第二周作业

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set for 24-Feb-2025

1.1 Problem 1

Let $\mathbb{F}$ be any field , and let $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ be bases of U and V, respectively . Define $\mathrm{Hom}_{\mathbb{F}}(U,V)$ as the set of $\mathbb{F}$ -linear maps from U to V .

1. Endow $\mathrm{Hom}_{\mathbb{F}}(U,V)$ with the structure of a vector space over $\mathbb{F}$.
2. Determine the dimension of $\mathrm{Hom}_{\mathbb{F}}(U,V)$ and provide a basis for it.

解答

1. 由于 $\mathrm{Hom}_{\mathbb{F}}(U,V)$ 是从 U 到 V 的线性映射的集合, 因此我们可以定义加法和数乘如下:

$$(f+g)(u) = f(u) + g(u)$$
$$(\lambda f)(u) = \lambda f(u)$$

   其中 $f,g \in \mathrm{Hom}_{\mathbb{F}}(U,V)$, $\lambda \in \mathbb{F}$, $u \in U$.

2. 由于 $\{u_i\}_{i=1}^n$ 和 $\{v_j\}_{j=1}^m$ 分别是 U 和 V 的基, 因此我们可以定义一个线性映射 f_{ij} 如下:

$$f_{ij}(u_k) = v_j \delta_{ik}$$

   显然, $\{f_{ij}\}_{i=1,j=1}^{n,m}$ 是 $\mathrm{Hom}_{\mathbb{F}}(U,V)$ 的一个基. 由此我们可以得到 $\mathrm{Hom}_{\mathbb{F}}(U,V)$ 的维数为 nm.

1.2 Problem 2

(Blank-Filling Question)Throughout , $\mathbb{F}$ is an arbitrary field, and $\{u_i\}_{i=1}^l$, $\{b_j\}_{j=1}^m$, and $\{c_k\}_{k=1}^n$ are bases of U, V and W , respectly. Carefully endow the following spaces with $\mathbb{F}$ -linear structures. and write down their dimensions along with the corrseponding distinguished bases.

1. $U \oplus V$ as subspace.
2. $U \times V$ as the usual Cartesian product.
3. $\mathrm{Hom}_{\mathbb{F}}(U,V)$ as the set of linear maps.
4. $\mathrm{Hom}_{\mathrm{Sets}}(U,V)$ as the set of maps.
5. $\mathrm{Hom}_{\mathbb{F}}(U \times V, W)$, where $U \times V$ is defined a priori.
6. $\mathrm{Bil}_{\mathbb{F}}(U, VlW)$, the set of bilinear maps from U and V to W.
7. $\mathrm{Hom}_{\mathbb{F}}(U, \mathrm{Hom}_{\mathbb{F}}(V,W))$,also known as the currying of bilinear maps.

解答

1. $U \oplus V$ 是 U 和 V 的直和,
$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \qquad \lambda(u_1, v_1) = (\lambda u_1, \lambda v_1)$$
维数为 $l + m$, 基为 $\{u_i\}_{i=1}^{l} \cup \{v_j\}_{j=1}^{m}$.
2. $U \times V$ 是 U 和 V 的笛卡尔积, 维数为 $l + m$, 基为 $\{(u_i, 0)\}_{i=1}^{l} \cup \{(0, v_j)\}_{j=1}^{m}$.
3. $\mathrm{Hom}_{\mathbb{F}}(U, V)$ 是从 U 到 V 的线性映射的集合, 维数为 lm, 基为 $\{f_{ij}\}_{i=1,j=1}^{l,m}$.
4. $\mathrm{Hom}_{\mathrm{Sets}}(V, W)$ 是从 U 到 V 的映射的集合, 维数为 m^l, 基为 $\{g_i\}_{i=1}^{l}$.
5. $\mathrm{Hom}_{\mathbb{F}}(U \times V, W)$ 是从 $U \times V$ 到 W 的线性映射的集合, 维数为 $(l + m)n$, 基为 $\{h_{ijk}\}_{i=1,k=1}^{l,n} \cup \{h_{ijk}\}_{j=1,k=1}^{m,n}$.
6. $\mathrm{Bil}_{\mathbb{F}}(U, V; W)$ 是从 U 和 V 到 W 的双线性映射的集合, 维数为 lmn, 基为 $\{f_{ijk}\}_{i=1,j=1,k=1}^{l,m,n}$.
7. $\mathrm{Hom}_{\mathbb{F}}(U, \mathrm{Hom}_{\mathbb{F}}(V, W))$ 是从 U 到 $\mathrm{Hom}_{\mathbb{F}}(V, W)$ 的线性映射的集合, 维数为 lmn, 基为 $\{g_{ijk}\}_{i=1,j=1,k=1}^{l,m,n}$.

2 Problem Set for 24-Feb-2025

2.1 Exercise 1.1

Let $P_n := \{f(x) \in \mathbb{F}[x] \mid \deg f(x) < n\}$. Pick $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that $a_i \neq a_j$ for any $i \neq j$. Show that
$$f_j(x) := \prod_{i \neq j} (x - a_i)(1 \leq i \leq n)$$
from a basis of P_n.

解答 Lagrange 插值公式告诉我们, 对于任意的 $f(x) \in P_n$, 我们有
$$f(x) = \sum_{i=1}^{n} f(a_i) \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} = \sum_{i=1}^{n} \frac{f(a_i) f_j(x)}{\prod_{j \neq i}(a_i - a_j)}$$

2.2 Exercise 4.1

Assume $f(x) = x^3 + px + q \in \mathbb{Z}[x]$ is irreducible and $\alpha \in \mathbb{C}$ is a root of f.

1. Show that $\mathbb{Q}[\alpha] := \{g(\alpha) \mid g(x) \in \mathbb{Q}[x]\}$ is a linear space over $\mathbb{Q}$ and $1, \alpha, \alpha^2$ from a basis.
2. Prove that $\phi : \beta \mapsto f'(\alpha)\beta$ gives a linear map on $\mathbb{Q}[\alpha]$ and find its matrix under $1, \alpha, \alpha^2$.

解答

1. $0 \in \mathbb{Q}[\alpha], (f + g)(\alpha) = f(\alpha) + g(\alpha), (\lambda f)(\alpha) = \lambda f(\alpha)$
For $\alpha^n (n \geq 3)$, 利用等式 $\alpha^3 = -p\alpha - q$ 反复代换可以表示成 $1, \alpha, \alpha^2$ 的线性组合.

2. 由于 $\phi(\beta) = \beta f'(\alpha)$, 因此我们有

$$\phi(1) = f'(\alpha), \quad \phi(\alpha) = \alpha f'(\alpha), \quad \phi(\alpha^2) = \alpha^2 f'(\alpha)$$

   因此我们有
$$\begin{pmatrix} f'(\alpha) \\ \alpha f'(\alpha) \\ \alpha^2 f'(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q & -p & -3 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}$$

2.3 Exercise 4.3

Prove that if $f, g \in \mathbb{Z}[x]$ are primitive, then so is fg.
解答 Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{i=0}^{m} b_i x^i$, then we have $fg(x) = \sum_{i=0}^{n+m} c_i x^i$, where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.
Suppose fg is not primitive, then there exists a prime p such that $p | c_i$ for all i. Let a_i is the smallest index such that $p \nmid a_i$, b_j is the smallest index such that $p \nmid b_j$, so c_{i+j} is not divide by p