Chapter 24

The Normal Distribution

The normal distribution is a very important example of a continuous probability distribution. Strictly speaking there are infinitely many different normal distributions, each determined by two parameters:

- μ, the mean, and
- σ^2, the variance.

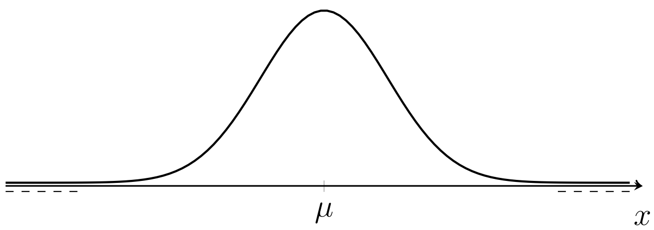
We say a random variable X is **normally distributed** with mean μ and variance σ^2 if it has the following probability density function (pdf):

$$f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} .$$

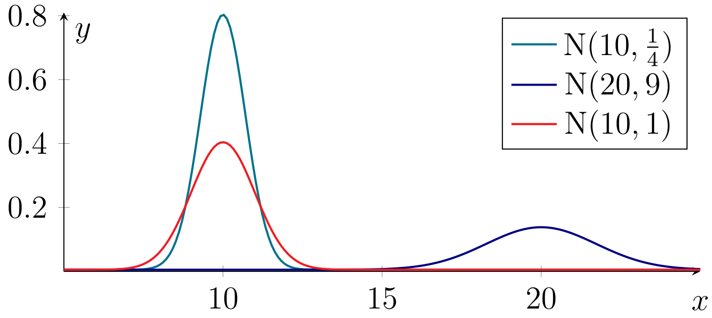
where $x \in \mathbf{R}$ and σ is the standard deviation. We write this as

$$X \sim N(\mu, \sigma^2).$$

The graph of this pdf has the following shape for any value of μ and σ:



The parameter μ gives the centre of the distribution, and σ^2 determines the 'width' of the distribution. Remember that the total area under the curve must be 1 (since f is a pdf) so as σ^2 increases, the function will spread horizontally, but decrease in height. The next figure provides some examples.



The pdf for any **normally distributed** random variable has the following features:

- it is non-zero for every value of x,
- it is symmetric about the mean μ. Thus the mean, the mode, and the median all coincide.
- 95.44% of the distribution lies within 2 standard deviations of the mean. That is,

$$\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.9544\,.$$

- 99.74% of the distribution lies within 3 standard deviations of the mean. That is,

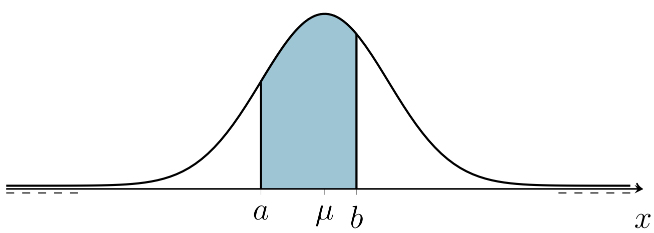
$$\Pr(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 0.9974\,.$$

24.1 Calculating Probabilities for the Normal Distribution

Recall that for a continuous random variable X

$$\Pr(a \leq X \leq b) = \int_a^b f(x)\,dx\,.$$

That is, the probability that X takes a value in the interval $[a, b]$ is equal to the area under the curve between a and b. Unfortunately, for a normally distributed random variable this is problematic, since the pdf for a normal distribution has no practical antiderivative.



$$\int_a^b f(x)\,dx \;= ?$$

To calculate probabilities for normally distributed random variables we instead need to make use of some special properties of the normal distribution. We start by focusing on the particular distribution

$$\boxed{\boldsymbol{Z \sim N(0,1)}}$$

which is known as the **standard normal distribution**.

- For the remainder of this chapter, the symbol Z will always represent a *standard* normal random variable (which has mean 0 and variance 1).

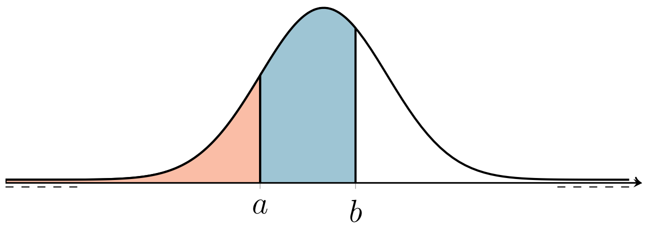
- The area under the standard normal distribution's pdf, for some intervals of the form $(-\infty, a)$, has been tabulated (see pages 10–11). These standard normal tables are also provided in the final Maths 1 exam. We can use these tables to calculate the probabilities of the form $\mathbf{Pr}(\boldsymbol{Z < a})$ simply by finding the value of a on the standard normal table.

- Recall (from the previous chapter) that, since Z is continuous, we must have

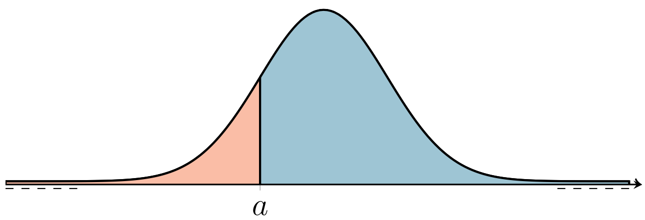
$$\mathbf{Pr}(\boldsymbol{Z < a}) \;=\; \mathbf{Pr}(\boldsymbol{Z \leq a}).$$

  Thus, knowing the probabilities for intervals of the form $(-\infty, a)$ is enough, since we can write

$$\Pr(a \leq Z \leq b) \;=\; \Pr(Z \leq b) - \Pr(Z < a)$$



  and
$$\Pr(Z \geq a) \;=\; 1 - \Pr(Z < a).$$



- It is very important to remember that the tables only give probabilities for

  standard normal random variables

  (that is, for normal random variables with a mean of 0 and variance of 1).

Example 1. Suppose that $X \sim N(0,1)$. Find

(a) $\Pr(X < -1.32)$ (b) $\Pr(X < -1.33)$ (c) $\Pr(X < -1.325)$

(d) $\Pr(X < 1.13)$ (e) $\Pr(X \le 1.13)$ (f) $\Pr(X = 1.13)$

(g) $\Pr(X > 1.13)$

Solution: Notice that X is a *standard* normal random variable
(so we can use the standard normal tables).

(a) From the tables on page 10, we see that $\Pr(X < -1.32) = 0.0934\,.$

(b) From the tables on page 10, we see that $\Pr(X < -1.33) = 0.0918\,.$

(c) Note that -1.325 is halfway between -1.32 and -1.33.
So (using our answers from (a) and (b)) it seems reasonable to expect that
we should have
$$\begin{aligned}\Pr(X < -1.325) &\approx \frac{1}{2}(0.0934 + 0.0918)\\ &= 0.0926\,.\end{aligned}$$

(d) From the tables on page 11, we see that $\Pr(X < 1.13) = 0.8708\,.$

(e) The random variable X is continuous so
$$\begin{aligned}\Pr(X \le 1.13) &= \Pr(X < 1.13)\\ &= 0.8708\,.\end{aligned}$$

(f) The random variable X is continuous so $\Pr(X = 1.13) = 0.$

(g) We have
$$\begin{aligned}\Pr(X > 1.13) &= 1 - \Pr(X \le 1.13)\\ &= 1 - 0.8708\\ &= 0.1292\,.\end{aligned}$$

$\square$

Example 2. Suppose that $X \sim N(0,1)$. Find $\Pr(-1.32 \le X \le 1.13)$.

Solution: Notice that X is a *standard* normal random variable (so we can use the standard normal tables). We have

$$\begin{aligned} \Pr(-1.32 \le X \le 1.13) &= \Pr(X \le 1.13) \; - \; \Pr(X < -1.32) \\ &= 0.8708 - 0.0934 \quad \text{(from the standard normal tables)} \\ &= 0.7774\,. \end{aligned}$$

$\square$

If $X \sim N(\mu, \sigma^2)$ is *not* a standard normal distribution (so $\mu \neq 0$ or $\sigma^2 \neq 1$), then we will need to use the following fact:

$$\boxed{\text{If } X \sim N(\mu, \sigma^2) \;\text{ then }\; \frac{X-\mu}{\sigma} \sim N(0,1).}$$

This allows us to convert non–standard normal random variables into **standard** normal random variables, so that we can use our normal tables.

- The symbol Z is commonly used to replace the fraction $\dfrac{X-\mu}{\sigma}$, which fits in with the convention (stated on page 3) that Z is used to represent a *standard* normal variable.

Example 3. If $X \sim N(2, 0.16)$ then find $\Pr(X \le 2.12)$.

Solution:

We have $\mu = 2$ and $\sigma = \sqrt{0.16} \; = \; 0.4$.

$$\begin{aligned} \text{Thus} \quad \Pr(X \le 2.12) &= \Pr\left(\frac{X-2}{0.4} \;\le\; \frac{2.12-2}{0.4}\right) \\ &= \Pr\left(Z \;\le\; 0.30\right) \quad \text{where } Z \sim N(0,1) \\ &= 0.6179 \quad \text{(from the standard normal tables).} \end{aligned}$$

$\square$

Example 4. A television station gets fined if its commercial time exceeds 20 minutes in any one hour. If the commercial time per hour is normally distributed, with mean 15 minutes and standard deviation 2 minutes, find the probability that a station is fined as a result of time devoted to commercials during a particular one hour program.

Solution:

Let X be the number of minutes of commercials shown during one hour.

We are told that $\mu = 15$ and $\sigma = 2$, and so we can write $X \sim N(15, 4)$.

We have

$$
\begin{aligned}
&\text{Pr(station fined for spending too long on commercials during that particular program)} \\
&= \text{Pr(commercial time during that particular program exceeds 20 minutes)} \\
&= \Pr(X > 20) \\
&= 1 - \Pr(X \leq 20) \\
&= 1 - \Pr\left(\frac{X - 15}{2} \leq \frac{20 - 15}{2}\right) \\
&= 1 - \Pr(Z \leq 2.50) \quad \text{where } Z \sim N(0, 1) \\
&= 1 - 0.9938 \quad \text{(obtained from the standard normal tables)} \\
&= 0.0062\,.
\end{aligned}
$$

Thus the probability that the television station gets fined for spending too long on commercials during the particular one hour program is 0.0062. □

Often we will consider *real-life applications* in which we are told that a particular variable has a normal distribution (as in Example 4, above), when strictly speaking we should say that the variable can be *approximated* by a normal random variable. In these types of examples, it is very common that the variable only takes values from some particular interval (not the entire set $\mathbf{R}$), and the pdf should correspondingly be zero for all the "impossible" values outside of that interval. Fortunately though, the probabilities that would be obtained by using the stated normal distribution for those "impossible" intervals are extremely tiny, and can be regarded as being completely insignificant, and the answers we get by "pretending" that the variable has the normal distribution are incredibly close to the true answers. This situation will be encountered again many times, without further comment (including in Example 6 of this current chapter, and in many of the homework exercises).

Example 5. A normal random variable, X, has mean 10, and standard deviation 2.

(a) Find a such that $\Pr(X \le a) = 0.9515$

(b) Find b such that $\Pr(X > b) = 0.8485$

Solution:

(a)

$$\begin{aligned} \text{First note that} \quad \Pr(X \le a) &= 0.9515 \\ \Rightarrow \quad \Pr\left(\frac{X-10}{2} \le \frac{a-10}{2}\right) &= 0.9515\,. \end{aligned}$$

We see (by looking for the probability 0.9515 within the standard normal table on page 11) that a standard normal random variable Z satisfies

$$\Pr(Z \le \mathbf{1.66}) = 0.9515\,.$$

$$\begin{aligned} \text{Thus we must have} \quad \frac{a-10}{2} &= 1.66\,. \\ \text{Then rearranging gives} \quad a-10 &= 3.32\,, \\ \text{and so we conclude that} \quad a &= 13.32\,. \end{aligned}$$

(b)

$$\begin{aligned} \Pr(X > b) &= 0.8485 \\ \Rightarrow \quad \Pr(X \le b) &= 1 - 0.8485 \\ &= 0.1515\,. \\ \text{Then} \quad \Pr\left(\frac{X-10}{2} \le \frac{b-10}{2}\right) &= 0.1515\,. \end{aligned}$$

We see (by looking for the probability 0.1515 within the standard normal table on page 10) that a standard normal random variable Z satisfies

$$\Pr(Z \le \mathbf{-1.03}) = 0.1515\,.$$

$$\begin{aligned} \text{Thus we must have} \quad \frac{b-10}{2} &= -1.03\,. \\ \text{Then rearranging gives} \quad b-10 &= -2.06\,, \\ \text{and so we conclude that} \quad b &= 7.94\,. \end{aligned}$$

□

Example 6. Suppose that the height of adults follows a normal distribution, with mean 168 cm and standard deviation 10 cm.

(a) Find the probability that a randomly chosen adult has height at least 172 cm.

(b) Find h such that exactly 30.15% of adults are of height at least h cm.

Solution: Let X = the height (in centimetres) of a randomly chosen adult.

We are told that $\mu = 168$ and $\sigma = 10$, and so we can write $X \sim N(168, 100)$.

(a)
$$\begin{aligned}
&\Pr(X \geq 172) \\
&= 1 - \Pr(X < 172) \\
&= 1 - \Pr\left(\frac{X - 168}{10} < \frac{172 - 168}{10}\right) \\
&= 1 - \Pr(Z < 0.40) \quad \text{where } Z \sim N(0,1) \\
&= 1 - 0.6554 \ \ \text{(obtained from the standard normal tables)} \\
&= 0.3446\,.
\end{aligned}$$

(b)
$$\begin{aligned}
\text{We want to find } h \text{ so that} \quad \Pr(\text{height is at least } h \text{ cm}) &= 0.3015\,. \\
\text{That is,} \quad \Pr(X \geq h) &= 0.3015\,. \\
\text{That is,} \quad 1 - \Pr(X < h) &= 0.3015\,. \\
\text{Rearranging gives} \quad \Pr(X < h) &= 0.6985\,, \\
\text{and thus} \quad \Pr\left(\frac{X - 168}{10} < \frac{h - 168}{10}\right) &= 0.6985\,.
\end{aligned}$$

Since $\dfrac{X - 168}{10} \sim N(0,1)$, we can use the standard normal tables. We see (by looking for the probability 0.6985 within the standard normal table on page 11) that a standard normal random variable Z satisfies

$$\Pr(Z < \mathbf{0.52}) = 0.6985\,.$$

$$\begin{aligned}
\text{Thus we must have} \quad \frac{h - 168}{10} &= 0.52\,. \\
\text{Then rearranging gives} \quad h - 168 &= 5.2\,, \\
\text{and so we conclude that} \quad h &= 173.2\,.
\end{aligned}$$

□

Exercises for Chapter 24

1. Suppose that the weights of packets of flour are normally distributed, with mean 1 kg, and standard deviation 50 grams. Find the probability that a randomly selected packet of flour, marked as weighing 1 kg, will actually weigh at most 900 grams.

2. The time taken for a man to travel to work varies according to a normal distribution with mean 30 minutes, and standard deviation 5 minutes. If he is due to arrive at his office by 9 AM each day, find the probability that he will **not** be late on a morning when he leaves home at 8.35 AM.

3. The resistances of heating elements are normally distributed with mean of 50 ohms, and standard deviation of 4 ohms.

   (a) Find the probability that a randomly selected element will have resistance less than 40 ohms.

   (b) If specifications require that acceptable elements shall have resistances between 45 and 55 ohms, find the probability that a randomly selected element fits these specifications.

4. Steel rods are manufactured to have a diameter of 5 cm. In fact, the diameters are normally distributed with a mean of 5 cm and a standard deviation of 0.02 cm. A steel rod is acceptable if its diameter lies between 4.958 and 5.042 cm. Find the percentage of rods made that will be unacceptable.

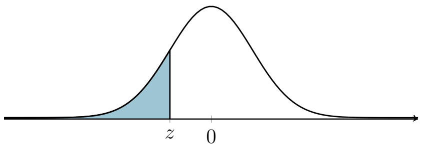
5. The life of a certain brand of car battery is known to be normally distributed with mean of 54 months, and standard deviation of 6 months. Find the value of l such that the probability of a battery only lasting l months *or less* is only 0.0495 .

6. If X is a normal random variable with

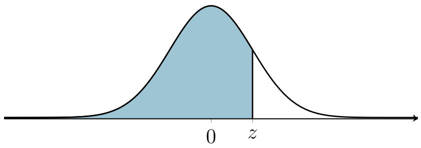
$$\Pr(X < 7.3) = 0.6217 \quad \text{and} \quad \Pr(X < 5.8) = 0.4522\,,$$

   then find the mean and standard deviation of X. Write your answers accurate to 2 decimal places.

Standard Normal Probabilities



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
–3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
–3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
–3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
–3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
–3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
–2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
–2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
–2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
–2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
–2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
–2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
–2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
–2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
–2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
–2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
–1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
–1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
–1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
–1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
–1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
–1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
–1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
–1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
–1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
–1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
–0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
–0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
–0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
–0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
–0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
–0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
–0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
–0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
–0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
–0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

24.2 Answers for the Chapter 24 Exercises

1. The probability that the packet of flour weighs at most 900 grams is 0.0228 .
2. The probability that the man will *not* be late is 0.1587 .
3. (a) The probability that the resistance will be less than 40 ohms is 0.0062 .
   (b) The probability that a heating element will fit the specifications is 0.7888 .
4. 3.58% of the rods made will be unacceptable.
5. The required value of l is 44.1 .
6. We have $\mu = 6.22$ (2 d.p.) and $\sigma = 3.49$ (2 d.p.).