Chapter 1

Algebra Review

Reference: "Calculus", by James Stewart.

1.1 Real Numbers

The real numbers include the integers, the fractions and decimals, and the irrational numbers.

- **integers:** $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
- **rational numbers:** These are the numbers which can be written in the form $\dfrac{p}{q}$, where p and q are integers, with $q \neq 0$.

  For example, $\frac{2}{5}$ and $4 = \frac{4}{1}$ and $3.87 = \frac{387}{100}$ are rational numbers.
  It is important to remember that

  we **must NEVER have zero as the denominator of a fraction**.

- **irrational numbers:** These are the real numbers which are **NOT** rational.

  (That is, these are the real numbers which CANNOT be written as fractions.)

  For example, $\sqrt{2}$ and π are irrational numbers.
  (Note that $\pi \neq \frac{22}{7}$ because $\pi = 3.14159\ldots$ whereas $\frac{22}{7} = 3.14285\ldots$)

We will use the following notation:

$\mathbf{R}$ = the set of all real numbers,
$\mathbf{R}^+$ = the set of **positive** real numbers, and
$\mathbf{R}^-$ = the set of **negative** real numbers.

Note: The set $\mathbf{R}$ of all real numbers does **NOT** include the square root of NEGATIVE numbers.
For example, $\sqrt{-16}$ is **NOT** a real number.

1.2 Quadratics

A **quadratic** equation has the form $ax^2 + bx + c = 0$.

If an equation has been **factorised** (i.e. **written as a product** of factors), then it is easy to solve the equation.

Example 1. Solve the quadratic equation $x^2 + 3x + 2 = 0$.

Solution: We can factorise $x^2 + 3x + 2$ as $(x+1)(x+2)$.
(This will be shown in Example 3.)
Thus we can rewrite the quadratic equation given in this example as

$$(x+1)(x+2) = 0.$$

Then we must have

$$x + 1 = 0 \quad \text{or} \quad x + 2 = 0.$$

That is,

$$x = -1 \ \text{ or } \ x = -2.$$

□

Notice that this method of solution depends on the fact that, for any real numbers a and b we have

$$ab = 0 \iff a = 0 \ \text{ or } \ b = 0.$$

It is essential that we have the number 0 in this statement, rather than some other value (such as $1, 2, \ldots$)

Note.

(a) **Factorise** means "write as a product of factors".

(b) **Solve** means "find the values of x that satisfy the equation".

Another method for solving quadratic equations is to use the following **quadratic formula**:

$$\boxed{ax^2 + bx + c = 0 \qquad \Longleftrightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The number $b^2 - 4ac$ in this formula is called the **discriminant** of the quadratic.

Example 2. Use the quadratic formula to solve $x^2 + 3x + 2 = 0$.

Solution: Since $a = 1$, $b = 3$ and $c = 2$, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2 \times 1} \\ &= \frac{-3 \pm \sqrt{1}}{2} \\ &= \frac{-3 + 1}{2} \quad \text{or} \quad \frac{-3 - 1}{2} \\ &= \frac{-2}{2} \quad \text{or} \quad \frac{-4}{2} \\ &= -1 \quad \text{or} \quad -2\,. \end{aligned}$$

$\square$

Consider the quadratic equation $ax^2 + bx + c = 0$.

- If $b^2 - 4ac > 0$ then the equation has two real solutions. (See Example 2 above.)
- If $b^2 - 4ac = 0$ then the equation has exactly one solution.
- If $b^2 - 4ac < 0$ then the equation has **no** real solutions (since the square root of a negative number is **NOT** real).

We have seen how to solve a quadratic equation (by using the quadratic formula). Now we will see how this can be used to obtain a factorisation.

Suppose we know that $ax^2 + bx + c = 0$ when $x = x_1$ or $x = x_2$.

We can use these solutions to **factorise** the quadratic expression $ax^2 + bx + c$.

In particular, we have $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

Example 3. Factorise the quadratic expression $x^2 + 3x + 2$.

Solution: We showed in Example 2 that $x^2 + 3x + 2 = 0$ when

$$x = -1 \quad \text{or} \quad x = -2.$$

Thus we can factorise $x^2 + 3x + 2$ as

$$\begin{aligned} x^2 + 3x + 2 &= (x - -1)(x - -2) \\ &= (x + 1)(x + 2) \end{aligned}$$

$\square$

Example 4. Factorise the quadratic expression $2x^2 + 9x - 6$.

Solution: By the quadratic formula, we have $2x^2 + 9x - 6 = 0$ if

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{81 - 4 \times 2 \times -6}}{2 \times 2} \\ &= \frac{-9 \pm \sqrt{129}}{4} \\ &= \frac{-9 + \sqrt{129}}{4} \quad \text{or} \quad \frac{-9 - \sqrt{129}}{4} \end{aligned}$$

Thus we have

$$\begin{aligned} 2x^2+9x-6 &= 2\left(x-\frac{-9+\sqrt{129}}{4}\right)\left(x-\frac{-9-\sqrt{129}}{4}\right) \\ &= 2\left(x+\frac{9-\sqrt{129}}{4}\right)\left(x+\frac{9+\sqrt{129}}{4}\right). \end{aligned}$$

$\square$

Note: If $b^2-4ac<0$ then $ax^2+bx+c=0$ has **no** real solutions.

In this case, the quadratic expression ax^2+bx+c **cannot** be factorised in $\mathbf{R}$.

Exercises

1. Solve the following quadratic equations for x :

   (a) $x^2+4x+3=0$ (b) $x^2+13x+42=0$ (c) $x^2+2x+1=0$
   (d) $x^2-2x-2=0$ (e) $2x^2-2x-2=0$ (f) $x^2+x+1=0$

2. Factorise the following quadratic expressions:

   (a) x^2+4x+3 (b) $x^2+13x+42$ (c) x^2+2x+1
   (d) x^2-2x-2 (e) $2x^2-2x-2$ (f) x^2+x+1

1.3 Cubics

We use factorisation to solve the **cubic** equation $ax^3 + bx^2 + cx + d = 0$.

Example 5. (a) Factorise $x^3 + 3x^2 + 2x$.

(b) Hence solve the cubic equation $x^3 + 3x^2 + 2x = 0$.

Solution: (a) $x^3 + 3x^2 + 2x = x(x^2 + 3x + 2) = x(x + 1)(x + 2)$.

(b) Thus

$$x^3 + 3x^2 + 2x = 0 \text{ if and only if } x(x + 1)(x + 2) = 0.$$

So

$$x = 0 \text{ or } x + 1 = 0 \text{ or } x + 2 = 0.$$

That is,

$$x = 0 \text{ or } x = -1 \text{ or } x = -2.$$

$\square$

Question: In general, how do we factorise a cubic?

Answer: We use the following fact, which is known as the **Factor Theorem**:

If f is a polynomial, and if $f(b) = 0$, then $x - b$ is a factor of $f(x)$

That is, if f is a polynomial and if $f(b) = 0$, then we can write

$$f(x) = (x - b) \times g(x)$$

for some polynomial g . (The polynomial g is usually found by using long division.)

Notes:

- A **polynomial** is an expression of the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0,$$

  where $a_0,\ a_1,\ a_2,\ \ldots,\ a_n$ are real numbers. The number n is known as the **degree** of the polynomial.

  Note that the powers of x must be positive integers (so that we have terms such as $x,\ x^2,\ x^3,\ \ldots$). In particular, we cannot have terms such as $\sqrt{x}$ as part of a polynomial.

- **Factorising** a polynomial involves rewriting the polynomial as a **product** of terms, where each term is either

  ⋆ linear (i.e., of the form $bx + c$),

  or else

  ⋆ quadratic with negative discriminant (i.e., of the form ax^2+bx+c with $b^2 - 4ac < 0$).

Example 6. (a) Factorise $x^3 + x^2 - 4x - 4$.

(b) Hence solve the cubic equation $x^3 + x^2 - 4x - 4 = 0$.

Solution: (a) Let $f(x) = x^3 + x^2 - 4x - 4$.

$$\begin{aligned} f(0) &= 0^3 + 0^2 - 4 \times 0 - 4 = -4 \neq 0 \\ f(1) &= 1^3 + 1^2 - 4 \times 1 - 4 = -6 \neq 0 \\ f(2) &= 2^3 + 2^2 - 4 \times 2 - 4 = 8 + 4 - 8 - 4 = 0. \end{aligned}$$

Thus, by the Factor Theorem, $x - 2$ is a factor of $f(x)$.

That is,

$$x^3 + x^2 - 4x - 4 = (x-2)g(x) \text{ for some polynomial } g .$$

To find $g(x)$, we write $g(x) = \dfrac{x^3 + x^2 - 4x - 4}{x - 2}$, and use long division[*] .

$$\begin{array}{rrrrrrrrrrr}
 & & & & x^2 & + & 3x & + & 2 & & \\ \cline{5-11}
x & - & 2 & \big) & x^3 & + & x^2 & - & 4x & - & 4 \\
 & & & - & (x^3 & - & 2x^2\,) & & & & \\ \cline{5-11}
 & & & & & & 3x^2 & - & 4x & - & 4 \\
 & & & & & - & (3x^2 & - & 6x) & & \\ \cline{7-11}
 & & & & & & & & 2x & - & 4 \\
 & & & & & & & - & (2x & - & 4) \\ \cline{9-11}
 & & & & & & & & & & 0
\end{array}$$

Thus

$$\begin{aligned}
x^3 + x^2 - 4x - 4 &= (x-2)(x^2 + 3x + 2) \\
&= (x-2)(x+1)(x+2).
\end{aligned}$$

(b) Then $x^3 + x^2 - 4x - 4 = 0$ if and only if $(x-2)(x+1)(x+2) = 0$,

i.e. if and only if $\quad x = 2, \;\; x = -1 \;\;$ or $\;\; x = -2$.

$\square$

Example 7. Solve the cubic equation $x^3 - 2x^2 + 1 = 0$.

Solution: Let $f(x) = x^3 - 2x^2 + 1$. Since $f(1) = 0$ then $x - 1$ is a factor of $f(x)$.

$$\begin{array}{rrrrrrrrrrr}
 & & & & x^2 & - & x & - & 1 & & \\ \cline{5-11}
x & - & 1 & \big) & x^3 & - & 2x^2 & + & 0x & + & 1 \\
 & & & - & (x^3 & - & x^2\,) & & & & \\ \cline{5-11}
 & & & & & & -x^2 & + & 0x & + & 1 \\
 & & & & & - & (-x^2 & + & x) & & \\ \cline{7-11}
 & & & & & & & & -x & + & 1 \\
 & & & & & & & - & (-x & + & 1) \\ \cline{9-11}
 & & & & & & & & & & 0
\end{array}$$

[*]The Maths 1 section of TCOLE has an animation of the long division used in this example.

We have $x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1)$ and so

$$\begin{aligned} x^3 - 2x^2 + 1 = 0 \quad &\iff \quad (x-1)(x^2-x-1) = 0 \\ &\iff \quad x - 1 = 0 \quad \text{or} \quad x^2 - x - 1 = 0 \\ &\iff \quad x = 1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{1+4}}{2} \\ &\iff \quad x = 1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{5}}{2} \,. \end{aligned}$$

$\square$

Note: We have not fully factorised $x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1)$. A full factorisation is not necessary because this question has only asked us to "solve" the equation.

Exercises

1. Factorise:

   (a) $x^3 + 4x^2 + 3x$ (b) $x^3 - 2x^2 - x + 2$ (c) $x^3 - 6x^2 + 12x - 8$

2. Solve for x :

   (a) $2x^3 - 4x^2 + 2 = 0$ (b) $x^3 + 3x^2 + 3x + 2 = 0$ (c) $x^3 + 4x^2 + 5x + 6 = 0$
   (d) $x^4 - 2x^2 - 3x - 2 = 0$ (e) $x^4 - 10x^2 + 9 = 0$

1.4 Special Factorisations

- We have

$$\boxed{x^2 - a^2 = (x+a)(x-a)}$$

This is known as the **difference of perfect squares** formula.

- Note that $x^2 + a^2$ **cannot** be factorised in $\mathbf{R}$.

- However, we **can** factorise $x^3 + a^3$:

Let $f(x) = x^3 + a^3$. Then $f(-a) = (-a)^3 + a^3 = -a^3 + a^3 = 0$.
Thus, by the Factor Theorem, $x + a$ is a factor of $x^3 + a^3$.

$$\begin{array}{rrrrrrrr}
 & & x^2 & - & ax & + & a^2 & \\ \cline{3-8}
x + a & \big) & x^3 & + & 0x^2 & + & 0x & + \; a^3 \\
 & - & (x^3 & + & ax^2) & & & \\ \cline{3-8}
 & & & & -ax^2 & + & 0x & + \; a^3 \\
 & & & - & (-ax^2 & - & a^2x) & \\ \cline{5-8}
 & & & & & & a^2x & + \; a^3 \\
 & & & & & - & (a^2x & + \; a^3) \\ \cline{7-8}
 & & & & & & & 0
\end{array}$$

Thus

$$\boxed{x^3 + a^3 = (x+a)\left(x^2 - ax + a^2\right)}$$

- By replacing a with $-a$ in the above factorisation, we have

$$\boxed{x^3 - a^3 = (x-a)\left(x^2 + ax + a^2\right)}$$

Exercises

Factorise:

(a) $x^2 - 1$ (b) $t^3 + 1$ (c) $x^3 - 27$ (d) $x^4 - 1$
(e) $x^3y - y^3x$ (f) $x^2 - 4y^2$

1.5 Other Factorisations

The Factor Theorem can be used to factorise **many** polynomials. However, it cannot be used for **every** polynomial, because we cannot always find a b–value which satisfies $f(b) = 0$.

Sometimes, however, we can factorise polynomials by making use of the result that

$$(m+n)^2 = m^2 + 2mn + n^2.$$

Example 8. Factorise $x^4 + 2x^2 + 4$.

Solution: Let $f(x) = x^4 + 2x^2 + 4$. We cannot find any real number b which satisfies $f(b) = 0$, so we cannot use the Factor Theorem.

Since $x^4 = (x^2)^2$ and $4 = 2^2$, let us consider the expression $(x^2+2)^2$. We know that

$$\begin{aligned} (x^2+2)^2 &= x^4 + 4x^2 + 4. \\ \text{That is,} \quad x^4 + 4x^2 + 4 &= (x^2+2)^2. \end{aligned}$$

Then

$$\begin{aligned} x^4 + 2x^2 + 4 &= (x^2+2)^2 - 2x^2 \\ &= (x^2+2)^2 - (\sqrt{2}\,x)^2 \quad \text{which is a difference of perfect squares} \\ &= ((x^2+2) + \sqrt{2}\,x)((x^2+2) - \sqrt{2}\,x) \\ &= (x^2 + \sqrt{2}\,x + 2)(x^2 - \sqrt{2}\,x + 2). \end{aligned}$$

Note that these quadratic terms cannot be factorised further within $\mathbf{R}$, because they each have negative discriminant. Thus we have finished the factorisation. □

Exercises

Factorise the following expressions:

(a) $x^4 + x^2 + 4$ (b) $36x^4 + 15x^2 + 4$

(c) $t^4 - t^2 + 1$ (d) $t^6 + 1$ **Hint:** Rewrite $t^6 + 1$ as $(t^2)^3 + 1^3$.

1.6 Pascal's Triangle and the Binomial Theorem

Pascal's triangle is constructed as follows:

1. We start with a triangle of 1's which can be extended down forever.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & & & 1 & & \\ & 1 & & & & & & 1 & \\ 1 & & & & & & & & 1 \\ & & & & \text{etc.} & & & & \end{array}$$

2. We fill in the triangle by adding pairs of adjacent numbers, and then writing the answer in the row below the pair of added numbers:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ & & & & & & \text{etc.} & & & & & & \end{array}$$

There is a result, known as the Binomial Theorem, which says that

$$(a+x)^n = a^n + {}^n\mathrm{C}_1\, a^{n-1}x + {}^n\mathrm{C}_2\, a^{n-2}x^2 + \ldots + {}^n\mathrm{C}_r\, a^{n-r}x^r + \ldots + x^n.$$

(This result is on the formula sheet which is provided in the Maths 1 exams.)

The numbers ${}^n\mathrm{C}_1$, ${}^n\mathrm{C}_2$, ... in the Binomial Theorem can be found using a particular button on your calculator. We will learn about the numbers ${}^n\mathrm{C}_1$, ${}^n\mathrm{C}_2$, ... later in the year.

For now we will see how we can easily write down some Binomial Theorem results by using the numbers from the rows of Pascal's Triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \text{etc.} & & & & & &
\end{array}
$$

We have $(a+x)^0 = 1$

We have $(a+x)^1 = 1a + 1x$

We have $(a+x)^2 = 1a^2 + 2ax + 1x^2$

We have $(a+x)^3 = 1a^3 + 3a^2x + 3ax^2 + 1x^3$

etc.

Example 9. Find the expanded form of

(a) $(a+x)^5$ (b) $(a-2x)^5$.

Solution: (a) The sixth row of Pascal's Triangle contains the numbers

$$1, \quad 5, \quad 10, \quad 10, \quad 5 \text{ and } 1.$$

These are the coefficients in the expanded form of $(a+x)^5$, and so we have

$$(a+x)^5 = 1a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + 1x^5.$$

(b) We use the same coefficients to obtain

$$
\begin{aligned}
(a-2x)^5 &= 1a^5 + 5a^4 \times -2x + 10a^3 \times (-2x)^2 + 10a^2 \times (-2x)^3 + \\
&\qquad 5a \times (-2x)^4 + 1 \times (-2x)^5 \\
&= a^5 - 10a^4x + 40a^3x^2 - 80a^2x^3 + 80ax^4 - 32x^5
\end{aligned}
$$

□

Exercises

Find the expanded form of each of the following expressions:

(a) $(a+x)^6$ (b) $(a-x)^6$ (c) $(1+x^2)^4$ (d) $(2-x)^4$

1.7 Completing the Square

Completing the square is the process of rewriting a quadratic expression x^2+bx+c in the form $(x+p)^2+q$. That is, we must find numbers p and q such that

$$x^2+bx+c = (x+p)^2+q \quad \text{for all values of } x\,.$$

We make use of the fact that

$$(x+p)^2 = x^2+2px+p^2$$

and so we choose p to be half the coefficient of x, i.e., $p=\frac{b}{2}$.

Example 10. Complete the square for the following quadratic expressions:

(a) x^2+6x+5

Solution: **Method 1.** Note that half of the coefficient of x is 3, and $(x+3)^2 = x^2+6x+9$. Subtracting 4 from both sides gives

$$x^2+6x+5=(x+3)^2-4$$

Method 2. Put $x^2+6x+5=(x+p)^2+q$ and solve for p and q. Note that $(x+p)^2+q=x^2+2px+p^2+q$, and so we must find the values of p and q that satisfy

$$x^2+6x+5=x^2+2px+p^2+q \quad \text{for all values of } x\,.$$

We see that we must have $6=2p$ (coefficients of x) and $5=p^2+q$ (constant terms).

Thus $p=3$ and $3^2+q=5$, i.e., $p=3$ and $q=5-9=-4$.

Hence $x^2+6x+5=(x+3)^2-4$. □

(b) $2x^2 - 4x + 7$

Solution: First write $2x^2 - 4x + 7 = 2\left(x^2 - 2x + \frac{7}{2}\right)$. Now complete the square for $x^2 - 2x + \frac{7}{2}$.

Note that $(x-1)^2 = x^2 - 2x + 1$. Adding $\frac{5}{2}$ to both sides gives

$$x^2 - 2x + \frac{7}{2} = (x-1)^2 + \frac{5}{2} .$$

Finally, $\quad 2x^2 - 4x + 7 = 2\left(x^2 - 2x + \frac{7}{2}\right) = 2\left((x-1)^2 + \frac{5}{2}\right)$.

Alternatively, we can write $\quad 2x^2 - 4x + 7 = 2(x-1)^2 + 5$. $\square$

Exercises

Complete the square for the following quadratic expressions:

(a) $x^2 + 4x + 7$ (b) $x^2 - 4x + 7$ (c) $3x^2 + 6x + 4$

1.8 Equations involving Square Roots

Note that if $x^2 = 9$ then $x = \pm 3$.

Thus the number 9 has **two** square roots, namely 3 and -3 . We write

$$\sqrt{9} = 3, \quad -\sqrt{9} = -3 \quad \text{and} \quad \pm\sqrt{9} = \pm 3.$$

That is,

- we use the symbol $\sqrt{\ }$ to denote the **positive** square root, and
- we use $-\sqrt{\ }$ to denote the **negative** square root, and
- we use $\pm\sqrt{\ }$ when we want to denote **both** square roots.

In this section we will solve some equations involving square roots. This can be done by

- writing the equation so that it has the square root term by itself on one side of the equation, and then
- squaring both sides of this equation.

Note:

When we square both sides of an equation, we

must check our solutions

because extra values might have been introduced.

Example 11. Solve $2 + \sqrt{x^2+1} = x + 1$.

Solution: We start by rewriting the equation as

$$\sqrt{x^2+1} = x - 1.$$

Then squaring both sides gives

$$\begin{aligned} (\sqrt{x^2+1})^2 &= (x-1)^2 \\ \text{That is,} \quad x^2+1 &= x^2-2x+1 \\ \text{That is,} \quad 0 &= -2x \\ \text{That is,} \quad x &= 0 \end{aligned}$$

Note that this x –value is the solution to the **squared** equation. We **must check** whether this value is also a solution of the **original** equation given in the question:

Note that when $x = 0$ then

$$\begin{aligned} 2 + \sqrt{x^2+1} &= 2 + \sqrt{0^2+1} = 2 + \sqrt{1} = 3 \\ \text{whereas } x+1 &= 0+1 = 1 \end{aligned}$$

Thus $x = 0$ is **not** a solution of the original equation.

Therefore, there are **no** solutions to the original equation. □

Example 12. Solve $\sqrt{x+1} = x - 1$.

Solution:

$$\begin{aligned} \text{If } \sqrt{x+1} &= x-1 \\ \text{then } \left(\sqrt{x+1}\right)^2 &= (x-1)^2 \\ \text{i.e. } \quad x+1 &= x^2-2x+1 \\ 0 &= x^2-3x \\ 0 &= x(x-3)\,. \end{aligned}$$

Thus, we know that

$$x = 0 \quad \text{or} \quad x = 3\,.$$

Next we must check whether these values work in the original equation. We need to check whether the values $x = 0$ and $x = 3$ satisfy

$$\sqrt{x+1} = x - 1$$

(which was the equation given in the question):

Check $x = 0$. When $x = 0$, we have

$\sqrt{x+1} = \sqrt{0+1} = \sqrt{1} = 1$, and

$x - 1 = 0 - 1 = -1$.

Thus $\sqrt{x+1} \neq x - 1$ when $x = 0$.

Check $x = 3$. When $x = 3$, we have

$\sqrt{x+1} = \sqrt{3+1} = \sqrt{4} = 2$, and

$x - 1 = 3 - 1 = 2$.

Thus $\sqrt{x+1} = x - 1$ when $x = 3$.

Therefore, the only correct answer is $x = 3$.

$\square$

Exercises

Solve for x :

(a) $\sqrt{x+1} = 1 - x$

(b) $\sqrt{3x-5} = x - 1$

(c) $\sqrt{2x+1} - \sqrt{x} = 1$

(d) $\sqrt{3x-2} = -x$

1.9 Sets

A **set** is a collection of objects. The objects in a set are called **elements**.

Example 13. Consider the set $A = \{1,\ 2,\ 3,\ 4\}$.

The elements of A are $1,\ 2,\ 3$ and 4 . When we write

$$1 \in A$$

we just mean that 1 **is an element of** A . That is, 1 **is in** the set A .

Similarly, when we write

$$2 \in A$$

we just mean that 2 **is in** the set A .

Consider any sets D and E . Then

$$\begin{aligned} D \cup E &= \text{ the set of elements which are in } D \textbf{ or } E \text{ (or both)} \\ &= D \textbf{ union } E, \text{ and} \end{aligned}$$

$$\begin{aligned} D \cap E &= \text{ the set of elements which are in } D \textbf{ and } E \\ &= D \textbf{ intersection } E. \end{aligned}$$

Example 14. Suppose that $D = \{1,\ 3,\ 4\}$ and $E = \{2,\ 4,\ 6,\ 7\}$. Then

$$\begin{aligned} D \cup E &= \{1,\ 2,\ 3,\ 4,\ 6,\ 7\} \\ D \cap E &= \{4\} \end{aligned}$$

Set difference.
If F and G are sets, then

$G \setminus F$ is the set of all elements in G which are **not** in F.

Example 15. $\mathbf{R} \setminus \{0\}$ is the set of all real numbers **except for** 0.

Exercises

Let $C = \{-2,\ -1,\ \frac{1}{2}\ ,\ 3\}$ and $D = \{-1,\ 3\}$. Find

(a) $C \cap D$ (b) $C \cup D$ (c) $C \setminus D$

1.10 Ordering Real Numbers

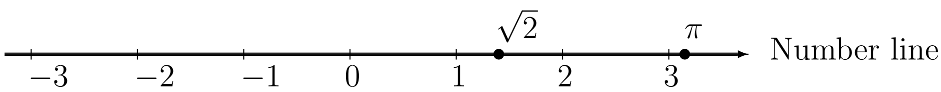
Consider any real numbers a and b.

Notation:

- We write $a < b$ (or $b > a$) whenever $b - a$ is positive.
- We write $a \leq b$ (or, alternatively, $b \geq a$) if $a < b$ or $a = b$.

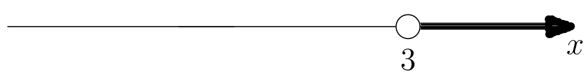
The number line

We can represent the real numbers with a number line:

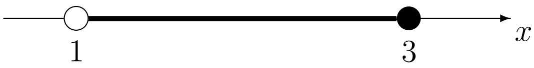


If $b > a$ then b lies to the right of a on the number line:

Example 16. The set $\{x \in \mathbf{R} \mid x > 3\}$ is the set of all real numbers which lie to the right of 3 on the number line:



Example 17. The set $\{x \in \mathbf{R} \mid 1 < x \leq 3\}$ is the set of all real numbers which lie to the right of 1 and to the left of (and including) 3 :



Intervals

An **interval** is a set of real numbers with "no gaps". We often denote intervals by using round and/or square brackets, as detailed below:

- A **round** bracket means that the corresponding endpoint is **not** included in the interval; that is, no "=" appears in the corresponding inequality symbol.

  On the number line this endpoint is represented by an **open** circle; that is, at the endpoint of the interval, we draw a small circle which is **not** coloured in.

In contrast,

- a **square** bracket means that the corresponding endpoint **is** included in the interval; that is, an "=" **does** appear in the corresponding inequality symbol.

  On the number line this endpoint is represented by an **closed** circle; that is, at the endpoint of the interval, we draw a small circle which **is** coloured in.

	Interval:	**Bracket Notation:**	**Interval on the number line:**
(a)	$\{x \mid a < x < b\}$	(a, b)	
(b)	$\{x \mid a \leq x \leq b\}$	$[a, b]$	
(c)	$\{x \mid a < x \leq b\}$	$(a, b]$	
(d)	$\{x \mid a \leq x < b\}$	$[a, b)$	

	Interval:	Bracket Notation:	Interval on the number line:
(e)	$\{x \mid x > a\}$	(a, ∞)	
(f)	$\{x \mid x \geq a\}$	$[a, \infty)$	
(g)	$\{x \mid x < b\}$	$(-\infty, b)$	
(h)	$\{x \mid x \leq b\}$	$(-\infty, b]$	
(i)	$\mathbf{R}$	$(-\infty, \infty)$	
(j)	$\mathbf{R}^+$	$(0, \infty)$	
(k)	$\mathbf{R}^-$	$(-\infty, 0)$	

Exercises

Represent the following intervals using bracket notation:

(a) $\{x \mid 4 \leq x < 12\}$ (b) $\{x \mid x \leq 4\}$ (c) $\{x \mid x \geq 4\}$

(d) $\{x \mid 4 \leq x \leq 12\} \cap \{x \mid 7 \leq x < 22\}$ (e) $\{x \mid 4 \leq x \leq 12\} \cup \{x \mid 7 \leq x < 22\}$

1.11 Inequalities

Expressions involving $<$, $\leq$, $>$ or $\geq$ are called **inequalities**. We need to know how to solve inequalities, and this depends on knowing the following three rules:

Suppose a, b and c are any given real numbers.

(I) If $a > b$ then $a + c > b + c$.

That is, adding (or subtracting) a number to both sides of an inequality does **not** change the direction of the inequality symbol.

(II) If $a > b$ and $c > 0$ then $ac > bc$.

That is, multiplying (or dividing) an inequality by a **positive** number does **not** change the direction of the inequality symbol.

(III) If $a > b$ and $c < 0$ then $ac < bc$.

That is, multiplying (or dividing) an inequality by a **negative** number **changes** the direction of the inequality symbol.

Example 18. (a) Solve the inequality $(x + 4)(x - 1) > 0$.

(b) Solve the inequality $(x + 4)(x - 1) < 0$.

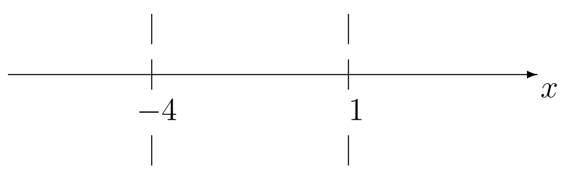
Solution: First we solve the corresponding **equation**:

$$(x + 4)(x - 1) = 0.$$

We obtain

$$x = -4 \text{ or } x = 1.$$

These two values break up the real line into three sections:



On each of these intervals, we determine the sign of the quadratic $(x + 4)(x - 1)$, as follows:

$(x+4)(x-1)$	$+$		$-$		$+$
x		-4		1	
	e.g. if $x=-5$ then $(x+4)(x-1)=6$, which is positive		e.g. if $x=0$ then $(x+4)(x-1)$ $=-4$, which is negative		e.g. if $x=2$ then $(x+4)(x-1)=6$, which is positive

(a) Since we want to solve $(x+4)(x-1)>0$, we look for the section(s) of the number–line where $(x+4)(x-1)$ is *positive.*

We see that $(x+4)(x-1)>0$ when $x<-4$ or $x>1$.

That is, we must have

$$x \in (-\infty, -4) \cup (1, \infty).$$

(b) Since we want to solve $(x+4)(x-1)<0$, we look for the section(s) of the number–line where $(x+4)(x-1)$ is *negative.*

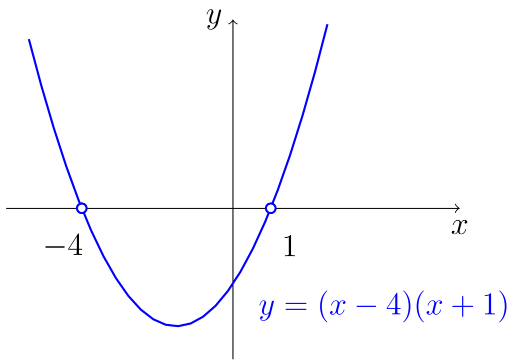
We see that $(x+4)(x-1)<0$ when $-4<x<1$.

That is, we must have

$$x \in (-4, 1).$$

$\square$

Note: We shall learn in Chapter 3 that the **graph** of $y=(x+4)(x-1)$ is as shown:



From this graph, it is immediately clear that

(a) $(x+4)(x-1)>0$ (i.e. $y>0$) when $x<-4$ or $x>1$.

(b) $(x+4)(x-1)<0$ (i.e. $y<0$) when $-4<x<1$.

This technique of solving an inequality by looking at a graph is certainly worth remembering! At the end of Chapter 2, we shall see some more examples in which we will **use graphs to help solve inequalities**.

Exercises

Solve the following inequalities for x :

(a) $1 + x < 7x + 5$ (b) $4 \leq 3x - 2 < 13$ (c) $2x + 1 \leq 4x - 3 \leq x + 7$

(d) $x^2 + 3x < 4$ (e) $x^2 + 5x > -6$

Inequalities involving fractions

The following examples illustrate that we must be **very careful** when we are trying to solve inequalities involving fractions.

Note that

$$\frac{f(x)}{3} \; < \; 4 \quad \Rightarrow \quad f(x) \; < \; 12.$$

(Notice that the direction of the inequality symbol did **NOT** change, because we multiplied the inequality by a positive number.)

In contrast, note that

$$\frac{f(x)}{-3} \; < \; 4 \quad \Rightarrow \quad f(x) \; > \; -12.$$

(Notice that the direction of the inequality symbol **DID** change, because we multiplied the inequality by a **negative** number.)

Finally, consider the inequality

$$\frac{f(x)}{g(x)} \; < \; 4.$$

If we multiply both sides of this inequality by $g(x)$, we **do not know** which way to make the inequality symbol point, because we **do not know** whether $g(x)$ is positive or negative. Two ways to approach this problem are given next.

Method 1: We work through two separate cases:

Case 1: If $g(x) > 0$ we obtain $f(x) \ < \ 4g(x)$.

Note that the direction of the inequality symbol does **NOT** change.

Case 2: If $g(x) < 0$ we obtain $f(x) \ > \ 4g(x)$.

Note that the direction of the inequality symbol **DOES** change.

Method 2: We multiply both sides of the inequality by $(g(x))^2$.

Since $(g(x))^2$ is **NEVER NEGATIVE**, the direction of the inequality symbol does **NOT** change. We have

$$\begin{aligned} \frac{f(x)}{g(x)} &< 4 \\ \Rightarrow \frac{f(x)}{g(x)} \times (g(x))^2 &< 4\,(g(x))^2 \\ \text{i.e. } f(x)g(x) &< 4\,(g(x))^2 \end{aligned}$$

This can then be solved in a similar way to the example on page 22.

Example 19. Solve the inequality $\dfrac{1+x}{1-x} \ \leq \ 1$.

Solution: **Method 1:**
Case 1:

If $1 - x > 0$ we get $1 + x \ \leq \ 1 - x$.

That is, we have $1 > x$ and $2x \ \leq \ 0$.

That is, we have $x < 1$ and $x \ \leq \ 0$.

That is, $x \ \leq \ 0$.

Case 2:

If $1 - x < 0$ we get $1 + x \ \geq \ 1 - x$.

That is, we have $1 < x$ and $2x \ \geq \ 0$.

That is, we have $x > 1$ and $x \ \geq \ 0$.

That is, $x \ > \ 1$.

Combining these two cases gives the solution set $(-\infty, 0] \cup (1, \infty)$

Method 2:

To avoid having to do separate cases, we instead multiply both sides of the inequality by the **non–negative** quantity $(1-x)^2$.

First note that if

$$\frac{1+x}{1-x} \leq 1$$

then we immediately know that we must have

$$x \neq 1$$

(so that the denominator of the fraction is NOT zero).

Now we will multiply both sides of the inequality

$$\frac{1+x}{1-x} \leq 1$$

by the **non–negative** quantity $(1-x)^2$.

We obtain

$$\begin{aligned}
\frac{1+x}{1-x}(1-x)^2 &\leq 1\,(1-x)^2 \\
(1+x)(1-x) &\leq 1-2x+x^2 \\
1-x^2 &\leq 1-2x+x^2 \\
0 &\leq 2x^2-2x \\
0 &\leq 2x(x-1)
\end{aligned}$$

Now note that $2x(x-1)=0 \iff x=0$ or $x=1$.

$$\begin{array}{c|c|c|c}
2x(x-1) & + & - & + \\
\hline
x & \quad 0 & \quad 1 & \\
 & \text{Try } x=-1 & \text{Try } x=\frac{1}{2} & \text{Try } x=2
\end{array}$$

Thus

$$(x \leq 0 \quad \text{or} \quad x \geq 1) \quad \text{and} \quad x \neq 1$$

$$x \leq 0 \quad \text{or} \quad x > 1\,.$$

Method 3:

Just like in Method 2, we will multiply both sides of the inequality by $(1-x)^2$. But first we will rearrange the inequality to get 0 on the right–hand–side. The advantage of following this method is that we can avoid the expanding step and the factorising step!

$$
\begin{aligned}
\frac{1+x}{1-x} &\leq 1 \\
\frac{1+x}{1-x} - 1 &\leq 0 \\
\frac{1+x}{1-x} - \frac{1-x}{1-x} &\leq 0 \\
\frac{1+x-(1-x)}{1-x} &\leq 0 \\
\frac{1+x-1+x}{1-x} &\leq 0 \\
\frac{2x}{1-x} &\leq 0 \\
\frac{2x}{1-x}\,(1-x)^2 &\leq 0\,(1-x)^2 \\
2x(1-x) &\leq 0
\end{aligned}
$$

	$x < 0$		$0 < x < 1$		$x > 1$
$2x(1-x)$	$-$	0	$+$	1	$-$
	Try $x=-1$		Try $x=\frac{1}{2}$		Try $x=2$

Thus

$$(x \leq 0 \quad \text{or} \quad x \geq 1) \quad \text{and} \quad x \neq 1$$

$$x \leq 0 \quad \text{or} \quad x > 1\,.$$

$\square$

Exercises

Solve the following inequalities for x :

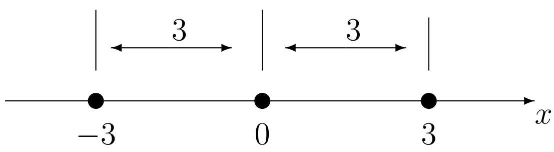
(a) $\dfrac{1+x}{1-x} > 1$ (b) $\dfrac{x}{3+x} < 4$

1.12 Absolute Values

The **absolute value** of x , denoted by $|x|$, is the distance from x to 0 on the number line. Since distances are always positive or 0 , we have

$$|x| \geq 0$$

$$|3| = 3$$
$$|-3| = 3$$
$$|0| = 0$$



Example 20.

In general, we have

$$\boxed{|x| = x \quad \text{if} \quad x \geq 0 \qquad \text{and} \qquad |x| = -x \quad \text{if} \quad x < 0}$$

Examples:

$$|3| = 3 \quad \text{since} \quad 3 \geq 0$$
$$|-3| = -(-3) = 3 \quad \text{since} \quad -3 < 0$$

Note: Absolute values can also be removed by using the following formula:

$$\boxed{|x| = \sqrt{x^2}}$$

This rule holds for any real number x .

Example:

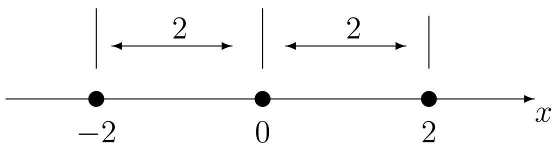
$$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$$

Note:

$$\boxed{\sqrt{x^2} = |x| \quad \textbf{but} \quad \left(\sqrt{x}\right)^2 = x}$$

Example 21. Solve $|x| = 2$.

Solution: We need to find all real numbers x such that
the distance between x and 0 is equal to 2.



Measuring a distance of 2 units from 0 leads to the numbers -2 and 2 .

Thus $\quad |x| = 2 \iff x = -2$ or $x = 2$.

□

In general, for any positive number a , we have

$$\boxed{|x| = a \quad \text{if and only if} \quad x = a \quad \text{or} \quad x = -a}$$

Warning: Solving the equation $|x| = 2$ leads to two answers: $x = \pm 2$.
However, finding $|2|$ leads to only one answer, namely $|2| = 2$.

Example 22. Solve $|2x-5|=3$.

Solution: Suppose that $|2x-5|=3$. By the above rule, we have

$$2x-5=3 \text{ or } 2x-5=-3.$$

Thus

$$2x=8 \text{ or } 2x=2,$$

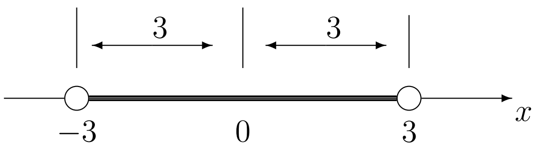
and so

$$x=4 \text{ or } x=1.$$

$\square$

Example 23. Solve $|x|<3$.

Solution: We want the distance between x and 0 to be less than 3.



We see that x can be any number between -3 and 3.
Therefore, $-3<x<3$. $\square$

In general, for any positive number a, we have

$$\boxed{|x|<a \quad \text{if and only if} \quad -a<x<a}$$

Similarly,

$$\boxed{|x|\le a \quad \text{if and only if} \quad -a\le x\le a}$$

Example 24. Solve $|x-5|<2$.

Solution: Suppose that $|x-5|<2$. By the above rule, we have

$$-2<x-5<2.$$

Hence

$$-2+5<x<2+5,$$

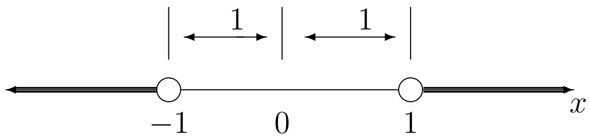
and so

$$3<x<7.$$

$\square$

Example 25. Solve $|x|>1$.

Solution: We want the distance between x and 0 to be greater than 1 .



We see that we must have $x<-1$ or $x>1$. $\square$

In general, for any positive number a , we have

$$\boxed{|x|>a \quad \text{if and only if} \quad x<-a \quad \text{or} \quad x>a}$$

Similarly,

$$\boxed{|x|\geq a \quad \text{if and only if} \quad x\leq -a \quad \text{or} \quad x\geq a}$$

Example 26. Solve $|3x+2| \geq 4$.

Solution: Suppose that $|3x+2| \geq 4$. By the above rule, we have

$$3x+2 \geq 4 \quad \text{or} \quad 3x+2 \leq -4.$$

Thus

$$3x \geq 2 \quad \text{or} \quad 3x \leq -6,$$

and so

$$x \geq \frac{2}{3} \quad \text{or} \quad x \leq -2.$$

$\square$

Further properties of absolute value

For all real numbers a, b and x, we have

(a) $|x-a| =$ the distance between x and a on the number line

(b) $|ab| = |a||b|$

(c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$

(d) $|x|^2 = x^2$

Exercises

Solve for x:

(a) $|3x-6| = 6$ (b) $|x-1| \leq 3$ (c) $|6x+1| > 7$

1.13 Answers to Chapter 1 Exercises

1.2: 1. (a) $-1,\ -3$ (b) $-6,\ -7$ (c) -1
(d) $1 \pm \sqrt{3}$ (e) $\frac{1\pm\sqrt{5}}{2}$ (f) No real solutions.

2. (a) $(x+1)(x+3)$ (b) $(x+6)(x+7)$ (c) $(x+1)^2$
(d) $(x-1-\sqrt{3})(x-1+\sqrt{3})$ (e) $2(x-\frac{1+\sqrt{5}}{2})(x-\frac{1-\sqrt{5}}{2})$
(f) Cannot be factorised within $\mathbf{R}$.

1.3: 1. (a) $x(x+1)(x+3)$ (b) $(x-2)(x+1)(x-1)$ (c) $(x-2)^3$

2. (a) $1,\ \frac{1\pm\sqrt{5}}{2}$ (b) -2 (c) -3 (d) $-1,\ 2$ (e) $\pm 1,\ \pm 3$

1.4: (a) $(x+1)(x-1)$ (b) $(t+1)(t^2-t+1)$ (c) $(x-3)(x^2+3x+9)$
(d) $(x^2+1)(x+1)(x-1)$ (e) $xy(x+y)(x-y)$ (f) $(x-2y)(x+2y)$

1.5: (a) $(x^2-\sqrt{3}x+2)(x^2+\sqrt{3}x+2)$
(b) $(6x^2-3x+2)(6x^2+3x+2)$
(c) $(t^2-\sqrt{3}\,t+1)(t^2+\sqrt{3}\,t+1)$
(d) $(t^2+1)(t^2-\sqrt{3}\,t+1)(t^2+\sqrt{3}\,t+1)$

1.6: (a) $a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$
(b) $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$
(c) $1+4x^2+6x^4+4x^6+x^8$
(d) $16-32x+24x^2-8x^3+x^4$

1.7: (a) $(x+2)^2+3$ (b) $(x-2)^2+3$ (c) $3[(x+1)^2+\frac{1}{3}]=3(x+1)^2+1$

1.8: (a) 0 (b) $2,3$ (c) $0,4$ (d) No solutions.

1.9: (a) $\{-1,3\}$ (b) C (c) $\{-2,\frac{1}{2}\}$

1.10: (a) $[4,12)$ (b) $(-\infty,4]$ (c) $[4,\infty)$ (d) $[7,12]$ (e) $[4,22)$

1.11: (a) $(-\frac{2}{3},\infty)$ (b) $[2,5)$ (c) $[2,3\frac{1}{3}]$ (d) $(-4,1)$ (e) $(-\infty,-3)\cup(-2,\infty)$

1.11: (a) $(0,1)$ (b) $(-\infty,-4)\cup(-3,\infty)$

1.12: (a) $4,0$ (b) $[-2,4]$ (c) $(-\infty,-\frac{4}{3})\cup(1,\infty)$