Supplementary: Graph-based SLAM-Aware Exploration with Prior Topo-metric Information

Abstract

This note provides the detailed derivation of Equation (5) in the paper "*Graph-based SLAM-Aware Exploration with Prior Topo-metric Information*", which relates the pose graph reliability in SLAM with the pose graph topology. It is important to note that the supplemental materials are not considered contributions of this paper; rather, they are intended to offer a more comprehensive background and facilitate a better understanding. Similar derivation can also be found in Chapter 10.2 of [Gao et al. 2017] or in [Placed and Castellanos 2023]. The Equation (5) is replicated as follows:

$$\mathbb{I}(\boldsymbol{x}) = \mathbf{H} = \frac{1}{2}\sum_{k=1}^{m} \mathbf{J}^\top \mathbf{\Sigma}^{-1}\mathbf{J} = \frac{1}{2}\sum_{k=1}^{m} \mathbf{B}_k \mathbf{B}_k^\top \otimes \widetilde{\mathbf{\Sigma}}_k^{-1},$$

1 Relationship between Pose Graph Reliability and Graph Topology

1.1 Formulation of Pose Graph Optimization

Pose graph is the simplest form of back-end optimization in SLAM, but it is efficient and reliable. Here we derive the uncertainty evaluation of the robot's pose graph in SLAM, especially establishing the relationship between the uncertainty of the pose graph and the pose graph topology. The uncertainty of the pose graph actually means the reliability of the pose estimation results from the pose graph optimization, using nonlinear optimization methods such as Gauss-Newton or LM methods. Intuitively, this requires each pose in the pose graph be well-constrained with other poses.

Consider a pose graph with n poses and m edges, the pose graph optimization problem can be formulated as follows:

$$\begin{aligned} \boldsymbol{x}^* &= \arg\min\nolimits_{\boldsymbol{x}} F(\boldsymbol{x}), \\ \text{s.t.} \quad F(\boldsymbol{x}) &= \tfrac{1}{2}\textstyle\sum_{k=1}^{m} F_k(\boldsymbol{x}) = \tfrac{1}{2}\sum_{j=1}^{m} \boldsymbol{e}_k^\top(\boldsymbol{x})\mathbf{\Sigma}_k^{-1}\boldsymbol{e}_k(\boldsymbol{x}). \end{aligned} \tag{1}$$

Here $\boldsymbol{x}$ represents a stacked column vector of robot poses, and each pose $\boldsymbol{x}_i$ is represented as a 3×1 column vector for 2D, or 6×1 column vector for 3D. $F_k(\boldsymbol{x})$ is the squared error corresponding to edge z_k in pose graph $\mathcal{G}$, where the $\boldsymbol{e}_k(\boldsymbol{x})$ can be computed as:

$$\boldsymbol{e}_k(\boldsymbol{x}) = \ln\left(\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j\right)^\vee. \tag{2}$$

And $\boldsymbol{\Sigma}_k$ is the covariance of measurement $\boldsymbol{T}_{ij}$. Here $\boldsymbol{T}_{ij}$ is the relative measurement between pose $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ (can be obtained from ICP-based method), $\boldsymbol{T}_i$ and $\boldsymbol{T}_j$ is the estimated transformation matrix of the pose $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ respectively. Note in Eq. (2), $\ln$ is taken over the transformation matrix $\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j$, and then the resulting matrix is transformed into vector space by $\vee$. Without extra measurement $\boldsymbol{T}_{ij}$ between pose $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, $\boldsymbol{T}_i^{-1}\boldsymbol{T}_j$ is the expected relative measurement according to current value of $\boldsymbol{T}_i$ and $\boldsymbol{T}_j$. However, since both $\boldsymbol{T}_i$ and $\boldsymbol{T}_j$ have some accumulated error in SLAM odometry, the actual measurement $\boldsymbol{T}_{ij}$ is slightly different from $\boldsymbol{T}_i^{-1}\boldsymbol{T}_j$. Such difference forms the error term $\boldsymbol{e}_k(\boldsymbol{x})$ in Eq. (1).

The pose graph optimization aims to find an optimization value of $\boldsymbol{x}$ so that the value of the objective function $F(\boldsymbol{x})$ is minimized.

1.2 Linearization of Pose Graph Optimization

First, let's describe the mathematical modeling of the pose graph optimization problem. For Eq. (2), we want to optimize the two poses $\boldsymbol{T}_i$ and $\boldsymbol{T}_j$, so that $\boldsymbol{e}_k(\boldsymbol{x})$ is minimized. In order to take the derivative, we use Lie algebra by applying left disturbance to these two transformation matrixes. So we have:

$$\boldsymbol{T}_i \to \exp(\boldsymbol{\xi}_i^\wedge)\boldsymbol{T}_i, \quad \boldsymbol{T}_j \to \exp(\boldsymbol{\xi}_j^\wedge)\boldsymbol{T}_j \tag{3}$$

Note here the perturbance $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$ is defined in the world frame, because it pre-multiplies $\boldsymbol{T}_i$ and $\boldsymbol{T}_j$ (which means we can write as $\boldsymbol{\xi}_i = \boldsymbol{\xi}_{wi}$). Now Eq. (2) becomes to:

$$\boldsymbol{e}_k(\boldsymbol{x}) = \ln(\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\exp(-\boldsymbol{\xi}_i^\wedge)\exp(\boldsymbol{\xi}_j^\wedge)\boldsymbol{T}_j)^\vee \tag{4}$$

We want to move all disturbance terms to the left or right side for better handling. This can be achieved by applying the adjoint property of the Lie algebra, i.e.,

$$\exp\left((\mathrm{Ad}_T\,\boldsymbol{\xi})^\wedge\right) = \boldsymbol{T}\exp(\boldsymbol{\xi}^\wedge)\boldsymbol{T}^{-1} \tag{5}$$

And further, we have:

$$\exp(\boldsymbol{\xi}^\wedge)\boldsymbol{T} = \boldsymbol{T}\exp\left((\mathrm{Ad}_{\boldsymbol{T}^{-1}}\,\boldsymbol{\xi})^\wedge\right) \tag{6}$$

By using this, we can move all disturbance terms in Eq. (4) to the right side of $\boldsymbol{T}_j$. So we get:

$$\begin{aligned}
\boldsymbol{e}_k(\boldsymbol{x}) &= = \ln\left(\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\exp\left((-\boldsymbol{\xi}_i)^\wedge\right)\exp\left(\boldsymbol{\xi}_j^\wedge\right)\boldsymbol{T}_j\right)^\vee \\
&= \ln\left(\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j\exp\left(\left(-\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge\right)\exp\left(\left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right)\right)^\vee && //Step\ 2 \\
&\approx \ln\left(\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j\left[\boldsymbol{I} - \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right]\right)^\vee && //Step\ 3 \\
&\approx \boldsymbol{e}_{ij} + \frac{\partial \boldsymbol{e}_k(\boldsymbol{x})}{\partial \boldsymbol{\xi}_i}\boldsymbol{\xi}_i + \frac{\partial \boldsymbol{e}_k(\boldsymbol{x})}{\partial \boldsymbol{\xi}_j}\boldsymbol{\xi}_j && //Step\ 4
\end{aligned} \tag{7}$$

Step 2 to 3: In Step 2, the two terms $\exp\left(\left(-\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge\right)$ and $\exp\left(\left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right)$ are close to identity matrix, and so can be approximated by the first order Taylor expansion:

$$\exp\left(\left(-\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge\right) = \boldsymbol{I} + \left(-\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge \tag{8}$$

$$\exp\left(\left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right) = \boldsymbol{I} + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge \tag{9}$$

Taking the multiplication of these two terms, we have:

$$\begin{aligned}\exp\left(\left(-\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge\right)\exp\left(\left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right) = \boldsymbol{I} &+ \left(-\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge \\ &+ \left(-\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge\left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\end{aligned} \tag{10}$$

The second-order term is omitted in Step 3 of Eq. (7).

Step 3 to 4: In Step 4, first we define the Lie algebra corresponds to $\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j$ to be $\boldsymbol{e}_{ij}$. So we have:

$$\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j = \exp\left(\boldsymbol{e}_{ij}^\wedge\right) \tag{11}$$

We also define $\boldsymbol{y}^\wedge$ as:

$$\exp\left(\boldsymbol{y}^\wedge\right) = \boldsymbol{I} + \left(-\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge \tag{12}$$

So we get:

$$\begin{aligned}\boldsymbol{e}_k(\boldsymbol{x}) &\approx \ln\left(\boldsymbol{T}_{ij}^{-1}\boldsymbol{T}_i^{-1}\boldsymbol{T}_j\left[\boldsymbol{I} - \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right]\right)^\vee \\ &= \ln\left(\exp\left(\boldsymbol{e}_{ij}^\wedge\right)\exp\left(\boldsymbol{y}^\wedge\right)\right)^\vee\end{aligned} \tag{13}$$

Here $\boldsymbol{y}^\wedge$ can be treated as a smaller term in the equation, so we obtain the following according to the BCH right-multiplication approximation formula:

$$\boldsymbol{e}_k(\boldsymbol{x}) = \ln\left(\exp\left(\boldsymbol{e}_{ij}^\wedge\right)\exp\left(\boldsymbol{y}^\wedge\right)\right)^\vee \approx \mathcal{J}_r^{-1}\left(\boldsymbol{e}_{ij}\right)x + \boldsymbol{e}_{ij} \tag{14}$$

Let's further evaluate the $\boldsymbol{y}^\wedge$ in Eq. (12). According to the Taylor expansion of $\ln$ function, for a matrix A, we have:

$$\ln(A) = (A - \boldsymbol{I}) - \frac{(A-\boldsymbol{I})^2}{2} + \frac{(A-\boldsymbol{I})^3}{3} + \cdots \tag{15}$$

We have:

$$\begin{aligned}\boldsymbol{y}^\wedge &= \ln\left(\boldsymbol{I} + \left(-\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\right) \\ &\approx \left(-\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j\right)^\wedge\end{aligned} \tag{16}$$

So we have:

$$\boldsymbol{y} \approx -\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_i + \mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\,\boldsymbol{\xi}_j \tag{17}$$

Take into Eq. (14), we have:

$$\begin{aligned}\boldsymbol{e}_k(\boldsymbol{x}) &\approx \mathcal{J}_r^{-1}\left(\boldsymbol{e}_{ij}\right)\boldsymbol{y} + \boldsymbol{e}_{ij} \\ &= \boldsymbol{e}_{ij}\underbrace{-\mathcal{J}_r^{-1}\left(\boldsymbol{e}_{ij}\right)\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}}_{\frac{\partial \boldsymbol{e}_{ij}}{\partial \boldsymbol{\xi}_i}}\boldsymbol{\xi}_i + \underbrace{\mathcal{J}_r^{-1}\left(\boldsymbol{e}_{ij}\right)\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}}_{\frac{\partial \boldsymbol{e}_{ij}}{\partial \boldsymbol{\xi}_j}}\boldsymbol{\xi}_j \\ &= \boldsymbol{e}_{ij} + \frac{\partial \boldsymbol{e}_{ij}}{\partial \boldsymbol{\xi}_i}\boldsymbol{\xi}_i + \frac{\partial \boldsymbol{e}_{ij}}{\partial \boldsymbol{\xi}_j}\boldsymbol{\xi}_j\end{aligned} \tag{18}$$

Note here $\frac{\partial \boldsymbol{e}_{ij}}{\partial \boldsymbol{\xi}_i}$ and $\frac{\partial \boldsymbol{e}_{ij}}{\partial \boldsymbol{\xi}_j}$ differ only in sign. Note in $\mathbf{SE}(3)$, the adjoint matrix is a 6×6 matrix with the form:

$$\mathrm{Ad}(\boldsymbol{T}) = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t}^{\wedge}\boldsymbol{R} \\ \mathbf{0} & \boldsymbol{R} \end{bmatrix} \tag{19}$$

Up to now, we formulate each error term as a linear function of $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$, according to Eq. (7). Then we can use the nonlinear optimization methods to iteratively update $\boldsymbol{\xi}_i$ (and so $\exp(\boldsymbol{\xi}_i^{\wedge})\boldsymbol{T}_i$) to minimize the total error in the pose graph.

For $\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})$ in above equation, it is the Jacobi $\mathcal{J}_r$ of $\boldsymbol{e}_{ij}$ in BCH approximation. Typically, it is approximated as:

$$\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij}) \approx \boldsymbol{I} + \frac{1}{2}\begin{bmatrix} \phi_e^{\wedge} & \rho_e^{\wedge} \\ 0 & \phi_e^{\wedge} \end{bmatrix} \tag{20}$$

If we take $\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij}) \approx \boldsymbol{I}$, then the Eq. (18) will be the same as in [Placed and Castellanos 2023]. The detailed definitions of ϕ_e and ρ_e are referred to Chapter 10.2 in [Gao et al. 2017].

The objective is to minimize the Eq. (1) so that measurement error is minimized. It becomes to:

$$\begin{aligned} F(\boldsymbol{x}) &= \frac{1}{2}\sum_{k=1}^{m} \boldsymbol{e}_k^{\top}(\boldsymbol{x})\boldsymbol{\Sigma}_k^{-1}\boldsymbol{e}_k(\boldsymbol{x}) \\ &= \frac{1}{2}\sum_{k=1}^{m}\left[\boldsymbol{e}_{ij} - \mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\boldsymbol{\xi}_i + \mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\boldsymbol{\xi}_j\right]^{\top}\boldsymbol{\Sigma}_k^{-1} \\ &\quad \left[\boldsymbol{e}_{ij} - \mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\boldsymbol{\xi}_i + \mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\boldsymbol{\xi}_j\right] \end{aligned} \tag{21}$$

If we represent $-\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\boldsymbol{\xi}_i + \mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}\boldsymbol{\xi}_j$ as

$$\begin{bmatrix} 0 & \ldots & -\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}} & \ldots & \mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}} & \ldots & 0 \end{bmatrix}_{6\times 6m} \begin{bmatrix} 0 \\ \ldots \\ \boldsymbol{\xi}_i \\ \ldots \\ \boldsymbol{\xi}_j \\ \ldots \\ 0 \end{bmatrix}_{6m\times 1} = \mathbf{G}^k\boldsymbol{\xi}^k \tag{22}$$

We can write Eq. (21) as:

$$\begin{aligned} F(\boldsymbol{x}) &= \frac{1}{2}\sum_{k=1}^{m}\left(\boldsymbol{e}_{ij}^{\top}\boldsymbol{\Sigma}_k^{-1}\boldsymbol{e}_{ij} + 2\boldsymbol{e}_{ij}^{\top}\boldsymbol{\Sigma}_k^{-1}\mathbf{G}^k\boldsymbol{\xi} + \left(\mathbf{G}^k\boldsymbol{\xi}\right)^{\top}\boldsymbol{\Sigma}_k^{-1}\left(\mathbf{G}^k\boldsymbol{\xi}\right)\right) \\ &= \frac{1}{2}\sum_{k=1}^{m}\left(\boldsymbol{e}_{ij}^{\top}\boldsymbol{\Sigma}_k^{-1}\boldsymbol{e}_{ij} + 2\boldsymbol{e}_{ij}^{\top}\boldsymbol{\Sigma}_k^{-1}\mathbf{G}^k\boldsymbol{\xi} + \boldsymbol{\xi}^{\top}(\mathbf{G}^k)^{\top}\boldsymbol{\Sigma}_k^{-1}\mathbf{G}^k\boldsymbol{\xi}\right) \end{aligned} \tag{23}$$

In the above formulation, the Hessian matrix is:

$$
\begin{aligned}
\mathbf{H} &= \frac{1}{2}\sum_{k=1}^{m}(\mathbf{G}^k)^\top \mathbf{\Sigma}_k^{-1}\mathbf{G}^k \\
&= \frac{1}{2}\sum_{k=1}^{m}\mathbf{B}_k\mathbf{B}_k^\top \otimes (\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}})^\top \mathbf{\Sigma}_k^{-1}(\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})\,\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}) \\
&= \frac{1}{2}\sum_{k=1}^{m}\mathbf{B}_k\mathbf{B}_k^\top \otimes \widetilde{\mathbf{\Sigma}}_k^{-1}
\end{aligned}
\tag{24}
$$

where $\mathbf{B}_k$ is the Laplacian factor of the k-th edge. Note here the disturbance $\boldsymbol{\xi}$ is defined in world coordinate, so we have adjoint terms $\mathrm{Ad}_{\boldsymbol{T}_j^{-1}}$ in H. Here $\mathcal{J}_r^{-1}(\boldsymbol{e}_{ij})$ can be approximated as $\boldsymbol{I}_{6\times 6}$ according to Eq. (20).

According to the definition of Fisher Information Matrix (FIM), $\mathbf{H}$ calculated at the optimal solution of Eq. (1) is the FIM. In fact, we can only get the maximum likelihood estimation (MLE) of Eq. (1), and the obtained $\mathbf{H}$ thus only approximates the actual FIM [Kitanov and Indelman 2019], called the *observed* FIM.

Note that Eq. (24) establishes the relationship between the pose graph uncertainty and graph Laplacian.

Bibliography

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