Chapter 18

An Introduction to Probability

- A **population** refers to the whole group of interest, whereas a **sample** refers to part of the population.

  Ideally a sample will be *representative* of the population. That is, ideally all the various subgroups of the population will appear in the sample *in similar proportions as they did in the population.* To achieve this goal, we will consider **random samples**:

  > In a **random sample**, each item is chosen randomly (entirely by chance) so that every item in the population has an equal chance of being chosen. Thus a **random sample of size n** is collected from a population by randomly choosing n items from a population; every set of n items in the population has an equal chance of being selected as the random sample.
  > (Random samples will be of particular interest to us in Chapters 20.4 and 25.)

- A statistical **experiment** is any happening in which the outcome is uncertain.
  The list of all possible outcomes for an experiment is called the **sample space**.
  An **event** is a subset of the sample space.

- In **probability** we assign a numerical value to the likelihood of a given event occurring.
  We denote the probability of the event A by $\Pr(A)$.

If all outcomes are equally likely, then

$$\boxed{\Pr(A) \;=\; \frac{\text{number of outcomes in the event } A}{\text{total number of possible outcomes}}}$$

Example 1. Suppose we toss a **\$1 coin** and a **\$2 coin**.

Then the possible outcomes are shown here:

\$1 coin	**\$2 coin**
heads	heads
tails	tails
heads	tails
tails	heads

If we let H represent that a coin shows 'heads', and T represent that a coin shows 'tails', then the sample space consists of these 4 possible outcomes:

$$\{\; H_{\$1}H_{\$2}, \quad T_{\$1}T_{\$2}, \quad H_{\$1}T_{\$2}, \quad T_{\$1}H_{\$2} \;\} \;.$$

Thus, if A is the event $\{H_{\$1}H_{\$2}\}$ (of receiving heads on both coins), then

$$\Pr(A) \;=\; \frac{\text{number of outcomes in this event } A}{\text{total number of outcomes}} \;=\; \frac{1}{4}\;.$$

□

Note that we always have $\boxed{\mathbf{0 \;\leq\; Pr(\boldsymbol{A}) \;\leq\; 1}\,.}$

Example 2. The letters of the word **AUSTRALIAN** are to be arranged in a row.

(a) Find the probability that the vowels will be together.

(b) Find the probability that all the **A**'s will occur together.

Solution:

(a) We will use the fact that

$$\Pr(\text{vowels are together}) = \frac{\text{number of arrangements with the vowels together}}{\text{total number of arrangements}}.$$

- First we list the letters: **AAAILNRSTU**.
  Note that there are 10 letters, of which 3 are **A**'s. Thus
  $$\text{the } \textit{total} \text{ number of arrangements} = \frac{10!}{3!} = 604\,800.$$
- To find out how many arrangements have the vowels together, consider "**AAAIU**" to be a single unit:

  $$\boxed{\textbf{AAAIU}}\ \textbf{LNRST}$$

  There are 6 different units to be arranged, and the 5 vowels (including 3 **A**'s) must also be arranged within their unit. Thus there are
  $$6! \times \frac{5!}{3!} = 14\,400 \text{ arrangements with the } \textit{vowels together}.$$

Therefore the probability that the vowels are together is $\dfrac{14\,400}{604\,800} = \dfrac{1}{42}$.

(b) We will use the fact that

$$\Pr(\text{all the } \mathbf{A}\text{'s occur together}) = \frac{\text{number of arrangements with all the } \mathbf{A}\text{'s together}}{\text{total number of arrangements}}.$$

- As above, the *total* number of arrangements = 604 800.
- To find out how many arrangements have all the **A**'s together, consider "**AAA**" to be a single unit:

  $$\boxed{\textbf{AAA}}\ \textbf{ILNRSTU}$$

  There are 8 different units to be arranged, with the 3 **A**'s needing to be arranged within their unit. Thus there are
  $$8! \times \frac{3!}{3!} = 40\,320 \text{ arrangements with all the } \mathbf{A}\text{'s together}.$$

Therefore the probability that all the **A**'s are together is $\dfrac{40\,320}{604\,800} = \dfrac{1}{15}$.

□

Example 3. Consider the following group of 5 people:

Adam, **B**eth, **C**hris, **D**onald and **E**ve.

Suppose two of these people are to be sent on holiday. Then the sample space consists of the following 10 pairs:

AB, AC, AD, AE
BC, BD, BE
CD, CE
DE

(a) Find the probability that **C**hris and **D**onald go on holiday.

(b) Find the probability that **B**eth goes on holiday.

Solution:

(a) There is only 1 outcome containing both **C** and **D**, out of the 10 possible outcomes. So the probability that **C**hris and **D**onald go on holiday is $\frac{1}{10}$.

(b) There are 4 outcomes containing **B** out of 10 possible outcomes. So the probability that **B**eth goes on holiday is $\frac{4}{10} = \frac{2}{5}$.

□

Suppose that A is an event. Then the **complement** of A is the event that A does **not** occur. We denote the complement of A by A' .

We have the following result: $\boxed{\Pr(A') = 1 - \Pr(A).}$

Example 4. For the example above, find the probability that **B**eth does **not** go on holiday.

Solution:

$$\begin{aligned} \Pr(\textbf{B}\text{eth does }\textbf{not}\text{ go on holiday}) &= 1 - \Pr(\textbf{B}\text{eth }\textit{does}\text{ go on holiday}) \\ &= 1 - \frac{2}{5} \quad \text{(using the answer from Example 3(b))} \\ &= \frac{3}{5} \end{aligned}$$

□

The notation $A \cap B$ means that A **and** B both occur.

Similarly, $A \cup B$ means that A **or** B (or *both* of them) occur.

Example 5. Suppose two (different) dice are to be tossed.

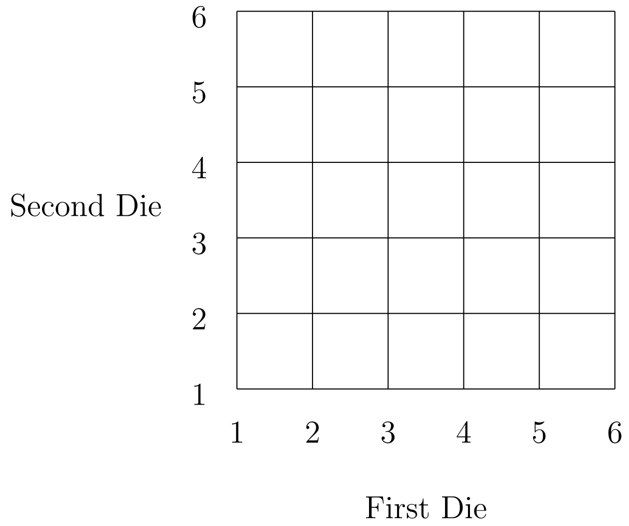
- Let A be the event {even number on both dice}.
- Let B be the event {two sixes}.
- Let C be the event {two fives}.

Find (a) $\Pr(A)$ (b) $\Pr(B)$ (c) $\Pr(C)$

(d) $\Pr(A \cup B)$ (e) $\Pr(A \cap B)$ (f) $\Pr(A \cup C)$.

Solution:

It is convenient to represent the possible outcomes when we toss two dice by using a grid, as follows:

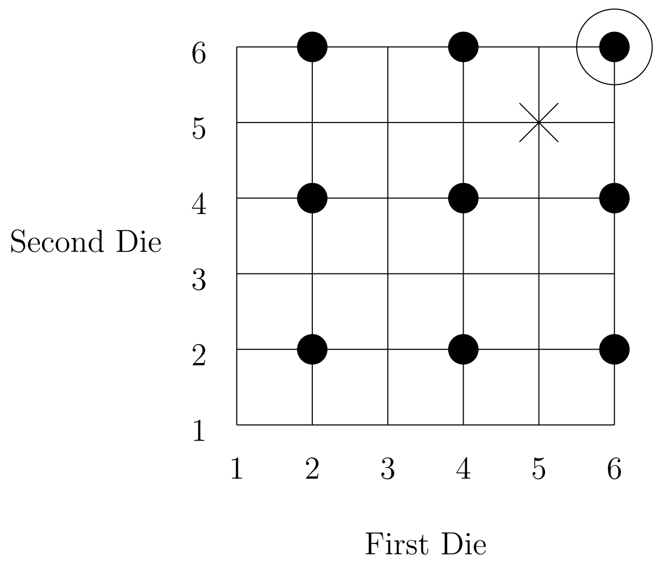


We see that there are 36 possible outcomes in total (corresponding to the 36 points on this grid).

Let $\bullet$ = event A,

and $\bigcirc$ = event B,

and $\times$ = event C.



(a) $\Pr(A) = \dfrac{9}{36} = \dfrac{1}{4}$.

(b) $\Pr(B) = \dfrac{1}{36}$.

(c) $\Pr(C) = \dfrac{1}{36}$.

(d) $\Pr(A \cup B) \;=\; \Pr(A \text{ or } B \text{ or } both) \;=\; \dfrac{9}{36} \;=\; \dfrac{1}{4}\,.$

(Note that although the outcome {two sixes} belongs to *both* A and B, it must only be counted *once*.)

(e) $\Pr(A \cap B) \;=\; \Pr(A \text{ and } B) \;=\; \dfrac{1}{36}\,.$

(f) $\Pr(A \cup C) \;=\; \Pr(A \text{ or } C \text{ or } both) \;=\; \dfrac{10}{36} \;=\; \dfrac{5}{18}\,.$ □

The following result, known as the **Addition Rule**, is included on the Formula Sheet given in the Mathematics 1 exams:

$$\boxed{\Pr(A \cup B) \;=\; \Pr(A) + \Pr(B) - \Pr(A \cap B)} \qquad \text{(Addition Rule)}$$

Note: In the previous example we can also use the Addition Rule to calculate $\Pr(A \cup B)$:

$$\begin{aligned}
\Pr(A \cup B) &= \Pr(A) \;+\; \Pr(B) \;-\; \Pr(A \cap B) \\
&= \frac{9}{36} \;+\; \frac{1}{36} \;-\; \frac{1}{36} \\
&= \frac{9}{36}\,.
\end{aligned}$$

Example 6. Suppose that two events A and B satisfy

$$\Pr(A) = 0.48, \quad \Pr(B) = 0.32 \;\text{ and }\; \Pr(A \cap B) = 0.25\,.$$

Find (a) $\Pr(A' \cap B')$ (b) $\Pr(A' \cup B')\,.$

Solution:

(a) We can calculate the answer easily by constructing a *table of intersections.*

We start by entering the given information. First, we are told that

- $\Pr(A) = 0.48$.
  This goes at the bottom of the column for A.

Next, we are told that

- $\Pr(B) = 0.32$.
  This goes at the right of the row for B.

$\cap$	A	A'	
B			0.32
B'			
	0.48		

$\Pr(B)$

$\Pr(A)$

The spaces in the *middle* of the table give the probabilities for the *intersection* of each pair of events.

So $\Pr(A \cap B)$ is the place where A and B cross, and we place 0.25 in that position:

$\cap$	A	A'	
B	0.25		0.32
B'			
	0.48		

$\Pr(A \cap B)$

Next, we can use the fact that $\Pr(A') = 1 - \Pr(A)$, and $\Pr(B') = 1 - \Pr(B)$, to calculate and enter the values for $\Pr(A')$ and $\Pr(B')$:

$\cap$	A	A'	
B	0.25		0.32
B'			0.68
	0.48	0.52	1

$\Pr(B')$

$\Pr(A')$

We can see that both the outside row and the outside column sum to 1.
Similarly, *the two entries inside each row and each column sum to the entry on the outside* (for example, $\Pr(A \cap B) + \Pr(A \cap B') = \Pr(A)$). We can now fill in the remaining spaces:

$\Pr(A' \cap B)$

$\Pr(A \cap B')$

$\cap$	A	A'	
B	0.25	0.07	0.32
B'	0.23	0.45	0.68
	0.48	0.52	1

$\Pr(A' \cap B')$

Now that the table is complete, we can easily write down that

$$\Pr(A' \cap B') = 0.45.$$

(b) Using the table of intersections that we have just constructed together with the Addition Rule, we have

$$\begin{aligned}\Pr(A' \cup B') &= \Pr(A') + \Pr(B') - \Pr(A' \cap B') \\ &= 0.52 + 0.68 - 0.45 \\ &= 0.75\end{aligned}$$

$\square$

Example 7. Suppose that 8 red balls, 7 white balls and 5 blue balls are in a bag. If one ball is to be taken at random from the bag, what is the probability that the ball is red or blue?

Solution:

8 red	7 white	5 blue

Note: There are 20 balls in total.

By the Addition Rule,

$$\begin{aligned}
\Pr(\text{ball is red } \textbf{or} \text{ blue}) &= \Pr(\text{ball is red}) + \Pr(\text{ball is blue}) - \Pr(\text{ball is red } \textbf{and} \text{ blue})^* \\
&= \frac{8}{20} + \frac{5}{20} - 0 \\
&= \frac{13}{20}\,.
\end{aligned}$$

So the probability that the ball is red or blue is $\dfrac{13}{20}$. $\square$

*Note: A ball cannot be red **and** blue at the same time, and so Pr(ball is red **and** blue) = 0.

Two events are said to be **mutually exclusive** (or disjoint) if their underlying sets are *disjoint* (which means that their intersection is the empty set). Informally, we often say that two events are **mutually exclusive** if they

cannot occur at the same time.

Note that if A and B are mutually exclusive events then

- $\Pr(A \cap B) = 0$, and hence
- the Addition Rule simplifies to give

$$\Pr(A \cup B) \;=\; \Pr(A) \,+\, \Pr(B)\,.$$

The next example is about a pack of cards. Details of the contents of a pack of cards are given on page 11 of this chapter.

Example 8. Suppose that one card is to be chosen from a pack of 52.

- Let A be the event {a heart}.
- Let B be the event {an ace of spades}.
- Let C be the event {an ace}.

$$\text{Then} \quad \Pr(A) = \frac{\text{number of hearts in the pack}}{\text{total number of cards in the pack}} = \frac{13}{52} = \frac{1}{4},$$

$$\text{and} \quad \Pr(B) = \frac{\text{number of aces of spades in the pack}}{\text{total number of cards in the pack}} = \frac{1}{52},$$

$$\text{and} \quad \Pr(C) = \frac{\text{number of aces in the pack}}{\text{total number of cards in the pack}} = \frac{4}{52} = \frac{1}{13}.$$

Note that the events A and B are mutually exclusive (since a card *cannot* be a heart and a spade at the same time). Thus $\Pr(A \cap B) = 0$, and so

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) \\ &= \frac{1}{4} + \frac{1}{52} \\ &= \frac{7}{26}. \end{aligned}$$

In contrast however, note that A and C are *not* mutually exclusive (since it *is* possible to be a heart and an ace at the same time!). In fact, we have

$$\Pr(A \cap C) = \frac{1}{52}$$

(since there is *one* ace of hearts in the pack), and thus

$$\begin{aligned} \Pr(A \cup C) &= \Pr(A) + \Pr(C) - \Pr(A \cap C) \\ &= \frac{1}{4} + \frac{1}{13} - \frac{1}{52} \\ &= \frac{4}{13}. \end{aligned}$$

□

Exercises for Chapter 18

1. It is known that for two particular events, A and B, we have

$$\Pr(A \cup B) = \frac{1}{2} \quad \text{and} \quad \Pr(A) = \frac{2}{3}\Pr(B).$$

   Evaluate $\Pr(A)$ if A and B are mutually exclusive.

2. A committee consisting of 4 people is to be chosen at random from a class containing 6 girls and 5 boys. What is the probability that the committee will contain 4 girls (and no boys)?

3. A committee of 6 is to be randomly selected from 10 people. What is the probability that the eldest and youngest people will both be on the committee?

4. The letters of the word TROGLODYTE are to be arranged at random. Find the probability that

   (a) the O's will be together.

   (b) the consonants will be together.

   (c) there will be at least 4 letters between the two O's.

5. Suppose A and B are events such that

$$\Pr(A) = 0.3, \quad \Pr(B) = 0.6 \quad \text{and} \quad \Pr(A \cup B) = 0.8\,.$$

   Find

   (a) $\Pr(A \cap B)$

   (b) $\Pr(A' \cap B)$

   (c) $\Pr(A \cap B')$

   (d) $\Pr(A' \cap B')$.

18.1 Answers for the Chapter 18 Exercises

1. $\Pr(A) = \frac{1}{5}$
2. The probability that the committee will contain 4 girls and 0 boys is $\frac{1}{22}$.
3. The probability that the youngest and the eldest people will be on the committee is $\frac{1}{3}$.
4. (a) The probability that the O's will be together is $\frac{1}{5}$.
   (b) The probability that the consonants will be together is $\frac{1}{30}$.
   (c) The probability that there will be at least 4 letters between the two O's is $\frac{1}{3}$.
5. (a) 0.1 (b) 0.5 (c) 0.2 (d) 0.2

The contents of a pack of cards

There are 52 cards in a pack.

Each of the 52 cards has one of the following four "suits":

- clubs (i.e. $\clubsuit$), diamonds (i.e. $\diamondsuit$), hearts (i.e. $\heartsuit$), spades (i.e. $\spadesuit$)

Each suit contains the following 13 cards:

2, 3, 4, 5, 6, 7, 8, 9, 10
J (i.e. "jack"), Q (i.e. "queen"), K (i.e "king"), A (i.e. "ace")

Thus the full pack is as follows:

2 of clubs	2 of diamonds	2 of hearts	2 of spades
3 of clubs	3 of diamonds	3 of hearts	3 of spades
4 of clubs	4 of diamonds	4 of hearts	4 of spades
5 of clubs	5 of diamonds	5 of hearts	5 of spades
6 of clubs	6 of diamonds	6 of hearts	6 of spades
7 of clubs	7 of diamonds	7 of hearts	7 of spades
8 of clubs	8 of diamonds	8 of hearts	8 of spades
9 of clubs	9 of diamonds	9 of hearts	9 of spades
10 of clubs	10 of diamonds	10 of hearts	10 of spades
J of clubs	J of diamonds	J of hearts	J of spades
Q of clubs	Q of diamonds	Q of hearts	Q of spades
K of clubs	K of diamonds	K of hearts	K of spades
A of clubs	A of diamonds	A of hearts	A of spades