Assignment 2

Quantum Computing/AI/Science

January 16, 2024

Notation: For all questions on this sheet, we set $\sigma_1 = X$, $\sigma_2 = Y$ and $\sigma_3 = Z$, and we denote the identity matrix of any dimension with $\mathbb{1}$.

Question 1 (Matrix exponential).

1. Given two matrices A and B, prove that $[A, B] = 0$ implies $e^{A+B} = e^A e^B$.
2. Give an example of two 2×2 matrices C and D for which $e^{C+D} \neq e^C e^D$.

Question 2 (Pauli Matrices).

1. Calculate σ_i^2 for all $i \in \{1, 2, 3\}$.
2. Calculate $[\sigma_i, \sigma_j]$ for all $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.
3. Let ϵ_{ijk} be the Levi-Civita symbol. By using this symbol write down a simple expression for $[\sigma_i, \sigma_j]$ which holds for all $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.

Question 3 (Hamiltonian time evolution).

1. Consider a quantum system described by a time-independent Hamiltonian H. Let the state of the system at time t be denoted by $|\psi(t)\rangle$. Let $0 \leq t_1 \leq t_2$. Write down the unitary operator $U(t_1, t_2)$ which maps $|\psi(t_1)\rangle$ to $|\psi(t_2)\rangle$.
2. Prove that the operator you wrote down in 2.1 is unitary.
3. Let $\vec{v}$ be a 3-dimensional unit vector, and define $\vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$. Prove that

$$e^{i\theta\vec{v}\cdot\vec{\sigma}} = \cos(\theta)\mathbb{1} + i\sin(\theta)\vec{v} \cdot \vec{\sigma} \tag{1}$$

Question 4 (Observables).

1. Let M be an observable, and let $M|\psi\rangle = m|\psi\rangle$. What is the average value observed when measuring M on state $|\psi\rangle$. What is the standard deviation?
2. Suppose we have a qubit in the state $|0\rangle$ and we measure the observable X. What is the average value of the measurement outcomes? What is the standard deviation?
3. Given a unit vector $\vec{v}$, prove that $\vec{v} \cdot \vec{\sigma}$ is a valid observable.
4. Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v}\cdot\vec{\sigma}$ given that the state prior to the measurement is $|0\rangle$. What is the state after the measurement if the result $+1$ was obtained?
5. Calculate the average value of the observable $X_1 Z_2$ when measuring the state $1/\sqrt{2}(|00\rangle + |11\rangle)$

Question 5 (Heisenberg Uncertainty Principle).

1. Let A and B be Hermitian operators and $|\psi\rangle$ a quantum state. Prove that

$$|\langle\psi|[A,B]|\psi\rangle|^2 \leq 4\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \tag{2}$$

   (Hint: Use the Cauchy-Schwarz inequality)

2. Let C and D be two observables. Use Eq.(2) to prove the Heisenberg Uncertainty principle

$$\Delta(C)\Delta(D) \geq \frac{|\langle\psi|[C,D]|\psi\rangle|}{2} \tag{3}$$

3. Provide an interpretation for Eq.(3).

4. Use Eq.(3) to provide a lower bound for $\Delta(X)\Delta(Y)$.

Question 6 (Density Matrices).

1. Prove that for all square matrices A and B of the same dimension we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

2. Let ρ be a density operator. Prove that $\mathrm{Tr}(\rho^2) \leq 1$ with equality if and only if ρ is a pure state.

3. Let $\{|\tilde{\psi}_i\rangle\}$ and $\{|\tilde{\phi}_i\rangle\}$ be ensembles of vectors (not necessarily normalized). Assume that

$$|\tilde{\psi}_i\rangle = \sum_j U_{ij}|\tilde{\phi}_j\rangle \tag{4}$$

   where U_{ij} are the entries of a unitary matrix U. Prove that

$$\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j| \tag{5}$$

4. Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$ where $|a\rangle$ is a pure state of system A and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

5. For each of the four Bell states, find the reduced density operator for each qubit.

Question 7 (Transverse field Ising model).

Define the transverse-field Ising model on a length N 1D chain via

$$H = -J\sum_{i=0}^{N-2} Z_i Z_{i+1} - h\sum_{i=0}^{N-1} X_i \tag{6}$$

1. Use the qiskit object `SparsePauliOp` to write a function called `getHamiltonian` which takes in a tuple (N, J, h) and returns the transverse field Ising model Hamiltonian as a `SparsePauliOp`.

2. Write a function which for any tuple (N, J, h) returns the ground state and ground state energy of the corresponding transverse field Ising model.