Recurrence of Schreier Graphs of the Iterated Monodromy Groups of Expanding Thurston Maps

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Background

On the dynamics side, the iterated monodromy groups provide powerful algebraic tools to encode the combinatorial information about the dynamical system. For instance, the Julia set of a map can be reconstructed from its iterated monodromy group[1]. On the group theory side, the iterated monodromy groups give rise to a rich class of groups with interesting properties, such as amenability, intermediate growth, etc.

In this work, we study the iterated monodromy groups of expanding Thurston maps. We first give a brief introduction to the main objects involved in our study.

Definition 1: An **expanding Thurston map** is a branched covering map $f : S^2 \to S^2$ that is postcritically finite and expanding. We have **cell decompositions** for such maps.

Definition 2: The **iterated monodromy group** of an expanding Thurston map f is constructed as follows:

We take a point $p \in S^2 \setminus \operatorname{post} f$ and a loop $\gamma \subseteq S^2 \setminus \operatorname{post} f$ based at p. Clearly $g = [\gamma]$ is an element in $G = \pi_1(S^2 \setminus \operatorname{post} f, p)$. For any $q \in f^{-n}(p) =: X_n$, the loop γ can be lifted by f^n to a path γ_q starting at q. We set $g(q)$ to be the endpoint of γ_q. This defines an action of G on the tree $X^* := \cup_{n=0}^{\infty} X_n$, which means that we have a group homomorphism $\varphi : G \to \operatorname{Aut}(X^*)$. The **iterated monodromy group** of f is $\operatorname{IMG}(f) := G/\ker(\varphi) \cong \varphi(G)$.

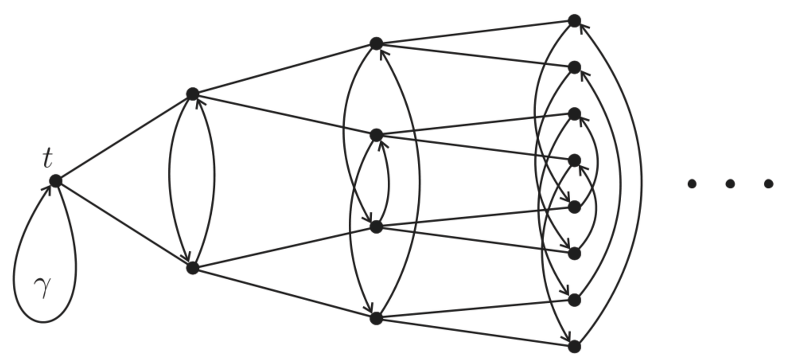

Figure 1: Iterated monodromy action *(This figure is from [1])*.

Definition 3: The **Schreier graph** is a generalization of Cayley graph. Let G be a group acting on a set X, and S a (symmetric) generating set of G. The Schreier graph $\Gamma(G, S, X)$ is the graph with vertex set X and edge set $\{\{sx, s\} : x \in X, s \in S\}$.

Main Result

We proved the following result:

Theorem 1: *Let f be an expanding Thurston map without periodic critical points. Then the Schreier graph $\Gamma_\infty := \Gamma(\operatorname{IMG}(f), S, X^\omega = \partial X^*)$ is recurrent, i.e. the simple random walk on it is recurrent.*

Key Tools

Solenoid

We can study the behavior of f at "infinity" using the solenoid.

Definition 4: Given an expanding Thurston map f, the **solenoid** $\mathcal{S}(f)$ is the inverse limit of the system:

$$\cdots \xrightarrow{f} S^2 \xrightarrow{f} S^2 \xrightarrow{f} S^2 \xrightarrow{f} S^2,$$

or more concretely, the set of all infinite sequences $x = (\ldots, x_2, x_1, x_0)$ such that $f(x_{i+1}) = x_i$ for all i.

The solenoid $\mathcal{S}(f)$ is connected but not path-connected. We call the connected components of $\mathcal{S}(f)$ **leaves**. Moreover, we can lift tiles to the solenoid and study the tile structure on the solenoid.

Benjamini-Schramm convergence

Another key tool is the Benjamini-Schramm convergence of planar graphs. In their paper [2], Benjamini and Schramm introduced the notion of **Benjamini-Schramm convergence** of graphs. Specifically, they proved the following seminal theorem:

Theorem 2 (Benjamini–Schramm[2]). *Let $M < +\infty$ and let G_n be pointed random finite planar graphs with degrees bounded by M such that $G_n \xrightarrow{B.S} (G, o)$. Then (G, o) is almost surely recurrent.*

Proof Strategy

The main idea is to take use of the combinatorial structure of tiles.

Step 1: Turn the Schreier graphs to adjacency graphs of tiles

Let $\mathcal{G}_n$ be the adjacency graph of n-tiles and $\Gamma_n := \Gamma(\operatorname{IMG}(f), S, X_n)$. We first show that Γ_n is a subgraph of $\mathcal{G}_n^2$. But the situation is more complicated for Γ_∞.

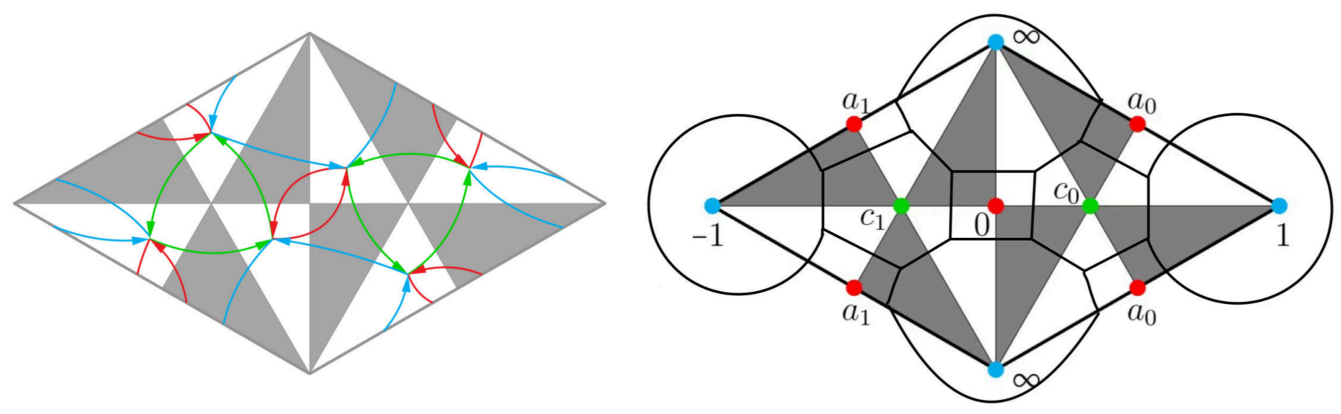

Figure 2: Schreier graph and adjacency graph *(This figure is originally from [3])*.

Step 2: Dealing with infinity: using solenoid

Let $\mathcal{G}_n^L$ be the adjacency graph of n-tiles on the leaf $L \subseteq \mathcal{S}(f)$. Actually, the level of tiles does not matter here, we have the following result:

Proposition 2: *Let L be a leaf of the solenoid $\mathcal{S}(f)$. Then $\mathcal{G}_n^L$ is isomorphic to $\mathcal{G}_0^L$ for all n. Therefore, we just write $\mathcal{G}^L$ when considering the adjacency graph on L.*

Given $x \in L \subseteq \mathcal{S}(f)$, let σ be a 0-tile on L containing x, then we have a pointed graph $(\mathcal{G}^L, \sigma)$. Then choose x with respect to an apporiate measure μ, we have a random pointed graph $(\mathcal{G}_\infty, \rho)$. We can associate Γ_∞ with $\mathcal{G}_\infty$ in similar ways. From now on, we can focus on the combinatorial behavior of tiles.

Step 3: Benjamini-Schramm convergence of planar graphs

We want to take use of the *Theorem 2* and we proved the following result:

Proposition 1: $\mathcal{G}_n \xrightarrow{B.S} (\mathcal{G}_\infty, \rho)$.

Thus as a corollary, we have that $(\mathcal{G}_\infty, \rho)$ is a.s recurrent.

Step 4: Moving from a.s recurrence to recurrence

To prove recurrence, we need to analyze the leaves of the solenoid more carefully. Actually, we proved that except for some "bad" leaves, the adjacency graphs of tiles on the leaves are all the same. More explicity, we have:

Proposition 2: *Let L_1, L_2 be leaves of the solenoid $\mathcal{S}(f)$. If they do not contain points of the form $p = (\ldots, p_2, p_1, p_0)$ with $p_i \in \operatorname{post} f$ for all i, then $\mathcal{G}^{L_1}$ and $\mathcal{G}^{L_2}$ are isomorphic.*

And for those "bad" leaves, we can show that only the tiles around the point p of the above form have different behavior, while the rest of the tiles have similar behavior. This completes the proof of *Theorem 1*.

Future Work

1. Compute the asymptotic entropy of $\operatorname{IMG}(f)$ (w.r.t a probability measure μ).
2. Compute the growth rate of IMG(f).
3. Use the language in [1] to put the results in a more general context.
4. **Main Goal**: Prove the amenability of $\operatorname{IMG}(f)$ of expanding Thurston maps.

Acknowledgements and References

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