2.6 Frequency-domain representation of discrete-time signals and Systems.

* sinusoidal and complex sequences play a particularly important role in representing discrete-time signals.
* Complex exponential sequences are eigenfunctions of LTI systems, and the response to a sinusoidal input is sinusoidal with the same frequency as the input and with amplitude and phase determined by the system.

* Frequency Response of the system $H(e^{jw})$

$$\text{input } x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n] = x[n] * h[n]$$

impulse $\delta[n]$ $\qquad y[n] = h[n]$ impulse Response

eigen function e^{jwn}, $-\infty < n < \infty$ $\qquad y[n] = H(e^{jw})\, e^{jwn}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] \\ &= \sum_{k} e^{jw(n-k)}\, h[k] \\ &= e^{jwn} \sum_{k=-\infty}^{\infty} e^{-jwk}\, h[k] \quad = e^{jwn}\, H(e^{jw}) \end{aligned}$$

e^{jwn} : eigen function of the system (since the same signal appears at the output)

$$\boxed{H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-jwk}}$$: eigen value of the system called Frequency Response.

* In general, $H(e^{jw})$ is complex and can be expressed in terms of its real and imaginary parts as:

$$H(e^{jw}) = H_R(e^{jw}) + j\,H_I(e^{jw})$$

OR in terms of magnitude and phase:

$$H(e^{jw}) = |H(e^{jw})|\, e^{j\angle H(e^{jw})}$$

<u>Example 2.14</u> Frequency Response of the ideal delay system.

$y[n] = x[n-n_d]$ $\quad$ n_d : fixed integer

* Let $x[n] = e^{jnw} \Rightarrow y[n] = e^{jw(n-n_d)} = e^{jwn}\, e^{-jwn_d}$

$$= e^{jwn}\, H(e^{jw})$$

$\Rightarrow$ Frequency Response $H(e^{jw}) = e^{-jwn_d}$

$$= \cos(wn_d) - j\sin(wn_d)$$

$$= H_R + j\,H_I$$

* Another way to obtain $H(e^{jw})$ from the impulse response

$h[n] = \delta[n-n_d]$

$$H(e^{jw}) = \sum_{-\infty}^{\infty} h[n]\, e^{-jwn} = \sum \delta[n-n_d]\, e^{-jwn}$$

$$= \sum \delta[n-n_d]\, e^{-jwn_d} = e^{-jwn_d} \sum_n \delta[n-n_d]$$

$$= e^{-jwn_d}$$, note that $\sum_n \delta[n-n_d] = \begin{cases} 1 & n = n_d \\ 0 & \text{else} \end{cases}$

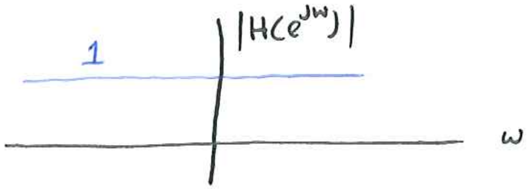
* in terms of magnitude and phase $|H(e^{jw})| = 1$, Mag. Response

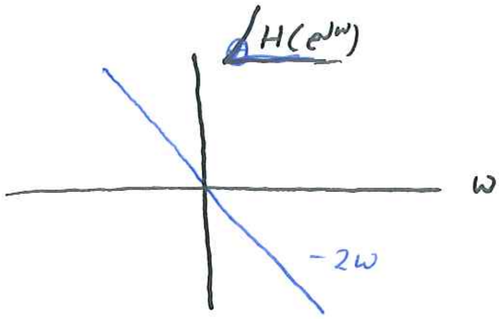
$\angle H(e^{jw}) = -wn_d$, Phase s.

Example : let $h[n] = \delta[n-2]$

if $x[n] = 3\cos\left(\frac{\pi}{10}n + \frac{\pi}{6}\right)$, find $y[n]$?

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-jwn} = e^{-jw.2} = 1 \angle -2w$$





$$\Rightarrow y[n] = 3\,|H(e^{jw_0})| \cos\left(\frac{\pi}{10}n + \frac{\pi}{6} + \angle H(e^{jw_0})\right)$$

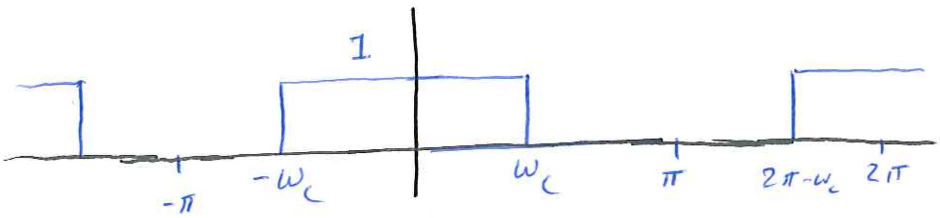
$$= 3.1.\cos\left(\frac{\pi}{10}n + \frac{\pi}{6} - 2 * \frac{\pi}{10}\right)$$

$$= 3\cos\left(\frac{\pi}{10}n + \frac{\pi}{6} - \frac{\pi}{5}\right)$$

$$= 3\cos\left(\frac{\pi}{10}[n-2] + \frac{\pi}{6}\right)$$

Example : let $|H(e^{jw})| = \begin{cases} 1 & |w| < w_c \\ 0 & \text{else} \end{cases}$

be a low pass Filter



then $y[n] = \begin{cases} A.1.\cos(nw_0 + \phi + \angle H(e^{jw_0})) & |w_0| < w_c \\ 0 & |w_0| > w_c \end{cases}$

* It can be shown that a broad class of signals can be represented as a linear combination of complex exponentials in the form

$$x[n] = \sum_k \alpha_k e^{jw_k n}$$

From the principle of superposition and $y[n] = H(e^{jw}) e^{jwn}$, the corresponding output of an LTI system is

$$y[n] = \sum_k \alpha_k H(e^{jw_k}) e^{jw_k n}$$

Example: 2.15 Sinusoidal Response of LTI system

$$x[n] = A\cos(w_0 n + \phi) \longrightarrow \boxed{H(e^{jw})} \longrightarrow y[n] = ??$$

$$x_1[n] = \frac{A}{2} e^{j\phi} e^{jw_0 n} \qquad\qquad y_1[n] = H(e^{jw_0}) \frac{A}{2} e^{j\phi} e^{jw_0 n}$$

$$x_2[n] = \frac{A}{2} e^{-j\phi} e^{-jw_0 n} \qquad\qquad y_2[n] = H(e^{-jw_0}) \frac{A}{2} e^{-j\phi} e^{-jw_0 n}$$

$$\Rightarrow y[n] = \frac{A}{2}\left[H(e^{jw_0}) e^{j\phi} e^{jw_0 n} + H(e^{-jw_0}) e^{-j\phi} e^{-jw_0 n} \right]$$

Note: if $h[n]$ is real, it can be shown that $H(e^{-jw_0}) = H^*(e^{jw_0})$

$$\begin{cases} |H(e^{-jw_0})| = |H(e^{jw_0})| & \text{Even symmetry} \\ \angle H(e^{-jw_0}) = \angle - H(e^{jw_0}) & \text{odd symmetry} \end{cases}$$

$$\Rightarrow y[n] = A\,|H(e^{jw_0})| \cos(w_0 n + \phi + \theta) \quad , \quad \theta = \angle H(e^{jw_0})$$

Discrete $\longleftrightarrow$ Periodic

Notes:-

* The frequency response of discrete-time LTI systems is always a periodic function of the frequency variable w with period 2π.

$$H(e^{j(w+2\pi)}) = \sum_n h[n]\, e^{-j(w+2\pi)n} = \sum h[n]\, e^{-jwn}\, e^{-j2\pi n}$$

$e^{\mp j2\pi n} = 1$ for n : integer

$$\Rightarrow H(e^{j(w+2\pi)}) = H(e^{jw}) \quad \text{for all } w.$$

more generally, $H(e^{j(w+2\pi r)}) = H(e^{jw})$ for r an integer

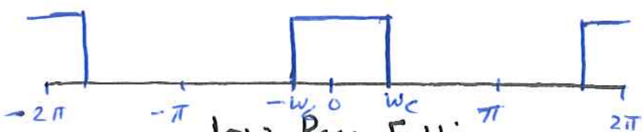
* we have shown earlier that e^{jwn} is indistinguishable from the sequence

$$e^{j(w+2\pi)n} \qquad -\infty < n < \infty$$

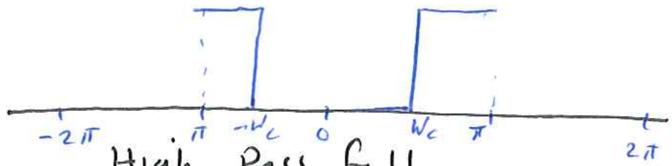
$\Rightarrow$ it follows that we need only specify $H(e^{jw})$ over an interval of length 2π, e.g., $0 \leq w \leq 2\pi$ or $-\pi < w \leq \pi$.

* Low frequencies are close to zero, $\pm 2\pi$, 4π, ..
* high " " close to $\pm\pi$, $\pm 3\pi$, ...

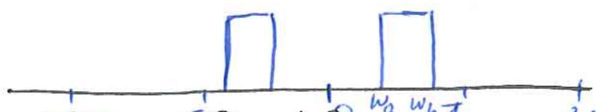
Ideal Frequency selective filters



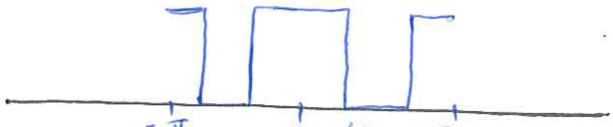
low Pass Filter



High Pass filter.



Band Pass



Band stop.

Example 2.16 Freq. Response of the Moving - Average System

$$y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$$

the impulse response $h[n] = \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \le n \le M_2 \\ 0 & \text{otherwise} \end{cases}$

the freq. response $H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$

$$= \frac{1}{M_1+M_2+1} \sum_{-M_1}^{M_2} e^{-jwn}$$

For the causal moving average system, $M_1 = 0$:

$$H(e^{jw}) = \frac{1}{M_2+1} \sum_{n=0}^{M_2} e^{-jwn}$$

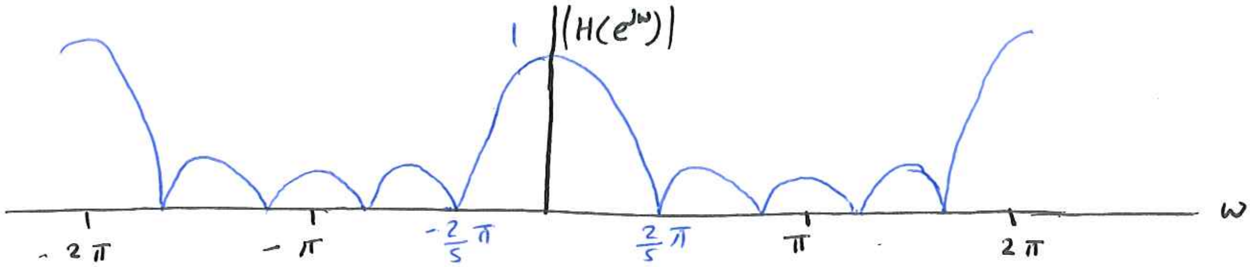
$$\boxed{\sum_{k=0}^{K} \alpha^k = \frac{1-\alpha^{K+1}}{1-\alpha} \quad \text{Geometric Series}}$$

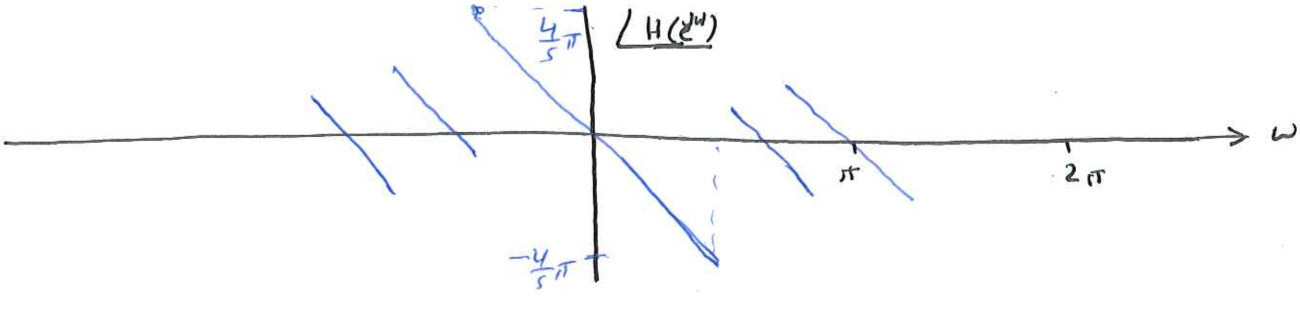
$$= \frac{1}{M_2+1} \frac{1-e^{-jw(M_2+1)}}{1-e^{-jw}}$$

$$= \frac{1}{M_2+1} \frac{\left[e^{+jw(\frac{M_2+1}{2})} - e^{-jw(\frac{M_2+1}{2})}\right]}{\left[e^{jw/2} - e^{-jw/2}\right]} \frac{e^{-jw(\frac{M_2+1}{2})}}{e^{-jw/2}}$$

$$= \frac{1}{M_2+1} \frac{\sin[w(M_2+1)/2]}{\sin(w/2)} e^{-jwM_2/2}$$

let $M_2 = 4 \Rightarrow |H(e^{jw})| = \frac{1}{5} \frac{\sin(\frac{5}{2}w)}{\sin(\frac{w}{2})}$ $\quad \angle H(e^{jw}) = -2w$





* Note that $|H(e^{jw})|$ is periodic and continuous
* the system response falls off at "high frequencies"

$\Rightarrow$ the system behaves as a low-pass filter.

* consider the previous example:

① Evaluate $y[n]$ when $x_1[n] = 15\cos\left(\frac{4}{10}\pi n + \frac{\pi}{4}\right)$

at $w_0 = \frac{4}{10}\pi$, $|H(e^{jw_0})| = 0 \Rightarrow y_1[n] = 0$

② $x_2[n] = 15\cos\left(\frac{2}{10}\pi n + \frac{\pi}{4}\right)$

at $w_0 = \frac{2}{10}\pi$, $|H(e^{jw_0})| \approx 0.65$

$\underline{\angle H(e^{jw_0})} = -2 * w_0 = -\frac{4}{10}\pi$

$\Rightarrow y_2[n] = 15 * 0.65 * \cos\left(\frac{2}{10}\pi n + \frac{\pi}{4} - \frac{2}{5}\pi\right)$

③ $x_3[n] = 15\cos\left(\frac{11}{5}\pi n + \frac{\pi}{4}\right)$

$= 15\cos\left(\left(\frac{2}{10}\pi + 2\pi\right)n + \frac{\pi}{4}\right)$

$= 15\cos\left(\frac{2}{10}\pi n + \frac{\pi}{4}\right)$

$\Rightarrow y_3[n] = y_2[n]$

2.7 Fourier Transform (Discrete-time Fourier Transform DTFT)

Many sequences can be represented by a Fourier integral of the form:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw \qquad \text{inverse FT (synthesis)}$$

where

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

w: continuous variable

n: discrete variable

* $x[n]$ can be represented as a superposition of infinitesimally small complex sinusoids of the form $\frac{1}{2\pi} X(e^{jw})\, e^{jwn}\, dw$ with $w \in [0, 2\pi]$.
  $X(e^{jw})$ determines the relative amount of each complex sinusoidal component.

* $X(e^{jw})$ is periodic with period 2π and continuous.

* if $x[n]$ is absolutely summable, then $X(e^{jw})$ exists

$$|X(e^{jw})| \le \sum_n |x[n]|\,|e^{-jwn}|$$
$$\le \sum_n |x[n]| < \infty \qquad \text{convergence condition}$$

<u>Example 2.17</u> let $x[n] = a^n u[n]$, Find the FT of $x[n] = ?$

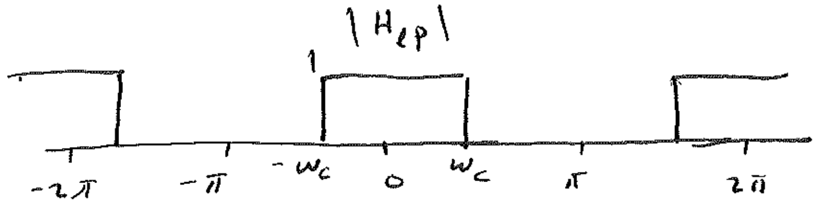
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = \sum_{n=0}^{\infty} a^n e^{-jwn} = \sum_{n=0}^{\infty} (a e^{-jw})^n$$

note that: $\sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|}$ if $|\alpha| < 1$

$$\Rightarrow X(e^{jw}) = \frac{1}{1 - a e^{-jw}} \qquad \text{if } |a e^{-jw}| < 1 \text{ or } |a| < 1$$

Example 2.18   Ideal Low Pass Filter

let $H_{lp}(e^{jw}) = \begin{cases} 1 & |w| < w_c \\ 0 & w_c \le |w| \le \pi \end{cases}$



then $$h_{lp}[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1 \, e^{jwn}\, dw$$

$$= \frac{1}{2\pi \cdot jn}\left[e^{jw_c n} - e^{-jw_c n} \right]$$

$$= \frac{\sin w_c n}{\pi n}, \quad -\infty < n < \infty$$

note that, $h_{lp}[n]$ is non causal ($h[n] \neq 0$ for $n < 0$)

$\Rightarrow$ LPF (ideal) is non causal.

Symmetry Properties :-

if $x[n]$ is a real sequence, then the Fourier Transform is conjugate Symmetric

i.e, $X(e^{jw}) = X^*(e^{-jw})$

* In Rectangular Form $X_R(e^{jw}) + j X_I(e^{jw}) = X_R(e^{-jw}) - j X_I(e^{-jw})$

$\Rightarrow X_R(e^{jw}) = X_R(e^{-jw})$ Even symm. and $X_I(e^{jw}) = -X_I(e^{-jw})$ odd symm.

* In Polar Form : $|X(e^{jw})| e^{j\angle H(e^{jw})} = |X(e^{-jw})| e^{-j\angle H(e^{-jw})}$

$\Rightarrow |X(e^{jw})| = |X(e^{-jw})|$ Even symm.

$\angle H(e^{jw}) = \angle -H(e^{-jw})$ odd symm.

2.9 Fourier Transform Theorems (Properties)

Notation :
$$X(e^{jw}) = \mathcal{F}\{x[n]\}$$
$$x[n] = \mathcal{F}^{-1}\{X(e^{jw})\}$$
$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

$\boxed{1}$ Linearity
$$a\,x_1[n] + b\,x_2[n] \xleftrightarrow{\mathcal{F}} a\,X_1(e^{jw}) + b\,X_2(e^{jw})$$

$\boxed{2}$ Time shifting $\quad x[n-n_d] \longleftrightarrow e^{-jwn_d}\,X(e^{jw})$

Frequency shifting $\quad e^{jw_0 n}\,x[n] \longleftrightarrow X(e^{j(w-w_0)})$

$\boxed{3}$ Time Reversal $\quad x[-n] \longleftrightarrow X(e^{-jw})$

conjugate

if $x[n]$ is real, then $\quad X(e^{-jw}) = X^*(e^{jw})$
$$\Rightarrow \quad x[-n] \longleftrightarrow X^*(e^{jw})$$

$\boxed{4}$ Differentiation in Frequency : $\quad n\,x[n] \longleftrightarrow j\,\dfrac{d\,X(e^{jw})}{dw}$

$\boxed{5}$ Parseval's Theorem :
$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw})|^2\,dw$$

* $|X(e^{jw})|^2$ is called the "Energy Density spectrum", since it determines how the energy is distributed in the frequency domain.
* Energy Density spectrum is defined only for finite-energy signals.

6 The Convolution Theorem

if

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= x[n] * h[n]$$

Convolution

then,

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

Multiplication

7 The Modulation or Windowing Theorem

if

$$x[n] \longrightarrow \underset{\substack{\uparrow \\ w[n]}}{\otimes} \longrightarrow y[n] = x[n]\, w[n]$$

then,

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

is called a periodic convolution (convolution of two periodic functions with the limits of integration extending over one period).

Duality for Discrete-Time sequence :-

* Discrete-time convolution of sequences (convolution sum) is equivalent to multiplication of corresponding periodic ~~Fourier transfor~~ transforms.
* Multiplication of sequences is equivalent to periodic convolution of the corresponding Fourier transforms.

Example 2.22 Determine The DTFT for $x[n] = a^n u[n-5]$

Solution:: from the table

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}} \qquad |a|<1$$

$$x[n] = a^5 a^{n-5} u[n-5]$$

$$X(e^{j\omega}) = a^5 \mathcal{F}\{a^{n-5} u[n-5]\}$$

$$= a^5 \frac{1}{1-ae^{-j\omega}} e^{-j\omega.5}$$

Example 2.23 Suppose $X(e^{j\omega}) = \dfrac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$

using Partial Fraction Expansion, we can expand $X(e^{j\omega})$ into

$$X(e^{j\omega}) = \frac{\alpha}{1-ae^{-j\omega}} + \frac{\beta}{1-be^{-j\omega}} = \frac{\alpha(1-be^{-j\omega})+\beta(1-ae^{-j\omega})}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

$$1 = \alpha(1-be^{-j\omega}) + \beta(1-ae^{-j\omega})$$

$$\alpha\Big|_{e^{-j\omega}=\frac{1}{a}} \Rightarrow 1 = \alpha\left(1-b.\frac{1}{a}\right) \Rightarrow \alpha = \frac{a}{a-b}$$

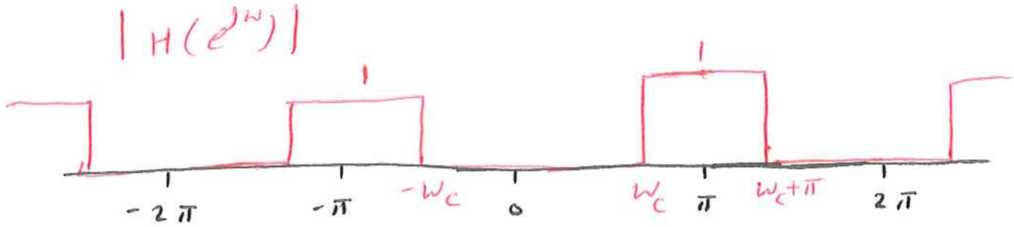
$$\beta\Big|_{e^{-j\omega}=\frac{1}{b}} \Rightarrow 1 = \beta\left(1-a.\frac{1}{b}\right) \Rightarrow \beta = \frac{-b}{a-b}$$

$$\Rightarrow X(e^{j\omega}) = \frac{a/(a-b)}{1-ae^{-j\omega}} - \frac{b/(a-b)}{1-be^{-j\omega}}$$

$$x[n] = \frac{1}{a-b}\left[a\, a^n u[n] - b\, b^n u[n]\right] = \frac{1}{a-b}\left[a^{n+1}u[n] - b^{n+1}u[n]\right]$$

Example 2.24 let

$$H(e^{jw}) = \begin{cases} e^{-jwn_d} & w_c < |w| < \pi \\ 0 & |w| < w_c \end{cases}$$



* this is a highpass filter with a linear phase.
* this frequency response can be expressed as

$$H(e^{jw}) = e^{-jwn_d}(1 - H_{lp}(e^{jw}))$$

$$= e^{-jwn_d} - e^{-jwn_d} H_{lp}(e^{jw})$$

where $H_{lp}(e^{jw})$ is periodic with period 2π

$$H_{lp}(e^{jw}) = \begin{cases} 1 & |w| < w_c \\ 0 & w_c < |w| < \pi \end{cases}$$ Low Pass Filter

from example (2.18) we have shown that $h_{lp}[n] = \dfrac{\sin w_c n}{\pi n}$

using
$$\delta[n-n_d] \longleftrightarrow e^{-jwn_d}$$
$$x[n-n_d] \longleftrightarrow X(e^{jw})\, e^{-jwn_d}$$

then
$$h[n] = \delta[n-n_d] - \frac{\sin w_c (n-n_d)}{\pi (n-n_d)}$$

Example 2.25

let $y[n] - \frac{1}{2} y[n-1] = x[n] - \frac{1}{4} x[n-1]$

be 1st order difference equation, Find the impulse response.

solution

set $x[n] = \delta[n]$

then,

$$h[n] - \frac{1}{2} h[n-1] = \delta[n] - \frac{1}{4} \delta[n-1]$$

apply $\mathcal{F}\{\}$

$$H(e^{jw}) - \frac{1}{2} H(e^{jw}) e^{-jw} = 1 - \frac{1}{4} e^{-jw}$$

$$H(e^{jw}) = \frac{1 - \frac{1}{4} e^{-jw}}{1 - \frac{1}{2} e^{-jw}}$$

$$h[n] = \mathcal{F}\{H(e^{jw})\}$$

$$= \frac{1}{1 - \frac{1}{2} e^{-jw}} - \frac{1/4}{1 - \frac{1}{2} e^{-jw}} e^{-jw}$$

from the pairs (table) and time shift property

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1 - \frac{1}{2} e^{-jw}}$$

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

* In ch. 3, we will see that the z-transform is more useful than the FT for dealing with difference equations.