高等代数 (荣誉) I 作业模板

请输入姓名

Monday 26$^{\text{th}}$ May, 2025

0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set for 19-May and 22-May

1.1 Exercise 2

Let J_A and J_B denote the Jordan form of A and B respectively (all matrices are in $\mathbb{F}^{n\times n}$).
1. Write down the Jordan form of the linear transformation $X \mapsto AXA^T$;
2. Write down the Jordan form of the linear transformation $X \mapsto AXA$;
3. Write down the Jordan form of the linear transformation $X \mapsto AX - XA$;
4. Write down the Jordan form of the linear transformation $X \mapsto AX - XA^T$;
5. Write down the Jordan form of the linear transformation $X \mapsto AXB$.
解答

1. $A\otimes A = (P^{-1}J_AP)\otimes(P^{-1}J_AP) = (P^{-1}\otimes P^{-1})(J_A\otimes J_A)(P\otimes P) = (P\otimes P)^{-1}(J_A\otimes J_A)(P\otimes P)$

2. $J_A \otimes J_A^T$

3. $J_A \otimes J_A \otimes I - I \otimes A$

4. $J_A \otimes J_A \otimes I - I \otimes A^T$

5. $J_A \otimes J_B^T$

1.2 Exercise 3

Show that $\dim\ker[X \mapsto (AX - XA^T)] \geq n$, and explain when the equality holds.
解答 考虑 $J_A \otimes I - I \otimes J_A^T$, 每个块 $J_i \otimes I - J_A^T$ 中一定有一个列为 0, 通过初等变换至少有 $n_i - 1$ 个新的列为 0, 如果存在特征值相同为 0 的列更多.
故 $\dim\ker[X \mapsto (AX - XA^T)] \geq n$ 取等条件为 A 的特征值各不相同或可对角化

1.3 Exercise 4

Show that there exists $\{0,1\}$-matrices A and B such that $A\otimes B$ and $B\otimes A$ are not similar.
解答 Let $T_1 : X \mapsto AXB^T$, $T_2 : Y \mapsto BYA^T$, $\phi : X \mapsto X^T$, then

$$\phi T_1 = T_2 \phi$$

so $A\otimes B \sim B\otimes A$

1.4 Exercise 5

Show by $1:1$ correspondence that, there exists a unique isomorphism

$$\Phi_{U,V,W} : U\otimes(V\otimes W) \to (U\otimes V)\otimes W,$$

sending simple tensor $u\otimes(v\otimes w)$ to $(u\otimes v)\otimes w$.
解答 $U\otimes(V\otimes W)$ 的基 $u_i\otimes(v_j\otimes w_k)$, $(U\otimes)V\otimes W$ 的基 $(u_i\otimes v_j)\otimes w_k$
Φ 把基一一映射到对应的另一组基, 显然是线性双射, 且唯一

1.5 Exercise 6

Show that for arbitrary linear surjection p, the linear map $\mathrm{id}_U \otimes p$ is also a surjection.
解答 p is surjective , so $\forall y \in U$, $\exists x \in U$ such that $y = p(x)$
$\forall (u, y) \in U \otimes U$, $\exists (u, x) \in U \otimes U$ such that $(\mathrm{id}_U \otimes p)(u, x) = u \otimes y$

1.6 Exercise 7

Show that for arbitrary linear injection i, the linear map $\mathrm{id}_U \otimes i$ is also an injection.
解答 i is injective, so if $i(x) = 0$, then $x = 0$
suppose that $(\mathrm{id}_U \otimes p)(u, x) = 0$, then $\mathrm{id}_U(u) = 0$ and $p(x) = 0$