Mathematics 1

Sheet 25: Probability Density Functions

1. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < -1 \\ \frac{3}{2}x^2 & \text{if} \quad -1 \leq x \leq 1 \\ 0 & \text{if} \quad x > 1\,. \end{cases}$$

   (a) Sketch the graph of $y = f(x)$.

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

2. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < -\frac{\pi}{6} \\ \cos x & \text{if} \quad -\frac{\pi}{6} \leq x \leq \frac{\pi}{6} \\ 0 & \text{if} \quad x > \frac{\pi}{6}\,. \end{cases}$$

   (a) Sketch the graph of $y = f(x)$.

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

3. Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < -\frac{\pi}{2} \\ \frac{1}{2}\cos x & \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & \text{if} \quad x > \frac{\pi}{2}\,. \end{cases}$$

   Show that f is a probability density function.

4. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} e^x & \text{if} \quad x \leq 0 \\ 0 & \text{if} \quad x > 0. \end{cases}$$

   (a) Sketch the graph of $y = f(x)$.

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

   (c) Find $\Pr(X \leq -1)$.

   (d) Find $\Pr(X \leq 0)$.

   (e) Find $\Pr(X \leq 1)$.

   (f) Find $\Pr(X \leq 2)$.

5. Consider the continuous random variable X with probability density function

$$f(x)=\begin{cases} 0 & \text{if} \quad x<0 \\ \sin x & \text{if} \quad 0\le x\le \frac{\pi}{2} \\ 0 & \text{if} \quad x>\frac{\pi}{2}\,. \end{cases}$$

   (a) Sketch the graph of $y=f(x)$.

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

   (c) Find $\Pr\left(X<\dfrac{\pi}{6}\right)$.

   (d) Find $\Pr\left(X\le\dfrac{\pi}{6}\right)$.

   (e) Find $\Pr\left(X<\dfrac{\pi}{4}\right)$.

   (f) Find $\Pr\left(X\le\dfrac{\pi}{4}\right)$.

   (g) Find $\Pr\left(X<\dfrac{\pi}{3}\right)$.

   (h) Find $\Pr\left(X\le\dfrac{\pi}{3}\right)$.

6. Consider the continuous random variable X with probability density function

$$f(x)=\begin{cases} 0 & \text{if} \quad x<0 \\ \frac{1}{72}x^2 & \text{if} \quad 0\le x\le k \\ 0 & \text{if} \quad x>k \end{cases}$$

   where k is a constant.

   (a) Find the value of k.

   (b) Find $\Pr(X>2)$.

   (c) Find $\Pr(3<X<4)$.

7. Consider the continuous random variable X with probability density function

$$f(x)=\begin{cases} 0 & \text{if} \quad x<\frac{1}{2} \\ \dfrac{1}{x} & \text{if} \quad \frac{1}{2}\le x\le k \\ 0 & \text{if} \quad x>k \end{cases}$$

   where k is a constant.

   (a) Find the value of k. Write your answer to 4 d.p.

   (b) Find $\Pr(X\ge 1)$. Write your answer to 4 d.p.

8. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} e^{2x} & \text{if } \ x \le k \\ 0 & \text{if } \ x > k \end{cases}$$

where k is a constant.

(a) Find the value of k.

(b) Find $\Pr(-\ln 2 \le X \le 0)$.

(c) Find $\Pr(X \le 0 \mid X \ge -\ln 2)$.

9. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{9}{8x^2} & \text{if } \ x < -3 \\ \dfrac{x^2}{72} & \text{if } \ -3 \le x \le 3 \\ \dfrac{9}{8x^2} & \text{if } \ x > 3. \end{cases}$$

(a) Find $\Pr(X < -3)$.

(b) Find $\Pr(X > 3)$.

(c) Find $\Pr(-3 \le X \le 3)$.

(d) Find $\Pr(X \le 1)$.

(e) Find $\Pr(-4 \le X \le 5)$.

10. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ 2x & \text{if } \ 0 \le x \le 1 \\ 0 & \text{if } \ x > 1. \end{cases}$$

(a) Find $\Pr\left(X < \dfrac{1}{2}\right)$.

(b) Find the median of X.

Touch here for a video about finding the median.

(c) Find $E(X)$.

(d) Find $E(X^2)$.

(e) Find $\mathrm{Var}(X)$.

(f) Find the standard deviation of X.

11. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} e^x & \text{if} \quad x \leq 0 \\ 0 & \text{if} \quad x > 0 \end{cases}$$

(taken from Exercise 4 of this Exercise Sheet).

(a) Find the median of X.

Give your answer to 4 decimal places.

(b) Find the value of p such that $\Pr(X \leq p) = \dfrac{1}{10}$.

Give your answer to 4 decimal places.

12. Consider the function

$$f(x) = \begin{cases} e^{2x} & \text{if} \quad x \leq 0 \\ \dfrac{1}{40}x^2 + \dfrac{11}{120} & \text{if} \quad 0 < x \leq k \\ 0 & \text{if} \quad x > k\,. \end{cases}$$

(a) Find the value of k so that f is a probability density function.

(b) Find p so that $\Pr(X \leq p) = \dfrac{9}{50}$.

(c) Find r so that $\Pr(X \leq r) = \dfrac{3}{5}$.

13. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \dfrac{1}{72}x^2 & \text{if} \quad 0 \leq x \leq 6 \\ 0 & \text{if} \quad x > 6 \end{cases}$$

(taken from Exercise 6 of this Exercise Sheet).

(a) Find the median of X.

Give your answer to 4 decimal places.

(b) Find the value of p such that

$$\Pr(X \geq p) = 0.875\,.$$

(c) Find $E(X)$.

(d) Find $E(X^2)$.

(e) Find $\mathrm{Var}(X)$.

(f) Find the standard deviation of X.

Give your answer to 4 decimal places.

14. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < -2 \\ \frac{1}{9}(x^2 - 2x + 1) & \text{if} \quad -2 \le x \le 1 \\ 0 & \text{if} \quad x > 1. \end{cases}$$

(a) Find $E(X)$.

(b) Find $E(-8X + 1)$.

(c) Find $E(X^2)$.

(d) Find $E(10X^2 - 13)$.

(e) Find $\text{Var}(X)$.

(f) Find $\text{Var}(8X - 21)$.

(g) Find the standard deviation of X.

Give your answer to 4 decimal places.

15. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ \frac{1}{x} & \text{if} \quad 1 \le x \le e \\ 0 & \text{if} \quad x > e. \end{cases}$$

Give your answer to 4 decimal places in each of the following:

(a) Find $\Pr(X \ge 2)$.

(b) Find the median of X.

(c) Find $E(X)$.

(d) Find $E(2X + 1)$.

(e) Find $\text{Var}(X)$.

(f) Find $\text{Var}(2X + 1)$.

16. Consider the continuous random variable X with probability density function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < -3 \\ \frac{1}{x^2} & \text{if} \quad -3 \le x \le -\frac{3}{4} \\ 0 & \text{if} \quad x > -\frac{3}{4}. \end{cases}$$

(a) Find the median of X.

(b) Find $E(X)$. Give your answer to 4 decimal places.

(c) Find $\text{Var}(X)$. Give your answer to 4 decimal places.

(d) Find the value of p such that $\Pr(X \le p) = \frac{1}{3}$.

(e) Find the value of q such that $\Pr(X > q) = \frac{5}{6}$.

17. Consider the continuous random variable X defined to be the number of hours between calls made to my phone. Suppose that the probability density function for this random variable X is given by $f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ e^{-x} & \text{if} \quad x \geq 0\,. \end{cases}$

   (a) Find the probability that there is less than 1 hour between calls made to my phone.
   Give your answer to 4 decimal places.

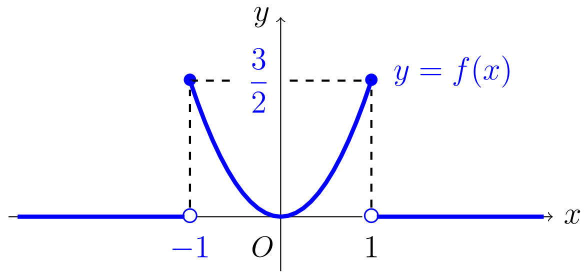
   (b) Find the probability that the gap between calls made to my phone is between 1 and 2 hours. Give your answer to 4 decimal places.

   (c) Find, to the nearest minute, the median length of time between calls made to my phone.

   (d) Suppose that on 10% of occasions there is more than a p hour break between calls made to my phone. Find the value of p.
   Give your answer to 2 decimal places.

Answers:

1. (a)



(b) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{2}x^2\,dx + \int_{1}^{\infty} 0\,dx$$

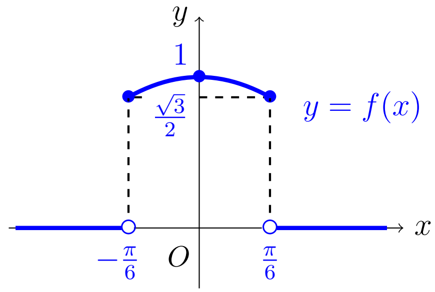
$$= 0 + \left[\frac{3}{2} \times \frac{x^3}{3}\right]_{-1}^{1} + 0$$

$$= \left[\frac{x^3}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2} - \left(\frac{-1}{2}\right)$$

$$= 1, \quad \text{as required.}$$

2. (a)



(b)
$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-\frac{\pi}{6}} f(x)\,dx + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} f(x)\,dx + \int_{\frac{\pi}{6}}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-\frac{\pi}{6}} 0\,dx + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos x\,dx + \int_{\frac{\pi}{6}}^{\infty} 0\,dx$$

$$= 0 + \Big[\sin x\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} + 0$$

$$= \sin\left(\frac{\pi}{6}\right) - \sin\left(-\frac{\pi}{6}\right)$$

$$= \frac{1}{2} - \left(-\frac{1}{2}\right)$$

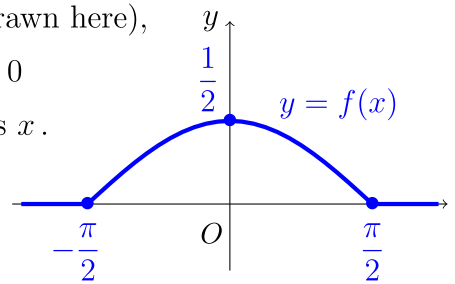
$$= 1, \quad \text{as required.}$$

3. We need to check three conditions:

(i) By taking the union of the three intervals given by the three conditions listed in the piecewise definition of f, we see that

$$\mathrm{dom}(f) = \left(-\infty, -\frac{\pi}{2}\right) \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \infty\right) = \mathbf{R}.$$

(ii) From the graph (drawn here), we see that $f(x) \geq 0$ for all real numbers x.



(iii) Finally, we show that the area under the curve $y = f(x)$ is equal to 1:

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-\frac{\pi}{2}} f(x)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-\frac{\pi}{2}} 0\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos x\,dx + \int_{\frac{\pi}{2}}^{\infty} 0\,dx$$

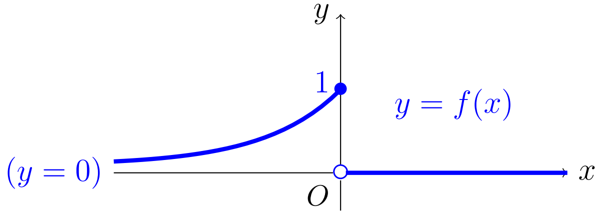
$$= 0 + \left[\frac{1}{2}\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$$

$$= \frac{1}{2} \times 1 - \frac{1}{2} \times -1$$

$$= 1, \quad \text{as required.}$$

4. (a)



(b) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

$$\begin{aligned}
&= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{0} e^x\,dx + \int_{0}^{\infty} 0\,dx \\
&= \lim_{a\to-\infty}\int_{a}^{0} e^x\,dx + 0 \\
&= \lim_{a\to-\infty}\Big[\,e^x\,\Big]_a^0 \\
&= \lim_{a\to-\infty}(e^0 - e^a) \\
&= e^0 - \lim_{a\to-\infty} e^a \\
&= 1 - 0 \\
&= 1, \quad \text{as required.}
\end{aligned}$$

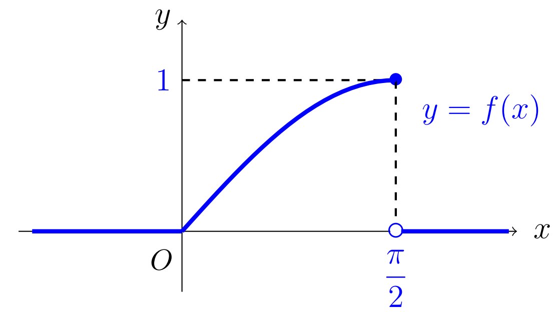
(c) $\dfrac{1}{e}$

(d) 1

(e) 1

(f) 1

5. (a)



(b) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

$$\begin{aligned}
&= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\frac{\pi}{2}} \sin x\,dx + \int_{\frac{\pi}{2}}^{\infty} 0\,dx \\
&= 0 + \Big[-\cos x\Big]_0^{\frac{\pi}{2}} + 0 \\
&= -\cos\left(\frac{\pi}{2}\right) - (-\cos 0) \\
&= -0 + 1 \\
&= 1, \quad \text{as required.}
\end{aligned}$$

(c) $1 - \dfrac{\sqrt{3}}{2}$ (d) $1 - \dfrac{\sqrt{3}}{2}$ (e) $1 - \dfrac{1}{\sqrt{2}}$

(f) $1 - \dfrac{1}{\sqrt{2}}$ (g) $\dfrac{1}{2}$ (h) $\dfrac{1}{2}$

6. (a) $k = 6$ (b) $\dfrac{26}{27}$ (c) $\dfrac{37}{216}$

7. (a) $k = 1.3591$ (4 d.p.) (b) 0.3069 (4 d.p.)

8. (a) $k = \dfrac{1}{2}\ln 2$ (b) $\dfrac{3}{8}$ (c) $\dfrac{3}{7}$

9. (a) $\frac{3}{8}$ (b) $\frac{3}{8}$ (c) $\frac{1}{4}$ (d) $\frac{109}{216}$ (e) $\frac{79}{160}$

10. (a) $\frac{1}{4}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{2}{3}$ (d) $\frac{1}{2}$ (e) $\frac{1}{18}$ (f) $\frac{1}{3\sqrt{2}}$

11. (a) -0.6931 (4 d.p.) (b) -2.3026 (4 d.p.)

12. (a) $k = 3$ (b) $p = \ln(0.6)$ (c) $r = 1$

13. (a) 4.7622 (4 d.p.) (b) 3 (c) 4.5
    (d) 21.6 (e) 1.35 (f) 1.1619 (4 d.p.)

14. (a) -1.25 (b) 11 (c) 1.9 (d) 6
    (e) 0.3375 (f) 21.6 (g) 0.5809 (4 d.p.)

15. (a) 0.3069 (4 d.p.) (b) 1.6487 (4 d.p.) (c) 1.7183 (4 d.p.)
    (d) 4.4366 (4 d.p.) (e) 0.2420 (4 d.p.) (f) 0.9681 (4 d.p.)

16. (a) -1.2 (b) -1.3863 (4 d.p.)
    (c) 0.3282 (4 d.p.) (d) -1.5 (e) -2

17. (a) The probability that the gap between calls is less than 1 hour is 0.6321 (4 d.p.).
    (b) The probability that the gap between calls is between 1 and 2 hours is 0.2325 (4 d.p.).
    (c) The median length of time between calls is 42 minutes.
    (d) $p = 2.30$ (2 d.p.)