Mathematics 1

Sheet 21: Probability

$$\Pr(A) = \frac{\text{number of outcomes in which } A \text{ occurs}}{\text{total number of possible outcomes}}$$

1. The letters from the word PROMISE are to be arranged randomly in a row.

   (a) How many arrangements are possible?

   (b) How many of these arrangements would have exactly 3 letters between the P and the R?

   (c) Hence find the probability that if the letters in the word PROMISE are randomly arranged there will be exactly 3 letters between the P and the R.

2. **Three** of the letters of the word SUNDAY are to be arranged in a row.

   (a) How many arrangements are possible?

   (b) How many arrangements begin with S?

   (c) What is the probability that an arrangement begins with S?

3. The letters of the word INDEPENDENCE are to be arranged in a row. What is the probability that the E's are all together?

4. In May of 2008, the leaders of the 12 South American nations signed a treaty which created the Union of South American Nations. Two of these leaders were women. If the leaders were randomly arranged in a row for the celebratory photograph, what is the probability that the two female leaders had more than two people standing between them?

5. From 7 teachers and 5 students a random selection of 7 people is to made.

   (a) How many different combinations are possible?

   (b) How many of these combinations contain at least 4 teachers?

   (c) What is the probability that a combination contains at least 4 teachers?

6. A box contains 6 white balls and 4 black balls. A sample of 5 balls is to be chosen *without* replacement. What is the probability that 3 of the chosen balls are white, and 2 are black?

   Hint: We have

   Pr(the sample contains 3 white balls and 2 black balls)

   $$= \frac{\text{number of samples containing 3 white and 2 black balls}}{\text{total number of samples (of 5 balls)}}$$

7. Suppose the two events A and B satisfy

   $\Pr(A) = 0.3,\ \Pr(B') = 0.25$ and $\Pr(A \cap B) = 0.1.$

   (a) Construct a table of intersections.

   (b) Hence find the following:

   (i) $\Pr(A \cup B)$ (ii) $\Pr(A' \cup B)$ (iii) $\Pr(A \cup B')$.

8. Suppose that two mutually exclusive events R and S satisfy $\Pr(R) = 0.4168$ and $\Pr(S) = 0.3275$.

   (a) Construct a table of intersections.

   (b) Are the events R' and S' mutually exclusive? (Justify your answer.)

9. Of the guests staying at the holiday park accommodation near Cradle Mountain on a particular evening, 70% could speak French and x% could speak Japanese. If 10% of the guests could speak neither language, what (in terms of x) is the probability that a randomly selected guest could speak both languages?

   Hint: Construct a table of intersections.

10. My father works in Zhengzhou and my mother works in Singapore. When they can, they meet on the 18^{th} of the month. The months in which they meet are randomly chosen from the standard 12 months of the year.

    (a) Suppose that they meet 5 times during the year. Find the probability that exactly 2 of the "meeting months" begin with the letter 'J'.

    (b) **Maths 1 Extension (Not examinable):** Suppose that they meet m times during the year, where $m \in \{2, 3, \ldots, 12\}$. Let $P(m)$ be the probability that exactly 2 of those m "meeting months" begin with the letter 'J'.

    (i) Show that

    $$P(m) = \frac{m(m-1)(12-m)}{440}.$$

    Hint: Use ${}^{n}\mathrm{C}_{r} = \dfrac{n!}{r!\,(n-r)!}$.

    (ii) Now consider the function

    $$f : [2, 12] \to \mathbf{R} \text{ where } f(x) = \frac{x(x-1)(12-x)}{440}.$$

    Find the x-value $x_{\max}$ such that $f(x_{\max})$ is the global maximum of f.

    (iii) Which value of m maximizes $P(m)$?

    Remember that m is a discrete variable.

    That is, $m \in \{2, 3, \ldots, 12\}$.

    Hint: We cannot put $m = x_{\max}$ since $x_{\max} \notin \{2, 3, \ldots, 12\}$, but we should test $m = \mathrm{int}(x_{\max})$ and $m = \mathrm{int}(x_{\max}) + 1$, where

    $$\begin{aligned} \mathrm{int}(x_{\max}) &= (\text{greatest integer less than } x_{\max}) \\ &= (\text{integer part of } x_{\max}\,.) \end{aligned}$$

11. **Maths 1 Extension (Not examinable):**
The foods that Carmen ate at a particular dinner were

avocado, asparagus, falafel, alfalfa, sprouts,
bread, butter, beetroot, cheese, pineapple.

If the letters from these foods are all put in one pile and then arranged in a row, what is the probability that all of the E's are together?

Hint: In total, the given list of words contains 71 letters, consisting of 12 a's, 3 b's, 2 c's, 2 d's, 10 e's, 4 f's, 1 g, 1 h, 1 i, 5 l's, 1 n, 5 o's, 5 p's, 5 r's, 5 s's, 5 t's, 3 u's and 1 v.

12. **Maths 1 Extension (Not examinable):**
Genetic code specifies which of the 20 standard amino acids is next to be synthesised. Three of these amino acids are called Alanine, Lysine and Serine. Let us denote these three amino acids with the letters A, L and S, and suppose that the other 17 amino acids are labelled with the symbols A_4, A_5, $\ldots$, A_{20}. Suppose that, when a sequence of amino acids is synthesised:

- All of the 20 amino acids A, L, S, A_4 $\ldots$, A_{20} are equally likely to appear.
- Repetitions are allowed.
- Order matters, so, for example, the sequence $ALSAAAA$ is not equal to $LASAAAA$.

(a) How many sequences of 3 amino acids can be formed such that exactly one amino acid in the sequence comes from the set $\{A, L, S\}$?

Hint: The sequence can be formed by filling three boxes

$$\boxed{}\boxed{}\boxed{}$$

with the letters A, L, S, A_4, A_5, $\ldots A_{20}$.
Let X denote A, L or S. Then exactly one of the boxes must contain X:

$$\boxed{X}\boxed{}\boxed{} \quad \text{or} \quad \boxed{}\boxed{X}\boxed{} \quad \text{or} \quad \boxed{}\boxed{}\boxed{X}$$

Now first place A, L or S in the position marked with X, and then place the remaining letters. Remember that repetitions are allowed!

(b) How many sequences of 4 amino acids can be formed such that exactly two amino acids in the sequence come from the set $\{A, L, S\}$?

Hint: The sequence can be formed by filling four boxes

$$\boxed{}\boxed{}\boxed{}\boxed{}$$

with the letters A, L, S, A_4, A_5, $\ldots A_{20}$.
Let X denote A, L or S. Then exactly two of the boxes must contain X:

$$\boxed{X}\boxed{X}\boxed{}\boxed{} \quad \text{or} \quad \boxed{X}\boxed{}\boxed{X}\boxed{} \quad \text{or} \quad \boxed{X}\boxed{}\boxed{}\boxed{X}$$

$$\text{or} \quad \boxed{}\boxed{X}\boxed{X}\boxed{} \quad \text{or} \quad \boxed{}\boxed{X}\boxed{}\boxed{X} \quad \text{or} \quad \boxed{}\boxed{}\boxed{X}\boxed{X}$$

Now first place A, L or S in the positions marked with an X, and then place the remaining letters. Remember that repetitions are allowed!

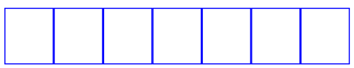
(c) In a sequence of 4 amino acids, what is the probability that exactly two amino acids in the sequence come from the set $\{A, L, S\}$?

Give your answer to four decimal places.

(d) **(Challenge)** In a sequence of 7 amino acids, what is the probability that exactly three amino acids in the sequence come from the set $\{A, L, S\}$?

Give your answer to four decimal places.

Hint: The sequence can be formed by filling seven boxes



with the letters A, L, S, A_4, A_5, $\ldots A_{20}$.

Let X denote A, L or S. Then exactly three of the boxes must contain X. Listing all the possibilities takes too long, so we look for a formula for the number of ways that we can place the 3 Xs ? Hint: in part (b), we can place the Xs in ${}^4\mathrm{C}_2 = 6$ ways.

Revision on Rates of Change

- Recall that $\dfrac{dy}{dx}$ can be thought of as the **rate of change** of y with respect to x. Furthermore, recall that

  ⋆ if a quantity is **increasing** then its derivative is **positive**,

  whereas

  ⋆ if a quantity is **decreasing** then its derivative is **negative**.

- Usually we are interested in rates of change **with respect to time** (in which case the phrase "with respect to time" is often omitted). For example,

  $$\frac{dV}{dt} \text{ is the rate of change of volume}$$

  (with respect to time).

- If we don't have a convenient formula linking the two variables in the required rate of change, then we use the chain rule.

  For example, sometimes it might be useful to write

  $$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

- Recall that to find a derivative, we need to have a formula which is expressed **only** in terms of the two variables listed in that particular derivative. For example, if $V = \pi r^2 h$ then we can't find $\frac{dV}{dr}$ unless we eliminate h from the formula first (or else if we are told that h is remaining constant).

Revision Exercises on Rates of Change

13. The volume, V (measured in m^3) of water in a reservoir at time t (measured in minutes) is given by

$$V = 3 + \sin\left(\frac{t}{4}\right).$$

   (a) Find, to three decimal places, the volume of water in the reservoir when $t = 10$.

   (b) Find, to two decimal places, the rate of change of the volume of water in the reservoir when $t = 10$.

14. The volume of a spherical balloon is increasing at a rate of $0.1\ \text{m}^3 \cdot \text{min}^{-1}$.

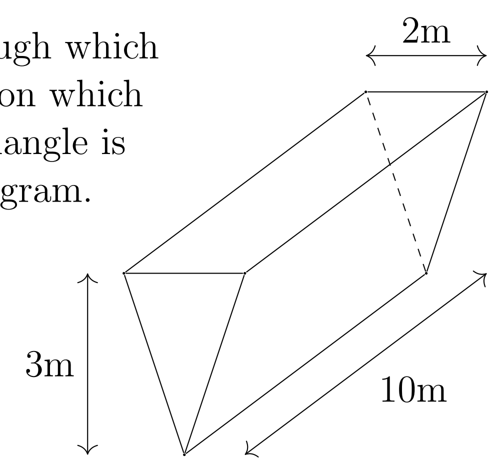
   (a) At what rate is the radius increasing, when the radius is 2.5 m?

   (b) At what rate is the surface area increasing, when the radius is 2.5 m?

   Note: The surface area of a sphere is given by $A = 4\pi r^2$.

15. An inverted cone has height 14 m, and has diameter of 7 m at the top. If water is leaking out of the bottom of the cone at the rate of $0.1\ \text{m}^3 \cdot \text{min}^{-1}$, find the rate at which the water level is falling when the water level is 10 m above the bottom. Write your answer in a sentence and accurate to four decimal places.

16. A horizontal trough which has a cross–section which is an isosceles triangle is shown in the diagram.

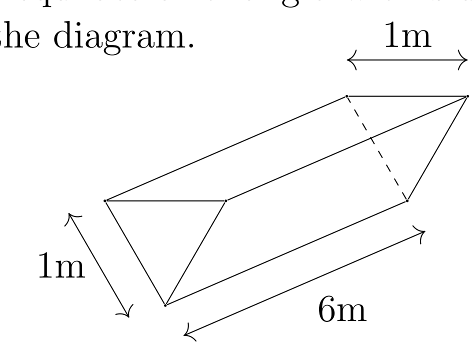


   The trough is being filled with water at the rate of $0.1\ \text{m}^3{\cdot}\text{min}^{-1}$. At what rate is the depth increasing

   (a) when the water's depth is 1.5 m?

   (b) when the trough is half–full?

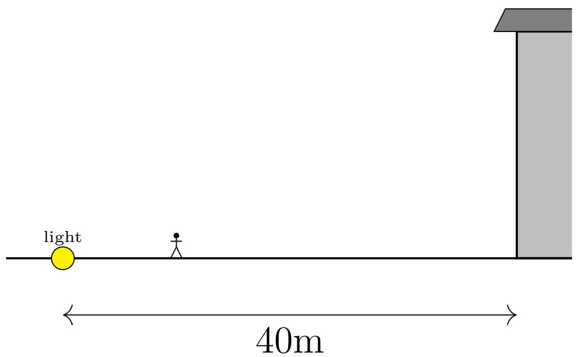
17. A trough is 6 m long, and has uniform cross–section of an equilateral triangle with sides 1 m, as shown in the diagram.



   Water leaks from the bottom of the trough, at a constant rate of $0.01\ \text{m}^3 \cdot \text{min}^{-1}$. Find, to four decimal places, the rate at which the water level is falling when the water is 0.2 m deep.

18. A floodlight at ground level shines on a building, and is located 40 m from the foot of the building.

A 2 m tall woman walks directly towards the building at 1.6 $\text{m}\cdot\text{s}^{-1}$. How fast is her shadow on the building shortening at the instant when she is 10 m from the building? Write your answer accurate to two decimal places.



Answers:

1. (a) There are $7! = 5040$ arrangements.
   (b) There are $6 \times 5! = 720$ arrangements with exactly 3 letters between the P and R.
   (c) The probability is $\dfrac{720}{5040} = \dfrac{1}{7}$.

2. (a) There are $6 \times 5 \times 4 = 120$ arrangements.
   (b) Of the arrangements, $1 \times 5 \times 4 = 20$ start with S.
   (c) The probability is $\dfrac{20}{120} = \dfrac{1}{6}$.

3. The probability that all the E's are together is $\dfrac{1}{55}$.
   (There are $\dfrac{12!}{4!\,3!\,2!}$ arrangements in total, of which $\dfrac{9!}{3!\,2!}$ have all the E's together.)

4. The probability is $\dfrac{6}{11}$.

5. (a) There are ${}^{12}\mathrm{C}_7 = 792$ combinations.
   (b) Of the combinations, 596 contain at least 4 teachers. This was found by calculating
   $${}^7\mathrm{C}_4 \times {}^5\mathrm{C}_3 + {}^7\mathrm{C}_5 \times {}^5\mathrm{C}_2 + {}^7\mathrm{C}_6 \times {}^5\mathrm{C}_1 + {}^7\mathrm{C}_7 \times {}^5\mathrm{C}_0 = 596.$$
   (c) The required probability is $\dfrac{596}{792} = \dfrac{149}{198}$.

6. The required probability is $\dfrac{{}^6\mathrm{C}_3 \times {}^4\mathrm{C}_2}{{}^{10}\mathrm{C}_5} = \dfrac{120}{252} = \dfrac{10}{21}$.

7. (a)

$\cap$	A	A'	
B	0.1	0.65	0.75
B'	0.2	0.05	0.25
	0.3	0.7	1

   (b) (i) 0.95
   (ii) 0.8
   (iii) 0.35

8. (a) Note that, since R and S are mutually exclusive, we must have $\Pr(R \cap S) = \mathbf{0}$. Thus the table of intersections is:

$\cap$	R	R'	
S	**0**	0.3275	0.3275
S'	0.4168	0.2557	0.6725
	0.4168	0.5832	1

   (b) We see that $\Pr(R' \cap S') \neq 0$, and so immediately conclude that R' and S' are ***not*** mutually exclusive.

9. The probability is $\dfrac{x}{100} - 0.2$.

10. (a) The probability is $\dfrac{7}{22}$.

(b) (i)

$$
\begin{aligned}
P(m) &= \frac{{}^{3}\mathrm{C}_{2} \times {}^{9}\mathrm{C}_{m-2}}{{}^{12}\mathrm{C}_{m}} \\
&= \frac{3 \times \dfrac{9!}{(m-2)!\,(9-(m-2))!}}{\dfrac{12!}{m!\,(12-m)!}} \\
&= 3 \times \frac{9!}{(m-2)!\,(11-m)!} \times \frac{m!\,(12-m)!}{12!} \\
&= 3 \times \frac{9!}{12!} \times \frac{m!}{(m-2)!} \times \frac{(12-m)!}{(11-m)!} \\
&= 3 \times \frac{1}{12 \times 11 \times 10} \times m(m-1) \times (12-m) \qquad (*) \\
&= \frac{1}{440} m(m-1)(12-m), \qquad \text{as required.}
\end{aligned}
$$

* Recall that $n! = n(n-1)(n-2)\ldots(2)(1)$.

(ii) $x_{\max} = \dfrac{13+\sqrt{133}}{3}$

(iii) $m = 8$.

11. The probability that the E's are together is $\dfrac{1}{7\,447\,387\,948}$.

12. (a) The number of ways is $3 \times 3 \times 17^2 = 2\,601$.

(b) The number of ways is $6 \times 3^2 \times 17^2 = 15\,606$.

(c) The probability is $\dfrac{15\,606}{20^4} = 0.0975$ (4 d.p.).

(d) The probability is $\dfrac{78\,927\,345}{1\,280\,000\,000} = 0.0617$ (4 d.p.).

13. (a) The volume is 3.598 m^3 (3 d.p.).
(Remember to use radians mode on your calculator!)

(b) The volume is *decreasing* at a rate of $0.20\,\mathrm{m}^3 \cdot \mathrm{min}^{-1}$ (2 d.p.).

14. (a) The radius is increasing at a rate of $\dfrac{1}{250\pi}$ $\mathrm{m} \cdot \mathrm{min}^{-1}$.

(b) The surface area is increasing at a rate of 0.08 $\mathrm{m}^2 \cdot \mathrm{min}^{-1}$.

15. The water level is falling at a rate of 0.0051 $\mathrm{m} \cdot \mathrm{min}^{-1}$ (4 d.p.).

16. (a) The depth is increasing at a rate of 0.01 $\mathrm{m} \cdot \mathrm{min}^{-1}$.

(b) The depth is increasing at a rate of $\dfrac{1}{100\sqrt{2}}$ $\mathrm{m} \cdot \mathrm{min}^{-1}$.

17. The water level is falling at a rate of 0.0072 $\mathrm{m} \cdot \mathrm{min}^{-1}$ (4 d.p.).

18. The woman's shadow is shortening by 0.14 $\mathrm{m} \cdot \mathrm{s}^{-1}$ (2 d.p.) when she is 10 m from the building.