高等代数 (荣誉) I 作业模板

请输入姓名

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

0.1 Exercise 1

Characterise the ceigenspace corresponding to the ceigenvalue 0.
解答 $\bar{A}$ 的零空间

0.2 Exercise 2

Demonstrate that if λ is a ceigenvalue, then so too is $e^{i\theta} \cdot \lambda$. Furthermore, show that for any ceigenvector v, there exists a ceigenvector associated with a ceigenvalue ≥ 0.
解答 考虑 $A\bar{v} = \lambda v$, $\Rightarrow A(\overline{e^{-i\theta}v}) = \lambda e^{2i\theta}(e^{-i\theta}v)$

0.3 Exercise 3

Suppose (v, σ) is a ceigenpair of A; then $(v, \sigma\overline{\sigma})$ is an eigenpair of $A\overline{A}$. In particular,
- if $A \in \mathbb{C}^{d\times d}$ possesses d ceigenpairs which are linearly independent over $\mathbb{C}$, then $A\overline{A}$ is diagonalisable with all eigenvalues non-negative.
Now consider the following **partial converse statement**:
- A has no ceigenvectors if and only if $A\overline{A}$ admits no eigenvalues in $\mathbb{R}_{\geq 0}$; - $A \in \mathbb{C}^{d\times d}$ has d linearly independent ceigenpairs over $\mathbb{C}$ if and only if $A\overline{A}$ is diagonalisable with non-negative eigenvalues.
解答 $A\bar{A}v = \sigma\bar{\sigma}v$,

0.4 Exercise 4

Elucidate **Exercise 3** (particularly the warning) with the following example:

$$A = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

解答 $A\bar{A} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 因此 v 是 $A\bar{A}$ 的特征向量, 但是可以计算 v 不是 A 的 c 特征向量.

0.5 Exercise 5

This serves to complete **Exercise 2**. Suppose $\{(v_i, \lambda_i)\}_{i=1}^n$ are ceigenpairs of A such that $|\lambda_i| \neq |\lambda_j|$; show that $\{v_i\}_{i=1}^n$ are linearly independent.
解答 $\{(v_i, |\lambda_i|^2)\}_{i=1}^n$ 也是 $A\bar{A}$ 的特征对, 由于 $|\lambda_i|^2 \neq |\lambda_j|^2$, 所以 $\{v_i\}_{i=1}^n$ 线性无关.

0.6 Exercise 6

We aim to identify the analogue of the Jordan canonical form in the c-version, namely, $X = PJ\overline{P}^{-1}$. We commence with a very special case:
- $P\overline{P} = I$ if and only if $P = Q\overline{Q}^{-1}$ for some invertible Q.

解答 "$\Rightarrow$": $P\bar{P} = I$, Let $Q = P + I$, then $P\bar{Q} = P\bar{P} + P = P + I + Q$

0.7 Exercise 7

The most elementary Jordan matrix is the diagonal matrix. Show that A is cdiagonalisable (i.e., $A = PD\overline{P}^{-1}$ for some P) if and only if $A\overline{A}$ is diagonalisable. In this scenario, $A\overline{A}$ has only non-negative eigenvalues.

解答 "$\leftarrow$": $A\bar{A} = P^{-1}DP$ and $D > 0$, then we can find P such that $A = P\sqrt{D}P^{-1}$

0.8 Exercise 8

(Nilpotent part). Show that there exists a matrix P such that $PA\overline{P}^{-1} = \begin{pmatrix} N & O \\ O & B \end{pmatrix}$, where N comprises nilpotent Jordan blocks and B is invertible.

解答 把 $A\bar{A}$ 进行 $Jordan$

0.9 Exercise 9

To analyse the Jordan form of A, we begin by studying the Jordan form of $A\overline{A}$. Demonstrate the following:

1. $A\overline{A}$ has a characteristic polynomial with real coefficients;
2. non-real Jordan blocks occur in conjugate pairs;
3. if $\dim\ker(\lambda I - A\overline{A}) = 1$ and $\lambda \in \mathbb{R}$, then $\lambda \geq 0$;

解答 1. Ex 2. $A\bar{A}$ 的特征值都是实数, 其特征多项式是实系数的.

2. Ex 12. A is csimilar to $\bar{A}$ and $\bar{A\bar{A}} = \bar{A}A$ 3.$v^*A\bar{A}v = \lambda v^*v$, 左为正, 故 $\lambda > 0$

0.10 Exercise 10

解答 对 $A\bar{A}$ 正特征值部分直接取对应特征向量, 对复数特征值部分, 交替选取 z 和 $\bar{z}$ 对应特征向量, 对负数特征值部分, 个数一定是偶数, 然后对对应 Jordan 块直和得到半-Jordan 块后再计算.