# 第十一周作业

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## 0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

## 目录

# 1 exercise 0

Show that $\varphi^*$ is injective when $\varphi$ is surjective.
解答 $\forall l_1, l_2 \in V^*$ 且 $\varphi^*(l_1) = \varphi^*(l_2)$ ，那么即 $\forall u \in U$ ，$\varphi^*(l_1)(u) = \varphi^*(l_2)(u)$ ，即 $l_1(\varphi(u)) = l_2(\varphi(u))$ ，
由于 $\varphi$ 满射, $\forall v \in V, \exists u \in U$ such that $\varphi(u) = v$ ,
所以 $\forall v \in V$ ，$l_1(v) = l_2(v)$ ，即 $l_1 = l_2$ .

# 2 exercise 1

Show that $\operatorname{ann}(\operatorname{im}\varphi)$ and $\ker(\varphi^*)$ are the same subspaces of $V^*$.
解答 $\operatorname{ann}(\operatorname{im}\varphi) = \{f \mid f(\operatorname{im}\varphi) = 0\}$
$\ker(\varphi^*) = \{l \mid \varphi^*(l) = 0\} = \{l \mid \forall u \in U, \varphi^*(l)(u) = 0\} = \{l \mid l(\varphi(u)) = 0, \forall u \in U\}$.

# 3 Exercise 2

When $\varphi$ is surjective, show that $\operatorname{im}(\varphi^*)$ and $\operatorname{ann}(\ker\varphi)$ are the same subspaces of $U^*$
解答 $\operatorname{ann}(\ker(\varphi)) = \{f \mid f(\ker\varphi) = 0\}$
$\operatorname{im}(\varphi^*) = \{l \circ \varphi \mid l \in V^*\}$
Obviously, $\operatorname{im}(\varphi^*) \subset \operatorname{ann}(\ker(\varphi))$
$\forall f \in \operatorname{ann}(\ker\varphi)$ , Let $g(v) = g(\varphi(u)) = f(u)$, 这是良定义的, 因为 $\varphi(u) = \varphi(u') = v$, $u - u' \in \ker\varphi$.
so $f = g \circ \varphi \in \operatorname{im}(\varphi^*)$

# 4 Exercise 3

When $\varphi : S \hookrightarrow V$ is injective, show that both $(V/S)^*$ and $\operatorname{ann}(S)$ are isomorphic to $\ker(\varphi^*)$.
解答 有 1, $\ker(\varphi^*) = \operatorname{ann}(\operatorname{im}\varphi)$, 由于 $\varphi$ 单射, $\operatorname{im}\varphi \cong S$
因此 $\ker(\varphi^*) \cong \operatorname{ann}(S)$
而令 $l : (V/S)^* \to \operatorname{ann}(S), \quad f \mapsto l(f), f(v+S) = g(v)$, 这是良定义的. 那么 $\ker l = 0$, 并且 $l$ 显然满.

# 5 Exercise 4

For any $V$, we define the evaluation as before:

$$\Phi_V : V \to V^{**}, \quad v \mapsto \begin{bmatrix} V^* & \to & \mathbb{F} \\ \ell & \mapsto & \ell(v) \end{bmatrix}.$$

We define $\varphi^{**} := (\varphi^*)^*$, that is, the pre-composition of $\ell : U^* \to \mathbb{F}$ by $\varphi^* : V^* \to U^*$. Show the equality of the compositions

$$\left[U \xrightarrow{f} V \xrightarrow{\Phi_V} V^{**}\right] = \left[U \xrightarrow{\Phi_U} U^{**} \xrightarrow{f^{**}} V^{**}\right].$$

In other words, $\Phi_V(f(u)) = f^{**}(\Phi_U(u))$ for any $u \in U$. This is why we say $\Phi$ is natural.

解答 等式左边为 $u \mapsto [l \mapsto l(f(u))]$
等式右边为 $[u \mapsto k(u)] \mapsto [l \mapsto l(v)] = u \mapsto [l \circ f \mapsto l \circ f(u)]$

# 6 Exercise 5

Show that

$$a : \mathbb{F}^{m\times n} \to (\mathbb{F}^{n\times m})^*, \quad M \mapsto \begin{bmatrix} \mathbb{F}^{n\times m} & \to & \mathbb{F} \\ X & \mapsto & \mathrm{trace}(M \cdot X) \end{bmatrix}$$

is a linear map. And show that $a$ is surjective.

解答 $a(M+N) = [X \mapsto tr((M+N)\cdot X)] = [X \mapsto M\cdot X] + [X \mapsto N \cdot X] = a(M) + a(N)$

$a(\lambda M) = [X \mapsto tr((\lambda M)\cdot X)] = \lambda[X \mapsto tr(M\cdot X)] = \lambda a(M)$

取 $(\mathbb{F}^{n\times m})^*$ 的基 $f(E_[ij]) = tr(M \cdot E_{ij})$. 显然对于每个映射 $f$ 都存在 $M$ 使得 $a(M) = f$.

# 7 Exercise 6

解答 Z1:$\mathrm{ann}(\mathrm{col}(A)) = \{y \mid y^T A x = 0, \forall x\} = \{y \mid A^T y = 0\} = \ker A^T$

Z3:$\mathrm{ann}(\ker A) = \{y \mid y^T x = 0 \forall Ax = 0\} = col(A^T)$

Z4:$\ker(\ker(A^T)) = \{z \mid y^T z = 0, \forall A^T y = 0\} = \mathrm{col}(A)$

Z5:$A$ 行满秩, 则 $A^T$ 列满秩

Z6:$A$ 列满秩, 则 $A^T$ 行满秩