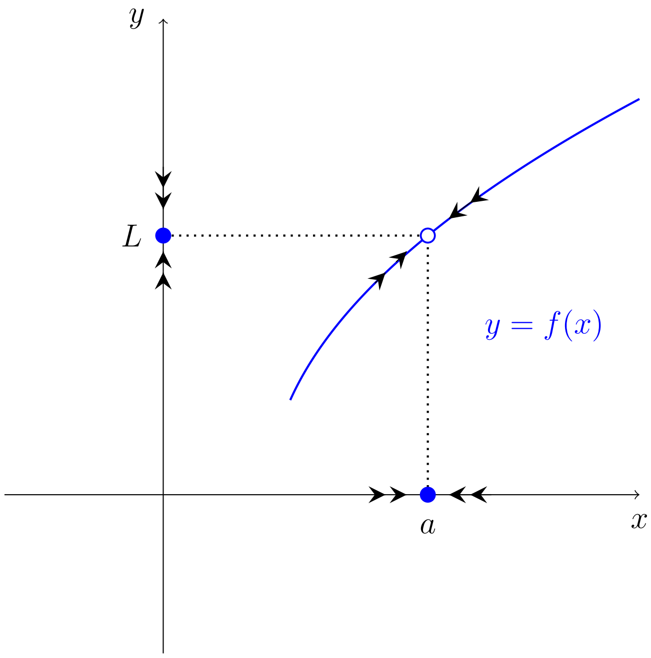
Chapter 7

Limits and Continuity

7.1 Limits – An informal treatment

We say that $\lim_{x \to a} f(x)$ **exists** if we can find a number, say L, such that $f(x)$ gets very close to L, as x gets very close to a (from **both** sides). We write $\lim_{x \to a} f(x) = L$, and we say that the limit of $f(x)$, as x approaches a, is L. This is called a **two–sided** limit (since x approaches a from **both** sides).



Suppose that $f(x) = x + 3$. We will investigate the values of $f(x)$ when x is close 2, so that we get an idea of the value of $\lim_{x \to 2} f(x)$.

x	$f(x)$
1.9	4.9
1.99	4.99
1.999	4.999

x	$f(x)$
2.1	5.1
2.01	5.01
2.001	5.001

As x gets close to 2 (from the left **and** from the right) then we see that $f(x)$ gets close to 5. Thus, we would expect that $\lim_{x \to 2} f(x) = 5$. This is indeed the value of the limit, but we shall soon state a result that allows us to calculate this kind of limit in a simple and mathematically correct manner.

Note also that $f(2) = 2 + 3 = 5$. So for $f(x) = x + 3$, we see that $\lim_{x \to 2} f(x) = f(2)$.

Suppose that $f(x) = x^2 - x + 2$. We will investigate the values of $f(x)$ when x is close 2, so that we get an idea of the value of $\lim_{x \to 2} f(x)$.

x	$f(x)$
1.9	3.71
1.99	3.9701
1.999	3.997001

x	$f(x)$
2.1	4.31
2.01	4.0301
2.001	4.003001

As x gets close to 2 (from the left **and** from the right) then we see that $f(x)$ gets close to 4. Thus, we would expect that $\lim_{x \to 2} f(x) = 4$. This is indeed the value of the limit, but we shall soon state a result that allows us to calculate this kind of limit in a simple and mathematically correct manner.

Note also that $f(2) = 2^2 - 2 + 2 = 4$. So for $f(x) = x^2 - x + 2$, we see that $\lim_{x \to 2} f(x) = f(2)$.

In the two examples that we have seen so far, we could have obtained the answer for the limit $\lim_{x \to a} f(x)$ simply by substituting the number a into the function. That is, we have had

$$\lim_{x \to a} f(x) \;=\; f(a).$$

We state the following two results without proof:

Result 1. If f is a polynomial then

$$\lim_{x \to a} f(x) \;=\; f(a).$$

Result 2. If f and g are polynomials, with $g(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} \;=\; \frac{f(a)}{g(a)} \;.$$

Example 1.

$$(a) \lim_{x \to 3} 3x$$
$$= 3 \times 3$$
$$= 9$$

$$\text{(b)} \lim_{x \to 3} \frac{3x}{x+2}$$
$$= \frac{3 \times 3}{3+2}$$
$$= \frac{9}{5}$$

Note that if f is **NOT** a polynomial (or a fraction of polynomials), then we **might** have

$$\lim_{x \to a} f(x) \;\neq\; f(a).$$

That is, it is possible that the limit **cannot** be found just by substituting the number a into the function. In particular, we should keep in mind that whenever we evaluate $\lim_{x \to a} f(x)$, we are actually interested in the value of $f(x)$ for x **close** to a (but **not equal** to a).

Example 2. (a) Find $\lim_{x \to 1}(x+1)$.

(b) Consider the function $f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1\,. \end{cases}$

(i) Find $f(1)$.

(ii) Find $\lim_{x \to 1} f(x)$.

Solution: (a) Since $x+1$ is a polynomial, we have

$$\begin{aligned} \lim_{x \to 1}(x+1) &= 1+1 \\ &= 2 \end{aligned}$$

(b) (i) We have $f(1) = \pi$.

(ii) We have $\lim_{x \to 1} f(x) = \lim_{x \to 1}(x+1) = 1+1 = 2$.

$\square$

Note that in Example 2(b), the limit **CANNOT** be found just by substituting $x = 1$ into $f(x)$. That is, in Example 2(b) we have

$$\lim_{x \to 1} f(x) \neq f(1)\,.$$

Suppose we need to calculate

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials. If $f(a) = 0$ and $g(a) = 0$, then it is useful to

(a) factorise $f(x)$ and/or $g(x)$, and

(b) cancel terms,

in order to calculate $\lim_{x \to a} \frac{f(x)}{g(x)}$.

Example 3.

$$\begin{aligned} \lim_{x \to 2} \frac{x^2-4}{x-2} &= \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} \\ &= \lim_{x \to 2} (x+2) \\ &= 2+2 \\ &= 4 \end{aligned}$$

Note that in Example 3, the limit **CANNOT** be found just by substituting $x = 2$ into $f(x)$ (since that would give 0 in the denominator of the fraction). That is, in Example 3, we have

$$\lim_{x \to 2} f(x) \neq f(2)$$

(since $f(2)$ is **not** defined).

Example 4.

(a)

$$\begin{aligned}
&\lim_{x \to 0} \frac{x^2 + 5x}{x} \\
&= \lim_{x \to 0} \frac{x(x+5)}{x} \\
&= \lim_{x \to 0} (x+5) \\
&= 0 + 5 \\
&= 5
\end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} \\
&= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \\
&= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \lim_{h \to 0} (6+h) \\
&= 6 + 0 \\
&= 6
\end{aligned}$$

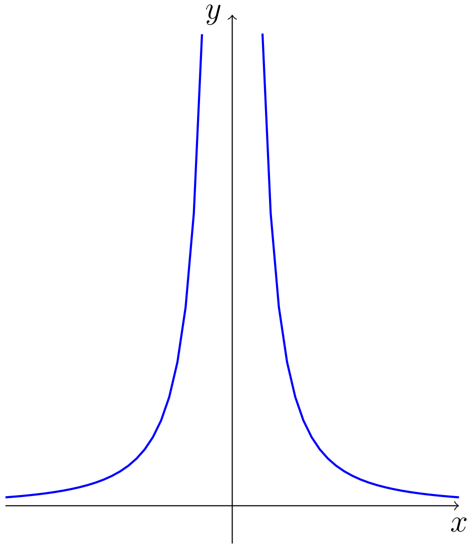
We now state (without proof) an important result, that you need to know:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

From this result, we are able to deduce important results regarding the derivatives of trigonometric functions.

Note: As usual, when we write $\sin x$ we are treating x as if it is an angle measured in radians, not degrees. The above limit when x is measured in degrees is a number that is not as simple as 1. This is one reason that mathematicians prefer radians to degrees.

Suppose that $f(x) = \dfrac{1}{x^2}$.



$y = \frac{1}{x^2}$

Note that $\lim_{x\to 0} f(x)$ **does not exist**, because $\dfrac{1}{x^2}$ does **not** approach any particular number as x gets closer and closer to 0 . In fact, we see from the above graph that as $x \to 0$ we have $\dfrac{1}{x^2} \to \infty$.

We write $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ (even though we have already stated that $\lim_{x\to 0} \dfrac{1}{x^2}$ does not exist).

In general, we write

$$\lim_{x\to a} f(x) = \infty$$

if $f(x)$ gets arbitrarily large (i.e. "goes to ∞ ") as x gets closer and closer to a . However we must remember that when we write $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} f(x)$ **does not exist** (since ∞ is **not a number**).

Similarly, if $\lim_{x\to b} g(x) = -\infty$ then $\lim_{x\to b} g(x)$ does not exist.

Exercises

Evaluate each of the following limits:

(a) $\lim_{x\to 1}(5x+3)$ (b) $\lim_{x\to 0}|x|$ (c) $\lim_{x\to \pi}\cos x$

(d) $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$ (e) $\lim_{x\to -2}\dfrac{x^3+8}{x+2}$ (f) $\lim_{t\to 4}\dfrac{\sqrt{t}-2}{t-4}$

Limit Laws

Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and that c is a constant. Then

1. $\lim_{x\to a}[f(x)+g(x)] = L+M$
2. $\lim_{x\to a}[f(x)-g(x)] = L-M$
3. $\lim_{x\to a}[f(x)g(x)] = LM$
4. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$.
5. if $M=0$ and $L\neq 0$ then $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ does not exist.
6. $\lim_{x\to a}[cf(x)] = cL$
7. $\lim_{x\to a} c = c$.
8. $\lim_{x\to a} x = a$.

From these laws, we can derive many rules for calculating limits. For example, we can use these laws to show the following results (which were stated on page 3).

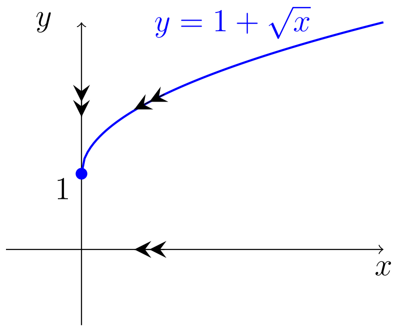
- Each **polynomial** f satisfies $\lim_{x\to a} f(x) = f(a)$.
- Similarly, if f and g are polynomials such that $g(a)\neq 0$ then $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$.

7.2 One–sided Limits

When we write $\lim_{x\to a^-} f(x)$, we only consider x–values which are **less** than a. This is called a **left–hand** limit. We say that $\lim_{x\to a^-} f(x)$ **exists**, and that $\lim_{x\to a^-} f(x) = L$ if there is a number L such that $f(x)$ gets very close to L, as x gets very close to a **from the left**.

Similarly, when we write $\lim_{x\to a^+} f(x)$, we only consider x–values which are **greater** than a. This is called a **right–hand** limit.

Let us consider how we might find $\lim_{x\to 0^+} \left(1+\sqrt{x}\,\right)$.



From the graph, it appears that as x approaches 0 from the right, then y approaches 1. That is, $1+\sqrt{x}$ approaches 1.

So it seems that

$$\lim_{x\to 0^+} \left(1+\sqrt{x}\,\right) = 1$$

This is in fact correct. It turns out that the square root function is another example of a function whose limits are found simply by substitution. That is we can calculate the above limit as follows:

$$\begin{aligned}\lim_{x\to 0^+} \left(1+\sqrt{x}\,\right) &= 1+\sqrt{0} \\ &= 1+0 \\ &= 1\end{aligned}$$

Example 5. Find $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3^+} f(x)$ where

$$f(x) = \begin{cases} x+2 & \text{if } x \le 3 \\ 4 & \text{if } x > 3\,. \end{cases}$$

Solution: We have

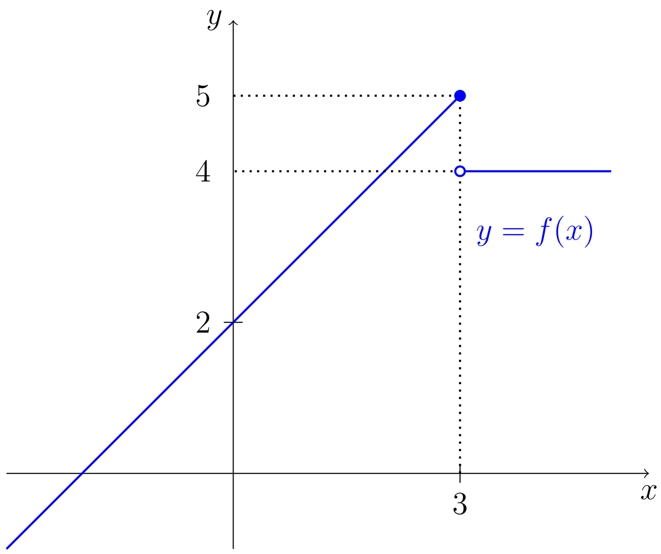
$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} (x+2) \\ &= 3+2 \\ &= 5 \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} 4 \\ &= 4\,. \end{aligned}$$

$\square$

Note that we can guess that these are the correct limits by looking at the function's graph:



Result 3. Let f be a function that is defined for all numbers near the number a *. Then

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

From this result, we deduce that

if the one–sided limits are **not equal** to each other then the two–sided limit does not exist.

We also deduce that

if one (or both) of the one–sided limits does not exist, then the two–sided limit also does not exist.

Example 6. Find $\lim_{x\to 3} f(x)$ where $f(x) = \begin{cases} x+2 & \text{if } x \le 3 \\ 4 & \text{if } x > 3\,. \end{cases}$

Solution: In Example 5 we saw that

$$\lim_{x\to 3^-} f(x) = 5 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = 4.$$

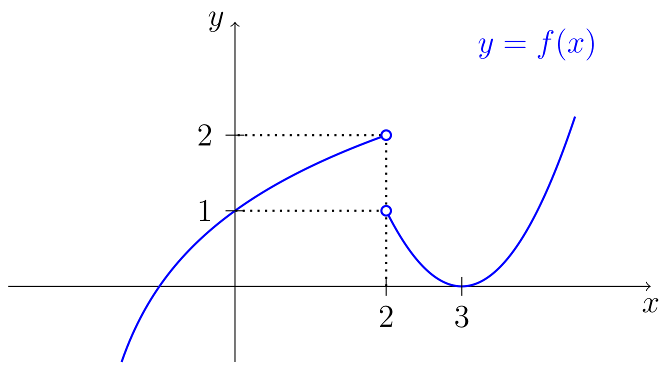
Since

$$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$$

then $\lim_{x\to 3} f(x)$ does **not** exist. □

If you are given the graph of a function but you are not given a rule for the function, then you may use the graph to evaluate limits.

Example 7. Consider the function f whose graph is given below.



*This means that there is a positive number ϵ such that $(a-\epsilon, a+\epsilon)\setminus\{a\} \subseteq \mathrm{dom}(f)$.

(a) From the graph we see that $\lim_{x \to 2^-} f(x) = 2$

(b) From the graph we see that $\lim_{x \to 2^+} f(x) = 1$

(c) We conclude that $\lim_{x \to 2} f(x)$ does **not** exist because

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

Example 8. Find $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \,. \end{cases}$

Solution:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(x^2 + 1\right) \\ &= 0^2 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (1) \\ &= 1 \end{aligned}$$

We see that

$$\lim_{x \to 0^+} f(x) = 1 \text{ and } \lim_{x \to 0^-} f(x) = 1$$

and therefore

$$\lim_{x \to 0} f(x) = 1.$$

□

Exercises

1. Consider the function $f(x) = \begin{cases} x^2 + 1 & \text{for } x \geq 2 \\ 2x - 1 & \text{for } x < 2 \end{cases}$

   (a) Find $\lim_{x \to 2^-} f(x)$.

   (b) Find $\lim_{x \to 2^+} f(x)$.

   (c) Find $\lim_{x \to 2} f(x)$.

2. Evaluate the following limits.

   (a) $\lim_{x \to 1} (2x - 5)$

   (b) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

   (c) $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

   (d) $\lim_{x \to 2} \dfrac{2x + 1}{x + 2}$

   (e) $\lim_{x \to 2} f(x)$ where $f(x) = \begin{cases} 2x - 1 & \text{for } x \geq 2 \\ x^2 - 1 & \text{for } x < 2 \end{cases}$

   (f) $\lim_{x \to 3} f(x)$ where $f(x) = \begin{cases} x^2 & \text{for } x < 3 \\ 3x - 1 & \text{for } x \geq 3 \end{cases}$

7.3 Continuity

We say that a function f is **continuous** at $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

In fact this single equation actually involves **three** conditions.
In particular, $f(x)$ is **continuous** at $x = a$ if

(i) $f(a)$ is defined; and

(ii) $\lim\limits_{x \to a} f(x)$ exists; and

(iii) $\lim\limits_{x \to a} f(x) = f(a)$.

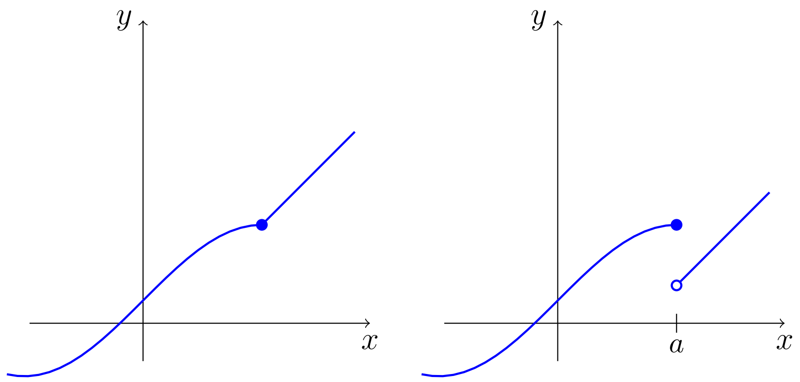
If one or more of these three properties does **not** hold, then $f(x)$ is **discontinuous** at $x = a$.

When we just say that f is **continuous** (without stating a particular x–value) then we mean that f is continuous at **every** point in its domain.

Note:

Roughly speaking f is continuous at $x = a$ if we do **not** need to lift our pens at $x = a$ when we are sketching a graph of $y = f(x)$. If we **do** need to lift our pen at $x = a$ when we are sketching the graph of $y = f(x)$ then f is discontinuous at $x = a$.

Note however, that **if we are asked to explain** why a function is continuous (or discontinuous) at a particular point, then we **must** refer to the conditions (i), (ii) and (iii) given above. It is **not** good enough to just talk about lifting your pen.



Continuous — Not continuous at $x = a$

Example 9. (a) Consider the function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.

Is f continuous at $x = 2$?

That is, does $\lim\limits_{x \to 2} f(x) \;=\; f(2)$?

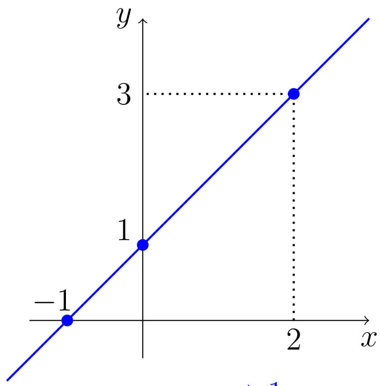
Solution: Note that $f(2)$ is **not** defined (since we cannot have 0 for the denominator of a fraction).

Thus f is **not** continuous at $x = 2$. $\square$

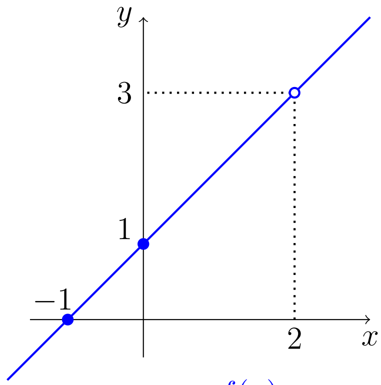
Note that

$$\begin{aligned} f(x) &= \frac{x^2 - x - 2}{x - 2} \\ &= \frac{(x-2)(x+1)}{x-2} \\ &= x + 1 \end{aligned}$$

but with $\operatorname{dom}(f) = \mathbf{R} \setminus \{2\}$.



$y = x + 1$



$y = f(x)$

We see that we have to lift our pen at $x = 2$ to jump over the hole in the graph of $y = f(x)$. (We are **not** allowed to trace around the "outside" of the hole.)

(In contrast, we do *not* need to lift our pen on the $y = x + 1$ graph, since $y = x + 1$ is continuous everywhere.)

(b) Consider the function $f(x) = \begin{cases} x^2+1 & \text{for } x \geq 0 \\ 1 & \text{for } x < 0\,. \end{cases}$

Is f continuous at $x = 0$?

That is, does $\lim_{x\to 0} f(x) \;=\; f(0)$?

Solution: (i) $f(0) = 0^2 + 1 = 1\,.$

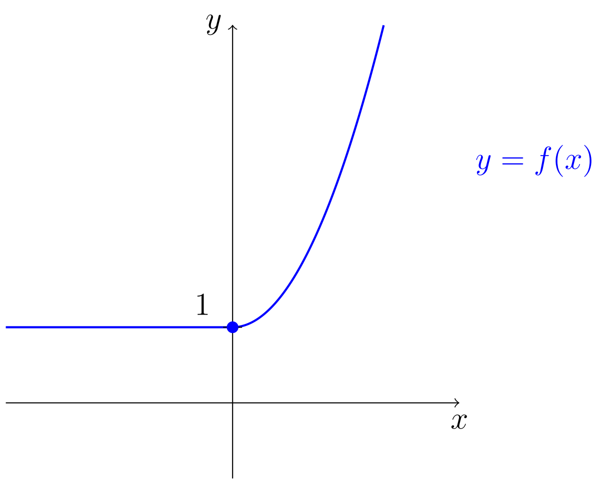
(ii) We saw in Example 8 that $\lim_{x\to 0} f(x)$ exists, with $\lim_{x\to 0} f(x) \;=\; 1$.

(iii) We see from (i) and (ii) above that

$$\lim_{x\to 0} f(x) \;=\; f(0).$$

Thus f **is** continuous at $x = 0$.

$\square$



Note: We can draw the graph of $y = f(x)$ without having to lift our pens.

(c) Consider the function $f(x) = \begin{cases} x^2 & \text{for } x < 3 \\ 3x - 1 & \text{for } x \geq 3\,. \end{cases}$

Is f continuous at $x = 3$?

That is, does $\lim_{x\to 3} f(x) = f(3)$?

Solution: (i)

$$\begin{aligned} f(3) &= 3 \times 3 - 1 \\ &= 8 \end{aligned}$$

(ii)

$$\begin{aligned} \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (3x - 1) \\ &= 3 \times 3 - 1 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} \left(x^2\right) \\ &= 3^2 \\ &= 9 \end{aligned}$$

Therefore $\lim_{x\to 3} f(x)$ does not exist, because $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$.

Therefore f is not continuous at $x = 3$ because $\lim_{x\to 3} f(x)$ does not exist.

$\square$

(d) Consider the function $f(x) = \begin{cases} x+2 & \text{for } x \neq 1 \\ 4 & \text{for } x = 1. \end{cases}$

Is f continuous at $x = 1$?

That is, does $\lim_{x \to 1} f(x) = f(1)$?

Solution: (i) $f(1) = 4$ (This is clear from the definition of the function.)

(ii)

$$\begin{aligned} \lim_{x \to 1} f(x) &= \lim_{x \to 1} (x+2) \\ &= 1+2 \\ &= 3 \end{aligned}$$

(iii) We see that

$$\lim_{x \to 1} f(x) \neq f(1).$$

Therefore f is not continuous at $x = 1$. □

Exercises

State whether the following functions are continuous. If a point of discontinuity occurs, explain why it is a point of discontinuity.

(a) $f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ x & \text{for } x \leq 0 \end{cases}$

(b) $f(x) = \begin{cases} x^2+1 & \text{for } x > 0 \\ x & \text{for } x \leq 0 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$

(d) $f(x) = \begin{cases} 2x-3 & \text{for } x \leq 2 \\ x^2-2x & \text{for } x > 2 \end{cases}$

Continuity Laws

Suppose that the functions f and g are continuous at a, and that c is a constant. Then

- $f + g$ is continuous at a.
- $f - g$ is continuous at a.
- fg is continuous at a.
- if $g(a) \neq 0$ then $\dfrac{f}{g}$ is continuous at a.
- cf is continuous at a.

It can be shown that the functions $h_1(x) = c$ and $h_2(x) = x$ are continuous everywhere. Then, by using the above laws, it can be shown that

each **polynomial** is continuous everywhere

It can also be shown that for all **polynomials** f and g, we have

$$\frac{f}{g} \text{ is continuous at each point } a \text{ such that } g(a) \neq 0$$

Two other useful results concerning continuity are as follows:

- If g is continuous at a, and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

  Recall that $f \circ g$ is the function defined by $f \circ g(x) = f(g(x))$.
- The square root function $f(x) = \sqrt{x}$ is continuous on its domain.

  Note (not examinable): a formal definition of limit only chooses values of x in the domain of the function $f(x)$, and so, for example

$$\lim_{x \to 0} \sqrt{x} = \lim_{x \to 0^+} \sqrt{x} = 0 .$$

7.4 Answers to Chapter 7 Exercises

7.1:

(a) 8 (b) 0 (c) -1

(d) 8 (e) 12 (f) $\frac{1}{4}$

7.2:

1. (a) 3 (b) 5 (c) The limit does not exist.

2. (a) -3 (b) 2 (c) 8

   (d) $\frac{5}{4}$ (e) 3 (f) The limit does not exist.

7.3:

(a) Continuous everywhere.

(b) Discontinuous at $x = 0$, since $\lim\limits_{x \to 0} f(x)$ does not exist.

(c) Continuous everywhere.

(d) Discontinuous at $x = 2$, since $\lim\limits_{x \to 2} f(x)$ does not exist.