Chapter 11

Antidifferentiation and Integration

Reference: "Calculus", by James Stewart.

11.1 Antidifferentiation

Antidifferentiation is the process of finding a function whose derivative is known.

Example 1. Find a function f which satisfies $f'(x) = 3x^2$.
That is, find an **antiderivative** of $3x^2$.

Solution: There are many answers. For example, we could have

$$f(x) = x^3 \quad \text{or} \quad f(x) = x^3 + 1 \quad \text{or} \quad f(x) = x^3 - 17 \quad \text{etc.}$$

$\square$

Note: All these answers are of the form

$$f(x) = x^3 + C \quad (\text{where } C \text{ is any real number}).$$

This is known as the **most general** antiderivative of $3x^2$.

Since $\dfrac{d}{dx}\left(\dfrac{1}{r+1}x^{r+1}+C\right)=x^r$, we conclude that

$$\boxed{\text{if } f'(x)=x^r \text{ , then } f(x)=\frac{1}{r+1}x^{r+1}+C}$$

as long as $r \neq -1$.

Example 2. Find the most general antiderivative of $\dfrac{x^3+2x^2+1}{x^2}$.

Solution:

$$\begin{aligned}
\text{If}\quad f'(x) &= \frac{x^3}{x^2} + \frac{2x^2}{x^2} + \frac{1}{x^2} \\
&= x + 2 + x^{-2} \\
&= x + 2x^0 + x^{-2}
\end{aligned}$$

$$\begin{aligned}
\text{then}\quad f(x) &= \frac{1}{2}\,x^2 + 2\times\frac{1}{1}\,x^1 + \frac{1}{-1}\,x^{-1} + C \\
&= \frac{x^2}{2} + 2x - \frac{1}{x} + C\,.
\end{aligned}$$

□

Example 3. Find the function f that satisfies $f'(x) = 4x+3$ and $f(2) = 15$.

Solution: We first antidifferentiate to obtain

$$f(x) = 2x^2+3x+C.$$

We used the fact that $f(2)=15$ to find C :

$$\begin{aligned}
f(2) &= 15 \\
\therefore 2\cdot 2^2+3\cdot 2+C &= 15 \\
\therefore 14+C &= 15 \\
\therefore C &= 1
\end{aligned}$$

Hence,

$$f(x) = 2x^2+3x+1.$$

□

Since $\dfrac{d}{dx}\left(-\dfrac{1}{k}\cos kx + c\right) = \sin kx,$ we conclude that

$$\boxed{\text{if } f'(x) = \sin kx \text{ , then } f(x) = -\frac{1}{k}\cos kx + C.}$$

Similarly, we have the following results:

If $f'(x) = \cos kx$, then $f(x) = \dfrac{1}{k}\sin kx + C$.

If $f'(x) = \sec^2 kx$, then $f(x) = \dfrac{1}{k}\tan kx + C$.

If $f'(x) = e^{kx}$, then $f(x) = \dfrac{1}{k}e^{kx} + C$.

If $f'(x) = \dfrac{1}{x}$, then $f(x) = \ln|x| + C$.

The next three formulae only apply when $a > 0$.

If $f'(x) = \dfrac{1}{\sqrt{a^2 - x^2}}$, then $f(x) = \sin^{-1}\left(\dfrac{x}{a}\right) + C$.

If $f'(x) = \dfrac{-1}{\sqrt{a^2 - x^2}}$, then $f(x) = \cos^{-1}\left(\dfrac{x}{a}\right) + C$.

If $f'(x) = \dfrac{a}{a^2 + x^2}$, then $f(x) = \tan^{-1}\left(\dfrac{x}{a}\right) + C$.

Note: The fourth formula can be checked by differentiating $\ln|x| + C$:

If $x > 0$ then

$$\begin{aligned} \frac{d}{dx}(\ln|x| + C) &= \frac{d}{dx}(\ln x + C) \\ &= \frac{1}{x}\,. \end{aligned}$$

If $x < 0$ then

$$\begin{aligned} \frac{d}{dx}(\ln|x| + C) &= \frac{d}{dx}(\ln(-x) + C) \\ &= \frac{1}{-x} \times -1 \\ &= \frac{1}{x}\,. \end{aligned}$$

Thus in both cases we have $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$, whence we conclude that the most general antiderivative of $\frac{1}{x}$ is $\ln|x| + C$, as given above.

Example 4. (a) Find the most general antiderivative of $x^5 + 2\sin(3x)$.

Solution:

$$\begin{aligned} \text{If } f'(x) &= x^5 + 2\sin(3x) \\ \text{then } f(x) &= \frac{1}{6}x^6 + 2\left(-\frac{1}{3}\cos(3x)\right) + C \\ &= \frac{1}{6}x^6 - \frac{2}{3}\cos(3x) + C\,. \end{aligned}$$

□

(b) Find the most general antiderivative of $e^{\frac{x}{2}}$.

Solution:

$$\text{If } f'(x) = e^{\frac{x}{2}}$$
$$\text{then } f(x) = 2e^{\frac{x}{2}} + C\,.$$

□

We have already seen that

$$\text{if } f'(x) = x^r \text{ , then } f(x) = \frac{x^{r+1}}{r+1} + C$$

as long as $r \neq -1$.

Similarly, it can be shown (and should be learnt) that

$$\boxed{\text{if } f'(x) = (ax+b)^r \text{ , then } f(x) = \frac{(ax+b)^{r+1}}{a(r+1)} + C}$$

as long as $r \neq -1$.

Example 5. (a) Find the most general antiderivative of $(3x+4)^7$.

Solution: If

$$f'(x) = (3x+4)^7$$

then

$$\begin{aligned} f(x) &= \frac{(3x+4)^8}{3 \times 8} + C \\ &= \frac{(3x+4)^8}{24} + C \end{aligned}$$

$\square$

(b) Find the most general antiderivative of $\dfrac{1}{(5x-1)^{12}}$.

Solution: If

$$\begin{aligned} f'(x) &= \frac{1}{(5x-1)^{12}} \\ &= (5x-1)^{-12} \end{aligned}$$

then

$$\begin{aligned} f(x) &= \frac{(5x-1)^{-11}}{5 \times -11} + C \\ &= \frac{-1}{55(5x-1)^{11}} + C\,. \end{aligned}$$

$\square$

We have already seen that

$$\text{if } f'(x) = \frac{1}{x} \quad \text{then } f(x) = \ln|x| + C.$$

Similarly, it can be shown (and should be learnt) that

$$\boxed{\text{if } f'(x) = \frac{1}{ax+b} \quad \text{then } f(x) = \frac{1}{a}\ln|ax+b| \; + \; C}$$

Example 6. (a) Find the most general antiderivative of $\dfrac{1}{4x+3}$.

Solution: If

$$f'(x) = \frac{1}{4x+3}$$

then

$$f(x) = \frac{1}{4}\ln|4x+3| \; + \; C.$$

$\square$

(b) Find the most general antiderivative of $\dfrac{6}{5x-2}$.

Solution: If

$$\begin{aligned} f'(x) &= \frac{6}{5x-2} \\ &= 6 \times \frac{1}{5x-2} \end{aligned}$$

then

$$\begin{aligned} f(x) &= 6 \times \frac{1}{5}\ln|5x-2| + C \\ &= \frac{6}{5}\ln|5x-2| + C \end{aligned}$$

$\square$

Exercises

Find the most general antiderivative of each of the following expressions:

(a) $x^3 + 8$

(b) $x^4 + 3x^{\frac{1}{2}} - 2x^2 + x^{-2}$

(c) $\cos\left(\frac{3}{2}x\right)$

(d) e^{4x}

(e) $\frac{2}{x}$

(f) $\frac{1}{2x}$

(g) $x^2 + 2x$

(h) $\cos 3x$

(i) $\frac{1}{x^2}$

(j) $\sin\left(\frac{1}{2}x\right)$

(k) $\sqrt{x}$

(l) $\sec^2\left(\frac{1}{2}x\right)$

(m) $(2x+7)^9$

(n) $\sqrt{6x-5}$

(o) $(1-3x)^{10}$

(p) $\frac{1}{(4x+5)^{10}}$

(q) $\frac{1}{9x+2}$

(r) $\frac{2}{2x+3}$

(s) $\frac{5}{x^2+25}$

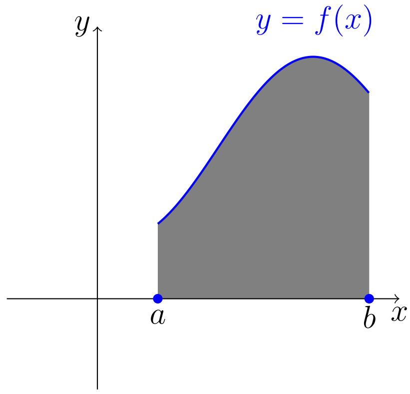
(t) $\frac{2}{x^2+25}$

11.2 Integral Calculus

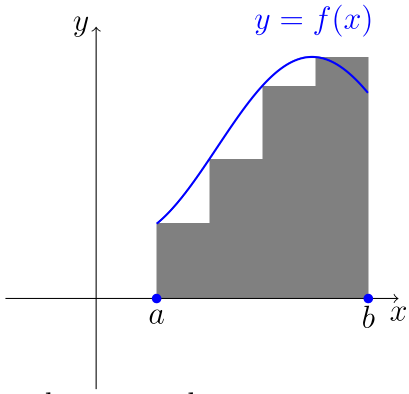
In **differential** calculus we used limits to find the slopes of tangents.

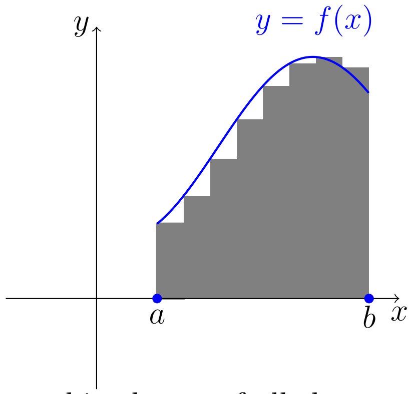
In **integral** calculus, limits are used to find areas under curves. In particular, areas can be approximated by rectangles, and the limit taken as the width of the rectangles decreases.

Suppose we are interested in the area bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.



We can approximate this area by rectangles as shown below:





As the rectangles get narrower, the combined area of all the rectangles approaches the area under the curve.

Suppose we approximate the area under the curve $y = f(x)$ in this way, for x between a and b. The limit of the combined areas of the rectangles as their width approaches 0 is denoted by the expression:

$$\boxed{\int_a^b f(x)\,dx}$$

This is called the **integral** of f from a to b.

The process of

1. adding the areas of the rectangles, and then
2. taking the limit as the width of the rectangles approaches 0

is known as **integration**.

The Fundamental Theorem of Calculus

Integration can be a very time-consuming process. In this section we shall see how the integration process can be avoided! In fact we shall see that we can find areas by working with the antiderivative of f .

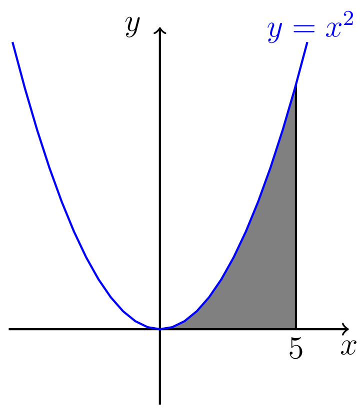
Fundamental Theorem of Calculus: Suppose f is continuous on $[a, b]$, and let F be any antiderivative of f . Then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Note: The antiderivative F is usually written inside square brackets.

Example 7. Find the area under $f(x) = x^2$ from $x = 0$ to $x = 5$.

Solution:

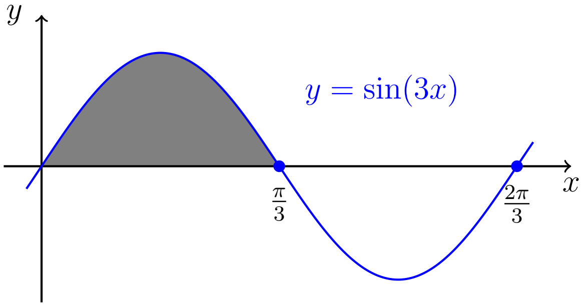


$$\begin{aligned} \text{Area} &= \int_0^5 x^2\,dx \\ &= \left[\frac{x^3}{3}\right]_0^5 \\ &= \frac{5^3}{3} - \frac{0^3}{3} \\ &= \frac{125}{3} \end{aligned}$$

$\square$

Example 8. Find the area under $f(x) = \sin(3x)$ from $x = 0$ to $x = \dfrac{\pi}{3}$.

Solution:

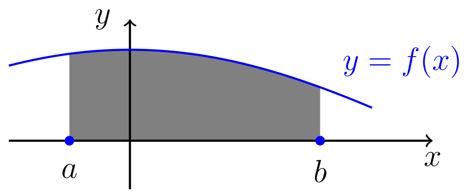


$$
\begin{aligned}
\text{Area} &= \int_0^{\frac{\pi}{3}} \sin(3x)\ dx \\
&= \left[-\frac{1}{3}\cos(3x) \right]_0^{\frac{\pi}{3}} \\
&= \left(-\frac{1}{3}\cos\pi\right) - \left(-\frac{1}{3}\cos 0\right) \\
&= \left(-\frac{1}{3}\times -1\right) - \left(-\frac{1}{3}\times 1\right) \\
&= \frac{2}{3}
\end{aligned}
$$

□

So far we have considered areas of regions which are **above** the x–axis.

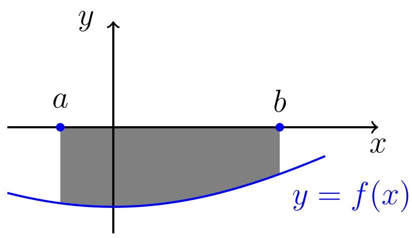
$$\text{Area} = \int_a^b f(x)\,dx$$



When a region is **below** the x–axis, the integral $\displaystyle\int_a^b f(x)\,dx$ will give a **negative** answer.

In this case, we can find the area by taking the **absolute value** of the integral.

$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$



Properties of Definite Integrals

The following properties help us to evaluate integrals.

(a) $\displaystyle\int_a^b 0\,dx \;=\; 0$

(b) $\displaystyle\int_a^b c\,dx \;=\; c(b-a)$

(c) $\displaystyle\int_a^b \Big[f(x)+g(x)\Big]\,dx \;=\; \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(d) $\displaystyle\int_a^b \Big[f(x)-g(x)\Big]\,dx \;=\; \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

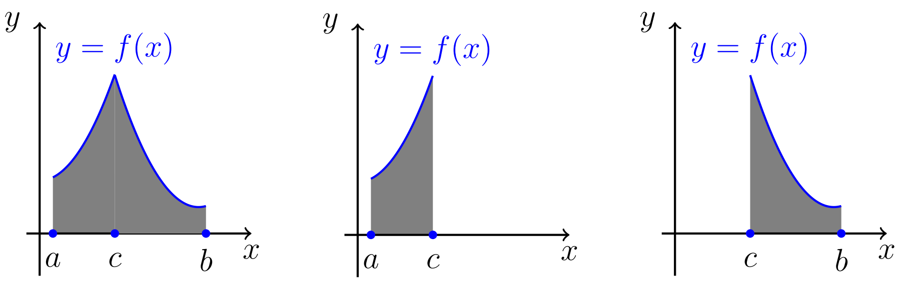
(e) $\displaystyle\int_a^b cf(x)\,dx \;=\; c\int_a^b f(x)\,dx$

(f) $\displaystyle\int_a^a f(x)\,dx \;=\; 0$

(g) $\displaystyle\int_a^b f(x)\,dx \;=\; -\int_b^a f(x)\,dx$

(h) $\displaystyle\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

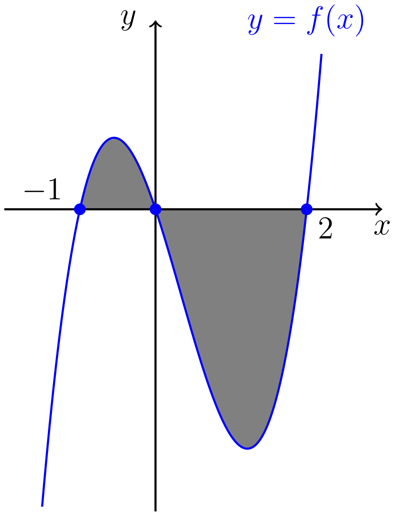
Property (h) shows us that an integral can be split into two separate integrals:



$$\int_a^b f(x)\,dx \qquad = \qquad \int_a^c f(x)\,dx \qquad + \qquad \int_c^b f(x)\,dx$$

Example 9. Find the area bounded by $f(x) = x(x-2)(x+1)$ and the x–axis.

Solution:



Note:

$$\begin{aligned} x(x-2)(x+1) &= x(x^2 - x - 2) \\ &= x(x-2)(x+1) \\ &= x^3 - x^2 - 2x \end{aligned}$$

$$\begin{aligned}
\text{Area} &= \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx \quad + \quad \left| \int_{0}^{2} (x^3 - x^2 - 2x)\, dx \right| \\
&= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^{0} \quad + \quad \left| \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{0}^{2} \right| \\
&= (0-0-0) - \left(\frac{1}{4} - \frac{-1}{3} - 1 \right) \quad + \quad \left| \left(\frac{16}{4} - \frac{8}{3} - 4 \right) - (0-0-0) \right| \\
&= 0 - \left(-\frac{5}{12} \right) \quad + \quad \left| \left(-\frac{8}{3} \right) - 0 \right| \\
&= \frac{5}{12} + \frac{8}{3} \\
&= \frac{37}{12}\,.
\end{aligned}$$

$\square$

Note: if we don't break the integral into two parts, then we get

$$\int_{-1}^{2} (x^3 - x^2 - 2x)\, dx \;=\; \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{2} \;=\; -\frac{9}{4}\,.$$

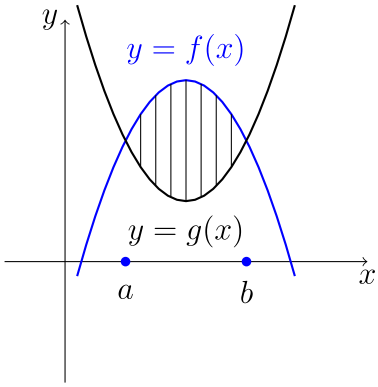
This is called the **net signed area**. It is the sum of the positive area above the x-axis together with the "negative" area below the x-axis.

Exercises

1. Find the area under $f(x) = \dfrac{1}{x}$ from $x = \dfrac{1}{2}$ to $x = 1$.
2. Find the area under $f(x) = x + e^{-x}$ from $x = 0$ to $x = 2$.
3. Find the area of the region bounded by the x–axis, $y = \sqrt{x}$ and $y = 6 - x$.
4. Find the area bounded by the $f(x) = (x-1)(x-2)(x-3)$ and the x–axis.
5. Find the area bounded by the $f(x) = x^4 - x^2$ and the x–axis.

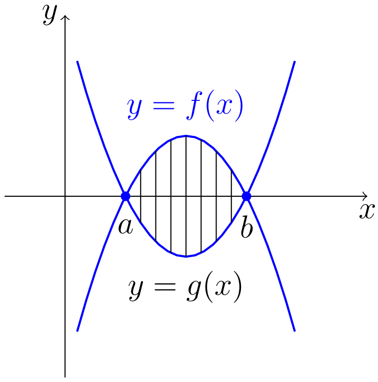
Area Between Two Curves

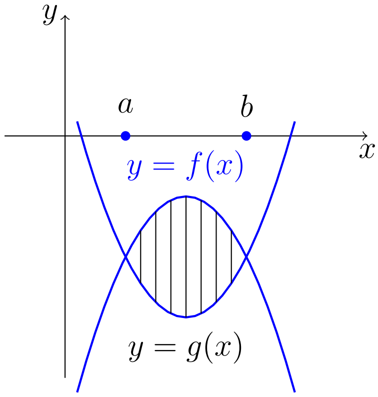
If we want to find the area **between two curves** we subtract the bottom function from the top function, and then integrate.



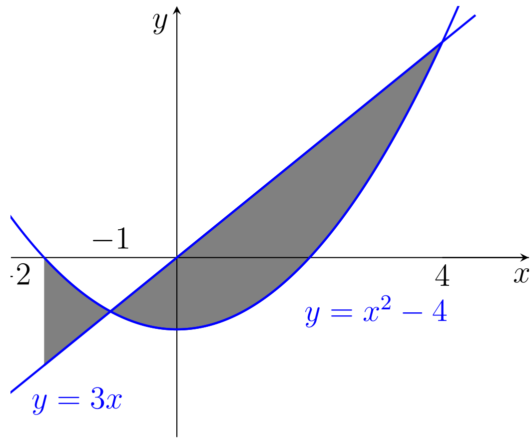
For example, if we want to find the area between f and g in any of the adjacent diagrams, we just need to find

$$\boxed{\int_a^b \Big(f(x) - g(x) \Big)\, dx}$$





Example 10. Find the area of the region between the graphs of $y = x^2 - 4$ and $y = 3x$ on the interval $-2 \le x \le 4$, as shaded in the diagram below.



Solution:

$$
\begin{aligned}
\text{Area} &= \int_{-2}^{-1} \big((x^2-4) - 3x\big)\ dx \ + \int_{-1}^{4} \big(3x - (x^2-4)\big)\ dx \\
&= \left[\frac{x^3}{3} - 4x - \frac{3x^2}{2}\right]_{-2}^{-1} + \left[\frac{3x^2}{2} - \frac{x^3}{3} + 4x\right]_{-1}^{4} \\
&= \left(\left(\frac{-1}{3} + 4 - \frac{3}{2}\right) - \left(\frac{-8}{3} + 8 - 6\right)\right) \\
&+ \left(\left(24 - \frac{64}{3} + 16\right) - \left(\frac{3}{2} - \frac{-1}{3} - 4\right)\right) \\
&= \left(\frac{13}{6} - \frac{-2}{3}\right) + \left(\frac{56}{3} - \frac{-13}{6}\right) \\
&= \frac{17}{6} + \frac{125}{6} \\
&= \frac{71}{3}
\end{aligned}
$$

$\square$

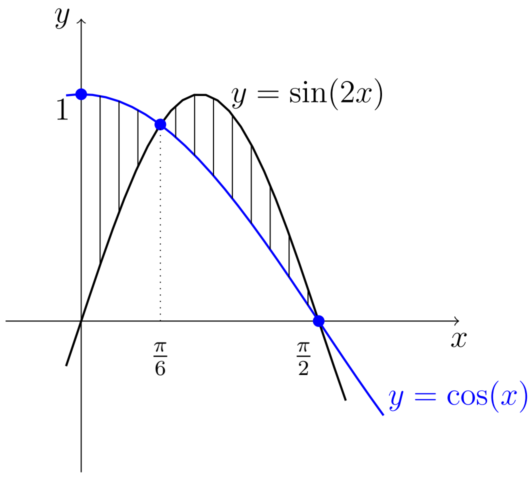
Exercises

Note:

When a question asks us to find the area of a region bounded by certain curves or lines, then we should draw a graph so that we can see

- whether or not the region needs to be broken into several parts, and
- which curve or line is on the "top", and which is on the "bottom".

6. Find the area bounded by $f(x) = x + 1$ and $g(x) = x^2 - x - 2$.
7. Find the area bounded by $f(x) = x + 3$ and $g(x) = 12 + x - x^2$.
8. Find the area bounded by $f(x) = 3x + 5$ and $g(x) = x^2 + 1$.
9. Find the area bounded by $f(x) = 3 - x^2$ and $g(x) = 2x^2$.
10. Find the area bounded by $f(x) = x^2$ and $g(x) = 3x$.
11. Using the same axes sketch $y = \sin x$ and $y = \sin 2x$ for $0 \leq x \leq \pi$. Calculate the smaller of the two areas bounded by the curves.
12. Find the area of the shaded region in the diagram given below.



11.3 Answers to Chapter 11 Exercises

11.1:
(a) $\frac{1}{4}x^4 + 8x + C$ (b) $\frac{1}{5}x^5 + 2x^{\frac{3}{2}} - \frac{2}{3}x^3 - x^{-1} + C$ (c) $\frac{2}{3}\sin\left(\frac{3}{2}x\right) + C$

(d) $\frac{1}{4}e^{4x} + C$ (e) $2\ln|x| + C$ (f) $\frac{1}{2}\ln|x| + C$

(g) $\frac{1}{3}x^3 + x^2 + C$ (h) $\frac{1}{3}\sin 3x + C$ (i) $-\frac{1}{x} + C$

(j) $-2\cos\left(\frac{1}{2}x\right) + C$ (k) $\frac{2}{3}x^{\frac{3}{2}} + C$ (l) $2\tan\left(\frac{1}{2}x\right) + C$

(m) $\frac{(2x+7)^{10}}{20} + C$ (n) $\frac{(6x-5)^{\frac{3}{2}}}{9} + C$ (o) $-\frac{(1-3x)^{11}}{33} + C$

(p) $-\frac{1}{36(4x+5)^9} + C$ (q) $\frac{1}{9}\ln|9x+2| + C$ (r) $\ln|2x+3| + C$

(s) $\tan^{-1}\left(\dfrac{x}{5}\right) + C$ (t) $\dfrac{2}{5}\tan^{-1}\left(\dfrac{x}{5}\right) + C$

11.2:
1. $\ln 2$ 2. $3 - e^{-2}$ 3. $\frac{22}{3}$ 4. $\frac{1}{2}$ 5. $\frac{4}{15}$

6. $\frac{32}{3}$ 7. 36 8. $\frac{125}{6}$ 9. 4 10. $\frac{9}{2}$

11. $\frac{1}{4}$ (See diagram below.) 12. $\frac{1}{2}$