Chapter 9

Differentiation by Rule

Reference: "Calculus", by James Stewart.

We have seen in the previous chapter that finding derivatives **from first principles** can be complicated, even for quite simple functions. In this chapter we introduce some **formulae** for differentiating. By using these formulae, we can differentiate many functions very easily.

9.1 The Power Rule

For **any** real number r , we have the following rule:

$$\boxed{\text{If } f(x) = x^r \text{ then } f'(x) = rx^{r-1}}$$

That is,

$$\boxed{\frac{d}{dx}(x^r) = rx^{r-1}}$$

Example 1. Using the Power Rule, find the derivative of the function $f(x) = x^4$.

Solution: We have $r = 4$, and so

$$\begin{aligned} f'(x) &= 4x^{4-1} \\ &= 4x^3 . \end{aligned}$$

$\square$

Example 2. Differentiate the following functions.

(a) $f(x) = \sqrt{x}$

Solution:

$$\begin{aligned} \text{Since} \quad f(x) &= x^{\frac{1}{2}} \\ \text{then} \quad f'(x) &= \frac{1}{2}\, x^{\frac{1}{2}-1} \\ &= \frac{1}{2}\, x^{-\frac{1}{2}} \end{aligned}$$

$\square$

(b) $f(x) = \dfrac{1}{x^2}$

Solution:

$$\begin{aligned} \text{Since} \quad f(x) &= x^{-2} \\ \text{then} \quad f'(x) &= -2x^{-3} \\ &= -\frac{2}{x^3} \end{aligned}$$

$\square$

9.2 Differentiation Laws

Suppose that f and g are differentiable functions, and that c is a constant. Then

- $\dfrac{d}{dx}(cf) = c\,\dfrac{df}{dx}$. That is, $(cf)' = cf'$.
- $\dfrac{d}{dx}c = 0$.

  This can be seen from the Power Rule as follows:

  Let $y = c = c \times 1 = cx^0$.

  Then $\dfrac{dy}{dx} = c \times 0x^{-1} = 0$.

- $\dfrac{d}{dx}(cx) = c$.

  This can be seen from the Power Rule as follows:

  Let $y = cx = cx^1$.

  Then $\dfrac{dy}{dx} = c \times 1x^0 = c \times 1 \times 1 = c$.

- $\dfrac{d}{dx}(f+g) = \dfrac{df}{dx} + \dfrac{dg}{dx}$. That is, $(f+g)' = f' + g'$.
- $\dfrac{d}{dx}(f-g) = \dfrac{df}{dx} - \dfrac{dg}{dx}$. That is, $(f-g)' = f' - g'$.

The rules for differentiating products and quotients are more complicated. We will consider those laws in Sections 9.3 and 9.4.

Example 3. Find the derivative of $f(x) = 3x^3 - 5x^2 + 2 + \dfrac{3}{x}$.

Solution:

$$
\begin{aligned}
\text{Since} \quad f(x) &= 3x^3 - 5x^2 + 2 + 3x^{-1} \\
\text{then} \quad f'(x) &= 3 \times 3x^2 - 5 \times 2x + 0 + 3 \times -1x^{-2} \\
&= 9x^2 - 10x - \frac{3}{x^2}.
\end{aligned}
$$

□

Exercises

Find $f'(x)$ for each of the following:

(a) $f(x) = 3x^2 + 6x - 5$

(b) $f(x) = 4x^5 + \dfrac{x}{2}$

(c) $f(x) = x^9 - 3x^4 + x^2$

(d) $f(x) = x - \dfrac{1}{2x} + \dfrac{4}{x^2}$

(e) $f(x) = \dfrac{2x^{\frac{3}{2}}}{3} + 7x^{-\frac{1}{5}}$

(f) $f(x) = 3x^2 - \dfrac{3}{\sqrt{x}} + \dfrac{4}{x^2}$

9.3 The Product Rule

Note that in general, $(fg)'(x) \neq f'(x)g'(x)$. Instead, the derivative of a product is given by the following rule (which is known as the **product rule**):

$$\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x)}$$

(as long as f and g are both differentiable functions).

This rule is often written without the variable x as follows:

$$\boxed{(fg)' = f'g + fg'}$$

Proof of the Product Rule: (Not examinable.)

Let $F(x) = f(x)g(x)$, where f and g are differentiable functions. Then

$$\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \times \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \times \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

Note: The product rule is often expressed as:

$$\boxed{(uv)' = u'v + uv'}$$

Example 4. Find the derivative of

$$f(x) = \left(x^3 + 2x - \frac{1}{x^2}\right)(3x^5 + x^2 - 8).$$

Solution:

$$\text{Let } u = x^3 + 2x - \tfrac{1}{x^2} = x^3 + 2x - x^{-2} \text{ and } v = 3x^5 + x^2 - 8.$$
$$\text{Then } u' = 3x^2 + 2 + 2x^{-3} \text{ and } v' = 15x^4 + 2x.$$

By the Product Rule,

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= \left(3x^2 + 2 + \frac{2}{x^3}\right)\left(3x^5 + x^2 - 8\right) + \left(x^3 + 2x - \frac{1}{x^2}\right)\left(15x^4 + 2x\right). \end{aligned}$$

□

9.4 The Quotient Rule

The derivative of a quotient is given by the following rule (which is known as the **quotient rule**):

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

(as long as f and g are both differentiable functions).

This rule is often written without the variable x as follows:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

Alternatively, we often express the quotient rule as:

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

Example 5. Find the derivative of $y = \dfrac{4x^2+7}{x^5-2x+6}$.

Solution: Let $u = 4x^2 + 7$ and $v = x^5 - 2x + 6$.

Then $u' = 8x$ and $v' = 5x^4 - 2$.

So

$$\begin{aligned}
\frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \text{(by the quotient rule)} \\
&= \frac{8x(x^5-2x+6) - (4x^2+7)(5x^4-2)}{(x^5-2x+6)^2} \\
&= \frac{8x^6 - 16x^2 + 48x - 20x^6 + 8x^2 - 35x^4 + 14}{(x^5-2x+6)^2} \\
&= \frac{-12x^6 - 35x^4 - 8x^2 + 48x + 14}{(x^5-2x+6)^2}
\end{aligned}$$

□

9.5 The Chain Rule

The chain rule is used to differentiate **composite functions**. It is a particularly useful rule, as it allows us to replace a complicated derivative with easier derivatives. The chain rule is as follows:

If the functions $y = f(u)$ and $u = g(x)$ are both differentiable, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$

Example 6. Differentiate $y = (2x-1)^4$.

Solution: **Method 1:** Let $u = 2x - 1$, so that $y = u^4$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx} \\ &= 4u^3 \times 2 \\ &= 8u^3 \\ &= 8(2x-1)^3 \end{aligned}$$

□

Solution: **Method 2:**

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\,(2x-1)^4 \\ &= 4\,(2x-1)^3 \times \frac{d}{dx}\,(2x-1) \\ &= 4\,(2x-1)^3 \times 2 \\ &= 8\,(2x-1)^3 \end{aligned}$$

□

Example 7. Differentiate $y = (x^3 + 2x)^3$.

Solution: **Method 1:** Let $u = x^3 + 2x$ so that $y = u^3$.

Solution: Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 3u^2 \times (3x^2 + 2) \\ &= 3(x^3 + 2x)^2(3x^2 + 2)\,. \end{aligned}$$

□

Method 2:

$$\begin{aligned} \frac{d}{dx}(x^3 + 2x)^3 &= 3(x^3 + 2x)^2 \times \frac{d}{dx}(x^3 + 2x) \\ &= 3(x^3 + 2x)^2 \times (3x^2 + 2)\,. \end{aligned}$$

□

Example 8. Differentiate $y = \left(\frac{x-2}{2x+1}\right)^9$.

Solution: We have

$$\begin{aligned} \frac{dy}{dx} &= 9\left(\frac{x-2}{2x+1}\right)^8 \times \frac{d}{dx}\left(\frac{x-2}{2x+1}\right) \\ &= 9\left(\frac{x-2}{2x+1}\right)^8 \times \left(\frac{u'v - uv'}{v^2}\right) \\ &= 9\left(\frac{x-2}{2x+1}\right)^8 \times \left(\frac{(1)(2x+1) - (x-2)(2)}{(2x+1)^2}\right) \\ &= 9\left(\frac{x-2}{2x+1}\right)^8 \times \left(\frac{2x+1-2x+4}{(2x+1)^2}\right) \\ &= 9\left(\frac{x-2}{2x+1}\right)^8 \times \frac{5}{(2x+1)^2} \\ &= \frac{45(x-2)^8}{(2x+1)^{10}}\,. \end{aligned}$$

□

Example 9. Differentiate $f(x) = x^2\sqrt{2x+5}$.

Solution: Let $u = x^2$ and $v = \sqrt{2x+5} = (2x+5)^{\frac{1}{2}}$.
Then $u' = 2x$ and $v' = \frac{1}{2}(2x+5)^{-\frac{1}{2}} \times 2 = \frac{1}{\sqrt{2x+5}}$.
By the product rule,

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= 2x\sqrt{2x+5} + \frac{x^2}{\sqrt{2x+5}} \end{aligned}$$

$\square$

Exercises

Find $f'(x)$ for each of the following:

(a) $f(x) = \sqrt{2x-1}$

(b) $f(x) = (2x-1)^5$

(c) $f(x) = \dfrac{1}{(x+1)^2}$

(d) $f(x) = \sqrt{1-x^3}$

(e) $f(x) = 3x^5 - 2x^2 + 1$

(f) $f(x) = 4x^2 + \sqrt{x} - \dfrac{1}{x}$

(g) $f(x) = (2-3x^3)^4$

(h) $f(x) = (2x-1)\sqrt{x^2-1}$

(i) $f(x) = \dfrac{4-3x^2}{2x^2+3}$

(j) $f(x) = \sqrt{4x^3-2x+5}$

(k) $f(x) = (2x-1)(x+2)^5$

(l) $f(x) = (4x^2-3)(5x-1)^8$

9.6 Derivatives of Trigonometric Functions

Recall, from Chapter 7, that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, as long as x is measured in **radians**.

We shall use this fact to obtain formulae for differentiating the trigonometric functions.

Note that the formulae which we obtain will only be valid if x is in radians.

Our first formula is

$$\boxed{\text{if } \ f(x) = \sin x \ \text{ then } \ f'(x) = \cos x}$$

when x is measured in **radians**.

Proof. (Not examinable.)

First note that

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \left[\frac{\cos h - 1}{h} \times \frac{\cos h + 1}{\cos h + 1}\right] \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \times \lim_{h\to 0} \frac{-\sin h}{\cos h + 1} \\
&= 1 \times \frac{0}{1+1} \\
&= 0.
\end{aligned}$$

Now, if we let $f(x) = \sin x$ we obtain

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \sin h}{h} + \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} \\
&= \cos x \lim_{h\to 0} \frac{\sin h}{h} + \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= \cos x \times 1 + \sin x \times 0 \quad \text{(using the results from above)} \\
&= \cos x\,.
\end{aligned}$$

$\square$

Similarly, it can be shown that

$$\boxed{\text{if } f(x) = \cos x \text{ then } f'(x) = -\sin x}$$

when x is measured in **radians**.

Example 10. Find the derivative of $y = x^2 \sin x$.

Solution: Let $u = x^2$ and $v = \sin x$.
Then $u' = 2x$ and $v' = \cos x$.
By the product rule,

$$\begin{aligned}
\frac{dy}{dx} &= u'v + uv' \\
&= 2x \sin x + x^2 \cos x
\end{aligned}$$

$\square$

Example 11. Find the derivative of $y = \cos^3 x$.

Solution:

$$\begin{aligned}
\text{Since } y &= (\cos x)^3 \\
\text{we have } \frac{dy}{dx} &= 3(\cos x)^2 \times -\sin x \\
&= -3\cos^2 x \, \sin x\,.
\end{aligned}$$

$\square$

Example 12. By applying the quotient rule, together with the differentiation rules for $\sin x$ and $\cos x$, show that

$$\boxed{\text{if } f(x) = \tan x \text{ then } f'(x) = \sec^2 x}$$

when x is measured in **radians**.

Solution: We have $f(x) = \dfrac{\sin x}{\cos x}$.

Let $u = \sin x$ and $v = \cos x$.

Then $u' = \cos x$ and $v' = -\sin x$.

By the Quotient Rule,

$$\begin{aligned}
f'(x) &= \frac{u'v - uv'}{v^2} \\
&= \frac{\cos x \times \cos x - \sin x \times - \sin x}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \left(\frac{1}{\cos x}\right)^2 \\
&= (\sec x)^2 \\
&= \sec^2 x .
\end{aligned}$$

$\square$

Example 13. Find the derivative of $y = \sin(2x)$.

Solution:

$$\frac{d}{dx}\sin\Big(\qquad\Big) = \cos\Big(\qquad\Big) \times \frac{d}{dx}\Big(\qquad\Big)$$

Therefore,

$$\begin{aligned} \frac{d}{dx}\sin(2x) &= \cos(2x) \times \frac{d}{dx}(2x) \\ &= 2\cos(2x) \ . \end{aligned}$$

$\square$

In general

$$\text{if } f(x) = \sin kx \text{ then } f'(x) = k\cos kx \ .$$

Similarly,

$$\text{if } f(x) = \cos kx \text{ then } f'(x) = -k\sin kx \ ,$$

and

$$\text{if } f(x) = \tan kx \text{ then } f'(x) = k\sec^2 kx \ .$$

These three formulae are given on the Formula Sheet which is provided in the Maths 1 exams.

Exercises

Find $\dfrac{dy}{dx}$ for the following:

(a) $y = \sin(3x - 2)$

(b) $y = x\cos x$

(c) $y = \sin^2 x$

(d) $y = \sin x \cos 3x$

(e) $y = \dfrac{\sin x}{x}$

(f) $y = \cos(4x^2)$

(g) $y = 4\tan(3x + 1)$

(h) $y = \tan(2x^3 - 3x)$

(i) $y = \sin 2x - \cos^2 x + x^4 - 2$

9.7 Derivatives of Exponential Functions

Consider the exponential function $f(x) = a^x$, where $a > 0$.

Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \,. \end{aligned}$$

- It can be shown that $\lim_{h\to 0} \dfrac{2^h - 1}{h} \approx 0.693$

  (obtained by finding $\dfrac{2^h - 1}{h}$ on a calculator when $h = \pm 0.00001$).
  Thus, if $f(x) = 2^x$ then

$$\begin{aligned} f'(x) &= 2^x \times \lim_{h\to 0} \frac{2^h - 1}{h} \\ &\approx 2^x \times 0.693 \end{aligned}$$

- Similarly, it can be shown that $\lim_{h\to 0} \dfrac{3^h - 1}{h} \approx 1.0986$

  (obtained by finding $\dfrac{3^h - 1}{h}$ on a calculator when $h = \pm 0.00001$).
  Thus, if $f(x) = 3^x$ then

$$\begin{aligned} f'(x) &= 3^x \times \lim_{h\to 0} \frac{3^h - 1}{h} \\ &\approx 3^x \times 1.0986 \end{aligned}$$

It seems likely that there should exist a number a between 2 and 3 such that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1 \,.$$

This special number a would satisfy

$$f(x) = a^x \quad \Rightarrow \quad f'(x) = a^x \times 1 = a^x \, .$$

The special number which has this property is approximately 2.7182818 , and is known as Euler's number. We denote this number by e . That is, the number e satisfies

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Thus we have the following result:

$$\boxed{\text{if } \ f(x) = e^x \ \text{ then } \ f'(x) = e^x}$$

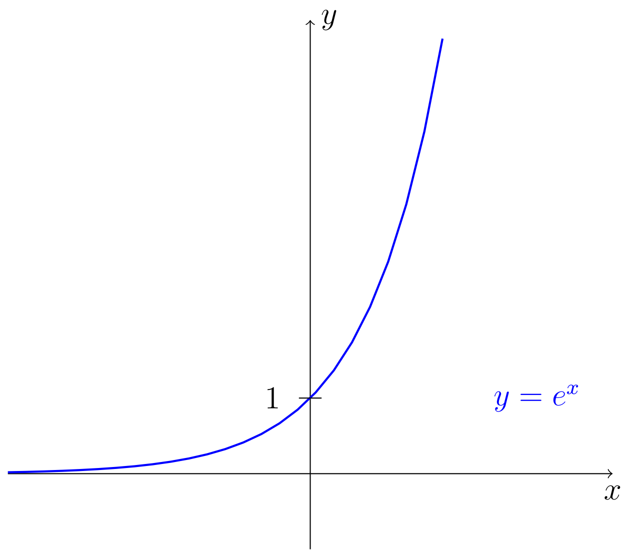
Warning: We **cannot** differentiate e^x by using the Power Rule. That is,

$$\text{if } \ f(x) = e^x \quad \text{then} \quad f'(x) \neq xe^{x-1}.$$

The Power Rule is used when we have x to the power of a **number**. In contrast, $f(x) = e^x$ does **not** have a number as the power, and does **not** have x as the base!

Note: The graph of $y = e^x$ has the same basic shape as the graphs of $y = 2^x$ and $y = 3^x$.

That is, the graph of $y = e^x$ looks like



Example 14. Find the derivative of $y = e^{2x}$.

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(e^{2x}\right)\\ &= e^{2x} \times \frac{d}{dx}(2x)\\ &= e^{2x} \times 2\\ &= 2e^{2x}\end{aligned}$$

□

In general,

$$\text{if } f(x) = e^{kx} \quad \text{then} \quad f'(x) = ke^{kx} .$$

This formula is given on the Formula Sheet which is provided in the Maths 1 exams.

Example 15. Find the derivative of $y = e^{5x}$.

Solution: Just put $k = 5$ in the above formula. This gives

$$\frac{dy}{dx} = 5e^{5x} .$$

□

Example 16. Find the derivative of $y = e^{-x^2}$.

Solution:

$$\begin{aligned}\frac{dy}{dx} &= e^{-x^2} \times -2x\\ &= -2x\, e^{-x^2}\end{aligned}$$

Alternative working:

Let $u = -x^2$ so that $y = e^u$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= e^u \times -2x\\ &= -2x\, e^{-x^2}\end{aligned}$$

$\square$

Example 17. Find the derivative of $y = xe^x$.

Solution: Let $u = x$ and $v = e^x$.
Then $u' = 1$ and $v' = e^x$.
By the product rule,

$$\begin{aligned} \frac{dy}{dx} &= u'v + uv' \\ &= e^x + xe^x \\ &= (1+x)e^x \end{aligned}$$

$\square$

Exercises

Find $\dfrac{dy}{dx}$ for each of the following:

(a) $y = e^{-3x}$	(b) $y = (2x^2+1)e^{3x}$	(c) $y = \dfrac{e^x}{e^x+1}$
(d) $y = (e^x + e^{2x})^8$	(e) $y = e^{(x^2)}$	(f) $y = \dfrac{x-1}{e^x}$
(g) $y = e^x + \sin x$	(h) $y = e^x \cos x$	(i) $y = \cos(e^x)$

9.8 Implicit Differentiation

Implicit differentiation is used to differentiate equations in which y is **not** written explicitly as a function of x. That is, implicit differentiation is used when the equation to be differentiated is **not** of the form $y = f(x)$ (with y on one side and the x–terms on the other side of the equation).

The steps involved in implicit differentiation are:

1. Differentiate both sides of the equation with respect to x.

2. Rearrange the resulting equation to get $\dfrac{dy}{dx}$ by itself.

As we shall see in the next few examples, implicit differentiation makes use of the **chain rule**. Often we need to use the product rule as well.

Example 18. Find $\dfrac{dy}{dx}$ for the equation $x^2 + y^2 = 1$.

Solution: First we need to differentiate both sides of the given equation with respect to x :

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}\,1 \\ \text{i.e. } \frac{d}{dx}\,x^2 + \frac{d}{dx}\,y^2 &= 0 \\ \text{i.e. } 2x + \frac{d}{dy}y^2\,\frac{dy}{dx} &= 0. \\ \text{Thus } 2x + 2y\,\frac{dy}{dx} &= 0\,. \end{aligned}$$

Next we need to rearrange to get $\dfrac{dy}{dx}$ by itself:

$$\begin{aligned} 2y\,\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\,\frac{x}{y}\,. \end{aligned}$$

$\square$

Example 19. Find $\dfrac{dy}{dx}$ for the equation $y^2 + \cos y = x$.

Solution:

$$\begin{aligned} \frac{d}{dx}\left(y^2 + \cos y\right) &= \frac{d}{dx}(x) \\ (2y - \sin y)\,\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y - \sin y} \end{aligned}$$

$\square$

Example 20. Find $\dfrac{dy}{dx}$ for the equation $x^6 + 2x^2y^3 + y^6 = 0$.

Solution: First we differentiate both sides of the given equation with respect to x . We obtain

$$\frac{d}{dx}\,x^6 \;+\; \frac{d}{dx}\,(2x^2\,y^3) \;+\; \frac{d}{dx}\,y^6 \;=\; \frac{d}{dx}\,0\,.$$

To differentiate $2x^2\,y^3$, we will need to use the product rule, with $u = 2x^2$ and $v = y^3$:

$$6x^5 + \left\{\left(\frac{du}{dx}\right) \times v + u \times \left(\frac{dv}{dx}\right)\right\} + 6y^5\,\frac{dy}{dx} = 0.$$

$$\text{i.e. } 6x^5 + \left\{\frac{d}{dx}(2x^2)y^3 + 2x^2\frac{d}{dx}(y^3)\right\} + 6y^5\frac{dy}{dx} = 0.$$

$$\text{i.e. } 6x^5 + \left\{4xy^3 + 2x^2 3y^2\frac{dy}{dx}\right\} + 6y^5\frac{dy}{dx} = 0.$$

$$\text{i.e. } 6x^5 + 4xy^3 + 6x^2y^2\frac{dy}{dx} + 6y^5\frac{dy}{dx} = 0.$$

Finally we need to rearrange to get $\dfrac{dy}{dx}$ by itself:

$$6x^2y^2\frac{dy}{dx} \;+\; 6y^5\frac{dy}{dx} \;=\; -6x^5 \;-\; 4xy^3$$

$$\text{i.e. } \frac{dy}{dx}\left(6x^2y^2 \;+\; 6y^5\right) \;=\; -6x^5 \;-\; 4xy^3$$

$$\text{i.e. } \frac{dy}{dx} \;=\; \frac{-6x^5 \;-\; 4xy^3}{6x^2y^2 \;+\; 6y^5}\,.$$

□

Exercises

For each of the following equations, find $\dfrac{dy}{dx}$ by using implicit differentiation:

(a) $x^2 - \cos y + y = \sin x$

(b) $3x^4 + xy - xy^4 = 3$

(c) $e^y + \sin y + 2x^2y = \dfrac{1}{x}$

(d) $5xy - e^x + \tan y = 0$

(e) $y + y^2 - y^3 = x + x^2 - x^3$

(f) $\sin(xy) + \cos(y^2) = 4x + 3$

9.9 Derivatives of Logarithmic Functions

Here we consider the logarithmic function with base e. That is, we consider the equation $y = \log_e x$. This is often written as $y = \ln x$, and is known as the *natural logarithm.*

We know that

$$y = \log_e x \quad \text{if and only if} \quad e^y = x\,.$$

Differentiating this second equation **implicitly** gives

$$\begin{aligned} \frac{de^y}{dx} &= \frac{dx}{dx} \\ \text{i.e.}\quad \frac{de^y}{dy}\frac{dy}{dx} &= 1 \\ \text{i.e.}\quad e^y\,\frac{dy}{dx} &= 1 \\ \text{i.e.}\quad \frac{dy}{dx} &= \frac{1}{e^y} \\ &= \frac{1}{x}\,. \end{aligned}$$

Thus

$$\boxed{\text{if} \quad f(x) = \log_e x \quad \text{then} \quad f'(x) = \frac{1}{x}}$$

Example 21. Find the derivative of $y = \ln(x^2 + x)$.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x^2+x} \times (2x+1) \\ &= \frac{2x+1}{x^2+x}\,. \end{aligned}$$

$\square$

Example 22. Find the derivative of $y = \ln(3x)$.

Method 1: $\dfrac{dy}{dx} = \dfrac{1}{3x} \times 3 = \dfrac{1}{x}$.

Method 2: Using a Log Law, we have

$$\begin{aligned} y &= \ln 3 + \ln x \\ \implies \frac{dy}{dx} &= 0 + \frac{1}{x} \\ &= \frac{1}{x} . \end{aligned}$$

Example 23. Find the derivative of $y = \ln\left(\dfrac{x^2+1}{x^6+5}\right)$.

Solution: **Method 1:**

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\frac{x^2+1}{x^6+5}} \times \frac{d}{dx}\left(\frac{x^2+1}{x^6+5}\right) \\ &= \dots \end{aligned}$$

Method 2: Using a Log Law, we can write

$$y = \ln(x^2+1) - \ln(x^6+5) .$$

$$\text{Thus } \frac{dy}{dx} = \frac{1}{x^2+1} \times (2x) - \frac{1}{x^6+5} \times (6x^5)$$

$$= \frac{2x}{x^2+1} - \frac{6x^5}{x^6+5} .$$

$\square$

Exercises

Find $\dfrac{dy}{dx}$ for each of the following:

(a) $y = x \ln x$	(b) $y = \ln(3x^2+1)$	(c) $y = \ln(e^x + e^{-x})$
(d) $y = \ln(\cos x)$	(e) $y = \dfrac{\ln x}{x}$	(f) $y = \cos(\ln x)$

9.10 Derivatives of Inverse Trigonometric Functions

In this section, we are going to learn how to differentiate

$$\sin^{-1} x,\ \cos^{-1} x \text{ and } \tan^{-1} x\,.$$

Derivative of $\sin^{-1} x$.

First of all, note that $\sin^{-1} x \neq (\sin x)^{-1}$ and so to differentiate $\sin^{-1} x$, we must use the fact that $\sin^{-1} x$ is the **inverse** of the function $\sin x$ with its domain restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let $y = \sin^{-1} x$. Then

$$\sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\,.$$

We find $\dfrac{dy}{dx}$ by using implicit differentiation:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\therefore \cos y \frac{dy}{dx} = \frac{dx}{dx}$$

$$\therefore \cos y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

Note that $\sin^2 y + \cos^2 y = 1 \implies \cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$.

Since $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$, we have $\cos y \geq 0$ and so $\cos y = \sqrt{1 - x^2}$.

Thus

$$y = \sin^{-1} x \quad \implies \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \quad \text{and so} \quad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}\,.$$

On the next page, we use the same method to find $\frac{d}{dx}\cos^{-1} x$ and $\frac{d}{dx}\tan^{-1} x$.

Derivative of $\cos^{-1} x$.

Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi$.
We find $\dfrac{dy}{dx}$ by using implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}(\cos y) &= \frac{d}{dx}(x) \\
\frac{d}{dy}(\cos y)\,\frac{dy}{dx} &= 1 \\
(-\sin y)\,\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{-\sin y} = \frac{-1}{\sin y}\,.
\end{aligned}$$

Note that $\sin^2 y + \cos^2 y = 1 \implies \sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-x^2}$.
Since $0 \le y \le \pi$, we have $\sin y \ge 0$ and so $\sin y = +\sqrt{1-x^2}$.
Thus

$$y = \cos^{-1} x \quad \implies \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad \text{and so} \quad \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}\,.$$

Derivative of $\tan^{-1} x$.

Let $y = \tan^{-1} x$. Then $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
We find $\dfrac{dy}{dx}$ by using implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\
\frac{d}{dy}(\tan y)\,\frac{dy}{dx} &= 1 \\
\left(\sec^2 y\right)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y}\,.
\end{aligned}$$

Note that

$$\begin{aligned}
\sin^2 y + \cos^2 y = 1 \quad &\Longrightarrow \quad \frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} \\
&\Longrightarrow \quad \tan^2 y + 1 = \sec^2 y \\
&\Longrightarrow \quad x^2 + 1 = \sec^2 y \, .
\end{aligned}$$

Thus

$$y = \tan^{-1} x \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2} \quad \text{and so} \quad \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \, .$$

Note.

We do **not** have to work through the previous steps whenever we want to differentiate $\sin^{-1} x$, $\cos^{-1} x$, or $\tan^{-1} x$. Instead, we can just use the results

$$\boxed{\begin{aligned}
f(x) = \sin^{-1} x \quad &\Longrightarrow \quad f'(x) = \frac{1}{\sqrt{1-x^2}} \\
f(x) = \cos^{-1} x \quad &\Longrightarrow \quad f'(x) = \frac{-1}{\sqrt{1-x^2}} \\
f(x) = \tan^{-1} x \quad &\Longrightarrow \quad f'(x) = \frac{1}{1+x^2}
\end{aligned}}$$

Example 24. Find the derivative of $f(x) = \sin^{-1}(3x)$.

Solution: Chain Rule:

$$\frac{d}{dx}\sin^{-1}\Big(\qquad\Big) = \frac{1}{\sqrt{1-(\qquad)^2}} \times \frac{d}{dx}\Big(\qquad\Big)$$

Thus

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(3x)^2}} \times \frac{d}{dx}(3x) \\ &= \frac{1}{\sqrt{1-9x^2}} \times 3 \\ &= \frac{3}{\sqrt{1-9x^2}}. \end{aligned}$$

□

Example 25. Let $y = \sin^{-1}\left(e^{4x}\right)$. Find $\dfrac{dy}{dx}$.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\sin^{-1}\left(e^{4x}\right)\right) \\ &= \frac{1}{\sqrt{1-(e^{4x})^2}} \times \frac{d}{dx}\left(e^{4x}\right) \\ &= \frac{1}{\sqrt{1-(e^{4x})^2}} \times 4e^{4x} \\ &= \frac{4e^{4x}}{\sqrt{1-(e^{4x})^2}} \end{aligned}$$

□

Example 26. Find the derivative of $f(x) = \cos^{-1}\left(\dfrac{x}{2}\right)$.

Solution:

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2} = \frac{-1}{\sqrt{4}\sqrt{1-\frac{x^2}{4}}} = \frac{-1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}} = \frac{-1}{\sqrt{4-x^2}}.$$

□

In general, if a is a constant such that $a > 0$ then

$f(x)$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\cos^{-1}\left(\dfrac{x}{a}\right)$	$\tan^{-1}\left(\dfrac{x}{a}\right)$
$f'(x)$	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\dfrac{-1}{\sqrt{a^2 - x^2}}$	$\dfrac{a}{a^2 + x^2}$

We must remember that a is **not** allowed to be a function of x in the above formulae.

Example 27. Find the derivative of $f(x) = \cos^{-1}\left(\dfrac{x}{2}\right)$.

Solution: Put $a = 2$. Then

$$f'(x) = \frac{-1}{\sqrt{2^2 - x^2}} = \frac{-1}{\sqrt{4 - x^2}} .$$

$\square$

Note: If we cannot put the function in the form given in the above table, then we should set $a = 1$ and use the chain rule.

Example 28. Find the derivative of $f(x) = \cos^{-1}\left(\dfrac{4}{x}\right)$.

Solution: Start with $\dfrac{d}{dx}\cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$ which we get by setting $a = 1$ in the formula for the derivative of $\cos^{-1}\left(\frac{x}{a}\right)$. The chain rule now gives us

$$
\begin{aligned}
\frac{d}{dx}\cos^{-1}\left(\frac{4}{x}\right) &= \frac{-1}{\sqrt{1-\left(\frac{4}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{4}{x}\right) \\
&= \frac{-1}{\sqrt{1-\frac{16}{x^2}}} \times \frac{d}{dx}\left(4x^{-1}\right) \\
&= \frac{-1}{\sqrt{1-\frac{16}{x^2}}} \times -4x^{-2} \\
&= \frac{-1}{\sqrt{1-\frac{16}{x^2}}} \times -\frac{4}{x^2} \\
&= \frac{4}{x^2\sqrt{1-\frac{16}{x^2}}} \\
&= \frac{4}{\sqrt{x^4(1-\frac{16}{x^2})}} \\
&= \frac{4}{\sqrt{x^4-16x^2}}\,.
\end{aligned}
$$

□

Exercises

Find the derivative of the following:

(a) $\sin^{-1}\left(\frac{x}{3}\right)$.

(b) $\cos^{-1}\left(\frac{x^2}{5}\right)$.

Hint: put $a = 1$ in the relevant formula and then use the chain rule.

(c) $\tan^{-1}(4x)$.

Hint: we can put $a = \frac{1}{4}$ or we can use the chain rule.

(d) $\cos^{-1}(-2x)$.

Hint: we **cannot** put $a = -\frac{1}{2}$ since a needs to be positive.

Put $a = 1$ in the relevant formula and then use the chain rule.

(e) $\sin^{-1}\left(\frac{3}{x}\right)$.

Hint: put $a = 1$ in the relevant formula and then use the chain rule.

9.11 Answers to Chapter 9 Exercises

9.2:

(a) $f'(x) = 6x + 6$

(b) $f'(x) = 20x^4 + \frac{1}{2}$

(c) $f'(x) = 9x^8 - 12x^3 + 2x$

(d) $f'(x) = 1 + \frac{1}{2x^2} - \frac{8}{x^3}$

(e) $f'(x) = x^{\frac{1}{2}} - \frac{7}{5}x^{-\frac{6}{5}}$

(f) $f'(x) = 6x + \frac{3}{2}x^{-\frac{3}{2}} - \frac{8}{x^3}$

9.5:

(a) $f'(x) = \frac{1}{\sqrt{2x-1}}$

(b) $f'(x) = 10(2x-1)^4$

(c) $f'(x) = \frac{-2}{(x+1)^3}$

(d) $f'(x) = \frac{-3x^2}{2\sqrt{1-x^3}}$

(e) $f'(x) = 15x^4 - 4x$

(f) $f'(x) = 8x + \frac{1}{2\sqrt{x}} + \frac{1}{x^2}$

(g) $f'(x) = -36x^2(2-3x^3)^3$

(h) $f'(x) = 2\sqrt{x^2-1} + \frac{x(2x-1)}{\sqrt{x^2-1}}$

(i) $f'(x) = \frac{-34x}{(2x^2+3)^2}$

(j) $f'(x) = \frac{6x^2-1}{\sqrt{4x^3-2x+5}}$

(k) $f'(x) = (x+2)^4(12x-1)$

(l) $f'(x) = 8(5x-1)^7(25x^2-x-15)$

9.6:

(a) $\frac{dy}{dx} = 3\cos(3x-2)$

(b) $\frac{dy}{dx} = \cos x - x\sin x$

(c) $\frac{dy}{dx} = 2\sin x\cos x \quad (= \sin 2x)$

(d) $\frac{dy}{dx} = \cos x\cos 3x - 3\sin x\sin 3x$

(e) $\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$

(f) $\frac{dy}{dx} = -8x\sin 4x^2$

(g) $\frac{dy}{dx} = 12\sec^2(3x+1)$

(h) $\frac{dy}{dx} = 3(2x^2-1)\sec^2(2x^3-3x)$

(i) $\frac{dy}{dx} = 2\cos 2x + \sin 2x + 4x^3$

9.7:

(a) $\frac{dy}{dx} = -3e^{-3x}$

(b) $\frac{dy}{dx} = e^{3x}(6x^2+4x+3)$

(c) $\frac{dy}{dx} = \frac{e^x}{(e^x+1)^2}$

(d) $\frac{dy}{dx} = 8(e^x+e^{2x})^7(e^x+2e^{2x})$

(e) $\frac{dy}{dx} = 2xe^{(x^2)}$

(f) $\frac{dy}{dx} = \frac{2-x}{e^x}$

(g) $\frac{dy}{dx} = e^x + \cos x$

(h) $\frac{dy}{dx} = e^x(\cos x - \sin x)$

(i) $\frac{dy}{dx} = -e^x\sin(e^x)$

9.8:

(a) $\frac{dy}{dx} = \frac{\cos x - 2x}{\sin y + 1}$

(b) $\frac{dy}{dx} = \frac{12x^3 + y - y^4}{x(4y^3 - 1)}$

(c) $\frac{dy}{dx} = \frac{-(1+4x^3y)}{x^2(e^y + \cos y + 2x^2)}$

(d) $\frac{dy}{dx} = \frac{e^x - 5y}{5x + \sec^2 y}$

(e) $\frac{dy}{dx} = \frac{1+2x-3x^2}{1+2y-3y^2}$

(f) $\frac{dy}{dx} = \frac{4 - y\cos(xy)}{x\cos(xy) - 2y\sin(y^2)}$

9.9:

(a) $\frac{dy}{dx} = \ln x + 1$

(b) $\frac{dy}{dx} = \frac{6x}{3x^2+1}$

(c) $\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \left(= \frac{e^{2x}-1}{e^{2x}+1} \right)$

(d) $\frac{dy}{dx} = -\tan x$

(e) $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

(f) $\frac{dy}{dx} = \frac{-\sin(\ln x)}{x}$

9.10:

(a) $f'(x) = \dfrac{1}{\sqrt{9 - x^2}}$

(b) $f'(x) = \dfrac{-2x}{\sqrt{25 - x^4}}$

(c) $f'(x) = \dfrac{4}{1 + 16x^2}$

(d) $f'(x) = \dfrac{2}{\sqrt{1 - 4x^2}}$

(e) $f'(x) = \dfrac{-3}{\sqrt{x^4 - 9x^2}}$