Mathematics 1

Exercise Sheet 18:
Triangular Matrix Method

Example: Use the triangular matrix method to solve the following system of linear equations:

$$\begin{aligned} x + 3y + 2z &= 5 \\ 2x \quad\quad - 3z &= 5 \\ y + 5z &= -3. \end{aligned}$$

Solution:

We can represent this system of equations in augmented matrix form, as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 0 & -3 & 5 \\ 0 & 1 & 5 & -3 \end{bmatrix}$$

We are going to use "row operations" to get a triangle of zeros below the main diagonal.

$$R_2 - 2R_1 \quad \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -6 & -7 & -5 \\ 0 & 1 & 5 & -3 \end{bmatrix}$$

$$6R_3 + R_2 \quad \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -6 & -7 & -5 \\ 0 & 0 & 23 & -23 \end{bmatrix}$$

Then R_3 implies that

$$\begin{aligned} 23z &= -23. \\ \text{That is, } z &= -1. \end{aligned}$$

Similarly, R_2 implies that

$$\begin{aligned} -6y - 7z &= -5. \\ \text{That is, } -6y &= -5 + 7z \\ &= -5 + 7 \times -1 \\ &= -12. \\ \text{That is, } y &= 2. \end{aligned}$$

Finally, R_1 implies that

$$\begin{aligned} x + 3y + 2z &= 5. \\ \text{That is, } x &= 5 - 3y - 2z \\ &= 5 - 3 \times 2 - 2 \times -1 \\ &= 1. \end{aligned}$$

Thus $(x, y, z) = (1, 2, -1)$.

□

Exercises:

1. Use the triangular matrix method to solve the following systems of linear equations.

(a)
$$\begin{aligned}
3x &- 2y + 5z = 23 \\
x &+ y + z = 4 \\
2x &+ 3y - 2z = -5
\end{aligned}$$

(b)
$$\begin{aligned}
5x &+ y - z = -22 \\
3x &- y + 2z = -20 \\
2x &+ 2y + z = -10
\end{aligned}$$

(c)
$$\begin{aligned}
4x &- 3y + 15z = 4 \\
2x &+ 6y - 10z = 1 \\
6x &+ 3y - 5z = 3
\end{aligned}$$

(d)
$$\begin{aligned}
7x &- 5y + 3z = 0 \\
3x &- 2y + 8z = -100 \\
2x &+ 3y + 4z = -25
\end{aligned}$$

(e)
$$\begin{aligned}
2x &+ 3y - 5z = -14 \\
3x &- 2y + 3z = 17 \\
4x &+ 3y - 2z = -1
\end{aligned}$$

(f)
$$\begin{aligned}
x &+ 2y - 5z = -6 \\
-2x &+ 3y - z = -13 \\
5x &+ 7y - 13z = -12
\end{aligned}$$

(g)
$$\begin{aligned}
3x &+ 6y - 6z = 9 \\
2x &- 5y + 4z = 6 \\
-x &+ 16y - 14z = -3
\end{aligned}$$

(h)
$$\begin{aligned}
x &+ y - z = 7 \\
4x &- y + 5z = 4 \\
6x &+ y + 3z = 20
\end{aligned}$$

(i)
$$\begin{aligned}
x &+ y - z = 0 \\
4x &- y + 5z = 0 \\
6x &+ y + 3z = 0
\end{aligned}$$

2. Use the triangular matrix method to solve the following systems of linear equations.

(a)
$$\begin{aligned}
3x &+ 2y + 4z + u = 10 \\
2x &+ 3y + 7z + u = 13 \\
x &+ 6y + 3z + u = 11 \\
2x &+ y + z + u = 5
\end{aligned}$$

(b)
$$\begin{aligned}
x &+ y - z + u = 7 \\
4x &- y + 5z + 9u = 4 \\
6x &+ y + 3z + 11u = 20 \\
2x &+ 2y - 2z + 2u = 14
\end{aligned}$$

Answers

1. (a) $(2, -1, 3)$

   (b) $(-5, 1, -2)$

   (c) $\left(\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{5} \right)$

   (d) $(10, 5, -15)$

   (e) $(2, -1, 3)$

   (f) $(3, -2, 1)$

   (g) $\left(3 + \dfrac{2}{9}k, \dfrac{8}{9}k, k \right)$ where $k \in \mathbf{R}$

   (h) No solution (inconsistent system)

   (i) $\left(-\dfrac{4}{5}k, \dfrac{9}{5}k, k \right)$ where $k \in \mathbf{R}$

2. (a) $(x, y, z, u) = (1, 1, 1, 1)$

   (b) No solution (inconsistent system)