CSC 249/449 Machine Vision: Homework 3, theoretical part

Term: Spring 2020
Instructor: Prof. Chenliang Xu
TA: Jing Shi, Zhiheng Li, Haitian Zheng, Yihang Xu, Sizhe Li, Xiaoning Gao, Tianyou Xiao, Weitao Tan
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Constraints: This assignment is to be carried out independently.

Problem 3 (15 pts): 1. SIFT feature has the scale invariant property. The critical part to enable this property, is that SIFT repetitively blur and down-sample the original image, to generate different scale space. We blur the image using Gaussian kernel to do convolution on the image.

$$G(x, y, \sigma) = \frac{1}{2\pi\sigma} e^{-(x^2+y^2)/2\sigma^2}, \tag{1}$$

where σ is 'scale', which controls how blur you want to get, and x, y are the coordinate location. Try to determine if applying gaussian blur twice to one image, equals to applying one-step Gaussian blur. If yes, what's the relationship between the scale of the two-step Gaussian blur and one-step Gaussain blur?

To put it simple, we focus on continuous 1-D unbounded signal and continuous 1-D unbounded gaussian kernel. Assume the input signal is $f(t)$, the first Gaussian blur kernel is

$$g_1(t) = \frac{1}{2\pi\sigma_1} e^{-\frac{t^2}{2\sigma_1^2}}, \tag{2}$$

and the second Gaussian blur signel is

$$g_2(t) = \frac{1}{2\pi\sigma_2} e^{-\frac{t^2}{2\sigma_2^2}}. \tag{3}$$

Now try to demonstrate if there exist Gaussian blur kernel $g_3(t)$ parameterized by σ_3, such that

$$f(t) * g_1(t) * g_2(t) = f(t) * g_3(t), \tag{4}$$

and the relationship of $\sigma_1, \sigma_2, \sigma_3$.
Hint: The convolution of two signal $f(t)$ and $g(t)$ is expressed as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau. \tag{5}$$

Problem 4 (15 pts): Least square linear fitting is one of the most fundamental fitting model. In the course, we have learned 2-D line fitting. Now let's extend it to 1) high dimensional points; 2) fitting with L2 weight penalty, which is also known as rigid regression. Now we have
- Data: $(\mathbf{x_n}, \mathbf{y_n})$, $n = 1, 2, ..., N$, $\mathbf{x_n} \in \mathbb{N}^d$, $\mathbf{y_n} \in \mathbb{N}^k$.
- Model: $\mathbf{y} = W\mathbf{x}$, where $W \in \mathbb{N}^{k \times d}$ is the weight of the linear model.
- Objective: find optimal W to minimize

$$E = ||Y - WX||^2 + \lambda ||W||^2, \tag{6}$$

where $Y = [\mathbf{y_1}, \mathbf{y_1}, ..., \mathbf{y_N}]$, $X = [\mathbf{x_1}, \mathbf{x_1}, ..., \mathbf{x_N}]$, $|| \cdot ||$ is Frobenius Norm, which means $||X|| = \sqrt{\mathrm{Tr}(X^T X)}$.
Try to prove that the optimal W is the

$$W = YX^T(XX^T + \lambda I)^{-1} \tag{7}$$