第三周作业

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set for 3th March 2025

1.1 Problem

Let $\mathbb{F}$ denote the ground field, and let S be any finite set.

1. Demonstrate that $\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F})$ forms a vector space.

2. Construct a linear bijection (hereinafter refeered to as a linear isomorphism)

$$\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) \to \mathbb{F}^{|S|}$$

3. Demonsrate that following function constitutes an injection of sets:

$$\varphi : S \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}), \mathbb{F})$$

$$s \mapsto \begin{bmatrix} \mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) & \to & \mathbb{F} \\ f & \mapsto & f(s) \end{bmatrix}$$

4. Demonstrate that the image $\varphi(S)$ forms a basis for $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}), \mathbb{F})$.

5. This is how we define

$$\mathbb{F}_{s_1} \oplus \mathbb{F}_{s_2} \oplus \cdots \oplus \mathbb{F}_{s_n} \qquad S = \{s_1, \ldots, s_n\}$$

解答

1. $\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) \neq \emptyset$
   $(f + g)(s) = f(s) + g(s)$
   $(\lambda f)(s) = \lambda f(s)$.

2. $\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) \to \mathbb{F}^{|S|}$
   $f \mapsto (f(s_1), f(s_2), \ldots, f(s_n))$

3. φ is injective.
   $\forall s \neq t$, $\varphi(s)$ is a map from $\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F})$ to $\mathbb{F}$.$\varphi(s)$,$\varphi(t)$ is two different maps.

4. $\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) \cong \mathbb{F}^{|S|}$By the iso we can see that $\varphi(S)$ is a basis for $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}), \mathbb{F})$.

5. $\mathbb{F}_{s_1} \oplus \mathbb{F}_{s_2} \oplus \cdots \oplus \mathbb{F}_{s_n} = \mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F})$

2 Problem Set for 6th March 2025

2.1 Exercise 1

Here is the compositon rules for lnear maps $U \to V \to W$. Show that

1. if f and g are linear injections, then so is $g \circ f$.

2. if $f \circ g$ is a linear injection, then so is f.

3. if f and g are linear surjections, then so is $g \circ f$.
4. if $f \circ g$ is a linear surjection, then so is g.

解答

1. $\forall x, y \in U$, $g(f(x)) = g(f(y)) \Rightarrow f(x) = f(y) \Rightarrow x = y$.
2. $\forall x, y \in U$, $f(x) = f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow x = y$.
3. $\forall z \in W$, $\exists x \in U$, $g(f(x)) = z$.
4. $\forall z \in W$, $\exists x \in U$, $g(f(x)) = z$.

2.2 Exercise 2

show that:

1. $\Phi(f_1, f_2)$ is an injection, only if f_1 and f_2 are injections.
2. $\Phi(f_1, f_2)$ is a surjection, if f_1 or f_2 are surjections.
3. $\Psi(f_1, f_2)$ is an injection, if f_1 or f_2 are injections.
4. $\Psi(f_1, f_2)$ is a surjection, only if f_1 and f_2 are surjections.

解答

1. $(\Phi(f_1, f_2))(u_1, u_2) = f_1(u_1) + f_2(u_2)$
   so
   $$f_1(u_1) = f_1(u_1') \Rightarrow (\Phi(f_1, f_2))(u_1, u_2) = (\Phi(f_1, f_2))(u_1', u_2) \Rightarrow u_1 = u_1'$$
   so f_1 and f_2 are injections.
2. $\forall z \in V$, $\exists u_1 \in U_1$, $\exists u_2 \in U_2$, $f_1(u_1) = z$ or $f_2(u_2) = z$.
3. $(\Psi(f_1, f_2))(u) = (f_1(u), f_2(u))$
   so
   $$\forall u_1 \neq u_2, \text{ only need } f_1(u_1) \neq f_1(u_2) \text{ or } f_2(u_1) \neq f_2(u_2)$$
   so f_1 or f_2 are injections.
4. $\forall z_1 \in V_1, z_2 \in V_2$, need $\exists u \in U$ such that $f_1(u) = z_1$ and $f_2(u) = z_2$.