[ES9] DIFFERENTIATION

Monday, May 4, 2020 6:43 AM

1. Find $f'(x)$ for each of the following functions:

(a) $f(x) = x^3$

$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$
$= \lim_{h\to 0} \frac{(x^2+2xh+h^2)(x+h) - x^3}{h}$
$= \lim_{h\to 0} \frac{(\cancel{x^3}+2x^2h+xh^2+x^2h+2xh^2+h^3)-\cancel{x^3}}{h}$
$= \lim_{h\to 0} \frac{3x^2\cancel{h}+3xh^2+h^3}{\cancel{h}}$
$= \lim_{h\to 0} 3x^2+3xh+h^2$
$= \lim_{h\to 0} 3x^2$
$= 3x^2$

(b) $f(x) = x^2$

$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x}{h}$
$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$
$= \lim_{h\to 0} \frac{2x\cancel{h}+h^2}{\cancel{h}}$
$= \lim_{h\to 0} 2x+h$
$= \lim_{h\to 0} 2x$
$= 2x$

(c) $f(x) = x^2 - 1$

$f'(x) = \lim_{h\to 0} \frac{((x+h)^2-1)-(x^2-1)}{h}$
$= \lim_{h\to 0} \frac{(x^2+2xh-1)-(x^2-1)}{h}$
$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh\cancel{-1}\cancel{-x^2}\cancel{+1}}{h}$
$= \lim_{h\to 0} \frac{2x\cancel{h}}{\cancel{h}}$
$= \lim_{h\to 0} 2x$
$= 2x$

(d) $f(x) = x - 1$

$f'(x) = \lim_{h\to 0} \frac{((x+h)-1)-(x-1)}{h}$
$= \lim_{h\to 0} \frac{\cancel{x}+h\cancel{-1}\cancel{-x}\cancel{+1}}{h}$
$= \lim_{h\to 0} \frac{h}{h}$
$= \lim_{h\to 0} 1$
$= 1$

(e) $f(x) = 2 - x^2$

$f'(x) = \lim_{h\to 0} \frac{(2-(x+h)^2)-(2-x^2)}{h}$
$= \lim_{h\to 0} \frac{2-(x^2+2xh+h^2)-2+x^2}{h}$
$= \lim_{h\to 0} \frac{\cancel{2}\cancel{-x^2}-2xh-h^2\cancel{-2}\cancel{+x^2}}{h}$
$= \lim_{h\to 0} \frac{-2x\cancel{h}-h^2}{\cancel{h}}$
$= \lim_{h\to 0} -2x$
$= -2x$

(f) $f(x) = 2x + 3$

$f'(x) = \lim_{h\to 0} \frac{2(x+h)+3-(2x+3)}{h}$
$= \lim_{h\to 0} \frac{\cancel{2x}+2h\cancel{+3}\cancel{-2x}\cancel{-3}}{h}$
$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}}$
$= \lim_{h\to 0} 2$
$= 2$

2. Find $f'(x)$ for each of the following functions:

(a) $f(x) = 4x^7$

$f'(x) = 28x^6$

(b) $f(x) = 4x^7 + 3$

$f'(x) = 28x^6$

(c) $f(x) = x^8 - x^2 + 2x$

$f'(x) = 8x^7 - 2x + 2$

(d) $f(x) = \dfrac{3}{x}$

$f'(x) = f'(3x^{-1})$
$= -3x^{-2}$
$= -\frac{3}{x^2}$

(e) $f(x) = \dfrac{1}{3}x - \dfrac{1}{\sqrt{x}} + 2$

$f'(x) = f'(\frac{1}{3}x - x^{-\frac{1}{2}} + 2)$
$= \frac{1}{3} + \frac{1}{2}x^{-\frac{3}{2}}$
$= \frac{1}{3} + \frac{1}{2x\sqrt{x}}$

(f) $f(x) = \sqrt[3]{x}$

$f'(x) = f'(x^{\frac{1}{3}})$
$= \frac{1}{3}x^{-\frac{2}{3}}$

3. Differentiate each of the following functions:

(a) $f(x) = 7x^6 - \dfrac{2}{3x^3} + 4$

$f'(x) = f'(7x^6 - \frac{2}{3}x^{-3} + 4)$
$= 42x^5 + 2x^{-4}$
$= 42x^5 + \frac{2}{x^4}$

(b) $f(x) = 12x^2 - 6\sqrt{x}$

$f'(x) = f'(12x^2 - 6x^{\frac{1}{2}})$
$= 24x - 3x^{-\frac{1}{2}}$
$= 24x - \frac{3}{\sqrt{x}}$

(c) $f(x) = \dfrac{x^2}{1 - x^3}$ u v

$f'(x) = \frac{(1-x^3)(2x)-(x^2)(-3x^2)}{(1-x^3)^2}$
$= \frac{2x - 2x^4 + 3x^4}{(1-x^3)^2}$
$= \frac{2x + x^4}{(1-x^3)^2}$

(d) $f(x) = \dfrac{4x+1}{2x^2+x+3}$ u v

$f'(x) = \frac{(2x^2+x+3)(4)-(4x+1)(4x+1)}{(2x^2+x+3)^2}$
$= \frac{8x^2+4x+12-(16x^2+8x+1)}{(2x^2+x+3)^2}$
$= \frac{-8x^2-4x+11}{(2x^2+x+3)^2}$

(e) $f(x) = (4x^3 - 3x + 2)^6$

$f'(x) = 6(4x^3 - 3x + 2)^5(12x - 3)$

(f) $f(x) = (2x+5)(3x+7)^6$ u v

$f'(x) = u\,dv + v\,du$
$= (2x+5)(6)(3x+7)^5(3) + (3x+7)^6(2)$
$= (36x+90)(3x+7)^5 + (3x+7)^5(6x+14)$
$= (42x+104)(3x+7)^5$
$= 2(3x+7)^5(21x+52)$

4. Differentiate each of the following functions:

(a) $y = \sin 4x - 3\cos 2x$

$\frac{dy}{dx} = 4\sin 4x + 6\sin 2x$

(b) $y = \tan x - 3\cos x$

Answers

1. (a) $f'(x) = 3x^2$ (b) $f'(x) = 2x$ (c) $f'(x) = 2x$
(d) $f'(x) = 1$ (e) $f'(x) = -2x$ (f) $f'(x) = 2$

2. (a) $f'(x) = 28x^6$ (b) $f'(x) = 28x^6$ (c) $f'(x) = 8x^7 - 2x + 2$
(d) $f'(x) = -\dfrac{3}{x^2}$ (e) $f'(x) = \dfrac{1}{3} + \dfrac{1}{2}x^{-\frac{3}{2}}$ (f) $f'(x) = \dfrac{1}{3}x^{-\frac{2}{3}}$

3. (a) $f'(x) = 42x^5 + \frac{2}{x^4}$ (b) $f'(x) = 24x - \frac{3}{\sqrt{x}}$
(c) $f'(x) = \frac{2x+x^4}{(1-x^3)^2}$ (d) $f'(x) = \frac{-8x^2-4x+11}{(2x^2+x+3)^2}$
(e) $f'(x) = 6(4x^3-3x+2)^5(12x^2-3)$ (f) $f'(x) = 2(3x+7)^5(21x+52)$

Comment for Question 3(f): When we use the Product Rule, we obtain $f'(x) = 2(3x+7)^6 + (2x+5)18(3x+7)^5$.
This can be simplified by bringing out the common factor of $2(3x+7)^5$.
We obtain $f'(x) = 2(3x+7)^5[(3x+7) + 9(2x+5)] = 2(3x+7)^5(21x+52)$.

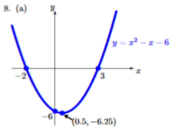
4. (a) $\frac{dy}{dx} = 4\cos 4x + 6\sin 2x$ (b) $\frac{dy}{dx} = \sec^2 x + 3\sin x$
(c) $\frac{dy}{dx} = 8\sin x\cos x + 3\sec^2 x$ (d) $\frac{dy}{dx} = -6\sin x\cos x(\cos x + 2\sin^2 x)$
(e) $\frac{dy}{dx} = \frac{\cos x}{2\sqrt{\sin x}}$ (f) $\frac{dy}{dx} = \frac{-\sin x}{2(1+\cos x)^{\frac{1}{2}}}$
(g) $\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$ (h) $\frac{dy}{dx} = 2x\sin(x^2)\sec^2(x^2)$

5. (a) $f'(x) = \sin x + x\cos x$ (b) $f'(x) = 6x\cos x - 3x^2\sin x$
(c) $f'(x) = 3\tan x + (3x+1)\sec^2 x$ (d) $f'(x) = 5\cos(5x+2)$
(e) $f'(x) = -0.8x^{-0.6}\sin(2x^{0.4}+3)$ (f) $f'(x) = 4x^3\sec^2(x^4)$
(g) $f'(x) = \sin(5x+2) + 5x\cos(5x+2)$ (h) $f'(x) = \cos x\tan(x^4) + 4x^3\sin x\sec^2(x^4)$
(i) $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$ (j) $f'(x) = \frac{6x\cos x + 3x^2\sin x}{\cos^2 x}$
(k) $f'(x) = \frac{x\cos(x+2) - \sin(x+2)}{x^2}$ (l) $f'(x) = \frac{4x^3(x^3-2)\sec^2(x^4) - 3x^2\tan(x^4)}{(x^3-2)^2}$

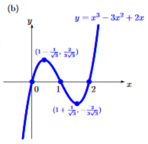
6. (a) $-1, -6, \frac{1}{3}$ (b) $6 - 2\sqrt{5}$

7. $[-3, 6]$

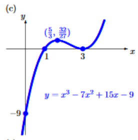
8. (a)



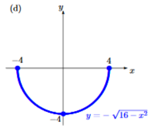
(b)



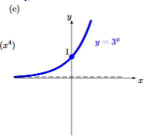
(c)



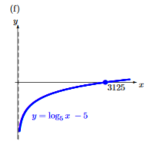
(d)



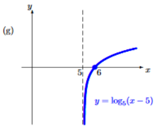
(e)



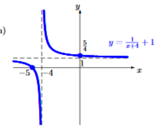
(f)



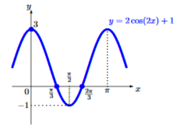
(g)



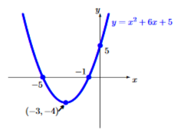
(h)



(i)



(j)



9. (a) $f(g(x)) = \sqrt{2-2x}$ (b) $x \in [-4, -1]$

10. (a) $f^{-1}(x) = \frac{1}{3}(x-2)$ $\text{dom}(f^{-1}) = \mathbf{R}$ $\text{ran}(f^{-1}) = \mathbf{R}$
(b) $f^{-1}(x) = \frac{1}{3}(x-2)$ $\text{dom}(f^{-1}) = (2, \infty)$ $\text{ran}(f^{-1}) = \mathbf{R}^+$
(c) $f^{-1}(x) = \frac{1}{3}(x-2)$ $\text{dom}(f^{-1}) = [20, \infty)$ $\text{ran}(f^{-1}) = [6, \infty)$

11. (a) 1 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{2}$

(c) $y = 4\sin^2 x + 3\tan x$

(d) $y = 2\cos^3 x - 3\sin^4 x$

(e) $y = \sqrt{\sin x}$

(f) $y = (1 + \cos x)^{\frac{1}{2}}$

(g) $y = \frac{\sin x}{x}$

(h) $y = \sec(x^2)$

5. Differentiate each of the following functions:

(a) $f(x) = x\sin x$

(b) $f(x) = 3x^2 \cos x$

(c) $f(x) = (3x + 1)\tan x$

(d) $f(x) = \sin(5x + 2)$

(e) $f(x) = \cos(2x^{0.4} + 3)$

(f) $f(x) = \tan(x^4)$

(g) $f(x) = x\sin(5x + 2)$

(h) $f(x) = \sin x \tan(x^4)$

(i) $f(x) = \frac{x}{\sin x}$

(j) $f(x) = 3x^2 \sec x$

(k) $f(x) = \frac{\sin(x + 2)}{x}$

(l) $f(x) = \frac{\tan(x^4)}{x^3 - 2}$

6. [Revision from Chapter 1]

(a) Solve $3x^3 + 11x = -2(10x^2 - 3)$ for x.

(b) Solve $x + 2\sqrt{x} = 4$ for x.

7. **[Revision from Chapter 3]**

Find the domain of the function $f(x) = \sqrt{3 - \sqrt{3 + x}}$.

8. **[Revision from Chapters 3–4]**

Sketch a graph for each of the following functions:

(a) $y = x^2 - x - 6$

(b) $y = x^3 - 3x^2 + 2x$

(c) $y = x^3 - 7x^2 + 15x - 9$

(d) $y = -\sqrt{16 - x^2}$

(e) $y = 3^x$

(f) $y = \log_5 x - 5$

(g) $y = \log_5(x - 5)$

(h) $y = \dfrac{1}{x + 4} + 1$

(i) $y = 2\cos(2x) + 1$

(j) $y = x^2 + 6x + 5$

9. **[Revision from Chapter 6]**

Consider the functions

$$f : [5, 17] \longrightarrow \mathbf{R} \quad \text{where } f(x) = \sqrt{x - 1}$$

and

$$g : [-4, 4] \longrightarrow \mathbf{R} \quad \text{where } g(x) = 3 - 2x.$$

(a) Find the rule for $f(g(x))$.

(b) For which values of x is $f(g(x))$ defined?

10. **[Revision from Chapter 6]**
Completely determine the inverse of each of the following functions. That is, for each of the following functions, find $\mathrm{dom}(f^{-1})$, $\mathrm{ran}(f^{-1})$ and the rule for f^{-1}.

(a) $f : \mathbf{R} \longrightarrow \mathbf{R}$ where $f(x) = 3x + 2$

(b) $f : \mathbf{R}^+ \longrightarrow \mathbf{R}$ where $f(x) = 3x + 2$

(c) $f : [6, \infty) \longrightarrow \mathbf{R}$ where $f(x) = 3x + 2$.