C6: COMPOSITION OF FUNCTIONS

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6.1 COMPOSITE FUNCTIONS

$f \circ g(x) = f(g(x))$

$\boxed{f \circ g \neq g \circ f.}$

when

$y = f(g(x))$

then

$\boxed{x \in \text{dom}(g) \text{ and } g(x) \in \text{dom}(f)}$

When y=

$g(f(x))$.

then

$\boxed{x \in \text{dom}(f) \text{ and } f(x) \in \text{dom}(g)}$

EXERCISES [pg5]

1. If $f(x) = x^2$ and $g(x) = 2x + 4$ then

(a) find the rule for $f(g(x))$.

$f(g(x)) = f(2x+4)$
$= (2x+4)^2$ ✓
$= 4x^2 + 16x + 16$

(b) find the rule for $g(f(x))$.

$g(f(x)) = g(x^2)$
$= 2x^2 + 4$ ✓

2. Consider the functions

$$f : [-4, 6] \longrightarrow \mathbf{R} \text{ where } f(x) = 10x - 2$$

and

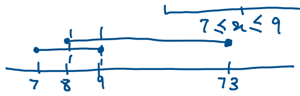
$$g : [8, 73] \longrightarrow \mathbf{R} \text{ where } g(x) = 5x - 39.$$

(a) Find dom$(f \circ g)$.

$x \in \text{dom}(g)$ and $g(x) \in \text{dom}(f)$
$x \in [8, 73]$ and $5x - 39 \in [-4, 6]$
$8 \leqslant x \leqslant 73$ $\quad -4 \leqslant 5x - 39 \leqslant 6$

$-4 \leqslant 5x - 39$ $\quad 5x - 39 \leqslant 6$
$35 \leqslant 5x$ $\quad 5x \leqslant 45$
$7 \leqslant x$ $\quad x \leqslant 9$

$7 \leqslant x \leqslant 9$



$\text{dom}(f \circ g) = [8, 9]$ ✓

(b) Find dom($g \circ f$).

(c) Find the rule for $f(g(x))$.

(d) Find the rule for $g(f(x))$.

3. If $f(x) = \begin{cases} x^2 - 2 & \text{if } x > 1 \\ 1 - x & \text{if } x \leq 1 \end{cases}$ and $g(x) = 2x$
then find the rule for $f(g(x))$.

6.2 INVERSE FUNCTIONS

$$f(g(x)) = x$$

and

$$g(f(x)) = x.$$

When f and g have the above property, we say that g is the **inverse** of f .

When g is the **inverse** of f , we write g as f^{-1}.

Using this notation, we can rewrite the above cancellation equations as

$$f(f^{-1}(x)) = x$$

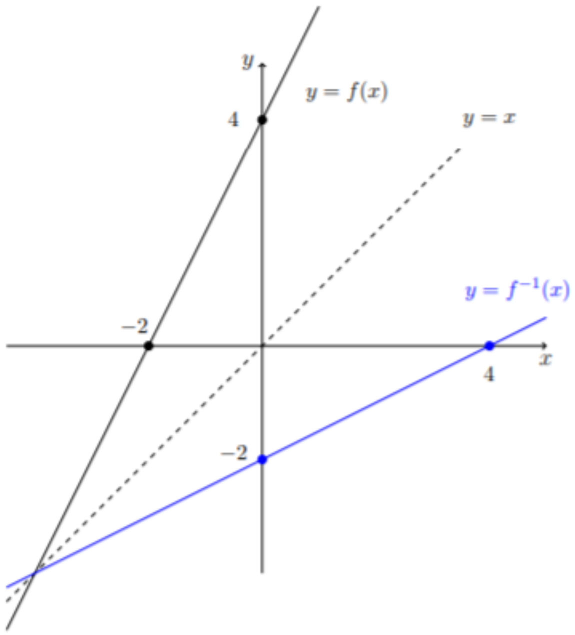
and

$$f^{-1}(f(x)) = x\,.$$

That is, f and f^{-1} ‘ **undo**’ (or cancel) each other.

GRAPHING INVERSE FUNCTIONS

Sketching the graphs for $y = f(x)$ and $y = f^{-1}(x)$ gives:



In the graph above we can see that the line $y = x$ acts like a mirror. This is a general property. That is,

the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$

A function has an inverse **function** if and only if it is one–one

DOMAIN AND RANGE

$\boxed{\mathrm{dom}(f^{-1}) = \mathrm{ran}(f)}$

$\boxed{\mathrm{ran}(f^{-1}) = \mathrm{dom}(f)}$

EXERCISES [pg10]

1. Find the inverse of each of the following functions:

(a) $f(x) = 2x + 1$

(b) $f(x) = 7x + 3$

(c) $f(x) = 3 - x$

(d) $f(x) = \dfrac{1}{x+1}$

(e) $f(x) = \dfrac{1}{x}$

2. (a) Find the smallest number b such that the function

$$f(x) = x^2 - 4 \text{ with } \operatorname{dom}(f) = [b, \infty)$$

has an inverse function. Find the rule for the inverse function.

(b) Find the largest number b such that the function

$$f(x) = (x+2)^2 \text{ with } \operatorname{dom}(f) = (-\infty, b]$$

has an inverse function. Find the rule for the inverse function.

3. Consider the function $f(x) = 1 + \sqrt{x+1}$.

(a) Find dom(f).

(b) Find ran(f).

(c) Find dom(f^{-1}).

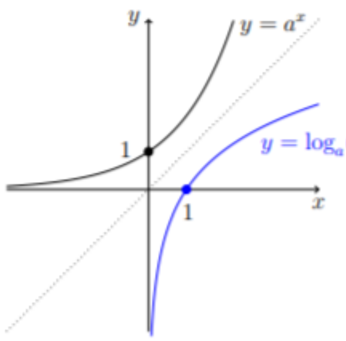
(d) Find ran(f^{-1}).

(e) Find the rule for f^{-1} .

(f) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

6.3 EXPONENTIALS AND LOGARITHMS

$$\boxed{\text{if } f(x) = a^x \text{ then } f^{-1}(x) = \log_a x}$$



logarithms and exponentials cancel each other,

(as long as they have the **same base**).

Example 9. Make x the subject of the following formula:

$$y = B \times 10^{\frac{ax}{b}}$$

(i.e. rearrange the formula to get x by itself).

Solution:

$$y = B \times 10^{\frac{ax}{b}}$$
$$\therefore \frac{y}{B} = 10^{\frac{ax}{b}}$$
$$\therefore \log_{10}\left(\frac{y}{B}\right) = \frac{ax}{b}$$
$$\therefore ax = b\log_{10}\left(\frac{y}{B}\right)$$
$$\therefore x = \frac{b}{a}\log_{10}\left(\frac{y}{B}\right)$$

EXERCISES [pg12]

1. Simplify the following expressions:

(a) $\log_3\left(3^{\sin x}\right)$

(b) $\log_{12}\left(12^{x-1}\right)$

(c) $4^{\log_4 \sqrt{x}}$

(d) $7^{2\log_7 x}$

(e) $3^{\frac{1}{2}\log_3(x+1)}$

2. Evaluate the following expressions:

(a) $\log_9 3$

(b) $\log_9\left(\dfrac{1}{27}\right)$

(c) $\log_4 0.25$

3. Make t the subject of the equation $y = k + Ca^{bt}$.

6.4 INVERSE TRIGONOMETRIC FUNCTIONS

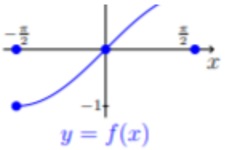
SINE AND ITS INVERSE

To turn sine into a **one–one** function, we restrict its domain to

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. That is, we consider the function

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbf{R}$$

where $f(x) = \sin x$.



We define $\sin^{-1}$ (or arcsin) to be the inverse of this function. That is, we write

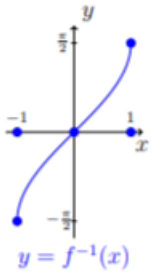
$$f^{-1}(x) = \sin^{-1}(x).$$

Note:

When we write $y = \sin^{-1} x$, we have $y \in \operatorname{ran}(f^{-1})$. Since

$$\operatorname{ran}(f^{-1}) = \operatorname{dom}(f) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

then we conclude that $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Thus we have

$$y = \sin^{-1} x$$

if and only if

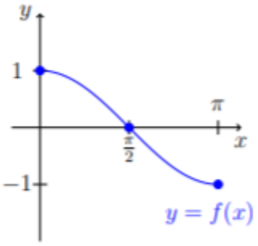
$$\sin y = x \quad \text{and} \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

COSINE AND ITS INVERSE

To turn cosine into a **one–one** function, we restrict its domain to $[0, \pi]$. That is, we consider the function

$$f : [0, \pi] \to \mathbf{R}$$

where $f(x) = \cos x$.



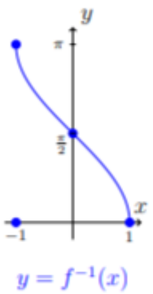
We define $\cos^{-1}$ (or arccos) to be the inverse of this function. That is, we write

$$f^{-1}(x) = \cos^{-1}(x).$$

Note:
When we write $y = \cos^{-1} x$, we have $y \in \operatorname{ran}(f^{-1})$. Since

$$\operatorname{ran}(f^{-1}) = \operatorname{dom}(f) = [0, \pi]$$

then we conclude that $y \in [0, \pi]$.



Thus we have

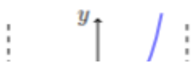
$$y = \cos^{-1} x$$

if and only if

$$\cos y = x \quad \text{and} \quad y \in [0, \pi].$$

TAN AND ITS INVERSE

To turn tan into a **one–one** function,

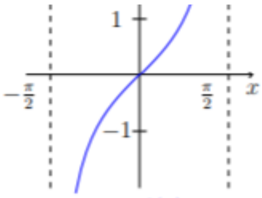


we restrict its domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

That is, we consider the function

$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbf{R}$$

where $f(x) = \tan x$.



$y = f(x)$

We define $\tan^{-1}$ (or arctan) to be the inverse of this function. That is, we write

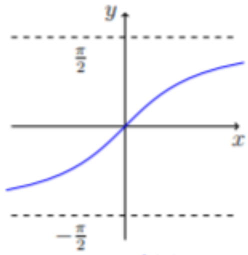
$$f^{-1}(x) = \tan^{-1}(x).$$

Note:

When we write $y = \tan^{-1} x$, we have $y \in \text{ran}(f^{-1})$. Since

$$\text{ran}(f^{-1}) = \text{dom}(f) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

then we conclude that $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



$y = f^{-1}(x)$

Thus we have

$$y = \tan^{-1} x$$

if and only if

$$\tan y = x \quad \text{and} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Warning:

Recall that usually

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

In particular,

$$\sin^{-1} x \neq \frac{1}{\sin x}.$$

That is,

$$\sin^{-1} x \neq (\sin x)^{-1}.$$

Similarly, note that

$$\cos^{-1} x \neq (\cos x)^{-1} \quad \text{and} \quad \tan^{-1} x \neq (\tan x)^{-1}.$$

EXERCISES [pg17]

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(b) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(c) $\cos^{-1}\left(-\frac{1}{2}\right)$

(d) $\cos^{-1}(1)$

(e) $\tan^{-1}(-\sqrt{3})$