3 Chapter Three

The Z- Transform

Motivation:

- Fourier Transform $X(e^{jw})$ does't converge for all sequences.
- Z- transform is a generalization that covers a broader class of signals.
- Z - transform notation is often more convenient than the Fourier transform notation

* The FT of a sequence $x[n]$ was defined as $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jnw}$

* The Z-transform of a sequence $x[n]$ is defined as

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

called bilateral Z-transform

z : complex variable (continuous)

notation : $x[n] \xleftrightarrow{Z} X(z)$

* There is a close relationship between FT and Z-transform:

$$X(e^{jw}) = X(z)\Big|_{z=e^{jw}}$$

FT, when it exists, is simply $X(z)$ with $z = e^{jw}$. This corresponds to restricting z to have unity magnitude ; i.e., for $|z| = 1$, the Z-transform corresponds to the FT.

More generally, we can express the complex variable z in polar form as

$$z = re^{jw}$$

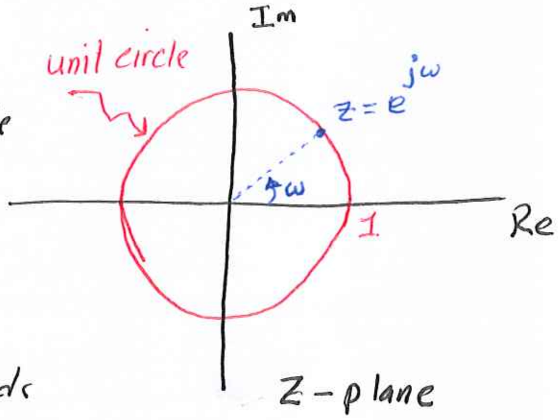
$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n] (re^{jw})^{-n} = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-jwn}$$

This equation can be interpreted as the FT of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}. If $r=1$, the equation reduces to the FT of $x[n]$.

Unit circle and z-plane

* since the z-transform is a function of a complex variable, it is convenient to describe and interpret it using the complex z-plane.
* In z-plane, the contour corresponding to $|z|=1$ is a circle of unit radius. It is referred to as the "unit circle".

* unit circle is the set of points $z = e^{jw}$, $0 \le w \le 2\pi$
* The z-transform evaluated on the unit circle corresponds to the FT.



if we evaluate $X(z)$ at points on the unit circle at $z=1$ ($w=0$), through $z=j$ ($w=\frac{\pi}{2}$) to $z=-1$ ($w=\pi$), we obtain the FT for $0 \le w \le \pi$.
Continuing around the unit circle corresponds to FT for $\pi \le w \le 2\pi$ or ($w = -\pi$ to 0)

In Ch. 2, The FT was displayed on a linear frequency axis.
⇒ With this interpretation, the inherent periodicity in frequency of the FT is captured naturally.

Region of Convergence (ROC)

* As we discussed in ch.2, the power series representing the FT, $X(e^{jw}) = \sum x[n] e^{-jnw}$ does't converge for all sequences.

* Similarly, the z-transform does't converge for all sequences or for all values of z.

* For any given sequence, the set of values for which the z-transform power series converges is called the "Region of convergence" (ROC), of the z-transform.

* If the sequence $x[n]$ is absolutely summable, the FT converges to a continuous function of w. Apply this criteria, leads to the condition

$$|X(re^{jw})| \leq \sum_{-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

for convergence of the z-transform.

$\Rightarrow$ because of the multiplication of $x[n]$ by a real exponential r^{-n}, it is possible for the z-transform to converge even if the FT $(r=1)$ does not.

For example:

$x[n] = u[n]$ is not absolutely summable, the FT power series does not converge absolutely.

However, $r^{-n} u[n]$ is absolutely summable if $r > 1$

This means that the z-transform for $u[n]$ exists with an ROC $r = |z| > 1$

$\Rightarrow$ r should be choosen so that $x[n] r^{-n}$ is abs. summable $\sum |x[n] r^{-n}| < \infty$, hence the z-transform converge.

Example :

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] \longrightarrow \sum_{-\infty}^{\infty} |x[n]| = \frac{1}{1-\frac{1}{2}} = 2$$

$\Rightarrow$ FT exists (converges)

$$x[n] = 2^n u[n] \longrightarrow \sum |x[n]| = \infty$$

$\Rightarrow$ FT Does't exist (Diverges)

* In general, exponential sequence $\alpha^n u[n]$ which decays (to right or left) has FT. While, the exponential sequence which grows exponentially (i.e $|\alpha| > 1$) has no FT.

* If we multiply the growing exponential with a decaying exponential (decaying fast than the growing exponential)

then $$X_r(e^{jw}) = \sum_{-\infty}^{\infty} \left(x[n]\, r^{-n}\right) e^{-jwn}$$

we choose r such that $(x[n]\, r^{-n})$ is absolutely summable.

$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \left(re^{+jw}\right)^{-n}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$, where $\underbrace{z = re^{jw}}_{\text{complex variable}}$

$|z| = r$, $\theta_z = w$.

* z-transform converges for some values of r such that

$$\sum_{-\infty}^{\infty} |x[n]\, r^{-n}| < \infty$$

Pole - Zero Plot

* The z-transform is most useful when the infinite sum can be expressed in a closed form.
* Among the most important and useful z-transforms are those for which $X(z)$ is equal to a rational function inside the ROC, i.e,

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z.

In general,

The values of z for which $X(z) = 0$ are the zeros of $X(z)$

" " " z " " $X(z) = \infty$ " " poles of $X(z)$

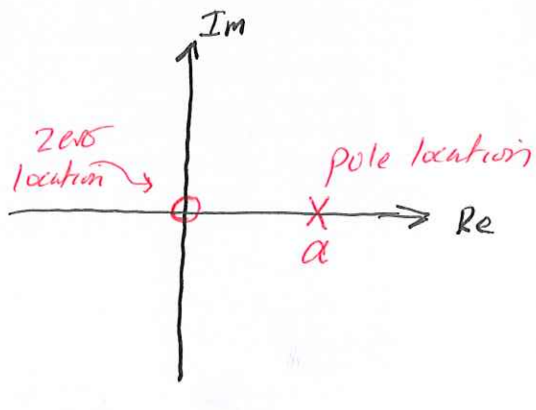
* For rational z-transform, a number of important relationships exist between the locations of the poles of $X(z)$ and the ROC of the z-transform.

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

$X(z) = 0 \Rightarrow z = 0$, zero (o)

$X(z) = \infty \Rightarrow z = a$, pole (x)

$\Rightarrow$ pole - zero plot



Example 3.1 Right-sided exponential sequence. $x[n] = a^n u[n]$

a: real or complex

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

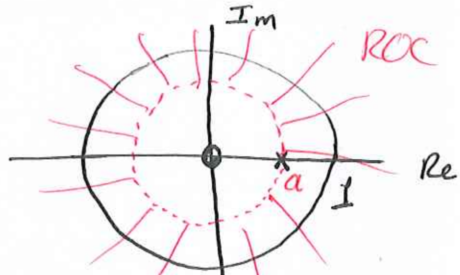
$$= \sum_{-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

For convergence of $X(z)$ we need $\sum_{n=0}^{\infty} |a z^{-1}|^n < \infty$

Thus, the ROC : $|a z^{-1}| < 1 \Rightarrow |z| > |a|$

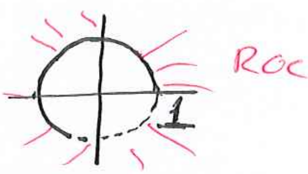
$\Rightarrow$ inside the ROC, the infinite series converges to

$$X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$



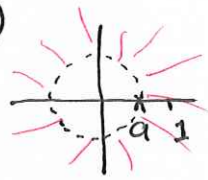
case 1, if $a = 1$, $x[n] = u[n]$

$$\Rightarrow X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

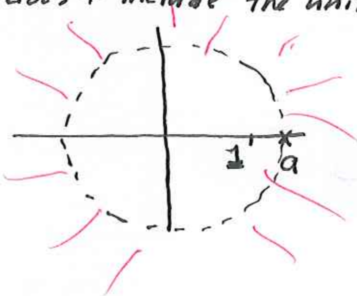


case 2, if $a < 1$, the FT of $x[n] = a^n u[n]$ converges to (since the ROC includes the unit circle)

$$X(e^{jw}) = \frac{1}{1 - a e^{-jw}}$$

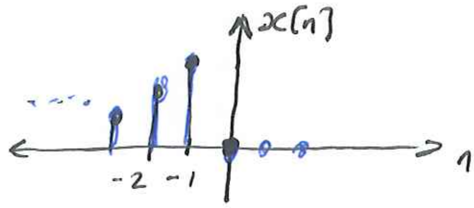


case 3, if $a > 1$, the FT of the right-sided exponential does not converge. (since the ROC does't include the unit circle)



Example 3.2 Left sided sequence $x[n] = -a^n u[-n-1]$

$$x[n] = \begin{cases} -a^n & n \le -1 \\ 0 & n > -1 \end{cases}$$



left sided $\equiv$ non zero only for $n \le -1$

$$X(z) = \sum_{-\infty}^{\infty} -a^n u[-n-1]\, z^{-n} = -\sum_{-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{-\infty}^{-1} (a z^{-1})^n$$

change of variable $n \longrightarrow -n$

$$X(z) = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

note that $\sum_{n=0}^{\infty} \alpha^n = 1 + \sum_{n=1}^{\infty} \alpha^n$

$$\Rightarrow X(z) = -\left[\sum_{n=0}^{\infty} (a^{-1} z)^n - 1 \right]$$

$$= -\left[\sum_{n=0}^{\infty} (a^{-1} z)^n \right] + 1$$

$$= 1 - \frac{1}{1 - a^{-1} z}$$

ROC

if $\boxed{\begin{array}{l} |a^{-1} z| < 1 \\ |z| < |a| \end{array}}$

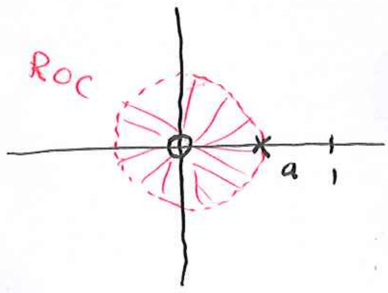
$$= \frac{1 - a^{-1} z - 1}{1 - a^{-1} z}$$

$$= \frac{-a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

The pole-zero plot :-

zero at 0

pole at a



$\ast$ if $|a| < 1$, the sequence $x[n]$ grows exponentially as $n \longrightarrow -\infty$

$\Rightarrow$ FT Does't exist

* if $|a| > 0$, the FT is $X(e^{jw}) = \dfrac{1}{1 - ae^{-jw}}$

* comparing $x[n] = a^n u[n]$ (right-sided sequence)

$$X(z) = \frac{1}{1 - az^{-1}} \qquad ROC \quad |z| > |a|$$

with $x[n] = -a^n u[-n-1]$ (left-sided sequence)

also $$X(z) = \frac{1}{1 - az^{-1}} \qquad ROC: \ |z| < |a|$$

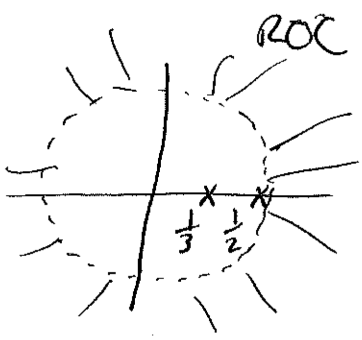
Note: they have the same expression but different ROC.

Example 3.3 $\quad x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

from $\quad a^n u[n] \longleftrightarrow \dfrac{1}{1 - az^{-1}}, \ |z| > |a|$

$$\Rightarrow X(z) = \underbrace{\frac{1}{1 - \frac{1}{2}z^{-1}}}_{ROC_1:\ |z| > \frac{1}{2}} + \underbrace{\frac{1}{1 + \frac{1}{3}z^{-1}}}_{ROC_2:\ |z| > \frac{1}{3}}$$

$$ROC = ROC_1 \cap ROC_2 : \ |z| > \frac{1}{2}$$

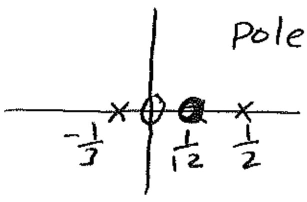


$$\Rightarrow X(z) = \frac{2(1 - \frac{1}{12}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} \qquad ROC: |z| > \frac{1}{2}$$

$$= \frac{2z(z - 1/12)}{(z - \frac{1}{2})(z + 1/3)}$$

poles: $\frac{1}{2}, -\frac{1}{3}$

zeros: $0, \frac{1}{12}$



Pole-zero plot

Example 3.5 Two sided exponential sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

Note that this sequence grows exponentially as $n \to \infty$.

using the z-transform pairs :-

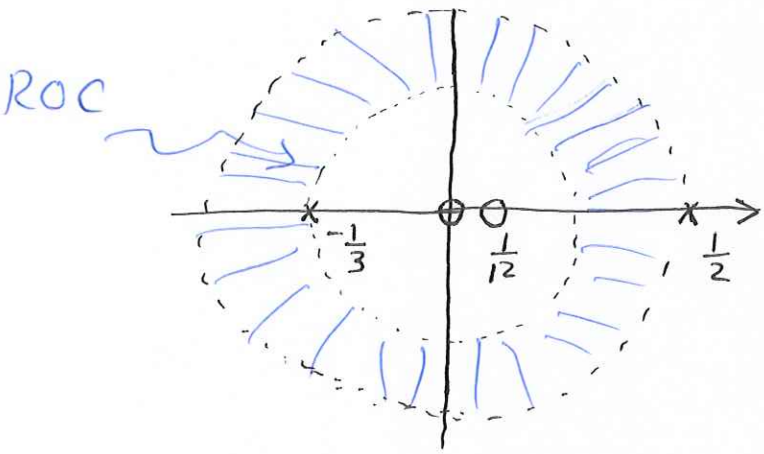
$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

$$-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

then

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$|z| > \frac{1}{3} \quad \text{and} \quad |z| < \frac{1}{2}$$



z-plane
pole-zero plot
and ROC

$$X(z) = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

zeros : $z = 0$, $z = \frac{1}{12}$

poles : $z = -\frac{1}{3}$, $z = \frac{1}{2}$

* The ROC is the annular region $\frac{1}{3} < |z| < \frac{1}{2}$
* The Rational function in this example is identical to the previous example, but the ROC is different
* Since the ROC doesn't contain the unit circle, then $x[n]$ doesn't have a Fourier Transform.

Notes:

* Previous examples show that infinitely long exponential sequences have Z-transforms that can be expressed as a rational functions of either z or z^{-1}.

* The case where the sequence has finite length also has a rather simple form.

* If the sequence is non zero only in the interval $N_1 \leq n \leq N_2$, the z-transform

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

has no problem of convergence, as long as each the terms of $|x[n] z^{-n}|$ is finite.

In general, it may not be posible to express the sum of a finite set of terms in a closed form.

Example: let $x[n] = \delta[n] + \delta[n-5]$

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} = \sum_{-\infty}^{\infty} (\delta[n] + \delta[n-5]) z^{-n}$$

$$= \sum_{-\infty}^{\infty} \delta[n] z^{-n} + \sum_{-\infty}^{\infty} \delta[n-5] z^{-n}$$

$$= \sum_{-\infty}^{\infty} \delta[n] z^{0} + \sum_{-\infty}^{\infty} \delta[n-5] z^{-5}$$

$$= 1 + z^{-5}$$ which is finite for $|z| > 0$

$\Rightarrow$ ROC: $|z| > 0$

Note: If $x[n]$ is finite, FT exist.

Examples: Find $X(z)$ for the following sequences:

① $x[n] = \{\underset{\uparrow}{5}, 0, 2, -1, 3\}$ right-sided sequence

$$X(z) = \sum_{n=0}^{4} x[n]\, z^{-n}$$

$$= 5 + 2z^{-2} - z^{-3} + 3z^{-4}$$

ROC: $|z| > 0$ all z-plane except $z = 0$

② $x[n] = \{5, 0, 2, -1, \underset{\uparrow}{3}\}$ left-sided sequence

$$X(z) = \sum_{n=-4}^{0} x[n]\, z^{-n}$$

$$= 3 - z + 2z^{+2} + 5z^{+4}$$

ROC: all z-plane except $z = \infty$

③ $x[n] = \{-6, +4, \underset{\uparrow}{-2}, 1, 2, 4, 6\}$ Two sided

$$X(z) = -6z^{2} + 4z^{1} - 2 + z^{-1} + 2z^{-2} + 4z^{-3} + 6z^{-4}$$

ROC: all z-plane except $z = 0$ and $z = \infty$

Example. 3.6 Finite length truncated exponential sequence.

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (a z^{-1})^n$$

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

$$= \frac{(a z^{-1})^0 - (a z^{-1})^N}{1 - a z^{-1}}$$

$$= \frac{1 - a^N z^{-N}}{1 - a z^{-1}} = \frac{z^{-N}(z^N - a^N)}{z^{-1}(z-a)}$$

$$= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

Poles : $z = a$, $z = 0$ ($N-1$ order pole)

Zeros : $z^N = a^N$, we have N zeros

Note that the zeros ~~at~~ are at z-plane locations

$$z_k = a e^{j(2\pi k/N)} \qquad k = 0, 1, \cdots, N-1$$

(these values z_k satisfy the equation $z^N = a^N$)

* ROC includes the entire z-plane except at the origin ($z=0$)
* The zero corresponding to $k=0$ cancels the pole at $z=a$.

⇒ only we have $N-1$ poles at the origin.

⇒ the remaining zeros are at z-plane locations

$$z_k = a e^{j(2\pi k/N)} \qquad k = 1, \cdots, N-1$$

For $N = 8$, and $|a| < 1$

$$z_k = a e^{j\frac{\pi}{4}k} \qquad k = 1, \cdots, 7$$

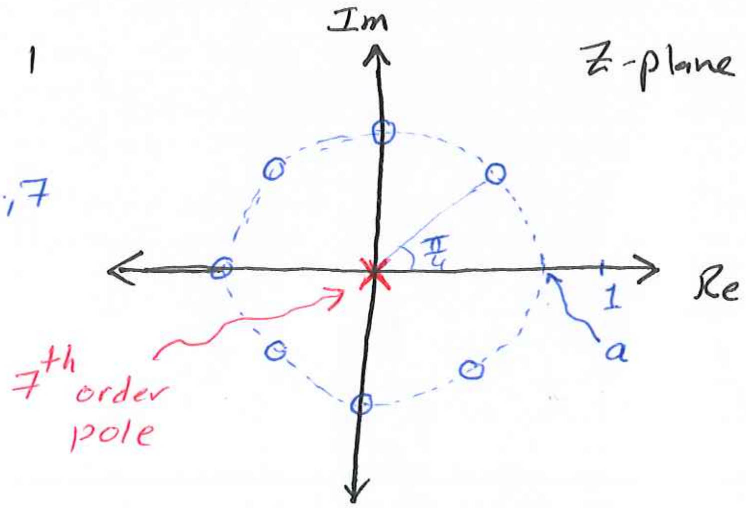


TABLE 1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

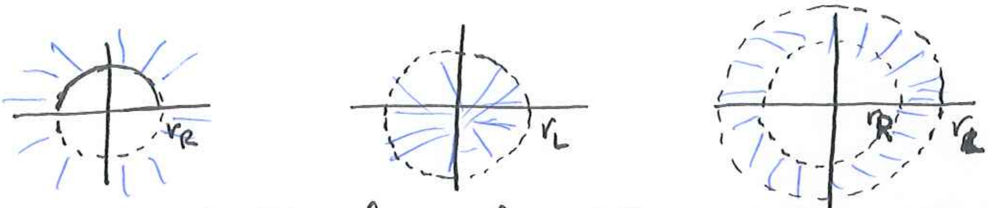
TABLE 2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

3.2 Properties of the ROC for The Z-Transform

Assuming that the algebraic expression for the z-transform is a rational function and $x[n]$ has finite amplitude.

1. The ROC will either be of the form $0 \leq r_R < |z|$, or $|z| < r_L \leq \infty$, or, in general the annulus $0 \leq r_R < |z| < r_L \leq \infty$.



2. The Fourier Transform of $x[n]$ converges absolutely if and only if the ROC includes the unit circle.

3. The ROC can't contain any poles.

4. If $x[n]$ is "a right-sided sequence", i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the outermost (i.e, largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

5. If $x[n]$ is "a left-sided sequence", i.e, a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the innermost (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

6. If $x[n]$ is "a finite-duration sequence", i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z=0$ or $z=\infty$.

7. A two-sided sequence is an infinite-duration that is neither right nor left sided. In such case the ROC will consist of a ring in the z-plane not containing any poles.

8. The ROC must be a connected Region.

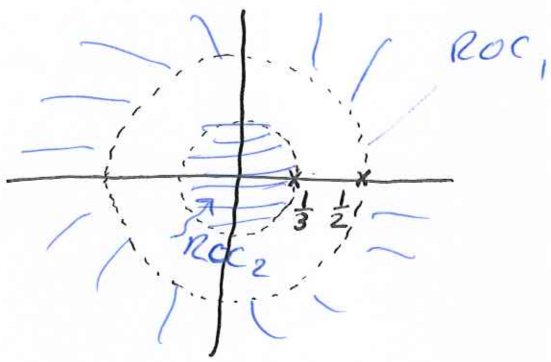
There is possiblility of no overlap between the ROCs of the right- and left-sided parts. In such cases, the z-transform of the sequence simply does't exist.

Example 3.7.

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{-1}{3}\right)^n u[-n-1]$$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - -\frac{1}{3} z^{-1}}$$

$$= \underbrace{\frac{1}{1 - \frac{1}{2} z^{-1}}}_{ROC_1 \; |z| > \frac{1}{2}} + \underbrace{\frac{1}{1 + \frac{1}{3} z^{-1}}}_{ROC_2 \; |z| < \frac{1}{3}}$$



Since, there is no overlap between $|z| > \frac{1}{2}$ and $|z| < \frac{1}{3}$, we conclude that $x[n]$ has no z-transform (no Fourier transform) representation.

Stability, Causality of the LTI system and the ROC

consider the system

$$y[n] - \frac{1}{3} y[n-1] = x[n]$$

$$Y(z) - \frac{1}{3} Y(z) z^{-1} = X(z)$$

$$Y(z)\left[1 - \frac{1}{3} z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

H(z) is called the system function

$$h[n] = Z^{-1}\{H(z)\}$$

to find h[n], ~~we~~ have ~~two~~ possibilities

① if the system is causal, then

$$h[n] = \left(\frac{1}{3}\right)^n u[n] \qquad ROC: \ |z| > \frac{1}{3}$$

the unit circle is included in the ROC

$\Rightarrow$ FT exist

$\Rightarrow$ h[n] is absolutely summable $\Rightarrow$ system is stable.

② if the system is not causal, then

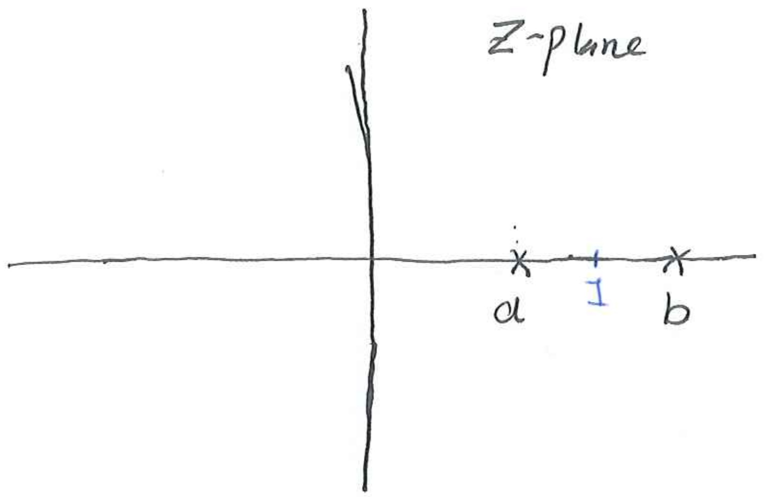
$$h[n] = -\left(\frac{1}{3}\right)^n u[-n-1] \qquad ROC: \ |z| < \frac{1}{3}$$

the unit circle is not included in the ROC

$\Rightarrow$ FT Does't exist.

$\Rightarrow$ h[n] is not absolutely summable $\Rightarrow$ Not stable.

Example: refer to example 3.8 in the textbook



Possible ROCs ?

$a < |z| < b$

* Two sided sequence.
* FT exists
* Non causal.

$|z| > b$

* Right sided sequence.
* FT does't exist
* Causal.

$|z| < a$

* Left sided sequence.
* FT does't exist
* Non causal

For the system to be causal and absolutely summable (FT exist) (stable system), all poles should be inside the unit circle and the sequence has to be right sided sequence.

3.3 The Inverse Z-transform

1. The inverse can be obtained by the following complex contour integral

$$x[n] = \frac{1}{2\pi} \oint_C X(z)\, z^{n-1}\, dz$$

C: closed contour within the ROC

To evaluate this integral, we need to learn the theory of complex variables.

2. Inspection Method (Tables)

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$-a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

$$\vdots$$

3. Partial Fraction Expansion (PFE)

usually, $X(z)$ is expressed as a ratio of polynomials in z^{-1}

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$X(z)$ can be expressed in the form:

$$X(z) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

c_k: non zero zeros

d_k: non zero poles of $X(z)$.

Using PFE

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-r}}_{\text{This term} = 0 \text{ if } M<N} + \sum_{\substack{k=1 \\ k\neq i}}^{N} \frac{A_k}{(1-d_k z^{-1})} + \underbrace{\sum_{m=1}^{S} \frac{C_m}{(1-d_i z^{-1})^m}}_{\text{This term} = 0 \text{ if there is no multiple order pole.}}$$

$$C_m = \frac{1}{(S-m)!\,(-d_i)^{S-m}} \left\{ \frac{d^{S-m}}{dw^{S-m}} \left[(1-d_i w)^S X(w^{-1}) \right] \right\}_{w=d_i^{-1}}$$

$$A_k = (1-d_k z^{-1})\, X(z) \Big|_{z=d_k}$$

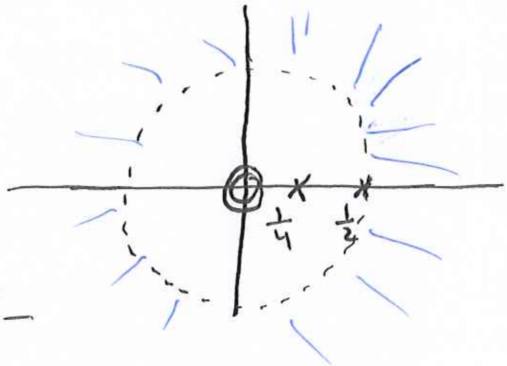
Example 3.9

$$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \qquad \text{ROC: } |z| > \frac{1}{2}$$

We have two poles : $z_1 = \frac{1}{4}$, $z_2 = \frac{1}{2}$

two zeros : $z = 0$

The pole-zero plot :



it is clear that $x[n]$ is a right-sided sequence.

poles are both 1^{st} order.

$$X(z) = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$x[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \right] u[n]$$

$$A_1 = \frac{1}{1-\frac{1}{2}z^{-1}} \Big|_{z^{-1}=4} = \frac{1}{1-\frac{1}{2}\times 4} = -1$$

$$A_2 = \frac{1}{1-\frac{1}{4}z^{-1}} \Big|_{z^{-1}=2} = \frac{1}{1-\frac{1}{4}\times 2} = 2$$

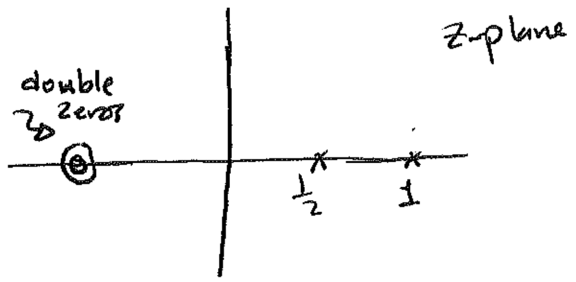
Example 3.10 Find $x[n]$.

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \qquad |z| > 1$$

Solution:-

$$X(z) = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}, \quad |z| > 1$$

The pole - zero plot



* From the ROC $|z| > 1$,
  It is clear that $x[n]$ is a right-sided sequence.
* since $M = N = 2$ and the poles are all 1^{st} order, $X(z)$ can be

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

B_0 can be obtained by long division

$$\begin{array}{r|l} & 2 \\ \hline \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \mp z^{-2} \pm 3z^{-1} \mp 2 \\ \hline & 5z^{-1} - 1 \end{array}$$

$$\Rightarrow X(z) = 2 + \frac{5z^{-1} - 1}{\frac{1}{2}z^{-1} - \frac{3}{2}z^{-1} + 1}$$

$$\frac{5z^{-1} - 1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} = \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - z^{-1})}$$

$$A_1 \Big|_{z^{-1} = 2} \quad 5 \times 2 - 1 = A_1(1 - 2) \Rightarrow A_1 = -9$$

$$A_2 \Big|_{z^{-1} = 1} \quad 5 \times 1 - 1 = A_2(1 - \tfrac{1}{2}) \Rightarrow A_2 = 8$$

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}} \quad , \; |z| > 1$$

$$x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8(1)^n u[n]$$

$$= 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8u[n]$$

Note:

When $X(z)$ is a rational function with high-degree polynomials in numerator and denominator, the computations to factor the denominator and compute the coefficients become much more difficult. In such cases, software tools such MATLAB is necessary.

$\boxed{4}$ Power Series Expansion

From the definition of z-transform, the sequence values $x[n]$ are the coefficients of z^{-n}. Thus, if $X(z)$ is given as a power series in the form

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} = \cdots\cdots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots\cdots$$

then $x[n] = \{\ldots x[-2], x[-1], x[0], x[1], x[2], \cdots\}$

This approach is useful for finite-length sequences.

Example 3.11

$$X(z) = z^2\left(1-\tfrac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right)$$

Solution :

$$\begin{aligned} X(z) &= \left(z^2 - \tfrac{1}{2}z\right)\left(1 - \cancel{z^{-1}} + \cancel{z^{-1}} - z^{-2}\right) \\ &= \left(z^2 - \tfrac{1}{2}z\right)\left(1 - z^{-2}\right) \\ &= z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1} \end{aligned}$$

hence, by inspection

$$x[n] = \left\{ 1, -\tfrac{1}{2}, \underset{\substack{\uparrow \\ n=0}}{-1}, \tfrac{1}{2} \right\}$$

or $\quad x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$

When $X(z)$ is the ratio of polynomials, it is sometimes useful to obtain a power series by long division of the polynomials.

Example 3.13 : $\quad X(z) = \dfrac{1}{1-az^{-1}} \qquad |z| > |a|$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} \\ \hline 1-az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & az^{-1} - a^2z^{-2} \\ \hline & a^2z^{-2} \\ & a^2z^{-2} - a^3z^{-3} \\ & \vdots \end{array}$$

$$\begin{aligned} \Rightarrow X(z) &= 1 + az^{-1} + a^2z^{-2} + \cdots \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \end{aligned}$$

$$\Rightarrow x[n] = a^n u[n]$$

since $\quad X(z) = \displaystyle\sum_{n=-\infty}^{\infty} x[n]\, z^{-1}$

If $|z| < |a|$, then

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 \\ \hline -az^{-1}+1 & 1 \\ & 1 - a^{-1}z \\ \hline & a^{-1}z \\ & a^{-1}z - a^{-2}z^2 \\ \hline & \vdots \end{array}$$

$$\Rightarrow X(z) = \sum_{-\infty}^{-1} -a^n z^{-n}$$

$$\Rightarrow x[n] = -a^n u[-n-1]$$

3.4 Z-Transform Properties

let

$$x[n] \xleftrightarrow{Z} X(z), \quad ROC = R_x$$
$$x_1[n] \longleftrightarrow X_1(z), \quad ROC = R_{x1}$$
$$x_2[n] \longleftrightarrow X_2(z), \quad ROC = R_{x2}$$

$\boxed{1}$ Linearity

$$a x_1[n] + b x_2[n] \xleftrightarrow{Z} a X_1(z) + b X_2(z)$$

$$ROC = R_{x1} \cap R_{x2}$$

$\boxed{2}$ Time shifting

$$x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z)$$

ROC $= R_x$ (except for the possible addition or deletion of $z=0$ or $z=\infty$)

proof:

if $y[n] = x[n-n_0]$

$$Y(z) = \sum_{-\infty}^{\infty} x[n-n_0] z^{-n}$$

let $m = n - n_0$

$$\Rightarrow Y(z) = \sum_{m=-\infty}^{\infty} x[m] z^{-m-n_0}$$
$$= z^{-n_0} \sum_{-\infty}^{\infty} x[m] z^{-m}$$
$$= z^{-n_0} X(z) \quad \#$$

Example 3.14 $\quad X(z) = \dfrac{1}{z - \frac{1}{4}}, \; |z| > \frac{1}{4}$, Find $x[n]$?

$\boxed{\text{Method I}}$ $X(z) = \dfrac{z^{-1}}{\cancel{1} - \frac{1}{4} z^{-1}} \Rightarrow \dfrac{X(z)}{z^{-1}} = \dfrac{1}{1 - \frac{1}{4} z^{-1}} = Y(z)$

$$X(z) = Y(z)\, z^{-1}$$

apply $Z^{-1}\{\ \}$:

$\Rightarrow x[n] = y[n-1]$, $\quad y[n] = \left(\frac{1}{4}\right)^n u[n]$

$\Rightarrow \boxed{x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]}$

$\boxed{\text{Method II}}$ $\quad X(z) = \dfrac{z^{-1}}{1-\frac{1}{4}z^{-1}} \qquad |z| > \frac{1}{4}$

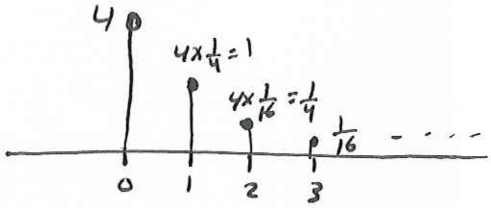
since $(M \geq N)$, we can apply long division:

$$\begin{array}{r|l} & -4 \\ \hline -\frac{1}{4}z^{-1}+1 & z^{-1} \\ & z^{-1}-4 \\ \hline & 4 \end{array}$$

$$X(z) = -4 + \frac{4}{1-\frac{1}{4}z^{-1}}$$

apply $z^{-1}\{\ \}$:

$$x[n] = -4\delta[n] + 4\left(\tfrac{1}{4}\right)^n u[n]$$



$4\delta[n] + \left(\frac{1}{4}\right)^{n-1} u[n-1]$

$$\Rightarrow x[n] = -4\delta[n] + 4\delta[n] + \left(\tfrac{1}{4}\right)^{n-1} u[n-1]$$
$$= \left(\tfrac{1}{4}\right)^{n-1} u[n-1]$$

$\boxed{3}$ Multiplication by an Exponential Sequence.

$$z_0^n x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right)$$

$ROC = |z_0| R_x$

shrinking or expansion of the z-plane. (ROC)

example: if $R_x: \quad r_R < |z| < r_L$

than $R_{z_0^n x[n]}: \ |z_0| r_R < |z| < |z_0| r_L$

proof:

$$\mathcal{Z}\{z_0^n x[n]\} = \sum x[n] z_0^n z^{-n}$$
$$= \sum x[n] \left(\frac{z}{z_0}\right)^{-n}$$
$$= X\left(\frac{z}{z_0}\right)$$

similar to

$$e^{jn\omega_0} x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

example 3.15 For $x[n] = r^n \cos(\omega_0 n)\, u[n]$ $\quad r > 0$

Find $X(z)$?

Solution ::

$$x[n] = r^n \left\{ \frac{e^{jnw_0} + e^{-jnw_0}}{2} \right\} u[n]$$

$$= \frac{1}{2}\left(re^{jw_0}\right)^n u[n] + \frac{1}{2}\left(re^{-jw_0}\right)^n u[n]$$

$$\Rightarrow X(z) = \frac{1/2}{1 - re^{jw_0} z^{-1}} + \frac{1/2}{1 - re^{-jw_0} z^{-1}}$$

ROC : $|z| > r$

Second Method

$$u[n] \longleftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

$$\frac{1}{2}(\underbrace{re^{jw_0}}_{z_0^n})^n u[n] \longleftrightarrow \frac{1}{1 - \left(\frac{z}{re^{jw_0}}\right)^{-1}} \qquad \text{ROC}: |z| > 1 * r$$

$$= \frac{1}{1 - \left(\frac{re^{jw_0}}{z}\right)} = \frac{1}{1 - re^{jw_0} z^{-1}}$$

also $\frac{1}{2}\left(re^{-jw_0}\right)^n u[n] = \frac{1}{2} z_0^n u[n] \longleftrightarrow \dfrac{1/2}{1 - re^{-jw_0} z^{-1}}$

Now $X(z) = \dfrac{1/2}{1 - re^{jw_0} z^{-1}} + \dfrac{1/2}{1 - re^{-jw_0} z^{-1}} = \dfrac{1 - r\cos w_0\, z^{-1}}{1 - 2r\cos(w_0) z^{-1} + r^2 z^{-1}}$

ROC: $|z| > r$

$\boxed{4}$ Differentiation of $X(z)$

$$n\,x[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} \qquad ROC = R_x$$

proof:

$$X(z) = \sum_{-\infty}^{\infty} x[n]\, z^{-n}$$

$$\frac{dX(z)}{dz} = \sum -n\,x[n]\, z^{-n-1}$$

mult. by z $\Rightarrow$ $$-z\frac{dX(z)}{dz} = \sum \{n x[n]\}\, z^{-n}$$

$$= \mathcal{Z}\{n x[n]\}$$

Example 3.16 $\quad X(z) = \log(1+az^{-1}) \qquad |z| > |a|$

Find $x[n]$?

$$\frac{d}{dz}X(z) = \frac{-az^{-2}}{(1+az^{-1})}$$

multiply by $(-z)$ both sides :

$$-z\frac{d}{dz}X(z) = \frac{az^{-1}}{(1+az^{-1})}$$

apply $\mathcal{Z}^{-1}\{\ \}$

$$n\,x[n] = a(-a)^{n-1}u[n-1]$$

$$x[n] = \frac{a}{n}a^{n-1}(-1)^{n-1}u[n-1]$$

$$= \frac{1}{n}a^n(-1)^{n-1}u[n-1]$$

$\left.\begin{matrix}(1)^{n-1}\\(1)^{n+1}\end{matrix}\right\}$ same

$$= \frac{a^n}{n}(-1)^{n+1}u[n-1]$$

$$\Rightarrow \quad (-1)^{n+1}\frac{a^n}{n}u[n-1] \longleftrightarrow \log(1+az^{-1}) \qquad |z|>|a|$$

Example 3.17 let $x[n] = n a^n u[n]$
Find $X(z)$.

Solution:

$$x[n] = n(a^n u[n])$$

$$X(z) = -z \frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) \qquad |z| > |a|$$

$$= -z \quad \frac{-(--az^{-2})}{(1-az^{-1})^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$\Rightarrow \; n a^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|.$$

$\boxed{5}$ Conjugate of a complex sequence

$$x^*[n] \overset{Z}{\longleftrightarrow} X^*(z^*) \qquad ROC = R_x$$

$\boxed{6}$ Time Reversal

$$x^*[-n] \longleftrightarrow X^*\left(\frac{1}{z^*}\right) \qquad ROC = \frac{1}{R_x}$$

if $R_{x[n]}$: $r_R < |z| < r_L$

the $R_{x^*[-n]}$: $\frac{1}{r_L} < |z| < \frac{1}{r_R}$

if $x[n]$ is real sequence, then

$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right) \qquad ROC = \frac{1}{R_x}$$

Example 3.18 let $x[n] = a^{-n} u[-n]$
Find $X(z)$.

solution

$x[n]$ is a time-reversed version of $a^n u[n]$

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

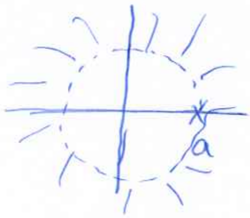
$$\Rightarrow X(z) = \frac{1}{1-a\left(\frac{1}{z}\right)^{-1}} \qquad |z| < \left|\frac{1}{a}\right|$$

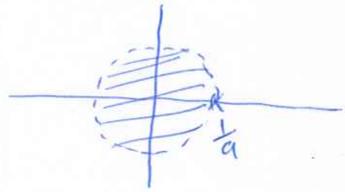
$$|z| < |a^{-1}|$$

$$= \frac{1}{1-az}$$

$$= \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

note that $a^n u[n]$ has a pole at $z = a$, while $X(z)$ has a pole at $z = 1/a$





$\boxed{7}$ Convolution of Sequences

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z) X_2(z)$$

ROC contains $R_{x_1} \cap R_{x_2}$

proof: $y[n] = \sum_k x_1[k] x_2[n-k]$

$$Y(z) = \sum_n y[n] z^{-n} = \sum_n \left\{ \sum_k x_1[k] x_2[n-k] \right\} z^{-n}$$

$$= \sum_k x_1[k] \sum_n x_2[n-k] z^{-n} \qquad \text{change of variable } m = n-k$$

$$= \sum_k x_1[k] \left\{ \sum x_2[m] z^{-m} \right\} z^{-k}$$

$$Y(z) = \sum_k x_1[k] \underbrace{X_2(z)}_{|z| \in R_{x_2}} z^{-k}$$

$$= X_2(z) \sum_k x_1[k] z^{-k}$$

$$\Rightarrow Y(z) = X_1(z) X_2(z)$$ ROC contains $R_{x_1} \cap R_{x_2}$

Example 3.19

$$x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$
$$x_2[n] = \delta[n] - \delta[n-1]$$

let $y[n] = x_1[n] * x_2[n]$
Find $y[n]$.

solution : .

$$Y(z) = X_1(z) X_2(z)$$

$$X_1(z) = 1 + 2z^{-1} + z^{-2}$$ R_{x_1} : all z plane except $z=0$ ($|z|>0$)

$$X_2(z) = 1 - z^{-1}$$ R_{x_2} : $|z| > 0$

$$Y(z) = (1 + 2z^{-1} + z^{-2})(1 - z^{-1})$$
$$= 1 + z^{-1} - z^{-2} - z^{-3}$$ ROC : $|z| > 0$

$$y[n] = \mathcal{Z}^{-1}\{Y(z)\}$$
$$= \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$$

$\boxed{\text{Table 3.2}}$ Page 161 in the book summarizes z-Transform properties.

Example 3.20 Convolution of infinite length sequences

Consider an LTI system with $h[n] = a^n u[n]$, $|a|<1$.
for $x[n] = A\,u[n]$

1. Find $y[n]$
2. Plot pole-zero diagram for $Y(z)$, specify the ROC

Solution:

$$H(z) = \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$X(z) = \frac{A}{1-z^{-1}} \qquad |z|>1$$

$$Y(z) = H(z)X(z)$$
$$= \frac{A}{(1-az^{-1})(1-z^{-1})} \qquad \text{ROC } |z|>1$$
$$= \frac{\alpha_1}{1-az^{-1}} + \frac{\alpha_2}{1-z^{-1}}$$

$$\alpha_1 = Y(z)(1-az^{-1})\Big|_{z^{-1}=a} = \frac{A}{1-z^{-1}}\Big|_{z^{-1}=a} = \frac{A}{1-\frac{1}{a}} = \frac{aA}{a-1} = \frac{-aA}{1-a}$$

$$\alpha_2 = (1-z^{-1})Y(z)\Big|_{z^{-1}=1} = \frac{A}{1-az^{-1}}\Big|_{z^{-1}=1} = \frac{A}{1-a}$$

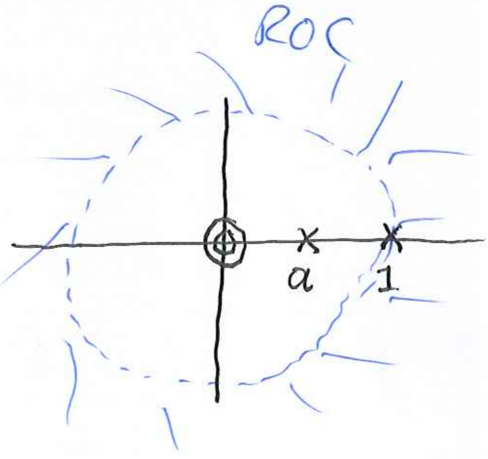
$$Y(z) = \frac{A}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \quad |z|>1$$

$$y[n] = \frac{A}{1-a}\left(u[n] - a\,a^n u[n]\right)$$

2. For pole-zero plot of $Y(z)$

$$Y(z) = \frac{Az^2}{(z-a)(z-1)} \qquad |z|>1$$

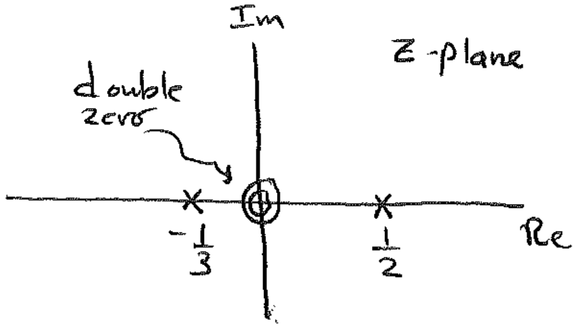
zeros at $z=0$ (Double)
poles at $z=a$, $z=1$



Extra Examples

Q. 1 The system function $H(z)$ of a causal LTI system has pole-zero plot shown below. If $H(z=1) = 3/4$

(a) Find $H(z)$.



$$H(z) = \frac{K z^2}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

$$H(1) = K / [\tfrac{1}{2} \cdot \tfrac{4}{3}] = \frac{K}{2/3} = \frac{3}{4} \Rightarrow K = \frac{1}{2}$$

since $H(z)$ is causal $\Rightarrow$ ROC : $|z| > \frac{1}{2}$

(b) Find $h[n]$.

$$H(z) = \frac{1/2}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} \qquad |z| > \frac{1}{2}$$

$$= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{3}z^{-1}}$$

$A\Big|_{z^{-1}=2}$ $\quad \frac{1}{2} = A(1 + \frac{1}{3}z^{-1}) + B(1 - \frac{1}{2}z^{-1}) \Rightarrow \frac{1}{2} = A(1 + \frac{2}{3})$

$\Rightarrow \frac{1}{2} = \frac{5}{3}A \Rightarrow A = \frac{3}{10}$

$B\Big|_{z^{-1}=3}$ $\quad \frac{1}{2} = B(1 - \frac{3}{2}) \Rightarrow \frac{1}{2} = -\frac{1}{2}B \Rightarrow B = -1$

$$\Rightarrow h[n] = 0.3 \left(\tfrac{1}{2}\right)^n u[n] + -1 \left(-\tfrac{1}{3}\right)^n u[n]$$

(c) Determine $y[n]$ when $x[n] = u[n] - \frac{1}{2}u[n-1]$

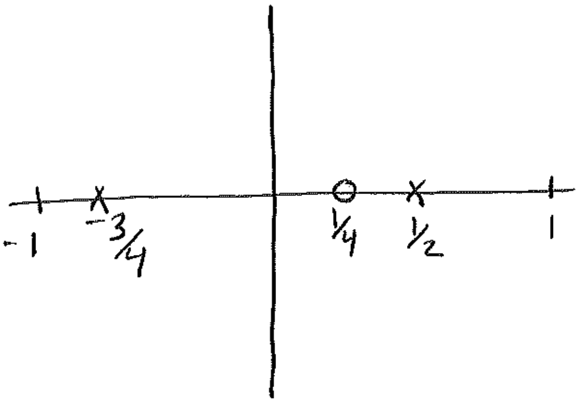
$$Y(z) = X(z)H(z) = \left(\frac{1}{1 - z^{-1}} - \frac{1/2\, z^{-1}}{1 - z^{-1}}\right) H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - z^{-1}} H(z)$$

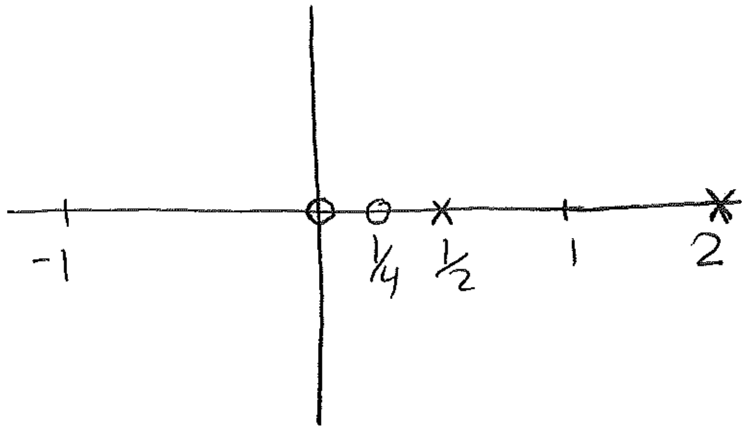
ROC : $|z| > 1$

$$Y(z) = \frac{1 - \frac{1}{2}z^{-1}}{(1 - z^{-1})} \cdot \frac{1/2}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}} + \frac{A_3}{1 + \frac{1}{3}z^{-1}}$$

$$y[n] = \mathcal{Z}^{-1}\{Y(z)\} \ \ldots\ldots$$

Q. 2 Consider an LTI system. Assume $y[n]$ is stable and its z-transform $Y(z)$ has a pole-zero diagram in the right figure, and $x[n]$ is also stable and its pole-zero diagram is shown in the left side figure.





(a) What is the ROC of $Y(z)$?

it is stable $\Rightarrow$ $\frac{1}{2} < |z| < 2$

(b) Is $y[n]$ left sided, right sided or two sided?

It is two sided.

(c) What is the ROC of $X(z)$?

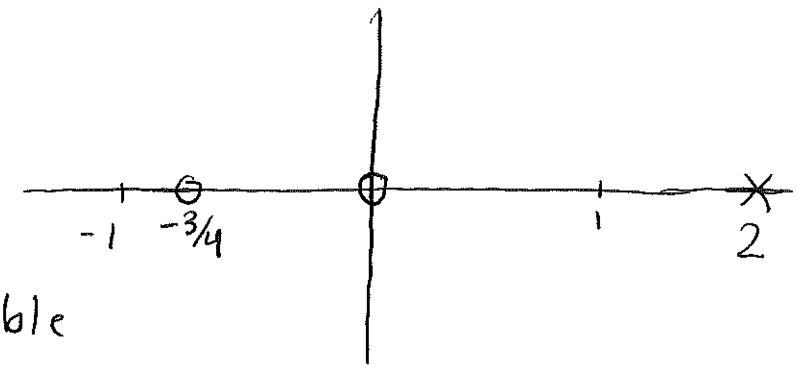
It is stable $\Rightarrow$ $|z| > 3/4$

(d) Is $x[n]$ a causal sequence?

Yes.

(e) Draw the pole-zero plot of $H(z)$ and specify its ROC. Find $H(z)$.

$$H(z) = \frac{k z (z + 3/4)}{z - 2}$$



ROC:

$|z| < 2$ $\Rightarrow$ it is stable

also R_y should include $R_x \cap R_h$

(f) Is $h[n]$ anticausal?

Yes. (left sided)

Q.3 consider an LTI system

let

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3} 2^n u[-n-1]$$

and the z-transform of $y[n]$ is

$$Y(z) = \frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}$$

(a) Find z-transform of $x[n]$

(b) Find $H(z)$, plot the zero-pole diagram and indicate the ROC.

(c) What is the ROC of $Y(z)$?

(d) Is the system stable?

Example (complex Roots)

$$H(z) = \frac{z+3}{(z+5)(z^2+4z+5)}$$

$z^2+4z+5=0$

$z_{1,2} = \frac{-4 \pm \sqrt{16-4\times1\times5}}{2.1}$

$= \frac{-4 \pm \sqrt{-4}}{2}$

$P_{1,2} = -2 \pm j$

complex conjugate

Method 1 (more compleeated)

using complex first order roots

$$H(z) = \frac{A}{z+5} + \frac{B}{(z-P_1)} + \frac{B^*}{(z-P_1^*)}$$

Method 2 using second order polynomial

$$H(z) = \frac{A}{z+5} + \frac{Bz+c}{z^2+4z+5}$$

$$A = (z+5)\,H(z)\Big|_{z=-5} = \frac{z+3}{z^2+4z+5}\Big|_{z=-5} = \boxed{-0.2 = A}$$

for B and c:

$A(z^2+4z+5) + (Bz+c)(z+5) = z+3$

$Az^2+4Az+5A + Bz^2+5Bz + cz + c.5 = z+3$

$(A+B)z^2 + (4A+5B+c)z + (5A+5c) = 0z^2+1.z+3$

$A+B=0 \Rightarrow \boxed{B=0.2}$

$5(A+c)=3 \Rightarrow c = 0.8$

$4A+5B+C=1 \Rightarrow$

$$H(z) = \frac{-0.2}{z+5} + \underbrace{\frac{0.2z+0.8}{z^2+4z+5}}_{\frac{C}{z-P_1} + \frac{C^*}{z-P_1^*}}$$

Find: C

Example Function (or exponential) in the numerator.

$$H(z) = \frac{z+3}{z^3+7z^2+10z} G(z), \quad \text{consider causal system}$$

let $G(z) = z^{-3}$

$\Rightarrow$ solve for

$$\frac{H(z)}{G(z)} = \frac{z+3}{z^3+7z^2+10z}$$

$$= \frac{z+3}{z(z+2)(z+5)}$$

$$= \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z+5}$$

$$= \frac{0.3}{z} + \frac{-1/6}{z+2} + \frac{-2/15}{z+5}$$

then

$$H(z) = \frac{0.3}{z} z^{-3} - \frac{1}{6} \cdot \frac{z^{-3}}{z+2} - \frac{2/15}{z+5} z^{-3}$$

$$= 0.3 z^{-4} - \frac{1}{6} \frac{z^{-4}}{1+2z^{-1}} - \frac{2/15}{1+5z^{-1}} z^{-4}$$

$$h[n] = 0.3\,\delta[n-4] - \frac{1}{6}(-2)^{n-4} u[n-4]$$

$$- \frac{2}{15}(-5)^{n-4} u[n-4].$$

Example (Repeated Roots)

$$H(z) = \frac{z+3}{z(z+2)^2(z+5)} \quad \text{, causal system.}$$

$$= \frac{A_1}{z} + \frac{B}{z+2} + \frac{C}{(z+2)^2} + \frac{D}{z+5}$$

$$A = z\,H(z)\Big|_{z=0} \Rightarrow A = \frac{3}{20}$$

$$C = (z+2)^2 H(z)\Big|_{z=-2} \Rightarrow \frac{z+3}{(z)(z+5)}\Bigg|_{z=-2} = \frac{\cancel{+}1}{(-2)(3)} = -\frac{1}{6}$$

$$D = (z+5)^2 H(z)\Big|_{z=-5} \Rightarrow \frac{z+3}{z(z+2)^2}\Bigg|_{z=-5} = \frac{+2}{45}$$

$$B = \frac{d}{dz}\left[(z+2)^2 H(z)\right]\Bigg|_{z=-2}$$

$$\frac{d}{dz}\left[\frac{z+3}{z(z+5)}\right] = \frac{d}{dz}\left[\frac{z+3}{z^2+5z}\right]$$

$$= \frac{(z^2+5z)(1) - (z+3)(2z+5)}{(z^2+5z)^2}\Bigg|_{z=-2}$$

$$= \frac{(4-10)-(1)(1)}{(4-10)^2}$$

$$= \frac{-7}{36}$$

$$H(z) = \frac{3/20}{z} + -\frac{7}{36}\,\frac{z^{-1}}{1+2z^{-1}} + -\frac{1}{6}\,\frac{z^{-2}}{(1+2z^{-1})^2} + \frac{2}{45}\,\frac{z^{-1}}{1+5z^{-1}}$$

$$h[n] = \frac{3}{20}\,\delta[n-1] - \frac{7}{36}(-2)^{n-1}u[n-1] - \frac{1}{6}(n-2)(-2)^{n-2}u[n-2]$$

$$+ \frac{2}{45}(-5)^{n-1}u[n-1]$$