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[MATHEMATICS 1] EXERCISE SHEET 14 : INDEFINITE INTEGRALS

1 a) $\int_0^m x^2\,dx = 72$

$\left[\frac{1}{3}x^3\right]_0^m = 72$

$\frac{1}{3}(m)^3 - \frac{1}{3}(0)^3 = 72$

$m^3 = 216$

$m = 6$

b) $\int_8^m x\,dx = 18$

$\left[\frac{1}{2}x^2\right]_8^m = 18$

$\frac{1}{2}(m)^2 - \frac{1}{2}(8)^2 = 18$

$\frac{1}{2}m^2 - 32 = 18$

$m^2 = 100$

$m = 10$

c) $\int_m^7 5\,dx = 25$

$\left[5x\right]_m^7 = 25$

$5(7) - 5(m) = 25$

$m = 2$

d) $\int_0^m (4-x)\,dx = 8$

$\left[4x - \frac{1}{2}x^2\right]_0^m = 8$

$\left(4(m) - \frac{1}{2}(m)^2\right) - \left(4(0) - \frac{1}{2}(0)^2\right) = 8$

$-\frac{1}{2}m^2 + 4m - 8 = 0$

$m^2 - 8m + 16 = 0$

$(m-4)(m-4) = 0$

$m = 4 \qquad m = 4$

$\therefore m = 4$

e) $\int_0^m e^{3x}\,dx = 21$

$\left[\frac{1}{3}e^{3x}\right]_0^m = 21$

$\frac{1}{3}e^{3m} - \frac{1}{3}e^{3(0)} = 21$

$e^{3m} = 64$

take $\ln$ of both sides:

$\ln e^{3m} = \ln 64$

$3m = \ln(2^6)$

$m = \frac{\cancel{6}^{2}\ln 2}{\cancel{3}}$

$m = 2\ln 2$

2a) i) $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

ii) $\frac{d}{dx}\left(\frac{1}{2}\sin 2x + C\right) = \cos 2x$

b) i) $\int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + C$

ii) $\frac{d}{dx}\left(\frac{1}{3}\tan 3x + C\right) = \sec^2 3x$

c) i) $\int (1+\sin x)\,dx = x - \cos x + C$

ii) $\frac{d}{dx}\left(x - \cos x + C\right) = 1 + \sin x$

3a) i) $\left(x + \frac{1}{x}\right)^2 = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)$

$= (x + x^{-1})(x + x^{-1})$

$= x^2 + x^{-2} + 1 + 1$

$= x^2 + \frac{1}{x^2} + 2$

ii) $\int \left(x + \frac{1}{x}\right)^2 dx = \int \left(x^2 + \frac{1}{x^2} + 2\right) dx$

$= \int (x^2 + x^{-2} + 2)\,dx$

$= \frac{1}{3}x^3 - x^{-1} + 2x + C$

$= \frac{1}{3}x^3 - \frac{1}{x} + 2x + C$

b) i) $\left(x^2+\frac{2}{x}\right)^3 = (x^2+2x^{-1})(x^2+2x^{-1})(x^2+2x^{-1})$

$= (x^4+4x^{-2}+4x)(x^2+2x^{-1})$

$= x^6+4+4x^3+2x^3+8x^{-3}+8$

$= x^6+6x^3+12+\frac{8}{x^3}$

ii) $\int \left(x^2+\frac{2}{x}\right)^2 dx = \int \left(x^6+6x+12+\frac{8}{x^3}\right) dx$

$= \frac{1}{7}x^7+2x^3+12x-\frac{4}{x^2}+C$

c) i) $(1+6e^{3x})^2 = (1+6e^{3x})(1+6e^{3x})$

$= 1+36e^{6x}+12e^{3x}$

ii) $\int (1+6e^{3x})^2 dx = \int (1+36e^{6x}+12e^{3x})\, dx$

$= x+6e^{6x}+4e^{3x}+C$

4a) i) $\frac{x+x^2}{2x} = \frac{x(1+x)}{2x}$

$= \frac{1+x}{2}$

ii) $\int \frac{x+x^2}{2x} dx = \int \frac{1+x}{2} dx$

$= \frac{1}{2}\int (1+x)\, dx$

$= \frac{1}{2}\left(x+\frac{1}{2}x^2+C\right)$

$= \frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{2}C$

b) i) $\frac{e^{5x}-e^{2x}}{e^{3x}} = \frac{e^{3x}-1}{e^x}$

$= e^{2x}-\frac{1}{e^x}$

ii) $\int \frac{e^{5x}-e^{2x}}{e^{3x}} dx = \int \left(e^{2x}-\frac{1}{e^x}\right) dx$

$= \frac{1}{2}e^{2x}+\frac{1}{e^x}+C$

c) i) $\left(\frac{1}{\cos 4x}\right)^2 = \left(\frac{1}{\cos 4x}\right)\left(\frac{1}{\cos 4x}\right)$

$= \frac{1}{\cos^2 4x}$

$= \sec^2 4x$

ii) $\int \left(\frac{1}{\cos 4x}\right)^2 dx = \int \sec^2 4x\, dx$

$= \frac{1}{4}\tan 4x+C$

5a) $\int (2x+1)^{20} dx = \frac{1}{2(20+1)}(2x+1)^{21}+C$

$= \frac{1}{42}(2x+1)^{21}+C$

b) $\int (2x+1)^{-1} dx = \frac{1}{2}\ln|2x+1|+C$

c) $\int \sqrt{4+9x}\, dx = \int (4+9x)^{\frac{1}{2}} dx$

$= \frac{2}{27}(4+9x)^{\frac{3}{2}}+C$

d) $\int (4+9x)^{-1} dx = \frac{1}{9}\ln|4+9x|+C$

e) $\int \sqrt[3]{6x+1}\, dx = \int (6x+1)^{\frac{1}{3}} dx$

$= \frac{1}{8}(6x+1)^{\frac{4}{3}}+C$

f) $\int \frac{1}{6x+1} dx = \frac{1}{6}\ln|6x+1|+C$

6a) $\int \frac{x+1}{x} dx = \int 1+\frac{1}{x}\, dx$

$= x+\ln|x|+C$

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b) $\int \frac{x}{x+1}\,dx = \int 1 - \frac{1}{x+1}\,dx$

$= x - \ln|x+1| + C$

7) $\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x + C\right)$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + \frac{1}{2}C$

8) a) $\int \frac{2}{4+x^2}\,dx = 2\int \frac{1}{2^2+x^2}\,dx$

$= 2\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C\right)$

$= \tan^{-1}\left(\frac{x}{2}\right) + 2C$

b) $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + C$

c) $\int \frac{-1}{\sqrt{16+x^2}}\,dx = \cos^{-1}\left(\frac{x}{4}\right) + C$

9) a) $f(x) = x$

$f_{ave} = \frac{1}{6-2}\int_2^6 x\,dx$

$= \frac{1}{4}\left[\frac{1}{2}x^2\right]_2^6$

$= \frac{1}{4}\left(\frac{1}{2}(6)^2 - \frac{1}{2}(2)^2\right)$

$= 4$

b) $f(x) = 1$

$f_{ave} = \frac{1}{6-2}\int_2^6 1\,dx$

$= \frac{1}{4}\left[x\right]_2^6$

$= \frac{1}{4}(6-2)$

$= 1$

c) $f(x) = x^3$

$f_{ave} = \frac{1}{6-2}\int_2^6 x^3\,dx$

$= \frac{1}{4}\left[\frac{1}{4}x^4\right]_2^6$

$= \frac{1}{4}\left(\frac{1}{4}(6)^4 - \frac{1}{4}(2)^4\right)$

$= 80$

10) a) i) $f(x) = \sin x$

$f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx$

$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$

$= \frac{1}{\pi}\left(-\cos(\pi) - -\cos(0)\right)$

$= \frac{2}{\pi}$

ii) $\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi}$

$= \left(-\cos(\pi) - -\cos(0)\right)$

Area $= 2$

b) i) $f(x) = \sin x$

$f_{ave} = \frac{1}{2\pi - 0}\int_0^{2\pi} \sin x\,dx$

$= \frac{1}{2\pi}\left[-\cos x\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left(-\cos(2\pi) - -\cos(0)\right)$

$= 0$

ii) Area $= A + B$


$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} \sin x\,dx$

$= \left[-\cos x\right]_0^{\pi} + \left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$

$= \left(-\cos(\pi) - -\cos(0)\right) + \left|\left(-\cos(2\pi) - -\cos(\pi)\right)\right|$

$= 2 + |-2|$

$= 4$

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[MATHEMATICS 1] EXERCISE SHEET 14: INDEFINITE INTEGRALS

1) i) $f(x) = \sin x$

$$f_{ave} = \frac{1}{2\pi - \pi} \int_{\pi}^{2\pi} \sin x \, dx$$

$$= \frac{1}{\pi}\left[-\cos x\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left(-\cos(2\pi) - -\cos(\pi)\right)$$

$$= \frac{-2}{\pi}$$

ii) $$\text{area} = \left|\int_{\pi}^{2\pi} \sin x \, dx\right|$$

$$= \left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$$

$$= \left|\left(-\cos(2\pi) - -\cos(\pi)\right)\right|$$

$$= |-2|$$

$$= 2$$

11) $D = 10 + 4\cos\left(\frac{\pi}{12}t\right)$

$$D_{ave} = \frac{1}{16-12}\int_{12}^{16}\left(10 + 4\cos\left(\frac{\pi}{12}t\right)\right) dt$$

$$= \frac{1}{4}\left[10t + \frac{48}{\pi}\sin\left(\frac{\pi}{12}t\right)\right]_{12}^{16}$$

$$= \frac{1}{4}\left(\left(10(16) + \frac{48}{\pi}\sin\left(\frac{\pi}{12}(16)\right)\right) - \left(10(12) + \frac{48}{\pi}\sin\left(\frac{\pi}{12}(12)\right)\right)\right)$$

$$= \frac{1}{4}\left(40 - \frac{48\sqrt{3}}{\pi 2}\right)$$

$$= 6.692026627$$

$$\approx 6.7m$$

12) $K = 300 + 200e^{-\frac{1}{2}t}$

a) when $t = 0$:

$$K = 300 + 200e^{-\frac{1}{2}(0)}$$

$$= 500$$

Kangaroo population at the start of 1990s: 500 000

b) when $t = 10$:

$$K = 300 + 200e^{-\frac{1}{2}(10)}$$

$$= 301.3475894$$

$$\approx 301$$

Kangaroo population at the end of 1990s: 301 000

c) $$K_{ave} = \frac{1}{10-0}\int_0^{10} 300 + 200e^{-\frac{1}{2}t} \, dt$$

$$= \frac{1}{10}\left[300t - 400e^{-\frac{1}{2}t}\right]_0^{10}$$

$$= \frac{1}{10}\left(\left(300(10) - 400e^{-\frac{1}{2}(10)}\right) - \left(300(0) - 400e^{-\frac{1}{2}(0)}\right)\right)$$

$$= 339.7304821$$

$$\approx 340$$

Average population is approximately 340 000

13) $T = 180 + 10\sin\left(\frac{\pi}{2}t\right)$

$$T_{ave} = \frac{1}{9-0}\int_0^9 180 + 10\sin\left(\frac{\pi}{2}t\right) dt$$

$$= \frac{1}{9}\left[180t - \frac{20}{\pi}\cos\left(\frac{\pi}{2}t\right)\right]_0^9$$

$$= \frac{1}{9}\left(\left(180(9) - \frac{20}{\pi}\cos\left(\frac{\pi}{2}(9)\right)\right) - \left(180(0) - \frac{20}{\pi}\cos\left(\frac{\pi}{2}(0)\right)\right)\right)$$

$$= 180.7073553$$

$$\approx 180.7$$

The average temperature is 180.7°C

14) $a(t) = 528(2t+1)^{10}$

a) $$v(t) = \int a(t) \, dt$$

$$= \int 528(2t+1)^{10} \, dt$$

$$= 24(2t+1)^{11} + C$$

when $t = 0, v = 0$:

$$0 = 24(2(0)+1)^{11} + C$$

$$C = -24$$

$$v(t) = 24(2t+1)^{11} - 24$$

b) $$x(t) = \int v(t) \, dt$$

$$= \int 24(2t+1)^{11} - 24 \, dt$$

$$= (2t+1)^{12} - 24t + C$$

when $t = 0, x = 0$:

$$0 = (2(0)+1)^{12} - 24(0) + C$$

$$C = -1$$

$$x(t) = (2t+1)^{12} - 24t - 1$$

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[MATHEMATICS I] EXERCISE SHEET 14: INDEFINITE INTEGRALS

15) $x(t) = 3\sin(10t)$

a) $v(t) = \frac{d}{dt}(x(t))$
$= \frac{d}{dt}(3\sin(10t))$
$= 30\cos(10t)$

$a(t) = \frac{d}{dt}(v(t))$
$= \frac{d}{dt}(30\cos(10t))$
$= -300\sin(10t)$

b) $v_{ave} = \frac{1}{2-0}\int_0^2 30 \quad (10t)\,dt$

$= \frac{1}{2}\left[3\sin(10t)\right]_0^2$

$= \frac{1}{2}\left(3\sin(10(2)) - 3\sin(10(0))\right)$

$= 1.369417876$

$\approx 1.37$

The average velocity during first 2 seconds is 1.37 cm/s

16) $v(t) = \frac{1}{2000}(t^3 - 90t^2 + 1800t)$

a) $a(t) = \frac{d}{dt}(v(t))$
$= \frac{d}{dt}\left(\frac{1}{2000}(t^3 - 90t^2 + 1800t)\right)$
$= \frac{1}{2000}(3t^2 - 180t + 1800)$

$x(t) = \int v(t)\,dt$
$= \frac{1}{2000}\int t^3 - 90t^2 + 1800t\,dt$
$= \frac{1}{2000}\left(\frac{1}{4}t^4 - \frac{90}{3}t^3 + \frac{1800}{2}t^2\right)$
$= \frac{1}{2000}\left(\frac{1}{4}t^4 - 30t^3 + 900t^2\right)$

b) when $t = 30$:
$x(t) = \frac{1}{2000}\left(\frac{1}{4}(30)^4 - 30(30)^3 + 900(30)^2\right)$
$= 101.25$

Height after 30 seconds is 101.25m

c) $v_{ave} = \frac{1}{30-0}\int_0^{30}\frac{1}{2000}(t^3 - 90t^2 + 1800t)\,dt$

$= \frac{1}{30}\left[\frac{1}{2000}\left(\frac{1}{4}t^4 - 30t^3 + 900t^2\right)\right]_0^{30}$

c) $v_{ave} = \frac{1}{30}\left(\frac{1}{2000}\left(\frac{1}{4}(30)^4 - 30(30)^3 + 900(30)^2\right) - \frac{1}{2000}\left(\frac{1}{4}(0)^4 - 30(0)^3 + 900(0)^2\right)\right)$

$= \frac{27}{8}$

$= 3.375$

Average velocity during first 30 seconds is 3.375 m/s.

17) GIVEN $r = 10$ cm
$\frac{dV}{dt} = (-)kr^2$ cm³/hr
when $t = 4$, $V = \frac{1}{2}V_0$
$\frac{dV}{dr} = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$
$V_0 = \frac{4}{3}\pi(10)^3$
$V_0 = \frac{4000}{3}\pi$

when $t = 4$
$V = \frac{1}{2}\times\frac{4000}{3}\pi$
$V = \frac{2000}{3}\pi$
$\frac{4}{3}\pi r^3 = \frac{2000}{3}\pi$
$r^3 = 500$
$r = \sqrt[3]{500}$

$\frac{\Delta V}{\Delta t} = \frac{\frac{2000}{3}}{4}$

$\frac{500}{3}$

19.5

$\frac{dv}{dt}\times\frac{dV}{dr}$

$\frac{dr}{dt}$

17) GIVEN $r = 10$, $\frac{dV}{dt} = kr^2$, when $t = 4$, $V = \frac{1}{2}V_0$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr}\times\frac{dr}{dt}$

$kr^2 = 4\pi r^2 \times \frac{dr}{dt}$

$(100 - \sqrt[3]{500})\pi r^2$

$\frac{dr}{dt} = (100 - \sqrt[3]{500})$

$r = \int\frac{dr}{dt}\,dt = \int(100 - \sqrt[3]{500})\,dt = (100 - \sqrt[3]{500})t$

$10 = (100 - \sqrt[3]{500})(0) + C$

$C = 10$

when $r = 0$:

$0 = (100 - \sqrt[3]{500})t + 10$

$t =$

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17) GIVEN:

initial radius = 10

$\frac{dv}{dt} \overset{(-)}{=} kr^2$ (decreasing)

when $t=4$, $v = \frac{1}{2} v_0$

$v = \frac{4}{3}\pi r^3$

INITIAL Volume:

when $r=10$:

$$v_0 = \frac{4}{3}\pi(10)^3$$

$$= \frac{4000}{3}\pi$$

WHEN $t=4$, $V_4 = \frac{1}{2} V_0$:

$$V_4 = \frac{1}{2} \times \frac{4000}{3}\pi$$

$$= \frac{2000}{3}\pi$$

FIND radius when $V_4$:

$$\frac{4}{3}\pi r^3 = \frac{2000}{3}\pi$$

$$r^3 = 500$$

$$r = \sqrt[3]{500}$$

$$\frac{dv}{dr} = \frac{dv}{dr} \times \frac{dr}{dt}$$

$$-kr^2 = 4\pi r^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = -\frac{k}{4\pi}$$

$$r(t) = \frac{-k}{4\pi} t + C$$

when $t=0$, $r=10$:

$$10 = C - \frac{k}{4\pi}(0)$$

$$c = 10$$

$$\therefore r(t) = 10 - \frac{kt}{4\pi}$$

$k = \pi(10 \sqrt[3]{500})$

when $v=0$, $r=0$

$$0 = 10 - \frac{\pi(100 - \sqrt[3]{500})t}{4\pi}$$

$$(10 - \sqrt[3]{500})t = 40$$

$$t = 19.38928841$$

It takes approximately 19 and a half hours.