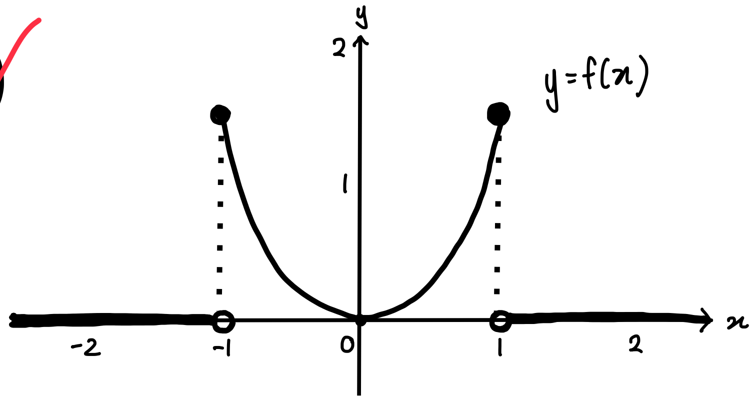
EXERCISE SHEET 25 <SELECTED QUESTIONS>

① a)



b) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{2}x^2\,dx + \int_{1}^{\infty} 0\,dx$$

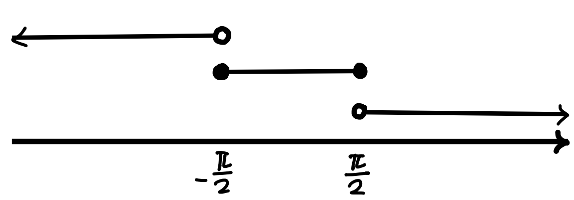
$$= 0 + \left[\frac{1}{2}x^3\right]_{-1}^{1} + 0$$

$$= \frac{1}{2}(1)^3 - \frac{1}{2}(-1)^3$$

$$= 1$$ <Shown>

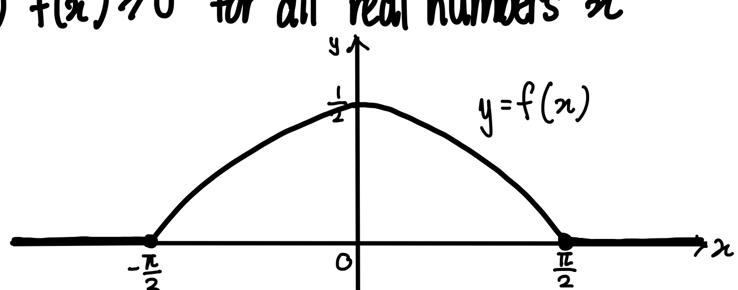
③ Properties of Probability Density Functions:

(a) Domain of f is a set of all real numbers



$$\left(-\infty, -\frac{\pi}{2}\right) \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \infty\right) = \mathbb{R}$$ <Shown>

(b) $f(x) \geq 0$ for all real numbers x



From the graph, $f(x) \geq 0$ for all $x \in \mathbb{R}$ <Shown>

(c) Show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-\frac{\pi}{2}} f(x)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} f(x)\,dx$$

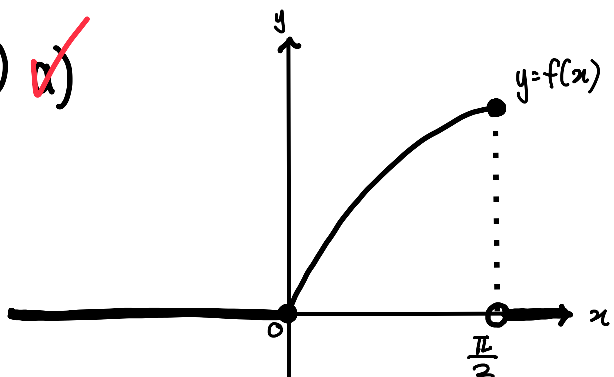
$$= \int_{-\infty}^{-\frac{\pi}{2}} 0 \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos x \, dx + \int_{\frac{\pi}{2}}^{\infty} 0 \, dx$$

$$= 0 + \left[\frac{1}{2}\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$= 1$$ <shown> //

⑤ a)



b) $$\int_{-\infty}^{\infty} f(x)\,dx) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\frac{\pi}{2}} \sin x \, dx + \int_{\frac{\pi}{2}}^{\infty} 0 \, dx$$

$$= 0 + \left[-\cos x\right]_{0}^{\frac{\pi}{2}} + 0$$

$$= -\cos\left(\frac{\pi}{2}\right) - -\cos(0)$$

$$= 1$$ <Shown> //

c) $$\Pr\left(X < \frac{\pi}{6}\right) = \int_{-\infty}^{\frac{\pi}{6}} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{6}} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\frac{\pi}{6}} \sin x \, dx$$

$$= 0 + \left[-\cos x\right]_{0}^{\frac{\pi}{6}}$$

$$= -\cos\left(\frac{\pi}{6}\right) - -\cos(0)$$

$$= 1 - \frac{\sqrt{3}}{2}$$ //

d) $Pr(X \leqslant \frac{\pi}{6}) = Pr(X < \frac{\pi}{6})$

$$= 1 - \frac{\sqrt{3}}{2} //$$

e) $Pr\left(X < \frac{\pi}{4}\right) = \int_{-\infty}^{\frac{\pi}{4}} f(x)\, dx$

$$= \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\frac{\pi}{4}} \sin x \, dx$$

$$= 0 + \left[-\cos x\right]_{0}^{\frac{\pi}{4}}$$

$$= -\cos\left(\frac{\pi}{4}\right) - -\cos(0)$$

$$= 1 - \frac{1}{\sqrt{2}} //$$

f) $Pr(X \leq \frac{\pi}{4}) = Pr(X < \frac{\pi}{4})$

$$= 1 - \frac{1}{\sqrt{2}} //$$

g) $Pr(X < \frac{\pi}{3}) = \int_{-\infty}^{\frac{\pi}{3}} f(x)\, dx$

$$= \int_{-\infty}^{0} 0 \, dx + \int_{\frac{\pi}{3}}^{0} \sin x \, dx$$

$$= 0 + \left[-\cos x\right]_{\frac{\pi}{3}}^{0}$$

$$= -\cos\left(\frac{\pi}{3}\right) - -\cos(0)$$

$$= \frac{1}{2} //$$

h) $Pr(X \leq \frac{\pi}{3}) = Pr(X < \frac{\pi}{3})$

$$= \frac{1}{2}$$ //

⑦ a) $\int f(x)\,dx = 1$

$$1 = \int_{-\infty}^{\frac{1}{2}} 0\,dx + \int_{\frac{1}{2}}^{k} \frac{1}{x}\,dx + \int_{k}^{\infty} 0\,dx$$

$$= \Big[\ln x\Big]_{\frac{1}{2}}^{k}$$

$$= \ln k - \ln\frac{1}{2}$$

$$\ln k = \ln\frac{e}{2}$$

$$k = \frac{e}{2}$$

$$= 1.3591 \langle 4\text{dp} \rangle$$

b) $\Pr(X \geqslant 1) = \int_{1}^{\infty} f(x)\,dx$

$$= \int_{1}^{\frac{e}{2}} \frac{1}{x}\,dx + \int_{\frac{e}{2}}^{\infty} 0\,dx$$

$$= \Big[\ln x\Big]_{1}^{\frac{e}{2}} + 0$$

$$= \ln\left(\frac{e}{2}\right) - \ln(1)$$

$$= 0.3069 \langle 4\text{dp} \rangle$$ //

⑩ a) $\Pr(X < \frac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\frac{1}{2}} 2x\,dx$$

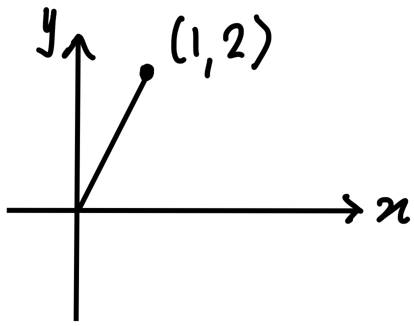
$$= 0 + \Big[x^2\Big]_{0}^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)^2 - (0)^2$$

$$= \frac{1}{4}$$ //

b) $Pr(X \leq m) = \frac{1}{2}$

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$$



$$\int_{0}^{m} 2x = \left[x^2\right]_0^m$$

$$= m^2 - (0)^2$$

$$m = \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}$$

c) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x(2x)\,dx + \int_{1}^{\infty} 0\,dx$$

$$= \left[\frac{2}{3}x^3\right]_0^1$$

$$= \frac{2}{3}$$ //

d) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_{0}^{1} 2x^3\,dx$$

$$= \left[\frac{1}{2}x^4\right]_0^1$$

$= \frac{1}{2}$ //

e) $Var(X) = E(X^2) - (E(X))^2$

$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$

$= \frac{1}{18}$ //

f) $SD = \sqrt{Var(X)}$

$= \sqrt{\frac{1}{18}}$

$= \frac{1}{3\sqrt{2}}$ //