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Notes on Partial Differential Equations

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Contents

1 The Wave Equation

1.1 The General Situation

Equation:

$$u_{tt} = c^2 u_{xx} \quad \text{for} -\infty < x < +\infty$$

Solution:

$$u(x,t) = f(x+ct) + g(x-ct)$$

1.2 Initial Value Problem

Equation:

$$\text{DE: } u_{tt} = c^2 u_{xx} \quad \text{for} -\infty < x < +\infty$$

$$\text{IC: } u(x,0) = \phi(x), u_t(x) = \psi(x)$$

Solution:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

1.3 Reflections of Waves

Equation:

$$\text{DE: } u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < +\infty,\ -\infty < t < +\infty$$

$$\text{IC: } u(x,0) = \phi(x), \quad u_t(x) = \psi(x) \quad \text{for } 0 < x < +\infty,\ t = 0$$

$$\text{BC: } u(0,t) = 0 \quad \text{for } x = 0,\ -\infty < t < +\infty.$$

Solution:

$$u(x,t) = \tfrac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \quad \text{for } x > c|t|$$
$$u(x,t) = \tfrac{1}{2}[\phi(ct+x) - \phi(ct-x)] + \tfrac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)ds \quad \text{for } x < c|t|$$

1.4 The Finite Interval

TBD

1.5 Waves with a Source

Equation:

$$\text{DE: } u_{tt} - c^2 u_{xx} = f(x,t) \quad \text{for} - \infty < x < +\infty$$

$$\text{IC: } u(x,0) = \phi(x), u_t(x) = \psi(x)$$

Solution:

$$u(x,t) = \tfrac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \tfrac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)ds + \tfrac{1}{2c} \iint\limits_{\Delta} f$$

Proof: *TBD* □

1.6 Boundary Problems: The Dirichlet Condition

Equation:

$$\text{DE: } u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l,\ -\infty < t < +\infty$$

$$\text{IC: } u(x,0) = \phi(x), \quad u_t(x) = \psi(x) \quad \text{for } 0 < x < l,\ t = 0$$

$$\text{BC: } u(0,t) = u(l,t) = 0 \quad \text{for } x = 0,\ -\infty < t < +\infty.$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right) \right) \sin\left(\frac{n\pi x}{l}\right)$$

$A_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$\frac{n\pi c}{l} B_n = \frac{2}{l} \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$ or

$B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$

1.7 Boundary Problems: The Neumann Condition

Equation:

$$\text{DE: } u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l,\ -\infty < t < +\infty$$

$$\text{IC: } u(x,0) = \phi(x), \quad u_t(x) = \psi(x) \quad \text{for } 0 < x < l,\ t = 0$$

$$\text{BC: } u_x(0,t) = u_x(l,t) = 0 \quad \text{for } x = 0,\ -\infty < t < +\infty.$$

Solution:

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right)\cos\left(\frac{n\pi x}{l}\right)$$

Thus we have the following equations:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\left(\frac{n\pi x}{l}\right)$$

Using the same Fourier series, the coefficients are:

$$A_n = \tfrac{2}{l}\int_0^l \phi(x)\cos\left(\tfrac{n\pi x}{l}\right)dx$$
$$B_n = \tfrac{2}{n\pi c}\int_0^l \psi(x)\cos\left(\tfrac{n\pi x}{l}\right)dx$$

Remark: the equations above also hold true for A_0 and B_0.

1.8 Boundary Problems: The Robin Condition

This is the most complicated case, we first list the equations and make some interpretation.

Equation:

$$\text{DE: } u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l,\ -\infty < t < +\infty$$

$$\text{IC: } u(x,0) = \phi(x), \quad u_t(x) = \psi(x) \quad \text{for } 0 < x < l,\ t = 0$$

$$\text{BC: } u_x(0,t) - a_0 u(0,t) = 0 \text{ and } u_x(l,t) + a_l u(l,t) = 0.$$

This means that we are solving $-X'' = \lambda X$ with the boundary conditions:

$$X' - a_0 X = 0 \text{ at } x = 0$$
$$X' + a_l X = 0 \text{ at } x = l.$$

1.8.1 Positive Eigenvalues

First, we have $\lambda = \beta^2 > 0$, and the general solution: $X(x) = C\cos\beta x + D\sin\beta x$. Using the boundary conditions, we can get:

$$\beta D - a_0 C = 0$$

$$(\beta D + a_l C)\cos\beta l + (-\beta C + a_l D)\sin\beta l = 0$$

Thus $D = \frac{a_0 C}{\beta}$. Next, we substitute for D, eliminate C, we have:

$$(a_0 + a_l)\cos\beta l + \left(-\beta + \frac{a_l a_0}{\beta}\right)\sin\beta l = 0$$

which is:

$$(\beta^2 - a_0 a_l)\tan\beta l = (a_0 + a_l)\beta$$

And we cannot ignore the *exceptional case*:

$$\cos\beta l = 0 \text{ and } \beta = \sqrt{a_0 a_l}$$

Next, we can solve the $X(x)$:

$$X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right)$$

case : $a_0 > 0$ and $a_1 > 0$

We can see from the graph that:

$$n\frac{\pi}{l} < \beta_n < (n+1)\frac{\pi}{l},\ n = 0, 1, 2, 3, ...$$

$$n^2\frac{\pi^2}{l^2} < \lambda_n < (n+1)^2\frac{\pi^2}{l^2},\ n = 0, 1, 2, 3, ...$$

Furthermore,

$$\lim_{n\to\infty} \beta_n - n\frac{\pi}{l} = 0$$

case : $a_0 < 0, a_l > 0$ and $a_0 + a_l > 0$

The two inequality in the case above still holds here, with a slightly different situation, that is there may not exist an eigenvalue λ_0 in the interval $\left(0, \frac{\pi^2}{l^2}\right)$. And there is an eigenvalue λ_0 in the interval $\left(0, \frac{\pi^2}{l^2}\right)$ iff:

$$a_0 + a_l > -a_0 a_l l$$

1.8.2 Zero Eigenvalue

To be added.

1.8.3 Negative Eigenvalue

We set $\lambda = -\gamma^2 < 0$ and wwrite the solution of the PDE as

$$X(x) = C\cosh\gamma x + D\sinh\gamma x.$$

Similarly, we can write the eigenvalue equation as

$$\tanh\gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

And the corresponding eigenfunction is

$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.$$

case : $a_0 > 0$ and $a_1 > 0$: No negative eigenvalue.

case : $a_0 < 0, a_1 > 0$ and $a_0 + a_l > -a_0 a_l l$: No negative eigenvalue.

case : $a_0 < 0, a_1 > 0$ and $a_0 + a_l < -a_0 a_l l$: One negative eigenvalue.

1.8.4 Summary

To be added.

2 The Diffusion Equation

The solution of diffusion equation is a little bit harder than the wave equation. But if you use the Fourier series, there are much similarity.

We will begin from the diffusion on the whole line.

2.1 Diffusion on the whole line

Equation:

$$u_t = k u_{xx}$$
$$u(x,0) = \phi(x)$$

The solution comes in several steps. First, we find a special solution:

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp.$$

which satiesfies:

$$Q(x,0) = 1, \text{for } x > 0 \quad Q(x,0) = 0, \text{for } x < 0.$$

Next, we define S to be Q_x, that is:

$$S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}.$$

Finally, we claim that the unique solution is given by:

Solution:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y) dy.$$

And in the real computation, we need some examples and exercies.

2.1.1 Examples

To be added.

2.2 Diffustion with a source

Equation:

$$u_t - ku_{xx} = f(x,t)$$
$$u(x,0) = \phi(x)$$

Solution:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$$

We will go back to the proof in later sections.

2.3 Boundary Problems: The Dirichlet Condition

The method we will use here is totally the same as we have done in wave equations, so we will just present the solutions.

Equation:

$$\text{DE: } u_t = ku_{xx} \quad \text{for } 0 < x < l,\ -\infty < t < +\infty$$
$$\text{IC: } u(x,0) = \phi(x), \quad \text{for } 0 < x < l,\ t = 0$$
$$\text{BC: } u(0,t) = u(l,t) = 0 \quad \text{for } x = 0,\ -\infty < t < +\infty.$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$
$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

2.4 Boundary Problems: The Neumann Condition

Equation:

$$\text{DE: } u_t = ku_{xx} \quad \text{for } 0 < x < l,\ -\infty < t < +\infty$$
$$\text{IC: } u(x,0) = \phi(x), \quad \text{for } 0 < x < l,\ t = 0$$
$$\text{BC: } u_x(0,t) = u_x(l,t) = 0 \quad \text{for } x = 0,\ -\infty < t < +\infty.$$

Solution:

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi x}{l}\right)$$

$$A_n = \frac{2}{l}\int_0^l \phi(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

2.5 Boundary Problems: The Robin Condition

This is also the same as before, which is quite technical, so we do not write the solution here.

3 Harmonic Functions

3.1 Laplace's Equation

Equation: **(Laplace equation)**

$$\Delta u = 0$$

A solution of the Laplace equation is called a ***harmonic function***.

Equation: **(Poisson Equation)**

$$\Delta u = f$$

3.2 Maximum Principle

Theorem 3.2.1: **(Maximum Principle)** Let D be a connected bounded open set. Let either $u(x,y)$ or $u(x,y,z)$ be a harmonic function in D that is continuous on $\overline{D} = D \cup (\partial D)$. Then the maximum and the minimum values of u are attained on ∂D and nowhere inside.

3.3 Invariance Property

The Laplace equation is invariant under all rigid motions (including translations and rotations). We omit the proof here, just to note some special forms of Laplace equation.

3.3.1 Two Dimensions

Equation:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

3.3.2 Three Dimensions

Equation:

$$\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right]$$

Note that the symbols ϕ and θ here are switched and is different to what we used to know.

3.4 Green's Identities

Our basic tool here is the divergence theorem:

Theorem 3.4.1:

$$\iiint_D \nabla\cdot\boldsymbol{F} d\boldsymbol{x} = \iint_{\partial D} \boldsymbol{F}\cdot\boldsymbol{n} dS$$

3.4.1 Green's First Identity

$$\iint_{\partial D} v\frac{\partial u}{\partial n}dS = \iiint_D \nabla v\cdot\nabla u d\boldsymbol{x} + \iiint_D v\Delta u d\boldsymbol{x}$$

where $\frac{\partial u}{\partial x} = \boldsymbol{n}\cdot\nabla u$

4 Miscellaneous

4.1 Notations

$$grad f = div f = \nabla f = \left(f_x, f_y, f_z\right)$$

$$div\boldsymbol{F} = \nabla\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\Delta u = div\ grad f = \nabla\cdot\nabla f = u_{xx} + u_{yy} + u_{zz}$$

$$|\nabla f|^2 = u_x^2 + u_y^2 + u_z^2$$

4.2 Initial and Boundary Conditions

4.2.1 Initial Conditions

An *initial condition* specifies the physical state at a particular time t_0.

Example: For the wave equation there is a pair of initial conditions:

$$u(\boldsymbol{x}, t_0) = \phi(\boldsymbol{x}) \text{ and } u_t(\boldsymbol{x}, t_0) = \psi(\boldsymbol{x}).$$

4.2.2 Boundary Conditions

There are 3 most important kinds of boundary conditions:

1. **Dirichlet Condition**: u is specified, for example:
2. **Neumann condition**: the normal dericative $\frac{\partial u}{\partial n}$ is specified.
3. **Robin condition**: $\frac{\partial u}{\partial n} + au$ is specified.