7.5 Design of FIR Filters by Windowing

Starting with an ideal desired frequency response

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]\, e^{-j\omega n}$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad \text{impulse response.}$$

many ideal systems (filters) are defined by piecewise-constant frequency response with discontinuity between bands.

⇒ as a result, these systems have impulse responses that are noncausal and infinitely long.

The most straightforward approach to obtaining FIR approximation is to truncate the ideal impulse response through the process referred to as windowing.

A particularly simple way to obtain a causal FIR filter from $h_d[n]$ is to truncate $h_d[n]$, i.e.,

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

More generally, $h[n] = h_d[n] \cdot w[n]$

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{o.w} \end{cases}$$

$w[n]$ finite duration rectangular window

M : order of the system function polynomial
$(M+1)$: length of the impulse response of an FIR filter.

It follows from modulation (or windowing) theorem.

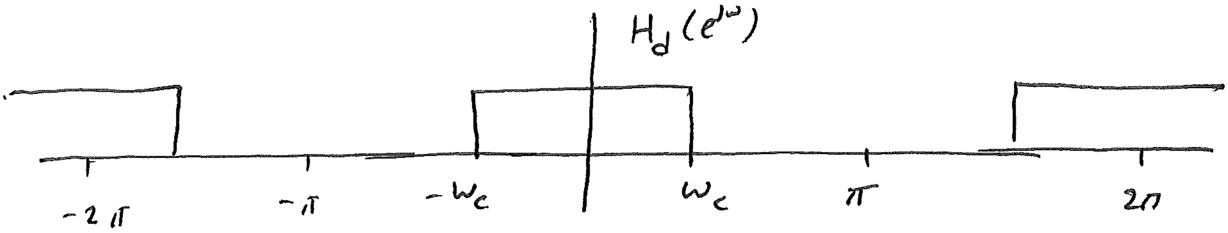
$$h[n] = h_d[n] . w[n] \iff H(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(w-\theta)})\, d\theta$$

where $H(e^{jw})$ is the periodic convolution.

---

let an ideal LPF of Bandwidth $w_c < \pi$ is given by

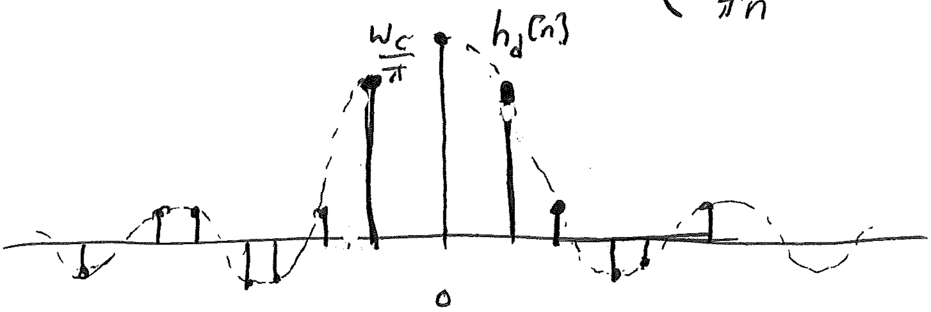
$$H_d(e^{jw}) = \begin{cases} 1 & |w| \leq w_c \\ 0 & w_c < |w| \leq \pi \end{cases}$$



then $h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{jw})\, e^{jwn}\, dw =$

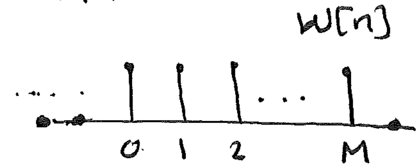
$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} 1 . e^{-jwn}\, dw = \begin{cases} \frac{w_c}{\pi} & n = 0 \\ \frac{\sin w_c n}{\pi n} & n \neq 0 \end{cases}$$

non causal
and infinite



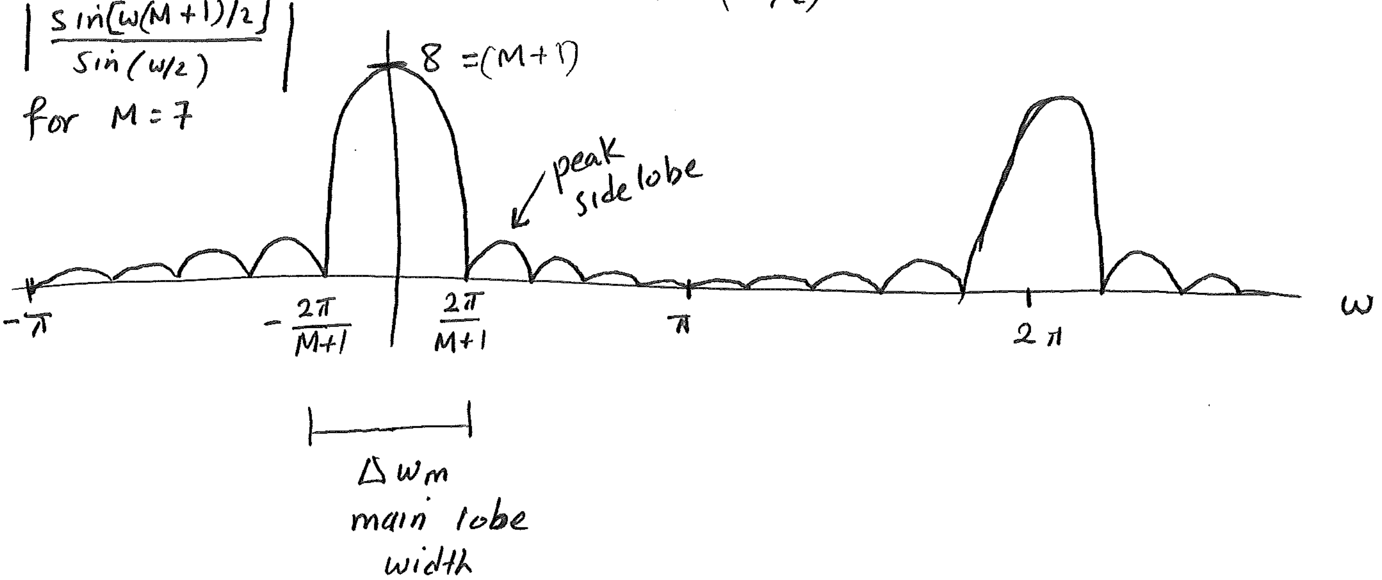
Now, consider a rectangular window of length M+1

$$w[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{o.w} \end{cases}$$



$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$= e^{-j\omega \frac{M}{2}} \frac{\sin(\omega[M+1]/2)}{\sin(\omega/2)}$$

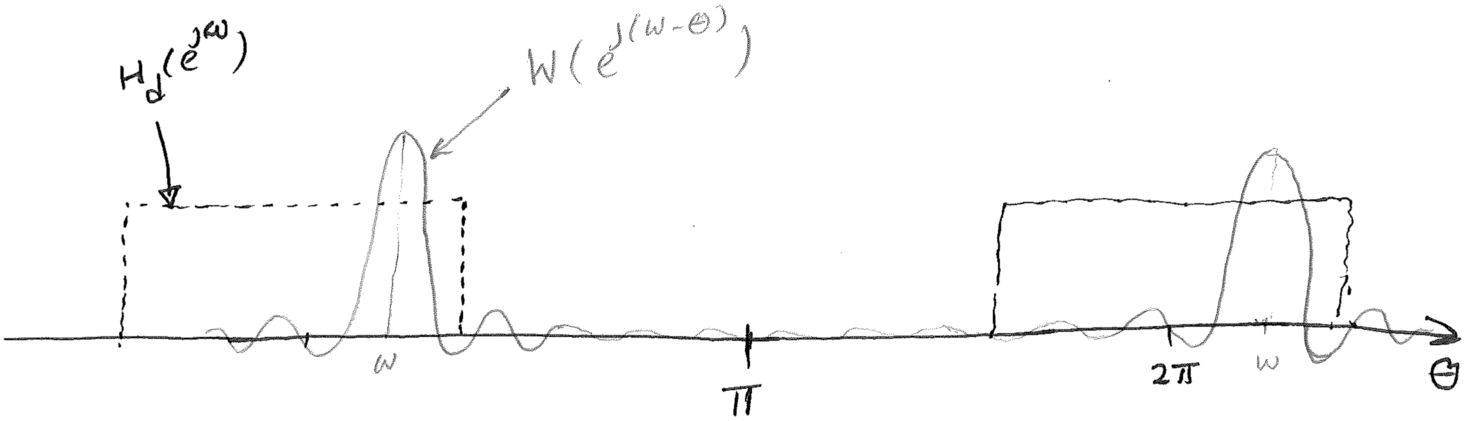


Notes:-

* As M increases, the width of the "main lobe" decreases. (the main lobe is the region between the first zero-crossings)
* For the rectangular window, the main lobe width is $\Delta\omega_m = \frac{4\pi}{M+1}$

* As M↑, the peak amplitudes of the main lobe and the side lobes grow in a manner such that the area under each lobe is a constant while the width of each lobe decreases with M.

Consequently, as M↑, the oscillations of $H(e^{j\omega})$ becomes more rapidly, but don't decrease in amplitude.



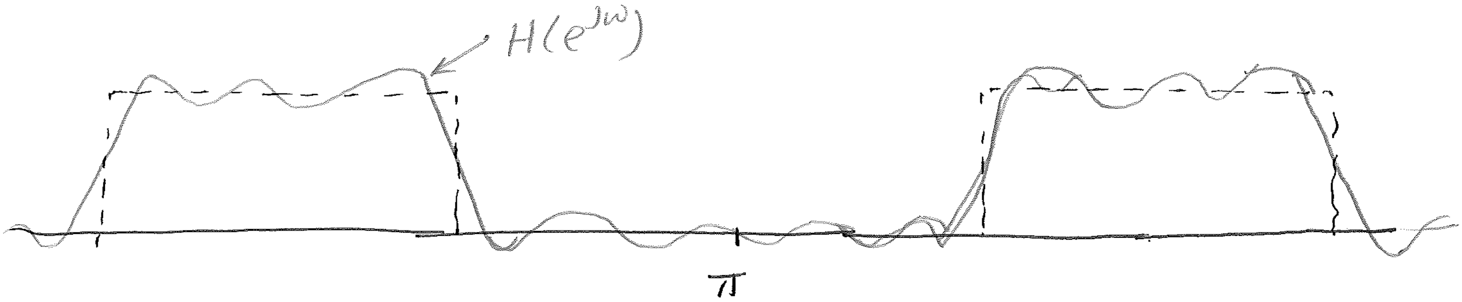
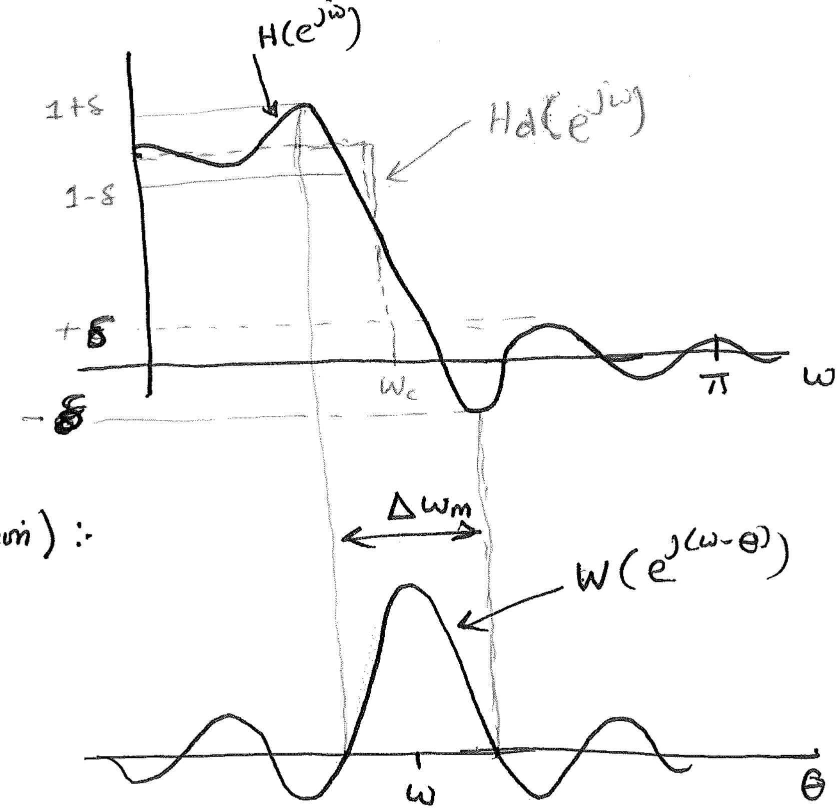


illustration of type of approximation obtained at discontinuity of the ideal frequency response.



* δ at the passband and the stopband is equal due to the symmetry.

* the peak approximation error is given by (Attenuation) :-

  $20 \log_{10} \delta$

* the distance between the peak ripples on either sides of discontinuity is approximately the main lobe width $\Delta\omega_m$.

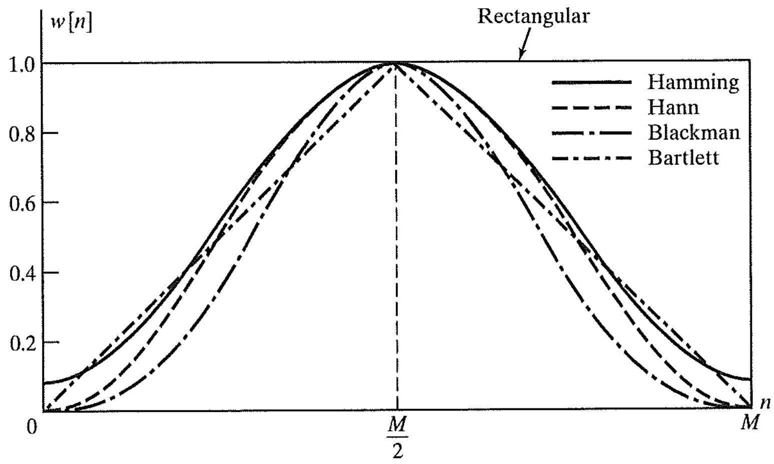


Figure 29 Commonly used windows.

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \ M \text{ even} \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60b}$$

Hann

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60c}$$

Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60d}$$

Blackman

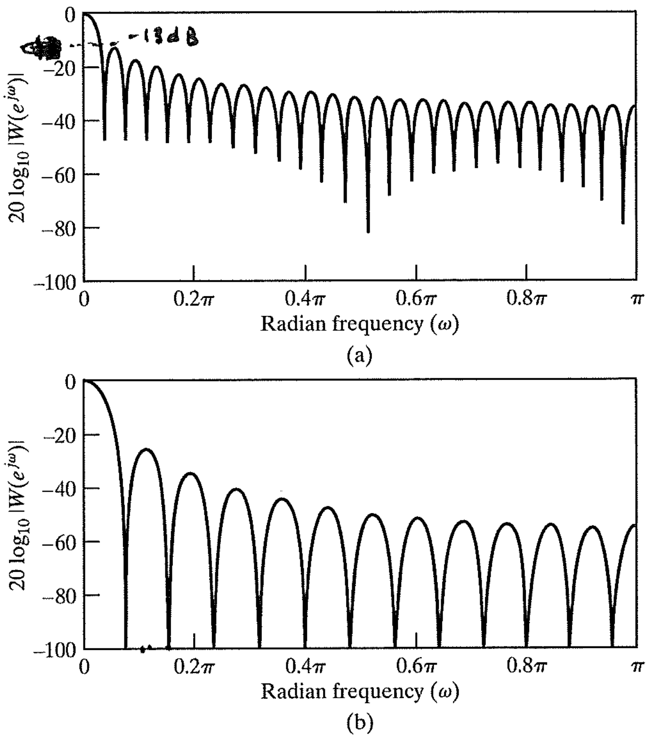
$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60e}$$

(For convenience, Figure 29 shows these windows plotted as functions of a continuous variable; however, as specified in Eq. (60), the window sequence is defined only at integer values of n.)

The Bartlett, Hann, Hamming, and Blackman windows are all named after their originators. The Hann window is associated with Julius von Hann, an Austrian meteorologist. The term "hanning" was used by Blackman and Tukey (1958) to describe the operation of applying this window to a signal and has since become the most widely used name for the window, with varying preferences for the choice of "Hanning" or "hanning." There is some slight variation in the definition of the Bartlett and Hann windows. As we have defined them, $w[0] = w[M] = 0$, so that it would be reasonable to assert that with this definition, the window length is really only $M - 1$ samples. Other definitions of the Bartlett and Hann windows are related to our definitions by a shift of one sample and redefinition of the window length.

The windows defined in Eq. (60) are commonly used for spectrum analysis as well as for FIR filter design. They have the desirable property that their Fourier transforms 1 are concentrated around $\omega = 0$, and they have a simple functional form that allows them 2 to be computed easily. The Fourier transform of the Bartlett window can be expressed as a product of Fourier transforms of rectangular windows, and the Fourier transforms of the other windows can be expressed as sums of frequency-shifted Fourier transforms of the rectangular window, as given by Eq. (59). (See Problem 43.)

The function $20 \log_{10} |W(e^{j\omega})|$ is plotted in Figure 30 for each of these windows with $M = 50$. The rectangular window clearly has the narrowest main lobe, and thus, for a given length, it should yield the sharpest transitions of $H(e^{j\omega})$ at a discontinuity of $H_d(e^{j\omega})$. However, the first side lobe is only about 13 dB below the main peak, resulting in oscillations of $H(e^{j\omega})$ of considerable size around discontinuities of $H_d(e^{j\omega})$. Table 2, which compares the windows of Eq. (60), shows that, by tapering the window smoothly to zero, as with the Bartlett, Hamming, Hann, and Blackman windows, the side lobes (second column) are greatly reduced in amplitude; however, the price paid is a much wider main lobe (third column) and thus wider transitions at discontinuities of $H_d(e^{j\omega})$. The other columns of Table 2 will be discussed later.



M = 50
narrowest main lobe
⇒ yield the sharpest transitions of $H(e^{j\omega})$ at the discontinuity of $H_d(e^{j\omega})$

Figure 30 Fourier transforms (log magnitude) of windows of Figure 29 with $M = 50$. (a) Rectangular. (b) Bartlett.

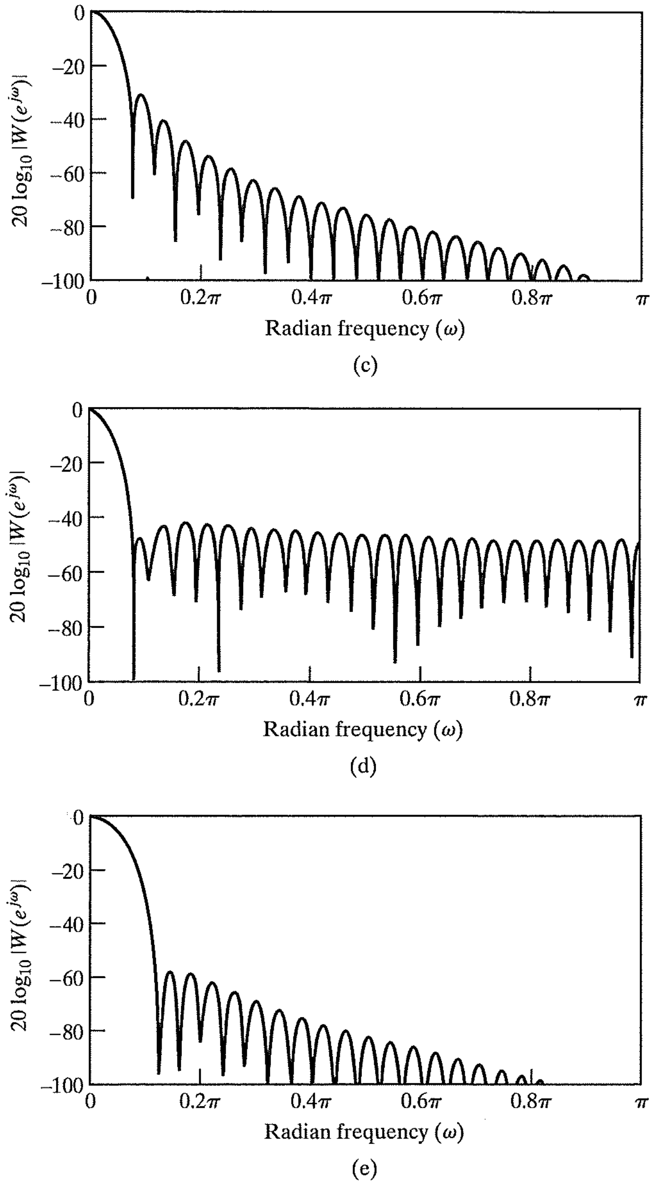


Figure 30 *(continued)* (c) Hann. (d) Hamming. (e) Blackman.

5.2 Incorporation of Generalized Linear Phase

In designing many types of FIR filters, it is desirable to obtain causal systems with a generalized linear-phase response. All the windows of Eq. (60) have been defined in anticipation of this need. Specifically, note that all the windows have the property that

$$w[n] = \begin{cases} w[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise;} \end{cases} \tag{61}$$

Table 2

Some times called Stopband attenuation (dB) (Maximum)

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20 \log_{10} \delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Linear phase system

* In designing many types of FIR filters, it is desirable to obtain causal systems with a generalized linear-phase response.

* All the windows (rectangular, triangular, Hann, ...) given in this chapter have been defined in anticipation of this need.
  specifically, note that all the windows have the property that

$$w[n] = \begin{cases} w[M-n] & 0 \leq n \leq M \\ 0 & \text{o.w.} \end{cases}$$

  i.e., they are symmetric about the point $\left(\frac{M}{2}\right)$

* A filter is said to have a generalized linear phase response if its frequency response can be expressed in the form

$$H(e^{j\omega}) = A(e^{j\omega})\, e^{-j(\alpha\omega - \beta)}$$

where α, β are constants
$A(e^{j\omega})$ is a real function of ω

if A is +ve, then the phase $\angle H(e^{j\omega}) = \beta - \alpha\omega$
if A is −ve, then " " $\angle H(e^{j\omega}) = \pi + \beta - \alpha\omega$

in either case, the phase is linear function of ω.

* it is common to restrict the filter to having a real-valued impulse response. $h[n]$, since this greatly simplifies the computational complexity in the implementation of the filter.

* A FIR system has linear phase if the impulse response satisfies either the even symmetric condition

$$h[n] = h[M-n]$$

or odd symmetric condition

$$h[n] = -h[M-n]$$

The system has different characteristic depending on whether M is odd or even. Saying this, we have four types of linear phase systems.

Example

consider having an odd number of samples in $h[n]$, and even symmetry. $M = 6$, then (length $= M+1 = 7$)
odd

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= h[0] + h[1] e^{-j\omega} + h[2] e^{-j2\omega} + h[3] e^{-j3\omega} + h[4] e^{-j4\omega} + h[5] e^{-j5\omega} + h[6] e^{-j6\omega}$$

$$= e^{-j3\omega}\left[h[0] e^{j3\omega} + h[1] e^{+j2\omega} + h[2] e^{+j\omega} + h[3] + h[4] e^{-j\omega} + h[5] e^{-j2\omega} + h[6] e^{-j3\omega} \right]$$

even symmetry $\Rightarrow$ $h[0] = h[6-0] = h[6]$

$h[1] = h[5]$, $h[2] = h[4]$

$$\Rightarrow H(e^{j\omega}) = e^{-j3\omega}\left(h[0]\left[e^{+j3\omega} + e^{-j3\omega} \right] + h[1]\left[e^{j2\omega} + e^{-j2\omega} \right] + \cdots \right)$$

$$= e^{-j3\omega}\left(2h[0] \cos 3\omega + 2h[1] \cos(2\omega) + 2h[2] \cos\omega + h[3] \right)$$

linear phase

$$= e^{-j3\omega} \sum_{n=0}^{3} a[n] \cos(\omega n)$$

with $a[0] = h[3]$

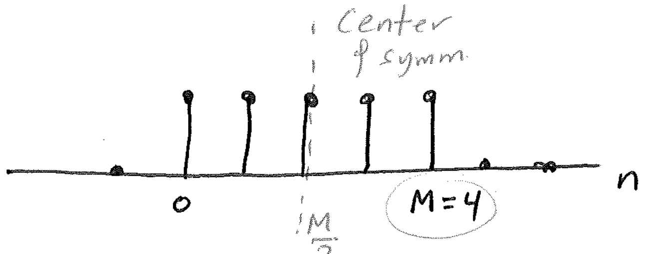
$a[n] = 2h[3-n]$, $n = 1,2,3$

refer to textbook page (357)

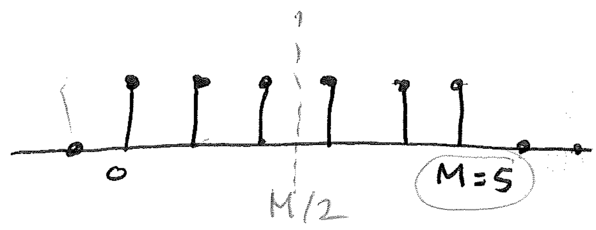
Based on M (even or odd) and the symmetry (even or odd) we have four types of linear phase systems.

Type	Symmetry	M	$H(e^{jw})$
I	Even	Even	$e^{-jw\frac{M}{2}}\left(\sum_{k=0}^{M/2} a[k]\cos wk\right)$
II	Even	odd	$e^{-jw\frac{M}{2}}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos(w-\frac{1}{2})\right\}$
III	odd	Even	$j\,e^{-jw\frac{M}{2}}\left\{\sum_{k=1}^{M/2} c[k]\sin(wk)\right\}$
IV	odd	odd	$j\,e^{-jw\frac{M}{2}}\left\{\sum_{k=1}^{(M+1)/2} d[k]\sin w(k-\frac{1}{2})\right\}$

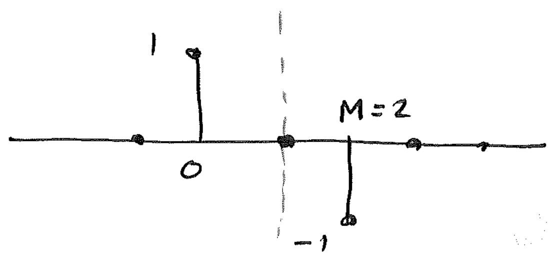
Example



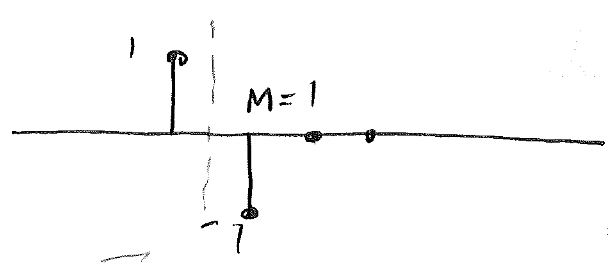
Type I , $h[n] = h[M-n]$



Type II



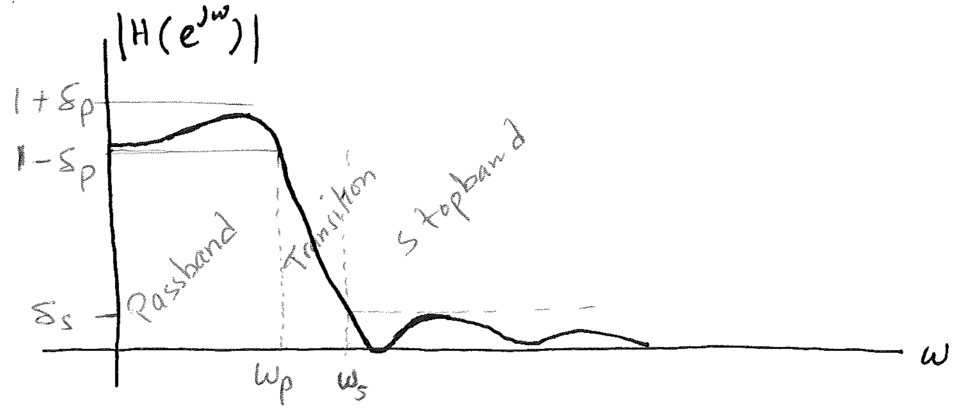
Type III , $h[n] = -h[M-n]$



Type IV

Example (Procedures)

1. First, the specification must be established.



* For window design, the resulting filter will have the same peak error $\delta = \delta_s = \delta_p$.

let $\omega_p = 0.4\pi$

$\omega_s = 0.6\pi$

stopband ripple: $\delta_s = 0.01$

passband ripple: $\delta_p = 0.05$

we choose $\delta = 0.01$ since windowing method inherently have $\delta_s = \delta_p = \delta$

2. Find the cutoff frequency.

Owing to the symmetry of the approximation at the the discontinuity of $H_d(e^{j\omega})$, we set

$$\omega_c = \frac{\omega_p + \omega_s}{2} = 0.5\pi$$

3. choose appropriate window

* peak approximation error (attenuation in stopband) $= 20\log_{10}\delta$

$$= 20\log_{10} 10^{-2} = -40\text{ dB}.$$

Hanning, Hamming, blackman, ... are all appropriate

we choose Hanning.

* Main lobe width $= \omega_s - \omega_p = 0.2\pi = \Delta\omega$

for Hanning $\Delta\omega = \dfrac{8\pi}{M}$

filter order $M = \dfrac{8\pi}{\Delta\omega} = \dfrac{8\pi}{0.2\pi} = 40$

filter order ≥ 40
filter length ≥ 41

$\boxed{4}$ design Linear - phase LPF

the desired frequency response is

generalized linear phase factor is incorporated.

$$H_d(e^{j\omega}) = H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

$$\Rightarrow h_d[n] = \frac{1}{2}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2}\, e^{j\omega n} = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}, \quad -\infty < n < \infty$$

note that, it is easily shown that $h_{lp}[M-n] = h_{lp}[n]$
so, if we use a symmetric window

$$\Rightarrow h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} \cdot w[n] = h_d[n] \cdot w[n]$$

where $w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M) & 0 \leq n \leq M \\ 0 & \text{o.w} \end{cases}$

Notes :

① since $M = 40$ is an Even integer, the resulting linear-phase system would be of Type I

② Observe that it is not necessary to plot either the phase or group delay, since we know the phase is precisely linear and the delay is $\frac{M}{2} = 20$ samples.

Example: Design a discrete-time LPF for a voice signal. The specification are

Passband $F_{pass} = 4$ KHZ with 0.8 dB ripple

Stopband $F_{stop} = 4.5$ KHZ with 50 dB attenuation

Sampling frequency $F_s = 22$ KHZ

(a) Determine the discrete-time Passband and Stop band frequencies

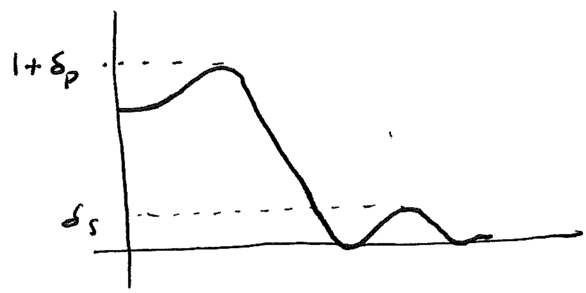
to map from analog to digital frequency

$$w = \frac{2\pi F}{F_s} = \Omega T$$

$$\Rightarrow w_p = \frac{2\pi * 4\,K}{22\,K} = 0.36\pi \text{ rad.}$$

$$w_s = \frac{2\pi * 4.5}{22} = 0.41\pi \text{ rad.}$$

(b) Determine the max. and min. values of $|H(e^{jw})|$ in the passband and stopband, where $H(e^{jw})$ is the filter frequency response.



0.8 ripple means that the freq. response in the passband is within the interval $1 \pm \delta_p$

$$\Rightarrow 20 \log(1+\delta) = 0.8$$

$$\Rightarrow \delta = 10^{(0.8/20)} - 1 = 0.096.$$

$\Rightarrow$ freq. response within passband $0.9035 < H(e^{jw}) < 1.096$

Similarly in the stopband, the max. value $H(e^{jw}) < 10^{-50/20}$ attenuation

< 0.0031

$\Rightarrow \omega_p = 0.36\pi \qquad \omega_{stop} = 0.41\pi$

$\Rightarrow$ Stopband attenuation $> 50\,dB$

$\Rightarrow$ We choose Hamming or Blackman.

If we consider Hamming $\Rightarrow$ main lobe width $= \frac{8\pi}{M}$

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & o.w \end{cases}$$

Minimum filter order: $M = \frac{8\pi}{\Delta\omega} = \frac{8\pi}{0.41\pi - 0.36\pi}$

$= 160$

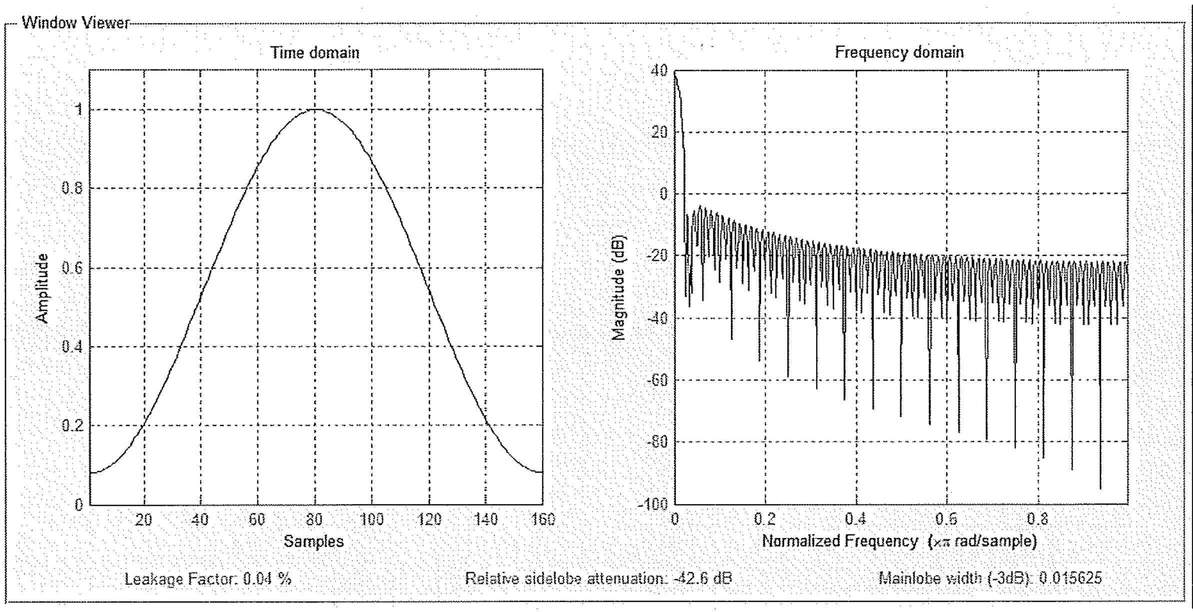
$h[n] = h_d[n]\, w[n]$

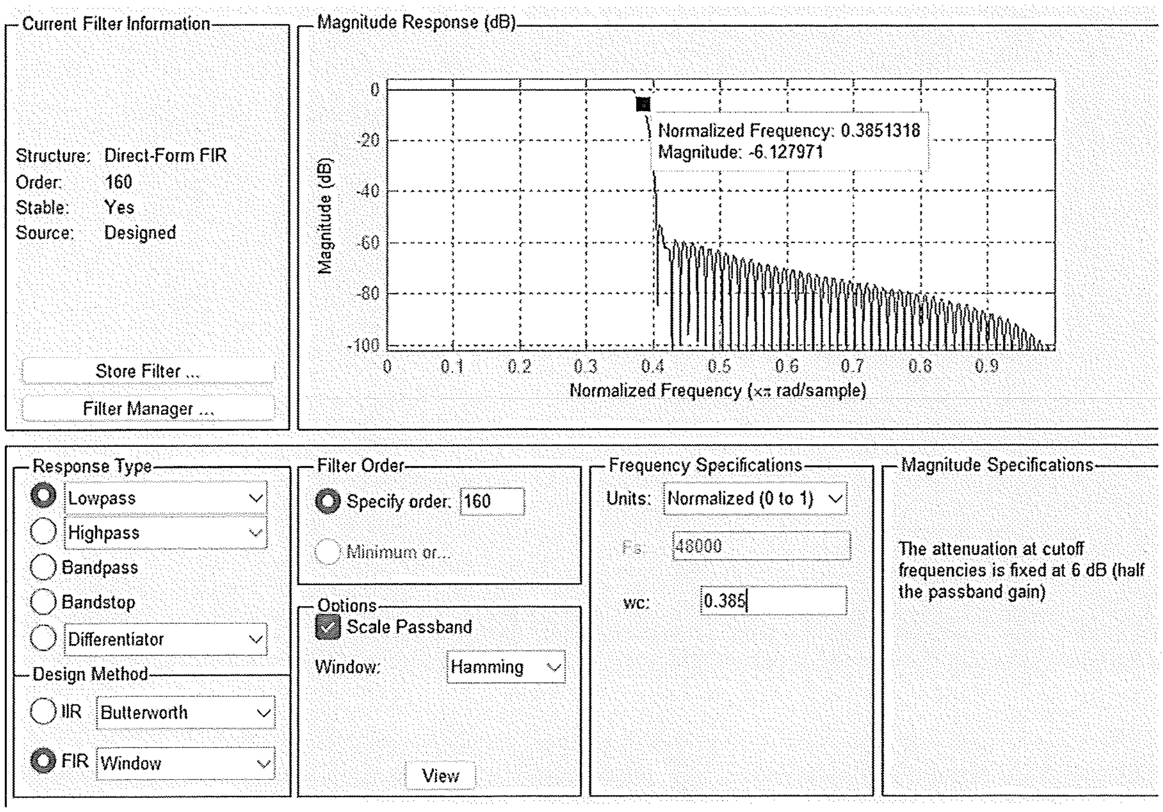
where $h_d[n] = \frac{\sin\left[\omega_c\left(n - \frac{M}{2}\right)\right]}{\pi\left(n - \frac{M}{2}\right)} \qquad -\infty < n < \infty$

* Since $h[n]$ is symmetrical, we only need to compute the values $h[0], h[1], \ldots, h[80]$, and then use the symmetry property to obtain other coefficients.

* ω_c is centred in transition band $[0.36\pi, 0.41\pi]$

$\Rightarrow \omega_c = \frac{\omega_s + \omega_p}{2} = \frac{(0.36 + 0.41)\pi}{2} = 0.385\pi.$





Filter coefficients: **h[0]=** 0.000186994356484 **h[1]=** 0.000312120135200 **h[2]=**0.000031250989472 **h[3]** =-0.000308654140016 … …. ..**h[81]=** 0.384787022651450 … . … **h[161]=** 0.000186994356484

Example: Design a decimator that down samples an input signal $x[n]$ by a factor $(M=2)$. Use Windowing method to determine the coefficients of the FIR filter that is down by at least 50 dB in the stop band. Let the transition bandwidth $\Delta w = 0.1\,\pi$.

Solution:

$$x[n] \xrightarrow{} \boxed{h[n]} \xrightarrow{} \boxed{2\downarrow} \xrightarrow{}$$

F_s $\quad$ F_s $\quad$ $F_s' = \frac{F_s}{2}$

The cutoff freq. for the LPF is $F_c = \frac{F_s'}{2} = \frac{F_s}{2M}$

$$w_c = \Omega_c T = 2\pi \frac{F_c}{F_s} = \frac{2\pi \, F_s/2M}{F_s}$$

$$\Rightarrow w_c = \frac{\pi}{M} = \frac{\pi}{2}$$

Since the attenuation $\lesssim 50$ dB, we use Hamming window.

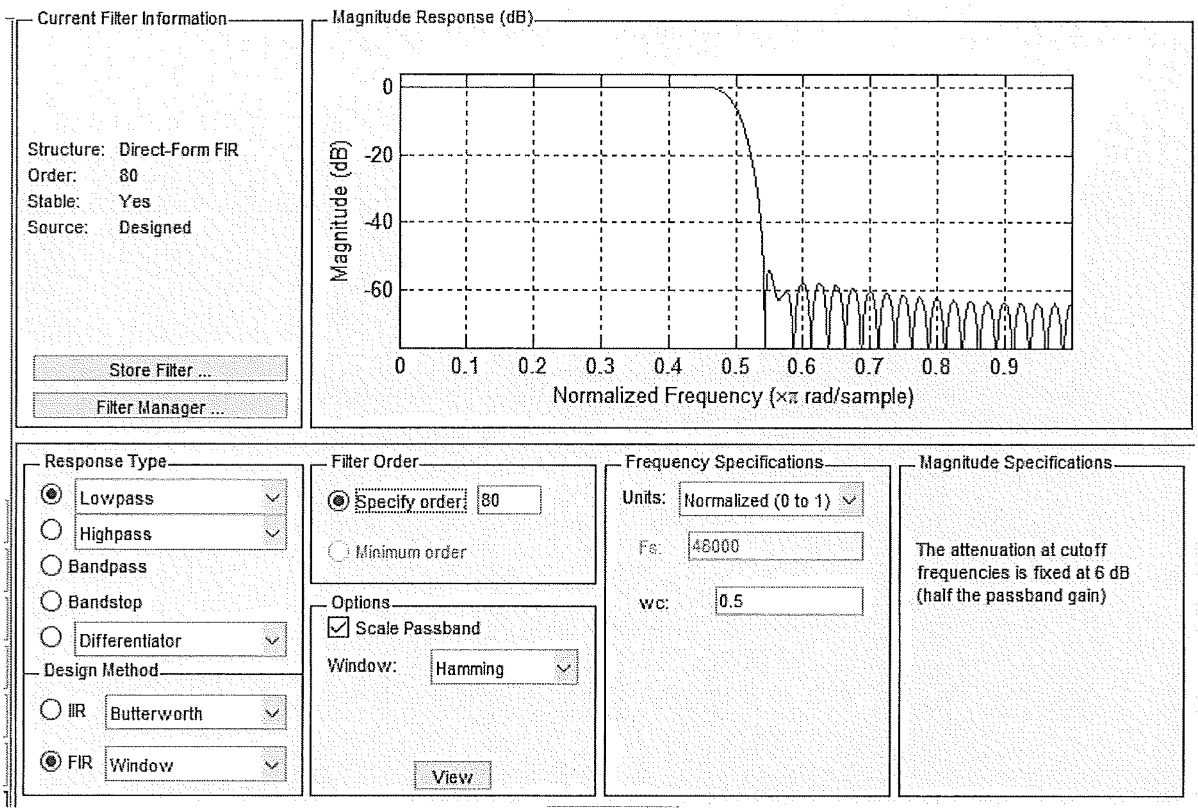
$$\Rightarrow \text{Min. Filter order}: \frac{8\pi}{M} = \Delta w$$

$$\Rightarrow M = \frac{8\pi}{\Delta w} = \frac{8\pi}{0.1\pi} = 80$$

$$h[n] = h_d[n]\, w[n]$$

$$h_d[n] = \frac{\sin[w_c(n-\frac{M}{2})]}{\pi(n-M/2)} = \frac{\sin[\frac{\pi}{2}(n-40)]}{\pi(n-40)}, \quad -\infty < n < \infty$$

$$w[n] = \begin{cases} 0.54 - 0.46\cos\frac{2\pi n}{M} & 0 \le n \le M \\ 0 & \text{ow} \end{cases}$$


Current Filter Information
Structure: Direct-Form FIR
Order: 80
Stable: Yes
Source: Designed
Store Filter ...
Filter Manager ...
Magnitude Response (dB)
0
-20
-40
-60
Magnitude (dB)
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
Normalized Frequency (×π rad/sample)
Response Type
Lowpass
Highpass
Bandpass
Bandstop
Differentiator
Design Method
IIR Butterworth
FIR Window
Filter Order
Specify order: 80
Minimum order
Options
Scale Passband
Window: Hamming
View
Frequency Specifications
Units: Normalized (0 to 1)
Fs: 48000
wc: 0.5
Magnitude Specifications
The attenuation at cutoff frequencies is fixed at 6 dB (half the passband gain)