Mathematics 1

Exercise Sheet 20: Permutations and Combinations

1. Without using your calculator, simplify

   (a) $8 \times 7!$ (b) $\dfrac{8!}{7!}$ (c) $\dfrac{9!}{7!2!}$ (d) $\dfrac{450!}{449!}$

2. Simplify

   (a) $\dfrac{n!}{(n-1)!}$ where $n \in \{1, 2, 3, \dots\}$

   (b) $\dfrac{n!}{(n-2)!}$ where $n \in \{2, 3, 4, \dots\}$

3. In order to travel from Melbourne to Brisbane, Joe is given the following choices. He can fly directly from Melbourne to Brisbane on one of four airlines, or he can fly from Melbourne to Sydney on one of five airlines and then travel from Sydney to Brisbane with one of six bus lines, or Joe can go on one of three bus lines directly from Melbourne to Brisbane. In how many ways can he travel from Melbourne to Brisbane?

4. (a) How many arrangements are there of two people taken from a set of 40 people?

   (b) In how many ways can two people be seated in a row of 40 empty seats?

   (c) In how many of the ways in part (b) are the two people sitting together?

   (d) In how many of the ways in part (b) are the two people **not** sitting together?

5. (a) In how many ways can the letters in the word VALUE be arranged?

   (b) In how many of these arrangements is there a vowel first?

6. In how many ways can 5 different maths books, 4 different physics books and 2 different biology books be arranged on a shelf if the books in each subject are to remain together?

   Note: When we are asked to keep certain items *together*, then we should group those items together as a "unit". We need to arrange the units, *and* we need to arrange the items *within* the units.

7. In how many ways can the letters of the word COOLANGATTA be arranged?

   Note: The number of arrangements of n objects of which p are of one kind, q of another kind and so on is given by $\dfrac{n!}{p!\,q!\,\dots}$.

8. (a) Find the number of ways of arranging all the letters of the word MINIMUM.

   (b) In how many of these arrangements do the four consonants all appear next to each other?

9. Consider the letters of the word STEEPLES.

   (a) In how many ways can these letters be arranged?

   (b) In how many of these arrangements do the three Es come together?

   (c) How many of the arrangements begin and end with S (with no conditions about whether or not the Es are together)?

10. Consider the letters in the word ECONOMICS.

    (a) In how many ways can these letters be arranged?

    (b) How many of these arrangements have the letters in the pattern CVCVCVCVC, where C denotes a consonant and V denotes a vowel?

11. How many **even** four–digit numbers can be formed with the digits 6, 7, 8, 9 if

    (a) repetitions are **not** allowed?

    (b) repetitions **are** allowed?

12. How many numbers greater than 4000 can be formed from the digits 3, 5, 7, 8 and 9, if repetitions are **not** allowed?

13. Consider the letters of the word APRICOTS. Find the number of arrangements with

    (a) exactly 3 letters between the P and the T.

    (b) at least 3 letters between the P and the T.

    Touch for a video solution of a similar problem.

14. Consider the letters of the word BOOMERANG. Find the number of arrangements with

    (a) exactly 4 letters between the B and the M.

    (b) exactly 5 letters between the two Os.

15. The letters from the word BALALAIKA are arranged in a row. Find the number of arrangements with

    (a) at least 4 letters between the B and the K.

    (b) exactly 4 letters between the B and the K.

    (c) more than 4 letters between the B and the K.

    (d) no more than 4 letters between the B and the K.

16. Before the start of a field hockey match, the teams line up to shake hands. The captain stands at one end of the line (nearest to the centre of the field) and the goalie stands at the other end of the line (nearest to the sideline). There are 11 players on a team. Suppose that Sarah and Felicity are two players in Trinity's team. Neither of them is the captain, neither of them is the goalie, and the goalie is not the captain either. How many ways are there for the team to line up

    (a) without any other restrictions?

    (b) with exactly four players standing between Sarah and Felicity?

    (c) with more than one player standing between Sarah and Felicity?

17. How many of the first million positive integers are palindromic numbers?

    Note: a palindromic number reads the same backward as forward.

18. How many arrangements of the letters in the word OSCILLOSCOPE

    (a) have the letters from the second half of the alphabet together?

    Hint: The second half of the alphabet is

    N, O, P, Q, R, S, T, U, V, W, X, Y, Z.

    (b) have the three O's together?

    (c) satisfy one or other (or both) of the previous conditions?

    Hint: Add the numbers of arrangements found in (a) and (b) together, but note that the intersection will be added twice, and so needs to be subtracted.

19. (a) Find the number of arrangements of the letters in the word ENGINEERING.

    (b) How many arrangements have the three E's together?

    (c) **Maths 1 Extension (Not examinable):**
    How many arrangements have **no** E's together?

20. A committee of 6 people is to be selected from the group $\{A, B, C, D, E, F, G, H, I, J\}$.

    How many committees can be formed

    (a) which contain both A and B?

    (b) which exclude both A and B?

21. (a) In how many ways can a committee of 3 men and 4 women be chosen from 6 men and 7 women?

    (b) How many of these committees contain a particular man and 2 particular women?

22. A class consists of 16 boys of whom 5 are prefects. How many committees of 7 can be formed if each consists of

    (a) exactly 3 prefects?

    (b) at least 2 prefects?

23. A team of 5 people is to be chosen from 7 engineers and 8 mathematicians. Calculate the number of ways this can be done if the team is to contain

    (a) 2 engineers and 3 mathematicians.

    (b) at least 1 engineer and at least 2 mathematicians.

24. A committee of 7 politicians is to be chosen from 10 Liberal members, 8 Labour members and 5 Independents. In how many ways can this be done so as to include exactly 1 Independent, at least 3 Liberal members, and at least 1 Labour member?

25. Suppose that students are required to study four out of ten available subjects. Suppose further that the subjects are classified into two groups, each containing five subjects.

    How many subject combinations are possible

    (a) if a student must take two subjects from each group?

    (b) if a student must take at least one subject from each group?

26. (a) Find the expanded form of

$$(a+x)(a+x)(a+x)(a+x)(a+x)(a+x)\,.$$

(b) Show that there are 64 terms in this expansion (some of which are the same).

Each term is a product of the x from some brackets and the a from the other brackets.

(c) Use ${}^nC_r$ notation to describe the number of a^4x^2 terms in this expansion.

(d) Use ${}^nC_r$ notation to describe the number of ax^5 terms in this expansion.

(e) Hence write down the expanded form of $(2+x)^{11}$ without using Pascal's triangle.

27. **Maths 1 Extension (Not examinable):**

If an executive of three people can be selected from the members of a larger council in 84 ways, how many members does the council have ?

28. **Maths 1 Extension (Not examinable):**

How many arrangements are there of four tiles from the Scrabble tiles below ?

K 5	**A** 1	**L** 1	**A** 1	**O** 1	**D** 2	**G** 2

Answers:

1. (a) 8! (b) 8 (c) 36 (d) 450
2. (a) n (b) $n(n-1)$
3. There are 37 ways that Joe can travel from Melbourne to Brisbane.
4. (a) There are 1560 arrangements.
   (b) There are 1560 ways.
   (c) There are 78 ways for the two people to sit together.
   (d) There are 1482 ways for the two people to sit apart.
5. (a) There are 120 possible arrangements.
   (b) Of these arrangements, 72 start with a vowel.
6. There are 34 560 ways to arrange the books with the books in each subject remaining together.
7. There are 1 663 200 possible arrangements.
8. (a) There are 420 possible arrangements.
   (b) There are 48 arrangements with the 4 consonants together.
9. (a) There are 3360 possible arrangements.
   (b) There are 360 arrangements if the three Es are together.
   (c) There are 120 arrangements if the arrangements begin and end with S.
10. (a) There are $\frac{9!}{2!2!} = 90\,720$ arrangements.
    (b) There are $\frac{5!4!}{2!2!} = 720$ arrangements with the pattern CVCVCVCVC.
11. (a) There are 12 suitable numbers if repetitions are not allowed.
    (b) There are 128 suitable numbers if repetitions are allowed.

12. There are 216 suitable numbers.

13. (a) There are 5760 possible arrangements.
    (b) There are 14400 possible arrangements.

14. (a) There are 20160 possible arrangements.
    (b) There are 15120 possible arrangements.

15. (a) There are 2100 suitable arrangements.
    (b) There are 840 suitable arrangements.
    (c) There are 1260 suitable arrangements.
    (d) There are 6300 suitable arrangements.

16. (a) There are $9! = 362\,880$ ways.
    (b) There are $40\,320$ ways.
    (c) There are $211\,680$ ways.

17. There are 1998 palindromic numbers in this set.

18. (a) There are $75\,600$ possible arrangements.
    (b) There are $453\,600$ possible arrangements.
    (c) There are $514\,080$ possible arrangements.

19. (a) There are $\frac{11!}{3!3!2!2!} = 277\,200$ arrangements.
    (b) There are $\frac{9!}{3!2!2!} = 15\,120$ arrangements with the E's together.
    (c) There are $141\,120$ arrangements with no E's together.

20. (a) 70 committees can be formed which contain both A and B.
    (b) 28 committees exclude both A and B.

21. (a) There are 700 committees containing 3 men and 4 women.
    (b) 100 of the committees contain a particular man and 2 particular women.

22. (a) There are 3300 possible committees containing exactly 3 prefects.
    (b) There are 8800 possible committees containing at least 2 prefects.

23. (a) There are 1176 possible teams containing exactly 2 engineers and 3 mathematicians.
    (b) There are 2646 possible teams containing at least 1 engineer and at least 2 mathematicians.

24. There are $73\,080$ suitable committees of the politicians.

25. (a) There are 100 possible subject combinations.
    (b) There are 200 possible subject combinations.

26. (a) $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.
    (b)(c)(d) Omitted.
    (e) $2048 + 11\,264x + 28\,160x^2 + 42\,240x^3 + 42\,240x^4 + 29\,568x^5 + 14\,784x^6 + 5280x^7 + 1320x^8 + 220x^9 + 22x^{10} + x^{11}$.

27. The council has 9 members.

28. There are 480 arrangements.