CHAPTER 4 Sampling of Continuous-Time Signals

* Discrete-time signals can arise in many ways, but they occur most commonly as representations of sampled continuous-time signals

* A sequence of samples, $x[n]$, is obtained from a continuous-time signal $x_c(t)$ according to the relation

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

$$\begin{tikzcd} x_c(t) \arrow[r] & \boxed{C/D} \arrow[r] & x[n] = x_c(nT) \\ & T : \text{sampling period} \arrow[u] & \end{tikzcd}$$

C/D : ideal continuous-to-Discrete converter.
T : sampling period (or T_s)
f_s : Sampling frequency (samples/second)
Ω_s : Sampling frequency ( ), measured in rad/second.

$$f_s = \frac{1}{T_s}, \quad \Omega_s = 2\pi f_s = \frac{2\pi}{T_s}$$

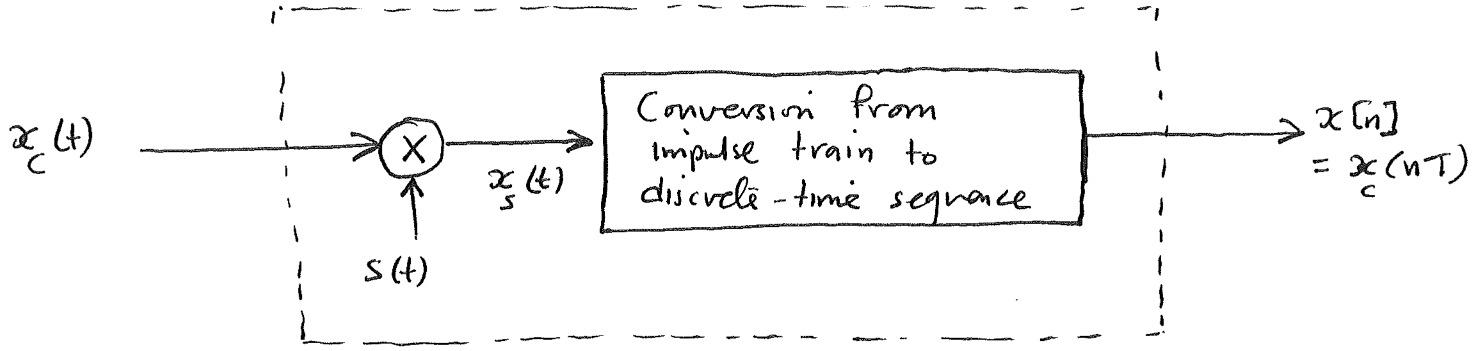
* In practical setting, the operation of sampling is implemented by analog-to-digital (A/D) converter.

* In addition to the sampling rate, important consideration in the implementation or choice of A/D converter include quantization of the output samples.

* The sampling process, mathematically, can be represented in two stages :-

$\boxed{1}$ Multiply by an impulse train (impulse train modulator)

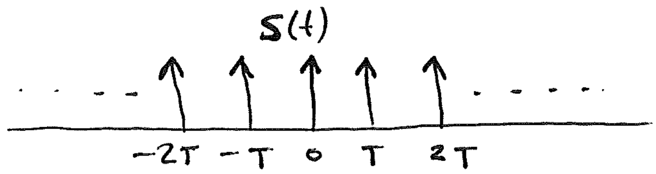
$\boxed{2}$ Conversion of the impulse train to a sequence.



C/D Converter

* The periodic impulse train : $s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$\delta(t)$ is the unit impulse function (Dirac Delta)



* The modulator output (product) is

$$x_s(t) = x_c(t) \cdot s(t) = \sum_{-\infty}^{\infty} x_c(t)\, \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t-nT)$$

i.e., the size (area) of the impulse at sample time (nT) is equal to the value of $x_c(t)$ at $t = nT$.

Remember: (shifting property of $\delta(t)$) :-

$x(t) \cdot \delta(t) = x(0)\, \delta(t)$

$x(t) \cdot \delta(t-t_0) = x(t_0)\, \delta(t-t_0)$

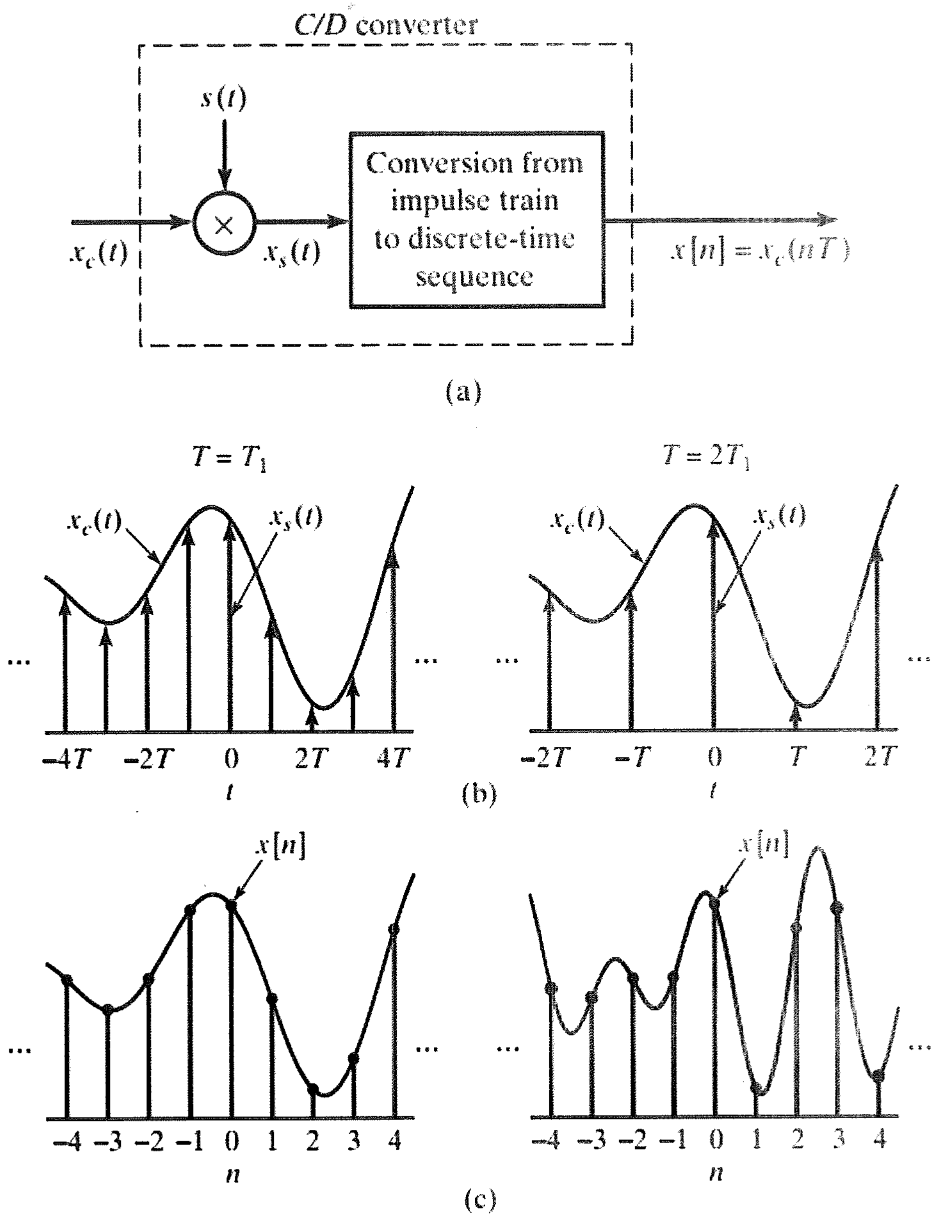


Figure Sampling with a periodic impulse train, followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.

- **The results of impulse train sampling for two different sampling rates. (Doubling the sapling time results into compression)**
- **The essential difference between $x_s(t)$ and x[n]**

  $x_s(t)$ is a continuous-time signal x[n] is a sequence that is indexed on the integer variable n.

  x[n] is obtained by normalizing the time index in $x_s[t]$ over T.

  Hence, the sequence x[n] contains no explicit information about the sampling period T.

4.2 Frequency - Domain Representation of Sampling

* In time - domain : (Multiplication)

$$x_s(t) = x_c(t) \cdot s(t)$$

$$s(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

* In frequency - domain : (Convolution)

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

where :

$$X_c(j\Omega) = \mathcal{F}\{x_c(t)\}$$

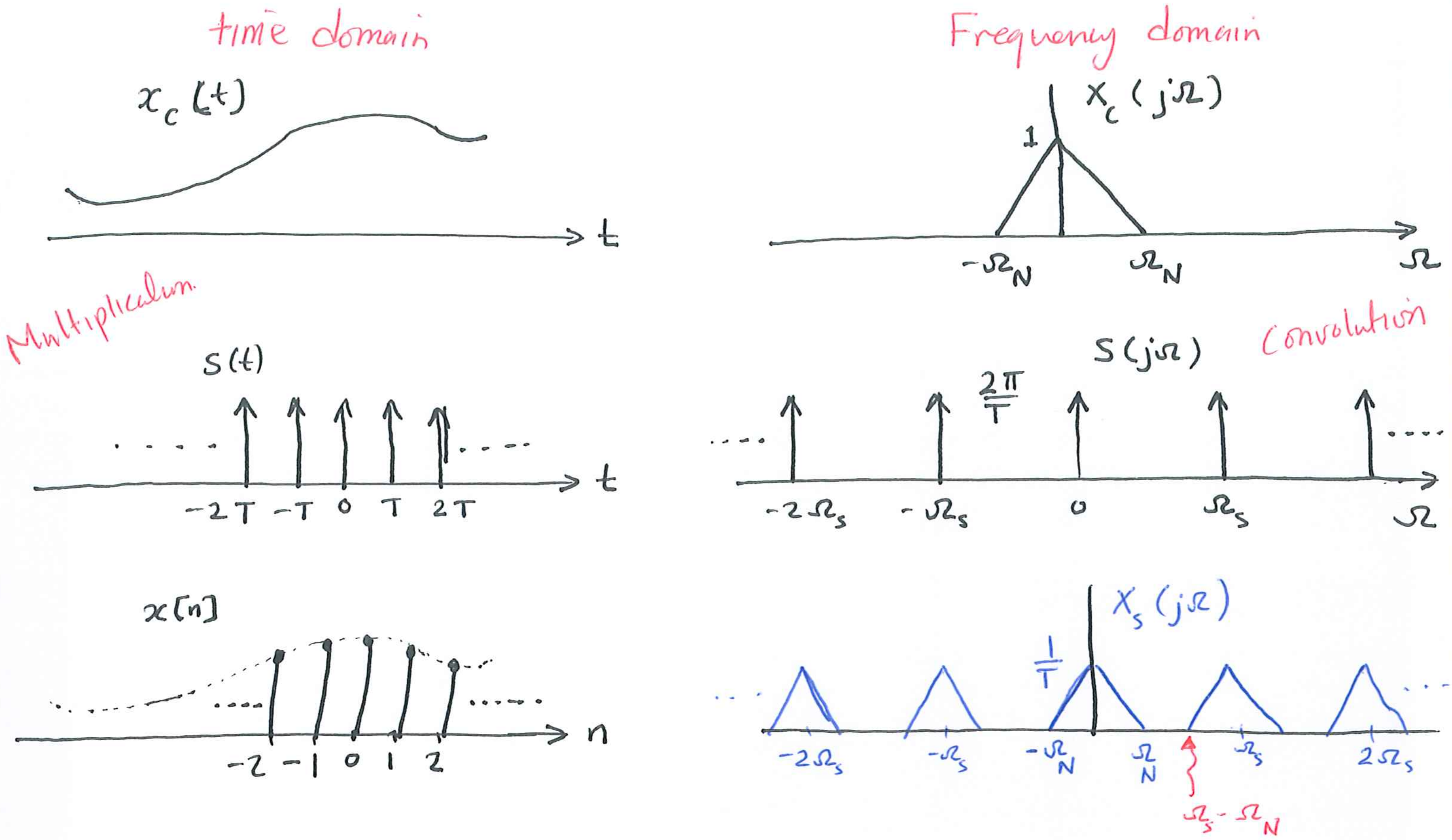
$$S(j\Omega) = \mathcal{F}\{s(t)\} = \underbrace{\frac{2\pi}{T}}_{=\Omega_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$\Omega_s = \frac{2\pi}{T}$: sampling frequency in rad/sec.

$$\Rightarrow X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$\boxed{X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - k\Omega_s)\right)}$$

* It states that the Fourier transform (FT) of $x_s(t)$ consists of periodically represented copies of $X_c(j\Omega)$.
* These copies are shifted by integer multiplies of the sampling frequency.

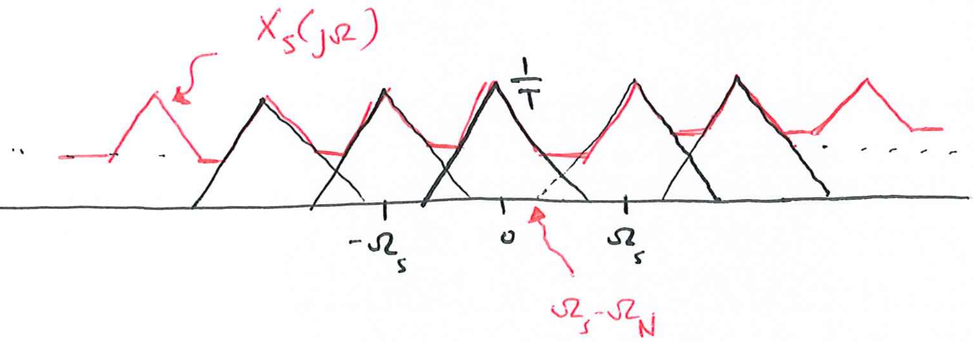


when $\Omega_s - \Omega_N \geq \Omega_N$ or $\Omega_s \geq 2\Omega_N$

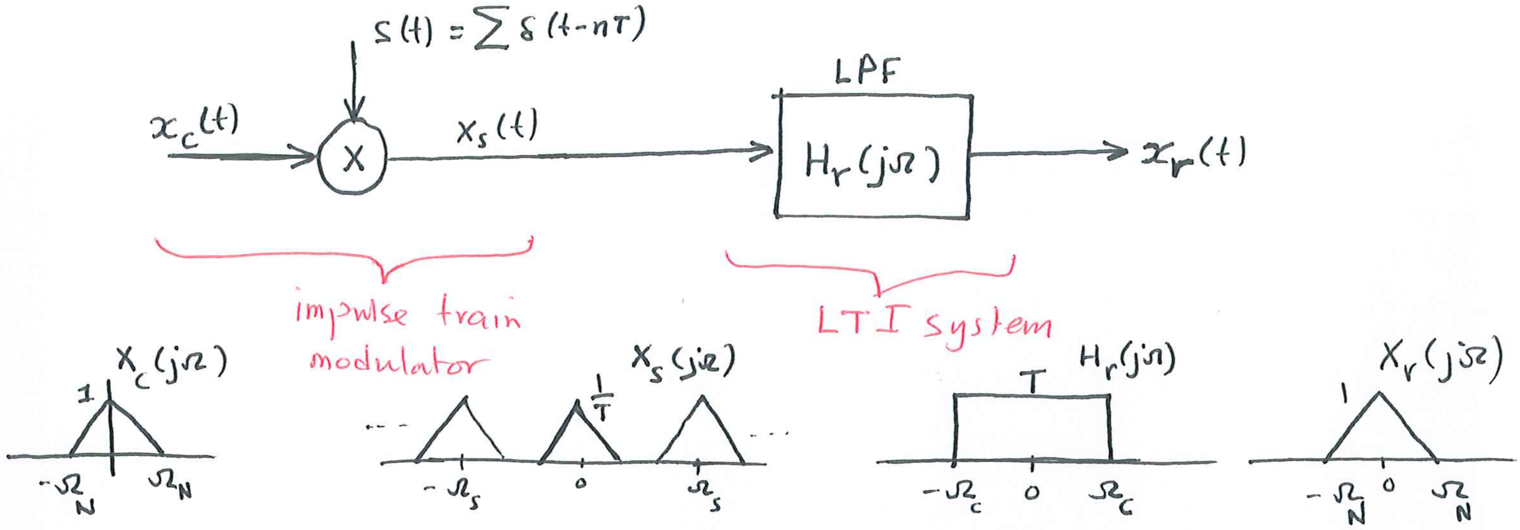
the replicas of $X_c(j\Omega)$ do not overlap,

$\Rightarrow$ consequently, $x_c(t)$ can be recovered from $x_s(t)$ with an ideal lowpass filter.

if $\Omega_s < 2\Omega_N \Rightarrow$ overlapping !!



Exact Recovery of a continuous-time from its samples using an ideal lowpass filter

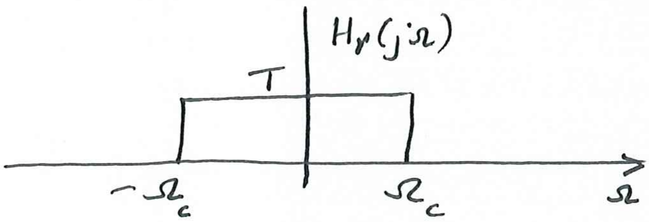


$x_r(t)$ is the recovered signal from its samples using an ideal LPF.

Assuming $\Omega_s > 2\Omega_N$, the output of the LTI system with frequency response $H_r(j\Omega)$ is

$$X_r(j\Omega) = H_r(j\Omega)\, X_s(j\Omega)$$

where $H_r(j\Omega) = \begin{cases} T & |\Omega| < \Omega_c \\ 0 & \text{else} \end{cases}$, Ω_c : cutoff freq.



$$\boxed{\Omega_N \leq \Omega_c \leq \Omega_s - \Omega_N}$$

then $X_r(j\Omega) = X_c(j\Omega) \Rightarrow x_r(t) = x_c(t)$

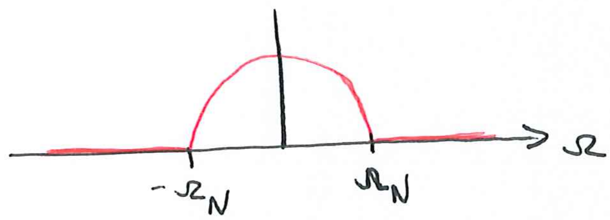
* If the inequality $\Omega_s \geq 2\Omega_N$ doesn't hold, (i.e, $\Omega_s < 2\Omega_N$), then the copies of $X_c(j\Omega)$ overlap, so that when they are added together, $X_c(j\Omega)$ is no longer recoverable using LPF.

* In this case, the reconstructed output $x_r(t)$ is related to the original continuous-time input through a distortion referred to as "aliasing distortion".

Nyquist - Shannon Sampling Theorem

Let $x_c(t)$ be a band limited signal with

$$X_c(j\Omega) = 0 \quad \text{for} \quad |\Omega| \geq \Omega_N$$



then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\Omega_s = \frac{2\pi}{T_s} = 2\pi f_s \geq 2\Omega_N$$

$2\Omega_N$ is called Nyquist rate

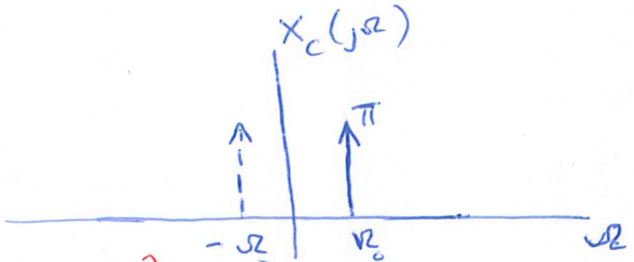
Ω_N is called Nyquist frequency (signal B.W).

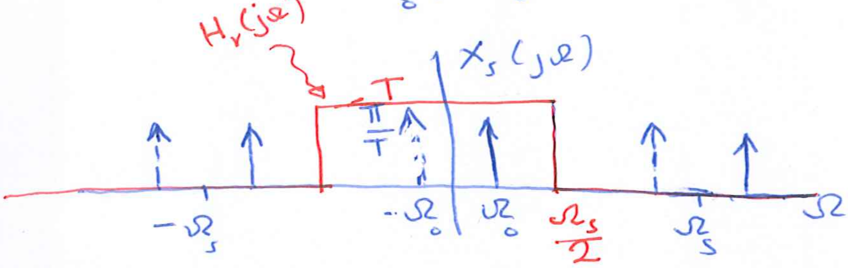
Example consider a simple case of a cosine signal

$$x_c(t) = \cos \Omega_0 t$$

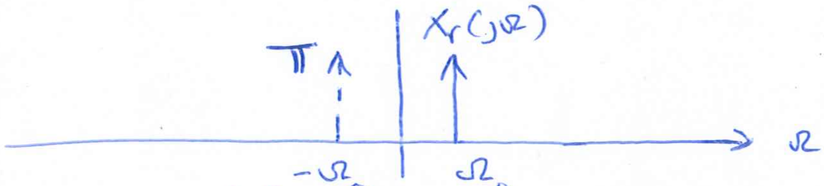
* the Fourier transform of $x_c(t)$ is

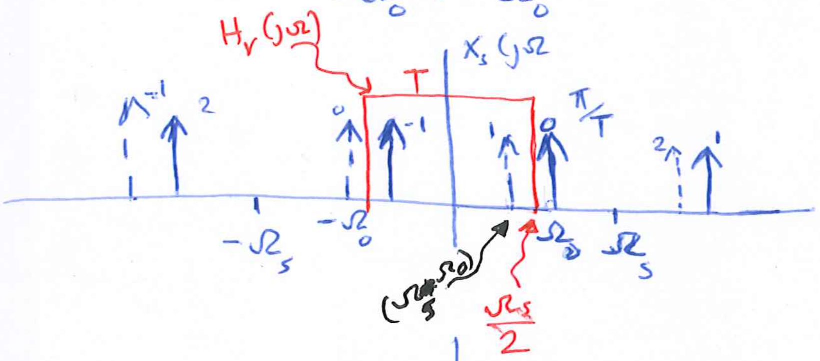
$$X_c(j\Omega) = \pi\,\delta(\Omega - \Omega_0) + \pi\,\delta(\Omega + \Omega_0)$$



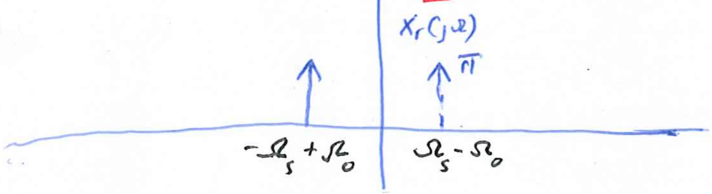


$\Omega_s > 2\Omega_0$ No aliasing





$\Omega_s < 2\Omega_0$



aliasing

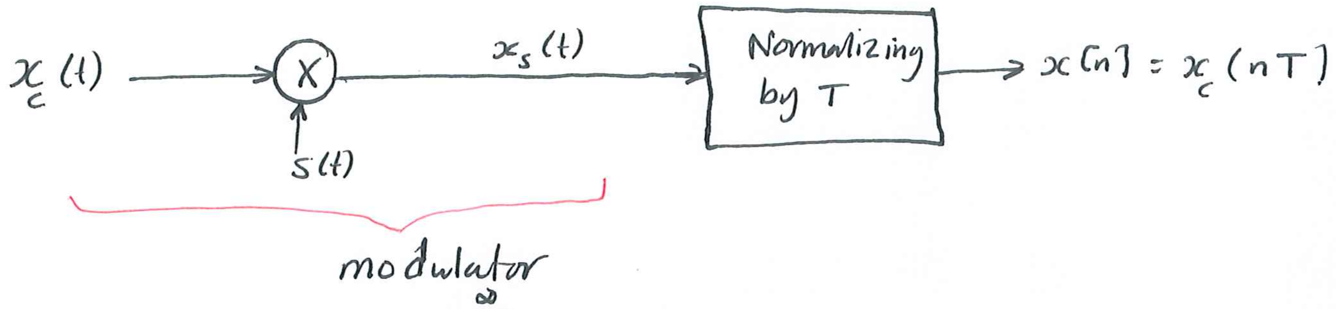
$x_r(t) = \cos(\Omega_0 t)$ reconstructed output (with no aliasing)

$x_r(t) = \cos((\Omega_s - \Omega_0)t)$ " " (with aliasing)

$x_r(t)$ with aliasing
lower freq. signal
(no. of samples is not enough!)

$X(e^{j\omega})$ in terms of $X_s(j\Omega)$ and $X_c(j\Omega)$

So far we considered only the impulse train modulator



$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j[\Omega - k\Omega_s])$$

our eventual objective is to express $X(e^{j\omega})$, DTFT, of the sequence $x[n]$, in terms of $X_s(j\Omega)$ and $X_c(j\Omega)$.

Start from $x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t - nT)$

apply the continuous-time Fourier Transform (CTFT),

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\Omega nT} \quad \text{———} \quad (*)$$

The DTFT of $x[n] = x_c(nT)$ is

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \text{———} \quad (**)$$

it follows that

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega = \Omega T} = X(e^{j\Omega T})$$

We have shown earlier that

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j[\Omega - k\Omega_s])$$

and since $X_s(j\Omega) = X(e^{j\Omega T})$

then,

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j[\Omega - k\Omega_s])$$

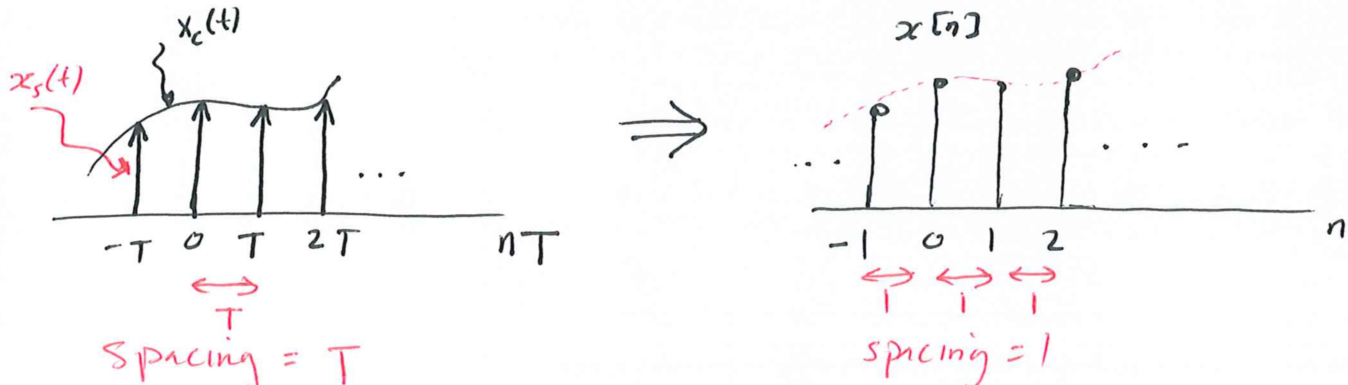
or equivalently $(\omega = \Omega T)$:-

$$\boxed{X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left[\frac{\omega}{T} - \frac{2\pi}{T}k\right]\right)}$$

* It is clear that $X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$ with a frequency scaling specified by $\omega = \Omega T = \frac{\Omega}{f_s}$

This frequency scaling (or normalization) is directly a result of the time normalization in the transformation from $x_s(t)$ to $x[n]$.

[* the t-axis is normalized by T $(\div T)$ divide
* the f-axis is " by $f_s = \frac{1}{T}$ $(* T)$ Mult.



Example 4.1 let $x_c(t) = \cos(4000\pi t)$

$T = \frac{1}{6000}$ sec.

(a) Find $x[n]$.

(b) Find and plot $X_c(j\Omega)$

(c) Find and plot $X_s(j\Omega)$

(d) Find and plot $X(e^{j\omega})$

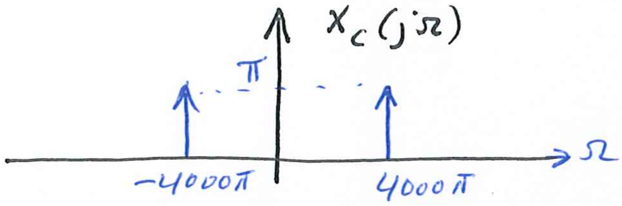
(e) Find $H_r(j\Omega)$ and $X_r(j\Omega)$

Solution:-

(a) $x[n] = x_c(nT) = \cos(4000\pi . nT) = \cos(\omega_0 n)$

$$\omega_0 = 4000\pi * \frac{1}{6000} = \frac{2}{3}\pi$$

(b) $X_c(j\Omega) = \pi\,\delta(\Omega - 4000\pi) + \pi\,\delta(\Omega + 4000\pi)$

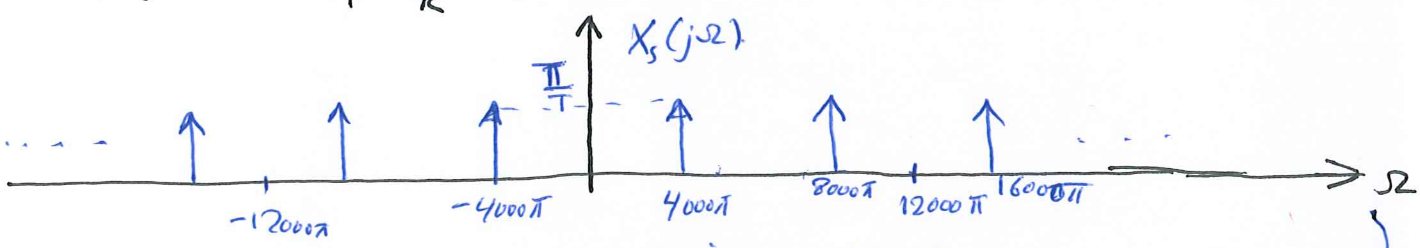


(c) the highest freq. in the signal is $\Omega_0 = \Omega_N = 4000\pi$ ($f = 2000\,HZ$)

the sampling freq. is $\Omega_s = \frac{2\pi}{T} = 12000\pi$ ($f_s = 6000\,HZ$)

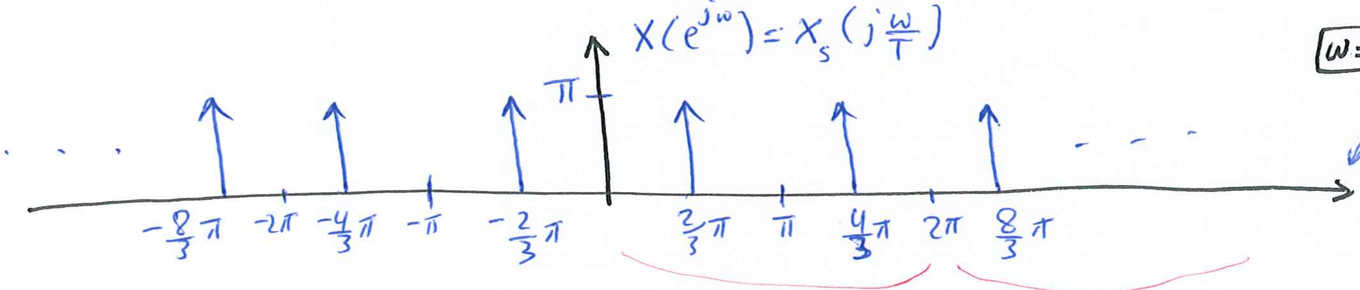
since $\Omega_s > 2\Omega_0 \Rightarrow$ No aliasing

$$X_s(j\Omega) = \frac{1}{T}\sum_k X_c(j[\Omega - k\Omega_s])$$



$\boxed{\omega = \Omega T}$

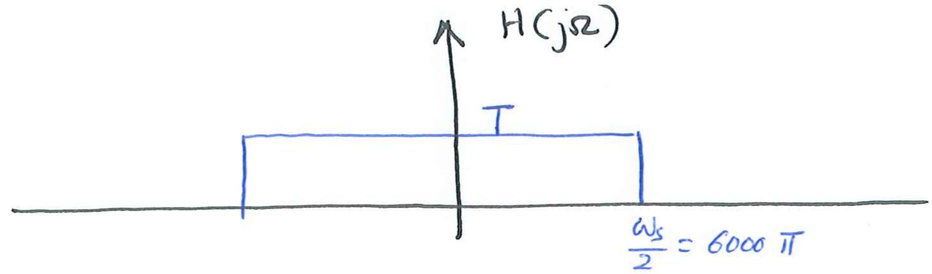
(d)

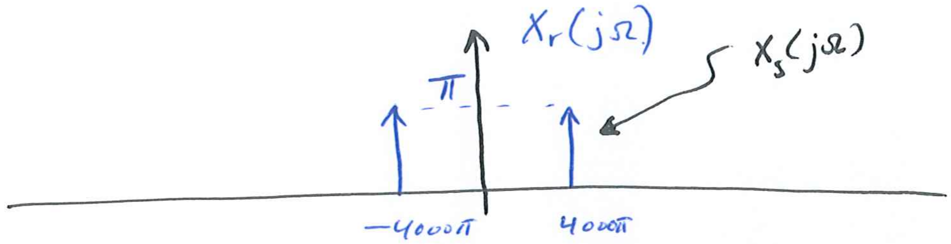


note that, we multiplied Ω by $T \Rightarrow \omega = \Omega T$

also, we have used the fact that scaling the independent variable of an impulse also scales it area, i.e., $\delta(\frac{\omega}{T}) = T\,\delta(\omega)$

(e)





$$\Rightarrow X_r(j\Omega) = \cos(4000\pi t)$$

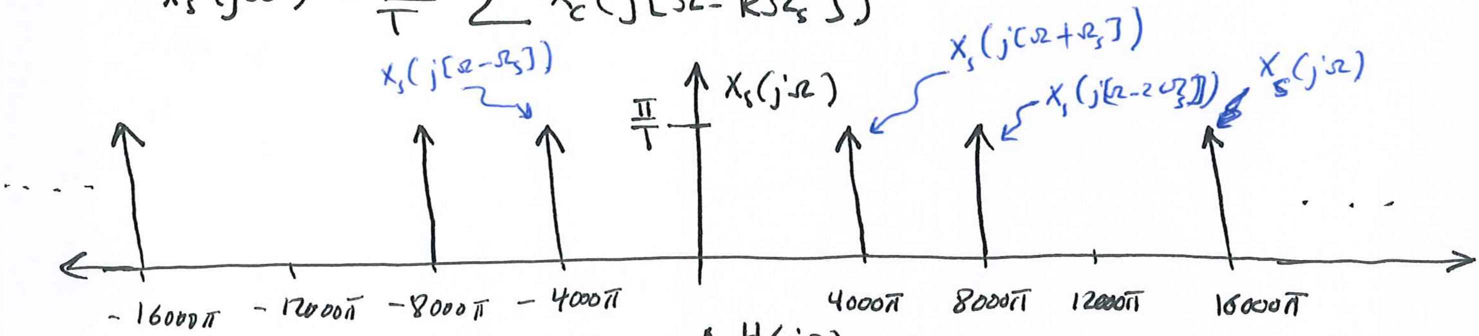
<u>Example 4.2</u> repeat example 4.2 but with $x_c(t) = \cos(16000\pi t)$ and use the same sampling period $T = \frac{1}{6000}$.

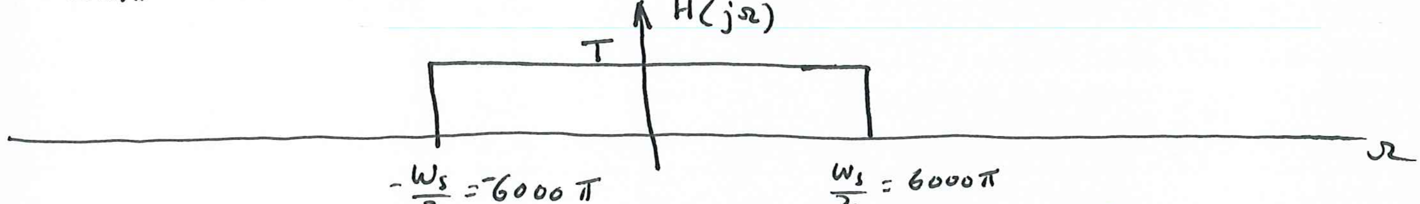
* This sampling period fails to satisfy the Nyquist criterion.

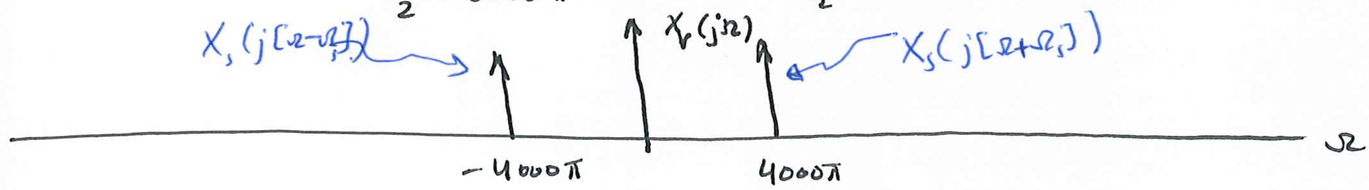
$$\Omega_s = \frac{2\pi}{T} = 12000\pi < 2\Omega_0 = 32000\pi.$$

$\Rightarrow$ We expect to see aliasing.

$$X_s(j\Omega) = \frac{1}{T}\sum X_c(j[\Omega - k\Omega_s])$$







Note that the FT $X_s(j\Omega)$ for both examples is identical. However, the impulse located at -4000 is from $X_s(j[\Omega-\Omega_s])$ and the impulse at $+4000$ is from $X_s(j[\Omega+\Omega_s])$ Rather than $X_s(j\Omega)$.

* That is the frequencies $\pm 4000\pi$ are alias frequencies.
* Plotting $X(e^{j\omega}) = X_s(j\frac{\omega}{T})$ as a function of ω yields the same graph as shown in the previous example (4.1)

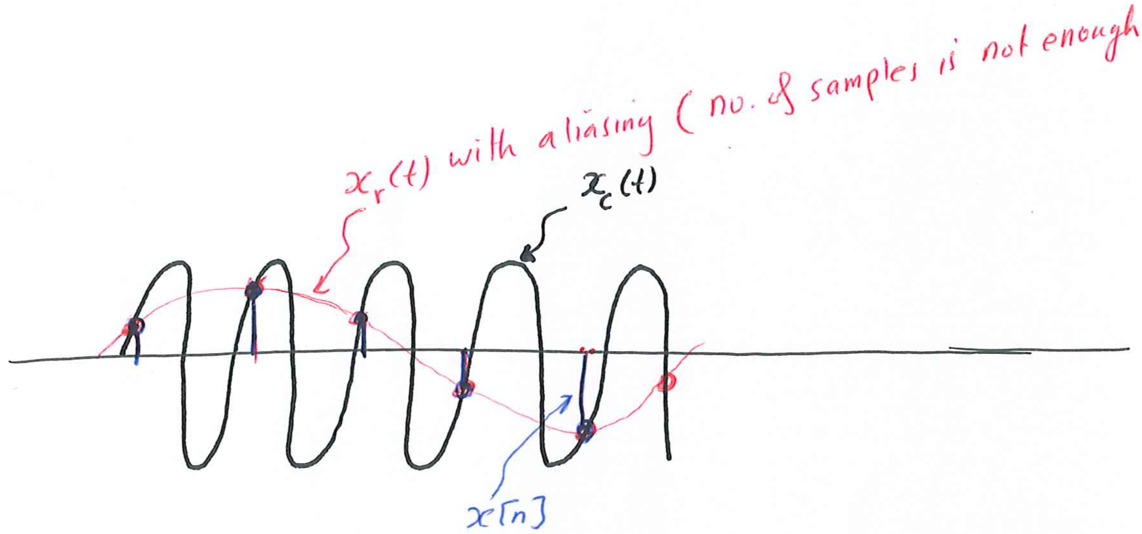
* the fundamental reason is that the sequence of samples is the same in both cases: i.e,

$x(t) = \cos(16000\pi t)$

$$x[n] = x(nT) = \cos\left(\frac{16000\pi n}{6000}\right) = \cos\left(\frac{8}{3}\pi n\right)$$

$$= \cos\left(2\pi n + \frac{2}{3}\pi n\right)$$

$$= \cos\left(\frac{2}{3}\pi n\right)$$



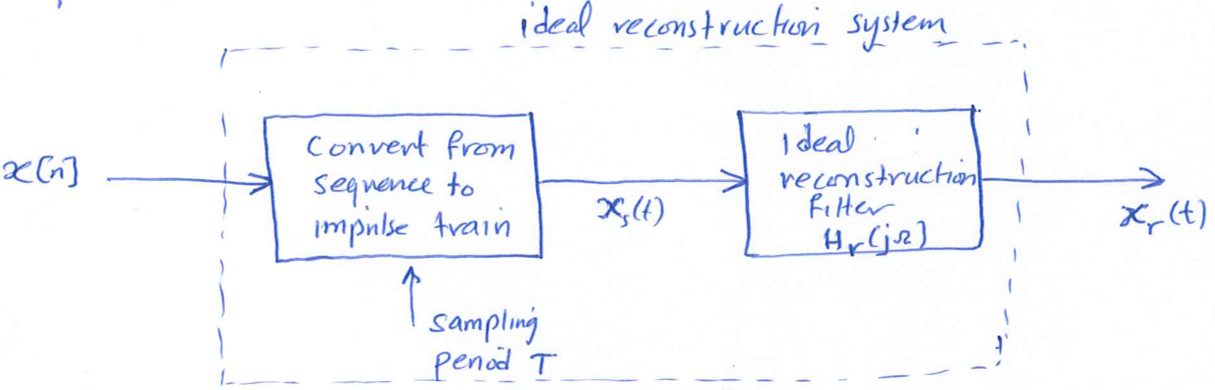
(Recall ~~that~~ we can add any integer multiply of 2π to the argument of the cosine without changing its value).

4.3 Reconstruction of a bandlimited signal from its Samples.

* According to the sampling theorem, samples of continuous-time bandlimited signal taken frequently enough are sufficient to represent the signal exactly. For this, we need to know the sampling period.
* Impulse train provides a convenient mean for understanding the process.
* If we are given the sequence of samples, $x[n]$, we can form an impulse train $x_s(t)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT) \qquad (1)$$

The n^{th} sample is ~~associate~~ associated with the impulse at $t = nT$. where T is the sampling period associated with the sequence $x[n]$.

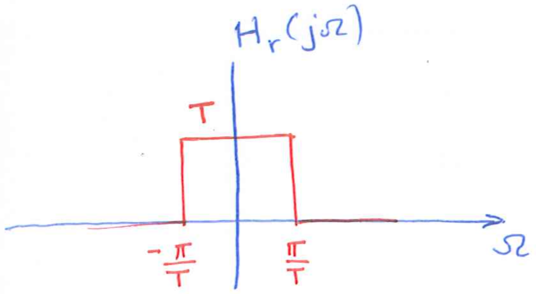


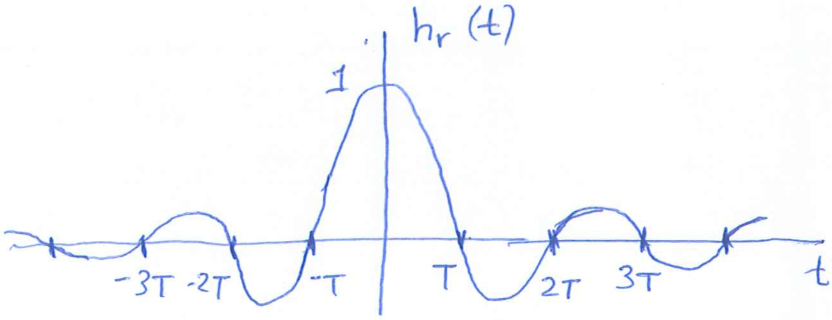
* If $x_s(t)$ is the input to an ideal low-pass continuous-time filter with frequency response $H_r(j\Omega)$ and impulse response $h_r(t)$ the output will be

$$x_r(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT) \qquad (2)$$

* Recall that the ideal reconstruction filter has a gain of T (to compensate for the factor of $1/T$ in the sampling process), and the cutoff frequency Ω_c: $\Omega_N < \Omega_c < \Omega_s - \Omega_N$

commonly used choice $\Omega_c = \frac{\Omega_s}{2} = \frac{\left(\frac{2\pi}{T}\right)}{2} = \frac{\pi}{T}$.





$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

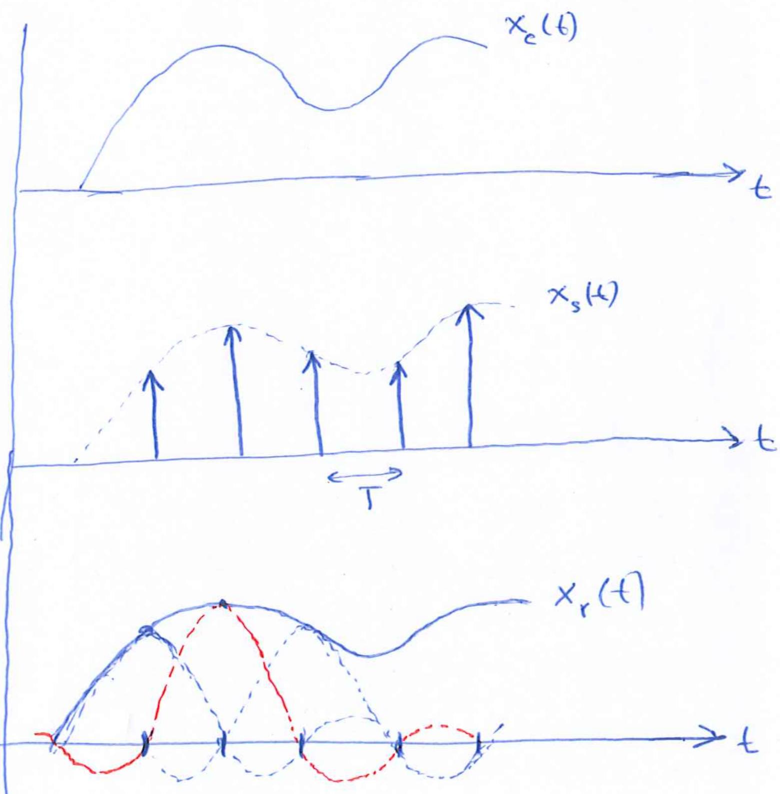
$$\Rightarrow \quad x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \, \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T} \qquad \text{—— (3)}$$

* equations (2) and (3) express the continuous-time signal in terms of a linear combination of basis function $h_r(t-nT)$ with the samples $x[n]$ playing the role of coefficients. (other choices of the basis functions are possible).

* note that
$$\left.\begin{aligned} &h_r(0) = 1 \\ &h_r(nT) = 0 \quad \text{for } n = \pm 1, \pm 2, \ldots \end{aligned}\right\} \text{—— (4)}$$

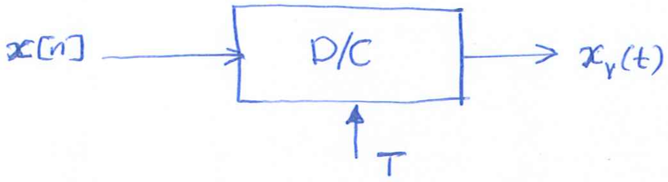
* considering equation (2) and based on (4) it follows that if $x[n] = x_c(nT)$ then.
$$x_r(mT) = x_c(mT) \quad \text{for all integers } m$$

$\Rightarrow$ That is, the signal that is reconstructed by equation (3) has the same values at the sampling times as the original continuous-time signal, independently of the sampling period T.



* The ideal LPF interpolates between the impulses of $x_s(t)$ to reconstruct $x_r(t)$.

* It is useful to formalize the preceding discussion by defining an ideal system for reconstructing a bandlimited signal from a sequence of samples.
* We will call this system the ideal discrete-to-continuous-time (D/C) converter.



* The properties of the ideal D/C converter are most easily seen in the frequency domain:-

from $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_r(t-nT)$

$$X_r(j\Omega) = \sum x[n]\, H_r(j\Omega)\, e^{-j\Omega T n}$$

$\Omega T = \omega$

$$\boxed{X_r(j\Omega) = H_r(j\Omega)\, X(e^{j\Omega T})} \quad \text{--- (5)}$$

LPF

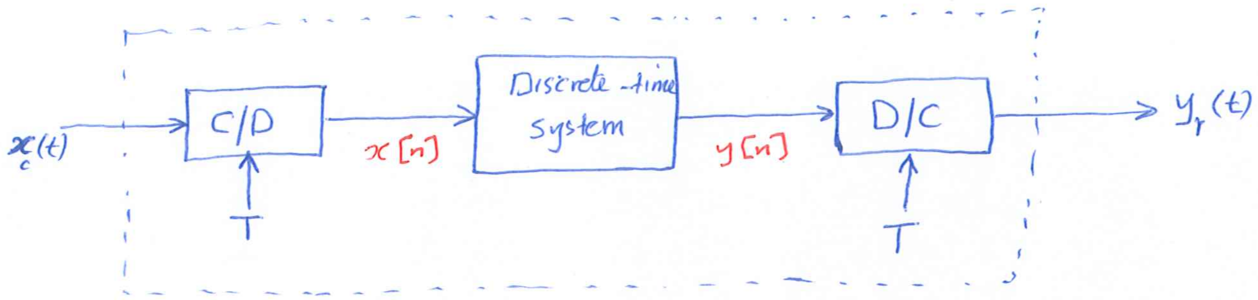
sampled version (periodic)

From (5), it is clear that the ideal LPF $H_r(j\Omega)$ selects the base period of the resulting periodic Fourier transform $X(e^{j\Omega T})$ and compensate for the $\left(\frac{1}{T}\right)$ scaling inherent in sampling.

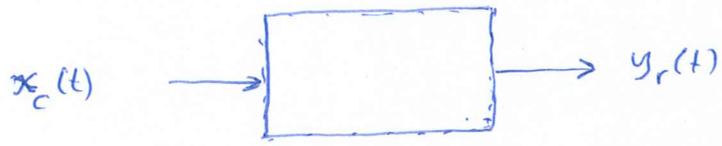
Thus, if $x[n]$ has been obtained by sampling a bandlimited signal at the Nyquist rate or higher, the reconstructed signal $x_r(t)$ will be equal to the original bandlimited signal.

4.4 Discrete-time Processing of Continuous-time Signals

* A major application of discrete-time systems is in the processing of continuous-time signals.



* The overall system is equivalent to a continuous-time system



* The properties of the overall system are dependent on the choice of the discrete-time system and the sampling rate.
* we assume that the C/D and D/C converters have the same T. This is not essential.
* The C/D converter produces a discrete-time signal

$$x[n] = x_c(nT)$$

** The DTFT of $x[n]$ is related to the continuous-time Fourier transform of $x_c(t)$

$$X(e^{j\omega}) = \frac{1}{T} \sum X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad \text{——— (6)}$$

*** The D/C converter creates a continuous-time output

$$y_r(t) = \sum y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

**** From (5)

$$Y_r(j\Omega) = H_r(j\Omega)\, Y(e^{j\Omega T}) \qquad \text{——— (7)}$$

$$= \begin{cases} T\, Y(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases}$$

* If the discrete-time system is linear and time-invariant, we have

$$Y(e^{jw}) = H(e^{jw})\, X(e^{jw}) \quad \text{———} \quad (8)$$

* combining (8) and (7)

$$Y_r(j\Omega) = H_r(j\Omega)\, Y(e^{jw}) = H_r(j\Omega)\, H(e^{j\Omega T})\, X(e^{j\Omega T})$$

* Next, using (6) with $\omega = \Omega T$,

$$Y_r(j\Omega) = H_r(j\Omega)\, H(e^{j\Omega T})\, \frac{1}{T} \sum X_c\left[j\left(\Omega - \frac{2\pi k}{T}\right)\right]$$

* if $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ (Band limited) then the ideal LPF cancels the factor $\frac{1}{T}$ and selects only the term for $k=0$:-

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})\, X_c(j\Omega) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

* Thus, if $X_c(j\Omega)$ is bandlimited and the sampling rate is at or above Nyquist rate, then

$$Y_r(j\Omega) = H_{eff}(j\Omega)\, X_c(j\Omega)$$

where

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases} \quad \text{———} \quad (9)$$

That is, the overall continuous-time system is equivalent to an LTI system whose effective frequency response is given by equation (9)

- conditions must be met for that

① the discrete-time system must be LTI

② the input signal must be bandlimited

③ the sampling rate must be high enough

Example: ideal continuous-time Low Pass filtering using a Discrete-time LPF.

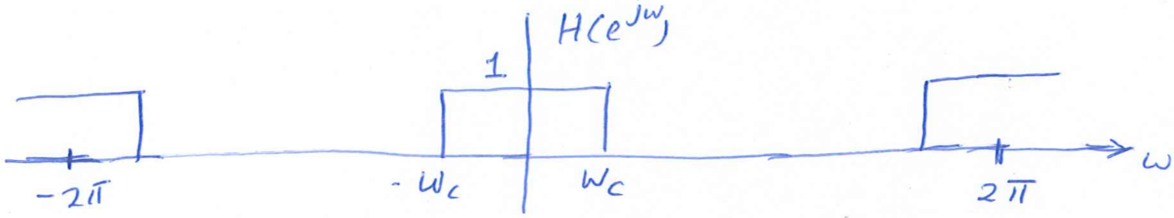
consider the discrete-time processing of continuous-time signals

$$x_c(t) \longrightarrow \boxed{C/D} \longrightarrow \underset{\text{Discret time syst.}}{\boxed{H(e^{j\omega})}} \longrightarrow \boxed{D/C} \longrightarrow y_r(t)$$

let the LTI discrete-time system have freq. response

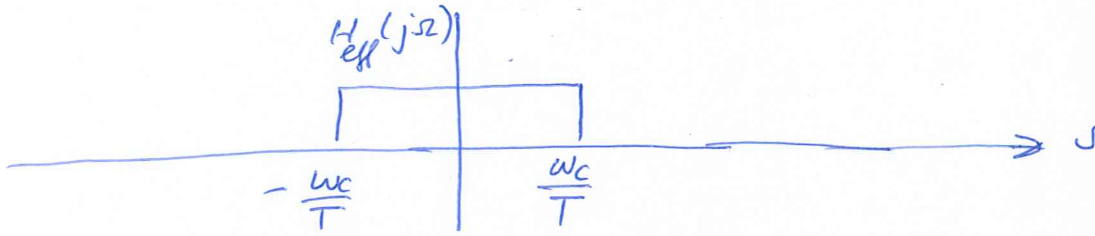
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

this freq. response is periodic with period 2π

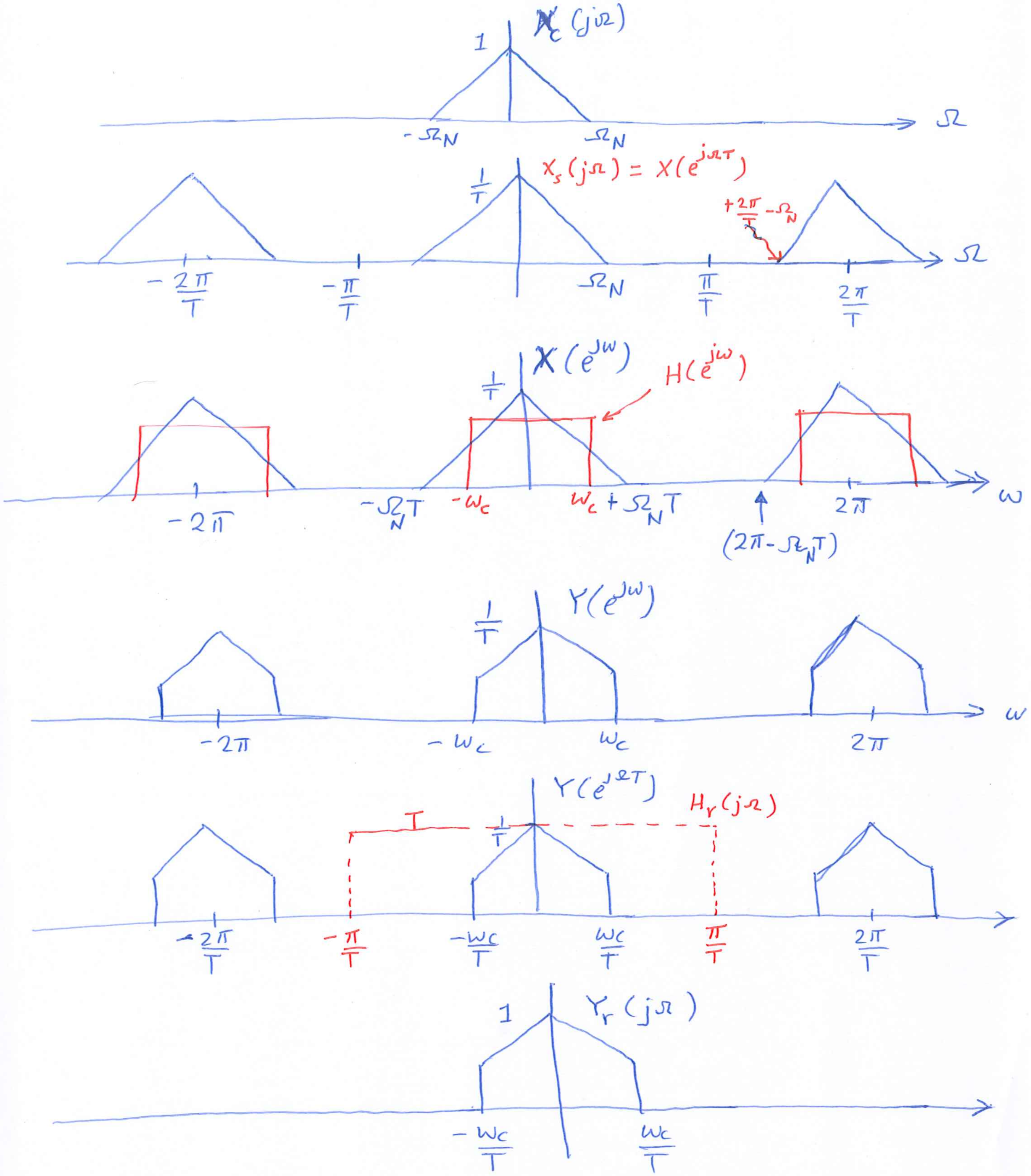


For bandlimited inputs sampled at or above the Nyquist rate, the overall system will behave as an LTI continuous-time system with freq. response

$$H_{eff}(j\Omega) = \begin{cases} 1 & |\Omega T| < \omega_c \quad \text{or} \quad |\Omega| < \frac{\omega_c}{T} \\ 0 & |\Omega T| \geq \omega_c \quad \text{or} \quad |\Omega| \geq \omega_c/T \end{cases}$$



* the next figure provides an interpretation of how this effective response is achieved.



X_c(jΩ)
1
-Ω_N
Ω_N
Ω
X_s(jΩ) = X(e^{jΩT})
1/T
+2π/T - Ω_N
-2π/T
-π/T
Ω_N
π/T
2π/T
X(e^{jω})
H(e^{jω})
1/T
-2π
-Ω_N T
-ω_c
ω_c + Ω_N T
(2π - Ω_N T)
2π
ω
Y(e^{jω})
1/T
-2π
-ω_c
ω_c
2π
ω
Y(e^{jΩT})
H_r(jΩ)
T
1/T
-2π/T
-π/T
-ω_c/T
ω_c/T
π/T
2π/T
Y_r(jΩ)
1
-ω_c/T
ω_c/T