C10: APPLICATIONS OF DIFFERENTIATION

Monday, May 11, 2020 8:21 AM

10.1 RATES OF CHANGE

a rate of change is given by the **derivative**.

the rate of change of V with respect to r is given by $\frac{dV}{dr}$.

the rate of change of s with respect to t is given by $\frac{ds}{dt}$.

EXERCISE (p.3)

1. ~~A circu~~lar oil ~~slick~~ is forming in a pool of water. ~~Find the~~ rate of increase of the area of the oil slick with respect to its radius,

(a) when the radius is 5 cm.

$A = \pi r^2$
$\frac{dA}{dr} = 2\pi r$
when $r = 5$:
$\frac{dA}{dr} = 2\pi(5)$
$= 10\pi$

(b) when the area of the slick is 36π cm^2 .

when $A = 36\pi$:
$36\pi = \pi r^2$
$r^2 = 36$
$r = 6$
when $r = 6$:
$\frac{dA}{dr} = 2\pi(6)$
$= 12\pi$

10.1 RELATED RATES OF CHANGE

Important Tips for (Related) Rates of Change Problems

Tip 1: Do **not** replace variables with numerical values until after differentiating. (Note that **constants** can be replaced with their numerical values at any time.)

Tip 2: **Before** we can find a derivative, we need to have an appropriate equation in which the **only two variables** are the two variables involved in the differentiation.

EXERCISES (p.9)

1. A spheri~~cal~~ balloon is expan~~ding in~~ such a way that it~~s radius is~~ increasing at a rate of 0.5 cm $\cdot$ s^{-1} .
At what rate is the balloon's volume increasing when the radius is 10 cm?

$\frac{dr}{dt} = +0.5$ cm/s
$V = \frac{4}{3}\pi r^3$
$\frac{dV}{dr} = 4\pi r^2$
when $r = 10$:
$\frac{dV}{dr} = 4\pi(10)^2$
$= 400\pi$
$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
$= 400\pi \times 0.5$
$= 200\pi$ cm^3/s

2. The sides of a cube are increasing at a rate of 0.2 cm $\cdot$ s^{-1} . At what rate is the volume of the cube increasing when the sides are 8 cm long?

$\frac{dl}{dt} = +0.2$
$V = l^3$
$\frac{dV}{dl} = 3l^2$
when $l = 8$:
$\frac{dV}{dl} = 3(8)^2$
$= 192$
$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$
$= 192 \times 0.2$
$= 38.4$ cm^3/s

3. A circular oil slick is forming on a pool of water in such a way that its radius is increasing at a rate of 2 cm $\cdot$ min^{-1} .

(a) At what rate is the area of the oil slick increasing when its radius is 5 cm?

$A = \pi r^2$
$\frac{dA}{dr} = 2\pi r$
$r = 5$
$\frac{dA}{dr} = 2\pi(5)$
$= 10\pi$
$\frac{dr}{dt} = 2$
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$
$= 10\pi \times 2$
$= 20\pi$ cm^2/s

(b) Show that the circumference of the oil slick is increasing at a constant rate.

$C = 2\pi r$
$\frac{dC}{dr} = 2\pi$
$\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$
$= 2\pi \times 2$
$= 4\pi$
$\therefore$ circumference is increasing at a constant rate of 4π cm/s

10.6 Answers to Chapter 10 Exercises

10.1:

1. (a) When $r = 5$ cm, the area is increasing at a rate (with respect to the radius) of 10π cm$^2 \cdot$ cm^{-1} .
   (b) When $A = 36\pi$ cm^2 , the area is increasing at a rate (with respect to the radius) of 12π cm$^2 \cdot$ cm^{-1} .

10.2:

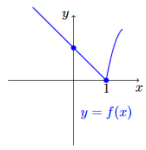
1. When $r = 10$ cm, the volume is increasing at a rate of 200π cm$^3 \cdot$ s^{-1} .
2. When $x = 8$ cm, the volume is increasing at a rate of 38.4 cm$^3 \cdot$ s^{-1} .
3. (a) When $r = 5$ cm, the area is increasing at a rate of 20π cm$^2 \cdot$ min^{-1} .
   (b) $\frac{dC}{dt} = 4\pi$ cm $\cdot$ min^{-1} , which is constant. (It does not rely on r or t .)
4. (a) When $r = 6$ cm, the volume is increasing at a rate of 216π cm$^3 \cdot$ min^{-1} .
   (b) When $r = 6$ cm, the radius is increasing at a rate of $\frac{5}{36\pi}$ cm $\cdot$ min^{-1} .
5. When $h = 5$ cm, the water level is rising at $\frac{2}{15\pi}$ cm $\cdot$ s^{-1} .
6. (a) When $r = 20$ cm, the radius is increasing at a rate of $\frac{9}{400\pi}$ cm $\cdot$ min^{-1}.
   (b) When $r = 20$ cm, the surface area is increasing at a rate of $\frac{18}{5}$ cm$^2 \cdot$ min^{-1}.
7. When there is still 5 m of rope out, the boat is approaching the pier at a rate of $\frac{2}{\sqrt{21}}$ m $\cdot$ s^{-1} .

10.3:

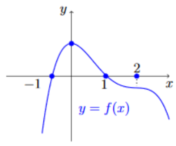
1. $y = 4x$ and $y = -4x + \frac{16}{3}$
2. The tangent is $y = -3x + \frac{\pi}{2}$, and the normal is $y = \frac{1}{3}\left(x - \frac{\pi}{6}\right)$.
3. $y = x + 2$
4. $a = 2$, $b = -6$ and $c = 11$.

10.4:

1. $p = -1$, $q = -8$, $r = 2$
2. $a = 1$, $b = -1$, $c = -5$, $d = -3$
3.



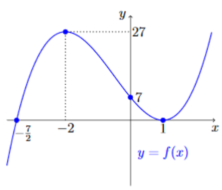
4.



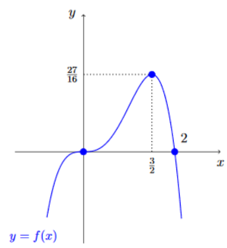
5. (a) Stationary point of inflection at $x = 2$.
   (b) Local maximum when $x = -\frac{8}{3}$, and local minimum when $x = 0$.
   (c) Local maximum when $x = 0$, and local minima when $x = \pm 2$.
   (d) Local maximum when $x = -1$, and local minimum when $x = 1$.

6.

(a)



(b)



(c)

4. Sand is being poured into a conical pile in such a way that the height of the pile is equal to its diameter.

(a) If the sand is being poured so that the radius of the pile is increasing at a rate of $3 \text{ cm} \cdot \text{min}^{-1}$, find the rate of increase of the volume when the radius is 6 cm.

(b) If the sand is being poured at a rate of $10 \text{ cm}^3 \cdot \text{min}^{-1}$, at what rate is the radius increasing when the radius is 6 cm?

5. When a particular bowl contains water to a depth of h cm, the volume V cm^3 of the water is given by $V = \pi h^2(10 - \frac{1}{3}h)$. Suppose that water is being poured into the bowl at a constant rate of $10 \text{ cm}^3 \cdot \text{s}^{-1}$. Find the rate at which the water level is rising when the depth of the water is 5 cm.

6. Air is being pumped into a spherical balloon at a rate of $36 \text{ cm}^3 \cdot \text{min}^{-1}$. When the radius is 20 cm,

(a) at what rate is the radius increasing?

(b) at what rate is the surface area increasing?

7. A child is standing on a pier and is pulling in a boat by means of a rope, which is being hauled in at the rate of $0.4 \text{ m} \cdot \text{s}^{-1}$. If the child's hands are 2 m above the level of the boat, at what rate is the boat approaching the pier when there is still 5 m of rope out?

10.3 EQUATIONS OF TANGENTS AND NORMALS

the **gradient of the tangent** to $y = f(x)$ is given by $f'(x)$.

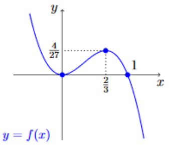
the **gradient of the curve** of $y = f(x)$ is also given by $f'(x)$.

EXERCISES (p.13)

1. Find the equations of the tangents to the parabola $y = 4x - 3x^2$ at the points where the parabola cuts the x-axis.

2. Find the equations of the tangent and the normal to $y = \cos 3x$ at the point where $x = \dfrac{\pi}{6}$.

3. Find the equation of the tangent to the curve $x^2 + 2xy - 2y^2 = -12$ at the point $(2, 4)$.



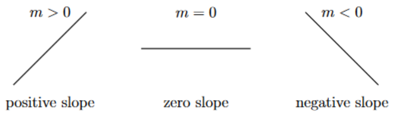
7. (a) $f'(x) = 3x^2,\ f''(x) = 6x$
   (b) $f'(0) = 0,\ f''(0) = 0$
   (c) Stationary point of inflection.
8. (a) $f'(x) = 4x^3,\ f''(x) = 12x^2$
   (b) $f'(0) = 0,\ f''(0) = 0$
   (c) Local minimum.
9. (a) $f'(x) = -4x^3,\ f''(x) = -12x^2$
   (b) $f'(0) = 0,\ f''(0) = 0$
   (c) Local maximum.
10. (a) The global maximum occurs when $x = 4$, and the global minimum occurs when $x = 3$.
    (b) The global maximum occurs when $x = 4$, and the global minimum occurs when $x = 1$.
    (c) The global maximum occurs when $x = 4$, and the global minimum occurs when $x = 0$.
    (d) The global maximum occurs when $x = -1$, and the global minimum occurs when $x = 0$.

10.5:

1. The largest area is 32 m^2.
2. The length of each edge is 1.5 cm.
3. The dimensions of the brick are 5 cm, 10 cm and $\frac{20}{3}$ cm.
4. The radius is 5 cm and the height is 10 cm.
5. (a) $a^2 + b^2 = 4$
   (b) $A = 2a\sqrt{4 - a^2}$
   (c) $a = \sqrt{2},\ b = \sqrt{2}$

4. The curve with equation $y = ax^2 + bx + c$ passes through the point $(1, 7)$ and has a gradient of 2 at the point $(2, 7)$. Find the values of a, b, c.

10.4 CURVE SKETCHING



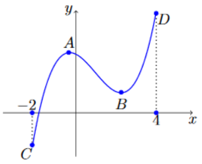
Basic Definitions

Suppose that f is a differentiable function.

- If $f'(x) > 0$ for all x in an interval then we say that f is **increasing** on that interval.
- If $f'(x) < 0$ for all x in an interval then we say that f is **decreasing** on that interval.

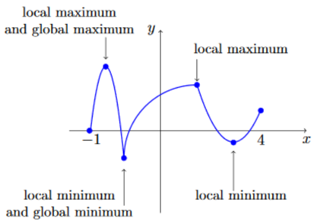
For example, the following graph is

- **increasing** on the sections CA and BD, and
- **decreasing** on the section AB.



Consider the graph given above.

- A is said to be a **local maximum**.
- B is said to be a **local minimum**.
- A and B are called **stationary points**.
- C is said to be the **global minimum** on the interval $[-2, 4]$.
- D is said to be the **global maximum** on the interval $[-2, 4]$.



We say that a function f has a **critical point** at $x = c$ if

- $f'(c) = 0$

or if

- c is a value in the domain of f such that $f'(c)$ does not exist.

The pictures below show critical points for which the derivative is zero:

The pictures below show critical points for which the derivative does not exist:

Note: If $f'(c) = 0$, then we also say that f has a **stationary point** at $x = c$.

EXERCISES (p.18)

1. The graph of $y = x^3 + px^2 + qx + r$ has a stationary point at $(2, -10)$ and a y-intercept of 2. Find p, q and r.

2. The graph of $y = ax^3 + bx^2 + cx + d$ has a stationary point at $(-1, 0)$. It cuts the y-axis at $y = -3$ and at this point it is parallel to the line $5x + y = 4$. Calculate the values of a, b, c and d.

3. Sketch the graph of a continuous function f which satisfies the following conditions:
   - $f(1) = 0$;
   - $f'(1)$ does not exist;

- $f'(x) < 0$ for $x < 1$; and
- $f'(x) > 0$ for $x > 1$.

4. Sketch the graph of a continuous function f which satisfies the following conditions:
   - $f(-1) = 0$ and $f(1) = 0$;
   - $f'(0) = 0$ and $f'(2) = 0$;
   - $f'(x) > 0$ for $x < 0$;
   - $f'(x) < 0$ for $0 < x < 2$, and
   - $f'(x) < 0$ for $x > 2$.

10.1 RATES OF CHANGE

First Derivative Test

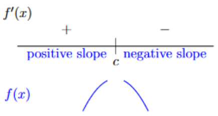
Local Maximum:

Suppose that a continuous function f has a **critical** point at $x = c$.
(That is, suppose that $f'(c) = 0$ or that $f'(c)$ does not exist.)

If

- $f'(x) > 0$ for all x immediately to the left of c , **and** if
- $f'(x) < 0$ for all x immediately to the right of c

then f has a local maximum at $x = c$.



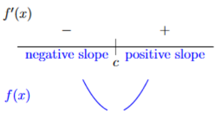
Local Minimum:

Suppose that a continuous function f has a **critical** point at $x = c$.

If

- $f'(x) < 0$ for all x immediately to the left of c , **and** if
- $f'(x) > 0$ for all x immediately to the right of c

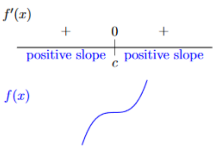
then f has a local minimum at $x = c$.



Stationary Point of Inflection:

Suppose that a continuous function f has a **stationary** point at $x = c$.
(That is, suppose that $f'(c) = 0$.) If

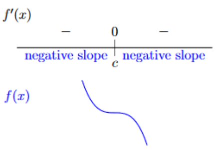
- $f'(x) > 0$ for all x immediately to the left of c , **and** if
- $f'(x) > 0$ for all x immediately to the right of c



or if

- $f'(x) < 0$ for all x immediately to the left of c , **and** if
- $f'(x) < 0$ for all x immediately to the right of c

then f has a stationary point of inflection at $x = c$.



EXERCISES (p.24)

5. For each of the following functions
   - find the x-coordinates of all the critical points, and
   - find the nature of all the critical points.

(a) $f(x) = (x-2)^3$

(b) $f(x) = x^2(x+4)$

(c) $f(x) = (x^2-4)^2$

(d) $f(x) = x + \frac{1}{x}$

6. Sketch the graph of $y = f(x)$ for each of the following functions by finding

- the x and y-intercepts;
- the coordinates of all the stationary points; and
- the nature of the stationary points.

(a) $f(x) = 2x^3 + 3x^2 - 12x + 7$

(b) $f(x) = x^3(2 - x)$

(c) $f(x) = x^2(1 - x)$

10.1 RATES OF CHANGE

Second Derivatives

The **second derivative** of a function f is obtained by differentiating $f'(x)$. We denote the second derivative of f by f'' or by $\frac{d^2y}{dx^2}$.

Second Derivative Test:

- If $f'(c) = 0$ and if $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f'(c) = 0$ and if $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f'(c) = 0$ and if $f''(c) = 0$ then the second derivative gives no information.

Note: It is often better to use the First Derivative Test (from the previous section) rather than the Second Derivative Test (particularly when f' is complicated to differentiate).

Warning: If $f'(c) = 0$ and $\boldsymbol{f''(c)} = 0$ you **cannot** make any conclusion about the nature of the stationary point at $x = c$. In such cases, the First Derivative Test should be used to identify whether you have a local maximum, a local minimum, or a stationary point of inflection.

EXERCISES (p.28)

7. Consider the function $f(x) = x^3$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Hence find $f'(0)$ and $f''(0)$.

(c) Using the **First** Derivative Test, identify the nature of the stationary point at $x = 0$.

8. Consider the function $f(x) = x^4$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Hence find $f'(0)$ and $f''(0)$.

(c) Using the **First** Derivative Test, identify the nature of the stationary point at $x = 0$.

9. Consider the function $f(x) = -x^4$.

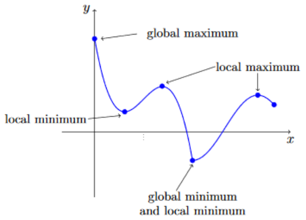
(a) Find $f'(x)$ and $f''(x)$.

(b) Hence find $f'(0)$ and $f''(0)$.

(c) Using the **First** Derivative Test, identify the nature of the stationary point at $x = 0$.

GLOBAL MAXIMA AND MINIMA

Global Maxima and Minima



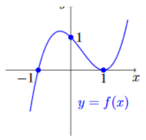
A function f has

- a **global** maximum at c if $f(c) \geq f(x)$ for all x in the function's domain.

Similarly, a function f has

- a **global** minimum at c if $f(c) \leq f(x)$ for all x in the function's domain.

Note that **not** every function has a global maximum or a global minimum. For example, consider the function $f(x) = x^3 - x^2 - x + 1$ whose graph was drawn in Example 15.



However, if f is a **continuous** function with domain of the form $[a, b]$, then we can be sure that f *does* have a global maximum and a global minimum. These points must occur either

- at critical points (i.e. points where $f'(x) = 0$ or where $f'(x)$ does not exist), or
- at the endpoints a, b of the domain.

EXERCISES (p.31)

10. Find the x-coordinates of the global maximum and the global minimum of

$$f(x) = 3x^4 - 20x^3 + 36x^2 + 4$$

for each of the following domains:

(a) $2 \le x \le 4$

(b) $1 \le x \le 4$

(c) $-1 \le x \le 4$

(d) $-1 \le x \le 2.5$

10.5 APPLIED MAX/MIN PROBLEMS

Steps for Solving Applied Max/Min Problems:

Step 1: Draw a diagram.
For many questions it is helpful to draw a diagram labelled by symbols representing the variables relevant to the problem.

Step 2: Write down a formula for the quantity to be maximized or minimized.

Step 3: Write the formula found in Step 2 as a **function of one variable.** This usually involves making use of

- geometric properties of the diagram, or
- information given in the question.

Step 4: Find all critical points.
That is, if f is the function to be maximized or minimized then find all x-values in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Step 5: Establish the nature of these critical points.
(For example, use the First Derivative Test.) Determine which critical point corresponds to the maximum or minimum required by the question.

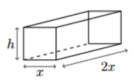
Step 6: Answer the given question.
Note: If the question was a "word question", then answer the question with a sentence.

EXERCISES (p.37-38)

1. A rectangular enclosure is to be built from 16 metres of fencing wire. The wire is only needed for three of the sides, since the fourth side is bounded by an existing wooden fence. What is the largest area of ground that the enclosure can cover?

2. A sheet of cardboard measures 15 cm by 7 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the length of the edge of the squares cut so that the box has maximum volume.

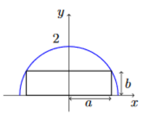
3. A rectangular brick is to be constructed so that
   - its total surface area is 300 cm^2; and
   - the length of its base is twice the width of its base.



Find the dimensions of the brick with maximum volume which satisfies these conditions.

4. A dog food manufacturer produces cylindrical cans with a volume of $250\pi \text{ cm}^3$. What are the dimensions of the can if the least amount of material is used in its construction? (That is, find the height and radius of the cylinder with minimum surface area.)

5. Suppose we have a rectangle inscribed in a semicircle of radius 2 (as shown in the diagram).



(a) Find an equation relating the variables a and b.

(b) Hence write the area of the rectangle in terms of a.

(c) Find a and b for the rectangle with largest area.

EXERCISES (p.28)

EXERCISES (p.28)