Chapter 3

Functions

Reference: "Calculus", by James Stewart.

3.1 Basic Definitions

A **function** f is a rule which maps each element x in some set D to a unique image $f(x)$.

The set D is called the **domain** of f. We write $\mathrm{dom}(f) = D$.

We often write

$$f : D \longrightarrow \mathbf{R}, \quad \text{together with the rule for } f,$$

to denote the function f whose domain is D.

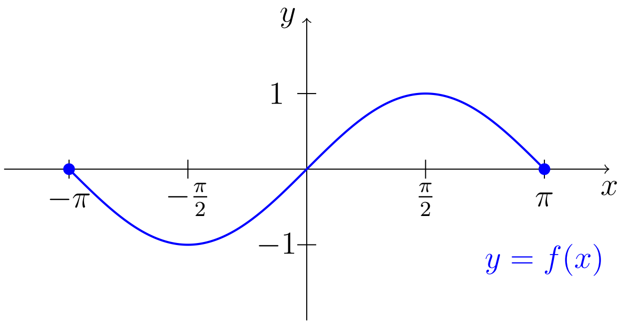
The **range** of f is the set which contains all the $f(x)$–values and **no other** values. In symbols:

$$\mathrm{ran}(f) = \{f(x) : x \in \mathrm{dom}(f)\}.$$

- **In practice, we find the range of a function by looking at which y–values are "used" in the function's graph.**

Example 1. Consider the function $f : [-\pi, \pi] \longrightarrow \mathbf{R}$ where $f(x) = \sin x$. Sketch a graph for $y = f(x)$, and state the domain and range of f .

Solution: The graph of $y = f(x)$ is drawn below:



From the notation $f : [-\pi, \pi] \longrightarrow \mathbf{R}$ we see that $\text{dom}(f) = [-\pi, \pi]$. By looking at which y –values are used in the graph, we see that $\text{ran}(f) = [-1, 1]$.
$\square$

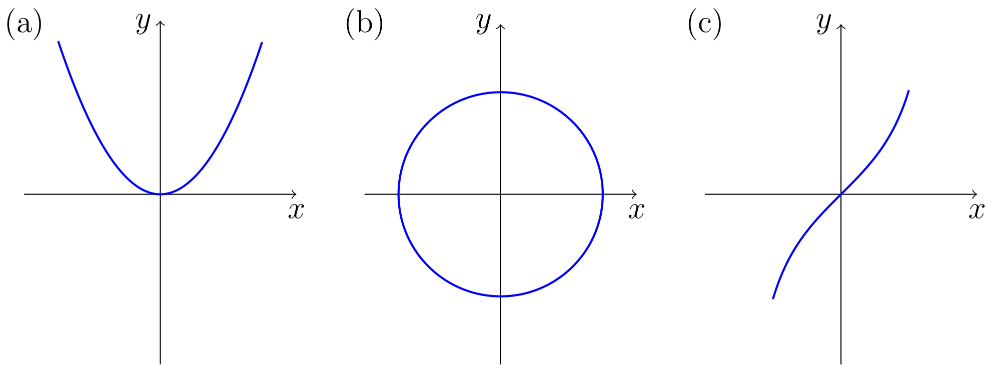
The graph of any function must satisfy the **Vertical Line Test**, which says that

each vertical line must cut the graph **at most once**.

Consider the graph of a function.

- If each **horizontal** line cuts the graph *at most once*, then we say that the function is **one–one**.
- If there are any **horizontal** lines which cut the graph *more* than once, then we say that the function is **many–to–one**.

Example 2. State whether the following graphs represent functions. For those graphs which represent functions, state whether the function is one–one or many–to–one.



Solution: (a) This is the graph of a function. It is many to one because there is a horizontal line that meets the graph more than once.

(b) This is not the graph of a function because there is a vertical line that meets the graph more than once.

(c) This is the graph of a function and the function is one-one.

$\square$

Example 3. Consider a function $f : \mathbf{R} \to \mathbf{R}$ which has rule $f(x) = x^2$. Find

(a) $f(3)$ (b) $f(A)$ (c) $f(2x)$ (d) $f(x+h)$

Solution: (a)

$$\begin{aligned} f(3) &= 3^2 \\ &= 9 \end{aligned}$$

(b) Assuming that A is a real number we have $f(A) = A^2$.

(c) Assuming that x is a real number we have

$$\begin{aligned} f(2x) &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

(d) Assuming that x and h are real numbers we have $f(x+h) = (x+h)^2$.

$\square$

Exercises

1. Sketch a graph, and state the domain and range for each of the following functions:

   (a) $f : [0, 2\pi] \longrightarrow \mathbf{R}$ where $f(x) = \sin x$

   (b) $f : [0, \pi] \longrightarrow \mathbf{R}$ where $f(x) = \sin x$

   (c) $f : [0, \pi) \longrightarrow \mathbf{R}$ where $f(x) = \sin x$

   (d) $f : (0, \pi) \longrightarrow \mathbf{R}$ where $f(x) = \sin x$

2. Are the following functions one–one?

   (a) $f \ : \ [0, 2\pi] \longrightarrow \mathbf{R}$ where $f(x) = \sin x$

   (b) $f \ : \ [0, \pi] \longrightarrow \mathbf{R}$ where $f(x) = \sin x$

   (c) $f \ : \ [0, 2\pi] \longrightarrow \mathbf{R}$ where $f(x) = \cos x$

   (d) $f \ : \ [0, \pi] \longrightarrow \mathbf{R}$ where $f(x) = \cos x$

3. Consider a function which has rule $f(x) = \sqrt{1-x}$. Find

   (a) $f(-3)$ (b) $f(A)$ (c) $f(2x)$

   (d) $f(x^3+1)$ (e) $f(x+h)$

4. Consider a function which has rule $f(x) = \sin 2x$. Find

   (a) $f(\frac{\pi}{2})$ (b) $f(A)$ (c) $f(2x)$

   (d) $f\left(\frac{1}{2}(x-\pi)\right)$ (e) $f(x+h)$

3.2 Maximal or Implied Domain

Often we are just told the rule for a function, and **not** told its domain. In this case, we assume that the domain is the set of all real numbers which are consistent with the given rule. That is, we assume that the domain is the set of

all x –values for which the function's rule "makes sense".

In particular,

- the denominator of any fraction can**not** be zero.

  That is, suppose $f(x) = \dfrac{g(x)}{h(x)}$. Then we need $h(x) \neq 0$.

- any expression under a square root (or any even root) must be positive or zero.

  That is, suppose $f(x) = \sqrt{g(x)}$. Then we need $g(x) \geq 0$.

We shall see later (on page 17) that

- any expression inside a logarithm must be strictly positive.
  That is, suppose $f(x) = \log_a(g(x))$. Then we need $g(x) > 0$.

Example 4. Find the maximal domain for the following functions:

(a) $f(x) = \dfrac{2x+6}{x+1}$

Solution: We need the denominator to be non-zero. That is,

$$\begin{aligned} x + 1 &\neq 0 \\ \therefore x &\neq -1. \end{aligned}$$

So $\mathrm{dom}(f) = \mathbf{R} \setminus \{-1\}$. $\square$

(b) $f(x) = \sqrt{x-1}$

Solution: We can only take square roots of non-negative numbers. That is,

$$\begin{aligned} x - 1 &\geq 0 \\ \therefore x &\geq 1. \end{aligned}$$

So $\mathrm{dom}(f) = [1, \infty)$. $\square$

Note: From now on, instead of writing 'maximal domain', we will just write 'domain'.

Exercises

Find the domain for the following functions:

(a) $f(x) = \dfrac{4x+1}{x+4}$

(b) $f(x) = \sqrt{2-x}$

(c) $f(x) = \dfrac{4}{1-x^2}$

(d) $f(x) = \sqrt{1-x^2}$

(e) $f(x) = \dfrac{4x}{\sqrt{1-x^2}}$

(f) $f(x) = 1 + x^2$

3.3 Some Basic Functions

Straight Lines

The rule for a straight line can be written in the form

$$\boxed{\quad y \;=\; mx + c \quad}$$

Sometimes this is written in the form $ax + by + d = 0$.

To sketch the graph of a straight line we sketch the line passing through two points satisfying the equation. The x and y intercepts are usually convenient points for this purpose.

- An x**–intercept** of a graph is a point where the graph cuts (or touches) the x–axis. To find any x–intercepts, put $y = 0$ into the graph's equation.

- A y**–intercept** of a graph is a point on the graph where the graph cuts (or touches) the y–axis. To find y–intercepts, put $x = 0$ into the graph's equation.

Example 5. Sketch $y \;=\; \frac{3}{5}x \;-\; 3$.

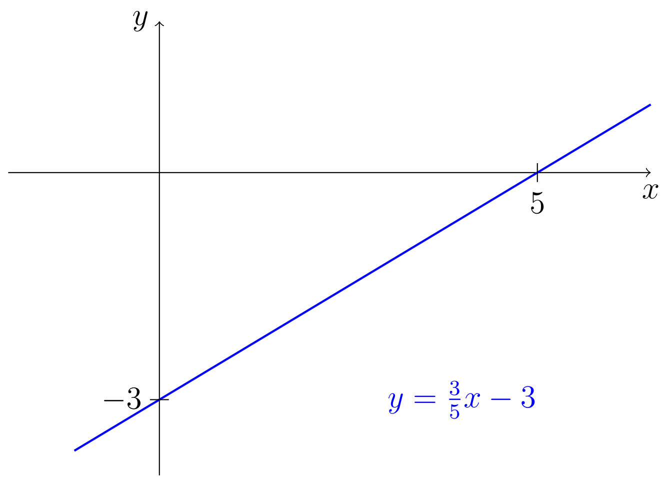
Solution: We first find the y–intercept by setting $x = 0$ in the equation for the line:

$$\begin{aligned} x = 0 \implies y &= \frac{3}{5} \times 0 - 3 \\ &= -3 \end{aligned}$$

We then find the x–intercept by setting $y = 0$ in the equation for the line:

$$\begin{aligned} y = 0 \implies &\frac{3}{5}x - 3 = 0 \\ &\implies x = 5 \end{aligned}$$

So the points $(0, 3)$ and $(5, 0)$ are points on the line. We simply plot those two points on the axes, and draw the straight line through those two points to obtain the following graph.



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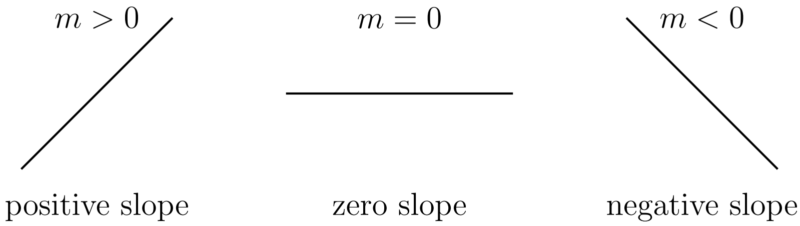
The **slope**, or **gradient**, of the line through the points (x_1, y_1) and (x_2, y_2) is given by

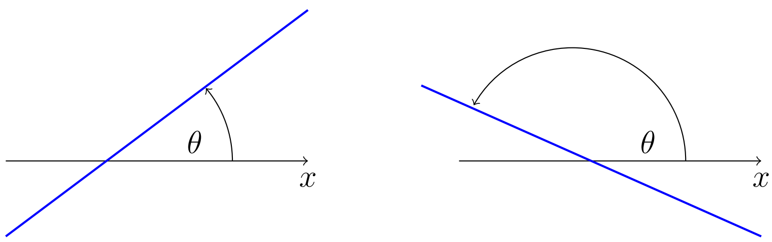
$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$



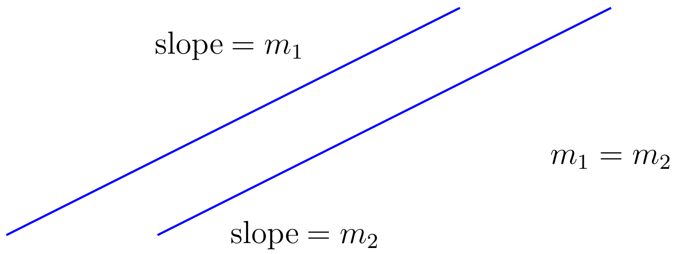
The slope, m, also satisfies

$$m = \tan\theta,$$

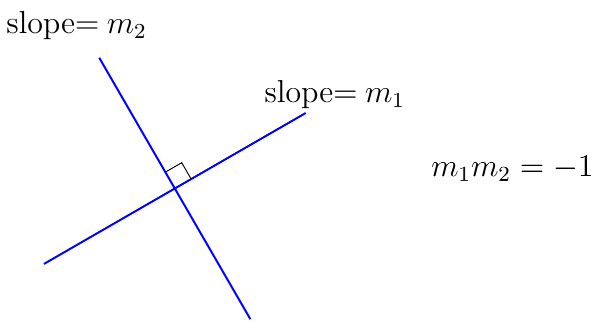
where θ is the angle which the line makes with the positive direction of the x-axis. We call θ the **angle of inclination**. Note that $0^\circ \leq \theta < 180^\circ$.



Two lines are **parallel** if they have the same slope.



Two lines are **perpendicular** to each other if they are at right angles (i.e. $90°$) to each other.



Result 1. Consider any two lines with slopes m_1 and m_2. Then the lines are perpendicular if and only if $m_1 m_2 = -1$.

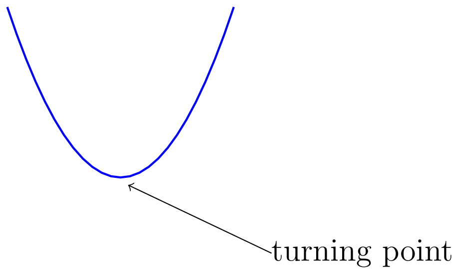
Parabolas

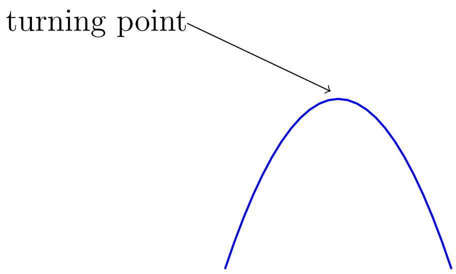
The rule for a parabola has the form

$$\boxed{y = ax^2 + bx + c}$$

If $a > 0$, a parabola has the basic shape:

If $a < 0$, a parabola has the basic shape:





To sketch the graph of a parabola we need to know

- the basic shape;
- the coordinates of any x and y intercepts, and
- the coordinates of the turning point.

The coordinates of the x and y intercepts and the turning point of

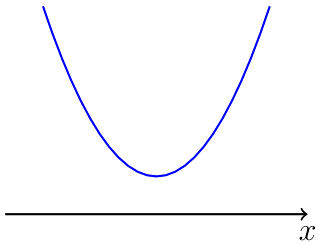
$$y = ax^2 + bx + c$$

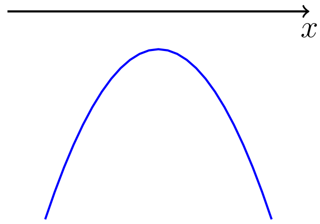
can be found as follows:

- To find any x–intercepts, we put $y = 0$ and solve $ax^2 + bx + c = 0$. Recall from Section 1.2 that the solutions of $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(the \textbf{quadratic formula})}.$$

  If the discriminant $b^2 - 4ac$ is negative, then there are no x–intercepts, and so the parabola lies entirely above or entirely below the x–axis.

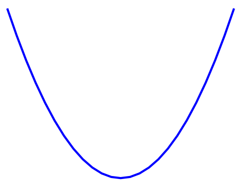




- To find the y –intercept, we put $x = 0$ into the equation, thus obtaining $y = c$.
- Since parabolas are symmetric, the x –value of the turning point lies halfway between the x –intercepts. Thus the x –value of the turning point is given by
$$x = \frac{-b}{2a} .$$
This holds even if the discriminant is negative.

Example 6. Sketch the graph of the parabola: $y = x^2 - 2x - 3$

Solution: **Basic shape:**



This is the basic shape because the coefficient of x^2 is 1 , which is positive.
x **–intercepts:** If $y = 0$ then we obtain

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ \text{i.e. } (x-3)(x+1) &= 0 \\ \text{So } x &= 3, \; -1 \end{aligned}$$

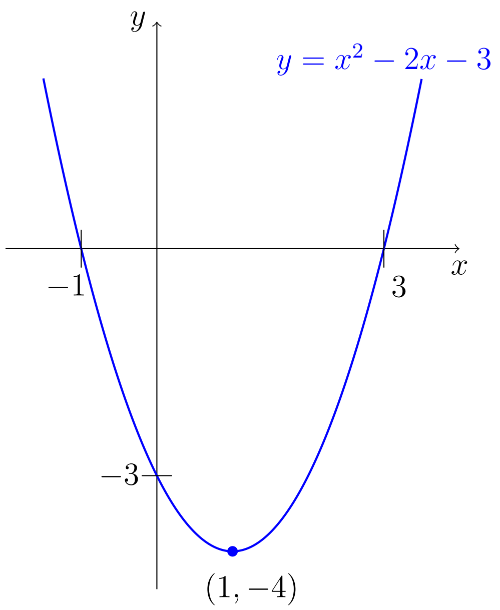
y **–intercept:** If $x = 0$ then we obtain

$$\begin{aligned} y &= 0^2 - 2 \times 0 - 3 \\ &= -3 . \end{aligned}$$

turning point:

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{2}{2} \\ &= 1 \\ \text{Note: } x = 1 \implies y &= 1^2 - 2(1) - 3 \\ &= -4 \end{aligned}$$

Therefore, the turning point is at $(1, -4)$. We now use all this information to sketch the required graph.



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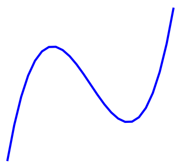
Cubics

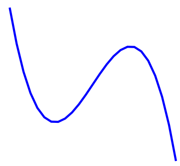
The rule for a cubic has the form

$$\boxed{y = ax^3 + bx^2 + cx + d}$$

If $a > 0$, a cubic has the shape:

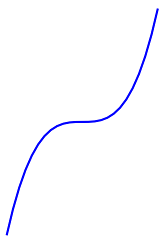


If $a < 0$, a cubic has the shape:

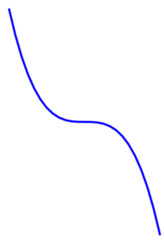


In the case when $b, c, d = 0$, the cubic's rule simplifies to $y = ax^3$ and its graph simplifies to:

if $a > 0$,



if $a < 0$.



To sketch the graph of a cubic we need to find

- the basic shape, and
- the coordinates of any x and y intercepts.

(In Chapter 10 we shall learn how to find the coordinates of any turning points.)

Example 7. Sketch a graph for the cubic $y = x^3 + x^2 - 4x - 4$.

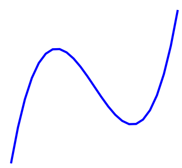
Solution:

$$\begin{aligned} x\text{–intercepts:} \qquad y = 0 \qquad &\Longleftrightarrow \qquad x^3 + x^2 - 4x - 4 = 0 \\ &\Longleftrightarrow \qquad (x-2)(x+1)(x+2) = 0 \end{aligned}$$

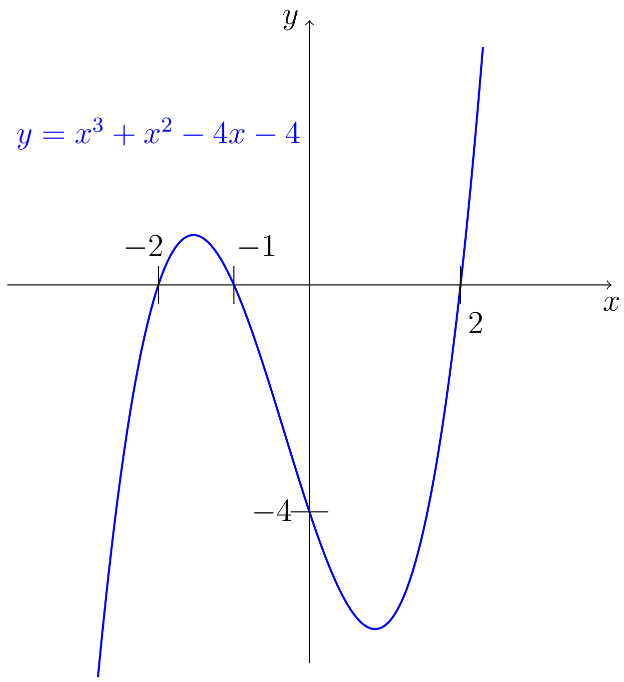
So the x–intercepts are at 2 , -1 and -2

$$y\text{–intercept: } x = 0 \implies y = 0^3 + 0^2 - 4 \times 0 - 4 = -4.$$

Basic shape:



We put this information together to obtain the graph:



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In Chapter 10 we will learn how to find the location of the turning points.

Absolute Value

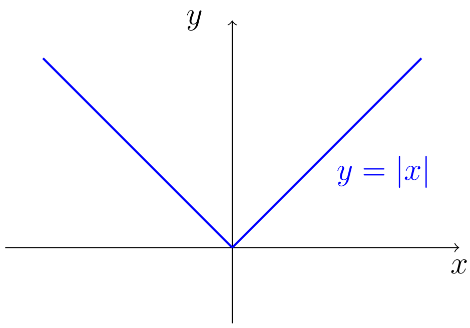
Consider the rule

$$\boxed{\; y \;=\; |x| \;}$$

Since

$$|x| = x \;\text{ if } x \geq 0 \quad \text{and} \quad |x| = -x \;\text{ if } x < 0$$

the graph of $y = |x|$ looks like:

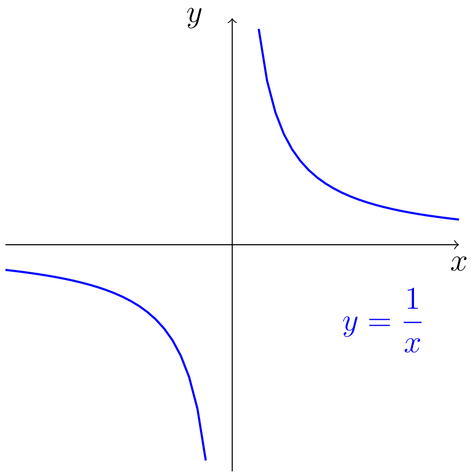


Hyperbolas

The simplest hyperbola has the rule

$$\boxed{\quad y = \frac{1}{x} \quad}$$

The graph of $y = \dfrac{1}{x}$ looks like:



Note that $y = \dfrac{1}{x}$ is not defined when $x = 0$. Thus the domain of the function $f(x) = \dfrac{1}{x}$ is $\mathbf{R} \setminus \{0\}$.

The graph shows that the *range* of $f(x) = \dfrac{1}{x}$ is also $\mathbf{R} \setminus \{0\}$.

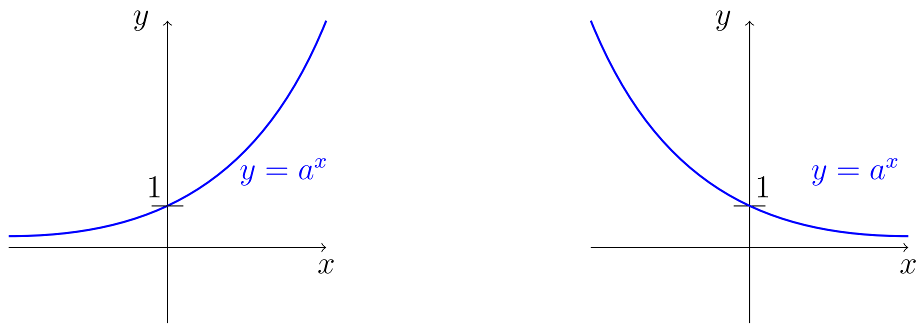
Exponential Functions

We will consider exponential functions of the form

$$\boxed{y = a^x, \quad \text{where } a > 0}$$

If $a > 1$, the graph of $y = a^x$ has the shape:

If $0 < a < 1$, the graph of $y = a^x$ has the shape:



$a > 1$ $\qquad\qquad$ $0 < a < 1$

The function $f(x) = a^x$ is defined for all x. That is, the domain of this function is $\mathbf{R}$.

The above graphs, show that the range of $f(x) = a^x$ is $(0, \infty)$.

Logarithmic Functions

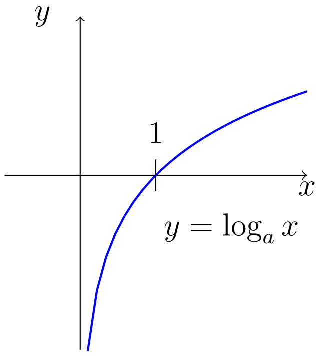
We will consider logarithmic functions

$$\boxed{y = \log_a x, \quad \text{where } a > 1}$$

The logarithmic function is defined by the rule

$$y = \log_a x \quad \text{if and only if} \quad x = a^y .$$

The graph of $y = \log_a x$ looks like



The function $f(x) = \log_a x$ is only defined for $x > 0$. That is, the domain of this function is $(0, \infty)$.

From the graph, we can see that the range of $f(x) = \log_a x$ is **R**.

Note: Suppose $f(x) = \log_a(g(x))$. Then we must have $g(x) > 0$.

Example 8. Find the domain of the function $f(x) = \log_3(5x + 10)$.

Solution:

$$\begin{aligned} \text{We must have } 5x + 10 &> 0 \\ \therefore 5x &> -10 \\ \therefore x &> -2 \\ \therefore \text{dom}(f) &= (-2, \infty) \end{aligned}$$

□

Semicircles

The graph corresponding to the equation

$$x^2 + y^2 = r^2$$

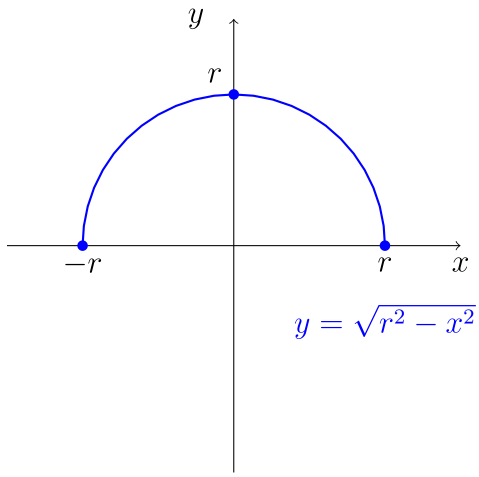
is a circle centred at the origin with radius r .

Thus (by the Vertical Line Test) $x^2+y^2 = r^2$ does **not** define a function. (See Example 2(b).)

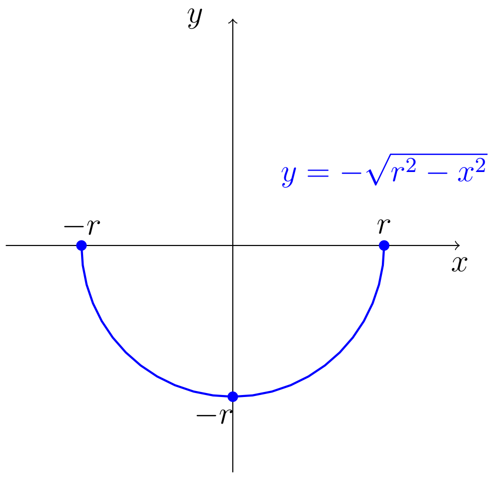
Rearranging the above equation gives $y^2 = r^2 - x^2$. That is,

$$y = \pm\sqrt{r^2 - x^2}\,.$$

The equation $y = \sqrt{r^2 - x^2}$ corresponds to the **upper semicircle**, which (by the Vertical Line Test) **does** represent a function.



Similarly, the equation $y = -\sqrt{r^2 - x^2}$ corresponds to the **lower semicircle**, which also represents a function.



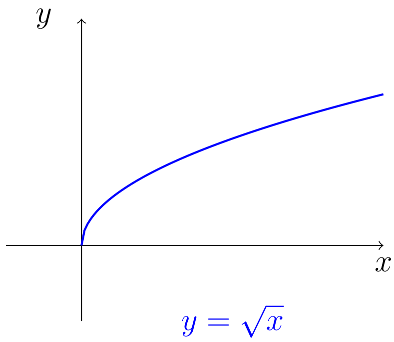
Note that the domain of each of these semicircular functions is $[-r, r]$.

The Square Root Graph

Consider the rule

$$\boxed{y = \sqrt{x}\,, \quad \text{where } x \geq 0}$$

The graph given by this equation looks like:



The domain and range of the function $f(x) = \sqrt{x}$ are both equal to $[0, \infty)$.

Exercises

1. On the same set of axes, sketch:

   (a) $y = 2x + 1$ (b) $y = 2x + 2$ (c) $y = 2x - 1$ (d) $y = 2x$

2. On the same set of axes, sketch: (a) $y = x$ (b) $y = -x$

3. Find the angle of inclination for each of the following straight lines. (Write your answers in degrees. In parts (c) and (d) give your answer to 3 decimal places.)

   (a) $y = x + 2$ (b) $x + y = 2$ (c) $y = 2x - 4$ (d) $y = -4x + 1$

4. State the domain and range for each of the following functions:

   (a) $f : \mathbf{R} \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (b) $f : [0, \infty) \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (c) $f : (0, \infty) \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (d) $f : \mathbf{R}^+ \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (e) $f : (-\infty, 0] \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (f) $f : \mathbf{R}^- \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (g) $f : [-3, 2) \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$
   (h) $f : \mathbf{R} \setminus \{0\} \longrightarrow \mathbf{R}$ $f(x) = 2x - 1$

5. Sketch the graphs for the following parabolas:

   (a) $y = x^2 - 1$ (b) $y = x^2 - 2x$ (c) $y = -x^2 + x + 2$

6. By using your graphs from 5, state the range of the following functions:

   (a) $f(x) = x^2 - 1$ (b) $f(x) = x^2 - 2x$ (c) $f(x) = -x^2 + x + 2$

7. Sketch the graphs for the following cubics:
   (a) $y = x^3 - x^2 - 2x$ (b) $y = -x^3 + 2x^2 + x - 2$ (c) $y = x^3 - 2x$

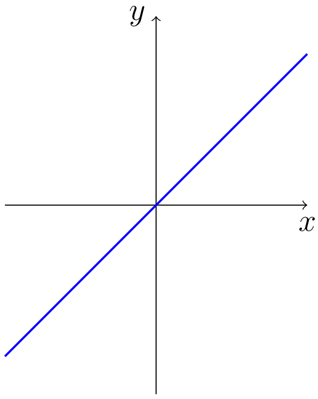
8. Find the domain of the following functions:

   (a) $f(x) = \log_2(2x - 1)$ (b) $f(x) = \log_{10}(1 - x)$
   (c) $f(x) = 2\log_4(x - 1)$

3.4 Chart of the Basic Graphs

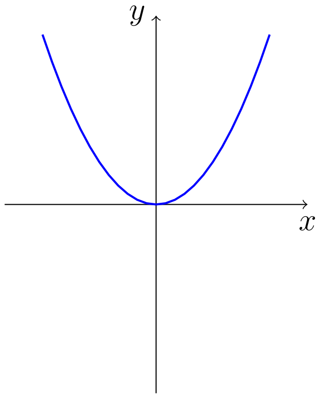
Here we provide a chart of each of the basic graphs introduced in this chapter. You **should learn these basic shapes**, as they will be needed for the more advanced graph sketching in the next chapter.

Straight Line:
$y = x$



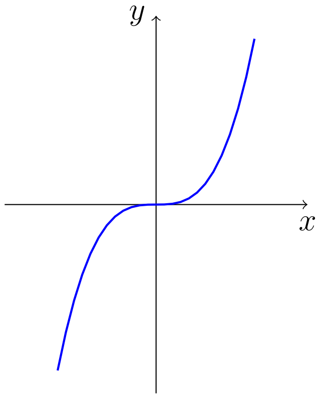
Domain = $\mathbf{R}$
Range = $\mathbf{R}$

Parabola:
$y = x^2$



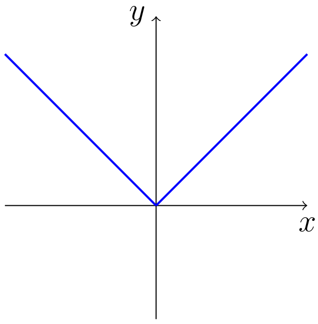
Domain = $\mathbf{R}$
Range = $[0, \infty)$

Cubic:
$y = x^3$



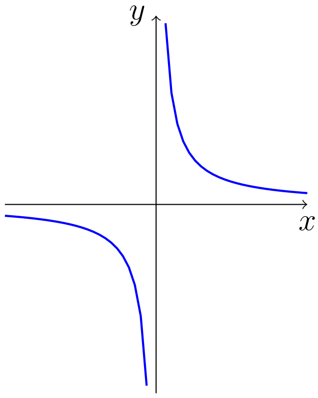
Domain = $\mathbf{R}$
Range = $\mathbf{R}$

Absolute Value:
$y = |x|$



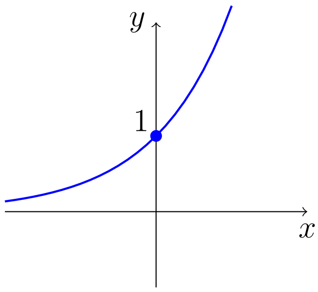
Domain $= \mathbf{R}$
Range $= [0, \infty)$

Hyperbola:
$y = \dfrac{1}{x}$



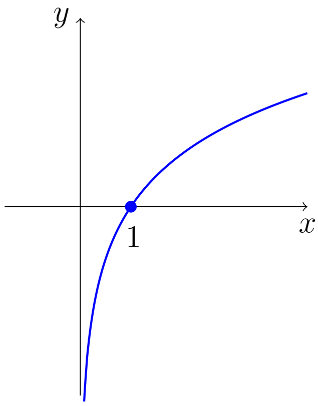
Domain $= \mathbf{R} \setminus \{0\}$
Range $= \mathbf{R} \setminus \{0\}$

Exponential:
$y = a^x$, $a > 1$



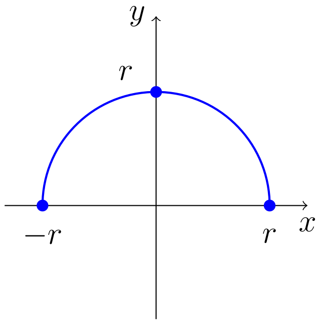
Domain $= \mathbf{R}$
Range $= (0, \infty)$

Logarithm:
$y = \log_a x$



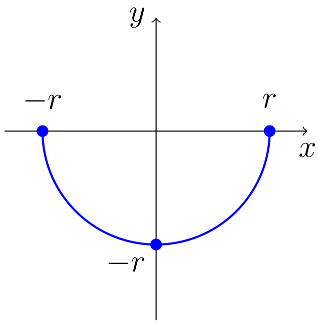
Domain $= (0, \infty)$
Range $= \mathbf{R}$

Upper Semicircle:
$y = \sqrt{r^2 - x^2}$



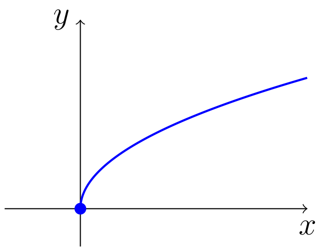
Domain $= [-r, r]$
Range $= [0, r]$

Lower Semicircle:
$y = -\sqrt{r^2 - x^2}$



Domain $= [-r, r]$
Range $= [-r, 0]$

Square Root:
$y = \sqrt{x}$



Domain $= [0, \infty)$
Range $= [0, \infty)$

3.5 Piecewise (or Hybrid) Functions

Often we want to consider a function which consists of pieces of different functions "stuck together". The standard way to do this is to list the various rules used, together with the corresponding domains, as illustrated in the following examples.

Example 9. Consider the function $f : \mathbf{R} \to \mathbf{R}$ with rule

$$f(x) = \begin{cases} x + 4 & \text{if } x < 1 \\ x^2 - 6x + 8 & \text{if } x \geq 1 \end{cases}$$

(a) Find $f(3)$.

(b) Find $f(0)$.

(c) Find $f(1)$.

(d) Sketch the graph of $y = f(x)$.

Solution:

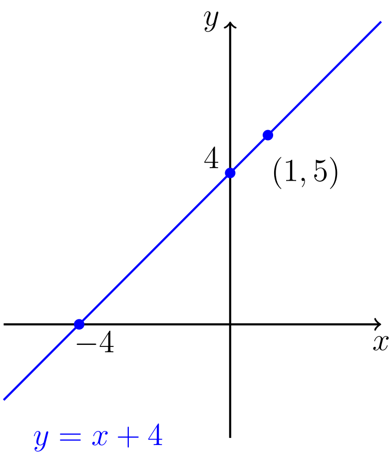
(a) To find $f(3)$ we put $x = 3$.
Since $x \geq 1$ we use the bottom part of the function.
Therefore, $f(3) = 3^2 - 6 \times 3 + 8 = -1$.

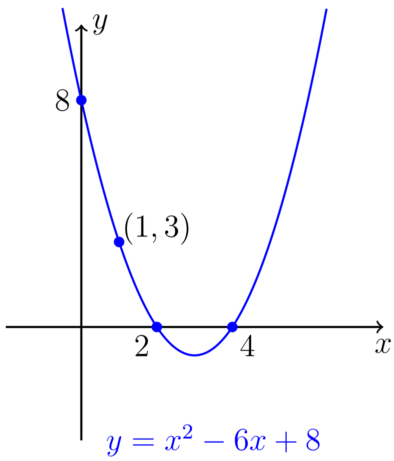
(b) To find $f(0)$ we put $x = 0$.
Since $x < 1$ we use the top part of the function.
Therefore, $f(0) = 0 + 4 = 4$.

(c) To find $f(1)$ we put $x = 1$.
Since $x \geq 1$ we use the bottom part of the function.
Therefore, $f(1) = 1^2 - 6 \times 1 + 8 = 3$.

(d) The sketches below show the graphs for

- the line $y = x + 4$ and
- the parabola $y = x^2 - 6x + 8 = (x - 2)(x - 4)$:

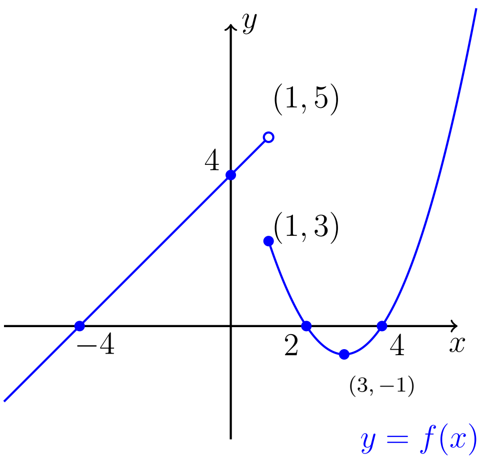




Thus the graph for

$$f(x) = \begin{cases} x+4 & \text{if } x < 1 \\ x^2 - 6x + 8 & \text{if } x \geq 1 \end{cases}$$

is as follows:



$\square$

Exercises

1. Consider the function $f(x) = \begin{cases} 2x+3 & \text{if } x < 0 \\ 2x-2 & \text{if } x \geq 0. \end{cases}$ Find

   (a) $f(-3)$ (b) $f(7)$ (c) $f(0)$

2. Sketch graphs for the following functions, and state their domains and ranges:

   (a) $f(x) = \begin{cases} 2x+2 & \text{if } x < 0 \\ 2x-2 & \text{if } x \geq 0 \end{cases}$

   (b) $f(x) = \begin{cases} 2x+2 & \text{if } -1 \leq x < 0 \\ 2x-2 & \text{if } 0 < x \leq 1 \end{cases}$

   (c) $f(x) = \begin{cases} x+2 & \text{if } x < -2 \\ 4-x^2 & \text{if } -2 \leq x \leq 2 \\ x-2 & \text{if } x > 2 \end{cases}$

3.6 Algebra of Functions

Consider functions f and g. Then the **sum** $f+g$ is defined in the obvious way, as follows:

$$(f+g)(x) = f(x) + g(x).$$

Suppose that we combine two functions together to make a new function (such as in the sum which was defined above). We need to be very careful when we determine the **domain** of the new function. In particular, we need to ensure that

all the steps in the working will make sense, and not just the final step.

We can make sure that **all** the steps in the working will make sense by realizing that

whenever we write a symbol (or expression) in the **brackets** of a function, then that symbol (or expression) **must** be in the **domain** of that function.

For example, when we write $f(x) + g(x)$, we have

x in the brackets of f, **and** x in the brackets of g.

Thus we need to have

$$x \in \mathrm{dom}(f) \textbf{ and } x \in \mathrm{dom}(g).$$

Example 10. Consider the functions

$$f : (-\infty, 0\,] \longrightarrow \mathbf{R} \text{ where } f(x) = \sqrt{1-x}$$

and

$$g : \mathbf{R} \setminus \{-2\} \longrightarrow \mathbf{R} \text{ where } g(x) = \frac{1}{x+2}\,.$$

Then

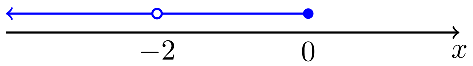
$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= \sqrt{1-x} + \frac{1}{x+2}\,.\end{aligned}$$

To find $\mathrm{dom}(f+g)$ we need

$$\begin{aligned} x \in \mathrm{dom}(f) &\textbf{ and } x \in \mathrm{dom}(g)\,. \\ \therefore x \in (-\infty, 0\,] &\textbf{ and } x \in \mathbf{R} \setminus \{-2\}\,. \\ \therefore x \leq 0 &\textbf{ and } x \neq -2\,. \\ \therefore \mathrm{dom}(f+g) &= (-\infty, 0\,] \setminus \{-2\}\,. \end{aligned}$$

Note that saying

$$x \leq 0 \quad \textbf{and} \quad x \neq -2$$



is the same as saying

$$x < -2 \quad \textbf{or} \quad -2 < x \leq 0 \,.$$

If you are asked to write the domain of $f+g$ as a union of intervals, then you should write:

$$\operatorname{dom}(f+g) = (-\infty,\, -2) \cup (-2,\, 0\,] \,.$$

The **difference** $f-g$, the **product** fg and the **quotient** $\dfrac{f}{g}$ are also defined in the obvious way, as follows:

- $(f-g)(x) = f(x) - g(x)$.

  Then $\operatorname{dom}(f-g)$ is equal to the set of x –values with

  $$x \in \operatorname{dom}(f) \quad \textbf{and} \quad x \in \operatorname{dom}(g).$$

- $(fg)(x) = f(x) \times g(x)$.

  Then $\operatorname{dom}(fg)$ is equal to the set of x –values with

  $$x \in \operatorname{dom}(f) \quad \textbf{and} \quad x \in \operatorname{dom}(g).$$

- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ as long as $g(x) \neq 0$.

  Then $\operatorname{dom}\left(\dfrac{f}{g}\right)$ is equal to the set of x –values with

  $$x \in \operatorname{dom}(f) \quad \textbf{and} \quad x \in \operatorname{dom}(g) \quad \text{and} \quad g(x) \neq 0.$$

Note:

$$\begin{aligned} (fg)(x) &= f(x) \times g(x) \\ &\neq f(g(x)) \quad \text{(We will learn about } f(g(x)) \text{ in Chapter 6)} \end{aligned}$$

Exercises

In each of the following, find the rules for $f+g,\ f-g,\ fg$ and $\dfrac{f}{g}$, and state the corresponding domains:

(a) $f(x) = \dfrac{x}{x-4}$ and $g(x) = \dfrac{1}{2x+3}$

(b) $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{x-1}$

(c) $f(x) = \log_2(x+2)$ and $g(x) = \dfrac{1}{x^2-1}$

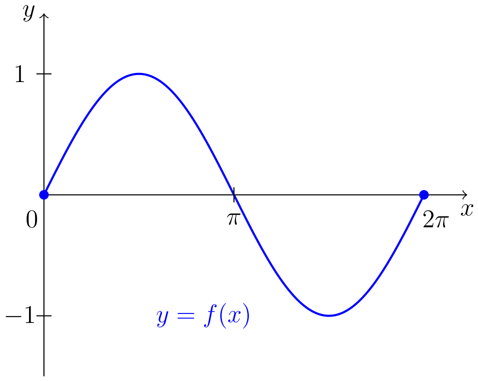
(d) $f(x) = x^2 - 2x + 1$ and $g(x) = \sqrt{x}$

(e) $f(x) = \sqrt{9-x^2}$ and $g(x) = 2x$

(f) $f(x) = \log_3 x$ and $g(x) = \sqrt{1-x}$

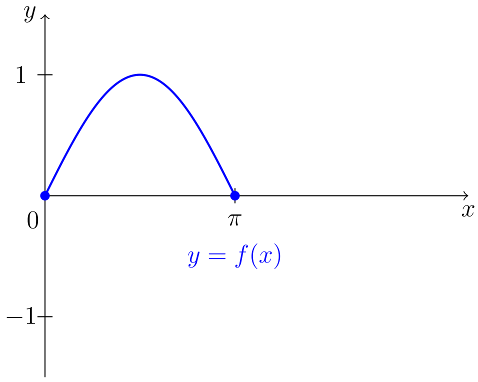
3.7 Answers to Chapter 3 Exercises

3.1: 1.(a)



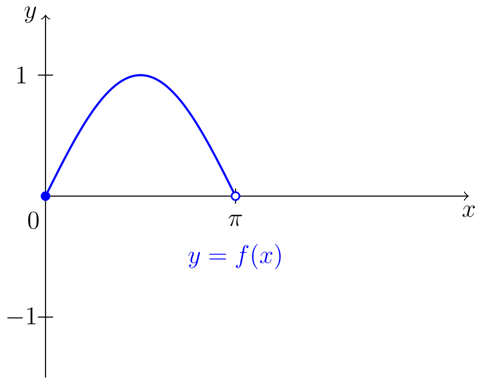
$\mathrm{dom}(f) = [0, 2\pi]$
$\mathrm{ran}(f) = [-1, 1]$

(b)

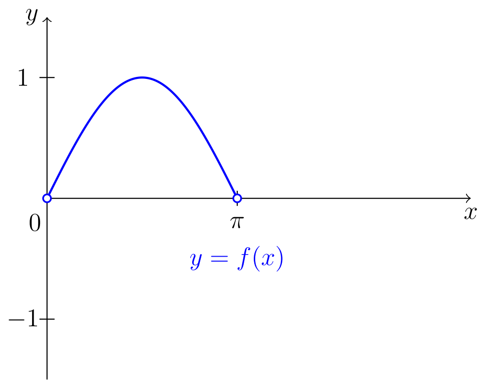


$\mathrm{dom}(f) = [0, \pi]$
$\mathrm{ran}(f) = [0, 1]$

(c)



$\mathrm{dom}(f) = [0, \pi)$
$\mathrm{ran}(f) = [0, 1]$

(d) 

$\mathrm{dom}(f) = (0, \pi)$
$\mathrm{ran}(f) = (0, 1]$

2.

(a) no (b) no (c) no (d) yes

3.

(a) 2 (b) $\sqrt{1-A}$ (c) $\sqrt{1-2x}$ (d) $\sqrt{-x^3}$

(e) $\sqrt{1-x-h}$

4.

(a) 0 (b) $\sin 2A$ (c) $\sin 4x$ (d) $\sin(x-\pi)$

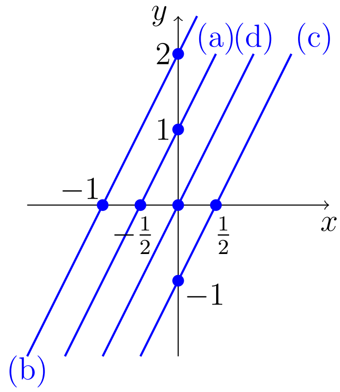
(e) $\sin(2(x+h))$

3.2: 1.

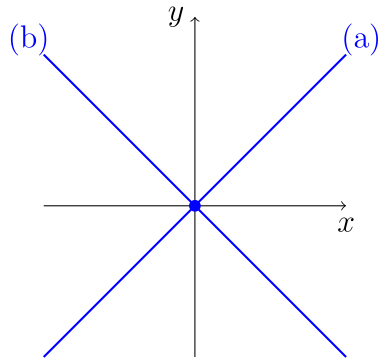
(a) $\mathbf{R} \setminus \{-4\}$ (b) $(-\infty, 2]$ (c) $\mathbf{R} \setminus \{\pm 1\}$

(d) $[-1, 1]$ (e) $(-1, 1)$ (f) $\mathbf{R}$

3.3: 1.



2.



3.

(a) $45°$ (b) $135°$ (c) $63.435°$ (d) $104.036°$

4.

(a) $\text{dom}(f) = \mathbf{R}$
$\text{ran}(f) = \mathbf{R}$

(b) $\text{dom}(f) = [0, \infty)$
$\text{ran}(f) = [-1, \infty)$

(c) $\text{dom}(f) = (0, \infty)$
$\text{ran}(f) = (-1, \infty)$

(d) $\text{dom}(f) = \mathbf{R}^+$
$\text{ran}(f) = (-1, \infty)$

(e) $\text{dom}(f) = (-\infty, 0]$
$\text{ran}(f) = (-\infty, -1]$

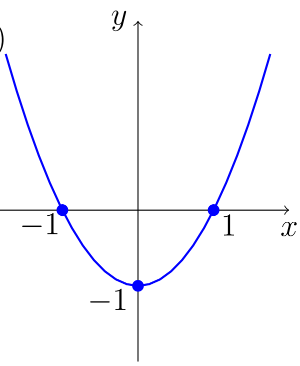
(f) $\text{dom}(f) = \mathbf{R}^-$
$\text{ran}(f) = (-\infty, -1)$

(g) $\text{dom}(f) = [-3, 2)$
$\text{ran}(f) = [-7, 3)$

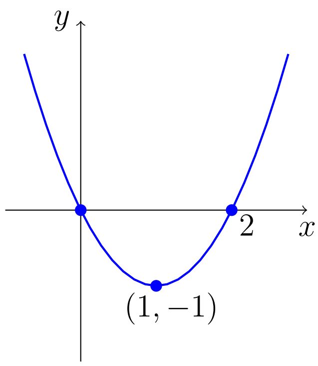
(h) $\text{dom}(f) = \mathbf{R} \setminus \{0\}$
$\text{ran}(f) = \mathbf{R} \setminus \{-1\}$

5.

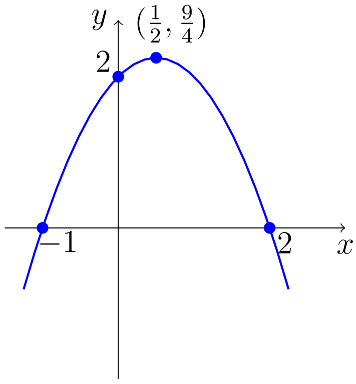
(a)



(b)



(c)



6.

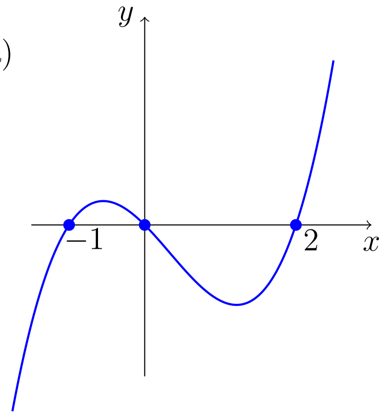
(a) $[-1, \infty)$

(b) $[-1, \infty)$

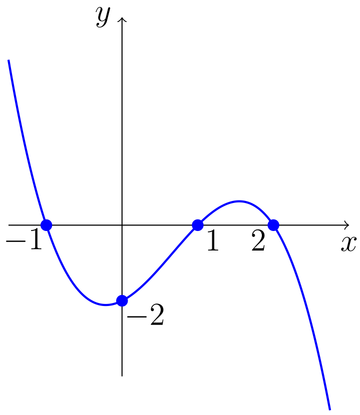
(c) $(-\infty, \frac{9}{4}]$

7.

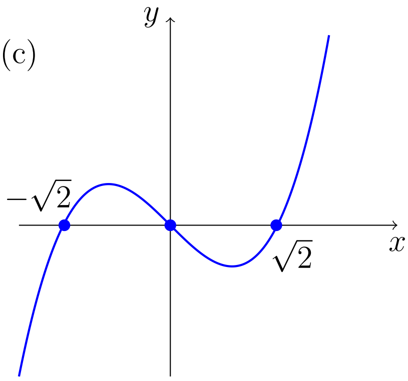
(a)



(b)



(c)



8.

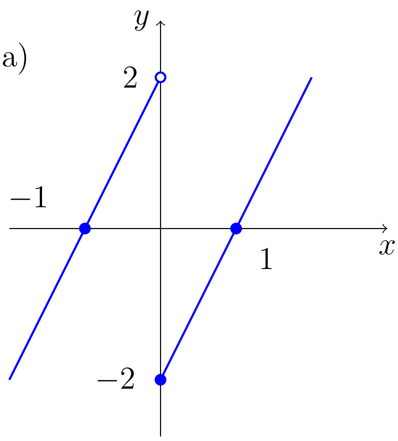
(a) $(\frac{1}{2}, \infty)$ (b) $(-\infty, 1)$ (c) $(1, \infty)$

3.5: 1.

(a) -3 (b) 12 (c) -2

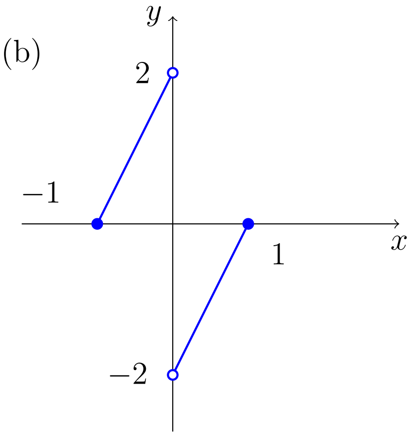
2.

(a)



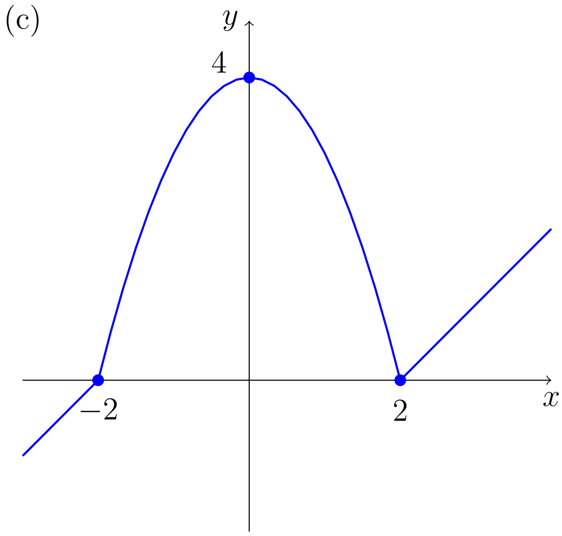
$\text{dom}(f) = \mathbf{R}$
$\text{ran}(f) = \mathbf{R}$

(b)



$$\text{dom}(f) = [-1,0) \cup (0,1]$$
$$\text{ran}(f) = (-2,2)$$

(c)



$$\text{dom}(f) = \mathbf{R}$$
$$\text{ran}(f) = \mathbf{R}$$

3.6: (a) $(f+g)(x) = \frac{x}{x-4} + \frac{1}{2x+3}$ $\quad$ $\text{dom}(f+g) = \mathbf{R} \setminus \{4, -\frac{3}{2}\}$

$(f-g)(x) = \frac{x}{x-4} - \frac{1}{2x+3}$ $\quad$ $\text{dom}(f-g) = \mathbf{R} \setminus \{4, -\frac{3}{2}\}$

$(fg)(x) = \frac{x}{(x-4)(2x+3)}$ $\quad$ $\text{dom}(fg) = \mathbf{R} \setminus \{4, -\frac{3}{2}\}$

$\left(\frac{f}{g}\right)(x) = \frac{x(2x+3)}{x-4}$ $\quad$ $\text{dom}(\frac{f}{g}) = \mathbf{R} \setminus \{4, -\frac{3}{2}\}$

(b) $(f+g)(x) = \sqrt{x+1} + \sqrt{x-1}$ $\quad$ $\text{dom}(f+g) = [1,\infty)$

$(f-g)(x) = \sqrt{x+1} - \sqrt{x-1}$ $\quad$ $\text{dom}(f-g) = [1,\infty)$

$(fg)(x) = \sqrt{(x+1)(x-1)}$ $\quad$ $\text{dom}(fg) = [1,\infty)$

$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x+1}{x-1}}$ $\quad$ $\text{dom}(\frac{f}{g}) = (1,\infty)$

(c) $(f+g)(x) = \log_2(x+2) + \frac{1}{x^2-1}$ $\quad$ $\operatorname{dom}(f+g) = (-2,\infty) \setminus \{\pm 1\}$

$(f-g)(x) = \log_2(x+2) - \frac{1}{x^2-1}$ $\quad$ $\operatorname{dom}(f-g) = (-2,\infty) \setminus \{\pm 1\}$

$(fg)(x) = \frac{\log_2(x+2)}{x^2-1}$ $\quad$ $\operatorname{dom}(fg) = (-2,\infty) \setminus \{\pm 1\}$

$\left(\frac{f}{g}\right)(x) = (x^2-1)\log_2(x+2)$ $\quad$ $\operatorname{dom}(\frac{f}{g}) = (-2,\infty) \setminus \{\pm 1\}$

(d) $(f+g)(x) = x^2 - 2x + 1 + \sqrt{x}$ $\quad$ $\operatorname{dom}(f+g) = [0,\infty)$

$(f-g)(x) = x^2 - 2x + 1 - \sqrt{x}$ $\quad$ $\operatorname{dom}(f-g) = [0,\infty)$

$(fg)(x) = x^{\frac{5}{2}} - 2x^{\frac{3}{2}} + \sqrt{x}$ $\quad$ $\operatorname{dom}(fg) = [0,\infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2-2x+1}{\sqrt{x}}$ $\quad$ $\operatorname{dom}(\frac{f}{g}) = (0,\infty)$

(e) $(f+g)(x) = \sqrt{9-x^2} + 2x$ $\quad$ $\operatorname{dom}(f+g) = [-3,3]$

$(f-g)(x) = \sqrt{9-x^2} - 2x$ $\quad$ $\operatorname{dom}(f-g) = [-3,3]$

$(fg)(x) = 2x\sqrt{9-x^2}$ $\quad$ $\operatorname{dom}(fg) = [-3,3]$

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{9-x^2}}{2x}$ $\quad$ $\operatorname{dom}(\frac{f}{g}) = [-3,0) \cup (0,3]$

(f) $(f+g)(x) = \log_3 x + \sqrt{1-x}$ $\quad$ $\operatorname{dom}(f+g) = (0,1]$

$(f-g)(x) = \log_3 x - \sqrt{1-x}$ $\quad$ $\operatorname{dom}(f-g) = (0,1]$

$(fg)(x) = (\log_3 x)\sqrt{1-x}$ $\quad$ $\operatorname{dom}(fg) = (0,1]$

$\left(\frac{f}{g}\right)(x) = \frac{\log_3 x}{\sqrt{1-x}}$ $\quad$ $\operatorname{dom}(\frac{f}{g}) = (0,1)$