Empirical Investigation of the Period of Oscillation of a Physical Ring Pendulum

Kyle Slinker (lab partner: Stefan Jeglinski)

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Abstract

We investigate how the period of oscillation T of a physical pendulum constructed from a ring depends on a number of factors: mass, amplitude, and diameter. We find that period does not depend on mass of the ring or initial amplitude (for small amplitudes), as predicted by our theory. We also obtain $T = (2.033 \pm 0.005)d^{(0.504\pm0.001)}$ (with T measured in seconds and d measured in meters) for the diameter dependence, which does not agree within experimental uncertainty with our theoretical model $T = 2\pi\sqrt{d/g}$. We discuss this discrepancy, concentrating on a swaying of the post holding the rings as a plausible cause.

Background and Theory

To analyze the motion of a physical pendulum constructed from a ring, we begin with the relationship between torque, angular acceleration, and moment of inertia

$$\tau = I\alpha. \tag{1}$$

The moment of inertia of a ring with radius r and mass m about an axis through its center point and perpendicular to its plane is $I_{\text{center}} = mr^2$. This is strictly only true for an infinitely thin ring, but our rings are very thin compared to their diameter so this approximation is reasonable. Using the parallel axis theorem, we find the moment of inertia of the ring about a point on its edge to be

$$I = mr^2 + I_{\text{center}} = 2mr^2. \tag{2}$$

The only torque acting on the ring is due to the gravitational force. When the ring is displaced from its initial rest position by an angle θ, the angle between the gravitational force and moment arm will be θ. Thus,

$$\tau = |\vec{\tau}| = |\vec{r} \times \vec{F}| = rmg\sin\theta. \tag{3}$$

Using the relationship $\alpha = \ddot{\theta}$ (where dots denote time derivatives), we have

$$rmg\sin\theta = 2mr^2\ddot{\theta} \tag{4}$$

$$g\sin\theta = 2r\ddot{\theta}. \tag{5}$$

The small angle approximation ($\sin\theta \approx \theta$) gives us a differential equation

$$\ddot{\theta} = \frac{g}{d}\theta, \tag{6}$$

where $d = 2r$ is the ring's diameter. The solution to this equation is

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{d}}t\right), \tag{7}$$

where θ_0 is the amplitude at which we release the ring. The period of oscillation is

$$T = 2\pi\sqrt{\frac{d}{g}}. \tag{8}$$

We can now model the period's dependence on d with the empirical equation

$$T = Ad^n. \tag{9}$$

Comparing (8) and (9), we expect $n = 1/2$ and $A = 2\pi/\sqrt{g}$.

Experimental

Various rings are hung from a knife edge and set into oscillatory motion. A photogate and timer are used to measure the period for the oscillation (with an uncertainty of 0.0001 s). We also characterize the rings in terms of a number of their physical parameters. Mass is measured on a digital balance (with 1% uncertainty). Radius is measured either with calipers or a meter stick, depending on the ring's diameter (with uncertainties of 0.02 mm or 2 mm, respectively). As mentioned in the theory section, we are approximating our rings as infinitely thin; we therefore measure both inner and outer diameter and average the two measurements to use as the ring's diameter. Amplitude is measured using a protractor (with an uncertainty of 2° to account for difficulty in holding the initial set-up).

Our theory suggests that the period should not depend on the initial angle, as long as it is small. As this is one of the more difficult parameters to account for, we begin by investigating this result empirically. As we will later see, our theory concerning release amplitude is confirmed. Thus, for subsequent sections of the investigation, we can safely disregard the effects of release amplitude and we will simply release the rings from some small amplitude.

Our theory also suggests that the period should not depend on the ring's mass. We will need to verify this empirically, as changing ring diameters in the next section will necessarily also change the ring masses. We will see later that our theory concerning mass is also confirmed.

Satisfied that amplitude and mass will not have an impact on the outcome of our variable diameter trials, we investigate the period's dependence on ring diameter. Our theory indicates that the period should depend on diameter, so we will fit our data to the empirical equation (9). Taking the logarithm of this equation yields

$$\ln T = n \ln d + \ln A. \tag{10}$$

In a plot of $\ln T$ as a function of $\ln d$, n will equal the slope and A can be obtained from the y-intercept by $A = e^{y\text{-intercept}}$.

Analysis

Physical data for each ring are shown in Table 1. Raw data uncertainties come from the instrument resolutions described in the **Experimental** section. Mean diameter is given by

$$d = \frac{d_o + d_i}{2}, \tag{11}$$

and its uncertainty is determined by quadrature

$$\sigma_d = \sqrt{\sigma_{d_o}^2 \left(\frac{\partial}{\partial d_o}\frac{d_o + d_i}{2}\right)^2 + \sigma_{d_i}^2 \left(\frac{\partial}{\partial d_i}\frac{d_o + d_i}{2}\right)^2} = \frac{1}{2}\sqrt{\sigma_{d_o}^2 + \sigma_{d_i}^2}. \tag{12}$$

As an example, we show the computations for the first ring:

$$d = \frac{32.42 \text{ mm} + 38.52 \text{ mm}}{2} = 35.47 \text{ mm} \tag{13}$$

and

$$\sigma_d = \frac{1}{2}\sqrt{(0.02 \text{ mm})^2 + (0.02 \text{ mm})^2} = 0.014 \text{ mm}. \tag{14}$$

Period data are shown in Table 2. Raw time measurements have uncertainties given by the resolution of the timer, 1 ms. Angle uncertainties are taken to be 2° to reflect the difficulty in holding and releasing the ring at a given angle. The last column shows the period for each ring, which is the average of the five individual measurements. The uncertainty in the period is given by the standard error. Average and standard deviations were computed in Excel. In this case, the standard error is the standard deviation divided by $\sqrt{5}$.

We now examine the period's dependence on starting amplitude. We note that the 5° and 10° periods are the same within uncertainty. Fig. 1 shows that there is some dependence of period on amplitude for

Ring	Mass (g)	Inner Diameter (mm)	Outer Diameter (mm)	Mean Diameter (m)
1	30.7 ± 0.3	32.42 ± 0.02	38.52 ± 0.02	0.03547 ± 0.00001
2	11.3 ± 0.1	64.06 ± 0.02	73.10 ± 0.02	0.06858 ± 0.00001
3	33.2 ± 0.3	63.48 ± 0.02	72.70 ± 0.02	0.06809 ± 0.00001
4	106 ± 1	63.40 ± 0.02	72.42 ± 0.02	0.06791 ± 0.00001
5	102 ± 1	139.10 ± 0.02	152.06 ± 0.02	0.14558 ± 0.00001
6	311 ± 3	284 ± 2	302 ± 2	0.293 ± 0.001
7	640 ± 6	428 ± 2	455 ± 2	0.442 ± 0.001

Table 1: Physical data for rings.

Ring	Amplitude (°)	T_1 (s)	T_2 (s)	T_3 (s)	T_4 (s)	T_5 (s)	T (s)
1		0.3791	0.3788	0.3791	0.3780	0.3775	0.3785 ± 0.0003
2		0.5268	0.5263	0.5256	0.5264	0.5258	0.5262 ± 0.0002
3		0.5253	0.5250	0.5268	0.5250	0.5257	0.5256 ± 0.0003
4		0.5280	0.5281	0.5263	0.5160	0.5278	0.525 ± 0.002
5		0.7707	0.7708	0.7707	0.7686	0.7701	0.7702 ± 0.0004
6	5 ± 2	1.0961	1.3788	1.3791	1.3780	1.3775	1.099 ± 0.004
6	10 ± 2	1.1014	1.0937	1.0974	1.1020	1.0968	1.098 ± 0.002
6	15 ± 2	1.1067	1.1070	1.1042	1.1033	1.1077	1.1058 ± 0.0009
6	20 ± 2	1.1129	1.1092	1.1109	1.1137	1.1080	1.111 ± 0.001
7		1.3441	1.3440	1.3437	1.3431	1.3436	1.3437 ± 0.0002

Table 2: Timing data. All raw time measurements have uncertainties given by the resolution of the timer, 1 ms.

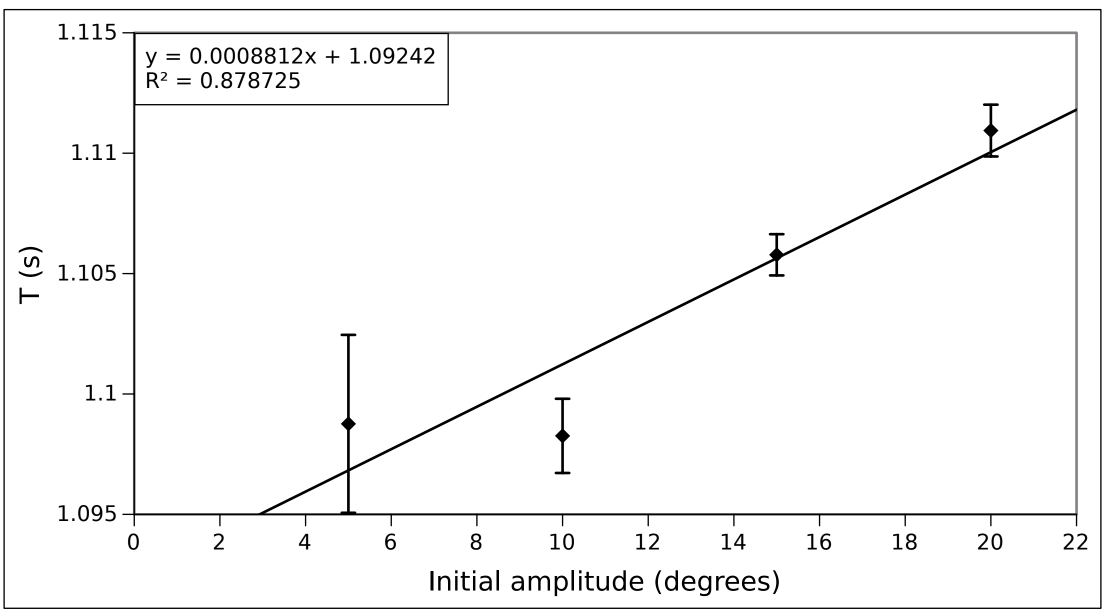

Figure 1: Dependence of period on starting amplitude. All data points are for Ring 6. Note that for small amplitude ($\leq 10°$), periods are experimentally indistinguishable. Horizontal error bars are too small to be seen.

larger amplitudes. Having $R^2 = 0.88$ indicates that this is not modeled well by a linear relationship, but that there is some correlation.

Fig. 2 shows that there is no dependence of period on mass, because all periods agree within uncertainties.

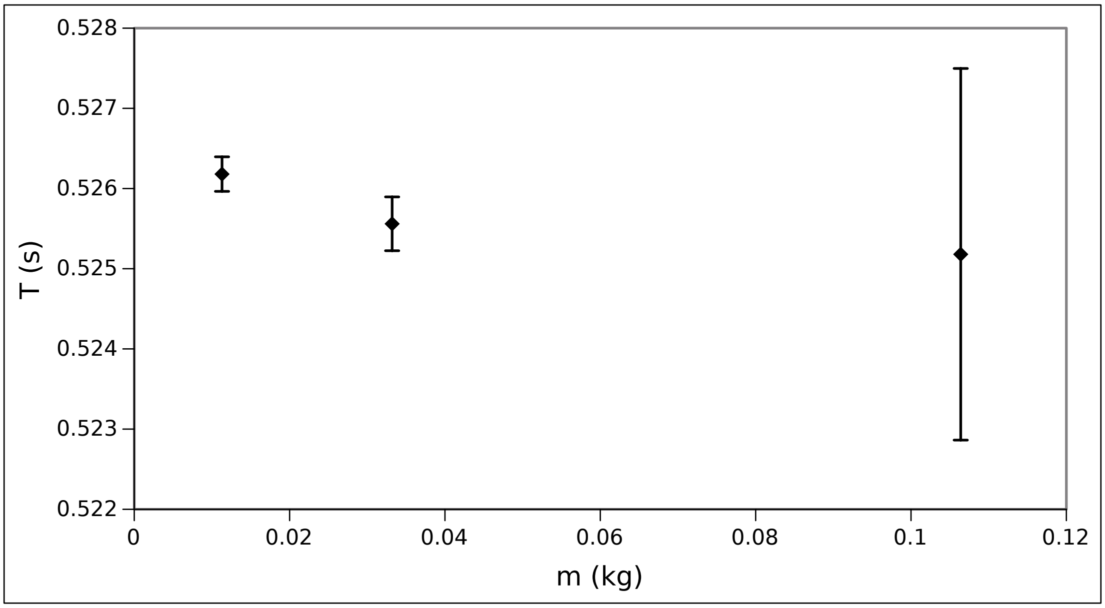

Figure 2: Dependence of period on mass. Data come from Rings 2, 3, and 4. Note that all periods are experimentally indistinguishable. Horizontal error bars are too small to be seen.

Fig. 3 shows the dependence of period on diameter. Having $R^2 = 0.99998$ indicates that this is modeled well by a linear relationship. The argument of a logarithm must be unitless; we therefore divide all periods by one second and all diameters by one meter. As an example, for the period of ring 1

$$\ln \frac{T}{1\ \mathrm{s}} = \ln \frac{0.3785\ \mathrm{s}}{1\ \mathrm{s}} = -0.9715. \tag{15}$$

From now on, the units within the argument of logarithms will be suppressed. To get the error bars, quadrature is used. As an example, for the period of ring 1

$$\sigma_{\ln T} = \sigma_T \frac{\partial \ln T}{\partial T} = \frac{\sigma_T}{T} = \frac{0.0003\ \mathrm{s}}{0.3785\ \mathrm{s}} = 0.00079. \tag{16}$$

Excel's `LINEST` gives the uncertainty of the slope and y-intercept

$$\sigma_{\text{slope}} = 0.0012 \tag{17}$$

$$\sigma_{y\text{-intercept}} = 0.0026. \tag{18}$$

The final result is $n = 0.504 \pm 0.001$; note that, as an exponent, this value is unitless. The coefficient in the empirical equation is computed by

$$A = e^{0.7093} = 2.0327\ \mathrm{s/m^{1/2}}. \tag{19}$$

The uncertainty is found using quadrature

$$\sigma_A = \sigma_b \frac{\partial e^b}{\partial b} = \sigma_b e^b = 0.003 e^{0.7093} = 0.0053\ \mathrm{s/m^{1/2}}, \tag{20}$$

where b is the y-intercept. Thus, $A = 2.033 \pm 0.005\ \mathrm{s/m^{1/2}}$. The units were determined by looking at the units of the expressions used in the logarithms.

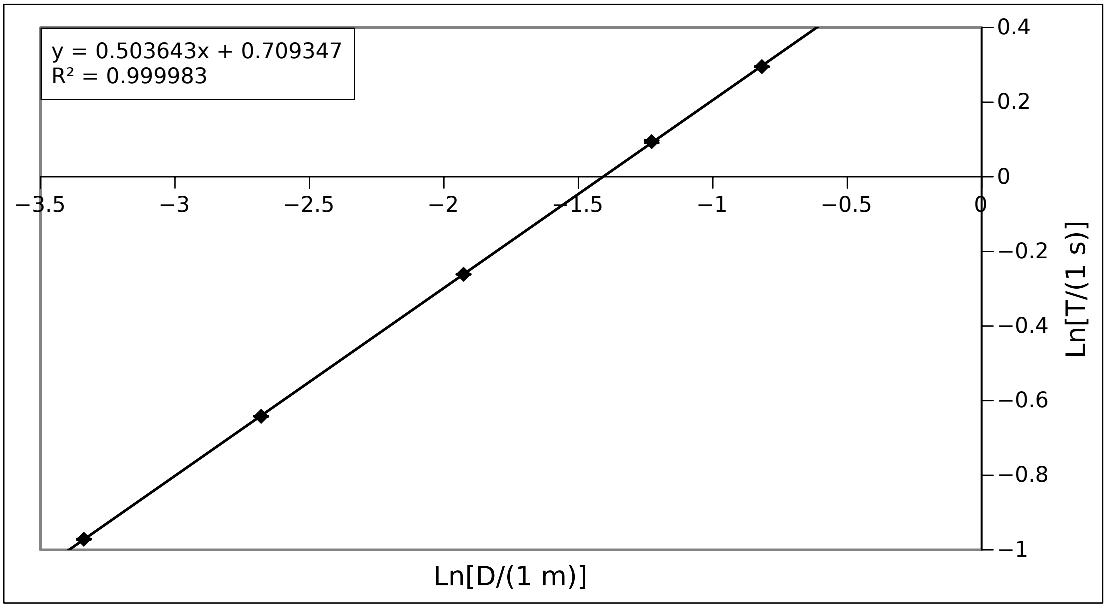

Figure 3: Dependence of period on diameter. Data come from Rings 1, 2, 5, 6 (from 5°), and 7. Horizontal and vertical error bars are too small to be seen.

Discussion

Our amplitude and mass results agree with our theory. Namely, there is a weak correlation between period and amplitude at large amplitude, but not at low amplitude; and there is no correlation between mass and period. Comparing our measured n and A to those from the theory section, $n = 0.504 \pm 0.001$ does not agree with the expected 1/2 to within experimental uncertainty and $A = (2.033 \pm 0.005)\ \mathrm{s/m}^{1/2}$ does not agree with $2\pi/\sqrt{g} = 2.00607\ \mathrm{s/m}^{1/2}$ to within experimental uncertainty, indicating that the measured differences are statistically significant. This does not mean our data or theory is invalid, just that our theory does not account for every effect we are measuring.

We propose a number of explanations of these discrepancies: friction, initial release, and the swaying post. There was noticeable friction, even when balancing the rings on a knife edge. When left swinging for some time, the amplitude decreased noticeably. There was also uncertainty stemming from the experimenter's ability to measure the initial angle and release the ring consistently. Given our theoretical and empirical findings, we conclude that the angle measurement likely did not contribute significantly to our final uncertainty, but releasing the rings inconsistently could. If not pulled straight sideways, the rings' motion becomes chaotic. We feel that the largest contribution to the final uncertainty came from the post that held the rings, which swayed visibly when the rings were oscillating. We note that both the potentially chaotic motion and swaying of the post were more pronounced for larger rings and larger amplitudes. The swaying post effectively creates a larger physical pendulum with greater moment of inertia. Considering our theory, this would effectively make our pendulum's diameter larger than we have accounted for. Therefore, this effect would cause all our measured periods to be too high.

We note that, even when the small angle approximation is broken, the period still does not depend strongly on initial amplitude. We see this in two ways: first, the small correlation coefficient and second, the difference in period over the range we investigated. Consider that, over our range of diameters, the period increased by 255% whereas, over our range of amplitudes, the period only increased by 1%. We therefore conclude that, although the period does depend on amplitude, a ring's diameter plays a much more important role in determining its period of oscillation.

Why This is a Good Report

Note that 118/119 does not require **Background and Theory** and **Experimental** sections, though some of this information may still make it into your report. Also, 118/119 does not require that quadrature be used (as was used in this report); upper/lower bound is fine.

Here is a breakdown of things physicists are specifically looking for in reports:

- Units are present for every number which is not unitless, which is essentially every number.
- Values are rounded appropriately when they are presented. In physics class, this means the uncertainty has one significant figure and then the value is rounded to match (for example, if the uncertainty ends in the tenth's place, the value also ends in the tenth's place). *The results of calculations are shown to one more digit than needed, to allow for rounding.*
- Note that this report concluded that the theoretical and measured values were not in agreement; *this does not mean that the theory, data, or conclusions are 'bad' or 'wrong.'* The data analyzed in this report is actually quite good, which is why the relative uncertainties obtained for both n and A are approximately 0.2%.
- Full sentences are used.
- This report was typeset using LaTeX. You are not expected to use something as fancy as the software used to typeset this report (other than one report for 281 students); software such as Microsoft Word, Open Office, or Apple Pages are fine. If you absolutely cannot use an equation editor, it is preferable for you to hand write your equations rather than resort to something like

  ```
  sigma_d=1/2*sqrt(sigma_d_o^2+sigma_d_i^2)=1/2*sqrt(0.02mm^2+0.02mm^2)=0.014mm
  ```

  which is difficult to read.

Abstract

- The abstract is succinct.
- The abstract is self-contained. If the abstract and report were separated, it would not change your ability to read the abstract.
- The abstract covers what was done, what was found, and sources of uncertainty.
- The abstract can still mention uncertainty in the event that your result was what you had expected. Just because your numbers agree does not mean there was no uncertainty.
- Remember the purpose of an abstract: for a real scientific article, the abstract allows researchers a quick glance into the paper so they can decide if they want to invest the time to read the entire thing. This is why it needs to be short and self-contained. It also means that not everything has to be in the abstract, just the most important points.

Note that it is less important that any specific point be in exactly the correct section of a report. It is more important that the point be in the report at all and in a place which makes logical sense in the context of how the experiment is being presented. The abstract is an exception. Here are more specific points grouped roughly by the section to which they generally apply:

Background and Theory

- The theory section presents all equations relevant for the report (other than those specifically for data and uncertainty analysis).
- Assumptions are spelled out.
- Enough context is given that the report's audience knows what's going on. In physics classes, you can assume that your audience knows the same physics you do.

Experimental

- The experimental section does not just list equipment used.
- The experimental section not only describes what was done, but why it was done that way.
- You do not need to explain the details of how to use the equipment; you can assume that your audience has at least a basic understanding of it (for example, this report did not explain how to read calipers).

Analysis

- The analysis begins with the raw data and ends with the desired results. Between these two points, it walks you through how the results were obtained from the raw data. This can include words, equations, tables, and plots.
- Not every number in a table has to have a unit right next to it, but a column or row does need to indicate the units of the values it contains if you leave them off the individual values.
- Tables and plots have short captions to give context. They do not have to be completely explained, as this is done more thoroughly in the body of the analysis.
- Plots have labeled axes that include units.
- Every time an equation is used, one sample calculation is shown. The rest of the values can then be computed in Excel and do not have to be shown explicitly.
- Statistical measures such as the average, standard deviation, R^2, uncertainty in slope, and uncertainty in y-intercept can be computed in Excel, but you must make it clear that this is how you obtained that information.

Discussion

- The discussion does not spend much space recapping things from earlier in the report.
- The final results are restated, but mostly so that their accuracy and precision can be considered.
- If results do not agree, the discussion does not try to make them agree.
- Words such as ‘close’ or ‘almost’ are not used. When you want to make this kind of point, use numbers to back it up and give context to show why this number suggests something is ‘close’ (for example, 1% may or may not be a large variation, but when compared to 255% the audience can see that it is not).
- Sources of uncertainty are discussed. When possible, specific examples of how the results are impacted by uncertainties are given (for example, reasoning that the swinging post should increase period).
- While deficiencies of the experimenters might be considered as contributing to uncertainty, the deficiencies are was not mistakes that the experimenters made, and are explained more fully than claiming ‘human error.’ This applies to timing, parallax distortions, counting large numbers, measuring moving objects, etc.
- The discussion can still mention uncertainties in the event that your result was what you had expected. Just because your numbers agree does not mean there was no uncertainty.