Investigating Resonance in a RLC Circuit

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***Abstract*— The electrical resonance of a CLR series circuit is tested. Specifically, investigation into properties of resonance angular frequency, Q-factor and phase difference for voltage across the capacitor in the CLR circuit. The experiment finds for larger resistance (R=1541Ω), resonance angular frequency(ω_{peak}) and Q-factor(Q) have experimental values of ω_{peak} = (195±2)×10^3 rads^{-1} and Q = 1.5±0.1 with relative difference of 1.27% and 22.7% with respect to theoretical values. For smaller resistance (R=525Ω), it is found that ω_{peak} = (214±2)×10^3 rads^{-1}, Q = 3.87±0.04 with relative difference of 0.4% and 24.8% in respect to theoretical values. Data measured suggests the voltage of the capacitor closely follows the theoretical resonance model but is further dampened by some external influences, especially around regions of resonance angular frequency.**

I. Introduction

A CLR series circuit is the joining of 3 components, a capacitor, an inductor and resistor together in a series circuit as seen below.

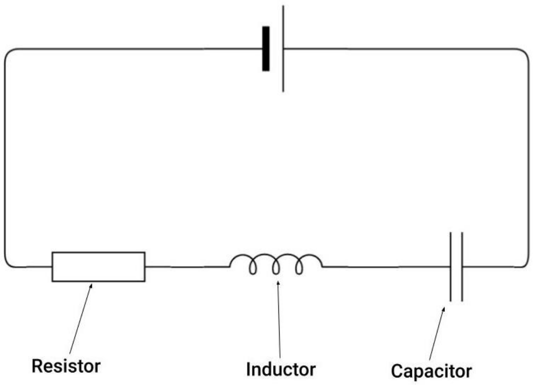

Figure 1. A circuit diagram of a CLR series circuit – a capacitor, inductor and resistor joined all together in one series loop.

In practice, an inductor is a coil of wire, when subject to current (denoted I), produces a magnetic field. By varying current, the magnetic field induces voltage which opposes change in current. Opposition towards change of current is quantified by 'inductance', denoted L (measured in Henries, H) [4]. A capacitor stores an amount of charge (denoted q) on 2 conductors (typically plates). Its defining property is 'capacitance', denoted C (measured in Farads, F), which measures the quantity of charge stored on conducting surfaces for a given voltage. A resistor is a component which resists current. A measure of how much it resists electric current is aptly named "resistance" (denoted R) and is measured in Ohms (Ω). These defining properties are critical in analysis of how voltage in the CLR circuit theoretically behaves.

Suppose a time dependent sinusoidal voltage is input into the circuit of the form

$$\mathrm{V(t)} = \mathrm{V_0}\cos(\omega t) \tag{1}$$

Where ω is the angular frequency (units of rads^{-1}) of input voltage and V_0 the peak voltage input. It is known that voltage across a resistor (V_R), inductor (V_L) and capacitor (V_C)[1] can be given by

$$V_R = IR \tag{2}$$

$$V_L = L\frac{dI}{dt} \tag{3}$$

$$V_C = \frac{q}{C} \tag{4}$$

From Kirchhoff's Second Law, EMF across the circuit must be equal to the sum of the potential difference over all components[5]. Considering current(I) as the derivative of q with respect to time, expressions of V_R, V_L can be expressed as functions of q and a differential equation is obtained (with R,L,C constant).

$$V_L + V_R + V_C = V_{EMF} \tag{5}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0\cos(\omega t) \tag{6}$$

The form of the differential equation is analogous to the forced damped harmonic oscillator. A solution can be expected of the form

$$q(t) = q_0\cos(\omega t + \phi) \tag{7}$$

A substitution of (7) into (6) finds expressions for q_0 and ϕ.

$$q_0 = -\frac{V_0\sin(\phi)}{R\omega} \tag{8}$$

$$\phi = \arctan\left(\frac{R}{\omega L - \frac{1}{\omega C}}\right) \tag{9}$$

The max voltage across the capacitor, V_{C0} can be determined.

$$V_{C0} = \frac{q_0}{C} = \frac{V_0}{C\left[\omega^2R^2 + L^2\left(\omega^2 - \frac{1}{LC}\right)^2\right]^{\frac{1}{2}}} \tag{10}$$

Provided ω can be varied, V_{C0} can be plotted as a function of ω.

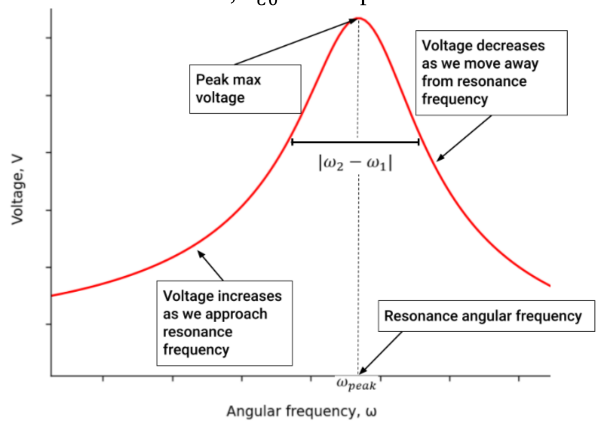

Figure 2. Graphed curve of (10), Max voltage(V_{C0}) as a function of angular frequency(ω) of the sinusoidal voltage input.

Fig 2. shows "resonance" in the circuit. Max voltage is low until it rises and peaks followed by a decline. Angular frequency at which max voltage peaks is denoted 'resonance angular frequency' (ω_{peak})[1]. It is given by

$$\omega_{peak} = \left[\frac{1}{LC} - \frac{R^2}{2L^2}\right]^{\frac{1}{2}} \tag{11}$$

'Sharpness' of peak is characterized by "Q-factor". Q-factor is defined by the value

$$Q = \frac{\omega_{peak}}{|\omega_2 - \omega_1|} \tag{12}$$

Where ω_1, ω_2 are the values of angular frequency at which the voltage is a factor of $\sqrt{2}$ smaller than the peak (see Fig.2). Larger Q-values indicate a sharp curve of narrow width while smaller Q-values suggest flatter curves.

Knowing values of R,L,C, one may determine Q-factor, ω_{peak} theoretically and compare to experimental values. The rationale behind measuring ω_{peak} and Q-factor is that it indicates the position of the curve and Q-factor quantitatively determines the shape of the curve. Understanding these values in comparison to the theoretical model provides numerical evidence for whether the RLC circuit behaves as derived. The data measured therefore gives a method of gauging how RLC circuits undergo resonance (useful in applications of signal processing[6]) and an evaluation of how good the model derived from the differential equation is.

II. Method

Values of voltage(V_{C0}) were measured for a range of angular frequencies along with experimentally determining ω_{peak} and Q-factor of an RLC circuit, once with larger resistance and again with a smaller resistance. ϕ (phase between input voltage and voltage across capacitor), is also measured. The measurement of ϕ is independent of voltage measurement and is a second source of data to record the behavior of the circuit.

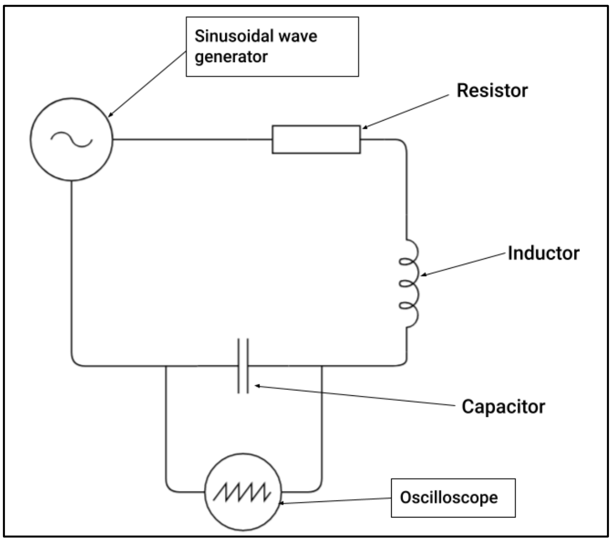

Figure 3. Schematic diagram of the RLC circuit being measured. One series with a wave generator and in parallel, an oscilloscope for measuring the voltage.

Prior to setup, values of resistance, inductance and capacitance are measured from the components. Considering components stray from being ideal, they may contain additional resistance or other properties which alter the circuit values of the system. For instance, the inductor was found to have a resistance of 21.3±0.1Ω. This is added to the resistance of the circuit.

First, the RLC circuit was built with a solderless breadboard. The resistor, inductor and capacitor are slotted into the breadboard in series. A wavefunction generator is required to input a sinusoidal voltage with angular frequency being controlled and an oscilloscope to measure the voltage. The setup can be described by the circuit diagram in Fig.3.

The voltage measuring mechanism was then implemented. This required a 'BNC to crocodile clips cable', the BNC part attached to the Channel 2 input of the oscilloscope. The crocodile clips are then clipped on both sides of the capacitor wire.

The setup is completed and with the wave-generator on, 2 voltage signals are seen, one is input signals from the wave generator (Channel 1) and the second is voltage signals seen from the capacitor (Channel 2).

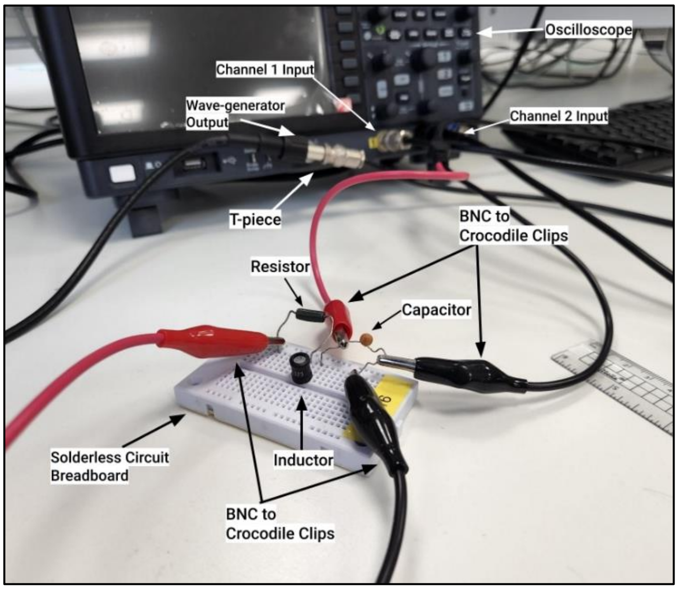

Figure 4: Real apparatus. The bottom crocodile clips connect the RLC circuit with the wave-generator. Together, the breadboard and RLC circuit components form the main series loop seen in Fig.3 The top pair of crocodile clips feed voltage across the capacitor to the oscilloscope via Channel 2 (this constitutes the parallel loop in Fig.3).

The oscilloscope is calibrated and peak-to-peak voltage and phase difference(ϕ) was measured through the cursor indicator measuring features[3]. This is repeated for varying ω, in small increments (between 6×10^4 to 2.9×10^5 rads^{-1}).

It was hypothesized the RLC circuit should follow the curve illustrated in Fig.2. It was expected that some small unaccounted resistance would deviate voltage values from the theoretical curve, even after measuring resistance from components.

III. RESULTS AND ANALYSIS

Measurements of quantities in the circuit are made. This is introduced in Table 1 below.

Measured properties	Corresponding Values
Input Voltage(V_0)	0.55±0.02 V
Inductance(L)	12.79±0.01 mH
Capacitance(C)	1.69±0.01 nF
Total Resistance(R)	Large:1541±1Ω Small: 525±1Ω

Table 1: Measured values of R,L,C and V_0. This includes 2 values of total resistance, one for a larger resistor (larger resistance) and one for a smaller resistor.

The data of max voltage across the capacitor(V_{C0}) and phase angle(ϕ) are plotted against angular frequency(ω).

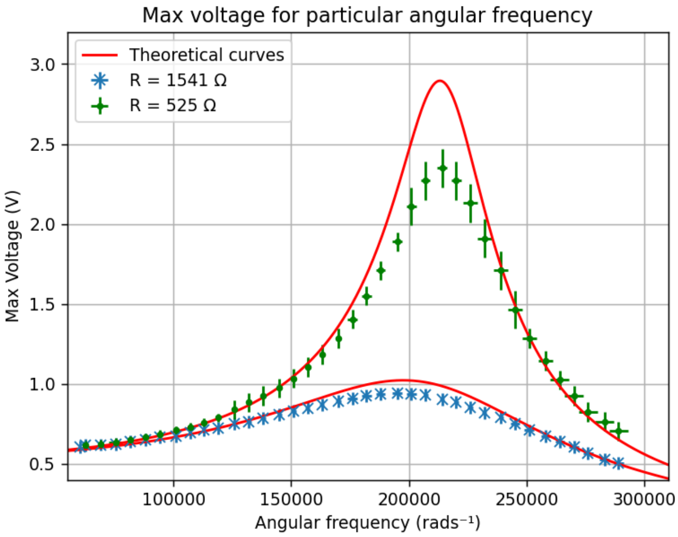

Figure 5: Graph of data of recorded voltage(V_{C0}) against angular frequency(ω). The curves are theoretical models based on (10), using values of C,L,R in Table 1.

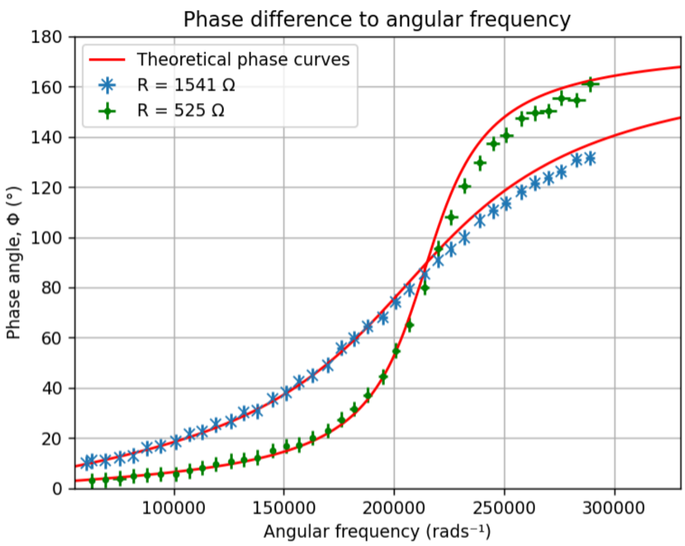

Figure 6: Plot of phase angle(ϕ) against angular frequency(ω). Curves plotted are based on (9), using C,L,R values from Table 1.

ω_{peak} and Q-factor were calculated through interpolation of data points and compared to theoretical values. Theoretical values are calculated using values in Table 1 and using (11) and (12).

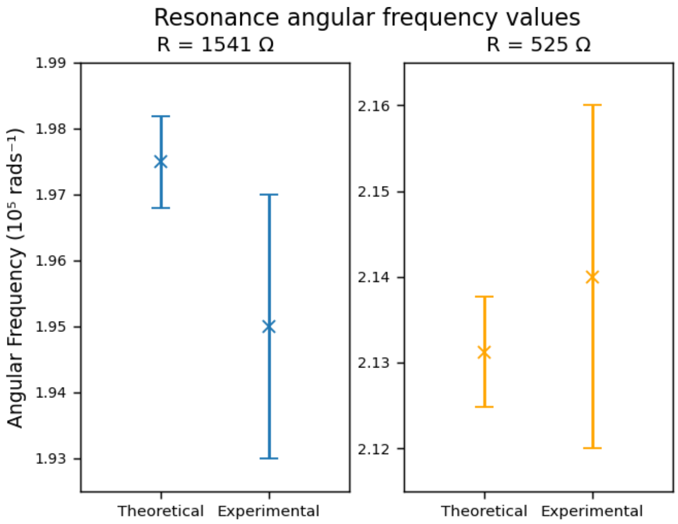

Figure 7: Experimental, theoretical values of ω_{peak}. For R = 1541Ω, an experimental value, (195 ± 2)×10^3 rads^{-1} and theoretical value, (197.0 ± 0.7)×10^3 rads^{-1} was found. For R = 525Ω, an experimental value, (214 ± 2)×10^3 rads^{-1} and theoretical value, (213.0 ± 0.6)×10^3 rads^{-1} was found.

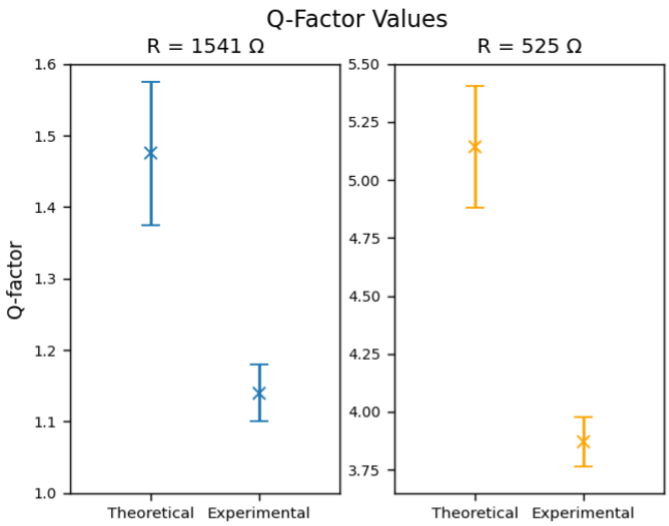

Figure 8: Experimental and theoretical values of Q-factor. For R = 1541Ω, an experimental value of 1.14±0.04 and theoretical value of 1.5±0.1 was found. For R = 525 Ω, an experimental value of 3.9±0.1 and theoretical value of 5.1±0.3 was found.

Firstly, an examination of Fig.5 shows the datapoints to be in the same position as the theoretical model with position of the peaks in very similar places. This is further backed by Fig.7 where experimental ω_{peak} values agree with the theoretical values within uncertainty and have small relative difference of 1.27% and 0.4% for the large and small resistance accordingly.

Qualitatively, Fig.5 shows for angular frequencies outside the range near resonance, the data behaves very well relative to the theoretical curve. This is also true for phase angle(ϕ) as seen in Fig.6. Roughly more than two-thirds of the data points and their uncertainties lie within the theoretical curves seen in Fig.5 and 6. This normally indicates a good fit and suggests that the RLC circuit does follow resonance phenomena closely and behaves as derived.

However, for regions close to the resonance angular frequency, the data in Fig.5 starts to deviate from what is theoretically predicted and the max voltage of those points are substantially smaller than what is expected. This is reflected in the Q-factor values calculated in Fig.8, for which experimental and theoretical values are in disagreement with each other, with differences in magnitude of 22.7% and 24.8% relative to theoretical values for large and small resistance respectively.

Although it is expected that the circuit is subject to additional resistance that was not measured which damps the voltage values, reduction in voltage values are substantially more than anticipated and can be seen as a discrepancy. This could be due to systematic errors which are not accounted for or the RLC circuit behaving differently to the model or both. But it is possible to quantify some systematic error and better determine the goodness of fit of the derived theoretical model despite disagreements seen in Fig.5.

Primarily, it can be determined that if there is systematic error, it is dominated by the resistance of the circuit. The argument is as follows:

- The calculation of ω_{peak} in (11) is dominated by the first term and thus dominated by L,C (first term is roughly an order of magnitude larger than the second term).
- Since observed ω_{peak} values agree within uncertainty with theoretical calculations, it is unlikely that L,C deviate from measured values. This is backed by intuition and observational analysis of the experiment, that components should not alter values of L and C significantly.
- It is very unlikely that the input voltage V_0 is incorrect as this is calibrated with the oscilloscope and by deduction, R values must deviate from what is recorded.

Quantitative estimates of unmeasured resistance can be done by applying a weighted non-linear regression of (10) to the data. The model presumes L is fixed with R,C as parameters. This assumption is reasonable as L=12.79mH is a very large value of inductance and inductance from other components should not contribute anything significant. C is allowed to vary but not expected to deviate much from the measured value.

Moreover, if the derived model is correct and most systematic error is due to unmeasured resistance, fits to both sets of data (large and small resistance) should return the same amount of change in R.

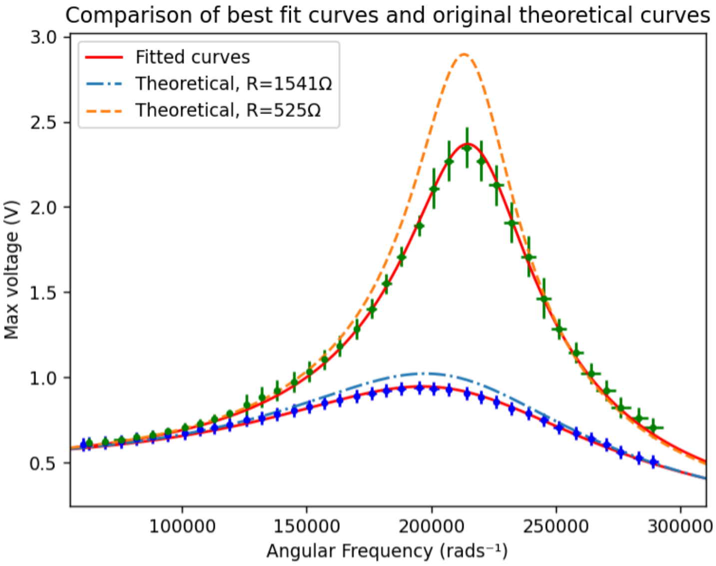

Figure 9: Fitted curves and the original theoretical curves for both sets of data. For larger resistance, the fit suggests C=1.65nF, R = 1700±10Ω. For smaller resistance, C=1.65nF, R = 651±2Ω. Uncertainty of C is not stated as it is negligible.

The fitted curves suggest new values for C and R. For C, the value suggested by both fits is the same with little deviation to the measured value(2% deviation), which is consistent with qualitative understanding of the experiment. Deviation of measured and fitted values of R is determined to be 155±10Ω for larger resistance and 126±2Ω for smaller resistance. Both fitted values of R give deviation against measured resistance which are similar but in disagreement. Arguably, a very shallow estimate of the unmeasured resistance would be between 124 Ω and 165Ω.

Assuming fitted curves represent the actual resistance, (10) represents the RLC circuit well. However reduced-Chi-squared values produced are 0.12 and 0.20 for large and small resistance respectively. This is indication that the fit may be poor[2]. We may look further considering normalized residuals.

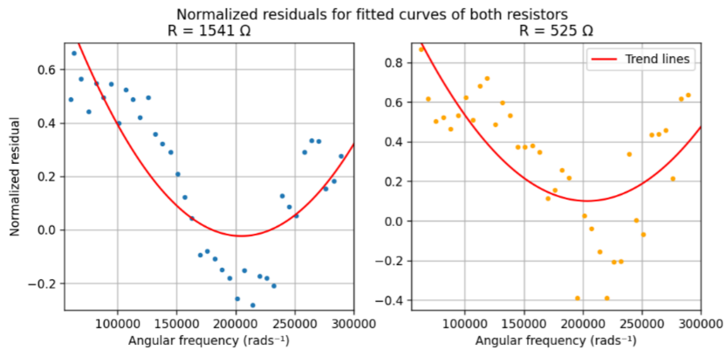

Figure 10. Plot of normalized residuals for fitted curves, recorded data (both large and small resistance)

Normalized residuals in Fig.10 are not randomly distributed. For both resistances, the residuals follow the same arc thus is unlikely that the trend is just coincidence and is random. This gives solid statistical evidence that the form of (10) is not quite right, there are likely other systematic uncertainties and phenomena besides unmeasured resistance altering values of data. Other systematic uncertainties may explain why resistance added to the fitted curves are in disagreement ($155\pm10\Omega$,$126\pm2\Omega$), but there is no concrete explanation.

For example, one explanation is the resistor behaving non-ideally and not obeying (3). As the voltage increases, the resistors temperature may increase, causing its' value of R to increase and produce additional resistance. This may explain why added resistance is in disagreement - the large resistor gains more resistance with increasing temperature than the small resistor.

IV. Conclusion

In summary, it seems if resistance of the RLC circuit is completely known, equations derived describe behavior of the voltage across the capacitor very closely. However, results in this experiment show there may be other forms of phenomena altering the behavior of the RLC circuit at least slightly. Unmeasured resistance in the system was estimated at 124-165Ω however an accurate or precise value cannot be determined as how other uncertainties affect the resistance value is not known.

Extensions to this experiment that are easy to setup would be to look at the RLC circuit with components in parallel and see how behavior differs to when in series. If one knows how the resistors' resistance changes as voltage changes, then it may also be possible to derive another differential equation which accounts for non-ideal resistors that vary depending on temperature/voltage. Such a differential equation may need to be solved numerically but can be tested by the experiment done in this paper.

References

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