EXERCISE SHEET 18: TRIANGULAR MATRIX METHOD

① g)
$$\begin{bmatrix} 3 & -2 & 5 & 23 \\ 1 & 1 & 1 & 4 \\ 2 & 3 & -2 & -5 \end{bmatrix}$$

h)
$$\begin{bmatrix} 4 & -3 & 15 & 4 \\ 2 & 6 & -10 & 1 \\ 6 & 3 & -5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 5 & 23 \\ 0 & 5 & -2 & -11 \\ 0 & 13 & -16 & -61 \end{bmatrix} \begin{matrix} 3R_2 - R_1 \\ 3R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 4 & -3 & 15 & 4 \\ 0 & 15 & -35 & -2 \\ 0 & -15 & 25 & 0 \end{bmatrix} \begin{matrix} 2R_2 - R_1 \\ R_3 - 3R_2 \end{matrix}$$

$$\begin{bmatrix} 3 & -2 & 5 & 23 \\ 0 & 5 & -2 & -11 \\ 0 & 0 & -54 & -162 \end{bmatrix} 5R_3 - 13R_2$$

$$\begin{bmatrix} 4 & -3 & 15 & 4 \\ 0 & 15 & -35 & -2 \\ 0 & 0 & -10 & -2 \end{bmatrix} R_3 + R_2$$

$-54z = -162$

$z = 3$

$5y - 2z = -11$

$5y - 2(3) = -11$

$y = -1$

$3x - 2y + 5z = 23$

$3x - 2(-1) + 5(3) = 23$

$x = 2$

$\boxed{(2, -1, 3)}$

$-10z = -2$

$z = \frac{1}{5}$

$15y - 35z = -2$

$15y - 35\left(\frac{1}{5}\right) = -2$

$y = \frac{1}{3}$

$4x - 3y + 15z = 4$

$4x - 3\left(\frac{1}{3}\right) + 15\left(\frac{1}{5}\right) = 4$

$x = \frac{1}{2}$

$\boxed{\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)}$

i)
$$\begin{bmatrix} 5 & 1 & -1 & -22 \\ 3 & -1 & 2 & -20 \\ 2 & 2 & 1 & -10 \end{bmatrix}$$

j)
$$\begin{bmatrix} 7 & -5 & 3 & 0 \\ 3 & -2 & 8 & -100 \\ 2 & 3 & 4 & -25 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & -1 & -22 \\ 0 & 8 & -13 & 34 \\ 0 & 8 & -1 & 10 \end{bmatrix} \begin{matrix} 3R_1 - 5R_2 \\ 3R_3 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 7 & -5 & 3 & 0 \\ 0 & 1 & 47 & -700 \\ 0 & 13 & -4 & 125 \end{bmatrix} \begin{matrix} 7R_2 - 3R_1 \\ 3R_3 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 5 & 1 & -1 & -22 \\ 0 & 8 & -13 & 34 \\ 0 & 0 & 12 & -24 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} 7 & -5 & 3 & 0 \\ 0 & 1 & 47 & -700 \\ 0 & 0 & -615 & 9225 \end{bmatrix} R_3 - 13R_2$$

$12z = -24$

$z = -2$

$8y - 13z = 34$

$8y - 13(-2) = 34$

$y = 1$

$5x + y - z = -22$

$5x + (1) - (-2) = -22$

$x = -5$

$\boxed{(-5, 1, -2)}$

$-615z = 9225$

$z = -15$

$y + 47(-15) = -700$

$y = 5$

$7x - 5y + 3z = 0$

$7x - 5(5) + 3(-15) = 0$

$x = 10$

$\boxed{(10, 5, -15)}$

(e)
$$\begin{bmatrix} 2 & 3 & -5 & -14 \\ 3 & -2 & 3 & 17 \\ 4 & 3 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & -6 & 9 \\ 0 & -27 & 24 & 0 \\ 0 & 5 & -24 & 0 \end{bmatrix} \begin{matrix} \\ 3R_2-2R_1 \\ 2R_3+R_2 \end{matrix}$$

$$\begin{bmatrix} 2 & 3 & -5 & -14 \\ 0 & -13 & 21 & 76 \\ 0 & 17 & -18 & -71 \end{bmatrix} \begin{matrix} \\ 2R_2-3R_1 \\ 3R_3-4R_2 \end{matrix}$$

$$\begin{bmatrix} 3 & 6 & -6 & 9 \\ 0 & -27 & 24 & 0 \\ 0 & 0 & -528 & 0 \end{bmatrix} \begin{matrix} \\ \\ 27R_3+5R_2 \end{matrix}$$

$0x + 0y - 528z = 0$

Let $k \in \mathbb{R}$

$z = k$

$-27y + 24(k) = 0$

$y = \frac{8}{9}k$

$3x + 6(\frac{8}{9}k) - 6(k) = 9$

$x = 3 + \frac{2}{9}k$

$$\begin{bmatrix} 2 & 3 & -5 & -14 \\ 0 & -13 & 21 & 76 \\ 0 & 0 & 123 & 369 \end{bmatrix} \begin{matrix} \\ \\ 13R_3+17R_2 \end{matrix}$$

$123z = 369$

$z = 3$

$-13y + 21(3) = 76$

$y = -1$

$2x + 3(-1) - 5(3) = -14$

$x = 2$

$\boxed{(2, -1, 3)}$

$\boxed{(3+\frac{2}{9}k, \frac{8}{9}k, k) \text{ where } k \in \mathbb{R}}$

(h)
$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 4 & -1 & 5 & 4 \\ 6 & 1 & 3 & 20 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 1 & 2 & -5 & -6 \\ -2 & 3 & -1 & -13 \\ 5 & 7 & -13 & -12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & -5 & 9 & -24 \\ 0 & 5 & -9 & 28 \end{bmatrix} \begin{matrix} \\ R_2-4R_1 \\ 2R_3-3R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & -6 \\ 0 & 7 & -11 & -25 \\ 0 & 29 & -31 & -89 \end{bmatrix} \begin{matrix} \\ R_2+2R_1 \\ 2R_3+5R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & -5 & 9 & -24 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \\ R_3+R_2 \end{matrix}$$

$0x + 0y + 0z = 4$

$0 = 4$ <inconsistent system>

$\boxed{\text{no solution}}$

$$\begin{bmatrix} 1 & 2 & -5 & -6 \\ 0 & 7 & -11 & -25 \\ 0 & 0 & 102 & 102 \end{bmatrix} \begin{matrix} \\ \\ 7R_3-29R_2 \end{matrix}$$

$102z = 102$

$z = 1$

$7y - 11(1) = -25$

$y = -2$

$x + 2(-2) - 5(1) = -6$

$x = 3$

$\boxed{(3, -2, 1)}$

(i)
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & -1 & 5 & 0 \\ 6 & 1 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 9 & 0 \\ 0 & -1 & -9 & 0 \end{bmatrix} \begin{matrix} \\ R_2-4R_1 \\ 2R_3-3R_2 \end{matrix}$$

(g)
$$\begin{bmatrix} 3 & 6 & -6 & 9 \\ 2 & -5 & 4 & 6 \\ -1 & 16 & -14 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 9 & 0 \\ 0 & 0 & -36 & 0 \end{bmatrix} \begin{matrix} \\ \\ 5R_3-R_2 \end{matrix}$$

$-36z = 0$

Let $k \in \mathbb{R}$

$z = k$

$-5y + 9k = 0$

$y = \frac{9}{5}k$

$x + \left(\frac{9}{5}k\right) - k = 0$

$x = -\frac{4}{5}k$

$\boxed{\left(-\frac{4}{5}k, \frac{9}{5}k, k\right) \text{ where } k = \mathbb{R}}$

② ⓐ
$$\begin{bmatrix} 3 & 2 & 4 & 1 & 10 \\ 2 & 3 & 7 & 1 & 13 \\ 1 & 6 & 3 & 1 & 11 \\ 2 & 1 & 1 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 4 & 1 & 10 \\ 0 & 5 & 13 & 1 & 19 \\ 0 & 9 & -1 & 1 & 9 \\ 0 & -2 & 6 & 0 & -8 \end{bmatrix} \begin{matrix} \\ 3R_2 - 2R_1 \\ 2R_3 - R_2 \\ R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 3 & 2 & 4 & 1 & 10 \\ 0 & 5 & 13 & 1 & 19 \\ 0 & 0 & -122 & -4 & -126 \\ 0 & 0 & -56 & 2 & -54 \end{bmatrix} \begin{matrix} \\ \\ 5R_3 - 9R_2 \\ 9R_4 + 2R_3 \end{matrix}$$

$$\begin{bmatrix} 3 & 2 & 4 & 1 & 10 \\ 0 & 5 & 13 & 1 & 19 \\ 0 & 0 & -122 & -4 & -126 \\ 0 & 0 & 0 & 468 & 468 \end{bmatrix} \begin{matrix} \\ \\ \\ 122R_4 - 56R_3 \end{matrix}$$

$468u = 468$

$u = 1$

$-122z - 4(1) = -126$

$z = 1$

$5y + 13(1) + 1(1) = 19$

$y = 1$

$3x + 2(1) + 4(1) + (1) = 10$

$x = 1$

$\boxed{(1, 1, 1, 1) = (x, y, z, u)}$

② ⓑ
$$\begin{bmatrix} 1 & 1 & -1 & 1 & 7 \\ 4 & -1 & 5 & 9 & 4 \\ 6 & 1 & 3 & 11 & 20 \\ 2 & 2 & -2 & 2 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 7 \\ 0 & -5 & 9 & 5 & -24 \\ 0 & 9 & -9 & -5 & 28 \\ 0 & 5 & -9 & -5 & 22 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_1 \\ 2R_3 - 3R_1 \\ 3R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 7 \\ 0 & -5 & 9 & 5 & -24 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 - R_3 \end{matrix}$$

$0x + 0y + 0z + 0u = -6$

$0 = -6$ ⟨inconsistent system⟩

$0x + 0y + 0z + 0u = 4$

$0 = 4$ ⟨inconsistent system⟩

$\boxed{\text{no solution}}$