リメディアル数学 (化学システム工学科) —— 第 4 回 2024/5/15 略解

問題 1.

(1) $f(x) = x^3$,

$$f(x+h) = (x+h)^3 = x^3 + 3hx^2 + 3h^2x + h^3.$$

(2) $$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{3ha^2 + 3h^2a + h^3}{h} = 3a^2 + 3ha + h^2.$$

(3) $$f'(a) = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0}(3a^2 + 3ha + h^2) = 3a^2.$$

問題 2.

(1) $f'(x) = 6x^2 + 10x - 1$.

(2) $f'(x) = 5x^4 + 8x^3$.

(3) $f'(x) = \dfrac{8}{x^3}$.

(4) $f'(x) = \sin x + x\cos x$.

(5) $$f'(x) = (3x^2-2)'(x^2+x+1) + (3x^2-2)(x^2+x+1)' = 6x(x^2+x+1) + (3x^2-2)(2x+1) = 12x^3 + 9x^2 + 2x - 2.$$

(6) $$f'(x) = (e^x)'\cos x + e^x(\cos x)' = e^x\cos x - e^x \sin x \quad (= e^x(\cos x - \sin x)).$$

(7) $f'(x) = (x)'\log x + x(\log x)' = \log x + 1$.

(8) $$f'(x) = \frac{(x^2)'(x+3) - x^2(x+3)7}{(x+3)^2} = \frac{2x(x+3) - x^2}{(x+3)^2} = \frac{x^2+6x}{(x+3)^2} \quad \left(= \frac{x(x+6)}{(x+3)^2}\right).$$

(9) $$f'(x) = \frac{(2x+1)'(x^2+1) - (2x+1)(x^2+1)'}{(x^2+1)^2} = \frac{2(x^2+1) - (2x+1)\cdot 2x}{(x^2+1)^2} = \frac{-2x^2 - 2x + 2}{(x^2+1)^2}.$$

(10) $$f'(x) = \frac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

(11) $$f'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

(12) $f'(x) = (2\log x)' = \dfrac{2}{x}$.

問題 3.

(1) $f'(x) = 3(2x^2+5)^2(2x^2+5)' = 12x(2x^2+5)$.

(2) $f'(x) = -\dfrac{3}{(x+1)^4}\cdot(x^2+1)' = -\dfrac{6x}{(x^2+1)^4}$.

(3) $$f'(x) = \sin\left(\frac{\pi}{4} - 2x\right)\cdot\left(\frac{\pi}{4} - 2x\right)' = 2\sin\left(\frac{\pi}{4} - 2x\right).$$

(4) $f'(x) = 2\sin x(\sin x)' = 2\sin x\cos x \quad (= \sin 2x)$.

(5) $f'(x) = \dfrac{1}{x^2+1}\cdot(x^2+1)' = \dfrac{2x}{x^2+1}$.

(6) $f'(x) = \dfrac{1}{\sin x}\cdot(\sin x)' = \dfrac{\cos x}{\sin x} \quad \left(= \dfrac{1}{\tan x}\right)$.

(7) $f'(x) = e^{-x^2}\cdot(-x^2)' = -2xe^{-x^2}$.

(8) $f'(x) = e^{\sin x}(\sin x)' = e^{\sin x}\cos x$.

(9) $f'(x) = \dfrac{1}{4}(x^3+1)^{-\frac{3}{4}}\cdot(x^3+1)' = \dfrac{3}{4}x^2(x^3+1)^{-\frac{3}{4}}$.

(10) $f'(x) = n(ax+b)^{n-1}(ax+b)' = an(ax+b)^{n-1}$.

(11) $$f'(x) = (e^{\log a^x})' = (e^{x\log a})' = e^{x\log a}\cdot(x\log a)' = a^x\log a.$$

(12) $f'(x) = \left(\dfrac{\log x}{\log a}\right)' = \dfrac{1}{x\log a}$.

問題 4.

(1) $\log y = x\log x$ より, 両辺を x で微分すると

$$\frac{y'}{y} = (x)'\log x + x(\log x)' = \log x + 1.$$

よって, $y' = y(\log x + 1) = x^x(\log x + 1)$.

(2) $\log y = \log x^{\log x} = (\log x)^2$ より, 両辺を x で微分すると

$$\frac{y'}{y} = 2\log x\cdot(\log x)' = \frac{2\log x}{x}.$$

よって, $y' = \dfrac{2\log x}{x}y = \dfrac{2x^{\log x}\log x}{x}$.

(3) $\log y = \dfrac{1}{2}\log(x^2+1) - \dfrac{1}{2}\log(x^2+2)$ より, 両辺を x で微分すると

$$\frac{y'}{y} = \frac{x}{x^2+1} - \frac{x}{x^2+2} = \frac{x}{(x^2+1)(x^2+2)}.$$

よって,

$$\begin{aligned} y' &= \frac{x}{(x^2+1)(x^2+2)}y \\ &= \frac{x}{\sqrt{(x^2+1)(x^2+2)}(x^2+2)}. \end{aligned}$$

問題 5.

(1) $f'(x) = \dfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(4-x^2)' = -\dfrac{x}{\sqrt{4-x^2}}.$

(2) $y' = -\dfrac{1}{(\tan x+1)^2}\cdot(\tan x+1)'$

$= -\dfrac{1}{(\tan x+1)^2}\cdot\dfrac{1}{\cos^2 x}$

$= -\dfrac{1}{(\sin x+\cos x)^2}.$

(3) $f'(x) = -\sin\dfrac{1}{x}\cdot\left(\dfrac{1}{x}\right)' = \dfrac{1}{x^2}\sin\dfrac{1}{x}.$

(4) $f'(x) = 4(e^x+1)^3(e^x+1)' - 2\cdot 2(e^x+1)(e^x+1)'$

$= 4e^x(e^x+1)^3 - 4e^x(e^x+1)$

$(= 4e^{2x}(e^x+1)(e^x+2)).$

(5) $f'(x) = \dfrac{1}{\frac{x-1}{x+1}}\left(\dfrac{x-1}{x+1}\right)'$

$= \dfrac{x+1}{x-1}\cdot\dfrac{(x-1)'(x+1)-(x-1)(x+1)'}{(x+1)^2}$

$= \dfrac{x+1}{x-1}\cdot\dfrac{2}{(x+1)^2}$

$= \dfrac{2}{(x+1)(x-1)}.$

問題 6.

(1) $f'(x) = \dfrac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\dfrac{x}{a}\right)'$

$= \dfrac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\dfrac{1}{a} = \dfrac{1}{\sqrt{a^2-x^2}}.$

(2) $f'(x) = -\dfrac{1}{\sqrt{1-x}}\cdot(\sqrt{x})'$

$= -\dfrac{1}{\sqrt{1-x}}\cdot\dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{x(x-1)}}.$

(3) $f'(x) = \dfrac{1}{a}\cdot\dfrac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\left(\dfrac{x}{a}\right)'$

$= \dfrac{1}{a^2+x^2}.$

(4) $f'(x) = (x)'\arctan x + x(\arctan x)'$

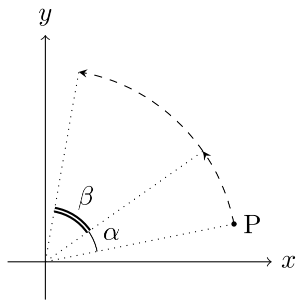
$= \arctan x + \dfrac{x}{1+x^2}.$

問題 7. 角度 $-\dfrac{\pi}{6}$ の回転行列は $\dfrac{1}{2}\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}$ である.

よって, 求める点は $\dfrac{1}{2}\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}\frac{2\sqrt{3}+3}{2}\\ \frac{-2+3\sqrt{3}}{2}\end{pmatrix}.$

問題 8.

(1) 下図の通り. (角度は適当でよい.)



(2) $A = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix},\ B = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}$
より,

$BA = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$

$= \begin{pmatrix}\cos\beta\cos\alpha - \sin\beta\sin\alpha & -(\cos\beta\sin\alpha + \sin\beta\cos\alpha)\\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha\end{pmatrix}$

$= \begin{pmatrix}\cos(\beta+\alpha) & -\sin(\beta+\alpha)\\ \sin(\beta+\alpha) & \cos(\beta+\alpha)\end{pmatrix}.$

この答えは BA が角度 $(\alpha+\beta)$ の回転行列である.
したがって, 本問の計算は「α だけ回転した後, β だけ回転する作用」と「$(\alpha+\beta)$ だけ回転させる作用」が変わらないことを表している.