Chapter 19

Conditional Probability

19.1 Conditional Probability

Conditional probability is the probability that a particular event occurs *if it is known (or assumed) that some other event occurs.*

- We write $\Pr(A \mid B)$ for the probability that

$$A \text{ occurs if it is known (or assumed) that } B \text{ occurs.}$$

We call it the probability of "A **given** B".

It is important to realize that usually $\boxed{\Pr(A \mid B) \neq \Pr(B \mid A).}$

Example 1. The distribution of voters in a street is as follows:

	Labor	Liberal	Green	Total
Male	20	21	9	50
Female	21	14	15	50
Total	41	35	24	100

Suppose that a voter will be randomly chosen from the street. Then

(a) $\Pr(\text{the chosen voter will vote for the Greens}) = \dfrac{24}{100} = 0.24$

(b) $\Pr(\text{the chosen voter will vote for the Greens} \mid \text{voter is female}) = \dfrac{15}{50} = 0.3$

(c) $\Pr(\text{the chosen voter will be female} \mid \text{votes for the Greens}) = \dfrac{15}{24} = 0.625$

By looking at our answers for (b) and (c), we notice that

$$\Pr(\text{Green} \mid \text{female}) \neq \Pr(\text{female} \mid \text{Green}).$$

□

Note: In general,

$$\boxed{\Pr(A \mid B) \;=\; \frac{\Pr(A \cap B)}{\Pr(B)}}$$

which means that

$$\Pr(A \cap B) \;=\; \Pr(A \mid B)\, \Pr(B)\,. \tag{19.1}$$

Similarly,

$$\Pr(B \mid A) \;=\; \frac{\Pr(B \cap A)}{\Pr(A)}$$

and so

$$\Pr(B \cap A) \;=\; \Pr(B \mid A)\, \Pr(A)\,. \tag{19.2}$$

Since $A \cap B \;=\; B \cap A$ we can conclude from Equations 19.1 and 19.2 that

$$\begin{aligned} \Pr(A \cap B) &= \Pr(A \mid B)\, \Pr(B) \\ &= \Pr(B \mid A)\, \Pr(A)\,. \end{aligned}$$

This formula is known as the **Multiplication Rule**, and is on the Formula Sheet which is provided in the Mathematics 1 exams.

Exercises for Section 19.1

1. If $\Pr(A) = 0.7$, $\Pr(B) = 0.2$ and $\Pr(A \cup B) = 0.8$, then find

   (a) $\Pr(A \cap B)$ (b) $\Pr(A \mid B)$.

2. If $\Pr(A) = 0.3$, $\Pr(B) = 0.8$ and $\Pr(A \cup B) = 0.9$, then find

   (a) $\Pr(A \mid B)$ (b) $\Pr(B \mid A)$.

3. If $\Pr(A) = 0.5$, $\Pr(A \cup B) = 0.8$ and $\Pr(B \mid A) = 0.2$, then find

   (a) $\Pr(A')$ (b) $\Pr(A \cap B)$ (c) $\Pr(B)$
   (d) $\Pr(A \mid B)$ (e) $\Pr(A \mid B')$.

4. The distribution of adults in a particular suburb is as follows:

	Employed	Unemployed	Home Duties	Student	Total
Male	17 683	2 917	48	964	21 612
Female	15 800	1 221	2 592	828	20 441
	33 483	4 138	2 640	1 792	42 053

   An adult will be randomly chosen from the suburb.

   (a) Find the probability that the chosen adult will be female.

   (b) Find the probability that the chosen adult is unemployed, *if it is known* (or assumed) that the adult is female.

19.2 Tree Diagrams

A tree diagram is a very useful way of representing probabilities when a problem involves several stages or several "layers" of information.

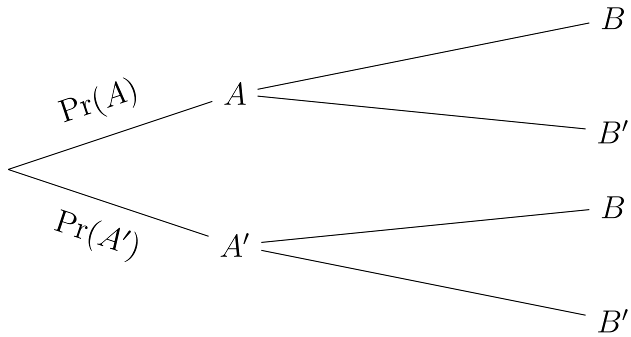
When we read a tree diagram, we always read from *left to right.*

Consider two events A and B, where the probability that B occurs depends on the outcome of A. The

- *top left* branch of the tree diagram drawn below tells us the probability that A occurs,

and the

- *bottom left* branch tells us the probability that A does *not* occur.



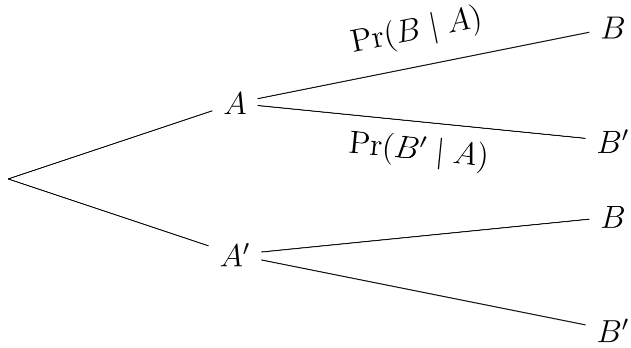
Of course, we must have $\Pr(A) + \Pr(A') = 1$.

The *top right* branches of the tree tells us

- the probability that B occurs *given* that A occurs,

and

- the probability that B does *not* occur *given* that A occurs.



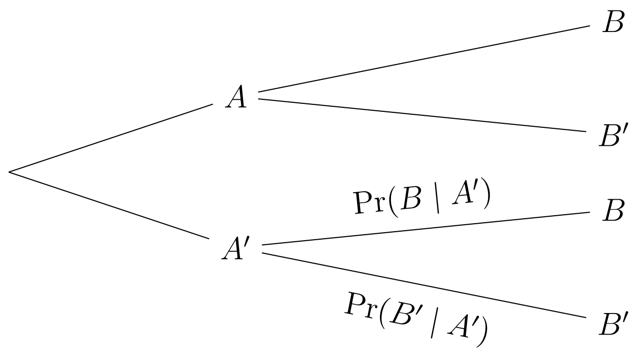
Because these events are *complement* to each other, we have $\Pr(B \mid A) + \Pr(B' \mid A) = 1$.

Similarly, the *bottom right* branches of the tree tells us

- the probability that B occurs *given* that A does *not* occur,

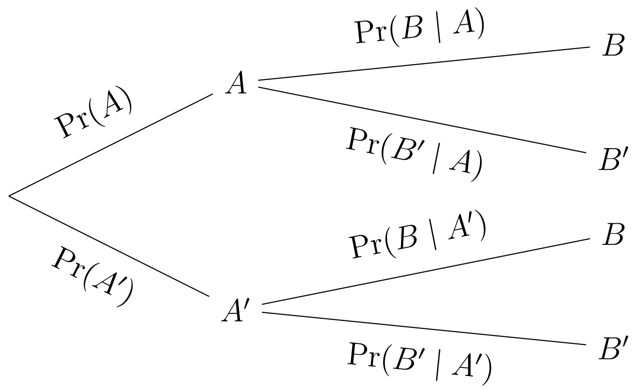
and

- the probability that B does *not* occur *given* that A does not occur.



Because these events are *complement* to each other, we have $\Pr(B \mid A') + \Pr(B' \mid A') = 1$.

Overall we have:



Tree diagrams allow us to calculate probabilities in a very intuitive way, which is based on the following result (known as the **Law of Total Probability**):

$$\Pr(B) \;=\; \Pr(B \mid A) \times \Pr(A) \;+\; \Pr(B \mid A') \times \Pr(A').$$

(This formula is *not* on the Formula Sheet.)

Because we always read a tree diagram from *left to right,* in practice we use a slightly rearranged version of this Law of Total Probability when calculating probabilities from a tree diagram:

$$\Pr(B) \;=\; \Pr(A) \times \Pr(B \mid A) \;+\; \Pr(A') \times \Pr(B \mid A').$$

Example 2. In a particular group of students, 60% are male and 40% are female. Suppose we know that 55% of the male students study maths, and 65% of the female students study maths.

(a) Draw a tree diagram for this example.

A student will be randomly selected.

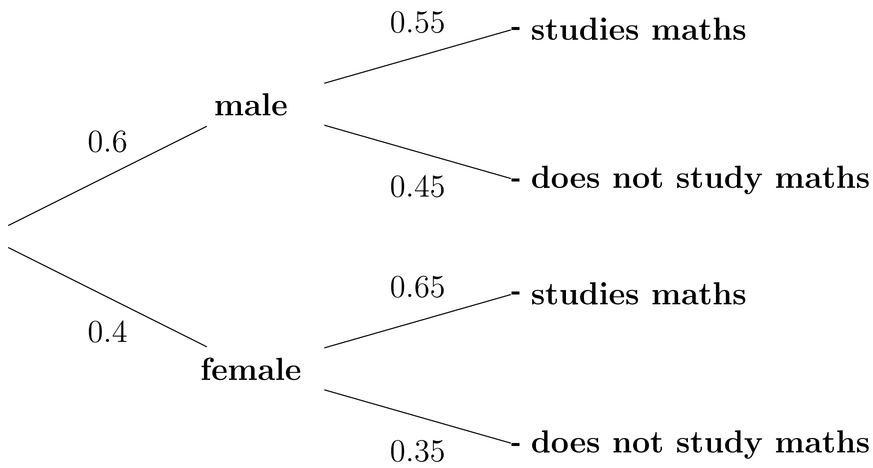
(b) Find the probability that the selected student studies maths and is female.

(c) What is the probability that the student is female *given* that the student studies maths? Give your answer as a decimal approximation to 4 decimal places.

(d) What is the probability that the student studies maths *given* that the student is female?

Solution:

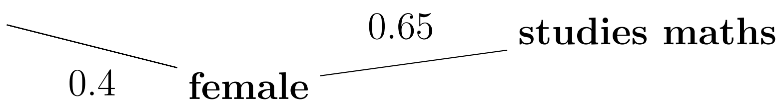
(a)



(b) Note that, by the Multiplication Rule, we have

$$\begin{aligned}\Pr(\text{studies maths } \textbf{and} \text{ female}) &= \Pr(\text{studies maths} \mid \text{female}) \times \Pr(\text{female}) \\ &= 0.65 \times 0.4 \\ &= 0.26\,.\end{aligned}$$

Alternatively, we can easily calculate this probability by realizing that the required probability is equivalent to finding Pr(female **and** studies maths), *and then following the relevant branches on the tree (from left to right).*



Again we see that

$$\begin{aligned}\Pr(\text{female } \textbf{and} \text{ studies maths}) &= 0.4 \times 0.65 \\ &= 0.26\,.\end{aligned}$$

(c) We have
$$\begin{aligned}
\Pr(\text{female} \mid \text{studies maths}) &= \frac{\Pr(\text{female } \textbf{and} \text{ studies maths})}{\Pr(\text{studies maths})} \\
&= \frac{0.4 \times 0.65}{0.6 \times 0.55 + 0.4 \times 0.65} \\
&= \frac{0.26}{0.59} \\
&= 0.4407 \quad (4 \text{ d.p.}).
\end{aligned}$$

(d) We have
$$\begin{aligned}
\Pr(\text{studies maths} \mid \text{female}) &= \frac{\Pr(\text{studies maths } \textbf{and} \text{ female})}{\Pr(\text{female})} \\
&= \frac{0.4 \times 0.65}{0.4} \\
&= 0.65.
\end{aligned}$$

□

Note that in (d), the "given" information is on the *left* part of the tree. When the "given" information is on the *left* of the tree, we can find the conditional probability *immediately* by simply looking at the relevant branch (or branches) of the tree, without having to use the formula for conditional probability.

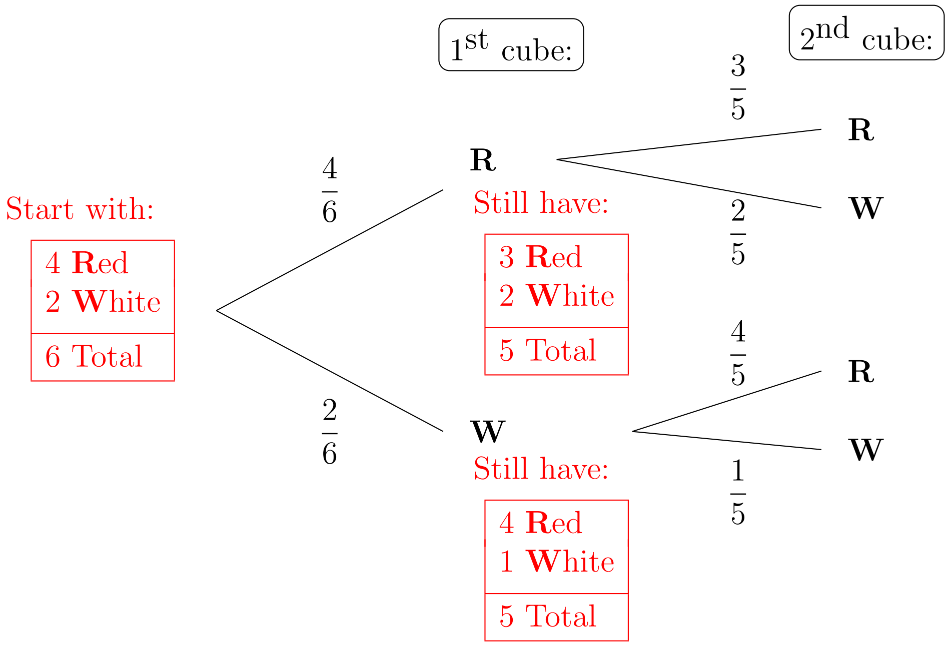
Example 3. Four red cubes and two white cubes are in a bag. If two cubes will be taken from the bag, without replacement, then find the probability

(a) of obtaining 2 red cubes.

(b) that the second cube taken will be red.

(c) that the first cube is red, given that the second cube is red.

Solution:



(a) $\Pr(\mathbf{RR}) = \frac{4}{6} \times \frac{3}{5} = \frac{12}{30} = \frac{2}{5}.$

(b)
$$\begin{aligned} \Pr(\text{second cube red}) &= \Pr(\mathbf{RR} \text{ or } \mathbf{WR}) \\ &= \frac{4}{6} \times \frac{3}{5} + \frac{2}{6} \times \frac{4}{5} \\ &= \frac{12}{30} + \frac{8}{30} = \frac{20}{30} = \frac{2}{3}. \end{aligned}$$

(c)
$$\begin{aligned} \Pr(\text{first cube red} \mid \text{second cube red}) &= \frac{\Pr(\text{first cube red } \textbf{and} \text{ second cube red})}{\Pr(\text{second cube red})} \\ &= \frac{2/5}{2/3} \quad \text{(using the answers from (a) and (b))} \\ &= \frac{3}{5}. \end{aligned}$$

□

Exercises for Section 19.2

1. Each morning a man watches television or reads a book. The probability that he watches television is 0.8. If he watches television there is a probability of 0.6 that he will fall asleep. However, if he reads a book the probability that he falls asleep is only 0.2. Find the probability that he falls asleep.

2. A student has a maths test every month. She calculates her chances of passing the next test according to the results on earlier tests.

   - If she passed the last test, she thinks her chances of passing the next test are 0.7.
   - If she failed the last test, she thinks her chances of passing the next test are 0.5.

   Given that she failed the test in February, find the probability that the student will

   (a) pass both of the tests in March and April.

   (b) pass the test in March, but fail the test in April.

   (c) fail the test in March, and pass the test in April.

   (d) fail both of the tests in March and April.

3. A farmer estimates that the probability of this season being a good one is $\frac{2}{3}$. Suppose that

   (a) if this season is good then the probability that the next season is good is $\frac{3}{5}$,

   (b) if this season is *not* good then the probability that the next season is good is $\frac{1}{2}$.

   Find the probability that

   (a) *both* this season and the next season are good.

   (b) that *only one* of the seasons is good.

4. From a box containing 6 black and 4 white balls, 2 balls are taken at random without replacement. Find the probability that

   (a) both balls are black.

   (b) one is black, and the other is white.

5. Thirty per cent of car drivers are classified by an insurance company as class A drivers, 50% as class B drivers, and 20% as class C drivers. The probability that a driver will make a claim in any one year is 0.01 for a class A driver, 0.05 for a class B driver, and 0.1 for a class C driver. Suppose that Tom Smith makes a claim for an accident. What is the probability that he is a class A driver?

6. Suppose that

   - Jane has a bag containing 5 chocolates and 4 marshmallows,
   - Peter has a bag containing 3 chocolates and 5 marshmallows, and
   - Rebecca has a bag containing 4 chocolates and 2 marshmallows.

   One of the bags is selected at random, and a selection is made from it.

   (a) What is the probability of selecting a chocolate?

   (b) Suppose that a chocolate is chosen. What is the probability that this chocolate came from Peter's bag?

7. Twenty per cent of a company's production is sent overseas. The overseas orders have 80% first grade items, while the local market has 65% first grade items.

   (a) What percentage of the company's production is first grade?

   (b) What is the probability that an article will be sent overseas, given that it has been found to be first grade?

8. A gambler has in his pocket a fair coin and a two–headed coin. He selects one of the coins at random, and tosses it.

   (a) If the coin shows heads, what is the probability that it is the fair coin?

   (b) Suppose he tosses the coin twice, and it shows heads both times. What is the probability that it is the fair coin?

9. A student travels to school by public transport on 3 out of every 5 days, and travels by car on 2 out of every 5 days. If the student travels by car, the probability that she is late is 0.2. If she travels by public transport, the probability that she is late is 0.1.

   (a) Find the probability that she is late for school.

   (b) Find the probability that she travelled to school by car given that she is late for school. Give your answer as a decimal approximation to 3 decimal places.

19.3 Independent Events

Two events are **independent** if the occurrence of one of them does *not* change the probability that the other will occur. In particular, A and B are independent if

$$\Pr(B \mid A) = \Pr(B) \text{ or if } \Pr(A \mid B) = \Pr(A).$$

We can use these formulae to simplify the Multiplication Rule. In fact, it can be shown that

$$\boxed{\text{A and B are independent if and only if } \Pr(A \cap B) = \Pr(A) \times \Pr(B).}$$

- This variant of the Multiplication Rule is useful when determining whether two events are indeed independent.

It can also be shown that if A and B are independent, then so are their complements.

Example 4. Suppose that the *independent* probabilities that Bart and Lisa will pass a maths test are 0.4 and 0.9 respectively. Calculate the probability that

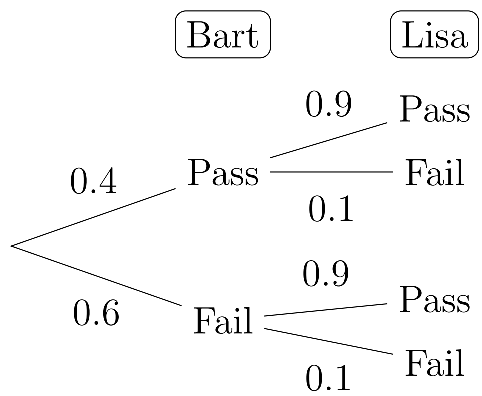
(a) both of them will pass the test.

(b) exactly one of them will pass the test.

(c) at least one of them will pass the test.

(d) Bart passes the test, if it is known that only one of them passes the test.

Solution:



(a) Because of the independence, we can write

$$\begin{aligned} \text{Pr(both of them will pass)} &= \text{Pr(Bart will pass } \textbf{and} \text{ Lisa will pass)} \\ &= \text{Pr(Bart will pass)} \times \text{Pr(Lisa will pass)} \\ &= 0.4 \times 0.9 \\ &= 0.36. \end{aligned}$$

Thus the probability that Bart and Lisa will both pass the test is 0.36.

(b) Pr(exactly one of them will pass)

$$\begin{aligned}
&= \Pr(\text{Bart will pass and Lisa will fail, or Bart will fail and Lisa will pass}) \\
&= 0.4 \times 0.1 \quad + \quad 0.6 \times 0.9 \\
&= 0.58.
\end{aligned}$$

So the probability that exactly one of them will pass the test is 0.58.

(c) Pr(at least one of them will pass)

$$\begin{aligned}
&= \Pr(\text{exactly one of them will pass}) \ + \ \Pr(\text{both of them will pass}) \\
&= 0.58 \quad + \quad 0.36 \quad (\text{from (a) and (b)}) \\
&= 0.94.
\end{aligned}$$

Alternative method:

$$\begin{aligned}
&\Pr(\text{at least one of them passes}) \\
&= 1 - \Pr(\text{neither of them pass}) \\
&= 1 - \Pr(\text{Bart fails and Lisa fails}) \\
&= 1 - 0.6 \times 0.1 \\
&= 0.94.
\end{aligned}$$

So the probability that at least one of them will pass the test is 0.94.

(d)

$$\begin{aligned}
&\Pr(\text{Bart passes} \ | \ \text{only one of them passes}) \\
&= \frac{\Pr(\text{Bart passes and only one passes})}{\Pr(\text{only one passes})} \\
&= \frac{\Pr(\text{Bart passes and Lisa fails})}{\Pr(\text{only one passes})} \\
&= \frac{0.4 \times 0.1}{0.58 \quad \text{from (b)}} \\
&= \frac{4}{58}
\end{aligned}$$

So the required probability is $\frac{4}{58}$.

□

Example 5. Suppose that A and B are events such that

$$\Pr(A) = 0.3, \;\; \Pr(B) = 0.6 \;\text{ and }\; \Pr(A \cap B) = 0.18.$$

Are A and B mutually exclusive, independent or neither?

Solution:

Since $\Pr(A \cap B) \neq 0$, we can immediately conclude that A and B are *not* mutually exclusive.

However, we can see that $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, and thus we conclude that A and B *are* independent.

$\square$

Exercises for Section 19.3

1. Suppose that A and B are events such that
$$\Pr(A) = \frac{1}{4}\,, \;\; \Pr(B) = b \;\text{ and }\; \Pr(A \cup B) = \frac{1}{3}\,.$$
   (a) Find b if A and B are mutually exclusive.

   (b) Find b if A and B are independent.

2. Two people with different complaints are treated with two different drugs, A and B respectively. (Thus these are independent events.) The probability of a cure with drug A is 0.8, and the probability of a cure with drug B is 0.7. Suppose that only one person is cured. Find the probability that the person treated with drug B is cured.

3. Two people with different complaints are treated with two different drugs. The probability that the person given drug x is cured is 0.7, and the probability that the person given drug y is cured is 0.6. Find the probability that

   (a) both people are cured.
   (b) only the person treated with drug x is cured.
   (c) at least one of the people is cured.
   (d) the person treated with drug x is cured, if it is known that only one person is cured.

4. The probability that milk bar A is open is 0.85, and the probability of milk bar B, which is independently owned and operated, being open is 0.9.

   (a) Find the probability that both milk bars are open.

   (b) Find the probability that only one of the milk bars is open.

5. It is known that for two particular events, A and B, we have
$$\Pr(A \cup B) = \frac{1}{2} \quad \text{and} \quad \Pr(B) = \frac{3}{2}\Pr(A).$$
   Evaluate $\Pr(A)$ if A and B are independent.

19.4 Answers for the Chapter 19 Exercises

19.1 1. (a) 0.1 (b) 0.5

2. (a) 0.25 (b) $\frac{2}{3}$

3. (a) 0.5 (b) 0.1 (c) 0.4 (d) 0.25 (e) $\frac{2}{3}$

4. (a) The probability that the adult is a female is $\frac{20\,441}{42\,053}$.
   (b) The probability that the woman is unemployed is $\frac{1\,221}{20\,441}$.

19.2 1. The probability that he falls asleep is 0.52.

2. (a) The probability that she passes the tests in both March and April is 0.35.
   (b) The probability of passing the March test and failing the April test is 0.15.
   (c) The probability of failing the March test and passing the April test is 0.25.
   (d) The probability of failing the tests in both March and April is 0.25.

3. (a) The probability that both seasons are good is $\frac{2}{5}$.
   (b) The probability that exactly one of the seasons is good is $\frac{13}{30}$.

4. (a) The probability that both balls are black is $\frac{1}{3}$.
   (b) The probability that one ball is black and the other is white is $\frac{8}{15}$.

5. The probability that he is a class A driver (given that he had an accident) is $\frac{1}{16}$.

6. (a) The probability of selecting a chocolate is $\frac{115}{216}$.
   (b) The probability that the selected chocolate came from Peter's bag is $\frac{27}{115}$.

7. (a) 68% of the production is first grade.
   (b) The probability that the article will go overseas, given that it is first grade, is $\frac{4}{17}$.

8. (a) The probability that the coin is fair, given that it shows heads, is $\frac{1}{3}$.
   (b) The probability that the coin is fair, given that it showed heads twice, is $\frac{1}{5}$.

9. (a) The probability that she is late is 0.14.
   (b) The probability that she travels by car, given that she is late, is 0.571 (3 d.p.).

19.3 1. (a) $b = \frac{1}{12}$ (b) $b = \frac{1}{9}$

2. The probability that the person treated with drug B is cured, given that only one person is cured, is $\frac{7}{19}$.

3. (a) The probability that both are cured is 0.42.
   (b) The probability that only the person treated with drug x is cured is 0.28.
   (c) The probability that at least one person is cured is 0.88.
   (d) The probability that if only one patient is cured then it is the one treated with x is $\frac{14}{23}$.

4. (a) The probability that both milk bars are open is 0.765.
   (b) The probability that only one milk bar is open is 0.22.

5. $\Pr(A) = \frac{5-\sqrt{13}}{6}$