Maths 1

Exercise Sheet 15: Limits and Integration to Infinity

1. (i) Find each of the following limits.
   (ii) State whether the limit exists.

   (a) $\lim\limits_{x\to\infty} x^2$ (b) $\lim\limits_{x\to\infty} \sqrt{x}$ (c) $\lim\limits_{x\to\infty} e^{-3x}$ (d) $\lim\limits_{x\to\infty} \dfrac{1}{x^{\frac{3}{2}}}$

   (e) $\lim\limits_{x\to\infty} e^{\frac{1}{2}x}$ (f) $\lim\limits_{x\to\infty} x^{\frac{5}{2}}$

2. (i) Find each of the following limits.
   (ii) State whether the limit exists.

   (a) $\lim\limits_{x\to-\infty} x^3$ (b) $\lim\limits_{x\to-\infty} x^4$ (c) $\lim\limits_{x\to-\infty} \dfrac{1}{x^3}$ (d) $\lim\limits_{x\to-\infty} \dfrac{1}{x^4}$

   (e) $\lim\limits_{x\to-\infty} e^{3x}$ (f) $\lim\limits_{x\to-\infty} e^{-2x}$ (g) $\lim\limits_{x\to-\infty} \dfrac{1}{e^x}$

3. (i) Find each of the following limits.
   (ii) State whether the limit exists.

   (a) $\lim\limits_{a\to\infty} \dfrac{1}{\sqrt{a}}$ (b) $\lim\limits_{b\to\infty} b^2$ (c) $\lim\limits_{c\to\infty} e^{-4c}$ (d) $\lim\limits_{d\to\infty} \ln d$

   (e) $\lim\limits_{p\to\infty} e^{2p}$ (f) $\lim\limits_{q\to\infty} q^{-3}$ (g) $\lim\limits_{y\to-\infty} y^5$ (h) $\lim\limits_{z\to-\infty} z^6$

4. Find the following limits.

   (a) $\lim\limits_{x\to\infty} \left(\dfrac{3}{x^2}\right)$ (b) $\lim\limits_{x\to\infty} \left(\dfrac{3}{x^2}+4\right)$ (c) $\lim\limits_{x\to\infty} (e^{-3x}+2)$

   (d) $\lim\limits_{x\to\infty} \left(\dfrac{5}{e^{-3x}+2}\right)$ (e) $\lim\limits_{x\to\infty} \left(\dfrac{1}{\frac{1}{x}+2}\right)$

   (f) $\lim\limits_{x\to\infty} \left(\dfrac{4+e^{-2x}}{\frac{4}{x}+12}\right)$

5. Find the following limits.

   (a) $\lim\limits_{x\to-\infty} (-e^{-x})$ (b) $\lim\limits_{x\to-\infty} \left(\dfrac{1}{x}-14\right)$ (c) $\lim\limits_{x\to-\infty} (7+e^{2x})$

   (d) $\lim\limits_{x\to-\infty} \left(\dfrac{6}{3+4e^x}\right)$ (e) $\lim\limits_{x\to-\infty} \left(\dfrac{1}{\frac{1}{x}+2}\right)$

   (f) $\lim\limits_{x\to-\infty} \left(\dfrac{5+e^{2x}}{\frac{2}{x^2}+15}\right)$

6. (a) Investigate $\lim\limits_{x\to-\infty} (\sin x)$.

   (b) Investigate $\lim\limits_{x\to\infty} (\cos x)$ and $\lim\limits_{x\to-\infty} (\cos x)$.

   (c) Investigate $\lim\limits_{x\to\infty} (4\cos x)$ and $\lim\limits_{x\to-\infty} (4\cos x)$.

7. Note that the Limit Laws are only valid if

$$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to\infty} g(x) \textit{ exist.}$$

Therefore, if

$$\lim_{x\to\infty} f(x) \text{ and/or } \lim_{x\to\infty} g(x) \text{ equal } \infty \text{ or } -\infty,$$

then we must try to rewrite our expression *before* using any of the Limit Laws.

Keeping this in mind, find the following limits.

(a) $\displaystyle\lim_{x\to\infty}\left(\frac{3+4x^2}{x^2}\right)$

Hint: Start by rewriting $\dfrac{3+4x^2}{x^2}$ as $\dfrac{3}{x^2}+4$.

(b) $\displaystyle\lim_{x\to\infty}\left(\frac{x^2}{2x^2+3}\right)$

Hint: Start by rewriting $\dfrac{x^2}{2x^2+3}$ as $\dfrac{1}{2+\frac{3}{x^2}}$.

(c) $\displaystyle\lim_{x\to\infty}\left(\frac{e^{2x}-3e^x}{4e^{2x}+e^x}\right)$

Hint: Start by rewriting $\dfrac{e^{2x}-3e^x}{4e^{2x}+e^x}$ as $\dfrac{1-\frac{3}{e^x}}{4+\frac{1}{e^x}}$.

(d) $\displaystyle\lim_{x\to\infty}\left(\frac{3x^3-2}{x^3+5x^2}\right)$ (e) $\displaystyle\lim_{x\to\infty}\left(\frac{2e^{3x}+e^x}{4-9e^{3x}}\right)$

(f) $\displaystyle\lim_{x\to\infty}\left(\frac{x+4}{x^2+10}\right)$ (g) $\displaystyle\lim_{x\to-\infty}\left(\frac{4x^2}{3-2x^2}\right)$

(h) $\displaystyle\lim_{x\to-\infty}\left(\frac{e^{2x}+4e^x}{2e^{2x}-12e^x}\right)$

Hint: Start by rewriting $\dfrac{e^{2x}+4e^x}{2e^{2x}-12e^x}$ as $\dfrac{e^x+4}{2e^x-12}$.

(i) $\displaystyle\lim_{x\to-\infty}\left(\frac{e^{-x}+1+6e^{-2x}}{3e^{-2x}+2}\right)$

Hint: Start by rewriting $\dfrac{e^{-x}+1+6e^{-2x}}{3e^{-2x}+2}$ as $\dfrac{e^x+e^{2x}+6}{3+2e^{2x}}$.

8. Evaluate the following improper integrals.

(a) $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ (b) $\displaystyle\int_{-\infty}^{-3} \frac{1}{x^2}\,dx$ (c) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

(d) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$ (e) $\displaystyle\int_0^\infty \frac{1}{(x+1)^2}\,dx$ (f) $\displaystyle\int_1^\infty \frac{1}{(x+1)^2}\,dx$

(g) $\displaystyle\int_{-\infty}^{1} e^x\,dx$ (h) $\displaystyle\int_{-\infty}^{\ln 3} e^x\,dx$

9. Evaluate the following improper integrals.

(a) $\displaystyle\int_{-\infty}^{0} e^{2x}\,dx$ (b) $\displaystyle\int_{-\infty}^{\ln 3} e^{2x}\,dx$ (c) $\displaystyle\int_0^\infty e^{-x}\,dx$

(d) $\displaystyle\int_2^\infty e^{-x}\,dx$ (e) $\displaystyle\int_0^\infty -e^{-2x}\,dx$ (f) $\displaystyle\int_{\ln 2}^\infty -e^{-2x}\,dx$

(g) $\displaystyle\int_2^\infty e^{-\frac{1}{2}x}\,dx$ (h) $\displaystyle\int_{\ln 100}^\infty e^{-\frac{1}{2}x}\,dx$

10. Evaluate the following improper integrals.

(a) $\int_0^{\infty} f(x)\,dx$ where

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \leq 1 \\ \frac{1}{x^3} & \text{if } x > 1 \end{cases}$$

(b) $\int_{-\infty}^{0} f(x)\,dx$ where

$$f(x) = \begin{cases} e^{2x} & \text{if } x \leq -1 \\ x & \text{if } x > -1 \end{cases}$$

(c) $\int_4^{\infty} f(x)\,dx$ where

$$f(x) = \begin{cases} \frac{1}{2x+1} & \text{if } 0 \leq x < 12 \\ e^{-x} & \text{if } x \geq 12 \end{cases}$$

(d) $\int_{-\infty}^{4} f(x)\,dx$ where

$$f(x) = \begin{cases} \frac{1}{x^4} & \text{if } x < -1 \\ x^2 & \text{if } x \geq -1 \end{cases}$$

(e) $\int_{-\infty}^{\infty} f(x)\,dx$ where

$$f(x) = \begin{cases} e^{4x} & \text{if } x \leq 0 \\ e^{-2x} & \text{if } x > 0 \end{cases}$$

(f) $\int_{-\infty}^{\infty} f(x)\,dx$ where

$$f(x) = \begin{cases} \frac{3}{x^2} & \text{if } x \leq -1 \\ 3 & \text{if } -1 < x \leq 1 \\ \frac{3}{x^4} & \text{if } x > 1 \end{cases}$$

Answers

1. (a) (i) ∞ (ii) Does not exist. (b) (i) ∞ (ii) Does not exist.
   (c) (i) 0 (ii) Exists. (d) (i) 0 (ii) Exists.
   (e) (i) ∞ (ii) Does not exist. (f) (i) ∞ (ii) Does not exist.

2. (a) (i) $-\infty$ (ii) Does not exist. (b) (i) ∞ (ii) Does not exist.
   (c) (i) 0 (ii) Exists. (d) (i) 0 (ii) Exists.
   (e) (i) 0 (ii) Exists. (f) (i) ∞ (ii) Does not exist.
   (g) (i) ∞ (ii) Does not exist.

3. (a) (i) 0 (ii) Exists. (b) (i) ∞ (ii) Does not exist.
   (c) (i) 0 (ii) Exists. (d) (i) ∞ (ii) Does not exist.
   (e) (i) ∞ (ii) Does not exist. (f) (i) 0 (ii) Exists.
   (g) (i) $-\infty$ (ii) Does not exist. (h) (i) ∞ (ii) Does not exist.

4. (a) 0 (b) 4 (c) 2 (d) $\frac{5}{2}$ (e) $\frac{1}{2}$ (f) $\frac{1}{3}$

5. (a) $-\infty$ (b) -14 (c) 7 (d) 2 (e) $\frac{1}{2}$ (f) $\frac{1}{3}$

6. (a) The function $\sin x$ oscillates between -1 and 1 as $x \to -\infty$, so $\sin x$ never converges to a particular number. Thus $\lim\limits_{x\to-\infty} \sin x$ does not exist.
   (b) The function $\cos x$ oscillates between -1 and 1 as $x \to \infty$ and as $x \to -\infty$, so $\cos x$ never converges to a particular number. Thus $\lim\limits_{x\to\infty} \cos x$ and $\lim\limits_{x\to-\infty} \cos x$ do not exist.
   (c) The function $4\cos x$ oscillates between -4 and 4 as $x \to \infty$ and as $x \to -\infty$, so $4\cos x$ never converges to a particular number. Thus $\lim\limits_{x\to\infty} (4\cos x)$ and $\lim\limits_{x\to-\infty} (4\cos x)$ do not exist.

7. (a) 4 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 3 (e) $-\frac{2}{9}$ (f) 0
   (g) -2 (h) $-\frac{1}{3}$ (i) 2 Video solution for 7h

8. (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$ (e) 1 (f) $\frac{1}{2}$
   (g) e (h) 3

9. (a) $\frac{1}{2}$ (b) $\frac{9}{2}$ (c) 1 (d) e^{-2} (e) $-\frac{1}{2}$ (f) $-\frac{1}{8}$
   (g) $2e^{-1}$ (h) $\frac{1}{5}$

10. (a) $\frac{2}{\pi} + \frac{1}{2}$ (b) $\frac{1}{2e^2} - \frac{1}{2}$ (c) $\ln\left(\frac{5}{3}\right) + e^{-12}$ (d) 22 (e) $\frac{3}{4}$ (f) 10