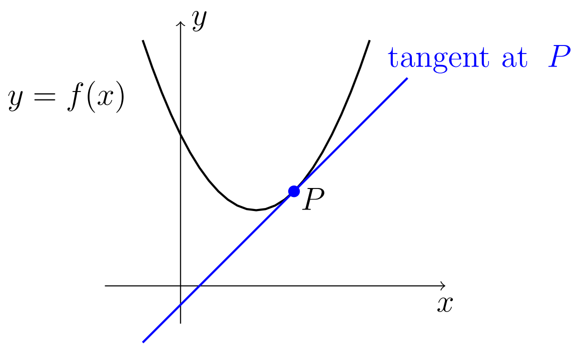
Chapter 8

Differentiability

Reference: "Calculus", by James Stewart.

8.1 Tangents

Consider the **tangent** to a curve $y = f(x)$ at point P, as shown in the adjacent diagram:

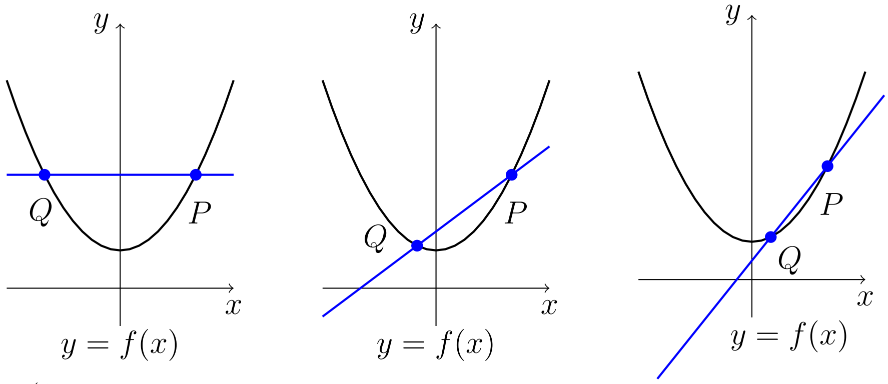


(We can see what the tangent looks like by imagining a car driving along the curve $y = f(x)$ at night. The front–lights and rear–lights of the car shine along the tangent.)

In this section we are going to develop a **mathematical** definition of a tangent.

Suppose Q is any point on the curve, other than P. We can draw a straight line through P and Q; this straight line is known as the **secant** PQ. We can see in the diagrams below that when we consider

Q **very close** to P, the secant PQ looks very similar to the tangent.

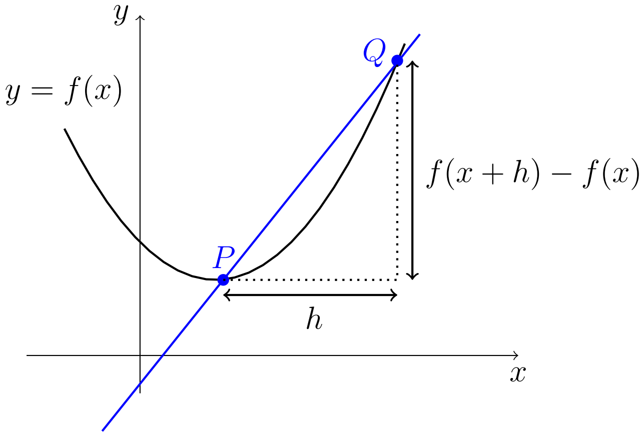


(Similarly, we can consider secants as Q approaches P from the right.)

If we let m denote the slope of the tangent at P, and m_{PQ} denote the slope of the secant line PQ, then we have

$$m = \lim_{Q \to P} m_{PQ}\,.$$

Let $(x, f(x))$ denote the coordinates of P, and let $(x+h, f(x+h))$ denote the coordinates of Q. Note that when Q is very close to P, then h will be very close to 0. That is, $Q \to P$ corresponds to $h \to 0$.



Recall that the slope of a straight line is given by $\dfrac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are any two points on the line.

Thus we can write

$$\begin{aligned} m_{PQ} &= \frac{f(x+h) \ - \ f(x)}{x+h \ - \ x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

and so

$$\boxed{m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

We are now ready to define a tangent:

> The **tangent** of the curve $y = f(x)$ at point $P = (x, f(x))$ is defined to be the line through P with slope given by the above limit, **provided that this limit exists.**

Note that at a particular point, if the above limit exists, then the curve has exactly one tangent at that point.

If, at a particular point the limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ does not exist, then either

- the curve has a vertical tangent at that point, or
- the curve has no tangent at that point.

In general,

- tangents do **not** exist at sharp corners, kinks, or sudden jumps in a curve.

Furthermore,

- $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \pm\infty$ when there is a vertical tangent,

8.2 Differentiability

The limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

considered in the previous section is known as the **derivative** of f . If this limit exists at $x=a$, we say that f is **differentiable** at $x=a$.

The derivative of f is usually denoted by $f'(x)$ or by $\dfrac{dy}{dx}$. So we have

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

(as long as this limit exists).
We have the following result:

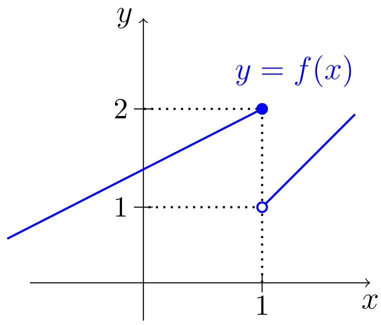
Let f be a "nice" function (i.e. let f be any function given in the Maths 1 course).

Then f is **differentiable** at $x=a$ if and only if

- f is continuous at $x=a$, and
- f does not have a sharp corner or kink at $x=a$, and
- the tangent of f at $x=a$ is not vertical.

Example 1. (a) Consider the function f whose graph is drawn below.

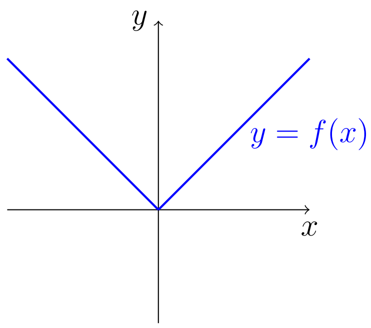
Note that f is not continuous at $x = 1$ (since $\lim_{x\to 1} f(x)$ does not exist).



The function f is **not** continuous at $x = 1$. Therefore f is **not** differentiable at $x = 1$.

(b) Consider the function $f(x) = |x|$, whose graph is drawn below.

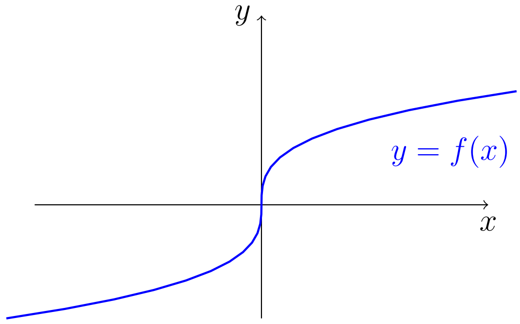
Note that f **is** continuous at $x = 0$. However, the graph of $y = f(x)$ has a sharp point at $x = 0$.



The graph of $y = f(x)$ has a **sharp point** at $x = 0$.
Therefore f is **not** differentiable at $x = 0$.

(c) Consider the function f whose graph is drawn below.

Note that f **is** continuous at $x = 0$, and does **not** have a sharp point at $x = 0$. However, the tangent at $x = 0$ is a vertical line, which means that f is **not** differentiable.



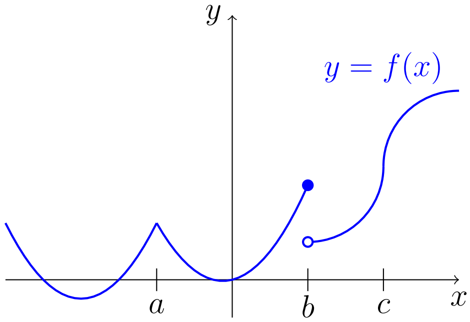
The tangent at $x = 0$ is vertical. Therefore f is **not** differentiable at $x = 0$.

Summary:

A function f is differentiable at $x = a$ if and only if the curve $y = f(x)$ has a tangent at $(a, f(a))$ (and the tangent is not a vertical line). In particular,

- a function is **not** differentiable at any points which correspond to sharp corners, kinks, or sudden jumps in its graph.
- A function **is** differentiable at those points which lie on a smooth curve, (as long as the tangent is not vertical).

Example 2. Consider the function f whose graph is drawn below.



- f is not differentiable at $x = a$ because there is a sharp point at $x = a$.
- f is not differentiable at $x = b$ because f is not continuous at $x = b$.
- f is not differentiable at $x = c$ because there is a vertical tangent at $x = c$.

Further Notes:

1. By its definition,

   $f'(x)$ is the gradient of the tangent (if it exists) to $y = f(x)$ at the point $(x, f(x))$.

   We also say that

   $f'(x)$ is the gradient **of the curve** of $y = f(x)$ at the point $(x, f(x))$.

2. $f'(x)$ also represents the **rate of change** of y with respect to x, at the point $(x, f(x))$ on the curve $y = f(x)$.

8.3 Differentiation, from First Principles

The process of finding the derivative of a function is called **differentiation**.

By definition, a function f is **differentiable** at $x = a$ if and only if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Establishing differentiability **from first principles** involves actually checking whether this limit exists.

Similarly, to find the derivative of $f(x)$ **using first principles**, we only use the definition of $f'(x)$. That is, we calculate $f'(x)$ by finding the following limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

Example 3. Differentiate the following functions from **first principles**.

(a) $f(x) = x - 5$

Solution: We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h-5) \; - \; (x-5)}{h} \\ &= \lim_{h \to 0} \frac{x+h-5-x+5}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= \lim_{h \to 0} 1 \\ &= 1 \end{aligned}$$

□

(b) $f(x) = 5 - 3x - 2x^2$

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{5 - 3(x+h) - 2(x+h)^2 \quad - \quad (5 - 3x - 2x^2)}{h} \\
&= \lim_{h \to 0} \frac{5 - 3x - 3h - 2(x^2 + 2xh + h^2) - 5 + 3x + 2x^2}{h} \\
&= \lim_{h \to 0} \frac{-3h - 2x^2 - 4xh - 2h^2 + 2x^2}{h} \\
&= \lim_{h \to 0} \frac{-3h - 4xh - 2h^2}{h} \\
&= \lim_{h \to 0} \frac{h(-3 - 4x - 2h)}{h} \\
&= \lim_{h \to 0} (-3 - 4x - 2h) \\
&= -3 - 4x - 2 \times 0 \\
&= -3 - 4x
\end{aligned}
$$

□

(c) $f(x) = \dfrac{2}{x}$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{2x - 2(x+h)}{(x+h)x} \times \frac{1}{h} \right) \\
&= \lim_{h\to 0} \left(\frac{2x - 2x - 2h}{(x+h)x} \times \frac{1}{h} \right) \\
&= \lim_{h\to 0} \left(\frac{-2h}{(x+h)x} \times \frac{1}{h} \right) \\
&= \lim_{h\to 0} \left(\frac{-2}{(x+h)x} \right) \\
&= \frac{-2}{(x+0)x} \\
&= -\frac{2}{x^2}\,.
\end{aligned}$$

□

Example 4. Differentiate $f(x) = \sqrt{2x+5}$ from first principles.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h} \times \frac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} \\
&= \lim_{h\to 0} \frac{2(x+h)+5-(2x+5)}{h(\sqrt{2(x+h)+5} + \sqrt{2x+5})} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2(x+h)+5} + \sqrt{2x+5})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} \\
&= \frac{2}{\sqrt{2x+5} + \sqrt{2x+5}} \\
&= \frac{2}{2\sqrt{2x+5}} \\
&= \frac{1}{\sqrt{2x+5}}
\end{aligned}
$$

$\square$

We can see from the above examples, that finding $f'(x)$ by first principles can be rather time–consuming (and tricky)! In the next chapter, we are going to learn how to find $f'(x)$ by using **formulae** (instead of using first principles).

Exercises

By differentiating from first principles, verify that

(a) when $f(x) = x$ then $f'(x) = 1$.

(b) when $f(x) = x^2$ then $f'(x) = 2x$.

(c) when $f(x) = x^3$ then $f'(x) = 3x^2$.

(d) when $f(x) = \sqrt{x}$ then $f'(x) = \dfrac{1}{2\sqrt{x}}$.

8.4 Answers to Chapter 8 Exercises

8.3: (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h\ -\ x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2\ -\ x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2\ -\ x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x \end{aligned}$$

(c)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 \ - \ x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3 \ - \ x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} \\
&= \lim_{h\to 0} (3x^2+3xh+h^2) \\
&= 3x^2
\end{aligned}$$

(d)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} \ - \ \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} \ - \ \sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x+h \ - \ x}{h(\sqrt{x+h}+\sqrt{x}\ )} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x}\ )} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$