NO: DATE:

[MATHEMATICS I] EXERCISE SHEET 16: LINEAR APPROXIMATIONS AND THE ANGLE BETWEEN TWO CURVES

1 a) i) $f(x) = x^2$ $f'(x) = 2x$

$f(3) = 9$ $f'(3) = 6$

$f(x) \approx 9 + 6(x-3)$

$= 9 + 6x - 18$

$= 6x - 9$

$f(3) \approx 6(3) - 9$

$= 9$

ii) $2.99^2 = f(2.99) \approx 6(2.99) - 9$

$= 8.94$

iii) $2.99^2 = 8.9401$

b) i) $f(x) = x^2$ $f'(x) = 2x$

$f(10) = 100$ $f'(10) = 20$

$f(x) \approx 100 + 20(x-10)$

$= 100 + 20x - 200$

$= 20x - 100$

$f(10) \approx 20(10) - 100$

$= 100$

ii) $10.05^2 = f(10.05) \approx 20(10.05) - 100$

$= 101$

iii) $10.05^2 = 101.0025$

c) i) $g(x) = \cos x$ $g'(x) = -\sin x$

$g(0) = 1$ $g'(0) = 0$

$g(x) \approx 1 + 0(x-0)$

$= 1$

$g(0) \approx 1$

ii) $\cos(0.1) = g(0.1) \approx 1$

iii) $\cos(0.1) = 0.9950$ (4dp)

d) i) $g(x) = \cos x$ $g'(x) = -\sin x$

$g(\frac{\pi}{2}) = 0$ $g'(\frac{\pi}{2}) = -1$

$g(x) \approx 0 + (-1)(x - \frac{\pi}{2})$

$= -x + \frac{\pi}{2}$

$g(\frac{\pi}{2}) \approx -\frac{\pi}{2} + \frac{\pi}{2}$

$= 0$

ii) $\cos(1.55) = g(1.55) \approx -1.55 + \frac{\pi}{2}$

$= 0.0208$ (4dp)

iii) $\cos(1.55) = 0.0208$ (4dp)

2 a) $f(x) = 2x + 7$ $f'(x) = 2$

$f(1) = 9$ $f'(1) = 2$

$f(x) \approx 9 + 2(x-1)$

$= 9 + 2x - 2$

$= 2x + 7$

b) $f(x) = 2x + 7$ $f'(x) = 2$

$f(5) = 17$ $f'(5) = 2$

$f(x) \approx 17 + 2(x-5)$

$= 17 + 2x - 10$

$= 2x + 7$

3 a) $f(x) = \ln(x+1)$ $f'(x) = \frac{1}{x+1}$

$f(0) = 0$ $f'(0) = 1$

$f(x) \approx 0 + 1(x-0)$

$= x$

$f(0) \approx 0$

b) i) $\ln(1.02) = \ln(0.02+1) = f(0.02) \approx 0.02$

ii) $\ln(1.02) = 0.0198$ (4dp)

c) i) $\ln(0.99) = \ln(-0.01+1) = f(-0.01) \approx -0.01$

ii) $\ln(0.99) = -0.0101$ (4dp)

4 a) $f(x) = \sqrt{x+1}$ $f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$f(0) = 1$ $f'(0) = \frac{1}{2}$

$f(x) \approx 1 + \frac{1}{2}(x-0)$

$= 1 + \frac{1}{2}x$

$f(0) \approx 1 + \frac{1}{2}(0)$

$= 1$

b) i) $\sqrt{1.1} = \sqrt{0.1+1} = f(0.1) \approx 1 + \frac{1}{2}(0.1)$

$= 1.05$

ii) $\sqrt{1.1} = 1.0488$ (4dp)

NO:.................... DATE:....................

[MATHEMATICS I] EXERCISE SHEET 16: LINEAR APPROXIMATIONS AND THE ANGLE BETWEEN TWO CURVES

c) i) $\sqrt{0.97} = \sqrt{-0.03+1} = f(-0.03) \approx 1 + \frac{1}{2}(-0.03)$
$= 0.985$

ii) $\sqrt{0.97} = 0.9849$ (4dp)

5 a) $f(x) = \frac{1}{x} = x^{-1}$ $\quad f'(x) = -x^{-2}$
$f(4) = \frac{1}{4} = 0.25$ $\quad f'(4) = -\frac{1}{16} = -0.0625$
$f(x) \approx 0.25 + (-0.0625)(x-4)$
$= 0.25 - 0.0625x + 0.25$
$= 0.5 - 0.0625x$
$f(4) \approx 0.5 - 0.0625(4)$
$= 0.25$

b) i) $\frac{1}{4.01} = f(4.01) \approx 0.5 - 0.625(4.01)$
$= 0.249375$

ii) $\frac{1}{4.2} = f(4.2) \approx 0.5 - 0.625(4.2)$
$= 0.2375$

iii) $\frac{1}{3.98} = f(3.98) \approx 0.5 - 0.625(3.98)$
$= 0.25125$

6 a) $g(x) = \frac{1}{4+x}$ $\quad g'(x) = -(4+x)^{-2}$
$g(0) = \frac{1}{4} = 0.25$ $\quad g'(0) = -0.0625$
$g(x) \approx 0.25 + (-0.625)(x-0)$
$= 0.25 - 0.0625x$
$g(0) \approx 0.25 - 0.0625(0)$
$= 0.25$

b) i) $\frac{1}{4.01} = \frac{1}{4+0.01} = g(0.01) \approx 0.25 - 0.0625(0.01)$
$= 0.249375$

ii) $\frac{1}{4.2} = \frac{1}{4+0.2} = g(0.2) \approx 0.25 - 0.0625(0.2)$
$= 0.2375$

iii) $\frac{1}{3.98} = \frac{1}{4-0.02} = g(-0.02) \approx 0.25 - 0.0625(-0.02)$
$= 0.25125$

7 $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
$f(100) = 10$ $\quad f'(100) = 0.05$
$f(x) \approx 10 + 0.05(x - 100)$
$= 0.05x + 5$

$g(x) = \sqrt{100+x}$ $\quad g'(x) = \frac{1}{2}(100+x)^{-\frac{1}{2}}$
$g(0) = 10$ $\quad g'(0) = 0.05$
$g(x) \approx 10 + 0.05(x - 0)$
$= 10 + 0.05x$
$h(x) = \sqrt{95+x}$ $\quad h'(x) = \frac{1}{2}(95+x)^{-\frac{1}{2}}$
$h(5) = 10$ $\quad h(5) = 0.05$
$h(x) \approx 10 + 0.05(x-5)$
$= 0.05x + 9.75$

i) $\sqrt{100.1} = \sqrt{100+0.1} = g(0.1) \approx 10 + 0.05(0.1)$
$= 10.005$

ii) $\sqrt{99.8} = \sqrt{100-0.02} = g(-0.02) \approx 10 + 0.05(-0.02)$
$= 9.999$

iii) $\sqrt{103} = f(103) \approx 0.05(103) + 5$
$= 10.15$

8 a) Let $f(x) = (1+x)^n$ $\quad f'(x) = n(1+x)^{n-1}$
$f(0) = 1$ $\quad f'(0) = n$
$f(x) \approx 1 + n(x-0)$
$= 1 + nx$

b) Pascal's Triangle

1
$(1+x)^1 \approx 1 + x \leftarrow$ 1 1 $\rightarrow (1+x)^1 = 1 + x$
$(1+x)^2 \approx 1 + 2x \leftarrow$ 1 2 1 $\rightarrow (1+x)^2 = 1 + 2x + x^2$
$(1+x)^3 \approx 1 + 3x \leftarrow$ 1 3 3 1 $\rightarrow (1+x)^3 = 1 + 3x + 3x^2 + x^3$

9 a) $f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad \Delta x = 1$
$f(100) = 10$ $\quad f'(100) = 0.05$

i) Approximate maximum error (f) $= |f'(x_0)\Delta x|$
$= |f'(100)(1)|$
$= 0.05$

ii) Approximate maximum percentage error (f) $= \left|\frac{0.05}{10}\right| \times 100\%$
$= 0.5\%$

10 d) $f(x) = 200 + 5\sin(\pi x)$ $\quad f'(x) = 5\pi\cos(\pi x)$
$f(100) = 200 + 5\sin(100\pi)$ $\quad f'(100) = 5\pi\cos(100\pi)$ $\quad \Delta x = 1$
$= 200$ $\quad = 5\pi$

i) Approximate maximum error (f) $= |5\pi(1)|$
$= 5\pi = 15.71$ (2dp)

ii) Approximate maximum percentage error (f) $= \left|\frac{5\pi}{200}\right| \times 100\%$
$= 7.85\%$

NO:.................... DATE:....................

[MATHEMATICS 1] EXERCISE SHEET 6: LINEAR APPROXIMATIONS AND THE ANGLE BETWEEN TWO CURVES

(11) a) $f(x) = 11 + e^{-3x}$ $f'(x) = -3e^{-3x}$

$f(0) = 12$ $f'(0) = -3$ $\Delta x = 0.2$

i) Approximate maximum error (f) $= |(-3)(0.2)|$

$= 0.6$

ii) Approximate maximum percentage error (f) $= \left|\frac{-0.6}{12}\right| \times 100\%$

$= 5\%$

(12) a) $f(x) = 7 + 5\cos x$ $f'(x) = -5\sin x$

$f(0) = 12$ $f'(0) = 0$ $\Delta x = 0.05$

i) Approximate maximum error (f) $= |(0)(0.05)|$

$= 0$

ii) Approximate maximum ~~percentage~~ error (f) $= \left|\frac{0}{12}\right| \times 100\%$

$= 0\%$

b) $f(0.8) = 7 + 5\cos(0.8)$ $f'(x) = -5\sin(0.8)$ $\Delta x = 0.05$

i) Approximate maximum error (f) $= |(-5\sin(0.8))(0.05)|$

$= 0.18$ (2dp)

ii) Approximate maximum percentage error (f) $= \left|\frac{0.036}{7+5\cos(0.8)}\right| \times 100\%$

$= 1.71\%$ (2dp)

~~(13) $f(x) = \frac{\sin x}{1+x^2}$ $f'(x) = (\sin x)(2x) - (1+x^2)(\cos x)$~~

~~$\Delta x = 0.1$ $f(3) = \frac{\sin 3}{10}$ $f'(3) = 6\sin 3 - 10\cos 3$~~

~~a) Approximate maximum error (f) $= |(6\sin 3 - 10\cos 3)(0.1)|$~~

~~$= 1.07$ (2dp)~~

~~b) Approximate maximum percentage error (f) $= \left|\frac{1.07}{\sin 3 \div 10}\right| \times 100\%$~~

~~$=$~~

(13) $f(x) = \frac{\sin x}{1+x^2}$ $f'(x) = \frac{-(\sin x)(2x) + (1+x^2)(\cos x)}{(1+x^2)^2}$

$\Delta x = 0.1$ $f(3) = \frac{\sin 3}{10}$ $f'(3) = \frac{-6\sin 3 + 10\cos 3}{100}$

a) Approximate maximum error (f) $= \left|\frac{-6\sin 3 + 10\cos 3}{100}(0.1)\right|$

$= 0.01$ (2dp)

b) Approximate maximum percentage error (f) $= \left|\frac{(-6\sin 3 + 10\cos 3)(0.1) \div 100}{(\sin 3) \div 10}\right| \times 100\%$

$= 76.15\%$ (2dp)

(14) $f(x) = \frac{e^x}{4+x}$ $f'(x) = \frac{(4+x)(e^x) - (e^x)(1)}{(4+x)^2}$

$f(16) = \frac{e^{16}}{20}$ $f'(x) = \frac{19e^{16}}{400}$

$\Delta x = 0.05$

a) Approximate maximum error (f) $= \left|\frac{19e^{16}}{400} \times 0.05\right|$

$= 21\,104.51$ (2dp)

b) Approximate maximum percentage error (f) $= \left|\frac{21104.51}{e^{16} \div 20}\right| \times 100\%$

$= 4.75\%$ (2dp)

(15) a) Let $f(x) = 4x$ $f'(x) = 4$ $\Delta x = 0.1$

$f(35) = 140$ $f'(35) = 4$

i) Approximate maximum error (f) $= |(4)(0.1)|$

$= 0.4$

The approximate maximum error in the perimeter of the square block of land is 0.4 metres.

ii) Approximate maximum percentage error (f) $= \left|\frac{0.4}{140}\right| \times 100\%$

$= 0.29\%$ (2dp)

The approximate maximum percentage error in the perimeter is 0.29%.

b) Let $g(x) = x^2$ $g'(x) = 2x$

$g(35) = 1225$ $g'(35) = 70$ $\Delta x = 0.1$

i) Approximate maximum error (g) $= |(70)(0.1)|$

$= 7$

The approximate maximum error in the area of the square block is 7 metres².

ii) Approximate maximum percentage error (g) $= \left|\frac{7}{1225}\right| \times 100\%$

$= 0.57\%$ (2dp)

The approximate maximum percentage error in the area of the square block of land is 0.57%.

(16) $V = \frac{2h^2 + 5h}{h+1}$ $V' = \frac{(h+1)(4h+5) - (2h^2+5h)}{(h+1)^2}$ $\Delta h = 0.5$

$V_{100} = \frac{20500}{101}$ $V'_{100} = \frac{20405}{10201}$

a) Approximate maximum error V $= \left|\frac{20405}{10201} \times 0.5\right|$

$= 1$

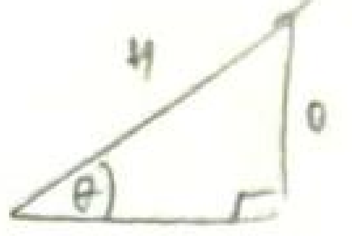
The approximate maximum error of the water's volume is 1 cm³.

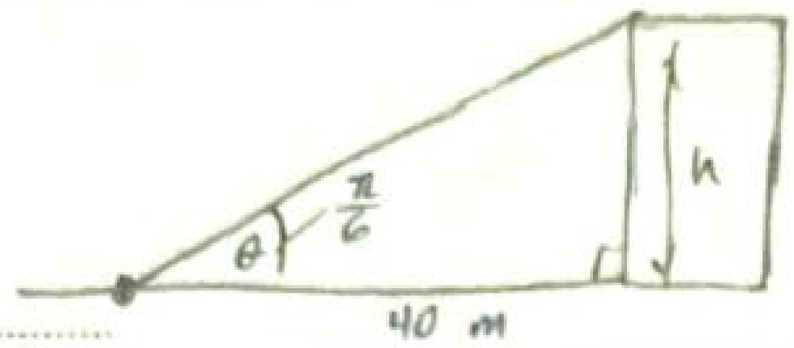
b) Approximate maximum percentage error V $= \left|\frac{1}{\frac{20500}{101}}\right| \times 100\%$

$= 0.49\%$ (2dp)

The approximate maximum percentage error of the water's volume is 0.49%.

$\tan\theta = \frac{O}{A}$





$\Delta\theta = 0.02$ rad

$h = 40 \times \tan\theta$

NO:

DATE:

[MATHEMATICS I] EXERCISE SHEET 16: LINEAR APPROXIMATIONS AND THE ANGLE BETWEEN TWO CURVES

(17) Let $f(x) = 40 \times \tan\theta$ $\quad f'(x) = 40\sec^2\theta$

$\Delta\theta = 0.02$ $\quad f(\frac{\pi}{6}) = \frac{40}{\sqrt{3}}$ $\quad f'(\frac{\pi}{6}) = \frac{160}{3}$

a) Approximate maximum error $(f) = \left|\frac{160}{3} \times 0.02\right|$

$= 1.07$ (2dp)

The approximate maximum error in the building's height is 1.07 metres.

b) Approximate maximum percentage error $(f) = \left|\frac{1.07}{40/\sqrt{3}}\right| \times 100\%$

$= 4.62\%$

The approximate maximum percentage error in the building's height is 4.62%.

(18) a) i) ① $y = x^2$, ② $y = x$ — substitute ① into ②

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$

substitute $x = 0$ & $x = 1$ into $y = x$:

$y = 0 \quad y = 1$

$(0,0)$, $(1,1)$ ← Points of intersection

ii) ① $y = x^2$

$\frac{dy}{dx} = 2x$

at $(0,0)$ $\quad$ at $(1,1)$

$m_1 = 2(0)$ $\quad m_2 = 2(1)$

$= 0$ $\quad = 2$

$0 = \tan\theta_1$ $\quad 2 = \tan\theta_2$

$\theta_1 = 0$ $\quad \theta_2 = 63.43494882$

② $y = x$

$\frac{dy}{dx} = 1$

at $(0,0)$ $\quad$ at $(1,1)$

$m_3 = 1$ $\quad m_4 = 1$

$1 = \tan\theta_3$ $\quad 1 = \tan\theta_4$

$\theta_3 = 45$ $\quad \theta_4 = 45$

$\theta_3 - \theta_1 = 45 - 0$ $\quad \theta_2 - \theta_4 = \tan^{-1}(2) - 45$

$= 45°$ $\quad = 18.43°$ (2dp)

Angle between curves at point(s) of intersection: 45° and 18.43°

b) i) ① $y = x^3$ ② $y = x^2 - x$

substitute ① into ②:

$x^3 = x^2 - x$

$x^3 - x^2 + x = 0$

$x(x^2 - x + 1) = 0$

$x = 0$ $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$= \frac{1 \pm \sqrt{-3}}{2}$ <rejected>

substitute $x = 0$ into $y = x^3$:

$y = 0$

$(0,0)$ ← Points of intersection

ii) ① $y = x^3$ $\quad$ ② $y = x^2 - x$

$\frac{dy}{dx} = 3x^2$ $\quad \frac{dy}{dx} = 2x - 1$

at $(0,0)$ $\quad$ at $(0,0)$

$m_1 = 3(0)^2$ $\quad m_2 = 2(0) - 1$

$= 0$ $\quad = -1$

$0 = \tan\theta_1$ $\quad -1 = \tan\theta_2$

$\theta_1 = 0$ $\quad \theta_2 = -45$

$\theta_1 - \theta_2 = 0 - -45$

$= 45°$

Angle between curves at the point of intersection $(0,0)$: 45°

c) i) ① $y = \sqrt{1+x}$ ② $y = \sqrt{1-x}$

substitute ① into ②:

$\sqrt{1+x} = \sqrt{1-x}$

$1 + x = 1 - x$

$2x = 0$

$x = 0$

when $x = 0$,

$y = \sqrt{1+0}$

$= 1$

Point of intersection: $(0,1)$

ii) ① $y = \sqrt{1+x}$ $\quad \frac{dy}{dx} = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

at $(0,1)$ $\quad \frac{1}{2} = \tan\theta_1$

$m_1 = \frac{1}{2}(1+0)^{-\frac{1}{2}}$ $\quad \theta_1 = 26.56505118$

$= \frac{1}{2}$

② $y = \sqrt{1-x}$ $\quad \frac{dy}{dx} = \frac{-1}{2}(1-x)^{-\frac{1}{2}}$

at $(0,1)$ $\quad -\frac{1}{2} = \tan\theta_2$

$m_2 = \frac{1}{2}(1-0)^{-\frac{1}{2}}$ $\quad \theta_2 = -26.56505118$

$= -\frac{1}{2}$

$\theta_1 - \theta_2 = 26.56505118 - -26.56505118$

$= 53.13°$ (2dp)

Angle between curves at $(0,1)$ is 53.13°