高等代数 (荣誉) I 作业模板

请输入姓名

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set For 07 April 2025

1.1 Exercise

Let V be a vector space over a field $\mathbb{F}$ and define $\mathrm{End}(V) := \mathrm{Hom}_{\mathbb{F}}(V, V)$.Define:

- $f \in \mathrm{End}(V)$ as almost zero (denoted $f \in \mathrm{AZ}(V)$), provided $\dim \mathrm{im}(f) < \infty$.
- $f \in \mathrm{End}(V)$ as almost isomorphism (denoted $f \in \mathrm{AI}(V)$), provided $\dim \ker(f) + \dim \frac{V}{\mathrm{im}\, f} < \infty$.

Now finish the following:

1. if dimV $< \infty$,determine $\mathrm{AZ}(V)$ and $\mathrm{AI}(V)$.
2. if $V = \mathbb{R}[x]$, find an injective $f in \mathrm{AI}(V)$ and surjective $g \in \mathrm{AI}(V)$.Additionally, find some $h \neq \mathrm{AI}(V) \cup \mathrm{AZ}(V)$.
3. Determine whether $\mathrm{AZ}(V)$ and $\mathrm{AI}(V)$ are subspaces of $\mathrm{End}(V)$.
4. Show that for arbitrary $f, g \in \mathrm{AZ}(V)$,one has $g \circ f \in \mathrm{AZ}(V)$
5. Prove that AZ is an ideal in $\mathrm{End}(V)$.
6. Show that for arbitrary $f, g \in \mathrm{AI}(V)$. One has $g \circ f \in \mathrm{AI}(V)$.
7. Demonsrate that if any two of $\{f, g, g \circ f\}$ belong to $\mathrm{AI}(V)$, then so does the third.
8. Show that for arbitrary $f \in \mathrm{AZ}(V)$, one has $(\mathrm{id}_V + f) \in \mathrm{AI}(V)$.

解答

1. 由于 dimV $< \infty$, 所以 $\dim \mathrm{im}(f) < \infty$ 说明 $f \in \mathrm{AZ}(V)$, 因此 $\mathrm{AZ}(V) = \mathrm{End}(V)$. 同理 $\mathrm{AI}(V) = \mathrm{End}(V)$
2. $f : x^n \mapsto x^{n+1}, n = 0, 1, \cdots$
   $g : x^n \mapsto nx^{n-1}, n = 0, 1, \cdots$
   $h : x^n \mapsto x^n \delta_{n,2[n/2]}, n = 0, 1, \cdots$.
3. $\mathrm{AZ}(V)$ 是子空间, 因为 $\dim \mathrm{im}(f + g) \leq \dim \mathrm{im}(f) + \dim \mathrm{im}(g) < \infty$.
   $\dim \mathrm{im}(\lambda f) = \dim \mathrm{im}(f) < \infty (\lambda \neq 0)$
   $\mathrm{AI}(V)$ 是子空间, 证明为 7
4. $\dim \mathrm{im}(g \circ f) \leq \dim \mathrm{im}(f) < \infty$ or $\dim \mathrm{im}(g \circ f) \leq \dim \mathrm{im}(g) < \infty$.
5. As above.
6. 见 7
7. $F : \ker g \circ f \to \ker g, \quad x \mapsto f(x)$. Then $\ker(g \circ f) \supset \ker f$, so $\ker F = \ker f$,and it is surjective.
   dually, $[V/\mathrm{im}\, g \to V/\mathrm{im}(g \circ f), \quad u + g(V) \mapsto u + g(f(V))]$ has kernel $V/(\mathrm{im}\, f + \mathrm{im}\, g)$.
   so $\dim \ker f + \dim \ker g \cap \mathrm{im}\, f = \ker g \circ f$, $\dim V/(\mathrm{im}\, f + \mathrm{im}\, g) + \dim V/\mathrm{im}(g \circ f) = \dim V/\mathrm{im}\, g$

2 Problem Set For 10 April 2025

2.1 Exercise 1

解答 Any $V \xrightarrow{f} U_2$, which factors through $V_1 \xrightarrow{\varphi} V_1$, factors through φ uniquely.
Every $U_2 \xrightarrow{f} V$ factors through $U_1 \xrightarrow{\psi} U_2$.
The subspace of (U, V_1) consisting of the linear maps whose kernel contains the image of φ.
The subspace of (U_2, V) consisting of the linear maps which factor through $U_2 \xrightarrow{\psi} U_1$.

2.2 Exercise 2

Suppose that $K \subseteq U$ is the kernel of $U \xrightarrow{f} V$. Denote the inclusion by the linear map $i : K \hookrightarrow U$. Show that for any linear space W, the linear map

$$(f \circ -) : (W, U) \to (W, V), \quad \varphi \mapsto f \circ \varphi$$

has kernel (W, K) along with the inclusion

$$(i \circ -) : (W, K) \hookrightarrow (W, U), \quad \varphi \mapsto i \circ \varphi.$$

Show in steps that

1. $(i \circ -)$ is indeed an injection, identifying (W, K) as a subspace of (W, U)
2. Show that $\ker(f \circ -)$ coincides with the image of $(i \circ -)$, that is, the subspace (W, K).

解答

1. 由于 i 是单射, 所以 $(i \circ -)$ 也是单射.
   $(i \circ -)$ 是线性映射, 所以 (W, K) 是 (W, U) 的子空间.
2. 对任意的 (W, K) 中的元素 φ, 由 $f \circ \varphi = 0$ 可知 $\varphi \in \ker(f \circ -)$.

2.3 Exercise 3

Suppose that $V \twoheadrightarrow \frac{V}{\operatorname{im} f}$ is the quotient map induced by a given $U \xrightarrow{f} V$. Denote the natural quotient map by the linear map $\pi : V \twoheadrightarrow \frac{V}{\operatorname{im} f}$. Show that for any linear space X, the linear map

$$(- \circ f) : (V, X) \to (U, X), \quad \psi \mapsto \psi \circ f$$

has kernel $(V/(\operatorname{im} f), X)$ along with the inclusion

$$(- \circ \pi) : (V/(\operatorname{im} f), X) \hookrightarrow (V, X), \quad \psi \mapsto \psi \circ \pi.$$

Show in steps that

1. $(- \circ \pi)$ is indeed a surjection, identifying $(V/(\operatorname{im} f), X)$ as a subspace of (V, X).

2. Show that $\ker(- \circ f)$ coincides with the image of $(- \circ \pi)$, that is, the subspace $(V/(\operatorname{im} f), X)$.

解答

1. 由于 π 是满射, 所以 $(- \circ \pi)$ 也是满射.
   $(- \circ \pi)$ 是线性映射, 所以 $(V/(\operatorname{im} f), X)$ 是 (V, X) 的子空间.

2. 对任意的 $(V/(\operatorname{im} f), X)$ 中的元素 φ, 由 $\varphi \circ \pi = 0$ 可知 $\varphi \in \ker(- \circ f)$.