Chapter 15

Matrices and Systems of Linear Equations

15.1 Definitions

A **matrix** is a rectangular array of numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

- We usually use *uppercase* letters (for example, A, B, C, ...) for the *names* of matrices, and we usually use *lowercase* letters (for example, a, b, c, ...) to represent the *numbers inside a matrix.*
- The numbers inside a matrix are called the **entries** or **elements** of the matrix.
- The sequence of all entries on a *horizontal* line is called a **row**, and the sequence of all entries on a *vertical* line is called a **column**. We number the rows from top to bottom, and the columns from left to right.
- The entry in the k^{th} row and l^{th} column of a matrix A is denoted by a_{kl}.
- A matrix with m rows and n columns is called an $m \times n$ matrix, or a matrix of **order** $m \times n$. We say this as "m by n".

Example 1.

$$A = \begin{bmatrix} 2 & -3 & 4 & 5 \\ 6 & 7 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \text{ is a } 3 \times 4 \text{ matrix since it has 3 rows and 4 columns.}$$

The entry in the 2^{nd} row and 4^{th} column of A is 3, and so we write $a_{24} = 3$.

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ is a } 4 \times 1 \text{ matrix.}$$

It can also be referred to as a **column matrix** or a **column vector**.

$$C = \begin{bmatrix} 1 & 2 & -3 & 4 & -1 \end{bmatrix} \text{ is a } 1 \times 5 \text{ matrix.}$$

It can also be referred to as a **row matrix** or **row vector**.

□

Further Definitions:

- If a matrix has order $n \times n$ (that is, if the number of rows equals the number of columns) then the matrix is called **square**.
- The **main diagonal** of a square matrix consists of the entries on the diagonal from the top left corner of the matrix down to the bottom right corner of the matrix.
- A **diagonal** matrix is a square matrix in which all of the entries which are *not* on the main diagonal must equal zero.
- A **triangular** matrix is a square matrix in which
  - all the entries which lie *below* the main diagonal are zero, and/or
  - all the entries which lie *above* the main diagonal are zero.

Example 2.

$$D = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$ is a 2×2 **square** matrix, and

$$E = \begin{bmatrix} 2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -6 \end{bmatrix}$$ is a 3×3 **square** matrix. E is also a 3×3 **diagonal** matrix, and a 3×3 **triangular** matrix.

$$F = \begin{bmatrix} 1 & 2 & -3 \\ \mathbf{0} & 4 & 5 \\ \mathbf{0} & \mathbf{0} & -6 \end{bmatrix}$$ and $$G = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 2 & 4 & \mathbf{0} \\ -3 & 5 & -6 \end{bmatrix}$$ are 3×3 **square** matrices, and are also 3×3 **triangular** matrices.

□

15.2 Matrix Operations

Matrix equality: Two matrices are equal if they have the same order and their corresponding entries are equal. For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$ if and only if $a = e$, $b = f$, $c = g$ and $d = h$.

Matrix addition: Two matrices can be added if they have the same size. We add two such matrices by adding the corresponding entries.

For example, we have $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix},$$

and so $$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 4 & \frac{1}{2} \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 5 & \frac{7}{2} \\ 1 & -4 \end{bmatrix}.$$

Scalar multiplication: Any matrix can be multiplied by a single number (scalar). We do this by multiplying all the entries of the matrix by that number.

For example, we have $$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix},$$

and so $$4 \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 8 & -4 \end{bmatrix}.$$

Matrix multiplication: The product AB of two matrices A and B is defined if and only if the number of columns of A is equal to the number of rows of B.

$$\begin{array}{c} n \text{ columns} \\ \begin{bmatrix} \vdots & \vdots & & \vdots & \vdots \\ & & & & \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix} \\ \\ A \end{array} \qquad \begin{array}{c} \begin{array}{c} n \\ \text{rows} \end{array} \begin{bmatrix} \cdots & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & \cdots & \cdots \end{bmatrix} \\ \\ B \end{array}$$

To calculate the entry in the (i, j) position (that is, in the i^{th} row and j^{th} column) of AB, we

$$\textbf{multiply the } \boldsymbol{i^{th}} \textbf{ row of } \boldsymbol{A} \textbf{ by the } \boldsymbol{j^{th}} \textbf{ column of } \boldsymbol{B}.$$

Example 3. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$ $\square$

$$\boxed{\begin{array}{c} \textbf{If } \boldsymbol{A} \textbf{ has order } \boldsymbol{m \times n} \textbf{ and } \boldsymbol{B} \textbf{ has order } \boldsymbol{n \times p} \\ \textbf{then } \boldsymbol{AB} \textbf{ has order } \boldsymbol{m \times p}\textbf{.} \end{array}}$$

Example 4. $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 3a+2d & 3b+2e & 3c+2f \\ 4a+1d & 4b+1e & 4c+1f \end{bmatrix}$ $\square$

Example 5.

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times 1 + -3 \times 4 & 1 \times 3 + 2 \times 6 + -3 \times 5 \\ 4 \times 2 + 5 \times 1 + 6 \times 4 & 4 \times 3 + 5 \times 6 + 6 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 \\ 37 & 72 \end{bmatrix}$$

$\square$

Transpose: The transpose A^T of a matrix A is obtained by

putting the $\boldsymbol{i^{th}}$ row of $\boldsymbol{A}$ into the $\boldsymbol{i^{th}}$ column of $\boldsymbol{A^T}$.

Note that if A has order $m \times n$ then A^T has order $n \times m$.

Example 6. If $A = \begin{bmatrix} 2 & 4 & 1 \\ 9 & 9 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 9 \\ 4 & 9 \\ 1 & 6 \end{bmatrix}$.

□

15.3 Properties of Matrix Operations

Addition Properties:

We always have

$$\begin{aligned} A + B &= B + A \\ A + (B + C) &= (A + B) + C \\ A + O &= A = O + A \\ A + -A &= O = -A + A \end{aligned}$$

where the matrix O is called the **zero** matrix, and
the matrix $-A$ is called the **negative** of A .

In the above properties, A, B, C, O and $-A$ all have the same order.
Thus, for example,

(a) if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$, whereas

(b) if $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ then $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $-A = \begin{bmatrix} -a & -b \\ -c & -d \\ -e & -f \end{bmatrix}$.

Multiplication Properties:

Provided that each of the following matrix products exists, we have

$$\begin{aligned} A(B+C) &= AB+AC \\ (B+C)A &= BA+CA \\ A(BC) &= (AB)C \\ \text{However, usually } AB &\neq BA. \end{aligned}$$

For a **square** matrix A, we have:

$$\begin{aligned} AO &= O = OA \\ \text{and } AI &= A = IA, \end{aligned}$$

where I is a diagonal matrix with

ones on the main diagonal and zeros elsewhere.

The matrix I is called the **identity matrix**, and must have the same order as A. Thus, for example,

if A is 2×2 then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, whereas

if A is 3×3 then $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Note: It is important to realize that, if we choose any two matrices A and B, then usually

$$\boxed{\boldsymbol{AB \neq BA}\,.}$$

For example, consider the matrices A and B given in the next example.

Example 7. Let $A = \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix}$.

$$\begin{aligned}
\text{Then} \quad AB &= \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix} \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -30+30 & -12+12 \\ 40-40 & 16-16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
\text{whereas} \quad BA &= \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix} \\
&= \begin{bmatrix} -30+16 & -60+32 \\ 15-8 & 30-16 \end{bmatrix} = \begin{bmatrix} -14 & -28 \\ 7 & 14 \end{bmatrix}.
\end{aligned}$$

In this example, we see that $\boxed{\boldsymbol{AB \neq BA}\,.}$ □

Note that

- multiplying on the *left* is called **pre**multiplying, whereas
- multiplying on the *right* is called **post**multiplying.

Exercises for Section 15.3

1. Calculate

   (a) $\begin{bmatrix} 2 & -2 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -1 & 0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 \\ -\frac{2}{3} & -1 \end{bmatrix} + \begin{bmatrix} 3 & 43 \\ -4 & 3 \end{bmatrix}$

2. Calculate

   (a) $-2\begin{bmatrix} 2 & -\frac{1}{2} \\ 3 & 0 \end{bmatrix}$ (b) $4\begin{bmatrix} 3 & \frac{1}{4} \\ -1 & 0 \end{bmatrix}$ (c) $3\begin{bmatrix} 5 & 8 \\ 6 & -4 \end{bmatrix}$

3. Calculate

   (a) $\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ \frac{1}{2} & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix}$

4. Given that $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & x \\ 2 & 6 & 5 \end{bmatrix}\begin{bmatrix} 2 & 7 & 1 \\ 8 & 2 & 8 \\ 1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 55 & 19 \\ 51 & 89 & 59 \\ 57 & 66 & 60 \end{bmatrix}$, find x.

15.4 The Determinant and Inverse of a 2×2 Matrix

The matrix B is the **inverse** of A if $AB = I = BA$.

Not all matrices have inverses.

When a matrix A does have an inverse B, then the inverse is *unique* (that is, it has no other inverses). Then we write A^{-1} instead of B, and so we have

$$AA^{-1} = I = A^{-1}A\,.$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define the **determinant** of A by

$$\boxed{\det(A) = ad - bc\,.}$$

We sometimes write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ instead of $\det(A)$.

If $\det(A) \neq 0$, then $\boxed{A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$

(We can check that this A^{-1} satisfies $AA^{-1} = I = A^{-1}A$.)

If $\det(A) = 0$, then A^{-1} does not exist, and we say that A is **singular**.

Example 8. Find the determinant and the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}.$$

Solution: We have $\det(A) = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 2 \times 0 - 3 \times 4 = -12$, and so

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1} = \frac{1}{-12} \begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 0 & 3 \\ 4 & -2 \end{bmatrix}.$$

□

Exercises for Section 15.4

1. Evaluate

(a) $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}$

2. Find the inverse of the following matrices:

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15.5 Solving Matrix Equations

Consider the matrix equation $\boldsymbol{AX} = \boldsymbol{C}$, where X is an unknown matrix. Suppose that our goal is to find X. Note that we *cannot divide* by A (since matrix division has *not* been defined). Instead, we

multiply both sides of the equation by $\boldsymbol{A^{-1}}$

(if A^{-1} exists).

In particular, if A^{-1} exists then we can solve the above equation for X, as follows:

$$\begin{aligned} AX = C \quad &\Rightarrow \quad A^{-1}AX = A^{-1}C \text{ (}\textbf{pre}\text{multiplying both sides of the equation by } A^{-1}\text{)} \\ &\Rightarrow \quad IX = A^{-1}C \\ &\Rightarrow \quad X = A^{-1}C\,. \end{aligned}$$

Similarly, if $XA = C$ then we can solve this equation for X, as follows:

$$\begin{aligned} XA = C \quad &\Rightarrow \quad XAA^{-1} = CA^{-1} \text{ (}\textbf{post}\text{multiplying both sides of the equation by } A^{-1}\text{)} \\ &\Rightarrow \quad XI = CA^{-1} \\ &\Rightarrow \quad X = CA^{-1}\,. \end{aligned}$$

Example 9. Find the matrix X such that $X\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix}$.

Solution: Since $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ is on the *right* of X, we **post**multiply by $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1}$.

$$\begin{aligned}
\text{We have } X\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} &= \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix} \\
\Rightarrow X\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1} &= \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1} \\
\Rightarrow XI &= \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{-12}\begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix}\right) \\
\Rightarrow X &= \frac{1}{-12}\begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix} \\
&= \frac{1}{-12}\begin{bmatrix} 0 & -27 \\ -4 & -1 \end{bmatrix} \\
&= \frac{1}{12}\begin{bmatrix} 0 & 27 \\ 4 & 1 \end{bmatrix}
\end{aligned}$$

□

Uniqueness of Solutions

Sometimes there can be more than one matrix X such that $AX = C$, or none. When there is only one X such that $AX = C$, we say that $AX = C$ has a *unique* solution.

Consider the matrix equation $AX = C$. We have the following result:

- When $\det(A) \neq 0$, then $AX = C$ has a *unique* solution.
  This unique solution is given by $X = A^{-1}C$.
- When $\det(A) = 0$, then $AX = C$ has either
  (a) *infinitely many* solutions, or
  (b) *no* solutions.

Exercises for Section 15.5

1. Find the matrix X such that

(a) $X\begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ -7 & 4 \end{bmatrix}$ (b) $X\begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ -13 & -1 \end{bmatrix}$

(c) $X\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $X\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

2. Find the matrix Y such that

(a) $\begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} Y = \begin{bmatrix} 9 & 2 \\ -7 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} Y = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix} Y = \begin{bmatrix} 8 & 6 \\ -13 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix} Y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3. Find the matrix Z such that

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} Z \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{bmatrix} Z \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$

15.6 Simultaneous Linear Equations with 2 Unknowns

An equation of the form $ax + by = p$ (where a, b and p are constants) represents a straight line in the x, y-plane.

Therefore the system of equations

$$\begin{array}{ccccc} ax & + & by & = & p \\ cx & + & dy & = & q \end{array}$$

represents two lines in the x, y-plane. When we solve these two equations *simultaneously*, we are finding the point of intersection of the two lines.

We can write the above system of linear equations in matrix form as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Example 10. Solve the following system of linear equations:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 8 \\ x & - & 4y & = & -7. \end{array}$$

Solution:

We need to find x and y such that *both* of the above equations are satisfied. To do this, we write the above system in matrix form:

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}.$$

Then **pre**multiplying by the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ gives

$$\begin{aligned} \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -7 \end{bmatrix}, \\ \text{and so } \begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{-11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \end{bmatrix} \\ &= \frac{1}{-11} \begin{bmatrix} -11 \\ -22 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \end{aligned}$$

Thus $x = 1$ and $y = 2$.

$\square$

Solving a System of 2 Linear Equations with 2 Unknowns

(a) Write the system of linear equations

$$\begin{array}{rcrcl} ax & + & by & = & p \\ cx & + & dy & = & q \end{array}$$

in matrix form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

(b) Calculate the determinant $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

(c) If $\Delta \neq 0$, then we have a *unique* solution given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(d) If $\Delta = 0$, then either

- there are *no* solutions.
  - In this case, the equations represent two parallel lines which don't intersect.

or

- there are *infinitely many* solutions.
  - In this case, the equations represent the same line, and so any point on this line is a solution.

Example 11. Solve the following system of linear equations:

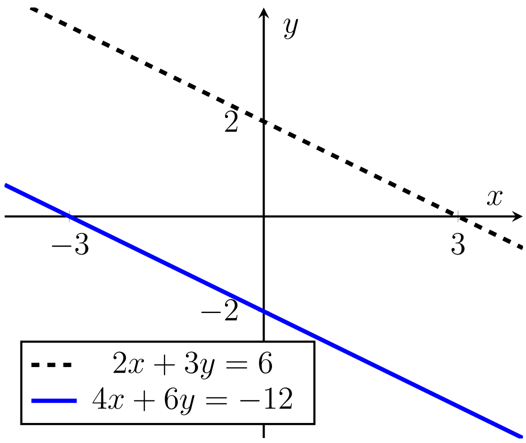
$$\begin{array}{rcrcr} 2x & + & 3y & = & 6 \\ 4x & + & 6y & = & -12. \end{array}$$

Solution: The relevant matrix equation is $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \end{bmatrix}.$

Note that the determinant $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2 \times 6 - 3 \times 4 = 12 - 12 = 0\,.$

Therefore, there is *no* solution or there are *infinitely many* solutions.

To decide which case we have, we can consider a *graph* of the two lines given by the two equations, as shown next.



From this graph, we see that $2x+3y = 6$ and $4x+6y = -12$ are non-intersecting parallel lines, and so we conclude that there are *no* solutions. $\square$

Example 12. Consider the following simultaneous system of linear equations, where k is a constant:

$$\begin{aligned} kx &+ 3y &= 4 \\ 3x &+ ky &= 5\,. \end{aligned}$$

(a) Write down the system of equations in matrix form $AX = B$.

(b) Calculate the determinant of the matrix A found in (a).

(c) Find the value(s) of k for which the system has a unique solution.

Solution:

(a) The relevant matrix equation is $\begin{bmatrix} k & 3 \\ 3 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

(b) We have $A = \begin{bmatrix} k & 3 \\ 3 & k \end{bmatrix}$, and so $\det(A) = \begin{vmatrix} k & 3 \\ 3 & k \end{vmatrix} = k^2 - 9$.

(c)
$$\begin{aligned} \text{There is a unique solution if and only if } \ & \det(A) \neq 0 \\ \iff & k^2 - 9 \neq 0 \\ \iff & (k-3)(k+3) \neq 0 \\ \iff & k \neq 3 \ \text{ and } \ k \neq -3 \\ \iff & k \in \mathbf{R} \setminus \{3, -3\}\,. \end{aligned}$$

$\square$

Exercises for Section 15.6

1. Solve the following systems of linear equations:

(a) $$\begin{array}{rcrcl} 2x & + & 3y & = & 8 \\ x & + & 4y & = & 9 \end{array}$$

(b) $$\begin{array}{rcrcr} 2x & - & y & = & -5 \\ x & + & 3y & = & 1 \end{array}$$

(c) $$\begin{array}{rcrcl} 2x & + & 3y & = & 6 \\ 4x & + & 6y & = & 3 \end{array}$$

(d) $$\begin{array}{rcrcr} 2x & + & 3y & = & 6 \\ 4x & + & 6y & = & 12 \end{array}$$

2. Consider the following simultaneous system of linear equations, where p is a constant:

$$\begin{array}{rcrcl} 3x & + & py & = & 9 \\ (p+1)x & + & 2y & = & 9\,. \end{array}$$

(a) Show that the system has a unique solution if and only if

$$p \in \mathbf{R} \setminus \{-3,\, 2\}.$$

(b) Find the value(s) of p for which the system has infinitely many solutions.

(c) Find the value(s) of p for which the system has no solution.

Hint for (b) and (c):

Consider the cases $p = -3$ and $p = 2$ separately.

That is, substitute $p = -3$ into the system and then try to solve the system.

Do the same for $p = 2$.

15.7 The Triangular Matrix Method

The triangular matrix method is an elimination method that will solve a system of linear equations with any number of unknowns.

We concentrate on the case of *three* unknowns x , y and z :

$$\begin{aligned} ax + by + cz &= p \\ dx + ey + fz &= q \\ gx + hy + iz &= r. \end{aligned}$$

These equations represent three planes in an $x - y - z$ system.

- When this system of three equations has a unique solution or has more than one solution, the equations are said to be **consistent**.
  - When there is a *unique solution*, the three planes meet in *one* common point.
  - When there are *infinitely many solutions*, the three planes meet in a common line (or plane).
- When the system has *no solution*, the equations are said to be **inconsistent**. In this case, there are three possibilities:
  - the planes are parallel to one another
  - two planes only are parallel
  - the three planes meet in three parallel lines, forming a triangular prism.

The triangular matrix method does not require us to find the inverse of a matrix. Instead, we use a form of elimination. The aim is to

operate on the rows of the matrix until we have a triangular matrix

(because then it becomes easy to solve for z, after which we can solve for y, and then x).

Example 13. Use the triangular matrix method to solve the system

$$\begin{array}{rcrcrcr} 2x & - & 3y & + & 4z & = & -3 \\ 4x & + & 2y & + & 5z & = & 5 \\ x & + & 6y & + & 3z & = & 5. \end{array}$$

Solution: We can write the system in matrix form as

$$\begin{bmatrix} 2 & -3 & 4 \\ 4 & 2 & 5 \\ 1 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}.$$

Alternatively we can represent this system more quickly by using **augmented** matrix form:

$$\begin{bmatrix} 2 & -3 & 4 & -3 \\ 4 & 2 & 5 & 5 \\ 1 & 6 & 3 & 5 \end{bmatrix}.$$

Then we perform **row operations** to obtain a *triangle of zeros below the main diagonal*. Note that there are *many* possible row operations that can be used; one of the (infinitely) many sequences of suitable row operations is shown here:

$$\begin{array}{r} \\ R_2 - 2R_1 \\ 2R_3 - R_1 \end{array} \begin{bmatrix} 2 & -3 & 4 & -3 \\ \mathbf{0} & 8 & -3 & 11 \\ \mathbf{0} & 15 & 2 & 13 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ 8R_3 - 15R_2 \end{array} \begin{bmatrix} 2 & -3 & 4 & -3 \\ \mathbf{0} & 8 & -3 & 11 \\ \mathbf{0} & \mathbf{0} & 61 & -61 \end{bmatrix}$$

Now that we have a *triangle of zeros below the main diagonal*, it is easy to find z.

$$\text{In particular, the last row tells us that} \quad 61z = -61$$

$$\text{and so} \quad \boldsymbol{z} = \mathbf{-1}.$$

Next we will use the *second row* to find y. In particular, the second row tells us that

$$\begin{aligned} 8y - 3z &= 11. \\ \text{That is,} \quad 8y &= 11 + 3z \\ &= 11 + 3 \times -1 \\ &= 8. \\ \text{Thus} \quad \boldsymbol{y} &= \mathbf{1}. \end{aligned}$$

Finally, the first row tells us that

$$\begin{aligned} 2x - 3y + 4z &= -3. \\ \text{That is,}\ \ 2x &= -3 + 3y - 4z \\ &= -3 + 3 \times 1 - 4 \times -1 \\ &= 4, \\ \text{and thus}\ \ \boldsymbol{x} &\boldsymbol{=} \mathbf{2}. \end{aligned}$$

Therefore the solution is $(x, y, z) = (2, 1, -1)$.

$\square$

Note: We can *check* these answers by substituting them back into the original equations.

The Allowable Row Operations:

- We are allowed to multiply or divide rows by non–zero numbers.
- We are allowed to add or subtract a multiple of one row to another row.
- We are allowed to swap rows.

Note: When we do several row operations within one step, we must ensure that each row operation *uses a row that has not already been used in that step.* For 3×3 examples, this means that we must not use the same pair of rows twice within one step.

Example 14. Use the triangular matrix method to solve the system

$$\begin{array}{rcrcrcl} 2x & + & y & + & 11z & = & 0 \\ 3x & + & 2y & + & 5z & = & 0 \\ 4x & + & 3y & - & z & = & 0. \end{array}$$

Solution: We first write the system in augmented matrix form

$$\begin{bmatrix} 2 & 1 & 11 & 0 \\ 3 & 2 & 5 & 0 \\ 4 & 3 & -1 & 0 \end{bmatrix}$$

and then perform row operations to obtain a *triangle of zeros below the main diagonal*. Just as in the previous example, there are infinitely many possible row operations that can be used. One of the infinitely many sequences of suitable row operations is shown here:

$$\begin{array}{r} \\ 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \begin{bmatrix} 2 & 1 & 11 & 0 \\ \mathbf{0} & 1 & -23 & 0 \\ \mathbf{0} & 1 & -23 & 0 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ R_3 - R_2 \end{array} \begin{bmatrix} 2 & 1 & 11 & 0 \\ \mathbf{0} & 1 & -23 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix}$$

Now that we have a *triangle of zeros below the main diagonal*, we will use the last row to try to find z. Notice that this R_3 just tells us that

$$\begin{aligned} 0x + 0y + 0z &= 0. \\ \text{That is, this } R_3 \text{ just tells us that } \ 0 &= 0. \\ \text{Since this is \textit{always satisfied}, we write } \ \boldsymbol{z} &= \boldsymbol{k} \ \text{ (where } k \text{ is any real number).} \end{aligned}$$

Next we will use the *second row* to find y. In particular, since R_2 tells us that

$$\begin{aligned} y - 23z &= 0, \\ \text{then we can immediately write } \ y &= 23z \ \text{ (where } z = k\text{).} \\ \text{Thus we have } \ \boldsymbol{y} &= \boldsymbol{23k}. \end{aligned}$$

$$\begin{aligned} \text{Finally, } R_1 \text{ tells us that } \ 2x + y + 11z &= 0. \\ \text{That is, } \ 2x &= -y - 11z \\ &= -23k - 11k \\ &= -34k, \\ \text{and so } \ \boldsymbol{x} &= \boldsymbol{-17k}. \end{aligned}$$

Therefore the solution is $(x, y, z) = (-17k,\ 23k,\ k)$, where $k \in \mathbf{R}$. □

Notice that in this example we have *infinitely many solutions* (since we have a solution for each value of k).

Example 15. Use the triangular matrix method to solve the system

$$\begin{array}{rcrcrcr} x & + & 3y & - & z & = & 2 \\ 3x & + & 5y & + & 2z & = & 0 \\ 4x & + & 12y & - & 4z & = & 5. \end{array}$$

Solution:

We first write the system in augmented matrix form:

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 3 & 5 & 2 & 0 \\ 4 & 12 & -4 & 5 \end{bmatrix}.$$

Then we perform row operations to obtain a *triangle of zeros below the main diagonal:*

$$\begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \end{array} \begin{bmatrix} 1 & 3 & -1 & 2 \\ \mathbf{0} & -4 & 5 & -6 \\ \mathbf{0} & \mathbf{0} & 0 & -3 \end{bmatrix}.$$

$$\begin{array}{rrcl} \text{Notice that this } R_3 \text{ tells us that} & 0x + 0y + 0z & = & -3. \\ \text{That is,} & 0 & = & -3, \end{array}$$

which we know is not true. Therefore, we must conclude that there is *no solution.* □

Exercises for Section 15.7

Use the triangular matrix method to solve the following systems of linear equations:

(a) $$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & -4 \\ 4x & + & 9y & - & 8z & = & -2 \\ -2x & - & 5y & + & 5z & = & 3 \end{array}$$

(b) $$\begin{array}{rcrcrcr} 2x & - & 3y & + & 4z & = & -3 \\ 4x & + & 2y & + & 5z & = & -11 \\ x & + & 6y & + & 3z & = & -10 \end{array}$$

(c) $$\begin{array}{rcrcrcr} 2x & - & 3y & + & 4z & = & 3 \\ 4x & + & 2y & + & 5z & = & 11 \\ x & + & 6y & + & 3z & = & 10 \end{array}$$

(d) $$\begin{array}{rcrcrcr} -x & - & 2y & - & z & = & -5 \\ 6x & + & y & + & 6z & = & 8 \\ x & - & y & + & z & = & 3 \end{array}$$

(e) $$\begin{array}{rcrcrcr} x & & & + & z & = & 1 \\ x & + & 2y & + & z & = & 6 \\ x & + & y & + & z & = & 4 \end{array}$$

(f) $$\begin{array}{rcrcrcr} x & & & + & z & = & 0 \\ x & + & 2y & + & z & = & 0 \\ x & + & y & + & z & = & 0 \end{array}$$

(g) $$\begin{array}{rcrcrcr} 3x & + & y & - & 2z & = & 7 \\ x & + & 2y & + & 3z & = & 1 \\ 2x & + & 3y & + & 4z & = & 3 \end{array}$$

(h) $$\begin{array}{rcrcrcr} x & - & y & + & 5z & = & 2 \\ 2x & - & y & + & 7z & = & 3 \\ x & + & 2y & - & 4z & = & -1 \end{array}$$

15.8 Answers for the Chapter 15 Exercises

15.3 1. (a) $\begin{bmatrix} 2 & 1 \\ 6 & 4.6 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 45 \\ -\frac{14}{3} & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} -4 & 1 \\ -6 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 12 & 1 \\ -4 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & 24 \\ 18 & -12 \end{bmatrix}$

3. (a) $\begin{bmatrix} 2 & 1 \\ -4 & 37 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 11 \\ 13 & -10 \end{bmatrix}$

4. $x = 9$

15.4 1. (a) 2 (b) 0 (c) 1

2. (a) $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15.5 1. (a) $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$ (b) $\frac{1}{7}\begin{bmatrix} 26 & 2 \\ -\frac{67}{2} & -12 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

2. (a) $\frac{1}{5}\begin{bmatrix} 23 & -6 \\ 34 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

(d) $\frac{1}{14}\begin{bmatrix} 17 \\ 4 \end{bmatrix}$

3. (a) $\frac{1}{4}\begin{bmatrix} 38 & -15 \\ -42 & 17 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} -3 & \frac{3}{2} \\ 22 & -3 \end{bmatrix}$

15.6 1. (a) $(1, 2)$ (b) $(-2, 1)$ (c) No solution

(d) Infinitely many solutions, namely all points on the line $2x + 3y = 6$.

2. (a) Solve $\begin{vmatrix} 3 & p \\ p+1 & 2 \end{vmatrix} \neq 0$ (b) $p = 2$ (c) $p = -3$

15.7 ***Note: The matrices given below might be different from your answers.***

(a) $(x, y, z) = (1, 2, 3)$. An augmented triangular matrix is $\begin{bmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 3 & 9 \end{bmatrix}$.

(b) $(x, y, z) = (-1, -1, -1)$.

An augmented triangular matrix is $\begin{bmatrix} 2 & -3 & 4 & -3 \\ 0 & 8 & -3 & -5 \\ 0 & 0 & 61 & -61 \end{bmatrix}$.

(c) $(x, y, z) = (1, 1, 1)$. An augmented triangular matrix is $\begin{bmatrix} 2 & -3 & 4 & 3 \\ 0 & 8 & -3 & 5 \\ 0 & 0 & 61 & 61 \end{bmatrix}$.

(d) No solution. An augmented triangular matrix is $\begin{bmatrix} -1 & -2 & -1 & -5 \\ 0 & 11 & 0 & 22 \\ 0 & 0 & 0 & -44 \end{bmatrix}$.

(e) No solution. An augmented triangular matrix is $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(f) Infinitely many solutions: $(x, y, z) = (-k, 0, k)$.

An augmented triangular matrix is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(g) $(x, y, z) = (4, -3, 1)$. An augmented triangular matrix is $\begin{bmatrix} 3 & 1 & -2 & 7 \\ 0 & 5 & 11 & -4 \\ 0 & 0 & 3 & 3 \end{bmatrix}$.

(h) Infinitely many solutions: $(x, y, z) = (1 - 2k, -1 + 3k, k)$.

An augmented triangular matrix is $\begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.