Chapter 21

Bernoulli Trials

21.1 Binomial Distribution

Example 1. The probability that a shooter hits the target is $\frac{4}{5}$. If the shooter is going to fire 4 shots, what is the probability of getting exactly 1 hit from those 4 shots?

Solution:

Let **H** represent a hit, and **M** represent a miss. Then

$$\begin{aligned}
&\Pr(\text{shooter gets } \mathbf{1\ hit} \text{ from the 4 shots}) \\
&= \Pr(\mathbf{H}\text{MMM}) + \Pr(\text{M}\mathbf{H}\text{MM}) + \Pr(\text{MM}\mathbf{H}\text{M}) + \Pr(\text{MMM}\mathbf{H}) \\
&= \frac{4}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{4}{5} \times \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \times \frac{4}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{4}{5} \\
&= 4 \times \frac{4}{5} \times \left(\frac{1}{5}\right)^3 \\
&= \frac{16}{625} .
\end{aligned}$$

So the probability that the shooter gets 1 hit from the 4 shots is $\frac{16}{625}$.

$\square$

Now suppose

$$X = \text{the number of hits obtained from the 4 shots (in the previous example).}$$

Then X has the probability distribution shown here:

x	$\Pr(X = x)$
0	$\Pr(\text{MMMM}) = \left(\frac{1}{5}\right)^4$
1	$\Pr(\textbf{H}\text{MMM or M}\textbf{H}\text{MM or MM}\textbf{H}\text{M or MMM}\textbf{H}) = 4 \times \frac{4}{5} \times \left(\frac{1}{5}\right)^3$
2	$\Pr(\textbf{HH}\text{MM or }\textbf{H}\text{M}\textbf{H}\text{M or }\textbf{H}\text{MM}\textbf{H}\text{ or M}\textbf{HH}\text{M or M}\textbf{H}\text{M}\textbf{H}\text{ or MM}\textbf{HH}) = 6 \times \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^2$
3	$\Pr(\textbf{HHH}\text{M or }\textbf{HH}\text{M}\textbf{H}\text{ or }\textbf{H}\text{M}\textbf{HH}\text{ or M}\textbf{HHH}) = 4 \times \left(\frac{4}{5}\right)^3 \times \frac{1}{5}$
4	$\Pr(\textbf{HHHH}) = \left(\frac{4}{5}\right)^4$

Notice that the probabilities in this probability distribution follow a pattern. In particular, for this probability distribution, we have

$$\Pr(X = x) = {}^4\mathrm{C}_x \left(\frac{4}{5}\right)^x \left(\frac{1}{5}\right)^{4-x}.$$

The variable X defined above is a **binomial** variable. Features of a binomial distribution are as follows:

- There are n independent trials.
- Each trial has two possible outcomes (called "success" and "failure").

  Note that, as previously mentioned in Section 20.4, the two words "success" and "failure" are <u>not</u> intended to imply any judgement about whether those particular outcomes are "good" or "bad".

- The probability of obtaining a "success" is *the same in each trial* (and, similarly, the probability of obtaining a "failure" remains *the same for each trial).*

Such trials are called **Bernoulli trials**.

Let $\quad X$ be the number of successes in the n Bernoulli trials,
(with *no* conditions on *where* the successes occur)

and let $\quad p$ be the (constant) probability of a "success".

Then X is a **binomial** variable, and we have

$$\boxed{\Pr(X = x) \;=\; {}^{n}\mathrm{C}_{x}\, p^{x}\, (1-p)^{n-x}\,.}$$

This formula is given on the Formula Sheet provided in the Maths 1 exams.

- It is worth noting that p is often given as a ***p**ercentage* of a large population (from which we consider a relatively small sample). We will elaborate on this idea in Chapter 25.2.
- Also note that a binomial random variable is *discrete*, with values $0, 1, 2, \ldots, n$.

For the probability distribution given on the previous page, we *could* use the general formula for expected value from Section 20.2 to calculate

$$\begin{aligned} E(X) &= \sum x \Pr(X = x) \\ &= 0 \times \left(\frac{1}{5}\right)^4 + 1 \times 4 \times \frac{4}{5}\left(\frac{1}{5}\right)^3 + 2 \times 6 \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^2 + 3 \times 4 \left(\frac{4}{5}\right)^3 \frac{1}{5} + 4 \times \left(\frac{4}{5}\right)^4 \\ &= \frac{16}{5}\,. \end{aligned}$$

However, whenever we have a binomial distribution, we do <u>not</u> need to use the general formula $E(X) = \sum x \Pr(X = x)$ from Section 20.2 to find the expected value. Instead we can use the following much easier formula:

If X has the *binomial* distribution, then $\boxed{E(X) \;=\; np\,.}$

Furthermore, if X has the *binomial* distribution, then

$$\boxed{\mathrm{Var}(X) \;=\; np(1-p)\,.}$$

These formulae are also provided on the Formula Sheet provided in the Maths 1 exams.

Thus we can use these formulae to easily find the expected value and the variance for the binomial variable on the previous page. In particular, since that variable had

$$n = 4 \text{ and } p = \frac{4}{5}\,,$$

$$\begin{aligned} \text{then} \quad E(X) &= np = 4 \times \frac{4}{5} = \frac{16}{5} \\ \text{and} \quad \mathrm{Var}(X) &= np(1-p) = 4 \times \frac{4}{5} \times \frac{1}{5} = \frac{16}{25}\,. \end{aligned}$$

Example 2. A container has 3 yellow cubes and 7 black cubes in it. Four cubes are taken at random, with each cube **being replaced** before the next is taken.

(a) Write the probability distribution table for the variable defined to be the number of yellow cubes taken from the container. Write the probabilities to 4 decimal places.

(b) Find the mean and variance of the number of yellow cubes taken from the container.

Solution:

Let X be the number of yellow cubes taken from the container.

Then X is a binomial variable, with

$$n = 4 \quad \text{and} \quad p = \Pr(\text{taking a yellow cube}) = \frac{3}{10} = 0.3\,.$$

(Notice that this probability is *constant*, because each chosen cube is **replaced** before the next cube is chosen.)

(a)

x	$\Pr(X = x)$
0	${}^4C_0 \times 0.3^0 \times 0.7^4 = 0.2401$ (4 d.p.)
1	${}^4C_1 \times 0.3^1 \times 0.7^3 = 0.4116$ (4 d.p.)
2	${}^4C_2 \times 0.3^2 \times 0.7^2 = 0.2646$ (4 d.p.)
3	${}^4C_3 \times 0.3^3 \times 0.7^1 = 0.0756$ (4 d.p.)
4*	${}^4C_4 \times 0.3^4 \times 0.7^0 = 0.0081$ (4 d.p.)

* It is possible to take 4 yellow cubes from the container (even though it only contains 3 yellow cubes) because each cube is **replaced** before the next is taken. (For example, the same yellow cube could be chosen each time!)

(b)

$$\begin{aligned} \text{We have} \quad E(X) &= np \\ &= 4 \times 0.3 \\ &= 1.2, \\[1em] \text{and} \quad \mathrm{Var}(X) &= np(1-p) \\ &= 4 \times 0.3 \times 0.7 \\ &= 0.84\,. \end{aligned}$$

That is, the mean is 1.2, and the variance is 0.84 .

□

Example 3. For a certain species of bird, there is a chance of 3 in 5 that a chick will survive the first month after hatching. From a brood of 10 chicks, what is the probability that

(a) *more than one* will survive. Write the answer accurate to 4 decimal places.

(b) the first two born will survive and none of the others will survive.
Write the answer accurate to 7 decimal places.

Note: Assume that the survival of each chick is independent of the survival of the other chicks.

Solution:

Let X be the number of chicks in the brood which survive.
Then X is binomial, with $n = 10$ and $p = \frac{3}{5} = 0.6$.

(a) Note that

$$\Pr(X > 1) = \Pr(X = 2) + \Pr(X = 3) + \ldots + \Pr(X = 10).$$

However, because calculating *so many terms* would be rather time-consuming, we choose to use the following alternative *much quicker* approach:

$$\begin{aligned} \Pr(X > 1) &= 1 - \Pr(X \leq 1) \\ &= 1 - \Pr(X = 0) - \Pr(X = 1) \\ &= 1 - {}^{10}C_0 \times 0.6^0 \times 0.4^{10} - {}^{10}C_1 \times 0.6^1 \times 0.4^9 \\ &= 0.9983 \quad \text{(4 d.p.)}. \end{aligned}$$

Thus the probability that more than one chick will survive is 0.9983 (4 d.p.).

(b) Let S (“success”) represent a survivor, and
let F (“failure”) represent a chick that dies.

$$\text{Then we have} \quad \Pr(\text{SSFFFFFFFF}) = (0.6)^2(0.4)^8$$
$$= 0.000\,2359 \quad \text{(7 d.p.)}.$$

So the required probability is 0.000 2359 (7 d.p.).

□

Notice that in (b) of this example, we did NOT use the Binomial formula. This is because it was specified that the successes and failures must occur in a particular order.

Binomial variables count ***how many*** *successes there are, but can never specify the* ***order*** *in which those successes occur.*

Example 4. For a certain species of bird, there is a chance of 3 in 5 that a chick will survive the first month after hatching. Consider a brood of n chicks. Find the smallest value of n so that there is *at least* a 99% chance that *at least* one chick in the brood will survive.

Solution:

Let X be the number of chicks in the brood which survive.

Then X is binomial, with $n = ?$ and $p = \dfrac{3}{5} = 0.6$.

We need to find the smallest n so that

$$\Pr(X \geq 1) \geq 0.99 .$$

Depending on the (unknown) size of n , it could be rather time-consuming (and complicated) to replace $\Pr(X \geq 1)$ with

$$\Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \ldots + \Pr(X = n).$$

Instead, we choose the simpler option of replacing $\Pr(X \geq 1)$ with

$$1 - \Pr(X < 1),$$

which simplifies to

$$1 - \Pr(X = 0).$$

Thus we can re-state this example as:

"Find the smallest n so that $1 - \Pr(X = 0) \geq 0.99$ "

and this leads to the following calculations:

$$\begin{aligned} 1 - \Pr(X = 0) &\geq 0.99 \\ \implies \Pr(X = 0) &\leq 0.01 . \\ \text{That is, } {}^{n}\mathrm{C}_0 \times 0.6^0 \times 0.4^n &\leq 0.01, \\ \text{which simplifies to give } 1 \times 1 \times 0.4^n &\leq 0.01 . \\ \text{Then } \ln(0.4^n) &\leq \ln(0.01). \\ \text{That is, } n \ln(0.4) &\leq \ln(0.01). \\ \text{Dividing by } \ln(0.4)\text{, which is } \textit{negative,} \text{ gives } n &\geq \frac{\ln(0.01)}{\ln(0.4)} . \\ \text{That is, } n &\geq 5.026 \quad \text{(3 d.p.)}. \end{aligned}$$

Since n is a whole number, the smallest possible value of n is 6. □

Exercises for Section 21.1

1. A machine manufacturing electronic components is known to produce 10% defectives. A batch of five components are chosen. Find the probability that there is no more than one defective in the batch.
2. An unbiased coin is tossed 6 times.
   (a) Find the probability distribution of the number of heads obtained.
   (b) Find the mean and variance of the number of heads obtained.
3. A machine produces tablets, of which 5% are chipped. The tablets are packaged in boxes of 20. What is the probability that a box contains at most one chipped tablet? Write the answer to 4 decimal places.
4. The quality control for the manufacture of transistors is such that 5% of all of the transistors produced are defective. In a sample of 10 transistors, what is the probability of there being more than one which is defective? Write the answer to 4 decimal places.
5. Sixty percent of people are known to be in favour of a certain proposal. Three people are selected at random.
   (a) Find the probability that exactly two people are in favour of the proposal.
   (b) Find the probability that at least two people are in favour of the proposal.
   (c) Find the probability that the first person asked is in favour of the proposal, and the second and third people are **not** in favour of the proposal.
6. If, on average, one person in 20 is left–handed, find the probability that in a group of 18 people, there are at least 3 left–handers. Write the answer to 4 decimal places.
7. A family has 6 children. Find the probability that there are fewer boys than girls, assuming that
   (a) the probability of a boy is 0.5
   (b) the probability of a boy is 0.55. Write this answer to 4 decimal places.
8. The probability of a female birth is 0.48. Find the probability that a couple with three children has at least one child of each sex.
9. A shooter has probability 0.95 of hitting a target at any particular shot. What is the probability that he hits the target at least 9 times in 10 shots? Write the answer to 4 decimal places.
10. A machine is making items of which some are defective. The machine is checked every hour, by drawing a random sample of 10 items from the hour's production. If the sample contains no defectives, the machine is left alone; otherwise it is adjusted. Find the probability that the machine is left alone when it is producing 10% defective items. Write the answer to 4 decimal places.
11. Given that a binomial variable has mean 12 and variance 8, find p and n .

21.2 Geometric Distribution

Now we will consider a random variable X defined to be

the number of failures which occur ***before*** *the first success*

in a sequence of Bernoulli trials, where (as in the previous section)

p is the (constant) probability of success in a single trial.

Then this X is known as a **geometric** variable. We have

$$\begin{aligned}
\Pr(X=0) &= \Pr(S) = p \\
\Pr(X=1) &= \Pr(FS) = (1-p)\,p \\
\Pr(X=2) &= \Pr(FFS) = (1-p)^2\,p \\
&\text{etc.}
\end{aligned}$$

Continuing this pattern leads to the following formula: $\boxed{\Pr(X=x) = (1-p)^x\,p}$
where $x = 0,\ 1,\ 2,\ \ldots$.

Note that

- X is the number of failures *before* the first success, *NOT including the success.*
- the number of trials is unlimited; thus there is no n–value in the geometric formula.
- X is *discrete*, with $X = 0, 1, 2, \ldots$.

Consider the probability distribution table for a geometric random variable X :

x	$\Pr(X=x)$
0	p
1	$(1-p)\,p$
2	$(1-p)^2\,p$
3	$(1-p)^3\,p$
4	$(1-p)^4\,p$
$\vdots$	$\vdots$

Notice that this is an *infinite* table.

Also notice that the probabilities in this table form what is known as a *geometric sequence.* (In particular, each probability in the table is obtained by multiplying the previous probability by the constant term $1-p$.)

It can be shown that, if X is a geometric variable, then

$$\boxed{E(X) = \frac{1-p}{p} \quad \text{and} \quad \mathrm{Var}(X) = \frac{1-p}{p^2}\ .}$$

The above formulae are given on the Formula Sheet provided in the Maths 1 exams.
Note that these formulae for $E(X)$ and $\mathrm{Var}(X)$ are simplified (but equivalent) versions of the formulae given for $E(X)$ and $\mathrm{Var}(X)$ in Sections 20.2 and 20.3 (and have the huge advantage of *not* requiring us to calculate the sum of infinitely many terms).

Next we consider probabilities of the form $\Pr(X \geq x)$, for a geometric random variable X .

This means that we need *at least* x failures before the first success. In particular,

- the first x outcomes must all be failures, and then
- the subsequent outcomes could either be *failures or successes* (or a mixture of both).

For example,

$$\begin{aligned}
\Pr(X \geq 2) &= \Pr(\text{there are } \textit{at least } 2 \text{ failures before the first success}) \\
&= \Pr(\text{the first outcome is a failure, and} \\
&\qquad\quad \text{the second outcome is also a failure, and} \\
&\qquad\qquad\quad \text{the remaining outcomes are either failures} \\
&\qquad\qquad\qquad \text{or successes, or a mixture of these}) \\
&= (1-p) \times (1-p) \times 1 \times 1 \times \ldots \\
&= (1-p)^2.
\end{aligned}$$

Using this type of argument leads to the following result:

If X is a geometric random variable, then $\boxed{\Pr(X \geq x) \;=\; (1-p)^x.}$

This very useful formula is **NOT** on the Formula Sheet provided in the Maths 1 exams.

Note that other inequalities can be rewritten to take advantage of this formula.
In particular, if X is a geometric variable, and if x is an integer, then we have

- $\Pr(X > x)$ $= \Pr(X \geq x+1)$ $= (1-p)^{x+1}$ e.g. $\Pr(X > 500)$ $= \Pr(X \geq 501)$ $= (1-p)^{501}$

- $\Pr(X < x)$ $= 1 - \Pr(X \geq x)$ $= 1-(1-p)^x$ e.g. $\Pr(X < 500)$ $= 1 - \Pr(X \geq 500)$ $= 1-(1-p)^{500}$

- $\Pr(X \leq x)$ $= 1 - \Pr(X > x)$ $= 1 - \Pr(X \geq x+1)$ $= 1-(1-p)^{x+1}$ e.g. $\Pr(X \leq 500)$ $= 1 - \Pr(X > 500)$ $= 1 - \Pr(X \geq 501)$ $= 1-(1-p)^{501}$

Example 5. A cube has 1 white face and 5 blue faces. It is tossed until the white face appears uppermost. Find the probability that

(a) exactly 4 blue faces appear before the first white face.

(b) fewer than 10 blue faces appear before the first white face.

(c) the cube is thrown 8 times.

Write each answer accurate to 4 decimal places.

Solution: Let X be the number of $\underbrace{\text{blue faces}}_{\textit{failures}}$ which appear before the first $\underbrace{\text{white face}}_{\textit{success}}$.

Then X is a geometric variable, with $p = \Pr(\text{white face}) = \dfrac{1}{6}$.

(a) We have $\Pr(X = 4) = (1-p)^4\, p = \left(\frac{5}{6}\right)^4 \frac{1}{6} = 0.0804$ (4 d.p.).

So the probability that exactly 4 blue faces appear before the first white face is 0.0804 (4 d.p.).

(b) We have

$$\begin{aligned} \Pr(X < 10) &= 1 - \Pr(X \geq 10) \\ &= 1 - (1-p)^{10} \\ &= 1 - \left(\frac{5}{6}\right)^{10} \\ &= 0.8385 \quad \text{(4 d.p.)}. \end{aligned}$$

So the probability that fewer than 10 blue faces appear before the first white face is 0.8385 (4 d.p.).

(c) Throwing the cube 8 times corresponds to having

7 blue faces appearing before the first white face appears.

That is, we are considering the following situation: $\overbrace{\underbrace{B\,B\,B\,B\,B\,B\,B}_{\text{7 blue faces}}\,W}^{\text{8 throws}}$

Thus we need to find

$$\Pr(X = 7) = (1-p)^7\, p = \left(\frac{5}{6}\right)^7 \frac{1}{6} = 0.0465 \text{ (4 d.p.)}.$$

So the probability that the cube is thrown 8 times is 0.0465 (4 d.p.).

□

Example 6. A punter picks a winner in 20% of the races that he bets on. If

$$X \;=\; \text{the number of "losses" before the first win,}$$

then find $\Pr(5 \le X \le 17)$. Write the answer accurate to 4 decimal places.

Solution:

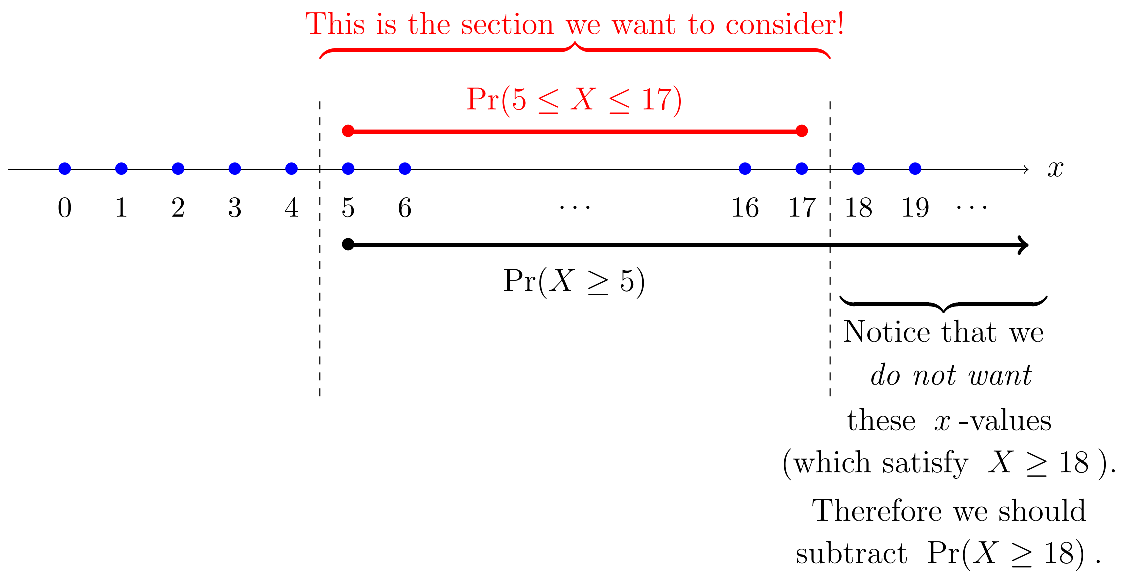
Note that X is a geometric variable, with $p = \Pr(\text{picks a winner}) = 0.2$.

It would be *rather slow* to find the answer by calculating

$$\Pr(5 \le X \le 17) \;=\; \Pr(X = 5) \;+\; \Pr(X = 6) \;+\; \ldots \;+\; \Pr(X = 17).$$

Therefore, we look for a quicker approach:

It helps to make use of the $\Pr(X \ge x) \;=\; (1-p)^x$ formula (from page 9). Of course, to use that formula, we need to rewrite the interval using terms of the form $\Pr(X \ge x)$.



$$
\begin{aligned}
\text{We have}\quad \Pr(5 \le X \le 17) &= \Pr(X \ge 5) \;-\; \Pr(X \ge 18) \\
&= (1-p)^5 \;-\; (1-p)^{18} \\
&= 0.8^5 \;-\; 0.8^{18} \\
&= 0.3097 \;\; \text{(4 d.p.).}
\end{aligned}
$$

$\square$

Exercises for Section 21.2

1. A fair die is rolled. What is the probability that the first six occurs on the sixth roll? Write your answer accurate to 5 decimal places.

2. Martin is not a very good driver and he has been learning for years. According to his driving instructor, he has a probability of 0.2 of passing the licence test. Furthermore, this probability will not change.

   (a) Find the probability that he passes the first time.

   (b) Find the probability that he fails fewer than 5 times before passing the test.

   (c) Specify the mean number of failures before he passes.

3. A punter finds that he is able to pick one winner in every 10 races. Let X denote the number of losses before the first win.

   (a) Find $\Pr(4 \leq X \leq 14)$, giving the answer accurate to 4 decimal places.

   (b) Find the expected value of X.

4. A *loaded* (or *weighted*) die is thrown until a six appears. Suppose the probability of getting at least 3 non–sixes before a six is $\dfrac{27}{64}$. Find the probability that a six will turn up on any one throw.

21.3 Answers for the Chapter 21 Exercises

21.1 1. The probability that there is at most one defective in the batch is 0.91854 .

2. (a)

x	$\Pr(X=x)$
0	$\frac{1}{64}$
1	$\frac{6}{64}$
2	$\frac{15}{64}$
3	$\frac{20}{64}$
4	$\frac{15}{64}$
5	$\frac{6}{64}$
6	$\frac{1}{64}$

(b) The mean is given by

$$E(X) \;=\; np \;=\; 6 \times \frac{1}{2} \;=\; 3$$

and the variance is given by

$$\mathrm{Var}(X) \;=\; np(1-p) \;=\; 6 \times \frac{1}{2} \times \frac{1}{2} \;=\; \frac{3}{2} \;.$$

3. The probability that a box contains no more than one chipped tablet is 0.7358 (4 d.p.).

4. The probability of having more than one defective is 0.0861 (4 d.p.).

5. (a) The probability that exactly two people are in favour is 0.432 .
   (b) The probability that at least two people are in favour is 0.648 .
   (c) The probability that only the first person asked is in favour is 0.096 .

6. The probability that there are at least 3 left–handers is 0.0581 (4 d.p.).

7. (a) The probability of 2 or fewer boys is $0.34\,375$.
   (b) The probability of 2 or fewer boys is now 0.2553 (4 d.p.).

8. The probability that there is at least one child of each sex is 0.7488 .

9. The probability that he hits the target at least 9 times in the 10 shots is 0.9139 (4 d.p.).

10. The probability that the machine is left alone is 0.3487 (4 d.p.).

11. $p = \dfrac{1}{3}$ and $n = 36$.

21.2 1. The probability that the first six occurs on the sixth roll is $0.06\,698$ (5 d.p.).

2. (a) The probability of passing the first time is 0.2 .
   (b) The probability of fewer than 5 failures is $0.67\,232$.
   (c) The mean number of failures is 4.

3. (a) The required probability is 0.4502 (4 d.p.).
   (b) The expected value is $\dfrac{1-p}{p} \;=\; \dfrac{1-0.1}{0.1} \;=\; 9$.

4. The probability of getting a six in any one throw is $\dfrac{1}{4}$.