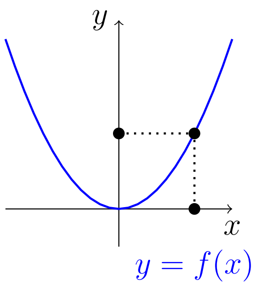
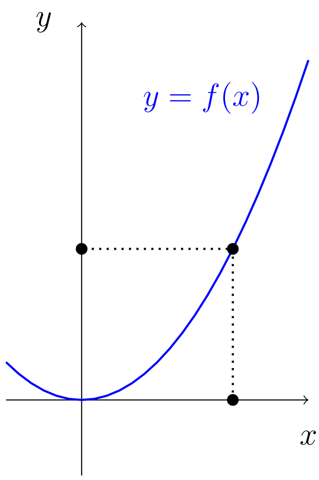
Chapter 14

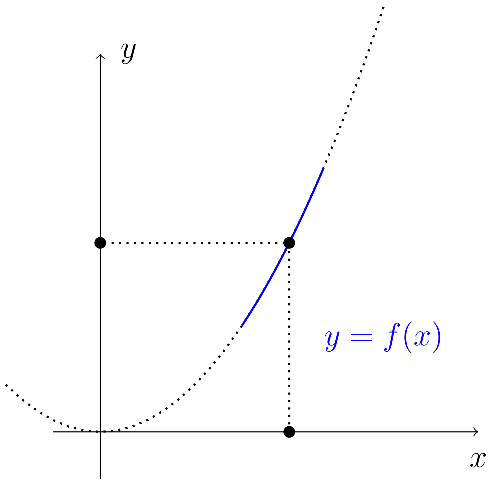
Further Applications of Differentiation

14.1 Tangents and Linear Approximations

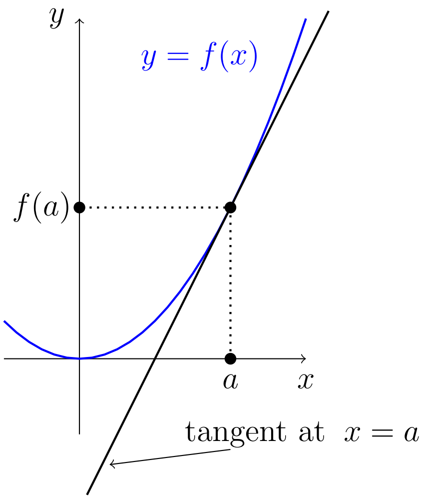
When we zoom in towards a point (x_0, y_0) on the graph of a differentiable function f, the graph starts to look like a straight line when x is close to x_0:







This straight line is actually the tangent to the curve $y = f(x)$ at that point, and the equation of this tangent can be found (as usual) as follows. Suppose that we wish to find the tangent at $x = a$.



The equation of the tangent will be

$$y = mx + C$$

where $m = f'(a)$ and C is chosen so that the tangent passes through the point $(a, f(a))$, i.e.,

$$\begin{aligned} f(a) &= f'(a)\, a \, + \, C \\ C &= f(a) - a f'(a)\,. \end{aligned}$$

Thus, the equation of the tangent to $y = f(x)$ at $x = a$ is

$$y = f'(a)x + f(a) - af'(a) .$$

Because the curve $y = f(x)$ is approximately the same as its tangent when x is close to a, we can use the tangent to approximate the value of $f(x)$ near $x = a$:

$$\begin{aligned} f(x) &\approx f'(a)x + f(a) - af'(a) \\ &= f(a) + f'(a)x - af'(a) \\ &= f(a) + f'(a)(x - a) \end{aligned}$$

i.e.

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

The above approximation is called the **linear approximation** of $f(x)$ at $x = a$. This approximation works best when x is close to a.

Example 1. Find the linear approximation of the function $f(x) = x^2$ at $x = 1$.

Solution: The derivative of $f(x) = x^2$ is $f'(x) = 2x$.
We have $f(1) = 1$ and $f'(1) = 2 \times 1 = 2$.
Therefore, the linear approximation of $f(x) = x^2$ at $x = 1$ is

$$\begin{aligned} f(x) &\approx 1 + 2(x - 1) \\ &= 1 + 2x - 2 \\ &= 2x - 1 . \end{aligned}$$

This means that the curve $y = x^2$ looks like the straight line $y = 2x - 1$ when x is close to 1. The first exercise on page 6 asks you to show this graphically. □

The linear approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

of a differentiable function f is useful because the linear approximation is always a very simple function, even if f is a complicated function.

Example 2. Find the linear approximation of $f(x) = \sin x$ at $x = 0$.

Solution: First note that $f(x) = \sin x$ and so $f'(x) = \cos x$.

Now we need to evaluate the function and its derivative at $x = 0$.

$$\begin{aligned} f(0) &= \sin(0) \\ &= 0 \\ \text{and } f'(0) &= \cos(0) \\ &= 1 \end{aligned}$$

Therefore the linear approximation of $f(x) = \sin x$ at $x = 0$ is

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x - 0) \\ &= \sin(0) + \cos(0) \cdot x \\ &= 0 + 1 \cdot x \\ &= x \end{aligned}$$

Thus $\sin x \approx x$ when x is close to 0. $\square$

Notice that x is much simpler to work with than $\sin x$. This is often used in physics to obtain approximate solutions to equations involving $\sin x$ when x is close to 0.

Example 3. (a) Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$.

(b) Use this linear approximation to find approximate values of $\sqrt{8.98}$ and $\sqrt{9.08}$. Write your answers to 4 decimal places.

Solution: (a) First note that

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \implies f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

At $x = 9$, we have $f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$.

Therefore, the linear approximation of $f(x)$ at $x = 9$ is

$$\begin{aligned} f(x) &\approx f(9) + f'(9)(x-9) \\ &= \sqrt{9} + \frac{1}{6}(x-9) \\ &= 3 + \frac{1}{6}(x-9) \\ &= \frac{3}{2} + \frac{x}{6}\,. \end{aligned}$$

Thus

$$\sqrt{x} \approx \frac{3}{2} + \frac{x}{6}$$

when x is close to 9.

(b) The simplest formula to use is $\sqrt{x} \approx 3 + \frac{1}{6}(x-9)$.

$$\begin{aligned} \sqrt{8.98} &\approx 3 + \frac{1}{6}(8.98-9) \\ &= 3 + \frac{1}{6}(-0.02) \\ &= 3 - \frac{2}{600} \\ &= 2.99666666\ldots \\ \sqrt{9.08} &\approx 3 + \frac{1}{6}(9.08-9) \\ &= 3 + \frac{1}{6}(0.08) \\ &= 3 + \frac{8}{600} \\ &= 3.01333333\ldots \end{aligned}$$

So, to 4 decimal places,

$$\sqrt{8.89} = 2.9967 \quad \text{and} \quad \sqrt{9.08} = 3.0133$$

$\square$

Note. By using the 'square root button' on our calculators, we find that

$$\begin{aligned}\sqrt{8.98} &= 2.99666481 \quad (8\,d.p.)\\ \sqrt{9.08} &= 3.01330384 \quad (8\,d.p.)\end{aligned}$$

which are very close to the approximate values found in (b).

However, if we try to use the linear approximation from (a) at an x-value that is *not* close to 9 then the approximation will probably *not* be accurate anymore. For example, consider $x = 20$. Using the linear approximation from (a), we obtain

$$\sqrt{20} \approx \frac{3}{2} + \frac{20}{6} = 4.8333 \quad (4\,d.p.)$$

However, using the 'square root button' on our calculators, we see that

$$\sqrt{20} = 4.4721 \quad (4\,d.p.)$$

Thus, we see that the linear approximation $\sqrt{x} \approx \frac{3}{2} + \frac{x}{6}$ is not so good when x is far away from 9.

Exercises

1. Sketch the graphs of $y = x^2$ and $y = 2x - 1$ on the same set of axes.

   (Note that, by Example 1, $y = 2x - 1$ is the linear approximation of $y = x^2$ at $x = 1$.)

2. Find the linear approximation to the following functions at $x = 0$.

   (a) $f(x) = x^2 + 3x$ (b) $f(x) = e^x$ (c) $f(x) = (1 + x)^{100}$.

3. Find the linear approximation to the following functions.

   (a) $f(x) = \sqrt{x}$ at $x = 1$ (b) $f(x) = \sin x$ at $x = \frac{\pi}{2}$.

4. (a) Find the linear approximation to x^5 at $x = 2$.

   (b) Use this to find approximate values for (i) $(1.99)^5$ (ii) $(2.001)^5$.

14.2 Linear approx. formula for error measurement

When we measure quantities, there are always inaccuracies in our measurements. When we then use these quantities in calculations, there is uncertainty (known as error) in our answers. Linear approximations give us an easy way of estimating the size of these errors.

Suppose that a student measures

the length of an edge of a cube to be $10\,\mathrm{cm}$, with a maximum error of $0.2\,\mathrm{cm}$.

Now suppose that this measured edge length is used to calculate the cube's volume.

Note that if we let x cm be the (unknown) *true* edge length, then the *true* volume is given by

$$V(x) = x^3\,\mathrm{cm}^3 .$$

Also note that, since the value of the *true* edge length x is unknown, then the value of the *true* volume $V(x)$ is also unknown.

Of course, when we use the *measured* edge length of $10\,\mathrm{cm}$, we calculate the volume to be

$$V(10) = 1000\,\mathrm{cm}^3 .$$

The difference between the (unknown) *true* volume and this *calculated* volume is

$$\big|V(x) - V(10)\big| .$$

Another way of saying this is that

the *calculated* volume has an (exact) error of $\big|V(x) - V(10)\big|$.

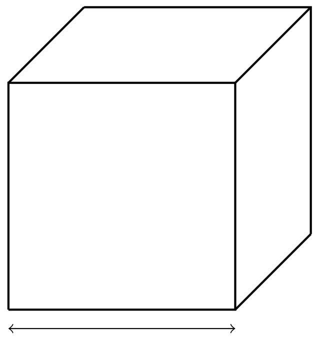
Unfortunately, since x is unknown, then $V(x)$ is unknown, and so this (exact) error in the *calculated* volume is also unknown.

As a sort of 'worst-case-scenario', we could instead aim to find the maximum possible error (rather than the exact error) in the calculated volume. A much simpler alternative is to use linear approximations to find an *estimate* of the maximum error.

Example 4. A student measures the length of an edge of a cube to be $10\,\mathrm{cm}$, with a maximum error of $0.2\,\mathrm{cm}$. Find

(a) the maximum error if the volume is calculated assuming an edge length of $10\,\mathrm{cm}$, and

(b) an estimate of this maximum error via the linear approximation.



Solution: (a) As before, let x cm be the *true* edge length, so that the *true* volume is given by

$$V(x) = x^3\,\mathrm{cm}^3\,.$$

We are told that

$$9.8 \le x \le 10.2,$$

and from this we can calculate that

$$(9.8 \text{ cm})^3 \;\le\; V(x) \text{ cm}^3 \;\le\; (10.2 \text{ cm})^3$$

i.e.

$$941.192 \text{ cm}^3 \;\le\; V(x) \text{ cm}^3 \;\le\; 1061.208 \text{ cm}^3\,.$$

Using the *measured* edge length of 10 cm we obtain a calculated volume 1000 cm^3. The maximum difference between the *true* volume $V(x)$ cm^3 and this *calculated* volume 1000 cm^3 is

$$1061.208 \text{ cm}^3 - 1000 \text{ cm}^3 = 61.208 \text{ cm}^3\,.$$

That is, the maximum error in the calculated volume is 61.208 cm^3.

Note that when stating error, the size of the error is given, but not its sign (i.e., no $\pm$).

(b) The linear approximation can be used to quickly gives us an *estimate* of the maximum error of the calculated volume. As before, consider the true volume $V(x)$ given by

$$V(x) = x^3 \text{ cm}^3 .$$

The linear approximation of $V(x)$ at $x = 10$ is

$$V(x) \approx V(10) + V'(10)(x - 10) .$$

Subtracting $V(10)$ from both sides and taking absolute values gives

$$\left|V(x) - V(10)\right| \approx \left|V'(10)(x - 10)\right|$$

i.e.

$$\left|V(x) - V(10)\right| \approx \left|V'(10)\right| \left|x - 10\right| \tag{14.1}$$

From $V(x) = x^3$ we have $V'(x) = 3x^2$, and so $V'(10) = 3(10)^2 = 300$.

Because the edge length was measured to be 10 cm with a maximum error of 0.2 cm , we have

$$\left| x - 10 \right| \leq 0.2$$

and so we can rewrite the approximation in statement (as

$$\begin{aligned} \left|V(x) - V(10)\right| &\approx \left|300\right| \times \left|x - 10\right| \\ &\leq 300 \times 0.2 \\ &= 60 . \end{aligned}$$

That is, an estimate for the maximum error of the calculated volume is

$$300 \times 0.2 = 60 \text{ cm}^3 .$$

□

We see that this estimate for the maximum error compares well with the (true) maximum error of 61.208 cm^3 found in (a).

In general, the approximate maximum error in evaluating $f(x)$ when x is measured as x_0 with a maximum error of Δx is given by

$$\boxed{\text{Approx. Max Error}(f) = \left| f'(x_0)\Delta x \right|}$$

Example 5. Consider the function $f(x) = 5x^4 - 10x^3 + 12$.

Suppose that x is measured to be equal to 1.6 with a maximum error of 0.2.

(a) Not examinable. Find the true maximum error in $f(x)$.

(b) Use a linear approximation to find the approximate maximum error in $f(x)$.

Solution: (a) First, we calculate $f(x)$ with the measured value of $x = 1.6$

$$f(1.6) \;=\; 5(1.6)^4 - 10(1.6)^3 + 12 \;=\; 3.808$$

The true maximum error is the largest difference between $f(x)$ and $f(1.6)$. This will occur either at the global maximum value of $f(x)$ or the global minimum value of $f(x)$.

The maximum error in x is 0.2, and so the relevant domain is

$$[\,1.6 - 0.2,\ 1.6 + 0.2\,] \;=\; [\,1.4,\ 1.8\,].$$

The global maximum value of $f(x)$ and the global minimum value of $f(x)$ will occur at a stationary point of $f(x)$, or an endpoint of the domain.

Stationary points: solve $f'(x) = 0$.

Note that

$$f(x) = 5x^4 - 10x^3 + 12 \qquad \Longrightarrow \qquad f'(x) = 20x^3 - 30x^2 = 10x^2(2x - 3)$$

and so

$$f'(x) = 0 \qquad \Longleftrightarrow \qquad 10x^2(2x - 3) = 0 \qquad \Longleftrightarrow \qquad x = 0 \ \text{ or } \ x = 1.5\,.$$

Of these, only $x = 1.5$ is in the domain.

Endpoints of domain: $x = 1.4$ or $x = 1.8$.

We now consider the y-values at the stationary point and at the end-points of the domain:

$$\begin{aligned}
x = 1.5 \quad &\Longrightarrow \quad f(x) = 3.5625 \quad \leftarrow \text{ global minimum value of } f(x) \\
x = 1.4 \quad &\Longrightarrow \quad f(x) = 3.768 \\
x = 1.8 \quad &\Longrightarrow \quad f(x) = 6.168 \quad \leftarrow \text{ global maximum value of } f(x)
\end{aligned}$$

Thus, when $x \in [\,1.4,\, 1.8\,]$,

$$3.5625 \leq f(x) \leq 6.168$$

i.e.,

$$-0.2455 \ \leq \ f(x) - f(1.6) \ \leq \ 2.36$$

and so the true maximum error in $f(x)$ is 2.36.

(b) We first calculate the derivative of $f(x)$:

$$f'(x) = 20x^3 - 30x^2$$

We have $x_0 = 1.6$ and $\Delta x = 0.2$.

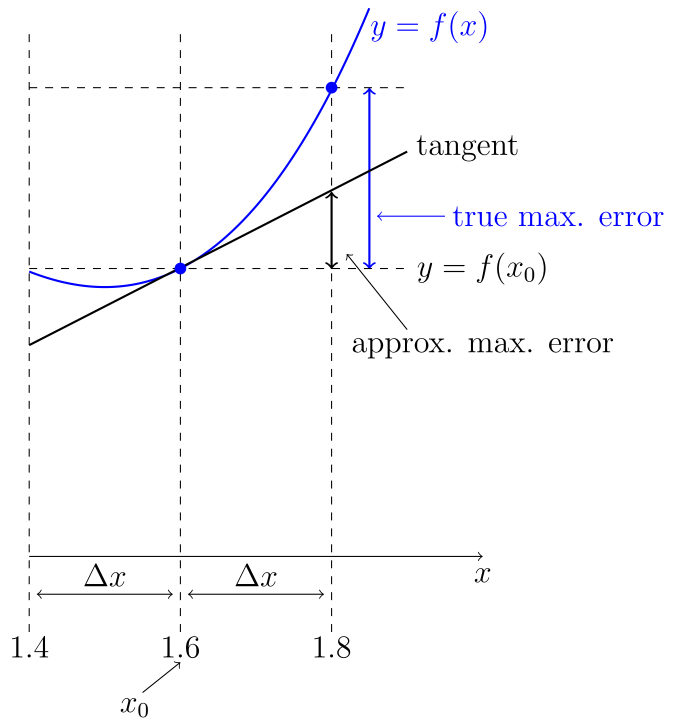
Therefore, the approximate maximum error in $f(x)$ is

$$\begin{aligned}
|f'(x_0)\Delta x| &= |(20(1.6)^3 - 30(1.6)^2) \times 0.2| \\
&= 1.024
\end{aligned}$$

□

The linear approximation to $f(x)$ at the point where $x = x_0$ is the tangent to the curve $y = f(x)$ at $x = x_0$. In estimating the maximum error in $f(x)$ via this linear approximation, we are finding the difference in the y-values between $(x_0, f(x_0))$ and the points on the tangent at the minimum and maximum values of the domain.

The various forms of error considered in this section are illustrated in the following diagram.



Example 6. Consider the function $f(x) = x^2 \sin(2\pi x)$.

Suppose that x is measured to be equal to 10 with a maximum error of 0.5 . Use the linear approximation formula for error measurement to find the approximate maximum error in $f(x)$.

Solution: We have $x_0 = 10$, $\Delta x = 0.5$ and

$$\begin{aligned} f'(x) &= 2x\sin(2\pi x) + x^2\cos(2\pi x) \times 2\pi \\ &= 2x\sin(2\pi x) + 2\pi x^2\cos(2\pi x)\,. \end{aligned}$$

Therefore, the approximate maximum error in $f(x)$ is

$$\begin{aligned} & \left| f'(10) \times 0.5 \right| \\ = & \left| \left[2(10)\sin(2\pi \times 10) + 2\pi(10^2)\cos(2\pi \times 10)\right] \times 0.5 \right| \\ = & \left| [20 \times 0 + 2\pi \times 100 \times 1] \times 0.5 \right| \\ = & \ 100\pi \\ = & \ 314.1593 \quad (4 \text{ d.p. }) \end{aligned}$$

□

Note. Do NOT assume that the maximum error of $f(x)$ occurs when the true value of x is as large as possible. For the above example, we have

$$\begin{aligned}
|f(x_0+\Delta x)-f(x_0)| &= |f(10+0.5)-f(10)| \\
&= \left|(10+0.5)^2\sin\left(2\pi(10+0.5)\right)-(10)^2\sin\left(2\pi(10)\right)\right| \\
&= \left|(10.5)^2\sin\left(21\pi\right)-(10)^2\sin\left(20\pi\right)\right| \\
&= |0-0| = 0
\end{aligned}$$

but the maximum error of the function $f(x)$ is not zero!

The final form of error considered in this section is the approximate maximum percentage error. This expresses the approximate maximum error as a percentage of the value $f(x_0)$ that has been calculated from the measured value x_0.

The percentage error is a useful indicator of the significance of the error.

For example: If the approximate maximum error was 1 and the value calculated for $f(x_0)$ was 10, then that's a 1-in-10 or 10% error. If, however, the value calculated for $f(x_0)$ was $10\,000$, an approximate maximum error of 1 would be the much less significant 1-in-10 000 or 0.01% error.

The approximate maximum percentage error in evaluating a function $f(x)$, when x is measured as x_0 with a maximum error of Δx, is given by

$$\boxed{\text{Approx. Max Percentage Error}(f) = \left|\frac{f'(x_0)\Delta x}{f(x_0)}\right| \times 100\%}$$

Example 7. Consider the function $f(x)=x^3+9x+2$. Suppose that x is measured to be equal to 2 with a maximum error of 0.2.

Use the linear approximation formula for error measurement to find

(a) the approximate maximum error in $f(x)$, and

(b) the approximate maximum percentage error in $f(x)$.

Solution: We have $x_0=2$, $\Delta x=0.2$ and $f'(x)=3x^2+9$.

(a) The approximate maximum error in $f(x)$ is

$$|f'(x_0)\Delta x| = \left|\left(3(2)^2+9\right)\times 0.2\right| = 4.2\,.$$

(b) The approximate maximum percentage error in $f(x)$ is

$$\begin{aligned}\left|\frac{f'(x_0)\Delta x}{f(x_0)}\right| \times 100\% &= \left|\frac{3(2)^2+9}{2^3+9(2)+2} \times 0.2\right| \times 100\% \\ &= \left|\frac{21}{28} \times 0.2\right| \times 100\% \\ &= 15\% \,.\end{aligned}$$

□

Exercises

1. Consider the function $f(x) = -x^2 + 5x - 6$.

   Suppose that x is measured to be equal to 2.4 with a maximum error of 0.2 .

   Use the linear approximation formula for error measurement to find the approximate maximum error in $f(x)$.

2. Consider the function $f(x) = x \sin x$.

   Suppose that x is measured to be equal to 7 with a maximum error of 0.1 .

   Use the linear approximation formula for error measurement to find

   (a) the approximate maximum error in $f(x)$, and

   (b) the approximate maximum percentage error in $f(x)$.

3. The radius of a sphere is measured and found to be equal to 10 cm with a maximum error of 0.8 cm .

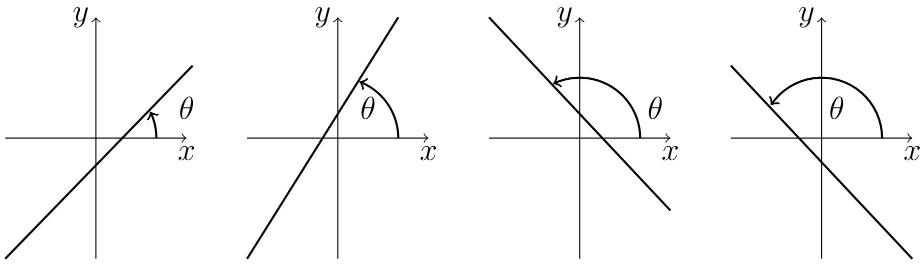
   Use the linear approximation formula for error measurement to estimate

   (a) the maximum possible error in the volume of the sphere when this radius is used to calculate the volume, and

   (b) the maximum possible error in the surface area of the sphere when this radius is used to calculate the surface area.

14.3 The Angle Between Two Curves

Recall from page that the angle of inclination of a straight line is just the angle which the line makes with the positive direction of the x–axis.



This angle is related to the gradient of the line by the formula

$$\boxed{m \;=\; \tan\theta}$$

Note that we always choose the angle of inclination so that

$$0^\circ \le \theta < 180^\circ \text{ when } \theta \text{ is measured in degrees}$$

or

$$0 \le \theta < \pi \text{ when } \theta \text{ is measured in radians .}$$

Example 8. Find the angle of inclination of following lines. Write your answer in degrees, to 2 decimal places.

(a) $y = 4x - 9$

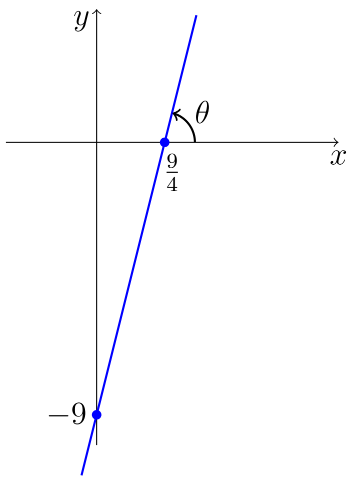
(b) $y = -2x + 9$

Solution:

(a) The line $y = 4x - 9$ has slope 4 . So the angle of inclination θ satisfies

$$\tan\theta = 4 \quad \text{where} \quad 0^\circ \le \theta < 180^\circ$$

$$\begin{aligned} \text{So } \theta &= 75.963756\ldots^\circ \\ &= 75.96^\circ \text{ (2 decimal places)} \end{aligned}$$

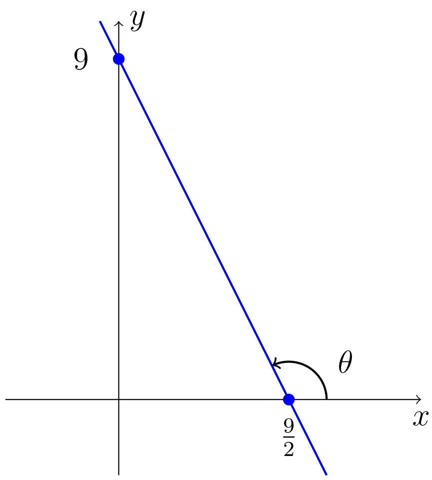


(b) The line $y = -2x + 9$ has slope -2 . So the angle of inclination θ satisfies

$$\tan\theta = -2 \quad \text{where} \quad 0^\circ \leq \theta < 180^\circ$$

Since $\tan\theta < 0$ we will first find the basic angle β .

$$\begin{aligned} \text{If } \tan\beta &= 2 \\ \text{then } \beta &= 63.434948\ldots^\circ \\ \text{So } \theta &= 180^\circ - 63.434948\ldots^\circ \\ &= 116.56505\ldots^\circ \ \text{(2 decimal places)} \end{aligned}$$

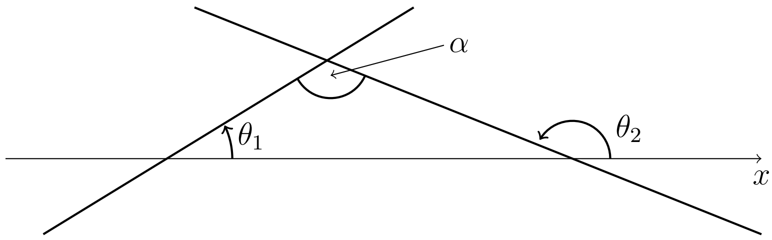


$\square$

To find the angle between two intersecting straight lines, we just need to

- find the angle of inclination of each of the lines, and then
- subtract the smaller angle from the larger angle.

We can see why this works, by considering the diagram below, in which the angle of intersection is labelled α :



Since the three angles in a triangle must add up to give $180°$, we have

$$\alpha + \theta_1 + (180° - \theta_2) = 180°.$$

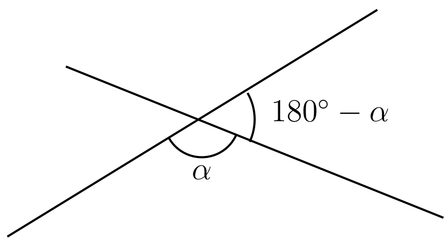
$$\text{That is,} \quad \alpha + \theta_1 - \theta_2 = 0.$$

Thus we see that

$$\alpha = \theta_2 - \theta_1$$

as required.

Note that there are actually two angles between the intersecting lines, namely α and $180° - \alpha$.



We only have to write down one of the two possible angles for our answer.

Example 9. Find the angle between the lines $y = 4x-9$ and $y = -2x+9$.

Solution: From the previous page, the angles of inclination for these two lines are

$$75.96° \text{ and } 116.57° .$$

Thus the angle between the lines is: $116.57° - 75.96° = 40.6°$ (1 d.p.)
Note: We can also write the answer as $180° - 40.6° = 139.4°$.

□

The angle between two intersecting curves is just the angle between the tangents to the two curves at the point of intersection. This angle can be found using the method shown on the previous page.

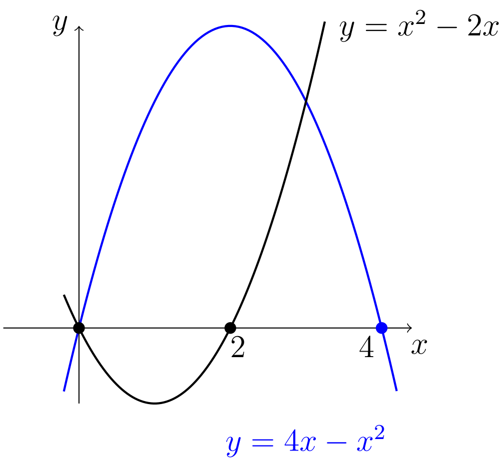
Example 10. (a) Find the points of intersection of the parabolas $y = x^2 - 2x$ and $y = 4x - x^2$.

(b) Find the angle between the curves at their point of intersection to the right of the origin.

Solution:

Note that

(a)
$$\begin{aligned} & x^2 - 2x = 4x - x^2 \\ \iff\quad & 2x^2 - 6x = 0 \\ \iff\quad & 2x(x-3) = 0 \\ \iff\quad & x = 0 \quad \text{or} \quad x = 3 . \end{aligned}$$



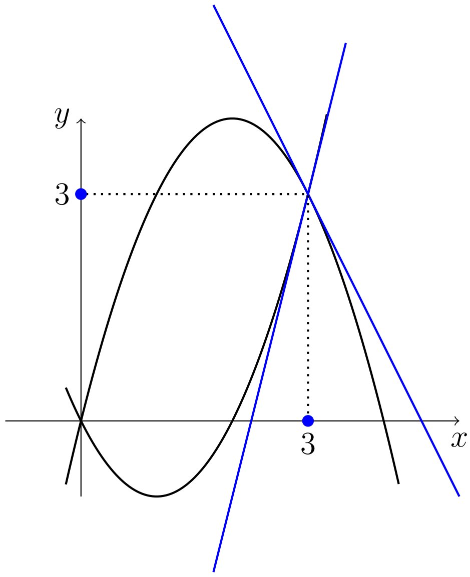
$$\text{Also,} \quad x = 0 \implies y = 0^2 - 2(0) = 0$$
$$\text{and} \quad x = 3 \implies y = 3^2 - 2(3) = 3 .$$

Therefore, the points of intersection are $(0,0)$ and $(3,3)$.

(b)

The intersection point to the right of the origin is $(3, 3)$.

$$\begin{aligned} y &= x^2 - 2x \\ \implies \frac{dy}{dx} &= 2x - 2 \\ x = 3 \implies \frac{dy}{dx} &= 2 \cdot 3 - 2 \\ &= 4 \end{aligned}$$



Let θ_1 be the angle of inclination. Then

$$\begin{aligned} \tan\theta_1 &= 4 \text{ where } 0^\circ \leq \theta_1 < 180^\circ \\ \theta_1 &= 75.96^\circ \text{ (2 decimal places)} \end{aligned}$$

$$\begin{aligned} y &= 4x - x^2 \\ \implies \frac{dy}{dx} &= 4 - 2x \\ x = 3 \implies \frac{dy}{dx} &= 4 - 2 \cdot 3 \\ &= -2 \end{aligned}$$

Let θ_2 be the angle of inclination. Then

$$\begin{aligned} \tan\theta_2 &= -2 \text{ where } 0^\circ \leq \theta_1 < 180^\circ \\ \theta_2 &= 116.57^\circ \text{ (2 decimal places)} \end{aligned}$$

The angle between the curves at $(3, 3)$ is

$$116.57^\circ - 75.96^\circ = 40.6^\circ \text{ (1 decimal place)}$$

$\square$

Exercises

Find the angle between the following curves at their point of intersection

(a) to the right of the origin.

(b) to the left of the origin.

Write your answer in degrees to 2 decimal places.

1. $y = x^2 - 1$ and $y = x + 1$

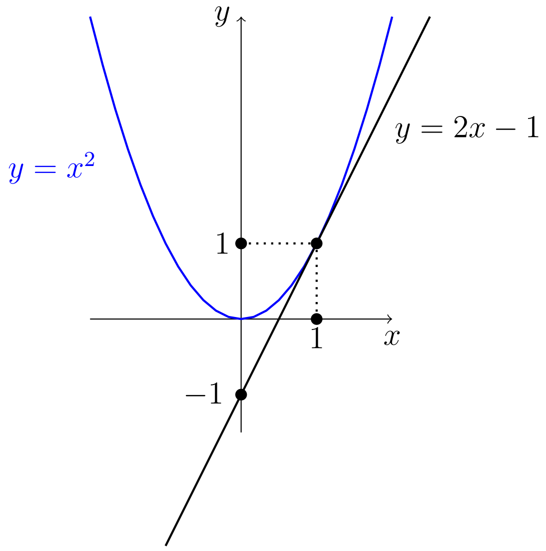
2. $y = 4x^2$ and $y = (x - 1)^2$

3. $y = x^2 - 4$ and $y = 4 - x^2$

4. $y = |x|$ and $y = \dfrac{1}{3}(4 - x)$

14.4 Answers to Chapter 14 Exercises

14.1: 1.



2. (a) $3x$ (b) $x + 1$ (c) $100x + 1$.
3. (a) $\frac{1}{2}x + \frac{1}{2}$ (b) 1
4. (a) $80x - 128$ (b) (i) 31.2 (ii) 32.08 .

14.2: 1. 0.04 .

2. (a) 0.5934 (4 d.p.) (b) 12.90% (2 d.p.)
3. (a) 320π cm^3 (b) 64π cm^2 .

14.3: 1. (a) $30.96°$ (or $149.04°$) (b) $71.57°$ (or $108.43°$)

2. (a) $57.43°$ (or $122.57°$) (b) $6.91°$ (or $173.09°$)
3. (a) $28.07°$ (or $151.93°$) (b) $28.07°$ (or $151.93°$)
4. (a) $63.43°$ (or $116.57°$) (b) $26.57°$ (or $153.43°$)