[ES11] TANGENTS, NORMALS, AND RATES OF CHANGE

Monday, May 18, 2020 7:03 AM

1. The radius of a circular oil slick is increasing at a rate of $0.2\ \text{m}\cdot\text{s}^{-1}$. Find the $\boxed{\text{rate}}$ at which the area of the oil slick is increasing when the radius is 100 m. Write your final answer in a $\boxed{\text{sentence}}$.

$\frac{dr}{dt} = 0.2$
$A = \pi r^2$
$\frac{dA}{dr} = 2\pi r$
when $r = 100$:
$\frac{dA}{dr} = 2\pi(100)$
$= 200\pi$
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$
$= 200\pi \times 0.2$
$= 40\pi$
When the radius is 100m, the area is increasing at the rate of $40\pi\ m^2 \cdot s^{-1}$ ✓

2. The radius of a cylinder is increasing at a rate of $0.5\ \text{cm}\cdot\text{min}^{-1}$, whilst its height is remaining constant at 20 cm. For each of the questions below, present your final answer in a $\boxed{\text{sentence.}}$

(a) At what $\boxed{\text{rate}}$ is its $\boxed{\text{volume}}$ increasing when the radius is 4 cm? $r=4$ $\frac{dV}{dt}$

$\frac{dr}{dt} = 0.5$
$V = \pi r^2 h$
sub $h = 20$
$V = 20\pi r^2$
$\frac{dV}{dr} = 40\pi r$
$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
$= 40\pi(4) \times 0.5$
$= 80\pi$

When the radius is 4cm, the volume increases at the rate of $80\pi\ cm^3 \cdot min^{-1}$ ✓

(b) At what $\boxed{\text{rate}}$ is its $\boxed{\text{surface area}}$ increasing when the radius is 4 cm?

Hint: You are meant to know that a cylinder's surface area is given by $A = 2\pi rh + 2\pi r^2$.

$A = 2\pi rh + 2\pi r^2$
sub $h = 20$:
$A = 2\pi r(20) + 2\pi r^2$
$= 40\pi r + 2\pi r^2$
$\frac{dA}{dr} = 40\pi + 4\pi r$
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$
$= [40\pi + 4\pi(4)] \times 0.5$
$= 28\pi$

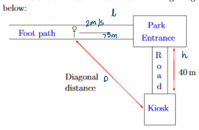
When the radius is 28π, the surface area increases at the rate of $28\pi\ cm^2 \cdot min^{-1}$ ✓

3. A 9 m long ladder is leaning against a vertical wall, with one end on horizontal ground. The lower end is slipping away from the wall at a constant speed of $2\ \text{m}\cdot\text{s}^{-1}$. Find the $\boxed{\text{rate}}$ at which the upper end of the ladder is sliding down the wall when the lower end is 3 m away from the wall. Write your final answer in a $\boxed{\text{sentence}}$

Hint: There is a similar example in your Course Notes!

$l = 9\ m$
$\frac{dx}{dt} = 2\ m/s$
$x = 3\ m$
Find $\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$
$l^2 = x^2 + h^2$
$h^2 = l^2 - x^2$
$h = \sqrt{l^2 - x^2}$
$= (l^2 - x^2)^{\frac{1}{2}}$
substitute $l = 9$ into h:
$h = (9^2 - x^2)^{\frac{1}{2}}$
$= (81 - x^2)^{\frac{1}{2}}$
$\frac{dh}{dx} = \frac{1}{2}(81 - x^2)^{-\frac{1}{2}} \times -2x$
$= \frac{-2x}{2\sqrt{81 - x^2}}$
when $x = 3$:
$\frac{dh}{dx} = \frac{-2(3)}{2\sqrt{81 - (3)^2}}$
$= \frac{-6}{2\sqrt{72}}$
$= \frac{-6}{2\cdot 3\cdot 2\cdot\sqrt{2}}$
$= -\frac{6}{12\sqrt{2}}$
$= -\frac{1}{2\sqrt{2}}$
$\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$
$= -\frac{1}{2\sqrt{2}} \times 2$
$= -\frac{1}{\sqrt{2}}$
The ladder is slipping down the wall at $\frac{1}{\sqrt{2}}\ m \cdot s^{-1}$ ✓

4. A child is walking along a footpath towards a park entrance at $2\ \text{m}\cdot\text{s}^{-1}$. A straight road, 40 m long, runs from the park entrance to the kiosk, and is at right angles to the footpath. The diagonal distance between the child and the kiosk is indicated in the diagram given below:



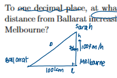
(a) Find the rate at which the diagonal distance between the child and the kiosk is changing when she is 75 m from the park entrance. Present your final answer in a $\boxed{\text{sentence,}}$ indicating whether the distance is increasing or decreasing, and accurate to three decimal places.

FIND $\frac{dD}{dt} = \frac{dD}{dl} \times \frac{dl}{dt}$
GIVEN $\frac{dl}{dt} = -2\ m/s$
$D^2 = l^2 + h^2$
$D = \sqrt{l^2 + h^2}$
substitute $h = 40$
$D = (l^2 + 1600)^{\frac{1}{2}}$
$\frac{dD}{dl} = \frac{1}{2}(l^2 + 1600)^{-\frac{1}{2}} \times 2l$
$= \frac{l}{\sqrt{l^2 + 1600}}$
substitute $l = 75$
$\frac{dD}{dl} = \frac{75}{\sqrt{75^2 + 1600}}$
$= \frac{15}{17}$
$\frac{dD}{dt} = \frac{15}{17} \times -2$
$= -1.764705882$
$\approx 1.765\ m/s$
When the child is 75m away from the park entrance, the distance between the child and the kiosk is decreasing at 1.765 m/s. ✓

(b) Find the $\boxed{\text{~~rate~~}}$ at which the $\boxed{\text{~~diagonal distance~~}}$ between the child and the kiosk is $\boxed{\text{changing}}$ when this ~~diagonal distance is 50 m~~.
Give your final answer in a $\boxed{\text{~~sentence,~~}}$ indicating whether the distance is $\boxed{\text{~~increasing or decreasing.~~}}$

FIND $\frac{dD}{dt} = \frac{dD}{dl} \times \frac{dl}{dt}$
GIVEN $\frac{dl}{dt} = -2\ m/s$
when $D = 50$:
$50^2 = l^2 + 40^2$
$l = 30$
$\frac{dD}{dl} = \frac{l}{\sqrt{l^2 + 1600}}$
substitute $l = 30$ into $\frac{dD}{dl}$
$\frac{dD}{dl} = \frac{30}{\sqrt{30^2 + 1600}}$
$= \frac{3}{5}$
$\frac{dD}{dt} = \frac{3}{5} \times -2$
$= -1.2\ m/s$
When the diagonal distance is 50m, the diagonal distance between the child and the kiosk is decreasing at the rate of 1.2m/s. ✓

5. Ballarat is 100 km due west of Melbourne. Sarah drives due north from Melbourne at a constant speed of 100 km/h.
To ~~one decimal place~~, at what $\boxed{\text{~~rate in km/h~~}}$ is Sarah's distance from Ballarat increasing when she is ~~20 km~~ from Melbourne?



FIND $\frac{dD}{dt} = \frac{dD}{dh} \times \frac{dh}{dt}$
GIVEN $\frac{dh}{dt} = 100\ km/h$
$D^2 = h^2 + l^2$
$D = \sqrt{h^2 + l^2}$
substitute $l = 100$
$D = (h^2 + 10000)^{\frac{1}{2}}$
$\frac{dD}{dh} = \frac{1}{2}(h^2 + 10000)^{-\frac{1}{2}} \times 2h$
$= \frac{h}{\sqrt{h^2 + 10000}}$
substitute $h = 20$

1. When the oil slick's radius is 100 m, its area is increasing at a rate of $40\pi\ \text{m}^2\cdot\text{s}^{-1}$.
2. (a) When the radius is 4 cm, the volume is increasing at a rate of $80\pi\ \text{cm}^3\cdot\text{min}^{-1}$.
   (b) When the radius is 4 cm, the surface area is increasing at a rate of $28\pi\ \text{cm}^2\cdot\text{min}^{-1}$
3. The upper end of the ladder is slipping down the wall at a rate of $\frac{1}{\sqrt{2}}\ \text{m}\cdot\text{s}^{-1}$.
4. (a) The distance between the child and the kiosk is decreasing at $1.765\ \text{m}\cdot\text{s}^{-1}$.
   (b) The distance between the child and the kiosk is decreasing at $1.2\ \text{m}\cdot\text{s}^{-1}$.
5. Sarah's distance from Ballarat is increasing at a rate of 19.6 km/h.
6. The person's shadow is increasing in length at a rate of $\frac{4}{3}\ \text{m}\cdot\text{s}^{-1}$.
   Touch here to see a solution video
7. The water level is rising at a rate of $0.1\ \text{cm}\cdot\text{s}^{-1}$.
8. The snowball's radius is decreasing at a rate of $\frac{1}{8\pi}\ \text{cm}\cdot\text{min}^{-1}$.
9. The depth is decreasing at a rate of $\frac{1}{3}\ \text{cm}\cdot\text{s}^{-1}$.
10. At the instant when there is $0.3\ \text{m}^3$ of grain in the heap, the height of the heap is increasing at a rate of $0.073\ \text{m}\cdot\text{min}^{-1}$.
11. When $\theta = 60°$, the light ray is moving along the beach at a speed of $240\pi\ \text{m}\cdot\text{min}^{-1}$.
12. (a) Tangent: $y = x - 3$ Normal: $y = -x + 3$
    (b) Tangent: $y = 3x - 4$ Normal: $y = -\frac{1}{3}x - 4$
    (c) Tangent: $y = 10x - 20$ Normal: $y = -\frac{1}{10}x + \frac{1}{5}$
    (d) Tangent: $y = 6x + 3 - \frac{3\pi}{2}$ Normal: $y = -\frac{1}{6}x + 3 + \frac{\pi}{24}$ 4

13. 14. 15. 16.

13. (a) Tangent: $y = x - 2$ Normal: $y = -x + 2$
    (b) Tangent: $y = -x + 3$ Normal: $y = x + 3$
    (c) Tangent: $y = 4x - 7$ Normal: $y = -\frac{1}{4}x + \frac{3}{2}$
14. $y = -\frac{5}{7}x - \frac{12}{7}$
15. $-\frac{36}{23}$
16. (a) $(-1, 3)$, $(4, 8)$ (b) $y = -4x - 1$, $y = 6x - 16$
    (c) 104.04°, 80.54°
17. The required point is $(3, 9)$.

[LECTURE] TITLE

CUE COLUMN Questions/Cues	NOTE TAKING COLUMN Key Ideas/Important facts/Repeated (stressed) information
	-

SUMMARY

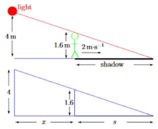
Term	Definition

REVIEW QUESTIONS
Link to Anki:

Question	Answer in white

6. A light is exactly 4 metres above the ground. A person of height 1.6 metres walks away from the light at a constant speed of $2 \text{ m} \cdot \text{s}^{-1}$. At what rate is the length of the person's shadow increasing? Write your final answer in a sentence.

Hint: We are interested in the following diagram:



7. A container has such a shape that when the depth of water in it is h cm, the volume of water is V cm^3, where $V = 108h + h^3$. Suppose that water is poured into the container at a constant rate of $30 \text{ cm}^3 \cdot \text{s}^{-1}$. At what rate is the water level rising when the depth is 8 cm? Present your final answer in a sentence.

8. A melting snowball which is always spherical in shape is decreasing in volume at a constant rate of $8 \text{ cm}^3 \cdot \text{min}^{-1}$. Find the rate at which the radius is changing when the snowball's radius is 4 cm.

Give your final answer in a sentence, indicating whether the radius is increasing or decreasing.

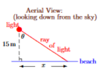
9. Water is flowing out through a hole at the vertex of an inverted right circular cone whose semi–vertical angle is $60°$. The rate of flow (measured in $\text{cm}^3 \cdot \text{s}^{-1}$) is π times the square of the depth of the water. At what rate is the depth decreasing? Give your final answer in a sentence.

Hint: From a diagram, we see that $\tan 60° = \frac{r}{h}$. That is, $r = \sqrt{3}h$.

10. Grain is being poured into a heap at a rate of $0.1 \text{ m}^3 \cdot \text{min}^{-1}$. The heap is in the shape of a circular cone with semi–vertical angle $45°$. Find the rate at which the height of the cone is increasing at the instant when the volume of the grain in the heap is 0.3 m^3. Write your final answer in a sentence and accurate to three decimal places.

Hint: When $V = 0.3$ we obtain $h^3 = \frac{0.9}{\pi}$.

11. A searchlight, mounted on a low rock 15 metres from the nearest point of a straight beach, revolves at two revolutions per minute. Find the speed at which the light is moving along the beach when $\theta = 60°$, giving your final answer in a sentence.



Hints:

- We want to find $\frac{dx}{dt}$.
- It is useful to use the Chain Rule: $\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$.
- The angle θ should be measured in radians.
- The rate $\frac{d\theta}{dt}$ should be measured in radians per minute.

Touch here to see a hint video

12. Find the equations of the tangent and the normal to the following curves at the points indicated:

(a) $y = x^2 - 5x + 6$ at $(3, 0)$.

(b) $y = 2x^2 + 3x - 4$ when $x = 0$.

(c) $y = 3x^3 - 7x^2 + 2x$ when $x = 2$.

(d) $y = 3\tan x$ when $x = \frac{\pi}{4}$.

13. Find the equations of the tangent and the normal to the following curves at the points indicated:

(a) $y = \ln(x - 1)$ when $x = 2$.

(b) $y = 3 - x - x^2$ where it crosses the y axis.

(c) $y = 2x^2 - 4x + 1$ where the gradient is 4.

14. Find the equation of the tangent to the curve

$$x^2 + 3xy + 2y^2 = 6$$

at the point $(-1, -1)$.

15. Find the gradient of the curve

$$x^2y - 2x^3 - y^3 + 1 = 0$$

at the point $(2, -3)$.

16. The line $y = x + 4$ cuts the parabola $y = x^2 - 2x$ at two points A and B.

(a) Find the coordinates of these two points of intersection A and B.

(b) Find the equations of the tangents to $y = x^2 - 2x$ at A and B.

(c) Find the angles of inclination for each of these tangents, giving your final answers in degrees and to two decimal places.
That is, find the angles which the tangents make with the positive direction of the x–axis.