第五周作业

董仕强

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set for 17 March 2025

1.1 Exercise

Prove the following(Whenever we write f_*X, it is assumed that the linear space X is a subspace of the domain, and similarly for f^*X):

1. $U \subset f^*f_*U$, when does equality hold for all U?
2. $f_*f^*V \subset V$, when does equality hold for all V?
3. $f_*f^*f_* = f_*$.
4. $f^*f_*f^* = f^*$.
5. $f_*(U+V) = f_*U + f_*V$.
6. $f^*(U \cap V) = f^*U \cap f^*V$.
7. Explain $f_*(U \cap V) \subset f_*U \cap f_*V$.
8. Explain $f^*(U+V) \supset f^*U + f^*V$.

解答

1. f 单射
2. 等号能取到吧
3. $\forall v$, suppose $f(U) = v$, then $f_*(f^*v) = f_*U = v$.
4. $\forall v$, suppose $f^*(v) = U$, then $f^*(f_*u) = f^*v = U$.
5. $f(u+v) = f(u) + f(v) \in f_*U + f_*V$, so $f_*(U+V) \subset f_*U + f_*V$.
   $\forall f(u) \in f_*U$ and $f(v) \in f_*V$, $f(u) + f(v) = f(u+v) \in f_*(U+V)$. So $f_*U + f_*V \subset f_*(U+V)$.
6. $\forall x \in U \cap V$, $f^*(x) = f^*(x) \in f^*U \cap f^*V$. So $f^*(U \cap V) \subset f^*U \cap f^*V$.
   $\forall x \in f^*U \cap f^*V$,$f(x) \in U \cap V$,$x \in f^*(U \cap V)$.so $f^*U \cap f^*V \subset f^*(U \cap V)$.
7. $\forall x \in U \cap V$, $f_*(x) = f_*(x) \in f_*U \cap f_*V$. So $f_*(U \cap V) \subset f_*U \cap f_*V$.
8. $\forall x \in f^*U + f^*V$, $x = f^*(u) + f^*(v) = f^*(u+v) \in f^*(U+V)$. So $f^*U + f^*V \subset f^*(U+V)$.

1.2 Problem

1. Show that $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[x], \mathbb{F}) \cong \mathbb{F}[[x]]$.
2. Show that $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[[x]], \mathbb{F})$ has a subspace which is iso to $\mathbb{F}[x]$.

解答

1. $(\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[x], \mathbb{F}), +, \cdot)$ is a linear space over $\mathbb{F}$.we can find a basis $\{\varphi(x^k)\}_{k\geq 1}$
   let $\sigma : \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[x], \mathbb{F}) \to \mathbb{F}[[x]], f \mapsto g(x)$ is a linear map.
   let $\sigma(\varphi(x^k)) = \sum_{n=1}^{\infty} \delta_{kn} x^n$
   let $\sigma(f) = \sigma(\sum_{k=0}^{\infty} a_k \varphi(x^k)) = \sum_{k=0}^{\infty} a_k x^k = 0 \Rightarrow a_k = 0 \Rightarrow f = 0$
   and σ is surjective obveriously.
   so σ is an isomorphism.

2. $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[[x]], \mathbb{F})$ is a linear space over $\mathbb{F}$.
   let $U = \{\sum_{k=0}^{\infty} a_k \varphi(x^k) | a_k = 0, k \geq 1\}$
   let $\sigma : U \to \mathbb{F}[x], f \mapsto g(x)$ is a linear map.
   let $\sigma(\sum_{k=0}^{\infty} a_k \varphi(x^k)) = \sum_{k=0}^{\infty} a_k x^k$
   let $\sigma(f) = \sigma(\sum_{k=0}^{\infty} a_k \varphi(x^k)) = \sum_{k=0}^{\infty} a_k x^k = 0 \Rightarrow a_k = 0 \Rightarrow f = 0$
   and σ is surjective obveriously.
   so σ is an isomorphism.