第六周作业

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set for 24 March 2025

1.1 Exercise

Prove Isomoprphism Theorem A
解答

1. Let $\phi : U/\ker\varphi \to \operatorname{im}\varphi$.
2. $\phi(u+\ker\varphi+v+\ker\varphi) = \phi(u+v+\ker\varphi) = \varphi(u+v) = \varphi(u)+\varphi(v) = \phi(u+\ker\varphi)+\phi(v+\ker\varphi)$
   $\phi(\lambda(u+\ker\varphi)) = \phi(\lambda u+\ker\varphi) = \varphi(\lambda u) = \lambda\varphi(u) = \lambda\phi(u+\ker\varphi)$
3. If $\phi(u+\ker\varphi) = 0$, then $\varphi(u) = 0$ and $u+\ker\varphi = \ker\varphi$. then $[u] = [0]$
4. For any $y \in \operatorname{im}\varphi$, $\exists u$ such that $\varphi(u) = y$, then $\phi(u+\ker\varphi) = \varphi(u) = y$, so ϕ is surjective.

1.2 Exercise

Use Isomoprphism Theorem A to Prove Isomoprphism Theorem C.
解答 Define the map:

$$f : W/U \to W/V, \quad w+U \mapsto w+V$$

the map is surjective, since any $y \in W/V$ is of the form $w+V$, hence $y = (w+V)$ has a preimage w.
The kernel of f consists of elements for which $f(w+U) = 0$, i.e. $w \in V$ .Thus , $\ker f = V \in U$.
By Isomoprphism Theorem A, we obtain:

$$\frac{W}{V} \simeq \frac{W/U}{V/U}$$

1.3 Exercise

Let $U_i \subset V_i$ be subspace, prove the isomoprphism,

$$\frac{V_1 \times V_2}{U_1 \times U_2} \simeq \frac{V_1}{U_1} \times \frac{V_2}{U_2}$$

解答 Define the map

$$f : V_1 \times V_2 \to \frac{V_1}{U_1} \times \frac{V_2}{U_2}, quad (v_1, v_2) \mapsto (v_1+U_1, v_2+U_2)$$

The map is surjective, since any $y \in \frac{V_1}{U_1} \times \frac{V_2}{U_2}$ is of the form (v_1+U_1, v_2+U_2) , hence $y = (v_1+U_1, v_2+U_2)$ has a preimage (v_1, v_2).
The kernel of f consists of elements for which $f(v_1+U_1, v_2+U_2) = 0$, i.e. $v_1 \in U_1$ and $v_2 \in U_2$. Thus , $\ker f = U_1 \times U_2$.
By Isomoprphism Theorem A, we obtain:

$$\frac{V_1 \times V_2}{U_1 \times U_2} \simeq \frac{V_1}{U_1} \times \frac{V_2}{U_2}$$

1.4 Exercise

Let $f : V \to V$ be a liunear map. Use the Isomoprphism Theorem A to show that

$$\frac{\operatorname{im} f}{\operatorname{im} f \cap \ker f} = \operatorname{im} f \circ f = \frac{\operatorname{im} f + \ker f}{\ker f}$$

解答 Define the map

$$f : V/\ker f \to \operatorname{im} f$$

The map is surjective, since any $y \in \operatorname{im} f$ is of the form $f(v)$, hence $y = f(v)$ has a preimage v. The kernel of f consists of elements for which $f(v+\ker f) = 0$, i.e. $v \in \ker f$ .Thus , $\ker f = \operatorname{im} f \cap \ker f$. By Isomoprphism Theorem A, we obtain:

$$\frac{\operatorname{im} f}{\operatorname{im} f \cap \ker f} = \operatorname{im} f \circ f = \frac{\operatorname{im} f + \ker f}{\ker f}$$

1.5 Exercise

Let be $X \to Y \to Z$ linear maps with no additional assumptions. Prove that

1. $g^{-1}(g(f(X))) = \operatorname{im} f + \ker g$
2. $f(f^{-1}g^{-1}(0)) = \operatorname{im} f \cap \ker g$
3. $\frac{g^*(0)}{f_*f^*g*0} \simeq \frac{f_*X}{g^*g_*f_*X}$

解答

1. $g^{-1}(g(f(X))) = g^{-1}(g(\operatorname{im} f + 0)) = g^{-1}g(\operatorname{im} f) + g^{-1}(0) = \operatorname{im} f + \ker g$
2. $g^{-1}(0) = \ker g$ and $g^{-1}(0) \in \operatorname{im} f$
3. 带入上述两问就是 isomorphism 定理 B 的应用.

2 Problem Set for 24 March 2025

2.1 Exercise 0

解答

2 $\operatorname{im} g/\operatorname{im} f \to \operatorname{im} fg, \quad v + \operatorname{im} f \mapsto w$

4 $\operatorname{Hom}(U,W) \to \operatorname{Hom}(\operatorname{Hom}(V,W), \operatorname{HGm}(U,V)), \quad f \mapsto g$

5 as above.

6 $\operatorname{Hom}_{\mathrm{Set}}(S,V) \to V^n, \quad f \mapsto v$

7 $\operatorname{Hom}_{\mathbb{F}}(\mathbb{F}[[x]], \mathbb{F}) \to \mathbb{F}[x]$.

2.2 Exercise 1

Show that $U \simeq \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, U)$ for any $\mathbb{F}$-linear space U.

解答

1. Define the map
$$f : U \to \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, U), \quad u \mapsto f_u$$
2. $f(\lambda u) = f_u(\lambda) = \lambda f_u(1) = \lambda f(u)$
$f(u + v) = f_{u+v}(1) = f_u(1) + f_v(1) = f(u) + f(v)$
3. Let $f(u) = 0$, then $f_u(1) = 0$. So $f_u(\lambda) = 0$ for all $\lambda \in \mathbb{F}$.So $u = 0$
4. For all $f_u \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, U)$, there has $u = f_u(1)$ in U sucj that $f(u) = f_u$
5. Thus, f is bijective.

2.3 Exercise 2

Show that $U \simeq \mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(U, F), F)$ if $\dim U < \infty$.

解答

1. Define $\Phi : U \simeq \mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(U, F), F), \quad u \mapsto [f \mapsto f(u)]$
2. $\Phi(\lambda u) = \lambda\Phi(u)$
$\Phi(u + v) = \Phi(u) + \Phi(v)$
3. Let $\Phi(u) = 0$, then $f(u) = 0$ for all $f \in \mathrm{Hom}_{\mathbb{F}}(U, F)$. So $u = 0$
4. For all $f \in \mathrm{Hom}_{\mathbb{F}}(U, F)$, there has $u = f(u)$ in U such that $\Phi(u) = f$
5. Thus, Φ is linear Isomoprphism.

2.4 Exercise 3

Let V be a linear space and $S \subset V$ is linearly independent (S is not necessary finite). SHow that

$$\mathrm{Hom}_{\mathbb{F}}(\mathrm{span}(S), \mathbb{F}) \simeq \mathrm{Hom}_{\mathrm{Sets}\,9S,\mathbb{F}}$$

解答

1. Define the map
$$\Phi : \mathrm{Hom}_{\mathbb{F}}(\mathrm{span}(S), \mathbb{F}) \to \mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}), \quad f \mapsto (f(s))_{s\in S}$$
2. $\Phi(f + g) = ((f + g)(s))_{s\in S} = (f(s) + g(s))_{s\in S} = \Phi(f) + \Phi(g)$
$\Phi(\lambda f) = ((\lambda f)(s))_{s\in S} = (\lambda f(s))_{s\in S} = \lambda\Phi(f)$
3. Let $\Phi(f) = 0$, then $f(s) = 0$ for all $s \in S$. So $f = 0$
4. For all $g \in \mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F})$, there has $f(s) = g(s)$ in $\mathrm{span}(S)$ such that $\Phi(f) = g$
5. Thus, Φ is linear Isomoprphism.

2.5 Exercise 4

Recall that $\mathbb{C}$-linear spaces are $\mathbb{R}$-linear spaces. Show that $\mathrm{Hom}_{\mathbb{R}}(U,V) \simeq (\mathrm{Hom}_{\mathbb{C}}(U,V))^2$.
解答

0. Let $f, g \in \mathrm{Hom}_{\mathbb{C}}(U,V)$, $\lambda \in \mathbb{R}$, then $f+g \in \mathrm{Hom}_{\mathbb{C}}(U,V)$ and $\lambda f \in \mathrm{Hom}_{\mathbb{C}}(U,V)$

1. Define the map
$$\Phi : \mathrm{Hom}_{\mathbb{R}}(U,V) \to (\mathrm{Hom}_{\mathbb{C}}(U,V))^2, \quad f \mapsto (f_1, f_2)$$
$f_1(u) := f(u) - if(iu)$,then
$(f_1+g_1)(u) = (f+g)u - i(f+g)(iu) = f(u) + if(iu) + g(u) + ig(iu) = f_1(u) + g_1(u)$
$(\lambda f_1)u = (\lambda f)u - i(\lambda f)(iu) = \lambda(f(u) - if(iu)) = \lambda f_1(u)$.

2. $\Phi(f+g) = (f_1+g_1, f_2+g_2) = (f_1,f_2) + (g_1,g_2) = \Phi(f) + \Phi(g)$
$\Phi(\lambda f) = (\lambda f_1, \lambda f_2) = (\lambda f_1, \lambda f_2) = \lambda\Phi(f)$

3. Let $\Phi(f) = 0$, then $f_1(u) = 0$ and $f_2(u) = 0$ for all $u \in U$. So $f = 0$

4. For all $(f_1, f_2) \in (\mathrm{Hom}_{\mathbb{C}}(U,V))^2$, there has $f(u) = f_1(u) + if_2(u)$ in V such that $\Phi(f) = (f_1, f_2)$

1. Thus, Φ is linear Isomoprphism.