十五周作业

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0 说明

可以将作业中遇到的问题标注在此. 如有, 请补充.

目录

1 Problem Set for 26-May and 29-May

1.1 Exercise 0

Show that $U \otimes V \simeq V \otimes U$ with the following steps.

1. Construct $\varphi : U \otimes V \to V \otimes U$ via a 1:1 correspondence (refer to the preceding assignment). Describe φ by tracing the image of simple tensors.
2. Define $\psi : V \otimes U \to U \otimes V$ analogously.
3. Establish via 1:1 correspondence that both $\psi \circ \varphi$ and $\varphi \circ \psi$ are identical mappings.

解答 定义双线性映射 $\Phi : U\&V \to V \otimes U, (u, v) \mapsto v \otimes u$, 这唯一确定了线性映射 $\varphi : U \otimes V \to V \otimes U$ 和 $\psi : V \otimes U \to U \otimes V$, 那么 $\varphi \circ \psi = I$ 且 $\psi \circ \varphi = I$, 因此 φ, ψ 是双射, 于是 $U \otimes V \simeq V \otimes U$.

1.2 Exercise 1

Show that the following is an isomorphism:

$$\mathrm{Hom}(U, V^*) \simeq \mathrm{Hom}(V, U^*), \quad [u \mapsto f_u] \mapsto [v \mapsto [u \mapsto f_u(v)]].$$

解答 $\mathrm{Hom}(U, V^*) \simeq (V \otimes U)^*$, $\mathrm{Hom}(V, U^*) \simeq (U \otimes V)^*$, 且 $U \otimes V \simeq V \otimes U$

1.3 Exercise 2

Let V be a finite-dimensional vector space with basis $\{e_i\}$. Let $\{f_i\}$ denote the dual basis of V^*. Define the following mappings:

- $\Delta : \mathbb{F} \to V \otimes V^*, \quad 1 \mapsto \sum_{i=1}^{n} e_i \otimes f_i;$
- $\nabla : V \otimes V^* \to \mathbb{F}, \quad \sum u_i \otimes \varphi_i \mapsto \sum \varphi_i(u_i).$

For $f, g : V \to V$, determine the composition

$$\mathbb{F} \xrightarrow{\Delta} V \otimes V^* \xrightarrow{(f \otimes g^*)} V \otimes V^* \xrightarrow{\nabla} \mathbb{F}.$$

解答 $1 \mapsto \sum e_i \otimes f_i \mapsto f(e_i) \otimes f_i g \mapsto \sum f_i g f e_i = tr(gf)$

1.4 Exercise 3

Demonstrate that, for finite-dimensional vector spaces, there exists an isomorphism determined via simple tensors:

$$\Phi : V_1^* \otimes V_2^* \otimes V_3 \xrightarrow{\sim} \mathrm{Hom}(V_1 \otimes V_2, V_3), \quad f \otimes g \otimes x \mapsto [a \otimes b \mapsto f(a)g(b)x].$$

Now take $\otimes = \otimes_{\mathbb{R}}$ and $V_i = \mathbb{C}$ (the two-dimensional vector space with basis $\{1, i\}$). We know that the usual multiplication defines a map in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C} \otimes \mathbb{C}, \mathbb{C})$:

$$\times : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}, \quad w \otimes z \mapsto w \cdot z.$$

What is $\Phi^{-1}(\times)$?
解答 考虑基的变换, $(1\otimes 1, 1\otimes i, i\otimes 1, i\otimes i)\mapsto(1,i,i,-1)$, 因此令 $1,i$ 的对偶基为 f,g,$\Phi^{-1}(\times)=f\otimes f\otimes 1+f\otimes g\otimes i+g\otimes f\otimes i+g\otimes g\otimes(-1)$

1.5 Exercise 5

Let $A\in\mathbb{C}^{m\times m}$ and $B\in\mathbb{C}^{n\times n}$ be normal matrices (i.e., $PP^H=P^HP$). Show that $AX=XB$ if and only if $A^HX=XB^H$.
解答 $\Rightarrow$: $AX=XB$, 得到 $A^HAXB^H=A^HXBB^H$, 因此 $A(A^HXB^H)=(A^HXB^H)B$,
$X\in\ker(A\otimes I-I\otimes B^T)$, we want to show $X\in\ker(A^H\otimes I-I\otimes\bar{B})$,
$A\otimes I-I\otimes B^T$ 也是正规矩阵,$(A\otimes I-I\otimes B^T)^H=P^H=A^H\otimes I-I\otimes\bar{B}$
$\ker(P)=\ker(PP^H)=\ker(P^HP)=\ker(P^H)$

1.6 Exercise 6

Let $A,B\in\mathbb{C}^{n\times n}$ be Hermitian positive-definite matrices, i.e., $A=A^H$ and $u^H\cdot A\cdot u\geq 0$ with equality if and only if $u=\mathbf{0}$. Define the matrix C via component-wise multiplication (the stupid multiplication):

$$C=(c_{i,j}),\quad c_{i,j}=a_{i,j}\cdot b_{i,j}.$$

Show that C is also Hermitian positive-definite
解答 $A=P^HP$, $B=Q^HQ$, $A\otimes B=(P^HP)\otimes(Q^HQ)=(P^H\otimes Q^H)(PQ)=(P\otimes Q)^H(P\otimes Q)$.
因此 $A\otimes B$ 厄尔米特正定, C 是 $A\otimes B$ 的一个主子式, 因此也正定.