Mathematics 1

Sheet 23: Discrete Random Variables

Probability Distributions

1. A random variable X has the following probability distribution:

x	$\Pr(X = x)$
1	0.1
2	0.2
3	0.3
4	0.2
5	0.1
6	0.1

Find the mean and variance of X.

2. For a particular discrete random variable X, we have

$$\Pr(X = x) = 0.1(x+1) \quad \text{where } x = 0,\ 1,\ 2,\ 3.$$

Find the mean and variance of X.

3. The number X of occupants of cars crossing a toll bridge has the following probability distribution:

x	$\Pr(X = x)$
1	0.6
2	0.3
3	0.03
4	0.05
5	0.02

(a) Find the mean and variance of X.

(b) If the toll is \$1.50 per car, plus 25 cents per occupant, find the mean toll charge a driver pays.

4. The random variable X has the following probability distribution:

x	$\Pr(X = x)$
0	0.2
1	0.15
2	0.25
3	0.4

Calculate

(a) $E(X)$

(b) $E(X^2)$

(c) $E(X^2) - [E(X)]^2$

(d) $E(2X - 1)$.

5. For a particular discrete random variable X we have $\Pr(X = x) = kx^2$ where $x = 1,\ 2,\ 3,\ 4$.

   (a) Complete this table:

x	$\Pr(X = x)$
1	
2	
3	
4	

   (b) Find the value of k.

   (c) Find $E(X)$.

   (d) Find $\mathrm{Var}(X)$.

6. A die is biased such that the number X appearing uppermost when the die is rolled is given by

x	$\Pr(X = x)$
1	0.1
2	0.2
3	0.3
4	0.1
5	0.1
6	0.2

   (a) Calculate $E(X)$.

   (b) Let Y denote the number appearing uppermost when a **fair** die is rolled. Calculate $E(Y)$.

   A die is **fair** if each of the 6 outcomes is equally likely.
   Thus $\Pr(Y = y) = \frac{1}{6}$ for each $y \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$.

   (c) Let Z denote the total number obtained when the two dice are rolled, so that $Z = X + Y$.

   i. Find $\Pr(Z = z)$ for each $z \in \{2,\ 3,\ \ldots,\ 12\}$.

   ii. Calculate $E(Z)$, and hence verify that
   $$E(Z) = E(X) + E(Y)\,.$$

7. Two students, A and B, try independently to solve a mathematical problem, and their respective probabilities of solving it are 0.8 and 0.6.

   Let X denote the number of students who solve the problem.

   (a) Find $\Pr(X = x)$ for $x = 0,\ 1,\ 2$.

   (b) Find $E(X)$.

   (c) Find $\mathrm{Var}(X)$.

8. (a) Use a formula for the variance of a random variable X to establish the following result:
   For any number a, $\quad \mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$.

   (b) Suppose that a discrete random variable X has the probability distribution below.

x	$\Pr(X = x)$
-4	0.05
-2	0.15
-1	0.2
0	0.1
3	0.1
5	0.4

   Use the result in part (a) to calculate the variance of the random variable Y for which $Y = 7X$.

Hypergeometric Distribution

9. From a group of 6 men and 4 women, a committee of 3 people is to be chosen. What is the probability that the committee contains exactly 2 women?

In Exercises 10 to 15, give your answers to four decimal places.

10. A carton contains 12 eggs, 3 of which are cracked. A cook chooses 3 eggs from the carton at random. What is the probability that exactly one cracked egg is among those eggs chosen?

11. In a group of 20 articles there are four defective items. If a sample of 5 is taken at random from this consignment, find the probability that it will contain

    (a) no defective articles

    (b) at least 3 defective articles.

12. A box contains 20 pens, of which 15 are blue and 5 are red. If 3 pens are selected at random, find the probability that at least 2 of the chosen pens are red.

13. A retail trader receives a consignment of 10 dish-washing machines, of which 3 are faulty. If he chooses 5 of the machines to put on display, what is the probability that there are exactly 2 faulty machines on display?

14. A committee of 3 is chosen from 5 men and 7 women. Find the probability that the committee contains exactly 2 men.

15. An investigation is being conducted into the quality of a certain brand of chocolate which is packed in boxes containing 10 blocks. The investigation is carried out as follows:

    A sample of 3 blocks is taken from a box and tested for flavour.

    - If all 3 blocks taste good, then the whole batch is accepted.
    - If exactly 1 of the blocks does not taste good, another 2 blocks are selected from the box. If both of these blocks from the second sample taste good, then the whole batch is accepted.
    - Otherwise the whole batch is rejected.

    Suppose that the box contains 3 blocks which do not taste good (and 7 blocks which do taste good). Find the probability that the batch is accepted.

16. From a box of 10 balls, a random sample of 2 is taken (without replacement). If the box contains a mixture of blue and black balls, and if the probability that the sample contains exactly 2 blue balls is $\frac{1}{15}$, find the number of blue balls in the box.

Revision on Antidifferentiation and Integration

Reference: Chapter 11

17. Find the most general antiderivative of the following expressions:

(a) $x^2 + 3 - \dfrac{2}{x}$ (b) $e^{-2x} + 3\sin 4x$

(c) $2\sin 2x - \cos 3x$

18. Evaluate the following integrals:

(a) $\displaystyle\int_1^2 \frac{1}{2x}\,dx$ (b) $\displaystyle\int_1^9 \frac{1}{2x}\,dx$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} (5x + \sin 2x)\,dx$ (d) $\displaystyle\int_1^2 \left(2 + \frac{1}{x}\right)^2 dx$

(e) $\displaystyle\int_1^3 (x^3 + 1)\,dx$ (f) $\displaystyle\int_1^4 \frac{x+1}{\sqrt{x}}\,dx$

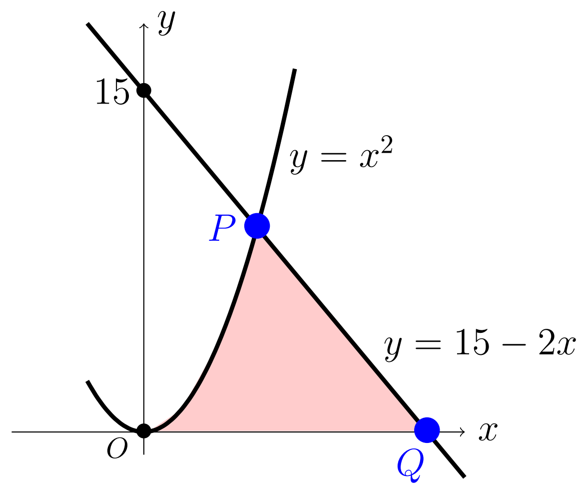
19. For each integral below, use an antiderivative to evaluate it and give your answer accurate to three decimal places.

(a) $\displaystyle\int_1^2 \left(e^{2x} + \frac{4}{x}\right) dx$

(b) $\displaystyle\int_1^2 \frac{e^{2x} - e^{-x}}{e^x}\,dx$

(c) $\displaystyle\int_0^{0.5} \sec^2(2x)\,dx$

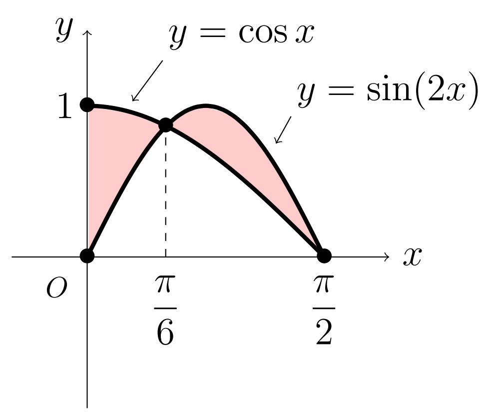
20. The graph drawn below shows the curve $y = x^2$ and the straight line $y = 15 - 2x$.



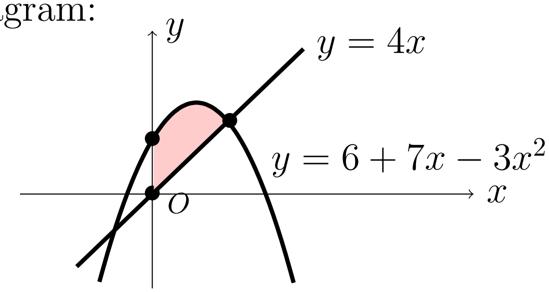
(a) Find the coordinates of P and Q.

(b) Find the area of the shaded region.

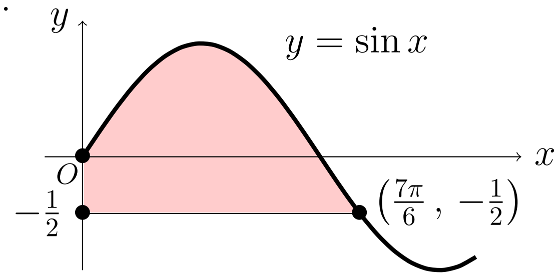
21. Find the area of the shaded region in the following diagram:



22. Find the area of the shaded region in the following diagram:



23. The graph drawn below shows the curve $y = \sin x$. Calculate, to three decimal places, the area of the shaded region.



24. (a) Sketch the graph of $f(x) = \dfrac{2}{1 + x^2}$.

Hint: Use the method for drawing graphs of reciprocal functions.

(b) On the same axes, sketch $g(x) = x^2$.

(c) Find the coordinates of the points of intersection of these graphs.

(d) Calculate the area of the region enclosed by the two graphs.

Answers:

1. $E(X) = 3.3, \quad \mathrm{Var}(X) = 2.01$

2. $E(X) = 2, \quad \mathrm{Var}(X) = 1$

3. (a) The mean is 1.59, and the variance is 0.8419.

   (b) The mean toll charge is
   $E(1.50 + 0.25X) = 1.50 + 0.25E(X) = \$1.8975$

4. (a) $E(X) = 1.85$ (b) $E(X^2) = 4.75$

   (c) $E(X^2) - [E(X)]^2 = 1.3275$ (d) $E(2X - 1) = 2.7$

5. (a)

x	$\Pr(X = x)$
1	k
2	$4k$
3	$9k$
4	$16k$

   (b) $k = \frac{1}{30}$,
   since we need $\sum \Pr(X = x) = 1$

   (c) $E(X) = \frac{10}{3}$

   (d) $\mathrm{Var}(X) = \frac{31}{45}$.

6. (a) $E(X) = 3.5$ (b) $E(Y) = 3.5$

   (c) (i)

z	$\Pr(Z = z)$
2	$\frac{1}{60}$
3	$\frac{3}{60}$
4	$\frac{6}{60}$
5	$\frac{7}{60}$
6	$\frac{8}{60}$
7	$\frac{10}{60}$
8	$\frac{9}{60}$
9	$\frac{7}{60}$
10	$\frac{4}{60}$
11	$\frac{3}{60}$
12	$\frac{2}{60}$

   (ii) We find that

   $E(Z) = \sum z \Pr(Z = z) = 7.$

   Notice also that

   $E(X) + E(Y) = 3.5 + 3.5 = 7.$

   Thus we see that in this example we have $E(Z) = E(X) + E(Y)$.

   That is, we have

   $E(X + Y) = E(X) + E(Y).$

   (Note that this result will ***always*** hold, because the expected value is a ***linear*** function.)

7. (a)

x	$\Pr(X = x)$
0	0.08
1	0.44
2	0.48

(b) $E(X) = 1.4$

(c) $\mathrm{Var}(X) = 0.4$

8. (a) Recall that we *always* have $\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$. Then, putting $Y = aX$, and using the linear property of expected value (twice), we have

$$\begin{aligned}\mathrm{Var}(aX) &= E\left((aX)^2\right) - (E(aX))^2 \\ &= E\left(a^2X^2\right) - (aE(X))^2 \\ &= a^2E(X^2) - a^2(E(X))^2 \\ &= a^2\left(E(X^2) - (E(X))^2\right) \\ &= a^2\,\mathrm{Var}(X) \quad \text{as required.}\end{aligned}$$

(b) 487.06

9. The probability that the committee contains exactly 2 women is 0.3.

10. The probability of choosing exactly one of the cracked eggs is 0.4909 (4 d.p.).

11. (a) The probability that the sample contains no defective articles is 0.2817 (4 d.p.).

(b) The probability that the sample contains at least 3 defective articles is 0.0320 (4 d.p.).

12. The probability that at least 2 of the pens are red is 0.1404 (4 d.p.).

13. The probability that exactly 2 faulty machines are on display is 0.4167 (4 d.p.).

14. The probability that the committee contains exactly two men is 0.3182 (4 d.p.).

15. The probability that the box is accepted is 0.5417 (4 d.p.).

16. There are 3 blue balls in the box.

17. (a) $\dfrac{x^3}{3} + 3x - 2\ln|x| + C$

(b) $-\dfrac{1}{2}e^{-2x} - \dfrac{3}{4}\cos 4x + C$

(c) $-\cos 2x - \dfrac{1}{3}\sin 3x + C$.

18. (a) $\ln\sqrt{2} = \frac{1}{2}\ln 2$ (b) $\ln 3$ (c) $\frac{5}{8}\pi^2 + 1$

(d) $\frac{9}{2} + 4\ln 2$ (e) 22 (f) $\frac{20}{3}$

19. (a) 26.377 (3 d.p.) (b) 4.612 (3 d.p.) (c) 0.779 (3 d.p.)

20. (a) $P = (3, 9)$ and $Q = (7.5, 0)$

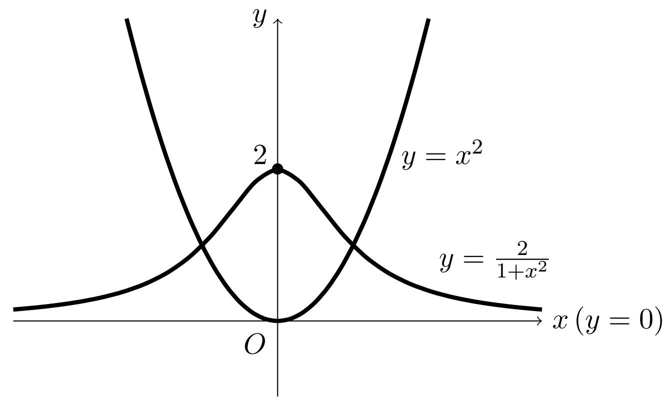
(b) 29.25 units^2.

21. 0.5 units^2

22. 10 units^2

23. 3.699 units^2 (3 d.p.)

24. (a) and (b):



(c) $(1, 1)$ and $(-1, 1)$

(d) $\pi - \frac{2}{3}$ units^2