Maths 1

Exercise Sheet 16: Linear Approximations and The Angle Between Two Curves

1. (a) (i) Find the linear approximation of $f(x) = x^2$ at $x = 3$.
   (ii) Use this linear approximation to find an approximate value of 2.99^2.
   (iii) Use your calculator to find the *true* value of 2.99^2.

   (b) (i) Find the linear approximation of $f(x) = x^2$ at $x = 10$.
   (ii) Use this linear approximation to find an approximate value of 10.05^2.
   (iii) Use your calculator to find the *true* value of 10.05^2.

   (c) (i) Find the linear approximation of $g(x) = \cos x$ at $x = 0$.
   (ii) Use this linear approximation to find an approximate value of $\cos(0.1)$.
   (iii) Use your calculator to find the *true* value of $\cos(0.1)$. Write your answer to 4 decimal places.

   (d) (i) Find the linear approximation of $g(x) = \cos x$ at $x = \frac{\pi}{2}$.
   (ii) Use this linear approximation to find an approximate value of $\cos(1.55)$. Write your answer to 4 decimal places.
   (iii) Use your calculator to find the *true* value of $\cos(1.55)$. Write your answer to 4 decimal places.

2. (a) Find the linear approximation of $f(x) = 2x+7$ at $x = 1$.
   (b) Find the linear approximation of $f(x) = 2x+7$ at $x = 5$.

3. (a) Find the linear approximation of $f(x) = \ln(x+1)$ at $x = 0$.

   (b) (i) Use this linear approximation to find an approximate value of $\ln(1.02)$.
   [Hint: Think carefully about what x-value should be used.]
   (ii) Use your calculator to find the *true* value of $\ln(1.02)$. Write your answer to 4 decimal places.

   (c) (i) Use the linear approximation from (a) to find an approximate value of $\ln(0.99)$.
   [Hint: Think carefully about what x-value should be used.]
   (ii) Use your calculator to find the *true* value of $\ln(0.99)$. Write your answer to 4 decimal places.

4. (a) Find the linear approximation of $f(x) = \sqrt{x+1}$ at $x = 0$.

   (b) (i) Use this linear approximation of $f(x)$ to find an approximate value of $\sqrt{1.1}$.
   (ii) Use your calculator to find the *true* value of $\sqrt{1.1}$. Write your answer to 4 decimal places.

(c) (i) Use the linear approximation of $f(x)$ to find an approximate value of $\sqrt{0.97}$.

(ii) Use your calculator to find the *true* value of $\sqrt{0.97}$. Write your answer to 4 decimal places.

In Questions 5 and 6, we will find approximate values for

$$\frac{1}{4.01}, \quad \frac{1}{4.2} \quad \text{and} \quad \frac{1}{3.98}.$$

We will see that there are various slightly different ways of setting up the problem. However they lead to the same final results.

5. (a) Find the linear approximation of $f(x) = \dfrac{1}{x}$ at $x = 4$.

(b) Use this linear approximation of $f(x)$ to find approximate values of

(i) $\dfrac{1}{4.01}$ (ii) $\dfrac{1}{4.2}$ (iii) $\dfrac{1}{3.98}$

6. (a) Find the linear approximation of $g(x) = \dfrac{1}{4+x}$ at $x = 0$.

(b) Use this linear approximation of $g(x)$ to find approximate values of

(i) $\dfrac{1}{4.01}$ (ii) $\dfrac{1}{4.2}$ (iii) $\dfrac{1}{3.98}$

7. Use the linear approximation of

$$\begin{aligned} & f(x) = \sqrt{x} \quad \text{at } x = 100, \\ \text{or} \quad & g(x) = \sqrt{100+x} \quad \text{at } x = 0, \\ \text{or} \quad & h(x) = \sqrt{95+x} \quad \text{at } x = 5, \\ \text{or} \quad & \ldots \end{aligned}$$

to find approximate values of

(i) $\sqrt{100.1}$ (ii) $\sqrt{99.8}$ (iii) $\sqrt{103}$

8. (a) Find the linear approximation to $(1+x)^n$ at $x = 0$ (where n is a constant).

(b) Compare this with the expansion of $(1+x)^n$ where n is a positive integer.

Hint: Use the Binomial Theorem or Pascal's Triangle, from Section 1.6 of the Course Notes.

9. Consider the function $f(x) = \sqrt{x}$. Suppose that x is measured to be 100, with a maximum error of 1.

(a) Use the linear approximation formula for error measurement to find

(i) the approximate maximum error in $f(x)$, and

(ii) the approximate maximum *percentage* error in $f(x)$.

(b) **Not examinable.** Find, to four decimal places, the *true* maximum error in $f(x)$. **Hint:** By looking at a graph of $y = f(x)$, it is clear that, *in this example*, the maximum error must occur at an endpoint of the interval $[99, 101]$.

The following exercise shows that the *true* maximum error does *not* always occur at an endpoint of the interval of possible x-values.

10. Consider the function $f(x) = 200 + 5\sin(\pi x)$. Suppose that x is measured to be 100, with a maximum error of 1.

    (a) Use the linear approximation formula for error measurement to find

    (i) the approximate maximum error in $f(x)$.
    Write your answer to 2 decimal places.

    (ii) the approximate maximum *percentage* error in $f(x)$.
    Write your answer to 2 decimal places.

    (b) **Not examinable.** Find the *true* maximum error in $f(x)$. **Hint:** Look at a graph of $y = f(x)$. We see that, in this example, the maximum error does *not* occur at an endpoint of the interval $[99, 101]$.

11. Consider the function $f(x) = 11 + e^{-3x}$. Suppose that x is measured to be 0, with a maximum error of 0.2 .

    (a) Use the linear approximation formula for error measurement to find

    (i) the approximate maximum error in $f(x)$, and

    (ii) the approximate maximum *percentage* error in $f(x)$.

    (b) **Not examinable.** Find, to two decimal places, the *true* maximum error in $f(x)$. **Hint:** By looking at a graph of $y = f(x)$, it is clear that, *in this example*, the maximum error must occur at an endpoint of the interval $[-0.2,\ 0.2]$.

12. Consider the function $f(x) = 7 + 5\cos x$.

    (a) Suppose that x is measured to be 0, with a maximum error of 0.05. Use the linear approximation formula for error measurement to find

    (i) the approximate maximum error in $f(x)$, and

    (ii) the approximate maximum *percentage* error in $f(x)$.

    (b) Suppose that x is measured to be 0.8, with a maximum error of 0.05. Use the linear approximation formula for error measurement to find

    (i) the approximate maximum error in $f(x)$.
    Write your answer to 2 decimal places.

    (ii) the approximate maximum *percentage* error in $f(x)$.
    Write your answer to 2 decimal places.

13. Consider the function $f(x) = \dfrac{\sin x}{1 + x^2}$.

    Suppose that x is measured to be 3, with a maximum error of 0.1. Use the linear approximation formula for error measurement to find

    (a) the approximate maximum error in $f(x)$.
    Write your answer to 2 decimal places.

    (b) the approximate maximum *percentage* error in $f(x)$.
    Write your answer to 2 decimal places.

14. Consider the function $f(x) = \dfrac{e^x}{4+x}$.

Suppose that x is measured to be 16, with a maximum error of 0.05. Use the linear approximation formula for error measurement to find, to two decimal places,

(a) the approximate maximum error in $f(x)$.

(b) the approximate maximum *percentage* error in $f(x)$.

15. A property-developer measures the length of an edge of a square block of land to be 35 metres, with a maximum error of 0.1 metres. Use the linear approximation formula for error measurement to find

(a) (i) the approximate maximum error in the *perimeter* of the square block of land when this side-length is used to calculate the perimeter, and

(ii) the approximate maximum *percentage* error in the perimeter of the square block of land when this side-length is used to calculate the perimeter.

Write your answer to (ii) to two decimal places.

(b) (i) the approximate maximum error in the *area* of the square block of land when this side-length is used to calculate the area, and

(ii) the approximate maximum *percentage* error in the area of the square block of land when this side-length is used to calculate the area.

Write your answer to (ii) to two decimal places.

16. A water tank is designed to store rain-water. The tank is oddly-shaped, and the volume of the water in the tank is given by

$$V = \frac{2h^2 + 5h}{h+1}$$

where

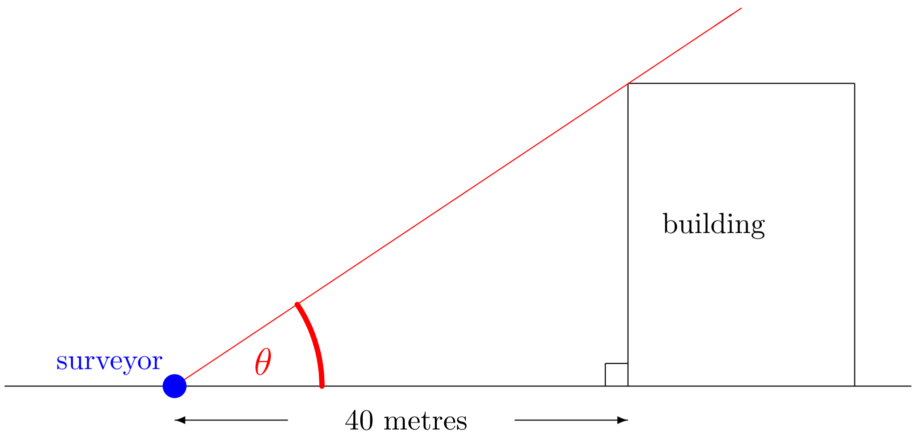
- V is the volume of the water, measured in cm^3, and
- h is the depth of the water measured in cm.

Suppose that the depth of the water in the tank is measured to be 100 cm, with a maximum error of 0.5 cm. Use the linear approximation formula for error measurement to find

(a) the approximate maximum error in the water's volume (to the nearest integer number of cubic centimetres), when this depth is used to calculate the volume.

(b) the approximate maximum *percentage* error in the water's volume (to two decimal places), when this depth is used to calculate the volume.

17. A surveyor is standing on flat ground exactly 40 metres from a building. Consider the angle of elevation, θ, shown in the diagram below. Suppose that the surveyor measures θ to be $\frac{\pi}{6}$ radians, with a maximum error of 0.02 radians, and uses this measurement to calculate the building's height.



Use the linear approximation formula for error measurement to find, to two decimal places,

(a) the approximate maximum error in the building's height, when the height is calculated using the angle θ.

(b) the approximate maximum *percentage* error in the building's height, when the height is calculated using the angle θ.

The Angle Between Two Curves

18. (i) Find the point(s) of intersection of the following curves.

(ii) Find the angle in degrees between the following curves at their point(s) of intersection.

When an answer is not an integer number of degrees, write it to two decimal places.

(a) $y = x^2, \quad y = x$

(b) $y = x^3, \quad y = x^2 - x$

(c) $y = \sqrt{1+x}, \quad y = \sqrt{1-x}$

Answers

1. (a) (i) $6x - 9$ (ii) 8.94 (iii) 8.9401
   (b) (i) $20x - 100$ (ii) 101 (iii) 101.0025
   (c) (i) 1 (ii) 1 (iii) 0.9950 (4 d.p.)
   (d) (i) $-x + \frac{\pi}{2}$ (ii) 0.0208 (4 d.p.) (iii) 0.0208 (4 d.p.)

2. (a) $2x + 7$ (b) $2x + 7$

   Note for Question 2: Since $f(x)$ is already a linear function (i.e. a straight line), then the linear approximation of $f(x)$ is just $f(x)$ itself.

3. (a) x (b)(i) 0.02 (b)(ii) 0.0198 (4 d.p.) (c)(i) -0.01 (c)(ii) -0.0101 (4 d.p.)

4. (a) $\frac{1}{2}x + 1$ (b)(i) 1.05 (b)(ii) 1.0488 (4 d.p.) (c)(i) 0.985 (c)(ii) 0.9849 (4 d.p.)

5. (a) $-0.0625x + 0.5$

   (b) (i) 0.249 375 [The true value is 0.249 377 (6 d.p.).]

   (ii) 0.2375 [The true value is 0.2381 (4 d.p.).]

   (iii) 0.25 125 [The true value is 0.25 126 (5 d.p.).]

6. (a) $-0.0625x + 0.25$

   (b) (i) 0.249 375 [The true value is 0.249 377 (6 d.p.).]

   (ii) 0.2375 [The true value is 0.2381 (4 d.p.).]

   (iii) 0.25 125 [The true value is 0.25 126 (5 d.p.).]

7. (a) 10.005 [The true value is 10.004 999 (6 d.p.).]

   (b) 9.99 [The true value is 9.98 999 (5 d.p.).]

   (c) 10.15 [The true value is 10.149 (3 d.p.).]

8. (a) $nx + 1$

   (b) The **Binomial Theorem** tells us that

$$(1+x)^n = 1 + {}^n\mathrm{C}_1\, x + \ldots + {}^n\mathrm{C}_r\, x^r + \ldots + x^n$$

where n is a positive integer.
Note that if x is close to 0, then the terms x^2, x^3, ... , x^n are *very* small. By omitting the very small terms involving x^2, x^3, ... , x^n, we can rewrite the Binomial Theorem as

$$(1+x)^n \approx 1 + {}^n\mathrm{C}_1\, x \quad \text{when } x \text{ is close to } 0.$$

Since ${}^n\mathrm{C}_1 = n$, we therefore have

$$(1+x)^n \approx 1 + nx \quad \text{when } x \text{ is close to } 0.$$

This is the same as the linear approximation of $(1+x)^n$ at $x = 0$, which we found in (a).

Similarly, recall how **Pascal's Triangle** helps us to expand expressions such as $(1+x)^n$, where n is a positive integer:

$$\begin{array}{ccccccccl}
 & & & 1 & & & & & \\
 & & 1 & & 1 & & & \longrightarrow & (1+x)^1 = 1 + x \\
 & 1 & & 2 & & 1 & & \longrightarrow & (1+x)^2 = 1 + 2x + x^2 \\
1 & & 3 & & 3 & & 1 & \longrightarrow & (1+x)^3 = 1 + 3x + 3x^2 + x^3 \\
 & & & \text{etc.} & & & & &
\end{array}$$

Once again, if x is close to 0, then the terms x^2, x^3, ... , x^n are *very* small. By omitting the very small terms involving x^2, x^3, ... , x^n, we can write

$$\begin{array}{ccccccccl}
 & & & 1 & & & & & \\
 & & 1 & & 1 & & & \longrightarrow & (1+x)^1 = 1 + x \\
 & 1 & & 2 & & 1 & & \longrightarrow & (1+x)^2 \approx 1 + 2x \\
1 & & 3 & & 3 & & 1 & \longrightarrow & (1+x)^3 \approx 1 + 3x \\
 & & & \text{etc.} & & & & &
\end{array}$$

Following this pattern gives $(1+x)^n \approx 1 + nx$ when x is close to 0, which (once again) is the same as the linear approximation obtained in (a).

9. (a)(i) 0.05 (a)(ii) 0.5%

   (b) 0.0501 (4 d.p.) (which is very close to the approximate maximum error).

10. (a)(i) 15.71 (2 d.p.) (a)(ii) 7.85%

   (b) 5 (which is *not* particularly close to the approximate maximum error).

11. (a)(i) 0.6 (a)(ii) 5%

   (b) 0.82 (2 d.p.)

12. (a)(i) 0 (a)(ii) 0%

   (b)(i) 0.18 (2 d.p.) (b)(ii) 1.71%

13. (a) 0.01 (2 d.p.) (b) 76.15% (2 d.p.)

14. (a) 21 104.51 (2 d.p.) (b) 4.75%

15. (a) (i) The approximate maximum error in the block's perimeter is 0.4 metres.

(ii) The approximate maximum percentage error in the block's perimeter is 0.29%.

(b) (i) The approximate maximum error in the block's area is 7 m^2.

(ii) The approximate maximum percentage error in the block's area is 0.57%.

16. (a) The approximate maximum error in the water's volume is 1 cm^3.

(b) The approximate maximum percentage error in the water's volume is 0.49%.

17. (a) The approximate maximum error in the building's height is 1.07 metres.

(b) The approximate maximum percentage error in the building's height is 4.62%.

18. (a) (i) $(0, 0)$ and $(1, 1)$ (ii) $45°$ (or $135°$) and $18.43°$ (or $161.57°$)

(b) (i) $(0, 0)$ (ii) $45°$ (or $135°$)

(c) (i) $(0, 1)$ (ii) $53.13°$ (or $126.87°$)