第十周作业

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Exercise

问题 1.1. 1. (V is non-trivial)** Demonstrate that V is not finite-dimensional.

解答 考虑 $a(x) = \begin{cases} e^{1/x^2-1}, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$, $a_n(x) = a(x-2n)$, 显然每个 $a_n(x)$ 都是 V 中的元素, 且 $a_n(x)$ 是线性无关的, 那么 V 中有无穷多个线性无关的元素, 所以 V 不是有限维的.

问题 1.2. 2. (Differential Operator)** Show that the operator $D : V \to V$, defined by $f \mapsto f'$, is well-defined. Determine the kernel $\ker D$, the image $\operatorname{im} D$, and the cokernel $\operatorname{coker} D$.

解答 因为 V 中的元素都是光滑函数, 所以它们的导数也是光滑函数. 且大于 0 的部分其导数也是 0, 因此 $D(f)$ 也在 V 中, 所以 D 是良定义的.
2. $\ker D = 0$, 因为 $D(f) = 0$ 当且仅当 f 是常数函数, 而常数函数在 V 中只有 0.
3. $\operatorname{im} D = f \in V \mid \exists x_0, \int_{-x_0}^{x_0} f \,\mathrm{d}\, x = 0$
4. $\operatorname{coker} D = C + \operatorname{im} D$, $C \in \mathbb{R}$

问题 1.3. 3. (Integrable Functionals)** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function that is Riemann integrable over any bounded interval $[a, b]$. Show that the following map defines an element of V^*:

$$V \to \mathbb{R}, \quad f \mapsto \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x.$$

解答 定义该映射为 σ, 那么 σ 是线性映射, 因为 $\int_{-\infty}^{\infty}(af_1 + bf_2)\varphi(x)\,\mathrm{d}x = a\int_{-\infty}^{\infty} f_1(x)\varphi(x)\,\mathrm{d}x + b\int_{-\infty}^{\infty} f_2(x)\varphi(x)\,\mathrm{d}x$, 所以 σ 是线性映射.

问题 1.4. (V^*, merging some functions, and introducing some non-functions)**. The preceding exercise defines a mapping

$$\Phi : \{\text{Locally Riemann-Integrable Functions}\} \to V^*, \quad f \mapsto \left[g \mapsto \int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x\right].$$

It is known that Φ is neither injective nor surjective. - Let f be the function defined by $f(x) = 0$ for all $x \neq 1$, and $f(1) = 1$. Then $f \in \ker \Phi$. - The Dirac delta functional δ, defined informally below, is not in the image of Φ.

解答 1. 令 $f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}$, 那么 $f(x)g(x) = \begin{cases} 0, & x \neq 1 \\ g(1), & x = 1 \end{cases}$, 由数学分析知识知道 $\int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}\,x = 0$, 也就是 $\Phi(f) = 0$, 但是 $f \neq 0$, 因此 Φ 不是单射
?f 不属于 V, 应该定义 $f(x) = \lim_{n\to\infty} exp(1 - 1/(1-(nx^2))) = \lim_{n\to\infty} b(nx)$, 别的地方全是 0?

2.Dirac delta function δ, 不能被写成某两个正常黎曼可积的乘积的无穷积分.?

问题 1.5. 5. $(V \hookrightarrow V^*)$** Show that $\ker \Phi \cap V = \{0\}$, and thus that V can be regarded as a subspace of V^*.

解答 令 $f \in V$ 且满足对任意 $\varphi \in V$ 都有 $\int_{-\infty}^{infty} f\varphi \,\mathrm{d}\, x = 0$, 令 $\varphi = f$, 再由于 f 连续可以得到 $f = 0$

问题 1.6. 6. (The Dirac Delta Functional)** The Dirac delta functional δ is informally described as:

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \quad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

Provide a formal definition of δ as an element of V^*.

解答 $\delta(f) = f(0)$, 那么 $\delta(af + bg) = (af + bg)(0) = af(0) + bg(0) = a\delta(f) + b\delta(g)$

问题 1.7. *7. (Generalised Derivatives)** Define φ as a piecewise linear function passing through the points:

$$(-\infty, 0) \to (-1, 0) \to (0, 2) \to (1, -1) \to (2, 0) \to (+\infty, 0).$$

Although φ is continuous, its derivative φ' is not classically defined. Use the identity

$$\int_{-\infty}^{\infty} f(x)g'(x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} f'(x)g(x)\,\mathrm{d}x$$

to define φ' in the distributional sense. Express φ'' explicitly as a linear combination of shifted Dirac delta functions $\delta(x - a)$.

解答 $\varphi' = 2\delta(x) - \delta(x - 1)$
$\varphi'' = 2\delta^2(x) - \delta^2(x - 1)$

问题 1.8. 8. (Generalised Limits)** For each $\varphi \in V \hookrightarrow V^*$, define a sequence of functions for each $n \in \mathbb{N}_+$ by:

$$(-)_n : V \to V, \quad \varphi \mapsto [x \mapsto n \cdot \varphi(nx)].$$

It is evident that:

$$\int_{-\infty}^{\infty} \varphi_n(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x.$$

The sequence $\{\varphi_n\}$ does not converge uniformly in V. However, show that there exists $L \in V^*$ such that:

$$L(f) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\varphi_n(x)\,\mathrm{d}x.$$

Your task is to show that the limit exists for arbitrary f, and that L is linear.

解答

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\varphi_n(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_{-\infty}^{\infty} nf(x)\varphi(nx)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x/n)\varphi(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
|\int_{-\infty}^{\infty}(f(x/n)-f(0))\varphi(x)\,\mathrm{d}\,x| &= |\int_{-A}^{A}(f(x/n)-f(0))\varphi(x)\,\mathrm{d}\,x| \\
&< |M_n\int_{-A}^{A}\varphi(x)\,\mathrm{d}\,x|.
\end{aligned}
$$

其中 $M_n = \max_{-A<x<A}|(f(x/n)-f(0))| = \max_{-A/n<x<A/n}|(f(x/n)-f(0))| \to 0$, $n \to \infty$
因此 $L(f) = f(0)\int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}\,x$ 他是线性的.