Chapter 13

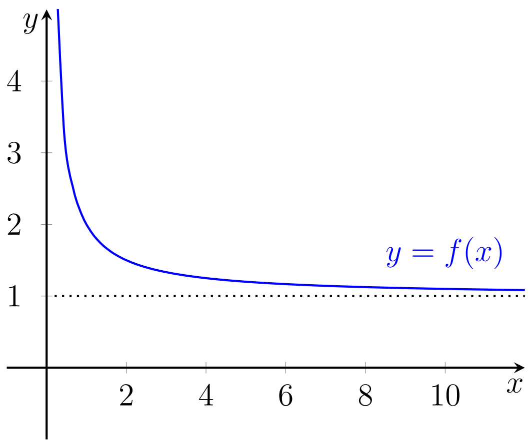
Limits and Integration to Infinity

Reference: “Calculus”, by James Stewart.

13.1 Limits to Infinity

Example 1. Consider the function $f(x) = \dfrac{1}{x} + 1\,.$

The graph of the curve $y = f(x)$ for $x > 0$ is drawn below:



Notice that $f(x)$ approaches 1 as x gets larger and larger. We write this in symbolic form as

$$\lim_{x \to \infty} f(x) = 1$$

or we might write “ $f(x) \to 1$ as $x \to \infty$ ” .

Example 2. Investigate $\displaystyle\lim_{x\to\infty} \frac{1}{x}$.

Solution: Note that

$$x = 100 \quad \Rightarrow \quad \frac{1}{x} = 0.01$$

$$x = 10000 \quad \Rightarrow \quad \frac{1}{x} = 0.0001$$

$$x = 1000000 \quad \Rightarrow \quad \frac{1}{x} = 0.000001$$

and so when x is large, $\frac{1}{x}$ is close to zero. In fact, by choosing x large enough, we can make $\frac{1}{x}$ as close to zero as we like, and so we can write

$$\lim_{x\to\infty} \frac{1}{x} \;=\; 0 \;.$$

□

Definition 1. When there is a real number L such that $f(x)$ can be made as close to L as we like for all sufficiently large x , the limit as x approaches infinity of $f(x)$ exists, and we write

$$\lim_{x\to\infty} f(x) = L\,.$$

In general, we have the following result:

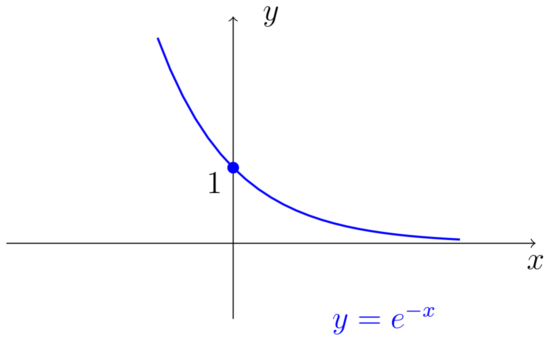
Result 1. For each constant r ,

$$\boxed{\text{if} \quad r > 0 \qquad \text{then} \qquad \lim_{x\to\infty} \frac{1}{x^r} \;=\; 0\;.}$$

We do not prove the above result.

Example 3. Investigate $\lim_{x\to\infty} e^{-x}$.

Solution: We can investigate this limit by considering the graph of $y = e^{-x}$.



From the graph, we see that y approaches zero as x approaches infinity. In symbols:

$$\lim_{x\to\infty} e^{-x} = 0$$

$\square$

Notes.

1. We also say that e^{-x} converges to 0 as x approaches infinity.
2. The existence of the limit is equivalent to a horizontal asymptote on the graph.

Limit Laws

Suppose that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$, (where $L, M \in \mathbf{R}$) and that c is a constant. Then

1. $\lim_{x\to\infty} \Big[f(x) + g(x)\Big] = L + M$.

2. $\lim_{x\to\infty} \Big[f(x) - g(x)\Big] = L - M$.

3. $\lim_{x\to\infty} \Big[f(x)g(x)\Big] = L \times M$.

4. if $M \neq 0$, then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

5. $\lim_{x\to\infty} c \;=\; c$.

6. $\lim_{x\to\infty} \Big[cf(x)\Big] \;=\; cL.$

Example 4. Find $\lim_{x\to\infty} \dfrac{5}{\sqrt{x}}$.

Solution: By Result 1, we have $\lim_{x\to\infty} \dfrac{1}{\sqrt{x}} \;=\; \lim_{x\to\infty} \dfrac{1}{x^{\frac{1}{2}}} \;=\; 0$.

Now by Limit Law 6, we have

$$\lim_{x\to\infty} \frac{5}{\sqrt{x}} \;=\; 5 \lim_{x\to\infty} \frac{1}{\sqrt{x}} \;=\; 5 \times 0 \;=\; 0 \ .$$

$\square$

Example 5. Find $\lim_{x\to\infty} \left(\dfrac{1}{x^2} + 7\right)$.

Solution: By Limit Laws 1 and 5, and Result 1 , we have

$$\lim_{x\to\infty} \left(\frac{1}{x^2} + 7\right) \;=\; \lim_{x\to\infty} \frac{1}{x^2} \;+\; \lim_{x\to\infty} 7 \;=\; 0 + 7 \;=\; 7 \ .$$

$\square$

Example 6. Find $\displaystyle\lim_{x\to\infty} \frac{2x^2+x+10}{3x^2+7x-12}$.

Solution: First note that, as $x \to \infty$, the polynomials $2x^2+x+10$ and $3x^2+7x-12$ do not approach a fixed number, and so we cannot use the limit laws immediately.

To evaluate this limit, we divide both the numerator and denominator by the highest power of x in the denominator:

$$\frac{2x^2+x+10}{3x^2+7x-12} = \frac{(2x^2+x+10)/x^2}{(3x^2+7x-12)/x^2} = \frac{2+\frac{1}{x}+\frac{10}{x^2}}{3+\frac{7}{x}-\frac{12}{x^2}}.$$

Now we can use the limit laws:

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2+x+10}{3x^2+7x-12} &= \lim_{x\to\infty} \frac{2+\frac{1}{x}+\frac{10}{x^2}}{3+\frac{7}{x}-\frac{12}{x^2}} \\
&= \frac{\displaystyle\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{10}{x^2}}{\displaystyle\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{7}{x} - \lim_{x\to\infty} \frac{12}{x^2}} \\
&= \frac{2+0+0}{3+0-0} = \frac{2}{3}.
\end{aligned}$$

$\square$

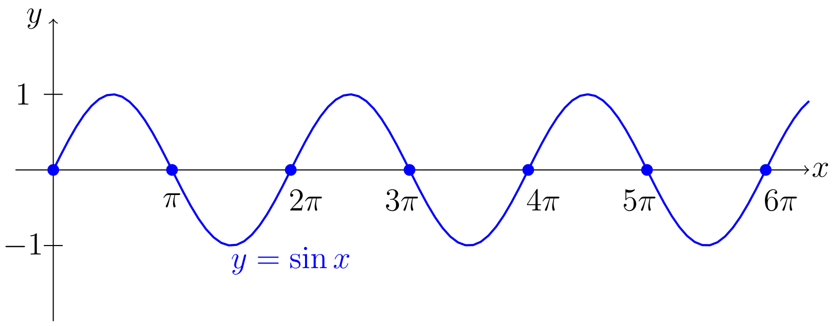
If $f(x)$ converges to a limit as $x \to \infty$, then it must be true that for sufficiently large values of x, the values of $f(x)$ will remain close to that limit. That is, we can bound the values of $f(x)$ as x approaches infinity by y-values above and below the limit.

Sometimes, however, a function remains bounded but does not converge to a particular number as x approaches infinity. In this case, the limit of $f(x)$ does not exist.

Example 7. Investigate $\lim_{x\to\infty} \sin x$.

Solution: The function $\sin x$ oscillates between -1 and 1 , as x goes to infinity, and so $\sin x$ never converges to a particular number. Thus

$$\lim_{x\to\infty} \sin x \quad \text{does not exist.}$$



□

Infinite Limits at Infinity

Definition 2. We write

$$\lim_{x\to\infty} f(x) = \infty$$

when $f(x)$ can be made as large as we like for all x sufficiently large.

We write

$$\lim_{x\to\infty} f(x) = -\infty$$

when $f(x)$ can be made as large **and negative** as we like for all x sufficiently large.

Notes.

1. It is important to remember that $-\infty$ and ∞ are **not** numbers.
2. If $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$ then $\lim_{x\to\infty} f(x)$ **does not exist** (since $f(x)$ is not approaching any particular number as $x \to \infty$).

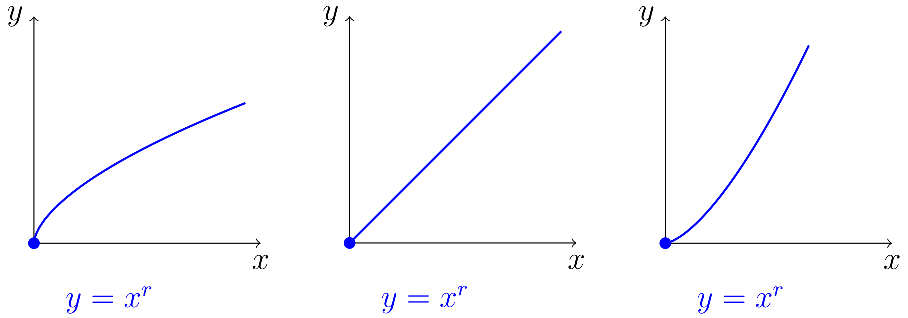
Result 2. For each constant r ,

$$\boxed{\text{if} \quad r > 0 \quad \text{then} \quad \lim_{x\to\infty} x^r \ = \ \infty \ .}$$

We do not prove the above result, but it is believable when we draw the graph of $y = x^r$.

Consider the following graphs of $y = x^r$ for the cases

$0 < r < 1$ $\qquad$ $r = 1$ $\qquad$ $r > 1$



In each case the graph suggests that

$$\lim_{x\to\infty} x^r = \infty$$

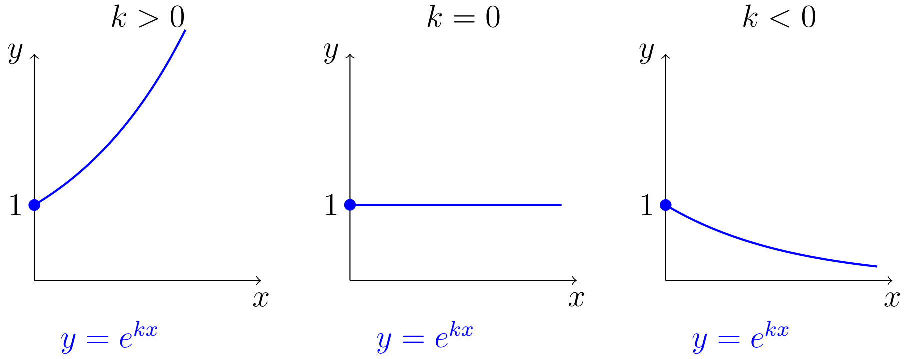
Result 3. For each constant k ,

$$\boxed{\lim_{x\to\infty} e^{kx} \ = \ \begin{cases} \infty & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k < 0\,. \end{cases}}$$

We do not prove the above result, but it is believable when we draw the graph of $y = e^{kx}$.

Consider the following graphs of $y = e^{kx}$ for the cases

$k > 0$ $\qquad$ $k = 0$ $\qquad$ $k < 0$



The above graphs suggest the following results:

- If $k > 0$, then $\lim_{x\to\infty} e^{kx} = \infty$.
- If $k = 0$, then $\lim_{x\to\infty} e^{kx} = 1$.
- If $k < 0$, then $\lim_{x\to\infty} e^{kx} = 0$.

Result 4. $\lim_{x\to\infty} \ln x = \infty$.

Proof. (Not examinable)
Consider any large number N . We need to make $\ln x > N$ by choosing x to be large enough. Note that $\ln x$ is an increasing function and so

$$x > e^N \quad \Longrightarrow \quad \ln x > \ln\left(e^N\right) = N\,.$$

Therefore, $\ln x$ is larger than N for all $x > e^N$. $\square$

Limits to Negative Infinity

Definition 3. For a real number L , we write

$$\lim_{x\to-\infty} f(x) = L$$

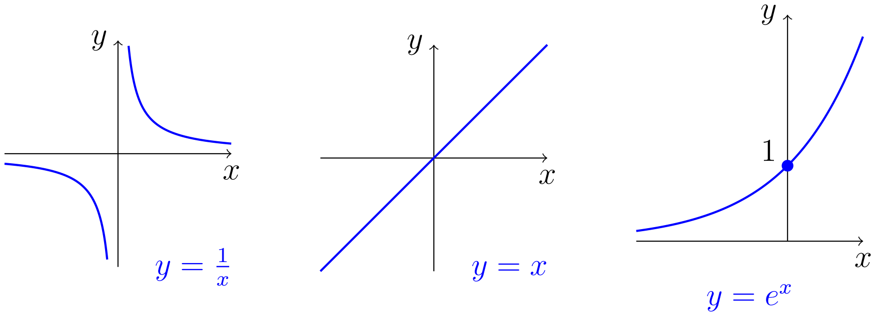
when $f(x)$ can be made as close to L as we like for all x sufficiently large **and negative**. In this case, we say that $\lim_{x\to-\infty} f(x)$ **exists.**

We define

$$\lim_{x\to-\infty} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = -\infty$$

in a similar way. Note that these infinite limits do not exist.

Example 8. Consider the following graphs:



By following the curves to left on these graphs, we conclude that

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} x = -\infty$$

$$\text{and} \quad \lim_{x \to -\infty} e^x = 0$$

The results and laws for limits to negative infinity are similar to those for limits to infinity. In particular, we have

1. All the limit laws for limits to ∞ also hold for limits to $-\infty$. For example,
   if $\lim_{x \to -\infty} f(x)$ exists and $\lim_{x \to -\infty} g(x)$ exists then
   $$\lim_{x \to -\infty} \Big[f(x) + g(x)\Big] = \lim_{x \to -\infty} f(x) + \lim_{x \to -\infty} g(x).$$
2. If k is a constant then $\lim_{x \to -\infty} e^{kx} = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \infty & \text{if } k < 0. \end{cases}$

For limits of powers of x, however, we need to be careful. Functions that are powers of x are not always defined on the negative numbers (e.g. $\sqrt{x}$).

If we consider only integer powers of x, we have

3. If $n \in \{1, 2, 3, \ldots\}$ then $\lim_{x \to -\infty} \frac{1}{x^n} = 0$.
4. If $n \in \{1, 2, 3, \ldots\}$ then $\lim_{x \to -\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$

Exercises

Find the following limits.

1. (a) $\lim_{x\to\infty} x^7$ (b) $\lim_{b\to\infty} e^{2b}$ (c) $\lim_{x\to\infty} \sqrt{x}$ (d) $\lim_{a\to\infty} \dfrac{1}{\sqrt{a}}$.

2. (a) $\lim_{x\to\infty} \dfrac{1}{e^{3x}}$ (b) $\lim_{b\to\infty} \dfrac{1}{b^7}$.

3. (a) $\lim_{x\to-\infty} x^2$ (b) $\lim_{x\to-\infty} \dfrac{1}{x^2}$.

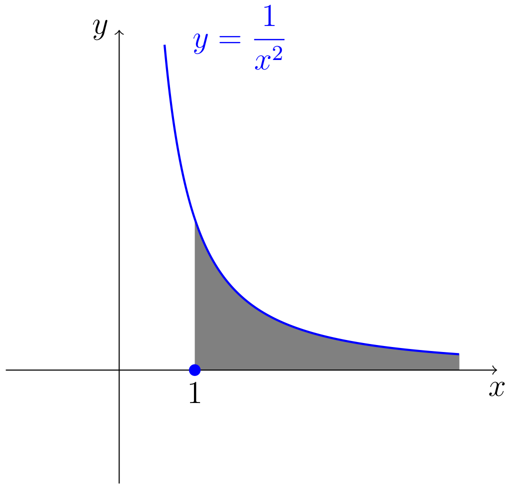
4. (a) $\lim_{x\to-\infty} e^{-x}$ (b) $\lim_{x\to-\infty} (1+e^x)$.

5. Let $f(x) = 3x$ and $g(x) = \dfrac{3}{x}$. Find

   (a) $\lim_{x\to\infty} f(x)$ (b) $\lim_{x\to\infty} g(x)$ (c) $\lim_{x\to\infty} \Big[f(x)g(x)\Big]$.

13.2 Integration to Infinity

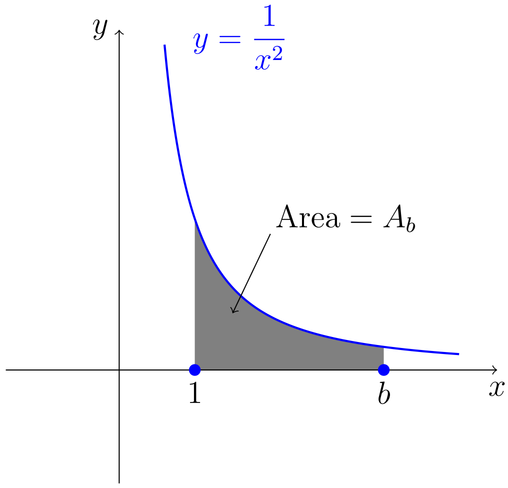
Example 9. Consider the infinite region under the curve $y = \dfrac{1}{x^2}$ that lies to the right of the line $x = 1$, as shown in the following diagram:



Since this region extends all the way to infinity (along the positive $x-$ axis), you might expect that the area of the region is infinite, but it turns out that the area is in fact a finite number.

Let us denote the area of the infinite region with the symbol A_∞ .
We now show how to find the value of A_∞ as a limit.

We first find the area A_b under the curve $y = \dfrac{1}{x^2}$ between the lines $x = 1$ and $x = b$.



This area is given by

$$A_b = \int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b = -\frac{1}{b} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{b}\,.$$

Notice that

- A_b is a function of b , and
- the area we want, A_∞ , is found as $A_\infty = \lim\limits_{b\to\infty} A_b\,.$

Now

$$A_\infty = \lim_{b\to\infty} A_b = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1 - \lim_{b\to\infty}\frac{1}{b} = 1 - 0 = 1\ .$$

Instead of writing A_∞ , we will now write just $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$, and so

$$\boxed{\int_1^\infty \frac{1}{x^2}\,dx \;=\; \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx \;=\; \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b \;=\; \ldots \;=\; 1\,.}$$

In general, we define

$$\boxed{\int_a^\infty f(x)\,dx \;=\; \lim_{b\to\infty}\int_a^b f(x)\,dx}$$

and

$$\boxed{\int_{-\infty}^b f(x)\,dx \;=\; \lim_{a\to-\infty}\int_a^b f(x)\,dx}$$

provided that the limits exist.

An integral of the form $\displaystyle\int_a^\infty f(x)\,dx$ or $\displaystyle\int_{-\infty}^b f(x)\,dx$ is called an **improper integral**.

An improper integral is the limit of definite integrals as one of the terminals approaches infinity (or negative infinity). Thus, although the notation does not include a “limit”, we **must** bring that in at the next step.

Here is an example where the improper integral does not exist:

Example 10. Find $\displaystyle\int_1^\infty \frac{1}{x}\,dx\,.$

Solution: Since

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \;=\; \lim_{b\to\infty}\big[\ln|x|\big]_1^b \;=\; \lim_{b\to\infty}\big(\ln|b| - \ln|1|\big) \;=\; \lim_{b\to\infty}\ln b \;=\; \infty\ ,$$

we have that $\displaystyle\int_1^\infty \frac{1}{x}\,dx \;=\; \infty\,.$

Thus, the corresponding area under the curve $y = \frac{1}{x}$ is infinite. $\square$

Splitting improper integrals into pieces

Improper integrals can be broken into two parts as follows:

$$\boxed{\int_a^\infty f(x)\,dx \;=\; \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx}$$

We choose the value of c in such a way that our calculations are as simple as possible.

Example 11. Consider the function $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ \dfrac{1}{x^2} & \text{if } x > 1\,. \end{cases}$

Find $\displaystyle\int_0^\infty f(x)\,dx\,.$

Solution:

$$
\begin{aligned}
\int_0^\infty f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx \\
&= \int_0^1 x\,dx + \int_1^\infty \frac{1}{x^2}\,dx \\
&= \left[\frac{1}{2}x^2\right]_0^1 + \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx \\
&= \left(\frac{1}{2}1^2 - \frac{1}{2}0^2\right) + \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b \\
&= \frac{1}{2} + \lim_{b\to\infty}\left(\frac{-1}{b} - \frac{-1}{1}\right) \\
&= \frac{1}{2} + 0 + 1 \\
&= \frac{3}{2}\,.
\end{aligned}
$$

□

Also, improper integrals to $-\infty$ can be broken up into two integrals:

$$\boxed{\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx}$$

We now define $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$. This integral includes two infinity symbols, and so must be found by taking two separate limits.

Suppose that both $\displaystyle\int_{-\infty}^{0} f(x)\,dx$ and $\displaystyle\int_{0}^{\infty} f(x)\,dx$ exist, i.e., that both of these limits are finite numbers.

Then we define

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx \;=\; \int_{-\infty}^{0} f(x)\,dx \;+\; \int_{0}^{\infty} f(x)\,dx}$$

(There is nothing special about 0 in the above definition, you are welcome to choose any number; the result always will be the same.)

Example 12. Find $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ if $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 2 \\ e^{-x} & \text{if } x \ge 2\,. \end{cases}$

Solution: We have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} 1\,dx + \int_{2}^{\infty} e^{-x}\,dx \\
&= 0 + \Big[x\Big]_0^2 + \lim_{b\to\infty} \int_{2}^{b} e^{-x}\,dx \\
&= (2-0) + \lim_{b\to\infty} \Big[-e^{-x}\Big]_2^b \\
&= 2 + \lim_{b\to\infty} \Big(-e^{-b} - \big(-e^{-2}\big)\Big) \\
&= 2 + \big(0 + e^{-2}\big) \\
&= 2 + e^{-2}\,.
\end{aligned}$$

□

Note. In the above example, the values of f at $x = 0$ and $x = 2$ do not affect the answer, since the area of a vertical line segment is zero. For example,

$$\int_2^2 1\,dx = \Big[x\Big]_2^2 = 2-2 = 0 \quad \text{and} \quad \int_2^2 e^{-x}\,dx = \Big[-e^{-x}\Big]_2^2 = -e^{-2}-\big(-e^{-2}\big) = 0\,.$$

So, for example, we would still evaluate the integral in exactly the same way and we would get the same answer if

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 2 \\ e^{-x} & \text{if } x > 2 \end{cases}$$

(notice that the value of f at $x = 2$ has been changed).

Exercises

Evaluate the following improper integrals.

1. (a) $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ (b) $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^3}}\,dx$ (c) $\displaystyle\int_2^\infty \frac{1}{(2x-3)^2}\,dx$ (d) $\displaystyle\int_0^\infty e^{-t}\,dt\,.$

2. (a) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2}\,dx$ (b) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$ (c) $\displaystyle\int_{-\infty}^{1} e^{3x}\,dx$ (d) $\displaystyle\int_{-\infty}^{1} e^{-x}\,dx\,.$

3. $\displaystyle\int_0^\infty f(x)\,dx$ where $f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ \dfrac{1}{x^2} & \text{if } x > 1\,. \end{cases}$

4. $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ where $f(x) = \begin{cases} e^x & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ e^{-x} & \text{if } x > 1\,. \end{cases}$

13.3 Answers to Chapter 13 Exercises

13.1: 1. (a) ∞ (b) ∞ (c) ∞ (d) 0.

2. (a) 0 (b) 0.
3. (a) ∞ (b) 0.
4. (a) ∞ (b) 1.
5. (a) ∞ (b) 0 (c) 9.

13.2: 1. (a) $\frac{1}{3}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{1}{2}$ (d) 1.

2. (a) 1 (b) $-\frac{1}{2}$ (c) $\frac{1}{3}e^3$ (d) ∞.
3. $\frac{4}{3}$.
4. $\frac{3}{2} + \frac{1}{e}$.