SH*T TO OPEN TOMORROW

1. **WINDOW 1**
   a. TCOLE
   b. Quiz
2. **WINDOW 2**
   a. Symbolab
   b. Desmos
   c. Matrix Multiplier
   d. Google
   e. Octave

2 COMPLEX NUMBERS (I)

1. Imaginary numbers, i

a. $i \equiv \sqrt{-1}$

$$i^2 = -1$$

b. $\sqrt{-c} = \sqrt{c \times -1} = \sqrt{c}\sqrt{-1} = \sqrt{c}i$

$$\sqrt{-c} = \sqrt{c}i$$

c. Powers of i

$i^2 = -1$
$i^3 = i^2 \times i = -i$
$i^4 = i^2 \times i^2 = 1$
$i^5 = i^4 \times i = i$

2. Complex numbers: Cartesian form

a. Cartesian form

$$z = x + iy,\ x, y \in \mathbb{R}$$

b. Solve with Complete The Square (CTS)

<example> Solve $P(z) = z^2 - 4z + 13$

COMPLETE THE SQUARE
$ax^2 + bx + c = a(x + (\frac{b}{2a})^2)^2 - (\frac{b}{2a})^2 + c$

$= (z-2)^2 + 9$
Turn $+$ into $--$
$= (z-x)^2 -- 9$
Turn $-c$ into $\sqrt{c}\,i$
$= (z-2)^2 - (3i)^2$

$$a^2 - b^2 = (a+b)(a-b)$$

$= (z - 2 + 3i)(z - 2 - 3i)$
$\therefore z = 2 \pm 3i$

c. Basic operations
   i. Equality
   $a + bi = c + di \leftrightarrow a = c \wedge b = d$
   ii. Addition/Subtraction
   $a + bi \pm c + di = (a \pm c) + i(b \pm d)$
   iii. Multiplication
   $(a + bi)(c + di) = (ac - bd) + i(ad + bc)$
   iv. Division
   $\frac{a+bi}{c+di} \times \frac{c-di}{c-di}$
   (^ conjugate)

d. Complex conjugate

$$\bar{z} = x - iy,\ x, y \in \mathbb{R}$$

e. Conjugate pair

$$z\bar{z} = x^2 + y^2$$

3. Complex Numbers: Polar Form

a. Argand Plane
   i. Quadrants
   <diagram>
   ii. Angles table

deg.	30	45	60	90	120	185	150	180
rad.	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
dec.	0.5235..	0.7853..	1.0471..	1.5707..	2.0943..	2.3561..	2.6179..	3.1415

   iii. sine and cosine table

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

$\frac{\sqrt{3}}{2} = 0.8660...$

$\frac{1}{\sqrt{2}} = 0.7071... = \frac{\sqrt{2}}{2}$

iv. Standard Triangles

b. Polar Form

i.

$$z = r(\cos\theta + i\sin\theta) = r\ cis\ \theta$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

ii. Argument of z

$$arg(z) = \theta\ \{or\ + 2\pi k, k \in \mathbb{Z}\}$$

iii. Principal Value of z

$$-\pi < P.V.arg(z) \leq \pi$$

c. Multiplication and Division

i. $cis\ \theta_1 \times cis\ \theta_2 = cis\ (\theta_1 + \theta_2)$

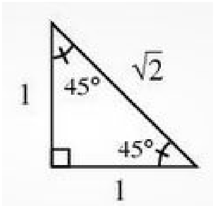
ii. $\frac{cis\ \theta_1}{cis\ \theta_2} = cis\ (\theta_1 - \theta_2)$

4. Cartesian to Polar Form

<example> Express $z = -2 - 2i$ in Polar Form

Find r with $r = |z| = \sqrt{x^2 + y^2}$

$r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$

<table>
<tr><td colspan="2">Method 1: Plot graph
<diagram>
Find Quadrant
$QIII:\ \theta = \pi + \alpha$
Find α & θ :</td><td rowspan="2">Method 2: Divide Cartesian form with r
$z = 2\sqrt{2}\,(\_\_ + i\,\_\_)$
$= -2 - 2i$
$= 2\sqrt{2}(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$
$\cos\theta = -\frac{1}{\sqrt{2}}$
$\sin\theta = -\frac{1}{\sqrt{2}}$
$\ominus\sin,\ \ominus\cos:\ QIII$
$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$</td></tr>
<tr><td>Standard Triangle

$\alpha = \frac{\pi}{4}$</td><td>$\tan\alpha = \frac{x}{y}$
$\tan\alpha = \frac{-2}{-2}$
$\alpha = \frac{\pi}{4}$
$\theta = \frac{5\pi}{4}$</td></tr>
</table>

$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$		

Sub into $z = r\ cis\ (\theta + 2\pi k)$

$z = 2\sqrt{2}cis(+2\pi k), k \in \mathbb{Z}$

5. Polar to Cartesian Form

<example> Express $z = 2cis(\frac{19\pi}{6})$ in Cartesian Form

Expand

$z = 2(\cos\frac{19\pi}{6} + i\sin\frac{19\pi}{6})$

Solve $\cos\theta$ and $\sin\theta$

$z = 2(-\frac{\sqrt{3}}{2} - i\frac{1}{2})$

Expand

$z = \sqrt{3} - i$

6. DeMoivre's Theorem

$$z^n = r^n cis(n\theta)$$

3 COMPLEX NUMBERS (II)

1. Roots of complex number

a.

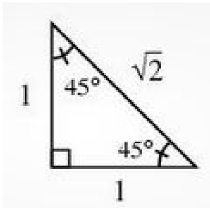
$$w = s\ cis\ \beta = r^{\frac{1}{n}}\ cis(\frac{\theta + 2\pi k}{n})$$

b. <example> Calculate square roots of $z = -1 + i$

Write z in polar form

$r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

<diagram>



$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$z = \sqrt{2}cis\ (\frac{3\pi}{4} + 2\pi k)$

Let $w = s\ cis\ \beta = r^{\frac{1}{n}}cis(\frac{\theta + 2\pi k}{n})$

$r = \sqrt{2},\ n = 2,\ \theta = \frac{3\pi}{4}$

$w = \sqrt{2}^{\frac{1}{2}}cis(\frac{\frac{3\pi}{4} + 2\pi k}{2}) = 2^{\frac{1}{4}}cis(\frac{\frac{3\pi}{4} + 2\pi k}{2})$

Calculate for $k = 0, 1, 2, ...n - 1$

$k=0:\ 2^{\frac{1}{4}}cis(\frac{\frac{3\pi}{4}+2\pi(0)}{2}) = 2^{\frac{1}{4}}cis(\frac{3\pi}{8})$
$k=1:\ 2^{\frac{1}{4}}cis(\frac{\frac{3\pi}{4}+2\pi(1)}{2}) = 2^{\frac{1}{4}}cis(\frac{11\pi}{8})$

2. Subsets of complex planes

a. <example> $Re(z) = -Im(z)$
b. <example>
c. <example>
d. <example>
e. Circles
   i.
   $|z - z_0| = r,\ z_0 = a + ib$
   Circle: radius=r, centre=(a,b)
   ii. <example> $|z + 3 - 3i| > 2$
   Find z_0
   $|z - (-3 + 3i)| > 2$
   Find radius and centre
   $z_0 = -3 + 3i$, centre=(-3,3)
   radius=2
   $>$ is outside the circle
   $<$ is inside the circle

4 MATRICES

1. Scalar Multiplication

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

2. Order

$$m(rows) \times n(columns)$$

Square Matrix
$m = n$

3. General Form of a Matrix

<diagram>

4. Equality

A and B are only equal if and only if:

a. They are of the same order
b. Corresponding elements are equal $a_{ij} = b_{ij}$ for all $\{i,j\}$

5. Matrix Addition

$A + B$ only exists if A and B have the same order

6. Matrix Multiplication

a. $A + B$ only exists if $A(m \times n)$ and $B(n \times p)$
b. <diagram>
c. $AB \neq BA$ (non-commutative)

7. Identity Matrix

<diagram>

8. Inverse Matrix

a.
$$\text{if } A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \frac{1}{|A|} \neq 0$$
b. Singular: $|A| = 0$
   Non-Singular/Regular: $|A| \neq 0$
c.
$$AA^{-1} = A^{-1}A = I$$
d. $AX = B$
   To find X:
   $A^{-1}AX = A^{-1}B$
   $X = A^{-1}B$

5 MATRIX TRANSFORMATIONS

1. Object and Image

a. **Object** points: (x, y)
b. **Image** points: (x', y')
c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$
d. Transformations are also known as **mappings**

2. General Linear Transformations

a. To find T, transform $(1, 0)$ and $(0, 1)$

3. Reflection in the x-axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

4. Reflection in the y-axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5. Reflection in the line $y = x$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

6. Reflection in the line $y = mx$

Where $m = \tan\theta$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

7. Rotation anticlockwise about the origin

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

8. Dilation/Contraction by a factor k parallel to the x-axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

9. Dilation/Contraction by a factor k parallel to the y-axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

10. Dilation/Contraction parallel to both x and y axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

11. Transformation inverse

Find the image of the parabola with equation $y = x^2$ after a dilation of factor 3 parallel to the x-axis followed by a rotation of $\frac{\pi}{2}$ radians. Use an inverse matrix.

$$T = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

$$^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

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Example 7.4 II

$$T^{-1} = \frac{1}{(0)(0)-(-1)(3)} \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 1/3\, y' \\ -x' \end{bmatrix}$$

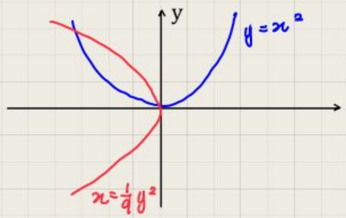
original line:

$$y = x^2$$

$$-x' = (\tfrac{1}{3} y')^2$$

$$x' = -\tfrac{1}{9} y'^2 \rightarrow x = \tfrac{1}{9} y^2$$

Sketch both the object and the image.



12. Combined Transformations

When a transformation represented by matrix A is followed by a transformation represented by matrix B the image $P''(x'', y'')$ is given by

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = BA \begin{bmatrix} x \\ y \end{bmatrix}$$

Remember it is important to get this order correct, since matrix multiplication is not commutative.

13. Degenerate Transformations

a. For matrices which are **singular** (determinant $\Delta = 0$)

b. ...

14. Isometry

a. An **isometry** is a transformation that preserves distance

i. Translations
ii. Reflections
iii. Rotations
iv. NOT Dilations

b. A **direct isometry** preserves orientation
i. Translations
ii. Rotations
iii. NOT Reflections

6 ELLIPSES AND HYPERBOLAS

1. Ellipse

$(h,k) : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

	Horizontal orientation	Vertical orientation
	$a > b$	$b > a$
centre	(h,k)	(h,k)
vertices	$(h \pm a, k)$	$(h, k \pm b)$
c	$\sqrt{a^2 - b^2}$	$\sqrt{b^2 - a^2}$
foci	$(h \pm c, k)$	$(h, k \pm c)$
eccentricity	$e = \frac{c}{a}$	$e = \frac{c}{b}$
major axis	$2a$	$2b$

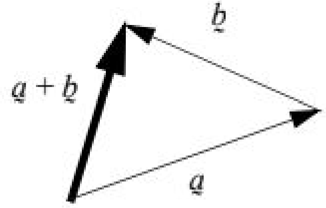
2. Hyperbola

$\pm \frac{(x-h)^2}{a^2} \mp \frac{(y-k)^2}{b^2} = 1$

	Horizontal orientation	Vertical Orientation
Equation		
Vertices		
c		
foci		
asymptotes		

7 VECTORS

1. Adding vectors

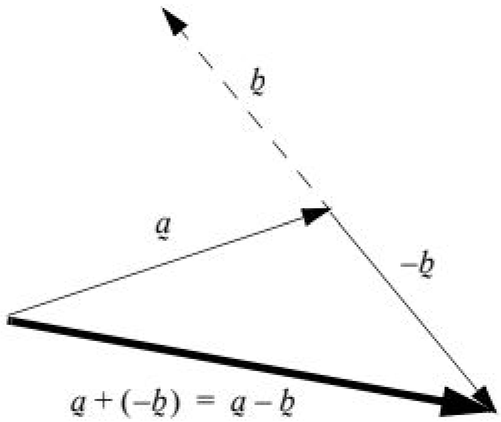


2. Scalar Multiplications

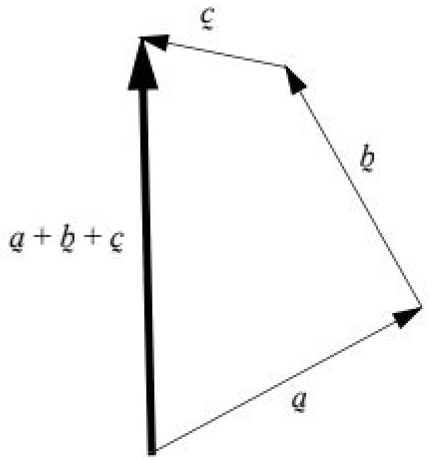
a. Parallel
$\tilde{a}$ and $\tilde{b}$ are parallel if $\tilde{b} = k\tilde{a}$, $k > 0$
$\tilde{a} // \tilde{b}$

b. Anti-parallel
$\tilde{a}$ and $\tilde{b}$ are anti-parallel if $\tilde{b} = k\tilde{a}$, $k < 0$
$\tilde{a}$ anti-// $\tilde{b}$

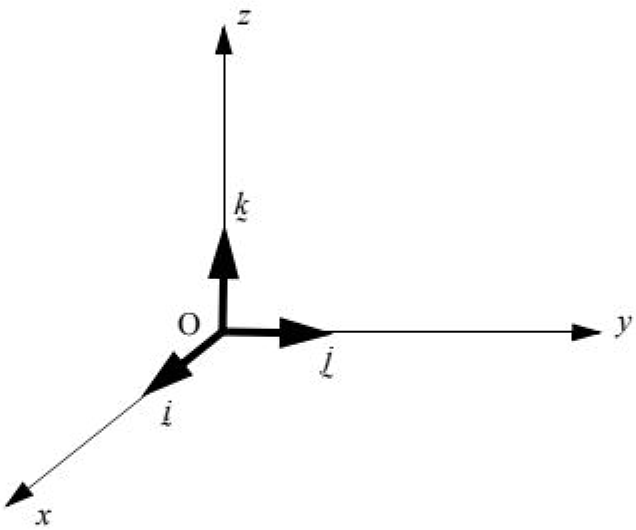
3. Vector Subtraction



4. Adding more than two vectors



5. Unit vectors



$\tilde{i}, \tilde{j}, \tilde{k}$ are unit vectors (have magnitude of 1)

In the direction of x, y, z

The formula

$$\overrightarrow{PQ} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j} + (z_2 - z_1)\underset{\sim}{k}$$

6. is useful for finding the **vector between two points** $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

7. To show that points P, Q, R are **collinear** using vectors, show that $\overrightarrow{PQ} // \overrightarrow{QR}$ and state that these two vectors have a common point Q.

8. Magnitude

$$|\underset{\sim}{a}| = \sqrt{x^2 + y^2 + z^2}.$$

9. Unit Vector

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}.$$

8 VECTOR MULTIPLICATION

1. Dot Product

AKA scalar product

for $\underset{\sim}{a} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j} + a_z\underset{\sim}{k}$

$\underset{\sim}{b} = b_x\underset{\sim}{i} + b_y\underset{\sim}{j} + b_z\underset{\sim}{k}$

$$\boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = a_x b_x + a_y b_y + a_z b_z}$$

2. Dot Product useful products

① $a \cdot a = |a|^2$

cummulative rule ② $a \cdot b = b \cdot a$

distributive rules

③ $a \cdot (b + c) = a \cdot b + a \cdot c$

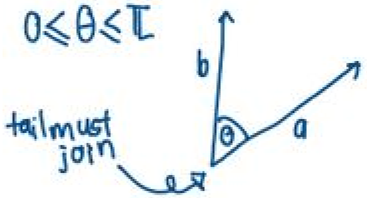
④ $(a + b) \cdot c = a \cdot c + b \cdot c$

⑤ $(a + b) \cdot (c + a) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$

3. Dot Product Theorem

$$\boxed{a \cdot b = |a||b| \cos\theta}$$

$0 \leq \theta \leq \pi$



$0 \leq \theta < \frac{\pi}{2}$	$\cos\theta > 0$	$a \cdot b > 0$
$\theta = \frac{\pi}{2}$	$\cos\theta = 0$	$a \cdot b = 0$
$\frac{\pi}{2} < \theta \leq \pi$	$\cos\theta < 0$	$a \cdot b < 0$

ORTHOGONAL means perpendicular

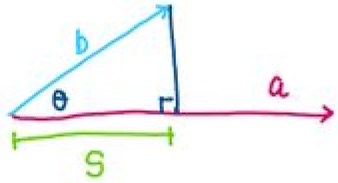
2 vectors are orthogonal iff & only if:

$a \cdot b = 0$

4. Angle between 2 vectors

$$\boxed{\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}} \Leftrightarrow \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

5. Scalar Projection



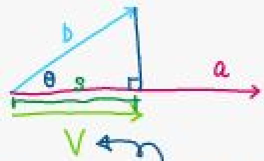
scalar projection of b onto a is S

$$\cos\theta = \frac{S}{|b|}$$

$$S = |\underset{\sim}{b}|\cos\theta = \frac{|a||b|\cos\theta}{|a|} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{a}} = \underset{\sim}{b} \cdot \hat{\underset{\sim}{a}}$$

$$\boxed{\therefore S = \underset{\sim}{b} \cdot \hat{\underset{\sim}{a}}}$$

6. Vector Projection



vector projection of b onto a

$v = S\hat{a}$

$$\boxed{v = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{a} \cdot \underset{\sim}{a}}\right)\underset{\sim}{a}}$$

9 VECTOR GEOMETRY

10 STATISTICAL INFERENCE

https://www.remnote.io/a/c10-statistical-inference/5f9be2fd59d095003f80db0c

1. Symbols

	Sample	Population
Mean	$\bar{x}$	μ
Standard Deviation	s	σ
Variance	s^2	σ^2

2. Expected Value

a. Aka the population mean

$$E(X) = \mu$$

3. Linear Function of a Random Variable

- **If** $W = aX + b$ **then:**
- $E(W) = E(aX + b) = aE(X) + b$
- $Var(W) = Var(aX + b) = a^2 Var(X)$
- $SD(W) = \sqrt{Var(W)} = \sqrt{a^2 SD^2(X)} = |a| SD(X)$

a.

4. Independent Random Variables

- **Let** X **and** Y **be independent random variables and let** $W = X + Y$
- $E(W) = E(X \pm Y) = E(X) \pm E(Y)$
- $Var(W) = Var(X \pm Y) = Var(X) + Var(Y)$

a.

5. Z-scores

a.

$$z = \frac{x - \mu}{\sigma}$$

b. Check Formula Sheet for z-scores

6. Confidence Intervals

a.

$$[\bar{x} - z\frac{x}{\sqrt{n}}, \bar{x} + z\frac{x}{\sqrt{n}}]$$

7. Hypothesis Testing

a. Null hypothesis, H_0
   i. Makes the claim that there is no difference $\mu = \mu_0$

b. Alternative hypothesis
   i. Makes the claim that there is a difference
      - One-tailed test: $\mu > \mu_0, \mu < \mu_0$
      - Two-tailed test: $\mu \neq \mu_0$

c. Type I error
   i. Occurs when we reject true null hypothesis

d. Type II error
   i. Occurs when we accept a false null hypothesis

11 TECHNIQUES OF ANTI-DIFFERENTIATION (I)

1. Antidifferentiation Rules and Formulae

(1) $\int kf(x)dx = k\int f(x)dx$

(2) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

(3) $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

(4) $\int 0dx = c$

(5) $\int 1dx = x + c$

(6) $\int (ax+b)^r dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c$

(7) $\int \frac{d}{ax+b}dx = \frac{d}{a}\ln|ax+b| + c$

(8) $\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + c$

(9) $\int \cos(kx)dx = \frac{1}{k}\sin(kx) + c$

(10) $\int \sec^2(kx)dx = \frac{1}{k}\tan(kx) + c$

(11) $\int e^{kx}dx = \frac{1}{k}e^{kx} + c$

(12) $\int \frac{b}{\sqrt{a^2-x^2}}dx = b\sin^{-1}(\frac{x}{a}) + c$

(13) $\int \frac{-b}{\sqrt{a^2-x^2}}dx = b\cos^{-1}(\frac{x}{a}) + c$

(14) $\int \frac{a}{a^2+x^2}dx = \tan^{-1}(\frac{x}{a}) + c$

(15) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$

2. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

3. Substitution with the Derivative Present

<example>

$\int 3x^2(x^3+1)^4 dx$

- Working
  - let $u = x^3 + 1$
  - then $\frac{du}{dx} = 3x^2$
  - then $dx = \frac{du}{3x^2}$
  - Substitute $(x^3+1) \rightarrow u$ and $dx \rightarrow \frac{du}{3x^2}$
  - $\int 3x^2(x^3+1)^4 dx = \int 3x^2(u)^4 \frac{du}{3x^2} = \int u^4 du$
  - Differentiate
  - $\int u^4 du = \frac{u^5}{5} + c$
  - Substitute $u \rightarrow (x^3+1)$
  - $\frac{(x^3+1)^5}{5} + c$
- Answer $\frac{(x^3+1)^5}{5} + c$

4. Even/Odd; sin and cos; tan and sec

$u = \sin x$	$\frac{du}{dx} = \cos x$	This substitution is helpful if there is an odd power of $\cos x$.
$u = \cos x$	$\frac{du}{dx} = -\sin x$	This substitution is helpful if there is an odd power of $\sin x$.
$u = \tan x$	$\frac{du}{dx} = \sec^2 x$	This substitution is helpful if there is an even power of $\sec x$.
$u = \sec x$	$\frac{du}{dx} = \sec(x)\tan(x)$	This substitution is helpful if there is an odd power of $\tan x$.

a.

5. Double-Angle Formula

a. If **both** sin and cos have **even powers**

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$$

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

6.

a.

12 TECHNIQUES OF ANTI-DIFFERENTIATION (II)

1. $\sqrt{K^2 - x^2}$ sub $x = K\sin(u)$ then $\sin^2(u) + \cos^2(u)$

a. <example> $\int \frac{1}{\sqrt{4-x^2}} dx; \quad -2 < x < 2$

2. Integration by parts

a. Product rule of differentiation

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

b. Make u whatever comes first on the list

i. **L**ogarithms ($\ln|x|$)

ii. **I**nverse trigonometric functions ($\sin^{-1}(x),\ \cos^{-1}(x),\ \tan^{-1}(x)$)

iii. **P**olynomials ($1,\ x,\ x^2...$)

iv. **E**xponential functions (e^{kx})

v. **T**rigonometric functions ($\sin(x),\ \cos(x),\ \tan(x),\ \sec(x)$)

b. <example> Find $\int x\sec^2(x)dx$

~P ; $\sec^2$ ~T; P comes before T in LIPET so $u = x$

Then $\frac{du}{dx} = 1$

And $\frac{dv}{dx} = \sec^2(x)$

Then $v = \tan(x)$

Sub into product rule of differentiation

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x\sec^2(x)dx = x\tan(x) - \int(\tan(x) \times 1)dx$$

$$= x\tan(x) - \int \frac{\sin(x)}{\cos(x)}dx$$

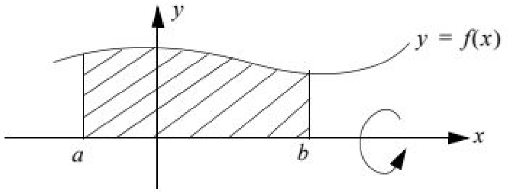
$$= x\tan(x) + \ln|\cos(x)| + c$$

3.

13 VOLUMES OF SOLIDS AND REVOLUTIONS

1. X-axis

1. Suppose the region to be rotated has the axis of rotation (i.e. the x axis) as one of its boundaries, as shown below.

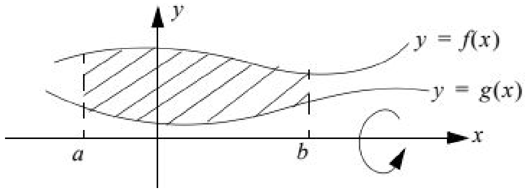


Then

$$V = \int_a^b \pi[f(x)]^2 dx.$$

2. X-axis with gap

2. Suppose there is a gap between the axis of rotation (i.e. the x axis) and the region, as shown below.



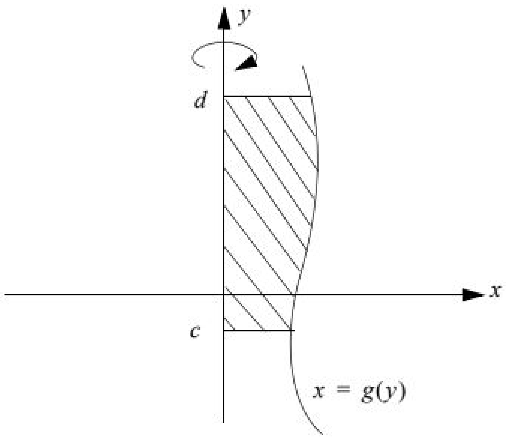
Then

$$V = \int_a^b \pi[f(x)]^2 dx - \int_a^b \pi[g(x)]^2 dx.$$

3. Y-axis

a. *change all equations to $x = \ldots$

3. Suppose the region to be rotated has the axis of rotation (i.e. the y axis) as one of its boundaries, as shown below.

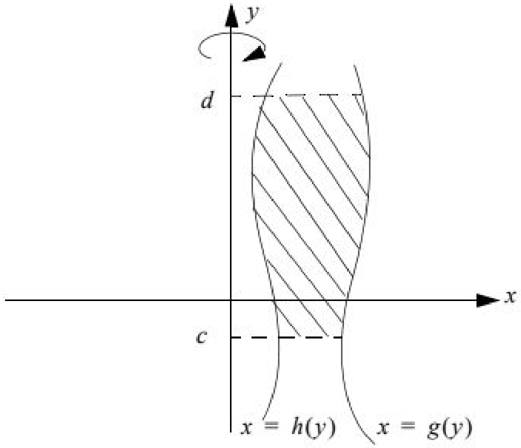


Then

$$V = \int_c^d \pi[g(y)]^2 dy.$$

4. Y-axis with gap

4. Suppose there is a gap between the axis of rotation (i.e. the y axis) and the region, as shown below.



Then $$V = \int_c^d \pi[g(y)]^2 dy - \int_c^d \pi[h(y)]^2 dy.$$

14 APPROXIMATION OF DEFINITE INTEGRALS

1. Trapezoidal Rule

a. Estimate

To approximate $\int_a^b f(x)dx$ use

$$T = \frac{h}{2}\{f(a) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(b)\}$$

where

$h = \frac{b-a}{n}$ is called the **step size**, and n is the number of steps (trapezoids).

b. Error Estimate

$U = max(|f''(x)|)$ on $[a, b]$

$$|E_T| \le \frac{b-a}{12}h^2 U,$$

where U is any upper bound for the values of $|f''|$ on $[a, b]$

2. Midpoint Rule

a. Estimate

To approximate $\int_a^b f(x)dx$, use

$$\begin{aligned} M &= hf(\bar{x}_1) + hf(\bar{x}_2) + \ldots hf(\bar{x}_n) \\ &= h[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)] \end{aligned}$$

where

$h = \frac{b-a}{n}$ is the step size

and

$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

are the midpoints of the intervals $[x_{i-1}, x_i]$.

b. Error Estimate

$$|E_M| \le \frac{b-a}{24}h^2 U$$

where U is any upper bound for the values of $|f''|$ on $[a, b]$.

15 DIFFERENTIAL EQUATIONS (I)

1.

16 DIFFERENTIAL EQUATIONS (II)

1.