№ 4.

$$p(\xi) = \frac{\theta}{2} \quad [(-1;1)/\{0\}] + \frac{1-\theta}{2} [0] + \frac{1-\theta}{2} [2]$$

a) 1. Омм   2. Омп

в) несмещ, сост – ?

с) эффективность через Крамера-Рао:

a) ОММ:

$$\alpha_1 = M\xi = \int_{-1}^{1} x p\, dx = \int_{-1}^{1} x\theta\, dx + 0 + 2(0{,}5-\theta) = 1 - 2\theta$$

$$\tilde{\alpha}_1 = \bar{x}$$

$$1 - 2\theta = \bar{x} \Rightarrow \hat{\theta}_1 = \frac{1-\bar{x}}{2}$$

ОМП:

$$L(\theta) = \prod_{i=1}^{n} p(x,\theta) = \theta^{n-m} \left(\frac{1}{2} - \theta\right)^m$$

$$\ln L(\theta) = (n-m)\ln\theta + m\ln\left(\frac{1}{2} - \theta\right)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{n-m}{\theta} + \frac{-m}{\frac{1}{2}-\theta} = \frac{n-m-2n\theta}{\theta - 2\theta^2} = 0 \to \tilde{\theta}_2 = \frac{1-p}{2}$$

в) Несм: $M[\hat{\theta}_1] = \frac{1}{2} M[1-\bar{x}] = \frac{1}{2} - \frac{1}{2}\sum M\xi = \frac{1}{2} - \frac{1}{2} + \theta = \theta \to$ несм

Сост: $D[\hat{\theta}_1] = \frac{1}{4} D[1-\bar{x}] = \frac{1}{4}\sum D\xi =$

$$\alpha_2 = \int_{-1}^{1} x^2 p\, dx = \int_{-1}^{1} x^2\theta\, dx + 2 - \theta = 2 - \frac{10}{3}\theta$$

$$\alpha_2 - \alpha_1^2 = \frac{2}{3}\theta - 4\theta^2 + 1$$

$$= \frac{\theta}{6n} - \frac{\theta^2}{n} + \frac{1}{4n} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост.}$$

Несм: $M[\hat{\theta}_2] = \frac{1}{2} M[1-p] = \frac{1}{2} - \frac{1}{2} M[p] = \frac{1}{2} - \frac{p}{2} = \theta \to$ несм

Сост: $D[\hat{\theta}_2] = \frac{1}{4} D[p] = \frac{\theta(1-2\theta)}{2n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост.

c) $D[\hat{\theta}_1] = \frac{\theta}{6n} - \frac{\theta^2}{n} + \frac{1}{4n}$

$\frac{\theta}{6n} - \frac{\theta^2}{n} + \frac{1}{4n} \geqslant \frac{\theta(1-2\theta)}{2n}$ непонятно.
про эффективность $\hat{\theta}_1$

$D[\hat{\tilde{\theta}}_2] = \frac{\theta(1-2\theta)}{2n} \leq \frac{\theta(1-2\theta)}{2n}$ $\Rightarrow \hat{\tilde{\theta}}_2$ эффективна.

№ 5

$\xi \sim R(\theta; 2\theta) \longrightarrow F(x,\theta) = \frac{x}{\theta} - 1 \quad (\theta; 2\theta) \qquad p(x;\theta) = \frac{1}{\theta}, \quad (\theta; 2\theta)$

$M\xi = \frac{3}{2}\theta \qquad D\xi = \frac{\theta^2}{12}$

a) 1. ОММ 2. ОМП

b) Несмещ., состоят. – ?, исправить

c) эффективность оценок?

d) построить точный интервал для θ.

e) построить асимпт. довер. интервал для θ.

a. ОММ

$$\alpha_1 = M\xi = \int_{\theta}^{2\theta} \frac{x}{\theta}\,dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = 2\theta - \frac{\theta}{2} = \frac{3}{2}\theta$$

$$\tilde{\alpha}_1 = \overline{x}$$

$$\frac{3}{2}\theta = \overline{x} \Rightarrow \boxed{\begin{aligned} \tilde{\theta}_1 &= \frac{2}{3}\overline{x} \\ \tilde{\theta}_1 &= \frac{2}{3n} x_i \end{aligned}}$$

ОМП

$$L(\theta) = \prod_{i=1}^{n} p(x,\theta) = \left(\frac{1}{\theta}\right)^n = \theta^{-n}$$

ф-я правдоподобия

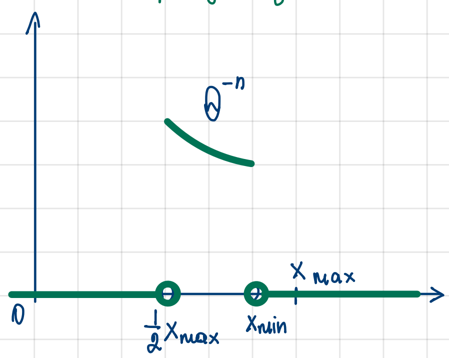
$(\theta \le x_i \le 2\theta)$

$\downarrow$

$\begin{pmatrix} x_{max} \le 2\theta \\ x_{min} \ge \theta \end{pmatrix} \quad \longrightarrow max$

$\downarrow$

$(\frac{1}{2}x_{max} \le \theta \le x_{min})$



$$\Rightarrow \tilde{\theta}_2 = \frac{1}{2}x_{max} \le x_{min}$$

b) Несм: $M[\tilde{\theta}_1] = M\left[\frac{2}{3n}\sum_{i=1}^{n} x_i\right] = \frac{2n}{3n}\sum M\xi = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta \longrightarrow$ несмещ.

Сост: $D[\tilde{\theta}_1] = D[\frac{2}{3n}\sum_{i=1}^{n} x_i] = \frac{4}{9n^2}\sum D\xi = \frac{4}{9n^2}\cdot n \cdot \frac{\theta^2}{12} \longrightarrow$

$\rightarrow D[\tilde{\theta}_1] = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0 \quad + \quad \theta_1$ – несмещ. $\Rightarrow \theta_1$ – сост.

Несм: $M[\tilde{\theta}_2] = M\left[\frac{1}{2} x_{max}\right] = \left\{ \varphi_{max}(x) = n(F(x))^{n-1} \cdot F'(x) \right\} =$

$= \frac{1}{2} \int\limits_{\theta}^{2\theta} xn\left(\frac{x}{\theta} - 1\right)^{n-1} \cdot \frac{1}{\theta} dx = \frac{2n+1}{2n+2} \cdot \theta \neq \theta \longrightarrow \tilde{\theta}_2$ – несмещ.

Можно исправить: $\tilde{\theta}_2' = \frac{n+1}{2n+1} x_{max}$

Сост: $D[\tilde{\theta}_2'] = \frac{(n+1)^2}{(2n+1)^2} D[x_{max}] = \left(\frac{n+1}{2n+1}\right)^2 \left(M x_{max}^2 - M^2 x_{max}\right) =$

$M x_{max}^2 = \int\limits_{\theta}^{2\theta} x^2 n\left(\frac{x}{\theta} - 1\right)^{n-1} \cdot \frac{1}{\theta} dx = \frac{4n^2 + 8n + 2}{n^2 + 3n + 2} \cdot \theta^2$

$M x_{max} = \left(\frac{2n+1}{n+1}\right)\theta \longrightarrow M^2 x_{max} = \frac{4n^2 + 1 + 4n}{n^2 + 1 + 2n}$

$= \left(\frac{n+1}{2n+1}\right)^2 \left(\frac{4n^2 + 8n + 2}{n^2 + 3n + 2} - \frac{4n^2 + 1 + 4n}{n^2 + 1 + 2n}\right)\theta^2 = \frac{(n+1)^2}{(2n+1)^2} \cdot \frac{n}{n^3 + 4n^2 + 5n + 2} \cdot \theta^2 =$

$= \frac{n\theta^2}{4n^3 + 12n^2 + 9n + 2} \Longrightarrow D[\tilde{\theta}_2'] = \frac{n\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n \to \infty]{} 0 \Longrightarrow \tilde{\theta}_2'$ – сост.

c) $D\tilde{\theta}_1 = \frac{\theta^2}{27n} \sim \frac{1}{27n}$

$D\tilde{\theta}_2' = \frac{n\theta^2}{(2n+1)^2(n+2)} \sim \frac{1}{4n^2}$

$\tilde{\theta}_2'$ – наиб. эффект.

d) $f(\vec{x}_n; \theta) \sim g(t)$

$\frac{x_{max}}{\theta} = y \sim \hat{F}(y)$

$\hat{F}(y) = P(x_{max} < \theta y) = \left\{ F(x) = \frac{x}{\theta} - 1 \right\} = (F(\theta y))^n = (y-1)^n \qquad y \in (1;2)$

β – доверит. вер-сть $\qquad \hat{\varphi}(y) = n(y-1)^{n-1}$

$t_1 = y_{\frac{1-\beta}{2}} \longrightarrow \int\limits_1^{t} n(y-1)^{n-1} dy = \frac{1-\beta}{2} \longrightarrow (y-1)^n \Big|_1^{t_1} = \frac{1-\beta}{2}$

$\left\{ t_1 = \sqrt[n]{\frac{1-\beta}{2}} + 1 \right\}$

$t_2 = y_{\frac{1+\beta}{2}} \longrightarrow \hat{F}(t_2) = \frac{1+\beta}{2} \longrightarrow t_2 = \sqrt[n]{\frac{1+\beta}{2}} + 1$

$\sqrt[n]{\frac{1-\beta}{2}} + 1 < \frac{x_{max}}{\theta} < \sqrt[n]{\frac{1+\beta}{2}} + 1$

$\frac{x_{max}}{\sqrt{\frac{1+\beta}{2}} + 1} < \theta < \frac{x_{max}}{\sqrt{\frac{1-\beta}{2}} + 1}$

e) $\dfrac{f(\tilde{\alpha}) - f(\alpha)}{\sigma(\tilde{\alpha})}\sqrt{n} \sim N(0;1)$ $\qquad \sigma(\tilde{\alpha}) = \sqrt{\nabla f^T K \nabla f}$

1. $\tilde{\theta} = f(\tilde{\alpha}) = \frac{2}{3}\bar{x}$ $\qquad \theta = f(\alpha) = \frac{2}{3}\alpha_1$

2. $\nabla f = \frac{2}{3}$

$$\left.\begin{aligned} \alpha_1 &= \bar{x} \\ \alpha_2 &= \bar{x}_2 \end{aligned}\right\} \longrightarrow K = \overline{x^2} - \bar{x}^2$$

$$\Rightarrow \sigma(\tilde{\alpha}) = \frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2} \Rightarrow \frac{\tilde{\theta} - \theta}{\frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2}} \sim N(0;1) = \frac{1}{\sqrt{2\pi}\cdot 1}\exp\left(-\frac{1}{2}x^2\right)$$

$$u_{\frac{1+\beta}{2}} = -u_{\frac{1-\beta}{2}} \Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u_{\frac{1-\beta}{2}}} e^{-\frac{x^2}{2}}dx = \frac{1-\beta}{2} \Rightarrow$$

$$\Rightarrow \frac{1}{2}\left(\operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + 1\right) = \frac{1-\beta}{2} \Longrightarrow \operatorname{erf}\left(\frac{u_{\frac{1-\beta}{2}}}{\sqrt{2}}\right) = -\beta \Rightarrow u_{\frac{1-\beta}{2}} = \sqrt{2}\operatorname{erf}^{-1}(-\beta) \Rightarrow$$

$$\beta = 0{,}95 \Rightarrow u_{\frac{1-\beta}{2}} \simeq -1{,}96 \qquad u_{\frac{1+\beta}{2}} \simeq 1{,}96$$

$$\Rightarrow -1{,}96 < \frac{\tilde{\theta} - \theta}{\frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2}}\sqrt{n} < 1{,}96 \Rightarrow \tilde{\theta} = \frac{2}{3}\bar{x} \Rightarrow$$

$$\Rightarrow -2{,}94\sqrt{\frac{\overline{x^2} - \bar{x}^2}{n}} + \frac{2}{3}\bar{x} < \theta < 2{,}94\sqrt{\frac{\overline{x^2} - \bar{x}^2}{n}} + \frac{2}{3}\bar{x}$$

N 6.

$$p(x) = \begin{cases} \dfrac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases} \qquad \theta > 1$$

$$p(x) = \frac{\theta - 1}{x^{\theta}} \qquad (1; +\infty) \qquad \theta > 1.$$

$$F(x) = 1 - \frac{1}{x^{\theta - 1}} \qquad (1; +\infty) \qquad \theta > 1$$

$$F^{-1}(y) = (1-y)^{-\frac{1}{\theta - 1}} \qquad y \in (0; 1)$$

$\vec{x_n}$ – выборка

a) Найти оценку парам θ ОМП

b) Доверит. инт. для медианы

c) Асимпт. доверит. инт. для θ

a) $$L(\theta) = \prod_{i=1}^{n} \left(\frac{\theta - 1}{x_i^{\theta}} \right) = (\theta - 1)^n \cdot \prod_{i=1}^{n} \frac{1}{x_i} \longrightarrow \max$$

$$\ln L(\theta) = n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i \longrightarrow \max$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\theta}_1 = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

b) ! $\hat{x}$ - медиана

$$\int_{-\infty}^{\hat{x}} p(x)\,dx = \frac{1}{2}$$

$$\int_{1}^{\hat{x}} (\theta - 1) \frac{1}{x^{\theta}}\,dx = (\theta - 1) \int_{1}^{\hat{x}} x^{-\theta}\,dx = (\theta - 1) \left. \frac{x^{-\theta + 1}}{-\theta + 1} \right|_1^{\hat{x}} = \left. - x^{-\theta + 1} \right|_1^{\hat{x}} = - \hat{x}^{-\theta + 1} + 1 = \frac{1}{2}$$

$$\Rightarrow \hat{x}^{-\theta + 1} = \frac{1}{2} \Rightarrow \hat{x}^{\theta - 1} = 2 \Rightarrow \hat{x} = 2^{\frac{1}{\theta - 1}}$$

Многомерное ЦПТ для ОМП

$$\frac{f(\tilde{\theta}) - f(\theta)}{\sigma(\tilde{\theta})} \sqrt{n} \rightsquigarrow N(0; 1)$$

$$\sigma(\tilde{\theta}) = \sqrt{\nabla f^{T}(\tilde{\theta})\, I^{-1}\, \nabla f(\tilde{\theta})}$$

$$f(\theta) = 2^{\frac{1}{\theta-1}} \Rightarrow \nabla f = 2^{\frac{1}{\theta-1}} \ln 2 \, \frac{1}{(\theta-1)^2}$$

$$f(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}}$$

! $$\tilde{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

$$I = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = \int_{-\infty}^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 \frac{\theta-1}{x^\theta} dx =$$

$$\frac{\partial\left[\ln(\theta-1) - \ln(x^\theta)\right]}{\partial \theta} = \frac{1}{\theta-1} - \ln x$$

$$= \int_1^\infty \frac{1}{\theta-1} \frac{1}{x^\theta} dx + \int_1^\infty \ln^2 x \cdot \frac{\theta-1}{x^\theta} dx - \int_1^\infty \frac{2\ln x}{x^\theta} dx = \frac{1}{(\theta-1)^2} + \frac{2}{(\theta-1)^2} - \frac{2}{(\theta-1)^2} =$$

$$= \frac{1}{(\theta-1)^2} \Rightarrow I^{-1}(\tilde{\theta}) = (\tilde{\theta}-1)^2 \Rightarrow \sigma(\tilde{\theta}) = \sqrt{2^{\frac{1}{\theta-1}} \cdot \ln 2 \cdot \left(\frac{1}{(\theta-1)^2}\right) \cdot (\theta-1)^2 \cdot (\theta-1)^2 \cdot 2^{\frac{1}{\theta-1}} \cdot \frac{1}{(\theta-1)^2}} = 2^{\frac{1}{\theta-1}} \cdot \ln 2 \cdot \frac{1}{\theta-1}$$

$$\Rightarrow \frac{f(\tilde{\theta}) - f(\theta)}{\sigma} \sqrt{n} \sim N(0;1) \Longrightarrow$$

$$-1{,}96 < \frac{2^{\frac{1}{\tilde{\theta}-1}} - 2^{\frac{1}{\theta-1}}}{2^{\frac{1}{\theta-1}} \cdot \ln 2 \cdot \frac{1}{\theta-1}} \sqrt{n} < 1{,}96$$

$$\beta = 0{,}95$$

$$-1{,}96 \cdot \frac{2^{\frac{1}{\hat{\theta}-1}} \cdot \ln 2}{(\hat{\theta}-1)\sqrt{n}} + 2^{\frac{1}{\hat{\theta}-1}} < \hat{x} < 1{,}96 \cdot \frac{2^{\frac{1}{\hat{\theta}-1}} \cdot \ln 2}{(\hat{\theta}-1)\sqrt{n}} + 2^{\frac{1}{\hat{\theta}-1}}$$

c) по ОМП

$$\int_1^\infty \frac{\partial}{\partial \theta} p(\vec{x}, \theta) dx = \int_1^\infty \frac{\partial}{\partial \theta}\left(\frac{\theta-1}{x^\theta}\right) dx = \int_1^\infty \frac{x^\theta - x^\theta \ln x (\theta-1)}{x^{2\theta}} dx =$$

$$= \int_1^\infty \frac{1 - \ln x(\theta-1)}{x^\theta} dx = x^{1-\theta} \ln x \Big|_1^\infty = 0 - 0 = 0 \Rightarrow$$ модель регулярна

$x \geq 1 \quad \theta > 1.$

$$\frac{f(\hat{\theta}) - f(\theta)}{\sigma(\hat{\theta})} \sqrt{n} \sim N(0;1) \qquad \sigma(\hat{\theta}) = \sqrt{\nabla f^T I^{-1} \nabla f}$$

$$f(\theta) = \theta \qquad \nabla f = 1 \qquad I^{-1} = (\hat{\theta} - 1)^2$$

$$\downarrow$$

$$f(\hat{\theta}) = \tilde{\theta} \qquad \sigma = \tilde{\theta} - 1$$

$$\frac{\tilde{\theta} - \theta}{\tilde{\theta} - 1} \sqrt{n} \sim N(0;1) \Rightarrow \beta = 0{,}95 \Rightarrow -1{,}96 < \frac{\tilde{\theta} - \theta}{\hat{\theta} - 1} \sqrt{n} < 1{,}96$$

$$-1{,}96\,\frac{\hat{\theta}-1}{\sqrt{n}}+\hat{\theta}<\theta<1{,}96\,\frac{\hat{\theta}-1}{\sqrt{n}}+\hat{\theta}$$

$$\tilde{\theta}=\frac{n}{\sum_{i=1}^{n}\ln x_i}+1.$$

по ОММ

$$\frac{f(\tilde{\alpha})-f(\alpha)}{\sigma(\tilde{\alpha})}\sqrt{n}\sim N(0;1)$$

$$\sigma(\tilde{\alpha})=\sqrt{\nabla f^{T}K\nabla f}$$

$$\alpha_1=M\xi=\int_1^{\infty}x\,\frac{\theta-1}{x^{\theta}}\,dx=\frac{\theta-1}{\theta-2}=1+\frac{1}{\theta-2}$$

$\theta>1$

$\theta>2$

$$\tilde{\alpha}_1=\bar{x}$$

$$1+\frac{1}{\theta-2}=\bar{x}\Rightarrow\tilde{\theta}_3=\frac{1}{\bar{x}-1}+2$$

$$f(\alpha)=\theta=\frac{1}{\alpha_1-1}+2$$

$$f(\tilde{\alpha}_1)=\tilde{\theta}=\frac{1}{\bar{x}-1}+2\Rightarrow\nabla f(\tilde{\alpha}_1)=\frac{-1}{(\bar{x}-1)^2}$$

$$k=k_{11}=\tilde{\alpha}_2-\tilde{\alpha}_1^{2}=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\bar{x}^2$$

$$\beta=0{,}95\Rightarrow-1{,}96<\frac{\frac{1}{\bar{x}-1}+2-\theta}{\frac{1}{(\bar{x}-1)^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2-\bar{x}^2}}\sqrt{n}<1{,}96$$

$$\frac{-1{,}96\sqrt{\frac{1}{n}\sum x_i^2-\bar{x}^2}}{(\bar{x}-1)^2\sqrt{n}}+2+\frac{1}{\bar{x}-1}<\theta<\frac{1{,}96\sqrt{\frac{1}{n}\sum x_i^2-\bar{x}^2}}{(\bar{x}-1)^2\sqrt{n}}+2+\frac{1}{\bar{x}}$$