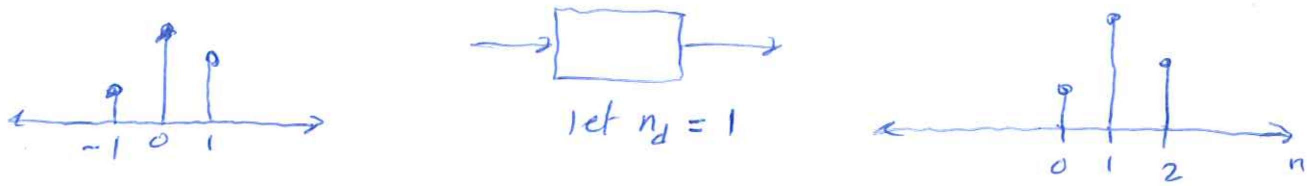
Discrete - Time Systems

* it is a transformation that maps an input sequence $x[n]$ into output sequence $y[n]$

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

$$y[n] = T\{x[n]\}$$

example : delay system $y[n] = x[n-n_d]$, $-\infty < n < \infty$



$y[n] = x[n-1]$

Properties :.

[1] Memoryless / Memory Systems :-

Memoryless : if the output $y[n]$ at every value of n depends only on input $x[n]$ at the same value of n.

Examples:- a) $y[n] = (x[n])^2$ memoryless

b) $y[n] = x[n-1]$ not memoryless

c) $y[n] = x[n+1]$ not memoryless.

d) $y[n] = x[n^2]$ " " "

$\boxed{2}$ Linear Systems:-

$$\begin{matrix} x_1[n] \\ x_2[n] \end{matrix} \longrightarrow \boxed{T\{\cdot\}} \longrightarrow \begin{matrix} y_1[n] \\ y_2[n] \end{matrix}$$

* The system is linear iff:-

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$
$$= y_1[n] + y_2[n]$$

and $T\{a x[n]\} = a T\{x[n]\} = a y[n]$, a: constant

* In general:

$$\left.\begin{aligned} \text{let } x_3[n] &= a x_1[n] + b x_2[n] \\ T\{x_3[n]\} &= T\{a x_1 + b x_2\} \\ &= a T\{x_1\} + b T\{x_2\} \end{aligned}\right\} \text{superposition principle}$$

Example:- Ideal delay system : $y[n] = x[n-n_d]$

let $x_3[n] = a x_1[n] + b x_2[n]$

$$y_3[n] = T\{x_3[n]\} = x_3[n-n_d]$$
$$= a x_1[n-n_d] + b x_2[n-n_d]$$
$$= a y_1[n] + b y_2[n]$$

$\Rightarrow$ Linear

Example: $y[n] = (x[n])^2$

$$y_3[n] = T\{x_3[n]\} = (x_3[n])^2 = (a x_1 + b x_2)^2$$
$$\neq a (x_1[n])^2 + b (x_2[n])^2$$
$$\neq a y_1[n] + b y_2[n]$$

It is not Linear

c) The accumulator system $y[n] = \sum_{k=-\infty}^{n} x[k]$

It is called so, since the output at time (n) is the accumulation (Sum) of the present and all past input samples

$$\text{let } x_3[n] = a x_1[n] + b x_2[n]$$

$$T\{x_3[n]\} = T\{a x_1[n] + b x_2[n]\}$$

$$= \sum_{k=-\infty}^{n} x_3[k]$$

$$= \sum_{k}^{n} a x_1[k] + b x_2[k]$$

$$= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k]$$

$$= a\, y_1[n] + b\, y_2[n]$$

The accumulator satisfies the superposition principle for all inputs $\Rightarrow$ it is Linear system.

d) $w[n] = \log_{10}(|x[n]|)$

This system is nonlinear, to prove that, one counter example is enough.

$$x_1 = 1 \longrightarrow w_1 = \log 1 = 0$$

$$x_2 = 10 \longrightarrow w_2 = \log 10 = 1$$

$$x_3 = x_1 + x_2 = 11 \longrightarrow w_3 = \log_{10} 11 \neq w_1 + w_2$$

[3] Time-Invariant (shift-Invariant) systems

It is a system for which a time shift (Delay) of the input sequence causes a corresponding (similar) shift in the output sequence.

if $\quad x[n] \longrightarrow \boxed{T\{.\}} \longrightarrow y[n]$

then $\quad x_1[n] = x[n-n_0] \longrightarrow \boxed{T\{.\}} \longrightarrow y_1[n] = y[n-n_0]$

for all n_0.

Test: we solve for both $y[n-n_0]$

$$y_1[n] = T\{x_1[n]\}$$

where $x_1[n] = x[n-n_0]$.

if $y[n-n_0] = y_1[n] \Longrightarrow$ The system is time-Invariant

Example: Accumulator $\quad y[n] = \sum_{k=-\infty}^{n} x[k]$

we Evaluate $\quad y[n-n_0] = \sum_{k=\infty}^{n-n_0} x[k]$

we define $\quad x_1[n] = x[n-n_0]$, then evaluate $y_1[n]$

$$\Rightarrow y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0]$$

change of variable $\quad m = k - n_0$

$$\Rightarrow y_1[n] = \sum_{m=-\infty}^{n-n_0} x[m]$$

(m) and (k) are dummy indicies

$\Rightarrow y_1[n] = y[n-n_0] \Rightarrow$ Accumulator is Time-Invariant

Examples:
1. $y[n] = x[n-n_0]$
2. $y[n] = (x[n])^2$
3. $y[n] = \log_{10}(|x[n]|)$

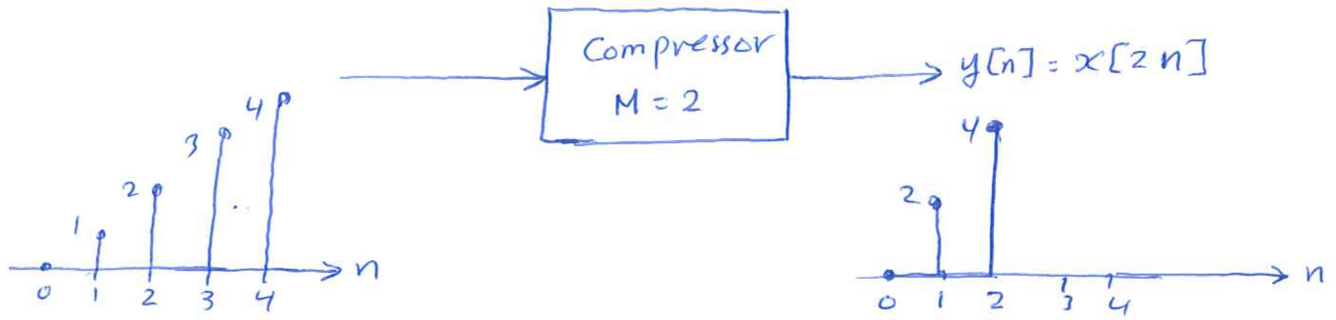
All are Time-Invariant

Example : Compressor System

$$y[n] = x[Mn] \qquad -\infty < n < \infty$$

M : positive integer

The compressor discards (M-1) samples out of M samples. It creates the output sequence by selecting every M^{th} samples.



Test: $x[2n] = y[n]$

$x_1[n] = x[n - n_0] \Rightarrow y_1[n] = x_1[2n] = x[2n - n_0]$

also $y[n - n_0] = x[2(n - n_0)] = x[2n - 2n_0] \neq y_1[n]$

$\Rightarrow y[n] = x[Mn]$ is not time-invariant

4 Causality

A system is causal if, for every choice of n_0, the output sequence value at $n = n_0$ depends only on the input sequence value for $n \leq n_0$.

Examples:- consider the following difference systems:-

① $y[n] = x[n+1] - x[n]$ (Forward difference system)

It is not causal, since the current value of the output depends on a future value of the input

② $y[n] = x[n] - x[n-1]$ (Backward difference System)

It is causal.

5 Stability

* A system is stable in the Bounded-Input, Bounded-Output (BIBO) sense if and only if every bounded input produces a bounded output.

* The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that $|x[n]| \leq B_x < \infty$ for all n.

* Stability requires that, for every bounded input, there exists a fixed +ve finite value B_y such that $|y[n]| \leq B_y < \infty$ for all n.

* If there is one bounded input for which the system property does not hold, then the system does not have the stability property.

Examples:: consider the following systems:

① $y[n] = \log_{10}(|x[n]|)$

It is unstable : let $x[n] = 0 < B_x < \infty$
then $y[n] = \infty$

② $y[n] = (x[n])^2$

It is stable : for every $|x[n]| \leq B_x$

$$|y[n]| = |x[n]|^2 \leq B_x^2$$

$$\Rightarrow B_y = B_x^2$$

③ Accumulator $y[n] = \sum_{-\infty}^{n} x[k]$

It is unstable ::

let $x[n] = u[n] \leq B_x$ where $B_x = 1$

then

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0 \end{cases}$$

there is no finite choice for B_y such that

$(n+1) \leq B_y < \infty$ for all n.

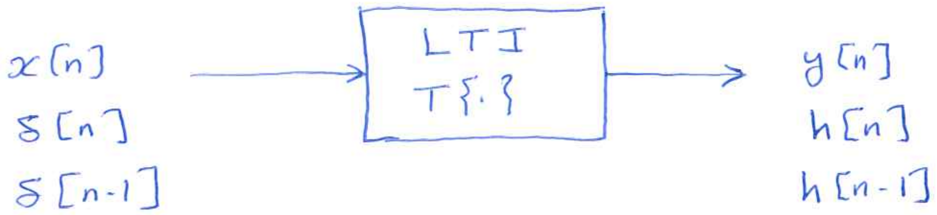
2.3 LTI Systems

* Linear - Time - Invariant (LTI) systems have significant Signal Processing applications

* Remember, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

* Consider the system $y[n] = T\{x[n]\}$



: impulse response

This due to Time-Invariant property

* In general, $y[n]$ for any sequence $x[n]$:

$$y[n] = T\{x[n]\} = T\Big\{\sum_{k} x[k]\,\delta[n-k]\Big\}$$

due to linearity:

$$\Rightarrow y[n] = \sum_{k} x[k]\,T\{\delta[n-k]\}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]} \quad \text{for all } n$$

$$\boxed{y[n] = x[n] * h[n]} \quad \text{Convolution Sum}$$

* Conclusion: LTI system is completely characterized by its impulse response $h[n]$ in the sense that, given $x[n]$ and $h[n]$ for all n, it is possible to compute each sample of the output sequence $y[n]$.

* properties:

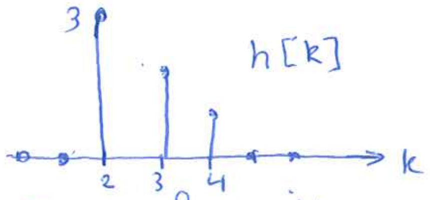
$$h[n-n_0] * x[n] = y[n-n_0]$$

$$h[n-n_1] * x[n-n_2] = y[n-n_1-n_2]$$

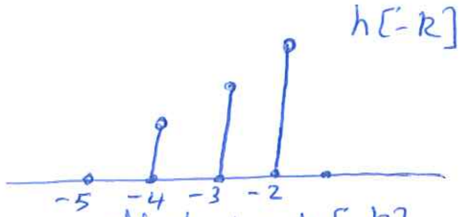
Example: Evaluate $y[n]$ for $h[n] = \{ \underset{\substack{\uparrow \\ n=2}}{3}, 2, 1 \}$

$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 1, 3 \}$

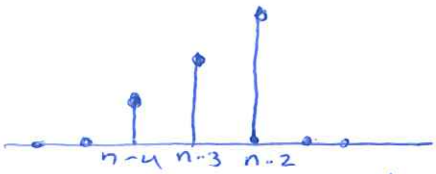
Solution: $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$



change of variable $n \to k$



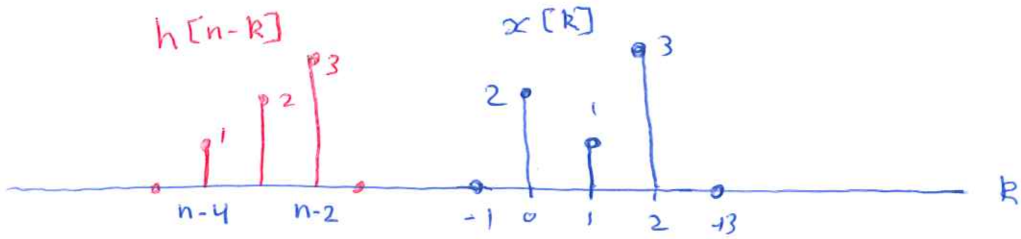
reflection: $h[-k]$



shift by n : $h[n-k]$

Then,

Sweep $\to$ Multiply $\to$ Sum



For $n-2 < 0 \Rightarrow n < +2$, $y[n] = 0$

For $n-2 = 0 \Rightarrow n = 2$, $y[n] = 3 * 2 = 6$

For $n-2 = 1 \Rightarrow n = 3$, $y[n] = 3*1 + 2*2 = 7$

$n-2 = 2 \Rightarrow n = 4$, $y[n] = 2*1 + 2*1 + 3*3 = 13$

$n-2 = 3 \Rightarrow n = 5$, $y[n] = 1*1 + 2*3 = 7$

$n-2 = 4 \Rightarrow n = 6$, $y[n] = 1*3 = 3$

$n-2 > 4 \Rightarrow n > 6$, $y[n] = 0$

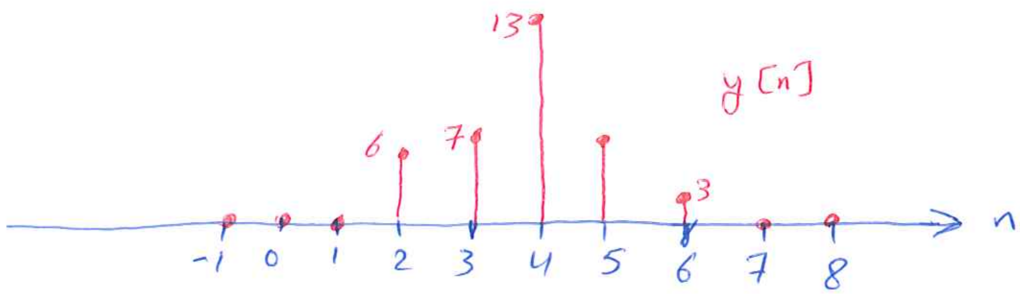


Table Method ①:-

$x[n]$ \ $h[n]$	3	2	1		
	$\nearrow 6$	$\nearrow 7$	$\nearrow 13$	$\nearrow 7$	$\nearrow 3$
2	$2*3=6$	$2*2=4$	$2*1=2$		
1	$1*3=3$	$1*2=2$	$1*1=1$		
3	$3*3=9$	$3*2=6$	$3*1=3$		

$$\Rightarrow y[n] = \{\underset{\uparrow\, n=2}{6}, 7, 13, 7, 3\}$$

Note that
$$N_h : [2, 4]$$
$$N_x : [0, 2]$$
$$N_y : [2+0, 4+2] = [2, 6]$$

Table Method ②:

n	2	3	4	5	6
$x[0] * h[n]$	$\cancel{3}\ 6$	$\cancel{2}\ 4$	$\cancel{1}\ 2$		
$x[1] * h[n-1]$		$\cancel{3}\ 3$	$\cancel{2}\ 2$	$\cancel{1}\ 1$	
$x[2] * h[n-2]$			$\cancel{3}\ 9$	$\cancel{2}\ 6$	$\cancel{1}\ 3$
$y[n] = \Sigma$	6	7	13	7	3

<u>OR</u>

n	2	3	4	5	6
$*3$	$\cancel{2}\ 6$	$\cancel{1}\ 3$	$\cancel{3}\ 9$		
$*2$		$\cancel{2}\ 4$	$\cancel{1}\ 2$	$\cancel{3}\ 6$	
$*1$			$\cancel{2}\ 2$	$\cancel{1}\ 1$	$\cancel{3}\ 3$
$y[n] = \Sigma$	6	7	13	7	3

Example let $h[n] = \{1, 2, 1, -1\}$, Determine $y[n]$
for $x[n] = \{1, 2, 3, 1\}$

$$N_y = [-1, 2] + [0, 3] = [-1, 5]$$

n	-1	0	1	2	3	4	5
$x[n-0]$	1	2	3	1			
$x[n-1]$		1	2	3	1		
$x[n-2]$			1	2	3	1	
$x[n-3]$				1	2	3	1
$h(0)\,x[n-0]$	1	2	3	1			
$h(1)\,x[n-1]$		2	4	6	2		
$h(2)\,x[n-2]$			1	2	3	1	
$h[3]\,x[n-3]$				-1	-2	-3	-1
$y[n] = \sum$	1	4	8	8	3	-2	-1

$$y[n] = \sum h[k]\,x[n-k]$$

$$y[-1] = \sum h[k]\,x[-1-k] = \cdots h[-1]\overset{0}{x}[0] + \overset{1}{\cancel{h[0]}}\,\overset{1}{\cancel{x[-1]}} + \overset{2}{\cancel{h[1]}}\,\overset{0}{\cancel{x[-2]}} +$$

$$= 1$$

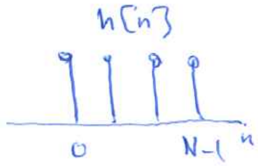
$\vdots$

$$\Rightarrow y[n] = \{1, 4, 8, 8, 3, -2, -1\}$$

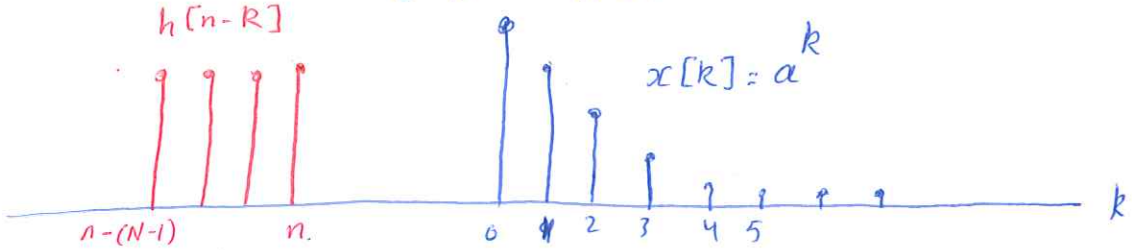
$n=0$

Example 2.11 Analytical Evaluation of the convolution sum.

$$h[n] = u[n] - u[n-N] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



$$x[n] = a^n u[n] = \begin{cases} a^n & n \ge 0 \\ 0 & n < 0 \end{cases}$$



we can find formulas for $y[n]$ for different sets of values of n.

$\boxed{\text{Case I}}$

for $n < 0$ no overlap $\Rightarrow y[n] = \sum_k x[k]\, h[n-k] = 0$

$\boxed{\text{Case II}}$

for $0 \le n$ and $n - N + 1 \le 0 \Rightarrow 0 \le n \le N-1$

$x[k]\, h[n-k] = a^k, \; 0 \le k \le n$, the mult. is non-zero

$$\Rightarrow y[n] = \sum_{k=0}^{n} a^k$$

using finite geometric series $\sum_{k=N_1}^{N_2} \alpha^k = \dfrac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$, $N_2 \ge N_1$

$$\Rightarrow y[n] = \frac{a^0 - a^{n+1}}{1-a} = \frac{1 - a^{n+1}}{1-a}$$

$\boxed{\text{Case III}}$ for $(n - N + 1) > 0 \Rightarrow n > N-1$

$x[k]\, h[n-k] = a^k$, $n - N + 1 \le k \le n$, the mult. is non zero.

$$\Rightarrow y[n] = \sum_{n-N+1}^{n} a^k = \frac{a^{n-N+1} - a^{n+1}}{1-a} = a^{n-N+1}\left(\frac{1-a^N}{1-a}\right)$$

$$\Rightarrow \quad y[n] = \begin{cases} 0 & n<0 \\ \dfrac{1-a^{n+1}}{1-a} & 0 \leq n \leq N-1 \\ a^{n-N+1}\left(\dfrac{1-a^{N}}{1-a}\right) & n > N-1 \end{cases}$$

note :-

since $h[n]$ or $x[n]$ is infinitè sequence, the analytical evaluation is the appropriate method to evaluate the convolution sum.

if Both $h[n]$ and $x[n]$ are finitè sequences, then the tabular method is easier.

Example : Find $y[n]$ for the LTI system

$$h[n] = a^n u[n] \quad , \quad |a| < 1$$

when $x[n] = u[n]$

Solution:

$$y[n] = \sum_{-\infty}^{\infty} x[n-k]\, h[k]$$

$$= \sum_{k=-\infty}^{\infty} u[n-k] \cdot a^k u[k]$$

$$u[k] = 0 \quad \text{for} \quad k < 0$$

$$u[n-k] = 0 \quad \text{for} \quad n-k < 0 \Rightarrow n < k$$

$$\Rightarrow y[n] = \sum_{k=0}^{n} a^k$$

$$= \begin{cases} \dfrac{1 - a^{n+1}}{1 - a} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\Rightarrow y[n] = \frac{1 - a^{n+1}}{1 - a} u[n]$$

note : $\lim_{n \to \infty} y[n] = \dfrac{1}{1-a}$

OR The step response can be obtained directly from $h[n]$

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{-\infty}^{n} h[k]$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{n} a^k u[k] = \sum_{k=0}^{n} a^k$$

2.4 Properties of LTI Systems

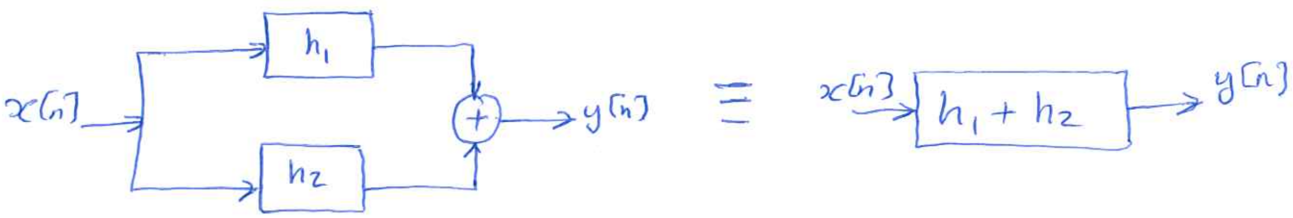
[1] Convolution operation is commutative

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$y[n] = \sum_{-\infty}^{\infty} x[n-m]\, h[m] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

[2] Convolution is Distributives over addition

$$y[n] = x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])$$



[3] Convolution satisfies the Associative Property:

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_2[n] * h_1[n])$$

[4] LTI system is stable if and only if the impulse response ($h[n]$) is absolutely summable

$$B_h = \sum_{-\infty}^{\infty} |h[k]| < \infty$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right| \leq \sum_{k} |h[k]|\,|x[n-k]|$$

if $x[k]$ is bounded, so that $|x[n]| \leq B_x$

$\Rightarrow |y[n]| \leq B_x B_h = B_y$ iff $h[n]$ is absolutely summable

$\boxed{5}$ LTI system is causal if $h[n] = 0$ for $n < 0$

It is sometimes convenient to refer to a sequence that is zero for $n < 0$ as a causal sequence.

i.e., $f[n]$ is a causal sequence if $f[n] = 0$ for $n < 0$.

proof. $y[n] = \sum_{-\infty}^{\infty} x[k]\, h[n-k]$

if $h[n-k]$ is causal then $h[n-k] = 0$ for $n-k < 0$
$n < k$

$$\Rightarrow y[n] = \sum_{-\infty}^{n} x[k]\, h[n-k]$$

LTI system	Impulse Response	Stability $\lvert \sum h[n] \rvert < B_h$
① $y[n] = x[n-n_d]$	$h[n] = \delta[n-n_d]$	Yes
② $y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$	$h[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} \delta[n-k] = \begin{cases} \frac{1}{M_1+M_2+1}, & -M_1 \leq n \leq M_2 \\ 0 & \text{else} \end{cases}$	Yes
③ $y[n] = x[n+1] - x[n]$	$h[n] = \delta[n+1] - \delta[n]$ $h[n] = \{1, \underset{\uparrow n=0}{-1}\}$	Yes
④ $y[n] = x[n] - x[n-1]$	$h[n] = \delta[n] - \delta[n-1]$ $h[n] = \{\underset{\uparrow n=0}{1}, -1\}$	Yes
⑤ $y[n] = \sum_{k=-\infty}^{n} x[k]$	$h[n] = \sum_{-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} = u[n]$	No, since $\sum_{k=0}^{\infty} u[k] = \infty$

$$\sum_{n=-\infty}^{n} |u[n]| = n+1 \qquad \text{as } n \to \infty \quad \Sigma \to \infty$$

Conclusions :-

* In general, a system with a finite-duration impulse Response (henceforth referred to as an FIR system) will always be stable, as long as the impulse response values is finite in magnitude. (see examples ① - ④)

* The impulse response of the accumulator (example ⑤) has infinite duration. This is an example of the class of systems referred to as infinite-duration impulse response (IIR) systems.

* An example of an IIR system that is stable is

$$h[n] = a^n u[n] \qquad \text{with} \quad |a| < 1$$

$$\Rightarrow B_h = \sum_{n=0}^{\infty} |a|^n \quad \text{(infinite geometric series)}$$

$$= \frac{1}{1-|a|} < \infty \qquad \text{for} \quad |a| < 1$$

$\Rightarrow$ it is stable when $|a| < 1$

however, if $|a| \geq 1$, then $B_h = \infty$

$\Rightarrow$ the system is unstable.

Impulse Response

Causal if $h[n]=0, n<0$

① $h[n] = \delta[n-n_d]$

Causal if $n_d \geq 0$
Non causal if $n_d < 0$

② $h[n] = \dfrac{1}{M_1+M_2+1} \displaystyle\sum_{k=-M_1}^{M_2} \delta[n-k]$

Causal if $M_1 \leq 0$
and $M_2 \geq 0$

③ $h[n] = \displaystyle\sum_{k=-\infty}^{n} \delta[k] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases}$

Causal

④ $h[n] = \delta[n+1] - \delta[n]$

since $h[-1] = \delta[0] - \delta[-1]$
$= 1 \neq 0$
$\Rightarrow$ Non causal.

⑤ $h[n] = \delta[n] - \delta[n-1]$

causal.

Note: The convolution of a shifted impulse sequence $\delta[n-n_d]$ with any sequence $x[n]$ is simply evaluated by shifting $x[n]$ by the displacement of the impulse

$$x[n] * \delta[n-n_d] = x[n-n_d].$$

Example:

$$x[n] \longrightarrow \underset{}{\overset{h_1}{\boxed{\text{Forward difference}}}} \longrightarrow \overset{h_2}{\boxed{\text{one-sample delay}}} \longrightarrow y[n]$$

$$h_1[n] = \delta[n+1] - \delta[n] \qquad\qquad h_2[n] = \delta[n-1]$$

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] \\ &= \left(\delta[n+1] - \delta[n]\right) * \left(\delta[n-1]\right) \\ &= \delta[n+1-1] - \delta[n-1] \\ &= \delta[n] - \delta[n-1] \end{aligned}$$

Backward difference system.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Note that the noncausal forward difference system is converted into causal system by cascading them with a delay.

$\Rightarrow$ In general, any noncausal FIR system can be made causal by cascading it with a sufficiently long delay.

$\boxed{6}$ Inverse System:

In general, if an LTI system has impulse response $h[n]$, then its inverse system, if it exists, has impulse response $h_i[n]$ defined by the relation

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]$$

In ch. 3, Z-transform provides a stright forward method of finding the inverse of an LTI system.

example:- Show that the accumulator is the inverse system of the backward difference system

$$\longrightarrow \boxed{\text{Accumulator}} \longrightarrow \boxed{\text{Backward}} \longrightarrow$$

$$\begin{aligned} h[n] &= u[n] * (\delta[n] - \delta[n-1]) \\ &= u[n] - u[n-1] \\ &= \delta[n] \end{aligned}$$

2.5 Linear Constant-Coefficient Difference equations

An important class of LTI systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an N^{th}-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Example 2.12 Difference equation representation of the Accumulator.

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + \sum_{-\infty}^{n-1} x[k]$$

$$= x[n] + y[n-1]$$

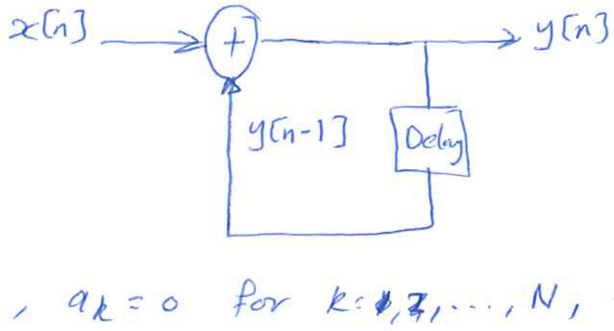
or $\quad y[n-0] - y[n-1] = x[n]$

$$a_0 y[n] + a_1 y[n-1] = b_0 x[n]$$

$\Rightarrow a_0 = 1, \quad a_1 = -1, \quad N = 1 \Rightarrow 1^{st}$ order Difference equation

$b_0 = 1, \quad M = 0$

This suggests a simple block diagram representation (Recursive representation)



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* In general, if we assume $a_0 = 1$, $a_k = 0$ for $k = 1, 2, \ldots, N$, then $y[n] = \sum_{m=0}^{M} b_m x[n-m]$
* the impulse response, $h[n] = \sum_{m=0}^{M} b_m \delta[n-m] = \begin{cases} b_n & n = 0, \ldots, M \\ 0 & \text{else} \end{cases}$
* it is causal

Example:- Determine the impulse response of the first order system given by :-

$$y[n] - a\,y[n-1] = x[n] \quad , \text{assume } y[o] = 0,\ n<0$$

(causal system)

Solution :-

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

$$y[n] = x[n] + a\,y[n-1]$$

$$y[0] = x[0] + a\,y[-1] = 1 = h[0]$$

$$y[1] = x[1] + a\,y[0] = 0 + a = h[1]$$

$$y[2] = x[2] + a\,y[1] = 0 + a.a = a^2 = h[2]$$

$$y[3] = x[3] + a\,y[2] = a^3 = h[3]$$

$\vdots$

$\vdots$

$$h[n] = a^n\,u[n] = \begin{cases} a^n & n \geq 0 \\ 0 & \text{else} \end{cases}$$

Let us assume that $y[n] = 0$ for $n > 0$ (noncausal system)

then $$y[n-1] = \frac{y[n] - x[n]}{a}$$

$$= a^{-1}\left(y[n] - x[n]\right)$$

$$x[n] = \delta[n]$$

$n=2$ $\quad y[1] = a^{-1}(y[2] - x[2]) = 0$

$n=1$ $\quad y[0] = a^{-1}(y[1] - x[1]) = 0$

$n=0$ $\quad y[-1] = a^{-1}(y[0] - x[0]) = a^{-1}(0-1) = -a^{-1}$

$n=-1$ $\quad y[-2] = a^{-1}(y[-1] - x[-1]) = a^{-1}.-a^{-1} = -a^{-2}$

$\vdots$

$$\Rightarrow h[n] = \begin{cases} -a^{-1} & n=0 \\ -a^{-2} & n=-1 \\ -a^{-3} & n=-2 \end{cases} = -a^{n-1}\,u[-n] = -a^{n}\,u[-n-1]$$