove the effect of the magnitude response

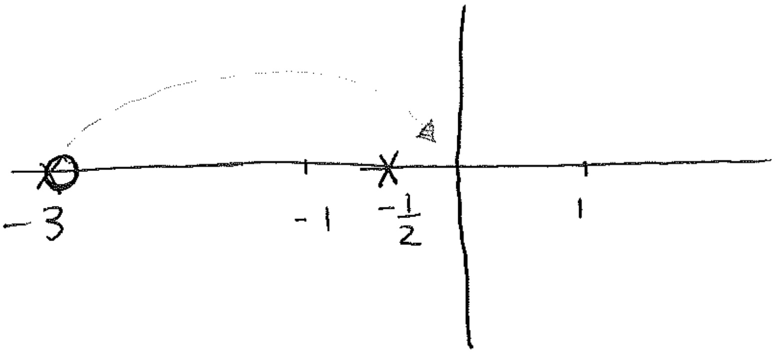


no magnitude distortion

* If we are interested to remove the effect of phase response of $H(z)$:

No phase distortion

Example 5.12 decompose $H(z) = \dfrac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$



$H(z)$ is not minimum phase

$$H(z) = \frac{3(z^{-1}+\frac{1}{3})}{1+\frac{1}{2}z^{-1}}$$

$$= \frac{3(z^{-1}+\frac{1}{3})}{1+\frac{1}{2}z^{-1}} \cdot \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}$$

$$= \left(3 \cdot \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}\right)$$

$$= H_{min} \cdot H_{ap}$$

Decompose $H(z) = \dfrac{9}{4}\dfrac{(z^{-1}+2/3\,e^{-j\frac{\pi}{4}})(z^{-1}+2/3\,e^{j\pi/4})}{1-\frac{1}{3}z^{-1}}$

It has two complex ~~poles~~ zeros outside the unit circle and a real pole inside

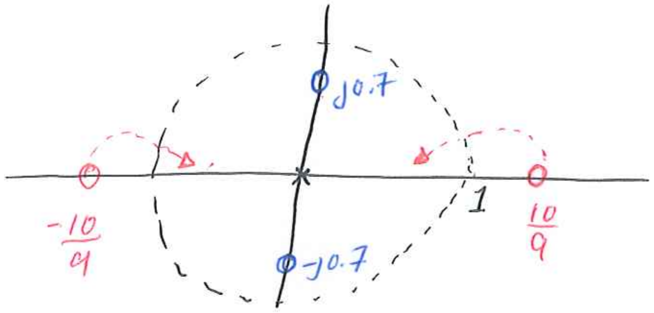
$$\Rightarrow H(z) = \frac{9}{4}\frac{(z^{-1}+2/3\,e^{-j\frac{\pi}{4}})(z^{-1}+\frac{2}{3}e^{j\pi/4})}{1-\frac{1}{3}z^{-1}} \cdot \left(\frac{(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1})(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1})}{(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1})(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1})}\right)$$

$$= \underbrace{\left[\frac{9}{4}\frac{(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1})(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1})}{1-\frac{1}{3}z^{-1}}\right]}_{H_{min}}\underbrace{\left[\frac{(z^{-1}+\frac{2}{3}e^{-j\frac{\pi}{4}})(z^{-1}+\frac{2}{3}e^{j\frac{\pi}{4}})}{(1+\frac{2}{3}e^{+j\pi/4}z^{-1})(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1})}\right]}_{H_{ap}}$$

Example: Decompose $H(z)$ into min. phase and all-pass

$$H(z) = \left(1 - \frac{10}{9} z^{-1}\right)\left(1 + \frac{10}{9} z^{-1}\right)\underbrace{\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)}$$

Direct to the min. phase since that the zero are inside $|z|<1$



to maintain $|H_{ap}(e^{j\omega})| = 1$, we express $H(z)$ zeros' that lies outside the unit circle in general form $\dfrac{z^{-1} - c^*}{\cancel{1} - c z^{-1}}$

$$\left(1 - \frac{10}{9} z^{-1}\right) \longrightarrow \left(-\frac{10}{9}\right)\left(z^{-1} - \frac{9}{10}\right) = -\frac{10}{9}\left(z^{-1} - \frac{9}{10}\right) \cdot \frac{1 - \frac{9}{10} z^{-1}}{1 - \frac{9}{10} z^{-1}}$$

$$= \underbrace{-\frac{10}{9}\left(1 - \frac{9}{10} z^{-1}\right)}_{\text{to the } H_{min}} \cdot \underbrace{\left(\frac{z^{-1} - 9/10}{1 - 9/10\, z^{-1}}\right)}_{\text{to the } H_{ap}(z)}$$

in the same way

$$1 + \frac{10}{9} z^{-1} \longrightarrow \frac{10}{9}\left(z^{-1} + 9/10\right) \cdot \frac{1 + \frac{9}{10} z^{-1}}{1 + \frac{9}{10} z^{-1}}$$

$$= \left(\frac{10}{9}\right)\left(1 + \frac{9}{10} z^{-1}\right) \cdot \left(\frac{z^{-1} + 9/10}{1 + \frac{9}{10} z^{-1}}\right)$$

$$\Rightarrow \quad H_{min}(z) = -\frac{100}{81}\left(1 - 0.9 z^{-1}\right)\left(1 + 0.9 z^{-1}\right)\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)$$

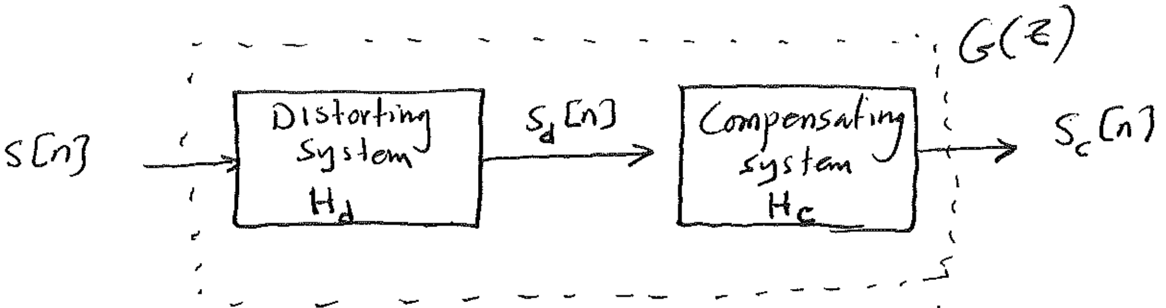
$$H_{ap}(z) = \frac{z^{-1} - 0.9}{1 - 0.9 z^{-1}} \cdot \frac{z^{-1} + 0.9}{1 + 0.9 z^{-1}}$$

Note that, the effect of $|H(e^{j\omega})|$ goes to H_{min}

" " " $\angle H(e^{j\omega})$ " " H_{ap}

Frequency-Response Compensation of Non-Minimum phase Systems

* In many signal-processing contexts, a signal has been distorted by an LTI system with an undesirable frequency response. It may then be of interest to process the distorted signal with a compensating system.



if perfect compensation is acheived, then $s_c[n] = s[n]$ such that

$H_c(z)$ is the inverse of $H_d(z)$.

However, if we assume that H_d is stable and causal and require H_c to be stable and causal, then perfect compensation is possible only if $H_d(z)$ is a minimum phase system, so that it has a stable, causal inverse

Assuming that $H_d(z)$ is known or approximated as a rational system function, we can form a minimum-phase $H_{d\,min}(z)$ by reflecting all zeros of $H_d(z)$ that are outside the unit circle to their conjugate reciprocal locations inside the unit circle

$\Rightarrow$ $H_d(z)$ and $H_{d\,min}(z)$ have the same freq. response magnitude and are related through an all-pass system such that

$$H_d(z) = H_{d\,min}(z)\, H_{ap}(z)$$

choose the compensating filter to be

$$H_c(z) = \frac{1}{H_{d\,min}(z)}$$

$\Rightarrow$ the overall system function relating $s[n]$ and $s_c[n]$ is

$$G(z) = H_d(z)\,H_c(z) = H_{ap}(z)$$

$\Rightarrow$ $G(z)$ corresponds to an all-pass system.

$\Rightarrow$ the frequency-response magnitude is eactly compensated for, whereas the phase response is modefied to $\underline{\angle H_{ap}(e^{j\omega})}$.

Example 5.13 Compensation of an FIR System

$$H_d(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-\tfrac{10}{8}e^{j0.8\pi}z^{-1})(1-\tfrac{10}{8}e^{-j0.8\pi}z^{-1})$$

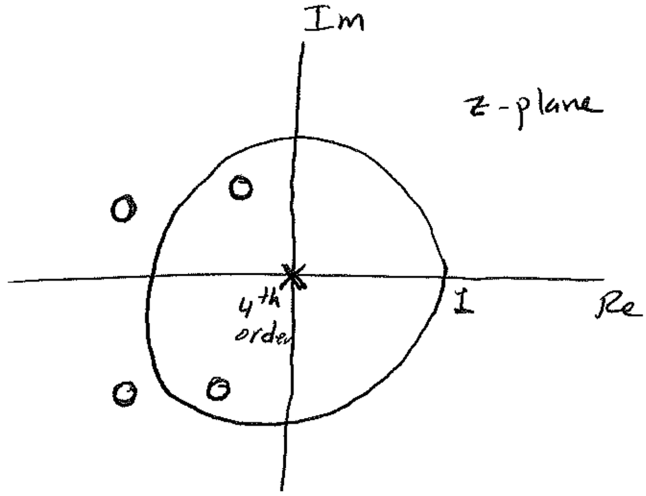
1. plot the pole-zero plot.
2. Find the ROC
3. Is the system minimum phase ?
4. Find the compensating system $H_c(z)$?

solution

1. The pole-zero plot

zeros at $0.9e^{j0.6\pi}$, $0.9e^{-j0.6\pi}$, $+1.25e^{j0.8\pi}$, $1.25e^{-j0.8\pi}$

poles at $z=0$ (4th order)



2. ROC :

since $H_d(z)$ is a polynomial with only negative powers of z,
$\Rightarrow$ the system is causal
also FIR $\Rightarrow$ stable.
ROC : all z-plane except $z=0$

3. since two of zeros are outside the unit circle, $H_d(z)$ is non minimum phase.

4. to obtain the minimum-phase system, we reflect the zeros that occurs at $z = 1.25e^{\pm j0.8\pi}$ to their conjugate reciprocal locations inside the unit circle

$$H_d(z) = (1-0.9e^{-j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})\left(-\tfrac{10}{8}\right)(e^{+j0.8})\left[z^{-1}-\tfrac{8}{10}e^{-j0.8}z^{-1}\right]$$
$$*\left(-\tfrac{10}{8}e^{-j0.8\pi}\right)\left[z^{-1}-\tfrac{8}{10}e^{j0.8\pi}z^{-1}\right]$$
$$=(1-0.9e^{-j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1.25)^2(z^{-1}-0.8e^{j0.8\pi})(z^{-1}-0.8e^{-j0.8\pi})$$

Remember: $\dfrac{z^{-1} - c^*}{1 - c z^{-1}}$

then,

$$H_d(z) = \left(1 - 0.9 e^{-j0.6\pi} z^{-1}\right)\left(1 - 0.9 e^{-j0.6\pi} z^{-1}\right)(1.25)^2$$
$$\times \left(z^{-1} - 0.8 e^{j0.8\pi}\right) \cdot \frac{1 - 0.8 e^{-j0.8\pi} z^{-1}}{1 - 0.8 e^{-j0.8\pi} z^{-1}} \cdot \left(z^{-1} - 0.8 e^{-j0.8\pi}\right) \cdot \frac{1 - 0.8 e^{j0.8\pi} z^{-1}}{1 - 0.8 e^{j0.8\pi} z^{-1}}$$

$$\Rightarrow H_{d\,min}(z) = \left(1 - 0.9 e^{-j0.6\pi} z^{-1}\right)\left(1 - 0.9 e^{j0.6\pi} z^{-1}\right)(1.25)^2$$
$$\times \left(1 - 0.8 e^{-j0.8\pi} z^{-1}\right)\left(1 - 0.8 e^{+j0.8\pi} z^{-1}\right)$$

and the all-pass system that relates $H_{d\,min}$ and H_d is

$$H_{ap}(z) = \frac{\left(z^{-1} - 0.8 e^{-j0.8\pi}\right)}{\left(1 - 0.8 e^{j0.8\pi} z^{-1}\right)} \cdot \frac{\left(z^{-1} - 0.8 e^{j0.8\pi}\right)}{\left(1 - 0.8 e^{-j0.8\pi} z^{-1}\right)}$$

then, the compensating system $H_c(z)$ is

$$H_c(z) = \frac{1}{H_{d\,min}(z)}$$