Chapter 10

Applications of Differentiation

Reference: "Calculus", by James Stewart.

10.1 Rates of Change

Recall that $\dfrac{dy}{dx}$ is the (instantaneous) **rate of change** of y with respect to x .

That is,

a rate of change is given by the **derivative**.

For example,

the rate of change of V with respect to r is given by $\dfrac{dV}{dr}$.

Similarly,

the rate of change of s with respect to t is given by $\dfrac{ds}{dt}$.

Rates of change **with respect to time** will be used so frequently that we shall often leave out the phrase "with respect to time". Whenever we see a question in which we are **not** told what the rate of change is with respect to, then we should assume that the rate is **with respect to time**.

For example,

$$\text{the rate of change of } s \text{ is given by } \frac{ds}{dt}\,.$$

Example 1. A spherical balloon is being inflated with air. Find the rate of change of the volume of the balloon with respect to its radius, when the radius is 4 cm.

Solution. Let r and V denote the radius and volume of the balloon respectively.
We want to find $\dfrac{dV}{dr}$ when $r = 4$ cm. The volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$.

(This formula is given on the Formula Sheet provided in the Maths 1 exams.)

$$\begin{aligned} \text{Thus } \frac{dV}{dr} &= \frac{4}{3}\pi\, 3r^2 \\ &= 4\pi r^2 \\ \text{So when } r = 4 \text{ cm, we have } \frac{dV}{dr} &= 4\pi 4^2 \\ &= 64\pi. \end{aligned}$$

That is, when $r = 4$ cm, the rate of change of the volume with respect to the radius is $64\pi\ \mathrm{cm}^3 \cdot \mathrm{cm}^{-1}$.

If a quantity is **decreasing**, then its derivative is **negative**.

Example 2. Suppose that the volume V of water left in a leaking tank is **decreasing** at a rate of $1\ \mathrm{cm}^3$ per second. Then $\dfrac{dV}{dt} = -1\ \mathrm{cm}^3 \cdot \mathrm{s}^{-1}$.

Example 3. Suppose that the distance x between the Earth and the sun **decreases** at a rate of 3 km per year. Then $\dfrac{dx}{dt} = -3\ \mathrm{km} \cdot \mathrm{year}^{-1}$.

Exercises

1. A circular oil slick is forming in a pool of water. Find the rate of increase of the area of the oil slick with respect to its radius,

   (a) when the radius is 5 cm.

   (b) when the area of the slick is $36\pi\ \text{cm}^2$.

10.2 Related Rates of Change

In this section we shall use the **Chain Rule** to find rates of change.

Example 4. A spherical balloon is being inflated and its radius is increasing at the constant rate of $3\ \mathrm{cm}\cdot\mathrm{min}^{-1}$. At what rate is its volume increasing when the radius of the balloon is 5 cm?

Solution: Let V be the balloon's volume in cm^3, and let r be the balloon's radius in cm.

Told: $\dfrac{dr}{dt} = 3\ \mathrm{cm}\cdot\mathrm{min}^{-1}$.

Know: For a sphere, we have $V = \dfrac{4}{3}\pi r^3$. Thus $\dfrac{dV}{dr} = 4\pi r^2$.

Find: $\dfrac{dV}{dt}$ when $r = 5$ cm.

By the chain rule, we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr}\,\frac{dr}{dt}\\ &= 4\pi r^2 \times 3\\ &= 12\pi r^2 .\end{aligned}$$

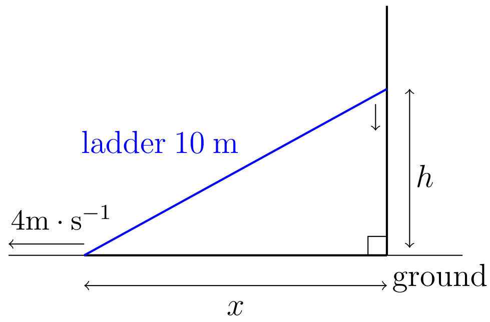
Thus when $r = 5$ cm we have

$$\frac{dV}{dt} = 12\pi \times 5^2 = 300\pi.$$

That is, when $r = 5$ cm, the volume is increasing at a rate of $300\pi\ \mathrm{cm}^3\cdot\mathrm{min}^{-1}$. □

Example 5. A 10 m long ladder has its upper end against a vertical wall, and its lower end on the horizontal ground. The lower end is slipping away from the wall at a constant rate of $4 \text{ m} \cdot \text{s}^{-1}$. Find the rate at which the upper end of the ladder is slipping down the wall when the lower end is 6 m from the wall.

Solution:



Let $x =$ distance between the lower end of the ladder and the wall (in m).
Let $h =$ height of the ladder (in m).

By the Chain Rule, we have

$$\frac{dh}{dt} = \frac{dh}{dx}\frac{dx}{dt} = \frac{dh}{dx} \times 4 .$$

To find $\frac{dh}{dx}$, we need to find a relationship between h and x .
By Pythagoras's Theorem, we have
$h^2 + x^2 = 100$ and so $h = (100 - x^2)^{\frac{1}{2}}$. Therefore,

$$\begin{aligned} \frac{dh}{dx} &= \frac{1}{2}(100 - x^2)^{-\frac{1}{2}} \times (-2x) \\ &= \frac{-x}{\sqrt{100 - x^2}} . \end{aligned}$$

Now $\frac{dh}{dt} = \frac{dh}{dx}\frac{dx}{dt} = \frac{dh}{dx} \times 4 = \frac{-x}{\sqrt{100 - x^2}} \times 4 .$

When $x = 6$,

$$\frac{dh}{dt} = \frac{-6}{\sqrt{100 - 36}} \times 4 = \frac{-6}{\sqrt{64}} \times 4 = \frac{-24}{8} = -3 .$$

Note: Since $\dfrac{dh}{dt} < 0$, we know that h is *decreasing*.

Finally, we answer with a **sentence**:

When $x = 6$ m, the top end of the ladder is slipping down the wall at $3\ \mathrm{m \cdot s^{-1}}$. $\square$

Important Tips for (Related) Rates of Change Problems

Tip 1: Do **not** replace variables with numerical values until after differentiating. (Note that **constants** can be replaced with their numerical values at any time.)

Tip 2: **Before** we can find a derivative, we need to have an appropriate equation in which the **only two variables** are the two variables involved in the differentiation.

For example, to find $\dfrac{dA}{dr}$ we need an equation in which A and r are the only two variables.

In particular, suppose A is the area of a circle. Then

$$A = \pi r^2.$$

Since A and r are the only two variables in this equation, we can find $\dfrac{dA}{dr}$ immediately. (We obtain $\dfrac{dA}{dr} = 2\pi r$.)

In contrast however, suppose A is the surface area of a cylinder. Then

$$A = 2\pi r^2 + 2\pi r h.$$

This equation contains h, as well as A and r. In this case, we can only find $\dfrac{dA}{dr}$ if

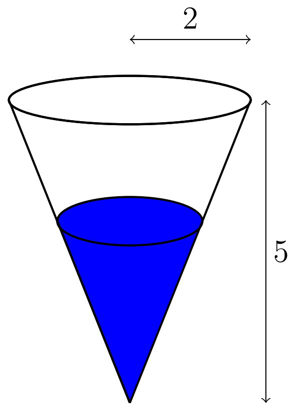
- we know that h is **constant**,

or if

- we eliminate h from our equation (using information given in the problem).

Example 6.

Consider an inverted right circular cone which has base radius 2 m and height 5 m. Suppose the cone contains some water, and suppose the water is leaking from the apex of the cone at the constant rate of $0.2 \text{ m}^3 \cdot \text{min}^{-1}$. Find the rate at which the water level is dropping when the depth of the water is 4 m. (Write your answer to 3 decimal places.)



Solution: Let h be the depth of the water,

let r be the radius of the water's surface, and

let V be the volume of the water contained in the cone.

We are told that $\dfrac{dV}{dt} = -0.2 \text{ m}^3 \cdot \text{min}^{-1}$.

We want to find $\dfrac{dh}{dt}$ when $h = 4$ m .

By the Chain Rule,

$$\frac{dh}{dt} = \frac{dh}{dV}\,\frac{dV}{dt}$$

$$= \frac{dh}{dV} \times -0.2$$

Before we can find $\dfrac{dh}{dV}$, we need an equation containing h and V

(and **no other variables**).

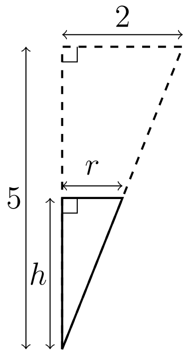
Note that the volume of the water in the cone is given by

$$V = \frac{1}{3}\pi r^2 h.$$

(Note: This formula is given on the formula sheet which is provided in the Maths 1 exams.)

Unfortunately, this formula contains an r (as well as an h and a V).

Before we can find $\dfrac{dh}{dV}$ we need to get rid of the r :



Using similar triangles, we find that

$$\frac{r}{h} = \frac{2}{5}$$
$$\therefore r = \frac{2}{5}h$$

$$\begin{aligned}
\text{Now } V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h \\
&= \frac{1}{3} \times \frac{4\pi}{25}h^3 \\
&= \frac{4\pi}{75}h^3 \\
\text{So } \frac{dV}{dh} &= \frac{4\pi h^2}{25} \\
\therefore \frac{dh}{dV} &= \frac{25}{4\pi h^2}
\end{aligned}$$

From the chain rule on the previous page we have

$$\begin{aligned}
\frac{dh}{dt} &= \frac{25}{4\pi h^2} \times (-0.2) \\
&= \frac{-5}{4\pi h^2}
\end{aligned}$$

When $h = 4$ we obtain

$$\begin{aligned}
\frac{dh}{dt} &= \frac{-5}{4\pi 4^2} \\
&= \frac{-5}{64\pi} \\
&= -0.025 \text{ (3 decimal places.)}
\end{aligned}$$

When the depth is 4 m, the water level is *dropping* by $0.025 \text{ m} \cdot \text{min}^{-1}$.

$\square$

Exercises

1. A spherical balloon is expanding in such a way that its radius is increasing at a rate of $0.5 \text{ cm} \cdot \text{s}^{-1}$.

   At what rate is the balloon's volume increasing when the radius is 10 cm?

2. The sides of a cube are increasing at a rate of $0.2 \text{ cm} \cdot \text{s}^{-1}$. At what rate is the volume of the cube increasing when the sides are 8 cm long?

3. A circular oil slick is forming on a pool of water in such a way that its radius is increasing at a rate of $2 \text{ cm} \cdot \text{min}^{-1}$.

   (a) At what rate is the area of the oil slick increasing when its radius is 5 cm?

   (b) Show that the circumference of the oil slick is increasing at a constant rate.

4. Sand is being poured into a conical pile in such a way that the height of the pile is equal to its diameter.

   (a) If the sand is being poured so that the radius of the pile is increasing at a rate of $3 \text{ cm} \cdot \text{min}^{-1}$, find the rate of increase of the volume when the radius is 6 cm.

   (b) If the sand is being poured at a rate of $10 \text{ cm}^3 \cdot \text{min}^{-1}$, at what rate is the radius increasing when the radius is 6 cm?

5. When a particular bowl contains water to a depth of h cm, the volume V cm^3 of the water is given by $V = \pi h^2(10 - \frac{1}{3}h)$. Suppose that water is being poured into the bowl at a constant rate of $10 \text{ cm}^3 \cdot \text{s}^{-1}$. Find the rate at which the water level is rising when the depth of the water is 5 cm.

6. Air is being pumped into a spherical balloon at a rate of $36 \text{ cm}^3 \cdot \text{min}^{-1}$. When the radius is 20 cm,

   (a) at what rate is the radius increasing?

   (b) at what rate is the surface area increasing?

7. A child is standing on a pier and is pulling in a boat by means of a rope, which is being hauled in at the rate of $0.4 \text{ m} \cdot \text{s}^{-1}$. If the child's hands are 2 m above the level of the boat, at what rate is the boat approaching the pier when there is still 5 m of rope out?

10.3 Equations of Tangents and Normals

Recall that

the **gradient of the tangent** to $y = f(x)$ is given by $f'(x)$.

This means that

the **gradient of the curve** of $y = f(x)$ is also given by $f'(x)$.

Example 7. Find the gradient of the curve $y = x^2 - x - 6$ when $x = 3$.

Solution: The gradient (or slope) of the curve is given by

$$\frac{dy}{dx} = 2x - 1.$$

Thus when $x = 3$, the gradient is equal to $2 \times 3 - 1 = 5$. $\square$

Example 8. For the function $y = x^3 - 6x$, find the x –coordinates of the points where the gradient is 6 .

Solution: The gradient is equal to 6 when $\frac{dy}{dx} = 6$.

$$\begin{aligned} & \frac{dy}{dx} = 6 \\ \iff \quad & 3x^2 - 6 = 6 \\ \iff \quad & 3x^2 = 12 \\ \iff \quad & x^2 = 4 \\ \iff \quad & x = \pm 2 \end{aligned}$$

$\square$

Recall that the rule of a straight line has the general form

$$\boxed{y = mx + c}$$

where m is the **slope** (or gradient) of the line.

Example 9. Find the equation of the tangent to the curve $y = x^4 - x^3 + x^2 - x + 1$ at the point $(1, 1)$.

Solution: First we find the value of m :
The slope is given by $\dfrac{dy}{dx} = 4x^3 - 3x^2 + 2x - 1$.

Thus when $x = 1$ we have $m = 4 \times 1^3 - 3 \times 1^2 + 2 \times 1 - 1 = 2$.

Thus the tangent at $x = 1$ has equation of the form

$$y = 2x + c.$$

Next we find the value of c :
Substituting the x –value and y –value for the point $(1, 1)$ into the tangent's equation gives $1 = 2 \times 1 + c.$

Thus $c = 1 - 2 = -1$.
Thus the tangent's equation at the point $(1, 1)$ is $y = 2x - 1.$ □

Definition: The **normal** to a curve at a point P is the line through P which is **perpendicular to the tangent** at P .

To find the slope of a normal, we make use of the following fact:

- two lines are perpendicular if the product of their gradients is -1 .

In particular, we have that

$$\text{the slope of the tangent} \times \text{the slope of the normal} = -1.$$

That is,

$$\text{the slope of the normal} = \frac{-1}{\text{the slope of the tangent}} .$$

Example 10. Find the equations of the tangent and the normal to the curve $y = x^2 - 4x + 4$ at the point where $x = 3$.

Solution: First note that $x = 3 \implies y = 3^2 - 4(3) + 4 = 1$, and so the point is $(3, 1)$.

Tangent.
The slope of the tangent to the curve is given by

$$\frac{dy}{dx} = 2x - 4 .$$

At the point $(3, 1)$, the tangent has slope $m = 2(3) - 4 = 2$.
Now, the equation of the tangent is

$$y = 2x + C .$$

Since the tangent passes through $(x, y) = (3, 1)$, we must have

$$1 = 2(3) + C$$
$$C = -5 .$$

Therefore, at the point $(3, 1)$, the equation of the **tangent** to $y = x^2 - 4x + 4$ is

$$y = 2x - 5 .$$

Normal.
At the point $(3, 1)$, the slope of the normal is

$$\frac{-1}{m} = -\frac{1}{2} .$$

Thus the equation of the normal is

$$y = -\frac{1}{2}x + C_2 .$$

Since the normal passes through $(x, y) = (3, 1)$, we must have

$$1 = -\frac{1}{2}(3) + C_2$$
$$C_2 = \frac{5}{2} .$$

Therefore, at the point $(3, 1)$, the equation of the **normal** to $y = x^2 - 4x + 4$ is

$$y = -\frac{1}{2}x + \frac{5}{2} .$$

□

Exercises

1. Find the equations of the tangents to the parabola $y = 4x - 3x^2$ at the points where the parabola cuts the x –axis.

2. Find the equations of the tangent and the normal to $y = \cos 3x$ at the point where $x = \dfrac{\pi}{6}$.

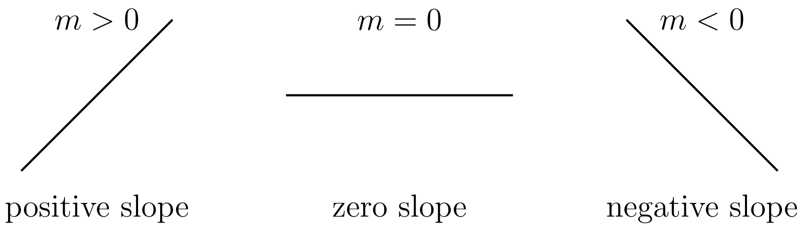
3. Find the equation of the tangent to the curve $x^2 + 2xy - 2y^2 = -12$ at the point $(2, 4)$.

4. The curve with equation $y = ax^2 + bx + c$ passes through the point $(1, 7)$ and has a gradient of 2 at the point $(2, 7)$. Find the values of a, b, c .

10.4 Curve Sketching

Recall that straight lines could have

- positive slope, or
- negative slope, or
- zero slope.



In this section we will be interested in the slopes of curves in general. We will be using the fact that

the slope of a curve at a point is given by the **derivative** at that point.

Basic Definitions

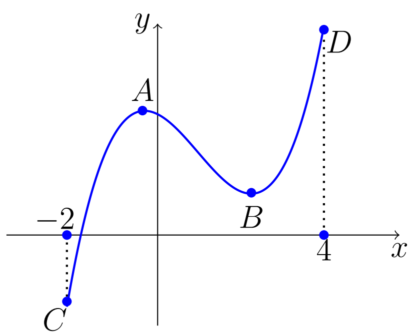
Suppose that f is a differentiable function.

- If $f'(x) > 0$ for all x in an interval then we say that f is **increasing** on that interval.

- If $f'(x) < 0$ for all x in an interval then we say that f is **decreasing** on that interval.

For example, the following graph is

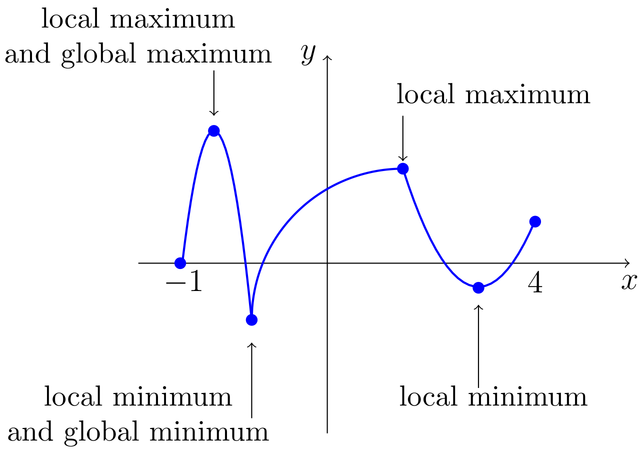
- **increasing** on the sections CA and BD, and
- **decreasing** on the section AB.



Consider the graph given above.

- A is said to be a **local maximum**.
- B is said to be a **local minimum**.
- A and B are called **stationary points**.
- C is said to be the **global minimum** on the interval $[-2, 4]$.
- D is said to be the **global maximum** on the interval $[-2, 4]$.

Example 11.



We say that a function f has a **critical point** at $x = c$ if

- $f'(c) = 0$

or if

- c is a value in the domain of f such that $f'(c)$ does not exist.

The pictures below show critical points for which the derivative is zero:

The pictures below show critical points for which the derivative does not exist:

Note: If $f'(c) = 0$, then we also say that f has a **stationary point** at $x = c$.

Example 12. Find the x –coordinates of all critical points of

$$f(x) = x^2 - 3x + 2.$$

Solution: Differentiating the given function gives $f'(x) = 2x - 3$.

(i) If $f'(x) = 0$ then

$$\begin{aligned} 2x - 3 &= 0 \\ \text{i.e. } x &= \frac{3}{2} \end{aligned}$$

Thus there is a critical point at $x = \frac{3}{2}$.

(ii) **Note:** We should also consider whether there are any x –values for which $f'(x)$ does not exist. We know that $f'(x) = 2x - 3$, which "makes sense" for **all** x –values. That is, in this example (and in most examples) $f'(x)$ **always** exists.

Thus the only critical point occurs when $x = \frac{3}{2}$. □

Example 13. Find the x –coordinates of all critical points of $f(x) = \sqrt[3]{x}$.

Solution: Since $f(x) = x^{\frac{1}{3}}$ then

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} .$$

(i) If $f'(x) = 0$ then

$$\frac{1}{3x^{\frac{2}{3}}} = 0.$$

That is, we have $1 = 0$, which is clearly impossible!

Thus we **cannot** put $f'(x) = 0$.

(ii) **Note:** We should also consider whether there are any x –values for which $f'(x)$ does not exist. We see that $f'(x)$ is not defined when $x = 0$ (since we cannot have 0 in the denominator of a fraction). Moreover, the value 0 **is** in the domain of f. Thus there is a critical point at $x = 0$.

So, in this example, the only critical point occurs when $x = 0$. □

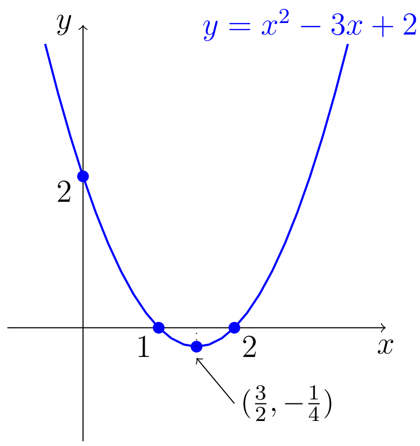
Fermat's Theorem:

If $f : [\,a,\, b\,] \longrightarrow \mathbf{R}$ has a local maximum or a local minimum at $c \in (a,\, b)$ then f has a critical point at $x = c$.

Thus, to find the local maxima and local minima of a function we should look at the critical points of that function.

Example 14. Consider the function $f(x) = x^2 - 3x + 2$.

From the graph, we see that there is a local minimum at $x = \frac{3}{2}$. This occurs at the critical point which we found in Example 12.



Exercises

1. The graph of $y = x^3 + px^2 + qx + r$ has a stationary point at $(2, -10)$ and a y–intercept of 2. Find p, q and r.

2. The graph of $y = ax^3 + bx^2 + cx + d$ has a stationary point at $(-1, 0)$. It cuts the y–axis at $y = -3$ and at this point it is parallel to the line $5x + y = 4$. Calculate the values of a, b, c and d.

3. Sketch the graph of a continuous function f which satisfies the following conditions:

   - $f(1) = 0$;
   - $f'(1)$ does not exist;
   - $f'(x) < 0$ for $x < 1$; and
   - $f'(x) > 0$ for $x > 1$.

4. Sketch the graph of a continuous function f which satisfies the following conditions:

   - $f(-1) = 0$ and $f(1) = 0$;
   - $f'(0) = 0$ and $f'(2) = 0$;
   - $f'(x) > 0$ for $x < 0$;
   - $f'(x) < 0$ for $0 < x < 2$, and
   - $f'(x) < 0$ for $x > 2$.

First Derivative Test

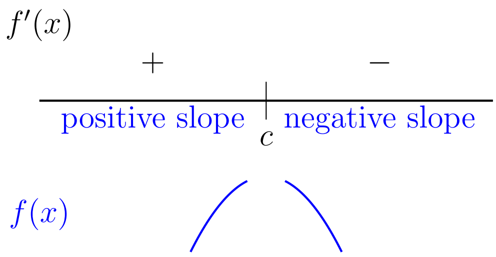
Local Maximum:

Suppose that a continuous function f has a **critical** point at $x = c$. (That is, suppose that $f'(c) = 0$ or that $f'(c)$ does not exist.)

If

- $f'(x) > 0$ for all x immediately to the left of c, **and** if
- $f'(x) < 0$ for all x immediately to the right of c

then f has a local maximum at $x = c$.



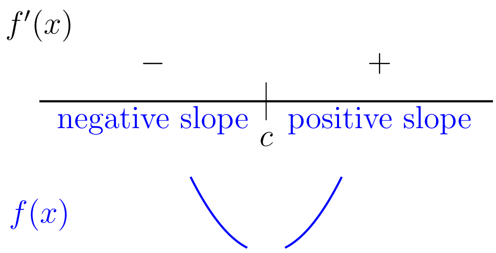
Local Minimum:

Suppose that a continuous function f has a **critical** point at $x = c$.

If

- $f'(x) < 0$ for all x immediately to the left of c, **and** if
- $f'(x) > 0$ for all x immediately to the right of c

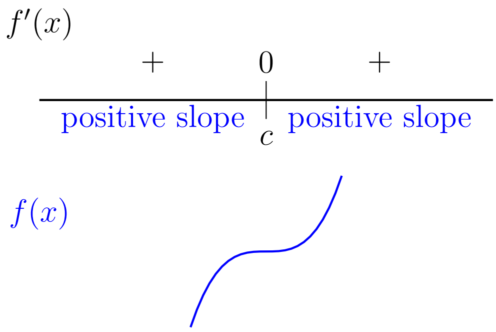
then f has a local minimum at $x = c$.



Stationary Point of Inflection:

Suppose that a continuous function f has a **stationary** point at $x = c$. (That is, suppose that $f'(c) = 0$.) If

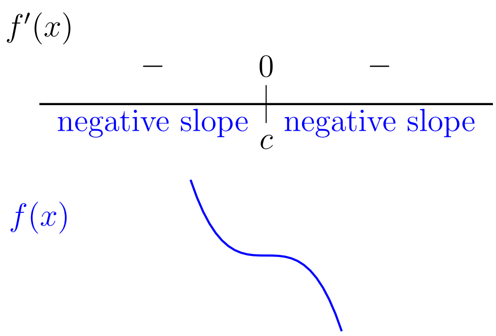
- $f'(x) > 0$ for all x immediately to the left of c, **and** if
- $f'(x) > 0$ for all x immediately to the right of c



or if

- $f'(x) < 0$ for all x immediately to the left of c, **and** if
- $f'(x) < 0$ for all x immediately to the right of c

then f has a stationary point of inflection at $x = c$.



Example 15. Sketch the graph of $y = x^3 - x^2 - x + 1$ by finding

- the x and y–intercepts;
- the coordinates of all the critical points; and
- the nature of all the critical points.

Solution:

x–intercepts: Factorising gives $y = (x-1)^2(x+1)$.

So if $y = 0$ then $x = 1, \ -1$.

y–intercept: If $x = 0$ then $y = 0^3 - 0^2 - 0 + 1 = 1$.

critical points: $\dfrac{dy}{dx} = 3x^2 - 2x - 1 = (3x+1)(x-1)$.

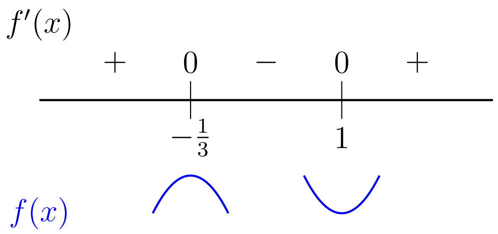
So $\dfrac{dy}{dx} = 0$ if $x = -\dfrac{1}{3}, \ 1$.

If $x = -\frac{1}{3}$ then $y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1 = \frac{32}{27}$.

If $x = 1$ then $y = 1 - 1 - 1 + 1 = 0$.

So the critical points are $\left(-\dfrac{1}{3}, \dfrac{32}{27}\right)$ and $(1, 0)$.

nature of the critical points:

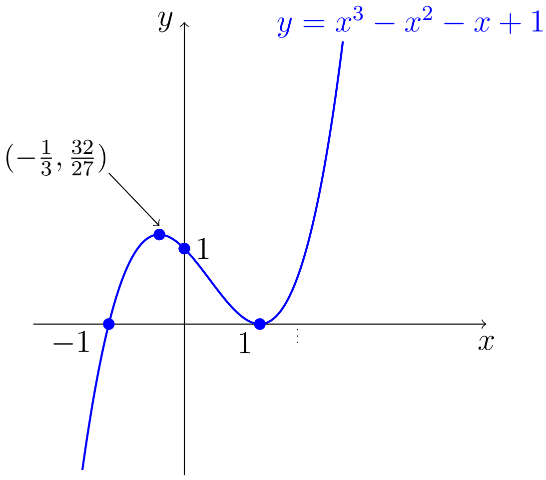


From the diagram we see that:

- If $x < -\frac{1}{3}$, then $f'(x) > 0$.
- If $x \in (-\frac{1}{3}, 1)$, then $f'(x) < 0$.
- If $x > 1$, then $f'(x) > 0$.

So by the first derivative test, there is a local maximum at $(-\frac{1}{3}, \frac{32}{27})$, and a local minimum at $(1, 0)$.

We now use this to sketch a graph of $y = f(x)$.



$\square$

In the next example, we see how to sketch a graph of $y = f'(x)$, when we are given a graph of $y = f(x)$. The steps to be used in such examples are as follows:

Step 1: Find the points where the slope of the given graph is **zero**.

At these points we have $f'(x) = 0$, and so we draw $y = 0$.

Step 2: Find any points where the given graph is **not** differentiable.

(For example, find any points for which the given graph is **not** continuous,

or where there is a sharp point.)

At these points $f'(x)$ does not exist, and so the graph $y = f'(x)$ has a hole

in it, and possibly a sudden jump.

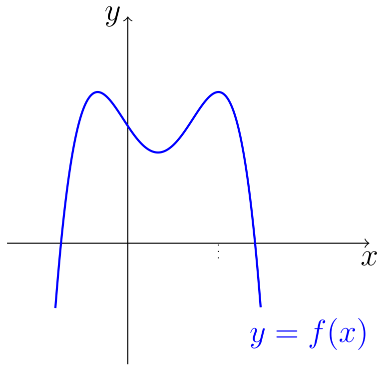
Step 3: Consider each "section" of the given graph (between the points found in Steps 1 and 2).

- If the slope of the given graph is **positive** (i.e. if $f'(x) > 0$), then draw **positive** y–values in that section.

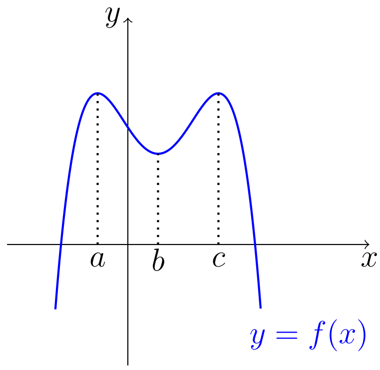
- If the slope of the given graph is **negative** (i.e. if $f'(x) < 0$), then draw **negative** y–values in that section.

Note that we shall not be concerned with the exact shape of our final graph, as long as the above three steps have been followed.

Example 16. The graph of $y = f(x)$ is drawn below. Sketch $y = f'(x)$.



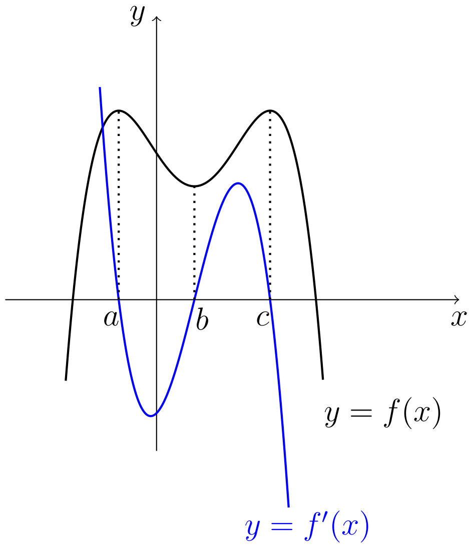
Solution: We first note that $f'(x) = 0$ whenever the tangent to $y = f(x)$ is horizontal. We mark the x –coordinates of these points on the graph as a , b and c .



Note that :

- $f'(x) = 0 \iff x = a, b, c$.
- f is increasing when $x < a$. Therefore $f'(x) > 0$ when $x < a$.
- f is decreasing on the interval (a, b) . Therefore $f'(x) < 0$ when $x \in (a, b)$.
- f is inreasing on the interval (b, c) . Therefore $f'(x) > 0$ when $x \in (b, c)$.
- f is decreasing when $x > c$. Therefore $f'(x) < 0$ when $x > c$.

Using this information, we sketch a graph of $y = f'(x)$.



$\square$

Exercises

5. For each of the following functions

   - find the x–coordinates of all the critical points, and
   - find the nature of all the critical points.

   (a) $f(x) = (x-2)^3$ (b) $f(x) = x^2(x+4)$

   (c) $f(x) = (x^2-4)^2$ (d) $f(x) = x + \dfrac{1}{x}$

6. Sketch the graph of $y = f(x)$ for each of the following functions by finding

   - the x and y–intercepts;
   - the coordinates of all the stationary points; and
   - the nature of the stationary points.

   (a) $f(x) = 2x^3 + 3x^2 - 12x + 7$ (b) $f(x) = x^3(2-x)$

   (c) $f(x) = x^2(1-x)$

Second Derivatives

The **second derivative** of a function f is obtained by differentiating $f'(x)$. We denote the second derivative of f by f'' or by $\dfrac{d^2y}{dx^2}$.

Example 17. Find $f''(x)$ for the following functions:

(a) $f(x) = 2x^3 + 3x^2 - x + 7$

Solution:

$$\begin{aligned} f'(x) &= 6x^2 + 6x - 1 \\ \therefore f''(x) &= 12x + 6 \end{aligned}$$

□

(b) $f(x) = \sin x + e^{3x}$

Solution:

$$\begin{aligned} f'(x) &= \cos(x) + 3e^{3x} \\ \therefore f''(x) &= -\sin(x) + 9e^{3x} \end{aligned}$$

□

In the previous section we saw how to identify local maxima, local minima and stationary points of inflection by examining the **first** derivative of a function. Here we shall see how the **second** derivative can sometimes be used to identify the nature of stationary points.

Second Derivative Test:

- If $f'(c) = 0$ and if $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f'(c) = 0$ and if $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f'(c) = 0$ and if $f''(c) = 0$ then the second derivative gives no information.

Example 18. Consider the function

$$f : \mathbf{R} \longrightarrow \mathbf{R} \text{ where } f(x) = x^3 - x.$$

(a) Find the stationary points of f .

Solution: We first find the derivative of $f(x)$:

$$f'(x) = 3x^2 - 1.$$

We then find the x –coordinates of the stationary points by solving $f'(x) = 0$.

$$\begin{aligned} & f'(x) = 0 \\ \iff & 3x^2 - 1 = 0 \\ \iff & \qquad x = \pm\frac{1}{\sqrt{3}} \end{aligned}$$

We now find the y –coordinates of the stationary points:

$$\begin{aligned} f\left(\frac{1}{\sqrt{3}}\right) &= \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} \\ &= \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} \\ &= -\frac{2}{3\sqrt{3}} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{1}{\sqrt{3}}\right) &= \left(-\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}} \\ &= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= \frac{2}{3\sqrt{3}} \end{aligned}$$

Hence the stationary points of f are

$$\left(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right) .$$

$\square$

(b) Use the Second Derivative Test to determine the nature of the stationary points of f .

Solution: We first find the second derivative:

$$f''(x) = 6x.$$

We substitute the x–coordinates of the stationary points into the second derivative:

$$f''\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} > 0,$$
$$f''\left(-\frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} < 0.$$

By the Second Derivative Test, there is a local minimum at $\left(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right)$ and a local maximum at $\left(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$. $\square$

Note: It is often better to use the First Derivative Test (from the previous section) rather than the Second Derivative Test (particularly when f' is complicated to differentiate).

Warning: If $f'(c) = 0$ and $\boldsymbol{f''(c)} = 0$ you **cannot** make any conclusion about the nature of the stationary point at $x = c$. In such cases, the First Derivative Test should be used to identify whether you have a local maximum, a local minimum, or a stationary point of inflection.

Exercises

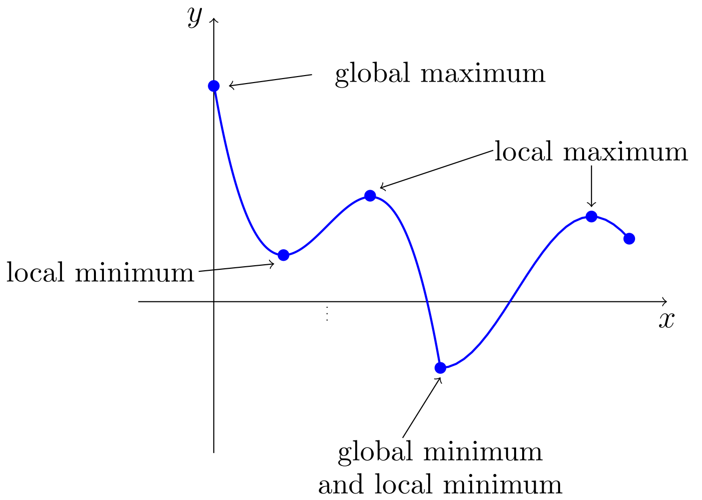
7. Consider the function $f(x) = x^3$.
   (a) Find $f'(x)$ and $f''(x)$.
   (b) Hence find $f'(0)$ and $f''(0)$.
   (c) Using the **First** Derivative Test, identify the nature of the stationary point at $x = 0$.

8. Consider the function $f(x) = x^4$.
   (a) Find $f'(x)$ and $f''(x)$.
   (b) Hence find $f'(0)$ and $f''(0)$.
   (c) Using the **First** Derivative Test, identify the nature of the stationary point at $x = 0$.

9. Consider the function $f(x) = -x^4$.
   (a) Find $f'(x)$ and $f''(x)$.
   (b) Hence find $f'(0)$ and $f''(0)$.
   (c) Using the **First** Derivative Test, identify the nature of the stationary point at $x = 0$.

Note: These three exercises show that if $f'(c) = 0$ and $f''(c) = 0$ then we cannot say what type of stationary point we have at $x = c$ without using other information.

Global Maxima and Minima



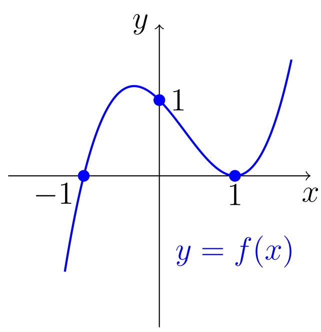
A function f has

- a **global** maximum at c if $f(c) \geq f(x)$ for all x in the function's domain.

Similarly, a function f has

- a **global** minimum at c if $f(c) \leq f(x)$ for all x in the function's domain.

Note that **not** every function has a global maximum or a global minimum. For example, consider the function $f(x) = x^3 - x^2 - x + 1$ whose graph was drawn in Example 15.



However, if f is a **continuous** function with domain of the form $[\boldsymbol{a}, \boldsymbol{b}]$, then we can be sure that f *does* have a global maximum and a global minimum. These points must occur either

- at critical points (i.e. points where $f'(x) = 0$ or where $f'(x)$ does not exist), or
- at the endpoints a, b of the domain.

Thus, to find the global maximum and the global minimum of f we just need to compare the values of f at its critical points and at the endpoints of its domain. The largest value obtained is the global maximum value, whereas the smallest value obtained is the global minimum value.

Example 19. Find the global extreme values of the function

$$f : [0, 2] \longrightarrow \mathbf{R} \text{ where } f(x) = x^3 - 3x + 2.$$

Solution: In this example, the endpoints of the domain are $x = 0$ and $x = 2$.

To find the critical points, we must first find $f'(x)$:

$$\begin{aligned} \text{Since} \quad f(x) &= x^3 - 3x + 2 \\ \text{then} \quad f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x + 1)(x - 1) \end{aligned}$$

If we put $f'(x) = 0$, we obtain the values $x = -1$ and $x = 1$. However, -1 is **not** part of the given domain, so we shall ignore this value. Thus, the critical point to be considered is at $x = 1$.

Critical point: $x = 1$ gives $f(x) = 1^3 - 3 \times 1 + 2 = 0$.

Endpoints: $x = 0$ gives $f(x) = 0^3 - 3 \times 0 + 2 = 2$.

$x = 2$ gives $f(x) = 2^3 - 3 \times 2 + 2 = 4$.

So

f has a global maximum value of 4, which occurs when $x = 2$,

and

f has a global minimum value of 0, which occurs when $x = 1$.

□

Exercises

10. Find the x–coordinates of the global maximum and the global minimum of

$$f(x) = 3x^4 - 20x^3 + 36x^2 + 4$$

for each of the following domains:

(a) $2 \leq x \leq 4$

(b) $1 \leq x \leq 4$

(c) $-1 \leq x \leq 4$

(d) $-1 \leq x \leq 2.5$

10.5 Applied Max/Min Problems

Steps for Solving Applied Max/Min Problems:

Step 1: Draw a diagram.
For many questions it is helpful to draw a diagram labelled by symbols representing the variables relevant to the problem.

Step 2: Write down a formula for the quantity to be maximized or minimized.

Step 3: Write the formula found in Step 2 as a **function of one variable.** This usually involves making use of

- geometric properties of the diagram, or
- information given in the question.

Step 4: Find all critical points.
That is, if f is the function to be maximized or minimized then find all x –values in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Step 5: Establish the nature of these critical points.
(For example, use the First Derivative Test.) Determine which critical point corresponds to the maximum or minimum required by the question.

Step 6: Answer the given question.
Note: If the question was a "word question", then answer the question with a sentence.

Examples

Example 20. Find two negative numbers whose product is 100 and whose sum is a maximum.

Solution:

Step 1: A diagram is not very relevant in this example.

Step 2:
We want to maximize

$$S = x + y \tag{10.1}$$

where x, y are negative numbers.

Step 3:
We are told that $xy = 100$. Thus

$$y = \frac{100}{x} , \tag{10.2}$$

which can be substituted into Equation 10.1 to give

$$\begin{aligned} S &= x + \frac{100}{x} \quad \text{where } x < 0. \\ &= x + 100x^{-1} \end{aligned}$$

Step 4:
Then

$$\frac{dS}{dx} = 1 - \frac{100}{x^2} ,$$

and this derivative exists for all x in the domain of the function S .

Thus any critical points must occur when $\dfrac{dS}{dx} = 0$.

Solving $\dfrac{dS}{dx} = 0$ gives

$$1 - \frac{100}{x^2} = 0.$$

That is,

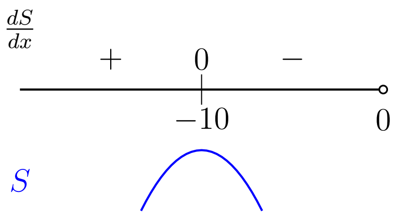
$$1 = \frac{100}{x^2} .$$

Thus we have $x^2 = 100$, whence $x = -10$.

(Note that we have chosen x to be the *negative* square root of 100, since x is supposed to be a *negative* number.)

Step 5:

By the First Derivative Test, we can deduce that we have a local maximum at $x = -10$. We see that this local maximum is also the **global** maximum.



Step 6:

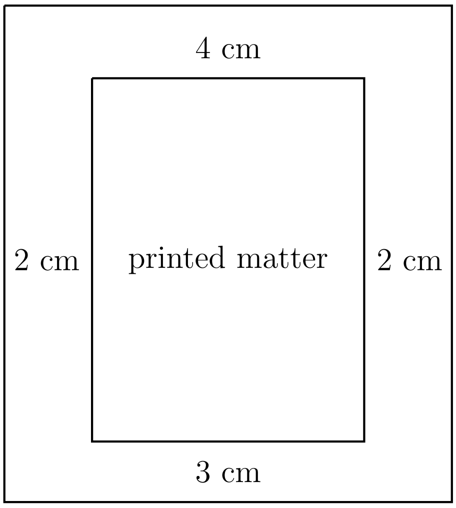
Substituting $x = -10$ into Equation 10.2, we see that y also equals -10.

Thus the two required numbers are -10 and -10.

□

Example 21. A rectangular poster (printed on one side only) is to contain 343 cm^2 of printed matter on it, with a 4 cm margin along the top, a 3 cm margin along the bottom, and a 2 cm margin along the sides. Find the dimensions of the poster with the smallest area.

Solution: **Step 1:**



Step 2:
We want to minimize
$$A = xy. \tag{10.3}$$

Step 3:
From the diagram, we see that the printed region of the poster has dimensions $y - 7$ cm and $x - 4$ cm. Thus we need

$$x > 4, \quad y > 7 \quad \text{and} \quad (x-4)(y-7) = 343.$$

$$\begin{aligned} \text{Then} \quad y - 7 &= \frac{343}{x-4}\,. \\ \text{That is,} \quad y &= \frac{343}{x-4} + 7 \\ &= \frac{7x+315}{x-4}\,. \end{aligned} \tag{10.4}$$

Substituting Equation 10.4 into Equation 10.3 gives

$$\begin{aligned} A &= \frac{x(7x+315)}{x-4} \\ &= \frac{7x^2+315x}{x-4} \quad \text{where } x > 4. \end{aligned}$$

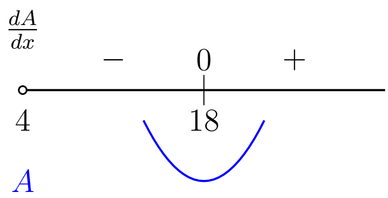
Step 4: Then, by the quotient rule for differentiation, we have

$$\begin{aligned} \frac{dA}{dx} &= \frac{(14x+315)(x-4) - 1(7x^2+315x)}{(x-4)^2} \\ &= \frac{14x^2+259x-1260-7x^2-315x}{(x-4)^2} \\ &= \frac{7x^2-56x-1260}{(x-4)^2} \\ &= \frac{7(x-18)(x+10)}{(x-4)^2}\,. \end{aligned}$$

This derivative exists for all x in the domain of the function A; thus any critical points must occur when $\frac{dA}{dx} = 0$.

Solving $\frac{dA}{dx} = 0$ gives $x = 18$ or $x = -10$. Since x is a *length*, we only need to consider *positive* x-values; thus we will only investigate the nature of the critical point at $x = 18$.

Step 5:
By the First Derivative Test, we can deduce that we have a local minimum at $x = 18$. We see that this local minimum is also the **global** minimum.



Step 6:
Substituting $x = 18$ into Equation 10.4, we see that

$$\begin{aligned} y &= \frac{7 \times 18 \ + \ 315}{18 - 4} \\ &= \frac{441}{14} \\ &= 31.5 \end{aligned}$$

Thus the poster with smallest area should have length 18 cm across the top, and sidelength 31.5 cm. □

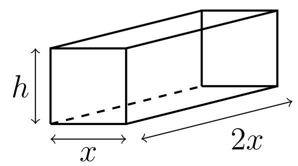
Exercises

1. A rectangular enclosure is to be built from 16 metres of fencing wire. The wire is only needed for three of the sides, since the fourth side is bounded by an existing wooden fence. What is the largest area of ground that the enclosure can cover?

2. A sheet of cardboard measures 15 cm by 7 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the length of the edge of the squares cut so that the box has maximum volume.

3. A rectangular brick is to be constructed so that

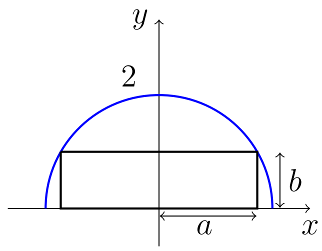
   - its total surface area is 300 cm^2 ; and
   - the length of its base is twice the width of its base.

   Find the dimensions of the brick with maximum volume which satisfies these conditions.

4. A dog food manufacturer produces cylindrical cans with a volume of 250π cm^3. What are the dimensions of the can if the least amount of material is used in its construction? (That is, find the height and radius of the cylinder with minimum surface area.)

5. Suppose we have a rectangle inscribed in a semicircle of radius 2 (as shown in the diagram).



(a) Find an equation relating the variables a and b.

(b) Hence write the area of the rectangle in terms of a.

(c) Find a and b for the rectangle with largest area.

10.6 Answers to Chapter 10 Exercises

10.1:

1. (a) When $r = 5$ cm, the area is increasing at a rate (with respect to the radius) of $10\pi \text{ cm}^2 \cdot \text{cm}^{-1}$.
   (b) When $A = 36\pi \text{ cm}^2$, the area is increasing at a rate (with respect to the radius) of $12\pi \text{ cm}^2 \cdot \text{cm}^{-1}$.

10.2:

1. When $r = 10$ cm, the volume is increasing at a rate of $200\pi \text{ cm}^3 \cdot \text{s}^{-1}$.
2. When $x = 8$ cm, the volume is increasing at a rate of $38.4 \text{ cm}^3 \cdot \text{s}^{-1}$.
3. (a) When $r = 5$ cm, the area is increasing at a rate of $20\pi \text{ cm}^2 \cdot \text{min}^{-1}$.
   (b) $\frac{dC}{dt} = 4\pi \text{ cm} \cdot \text{min}^{-1}$, which is constant. (It does not rely on r or t .)
4. (a) When $r = 6$ cm, the volume is increasing at a rate of $216\pi \text{ cm}^3 \cdot \text{min}^{-1}$.
   (b) When $r = 6$ cm, the radius is increasing at a rate of $\frac{5}{36\pi} \text{ cm} \cdot \text{min}^{-1}$.
5. When $h = 5$ cm, the water level is rising at $\frac{2}{15\pi} \text{ cm} \cdot \text{s}^{-1}$.
6. (a) When $r = 20$ cm, the radius is increasing at a rate of $\frac{9}{400\pi} \text{ cm} \cdot \text{min}^{-1}$.
   (b) When $r = 20$ cm, the surface area is increasing at a rate of $\frac{18}{5} \text{ cm}^2 \cdot \text{min}^{-1}$.
7. When there is still 5 m of rope out, the boat is approaching the pier at a rate of $\frac{2}{\sqrt{21}} \text{ m} \cdot \text{s}^{-1}$.

10.3:

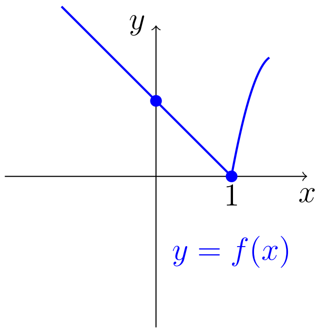
1. $y = 4x$ and $y = -4x + \frac{16}{3}$
2. The tangent is $y = -3x + \frac{\pi}{2}$, and the normal is $y = \frac{1}{3}\left(x - \frac{\pi}{6}\right)$.
3. $y = x + 2$
4. $a = 2,\ b = -6$ and $c = 11$.

10.4:

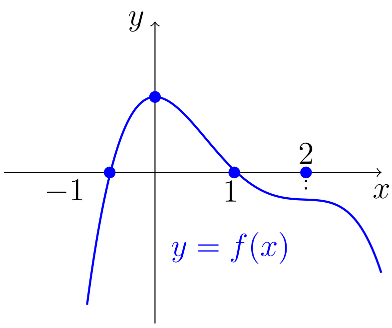
1. $p = -1,\ q = -8,\ r = 2$

2. $a = 1,\ b = -1,\ c = -5,\ d = -3$

3.



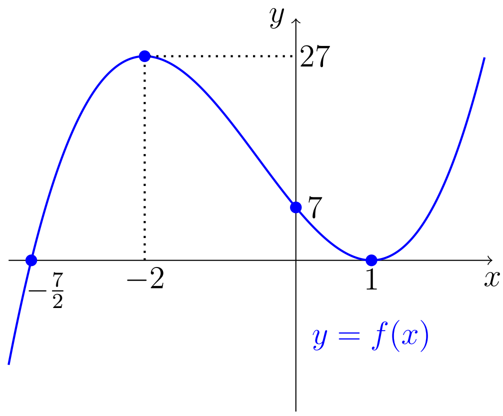
4.



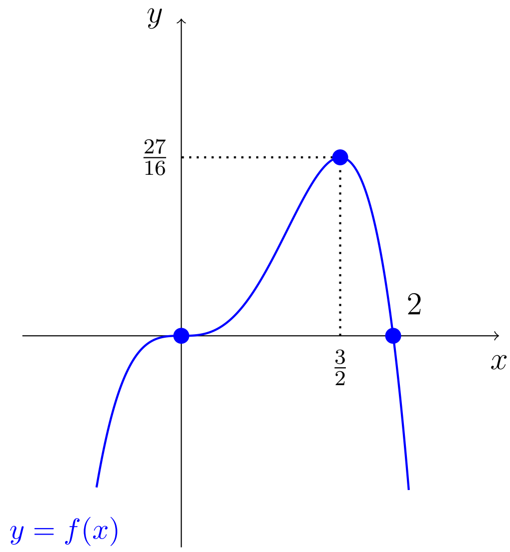
5. (a) Stationary point of inflection at $x = 2$.
   (b) Local maximum when $x = -\frac{8}{3}$, and local minimum when $x = 0$.
   (c) Local maximum when $x = 0$, and local minima when $x = \pm 2$.
   (d) Local maximum when $x = -1$, and local minimum when $x = 1$.

6.

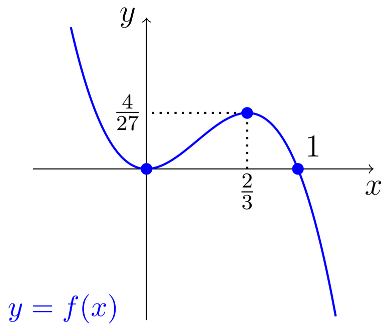
(a)



(b)



(c)



7. (a) $f'(x) = 3x^2,\ f''(x) = 6x$
   (b) $f'(0) = 0,\ f''(0) = 0$
   (c) Stationary point of inflection.

8. (a) $f'(x) = 4x^3,\ f''(x) = 12x^2$
   (b) $f'(0) = 0,\ f''(0) = 0$
   (c) Local minimum.

9. (a) $f'(x) = -4x^3,\ f''(x) = -12x^2$
   (b) $f'(0) = 0,\ f''(0) = 0$
   (c) Local maximum.

10. (a) The global maximum occurs when $x = 4$, and the global minimum occurs when $x = 3$.
    (b) The global maximum occurs when $x = 4$, and the global minimum occurs when $x = 1$.
    (c) The global maximum occurs when $x = 4$, and the global minimum occurs when $x = 0$.
    (d) The global maximum occurs when $x = -1$, and the global minimum occurs when $x = 0$.

10.5:

1. The largest area is 32 m^2 .
2. The length of each edge is 1.5 cm.
3. The dimensions of the brick are 5 cm, 10 cm and $\frac{20}{3}$ cm.
4. The radius is 5 cm and the height is 10 cm.
5. (a) $a^2 + b^2 = 4$

   (b) $A = 2a\sqrt{4 - a^2}$

   (c) $a = \sqrt{2}, \quad b = \sqrt{2}$