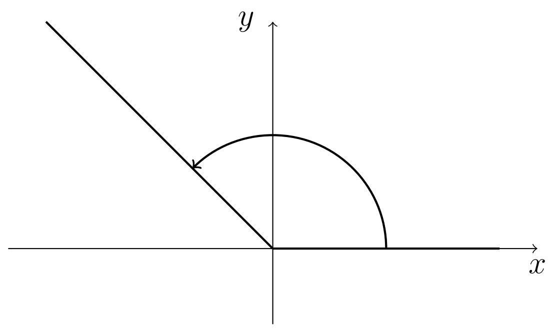
Chapter 2

Trigonometry Review

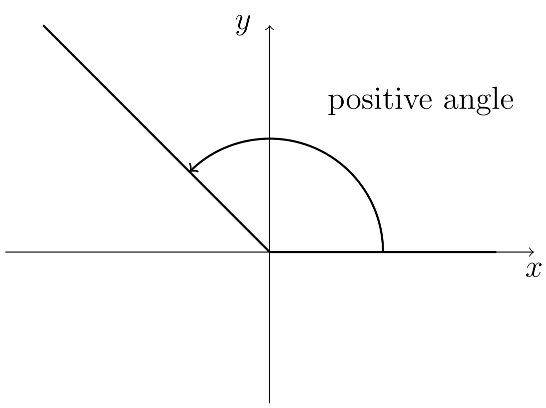
Reference: "Calculus", by James Stewart.

Conventions for measuring angles:

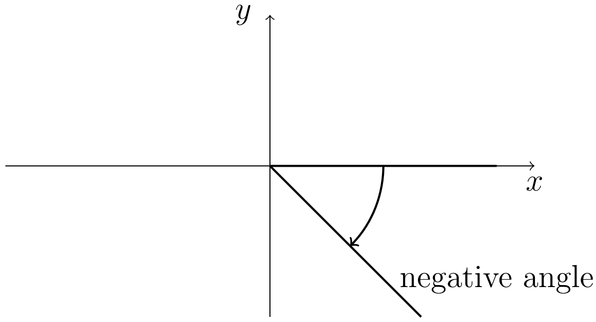
- Angles are usually measured from the positive x –axis.



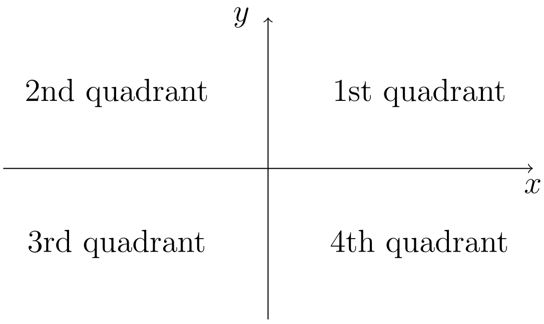
- Angles measured anticlockwise are positive.



- Angles measured clockwise are negative.

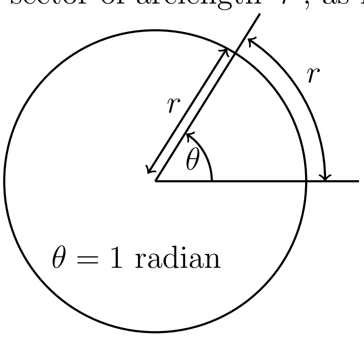


- The Cartesian Plane is divided into 4 quadrants.



2.1 Radians

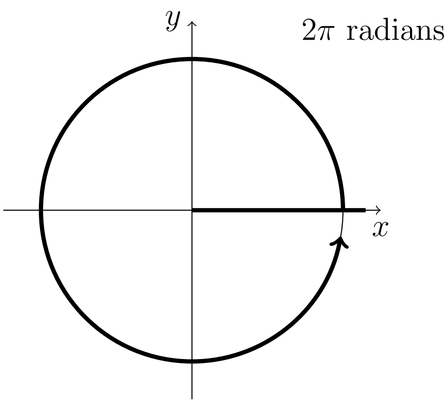
Consider a circle of radius r. A **radian** is defined to be the angle within the sector of arclength r, as illustrated in the diagram given below

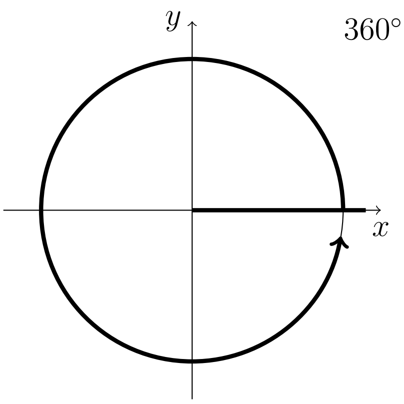


1 radian $\rightarrow$ arclength r units

2π radians $\rightarrow$ arclength $2\pi r$ units

The angle corresponding to a complete revolution is 2π radians.



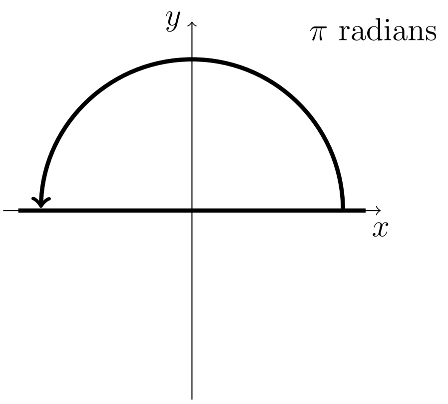


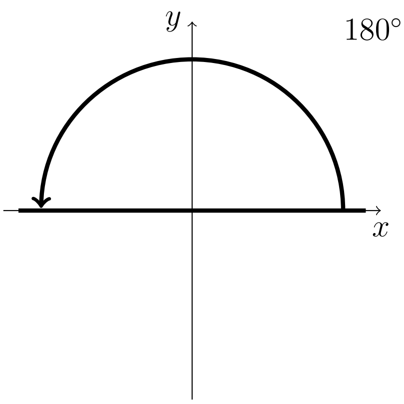
That is,

$$2\pi \text{ radians} = 360°,$$

and so

$$\boxed{\pi \text{ radians} = 180°}$$





Example 1. Convert the following angles from degrees to radians:

(a) $45°$ (b) $300°$

Solution: First note that $180° = \pi$ radians $\Longrightarrow$ $1° = \frac{\pi}{180}$ radians .

(a) Multiplying both sides by 45 gives

$$\begin{aligned} 45 \times 1° &= 45 \times \frac{\pi}{180} \text{ radians} \\ \therefore 45° &= \frac{\pi}{4} \text{ radians.} \end{aligned}$$

(b) Multiplying both sides by 300 gives

$$\begin{aligned} 300 \times 1° &= 300 \times \frac{\pi}{180} \text{ radians} \\ \therefore 300° &= \frac{5\pi}{3} \text{ radians.} \end{aligned}$$

□

Example 2. Convert the following angles from radians to degrees:

(a) $\dfrac{3\pi}{2}$ radians (b) $\dfrac{7\pi}{4}$ radians

Solution: (a)

$$\pi \text{ radians} = 180° \Longrightarrow \frac{3\pi}{2}\text{radians} = \frac{3}{2} \times 180° = 270°.$$

(b)

$$\pi \text{ radians} = 180° \Longrightarrow \frac{7\pi}{4}\text{radians} = \frac{7}{4} \times 180° = 315°.$$

□

Warning!

Whilst it is true that

- the **angle** π radians is equal to the **angle** $180°$,

you should keep in mind that

- the **number** π is **NOT** equal to the **number** 180.

(Instead we have $\pi = 3.14159 \ldots \neq 180$.)

Exercises

1. Convert the following angles from degrees to radians:

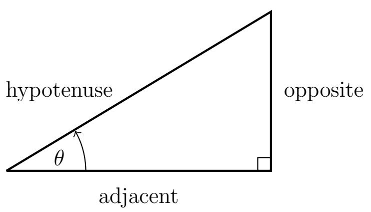
   (a) $30°$ (b) $90°$ (c) $120°$ (d) $135°$ (e) $270°$ (f) $360°$

2. Convert the following angles from radians to degrees:

   (a) $\frac{\pi}{4}$ radians (b) $\frac{\pi}{3}$ radians (c) $\frac{5\pi}{6}$ radians (d) π radians

2.2 The Trigonometric Functions

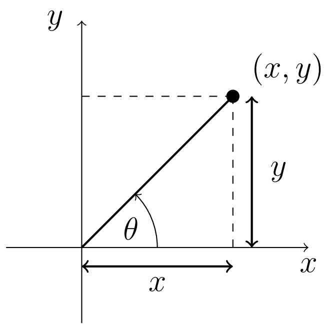
Perhaps you are familiar with the following use of a right-angled triangle in the definitions of the trigonometric functions:



$$\left.\begin{aligned} \sin\theta &= \frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}} \\ \cos\theta &= \frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}} \\ \tan\theta &= \frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}} \end{aligned}\right\} \quad \textbf{“SohCahToa”}$$

Note that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

It turns out to be very useful to introduce **coordinates**, and to define the trigonometric functions as follows:



First we define $r = \sqrt{x^2 + y^2}$.

Then we define

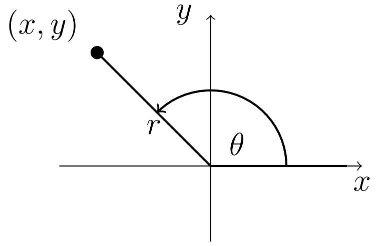
$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} \quad \text{for } x \neq 0.$$

As before, we have $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (as long as $\cos\theta \neq 0$).

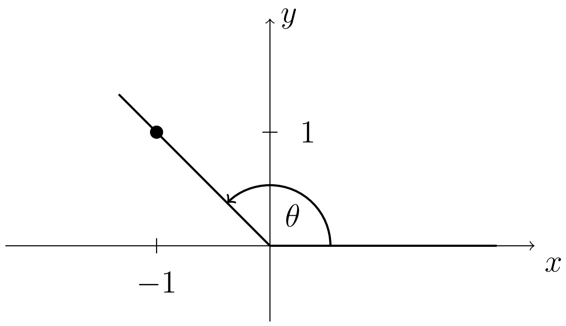
The advantage of these **"coordinate definitions"** is that the same formulae are used to compute $\sin\theta$, $\cos\theta$ and $\tan\theta$ **for any angle** θ (rather than only for angles between $0°$ and $90°$).



Example 3. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$, for an angle θ corresponding to the point $(x, y) = (-1, 1)$.

Solution: For the point $(-1, 1)$ we have

$$x = -1,\ y = 1 \text{ and } r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2}.$$



Now

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \qquad \cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}} \qquad \tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$$

□

Thus we see that the trigonometric functions sometimes give *negative* results. The following diagrams are useful for remembering the signs of the results of trigonometric calculations.

2nd quadrant	1st quadrant
$y, r > 0$ $x < 0$	$x, y, r > 0$
$r > 0$ $x, y < 0$	$y < 0$ $x, r > 0$
3rd quadrant	4th quadrant

S	**A**
$\sin\theta > 0$ $\cos\theta, \tan\theta < 0$	all ratios > 0
$\tan\theta > 0$ $\sin\theta, \cos\theta < 0$	$\cos\theta > 0$ $\sin\theta, \tan\theta < 0$
T	**C**

So far we have defined the trigonometric functions sin, cos and tan. There are three other trigonometric functions which we need to know about, and we define these next.

$$\begin{aligned}
\operatorname{cosec}\theta &= \frac{1}{\sin\theta} \quad \text{whenever } \sin\theta \neq 0 \\
\sec\theta &= \frac{1}{\cos\theta} \quad \text{whenever } \cos\theta \neq 0 \\
\cot\theta &= \frac{\cos\theta}{\sin\theta} \quad \text{whenever } \sin\theta \neq 0
\end{aligned}$$

Note that $\cot\theta = \dfrac{1}{\tan\theta}$ if $\sin\theta \neq 0$ and $\cos\theta \neq 0$.

Exercises

For each of the angles θ considered below, find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$.

(a) Consider the angle θ corresponding to the point $(1, \sqrt{3})$.

(b) Consider the angle θ corresponding to the point $(\sqrt{3}, 1)$.

(c) Consider the angle θ corresponding to the point $(1, 1)$.

(d) Consider the angle θ corresponding to the point $(0, 1)$.

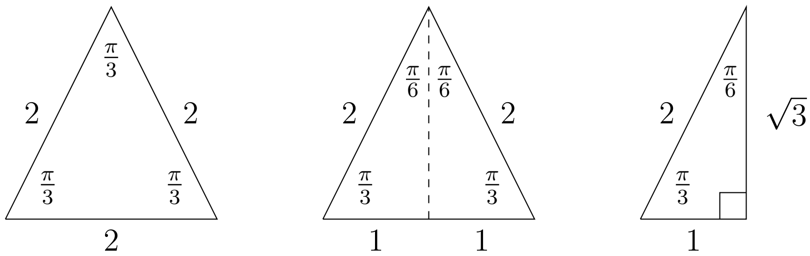
(e) Consider the angle θ corresponding to the point $(-1, 0)$.

(f) Consider the angle θ corresponding to the point $(-1, \sqrt{3})$.

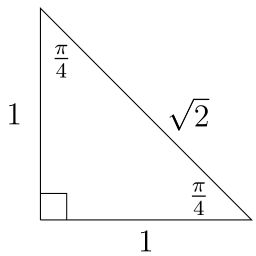
Evaluating Trigonometric Functions Exactly

For certain angles θ it is possible to calculate *exact* values for $\sin\theta$, $\cos\theta$ and $\tan\theta$, by using triangles as shown below.

- $\theta = \dfrac{\pi}{6}$ or $\dfrac{\pi}{3}$ (that is, $\theta =$ 30° or 60°):



- $\theta = \dfrac{\pi}{4}$ (that is, $\theta = 45°$):

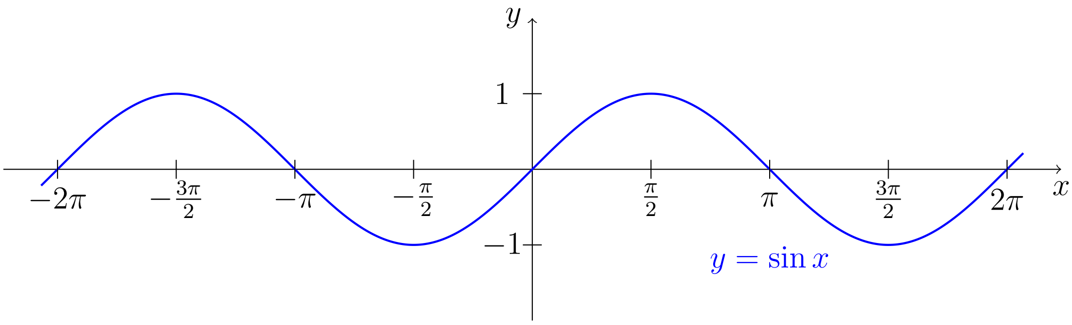


θ	$\frac{\pi}{6}$, i.e. 30°	$\frac{\pi}{4}$, i.e. 45°	$\frac{\pi}{3}$, i.e. 60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

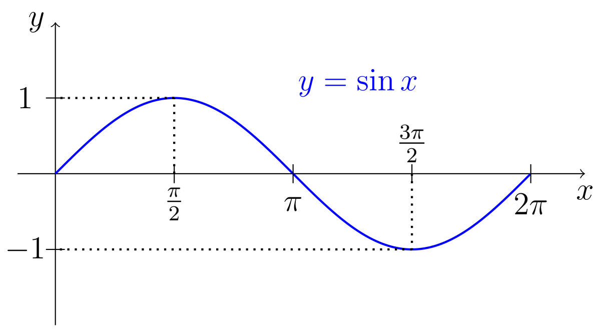
These triangles, and the table of exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ are given on the formula sheet provided in the Mathematics 1 exams.

Graphs for the Basic Trigonometric Functions

Consider the graph for $y = \sin x$:



You should learn that the graph for $y = \sin x$ for $0 \leq x \leq 2\pi$ looks like:

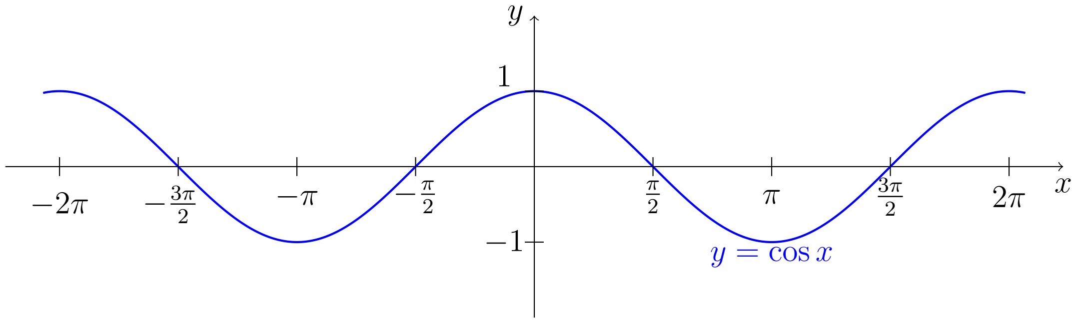


and that this graph repeats itself every 2π . We say that $y = \sin x$ has **period** 2π .

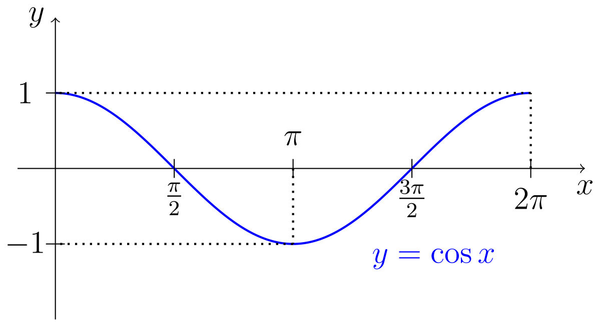
Note that from this graph we can easily see that $\sin x$ takes the following exact values:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0

The graph for $y = \cos x$ is given next:



You should learn that the graph for $y = \cos x$ for $0 \leq x \leq 2\pi$ looks like:

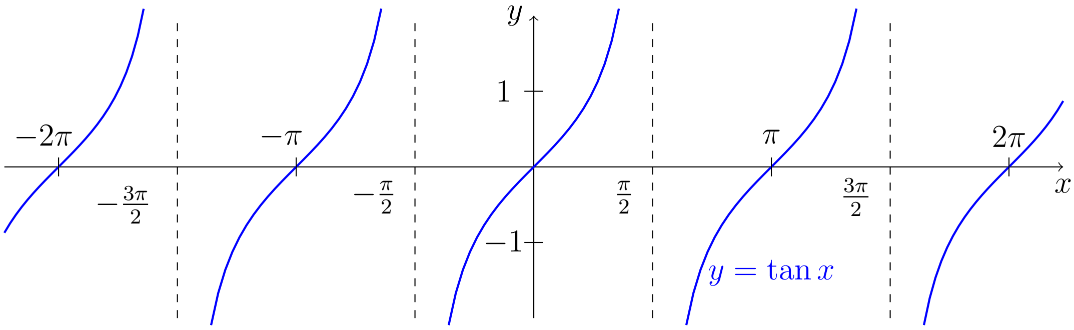


and that this graph repeats itself every 2π . We say that $y = \cos x$ has **period** 2π .

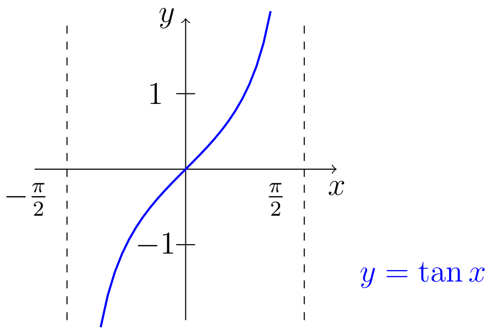
Note that from this graph we can easily see that $\cos x$ takes the following exact values:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	-1	0	1

Finally, we give the graph for $y = \tan x$:



You should learn that the graph for $y = \tan x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ looks like:



and that this graph repeats itself every π . We say that $y = \tan x$ has **period** π .

Recall that $\tan x$ is not defined when $\cos x = 0$ (for example, when $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$)

The vertical lines sketched at such values of x are called **asymptotes**.

Note that, just as in the section on inequalities, knowing the graphs will help us to solve inequalities.

2.3 Degrees versus Radians

So far we have considered the trigonometric expressions $\sin\theta$, $\cos\theta$ and $\tan\theta$ (as well as $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$) where θ is an **angle**.

Note that, by convention, we assume that the angle θ is **measured in radians** unless the symbol $^\circ$ also appears in the expression.

In fact, we will also want to define trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ (as well as $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$), where θ **is just a number**. This is done as follows:

- For any **number** θ, we define $\sin\theta$ to be the sine of the angle θ measured in **radians**.

Similarly we define the other trigonometric functions so that the number θ corresponds to the angle θ **measured in radians**.

Note: By defining trigonometric functions in terms of **radians** (rather than degrees) we end up with much simpler formulae when we study calculus later in the year.

Exercises

1. Using your calculator find the value (to 4 decimal places) of the following:

   (a) $\cos 4.86$ (b) $\sin 5.78$ (c) $\tan 49^\circ$ (d) $\cot 3.64$ (e) $\cos 5.316$

2. The depth of water in a particular part of a bay varies with time according to the following formula:

$$D = 20 + 3\sin\left(\frac{\pi}{12}t\right)$$

   where

   - D is the depth measured in metres, and
   - t is the number of hours after midnight, with $0 \le t \le 24$.

   (a) Find the depth when $t = 0$.

   (b) Find the depth at 1 am. Write your answer to 2 decimal places.

   (c) Find the depth at 2 am.

   (d) Find the maximum depth.

   (e) Find the minimum depth.

2.4 Trigonometric Identities

It can easily be shown that for all angles θ we have

$$\boxed{\sin^2\theta + \cos^2\theta \ = \ 1.}$$

From this it follows that

$$\tan^2\theta + 1 = \sec^2\theta.$$

Example 4. If $\sin\theta = \dfrac{3}{5}$ and $\dfrac{\pi}{2} < \theta < \pi$ find the value of $\cos\theta$ and $\tan\theta$.

Solution:

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\implies \left(\frac{3}{5}\right)^2 + \cos^2\theta &= 1 \\
\implies \cos^2\theta &= 1 - \frac{9}{25} \\
&= \frac{16}{25} \\
\implies \cos\theta &= \pm\sqrt{\frac{16}{25}} \\
&= \pm\frac{4}{5}
\end{aligned}$$

Since θ is in the second quadrant, we know that $\cos\theta < 0$ and so

$$\cos\theta = -\frac{4}{5}\,.$$

$\square$

Other useful identities are the **addition formulae** and the **subtraction formulae**, which are given next. These formulae appear on the formula sheet provided with the Maths 1 exams.

$$\begin{aligned}
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \quad \text{and} \\
\tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \quad \text{for all } x, y \text{ such that } 1 \mp \tan x \tan y \neq 0.
\end{aligned}$$

Putting $x = y$ into the addition formulae gives the so-called **double–angle formulae**, which also appear on the formula sheet given in the Maths 1 exams:

$$\begin{aligned}
\sin 2x &= 2\sin x \cos x, \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x \quad \text{and} \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \quad \text{for all } x \text{ such that } 1 - \tan^2 x \neq 0.
\end{aligned}$$

Example 5. Prove that $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} = \tan\theta$.

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} \\
&= \frac{2\sin\theta\cos\theta + \sin\theta}{2\cos^2\theta - 1 + \cos\theta + 1} \\
&= \frac{\sin\theta\,(2\cos\theta + 1)}{\cos\theta\,(2\cos\theta + 1)} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta \\
&= \text{RHS, as required.}
\end{aligned}$$

□

Note: When proving a trig. identity, we assume that we are proving the identity for all values of θ for which **both** sides of the equality sign are defined. In particular, in the above example, we are proving the identity for all values of θ for which $\cos\theta \neq 0$ and $\cos\theta \neq -\frac{1}{2}$.

Strategy for proving identities:

- Start with the more complicated side of the identity.
- Use the trig identities listed in this section (as well as expanding and/or factorising techniques) to make the more complicated side look more like the simpler side.

Example 6. Express $\sin\left(\dfrac{3\pi}{2} - \beta\right)$ and $\cos\left(\dfrac{3\pi}{2} - \beta\right)$ in terms of $\sin\beta$ and/or $\cos\beta$.

Solution: The subtraction formulae on page 13 give:

$$\begin{aligned}
\sin\left(\frac{3}{2}\pi - \beta\right) &= \sin\left(\frac{3}{2}\pi\right)\cos\beta \ - \ \cos\left(\frac{3}{2}\pi\right)\sin\beta \\
&= -1 \times \cos\beta \ - \ 0 \times \sin\beta \\
&= -\cos\beta
\end{aligned}$$

and

$$\begin{aligned}
\cos\left(\frac{3}{2}\pi - \beta\right) &= \cos\left(\frac{3}{2}\pi\right)\cos\beta \ + \ \sin\left(\frac{3}{2}\pi\right)\sin\beta \\
&= 0 \times \cos\beta \ + \ -1 \times \sin\beta \\
&= -\sin\beta\,.
\end{aligned}$$

$\square$

Similarly we can show that

$$\begin{aligned}
\sin(\pi - \beta) &= + \sin\beta \\
\cos(\pi - \beta) &= - \cos\beta \\
\tan(\pi - \beta) &= - \tan\beta \\[1em]
\sin(\pi + \beta) &= - \sin\beta \\
\cos(\pi + \beta) &= - \cos\beta \\
\tan(\pi + \beta) &= + \tan\beta \\[1em]
\sin(2\pi - \beta) &= - \sin\beta \\
\cos(2\pi - \beta) &= + \cos\beta \\
\text{and} \quad \tan(2\pi - \beta) &= - \tan\beta.
\end{aligned}$$

Exercises

1. Express the following in terms of $\sin\theta,\ \cos\theta,\ \tan\theta,\ \operatorname{cosec}\theta,\ \sec\theta$ or $\cot\theta$:

   (a) $\sin(2\pi-\theta)$ (b) $\cos(2\pi-\theta)$ (c) $\sin\left(\frac{\pi}{2}-\theta\right)$ (d) $\cos\left(\frac{\pi}{2}-\theta\right)$

   (e) $\sec\left(\frac{\pi}{2}-\theta\right)$ (f) $\tan(\pi-\theta)$ (g) $\cos\left(\frac{\pi}{2}+\theta\right)$ (h) $\sin\left(\frac{\pi}{2}+\theta\right)$

2. Prove the following identities:

   a) $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

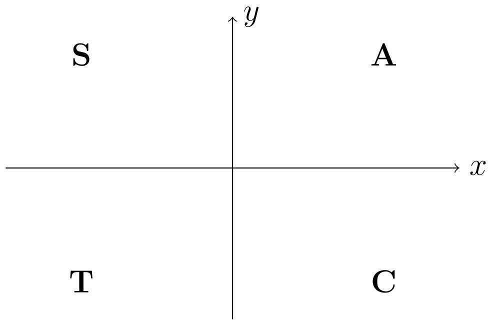
   b) $\sin^2\theta\cos^2\beta-\cos^2\theta\sin^2\beta=\sin^2\theta-\sin^2\beta$

   c) $(1-\tan\theta)^2+(1+\tan\theta)^2=2\sec^2\theta$

   d) $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

2.5 Evaluating Trigonometric Expressions in the 2nd, 3rd or 4th Quadrants

We can use the "ASTC" diagram below (from page 7)



to help us to easily evaluate trigonometric expressions outside the first quadrant.

The key is to express our 2nd, 3rd or 4th quadrant angles in the form

$$\pi - \beta, \quad \pi + \beta \quad \text{or} \quad 2\pi - \beta$$

where β is a **first** quadrant angle. In particular,

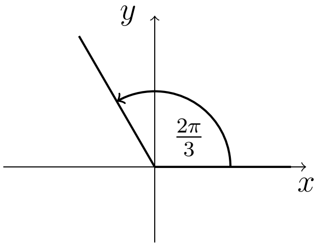
- we write a 2nd quadrant angle in the form $\pi - \beta$,
- we write a 3rd quadrant angle in the form $\pi + \beta$, and
- we write a 4th quadrant angle in the form $2\pi - \beta$.

We then leave out the " $\pi-$ ", " $\pi+$ " or " $2\pi-$ " parts of the expression, and decide whether the final answer is positive or negative by referring to the "ASTC" diagram.

Note: The above method works because of the formulae listed on page 15.

Example 7. Give exact values for $\sin\left(\frac{2\pi}{3}\right)$, $\cos\left(\frac{2\pi}{3}\right)$ and $\tan\left(\frac{2\pi}{3}\right)$.

Solution: First note that $\frac{2\pi}{3}$ is in the second quadrant, so we replace $\frac{2\pi}{3}$ with $\pi - \frac{\pi}{3}$.



$$\begin{aligned}
\sin\left(\frac{2\pi}{3}\right) &= \sin\left(\pi - \frac{\pi}{3}\right) \\
&= \sin\left(\frac{\pi}{3}\right) \quad \text{(using a formula given on page 15)} \\
&= \frac{\sqrt{3}}{2} \\
\cos\left(\frac{2\pi}{3}\right) &= \cos\left(\pi - \frac{\pi}{3}\right) \\
&= -\cos\left(\frac{\pi}{3}\right) \\
&= -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\tan\left(\frac{2\pi}{3}\right) &= \tan\left(\pi - \frac{\pi}{3}\right) \\
&= -\tan\left(\frac{\pi}{3}\right) \\
&= -\sqrt{3}
\end{aligned}$$

$\square$

Exercises

1. For each of the following angles θ, find exact values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ *without the use of a calculator.*

   (a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) $\frac{5\pi}{4}$ (e) $\frac{4\pi}{3}$ (f) $\frac{5\pi}{3}$ (g) $\frac{7\pi}{4}$ (h) $\frac{11\pi}{6}$

2. Without using a calculator, find the exact values of the expressions given below:

   (a) $\cot\frac{7\pi}{4}$ (b) $\cos\frac{11\pi}{6}$ (c) $\sin\frac{5\pi}{3}$ (d) $\operatorname{cosec}\frac{5\pi}{3}$ (e) $\sin 2\pi$

   (f) $\sin\frac{13\pi}{6}$ (g) $\tan\frac{9\pi}{4}$ (h) $\cot\frac{7\pi}{3}$ (i) $\cos\frac{13\pi}{6}$ (j) $\cos\frac{5\pi}{2}$

2.6 Trigonometric Equations and Inequalities

Case 1: Solve $\sin\theta = b$, where b is positive and $0 \le \theta \le 2\pi$.

Step 1: Find the first quadrant angle β which satisfies $\sin\beta = b$.
(We call β the **basic angle**.)

Step 2: Since we want $\sin\theta$ to be positive, we choose solutions

$$\theta = \underbrace{\beta}_{\text{1st quadrant}} \quad \text{and} \quad \theta = \underbrace{\pi - \beta.}_{\text{2nd quadrant}}$$

Example 8. Solve $\sin\theta \;=\; \dfrac{\sqrt{3}}{2}$ for $0 \le \theta \le 2\pi$.

Solution: Basic angle: $\dfrac{\pi}{3}$ (from the formula sheet)

We want $\sin\theta$ to be **positive** (since $\frac{\sqrt{3}}{2}$ is a positive number).

Thus we choose θ in the 1st and 2nd quadrants.

That is, we put

$$\theta \;=\; \frac{\pi}{3}\,,\;\; \pi - \frac{\pi}{3}$$

That is,

$$\theta \;=\; \frac{\pi}{3}\,,\;\; \frac{2\pi}{3}$$

(Note that these two answers are both between 0 and 2π .) □

Case 2: Solve $\sin\theta = -b$, where b is positive and $0 \le \theta \le 2\pi$.

Step 1: Find the first quadrant angle β which satisfies $\sin\beta = +b$.
(We call β the **basic angle**.)

Step 2: Since we want $\sin\theta$ to be negative, we choose solutions

$$\theta = \underbrace{\pi + \beta}_{\text{3rd quadrant}} \qquad \text{and} \qquad \theta = \underbrace{2\pi - \beta.}_{\text{4th quadrant}}$$

Example 9. Solve $\sin\theta \;=\; -\dfrac{\sqrt{3}}{2}$ for $0 \le \theta \le 2\pi$.

Solution: The basic angle β satisfies

$$\sin\beta = \frac{\sqrt{3}}{2} \qquad \text{(ignore minus sign when finding a basic angle)}$$
$$\beta = \frac{\pi}{3}.$$

Since $\sin\theta$ is negative, the solutions are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \qquad \text{and} \qquad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$\square$

Case 3: Solve $\cos\theta = b$, where b is positive and $0 \le \theta \le 2\pi$.

Step 1: Find the first quadrant angle β which satisfies $\cos\beta = b$. (We call β the **basic angle**.)

Step 2: Since we want $\cos\theta$ to be positive, we choose solutions

$$\theta = \underbrace{\beta}_{\text{1st quadrant}} \qquad \text{and} \qquad \theta = \underbrace{2\pi - \beta.}_{\text{4th quadrant}}$$

Case 4: Solve $\cos\theta = -b$, where b is positive and $0 \le \theta \le 2\pi$.

Step 1: Find the first quadrant angle β which satisfies $\cos\beta = +b$. (We call β the **basic angle**.)

Step 2: Since we want $\cos\theta$ to be negative, we choose solutions

$$\theta = \underbrace{\pi - \beta}_{\text{2nd quadrant}} \qquad \text{and} \qquad \theta = \underbrace{\pi + \beta.}_{\text{3rd quadrant}}$$

Case 5: Solve $\tan\theta = b$, where b is positive and $0 \le \theta \le 2\pi$.

Step 1: Find the first quadrant angle β which satisfies $\tan\beta = b$. (We call β the **basic angle**.)

Step 2: Since we want $\tan\theta$ to be positive, we choose solutions

$$\theta = \underbrace{\beta}_{\text{1st quadrant}} \qquad \text{and} \qquad \theta = \underbrace{\pi + \beta.}_{\text{3rd quadrant}}$$

Case 6: Solve $\tan\theta = -b$, where b is positive and $0 \le \theta \le 2\pi$.

Step 1: Find the first quadrant angle β which satisfies $\tan\beta = +b$.
(We call β the **basic angle**.)

Step 2: Since we want $\tan\theta$ to be negative, we choose solutions

$$\theta = \underbrace{\pi - \beta}_{\text{2nd quadrant}} \quad \text{and} \quad \theta = \underbrace{2\pi - \beta.}_{\text{4th quadrant}}$$

Note:

- To obtain any bigger answers, we just need to **add** suitable multiples of 2π to the two values obtained in Cases 1–6 on pages 19–22.

Similarly,

- to obtain smaller answers, we just need to **subtract** suitable multiples of 2π from the two values obtained in Cases 1–6 on pages 19–22.

Example 10. Solve $\sin\theta = \dfrac{\sqrt{3}}{2}$ for $-4\pi \le \theta \le 3\pi$.

Solution: As in Example 8 we obtain the two solutions

$$\theta = \frac{\pi}{3}, \ \frac{2\pi}{3}.$$

(These two values both lie between -4π and 3π.)

To find any other solutions we add/subtract multiples of 2π to those solutions already found, and check whether these new answers lie in the given interval for θ.

In particular, we obtain

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \quad \text{and} \quad \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3},$$

which **are** in the given interval; i.e. these values **are** between -4π and 3π.

Adding 2π again gives

$$\frac{7\pi}{3} + 2\pi = \frac{13\pi}{3} \qquad \text{and} \qquad \frac{8\pi}{3} + 2\pi = \frac{14\pi}{3} \ ,$$

which are **not** in the given interval; i.e. these values are **not** between -4π and 3π .

Note that since the values just obtained are already too big, there is no point in adding further multiples of 2π .

Similarly, if we subtract 2π from the original two solutions we obtain

$$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3} \qquad \text{and} \qquad \frac{2\pi}{3} - 2\pi = -\frac{4\pi}{3} \ ,$$

which **are** in the given interval; i.e. these values **are** between -4π and 3π .

Subtracting 2π again gives

$$-\frac{5\pi}{3} - 2\pi = -\frac{11\pi}{3} \qquad \text{and} \qquad -\frac{4\pi}{3} - 2\pi = -\frac{10\pi}{3} \ ,$$

which **are still** in the given interval; i.e. these values **are still** between -4π and 3π .

However, subtracting 2π again gives

$$-\frac{11\pi}{3} - 2\pi = -\frac{17\pi}{3} \qquad \text{and} \qquad -\frac{10\pi}{3} - 2\pi = -\frac{16\pi}{3} \ ,$$

which are **not** in the given interval; i.e. these values are **not** between -4π and 3π .

Thus the solutions to the given equation are

$$\theta = -\frac{11\pi}{3}, -\frac{10\pi}{3}, -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3} \, .$$

□

Example 11. Solve $\tan\left(2\theta + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{3}}$ for $0 \le \theta \le \pi$.

Solution: First we simplify the problem by replacing $2\theta + \dfrac{\pi}{4}$ with a new variable.
For example, let $x = 2\theta + \dfrac{\pi}{4}$. Note that

- if $\theta = 0$ then $x = 2 \times 0 + \dfrac{\pi}{4} = \dfrac{\pi}{4}$, and
- if $\theta = \pi$ then $x = 2 \times \pi + \dfrac{\pi}{4} = \dfrac{9\pi}{4}$.

Thus we can rewrite the problem as follows:

$$\text{Solve} \quad \tan x = \frac{1}{\sqrt{3}} \quad \text{for} \quad \frac{\pi}{4} \le x \le \frac{9\pi}{4} .$$

The basic angle β satisfies $\tan\beta = \frac{1}{\sqrt{3}}$. This

$$x = \frac{\pi}{6} \text{ (too small) or } x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Then add 2π :

$$x = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6} \text{ or } x = 2\pi + \pi + \frac{\pi}{6} = 3\pi + \frac{\pi}{6} \text{ (too big)} .$$

Now solve for θ :

$$\begin{aligned} x &= \frac{7\pi}{6}, \frac{13\pi}{6} \\ \therefore 2\theta + \frac{\pi}{4} &= \frac{7\pi}{6}, \frac{13\pi}{6} \\ \therefore 2\theta &= \frac{11\pi}{12}, \frac{23\pi}{12} \\ \therefore \theta &= \frac{11\pi}{24}, \frac{23\pi}{24} \end{aligned}$$

□

The last two examples in this chapter concern solving trigonometric **inequalities**. We can solve trigonometric inequalities by

- solving the corresponding **equation** first, and then
- seeing the solution to the inequality by looking at an appropriate graph.

Example 12. Solve $\cos x \geq \dfrac{1}{2}$ for $x \in [0, 2\pi]$.

Solution: **First we solve the equation** $\cos x = \dfrac{1}{2}$ **.**

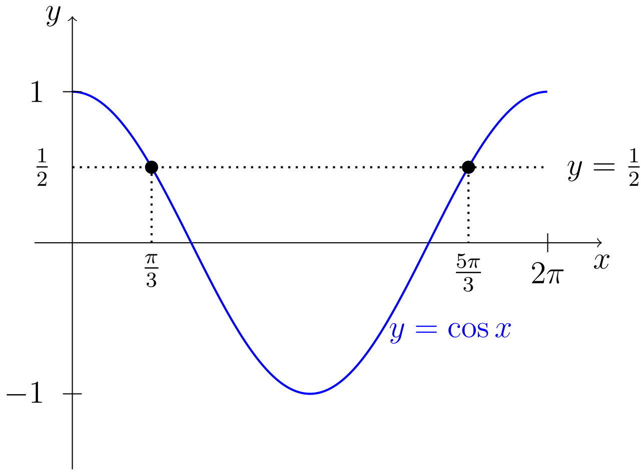
The basic angle is $\dfrac{\pi}{3}$.

Since we want $\cos x$ to be positive, we choose x in the first and fourth quadrants. That is, we choose

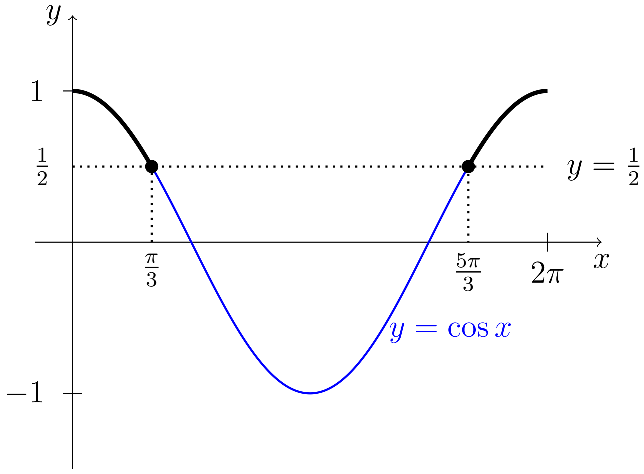
$$x = \frac{\pi}{3} \quad \text{or} \quad 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \ .$$

Note that adding or subtracting 2π to these solutions gives answers which are too big or too small respectively. Thus the only solutions to the equation $\cos x = \frac{1}{2}$ are the two values obtained above.

Next we use a graph of $y = \cos x$ and $y = \dfrac{1}{2}$, labelling the solutions to the equation $\cos x = \dfrac{1}{2}$.



Since we want $\cos x$ to be **greater than or equal to** $\dfrac{1}{2}$, we want the pieces of the $\cos x$ graph which are **above or on** the line $y = \dfrac{1}{2}$.



Thus we see that the solution set to the inequality $\cos x \geq \dfrac{1}{2}$ (with $x \in [0, 2\pi]$) is

$$\left[0\ ,\ \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}\ ,\ 2\pi\right].$$

$\square$

Example 13. Solve $\sin\left(2\theta + \frac{\pi}{4}\right) < \frac{\sqrt{3}}{2}$ for $0 \le \theta \le \frac{3\pi}{2}$.

Solution: First we simplify the problem by replacing $2\theta + \frac{\pi}{4}$ with a new variable.
For example, let $x = 2\theta + \frac{\pi}{4}$.

Note that

- if $\theta = 0$ then $x = 2 \times 0 + \frac{\pi}{4} = \frac{\pi}{4}$, and
- if $\theta = \frac{3\pi}{2}$ then $x = 2 \times \frac{3\pi}{2} + \frac{\pi}{4} = \frac{13\pi}{4}$.

Thus we can rewrite the original problem as follows:

$$\text{Solve } \sin x < \frac{\sqrt{3}}{2} \quad \text{for} \quad \frac{\pi}{4} \le x \le \frac{13\pi}{4} .$$

To solve this simplified inequality, we first should concentrate on the corresponding **equation**. That is, we consider the following problem:

$$\text{Solve } \sin x = \frac{\sqrt{3}}{2} \quad \text{for} \quad x \in \left[\frac{\pi}{4}, \frac{13\pi}{4}\right] .$$

The basic angle is $\frac{\pi}{3}$.
Since we want $\sin x$ to be positive, we choose x in the first and second quadrants. That is, we choose

$$x = \frac{\pi}{3} \quad \text{or} \quad \pi - \frac{\pi}{3} = \frac{2\pi}{3} .$$

Adding 2π to these two solutions gives two more solutions, namely

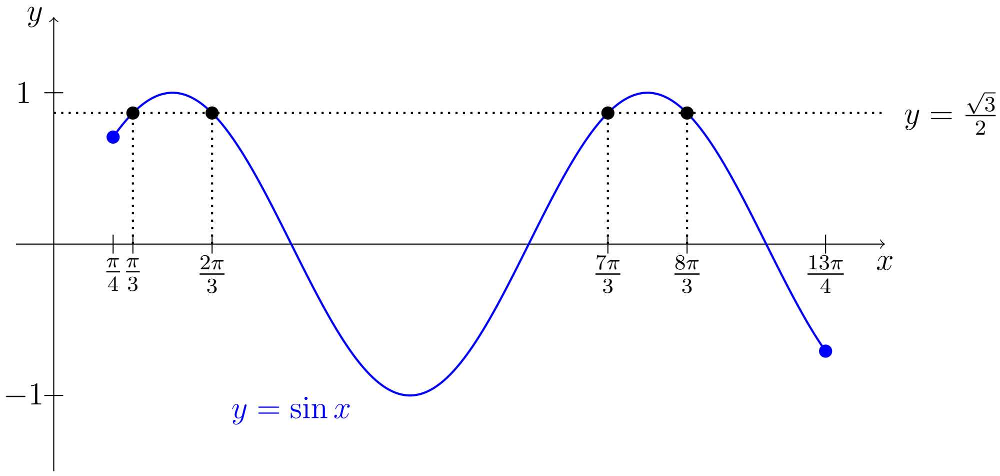
$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \quad \text{and} \quad \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3} .$$

Note that subtracting or adding further multiples of 2π gives us answers which are either too small or too big.

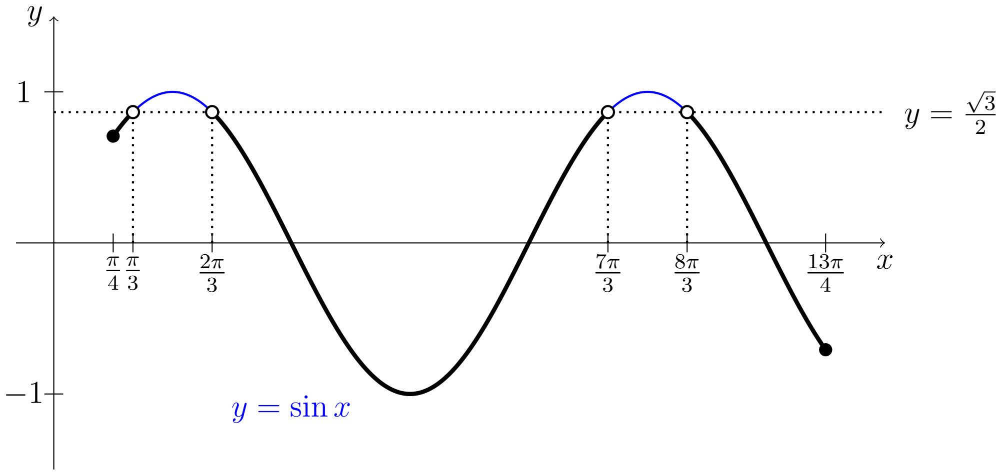
Thus the only solutions to the equation $\sin x = \frac{\sqrt{3}}{2}$ for $x \in \left[\frac{\pi}{4}, \frac{13\pi}{4}\right]$ are

$$x = \frac{\pi}{3} , \frac{2\pi}{3} , \frac{7\pi}{3} \text{ and } \frac{8\pi}{3} .$$

To solve the **inequality** $\sin x < \dfrac{\sqrt{3}}{2}$ we will sketch a graph of $y = \sin x$ and $y = \dfrac{\sqrt{3}}{2}$ $\left(\text{with } \dfrac{\pi}{4} \leq x \leq \dfrac{13\pi}{4}\right)$, labelling the solutions to the equation $\sin x = \dfrac{\sqrt{3}}{2}$.



Since we want $\sin x$ to be **less than** $\dfrac{\sqrt{3}}{2}$ we want the pieces of the $\sin x$ graph which are **strictly below** the line $y = \dfrac{\sqrt{3}}{2}$.



Thus we see that the solution set to the inequality

$$\sin x \ < \ \frac{\sqrt{3}}{2} \quad \left(\text{with } \ \frac{\pi}{4} \le x \le \frac{13\pi}{4}\right) \quad \text{is}$$

$$x \ \in \ \left[\frac{\pi}{4} \ , \ \frac{\pi}{3}\right) \ \cup \ \left(\frac{2\pi}{3} \ , \ \frac{7\pi}{3}\right) \ \cup \ \left(\frac{8\pi}{3} \ , \ \frac{13\pi}{4}\right] .$$

Since $\ x = 2\theta + \dfrac{\pi}{4}$, we have

$$2\theta + \frac{\pi}{4} \ \in \ \left[\frac{\pi}{4} \ , \ \frac{\pi}{3}\right) \ \cup \ \left(\frac{2\pi}{3} \ , \ \frac{7\pi}{3}\right) \ \cup \ \left(\frac{8\pi}{3} \ , \ \frac{13\pi}{4}\right] .$$

So,

$$2\theta \ \in \ \left[0 \ , \ \frac{\pi}{12}\right) \ \cup \ \left(\frac{5\pi}{12} \ , \ \frac{25\pi}{12}\right) \ \cup \ \left(\frac{29\pi}{12} \ , \ 3\pi\right] .$$

Thus the solution to the original problem is

$$\theta \ \in \ \left[0 \ , \ \frac{\pi}{24}\right) \ \cup \ \left(\frac{5\pi}{24} \ , \ \frac{25\pi}{24}\right) \ \cup \ \left(\frac{29\pi}{24} \ , \ \frac{3\pi}{2}\right] .$$

$\square$

Exercises

1. (a) Solve $\cos x = \dfrac{1}{2}$ for $0 \le x \le 2\pi$.

   (b) Solve $\cos x = \dfrac{1}{2}$ for $0 \le x \le 3\pi$.

   (c) Solve $\cos x = \dfrac{1}{2}$ for $-2\pi \le x \le 3\pi$.

2. Solve the following equations for $\theta \in [0, 2\pi]$:

   (a) $\sin 2\theta = -\dfrac{1}{2}$

   Hint: Let $x = 2\theta$ and solve $\sin x = -\frac{1}{2}$ for $x \in [0, 4\pi]$.

   (b) $\tan\left(3\theta + \dfrac{\pi}{4}\right) = -1$

   Hint: Let $x = 3\theta + \frac{\pi}{4}$ and solve $\tan x = -1$ for $x \in \left[\frac{\pi}{4} \ , \ \frac{25\pi}{4}\right]$.

(c) $\sin\theta + \sqrt{3}\cos\theta = 0$

Hint: Rewrite as $\sin\theta = -\sqrt{3}\cos\theta$.

Then rewrite as $\frac{\sin\theta}{\cos\theta} = -\sqrt{3}$. That is, $\tan\theta = -\sqrt{3}$.

3. Solve $\tan\left(2\theta + \dfrac{\pi}{4}\right) = 1$ for $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

4. (a) Using the same set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = 2\pi$.

   (b) Find the x–coordinates of the points of intersection of the two graphs.

   (c) Hence find the values of x between 0 and 2π for which $\sin x > \cos x$.

5. Solve the following inequalities, for $\theta \in [0, 2\pi]$:

   (a) $\sin\left(\theta - \dfrac{\pi}{6}\right) - \dfrac{1}{2} < 0$

   Hint: Rewrite as $\sin\left(\theta - \frac{\pi}{6}\right) < \frac{1}{2}$. Let $x = \theta - \frac{\pi}{6}$ and solve $\sin x < \frac{1}{2}$ for $x \in \left[-\frac{\pi}{6}, \frac{11\pi}{6}\right]$.

   (b) $\sqrt{2}\cos\left(\theta - \dfrac{\pi}{4}\right) + 1 \geq 0$

   Hint: Rewrite as $\cos\left(\theta - \frac{\pi}{4}\right) \geq -\frac{1}{\sqrt{2}}$.

   Let $x = \theta - \frac{\pi}{4}$ and solve $\cos x \geq -\frac{1}{\sqrt{2}}$ for $x \in \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$.

   (c) $\tan 2\left(\theta + \dfrac{\pi}{2}\right) \leq 1$

   Hint: Let $x = 2\left(\theta + \frac{\pi}{2}\right)$ and solve $\tan x \leq 1$ for $x \in [\pi, 5\pi]$.

2.7 Answers to Chapter 2 Exercises

2.1: 1. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$ (e) $\frac{3\pi}{2}$ (f) 2π
2. (a) $45°$ (b) $60°$ (c) $150°$ (d) $180°$

2.2:

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
(a)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
(b)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
(c)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
(d)	1	0	undefined	1	undefined	0
(e)	0	-1	0	undefined	-1	undefined
(f)	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$

2.3: 1. (a) 0.1471 (b) −0.4822 (c) 1.1504 (d) 1.8374 (e) 0.5676

2.

(a) When $t = 0$ the depth is 20 m.

(b) At 1 am the depth is ≈ 20.78 m.

(c) At 2 am the depth is 21.5 m.

(d) The maximum depth is 23 m.

(e) The minimum depth is 17 m.

2.4: 1. (a) $-\sin\theta$ (b) $\cos\theta$ (c) $\cos\theta$ (d) $\sin\theta$
(e) $\operatorname{cosec}\theta$ (f) $-\tan\theta$ (g) $-\sin\theta$ (h) $\cos\theta$
2. Omitted.

2.5: 1.

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$		θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
(a)	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	(b)	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
(c)	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	(d)	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
(e)	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	(f)	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
(g)	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	(h)	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$

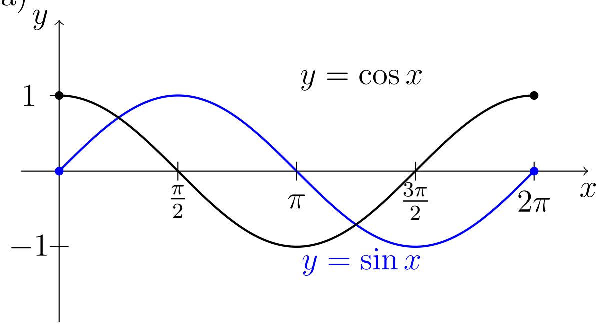
2. (a) -1 (b) $\frac{\sqrt{3}}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $-\frac{2}{\sqrt{3}}$ (e) 0 (f) $\frac{1}{2}$
(g) 1 (h) $\frac{1}{\sqrt{3}}$ (i) $\frac{\sqrt{3}}{2}$ (j) 0

2.6: 1. (a) $\frac{\pi}{3},\ \frac{5\pi}{3}$ (b) $\frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3}$ (c) $-\frac{5\pi}{3},\ -\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3}$

2. (a) $\frac{7\pi}{12},\ \frac{11\pi}{12},\ \frac{19\pi}{12},\ \frac{23\pi}{12}$ (b) $\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{3\pi}{2},\ \frac{11\pi}{6}$ (c) $\frac{2\pi}{3},\ \frac{5\pi}{3}$

3. $-\frac{\pi}{2},\ 0,\ \frac{\pi}{2}$

4. (a)



(b) $\frac{\pi}{4},\ \frac{5\pi}{4}$

(c) $\frac{\pi}{4} < x < \frac{5\pi}{4}$

5. (a) $\left[0, \frac{\pi}{3}\right) \cup (\pi, 2\pi]$ (b) $[0, \pi] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

(c) $\left[0, \frac{\pi}{8}\right] \cup \left(\frac{\pi}{4}, \frac{5\pi}{8}\right] \cup \left(\frac{3\pi}{4}, \frac{9\pi}{8}\right] \cup \left(\frac{5\pi}{4}, \frac{13\pi}{8}\right] \cup \left(\frac{7\pi}{4}, 2\pi\right]$