Chapter 12

Indefinite Integrals and Further Applications of Integration

12.1 Indefinite Integrals

Recall that if f is continuous on $[a, b]$ and if F is any antiderivative of f , then

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a)\,.}$$

This result is known as the **Fundamental Theorem of Calculus**.

Example 1. Evaluate the integral $\displaystyle\int_0^5 x^2\ dx$.

Solution: Since an antiderivative of x^2 is $\frac{1}{3}x^3$ then we can write

$$\begin{aligned}\int_0^5 x^2\ dx &= \left[\frac{1}{3}x^3\right]_0^5 \\ &= \frac{5^3}{3} - 0 \\ &= \frac{125}{3}\,.\end{aligned}$$

□

Because of the Fundamental Theorem of Calculus, it is natural to use the expression

$$\boxed{\int f(x)\,dx}$$

to represent an antiderivative of f. We shall use the convention that $\int f(x)\,dx$ represents the **most general antiderivative** of f.

Example 2. Find $\int x^2\,dx$.

Solution:

$$\int x^2\,dx = \frac{1}{3}x^3 + C \text{ where } C \text{ is an arbitrary constant.}$$

□

Integrals of the form $\int_a^b f(x)\,dx$ are known as **definite** integrals. The answers have no x–terms and usually are numbers, i.e., definite values.

Integrals of the form $\int f(x)\,dx$ are known as **indefinite** integrals. The answers have x–terms and are not specific (there is always an arbitrary constant).

- When finding indefinite integrals, you can **check your answers** by differentiating them. (This should give back the original expression.)
- Don't forget to include the **arbitrary constant** when finding indefinite integrals.

Antidifferentiation Rules and Formulae

1. For any non-zero constant k, and for any integrable function f, we have
$$\int kf(x)\,dx \;=\; k\int f(x)\,dx\,.$$
2. For any integrable functions f and g we have
$$\int \big[f(x)+g(x)\big]\,dx \;=\; \int f(x)\,dx + \int g(x)\,dx\,.$$

3. For any integrable functions f and g we have

$$\int \big[f(x) - g(x)\big]\, dx \;=\; \int f(x)\, dx - \int g(x)\, dx\,.$$

4.
$$\int 0\, dx \;=\; C\,.$$

5.
$$\int 1\, dx \;=\; x + C\,.$$

6. For all $r \neq -1$ we have

$$\int (ax+b)^r\, dx \;=\; \frac{1}{a(r+1)}\,(ax+b)^{r+1} + C\,.$$

7.
$$\int \frac{1}{ax+b}\, dx \;=\; \frac{1}{a}\ln\big|\,ax+b\,\big| + C\,.$$

8.
$$\int \sin kx\, dx \;=\; -\frac{1}{k}\cos kx + C\,.$$

9.
$$\int \cos kx\, dx \;=\; \frac{1}{k}\sin kx + C\,.$$

10.
$$\int \sec^2 kx\, dx \;=\; \frac{1}{k}\tan kx + C\,.$$

11.
$$\int e^{kx}\, dx \;=\; \frac{1}{k}e^{kx} + C\,.$$

12. For all $a > 0$ we have

(a) $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx \;=\; \sin^{-1}\left(\frac{x}{a}\right) + C$

(b) $\displaystyle\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx \;=\; \cos^{-1}\left(\frac{x}{a}\right) + C$

(c) $\displaystyle\int \frac{a}{a^2 + x^2}\, dx \;=\; \tan^{-1}\left(\frac{x}{a}\right) + C\,.$

Example 3. Find $\displaystyle\int \left(\frac{1}{(x+2)^2} - 3x \right) dx$.

Solution:

$$\begin{aligned}
\int \left(\frac{1}{(x+2)^2} - 3x \right) dx &= \int \frac{1}{(x+2)^2}\,dx - \int 3x\,dx \\
&= \int (x+2)^{-2}\,dx - 3\int x\,dx \\
&= -(x+2)^{-1} + C_1 - 3\left(\frac{1}{2}x^2 + C_2\right) \\
&= -(x+2)^{-1} + C_1 - \frac{3}{2}x^2 - 3C_2 \\
&= -(x+2)^{-1} - \frac{3}{2}x^2 + C \quad \text{where } C = -3C_2 + C_1\,.
\end{aligned}$$

$\square$

Since all the constants are arbitrary, it is okay to just add one "$+C$" when the final integral is found, as shown in the next example:

Example 4. Find $\displaystyle\int \frac{e^{2x}+7}{e^{3x}}\,dx$.

Solution:

$$\begin{aligned}
\int \frac{e^{2x}+7}{e^{3x}}\,dx &= \int \left(\frac{e^{2x}}{e^{3x}} + \frac{7}{e^{3x}} \right) dx \\
&= \int \left(e^{-x} + 7e^{-3x} \right) dx \\
&= \int e^{-x}\,dx + \int 7e^{-3x}\,dx \\
&= -e^{-x} + 7\int e^{-3x}\,dx \\
&= -e^{-x} - \tfrac{7}{3}e^{-3x} + C\,.
\end{aligned}$$

$\square$

Exercises for Section 12.1

Find the following integrals.

1. (a) $\int_0^3 x\,dx$ (b) $\int_0^3 1\,dx$ (c) $\int_0^3 0\,dx$

   (d) $\int_0^3 -3x^2\,dx$ (e) $\int_0^3 2x^3\,dx$.

2. (a) $\int \frac{1}{x}\,dx$ (b) $\int \frac{1}{x^2}\,dx$ (c) $\int \frac{1}{\sqrt{x}}\,dx$.

3. (a) $\int \left(x + x^2\right)dx$ (b) $\int \left(3x + \sqrt{x}\right)dx$.

4. (a) $\int \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)dx$ (b) $\int \left(7x^4 + \sin(2x)\right)dx$.

5. (a) $\int \frac{e^{7x} - e^x}{e^{2x}}\,dx$ (b) $\int \frac{x^2 + 5x}{x^2}\,dx$.

6. (a) $\int \sqrt{3x+1}\,dx$ (b) $\int \frac{1}{7x+2}\,dx$.

7. (a) $\int \frac{1}{9+x^2}\,dx$ (b) $\int \frac{7}{\sqrt{4-x^2}}\,dx$.

12.2 Further Applications of Integration

Average Value of a Function

The **average** (or **mean**) value of a function f over the interval $[a, b]$ is defined by

$$\boxed{f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Example 5. Find the average value of $f(x) = 3x^2$ over the interval $[0, 2]$.

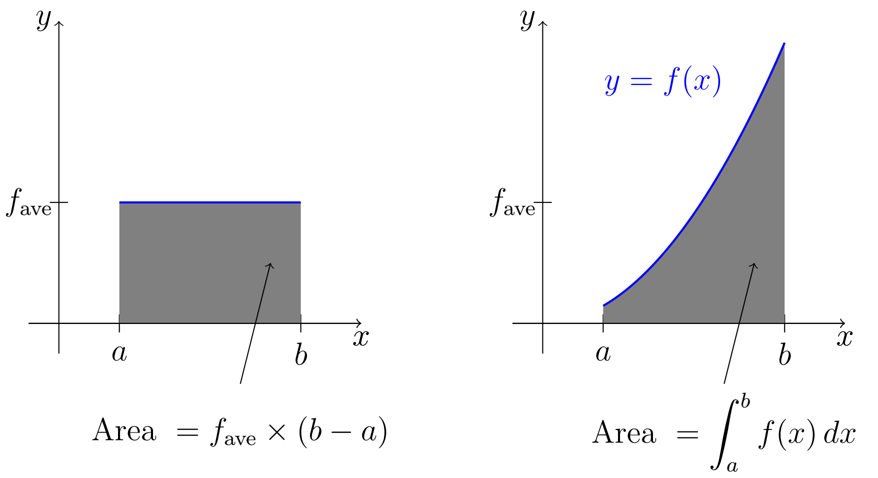
Solution: We have $a = 0$, $b = 2$ and $f(x) = 3x^2$. Thus

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2-0}\int_0^2 3x^2\,dx \\ &= \frac{1}{2}\left[x^3\right]_0^2 \\ &= \frac{1}{2}\left[2^3 - 0^3\right] \\ &= 4\,. \end{aligned}$$

□

Note that **if** $f(x) \geq 0$ on $[a, b]$, then f_{ave} is the height of a rectangle on $[a, b]$ with area that is the same as the area under $y = f(x)$. This follows directly from the definition of f_{ave} because

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \Longrightarrow \quad f_{\text{ave}} \times (b-a) = \int_a^b f(x)\,dx\,.$$



$$\begin{aligned}
\text{Area of rectangle } &= f_{\text{ave}} \times (b-a) \\
&= \int_a^b f(x)\,dx \\
&= \text{Area under } f(x) \text{ on } [\,a,\,b\,]\,.
\end{aligned}$$

Example 6. A cup of coffee is cooling in such a way that the temperature of the coffee is given by

$$T \;=\; 75e^{-0.1t} + 20$$

where T is measured in degrees Celsius and t is time measured in minutes.

Find the average temperature of the coffee, to two decimal places, over the first 10 minutes.

Solution: The first ten minutes occur between $t = 0$ and $t = 10$.

$$\begin{aligned}
T_{\text{ave}} &= \frac{1}{10-0}\int_0^{10} \big(75e^{-0.1t} + 20\big)\;dt \\
&= \frac{1}{10}\left[\frac{75}{-0.1}e^{-0.1t} + 20t\right]_0^{10} \\
&= \frac{1}{10}\big(\big(-750e^{-1} + 200\big) - \big(-750e^{0} + 0\big)\big) \\
&= \frac{1}{10}\big(-750e^{-1} + 200 + 750\big) \\
&= 95 - 75e^{-1} \\
&= 67.41 \text{ (2 decimal places.)}
\end{aligned}$$

The average temperature of the coffee over the first 10 minutes is 67.41° Celsius. $\square$

Time integrals

Consider a particle which moves along the x -axis with position $x(t)$, velocity $v(t)$ and acceleration $a(t)$, where t is time. Then

$$\frac{d}{dt}(x(t)) = v(t) \quad \text{and} \quad \frac{d}{dt}(v(t)) = a(t)$$

and so $x(t)$ is an antiderivative of $v(t)$, and $v(t)$ is an antiderivative of $a(t)$. Thus

$$\boxed{x(t) = \int v(t)\,dt \quad \text{and} \quad v(t) = \int a(t)\,dt}$$

Example 7. Suppose the acceleration of a particle moving along the x -axis, where x is measured in centimetres and time t is measured in seconds, is given by $a(t) = 3\ \text{cm}\cdot\text{s}^{-2}$ and suppose that initially (i.e., at time $t = 0\,\text{s}$)

- the particle is at $x = 1$ cm , and
- the velocity of the particle is $v = -2\ \text{cm}\cdot\text{s}^{-1}$.

Find the position (in cm) and velocity (in $\text{cm}\cdot\text{s}^{-1}$) of the particle as functions of time t , where t is measured in seconds.

Solution: The velocity of the particle is given by

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int 3\,dt \\ &= 3t + C_1 . \end{aligned}$$

Initially, the velocity of the particle is $v = -2$ and so

$$\begin{aligned} v(0) &= -\,2 \\ \therefore 3(0) + C_1 &= -\,2 \\ \therefore C_1 &= -\,2 . \end{aligned}$$

Therefore, after t seconds, the velocity of the particle is $v(t) = 3t - 2\ \text{cm}\cdot\text{s}^{-1}$.

The position of the particle is given by

$$\begin{aligned} x(t) &= \int v(t)\,dt \\ &= \int (3t-2)\ dt \\ &= \frac{3}{2}t^2 - 2t + C_2\,. \end{aligned}$$

Since the particle is initially at $x = 1$, we have

$$1 = \frac{3}{2}(0)^2 - 2(0) + C_2 \quad \text{and so} \quad C_2 = 1\,.$$

Therefore, after t seconds, the particle is at $x(t) = \frac{3}{2}t^2 - 2t + 1$ cm. □

Example 8. An arrow is moving in such a way that the vertical distance between the arrow and the ground is given by

$$x = 30t - \frac{9.8}{2}t^2$$

where x is measured in metres and t is measured in seconds.

Find the average velocity of the arrow in the first 3 seconds.

Solution: The first 3 seconds occur between $t = 0$ and $t = 3$. The velocity of the arrow is given by

$$v = \frac{dx}{dt} = 30 - 9.8t\,,$$

and so the average velocity in the first 3 seconds is given by

$$\begin{aligned} v_{\text{ave}} &= \frac{1}{3-0}\int_0^3 (30 - 9.8t)\,dt \\ &= \frac{1}{3}\left[30t - \frac{9.8}{2}t^2\right]_0^3 \\ &= \frac{1}{3}\left(30(3) - \frac{9.8}{2}(3^2) - 0\right) \\ &= \frac{45.9}{3} \\ &= 15.3 \end{aligned}$$

Thus, the average velocity in the first three seconds is $15.3\ \mathrm{m \cdot s^{-1}}$. □

Notice that, in the previous example, we have differentiated x to find v and then antidifferentiated to get x again.

In general, the average velocity between $t = a$ and $t = b$ is given by

$$\begin{aligned}
\text{average velocity} &= \frac{1}{b-a}\int_a^b v(t)\,dt \\
&= \frac{1}{b-a}\Big[x(t)\Big]_a^b \\
&= \frac{x(b)-x(a)}{b-a} \\
&= \frac{\text{change in position}}{\text{change in time}}\,.
\end{aligned}$$

Example 9. A spherical snowball of volume $V\,\mathrm{cm}^3$ and radius $r\,\mathrm{cm}$ melts in such a way that its volume decreases at a rate which is proportional to its surface area $A\,\mathrm{cm}^2$:

$$\frac{dV}{dt} = -kA\,.$$

One model for melting of snowballs suggests that when t is measured in seconds we have $k = 0.02$. Suppose that initially a particular snowball has a radius of $10\,\mathrm{cm}$. Let $r(t)$ be the radius of the snowball as a function of time t , where t is measured in seconds.

(a) Find $\dfrac{dr}{dt}$.

(b) Find $r(t)$.

(c) How long does it take for the snowball to melt completely?

Solution: (a) Since the snowball is spherical, we have

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2\,.$$

We are told that $\dfrac{dV}{dt} = -kA$, and that $k = 0.02$.

Thus we can write $\dfrac{dV}{dt} = -0.02 \times 4\pi r^2$.

Also, by the Chain Rule, we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= 4\pi r^2 \frac{dr}{dt}. \end{aligned}$$

Thus we have

$$4\pi r^2 \frac{dr}{dt} = -(0.02)4\pi r^2$$

$$\text{and so} \qquad \frac{dr}{dt} = -0.02\,.$$

(b) Now $r = \displaystyle\int \frac{dr}{dt}\,dt = \int -0.02\,dt = -0.02t + C$.

Since the snowball initially has radius 10 cm, we have

$$\begin{aligned} 10 &= -0.02(0) + C \\ 10 &= C\,. \end{aligned}$$

Therefore $r(t) = -0.02t + 10$.

(c) The snowball has melted completely when

$$\begin{aligned} r &= 0 \\ -0.02t + 10 &= 0 \\ 10 &= 0.02t \end{aligned}$$

$$t = \frac{10}{0.02} = 500\,.$$

Therefore, it takes 500 seconds for the snowball to melt.

$\square$

Exercises for Section 12.2

1. Find the average value of the following functions over the indicated intervals.

   (a) $f(x) = 3x + 7$ over the interval $[0, 9]$.

   (b) $f(x) = \sqrt{x}$ over the interval $[1, 16]$.

   (c) $f(x) = x^3$ over the interval $[-2, 2]$.

   (d) $f(x) = \sin x$ over the interval $[0, 2\pi]$.

2. An object moves along the x-axis with velocity $v(t) = 8t - 3$. Suppose that the object's initial position (at time $t = 0$) is at $x = 2$.

   (a) Find $a(t)$, the acceleration of the object as a function of time t.

   (b) Find $x(t)$, the position of the object as a function of time t.

   (c) Find the position of the object at time $t = 4$.

3. An object moves along the x-axis with acceleration $a(t) = -10$. Suppose that the object's initial position (at time $t = 0$) is at $x = 100$, and the object's initial velocity is $v = 4$.

   (a) Find $v(t)$, the velocity of the object as a function of time t.

   (b) Find $x(t)$, the position of the object as a function of time t.

   (c) Find the position of the object at time $t = 3$.

   (d) Find the time at which the object is not moving, i.e., find the value of t such that $v(t) = 0$.

12.3 Answers to Chapter 12 Exercises

12.1

1. (a) $\frac{9}{2}$ (b) 3 (c) 0 (d) -27 (e) $\frac{81}{2}$.
2. (a) $\ln|x| + C$ (b) $-\frac{1}{x} + C$ (c) $2x^{\frac{1}{2}} + C$.
3. (a) $\frac{1}{2}x^2 + \frac{1}{3}x^3 + C$ (b) $\frac{3}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + C$.
4. (a) $\ln|x| - 2x^{\frac{1}{2}} + C$ (b) $\frac{7}{5}x^5 - \frac{1}{2}\cos(2x) + C$.
5. (a) $\frac{1}{5}e^{5x} + e^{-x} + C$ (b) $x + 5\ln|x| + C$.
6. (a) $\frac{2}{9}\sqrt{(3x+1)^3} + C$ (b) $\frac{1}{7}\ln|7x+2| + C$.
7. (a) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$ (b) $7\sin^{-1}\left(\frac{x}{2}\right) + C$.

12.2

1. (a) $\frac{41}{2}$ (b) $\frac{14}{5}$ (c) 0 (d) 0 .
2. (a) $a(t) = 8$ (b) $x(t) = 4t^2 - 3t + 2$
   (c) $x = 54$.
3. (a) $v(t) = -10t + 4$ (b) $x(t) = -5t^2 + 4t + 100$
   (c) $x = 67$ (d) $t = 0.4$.