$\xi \sim R[0;\theta]$ $\quad \tilde{\theta}_1 = 2\bar{x}$ $\quad \tilde{\theta}_3 = x_{max}$

x_n – выборка $\quad \tilde{\theta}_2 = x_{min}$ $\quad \tilde{\theta}_5 = \left(x_1 + \frac{\sum_{k=2}^{n} x_k}{(n-1)}\right)$

а) Несмещенность, состоятельность. (исправить)

б) Эффективность – ?

$\tilde{\theta}_1$: $\quad \tilde{\theta}_1 = 2\bar{x} = 2\,\frac{1}{n}\sum_{i=1}^{n} x_i$ $\quad x_i \sim R(0;\theta) \Rightarrow M\xi = \frac{\theta}{2}$

$M\tilde{\theta}_1 = M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n}\sum_{i=1}^{n} Mx_i = \frac{2}{n}\cdot n\cdot\frac{\theta}{2} = \theta \Rightarrow \theta_1$ – несмещ. $\Rightarrow$ асимпт. несм.

$D\tilde{\theta}_1 = D\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{4}{n^2}\sum_{i=1}^{n} Dx_i = \frac{4}{n^2}\, nD\xi = \frac{4}{n}\cdot\frac{\theta^2}{12} = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

$\xi \sim R(a;b) \longrightarrow D\xi = \frac{(b-a)^2}{12}$

$\Rightarrow$ достаточное условие $\tilde{\theta}_1$ – состоят.

$\tilde{\theta}_2$: $\quad \tilde{\theta}_2 = x_{min}$ $\quad y = x_{min} \sim \Phi(y)$ ← ф-я распр-я

$\sim \varphi(y)$ ← пл-сть распр-я

$M\tilde{\theta}_2 = M[x_{min}] = \int_{-\infty}^{+\infty} y\,\varphi(y)\,dy$

$\Phi(y) = P(y<y) = P(x_{min}<y) = 1 - P(x_{min}\geq y) = 1 - P(x_1\geq y, x_2 \geq y, \dots, x_n \geq y) =$

$x_i = \xi$ – независ. $= 1 - \prod_{i=1}^{n} P(x_i \geq y) = 1 - \prod_{i=1}^{n}(1 - P(x_i < y)) = 1 - \prod_{i=1}^{n}(1 - F_i(y)) =$

$= 1 - (1-F(y))^n$ — ф-я распр-я.

$\Rightarrow \varphi(y) = n(1-F(y))^{n-1}\cdot F'(y) = n\left(1-\frac{y}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}$ $\quad (0;\theta)$

$F(y) = \frac{y}{b-a} = \frac{y}{\theta}$ $\quad F'(y) = p(y) = \frac{1}{\theta}$

$\Rightarrow M\tilde{\theta}_2 = \int_0^{\theta} y\,n\left(1-\frac{y}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\,dy =$ $\quad \frac{y}{\theta} = t, \quad dy = \theta\,dt$ $\quad =$

$= \int_0^1 t\,n(1-t)^{n-1}\theta\,dt = n\theta B(2,n) = n\theta\,\frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)} = \theta\,\frac{1!\,\Gamma(n)}{(n+1)\Gamma(n)} = \frac{\theta}{n+1}$ $\quad \neq \theta$, $\quad \not\xrightarrow[n\to\infty]{} \theta$

$\tilde{\theta}_2$ – смещ., $\tilde{\theta}_2$ – не асимпт. несмещ.

! Можно: $\quad \tilde{\theta}_2{}' = (n+1)x_{min}$ — несмещ. $\rightarrow$ асимпт. несмещ.

? $\forall\varepsilon>0 \quad P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0 \longleftrightarrow \tilde{\theta}_2 \xrightarrow{P} \theta$

$P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) = P(\tilde{\theta}_2 - \theta \geq \varepsilon) + P(\tilde{\theta}_2 - \theta \leq -\varepsilon) = P(\tilde{\theta}_2 \leq \theta - \varepsilon) + P(\tilde{\theta}_2 \geq \theta + \varepsilon) =$

$= F_{min}(\theta - \varepsilon + \delta) =$ $\quad \delta = \varepsilon$ $\quad = F_{min}(\theta) = 1 - (1 - F(\theta))^n =$ $\quad F(\theta) = \frac{\theta}{\theta} = 1$ $\quad = 1 - 0^n =$

$= 1 \not\xrightarrow[n\to\infty]{} 0 \Rightarrow \tilde{\theta}_2$ – не сост.

? $\tilde{\theta}_2' \xrightarrow{P} \theta \iff \forall \varepsilon > 0 \quad P(|\tilde{\theta}_2'{}_n - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$ 

$\Rightarrow$

$$\Rightarrow P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta + \varepsilon) = P\left(X_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = P(x_1 \geq \dots,\ x_2 \geq \dots \quad x_n \geq \dots) =$$

$$= \prod_{i=1}^{n} P\left(x_i \geq \frac{\theta+\varepsilon}{n+1}\right) = \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n = \left(1 - \frac{\theta+\varepsilon}{(n+1)\theta}\right)^n = \left(1 - \frac{1}{A}\right)^{-A\left(-\frac{\theta+\varepsilon}{\theta}\right)-1} \longrightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \qquad \frac{1}{A} \qquad -\frac{(n+1)\theta(-1)(\theta+\varepsilon)}{(\theta+\varepsilon)\,\theta} - 1$$

$$\xrightarrow[n\to\infty]{} \exp\left(-\frac{\theta+\varepsilon}{\theta}\right) > 0 \Rightarrow \tilde{\theta}_2' \text{ — не сост.}$$

$\theta_3 = X_{max}$

$X_i = \xi \sim F(x)$

$X_{max} = \omega \sim \Psi(z)$ — ф-я распр-я

$\sim \psi(z)$ — пл-сть распр-я

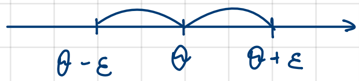
$$\Psi(z) = P(\omega < z) = P(\xi_1 < z, \dots, \xi_n < z) = \overset{F(z)}{P(\xi_1 < z)} \cdot \ldots \cdot \overset{F(z)}{P(\xi_n < z)} = (F(z))^n$$

$$\psi(z) = \Psi'(z) = n(F(z))^{n-1} \cdot F'(z) = n \cdot F^{n-1}(z) \cdot \frac{1}{\theta} \qquad (0;\theta)$$

$$M\tilde{\theta}_3 = \int_0^\theta zn\left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\, dz = \frac{n}{\theta^n}\int_0^\theta z^n\, dz = \frac{n\theta}{n+1} \neq 0 \xrightarrow[n\to\infty]{} \theta$$

$\tilde{\theta}_3$ — смещ, $\tilde{\theta}_3$ — асимпт. несм.

! Можно: $\tilde{\theta}_3' = \frac{n+1}{n} X_{max}$ — несм. $\longrightarrow$ асимпт. несмещ.

$\tilde{\theta}_3 \xrightarrow{P} \theta \iff \forall \varepsilon > 0 \quad P(|\tilde{\theta}_3 - \theta| < \varepsilon) \xrightarrow[n\to\infty]{} 1$ 

$$P(|\tilde{\theta}_3 - \theta| < \varepsilon) \leq P(\theta_3 < \theta + \varepsilon) = P(X_{max} < \theta + \varepsilon) =$$

$$= P(x_1 < \theta+\varepsilon,\ x_2 < \theta+\varepsilon,\ \dots,\ x_n < \theta+\varepsilon) = \prod_{i=1}^{n} P(x_i < \theta+\varepsilon) = (F(\theta+\varepsilon))^n =$$

$$= \left(\frac{\theta+\varepsilon}{\theta}\right)^n = \left(1 + \frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{\theta=const\geq 1} \infty > 0 \Longrightarrow \tilde{\theta}_3 \text{ — не сост.}$$

$$\tilde{\theta}_3' : \quad M[\tilde{\theta}_3^2] = \int_0^\theta z^2 n\left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\, dz = \frac{n}{\theta^n}\int_0^\theta z^{n+1}\, dz = \frac{\theta^2 n}{n+2}$$

$$\Rightarrow D[\tilde{\theta}_3] = \frac{\theta^2 n}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \theta^2\,\frac{n^3+2n^2+n-n^3-2n^2}{(n+2)(n+1)^2} = \theta^2\,\frac{n}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2}\,\frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$ Достаточное условие:

$\Rightarrow \tilde{\theta}_3'$ — сост.

$$\tilde{\theta}_5 = x_1 + \frac{\sum_{k=2}^{n} x_k}{(n-1)}$$

$$M\tilde{\theta}_5 = M[x_1] + \frac{n-1}{n-1} M[x_2] = \frac{\theta}{2}\left(1 + \frac{n-1}{n-1}\right) = \theta \Rightarrow \tilde{\theta}_5 \text{ - несмещ} \Rightarrow$$

$\Rightarrow$ ассимпт. несмещ.

$$D\tilde{\theta}_5 = D[x_1] + \frac{n-1}{n-1} D[x_2] = \frac{\theta^2}{12}\left(1 + \frac{1}{n-1}\right) \xrightarrow[n\to\infty]{} \frac{\theta^2}{12} \neq 0$$

"достаточного условия „не хватило"

$$\tilde{\theta}_5 \xrightarrow{P} \theta$$

(T) Крамера: $\begin{cases} \xi_n \xrightarrow{P} \xi \\ \eta_n \xrightarrow{P} \eta \end{cases} \rightarrow \xi_n + \eta_n \xrightarrow{P} \xi + \eta$

1. $x_1 \xrightarrow{P} x_1$

2. $\frac{\sum_{k=2}^{n} x_k}{n-1} \xrightarrow[n\to\infty]{} \frac{\tilde{\theta}_1}{2} \xrightarrow{P} \frac{\theta}{2}$

$\Rightarrow \tilde{\theta}_5 \xrightarrow{P} x_1 + \frac{\theta}{2} \Rightarrow \tilde{\theta}_5$ не сост.

! Можем свести к $\tilde{\theta}_1$: $\tilde{\theta}_5 = 2 \cdot \frac{\sum_{k=2}^{n} x_k}{n-1}$

Эффективность:

1. $D\tilde{\theta}_1 = \frac{\theta^2}{3n}$
2. $D\tilde{\theta}_2' = \frac{n\theta^2}{n+2}$
3. $D\tilde{\theta}_3' = \frac{\theta^2}{n(n+2)}$
4. $D\tilde{\theta}_5 = \frac{\theta^2}{2}\cdot\left(1 + \frac{1}{n-1}\right)$

1 и 3: $\frac{1}{3n} > \frac{1}{n(n+2)}$ $n > 1 \Rightarrow$

$\Rightarrow \tilde{\theta}_3'$ более эффективная.

2 и 4: $\frac{n}{n+2} > \frac{1}{2}\left(1 + \frac{1}{n-1}\right)$ $n > 1$.

$\Rightarrow \tilde{\theta}_5$ более эффективная.

$\frac{1}{n(n+2)} < \frac{1}{2}\cdot\left(1 + \frac{1}{n-1}\right)$ $n > 1 \Rightarrow \tilde{\theta}_3'$ самая эффект.

№ 3.

$$p(x) = \begin{cases} \dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \theta > 0 \Rightarrow F(x) = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$x_n = x_3$

$$\tilde{\theta}_1 = \overline{x} = \frac{x_3^{(1)} + x_3^{(2)} + x_3^{(3)}}{3} \qquad \tilde{\theta}_3 = med(x_3)$$

а) 1. $M\tilde{\theta}_1 = \frac{1}{3} \cdot 3 M\xi = M\xi = \int_0^\infty \frac{x}{\theta} e^{-\frac{x}{\theta}} dx = \theta \int_0^\infty t e^{-t} dt = \theta\Gamma(2) = \theta \Rightarrow$

$\Rightarrow \theta_1$ — несмещ.

2. $\exists$ вариац. ряд: $x^{(1)}, x^{(2)}, x^{(3)} \Rightarrow \theta_3 = x^{(2)}$

$$\underbrace{P(x \leq x^{(2)} < x + \Delta x)}_{\Phi(x+\Delta x) - \Phi(x)} \underset{=\Delta x}{=} C_3^1 P(x \leq \xi < x + \Delta x) C_2^1 P(\xi < x) C_1^1 P(\xi \geq x + \Delta x) =$$

$$= 6\underbrace{(F(x+\Delta x) - F(x))}_{\Delta x} \cdot F(x)(1 - F(x+\Delta x))$$

$$\Rightarrow \Phi'(x) = \varphi(x) \qquad 6p(x)F(x)(1-F(x))$$

$$\downarrow$$

$$\varphi(x) = 6p(x)F(x)(1-F(x)) \qquad (0; \infty)$$

$$M[\tilde{\theta}_3] = 6\int_0^\infty x \frac{e^{-\frac{x}{\theta}}}{\theta} (1 - e^{-\frac{x}{\theta}})(1 - 1 + e^{-\frac{x}{\theta}}) dx =$$

$$= \frac{6}{\theta}\left[\int_0^\infty x e^{-2\frac{x}{\theta}} dx - \int_0^\infty x e^{-3\frac{x}{\theta}} dx\right] = \frac{5\theta}{6} \Rightarrow \tilde{\theta}_3 \text{ — смещ.}$$

Исправление: $\tilde{\theta}_3' = \frac{6}{5} x^{(2)}$ — несмещ.

б) 3. $D\tilde{\theta}_1 = D[\frac{1}{3}\Sigma x] = \frac{1}{9} D[\Sigma x] = \frac{1}{3} D\xi = \frac{1}{3}(M\xi^2 - M^2\xi) =$

$= \frac{1}{3}\left(\int_0^\infty \frac{x^2 e^{-\frac{x}{\theta}}}{\theta} dx - \theta^2\right) = \frac{1}{3}\theta^2 \Rightarrow D\tilde{\theta}_1 = \frac{\theta^2}{3}.$

4. $D\tilde{\theta}_3'$:

$$\Phi(y) = P\left(\tfrac{6}{5}x^{(2)} < y\right) = P\left(\tfrac{6}{5}x^{(1)} < y, \tfrac{6}{5}x^{(2)} < y\right) + P\left(\tfrac{6}{5}x^{(2)} < y, \tfrac{6}{5}x^{(3)} < y\right) +$$

$$+ P\left(\tfrac{6}{5}x^{(3)} < y, \tfrac{6}{5}x^{(1)} < y\right) - 2P\left(\tfrac{6}{5}x^{(1)} < y, \tfrac{6}{5}x^{(2)} < y, \tfrac{6}{5}x^{(3)} < y\right) =$$

$$= 3(1 - e^{-\frac{5y}{6\theta}}) - 2(1 - e^{-\frac{5y}{6\theta}})^3$$

$$\varphi(y) = \frac{5}{6}\left(e^{-\frac{5y}{3\theta}} - e^{-\frac{5y}{2\theta}}\right) \Rightarrow D\tilde{\theta}_3' = M\tilde{\theta}_3'^2 - M^2\tilde{\theta}_3' =$$

$$= \theta^2\left(\frac{38}{25}-1\right) = \frac{13}{25}\theta^2 \quad \to D\tilde{\theta}_3' = \frac{13}{25}\theta^2$$

$$\left.\begin{aligned} D\tilde{\theta}_1 &= \frac{\theta^2}{3} \\ D\tilde{\theta}_3' &= \frac{13}{25}\theta^2 \end{aligned}\right\} \to \tilde{\theta}_1 - \text{самая эффективная.}$$

в) Регулярность

5. $0 = \int \frac{\partial}{\partial\theta_i} p(x,\theta)\,dx = \int \frac{\partial(\ln p(x,\theta))}{\partial\theta_i} p(x,\theta)\,dx_i$

$$\int_0^\infty \frac{\partial}{\partial\theta}\left(\frac{1}{\theta}e^{-\frac{x}{\theta}}\right)dx = \int_0^\infty -\frac{1}{\theta^2}e^{-\frac{x}{\theta}} + \frac{x}{\theta^2}e^{-\frac{x}{\theta}}\,dx = \frac{1}{\theta}(-1+\Gamma(2)) = 0. \;//$$

$p(x,\theta) = \frac{1}{\theta}e^{-\frac{x}{2}}$ — непр. дифф по θ.

$$I = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = \int_0^\infty \left(\frac{x}{\theta^2}-\frac{1}{\theta}\right)^2 e^{-\frac{x}{\theta}}\frac{1}{\theta}\,dx = \frac{1}{\theta^2}$$

$$\ln p = -\frac{x}{\theta} - \ln\theta$$

$\Rightarrow$ модель регулярна

6. $D[\tilde{g}_{z-eff}(x_n)] = \inf D[g(x)] = \left(\frac{g'(\theta)}{nI(\theta)}\right)^2$

$g(\theta) = \theta \Rightarrow g'(\theta) = 1 \Rightarrow \inf D[g(x)] = \frac{\theta^2}{3}.$

$$\begin{aligned} D\tilde{\theta}_1 &= \frac{\theta^2}{3} \\ D\tilde{\theta}_3' &= \frac{13}{25}\theta^2 \end{aligned} \qquad \tilde{\theta}_1 - r\text{-eff.}$$