[ES10] DIFFERENTIATION

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1. Differentiate each of the following functions:

(a) $f(x) = e^{3x+1}$

$f'(x) = 3e^{3x+1}$

(b) $f(x) = e^{x-x^3}$

$f'(x) = (1-3x^2)e^{x-x^3}$

(c) $f(x) = e^{\sqrt{x}}$

$= e^{x^{\frac{1}{2}}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

$= \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

(d) $f(x) = xe^{-x}$

$f'(x) = udv + vdu$

$= x(-1)e^{-x} + e^{-x}(1)$

$= -xe^{-x} + e^{-x} \quad = e^{-x}(-x+1)$

(e) $f(x) = xe^{3x} + x^3 - \pi$

$f'(x) = udv + vdu + 3x^2$

$= x(3)e^{3x} + e^{3x}(1) + 3x^2$

$= 3xe^{3x} + e^{3x} + 3x^2$

(f) $f(x) = xe^{x} + e$

$f'(x) = udv + vdu$

$= x(1)e^{x} + e^{x}(1)$

$= xe^{x} + e^{x}$

$= e^{x}(x+1)$

(g) $f(x) = xe^{-x^2}$

$f'(x) = udv + vdu$

$= x(-2x)e^{-x^2} + e^{-x^2}(1)$

$= -2x^2e^{-x^2} + e^{-x^2}$

$= e^{-x^2}(1-2x^2)$

(h) $f(x) = e^{x}\sin(x^2)$

$f'(x) = udv + vdu$

$= e^{x}\cos(x^2)(2x) + \sin(x^2)e^{x}$

$= e^{x}(2x\cos(x^2) + \sin(x^2))$

(i) $f(x) = e^{x}\cos(2x^{0.4}+3)$

$f'(x) = udv + vdu$

$= e^{x} \times -\sin(2x^{0.4}+3) \times 0.8x^{-0.6} + \cos(2x^{0.4}+3) \times e^{x}$

$= e^{x}(-0.8x^{-0.6}(\sin(2x^{0.4}+3)) + \cos(2x^{0.4}+3))$

(j) $f(x) = \dfrac{3x+1}{e^{x}+\tan x}$

$f'(x) = \dfrac{vdu - udv}{v^2}$

$= \dfrac{(e^{x}+\tan x)(3) - (3x+1)(e^{x}+\sec^2 x)}{(e^{x}+\tan x)^2}$

(k) $f(x) = \dfrac{\cos(2x^{0.4}+3)}{e^{x}-4}$

$f'(x) = \dfrac{vdu - udv}{v^2}$

$= \dfrac{(e^{x}-4)(-\sin(2x^{0.4}+3))(0.8x^{-0.6}) - (\cos(2x^{0.4}+3))(e^{x})}{(e^{x}-4)^2}$

2. Differentiate each of the following functions:

(a) $y = e^{\cos x} - e^{\tan x}$

$\frac{dy}{dx} = -\sin x\, e^{\cos x} - \sec^2 x\, e^{\tan x}$

(b) $y = \dfrac{\cos(2x)}{e^{2x}+1}$

$\frac{dy}{dx} = \dfrac{vdu - udv}{v^2}$

$= \dfrac{(e^{2x}+1)(2)(\cos(2x)) - (\cos(2x))(2e^{2x})}{(e^{2x}+1)^2}$

$= \dfrac{(2e^{2x}+2)(\cos(2x)) - (\cos(2x))(2e^{2x})}{(e^{2x}+1)^2}$

(c) $y = (3x^2 - e^{x})^5 \tan x$.

$\frac{dy}{dx} = udv + vdu$

$= (3x^2-e^{x})^5(\sec^2 x) + (\tan x)(5)(3x^2-e^{x})^4(6x-e^{x})$

$= (3x^2-e^{x})^4[(3x^2-e^{x})(\sec^2 x) + 5\tan x(6x-e^{x})]$

3. (a) Consider the curve given by $4x^2 + y^2 = 1$. Use implicit differentiation to find $\dfrac{dy}{dx}$.

Hint: Differentiate both sides with respect to x and use the Chain Rule:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

$4x^2 + y^2 = 1$

$\frac{d}{dx}4x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$

$8x + \frac{d}{dy}y^2\frac{dy}{dx} = 0$

$8x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -8x$

$\frac{dy}{dx} = -\frac{8x}{2y}$

$= -\frac{4x}{y}$

(b) Consider the curve given by $x^2 + xy^2 = 7$. Use implicit differentiation to find $\dfrac{dy}{dx}$.

Hint: Use the Product Rule to differentiate xy^2 with respect to x:

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(x) \times y^2 + x \times \frac{d}{dx}(y^2).$$

$x^2 + xy^2 = 7$

→ differentiate both sides with respect to x

$\frac{d}{dx}(x^2 + xy^2) = \frac{d}{dx}7$

$\frac{d}{dx}x^2 + \frac{d}{dx}(xy^2) = \frac{d}{dx}7$

$2x + (x)(\frac{d}{dx}y^2) + (y^2)(1) = 0$

$2x + y^2 + x(\frac{d}{dy}y^2\frac{dy}{dx}) = 0$

$x(2y\frac{dy}{dx}) = -2x - y^2$

$2y\frac{dy}{dx} = \frac{-2x-y^2}{x}$

$\frac{dy}{dx} = \frac{-2x-y^2}{2yx}$

Answers:

1. (a) $f'(x) = 3e^{3x+1}$ (b) $f'(x) = (1-3x^2)e^{x-x^3}$
   (c) $f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ (d) $f'(x) = e^{-x}(1-x)$
   (e) $f'(x) = e^{3x} + 3xe^{3x} + 3x^2$ (f) $f'(x) = e^{x}(1+x)$
   (g) $f'(x) = e^{-x^2}(1-2x^2)$ (h) $f'(x) = e^{x}(\sin(x^2) + 2x\cos(x^2))$
   (i) $f'(x) = e^{x}\cos(2x^{0.4}+3) - 0.8e^{x}x^{-0.6}\sin(2x^{0.4}+3)$
   (j) $f'(x) = \dfrac{3(e^{x}+\tan x) - (3x+1)(e^{x}+\sec^2 x)}{(e^{x}+\tan x)^2}$
   (k) $f'(x) = \dfrac{-0.8x^{-0.6}\sin(2x^{0.4}+3)(e^{x}-4) - e^{x}\cos(2x^{0.4}+3)}{(e^{x}-4)^2}$
2. (a) $\frac{dy}{dx} = -(\sin x)e^{\cos x} - (\sec^2 x)e^{\tan x}$
   (b) $\dfrac{dy}{dx} = \dfrac{-2\sin(2x)(e^{2x}+1) - 2e^{2x}\cos(2x)}{(e^{2x}+1)^2}$
   (c) $\frac{dy}{dx} = (3x^2-e^{x})^4[5(6x-e^{x})\tan x + (3x^2-e^{x})\sec^2 x]$
3. (a) $\dfrac{dy}{dx} = -\dfrac{4x}{y}$ (b) $\dfrac{dy}{dx} = \dfrac{-2x-y^2}{2xy}$
4. (a) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ (b) At $(x,y) = (1,1)$, we have $\dfrac{dy}{dx} = 1$ (c)(i) $\dfrac{dy}{dx} = 1$
   (c)(ii) At $(x,y) = (0,0)$, we have $\dfrac{dy}{dx} = 1$
   (d) Suppose that $(x,y) \neq (0,0)$ and $x^3 - y^3 = 0$. Then $x = y$ and so from (a) we have
   $\dfrac{dy}{dx} = \dfrac{x^2}{y^2} = \dfrac{x^2}{x^2} = 1$ which is the same formula for $\dfrac{dy}{dx}$ as in (c).
5. (a) $\dfrac{dy}{dx} = \dfrac{\pi x^{\pi-1}}{2y(1+20y^6)}$ (b) $\dfrac{dy}{dx} = \dfrac{2\sin 2x}{3y^2 - e^{y}}$
   (c) $\dfrac{dy}{dx} = \dfrac{2x(\cos(x^2+4) - 3y)}{3x^2 - \sin y}$ (d) $\dfrac{dy}{dx} = \dfrac{-\cos(x+y)}{2+\cos(x+y)}$
   (e) $\dfrac{dy}{dx} = \dfrac{y(y - \sin x - 4xe^{(2x^2)})}{1-xy}$ (f) $\dfrac{dy}{dx} = -\dfrac{y}{x}$
6. (a) $f'(x) = \dfrac{-4}{7-4x}$ (b) $f'(x) = x(1+2\ln 3x)$
   (c) $f'(x) = \ln x + 1 + \cos x$ (d) $f'(x) = \dfrac{2}{x}\ln x$
   (e) $f'(x) = \dfrac{3x^2+2}{x^3+2x+1}$ (f) $f'(x) = \dfrac{2x+e^{x}}{x^2+e^{x}}$
   (g) $f'(x) = 0$ (h) $f'(x) = -\tan x$
7. (a) $f'(x) = \dfrac{1}{2x}$
   Using a log law, the original function can be rewritten as $f(x) = \dfrac{1}{2}\ln x$
   (b) $f'(x) = \dfrac{1}{x} + \dfrac{1}{2(x-1)}$
   Using a log law, the original function can be rewritten as $f(x) = \ln x + \frac{1}{2}\ln(x-1)$
   (c) $f'(x) = 2x$
   Because logs and exponentials cancel each other, the original function can be rewritten as $f(x) = x^2$
   (d) $f'(x) = 1$
   Because logs and exponentials cancel each other, the original function can be rewritten as $f(x) = x$
   (e) $f'(x) = 1$
   Because logs and exponentials cancel each other, the original function can be rewritten as $f(x) = x$
   (f) $f'(x) = \dfrac{e^{x}}{e^{x}+1} - \dfrac{e^{x}}{e^{x}-1}$
   Using a log law, the original function can be rewritten as $f(x) = \ln(e^{x}+1) - \ln(e^{x}-1)$
8. (a) $f'(x) = \frac{2}{1+4x^2}$ (b) $f'(x) = -\frac{e^{x}}{\sqrt{1-e^{2x}}}$ (c) $f'(x) = \frac{-3}{\sqrt{4-9x^2}}$
   (d) $f'(x) = 2\sin^{-1}x + \frac{2x}{\sqrt{1-x^2}}$ (e) $f'(x) = \frac{1}{\sqrt{-x(x+1)}}$
   Comment for Question 8(e): When we use the chain rule, we obtain $f'(x) = \frac{2}{\sqrt{1-(2x+1)^2}}$. This can be rewritten as
   $f'(x) = \frac{2}{\sqrt{1-(4x^2+4x+1)}} = \frac{2}{\sqrt{-4x^2-4x}} = \frac{2}{\sqrt{-4x(x+1)}} = \frac{2}{2\sqrt{-x(x+1)}} = \frac{1}{\sqrt{-x(x+1)}}$.
9. (a) There are 1000 bacteria when $t = 0$.
   (b) There are 1000e (i.e. approximately 2718) bacteria when $t = 10$.
   (c) When $t = 10$, the number of bacteria is increasing at the rate of 100e (i.e. approximately 272) bacteria per second.
10. (a) $3^{x}\ln 3$ (b) $-\ln(2)\,2^{-x}$ (c) $2^{\sin x}\ln(2)\cos(x)$.

4. Consider the curve given by $x^3 - y^3 = 0$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

(b) Find $\frac{dy}{dx}$ at the point $(1, 1)$.

(c) Notice that we cannot use the formula in (a) to find $\frac{dy}{dx}$ at $(0, 0)$ since the dominator is zero when $(x, y) = (0, 0)$. However, this doesn't mean that $\frac{dy}{dx}$ doesn't exist at $(0, 0)$; we can find $\frac{dy}{dx}$ by rewriting

$$x^3 - y^3 = 0 \iff y^3 = x^3 \iff y = x \qquad (*)$$

Now find

(i) a simpler expression for $\frac{dy}{dx}$.

(ii) $\frac{dy}{dx}$ at $(0, 0)$.

(d) Verify that for $(x, y) \neq (0, 0)$, if (x, y) satisfy $(*)$ then the two expressions for $\frac{dy}{dx}$ in (a) and (c) are equal.

5. Find $\frac{dy}{dx}$ for each of the following equations:

(a) $y^2 + 5y^8 = x^\pi - 4$

(b) $\cos(2x) + y^3 - e^y = 4$

(c) $3x^2y + \cos y = \sin(x^2 + 4)$

(d) $\sin(x + y) + 2y = 0$

(e) $\ln y - xy = \cos x - e^{2x^2}$

(f) $\sin(xy) = 0$

6. Differentiate each of the following functions:

(a) $f(x) = \ln(7 - 4x)$

(b) $f(x) = x^2 \ln 3x$

(c) $f(x) = x \ln x + \sin x$

(d) $f(x) = (\ln x)^2$

(e) $f(x) = \ln(x^3 + 2x + 1)$

(f) $f(x) = \ln(x^2 + e^x)$

(g) $f(x) = \ln 7$

(h) $f(x) = \ln(\cos x)$

7. Differentiate each of the following functions:

Hint: It is easiest if we use logarithmic properties to rewrite these functions *before* we do the differentiation!

(a) $f(x) = \ln \sqrt{x}$

(b) $f(x) = \ln(x\sqrt{x-1})$

(c) $f(x) = \ln\left(e^{(x^2)}\right)$

(d) $f(x) = e^{\ln x}$

(e) $f(x) = \ln(e^x)$

(f) $f(x) = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$

8. Differentiate each of the following functions:

(a) $f(x) = \tan^{-1}(2x)$

(b) $f(x) = \cos^{-1}(e^x)$

(c) $f(x) = \cos^{-1}\left(\frac{3x}{2}\right)$

(d) $f(x) = 2x\sin^{-1} x$

(e) $f(x) = \sin^{-1}(2x + 1)$

9. The number N of bacteria in a particular solution at time t seconds (for $t \geq 0$) is given by $N = 1000e^{0.1t}$.

(a) Find the number of bacteria at time $t = 0$.

(b) Find the number of bacteria at time $t = 10$.

(c) At what rate is the number of bacteria increasing at time $t = 10$?