Module 7: Models with changing variance and frequency models

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Learning outcomes

- Understand how to fit ARCH, GARCH and SVM models in JAGS
- Know how to check assumptions for these methods
- Understand the basis of seasonal models
- Know the difference between time and frequency domain models and be able to implement a Fourier model

Relevant JAGS file:

```
jags_ARCH.R
jags_GARCH.R
jags_SVM.R
jags_Fourier.R
```

General principles of models for changing variance

- So far we have looked at models where the mean changes but the variance is constant:

$$y_t \sim N(\mu_t, \sigma^2)$$

- In this module we look at methods where instead:

$$y_t \sim N(\alpha, \sigma_t^2)$$

- These are:
- Autoregressive Conditional Heteroskedasticity (ARCH)
- Generalised Autoregressive Conditional Heteroskedasticity (ARCH)
- Stochastic Volatility Models (SVM)

Extension 1: ARCH

- An ARCH(1) Model has the form:

$$\sigma_t^2 = \gamma_0 + \gamma_1 \epsilon_{t-1}^2$$

  where ϵ_t is the residual, just like an MA model
- Note that $\epsilon_t = y_t - \alpha$ so the above can be re-written as:

$$\sigma_t^2 = \gamma_0 + \gamma_1 (y_{t-1} - \alpha)^2$$

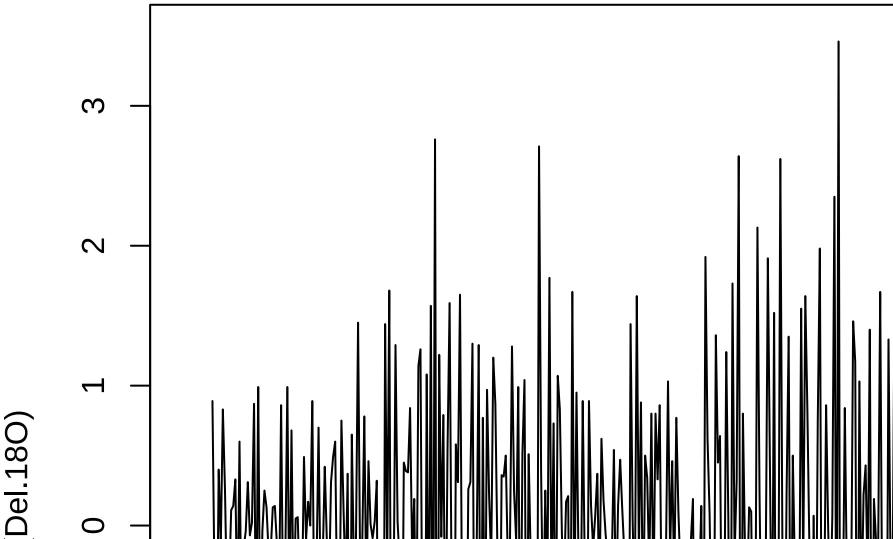
- The variance at time t thus depends on the previous value of the series (hence the autoregressive in the name)
- The residual needs to be squared to keep the variance positive.
- The parameters γ_0 and γ_1 also need to be positive, and usually $\gamma_1 \sim U(0, 1)$

JAGS code for ARCH models

```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(alpha, tau[t])
    tau[t] <- 1/pow(sigma[t], 2)
  }
  sigma[1] ~ dunif(0, 10)
  for(t in 2:T) {
    sigma[t] <- sqrt( gamma_1 + gamma_2 * pow(y[t-1] - alph
  }

  # Priors
  alpha ~ dnorm(0.0, 0.01)
  gamma_1 ~ dunif(0, 10)
  gamma_2 ~ dunif(0, 1)
}
```

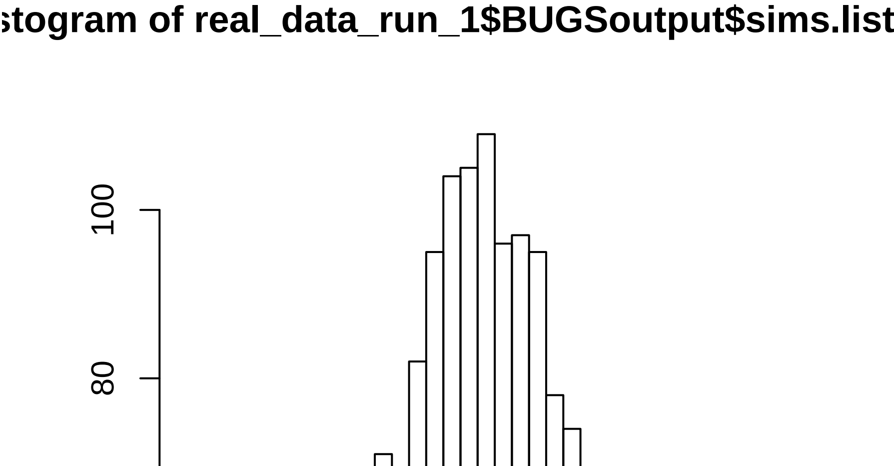
Example: ice core data

```r
with(ice2,plot(Age[-1],diff(Del.18O),type='l'))
```



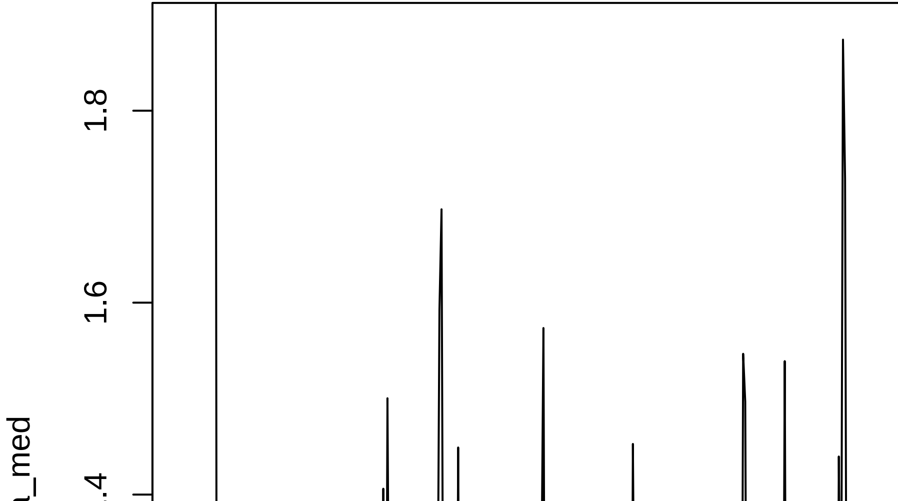
Example: ice core output

```
par(mfrow=c(1,2))
hist(real_data_run_1$BUGSoutput$sims.list$gamma_1, breaks=3
hist(real_data_run_1$BUGSoutput$sims.list$gamma_2, breaks=3
```



Example: posterior standard deviations

```
sigma_med = apply(real_data_run_1$BUGSoutput$sims.list$sign
plot(ice2$Age[-1],sigma_med,type='l',ylim=range(c(sigma_med
```



From ARCH to GARCH

- The Generalised ARCH model works by simply adding the previous value of the variance, as well as the previous value of the observation
- The GARCH(1,1) model thus has:

$$\sigma_t^2 = \gamma_1 + \gamma_2(y_{t-1} - \alpha)^2 + \gamma_3\sigma_{t-1}^2$$

- Strictly speaking $\gamma_1 + \gamma_2 < 1$ though like the stationarity conditions in ARIMA models we can relax this assumption and see if the data support it
- It's conceptually easy to extend to general GARCH(p,q) models which add in extra previous lags

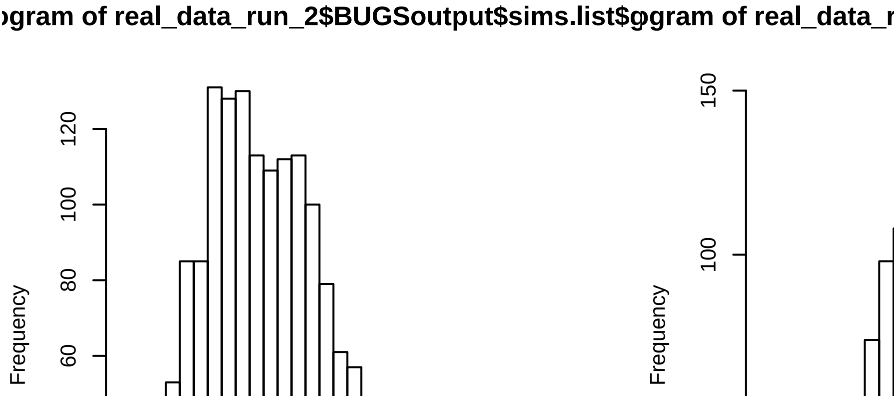
Example of using the GARCH(1,1) model

```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(alpha, tau[t])
    tau[t] <- 1/pow(sigma[t], 2)
  }
  sigma[1] ~ dunif(0,10)
  for(t in 2:T) {
    sigma[t] <- sqrt( gamma_1 + gamma_2 * pow(y[t-1] - alph
  }

  # Priors
  alpha ~ dnorm(0.0, 0.01)
  gamma_1 ~ dunif(0, 10)
  gamma_2 ~ dunif(0, 1)
  gamma_3 ~ dunif(0, 1)
```

Using the ice core data again

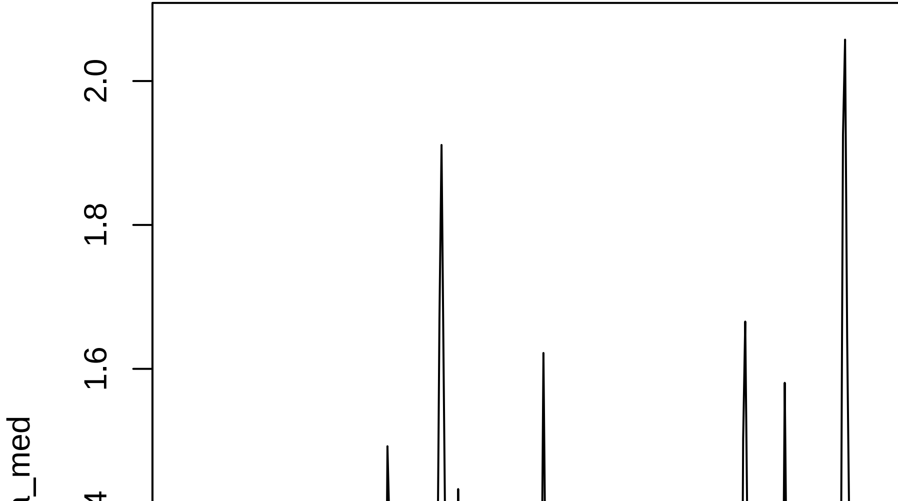
Have a look at the ARCH parameters;

```
par(mfrow=c(1,3))
hist(real_data_run_2$BUGSoutput$sims.list$gamma_1, breaks=3
hist(real_data_run_2$BUGSoutput$sims.list$gamma_2, breaks=3
hist(real_data_run_2$BUGSoutput$sims.list$gamma_3, breaks=3
```



Posterior median standard deviation

```
sigma_med = apply(real_data_run_2$BUGSoutput$sims.list$sign
plot(ice2$Age[-1], sigma_med, type='l', ylim=range(c(sigma_
```



Compare with DIC

```r
with(r_1, print(c(DIC, pD)))
```

```
## [1] 2153.082912    3.798069
```

```r
with(r_2, print(c(DIC, pD)))
```

```
## [1] 2141.421860    6.792948
```

- Suggests full GARCH model is best, despite the extra parameters

Stochastic Volatility Modelling

- Both ARCH and GARCH propose a deterministic relationship for the current variance parameter
- By contrast a Stochastic Volatility Model (SVM) models the variance as its own *stochastic process*
- SVMs, ARCH and GARCH are all closely linked if you go into the bowels of the theory
- The general model structure is often written as:

$$y_t \sim N(\alpha, \exp(h_t))$$

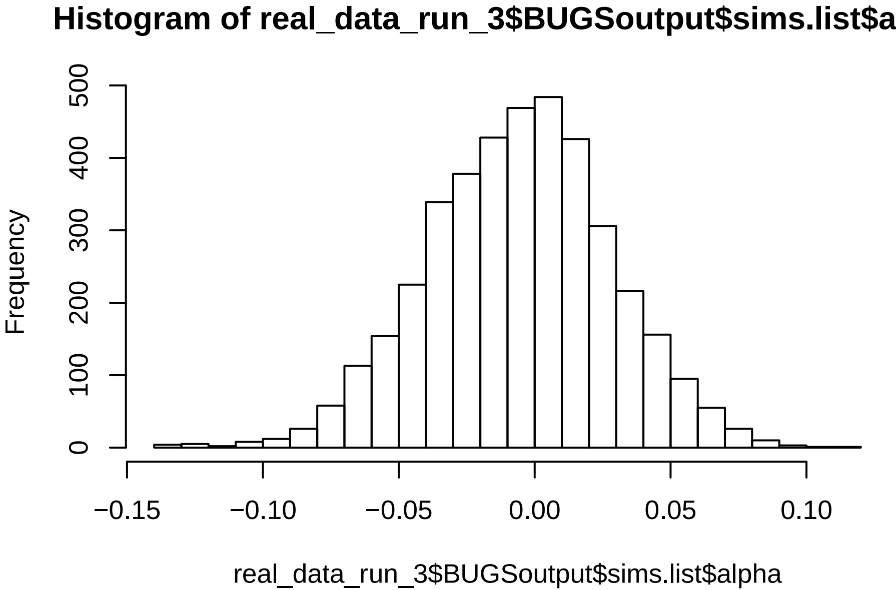
$$h_t \sim N(\mu + \phi(h_{t-1} - \mu), \sigma^2)$$

- You can think of an SVM being like a GLM but with a log link on the variance parameter

JAGS code for the SVM model

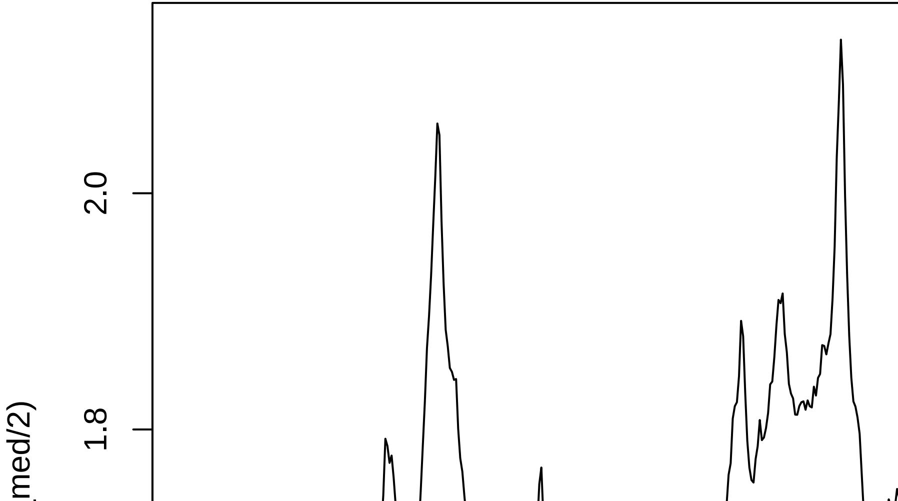
```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(alpha, tau_h[t])
    tau_h[t] <- 1/exp(h[t])
  }
  h[1] <- mu
  for(t in 2:T) {
    h[t] ~ dnorm(mu + phi * (h[t-1] - mu), tau)
  }

  # Priors
  alpha ~ dnorm(0, 0.01)
  mu ~ dnorm(0, 0.01)
  phi ~ dunif(-1, 1)
  tau <- 1/pow(sigma, 2)
```

Example of SVMs and comparison of DIC



Plot of h

```
h_med = exp(apply(real_data_run_3$BUGSoutput$sims.list$h,2,
plot(ice2$Age[-1],exp(h_med/2),type='l',ylim=range(c(exp(h_
```



Comparison with previous models

```r
with(r_1, print(c(DIC, pD)))
```

```
## [1] 2153.082912    3.798069
```

```r
with(r_2, print(c(DIC, pD)))
```

```
## [1] 2141.421860    6.792948
```

```r
with(r_3, print(c(DIC, pD)))
```

```
## [1] 2237.5672  161.1591
```

Note quite as good, and many extra parameters due to *h*!

Seasonal time series

- So far we haven't covered how to deal with data that are *seasonal* in nature
- These data generally fall into two categories:
  1. Data where we know the frequency or frequencies (e.g. monthly data on a yearly cycle, frequency $= 12$)
  2. Data where we want to estimate the frequencies (e.g. climate time series, animal populations, etc)
- The former are much simpler, as we can e.g. use month as a covariate in an ARIMAX model, perform a seasonal difference with an ARIMA, or fit a full seasonal ARIMA model (though the JAGS code for this gets complicated)
- The latter are much more interesting. The ACF and PACF can help, but we can usually do much better by creating a *power spectrum*

Methods for estimating frequencies

- The most common way to estimate the frequencies in a time series is to decompose it in a *Fourier Series*
- We write:

$$y_t = \alpha + \sum_{k=1}^{K} \left[\beta_k \sin(2\pi t f_k) + \gamma_k \cos(2\pi t f_k)\right] + \epsilon_t$$

- Each one of the terms inside the sum is called a *harmonic*. We decompose the series into a sum of sine and cosine waves rather than with AR and MA components
- Each sine/cosine pair has its own frequency f_k. If the corresponding coefficients β_k and γ_k are large we might believe this frequency is important

Estimating frequencies via a Fourier model

- It's certainly possible to fit the model in the previous slide in JAGS, as it's just a linear regression model with clever explanatory variables
- However, it can be quite slow to fit and, if the number of frequencies K is high, or the frequencies are close together, it can struggle to converge
- More commonly, people repeatedly fit the simpler model:

$$y_t = \alpha + \beta \sin(2\pi t f_k) + \gamma \cos(2\pi t f_k) + \epsilon_t$$

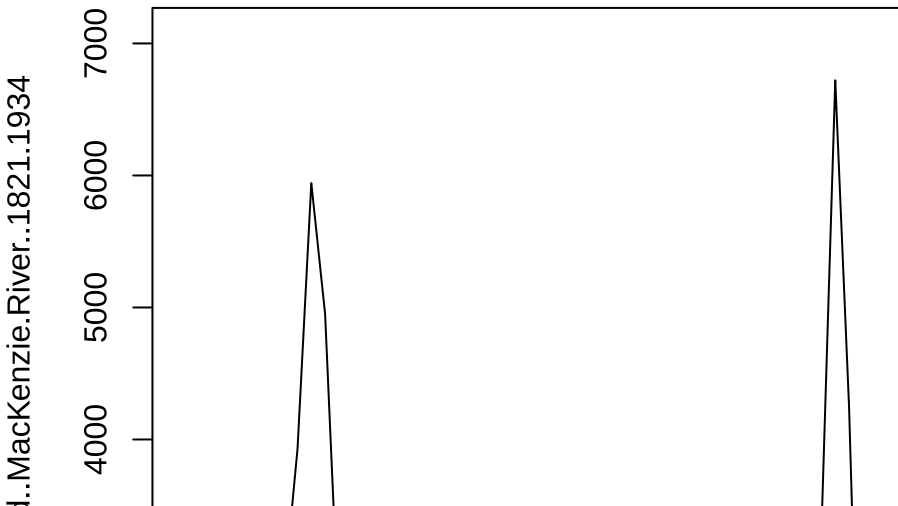
  for lots of different values of f_k. Then calculate the *power spectrum* as $P(f_k) = \frac{\beta^2+\gamma^2}{2}$. Large values of the power spectrum indicate important frequencies
- It's much faster to do this outside of JAGS, using other methods, but we will stick to JAGS

JAGS code for a Fourier model

```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(mu[t], tau)
    mu[t] <- alpha + beta * cos( 2 * pi * t * f_k ) + gamma
  }
  P = ( pow(beta, 2) + pow(gamma, 2) ) / 2

  # Priors
  alpha ~ dnorm(0.0,0.01)
  beta ~ dnorm(0.0,0.01)
  gamma ~ dnorm(0.0,0.01)
  tau <- 1/pow(sigma,2) # Turn precision into standard devi
  sigma ~ dunif(0.0,100.0)
}
'
```

Example: the Lynx data

```
lynx = as.ts(dmseries('http://data.is/Ky69xY'))
plot(lynx)
```

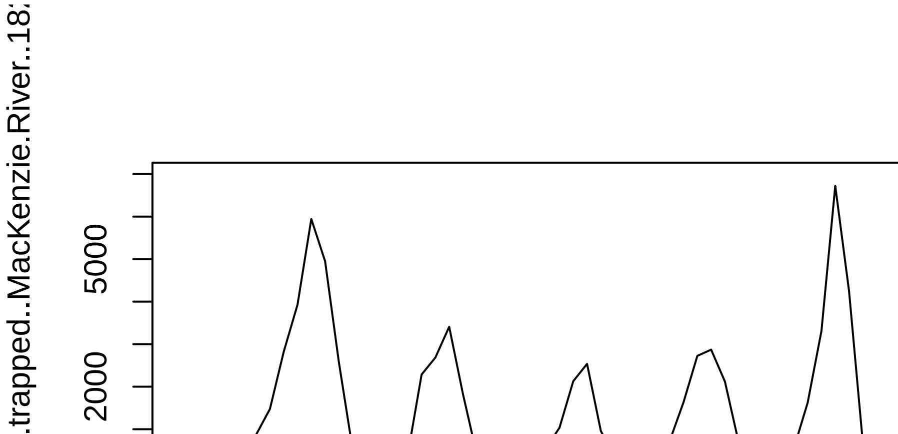


Code to run the JAGS model repeatedly

```
periods = 5:40
K = length(periods)
f = 1/periods
Power = rep(NA,K)
model_parameters =  c("P")

for (k in 1:K) {
  curr_model_data = list(y = as.vector(lynx[,1]),
                         T = length(lynx),
                         f_k = f[k],
                         pi = pi)

  model_run = jags(data = curr_model_data,
                   parameters.to.save = model_parameters,
                   model.file=textConnection(model_code),
                   n.chains=4,
                   n.iter=1000,
                   n.burnin=200,
```

Plotting the periodogram

```
par(mfrow = c(2, 1))
plot(lynx)
plot(f, Power, type='l')
axis(side = 3, at = f, labels = periods)
```



Bayesian vs traditional frequency analysis

- For quick and dirty analysis, there is no need to run the full Bayesian model, the R function `periodogram` in the TSA package will do the job
- However, the big advantage (as always with Bayes) is that we can also plot the uncertainty in the periodogram, or combine the Fourier model with other modelling ideas (e.g. ARIMA)
- There are much fancier versions of frequency models out there (e.g. Wavelets, or frequency selection models) which can also be fitted in JAGS but require a bit more time and effort
- These Fourier models work for continuous time series too

Summary

- We now know how to fit models with changing variance using a variety of techniques
- We can combine these new models with all the techniques we have previously learnt
- With known periodicity we can use seasonal differencing (or seasonal AR terms, not covered here but can be upon request)
- We've seen a basic Fourier model for estimating frequencies via the Bayesian periodogram