Chapter 5

Exponentials and Logarithms

5.1 Exponential Laws and Definitions

If $a,\ b > 0$ and if $r,\ s \in \mathbf{R}$ then

- $a^r a^s = a^{r+s}$
- $\dfrac{a^r}{a^s} = a^{r-s}$
- $(ab)^r = a^r b^r$
- $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$
- $(a^r)^s = a^{rs}$
- $a^{-r} = \dfrac{1}{a^r}$
- $a^0 = 1$
- $a^{\frac{1}{2}} = \sqrt{a}$

5.2 Logarithm Laws

If $x,\ y > 0$ and $a > 1$ and $r \in \mathbf{R}$ then

- **Log Law 1:** $\log_a(xy) = \log_a x + \log_a y$,
- **Log Law 2:** $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$,
- **Log Law 3:** $\log_a(x^r) = r\log_a x$, and
- **Log Law 4:** $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

Example 1. Use your calculator to evaluate $\log_2 7$.
Write your answer to 3 decimal places.

Solution: By using Log Law 4, and the $\boxed{\log} = \log_{10}$ button on our calculator, we have

$$\begin{aligned}
\log_2 7 &= \frac{\log_{10} 7}{\log_{10} 2} \\
&= \boxed{(}\,\boxed{\log}\,\boxed{7}\,\boxed{)}\,\boxed{\div}\,\boxed{(}\,\boxed{\log}\,\boxed{2}\,\boxed{)} \\
&= 2.807 \quad (3 \quad \text{d.p.})
\end{aligned}$$

□

Example 2. Using Log Laws 1–3, simplify the following expressions:

(a) $\dfrac{\log_5 8}{\log_5 4}$ (b) $\log_{10} 3 + 2\log_{10}\left(\frac{5}{4}\right) - \log_{10}\left(\frac{25}{32}\right)$.

Solution: (a)

$$\begin{aligned}
\frac{\log_5 8}{\log_5 4} &= \frac{\log_5(2^3)}{\log_5(2^2)} \\
&= \frac{3\log_5 2}{2\log_5 2} \qquad \text{(by Log Law 3)} \\
&= \frac{3}{2}.
\end{aligned}$$

Note:	$\dfrac{\log_5 8}{\log_5 4} \neq \dfrac{8}{4}$	and	$\dfrac{\log_5 8}{\log_5 4} \neq \log_5\left(\dfrac{8}{4}\right)$

(b)

$$
\begin{aligned}
& \log_{10} 3 + 2\log_{10}\left(\frac{5}{4}\right) - \log_{10}\left(\frac{25}{32}\right) \\
= \; & \log_{10} 3 + \log_{10}\left(\left(\frac{5}{4}\right)^2\right) - \log_{10}\left(\frac{25}{32}\right) \qquad \text{(by Log Law 3)} \\
= \; & \log_{10}\left(3 \times \left(\frac{5}{4}\right)^2\right) - \log_{10}\left(\frac{25}{32}\right) \qquad \text{(by Log Law 1)} \\
= \; & \log_{10}\left(\frac{75}{16}\right) - \log_{10}\left(\frac{25}{32}\right) \\
= \; & \log_{10}\left(\frac{75}{16} \div \frac{25}{32}\right) \qquad \text{(by Log Law 2)} \\
= \; & \log_{10}\left(\frac{75}{16} \times \frac{32}{25}\right) \\
= \; & \log_{10}(3 \times 2) \\
= \; & \log_{10} 6\,.
\end{aligned}
$$

$\square$

Exercises

1. Without using your calculator, simplify

$$\log_{10}\left(\frac{5}{32}\right) - 4\log_{10}\left(\frac{5}{4}\right) + 3\log_{10}\left(\frac{9}{2}\right) - 4\log_{10}\left(\frac{3}{5}\right)$$

2. **Maths 1 Extension (Not Examinable)**:

   Consider the functions defined by

$$f(x) = 2\log_5 x \quad \text{and} \quad g(x) = \log_5(x^2).$$

   These functions are **not** identical to each other. Why?

   Hint: Consider the domains of f and g .

5.3 Solving Exponential Equations

Example 3. Solve $3^x = 5$. Write your answer to 3 decimal places.

Solution.

$$\begin{aligned}
3^x &= 5 \\
\iff \quad \log_{10}(3^x) &= \log_{10} 5 \\
\iff \quad x\log_{10} 3 &= \log_{10} 5 \qquad \text{(by Log Law 3)} \\
\iff \quad x &= \frac{\log_{10} 5}{\log_{10} 3} \\
\iff \quad x &= 1.465 \quad (3 \quad \text{d.p.})
\end{aligned}$$

Example 4. Solve $2^x + 2^{-x} = 5$. Write your answer to 3 decimal places.

Solution:

> **Note:** It is **not** useful to take the logarithm of both sides since
>
> $$\log_{10}\left(2^x + 2^{-x}\right) \;=\; \log_{10} 5$$
>
> cannot be simplified:
>
> $$\log_{10}\left(2^x + 2^{-x}\right) \;\neq\; \log_{10}\left(2^x\right) + \log_{10}\left(2^{-x}\right)\ .$$

We need to rewrite the equation in a different form, so that it is easier to solve.

First of all, note that $2^{-x} = \dfrac{1}{2^x}$ and so we need to solve

$$2^x + \frac{1}{2^x} = 5\,.$$

Now, instead of solving for x , let us solve for 2^x .
Let $y = 2^x$. Then we need to solve

$$y + \frac{1}{y} = 5\,.$$

Multiplying both sides by y gives $y^2 + 1 = 5y$.
We can use the quadratic formula to solve this equation for y :

$$y^2 - 5y + 1 = 0 \qquad \Longleftrightarrow \qquad y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)} = \frac{5 \pm \sqrt{21}}{2}\,.$$

Thus

$$y = \frac{5 + \sqrt{21}}{2} \qquad \text{or} \qquad y = \frac{5 - \sqrt{21}}{2}\,.$$

Since $y = 2^x$, we must have

$$2^x = \frac{5 + \sqrt{21}}{2} \qquad \text{or} \qquad 2^x = \frac{5 - \sqrt{21}}{2}\,.$$

Now, to solve for x , we take the logarithm of both sides:

$$\log_{10}(2^x) = \log_{10}\left(\frac{5+\sqrt{21}}{2}\right) \quad \text{or} \quad \log_{10}(2^x) = \log_{10}\left(\frac{5-\sqrt{21}}{2}\right)$$

$$x\log_{10} 2 = \log_{10}\left(\frac{5+\sqrt{21}}{2}\right) \quad \text{or} \quad x\log_{10} 2 = \log_{10}\left(\frac{5-\sqrt{21}}{2}\right)$$

$$x = \frac{\log_{10}\left(\frac{5+\sqrt{21}}{2}\right)}{\log_{10} 2} \quad \text{or} \quad x = \frac{\log_{10}\left(\frac{5-\sqrt{21}}{2}\right)}{\log_{10} 2}\,.$$

So, to 3 decimal places, either

$$x = 2.260 \quad \text{or} \quad x = -2.260\,.$$

$\square$

Exercises

1. Use your calculator to evaluate the following (write your answer to 4 decimal places):

   (a) $\log_2 10$ (b) $\log_7 3$ (c) $\log_8 8192$

2. Find x to 3 decimal places:

   (a) $3^x = 0.2$ (b) $2^x = 5$ (c) $3^{x-1} = 7$ (d) $10^x + 10^{-x} = 5$

3. Solve

   (a) $2^{-x} = 32$ (b) $2^{2x} - 6 \times 2^x + 8 = 0$

4. Show that $\log_{10}\left(\frac{5-\sqrt{21}}{2}\right) = -\log_{10}\left(\frac{5+\sqrt{21}}{2}\right)$.

   Hint: Use the fact that $\dfrac{5-\sqrt{21}}{2} = \dfrac{\left(5-\sqrt{21}\right)\left(5+\sqrt{21}\right)}{2\left(5+\sqrt{21}\right)}$ and expand the brackets, simplify, and then use Log Law 3.

5.4 Answers to Chapter 5 Exercises

5.2: 1. $\log_{10} 45$

2. f and g are not identical because they have different domains.

In particular, $\text{dom}(f) = (0, \infty)$ whereas $\text{dom}(g) = \mathbf{R} \setminus \{0\}$.

Thus, for example, $g(-5) = \log_5(25) = 2$ whereas $f(-5)$ is not defined.

5.3: 1. (a) 3.3219 (b) 0.5646 (c) 4.3333

2. (a) -1.465 (b) 2.322 (c) 2.771 (d) ± 0.680

3. (a) -5 (b) 1, 2

4. Omitted.