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Coupling Stokes and Darcy equations

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Abstract

We study an interface problem between a fluid flow, governed by Stokes equations, and a flow in a porous medium, governed by Darcy equations. We consider a weak formulation of the coupled problem which allows to use classical Stokes finite elements in the fluid domain, and standard continuous piecewise polynomials in the porous medium domain. Meshes do not need to match at the interface. The formulation of Stokes equations is standard while a Galerkin least-squares formulation is used for a mixed form of Darcy equations. We prove the well-posedness of the coupled problem for this formulation and the convergence for some finite element approximations. A two-dimensional numerical example is also given.

Keywords: Stokes equations; Darcy equations; Beavers–Joseph–Saffman interface condition; Galerkin least-squares method; Mixed finite elements

1. Introduction

Let Ω_1 and Ω_2 be two non-intersecting open bounded subsets of $\mathbb{R}^n$, $n = 2$ or 3, with lipschitzian boundaries $\partial\Omega_1$ and $\partial\Omega_2$. We assume that their boundaries have a non-empty intersection denoted by $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ and to which we refer as the *interface* between Ω_1 and Ω_2. The remaining parts of the boundaries are $\Gamma_1 := \partial\Omega_1 - \Gamma$ and $\Gamma_2 := \partial\Omega_2 - \Gamma$ (see Fig. 1).

A viscous incompressible fluid occupies the whole domain Ω_1 as well as the interstitial space of the porous domain Ω_2. In Ω_1, we assume that the flow is governed by the stationary Stokes equations

$$-\nabla \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) = \boldsymbol{f}_1, \quad \text{in } \Omega_1, \tag{1}$$

$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \text{in } \Omega_1, \tag{2}$$

where $\boldsymbol{T} := -p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1)$ is the stress tensor, $\boldsymbol{D}(\boldsymbol{u}_1) := \frac{1}{2}(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_2^T)$ is the deformation rate tensor and $\mu > 0$ is the kinematic viscosity of the fluid. Space averaged velocity and pressure in the porous region Ω_2 are governed by Darcy equations

$$\mu\boldsymbol{u}_2 + K\nabla p_2 = 0, \quad \text{in } \Omega_2, \tag{3}$$

$$\nabla \cdot \boldsymbol{u}_2 = f_2, \quad \text{in } \Omega_2, \tag{4}$$

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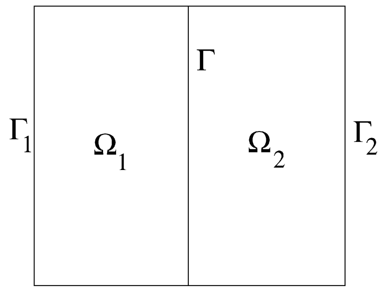

Fig. 1. Γ is the interface between Ω_1 and Ω_2. The remaining parts of their respective boundaries are Γ_1 and Γ_2.

where $K > 0$ is the Darcy permeability. For the simplicity of the exposition we assume that K is a constant, but a non-constant and sufficiently smooth K could be handled without essential modifications in the remaining of the paper. At this point we already notice that elimination of $\boldsymbol{u}_2$ in (4) by using (3) gives the Poisson equation

$$K\Delta p_2 = -\mu f_2, \quad \text{in } \Omega_2. \tag{5}$$

The following boundary conditions are considered on Γ_1 and Γ_2,

$$\boldsymbol{u}_1 = 0, \qquad \text{on } \Gamma_1, \tag{6}$$

$$p_2 = 0, \qquad \text{on } \Gamma_{2,D}, \tag{7}$$

$$\boldsymbol{u}_2 \cdot \boldsymbol{\nu}_2 = 0, \quad \text{on } \Gamma_{2,N}, \tag{8}$$

where $\boldsymbol{\nu}_i$ is the outgoing unit normal vector to $\partial\Omega_i$, $i = 1, 2$, and $\{\Gamma_{2,D}, \Gamma_{2,N}\}$ is a partition of Γ_2. Boundary conditions (6)–(8) are chosen to be homogeneous only for simplicity of the exposition and other type of boundary conditions could have been chosen as well.

On the interface we consider the transmission conditions

$$\boldsymbol{u}_1 \cdot \boldsymbol{\nu}_1 = \boldsymbol{u}_2 \cdot \boldsymbol{\nu}_1, \qquad \text{on } \Gamma, \tag{9}$$

$$-\boldsymbol{\nu}_1 \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) \cdot \boldsymbol{\nu}_1 = p_2, \quad \text{on } \Gamma, \tag{10}$$

which express mass conservation and equilibrium of normal forces across Γ respectively.

Another boundary condition remains to be prescribed at the interface to the solution of Stokes equations. At least for porous media of moderate porosity the no-slip boundary condition for the tangential fluid velocity may not be sufficiently appropriate. Based on experiments, Beavers and Joseph [2] proposed the following relation:

$$-\boldsymbol{\nu}_1 \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) \cdot \boldsymbol{\tau}_i = \frac{\alpha_i}{K^{1/2}}(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \boldsymbol{\tau}_i, \quad i = 1, n-1, \quad \text{on } \Gamma,$$

where α_i, $i = 1, n-1$, are positive non-dimensional constants, and $\boldsymbol{\tau}_i$, $i = 1, n-1$, are unit orthogonal vectors spanning the plane tangent to $\partial\Omega_i$. A mathematical justification of this interface condition was obtained by Saffman [21] who further observed that the seepage velocity $\boldsymbol{u}_2$ can be neglected. A rigorous justification was finally given using homogenization theory in [12]. We therefore consider the Beavers–Joseph–Saffman boundary condition

$$-\boldsymbol{\nu}_1 \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) \cdot \boldsymbol{\tau}_i = \frac{\alpha_i}{K^{1/2}}\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_i, \quad i = 1, n-1, \quad \text{on } \Gamma. \tag{11}$$

This coupled Stokes–Darcy problem (1)–(4), (6)–(11) has been studied recently from mathematical and numerical analysis viewpoints [13,8,6,1,18,19,5]. Finite element approximations firstly differ in the weak formulation of the coupled problem. While Stokes equations are generally handled using the standard mixed formulation, several approaches have been used for Darcy equations, including a mixed formulation [13] and the standard variational formulation of the equivalent Poisson equation (5) [8,6].

When a mixed formulation is used for Darcy equations, formulations of the coupled problem can differ depending on how the interface conditions are handled. Without entering into too much details, let us report that these approximations for Darcy equations involve classical mixed finite element methods (with finite elements such as those of

Raviart–Thomas, Brezzi–Douglas–Fortin–Marini, ...) [13,1,19], discontinuous Galerkin methods [18], or edge stabilized methods [5]. In every cases the pressure and velocity approximations in the porous domain are discontinuous, except in two of these works. In [1], a continuous finite element approximation of the velocity is obtained but using a very special finite element (defined on rectangles only) which moreover needs a modification in the vicinity of the interface, and in [5] the weak formulation uses stabilizing terms involving pressure jumps at the inter-element interfaces.

If the classical variational formulation of the Poisson equation (5) is used in the porous medium domain then continuous piecewise polynomials of a given order are standard finite elements for the pressure and work for the coupled problem as well [8]. As we shall see later, another advantage of this formulation is that the coupling can be formulated in a very easy and *natural* way. Nevertheless, in this case, the velocity obtained using (3) is discontinuous, at least if no post-processing is applied.

The motivation of the present work is to devise a finite element approximation of the coupled problem which combines usual Stokes finite elements (MINI, Taylor–Hood, conforming Crouzeix–Raviart, ...) in the free fluid domain with simple and continuous finite elements for both pressure and velocity in Darcy equations, namely P1 or P2 finite elements. These finite elements have the benefit of their simplicity and are generally already implemented in every house-made or commercial finite element code. Moreover, continuous velocity approximations are more satisfactory when used as inputs for the simulation of convective mass transport.

To this end, we use a mixed weak formulation which differs from the *dual* mixed formulation used in the previously cited literature, and which has more common features with the standard variational formulation of Poisson equation (5). For this formulation, interface conditions (9)–(10) at the interface are *natural*. This mixed formulation is a standard *primal* mixed variational formulation, studied for instance in [20], but augmented with a Galerkin least-squares term in the spirit of [9,15], so that the pair of velocity–pressure finite element spaces have no inf–sup condition to satisfy as with many mixed formulations.

The paper is organized as follows. In Section 2, after recalling the different formulations found in the literature, we introduce a new formulation using the *primal* mixed formulation of Darcy equations augmented with a suitable stabilizing residual term. In Section 3 we prove the well-posedness of the new weak formulation and an approximation result. We also give error estimates for a large family of finite element approximation spaces. Finally, in Section 4, we present the approximation errors obtained with various choices of finite elements in a numerical example.

Before closing this introduction, let us introduce some notations that will be used throughout the remaining of this article.

For $i = 1, 2$, let $L^2(\Omega_i)$ and $H^1(\Omega_i) := \{q \in L^2(\Omega_i);\ \frac{\partial q}{\partial x_j} \in L^2(\Omega_i)\}$ be the usual Sobolev spaces that we equip respectively with their usual norms

$$\|q\|_{0,\Omega_i} := \left(\int_{\Omega_i} q^2\, d\Omega\right)^{1/2} \quad \text{and} \quad \|q\|_{1,\Omega_i} := \left(\|q\|^2_{0,\Omega_i} + \sum_{j=1}^{n} \left\|\frac{\partial q}{\partial x_j}\right\|^2_{0,\Omega_i}\right)^{1/2}.$$

Define $H^1_{\Gamma_{2,D}}(\Omega_2) = \{q \in H^1(\Omega_2);\ q|_{\Gamma_{2,D}} = 0\}$. If $\Gamma_{2,D} \neq \emptyset$, we equip $H^1_{\Gamma_{2,D}}(\Omega_2)$ with the norm $|\cdot|_{1,\Omega_i}$ defined by

$$|q|_{1,\Omega_i} := \left(\sum_{j=1}^{n} \left\|\frac{\partial q}{\partial x_j}\right\|^2_{0,\Omega_i}\right)^{1/2}.$$

The same notation will be use throughout for the norms of the product spaces $\boldsymbol{L}^2(\Omega_i) := (L^2(\Omega_i))^n$ and $\boldsymbol{H}^1(\Omega_i) = (H^1(\Omega_i))^n$ and their subspaces, the notation of these product spaces being in bold. For instance, for every $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_i)$, we denote by $\|\boldsymbol{v}\|_{0,\Omega_i}$ the $\boldsymbol{L}^2(\Omega_i)$-norm defined by

$$\|\boldsymbol{v}\|_{0,\Omega_i} = \left(\sum_{j=1}^{n} \|(\boldsymbol{v})_j\|^2_{0,\Omega_i}\right)^{1/2},$$

where $(\boldsymbol{v})_j$ is the jth component of $\boldsymbol{v}$. We also define

$$\boldsymbol{H}^1_{\Gamma_1}(\Omega_1) = \left(H^1_{\Gamma_1}(\Omega_1)\right)^n = \left\{\boldsymbol{v} \in \left(H^1(\Omega_1)\right)^n:\ \boldsymbol{v}|_{\Gamma_1} = 0\right\},$$

as well as $H(\mathrm{div}, \Omega_2) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_2);\ \nabla \cdot \boldsymbol{v} \in L^2(\Omega_2)\}$ that we equip with the norm

$$\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega_2)} = \left(\|\boldsymbol{v}\|^2_{0,\Omega_2} + \sum_{j=1}^{n} \left\| \frac{\partial(\boldsymbol{v})_j}{\partial x_j} \right\|^2_{0,\Omega_2} \right)^{1/2}.$$

Similarly, we define $H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2) = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega_2);\ \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_{2,N}\}$, equipped with the norm $\|\cdot\|_{H(\mathrm{div},\Omega_2)}$.

Finally, let denote by $H^{1/2}(\Gamma)$ (respectively $H^{1/2}_{00}(\Gamma)$) the space of traces in Γ of functions in $H^1(\Omega_i)$ (respectively in $H^1(\Omega_i)$ and vanishing on Γ_i), the definition being independent of the choice of i (see [14]).

2. Weak formulation

In this section we first present weak formulations of the coupled Darcy–Stokes problem that have been used in the literature. This enables us to highlight the differences between them and the formulation that we introduce in the second part of this section.

2.1. Brief review of the literature

Multiplying (1) and (2) by test functions $\boldsymbol{v}_1$ and q_1 respectively, integrating by parts over Ω_1 the term involving ∇p_1 and using boundary conditions (10) and (11) yield the usual variational form of Stokes equations,

$$2\mu \int_{\Omega_1} D(\boldsymbol{u}_1) : D(\boldsymbol{v}_1)\, d\Omega + \int_{\Omega_1} p_1 \nabla \cdot \boldsymbol{v}_1\, d\Omega + \frac{1}{K^{1/2}} \int_{\Gamma} \Sigma_i \alpha_i (\tau_i \cdot \boldsymbol{u}_1)(\boldsymbol{v}_1 \cdot \tau_i)\, d\Gamma + \int_{\Gamma} p_2 (\boldsymbol{v}_1 \cdot \nu_1)\, d\Gamma$$
$$= \int_{\Omega_1} \boldsymbol{f}_1 \boldsymbol{v}_1\, d\Omega, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1), \tag{12}$$

$$\int_{\Omega_1} \nabla \cdot \boldsymbol{u}_1 q_1\, d\Omega = 0, \quad \forall q_1 \in L^2(\Omega_1). \tag{13}$$

A similar treatment applied to Darcy equations (3)–(4), integrating by parts the term involving ∇p_2 and using boundary conditions (7) and (8), yields the variational form of Darcy equations, also referred as its *dual* mixed formulation (see [20]),

$$\mu \int_{\Omega_2} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2\, d\Omega - K \int_{\Omega_2} p_2 \nabla \cdot \boldsymbol{v}_2\, d\Omega + K \int_{\Gamma} p_2 (\boldsymbol{v}_2 \cdot \nu_2)\, d\Gamma = 0, \quad \forall \boldsymbol{v}_2 \in H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2), \tag{14}$$

$$\int_{\Omega_2} \nabla \cdot \boldsymbol{u}_2\, q_2\, d\Omega = \int_{\Omega_2} f_2 q_2\, d\Omega, \quad \forall q_2 \in L^2(\Omega_2). \tag{15}$$

Note that boundary condition (8) and interface condition (9) have not been prescribed yet. They are not *natural* boundary conditions, at least for this weak formulation of Darcy equations. They can be enforced in a strong way by looking at the solution in a space of functions satisfying the interface condition. While this treatment is standard and easily handled numerically for boundary conditions like (8), it is less obvious for an interface condition like (9) especially if different velocity approximation spaces are used in each side of the interface. As done by Layton et al. [13], a way to incorporate it in a weak form is to introduce the Lagrange multiplier λ such that

$$\lambda = -\boldsymbol{\nu}_1 \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) \cdot \boldsymbol{\nu}_1 = p_2, \quad \text{on } \Gamma. \tag{16}$$

Variational equations (12)–(13) are consequently modified to give

$$2\mu \int_{\Omega_1} D(\boldsymbol{u}_1) : D(\boldsymbol{v}_1)\, d\Omega + \int_{\Omega_1} p_1 \nabla \cdot \boldsymbol{v}_1\, d\Omega + \frac{1}{K^{1/2}} \int_{\Gamma} \Sigma_i \alpha_i (\tau_i \cdot \boldsymbol{u}_1)(\boldsymbol{v}_1 \cdot \tau_i)\, d\Gamma + \int_{\Gamma} \lambda \boldsymbol{v}_1 \cdot \nu_1\, d\Gamma$$
$$= \int_{\Omega_1} \boldsymbol{f}_1 \boldsymbol{v}_1\, d\Omega, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1), \tag{17}$$

$$\int_{\Omega_1} \nabla \cdot \boldsymbol{u}_1 q_1\, d\Omega = 0, \quad \forall q_1 \in L^2(\Omega_1), \tag{18}$$

and interface condition (9) is written in the weak form

$$\int_{\Gamma} (\boldsymbol{v}_1 \cdot \nu_1 + \boldsymbol{v}_2 \cdot \nu_2)\tilde{\lambda}\, d\Gamma = 0, \quad \forall \tilde{\lambda} \in H^{1/2}_{00}(\Gamma). \tag{19}$$

Layton et al. [13] proved that Eqs. (14)–(15), (17)–(19) admit a unique solution with $(\boldsymbol{u}_1, p_1, \lambda) \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times H^{1/2}_{00}(\Gamma)$ and $(\boldsymbol{u}_2, p_2) \in H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2) \times L^2(\Omega_2)$, but under the assumption that

$$H^{1/2}_{00}(\Gamma) = H^{1/2}(\Gamma).$$

Such condition holds for instance if $\Gamma = \partial\Omega_1$ or $\Gamma = \partial\Omega_2$. Under this assumption on Γ, Layton et al. [13] proved these weak equations to be equivalent to the following ones which are the base of their finite element approximation results. Define

$$X = \boldsymbol{H}^1_{\Gamma_1}(\Omega_1) \times H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2) \quad \text{and} \quad M = L^2(\Omega_1) \times L^2(\Omega_2).$$

For $\boldsymbol{v} \in X$ (respectively $p \in M$) and $i = 1, 2$, we denote by $\boldsymbol{v}_i$ (respectively p_i) the function defined on Ω_i such that $p = (p_1, p_2)$ and we define

$$V = \left\{ \boldsymbol{v} \in X \colon \int_{\Gamma_1} (\boldsymbol{v}_1 \cdot \nu_1 + \boldsymbol{v}_2 \cdot \nu_2)\tilde{\lambda}\, d\Gamma = 0,\ \forall \tilde{\lambda} \in H^{1/2}_{00}(\Gamma) \right\}.$$

Then a new formulation of the problem simply writes: find $(\boldsymbol{u}, p) = ((\boldsymbol{u}_1, \boldsymbol{u}_2), (p_1, p_2)) \in V \times M$ such that

$$2\mu \int_{\Omega_1} D(\boldsymbol{u}_1) : D(\boldsymbol{v}_1)\, d\Omega + \frac{1}{K^{1/2}} \int_{\Gamma} \Sigma_i \alpha_i (\tau_i \cdot \boldsymbol{u}_1)(\boldsymbol{v}_1 \cdot \tau_i)\, d\Gamma + \mu \int_{\Omega_2} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2\, d\Omega$$
$$+ \int_{\Omega_1} p_1 \nabla \cdot \boldsymbol{v}_1\, d\Omega - K \int_{\Omega_2} p_2 \nabla \cdot \boldsymbol{v}_2\, d\Omega = \int_{\Omega_1} \boldsymbol{f}_1 \boldsymbol{v}_1\, d\Omega, \quad \forall \boldsymbol{v} \in V, \tag{20}$$

$$\int_{\Omega_1} \nabla \cdot \boldsymbol{u}_1 q_1\, d\Omega + \int_{\Omega_2} \nabla \cdot \boldsymbol{u}_2 q_2\, d\Omega = \int_{\Omega_2} f_2 q_2\, d\Omega, \quad \forall q \in M. \tag{21}$$

Approximation results were established in [13] with combinations finite elements which are stable for Stokes equations (MINI, Taylor–Hood, conforming Crouzeix–Raviart, etc., see [3]) and finite elements which are stable for the *dual* mixed formulation of Darcy equations (like Raviart–Thomas, Brezzi–Douglas–Marini, etc., see [3]). Unlike the reported finite elements which are usual for Stokes equations, those for Darcy equations provide discontinuous velocity and pressure. On another hand, they ensure continuity of fluxes and mass conservation locally.

Starting also from variational equations (20)–(21) of the coupled problem, other recent works have focused on methods using the same finite elements in the two regions. Although usual finite elements for Stokes equations do not work for the mixed form of Darcy equations, Arbogast and Brunson [1] successfully applied a finite element due to Fortin [11] which gives a continuous velocity approximation. Since the tangential component of the velocity does not need to be continuous at the interface, the finite elements surrounding the interface are modified. Unfortunately, this finite element method is only defined for rectangular meshes.

Another possibility is to apply a discontinuous Galerkin method in both regions (see Rivière [18]), but this method would require special data structures and computational techniques to fully use its advantages.

A somewhat different approach is that of Burman and Hansbo [5] who start from the usual formulations (12)–(13) and (14)–(15) of Stokes and Darcy equations. The interface condition (9) is prescribed by the method of Nitsche [16] which amounts to introduce several boundary integral terms in the weak equations. For the finite element approximation they chose to use a stabilized method which adds terms involving pressure jumps at inter-element interfaces in the weak formulation. Despite of the unusual character of the method and the discontinuity of the pressure approximation, the velocity approximation is continuous (and piecewise linear) in both domains.

All the reported variational formulations of the coupled problem are based on a mixed formulation of Darcy equations. One alternative is to use the classical variational formulation of the equivalent Poisson equation (5). Applied to our problem, this usual weak form writes

$$\int_{\Omega_2} \nabla p_2 \cdot \nabla q_2 \, d\Omega - \frac{\mu}{K} \int_{\Gamma} (\boldsymbol{u}_1 \cdot \boldsymbol{\nu}_1) q_2 \, d\Gamma = \frac{\mu}{K} \int_{\Omega_2} f_2 q_2 \, d\Omega, \quad \forall q_2 \in H^1_{\Gamma_{2,D}}(\Omega_2). \tag{22}$$

Discacciati et al. [6] proved the coupled problem (12)–(13) and (22) to be well-posed, with a solution such that $(\boldsymbol{u}_1, p_1) \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1)$ and $p_2 \in H^1_{\Gamma_{2,D}}(\Omega_2)$.

Comparatively to previous formulations, now p_2 lies in $H^1(\Omega_2)$ instead of $L^2(\Omega_2)$, and $\boldsymbol{u}_2$ is in $\boldsymbol{L}^2(\Omega_2)$ instead of $H(\mathrm{div}, \Omega_2)$, so that regularities are completely different in both sides of the interface. Nevertheless, transmission conditions at the interface are now *natural* as they are enforced in a weak form. In addition, Discacciati et al. [7,8] showed that classical finite elements for Stokes equations combined with classical conforming finite elements (typically P_1 or P_2 finite elements) for the variational Poisson equation (22) gives a convergent finite element approximation of the coupled problem. Note that if continuous P_1 or P_2 finite elements are used for the pressure approximation in (22) then the velocity output in the corresponding domain that results from relation (3) is by no means continuous.

2.2. Formulations using the primal mixed variational Darcy equations

Our motivation is to derive finite element approximations of the coupled problem using classical finite elements in the fluid region combined with standard and smooth approximations for both pressure and velocity unknowns in the porous medium domain. Moreover, we look for a formulation for which the interface conditions are easy to implement numerically.

As a result, we consider an alternative mixed formulation to (14)–(15), that is the *primal* mixed formulation

$$\mu \int_{\Omega_2} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 \, d\Omega + K \int_{\Omega_2} \nabla p_2 \cdot \boldsymbol{v}_2 \, d\Omega = 0, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{L}^2(\Omega_2), \tag{23}$$

$$\int_{\Omega_2} \boldsymbol{u}_2 \cdot \nabla q_2 \, d\Omega - \int_{\Gamma} (\boldsymbol{u}_1 \cdot \boldsymbol{\nu}_1) q_2 \, d\Gamma = \int_{\Omega_2} f_2 q_2 \, d\Omega, \quad \forall q_2 \in H^1(\Omega_2), \tag{24}$$

which is obtained from (3)–(4) in the same way that the *dual* mixed formulation (14)–(15) but with the integration by parts performed in the term containing $\nabla \cdot \boldsymbol{u}_2$ instead.

In the new weak form of the coupled Stokes–Darcy equations (12)–(13) and (23)–(24), interface conditions (9) and (10) are imposed in a weak and *natural* way.

In this new mixed formulation of Darcy equations, solutions have the same regularity than the solutions of the standard variational formulation (22): $(\boldsymbol{u}_2, p_2) \in \boldsymbol{L}^2(\Omega_2) \times H^1(\Omega_2)$. Convergent finite element discretizations for *primal* mixed formulation are for instance discussed in [20]. Examples are continuous piecewise polynomial of degree $k \geqslant 1$ for the pressure and piecewise polynomial (non-necessarily continuous) of degree $\ell \geqslant k-1$ for the velocity components. Unfortunately, velocity approximations are discontinuous (not even in $H(\mathrm{div}, \Omega_2)$). Moreover, we can show that these approximations are equivalent to discretize the variational formulation (22) of Poisson equation with continuous piecewise polynomials of degree k and to interpolate the gradient of the resulting pressure approximation with piecewise polynomial (non-necessarily continuous) of degree ℓ. In addition, as discussed and illustrated numerically in [22], there is no gain in taking any other value than $k-1$ for ℓ.

In [22] we tested numerically other velocity approximation spaces including the space of continuous piecewise linear functions, whereas the pressure was also approximated with continuous piecewise linear functions. Tests were encouraging but without theoretical justification. Moreover we know that such a couple of approximation spaces make spurious pressure modes to appear when applied to the *dual* mixed formulation (see for instance [17] or [10, Section II.4.2.3]). For our problem, if we denote them by $Q_h^2 \subset H^1_{\Gamma_{2,D}}(\Omega_2)$ and $V_h^2 \subset \boldsymbol{L}^2(\Omega_2)$, due to the saddle point structure of (23)–(24) they need to satisfy the stability condition (see for instance [20])

$$\inf_{q_h \in Q_h^2} \sup_{\boldsymbol{v}_h \in V_h^2} \frac{\int_{\Omega_2} \boldsymbol{v}_h \cdot \nabla q_h \, d\Omega}{|q_h|_{1,\Omega_2} \|\boldsymbol{v}_h\|_{0,\Omega_2}} > \delta > 0. \tag{25}$$

A way to circumvent this stability condition and allow standard finite element spaces is to work with augmented formulations, also known as Galerkin least-squares methods (see for instance [3, Sections I.5, V.7] and references therein). Here we apply a particular method used by Masud and Hughes [15] which consists in adding precisely the extra term $\frac{1}{2\mu}\int_{\Omega_2}(\mu \boldsymbol{u}_2 + K\nabla p_2)\cdot(-\mu \boldsymbol{v}_2 + K\nabla q_2)\, d\Omega$ to the *primal* mixed form (23)–(24), resulting in

$$\begin{aligned}
&\mu \int_{\Omega_2} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 \, d\Omega + K \int_{\Omega_2} \nabla p_2 \cdot \boldsymbol{v}_2 \, d\Omega - K \int_{\Omega_2} \boldsymbol{u}_2 \cdot \nabla q_2 \, d\Omega \\
&\qquad - K \int_{\Gamma} (\boldsymbol{u}_1 \cdot \boldsymbol{\nu}_1) q_2 \, d\Gamma + \frac{1}{2\mu} \int_{\Omega_2} (\mu \boldsymbol{u}_2 + K\nabla p_2)\cdot(-\mu \boldsymbol{v}_2 + K\nabla q_2)\, d\Omega \\
&\qquad = K \int_{\Omega_2} f_2 q_2 \, d\Omega, \quad \forall (\boldsymbol{v}_2, q_2) \in \boldsymbol{L}^2(\Omega_2) \times H^1_{\Gamma_{2,D}}(\Omega_2).
\end{aligned} \tag{26}$$

As it will be shown in the next section, this weak form of Darcy equations involves a continuous and elliptic bilinear form on one side of the equality and a linear form on the other side, both forms being defined on the velocity–pressure space $\boldsymbol{L}^2(\Omega_2) \times H^1(\Omega_2)$. As a result, the well-posedness of this Darcy problem alone is an easy matter for given suitable data $\boldsymbol{u}_1$ and $\boldsymbol{f}_2$ and there is no inf–sup condition to be satisfied.

In [15] this Galerkin least-squares method was in fact applied to the *dual* mixed variational formulation of Darcy equations, but the authors provided stability estimates in $\boldsymbol{L}^2(\Omega_2) \times H^1(\Omega_2)$ and not $H(\mathrm{div}, \Omega_2) \times L^2(\Omega_2)$ which is the function space normally associated to the *dual* formulation. Moreover the approximation space for the velocity field was forced to be included in $H(\mathrm{div}, \Omega_2)$ whereas $\boldsymbol{L}^2(\Omega_2)$ is sufficient. This stabilization is actually more natural for the *primal* formulation.

Although it is not investigated numerically in the present work, we can add a second stabilization term involving the divergence of the velocity in the manner of [4] or [15] (although in [15] this term is mesh dependent), $\int_{\Omega_2} \mathrm{div}\, \boldsymbol{u}_2 \,\mathrm{div}\, \boldsymbol{v}_2 \, d\Omega$, resulting in a variational equation in the velocity–pressure space $H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2) \times H^1_{\Gamma_{2,D}}(\Omega_2)$:

$$\begin{aligned}
&\frac{\mu}{2} \int_{\Omega_2} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 \, d\Omega + \frac{K}{2} \int_{\Omega_2} \nabla p_2 \cdot \boldsymbol{v}_2 \, d\Omega - \frac{K}{2} \int_{\Omega_2} \boldsymbol{u}_2 \cdot \nabla q_2 \, d\Omega - K \int_{\Gamma} (\boldsymbol{u}_1 \cdot \boldsymbol{\nu}_1) q_2 \, d\Gamma \\
&\qquad + \alpha \int_{\Omega_2} \mathrm{div}\, \boldsymbol{u}_2 \,\mathrm{div}\, \boldsymbol{v}_2 \, d\Omega + \frac{K^2}{2\mu} \int_{\Omega_2} \nabla p_2 \cdot \nabla q_2 \, d\Omega \\
&\qquad = \frac{K}{2} \int_{\Omega_2} f_2 q_2 \, d\Omega, \quad \forall (\boldsymbol{v}_2, q_2) \in H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2) \times H^1_{\Gamma_{2,D}}(\Omega_2),
\end{aligned} \tag{27}$$

here, α is a non-negative constant. For the approximation, the inf–sup condition (25) is again circumvented and continuous P_1 finite elements can be used, but optimal finite element spaces are those of Raviart–Thomas, Brezzi–Douglas–Marini, etc., ..., where the continuity of the flux is ensured at each inter-element face. Since these finite elements are not necessarily the ones we want to work with we do not pursue this path.

3. Well-posedness and finite element approximation of the coupled problem

In this section we study the well-posedness and the finite-dimensional approximation of the coupled Stokes–Darcy equations under the weak form (12)–(13) and (26) (case $\alpha = 0$). As a complement we also consider the weak formulation (12)–(13) and (27) with $\alpha = 1$, without any loss of generality over the case $\alpha > 0$. We also provide examples of finite element approximation spaces in the case $\alpha = 0$.

3.1. Well-posedness

In the following, for simplicity of the exposition, we assume that $\Gamma_1 \neq \emptyset$ and $\Gamma_{2,D} \neq \emptyset$. These assumptions allow us to state that $|\cdot|_{1,\Omega_1}$ and $|\cdot|_{2,\Omega_1}$ are convenient norms for $\boldsymbol{H}^1_{\Gamma_1}(\Omega_1)$ and $H^1_{\Gamma_{2,D}}(\Omega_2)$ respectively.

We introduce the following notation:

$$V_1 = \boldsymbol{H}^1_{\Gamma_1}(\Omega_1), \qquad V_2 = \begin{cases} \boldsymbol{L}^2(\Omega_2) & \text{if } \alpha = 0, \\ H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2) & \text{if } \alpha = 1, \end{cases}$$

$$Q_1 = L^2(\Omega_1), \qquad Q_2 = H^1_{\Gamma_{2,D}}(\Omega_2),$$

as well as

$$V = V_1 \times V_2, \quad \underline{V} = V \times Q_2.$$

We also adopt the following convention: every $\underline{\boldsymbol{u}} \in \underline{V}$ (respectively $\underline{\boldsymbol{v}} \in \underline{V}$) can be written $\underline{\boldsymbol{u}} = (\boldsymbol{u}, p_2) = (\boldsymbol{u}_1, \boldsymbol{u}_2, p_2)$ (respectively $\underline{\boldsymbol{v}} = (\boldsymbol{v}, q_2) = (\boldsymbol{v}_1, \boldsymbol{v}_2, q_2)$) with $(\boldsymbol{u}_1, \boldsymbol{u}_2) \in V$ and $p_2 \in Q_2$ (respectively $(\boldsymbol{v}_1, \boldsymbol{v}_2) \in V$ and $q_2 \in Q_2$). V and $\underline{V}$ are endowed with the usual norms of product spaces:

$$\|\boldsymbol{v}\|_V^2 = |\boldsymbol{v}_1|^2_{1,\Omega_1} + \|\boldsymbol{v}_2\|^2_{0,\Omega_2} + \alpha^2 \|\operatorname{div} \boldsymbol{v}_2\|^2_{0,\Omega_2}, \tag{28}$$

$$\|\underline{\boldsymbol{v}}\|_{\underline{V}}^2 = \|\boldsymbol{v}\|_V^2 + |p_2|^2_{1,\Omega_2}. \tag{29}$$

Let us introduce the bilinear forms

$$a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) = 2\mu \int_{\Omega_1} D(\boldsymbol{u}_1) : D(\boldsymbol{v}_1)\, d\Omega, \qquad a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) = \mu \int_{\Omega_2} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2\, d\Omega,$$

$$b_1(\boldsymbol{v}_1, p_1) = \int_{\Omega_1} p_1 \nabla \cdot \boldsymbol{v}_1\, d\Omega, \qquad b_2(\boldsymbol{v}_2, p_2) = K \int_{\Omega_2} \nabla p_2 \cdot \boldsymbol{v}_2\, d\Omega,$$

$$a_{1,\Gamma}(\boldsymbol{u}_1, \boldsymbol{v}_1) = \frac{1}{K^{1/2}} \int_{\Gamma} \Sigma_i \alpha_i \tau_i \cdot \boldsymbol{u}_1 \boldsymbol{v}_1 \cdot \tau_i\, d\Gamma, \qquad b_\Gamma(\boldsymbol{v}_1, p_2) = \int_{\Gamma} p_2 \boldsymbol{v}_1 \cdot \nu_1\, d\Gamma,$$

$$c_2(p_2, q_2) = \frac{K^2}{2\mu} \int_{\Omega_2} \nabla p_2 \cdot \nabla q_2\, d\Omega,$$

as well as the linear forms

$$l_1(\boldsymbol{v}_1) = \int_{\Omega_1} \boldsymbol{f}_1 \boldsymbol{v}_1\, d\Omega, \qquad l_2(q_2) = \int_{\Omega_2} f_2 q_2\, d\Omega \tag{30}$$

defined for all $\boldsymbol{u}_i, \boldsymbol{v}_i \in V_i$, $p_i, q_i \in Q_i$, $i = 1, 2$. They are used to define the following bilinear and linear forms,

$$\begin{aligned} a(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) = {} & K a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) + K a_{1,\Gamma}(\boldsymbol{u}_1, \boldsymbol{v}_1) + \frac{1}{2} a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) + c_2(p_2, q_2) \\ & + \frac{1}{2}\big(b_2(\boldsymbol{v}_2, p_2) - b_2(\boldsymbol{u}_2, q_2)\big) + K\big(b_\Gamma(\boldsymbol{v}_1, p_2) - b_\Gamma(\boldsymbol{u}_1, q_2)\big) + \alpha a_{\mathrm{div}}(\boldsymbol{u}_2, \boldsymbol{v}_2), \end{aligned} \tag{31}$$

$$b(\underline{\boldsymbol{v}}, p_1) = b_1(\boldsymbol{v}_1, p_1), \tag{32}$$

$$l(\underline{\boldsymbol{v}}) = K l_1(\boldsymbol{v}_1) + l_2(q_2), \tag{33}$$

defined for all $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \underline{V}$, $p_1 \in Q_1$. Therefore, weak equations (12)–(13) and (27) can then be put in the abstract form

$$a(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) + b(\underline{\boldsymbol{v}}, p_1) = l(\underline{\boldsymbol{v}}), \quad \forall \underline{\boldsymbol{v}} \in \underline{V}, \tag{34}$$

$$b(\underline{\boldsymbol{u}}, q_1) = 0, \quad \forall q_1 \in Q_1, \tag{35}$$

which enables us to prove easily the following existence result for this weak formulation of the coupled Stokes–Darcy equations.

Theorem 1. *Let* $f_1 \in L^2(\Omega_1)$ *and* $f_2 \in L^2(\Omega_2)$.

- *Eqs.* (12), (13), (26) *have a unique solution such that*
$$(\boldsymbol{u}_1, \boldsymbol{u}_2) \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1) \times \boldsymbol{L}^2(\Omega_2), \qquad (p_1, p_2) \in L^2(\Omega_1) \times H^1_{\Gamma_{2,D}}(\Omega_2).$$
- *Eqs.* (12), (13), (27) *have a unique solution such that*
$$(\boldsymbol{u}_1, \boldsymbol{u}_2) \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1) \times H_{\Gamma_{2,N}}(\mathrm{div}, \Omega_2), \qquad (p_1, p_2) \in L^2(\Omega_1) \times H^1_{\Gamma_{2,D}}(\Omega_2).$$

For any of the two solutions, corresponding to the cases $\alpha = 0$ *and* $\alpha = 1$ *respectively, there exists* $M = M(\alpha) > 0$ *which does not depend on the data nor on* $\underline{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, p_2)$ *and* p_1 *such that*

$$\|\underline{u}\|_{\underline{V}} + \|p_1\|_{0,\Omega_1} \leqslant M\big(\|\boldsymbol{f}_1\|_{0,\Omega_1} + \|\boldsymbol{f}_2\|_{0,\Omega_2}\big).$$

Proof. With applications of Cauchy–Schwarz, triangle and Sobolev trace inequalities we easily verify the continuity of $a(\cdot\,,\cdot)$ and $b(\cdot\,,\cdot)$: there exist $C_1, C_2 > 0$ such that

$$\big|a(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})\big| \leqslant C_1 \|\underline{\boldsymbol{u}}\|_{\underline{V}} \|\underline{\boldsymbol{v}}\|_{\underline{V}}, \quad \forall \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \underline{V}, \tag{36}$$
$$\big|b(\underline{\boldsymbol{u}}, q)\big| \leqslant C_2 \|\underline{\boldsymbol{u}}\|_{\underline{V}} \|q\|_{0,\Omega_1}, \quad \forall \underline{\boldsymbol{u}} \in \underline{V},\ \forall q \in Q_1. \tag{37}$$

On another hand, we have

$$a(\underline{\boldsymbol{u}}, \underline{\boldsymbol{u}}) = K a_1(\boldsymbol{u}_1, \boldsymbol{u}_1) + K a_{1,\Gamma}(\boldsymbol{u}_1, \boldsymbol{u}_1) + \frac{1}{2} a_2(\boldsymbol{u}_2, \boldsymbol{u}_2) + a_{\mathrm{div}}(\boldsymbol{u}_2, \boldsymbol{u}_2) + c_2(p_2, p_2).$$

From Korn inequality applied in Ω_1 and Poincaré inequality applied in Ω_2, and from the definition of the norm in $H(\mathrm{div}, \Omega_2)$ in the case $\alpha = 1$, it easily follows that $a(\cdot\,,\cdot)$ is a coercive bilinear form on $\underline{V}$: there exists $c > 0$ such that

$$\big|a(\underline{\boldsymbol{u}}, \underline{\boldsymbol{u}})\big| \geqslant c \|\underline{\boldsymbol{u}}\|^2_{\underline{V}}, \quad \forall \underline{\boldsymbol{u}} \in \underline{V}.$$

Moreover, regularity assumptions on f_1 and f_2 imply that ℓ is a linear continuous form on $\underline{V}$. It only remains to show that the following inf–sup condition is satisfied:

$$\inf_{\{q_1 \in Q_1;\ |q_1|_{0,\Omega_1}=1\}} \ \sup_{\{\underline{\boldsymbol{v}} \in \underline{V};\ \|\underline{\boldsymbol{v}}\|_{\underline{V}}=1\}} b(\underline{\boldsymbol{v}}, q_1) > \delta > 0. \tag{38}$$

This is an obvious consequence of the well-known inf–sup property for Stokes equation (see [3]):

$$\inf_{\{q_1 \in Q_1;\ |q_1|_{0,\Omega_1}=1\}} \ \sup_{\{\boldsymbol{v}_1 \in \boldsymbol{H}^1_{\Gamma_1}(\Omega_1);\ |\boldsymbol{v}_1|_{1,\Omega_1}=1\}} \int_{\Omega} q_1 \nabla \cdot \boldsymbol{v}_1 \, d\Omega > \delta > 0. \tag{39}$$

We conclude with the classical existence theory for saddle point problems which gives the desired result with a stability constant M depending on C_1, C_2, c and δ (see [3]). □

3.2. Approximation

Let $\underline{V}_h \subset \underline{V}$ and $Q_{1,h} \subset Q_1$ be conforming finite-dimensional approximation spaces with $\underline{V}_h = V_{1,h} \times V_{2,h} \times Q_{2,h}$, h being a discretization parameter. Consider the Galerkin approximation problem

$$a(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b(\underline{\boldsymbol{v}}_h, p_h) = l(\underline{\boldsymbol{v}}_h), \quad \forall \underline{\boldsymbol{v}}_h \in \underline{V}_h, \tag{40}$$
$$b(\underline{\boldsymbol{u}}_h, q_h) = l_2(q_h), \qquad \forall q_h \in Q_{1,h}. \tag{41}$$

Since the approximation spaces are conforming and $a(\cdot\,,\cdot)$ is coercive on the whole space $\underline{V}$ a unique solution to (40)–(41) is ensured once the discrete inf–sup condition is satisfied:

$$\alpha_h = \inf_{\{q_h \in Q_{1,h};\ \|q_h\|_{0,\Omega_1}=1\}} \ \sup_{\{\underline{\boldsymbol{v}}_h \in \underline{V}_h;\ \|\underline{\boldsymbol{v}}_h\|_{\underline{V}}=1\}} b(\underline{\boldsymbol{v}}_h, q_h) > 0. \tag{42}$$

Under this assumption we have the estimate (see [3])

$$\|\underline{\boldsymbol{u}}-\underline{\boldsymbol{u}}_h\|_{\underline{V}}+\|p_1-p_{1,h}\|_{0,\Omega_1}\leqslant C\Big(\inf_{\underline{\boldsymbol{v}}_h\in\underline{V}}\|\underline{\boldsymbol{u}}-\underline{\boldsymbol{v}}_h\|_{\underline{V}}+\inf_{p_h\in Q_{1,h}}\|p-p_h\|_{0,\Omega_1}\Big). \tag{43}$$

Moreover, the constant C does not depend on h when the inf–sup condition is satisfied uniformly with respect to the discretization parameter: $\alpha_h\geqslant\alpha>0$.

As for the continuous problem, we easily see that a key condition on the well-posedness of the discrete problem is the inf–sup condition in the fluid domain.

Theorem 2. *Assume that approximation spaces $V_{1,h}$ and $Q_{1,h}$ are such that*

$$\inf_{\{q_h\in Q_{1,h};\ \|q_h\|_{0,\Omega_1}=1\}}\ \sup_{\{\boldsymbol{v}_{1,h}\in V_{1,h};\ |\boldsymbol{v}_{1,h}|_{1,\Omega_1}=1\}} b_1(\boldsymbol{v}_{1,h},q_h)>\alpha_{1,h}>0. \tag{44}$$

Then (40)–(41) *admits a unique solution* $(\underline{\boldsymbol{u}},p_{1,h})$. *Moreover, if there exists $\alpha_1>0$ such that $\alpha_{1,h}>\alpha_1$, $\forall h>0$, then estimate* (43) *holds with C independent of h.*

Proof. We easily verify that (44) implies the same inf–sup property with $V_{1,h}$ and $b_1(\cdot,\cdot)$ replaced by $\underline{V}_h$ and b respectively. The general approximation theory of saddle point problems gives the desired results (see [3]). □

3.3. Examples of finite element approximations

We assume that Ω_1 and Ω_2 are polygons or polyhedra. On each domain Ω_i, $i=1,2$, we introduce a quasi-uniform family of meshes $(\mathcal{T}_h^i)_{h0}$ consisting of triangulations in *n-simplices* such that any intersection of two simplices is either empty, a vertex, an edge or a face. Note that we do not assume that these meshes match at the interface.

For a general domain Ω with mesh $\mathcal{T}_h$, and a functional space Q, we say that we use P_r finite elements if the approximation space $Q_h\subset Q$ is composed of the piecewise polynomial functions of degree r: $Q_h=\{q\in L^2(\Omega);\ q|_T\in\mathbb{P}(T),\ \forall T\in\mathcal{T}_h\}$. We keep the same notation, P_r, when the functional space is made of vectorial functions and P_r finite elements are use for each one of their components. Similarly, we say that we use P_r^c finite elements if the approximation space is only composed of continuous functions: $Q_h=C^0(\Omega)\cap\{q\in Q;\ q|_T\in\mathbb{P}(T),\ \forall T\in\mathcal{T}_h\}$.

We give here some examples of finite element approximation spaces for the coupled problem in the case $\alpha=0$, that is when $V_2=\boldsymbol{L}^2(\Omega_2)$. In this case, it is well known that vectorial approximations made of continuous piecewise polynomials of a given degree give interpolation errors that are optimal in $\boldsymbol{L}^2(\Omega_2)$ with respect to the degree of the polynomials. In the case $\alpha=1$, that is when $V_2=H_{\Gamma_{2,N}}(\mathrm{div},\Omega_2)$, optimal interpolation errors are known for finite element approximation spaces such as Raviart–Thomas spaces. Since we are not interested in these kind of approximation spaces, we consider the case $\alpha=0$ only.

Due to the coupling between the two domains, *a priori* the finite element approximation space for any of the four unknowns affects the approximation errors for the remaining ones. Ideal approximation spaces are for instance those combining approximation spaces of the same order r for each sub-problem, Stokes or Darcy problem. From an interpolation point of view, if approximation spaces satisfy

$$\inf_{\boldsymbol{v}_h\in V_{1,h}}\|\boldsymbol{u}_1-\boldsymbol{v}_h\|_{V_1}+\inf_{p_h\in Q_{1,h}}\|p_1-p_h\|_{Q_1}\leqslant Ch^r, \tag{45}$$

$$\inf_{\boldsymbol{v}_h\in V_{2,h}}\|\boldsymbol{u}_2-\boldsymbol{v}_h\|_{V_2}+\inf_{p_h\in Q_{2,h}}\|p_2-p_h\|_{Q_2}\leqslant Ch^s, \tag{46}$$

with C independent of h (but depending on suitable norms of $\boldsymbol{u}_1,\boldsymbol{u}_2$, p_1 and p_2) then they satisfy:

$$\inf_{\underline{\boldsymbol{v}}_h\in\underline{V}_h}\|\underline{\boldsymbol{u}}-\underline{\boldsymbol{v}}_h\|_{\underline{V}}+\inf_{p_h\in Q_{1,h}}\|p_1-p_h\|_{Q_1}\leqslant Ch^t \tag{47}$$

with $t=\min\{r,s\}$.

In the fluid domain, we consider usual finite elements for Stokes equations satisfying the inf–sup condition (42), like the MINI finite element (the first order approximation method), the Taylor–Hood or the conforming Crouzeix–Raviart (second order approximation methods) finite elements. For Darcy equations we chose usual finite element spaces such as continuous or discontinuous P_1 or P_2 finite elements for the pressure and the velocity.

Let us discuss some possible choices depending first on the choice of the finite element method for Stokes equations.

1st order approximation spaces. The MINI finite element gives the 1st order method for Stokes equations. Recall that in this case $V_{1,h} \times Q_{1,h}$ is constructed with the $(P_1^c + \text{bubbles}) - P_1^c$ finite element. Let us give simple choices of finite elements for Darcy approximation spaces $V_{2,h} \times Q_{2,h}$.

- $P_0 - P_1^c$. We know that it gives the 1st order method for the *primal* mixed formulation of Darcy equations (see [20]). Since it satisfies (46) with $r = 1$ (see [3]), this order is conserved in the approximation of the coupled problem with the augmented formulation (26). Nevertheless, compared to the application of the P_1^c finite element directly to the variational formulation of the associated Poisson equation, it brings nothing more in terms of smoothness or convergence order for instance.
- $P_1^c - P_1^c$. This is still the 1st order approximation method. The main drawback is that the number of degrees of freedom is increased (for the same h) with respect to the previous choice, without increasing the approximation order, at least theoretically. Nevertheless, a clear advantage is that the velocity approximation is continuous.

2nd order approximation spaces. Taylor–Hood and Crouzeix–Raviart finite elements give 2nd order approximations for Stokes equations. Recall that the Taylor–Hood element is $P_2^c - P_1^c$, whereas the Crouzeix–Raviart element is $(P_2^c + \text{bubble}) - P_0$. Let us give some examples for Darcy approximation spaces $V_{2,h} \times Q_{2,h}$.

- $P_1 - P_2^c$. It is the 2nd order approximation method for the *primal* mixed formulation of Darcy equations (see [20]). Since the corresponding approximation space satisfies (46) with $r = 2$ (see [3]), this order is conserved in the approximation of the coupled problem with the augmented formulation (26). But, as for $P_1 - P_2^c$, it does not bring nothing substantial compared to the P_2^c finite element approximation of the direct variational formulation of the equivalent Poisson equation (see [8]).
- $P_1^c - P_2^c$. With continuous piecewise linear velocity approximations, the approximation order is still 2 and the number of unknowns is decreased when compared to the previous choice. Moreover, the velocity approximation is continuous.
- $P_2^c - P_2^c$. In this case, the approximation order should not be higher than 2, even for the velocity. We shall verify this in one numerical example.

4. A numerical example

In the domain $\Omega = (0,1) \times (0,1)$ with $\Omega_1 = (0,1/2) \times (0,1)$, $\Omega_2 = (1/2,1) \times (0,1)$, we consider the following velocity and pressure fields:

$$\boldsymbol{u}_1(x,y) = \left(y^4 e^x, e^y \cos(2x)\right), \quad p_1(x,y) = -y^4 e^x, \quad \forall (x,y) \in \Omega_1,$$
$$\boldsymbol{u}_2(x,y) = \left(y^4 e^x, 4y^3 e^x\right), \quad p_2(x,y) = -y^4 e^x, \quad \forall (x,y) \in \Omega_2.$$

They are solutions of the Darcy–Stokes system (1)–(4), (6)–(11) with additional forcing terms, notably in the interface conditions (10)–(11), and in the continuity equations (2) and (4) (velocities are not divergence-free).

The structured meshes contain $\frac{2}{h^2}$ triangles, for $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$. See Fig. 2 for the $2 \times 10 \times 10$ triangular mesh and the velocity field which is also depicted.

We first consider the Taylor–Hood finite element approximation for the Stokes equations. For the Darcy equations, we consider the $P_1 - P_2^c$ and the $P_1^c - P_2^c$ finite elements. We already know that the $P_1 - P_2^c$ finite element provides a convergent approximation for Darcy equations (see [20]), while for the latter, nothing is known when applied to the *primal* mixed formulation of the Darcy equations. Approximation errors are depicted in Tables 1–4.

As expected, with the $P_1 - P_2^c$ finite element the convergence rate in the whole domain Ω is 2. When using $P_2^c - P_2^c$ instead of $P_1 - P_2^c$, convergence rates stay the same for almost all variables in every sub-domain. Replacing discontinuous P_1 by P_2^c for approximating the velocity in the Darcy equations leads to an improvement in the approximation of this variable for the given meshes, even if the convergence rate is very slightly deteriorated (Table 3). Unfortunately, for the pressure in the Darcy domain Ω_2, the convergence rate (in the $H^1(\Omega_2)$-norm) is no more 2. It is even less than 1.5 (Table 2).

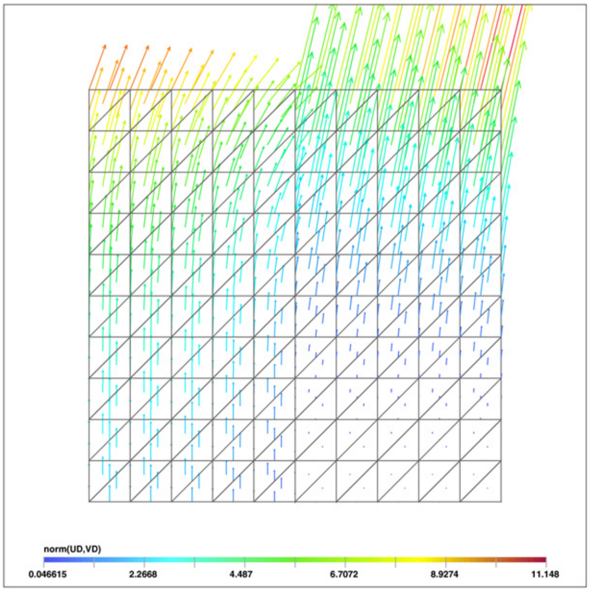

Fig. 2. The $2 \times 10 \times 10$ mesh and the velocity field.

Table 1
$\|p_2 - p_{2,h}\|_{1,\Omega_2}$. Taylor–Hood finite element is used for Stokes equations

	FE approximation spaces for Darcy equations		
	$P_1 - P_2^c$	$P_1^c - P_2^c$	$P_2^c - P_2^c$
$h = 1/10$	0.0141428	0.0142745	0.0142459
$h = 1/20$	0.00356351	0.00357584	0.0035742
$h = 1/40$	0.000893762	0.000895009	0.000894778
$h = 1/80$	0.000223762	0.000223763	0.000223862
Convergence rate	2	2	2

Table 2
$\|p_1 - p_{1,h}\|_{0,\Omega_1}$. Taylor–Hood finite element is used for Stokes equations

	FE approximation spaces for Darcy equations		
	$P_1 - P_2^c$	$P_1^c - P_2^c$	$P_2^c - P_2^c$
$h = 1/10$	0.00274511	0.00274633	0.00274574
$h = 1/20$	0.000643128	0.000643167	0.000643149
$h = 1/40$	0.000157842	0.000157843	0.000157842
$h = 1/80$	0.0000392714	3.92714e–05	3.92714e–05
Convergence rate	2	2	2

Table 3
$\|\boldsymbol{u}_2 - \boldsymbol{u}_{2,h}\|_{0,\Omega_2}$. Taylor–Hood finite element is used for Stokes equations

	FE approximation spaces for Darcy equations		
	$P_1 - P_2^c$	$P_1^c - P_2^c$	$P_2^c - P_2^c$
$h = 1/10$	0.141428	0.0126284	0.0125497
$h = 1/20$	0.00356342	0.00316145	0.0032401
$h = 1/40$	0.000893756	0.000790623	0.000820189
$h = 1/80$	0.000223762	0.000223762	0.000206027
Convergence rate	2	1.9	1.9

Table 4
$\|\boldsymbol{u}_1 - \boldsymbol{u}_{1,h}\|_{1,\Omega_1}$. Taylor–Hood finite element is used for Stokes equations

	FE approximation spaces for Darcy equations		
	$P_1 - P_2^c$	$P_1^c - P_2^c$	$P_2^c - P_2^c$
$h = 1/10$	0.0091766	0.0091766	0.00917663
$h = 1/20$	0.00231811	0.00231811	0.00231811
$h = 1/40$	0.000582742	0.000582742	0.000582742
$h = 1/80$	0.000146102	0.000146102	0.000146102
Convergence rate	2	2	2

Table 5
$\|p_2 - p_{2,h}\|_{1,\Omega_2}$. MINI finite element is used for Stokes equations

	FE approximation spaces for Darcy equations	
	$P_0 - P_1^c$	$P_1^c - P_1^c$
$h = 1/10$	0.329196	0.328502
$h = 1/20$	0.165157	0.164305
$h = 1/40$	0.0826304	0.0820384
$h = 1/80$	0.0413163	0.0409737
Convergence rate	1	1

Table 6
$\|p_1 - p_{1,h}\|_{0,\Omega_1}$. MINI finite element is used for Stokes equations

	FE approximation spaces for Darcy equations	
	$P_0 - P_1^c$	$P_1^c - P_1^c$
$h = 1/10$	0.0868604	0.0871625
$h = 1/20$	0.0380175	0.0288015
$h = 1/40$	0.018043	0.00990954
$h = 1/80$	0.00880418	0.00341653
Convergence rate	1	1.5

Table 7
$\|\boldsymbol{u}_2 - \boldsymbol{u}_{2,h}\|_{0,\Omega_2}$. MINI finite element is used for Stokes equations

	FE approximation spaces for Darcy equations	
	$P_0 - P_1^c$	$P_1^c - P_1^c$
$h = 1/10$	0.329022	0.0926356
$h = 1/20$	0.165109	0.0325165
$h = 1/40$	0.0826116	0.0114042
$h = 1/80$	0.0413075	0.00400755
Convergence rate	1	1.5

Table 8
$\|\boldsymbol{u}_1 - \boldsymbol{u}_{1,h}\|_{1,\Omega_1}$. MINI finite element is used for Stokes equations

	FE approximation spaces for Darcy equations	
	$P_0 - P_1^c$	$P_1^c - P_1^c$
$h = 1/10$	0.25461	0.254118
$h = 1/20$	0.127354	0.127045
$h = 1/40$	0.0636649	0.0634915
$h = 1/80$	0.0318268	0.0317359
Convergence rate	1	1

We now perform the computations for the same example but with the MINI finite element for the Stokes equations, and with the $P_0 - P_1^c$ and $P_1^c - P_1^c$ finite elements successively for the Darcy equations. Expected convergence rates are 1 for the norms associated to the variational form of the coupled problem (34)–(35). Errors are shown in Tables 5–8.

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