技術者リテラシー I (機械工学科) —— 第 9 回 2024/11/20 略解

問題 1.

(1) $\dfrac{1}{x} + \dfrac{1}{x^2 + 4}$

(2) $\dfrac{5}{x-1} + \dfrac{3}{(x-1)^2} + \dfrac{4}{x+2}$.

(3) $1 - \dfrac{3x^2 + 9}{x^2(x+3)} = 1 + \dfrac{1}{x} - \dfrac{3}{x^2} - \dfrac{4}{x+3}$.

問題 2.

(1) $\dfrac{1}{x^2 - 1} = \dfrac{1}{2}\left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right)$ より,

$$\int \frac{1}{x^2 - 1}\, dx = \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\, dx$$
$$= \frac{1}{2}\left(\log|x-1| - \log|x+1|\right) + C$$
$$= \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

(2) $\dfrac{x}{x^2 - 3x + 2} = \dfrac{2}{x-2} - \dfrac{1}{x-1}$ より,

$$\int \frac{x}{x^2 - 3x + 2}\, dx = \int \frac{x}{(x-2)(x-1)}\, dx$$
$$= \int \left(\frac{2}{x-2} - \frac{1}{x-1}\right)\, dx$$
$$= 2\log|x-2| - \log|x-1| + C$$
$$= \log \frac{(x-2)^2}{|x-1|} + C.$$

(3) $\dfrac{3x+3}{x^3 - 1} = \dfrac{2}{x-1} - \dfrac{2x+1}{x^2 + x + 1}$ より,

$$\int \frac{3x+3}{x^3 - 1}\, dx = \int \frac{3x+3}{(x-1)(x^2 + x + 1)}\, dx$$
$$= \int \left(\frac{2}{x-1} - \frac{2x+1}{x^2 + x + 1}\right)\, dx$$
$$= 2\log|x-1| - \log(x^2 + x + 1) + C$$
$$= \log \frac{(x-1)^2}{x^2 + x + 1} + C.$$

問題 3.

(1) $\displaystyle \int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

(2) $\displaystyle \int \sin x \cos x\, dx = \frac{1}{2}\int \sin 2x\, dx = -\frac{1}{4}\cos 2x + C.$

(3) $\sin 3x \cos x = \dfrac{1}{2}(\sin 4x + \sin 2x)$ より,

$$\int \sin 3x \cos x\, dx = \frac{1}{2}\int (\sin 4x + \sin 2x)\, dx$$
$$= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C.$$

(4) $\cos 3x \cos 2x = \dfrac{1}{2}(\cos 5x + \cos x)$ より,

$$\int \cos 3x \cos 2x\, dx = \frac{1}{2}\int (\cos 5x + \cos x)\, dx$$
$$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C.$$

(5) $t = \tan \dfrac{x}{2}$ とおくと, $\sin x = \dfrac{2t}{1 + t^2}$, $dx = \dfrac{2}{1+t^2}dt$ より,

$$\int \frac{1}{\sin x}\, dx = \int \frac{1 + t^2}{2t} \cdot \frac{2}{1+t^2}dt = \int \frac{1}{t}\, dt$$
$$= \log|t| + C = \log\left|\tan \frac{x}{2}\right| + C.$$

問題 4. 以下では, ⓝ, △ と書いて, それぞれ行列の n 行目, n 列目を表すものとする.

(1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{②+①\times(-3)} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{②\times(-\frac{1}{2})} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\xrightarrow{①+②\times(-2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. よって, rank$(A) = 2$.

(2) $A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow[③+①\times(-1)]{②+①\times 2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$ $\xrightarrow[△+△\times(-1)]{△+△\times 2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow{②\times(-\frac{1}{3})} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{pmatrix}$ $\xrightarrow{③+②\times(-3)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{△+△\times 1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

よって, rank$(A) = 2$.

(3) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 15 \end{pmatrix} \xrightarrow[③+①\times(-5)]{②+①\times(-2)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[△+△\times(-3)]{△+△\times(-2)}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. よって, rank$(A) = 1$.

(4) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 7 & 11 \end{pmatrix} \xrightarrow[\substack{②+①\times(-2) \\ ③+①\times(-3)}]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{pmatrix}$

$\xrightarrow[②\times(-1)]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[\substack{①+②\times(-2) \\ ③+②\times(-1)}]{} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix}$

$\xrightarrow[③\times(-1)]{} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{①+③\times3 \\ ②+③\times(-3)}]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

よって, $\mathrm{rank}(A) = 3$.

(5) $A = \begin{pmatrix} 1 & a & 1 & 1 \\ a & 1 & a & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & 1 \end{pmatrix}$

$\xrightarrow[\substack{②+①\times(-a) \\ ③+①\times(-1) \\ ④+①\times(-1)}]{} \begin{pmatrix} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 0 & 1-a \\ 0 & 0 & 0 & a-1 \\ 0 & 1-a & a-1 & 0 \end{pmatrix}$

$\xrightarrow[\substack{②+①\times(-a) \\ ③+①\times(-1) \\ ④+①\times(-1)}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a^2 & 0 & 1-a \\ 0 & 0 & 0 & a-1 \\ 0 & 1-a & a-1 & 0 \end{pmatrix}$

$\xrightarrow[②\leftrightarrow④]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a & a-1 & 0 \\ 0 & 0 & 0 & a-1 \\ 0 & 1-a^2 & 0 & 1-a \end{pmatrix}$

$\xrightarrow[④+②\times(-a-1)]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a & a-1 & 0 \\ 0 & 0 & 0 & a-1 \\ 0 & 0 & 1-a^2 & 1-a \end{pmatrix}$

$\xrightarrow[③+②\times1]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a & 0 & 0 \\ 0 & 0 & 0 & a-1 \\ 0 & 0 & 1-a^2 & 1-a \end{pmatrix}$

$\xrightarrow[③\leftrightarrow④]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a & 0 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 1-a & 1-a^2 \end{pmatrix}$

$\xrightarrow[④+③\times1]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a & 0 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 1-a^2 \end{pmatrix}$

$\xrightarrow[\substack{②\times(-1) \\ ④\times(-1)}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & (a-1)(a+1) \end{pmatrix}.$

よって, $\mathrm{rank}(A) = \begin{cases} 1 & (a = 1), \\ 3 & (a = -1), \\ 4 & (a \neq \pm 1). \end{cases}$