第一周作业

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

0 说明 **0**

1 Problem Set for 17-Feb-2025 **1**

1.1 Problem 1 1

2 Problem Set for 20-Feb-2025 **1**

2.1 Problem 1 1

2.2 Problem 2 2

2.3 Problem 3 2

2.4 Problem 4 3

1 Problem Set for 17-Feb-2025

1.1 Problem 1

Let $\mathbb{F}$ be an arbitrary field, and let $\mathbb{F}[x]$ denote the polynomial ring(algebra) in one indeterminate. For the sake of convention, assume that $x^0 = 1$.

问题 1.1. Demonastrate that $|FF[x]$ forms a vactor space over $\mathbb{F}$ with the basis $\{x^n\}_{n\geq 0}$.

解答 $\forall f, g, h \in \mathbb{F}[x]$, $f + g = g + f$ and $(f + g) + h = f + (g + h)$.
$\forall f \in \mathbb{F}[x], a.b \in \mathbb{F}, a(bf) = (ab)f$ and $(a + b)f = af + bf$.
$\forall f, g \in \mathbb{F}[x], a \in \mathbb{F}, a(f + g) = af + ag$.
$\forall f \in \mathbb{F}[x], 1 \cdot f = f$.
$\forall f \in \mathbb{F}[x], 0 \cdot f = 0$.
$\forall f \in \mathbb{F}[x], \exists g \in \mathbb{F}[x]$ such that $f + g = 0$.
$\forall f \in \mathbb{F}[x], \exists a_i \in \mathbb{F}$ such that $f = \sum_{i=0}^{n} a_i x^i$.

问题 1.2. Determine whether the set $\{x^n + 2 \cdot x^{n-1}\}_{n\geq 1}$ constitutes a basis for $\mathbb{F}[x]$,and provide your reasoning.

解答 no.
For example, 1 can not be expressed as a linear combination of $\{x^n + 2 \cdot x^{n-1}\}_{n\geq 1}$.

问题 1.3. Investigate whether the series $\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ belongs to $\mathbb{F}[x]$.

解答 no.
For $f \in \mathbb{F}[x]$,$f = \sum_{i=0}^{\infty} a_i x^i$, only a finite number of a_i are non-zero.

问题 1.4. Is it posible to define a linear map $\mathcal{L} : \mathbb{F}\langle x \rangle \to \mathbb{F}$ such that $\mathcal{L}(f) = l(f)$ for any $f \in \mathbb{F}[x]$.

解答 yes.

$$\mathcal{L}(f) = \lim_{n\to infty} \sum_{i=0}^{n} a_i$$

if f is a finate degree polynomial, higher degree terms are zero.

2 Problem Set for 20-Feb-2025

2.1 Problem 1

Find two linear maps

$$\alpha, \beta : \mathbb{F}[x] \to \mathbb{F}[x],$$

such that

$$\alpha(\beta(f)) - \beta(\alpha(f)) = f$$

for any $f \in \mathbb{F}[x]$,
is it posible to find such $\alpha, \beta : V \to V$ when V is of finite dimension?

解答

1. Let $\alpha(f) = f'$, $\beta(f) = xf$.

2. no.for finite V ,$\alpha : V \to V$ can be represented by a matrix A , $\beta : V \to V$ can be represented by a matrix B , then $\alpha(\beta(f)) - \beta(\alpha(f)) = f$ is equivalent to $AB - BA = I$, but $tr(AB - BA) = tr(AB) - tr(BA) = 0 \neq n$

2.2 Problem 2

Prove the following:

1. If f is irreducible in $\mathbb{Z}[x]$,then it is also irreducible in $\mathbb{Q}[x]$;

2. If f is irreducible in $\mathbb{R}[x]$,then it is also irreducible in $\mathbb{Q}[x]$.

解答

1. If f is reducible in $\mathbb{Q}[x]$, then $f = gh$, $g, h \in \mathbb{Q}[x]$, g, h can be written as $g = \frac{a}{b}g_1$, $h = \frac{c}{d}h_1$, $a, b, c, d \in \mathbb{Z}$, $g_1, h_1 \in \mathbb{Z}[x]$, then $f = \frac{ac}{bd}g_1h_1$, f is reducible in $\mathbb{Z}[x]$.

2. If f is reducible in $\mathbb{Q}[x]$, then $f = gh$, $g, h \in \mathbb{Q}[x]$, g, h can be written as $g = \frac{a}{b}g_1$, $h = \frac{c}{d}h_1$, $a, b, c, d \in \mathbb{R}$, $g_1, h_1 \in \mathbb{R}[x]$, then $f = \frac{ac}{bd}g_1h_1$, f is reducible in $\mathbb{R}[x]$.

2.3 Problem 3

1. Let $f \in \mathbb{Z}[x]$ be a monic polynomial of degree n. Denote the zaros of f in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if there is existly one z_i such that $|z_i| \geq 1$ and $f(0) \neq 0$, then f is irreducible in $\mathbb{Q}[x]$.

2. Let $f \in \mathbb{Z}[x]$ be a polynomial such that $f(0)$ is a prime. DEnote the zeros of f in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if $|z_i| > 1$ for all i , then f is irreducible.

3. Let $f(x) = \sum_{k=0}^n a_k \cdot x^k \in \mathbb{Z}[x]$ be a polynomial with $f(0)$ prime. Suppose that $|a_0| > \sum_{k=1}^n |a_k|$. Show that F is irreducible.

解答

1. only prove f is irreducible in $\mathbb{Z}[x]$.(Similarly hereinafter)
   If f is reducible in $\mathbb{Z}[x]$, then $f = gh$, $g, h \in \mathbb{Z}[x]$ .Let the absolute value of all the zeros of g in $\mathbb{C}$ is less than 1, then $|g(0)| = |g(z_1)g(z_2) \cdots g(z_n)| < 1$, and $g(0) \in \mathbb{Z}$, so $g(0) = 0$,$f(0) = g(0)h(0) = 0$.

2. If f is reducible in $\mathbb{Z}[x]$, then $f = gh$, $g, h \in \mathbb{Z}[x]$.$|f(0)| = |g(0)||h(0)|$ and $|f(0)|$ is a prime. Let $|g(0)| = 1$, so at least one of absolute value of zeros of g is less than 1.

3. If exists $z \in \mathbb{C}$ such that $f(z) = 0$ and $|z| \leq 1$, then $|a_0| = |\sum_{i=1}^{n} a_i z^i| \leq \sum_{i=1}^{n} |a_i|$.

2.4 Problem 4

1. Is there any irreducible $f(x) \in \mathbb{Z}[x]$ such that $f(f(x))$ is reducible?

2. Prove that $1 + \prod_{k=1}^{2025}(x-k)^2$ is irreducible in $\mathbb{Z}[x]$;

3. Prove that $\prod_{k=1}^{n}(x - x_k) + 1$is either irreducible in $\mathbb{Z}[x]$,or a perfect square;

4. ($f \in \mathbb{Z}[x]$)Prove that if $f(x) = 1$ has ≥ 4 solutions in $\mathbb{Z}$,then $f(x) = -1$ has no solutions in $\mathbb{Z}$.

5. Prove that the partial sum $(\mathrm{e}^x)_{\deg \leq n}$ is always irreducible in $\mathbb{Q}[x]$.

解答

1. Noticed thst $f(x) = x^2 + 10x + 17$ satisfies the condition.

2. can't solve

3. Let $f(x) = \prod_{k=1}^{n}(x - x_k) + 1$, $(x_1 < x_2 < \cdots < x_n)$. If f is reducible , Let $f = gh$,$g, h \in \mathbb{Z}[x]$ and $0 < \deg(g), \deg(h) < n$, then $\forall k$, $g(x_k)h(x_k) = 1$, so $g(x_k) = h(x_k) = 1$, so $g = h$. That means $f = g^2$.

4. If $f(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)g(x)+1$, $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, $g \in \mathbb{Z}[x]$, then $f(k) = -1 \Leftrightarrow (k-a_1)(k-a_2)(k-a_3)(k-a_4)g(k) = -2$. so -2 must be divisible by $(k-a_1)(k-a_2)(k-a_3)(k-a_4)$, but -2 is not divisible by 4.

5. Let $f(x) = 1 + \sum_{k=1}^{n} \frac{x^k}{k!}$, $n! \cdot f(x) = x^n + nx^{n-1} + \cdots + \frac{n!}{2!}x^2 + n!x + n! = g(x)$ is irreducible in $\mathbb{Z}[x]$.
   Schur?