C7: LIMITS AND CONTINUITY

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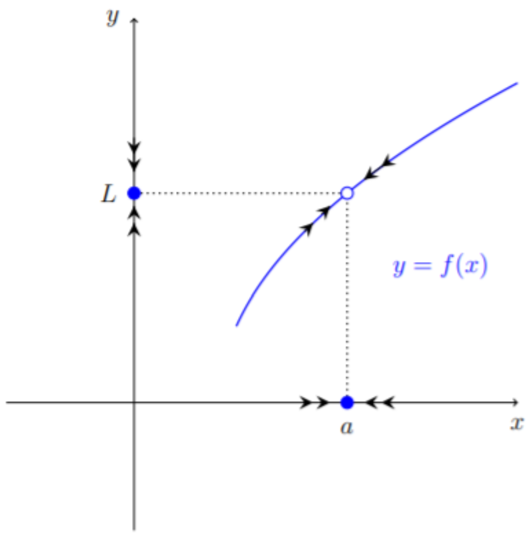
7.1 LIMIRS - AN INFORMAL TREATMENT

7.1 Limits – An informal treatment

We say that $\lim_{x \to a} f(x)$ **exists** if we can find a number, say L, such that $f(x)$ gets very close to L, as x gets very close to a (from **both** sides). We write $\lim_{x \to a} f(x) = L$, and we say that the limit of $f(x)$, as x approaches a, is L. This is called a **two–sided** limit (since x approaches a from **both** sides).



Suppose that $f(x) = x + 3$. We will investigate the values of $f(x)$ when x is close 2, so that we get an idea of the value of $\lim_{x \to 2} f(x)$.

x	$f(x)$
1.9	4.9
1.99	4.99
1.999	4.999

x	$f(x)$
2.1	5.1
2.01	5.01
2.001	5.001

As x gets close to 2 (from the left **and** from the right) then we see that $f(x)$ gets close to 5. Thus, we would expect that $\lim_{x \to 2} f(x) = 5$. This is indeed the value of the limit, but we shall soon state a result that allows us to calculate this kind of limit in a simple and mathematically correct manner.

Note also that $f(2) = 2 + 3 = 5$. So for $f(x) = x + 3$, we see that $\lim_{x \to 2} f(x) = f(2)$.

Result 1. If f is a polynomial then

$$\lim_{x \to a} f(x) = f(a).$$

Result 2. If f and g are polynomials, with $g(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

Note that if f is **NOT** a polynomial (or a fraction of polynomials), then we **might** have

$$\lim_{x \to a} f(x) \neq f(a).$$

That is, it is possible that the limit **cannot** be found just by substituting the number a into the function. In particular, we should keep in mind that whenever we evaluate $\lim_{x \to a} f(x)$, we are actually interested in the value of $f(x)$ for x **close** to a (but **not equal** to a).

Example 2. (a) Find $\lim_{x \to 1}(x + 1)$.

(b) Consider the function $f(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \end{cases}$

(b) Consider the function $f(x) = \begin{cases} \\ \pi & \text{if } x = 1. \end{cases}$

(i) Find $f(1)$.

(ii) Find $\lim_{x \to 1} f(x)$.

Solution: (a) Since $x + 1$ is a polynomial, we have

$$\begin{aligned} \lim_{x \to 1}(x + 1) &= 1 + 1 \\ &= 2 \end{aligned}$$

(b) (i) We have $f(1) = \pi$.

(ii) We have $\lim_{x \to 1} f(x) = \lim_{x \to 1}(x + 1) = 1 + 1 = 2$.

□

Note that in Example 2(b), the limit **CANNOT** be found just by substituting $x = 1$ into $f(x)$. That is, in Example 2(b) we have

$$\lim_{x \to 1} f(x) \neq f(1).$$

Suppose we need to calculate

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials. If $f(a) = 0$ and $g(a) = 0$, then it is useful to

(a) factorise $f(x)$ and/or $g(x)$, and

(b) cancel terms,

in order to calculate $\lim_{x \to a} \frac{f(x)}{g(x)}$.

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

EXERCISES [pg7]

Evaluate each of the following limits:

(a) $\lim_{x \to 1}(5x + 3)$

(b) $\lim_{x \to 0} |x|$

(c) $\lim_{x \to \pi} \cos x$

(d) $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

(e) $\lim_{x \to -2} \frac{x^3 + 8}{x + 2}$

(f) $\lim_{t \to 4} \frac{\sqrt{t} - 2}{t - 4}$

LIMIT LAWS

Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and that c is a constant.
Then

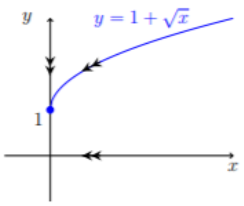
1. $\lim_{x\to a}[f(x)+g(x)] = L+M$
2. $\lim_{x\to a}[f(x)-g(x)] = L-M$
3. $\lim_{x\to a}[f(x)g(x)] = LM$
4. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$.
5. if $M = 0$ and $L \neq 0$ then $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ does not exist.
6. $\lim_{x\to a}[cf(x)] = cL$
7. $\lim_{x\to a} c = c$.
8. $\lim_{x\to a} x = a$.

7.2 ONE-SIDED LIMITS

When we write $\lim_{x\to a^-} f(x)$, we only consider x-values which are **less** than a. This is called a **left–hand** limit. We say that $\lim_{x\to a^-} f(x)$ **exists**, and that $\lim_{x\to a^-} f(x) = L$ if there is a number L such that $f(x)$ gets very close to L, as x gets very close to a **from the left**.

Similarly, when we write $\lim_{x\to a^+} f(x)$, we only consider x-values which are **greater** than a. This is called a **right–hand** limit.

Let us consider how we might find $\lim_{x\to 0^+}(1+\sqrt{x})$.



From the graph, it appears that as x approaches 0 from the right, then y approaches 1. That is, $1+\sqrt{x}$ approaches 1.
So it seems that

$$\lim_{x\to 0^+}(1+\sqrt{x}) = 1$$

This is in fact correct. It turns out that the square root function is another example of a function whose limits are found simply by substitution. That is we can calculate the above limit as follows:

$$\begin{aligned}\lim_{x\to 0^+}(1+\sqrt{x}) &= 1+\sqrt{0}\\ &= 1+0\\ &= 1\end{aligned}$$

Result 3. Let f be a function that is defined for all numbers near the number a *. Then

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

From this result, we deduce that

if the one–sided limits are **not equal** to each other
then the two–sided limit does not exist.

We also deduce that

if one (or both) of the one–sided limits does not exist,
then the two–sided limit also does not exist.

EXERCISES [pg12]

1. Consider the function $f(x) = \begin{cases} x^2+1 & \text{for } x \geq 2 \\ 2x-1 & \text{for } x < 2 \end{cases}$

(a) Find $\lim_{x\to 2^-} f(x)$

(b) Find $\lim_{x\to 2^+} f(x)$.

(c) Find $\lim_{x\to 2} f(x)$.

2. Evaluate the following limits.

(a) $\lim_{x\to 1}(2x-5)$

(b) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

(c) $\lim_{x\to 4} \dfrac{x^2-16}{x-4}$

(d) $\lim_{x\to 2} \dfrac{2x+1}{x+2}$

(e) $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} 2x-1 & \text{for } x \geq 2 \\ x^2-1 & \text{for } x < 2 \end{cases}$

(f) $\lim_{x\to 3} f(x)$ where $f(x) = \begin{cases} x^2 & \text{for } x < 3 \\ 3x-1 & \text{for } x \geq 3 \end{cases}$

7.3 CONTINUITY

We say that a function f is **continuous** at $x = a$ if

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

In fact this single equation actually involves **three** conditions.
In particular, $f(x)$ is **continuous** at $x = a$ if

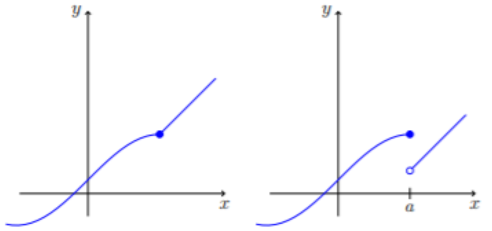
(i) $f(a)$ is defined; and

(ii) $\lim_{x\to a} f(x)$ exists; and

(iii) $\lim_{x\to a} f(x) = f(a)$.

If one or more of these three properties does **not** hold, then $f(x)$ is **discontinuous** at $x = a$.

When we just say that f is **continuous** (without stating a particular x-value) then we mean that f is continuous at **every** point in its domain.



Continuous   Not continuous at $x = a$

(c) Consider the function $f(x) = \begin{cases} x^2 & \text{for } x < 3 \\ 3x - 1 & \text{for } x \geq 3. \end{cases}$

Is f continuous at $x = 3$?

That is, does $\lim_{x \to 3} f(x) = f(3)$?

Solution: (i)

$$\begin{aligned} f(3) &= 3 \times 3 - 1 \\ &= 8 \end{aligned}$$

(ii)

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} (3x - 1) \\ &= 3 \times 3 - 1 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} (x^2) \\ &= 3^2 \\ &= 9 \end{aligned}$$

Therefore $\lim_{x \to 3} f(x)$ does not exist, because $\lim_{x \to 3^+} f(x) \neq \lim_{x \to 3^-} f(x)$.

Therefore f is not continuous at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist.

(d) Consider the function $f(x) = \begin{cases} x + 2 & \text{for } x \neq 1 \\ 4 & \text{for } x = 1. \end{cases}$

Is f continuous at $x = 1$?

That is, does $\lim_{x \to 1} f(x) = f(1)$?

Solution: (i) $f(1) = 4$ (This is clear from the definition of the function.)

(ii)

$$\begin{aligned} \lim_{x \to 1} f(x) &= \lim_{x \to 1} (x + 2) \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

(iii) We see that

$$\lim_{x \to 1} f(x) \neq f(1).$$

Therefore f is not continuous at $x = 1$. □

EXERCISES [pg17]

State whether the following functions are continuous. If a point of discontinuity occurs, explain why it is a point of discontinuity.

(a) $f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ x & \text{for } x \leq 0 \end{cases}$

(b) $f(x) = \begin{cases} x^2+1 & \text{for } x > 0 \\ x & \text{for } x \le 0 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$

(d) $f(x) = \begin{cases} 2x-3 & \text{for } x \le 2 \\ x^2-2x & \text{for } x > 2 \end{cases}$

CONTINUITY LAWS

Suppose that the functions f and g are continuous at a, and that c is a constant. Then

- $f+g$ is continuous at a.
- $f-g$ is continuous at a.
- fg is continuous at a.
- if $g(a) \neq 0$ then $\dfrac{f}{g}$ is continuous at a.
- cf is continuous at a.

It can be shown that the functions $h_1(x) = c$ and $h_2(x) = x$ are continuous everywhere. Then, by using the above laws, it can be shown that

$$\boxed{\text{each } \textbf{polynomial} \text{ is continuous everywhere}}$$

It can also be shown that for all **polynomials** f and g, we have

$$\boxed{\frac{f}{g} \text{ is continuous at each point } a \text{ such that } g(a) \neq 0}$$

Two other useful results concerning continuity are as follows:

- If g is continuous at a, and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

  Recall that $f \circ g$ is the function defined by $f \circ g(x) = f(g(x))$.
- The square root function $f(x) = \sqrt{x}$ is continuous on its domain.

  Note (not examinable): a formal definition of limit only chooses values of x in the domain of the function $f(x)$, and so, for example

$$\lim_{x \to 0} \sqrt{x} = \lim_{x \to 0^+} \sqrt{x} = 0.$$