Chapter 4

Transformations of Graphs

Reference: “Calculus”, by James Stewart.

Suppose that we know the graph of $y = f(x)$. The aim of this chapter is to learn how to sketch graphs corresponding to the following equations:

Column A	**Column B**
$y = f(x) + c$	$y = f(x + c)$
$y = f(x) - c$	$y = f(x - c)$
$y = cf(x)$	$y = f(cx)$
$y = -f(x)$	$y = f(-x)$
$y = \|f(x)\|$	
$y = (f(x))^2$	
$y = \dfrac{1}{f(x)}$	

One crucial distinction which we make between the various transformations is whether a transformation is ‘**inside**’ the function or ‘**outside**’ the function. For example,

- in each of the equations listed in Column A, the transformation is ‘**outside**’ the function. That is, the extra symbols are outside the brackets for the function $f(x)$.

In contrast,

- in each of the equations in Column B, the transformation is '**inside**' the function. That is, the extra symbols are inside the brackets for the function $f(x)$.

4.1 Translations, Reflections and Dilations 'Outside' the Function

If a transformation is '**outside**' the function then we need to change $\boldsymbol{y}$.

We apply the indicated operation to $\boldsymbol{y}$. In the following table we assume that $c > 0$.

Equation	**Action**	**Mathematical Description**
$y = f(x) + c$	Add c to the y–values.	A translation up by c.
$y = f(x) - c$	Subtract c from the y–values.	A translation down by c.
$y = cf(x)$	Multiply the y–values by c.	A dilation by a factor of c parallel to the y–axis.
$y = -f(x)$	Multiply the y–values by -1.	A reflection in the x–axis.

Example 1.

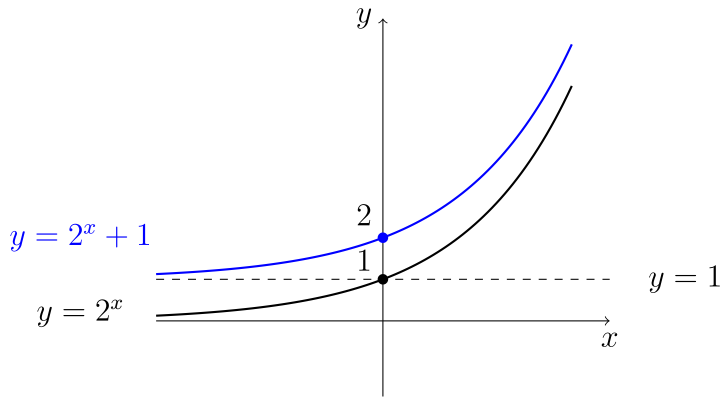
(a) Sketch the graph of $y = 2^x + 1$.

(b) Sketch the graph of $y = 2^x - 1$.

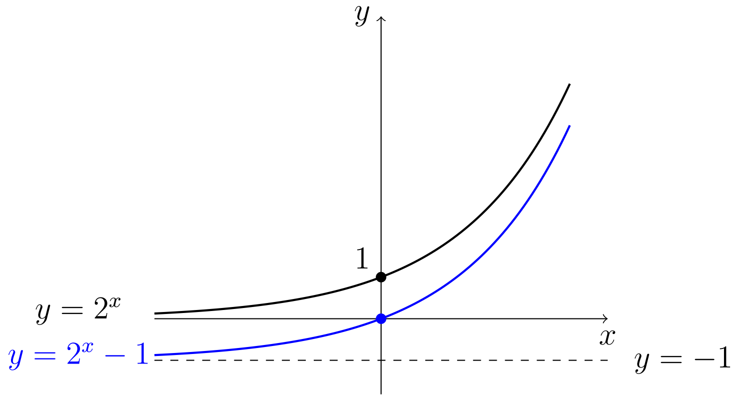
(c) Sketch the graph of $y = 2\sin x$.

(d) Sketch the graph of $y = -(2^x)$.

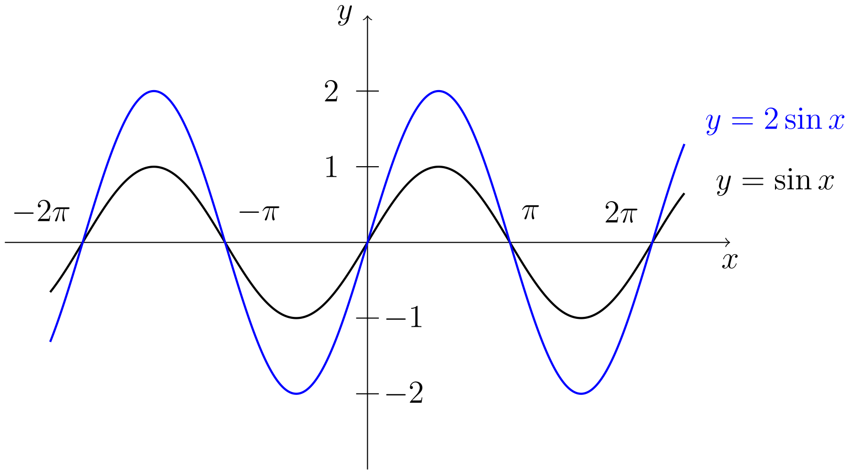
Solution: (a) We start by sketching the graph of $y = 2^x$ (which is one of the basic graphs from the previous chapter). We then observe that the graph of $y = 2^x + 1$ is obtained by translating the graph of $y = 2^x$ up one unit. Note that since the x-axis is a horizontal asymptote for the graph of $y = 2^x$, then there is a horizontal asymptote for the graph of $y = 2^x + 1$ which is obtained by translating the line $y = 0$ up one unit.



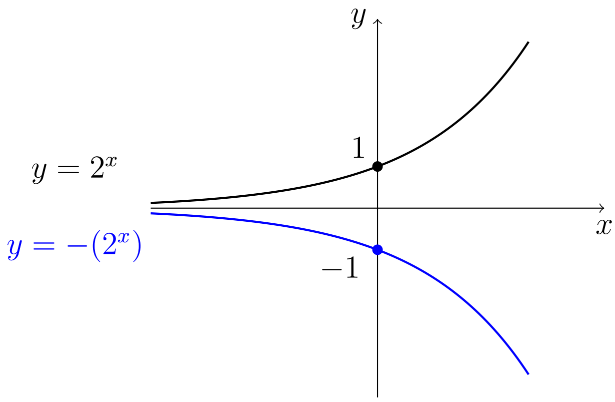
(b) We start by sketching the graph of $y = 2^x$. Observe that the graph of $y = 2^x - 1$ is obtained by translating the graph of $y = 2^x$ down one unit. Note that since the x-axis is a horizontal asymptote for the graph of $y = 2^x$, then there is a horizontal asymptote for the graph of $y = 2^x - 1$ which is obtained by translating the line $y = 0$ down one unit.



(c) We start by sketching the graph of $y = \sin x$. Observe that the graph of $y = 2\sin x$ is obtained by dilating the graph of $y = \sin x$ by a factor of two in the vertical direction.



(d) We start with the graph of $y = 2^x$. The graph of $y = -(2^x)$ is obtained by reflecting the graph of $y = 2^x$ in the x-axis.



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4.2 Translations, Reflections and Dilations 'Inside' the Function

If a transformation is '**inside**' the function then we need to change $\boldsymbol{x}$.

We apply the **opposite** of the indicated operation to $\boldsymbol{x}$.

Equation	**Action**	**Mathematical Description**
$y = f(x+c)$	Subtract c from the x–values.	A translation to the left by c.
$y = f(x-c)$	Add c to the x–values.	A translation to the right by c.
$y = f(cx)$	Divide the x–values by c.	A dilation by a factor of $\frac{1}{c}$ parallel to the x–axis.
$y = f(-x)$	Divide the x–values by -1.	A reflection in the y–axis.

Example 2. (a) Sketch the graph of $y = 2^{x+1}$.

(b) Sketch the graph of $y = 2^{x-1}$.

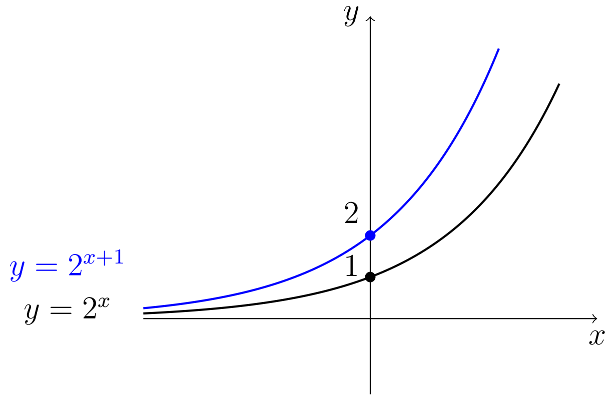
(c) Sketch the graph of $y = \sin(2x)$.

(d) Sketch the graph of $y = 2^{-x}$.

Solution: (a) We start by sketching the graph of $y = 2^x$. Observe that the graph of $y = 2^{x+1}$ is obtained by translating the graph of $y = 2^x$ one unit **to the left**. We find the y –intercept of $y = 2^{x+1}$ as follows:

$$x = 0 \qquad \Longrightarrow y = 2x + 1$$
$$= 2^{0+1}$$
$$= 2$$

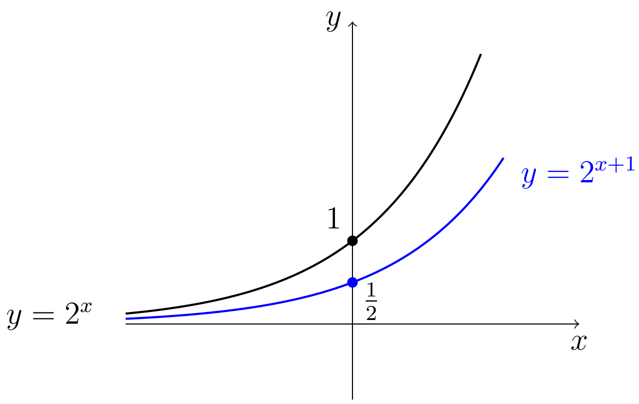
Note that x –axis is the horizontal asymptote for $y = 2^{x+1}$.



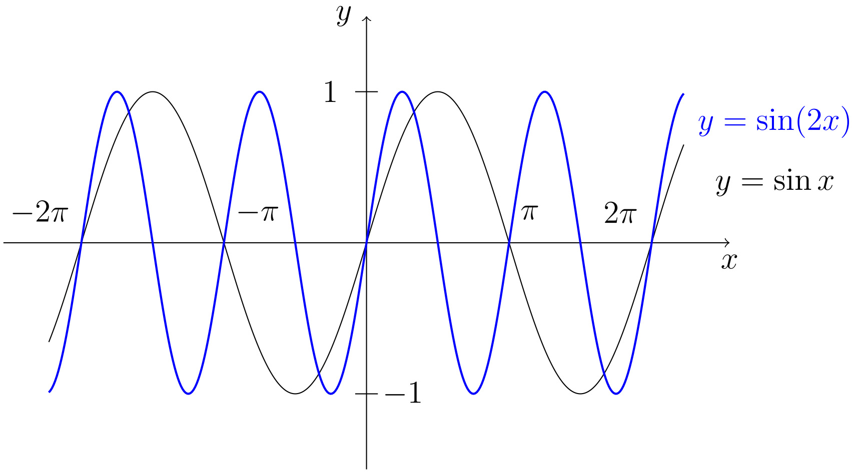
(b) We start by sketching the graph of $y = 2^x$. The graph of $y = 2^{x-1}$ is obtained by translating the graph of $y = 2^x$ one unit **to the right**. We find the y–intercept of $y = 2^{x-1}$ as follows:

$$x = 0 \qquad\qquad \Longrightarrow y = 2^{x-1}$$
$$= 2^{0-1}$$
$$= \frac{1}{2}$$

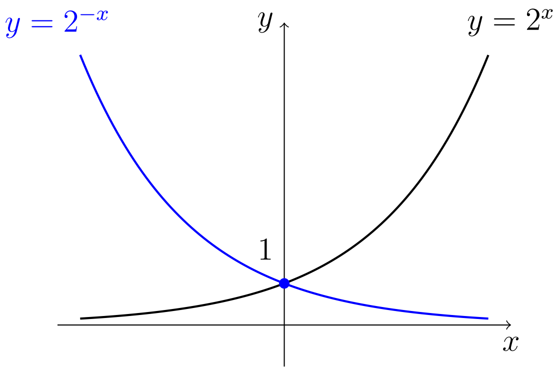
Note that x–axis is the horizontal asymptote for $y = 2^{x-1}$.



(c) We start by sketching the graph of $y = \sin x$. Observe that the graph of $y = \sin(2x)$ is obtained by dilating the graph of $y = \sin x$ by a factor a half in the horizontal direction.



(d) We start with the graph of $y = 2^x$. The graph of $y = 2^{-x}$ is obtained by reflecting the graph of $y = 2^x$ in the y–axis.



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Exercises

On the same set of axes, sketch:

(a) $y = \cos x$ and $y = -\cos x$

(b) $y = \log_2 x$ and $y = \log_2(-x)$

(c) $y = x^2$ and $y = x^2 - 2$

(d) $y = x^2$ and $y = (x-3)^2$

(e) $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x+2}$

(f) $y = x^3$ and $y = x^3 - 8$

4.3 Combinations of Transformations

Often we need to sketch graphs for equations which involve more than one transformation. The method is to build the complicated–looking functions from easier functions, using the transformations considered earlier.

The transformations which are "outside" the function should be applied in the usual arithmetic order.

For example, suppose that we want to sketch the graph of $y = af(x)+d$.

The arithmetic order of operations is to multiply by a first, and then to add d . Thus we should

- first apply the transformation involving the a , and
- then apply the transformation involving the d .

Suppose that we want to sketch the graph of $y = m\,(f(x) + r)$.

The arithmetic order of operations is to add r first, and then to multiply by m . Thus we should

- first apply the transformation involving the r , and
- then apply the transformation involving the m .

The transformations which are "inside" the function should be applied in the OPPOSITE of the usual arithmetic order.

For example, suppose that we want to sketch the graph of $y = f(bx+c)$.

The arithmetic order of operations is to multiply by b first, and then to add c . We should apply these operations in the opposite order. That is, we should

- first apply the transformation involving the c , and
- then apply the transformation involving the b .

Finally, suppose that we want to sketch the graph of $y = f(n(x + p))$.

The arithmetic order of operations is to add p first, and then to multiply by n . We should apply these operations in the opposite order. That is, we should

- first apply the transformation involving the n , and
- then apply the transformation involving the p .

If the function involves some transformations which are "outside" the function, and some transformations which are "inside" the function, then it does not matter whether the "outside" transformations are considered before the "inside" transformations or vice versa.

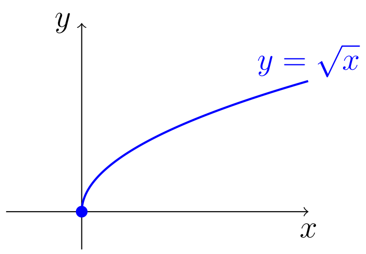
Example 3. Sketch the graph of $y = \sqrt{3x - 12}$.

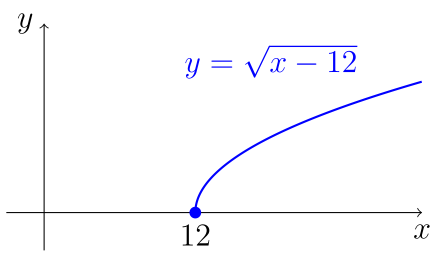
Solution: We start with $y = \sqrt{x}$. The graph of $y = \sqrt{3x - 12}$ is obtained by applying two "inside" transformations. To sketch the graph we need to apply these in the order that is opposite to the usual arithmetic order. We will sketch graphs in the following order:

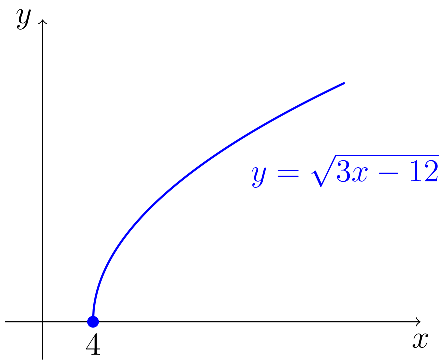
(1) $y = \sqrt{x}$ (One of the basic graphs.)

(2) $y = \sqrt{x - 12}$ (This is obtained from graph (1) by translating 12 units to the right.)

(3) $y = \sqrt{3x - 12}$ (This is obtained from graph (2) by dilating horizontally by a factor of $\frac{1}{3}$.)







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Exercises

Sketch graphs for each of the following functions:

1. (a) $y = x^2$ (b) $y = (x-1)^2$ (c) $y = (x-1)^2 + 1$
2. (a) $y = 2x^2$ (b) $y = 2(x-2)^2$ (c) $y = 2(x-2)^2 - 4$
3. (a) $y = \dfrac{1}{x}$ (b) $y = \dfrac{1}{x+1}$ (c) $y = \dfrac{1}{x+1} - 3$
4. (a) $y = \sin x$ (b) $y = \sin(2x)$ (c) $y = \sin(-2x)$
5. (a) $y = 3^x$ (b) $y = 3^{-x}$ (c) $y = 3^{-x} + 2$
6. (a) $y = \log_2 x$ (b) $y = \log_2(x+1)$ (c) $y = -\log_2(x+1)$
7. (a) $y = x^3$ (b) $y = (x+1)^3$ (c) $y = 1 + (x+1)^3$

4.4 Absolute Values

We want to sketch graphs for functions of the form $y = |f(x)|$.

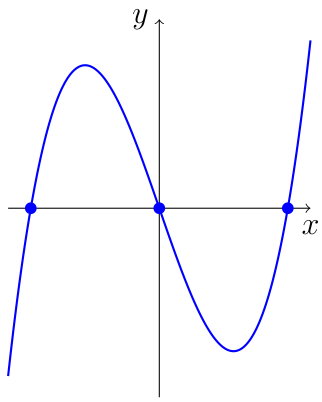
Since the absolute value symbols are '**outside**' the function, then (just like in Section 4.1) we need to apply the indicated operation to the $\boldsymbol{y}$–values in the $y = f(x)$ graph. That is, we just need to

take the absolute value of the y–values

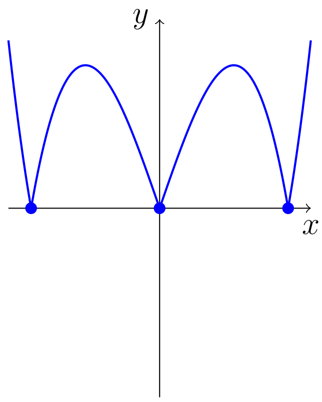
in the $y = f(x)$ graph.

Recall that absolute value satisfies the following rules:

- If $y \geq 0$ then $|y| = y$. That is, the positive y–values do not change.
- If $y < 0$ then $|y| = -y$. That is, the negative y–values change from negative to positive.



$y = f(x)$



$y = |f(x)|$

4.5 Squaring

We want to sketch graphs for functions of the form $y = \left[f(x)\right]^2$.

Since the squaring symbol is '**outside**' the function, then (just like in Section 4.1) we need to apply the indicated operation to the $\boldsymbol{y}$ –values in the $y = f(x)$ graph. That is, we just need to

square all the y –values

in the $y = f(x)$ graph.

We should start with the y –values which remain unchanged. Note that

$$\begin{aligned} y^2 = y \quad &\Rightarrow \quad y^2 - y = 0. \\ \text{i.e.} \quad & \quad y(y-1) = 0. \\ \text{i.e.} \quad & \quad y = 0 \ \text{ or } \ y = 1. \end{aligned}$$

So we have two special y –values, namely 0 and 1.

- If $y = 0$ then $y^2 = y$.

  Thus any point with $y = 0$ will **not** change if we square the function.

Similarly,

- if $y = 1$ then $y^2 = y$.

  Thus any point with $y = 1$ will **not** change if we square the function.

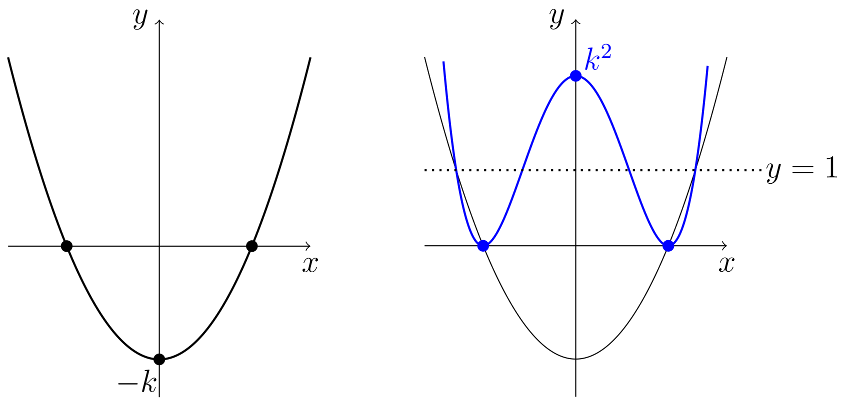
We will also need to consider other points on the graph (which are *not* special). It is helpful to consider at least one point from each "section" of the graph.

- Note that we always have $y^2 \geq 0$.

  That is, the graph of $y = \big[f(x)\big]^2$ will be in the upper half of the plane.

- If $y > 1$ then $y^2 > y$ e.g. 2^2 is **bigger** than 2.

- If $0 < y < 1$ then $y^2 < y$ e.g. $(\frac{1}{2})^2 = \frac{1}{4}$ is **smaller** than $\frac{1}{2}$.



$y = f(x)$ $y = [f(x)]^2$

4.6 Reciprocals

We want to sketch graphs for functions of the form $y = \dfrac{1}{f(x)}$.

Since the reciprocal symbol is '**outside**' the function, then (just like in Section 4.1) we need to apply the indicated operation to the $\boldsymbol{y}$–values in the $y = f(x)$ graph. That is, we need to

take the reciprocal of all the y–values

in the $y = f(x)$ graph.

We should start with the special y–values.

- The first special y –value is 0. Note that if $y = 0$ then $\dfrac{1}{y}$ is not defined.

  Therefore, we draw a **vertical asymptote** through any point with $y = 0$.

The other special y –values are those which remain unchanged. Note that

$$\frac{1}{y} = y \quad \Rightarrow \quad 1 = y^2.$$

$$\text{i.e.} \quad y = \pm 1.$$

- If $y = 1$ then $\dfrac{1}{y} = y$. Thus $y = 1$ remains unchanged.
- If $y = -1$ then $\dfrac{1}{y} = y$. Thus $y = -1$ remains unchanged.

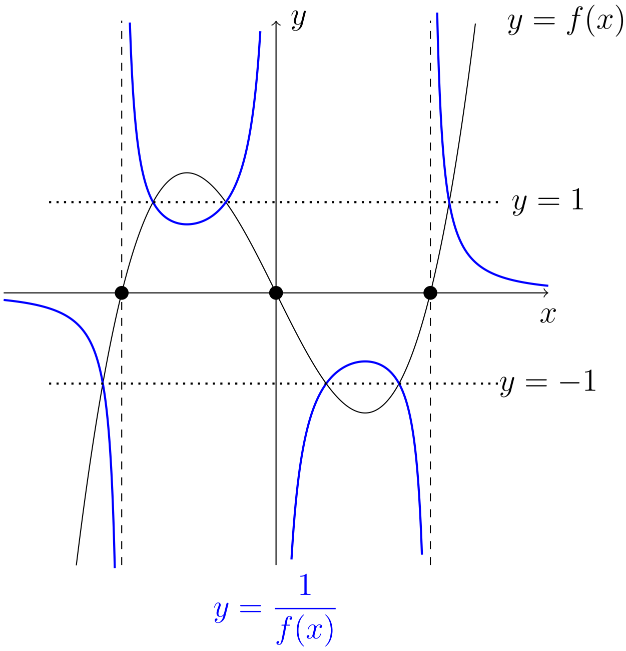
We will also need to consider other points on the graph (which are *not* special).
It is helpful to consider at least one point from each "section" of the graph.

Note that

- if y is large then $\frac{1}{y}$ is small. e.g. 10 is large but $\frac{1}{10}$ is small.

Similarly,

- if y is small then $\frac{1}{y}$ is large. e.g. $\frac{1}{4}$ is small but 4 is large.



4.7 Addition of Ordinates

We want to sketch graphs for functions of the form $y = f(x) + g(x)$.

Since the plus symbol is '**outside**' the functions, we should apply the indicated operation to the $\boldsymbol{y}$ –values. That is, we just add the y –values in the $y = f(x)$ and $y = g(x)$ graphs.

An easy way to add ordinates is to draw arrows, as described below:

- Starting on the x –axis, draw a vertical arrow to **either one** of the two graphs.

  (Note that it is neater if you draw the arrow to the closer graph.)

- Then copy that arrow onto the other graph, without changing the x –value!

  This second arrow now points to where the graph of $y = f(x) + g(x)$ should be.

See the Addition of Ordinates examples on TCOLE.

Exercises

On the same set of axes, sketch

(a) $y = x^2 - 3x$ and $y = |x^2 - 3x|$

(b) $y = x - 2$ and $y = |x - 2|$

(c) $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x^2}$

(d) $y = x^2$ and $y = x^4$

(e) $y = x^3$ and $y = \dfrac{1}{x^3}$

(f) $y = x^2 - 4x$ and $y = \dfrac{1}{x^2 - 4x}$

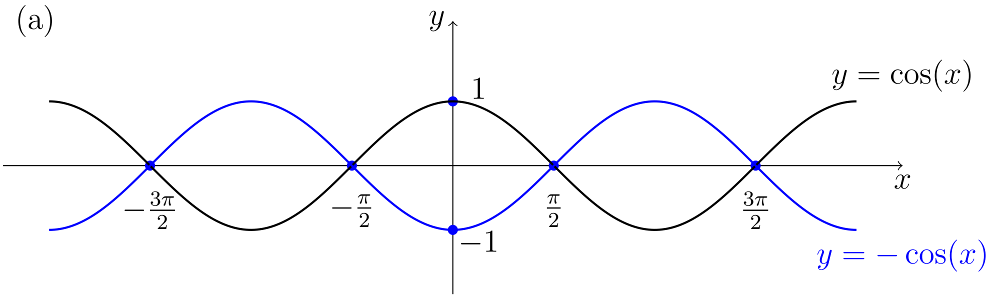
(g) $y = 2^x$, $y = 2^{-x}$ and $y = 2^x + 2^{-x}$

(h) $y = x^2 - 1$, $y = \dfrac{1}{x+1}$ and $y = x^2 - 1 + \dfrac{1}{x+1}$

4.8 Answers to Chapter 4 Exercises

4.2:

(a)



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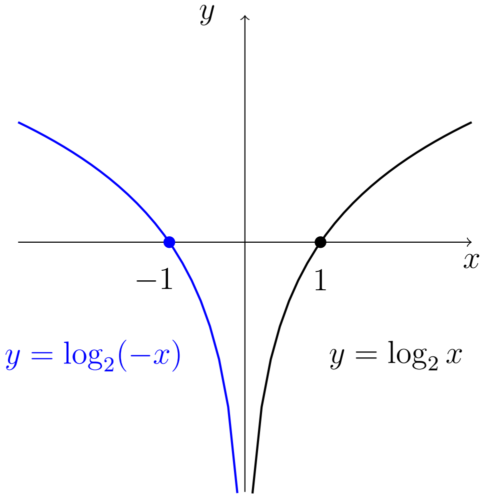
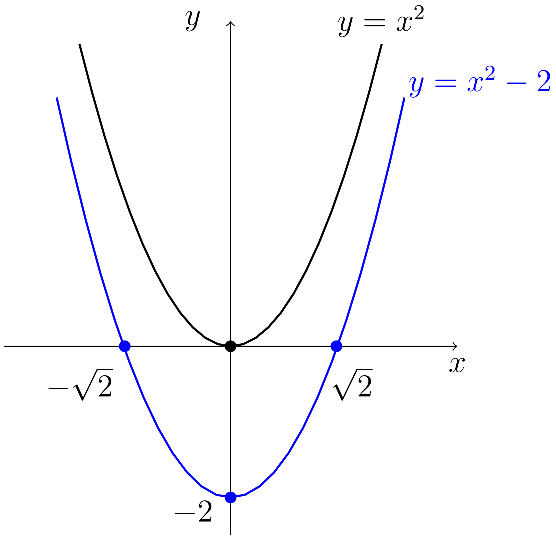
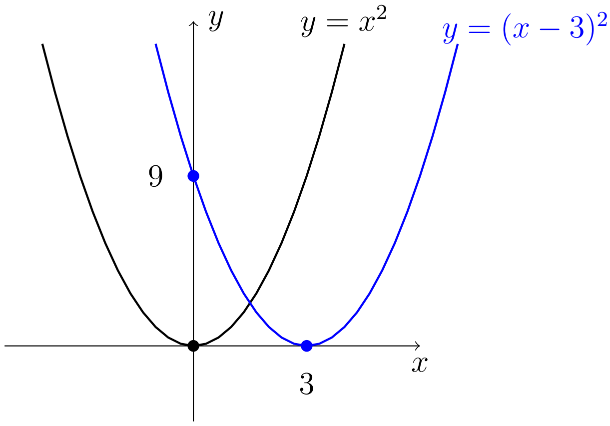


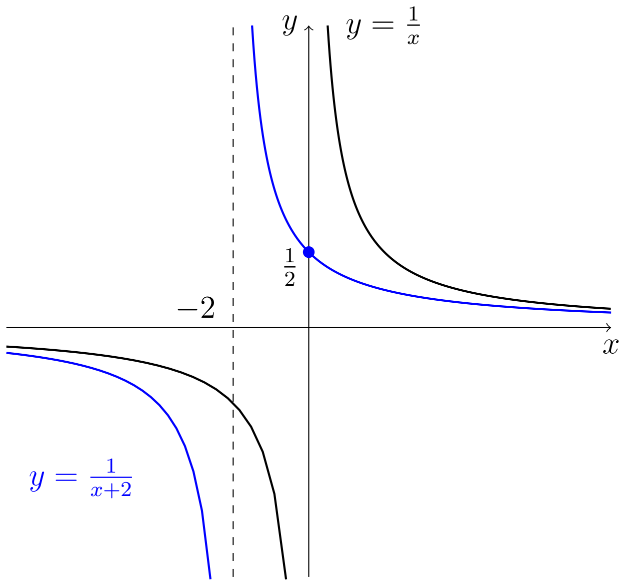
(c)



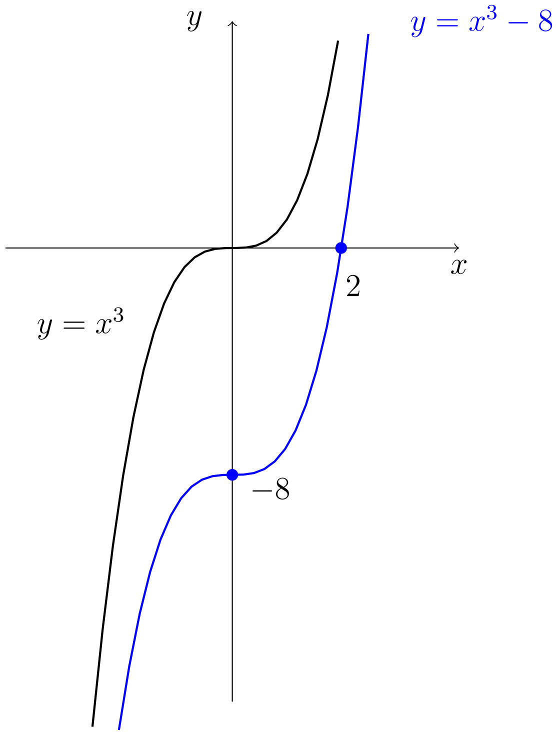
(d)



(e)

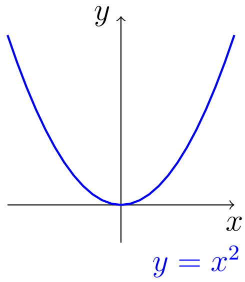


(f)

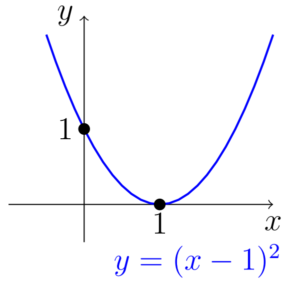


4.3

1. (a)



(b)



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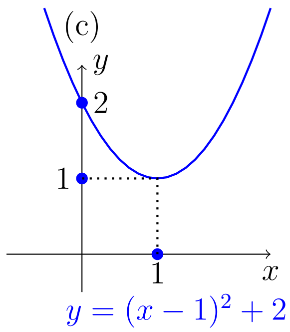
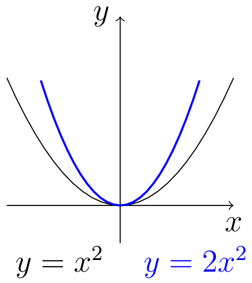


2. (a)



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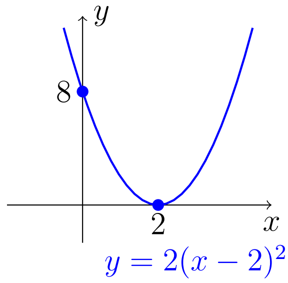

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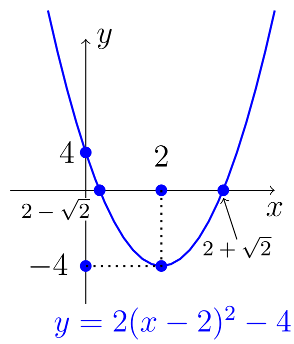
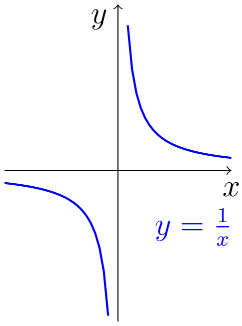
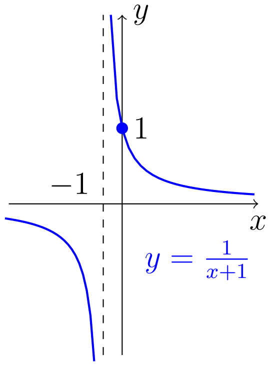


3. (a)



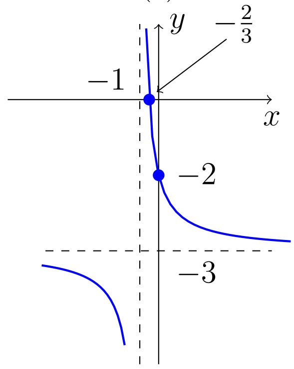
$y = \frac{1}{x}$

(b)



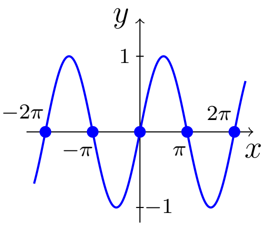
$y = \frac{1}{x+1}$

(c)



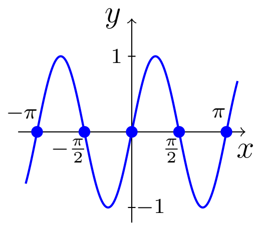
$y = \frac{1}{x+1} - 3$

4. (a)



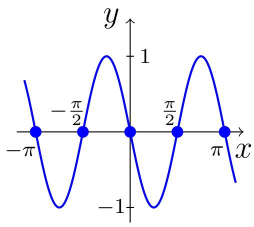
$y = \sin(x)$

(b)



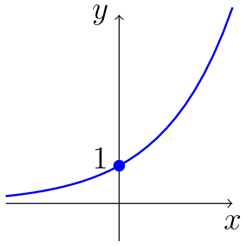
$y = \sin(2x)$

(c)



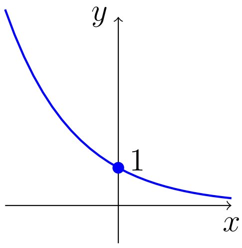
$y = \sin(-2x)$

5. (a)



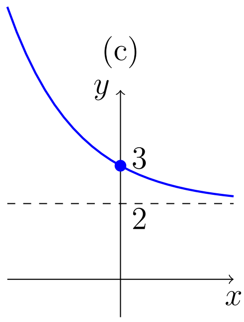
$y = 3^x$

(b)



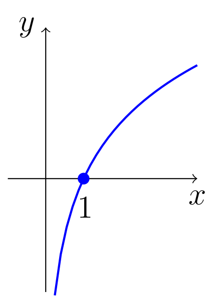
$y = 3^{-x}$

(c)



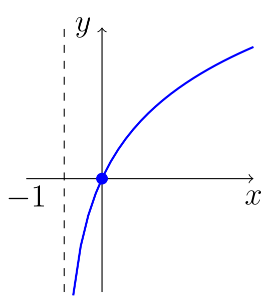
$y = 3^{-x} + 2$

6. (a)



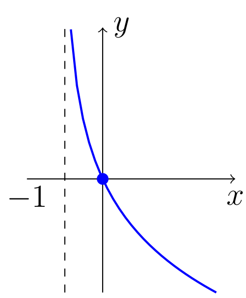
$y = \log_2(x)$

(b)



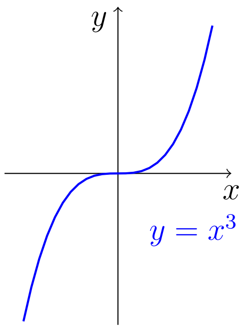
$y = \log_2(x+1)$

(c)

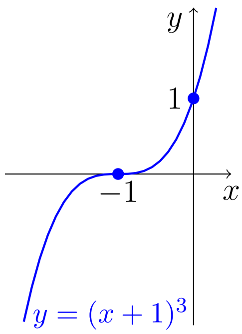


$y = -\log_2(x+1)$

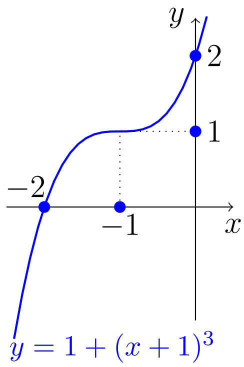
7. (a)



(b)

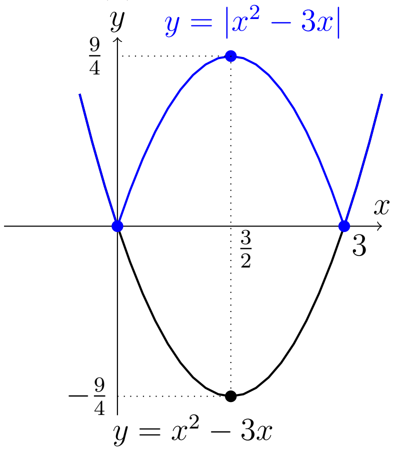


(c)

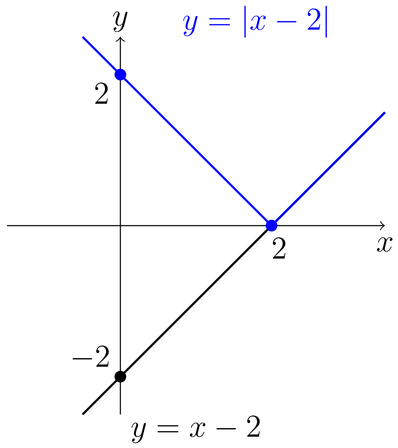


4.7

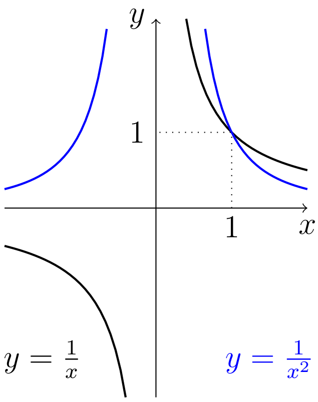
1. (a)



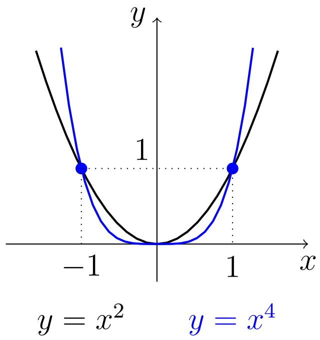
(b)



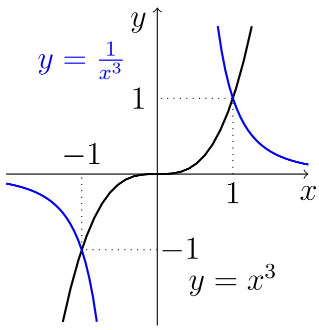
(c)



(d)

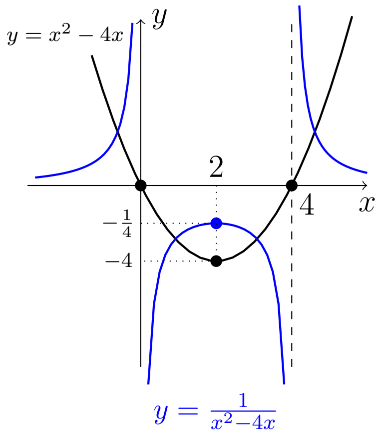


(e)



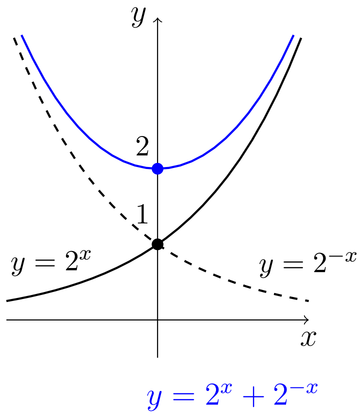
y
$y = \frac{1}{x^3}$
1
-1
1
x
-1
$y = x^3$

(f)



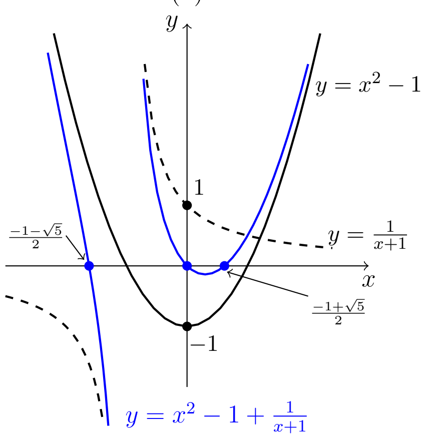
$y = x^2 - 4x$
y
2
4
x
$-\frac{1}{4}$
-4
$y = \frac{1}{x^2-4x}$

(g)



y
2
1
$y = 2^x$
$y = 2^{-x}$
x
$y = 2^x + 2^{-x}$

(h)



y
$y = x^2 - 1$
1
$y = \frac{1}{x+1}$
$\frac{-1-\sqrt{5}}{2}$
x
$\frac{-1+\sqrt{5}}{2}$
-1
$y = x^2 - 1 + \frac{1}{x+1}$