Maths 1

Exercise Sheet 14: Indefinite Integrals

1. Solve for m, where $m > 0$.

   (a) $\displaystyle\int_0^m x^2\,dx = 72$ (b) $\displaystyle\int_8^m x\,dx = 18$

   (c) $\displaystyle\int_m^7 5\,dx = 25$ (d) $\displaystyle\int_0^m (4-x)\,dx = 8$

   (e) $\displaystyle\int_0^m e^{3x}\,dx = 21$.

2. (a) (i) Find $\displaystyle\int \cos 2x\,dx$.

   (ii) Check your answer to (i) by differentiating it.

   (b) (i) Find $\displaystyle\int \sec^2 3x\,dx$.

   (ii) Check your answer to (i) by differentiating it.

   (c) (i) Find $\displaystyle\int (1+\sin x)\,dx$.

   (ii) Check your answer to (i) by differentiating it.

3. (a) (i) Expand $\left(x+\dfrac{1}{x}\right)^2$.

   (ii) Hence find $\displaystyle\int \left(x+\frac{1}{x}\right)^2\,dx$.

   (b) (i) Expand $\left(x^2+\dfrac{2}{x}\right)^3$.

   (ii) Hence find $\displaystyle\int \left(x^2+\frac{2}{x}\right)^3\,dx$.

   (c) (i) Expand $\left(1+6e^{3x}\right)^2$.

   (ii) Hence find $\displaystyle\int \left(1+6e^{3x}\right)^2\,dx$.

4. (a) (i) Simplify $\dfrac{x+x^2}{2x}$.

   (ii) Hence find $\displaystyle\int \frac{x+x^2}{2x}\,dx$.

   (b) (i) Simplify $\dfrac{e^{5x}-e^{2x}}{e^{3x}}$.

   (ii) Hence find $\displaystyle\int \frac{e^{5x}-e^{2x}}{e^{3x}}\,dx$.

   (c) (i) Simplify $\left(\dfrac{1}{\cos 4x}\right)^2$.

   (ii) Hence find $\displaystyle\int \left(\frac{1}{\cos 4x}\right)^2\,dx$.

5. Find the following integrals:

   (a) $\displaystyle\int (2x+1)^{20}\,dx$ (b) $\displaystyle\int (2x+1)^{-1}\,dx$

   (c) $\displaystyle\int \sqrt{4+9x}\,dx$ (d) $\displaystyle\int (4+9x)^{-1}\,dx$

   (e) $\displaystyle\int \sqrt[3]{6x+1}\,dx$ (f) $\displaystyle\int \frac{1}{6x+1}\,dx$.

6. (a) Find $\int \frac{x+1}{x}\,dx$.

   Hint: Similar to Question 4(a).

   (b) Find $\int \frac{x}{x+1}\,dx$.

   Hint: Use long division to rewrite $\frac{x}{x+1}$ as $1-\frac{1}{x+1}$.

7. Find $\int \cos^2 x\,dx$.

   Hint: Use a double-angle formula (from the formula sheet) to rewrite $\cos^2 x$ as $\frac{1}{2}(1+\cos 2x)$.

8. Find the following integrals:

   (a) $\int \frac{2}{4+x^2}\,dx$ (b) $\int \frac{1}{\sqrt{9-x^2}}\,dx$

   (c) $\int \frac{-1}{\sqrt{16-x^2}}\,dx$.

Applications of Integration

9. Find the average value of the following functions over the interval $[2,6]$:

   (a) $f(x)=x$ (b) $f(x)=1$ (c) $f(x)=x^3$.

10. (a) (i) Find the average value of $f(x)=\sin x$ over the interval $[0,\pi]$.

    (ii) Find the area bounded by $f(x)=\sin x$ and the x-axis, for x between 0 and π.

    (b) (i) Find the average value of $f(x)=\sin x$ over the interval $[0,2\pi]$.

    (ii) Find the area bounded by $f(x)=\sin x$ and the x-axis, for x between 0 and 2π.

    (c) (i) Find the average value of $f(x)=\sin x$ over the interval $[\pi,2\pi]$.

    (ii) Find the area bounded by $f(x)=\sin x$ and the x-axis, for x between π and 2π.

11. The depth of water at a certain part of Port Phillip Bay satisfies $D = 10+4\cos\left(\frac{\pi}{12}t\right)$, where

    - D is the depth, measured in metres, and
    - t is the number of hours after midnight.

    Find the average depth between 12 noon and 4 pm to the nearest one-tenth of a metre.

12. During the decade starting in 1990, drought caused Gippsland's kangaroo population to decline according to the formula

$$K = 300 + 200e^{-\frac{1}{2}t}$$

where

- K is the number of kangaroos, measured in thousands.
- t is measured in years, with $0 \leq t \leq 10$.

(a) Find the kangaroo population of Gippsland at the *start* of the 1990s (when $t = 0$).

(b) Find the kangaroo population of Gippsland at the *end* of the 1990s (when $t = 10$), to the nearest thousand.

(c) Find the *average* kangaroo population in Gippsland during the 10 year period, to the nearest thousand.

13. A baker switches an oven on, and waits while the oven heats up. When the oven's temperature reaches 180°C, the temperature starts to fluctuate according to the formula

$$T = 180 + 10\sin\left(\frac{\pi}{2}\,t\right),$$

where

- T is the temperature, measured in degrees Celsius.
- t is the number of minutes that have elapsed since the oven's temperature first reaches 180°.

The baker puts some biscuits into the oven when the oven's temperature first reaches 180°C, and the baker leaves them in the oven for exactly 9 minutes. Find the average temperature of the oven during the 9 minute interval when the biscuits are in the oven. Give your answer in a sentence and accurate to one decimal place.

14. An object moves along the x-axis with acceleration given by

$$a(t) = 528(2t + 1)^{10}.$$

Suppose that the object's initial position (at time $t = 0$) is at $x = 0$, and the object's initial velocity is $v = 0$.

(a) Find $v(t)$, the velocity of the object as a function of time.

(b) Find $x(t)$, the position of the object as a function of time.

15. Suppose that a piston is moving forwards and backwards along a horizontal straight line in such a way that its position satisfies $x(t) = 3\sin(10t)$, where

- x is measured in centimetres, and
- t is measured in seconds, with $t \geq 0$.

(a) Find the velocity and the acceleration of the piston as functions of time t.

(b) Find the average velocity of the piston during the first 2 seconds. Provide your answer in a sentence, rounded to two decimal places.

16. Suppose that a 60 second thrill-ride at an amusement park moves along a vertical track in such a way that its velocity is given by

$$v(t) \;=\; \frac{1}{2000}\left(t^3 - 90t^2 + 1800t\right),$$

where

- t is measured in seconds (with $0 \le t \le 60$), and
- $v(t)$ is measured in $\mathrm{m}\cdot\mathrm{s}^{-1}$.

Suppose that the initial height of the ride is 0 metres.

(a) Find the acceleration $a(t)$ and the height $x(t)$ of the ride as functions of time t.

(b) Find the height of the ride after 30 seconds.

(c) Find the average velocity of the ride during the first 30 seconds.

17. Suppose that a spherical snowball of radius 10 cm melts so that its volume **decreases** at a rate (with respect to time t) of $kr^2\ \mathrm{cm}^3\cdot\mathrm{hr}^{-1}$, where r is the snowball's radius, and where k is a positive constant. If it takes 4 hours for the snowball's volume to decrease to half the original volume, how long (to the nearest quarter-hour) does it take for the snowball to melt completely?

Answers

1. (a) $m=6$ (b) $m=10$ (c) $m=2$ (d) $m=4$ (e) $m=\ln 4$

2. (a) (i) $\frac{1}{2}\sin(2x)+C$ (ii) $\frac{d}{dx}\left(\frac{1}{2}\sin(2x)+C\right)=\frac{1}{2}\times 2\cos(2x)+0=\cos(2x)$

   (b) (i) $\frac{1}{3}\tan(3x)+C$ (ii) $\frac{d}{dx}\left(\frac{1}{3}\tan(3x)+C\right)=\frac{1}{3}\times 3\sec^2(3x)+0=\sec^2(3x)$

   (c) (i) $x-\cos x+C$ (ii) $\frac{d}{dx}(x-\cos x+C)=1-(-\sin x)+0=1+\sin x$

3. (a) (i) $x^2+2+\frac{1}{x^2}$ (ii) $\frac{x^3}{3}+2x-\frac{1}{x}+C$

   (b) (i) $x^6+6x^3+12+\frac{8}{x^3}$ (ii) $\frac{x^7}{7}+\frac{3}{2}x^4+12x-\frac{4}{x^2}+C$

   (c) (i) $1+12e^{3x}+36e^{6x}$ (ii) $x+4e^{3x}+6e^{6x}+C$

4. (a) (i) $\frac{1}{2}+\frac{1}{2}x$ (ii) $\frac{1}{2}x+\frac{1}{4}x^2+C$

   (b) (i) $e^{2x}-e^{-x}$ (ii) $\frac{1}{2}e^{2x}+e^{-x}+C$

   (c) (i) $\sec^2 4x$ (ii) $\frac{1}{4}\tan 4x+C$

5. (a) $\frac{(2x+1)^{21}}{42}+C$ (b) $\frac{1}{2}\ln|2x+1|+C$ (c) $\frac{2}{27}(4+9x)^{\frac{3}{2}}+C$

   (d) $\frac{1}{9}\ln|4+9x|+C$ (e) $\frac{1}{8}(6x+1)^{\frac{4}{3}}+C$ (f) $\frac{1}{6}\ln|6x+1|+C$

6. (a) $x+\ln|x|+C$ (b) $x-\ln|x+1|+C$

7. $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

8. (a) $\tan^{-1}\left(\frac{x}{2}\right)+C$

   (b) $\sin^{-1}\left(\frac{x}{3}\right)+C$

   (Alternatively, this answer can be written as $-\cos^{-1}\left(\frac{x}{3}\right)+C_1$)

   (c) $\cos^{-1}\left(\frac{x}{4}\right)+C$

   (Alternatively, this answer can be written as $-\sin^{-1}\left(\frac{x}{4}\right)+C_1$)

9. (a) 4 (b) 1 (c) 80

10. (a) (i) $f_{ave}=\frac{2}{\pi}$ (ii) Area $=2$

    (b) (i) $f_{ave}=0$ (ii) Area $=4$

    (c) (i) $f_{ave}=-\frac{2}{\pi}$ (ii) Area $=2$

11. The average depth between 12 noon and 4pm is 6.7 metres.

12. (a) The initial number of kangaroos was 500 000.
    (b) At the end of the 1990s, the number of kangaroos in Gippsland was approximately 301 000.
    (c) The average population during the 10 year period was approximately 340 000.

13. The average temperature is 180.7°C.

14. (a) $v(t) = 24(2t+1)^{11} - 24$
    (b) $x(t) = (2t+1)^{12} - 24t - 1$ Video solution

15. (a) $v(t) = 30\cos(10t)$ $\text{cm} \cdot \text{s}^{-1}$ and $a(t) = -300\sin(10t)$ $\text{cm} \cdot \text{s}^{-2}$
    (b) The average velocity during the first 2 seconds is 1.37 $\text{cm} \cdot \text{s}^{-1}$.

16. (a) $a(t) = \frac{1}{2000}(3t^2 - 180t + 1800)$ $\text{m} \cdot \text{s}^{-2}$ and
    $x(t) = \frac{1}{8000}(t^4 - 120t^3 + 3600t^2)$ m
    (b) After 30 seconds, the ride's height is 101.25 metres.
    (c) The average velocity of the ride during the first 30 seconds is 3.375 $\text{m}\cdot\text{s}^{-1}$.

17. It takes approximately 19 and a half hours for the snowball to melt completely.