第十三周作业

董仕强

Monday 19$^{\text{th}}$ May, 2025

0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

0.1 Exercise 1

Determine whether the following are inner product spaces (简要回答即可).

1. V is the vector space continuous functions over $\mathbb{R}$, and the pairing is

$$(f,g) \mapsto \int_0^1 f(x)g(x)\,\mathrm{d}\,x.$$

2. $V = \mathbb{R}[x]$, and the pairing is

$$(f,g) \mapsto \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,\mathrm{d}\,x.$$

3. $V = \mathbb{R}^{n\times n}$, and the pairing is

$$(A,B) \mapsto \det(A^T \cdot B).$$

解答 1. 是

2. 是, e^{-x^2} 保证了收敛性

3. 不是, 没有双线性

0.2 Exercise 2

Show that every real matrix is similar to a block diagonal matrix, wherein the candidates of the blocks are

1. the Jordan form of a real eigenvalue;

2. the block of the form

$$\begin{bmatrix} H_{r,\theta} & Q & & & \\ & H_{r,\theta} & Q & & \\ & & \ddots & \ddots & \\ & & & H_{r,\theta} & Q \\ & & & & H_{r,\theta} \end{bmatrix},$$

where

1. $H_{r,\theta} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$, and 2. $Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

解答 对实矩阵进行 Jordan 分解, 其特征值为实数的保持不变

对于复数特征值 $\lambda = re^{i\theta}$, 可以把复数写成矩阵形式

0.3 Exercise 2.5

Show that every real matrix is a product of two real symmetric matrices.

解答 $A = P^{-1}JP$, 然后把每个块 J_i 乘次对角线全为 1 的矩阵得到 J_i', 那么 $A = P^{-1}J'(P^{-1})^T \cdot P^T I' P$

0.4 Exercise 3

Let $(V, (-,-))$ be a finite dimensional real inner product space. For unit vector v_0 (i.e., $(v_0, v_0) = 1$), we define the reflection

$$\varphi : V \to V, \quad x \mapsto x - 2(x, v_0)\cdot v_0.$$

Show that for isometric transform ψ (i.e., $(\psi(u),\psi(v))=(u,v)$), 1 is an eigenvalue for ψ or $\psi\circ\varphi$. Overall, $1\in\sigma(\psi)\cup\sigma(\psi\circ\varphi)$.

解答 假设 1 不是 ψ 的特征值, 令 $(\psi\circ\phi)(x)=x$, 即寻找非 0 的 x 使得 $(\psi-I)(x)=2(x,v_0)\psi(v_0)$, 即 $x=2(x,v_0)(\psi-I)^{-1}\psi(v_0)$,⇒$(x,v_0)=2(x,v_0)((\psi-I)^{-1}\psi(v_0),v_0)$

所以 $(\psi-I)^{-1}\psi(v_0)$ 非 0, x 非 0

0.5 Exercise 4

Set $V:=\mathbb{R}[x]$ and $V_0:=\{f\in\mathbb{R}[x]\mid f(0)=f(1)\}$.

1. Prove that $V\times V\to\mathbb{R},\quad (f,g)\mapsto\int_0^1 f(x)g(x)\,\mathrm{d}\,x$ is an inner product.
2. Set $\mathscr{D}:V_0\to V,\quad f(x)\mapsto f'(x)$. Find $\dim\ker(\mathscr{D})$ and $\dim\operatorname{coker}(\mathscr{D})$.
3. Define the inner product restricted on the subspace

$$(\cdot,\cdot)_0:V_0\times V_0\to\mathbb{R},\quad (f,g)\mapsto\int_0^1 f(x)g(x)\,\mathrm{d}\,x.$$

Is there any linear map $\mathscr{D}^*:V\to V_0$ such that for any $h\in V_0$ and $g\in V$,

$$(\mathscr{D}^*g,h)_0=(g,\mathscr{D}h)?$$

解答 1. 双线性, 对称, 正定

2. $\dim\ker(\mathscr{D})=1$, $\dim\operatorname{coker}(\mathscr{D})=1$

3. $\mathscr{D}^*=-\mathscr{D}$

0.6 Exercise 5

Let $(V,(,))$ be an inner product space, where V is not necessary finite dimensional.

- Say $\varphi^\star$ is the adjoint of $\varphi\in\operatorname{Hom}_{\mathbb{R}}(V,V)$, provided $(\varphi^\star(x),y)=(x,\varphi(y))$ for arbitrary $x,y\in V$.
- Let U be a subspace of V. Set

$$U^\perp:=\{v\in V\mid (u,v)=0,\forall u\in U\}=\bigcap_{u\in U}\ker((u,-))$$

Now we assume that φ is invertible, and $\varphi^\star$ exists.

1. Show that $\varphi^\star$ is injective, and $(\operatorname{im}(\varphi^\star))^\perp=0$.
2. Show that $\varphi^\star$ is surjective $\implies$ $(\varphi^{-1})^\star$ exists and $(\varphi^{-1})^\star=(\varphi^\star)^{-1}$.
3. Show that $(\varphi^{-1})^\star$ exists $\implies$ $\varphi^\star$ is invertible and $(\varphi^{-1})^\star=(\varphi^\star)^{-1}$.
4. Let $V=\mathbb{R}[x]$ and $(f,g):=\sum_{n\geq 0}f_ig_i$, where $h=\sum h_i\cdot x^i$. Show that
1. $(V,(,))$ is an inner product space;
2. $L:f\mapsto\frac{f-f(0)}{x}$ is the linear map of moving left. Show that $(\mathrm{id}+L)$ is invertible;
3. Show that $\varphi:=(\mathrm{id}+L)^{-1}$ has no adjoint $\varphi^\star$.

解答 1. 令 $\phi^*(x)=0$, 那么 $(\phi^*(x),y)=(x,\phi(y))=0$ 对任意的 y 都成立, 所以 $x=0$

令 $y\in(\operatorname{im}(\phi^*))^\perp$, 那么对任意 x , $(\ ,\phi(y))=0$, $\phi(y)=0$, $y=0$.

2. ϕ^* bij, $\exists z,\phi(z)=x$, $(\phi^*((\phi^{-1})^*(x)),y)=((\phi^{-1})^*(x),\phi(y))=(x,y)$, 因此 $(\varphi^{-1})^\star=(\varphi^\star)^{-1}$

3. $(\phi^*((\phi^{-1})^*(x)),y)=((\phi^{-1})^*(x),\phi(y))=(x,y)$, 因此 $(\varphi^{-1})^\star=(\varphi^\star)^{-1}$

4.1. 双线性, 对称, 正定

4.2. $(id+L)(f) = \sum_{i=0}^{n-1}(f_i + f_{i+1})x^i + f_n x^n$, 显然可逆.

4.3. $(id+L)^{-1}(f) = \sum_{i=0}^{n-1}(\sum_{j=i}^{n}(-1)^{j-1}f_j)x^i + f_n x^n$

假设 $\phi = (id+L)^{-1}$ 存在 ϕ^*, 那么取 $f = 1$, $(\phi^*(1), g) = (1, \phi(g)) = g_0 - g_1 + g_2 - \cdots$, 取不同的 g, 得到的 $\phi^*(1)$ 不同, 矛盾.