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[MATHEMATICS 1] EXERCISE SHEET 15: LIMITS AND INTEGRATION TO INFINITY

SELECTED QUESTIONS

④ a) $\lim_{x\to\infty}\left(\frac{3}{x^2}\right) = 0$

c) $\lim_{x\to\infty}(e^{-3x}+2) = 2$

e) $\lim_{x\to\infty}\left(\frac{1}{\frac{1}{x}+2}\right) = \frac{1}{0+2} = \frac{1}{2}$

⑤ a) $\lim_{x\to-\infty}(-e^{-x}) = -\infty$

b) $\lim_{x\to-\infty}\left(\frac{1}{x}-14\right) = -14$

d) $\lim_{x\to\infty}\left(\frac{6}{3+4e^{x}}\right) = \frac{6}{3+4(0)}$

$= 2$

~ ⑥ a) $\lim_{x\to\infty}(\sin x)$ = does not exist

The function $\sin x$ osallates between -1 and 1 as $x\to-\infty$, so $\sin x$ never converges to a particular number

⑦ b) $\lim_{x\to\infty}\left(\frac{x^2}{2x^2+3}\right) = \lim_{x\to\infty}\left(\frac{1}{2+\frac{3}{x^2}}\right)$

$= \frac{1}{2+0}$

$= \frac{1}{2}$

c) $\lim_{x\to\infty}\left(\frac{e^{2x}-3e^{x}}{4e^{2x}+e^{x}}\right) = \lim_{x\to\infty}\left(\frac{1-\frac{3}{e^{x}}}{4+\frac{1}{e^{x}}}\right)$

$= \frac{1-0}{4+0}$

$= \frac{1}{4}$

⑧ a) $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_2^b$

$= \lim_{b\to\infty}\left(-\frac{1}{b} - \frac{-1}{2}\right)$

$= 0 + \frac{1}{2}$

$= \frac{1}{2}$

d) $\int_{-\infty}^{-1}\frac{1}{x^3}\,dx = \lim_{b\to-\infty}\int_b^1 \frac{1}{x^3}\,dx$

$= \lim_{b\to-\infty}\left[\frac{-1}{2x^2}\right]_b^1$

$= \lim_{b\to-\infty}\left(\frac{-1}{2(1)^2} - \frac{-1}{2b^2}\right)$

$= \lim_{b\to-\infty}\left(\frac{-1}{2} + \frac{1}{2b^2}\right)$

$= \frac{-1}{2} + 0$

$= -\frac{1}{2}$

f) $\int_{-\infty}^{1} e^{x}\,dx = \lim_{b\to-\infty}\int_b^1 e^{x}\,dx$

$= \lim_{b\to-\infty}\left[e^{x}\right]_b^1$

$= \lim_{b\to-\infty}\left(e^{1} + e^{b}\right)$

$= e + 0$

$= e$

OTHER QUESTIONS

① a) $\lim_{x\to\infty} x^2 = \infty$, the limit does not exist

b) $\lim_{x\to\infty}\sqrt{x} = \infty$, the limit does not exist

c) $\lim_{x\to\infty} e^{-3x} = \lim_{x\to\infty}\frac{1}{e^{3x}}$

$= 0$, the limit exists

d) $\lim_{x\to\infty}\frac{1}{x^{1/2}} = 0$, the limit exists

e) $\lim_{x\to\infty} e^{\frac{1}{3}x} = \infty$, the limit does not exist

f) $\lim_{x\to\infty} x^{5/2} = \infty$, the limit does not exist

③ a) $\lim_{x\to-\infty} x^3 = -\infty$, the limit does not exist

b) $\lim_{x\to-\infty} x^4 = \infty$, the limit does not exist

c) $\lim_{x\to-\infty}\frac{1}{x^3} = 0$, the limit exists

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d) $\lim_{x\to-\infty} \frac{1}{x^4} = 0$, the limit exists

e) $\lim_{x\to-\infty} e^{3x} = 0$, the limit exists

f) $\lim_{x\to-\infty} e^{-2x} = \infty$, the limit does not exist

g) $\lim_{x\to-\infty} \frac{1}{e^x} = \infty$, the limit does not exist

3 a) $\lim_{a\to\infty} \frac{1}{\sqrt{a}} = 0$, the limit exists

b) $\lim_{b\to\infty} b^2 = \infty$, the limit does not exist

c) $\lim_{c\to\infty} e^{-4c} = 0$, the limit exists

d) $\lim_{d\to\infty} \ln d = \infty$, the limit does not exist

e) $\lim_{p\to\infty} e^{2p} = \infty$, the limit does not exist

f) $\lim_{q\to\infty} q^{-3} = \lim_{q\to\infty} \frac{1}{q^3}$
$= 0$, the limit exists

g) $\lim_{y\to-\infty} y^5 = -\infty$, the limit does not exist

h) $\lim_{z\to-\infty} z^6 = \infty$, the limit does not exist

4 a) <selected question>

b) $\lim_{x\to\infty} \left(\frac{3}{x^2} + 4\right) = 0+4$
$= 4$

c) <selected question>

d) $\lim_{x\to\infty} \left(\frac{5}{e^{-3x}+2}\right) = \frac{5}{0+2}$
$= \frac{5}{2}$

e) <selected question>

f) $\lim_{x\to\infty} \left(\frac{4+e^{-2x}}{\frac{4}{x}+12}\right) = \frac{4+0}{0+12}$
$= \frac{1}{3}$

5 a) <selected question>

b) ~~$\lim_{x\to\infty} \left(\frac{1}{x}+14\right)$~~ = <selected question>

c) $\lim_{x\to-\infty} (7+e^{2x}) = 0+7$
$= 7$

d) $\lim_{x\to-\infty} \left(\frac{6}{3+4e^x}\right) = \frac{6}{3+0}$ <selected question>
$= 2$

e) $\lim_{x\to-\infty} \left(\frac{1}{\frac{1}{x}+2}\right) = \frac{1}{0+2}$
$= \frac{1}{2}$

f) $\lim_{x\to\infty} \left(\frac{5+e^{2x}}{\frac{3}{x^2}+15}\right) = \frac{5+0}{0+15}$
$= \frac{1}{3}$

6 a) <selected question>

"Investigate" must give explanation
if just "find" then don't need explanation

b) cos x oscillates between −1 and 1 as $x\to\infty$ and as $x\to-\infty$, so cos x never converges to a particular number. Thus, $\lim_{x\to\infty}(\cos x)$ and $\lim_{x\to-\infty}(\cos x)$ do not exist

c) 4 cos x oscillates between −4 and 4 as $x\to\infty$ and as $x\to-\infty$ so 4cos x never converges to a particular number. Thus, $\lim_{x\to\infty}(4\cos x)$ and $\lim_{x\to-\infty}(4\cos x)$ do not exist

7 a) $\lim_{x\to\infty}\left(\frac{3+4x^2}{x^2}\right) = \lim_{x\to\infty}\left(\frac{3}{x^2}+4\right)$
$= 0+4$
$= 4$

b) <selected question>

c) <selected question>

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7d) $\lim_{x\to\infty}\left(\frac{3x^3-2}{x^3+5x^2}\right) = \lim_{x\to\infty}\left(\frac{3-\frac{2}{x^3}}{1+\frac{5}{x}}\right)$

$= \frac{3-0}{1+0}$

$= 3$

e) $\lim_{x\to\infty}\left(\frac{2e^{3x}+e^{x}}{4-9e^{3x}}\right) = \lim_{x\to\infty}\left(\frac{2+\frac{1}{e^{2x}}}{\frac{4}{e^{3x}}-9}\right)$

$= \frac{2+0}{0-9}$

$= -\frac{2}{9}$

f) $\lim_{x\to\infty}\left(\frac{x+4}{x^2+10}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x}+\frac{4}{x^2}}{1+\frac{10}{x^2}}\right)$

$= \frac{0+0}{1+0}$

$= 0$

g) $\lim_{x\to-\infty}\left(\frac{4x^2}{3-2x^2}\right) = \lim_{x\to-\infty}\left(\frac{4}{\frac{3}{x^2}-2}\right)$

$= \frac{4}{0-2}$

$= -2$

h) $\lim_{x\to-\infty}\left(\frac{e^{2x}+4e^{x}}{2e^{2x}-12e^{x}}\right) = \lim_{x\to-\infty}\left(\frac{e^{x}+4}{2e^{x}-12}\right)$

$= \frac{0+4}{0-12}$

$= -\frac{1}{3}$

i) $\lim_{x\to-\infty}\left(\frac{e^{-x}+1+6e^{-2x}}{3e^{-2x}+2}\right) = \lim_{x\to-\infty}\left(\frac{e^{x}+e^{2x}+6}{3+2e^{2x}}\right)$

$= \frac{0+0+6}{3+0}$

$= 2$

8a) $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx$ <selected question>

$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_2^b$

8a) $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{b} - -\frac{1}{2}\right)$

$= 0+\frac{1}{2}$

$= \frac{1}{2}$

b) $\int_{-\infty}^{-3}\frac{1}{x^2}\,dx = \lim_{b\to-\infty}\int_b^{-3}\frac{1}{x^2}\,dx$

$= \lim_{b\to-\infty}\left[-\frac{1}{x}\right]_b^{-3}$

$= \lim_{b\to-\infty}\left(-\frac{1}{-3} - -\frac{1}{b}\right)$

$= \frac{1}{3}+0$

$= \frac{1}{3}$

c) $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx$

$= \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b$

$= \lim_{b\to\infty}\left(-\frac{1}{2b^2} - -\frac{1}{2(1)^2}\right)$

$= 0+\frac{1}{2}$

$= \frac{1}{2}$

d) <selected question>

e) $\int_0^\infty \frac{1}{(x+1)^2}\,dx = \lim_{b\to\infty}\int_0^b (x+1)^{-2}\,dx$

$= \lim_{b\to\infty}\left[-(x+1)^{-1}\right]_0^b$

$= \lim_{b\to\infty}\left(-\frac{1}{b+1} - -\frac{1}{0+1}\right)$

$= 0+1$

$= 1$

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i) $\int_1^\infty \frac{1}{(x+1)^2}\,dx = \lim_{b\to\infty}\int_1^b (x+1)^{-2}\,dx$

$= \lim_{b\to\infty}\left[-(x+1)^{-1}\right]_1^b$

$= \lim_{b\to\infty}\left(-\frac{1}{b+1} - -\frac{1}{1+1}\right)$

$= 0 + \frac{1}{2}$

$= \frac{1}{2}$

c) $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$

$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$

$= \lim_{b\to\infty}\left(-e^{-b} - -e^{-0}\right)$

$= \lim_{b\to\infty}\left(-e^{-b} + 1\right)$

$= 0+1$

$= 1$

9) <selected question>

b) $\int_{-\infty}^{\ln 3} e^{x}\,dx = \lim_{b\to-\infty}\int_b^{\ln 3} e^{x}\,dx$

$= \lim_{b\to-\infty}\left[e^{x}\right]_b^{\ln 3}$

$= \lim_{b\to-\infty}\left(e^{\ln 3} - e^{b}\right)$

$= 3 - 0$

$= 3$

d) $\int_2^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_2^b e^{-x}\,dx$

$= \lim_{b\to\infty}\left[-e^{-x}\right]_2^b$

$= \lim_{b\to\infty}\left(-e^{-b} - -e^{-2}\right)$

$= 0 + \frac{1}{e^2}$

$= \frac{1}{e^2}$

9 a) $\int_{-\infty}^{0} e^{2x}\,dx = \lim_{b\to-\infty}\int_b^0 e^{2x}\,dx$

$= \lim_{b\to-\infty}\left[\frac{1}{2}e^{2x}\right]_b^0$

$= \lim_{b\to-\infty}\left(\frac{1}{2}e^{2(0)} - \frac{1}{2}e^{2b}\right)$

$= \frac{1}{2} - 0$

$= \frac{1}{2}$

c) $\int_0^\infty -e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b -e^{-2x}\,dx$

$= \lim_{b\to\infty}\left[\frac{1}{2}e^{-2x}\right]_0^b$

$= \lim_{b\to\infty}\left(\frac{1}{2}e^{-2b} - \frac{1}{2}e^{-2(0)}\right)$

$= 0 - \frac{1}{2}$

$= -\frac{1}{2}$

e) $\int_{-\infty}^{\ln 3} e^{2x}\,dx = \lim_{b\to-\infty}\int_b^{\ln 3} e^{2x}\,dx$

$= \lim_{b\to-\infty}\left[\frac{1}{2}e^{2x}\right]_b^{\ln 3}$

$= \lim_{b\to-\infty}\left(\frac{1}{2}e^{2(\ln 3)} - \frac{1}{2}e^{2b}\right)$

$= \lim_{b\to-\infty}\left(\frac{1}{2}e^{\ln 3^2} - \frac{1}{2}e^{2b}\right)$

$= \lim_{b\to-\infty}\left(\frac{9}{2} - \frac{1}{2}e^{2b}\right)$

$= \frac{9}{2} - 0 = \frac{9}{2}$

f) $\int_{\ln 2}^\infty -e^{-2x}\,dx = \lim_{b\to\infty}\int_{\ln 2}^b -e^{-2x}\,dx$

$= \lim_{b\to\infty}\left[\frac{1}{2}e^{-2x}\right]_{\ln 2}^b$

$= \lim_{b\to\infty}\left(\frac{1}{2}e^{-2b} - \frac{1}{2}e^{-2(\ln 2)}\right)$

$= 0 - \frac{1}{8}$

$= \frac{-1}{8}$

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g) $\int_2^\infty e^{-\frac{1}{2}x}\,dx = \lim_{b\to\infty}\int_2^b e^{-\frac{1}{2}x}\,dx$

$= \lim_{b\to\infty}\left[-2e^{-\frac{1}{2}x}\right]_2^b$

$= \lim_{b\to\infty}\left(-2e^{-\frac{1}{2}b} - -2e^{-\frac{1}{2}(2)}\right)$

$= 0 + \frac{2}{e}$

$= \frac{2}{e}$

b) $\int_{-\infty}^0 f(x)\,dx$ where $f(x) = \begin{cases} e^{2x} & \text{if } x \le -1 \\ x & \text{if } x > -1 \end{cases}$

$\int_{-\infty}^0 f(x)\,dx = \int_{-\infty}^{-1} e^{2x}\,dx + \int_{-1}^0 x\,dx$

$= \lim_{b\to-\infty}\left[\frac{1}{2}e^{2x}\right]_b^{-1} + \left[\frac{1}{2}x^2\right]_{-1}^0$

$= \lim_{b\to-\infty}\left(\frac{1}{2}e^{2(-1)} - \frac{1}{2}e^{2b}\right) + \left(\frac{1}{2}(0)^2 - \frac{1}{2}(-1)^2\right)$

$= \frac{1}{2e^2} - \frac{1}{2}$

h) $\int_{\ln 100}^\infty e^{-\frac{1}{2}x}\,dx = \lim_{b\to\infty}\int_{\ln 100}^b e^{-\frac{1}{2}x}\,dx$

$= \lim_{b\to\infty}\left[-2e^{-\frac{1}{2}x}\right]_{\ln 100}^b$

$= \lim_{b\to\infty}\left(-2e^{-\frac{1}{2}(b)} - -2e^{-\frac{1}{2}(\ln 100)}\right)$

$= \lim_{b\to\infty}\left(-2e^{-\frac{1}{2}b} + 2\cancel{e^{-\ln 10}}\right)$

$= \cancel{0} + \frac{2}{10}$

$= \frac{1}{5}$

c) $\int_4^\infty f(x)\,dx$ where $f(x) = \begin{cases} \frac{1}{2x+1} & \text{if } 0 \le x < 12 \\ e^{-x} & \text{if } x \geq 12 \end{cases}$

$\int_4^\infty f(x)\,dx = \int_4^{12} \frac{1}{2x+1}\,dx + \int_{12}^\infty e^{-x}\,dx$

$= \left[\frac{1}{2}\ln|2x+1|\right]_4^{12} + \lim_{b\to\infty}\left[-e^{-x}\right]_{12}^b$

$= \left(\frac{1}{2}\ln|2(12)+1| - \frac{1}{2}\ln|2(4)+1|\right) + \left(\lim_{b\to\infty}\left(-e^{-b} - -e^{-12}\right)\right)$

$= \frac{1}{2}\ln 25 - \frac{1}{2}\ln 9 + 0 + e^{-12}$

$= \ln 5 - \ln 3 + e^{-12}$

$= \ln\frac{5}{3} + e^{-12}$

10 a) $\int_0^\infty f(x)\,dx$ where $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \le 1 \\ \frac{1}{x^3} & \text{if } x > 1 \end{cases}$

$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$

$= \int_0^1 \sin(\pi x)\,dx + \int_1^\infty \frac{1}{x^3}\,dx$

$= \left[\frac{-1}{\pi}\cos(\pi x)\right]_0^1 + \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx$

$= \left(\frac{-1}{\pi}\overbrace{\cos(\pi(1))}^{-1} - \frac{-1}{\pi}\overbrace{\cos(\pi(0))}^{1}\right) + \lim_{b\to\infty}\left[\frac{-1}{2x^2}\right]_1^b$

$= \frac{2}{\pi} + \lim_{b\to\infty}\left(\frac{-1}{2b^2} - \frac{-1}{2(1)^2}\right)$

$= \frac{2}{\pi} + 0 + \frac{1}{2}$

$= \frac{2}{\pi} + \frac{1}{2}$

d) $\int_{-\infty}^4 f(x)\,dx$ where $f(x) = \begin{cases} \frac{1}{x^4} & \text{if } x < -1 \\ x^2 & \text{if } x \geq -1 \end{cases}$

$\int_{-\infty}^4 f(x)\,dx = \int_{-\infty}^{-1} \frac{1}{x^4}\,dx + \int_{-1}^4 x^2\,dx$

$= \lim_{b\to-\infty}\left[\frac{-1}{3x^3}\right]_b^{-1} + \left[\frac{1}{3}x^3\right]_{-1}^4$

$= \lim_{b\to-\infty}\left(\frac{-1}{3(-1)^3} - \frac{-1}{3b^3}\right) + \left(\frac{1}{3}(4)^3 - \frac{1}{3}(-1)^3\right)$

$= \frac{1}{3} - 0 + \frac{65}{3}$

$= 22$

$3x^{-2}$

$3x^{-1}$

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√e) $\int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = \begin{cases} e^{4x} & \text{if } x \le 0 \\ e^{-2x} & \text{if } x > 0 \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} e^{4x}\,dx + \int_{0}^{\infty} e^{-2x}\,dx$$

$$= \lim_{b\to-\infty}\left[\frac{1}{4}e^{4x}\right]_b^0 + \lim_{a\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_0^a$$

$$= \lim_{b\to-\infty}\left(\frac{1}{4}e^{4(0)} - e^{4b}\right) + \lim_{a\to\infty}\left(-\frac{1}{2}e^{-2a} - -\frac{1}{2}e^{-2(0)}\right)$$

$$= \frac{1}{4} - 0 + 0 + \frac{1}{2}$$

$$= \frac{3}{4}$$

√f) $\int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = \begin{cases} \frac{3}{x^2} & \text{if } x \le -1 \\ 3 & \text{if } -1 < x \le 1 \\ \frac{3}{x^4} & \text{if } x > 1 \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} \frac{3}{x^2}\,dx + \int_{-1}^{1} 3\,dx + \int_{1}^{\infty} \frac{3}{x^4}\,dx$$

$$= \lim_{a\to-\infty}\left[-\frac{3}{x}\right]_a^{-1} + \left[3x\right]_{-1}^{1} + \lim_{b\to\infty}\left[\frac{-\cancel{3}}{\cancel{3}x^3}\right]_1^b$$

$$= \lim_{a\to-\infty}\left(\frac{-3}{-1} - \frac{-3}{a}\right) + \left(3(1) - 3(-1)\right) + \lim_{b\to\infty}\left(\frac{-1}{b^3} - \frac{-1}{(1)^3}\right)$$

$$= \lim_{a\to-\infty}\left(3 + \frac{3}{a}\right) + 6 + \lim_{b\to\infty}\left(-\frac{1}{b^3} + 1\right)$$

$$= 3 + 0 + 6 + 0 + 1$$

$$= 10$$