Mathematics 1

Sheet 26: Normal Distribution

1. Suppose that Z is a standard normal variable. Use "normal tables" to determine
   (a) the value of z for which $\Pr(Z < z) = 0.1020$,
   (b) the value of c for which $\Pr(Z \geq c) = 0.0102$,
   (c) $\Pr(-2 \leq Z \leq 2)$, and
   (d) the value of z for which
   $$\Pr(-z \leq Z \leq z) = 0.4108\,.$$

   Touch here to see a video of a similar problem to 1 (c).

2. A machine makes electrical resistors which have a mean resistance of 50 ohms with a standard deviation of 2 ohms. Assuming that the distribution is normal,
   (a) find the percentage of resistors made whose resistance is less than 47.5 ohms.
   (b) calculate the value of r such that there is a probability of 0.998 that a resistance is at most r ohms.

3. The local authorities in Melbourne have installed electric lamps in the streets of the city. Suppose the life of the lamps is normally distributed, with an average life of 1000 hours and a standard deviation of 200 hours.
   (a) Find the percentage of lamps which will fail in the first 700 hours.
   (b) Find h so that 10.03% of the lamps fail in the first h hours.

4. Steel rods are manufactured to have a diameter of 4 cm, but they are acceptable if the diameter is between 3.98 and 4.02 cm. The manufacturer has noticed that 3.01% of the rods are rejected for being **too small**.

   If the diameters are normally distributed, with mean 4 cm, find the distribution's standard deviation. Give your answer to two decimal places.

5. Suppose that X is a normally distributed random variable with mean 5. If the probability that X is greater than 7 is 0.1056, find
   (a) the standard deviation of X,
   (b) $\Pr(X < 3.992)$,
   (c) $\Pr(X < 3.992 \mid X < 7)$, to four decimal places, and
   (d) the value of x for which $\Pr(X > x) = 0.8023$.

6. Speedometers of cars are not accurate. When the speedometer of a car registers $35\ \text{km} \cdot \text{h}^{-1}$, the actual speed of the car is a variable having a normal distribution with mean $\mu = 33$ and standard deviation $\sigma = 2$.
   (a) What percentage of cars are exceeding $35\ \text{km} \cdot \text{h}^{-1}$ when their speedometers measure $35\ \text{km} \cdot \text{h}^{-1}$?
   (b) If this percentage was changed to 1.02% by reducing μ (but keeping $\sigma = 2$), then what value of μ would be needed?

7. The life of a certain type of car battery is known to be normally distributed with mean 30 months and standard deviation 6 months. The batteries cost the manufacturer \$40 each to make, and he sells them for \$60 each. If he refunds \$30 for any battery which lasts less than 24 months, what is the mean profit he will make per battery?

8. A fruitgrower produces peaches whose weights are normally distributed with a mean of 180 g and a standard deviation of 20 g.

   - Peaches whose weights exceed 200 g are sold to canneries yielding a profit of 40 cents per peach.
   - Peaches whose weights are between 150 g and 200 g are sold to wholesale markets at a profit of 20 cents per peach.
   - Peaches whose weights are less than 150 g are sold for jam at a profit of 10 cents per peach.

   (a) Calculate the percentage of peaches that are sold to canneries.
   (b) Calculate the percentage of peaches that are sold to wholesale markets.
   (c) Calculate the mean profit per peach, to the nearest tenth of a cent.

9. The weights of trout in a pool at a trout farm are normally distributed with mean 2 kg and standard deviation 0.4 kg.
   (a) Find the probability that a trout, selected at random from the pool, weighs less than 1.5 kg.
   (b) Find the weight which is exceeded by 99.01% of the trout in the pool.
   (c) A restaurant owner requires trout with weights in the range 1 to 2.5 kg. Find the percentage of trout in the pool that satisfy these requirements.

10. The weight of a certain type of apple is normally distributed with mean 40 grams, and variance 9 grams^2. What is the probability that an apple, selected at random, will weigh at least 46 grams?

Tangents, Normals, Angles Between Curves

11. Find the equations of the tangent and normal to the curve $y = 2x^2$ at $x = 1$.

12. Find the equations of the tangent and normal to the curve $y = e^{2x+1}$ at $x = 0$.

13. Find the equations of the tangent and normal to the curve $y = \sin(x^2) + x$ at $x = 0$.

14. Find the equations of the tangent and normal to the curve $x^2 + xy + y^2 = 7$ at the point $(1, 2)$.

15. The graphs with equations
$$y = x^2 - 2x \text{ and } y = x$$
intersect at the point $(3, 3)$. Find the angle in degrees between the curves at this point, giving your answer to two decimal places.

16. Find the angle (in degrees) which the curve $y = \ln x$ makes with the x–axis (at their point of intersection).

17. The curve $y = ax + \dfrac{b}{x^2}$ cuts the x–axis at $(2, 0)$, with an angle of inclination of $45°$.

    Find the value of a and b.

Limits and Continuity

18. Evaluate the following limits:

    (a) $\displaystyle\lim_{x\to 0} \frac{x^2 + 6x}{x}$

    (b) $\displaystyle\lim_{x\to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$

    (c) $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$

19. Consider the function
$$f(x) = \begin{cases} x^2 - x & \text{for } x \le 2 \\ 2x - 2 & \text{for } x > 2 \end{cases}$$
Is f continuous at $x = 2$?

    Justify your answer.

20. Let $f : (0, 6) \longrightarrow \mathbf{R}$ where
$$f(x) = \begin{cases} x^2 - 1 & \text{for } 0 < x < 2 \\ 3 & \text{for } 2 \le x < 4 \\ 5x - 25 & \text{for } 4 \le x < 6 \end{cases}$$

    (a) Sketch the graph of $y = f(x)$.

    (b) Find any values of $x \in \text{dom}(f)$ for which f is not continuous.

    Justify your answer.

21. Let $f : (-1, 2) \longrightarrow \mathbf{R}$ where

$$f(x) = \begin{cases} |x| & \text{for } -1 < x \leq 0 \\ x & \text{for } 0 < x < 1 \\ x^2 & \text{for } 1 \leq x < 2 \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

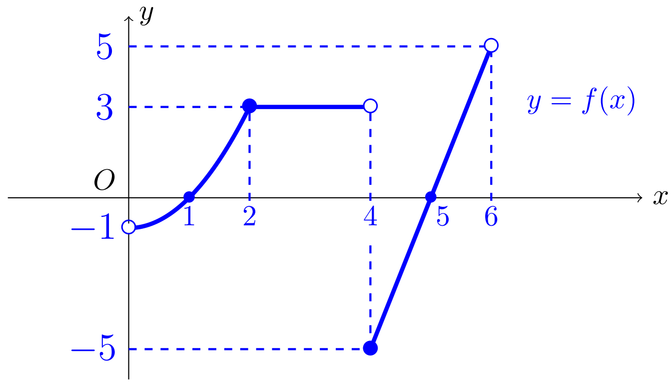
(b) Find any values of $x \in \text{dom}(f)$ for which f is not continuous.

Justify your answer.

Answers:

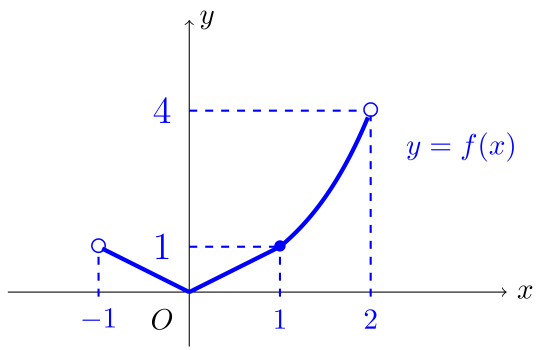
1. (a) $z = -1.27$ (b) $c = 2.32$ (c) 0.9544 (d) $z = 0.54$
2. (a) 10.56% of the resistors have resistance < 47.5 ohms.
   (b) The value of r is 55.76
3. (a) 6.68% of the lamps will fail in the first 700 hours.
   (b) 10.03% of the lamps fail within 744 hours, so $h = 744$.
4. The standard deviation is $\sigma = 0.01$ (2 d.p.).
5. (a) $\sigma = 1.6$ (b) 0.2643 (c) 0.2955 (4 d.p.)
   (d) $x = 3.64$
6. (a) 15.87% of the cars would be exceeding 35 $\text{km} \cdot \text{h}^{-1}$.
   (b) We would need to take $\mu = 30.36$
7. The mean profit per battery is \$15.24
   (The answer \$15.239 is also acceptable.)
8. (a) 15.87% of the peaches are sold to the canneries.
   (b) 77.45% of the peaches are sold to the wholesale markets.
   (c) The mean profit per peach is 22.5 cents.
9. (a) The probability that a trout weighs < 1.5 kg is 0.1056
   (b) 99.01% of the trout weigh more than 1.068 kg.
   (c) 88.82% of the trout satisfy the restaurant's requirements.
10. The probability that the apples's weight is at least 46 grams is 0.0228
11. Tangent: $y = 4x - 2$ Normal: $x + 4y = 9$
12. Tangent: $y = 2ex + e$ Normal: $y = -\frac{1}{2e}x + e$
13. Tangent: $y = x$ Normal: $y = -x$
14. Tangent: $5y = -4x + 14$ Normal: $4y = 5x + 3$
15. $30.96°$ (or $149.04°$)
16. $45°$ (or $135°$)
17. $a = \frac{1}{3}$ and $b = -\frac{8}{3}$
18. (a) 6 (b) $\frac{11}{9}$
    (c) The limit does not exist, since $\lim_{x \to 0^-} \frac{|x|}{x} = -1$ and $\lim_{x \to 0^+} \frac{|x|}{x} = 1$.
19. f is continuous at $x = 2$ since $\lim_{x \to 2} f(x) = f(2)$.

20. (a)



(b) f is not continuous at $x = 4$,
since $\lim_{x \to 4} f(x)$ does not exist.

21. (a)



(b) f is continuous everywhere in its domain, since

- each of the three parts of f (namely $y = |x|$, $y = x$ and $y = x^2$) are continuous everywhere, and also
- $\lim_{x \to 0} f(x) = f(0)$ and $\lim_{x \to 1} f(x) = f(1)$ (which shows that f is also continuous at those points where it changes from one part to another part).