第四周作业

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Sunday 16th March, 2025

0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

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1 Problem Set For March 2025

1.1 Exercise 1

Let $V = \mathbb{C}[x]$, and let $w = \frac{-1+\sqrt{3}\,\mathrm{i}}{2}$ denote a cubic root of unite.

1. Prove that each $V_k := \{f \in \mathbb{C}[x] \mid f(x) = w^k \cdot f(wx)\}$ is a linear subspace of V.

2. Prove that $V = V_0 \oplus V_1 \oplus V_2$.

3. Prove that the mapping

$$L : V \to V, \qquad f(x) \mapsto w \cdot f(wx) - w^2 f(w^2 x)$$

   is a linear transformation.

4. Compute $\mathrm{im}(L)$ and $\ker(L)$.

5. Say $(u, \lambda \in V \times \mathbb{C})$ is an eigenppair of L, provide $u \neq 0$ and $Lu = \lambda u$.Determine all eigenpairs of L.

解答

1. $V_k \neq 0$.
   $\forall f, g \in V_k, f(x) + g(x) = w^k \cdot f(wx) + w^k \cdot g(wx) = w^k \cdot (f(wx) + g(wx)) \in V_k$
   $\forall f \in V_k, \lambda \in \mathbb{C}, \lambda f(x) = \lambda w^k \cdot f(wx) = w^k \cdot \lambda f(wx) \in V_k$.

2. Suppose $f \in V_0 \cap V_1$, then $f(x) = w^0 \cdot f(wx) = w^1 \cdot f(wx)$, which implies $f(x) = 0$.
   $\forall f(x) = \sum_{i=0}^{n} a_i x^i$, let $f_m(x) = \sum_{k=0}^{s_m} a_{3k+m} x^{3k+m}, m = 0, 1, 2 \Rightarrow f_m \in V_m$.

3. $L(af + bg)$
   $= w \cdot (af + bg)(wx) - w^2 \cdot (af + bg)(w^2 x)$
   $= a(w \cdot f(wx) - w^2 \cdot f(w^2 x)) + b(w \cdot f(wx) - w^2 \cdot f(w^2 x))$
   $= aL(f) + bL(g)$.

4. $\mathrm{im}(L)$:
   $\forall f(x) = \sum_{i=0}^{n} a_i x^i$, $L(f) = w \cdot f(wx) - w^2 \cdot f(w^2 x)$
   $= w \cdot \sum_{i=0}^{n} a_i (wx)^i - w^2 \cdot \sum_{i=0}^{n} a_i (w^2 x)^i$
   $= \sum_{i=0}^{n} (w^{i+1} - w^{2i+2}) a_i x^i$
   $= \sum_{k=0}^{s} (w - w^2)(a_{3k} x^{3k} - a_{3k+1} x^{3k+1})$
   so , $\mathrm{im}(L) = V_0 \oplus V_1$
   $\ker(L)$:
   $\forall f(x) = \sum_{i=0}^{n} a_i x^i$, $L(f) = 0$
   $\Rightarrow \sum_{i=0}^{n} (w^{i+1} - w^{2i+2}) a_i x^i = 0$
   $\Rightarrow a_{3k} = a_{3k+1} = 0$
   so, $\ker(L) = V_2$.

5. $L(f(x)) = wf(wx) - w^2 f(w^2)x$, $L(f(wx)) = wf(w^2x) - w^2 f(x)$, $L(f(w^2x)) = wf(x) - w^2 f(wx)$
   $$A = \begin{pmatrix} 0 & w & -w^2 \\ -w^2 & 0 & w \\ w & -w^2 & 0 \end{pmatrix}$$

1.2 Exercise 2

解答

1. $V = \mathbb{F}[x]$, $f : g(x) \mapsto x \cdot g(x)$
2. $V = \mathbb{F}[x]$, $f : g(x) \mapsto g'(x)$
3. ?
4. $V = \mathbb{F}[x]$, $f : h(x) \mapsto xh(x)$, $g : h(x) \mapsto \frac{h(x)-h(0)}{x}$.

2 Problem Set for 13 March 2025

2.1 Exercise 1

Let $f : U \to V$ be linear map and $\dim U = n, \operatorname{din} V = m$.

1. Let $E \subset V$ be an k-dimensional subspace. Prove that
   $$f^{-1}(E) \geq n - m + k$$
   .
2. When f is surjective , the equality holds.

解答

1. Let $g : f^{-1}(E) \to E$, then $\ker g = \ker f$, $\operatorname{im}(g) = \operatorname{im}(f) \cap E$.
   $\dim(f^{-1}(E)) = \dim(\operatorname{im}(g)) + \dim(\ker(g))$.
   $\dim(E \cap \operatorname{im} f) \geq \dim(E) + \dim(\operatorname{im}(f)) - \dim(V)$
   $\dim(V) = \dim(\operatorname{im}(f)) + \dim(\ker(f))$.
   Q.E.D
2. f is surjective , $\operatorname{im}(f) = V$, $E + \operatorname{im} f = V$

2.2 Exercise 2

解答 有限维数的线性映射和矩阵密切联系，在抽象的线性映射和运算中，几乎都能找到具体的矩阵和运算进行类比. 矩阵也可以看作一种线性映射