Chapter 6

Composition of Functions

Reference: "Calculus", by James Stewart.

6.1 Composite Functions

Example 1. Consider the functions $f(x) = 2x$ and $g(x) = 2x + 1$.

Then the rule of the **composite** function $f \circ g$ is given by

$$\begin{aligned} f \circ g(x) = f(g(x)) &= f(2x+1) \\ &= 2(2x+1) \\ &= 4x+2. \end{aligned}$$

Note that when we form $f \circ g$ it is the function $\boldsymbol{g}$ which is applied first!

Similarly, the rule of the composite function $g \circ f$ is given by

$$\begin{aligned} g \circ f(x) = g(f(x)) &= g(2x) \\ &= 2(2x)+1 \\ &= 4x+1. \end{aligned}$$

Note that when we form $g \circ f$ it is the function $\boldsymbol{f}$ which is applied first!

Note that (as in the above example) we usually have

$$\boxed{f \circ g \neq g \circ f.}$$

Example 2. Consider the functions $f(x) = -x^2$ and $g(x) = \sqrt{x}$. Then

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(\sqrt{x}) \\ &= -(\sqrt{x})^2 \\ &= -x\,. \end{aligned}$$

Notes:

- For this calculation to make sense we must have $x \geq 0$
(since we cannot form $\sqrt{x}$ if x is negative).
- Note further that it is **not** obvious just by looking at the final rule $f \circ g(x) = -x$ that we need to make the restriction $x \geq 0$.
- Thus, to find the domain of a composite function it is important to consider the intermediate steps, rather than just look at the final rule!

Suppose we wish to find the domain of the function $y = f(g(x))$.
We need

x in the brackets of g , and $g(x)$ in the brackets of f .

This means we need to make sure that

$$\boxed{x \in \mathrm{dom}(g) \text{ and } g(x) \in \mathrm{dom}(f)}$$

Similarly, suppose we wish to find the domain of the function $y = g(f(x))$.
We need

x in the brackets of f , and $f(x)$ in the brackets of g .

This means we need to make sure that

$$\boxed{x \in \mathrm{dom}(f) \text{ and } f(x) \in \mathrm{dom}(g)}$$

Example 3. Consider the functions

$$f : \mathbf{R} \setminus \{0\} \longrightarrow \mathbf{R} \quad \text{where} \quad f(x) \ = \ \frac{1}{x}$$

and

$$g : \mathbf{R} \setminus \{-2\} \longrightarrow \mathbf{R} \quad \text{where} \quad g(x) \ = \ \frac{1}{x+2} \ .$$

Find the rule and the domain of the composite function $g \circ f$.

Solution: **Rule:** We have

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g\left(\frac{1}{x}\right) \\ &= \frac{1}{\frac{1}{x}+2} \\ &= \frac{1}{\frac{1+2x}{x}} \\ &= \frac{x}{1+2x} \ . \end{aligned}$$

Domain: We want to find $\mathrm{dom}(g \circ f)$.
That is, we want to find the x–values for which $g(f(x))$ is defined.
We need

$$x \in \mathrm{dom}(f) \quad \text{and} \quad f(x) \in \mathrm{dom}(g)$$

$$x \in \mathbf{R} \setminus \{0\} \quad \text{and} \quad \frac{1}{x} \in \mathbf{R} \setminus \{-2\}$$

$$x \neq 0 \quad \text{and} \quad \frac{1}{x} \neq -2$$

$$x \neq 0 \quad \text{and} \quad x \neq -\frac{1}{2} \, .$$

Therefore, $\mathrm{dom}(g \circ f) = \mathbf{R} \setminus \left\{0, -\frac{1}{2}\right\}$.

□

Example 4. Consider the functions

$$f \ : \ [2,8] \longrightarrow \mathbf{R} \qquad \text{where } f(x) = 2x+3$$

and

$$g \ : \ (0,2) \longrightarrow \mathbf{R} \qquad \text{where } g(x) = 3x-1\,.$$

Find the rule and the domain of the composite function $f \circ g$.

Solution: **Rule:**

We have

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(3x-1) \\ &= 2(3x-1)+3 \\ &= 6x-2+3 \\ &= 6x+1\,. \end{aligned}$$

Domain:

We want to find $\mathrm{dom}(f \circ g)$.

That is, we want to find the x–values for which $f(g(x))$ is defined.

We need

$$\begin{array}{rcl} x \in \mathrm{dom}(g) & \textbf{and} & g(x) \in \mathrm{dom}(f) \\ x \in (0,2) & \text{and} & 3x-1 \in [2,8] \\ 0 < x < 2 & \text{and} & 2 \le 3x-1 \le 8 \\ 0 < x < 2 & \text{and} & 3 \le 3x \le 9 \\ 0 < x < 2 & \text{and} & 1 \le x \le 3 \end{array}$$

$$\begin{aligned} \text{Therefore, } \mathrm{dom}(f \circ g) &= (0,2) \cap [1,3] \\ &= [\,1,\,2\,) \end{aligned}$$

$\square$

Exercises

1. If $f(x) = x^2$ and $g(x) = 2x + 4$ then

   (a) find the rule for $f(g(x))$.

   (b) find the rule for $g(f(x))$.

2. Consider the functions

$$f : [-4, 6] \longrightarrow \mathbf{R} \text{ where } f(x) = 10x - 2$$

   and

$$g : [8, 73] \longrightarrow \mathbf{R} \text{ where } g(x) = 5x - 39.$$

   (a) Find dom($f \circ g$).

   (b) Find dom($g \circ f$).

   (c) Find the rule for $f(g(x))$.

   (d) Find the rule for $g(f(x))$.

3. If $f(x) = \begin{cases} x^2 - 2 & \text{if } x > 1 \\ 1 - x & \text{if } x \leq 1 \end{cases}$ and $g(x) = 2x$
   then find the rule for $f(g(x))$.

6.2 Inverse Functions

Example 5. Consider the functions $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1}{x} - 2$.

(a) Find the rule for $f(g(x))$.

(b) Find the rule for $g(f(x))$.

Solution: (a)

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x} - 2\right) \\ &= \frac{1}{\left(\frac{1}{x} - 2\right) + 2} \\ &= \frac{1}{1/x} \\ &= x \end{aligned}$$

(b)

$$
\begin{aligned}
g(f(x)) &= g\left(\frac{1}{x+2}\right) \\
&= \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\
&= x + 2 - 2 \\
&= x
\end{aligned}
$$

$\square$

Note that in Example 5 we have

$$f(g(x)) = x$$

and

$$g(f(x)) = x.$$

That is, the functions ' **undo**' (or cancel) each other.

When f and g have the above property, we say that g is the **inverse** of f .

When g is the **inverse** of f , we write g as f^{-1}.

Using this notation, we can rewrite the above cancellation equations as

$$f(f^{-1}(x)) = x$$

and

$$f^{-1}(f(x)) = x\,.$$

That is, f and f^{-1} ' **undo**' (or cancel) each other.

Note: $f^{-1}(x)$ does **not** mean $\dfrac{1}{f(x)}$

Suppose that we know the equation of $y = f(x)$, and suppose that we want to find the equation of $y = f^{-1}(x)$. We can use **either** of the following methods:

Method 1: Swap x and y in the equation $y = f(x)$, and then rearrange to get y by itself.

Method 2: Rearrange the equation $y = f(x)$ to get x by itself, and then swap x and y.

Example 6. Consider the function $f(x) = \dfrac{1}{x+2}$ from Example 5. Find the equation of f^{-1}.

Solution:

$$
\begin{aligned}
y &= \frac{1}{x+2} \\
&\text{We first rearrange the equation to make } x \text{ the subject:} \\
x + 2 &= \frac{1}{y} \\
\therefore x &= \frac{1}{y} - 2 \\
&\text{Now swap } x \text{ and } y\text{:} \\
y &= \frac{1}{x} - 2 \\
\therefore f^{-1}(x) &= \frac{1}{x} - 2
\end{aligned}
$$

□

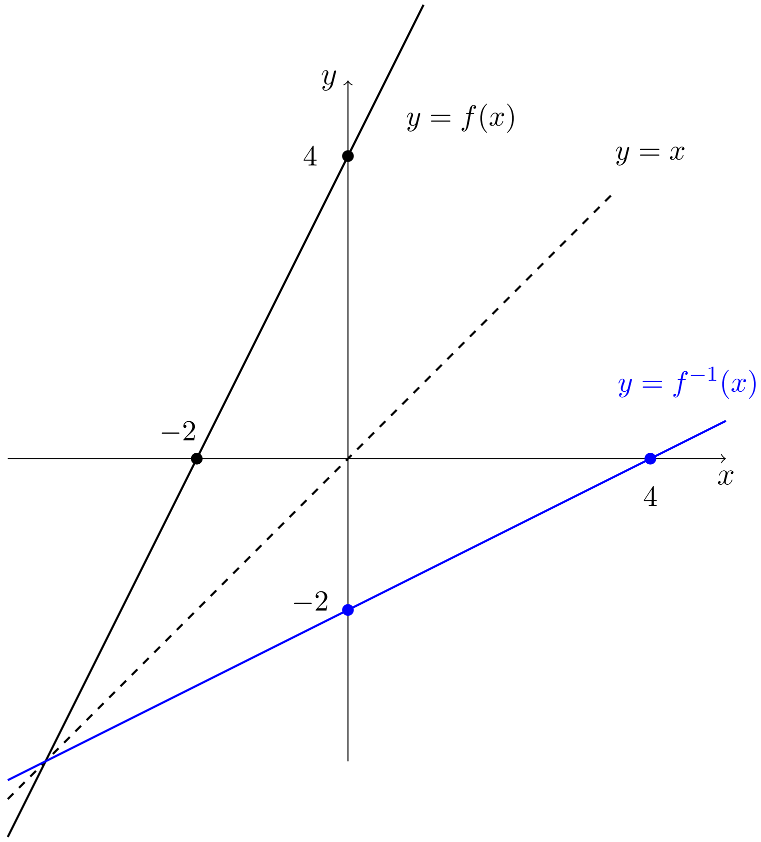
Example 7. If $f(x) = 2x + 4$ find $f^{-1}(x)$.

Solution:

$$
\begin{aligned}
&\text{Let } y = 2x + 4 \\
&\qquad \text{We first rearrange this equation to make } x \text{ the subject:} \\
y - 4 &= 2x \\
\therefore x &= \tfrac{1}{2}(y - 4) \\
&\text{Now swap } x \text{ and } y\text{:} \\
y &= \tfrac{1}{2}(x - 4) \\
\therefore f^{-1}(x) &= \tfrac{1}{2}(x - 4)
\end{aligned}
$$

□

Sketching the graphs for $y = f(x)$ and $y = f^{-1}(x)$ gives:

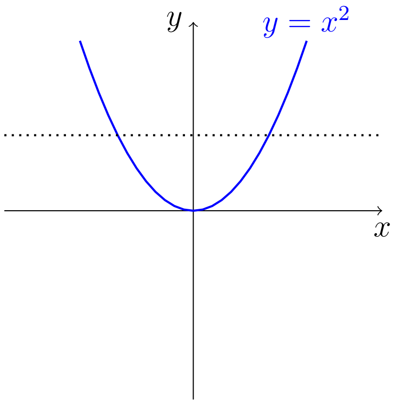


In the graph above we can see that the line $y = x$ acts like a mirror. This is a general property. That is,

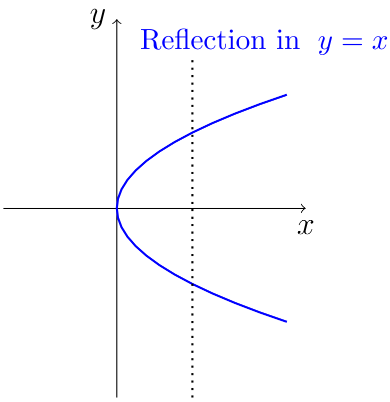
$$\boxed{\text{the graph of } f^{-1} \text{ is obtained by reflecting the graph of } f \text{ in the line } y = x}$$

If a function is **not** one–one, then its reflection in the line $y = x$ will have the property that some **vertical** lines will cut the graph **more than once**. That is,

if a function is **not** one–one then its inverse is **not a function**.



Not one-one



Not a function

In the Maths 1 course, we **only** want to study functions. Thus we will **only** take inverses of **one–one** functions.

A function has an inverse **function** if and only if it is one–one

Recall that when we find the inverse of a (one–one) function, we swap the x and y–values. This swapping of the x and y–values results in the following properties:

$$\boxed{\operatorname{dom}(f^{-1}) = \operatorname{ran}(f)}$$

and

$$\boxed{\operatorname{ran}(f^{-1}) = \operatorname{dom}(f)}$$

Example 8. If $f : (-\infty, 2] \longrightarrow \mathbf{R}$ where $f(x) = (x-2)^2$, find $f^{-1}(x)$.

Solution:

$$\text{Let } y = (x-2)^2$$

Swap x and y and then make y the subject of the equation:

$$\begin{aligned} x &= (y-2)^2 \\ y - 2 &= \pm\sqrt{x} \\ \therefore y &= 2 \pm \sqrt{x} \end{aligned}$$

Now we need to determine whether $f^{-1}(x) = 2+\sqrt{x}$ or $f^{-1}(x) = 2-\sqrt{x}$.

$$\begin{aligned} \text{If } y &= f^{-1}(x) \\ \text{then } y &\in \operatorname{ran}(f^{-1}) \\ \therefore y &\in \operatorname{dom}(f) \\ \therefore y &\in (-\infty, 2] \end{aligned}$$

This shows that y must be less than or equal to 2. Therefore

$$f^{-1}(x) = 2 - \sqrt{x}\,.$$

□

Exercises

1. Find the inverse of each of the following functions:

   (a) $f(x) = 2x + 1$ (b) $f(x) = 7x + 3$ (c) $f(x) = 3 - x$

   (d) $f(x) = \dfrac{1}{x+1}$ (e) $f(x) = \dfrac{1}{x}$

2. (a) Find the smallest number b such that the function

   $$f(x) = x^2 - 4 \text{ with } \operatorname{dom}(f) = [b, \infty)$$

   has an inverse function. Find the rule for the inverse function.

   (b) Find the largest number b such that the function

   $$f(x) = (x+2)^2 \text{ with } \operatorname{dom}(f) = (-\infty, b]$$

   has an inverse function. Find the rule for the inverse function.

3. Consider the function $f(x) = 1 + \sqrt{x+1}$.

   (a) Find dom(f). (b) Find ran(f). (c) Find dom(f^{-1}).

   (d) Find ran(f^{-1}). (e) Find the rule for f^{-1} .

   (f) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

6.3 Exponentials and Logarithms

Suppose that $a > 0$ and $a \neq 1$.

Recall that

$$y = \log_a x \quad \text{if and only if} \quad x = a^y .$$

For example, this means that

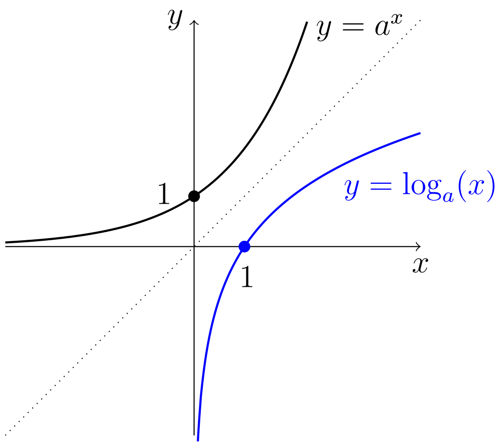
- $\log_a a = 1$ (since $a^1 = a$), and
- $\log_a 1 = 0$ (since $a^0 = 1$).

Here we examine the relationship between logarithmic and exponential functions in more detail.

Since the exponential function $f(x) = a^x$ is one–one, it has an inverse function. In fact, the inverse of $f(x) = a^x$ is the logarithmic function to the base a .

That is,

$$\boxed{\text{if } f(x) = a^x \text{ then } f^{-1}(x) = \log_a x}$$



In particular, this means that

$$\log_a(a^x) = x \quad \text{for all} \quad x \in \mathbf{R} \quad (\text{i.e. } f^{-1}(f(x)) = x)$$

and

$$a^{\log_a x} = x \quad \text{for all} \quad x > 0 \quad (\text{i.e. } f(f^{-1}(x)) = x)\,.$$

That is,

logarithms and exponentials cancel each other,

(as long as they have the **same base**).

Example 9. Make x the subject of the following formula:

$$y = B \times 10^{\frac{ax}{b}}$$

(i.e. rearrange the formula to get x by itself).

Solution:

$$\begin{aligned} y &= B \times 10^{\frac{ax}{b}} \\ \therefore \frac{y}{B} &= 10^{\frac{ax}{b}} \\ \therefore \log_{10}\left(\frac{y}{B}\right) &= \frac{ax}{b} \\ \therefore ax &= b \log_{10}\left(\frac{y}{B}\right) \\ \therefore x &= \frac{b}{a} \log_{10}\left(\frac{y}{B}\right) \end{aligned}$$

□

Exercises

1. Simplify the following expressions:

   (a) $\log_3\left(3^{\sin x}\right)$ (b) $\log_{12}\left(12^{x-1}\right)$ (c) $4^{\log_4 \sqrt{x}}$

   (d) $7^{2\log_7 x}$ (e) $3^{\frac{1}{2}\log_3(x+1)}$

2. Evaluate the following expressions:

   (a) $\log_9 3$ (b) $\log_9\left(\frac{1}{27}\right)$ (c) $\log_4 0.25$

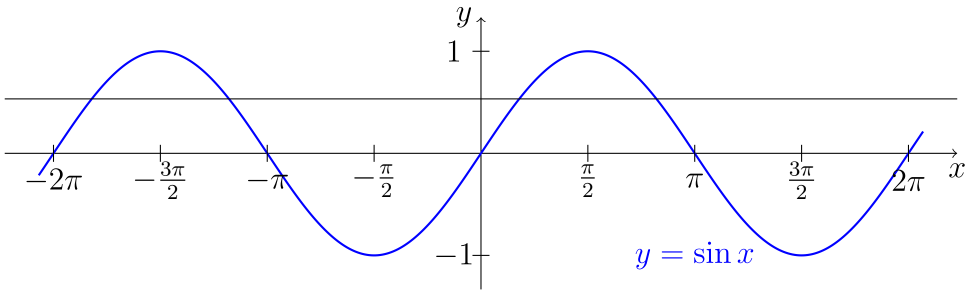
3. Make t the subject of the equation $y = k + Ca^{bt}$.

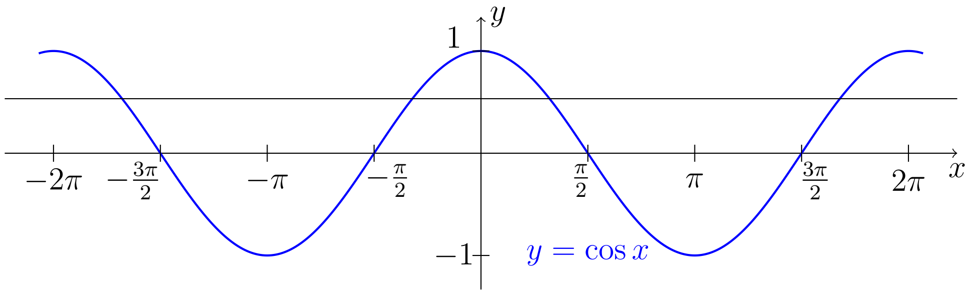
6.4 Inverse Trigonometric Functions

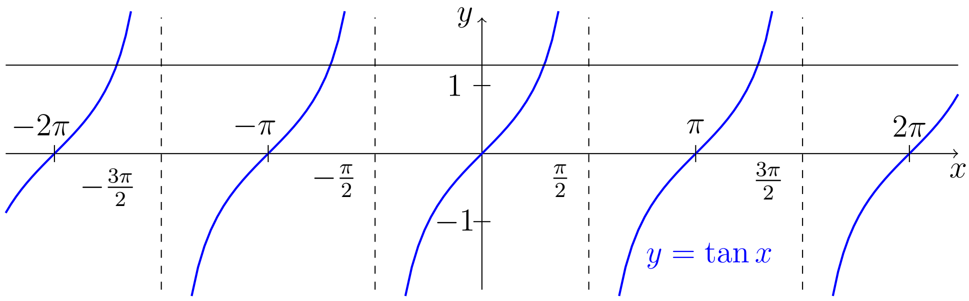
Reference: "Calculus", by James Stewart.

Recall that a function f has an **inverse function** if and only if f is one–one.

Clearly the trigonometric functions $\sin$, $\cos$ and $\tan$ are **not** one–one.







However if we restrict the domains of these functions to **make them one–one** then inverse functions will exist.

Recall also that an inverse function f^{-1} satisfies

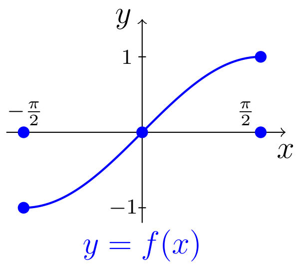
$$\text{dom}(f^{-1}) = \text{ran}(f) \quad \text{and} \quad \text{ran}(f^{-1}) = \text{dom}(f).$$

Sine and its Inverse

To turn sine into a **one–one** function, we restrict its domain to $\left[-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right]$. That is, we consider the function

$$f:\left[-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right]\to\mathbf{R}$$

where $f(x)=\sin x$.



We define $\sin^{-1}$ (or $\arcsin$) to be the inverse of this function. That is, we write

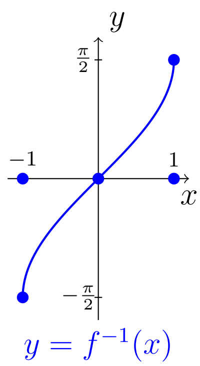
$$f^{-1}(x)=\sin^{-1}(x).$$

Note:

When we write $y=\sin^{-1}x$, we have $y\in\operatorname{ran}(f^{-1})$. Since

$$\operatorname{ran}(f^{-1})\;=\;\operatorname{dom}(f)\;=\;\left[-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right]$$

then we conclude that $y\in\left[-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right]$.



Thus we have

$$y=\sin^{-1}x$$

if and only if

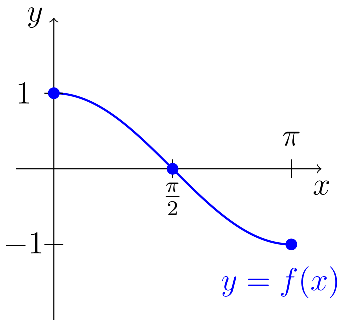
$$\sin y=x\quad\text{and}\quad y\in\left[-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right].$$

Cosine and its Inverse

To turn cosine into a **one–one** function, we restrict its domain to $[0, \pi]$. That is, we consider the function

$$f : [0, \pi] \to \mathbf{R}$$

where $f(x) = \cos x$.



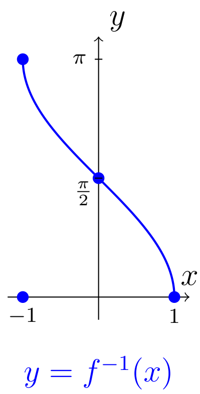
We define $\cos^{-1}$ (or arccos) to be the inverse of this function. That is, we write

$$f^{-1}(x) = \cos^{-1}(x).$$

Note:
When we write $y = \cos^{-1} x$, we have $y \in \operatorname{ran}(f^{-1})$. Since

$$\operatorname{ran}(f^{-1}) \;=\; \operatorname{dom}(f) \;=\; [0, \pi]$$

then we conclude that $y \in [0, \pi]$.



Thus we have

$$y = \cos^{-1} x$$

if and only if

$$\cos y = x \quad \text{and} \quad y \in [0, \pi].$$

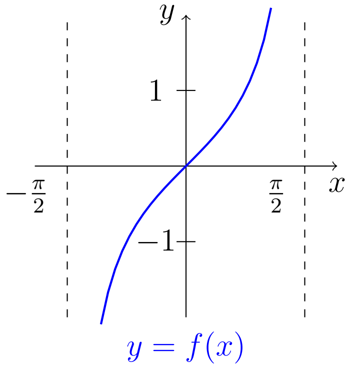
Tan and its Inverse

To turn tan into a **one–one** function, we restrict its domain to $\left(-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right)$.

That is, we consider the function

$$f : \left(-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right) \to \mathbf{R}$$

where $f(x) = \tan x$.



We define $\tan^{-1}$ (or arctan) to be the inverse of this function. That is, we write

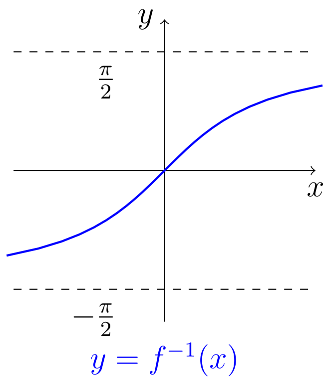
$$f^{-1}(x) = \tan^{-1}(x).$$

Note:

When we write $y = \tan^{-1} x$, we have $y \in \operatorname{ran}(f^{-1})$. Since

$$\operatorname{ran}(f^{-1}) \;=\; \operatorname{dom}(f) \;=\; \left(-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right)$$

then we conclude that $y \in \left(-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right)$.



Thus we have

$$y = \tan^{-1} x$$

if and only if

$$\tan y = x \quad \text{and} \quad y \in \left(-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right).$$

Warning:

Recall that usually

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

In particular,

$$\sin^{-1} x \neq \frac{1}{\sin x}.$$

That is,

$$\sin^{-1} x \neq (\sin x)^{-1}.$$

Similarly, note that

$$\cos^{-1} x \neq (\cos x)^{-1} \quad \text{and} \quad \tan^{-1} x \neq (\tan x)^{-1}.$$

Exercises

Evaluate the following:

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (c) $\cos^{-1}\left(-\frac{1}{2}\right)$

(d) $\cos^{-1}(1)$ (e) $\tan^{-1}(-\sqrt{3})$

6.5 Answers to Chapter 6 Exercises

6.1:

1. (a) $f(g(x)) = (2x+4)^2$ (b) $g(f(x)) = 2x^2 + 4$

2. (a) $[8, 9]$ (b) $[1, 6]$
   (c) $f(g(x)) = 50x - 392$ (d) $g(f(x)) = 50x - 49$

3.
$$f(g(x)) = \begin{cases} 4x^2 - 2 & \text{if } x > \frac{1}{2} \\ 1 - 2x & \text{if } x \leq \frac{1}{2} \end{cases}$$

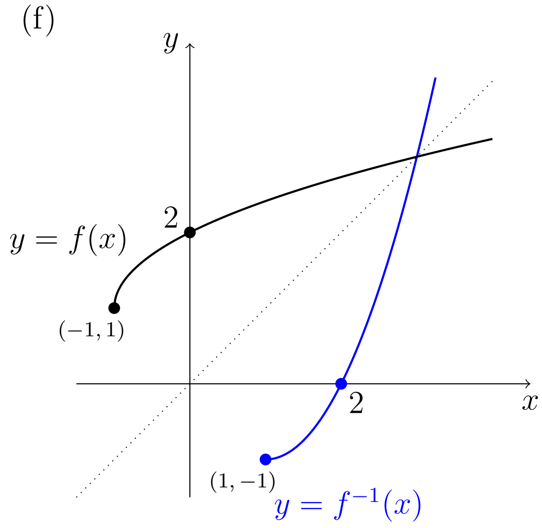
6.2:

1. (a) $f^{-1}(x) = \frac{1}{2}(x-1)$ (b) $f^{-1}(x) = \frac{1}{7}(x-3)$
   (c) $f^{-1}(x) = 3 - x$ (d) $f^{-1}(x) = \frac{1}{x} - 1$
   (e) $f^{-1}(x) = \frac{1}{x}$

2. (a) $b = 0, \quad f^{-1}(x) = \sqrt{x+4}$
   (b) $b = -2, \quad f^{-1}(x) = -\sqrt{x} - 2$

3. (a) $[-1, \infty)$ (b) $[1, \infty)$
   (c) $[1, \infty)$ (d) $[-1, \infty)$
   (e) $f^{-1}(x) = x^2 - 2x$

   (f)



6.3:

1. (a) $\sin x$ (b) $x-1$ (c) $\sqrt{x}$
   (d) x^2 (e) $\sqrt{x+1}$

2. (a) 0.5 (b) -1.5 (c) -1

3. $\frac{1}{b}\log_a\left(\frac{y-k}{C}\right)$

6.4:

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$

(d) 0 (e) $-\frac{\pi}{3}$