Pole Zero Placement Method

Geometric Interpretation of Pole/zero location :-

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} =$$

$H(z)$ in pole/zero form $\quad H(z) = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} (1 - c_k z^{-1})}{\prod_{k=0}^{N} (1 - d_k z^{-1})}$

Stable system means that the ROC of $H(z)$ includes $|z| = 1$

$\Rightarrow$ The freq. Response

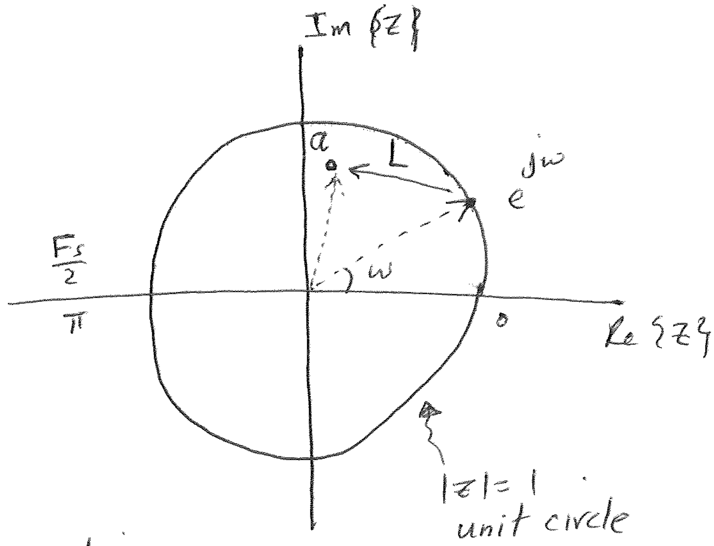
$$H(e^{jw}) = H(z)\Big|_{z=e^{jw}} = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} (1 - c_k e^{-jw})}{\prod_{k=0}^{N} (1 - d_k e^{-jw})}$$

The Mag. Response

$$|H(e^{jw})| = \frac{|b_0|}{|a_0|} \left| e^{-jw(M-N)} \right| \frac{\prod_{k=0}^{M} |e^{+jw} - c_k|}{\prod_{k=0}^{N} |e^{jw} - d_k|}$$

$$= \frac{|b_0|}{|a_0|} \frac{\prod^{M} |e^{jw} - c_k|}{\prod^{N} |e^{jw} - d_k|}$$

Note that $|H(e^{jw})|$ depends on terms $|e^{jw} - a|$



* $e^{jw} - a$ is a vector connecting e^{jw} to a

* $|L| = |e^{jw} - a|$ is the length of the vector L

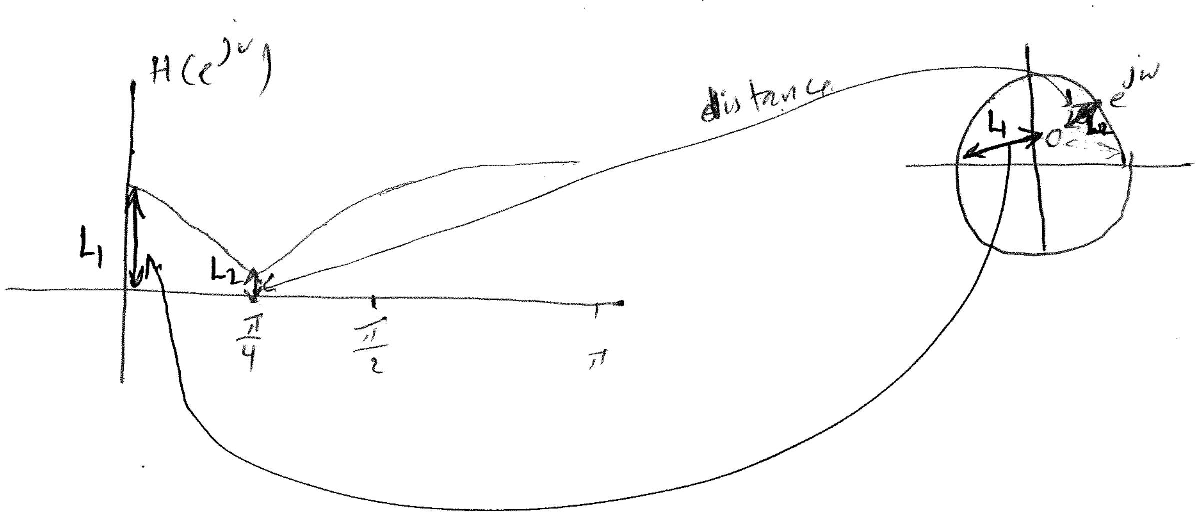
$$\Rightarrow |H(e^{jw})| = \frac{|b_0|}{|a_0|} \frac{\prod_{k=0}^{M} |e^{jw} - c_k|}{\prod^{N} |e^{jw} - d_k|}$$

$|e^{jw} - c_k|$ → distance from e^{jw} to c_k

$|e^{jw} - d_k|$ → distance from e^{jw} to d_k

$$= \frac{|b_0|}{|a_0|} \frac{\prod^{M} \text{"Distance from } e^{jw} \text{ to zeros"}}{\prod^{N} \text{"Distance from } e^{jw} \text{ to poles"}}$$

* When e^{jw} close to a zero, $|H(e^{jw})|$ is "small"
* " e^{jw} " " a pole, $|H(e^{jw})|$ is "large"

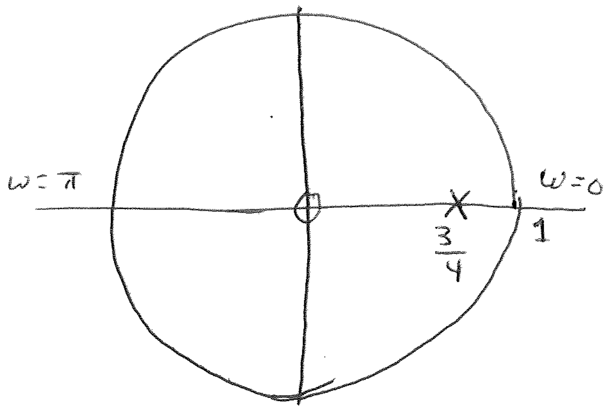


Example. let $H(z) = \dfrac{1}{1 - \frac{3}{4} z^{-1}}$, sketch $|H(e^{j\omega})|$

a Evaluate $|H(e^{j\omega})|$ at $\omega = 0, \frac{\pi}{4}, \pi$.

$$H(z) = \frac{z}{z - \frac{3}{4}}$$, pole at $z = 3/4$
Zero at $z = 0$

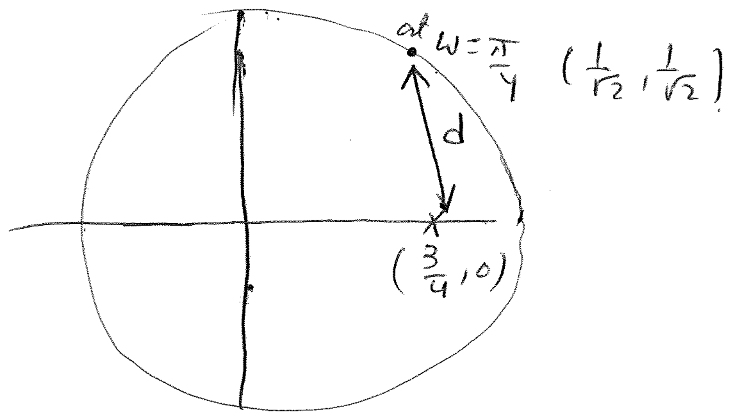
$|H(e^{j\omega})|$ at $\omega = 0$ (DC value)



the distance to the pole is $(1 - \frac{3}{4}) = \frac{1}{4}$

the distance to the zero $= 1$

$$\Rightarrow |H(e^{j\omega})|_{\omega=0} = \frac{1}{\frac{1}{4}} = 4$$



at $\omega = \frac{\pi}{4}$

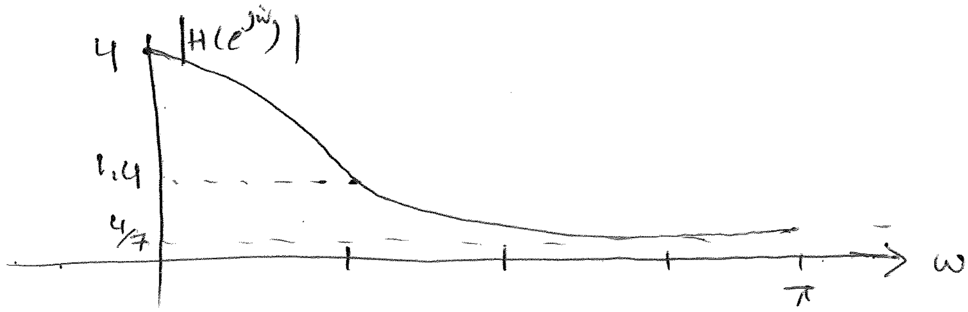
distance to the pole

$$d = \sqrt{\left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\approx 0.708$$

$\Rightarrow H(e^{j\omega})$ at $\omega = \frac{\pi}{4}$ is $\dfrac{1}{0.708} = 1.412 \approx \sqrt{2}$

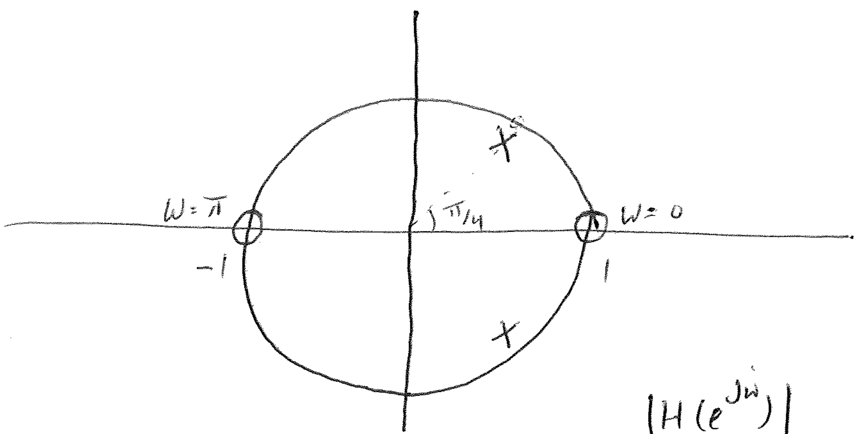
at $\omega = \pi \Rightarrow \dfrac{\text{distance to the zero}}{\text{distance to the pole}} = \dfrac{1}{1 + \frac{3}{4}} = \dfrac{4}{7}$

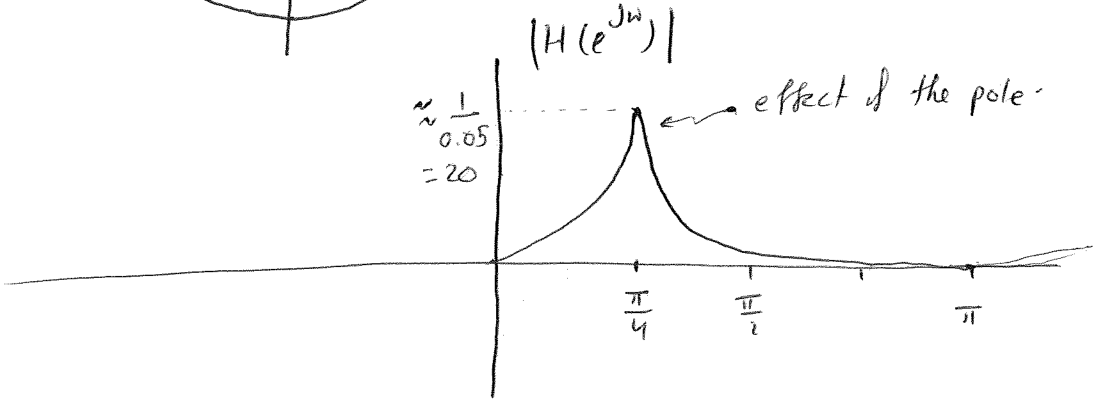


Example $\quad H(z) = \dfrac{1 - z^{-2}}{(1 - 0.95\, e^{j\pi/4} z^{-1})(1 - 0.95\, e^{-j\pi/4} z^{-1})}$

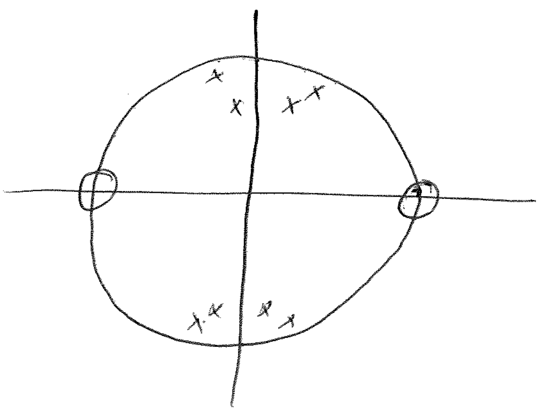
Find $|H(e^{jw})|$ at $w = 0, \frac{\pi}{4}, \pi$

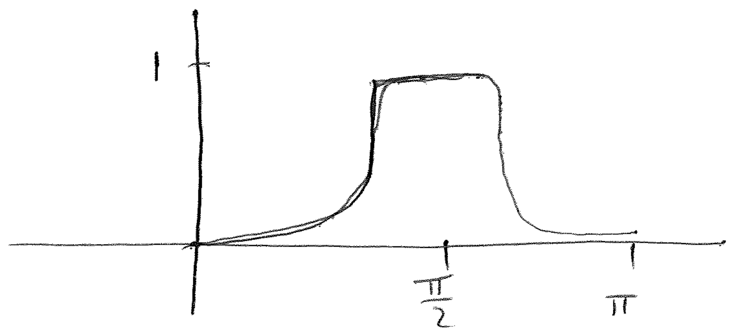
Two poles at $z = 0.95\, e^{j\pi/4}$ & $0.95\, e^{-j\pi/4}$
Two zero at $z = +1, -1$





Example Infer filter characteristics from pole/zero plot





A Narrowband Bandpass Filter (resonator)

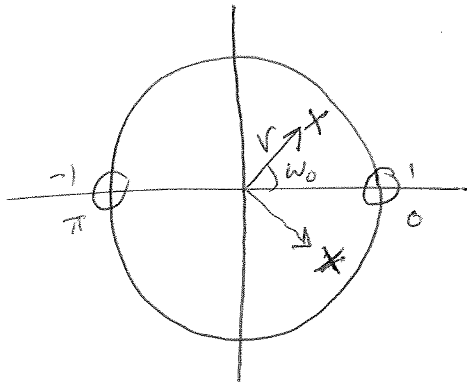
* Pass a single freq. F_0, $0 < F_0 < \frac{F_s}{2}$
* place a pole at the point inside the unit circle that corresponds to the resonant freq. F_0 (i.e w_0)

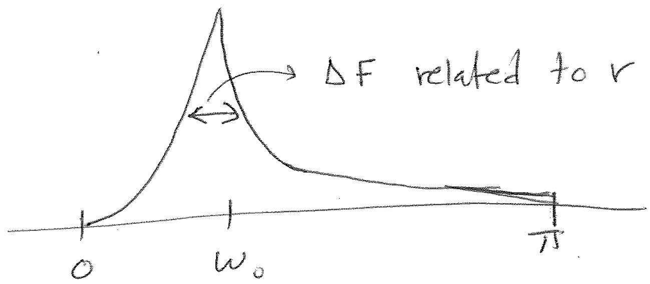
$$\Rightarrow w_0 = 2\pi \frac{F_0}{F_s}$$

* place zeros at the two end frequencies

i.e., $Z = 1$ $(F = 0)$ corresponds to $w = 0$

$Z = -1$ $(F = \frac{F_s}{2})$ " to $w = \pi$





$$H_r(Z) = \frac{K(Z-1)(Z+1)}{(Z-re^{jw_0})(Z-re^{-jw})} = \frac{K(Z^2-1)}{Z^2 - 2r\cos w_0 Z + r^2}$$

unknowns :-

① r is chosen such that the filter is highly selective (i.e., small 3-dB of the pass band, ΔF)

$$r \approx 1 - \frac{\Delta F}{F_s}\pi$$

② The gain factor K is inserted to ensure that the passband gain is 1, i.e, no amplification

$$K = \frac{|e^{j2w_0} - 2r\cos w_0\, e^{jw_0} + r^2|}{|e^{j2w_0} - 1|} = \frac{(1-r)\sqrt{1-2r\cos 2w_0 + r^2}}{2|\sin w_0|}$$

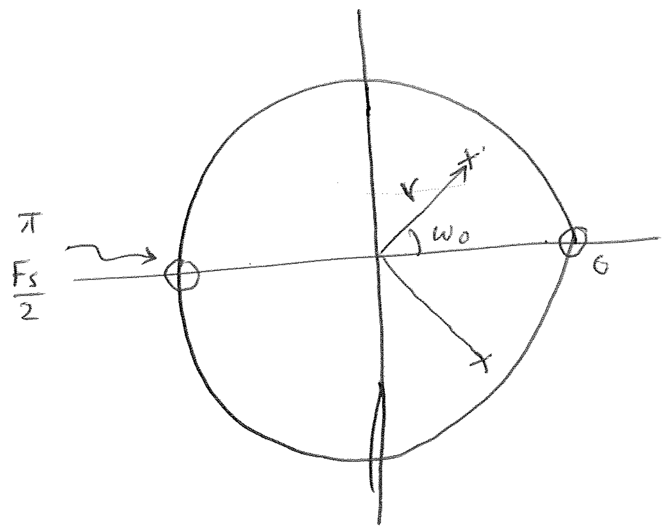
<u>Example</u> Design a second-order band pass filter using the pole-zero placement method & satisfying the following specifications:

$$F_s = 8000 \text{ HZ}$$

$$3\text{-dB BW} = 200 \text{ HZ} = \Delta F$$

$$\text{Pass band center freq.} = 1000 \text{ HZ} = F_0$$

$$\text{Zero gain at zero } \& \ 4000 \text{ HZ} = \frac{F_s}{2}$$



$$w_0 = 2\pi \frac{F_0}{F_s} = 2\pi \frac{1000}{8000} = \frac{\pi}{4} = 0.785 \text{ rad}$$

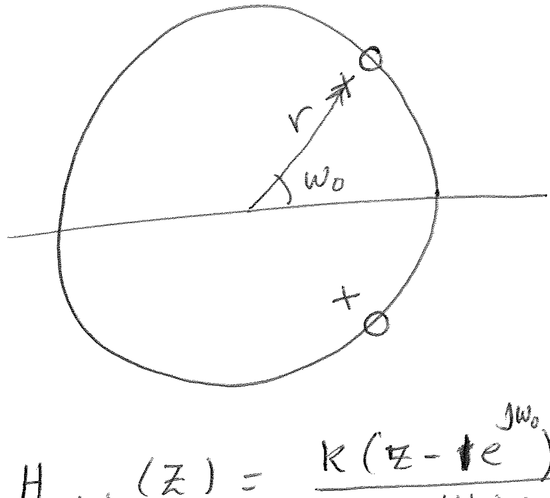
$$r = 1 - \frac{\Delta F}{F_s}\pi = 1 - \frac{200}{8000}\pi = 1 - \frac{\pi}{40} = 1 - \frac{3.14}{40} = 0.9215$$

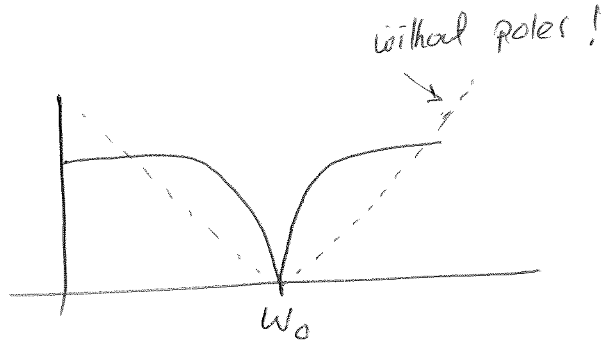
$$K = \frac{(1-r)\sqrt{1 - 2r\cos(2 \cdot \frac{\pi}{4}) + r^2}}{2\left|\sin \frac{\pi}{4}\right|} = 0.0755$$

$$H(z) = \frac{K(z^2 - 1)}{z^2 - 2r\cos w_0 z + r^2}$$

A Narrowband Band Stop Filter (Notch filter)

* Remove a single freq., $0 < F_0 < \frac{F_s}{2}$
* place zeros at the points on the unit circle that corresponds to the notch freq. F_0
* The corresponding angle is $\omega_0 = \frac{2\pi F_0}{F_s}$
* place poles at the points inside the unit circle that correspond to the notch freq. F_0





$$H_{notch}(z) = \frac{k(z - e^{j\omega_0})(z - e^{-j\omega_0})}{(z - re^{j\omega_0})(z - re^{-j\omega_0})} = \frac{k(z^2 - 2\cos\omega_0 z + 1)}{z^2 - 2r\cos\omega_0 z + r^2}$$

unknowns :-

* r is chosen such that the filter is highly selective i.e., small 3-dB B.W of passband, ΔF

$$r \approx 1 - \frac{\Delta F}{F_s}, \quad 0.9 < r < 1$$

* K is inserted to ensure that the pass band gain is 1

K is chosen such that no amplification

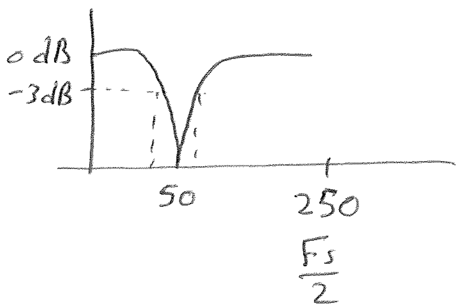
$$K = \frac{|1 - 2r\cos w_0 + r^2|}{2|1-\cos w_0|}$$

<u>Example</u> use pole-zero placement Method. to obtain the transfer function of a sample digital notch filter (see figure) that meets the following specifications

Notch freq. = 50 HZ

3 dB width = ± 5 HZ

Sampling Freq. F_s = 500 HZ



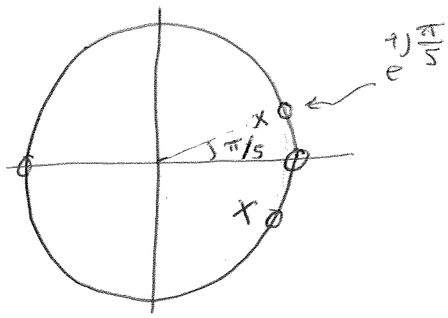
<u>solution</u>:

$$w_0 = 2\pi \frac{F_0}{F_s} = \frac{2\pi * 50}{500} = \frac{\pi}{5}$$

$$\Delta F = 10 \text{ HZ}$$

$$r = 1 - \frac{\Delta F}{F_s}\pi$$

$$= 1 - \frac{10}{500}\pi =$$



$$K = \frac{|1 - 2r\cos w_0 + r^2|}{2|1-\cos w_0|}$$

$$\Rightarrow H_{Notch}(Z) = \frac{K(Z - e^{j\frac{\pi}{5}})(Z - e^{-j\pi/5})}{(Z - re^{j\pi/5})(Z - re^{-j\pi/5})}$$

Example Design a second-order Notch Filter using pole-zero placement method :

$F_s = 8000$ HZ

3dB - BW = 100 HZ

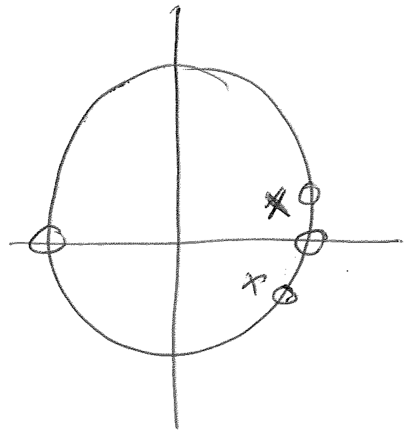
Stopband ~~ctr~~ center freq. $F_o = 1500$ HZ

Solution

$$\omega_o = 2\pi \frac{F_o}{F_s} = 2\pi \cdot \frac{15\cancel{00}}{80\cancel{00}} = \frac{3}{8}\pi$$

$$r = 1 - \frac{\Delta F}{F_s}\pi$$

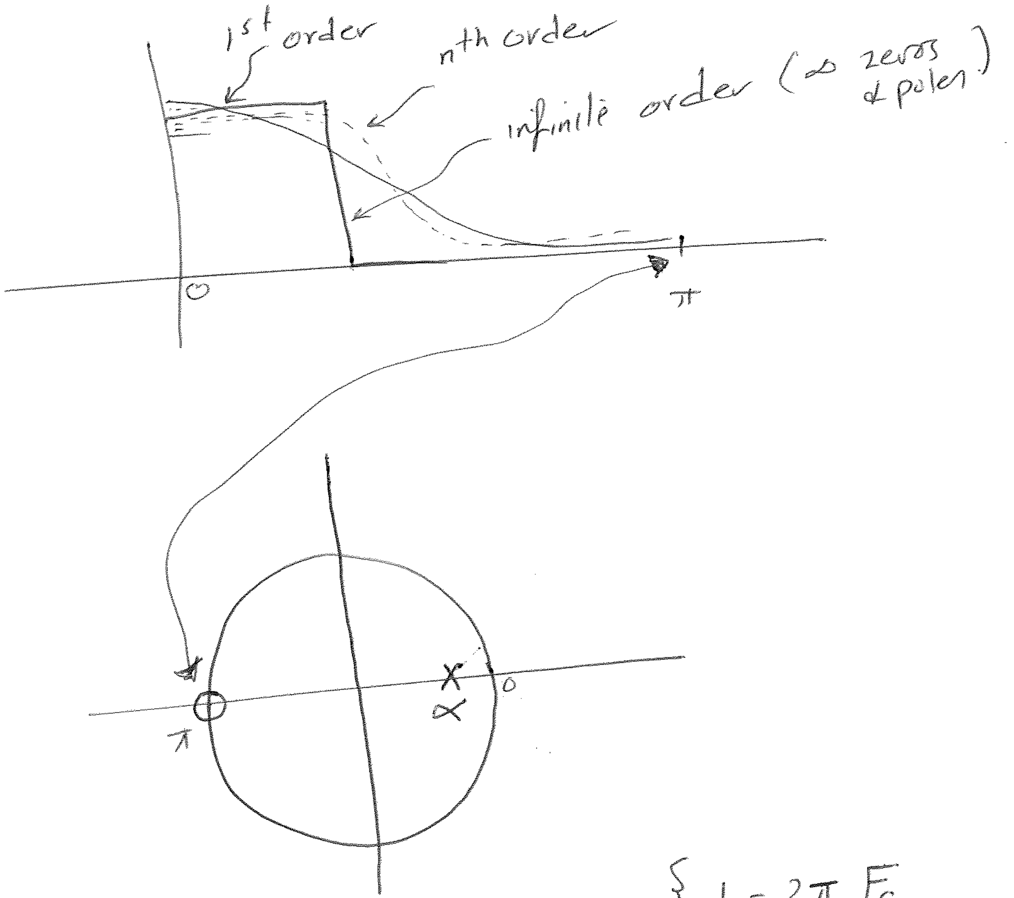
$$= 1 - \frac{100}{8000}\pi$$



$K =$

A First-Order Lowpass Filter (LPF)

* A wide band Filter
* Pass freq. components from 0 to F_c (cutoff freq).
* place a zero on the unit circle at $z = -1$
* " a pole on the real axis & inside the unit circle



$$H_{LP}(z) = \frac{k(z+1)}{z-\alpha}, \quad \alpha = \begin{cases} 1 - 2\pi \dfrac{F_c}{F_s} & F_c < \dfrac{F_s}{4} \\ \pi - 1 - 2\pi \dfrac{F_c}{F_s} & F_c > \dfrac{F_s}{4} \end{cases}$$

k is inserted to ensure that the pass band gain is 1 (No Amplification)

$$k = \frac{1-\alpha}{2}$$

Example

Design a 1^{st} order LPF with the following specifications :-

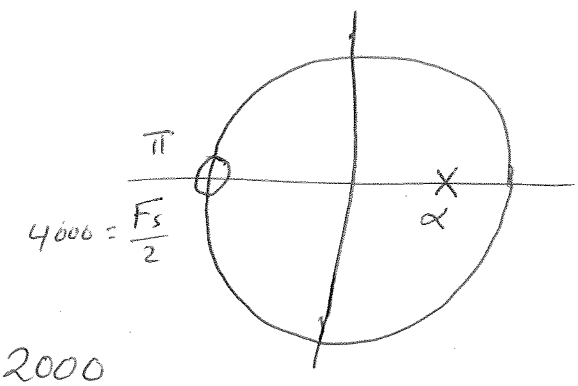
$F_s = 8000$ HZ

3-dB cutoff freq. $F_c = 100$ HZ

Zero gain at 4000 HZ

solution

$$H_{LP}(Z) = \frac{k(Z+1)}{Z-\alpha}$$



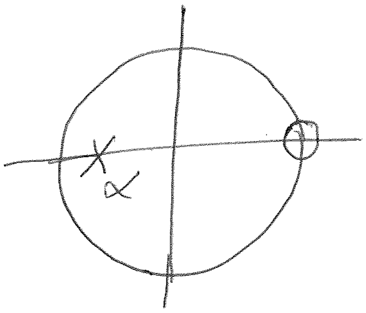
$$F_c = 100 \text{ HZ} < \frac{F_s}{4} = 2000$$

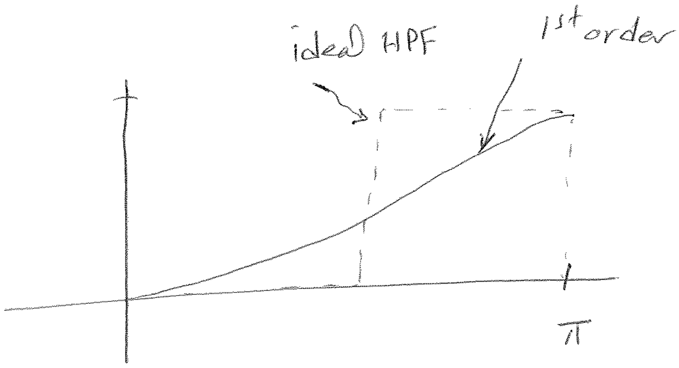
$$\Rightarrow \alpha = 1 - 2\pi \frac{F_c}{F_s} = 1 - 2\pi \frac{100}{8000} = 1 - \frac{\pi}{40}$$

$$K = \frac{1-\alpha}{2}$$

A First-order HPF

* A wide band Filter
* Suppress freq. components from 0 to F_c (cutoff freq)
* place a zero on the unit circle at $z = 1$
* " a pole on the real axis & inside the unit circle





$$H_{HP}(z) = \frac{K(z-1)}{z-\alpha}$$

$$\alpha = \begin{cases} 1 - 2\pi \dfrac{F_c}{F_s} & F_c < \dfrac{F_s}{4} \\ \pi - 1 - 2\pi \dfrac{F_c}{F_s} & F_c > \dfrac{F_s}{4} \end{cases}$$

K is inserted to ensure that the pass band gain is 1, i.e, no amplification

$$K = \frac{1+\alpha}{2}$$

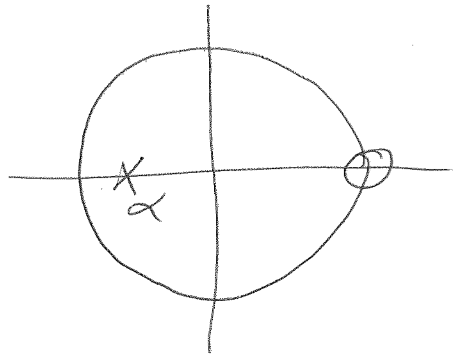
Example Design a 1^{st} order HPF

$F_s = 8000$ HZ

3-dB cut-off freq. $F_c = 3800$ HZ

zero gain at zero HZ

Solution



$F_c = 3800 > \frac{F_s}{4} = 2000$

$\Rightarrow \alpha = \pi - 1 - 2\pi \frac{F_c}{F_s}$

$= \pi - 1 - 2\pi \frac{3800}{8000}$

$\Rightarrow K = \frac{1+\alpha}{2}$

$H(z) = \frac{K(z-1)}{z-\alpha}$