WORKSPACE SETUP

1. **WINDOW 1**
   a. TCOLE
   b. Quiz
2. **WINDOW 2**
   a. Symbolab
   b. Desmos
   c. Matrix Multiplier
   d. Google

12 INDEFINITE INTEGRALS AND FURTHER APPLICATIONS

1. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

2. Antidifferentiation Rules and Formulae

a. $\int kf(x)dx = k\int f(x)dx$

b. $\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

c. $\int[f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

d. $\int 0dx = c$

e. $\int 1dx = x + c$

f. $\int (ax+b)^r dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c$

g. $\int \frac{d}{ax+b}dx = \frac{d}{a}\ln|ax+b| + c$

h. $\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + c$

i. $\int \cos(kx)dx = \frac{1}{k}\sin(kx) + c$

j. $\int \sec^2(kx)dx = \frac{1}{k}\tan(kx) + c$

k. $\int e^{kx}dx = \frac{1}{k}e^{kx} + c$

l. $\int \frac{b}{\sqrt{a^2-x^2}}dx = b\sin^{-1}(\frac{x}{a}) + c$

m. $\int \frac{-b}{\sqrt{a^2-x^2}}dx = b\cos^{-1}(\frac{x}{a}) + c$

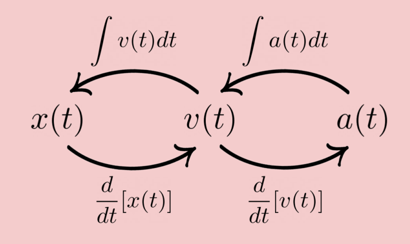
n. $\int \frac{a}{a^2+x^2}dx = \tan^{-1}(\frac{x}{a}) + c$

o. $\int \frac{1}{a^2+x}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$

3. Average Value of Function over interval [a,b]

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

4. Time Integrals



13 LIMITS AND INTEGRATION TO INFINITY

1. Limits to Infinity

a. If $r > 0$ then $\lim_{x\to\infty} \frac{1}{x^r} = 0$

b. If $r > 0$ then $\lim_{x\to\infty} x^r = \infty$

c. $$\lim_{x\to\infty} e^{kx} = \begin{cases} \infty & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k < 0\,. \end{cases}$$

2. Limits to Negative Infinity

a. $\lim_{x\to-\infty} \frac{1}{x} = 0$

b. $\lim_{x\to-\infty} x = -\infty$

c. $\lim_{x\to-\infty} e^x = 0$

d. $$\lim_{x\to-\infty} e^{kx} = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \infty & \text{if } k < 0. \end{cases}$$

e. $\lim_{x\to-\infty} \frac{1}{x^n} = 0$

f. $$\lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$$

3. Limit Laws

Suppose that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$, (where $L, M \in \mathbf{R}$) and that c is a constant. Then

1. $\lim_{x\to\infty} \Big[f(x) + g(x)\Big] = L + M$.

2. $\lim_{x\to\infty} \Big[f(x) - g(x)\Big] = L - M$.

3. $\lim_{x\to\infty} \Big[f(x)g(x)\Big] = L \times M$.

4. if $M \neq 0$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$.

5. $\lim_{x\to\infty} c = c$.

6. $\lim_{x\to\infty} \Big[cf(x)\Big] = cL$.

4. Integration to Infinity

a. $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \ldots = 1.$$

b. $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

c. $$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

d. $$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

e. $$\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx$$

f. $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

Example

Example 12. Find $\int_{-\infty}^\infty f(x)\,dx$ if $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 2 \\ e^{-x} & \text{if } x \ge 2. \end{cases}$

Solution: We have

$$\begin{aligned}
\int_{-\infty}^\infty f(x)\,dx &= \int_{-\infty}^0 f(x)\,dx + \int_0^2 f(x)\,dx + \int_2^\infty f(x)\,dx \\
&= \int_{-\infty}^0 0\,dx + \int_0^2 1\,dx + \int_2^\infty e^{-x}\,dx \\
&= 0 + \Big[x\Big]_0^2 + \lim_{b\to\infty} \int_2^b e^{-x}\,dx \\
&= (2-0) + \lim_{b\to\infty} \Big[-e^{-x}\Big]_2^b \\
&= 2 + \lim_{b\to\infty} \Big(-e^{-b} - (-e^{-2})\Big) \\
&= 2 + (0 + e^{-2}) \\
&= 2 + e^{-2}.
\end{aligned}$$

14 FURTHER APPLICATIONS OF DIFFERENTIATION

1. Linear Approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

2. Approx. Max Error

$$\text{Approx. Max Error}(f) = |f'(x_0)\Delta x|$$

3. Approx. Max Percentage Error

$$\text{Approx. Max Percentage Error}(f) = |\frac{f'(x_0)\Delta x}{f(x_0)}| \times 100\%$$

4. Angle of Inclination

$$m = \tan\theta$$

5. Angle between two curves

a. Find the angle of inclination of each of the lines, and then
b. Subtract the smaller angle from the larger angle

15 MATRICES AND SYSTEMS OF LINEAR EQUATIONS

1. Scalar Multiplication

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

2. Order

$$m(rows) \times n(columns)$$

Square Matrix
$m = n$

3. General Form of a Matrix

<diagram>

4. Equality

A and B are only equal if and only if:

a. They are of the same order
b. Corresponding elements are equal $a_{ij} = b_{ij}$ for all $\{i,j\}$

5. Matrix Addition

$A + B$ only exists if A and B have the same order

6. Matrix Multiplication

a. $A + B$ only exists if $A(m \times n)$ and $B(n \times p)$
b. <diagram>
c. $AB \neq BA$ (non-commutative)

7. Identity Matrix

<diagram>

8. Inverse Matrix

a.

$$\text{if } A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \frac{1}{|A|} \neq 0$$

b. Singular: $|A| = 0$
c. Non-Singular/Regular: $|A| \neq 0$
d.

$$AA^{-1} = A^{-1}A = I$$

e. $AX = C$
To find X:
$A^{-1}AX = A^{-1}C$
$X = A^{-1}C$

- When $det(A) \neq 0$ then $AX = C$ has a unique solution $(X = A^{-1}C)$
- When $det(A) = 0$ then $AX = C$ has either
  - Infinitely many solutions, or
  - No solutions

9. Solving a System of 2 Linear Equations with 2 Unknowns

16 STATISTICS

1. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

2.

17 PERMUTATIONS AND COMBINATIONS

1. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

2.

18 AN INTRODUCTION TO PROBABILITY

1. **Fundamental Theorem of Calculus**

$$\int_a^b f(x)dx = F(b) - F(a)$$

2.

19 CONDITIONAL PROBABILITY

1. **Multiplication Rule**

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

2. **Independent Events**

A and B are independent if and only if $Pr(A \cap B) = Pr(A) \times Pr(B)$

20 DISCRETE RANDOM VARIABLES

1. **Expected Value of Random Variable**

$$E(X) = \mu = \sum x Pr(X = x)$$

2. **Variance of a Random Variable**

$$Var(X) = E((X - \mu)^2)$$

If X is a random variable with mean μ and standard deviation σ, then we often have

$$\boxed{\Pr\left(\mu - 2\sigma \le X \le \mu + 2\sigma\right) \approx 0.95}$$

1.

21 BERNOULLI TRIALS

Now suppose

X = the number of hits obtained from the 4 shots (in the previous example).

Then X has the probability distribution shown here:

x	$\Pr(X = x)$
0	$\Pr(\text{MMMM}) = \left(\frac{1}{5}\right)^4$
1	$\Pr(\text{HMMM or MHMM or MMHM or MMMH}) = 4 \times \frac{4}{5} \times \left(\frac{1}{5}\right)^3$
2	$\Pr(\text{HHMM or HMHM or HMMH or MHHM or MHMH or MMHH}) = 6 \times \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^2$
3	$\Pr(\text{HHHM or HHMH or HMHH or MHHH}) = 4 \times \left(\frac{4}{5}\right)^3 \times \frac{1}{5}$
4	$\Pr(\text{HHHH}) = \left(\frac{4}{5}\right)^4$

Notice that the probabilities in this probability distribution follow a pattern. In particular, for this probability distribution, we have

$$\Pr(X = x) = {}^4\mathrm{C}_x \left(\frac{4}{5}\right)^x \left(\frac{1}{5}\right)^{4-x}.$$

- The probability of obtaining a "success" is *the same in each trial* (and, similarly, the probability of obtaining a "failure" remains *the same for each trial)*.

Such trials are called **Bernoulli trials**.

22 REVISION OF BINOMIAL, HYPERGEOMETRIC AND GEOMETRIC PROBABILITY DISTRIBUTIONS

1. **Binomial Distribution**
   a. X =number of successes in sequence n
   b. No conditions for order
   c. p =probability of success (constant); given as percentage of large population
   d. $X \le n$

$$Pr(X = x) = {}^nC_x p^x (1 - p)^{n-x}$$

   e. Mean

$$E(X) = np$$

   f. Variance

$$Var(X) = np(1-p)$$

2. Hypergeometric Distribution

a. X =number of items with a particular property in sample size n
b. Choosing **without replacement**
c. No p
d. Overall group: $\begin{cases} D \text{ items have property} & \text{and} \quad N-D \text{ do not.} \\ & N \text{ items in total} \end{cases}$

Sample: $\begin{cases} x \text{ items have property} & \text{and} \quad n-x \text{ do not.} \\ & n \text{ items in sample} \end{cases}$

$$Pr(X=x) = \frac{{}^{D}C_x \times {}^{N-D}C_{n-x}}{{}^{N}C_n}$$

e. Mean

$$E(X) = n\frac{D}{N}$$

f. Variance

$$Var(X) = n\frac{D}{N}(1-\frac{D}{N})\frac{N-n}{N-1}$$

3. Geometric Distribution

a. X =number of failures **before/until** success
b. No max X
c. p =probability of success (constant); given as percentage of large population

$$Pr(X=x) = (1-p)^x p$$

$$Pr(X \geq x) = (1-p)^x$$

d. Mean

$$E(X) = \frac{1-p}{p}$$

e. Variance

$$Var(X) = \frac{1-p}{p^2}$$

23 CONTINUOUS RANDOM VARIABLES

For every *continuous* random variable X and for every real number b, we have

1.
$$\boxed{\mathbf{Pr(X = b) \; = \; 0.}}$$

a. $Pr(X \leq b) = Pr(X < b)$
b. $Pr(X \geq a) = Pr(x > a)$
c. $Pr(a \leq X \leq b) = Pr(a < X < b)$

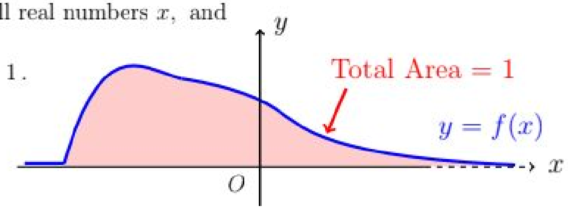
2. Probability Density Functions

A **probability density function** (often referred to as a **pdf**) is a function f with the following properties:

(a) The domain of f is the set of all real numbers; and

(b) $f(x) \geq 0$ for all real numbers x, and

(c) $\int_{-\infty}^{\infty} f(x)\,dx \; = \; 1$.



a.

$$Pr(a \leq X \leq b) = \int_a^b f(x)dx$$

$$\begin{aligned} \Pr(X \leq b) &= \int_{-\infty}^{b} f(x)\,dx \\ &= \lim_{l \to -\infty} \int_l^b f(x)\,dx \\ &= \ldots \end{aligned}$$

$$\begin{aligned} \Pr(X \geq a) &= \int_a^{\infty} f(x)\,dx \\ &= \lim_{n \to \infty} \int_a^n f(x)\,dx \\ &= \ldots . \end{aligned}$$

b.

3. Median

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2} \quad \text{or} \quad \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}.$$

4. Expected Value

$$E(X) \int_{-\infty}^{\infty} xf(x)dx$$

5. Properties of Expected Value

a. $$E(aX+b) = aE(X) + b.$$
   i. ^ Same for discrete random variables

b. $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

c. $$E\big(g(X)\big) = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

6. Variance

$$\text{Var}(X) = E\left((X-\mu)^2\right).$$

$$\text{Var}(X) = E\left(X^2\right) - \mu^2.$$

7. Properties of Variance and Standard Deviation

a. $$\text{Var}(aX+b) = a^2\,\text{Var}(X).$$
   i. ^ same for discrete random variables

24 THE NORMAL DISTRIBUTION

1. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

2.

25 STATISTICAL INFERENCE

1. Population proportion

$$p = \frac{\textit{number of xxx in population}}{\textit{total population}}$$

2. Sample proportion

$$p = \frac{\textit{number of xxx in sample}}{\textit{total sample}}$$

Note: The sample proportion from any particular sample is very unlikely to *equal* the population proportion.

3.

4. Random Variable for sample proportion

$$\hat{P} = \frac{X}{n}$$

5. Approximating X

If the population is 'large' (in comparison to the sample size), then we may approximate X well by assuming it has a binomial distribution. That is, the answers obtained by using the binomial formulae (by pretending that the probability of success remains constant) are similar to the answers obtained by using the hypergeometric formulae.

the sample size is less than one-tenth of the size of the population;

$$\text{that is, if} \quad n < \frac{\text{size of the population}}{10}.$$