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AN IMMERSED LINEAR FINITE ELEMENT METHOD WITH INTERFACE FLUX CAPTURING RECOVERY

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Abstract. A flux recovery technique is introduced for the computed solution of an immersed finite element method for one dimensional second-order elliptic problems. The recovery is by a cheap formula evaluation and is carried out over a single element at a time while ensuring the continuity of the flux across the interelement boundaries and the validity of the discrete conservation law at the element level. Optimal order rates are proved for both the primary variable and its flux. For piecewise constant coefficient problems our method can capture the flux at nodes and at the interface points exactly. Moreover, it has the property that errors in the flux are all the same at all nodes and interface points for general problems. We also show second order pressure error and first order flux error at the nodes. Numerical examples are provided to confirm the theory.

1. **Introduction.** The second order interface elliptic problem of interest in this paper is a variable coefficient Poisson equation in the unknown p with a jump in the coefficient across the interface in the domain. The primary variable p can be interpreted as temperature, pressure and so on. When one solves this equation numerically, the mesh can be either fitted or unfitted with the interface. A method allowing unfitted meshes would be very efficient when one has to follow a moving interface in a temporal problem. For an in-depth exposition of the numerics and applications of interface problems, we refer the reader to [4] and the references therein. In an immersed finite element (IFE) method, the mesh is made up of interface elements where the interface intersects elements and noninterface elements where the interface is absent. On a noninterface element one uses standard local shape functions while on an interface element one uses piecewise standard local shape functions subject to continuity and jump conditions. Representative works on IFE methods can be found in [2, 3, 4, 5, 6, 7], among others. We are interested in studying IFE methods that can produce accurate approximate flux u_h of p once an approximate p_h has been obtained, particularly those that can recover flux without having to solve a system of equations. Chou and Tang [1] initiated such methods when the mesh is fitted. In this paper we generalize them to the immersed interface mesh case. For simplicity we concentrate on one dimensional case, but express formulas in such a way that they still make sense in higher dimensions.

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Consider a one dimensional second-order interface boundary value problem on a finite interval (a, b)

$$\begin{cases} -(\beta(x)p'(x))' = f(x), & x \in (a,b), \\ p(a) = p(b) = 0, \end{cases} \tag{1.1}$$

where $\beta = \beta(x)$ is a positive function, $\beta \in C[a, \alpha)$ and $\beta \in C(\alpha, b]$ with a possible finite jump at $\alpha \in (a, b)$. We ask the solution p to be continuous at α and satisfy a jump condition in the derivative. Classically this is equivalent to two separated problems:

$$\begin{cases} -(\beta(x)p(x)')' = f(x), & x \in (a,\alpha), \\ p(a) = 0, & p(\alpha) = p_\alpha. \end{cases} \tag{1.2}$$

and

$$\begin{cases} -(\beta(x)p'(x))' = f(x) & x \in (\alpha,b), \\ p(b) = 0, & p(\alpha) = p_\alpha. \end{cases} \tag{1.3}$$

Thus if f is smooth enough, e.g., $f \in C(a, b)$, then $p \in C^2(a, \alpha)$ and $p \in C^2(\alpha, b)$. Globally $p \in H^1(a, b)$. Furthermore,

$$[p\,]_\alpha := p(\alpha^+) - p(\alpha^-) = 0, \tag{1.4}$$

$$[\beta p']_\alpha := \beta^+ p'(\alpha^+) - \beta^- p'(\alpha^-) = 0, \tag{1.5}$$

where $\beta^+ = \lim_{x\to\alpha^+} \beta(x)$ and $\beta^- = \lim_{x\to\alpha^-} \beta(x)$. We will use $[s]_\alpha$ to denote the jump of the quantity s across α. The related weak formulation is: given $f \in L^2(a, b)$, find $p \in H^1(a, b)$ such that

$$a(p, q) = (f, q) \qquad \forall q \in H_0^1(a, b)$$

where

$$a(p, q) := \int_a^b \beta(x)p'q'dx, \qquad (f, q) := \int_a^b f(x)q(x)dx.$$

If the smoothness of β is as above, we then have $p \in H^2(a, \alpha)$, $p \in H^2(\alpha, b)$, and $p \in H^1(a, b)$. In what follows we will call p, the pressure and $-\beta p'$ the flux. The rest of the paper is organized as follows. In Section 2, we introduce the approximate pressure space which is a generalization of the one in [3]. Its approximation properties are shown in Section 3. We then present our flux recovery scheme in Section 4 and show that for piecewise constant coefficient problems our method can capture the flux at nodes and at the interface points exactly. Moreover it has uniform error distribution over all nodes for general problems. Second order pressure error and first order flux error are shown at the nodes. The optimal convergence rates of the pressure and flux are shown in Section 5. Finally, numerical examples are provided in Section 6 to confirm the theory.

2. **Approximate pressure space.**

Let $a = x_0 < x_1 < \ldots < x_j < x_{j+1} < \ldots < x_{n-1} < x_n = b$ be a partition of $I = [a, b]$ and $\alpha \in (x_j, x_{j+1}]$. Let $h := \max_{1\le i\le n}(x_i - x_{i-1})$. If $\alpha \notin (x_{i-1}, x_i) \cup (x_i, x_{i+1})$, then ϕ_i denotes the usual piecewise linear (Lagrange nodal) basis function associated with x_i, i.e., ϕ_i is linear in each (x_k, x_{k+1}) and $\phi_i(x_k) = \delta_{ik}$. For x_j and x_{j+1}, the associated basis functions are defined to enforce

$$[\phi_j]_\alpha = 0, \quad [\bar\beta\phi_j']_\alpha = 0, \quad [\phi_{j+1}]_\alpha = 0, \quad [\bar\beta\phi_{j+1}']_\alpha = 0, \tag{2.1}$$

where $\bar\beta$ is the average function, i.e.,

$$\bar\beta = \begin{cases} \bar\beta_{\alpha l}, \text{ the average of } \beta \text{ over } [x_j, \alpha] \\ \bar\beta_{\alpha r}, \text{ the average of } \beta \text{ over } [\alpha, x_{j+1}] \end{cases} \tag{2.2}$$

$$\phi_j(x) = \begin{cases} 0, & a \le x < x_{j-1}, \\ \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} \le x < x_j, \\ \dfrac{x_j - x}{D} + 1, & x_j \le x < \alpha, \\ \dfrac{\rho(x_{j+1} - x)}{D}, & \alpha \le x < x_{j+1}, \\ 0, & x_{j+1} \le x \le b, \end{cases} \tag{2.3}$$

$$\phi_{j+1}(x) = \begin{cases} 0, & a \le x < x_j, \\ \dfrac{x - x_j}{D}, & x_j \le x < \alpha, \\ \dfrac{\rho(x - x_{j+1})}{D} + 1, & \alpha \le x < x_{j+1}, \\ \dfrac{x_{j+2} - x}{x_{j+2} - x_{j+1}}, & x_{j+1} \le x < x_{j+2}, \\ 0, & x_{j+2} \le x \le b, \end{cases} \tag{2.4}$$

where

$$\rho := \frac{\bar\beta_{\alpha l}}{\bar\beta_{\alpha r}}, \quad D := (x_{j+1} - x_j) - \frac{\bar\beta_{\alpha r} - \bar\beta_{\alpha l}}{\bar\beta_{\alpha r}}(x_{j+1} - \alpha). \tag{2.5}$$

Let $V_h := \text{span}\{\phi_i\}_{i=1}^{n-1}$ be the immersed finite element space for approximating p.

Remark 1. The above space is proposed and used in [3] when the function β is piecewise constant. In defining the present immersed finite element space we used the average $\bar\beta$ instead of the point limit values $\beta^{\pm}$. There are two reasons for this. One is that in higher dimensional extension, the interface is not just a point and so using average is more natural. More importantly, this expression comes out naturally in the flux recovery scheme below, when the function β is not piecewise constant.

We propose the following immersed interface method for problem (1.1): find $p_h \in V_h \subset H_0^1(a, b)$ such that

$$a(p_h, q_h) = (f, q_h) \qquad \forall q_h \in V_h. \tag{2.6}$$

3. **Approximation property of the immersed finite element space.** As usual, we study the approximation property of the IFE space V_h so that we can bound the error in the finite element solution by the approximation error of the interpolant to the exact solution.

For simplicity in the rest of the paper we shall assume there is only one interface element and the jump occurs at $\alpha \in (x_j, x_{j+1}]$. Given $p \in H^1(a, b)$, we define its interpolant $\hat I_h p \in V_h$ as follows:

$$\hat I_h p(x) = \begin{cases} \dfrac{x_{i-1} - x}{h} p(x_i) + \dfrac{x - x_i}{h} p(x_{i+1}), & i \ne j,\ x_i \le x \le x_{i+1}, \\ p(x_j) + \kappa(x - x_j), & x_j \le x < \alpha, \\ p(x_{j+1}) + \kappa\rho(x - x_{j+1}), & \alpha \le x \le x_{j+1}, \end{cases} \tag{3.1}$$

where

$$\kappa = \frac{p(x_{j+1}) - p(x_j)}{\alpha - x_j - \rho(\alpha - x_{j+1})}.$$

It is easy to verify that $\hat{I}_h p(x_i) = p(x_i), i = 0, 1, \ldots, n$, $[\hat{I}_h p]_\alpha = 0$, and $[\bar{\beta}\hat{I}_h p']_\alpha = 0$.

We now turn to the estimation of $\|p - \hat{I}_h p\|_{0,I}$ and $|p - \hat{I}_h p|_{1,I}$. Let $\bar{I}_h p$ be the continuous piecewise linear function on $I = [a, b]$ such that $\bar{I}_h p(x_i) = p(x_i), i = 0, 1, \ldots, n$ and $\bar{I}_h p(\alpha) = p(\alpha)$. Note that for $i \neq j$, we have $\bar{I}_h p(x) = \hat{I}_h p(x), \forall x \in I_{i+1} = (x_i, x_{i+1})$. From the classical approximation theory, we have

$$\|p - \bar{I}_h p\|_{0,I_{i+1}} \leq Ch^2 ||p''||_{0,I_{i+1}}, \tag{3.2}$$

$$|p - \bar{I}_h p|_{1,I_{i+1}} \leq Ch ||p''||_{0,I_{i+1}}. \tag{3.3}$$

For $I_{\alpha l} = (x_j, \alpha)$ we have

$$\begin{aligned}
\bar{I}_h p(x) - \hat{I}_h p(x) &= \left[\frac{p(\alpha) - p(x_j)}{\alpha - x_j} - \frac{p(x_{j+1}) - p(x_j)}{\alpha - x_j + \rho(x_{j+1} - \alpha)}\right](x - x_j) \\
&= \frac{\frac{\rho(x_{j+1} - \alpha)}{\alpha - x_j}[p(\alpha) - p(x_j)] - [p(x_{j+1}) - p(\alpha)]}{\alpha - x_j + \rho(x_{j+1} - \alpha)}(x - x_j) \\
&= \frac{\frac{\bar{\beta}_{\alpha l}(x_{j+1} - \alpha)}{\alpha - x_j}\int_{x_j}^{\alpha} p'(x)dx - \bar{\beta}_{\alpha r}\int_{\alpha}^{x_{j+1}} p'(x)dx}{\bar{\beta}_{\alpha r}[\alpha - x_j + \rho(x_{j+1} - \alpha)]}(x - x_j) \\
&= \frac{J_1 - J_2}{\bar{\beta}_{\alpha r}[\alpha - x_j + \rho(x_{j+1} - \alpha)]}(x - x_j),
\end{aligned}$$

where

$$J_1 = \left[\bar{\beta}_{\alpha l} p'(\alpha^-) - \bar{\beta}_{\alpha r} p'(\alpha^+)\right](x_{j+1} - \alpha), \tag{3.4}$$

and

$$J_2 = \frac{\bar{\beta}_{\alpha l}(x_{j+1} - \alpha)}{\alpha - x_j}\int_{x_j}^{\alpha}\int_{x}^{\alpha} p''(y)dydx - \bar{\beta}_{\alpha r}\int_{\alpha}^{x_{j+1}}\int_{\alpha}^{x} p''(y)dydx. \tag{3.5}$$

Let us estimate the term $[\bar{\beta}_{\alpha l} p'(\alpha^-) - \bar{\beta}_{\alpha r} p'(\alpha^+)]$ in (3.4)

$$\begin{aligned}
|\bar{\beta}_{\alpha l} p'(\alpha^-) - \bar{\beta}_{\alpha r} p'(\alpha^+)| &= |[\bar{\beta}_{\alpha l} p'(\alpha^-) - \bar{\beta}_{\alpha r} p'(\alpha^+)] - [\beta^- p'(\alpha^-) - \beta^+ p'(\alpha^+)]| \\
&\leq |\bar{\beta}_{\alpha l} - \beta^-|\,|p'(\alpha^-)| + |\bar{\beta}_{\alpha r} - \beta^+|\,|p'(\alpha^+)| \\
&\leq C|\bar{\beta}_{\alpha l} - \beta^-|\,||p||_{2,I_{\alpha l}} + C|\bar{\beta}_{\alpha r} - \beta^+|\,||p||_{2,I_{\alpha r}},
\end{aligned}$$

where we have used the one dimensional Sobolev imbedding theorem.

On the other hand,

$$|\bar{\beta}_{\alpha r} - \beta^+| = |\beta(\eta) - \beta(\alpha^+)| = |\int_{\alpha}^{\eta} \beta'(x)dx| \leq C(x_{j+1} - \alpha)^{1/2}|\beta|_{1,\alpha r},$$

where $\eta \in (\alpha, x_{j+1})$. Similarly,

$$|\bar{\beta}_{\alpha l} - \beta^-| = |\beta(\xi) - \beta(\alpha^-)| = |\int_{\xi}^{\alpha} \beta'(x)dx| \leq C(\alpha - x_j)^{1/2}|\beta|_{1,\alpha l},$$

where $\xi \in (x_j, \alpha)$.

We have

$$|[\bar{\beta}_{\alpha l} p'(\alpha^-) - \bar{\beta}_{\alpha r} p'(\alpha^+)]| \leq Ch^{1/2}[|\beta|_{1,\alpha l}||p||_{2,\alpha l} + |\beta|_{1,\alpha r}||p||_{2,\alpha r}].$$

Clearly,

$$
\begin{aligned}
|\bar{\beta}_{\alpha r}\int_{\alpha}^{x_{j+1}}\int_{\alpha}^{x} p''(y)dydx| &\leq \bar{\beta}_{\alpha r}\int_{\alpha}^{x_{j+1}}\int_{\alpha}^{x} |p''(y)|dydx \\
&= \bar{\beta}_{\alpha r}(x_{j+1}-\alpha)\int_{\alpha}^{x_{j+1}} |p''(y)|dy \\
&\leq \bar{\beta}_{\alpha r}(x_{j+1}-\alpha)^{\frac{3}{2}}\|p''\|_{0,I\alpha r}, \quad I_{\alpha r}=(\alpha,x_{j+1})
\end{aligned} \tag{3.6}
$$

and

$$
|\frac{\bar{\beta}_{\alpha l}(x_{j+1}-\alpha)}{\alpha-x_j}\int_{x_j}^{\alpha}\int_{x}^{\alpha} p''(y)dydx| \leq \bar{\beta}_{\alpha l}(x_{j+1}-\alpha)(\alpha-x_j)^{\frac{1}{2}}||p''||_{0,I\alpha r}. \tag{3.7}
$$

Noting that

$$
\alpha-x_j+\rho(x_{j+1}-\alpha)\geq \min\{\frac{1}{2}(x_{j+1}-x_j),\frac{1}{2}(x_{j+1}-x_j)\rho\},
$$

we have from (3.6)–(3.7) that

$$
\begin{aligned}
||\bar{I}_h p-\hat{I}_h p||_{0,I_{\alpha l}} &\leq Ch^{\frac{1}{2}}||x-x_j||_{0,I_\alpha}(||p''||_{0,I_{\alpha l}}+||p''||_{0,I_{\alpha r}}) &(3.8)\\
&\leq Ch^2(||p''||_{0,I_{\alpha l}}+||p''||_{0,I_{\alpha r}}). &(3.9)
\end{aligned}
$$

Still using (3.6)–(3.7), we get easily

$$
|\bar{I}_h p-\hat{I}_h p|_{1,I_{\alpha l}} \leq Ch(||p''||_{0,I_{\alpha l}}+||p''||_{0,I_{\alpha r}}). \tag{3.10}
$$

Similarly, we can estimate $||\bar{I}_h p-\hat{I}_h p||_{0,I_{\alpha r}}$ and $|\bar{I}_h p-\hat{I}_h p|_{1,I_{\alpha r}}$ and the same results hold.

Thus we have from (3.8)–(3.10) and the classical approximation theory that

$$
\begin{aligned}
||p-\hat{I}_h p||_{0,I_{\alpha l}} &\leq ||p-\bar{I}_h p||_{0,I_{\alpha l}}+||\bar{I}_h p-\hat{I}_h p||_{0,I_{\alpha l}} \\
&\leq Ch^2(||p''||_{0,I_{\alpha l}}+||p''||_{0,I_{\alpha r}}), &(3.11)\\
|p-\hat{I}_h p|_{1,I_{\alpha l}} &\leq |p-\bar{I}_h p|_{1,I_{\alpha l}}+|\bar{I}_h p-\hat{I}_h p|_{1,I_{\alpha l}} \\
&\leq Ch(||p''||_{0,I_{\alpha l}}+||p''||_{0,I_{\alpha r}}). &(3.12)
\end{aligned}
$$

Finally, we obtain from (3.11)-(3.12) that

$$
\begin{aligned}
||p-\hat{I}_h p||_{0,I} &\leq Ch^2||p''||_{0,I}, &(3.13)\\
|p-\hat{I}_h p|_{1,I} &\leq Ch||p''||_{0,I}, &(3.14)
\end{aligned}
$$

where

$$
||p''||_{0,I}^2 := ||p||_{2,(a,\alpha)}^2+||p||_{2,(\alpha,b)}^2.
$$

4. **Construction of the approximate flux.** In this section we will do the flux recovery, i.e., knowing p_h we will evaluate the approximate flux by a simple formula. Let us begin with the definition of an approximate flux u_h of the exact flux $u=-\beta p'$. We will then motivate it. Let $\bar{\beta}_{i+1/2}$ and $\bar{f}_{i+1/2}$ denote the average of $\beta(x)$ and $f(x)$ over the interval $I_{i+1}=[x_i,x_{i+1}]$, respectively. Let $x_{i+1/2}=\frac{1}{2}(x_i+x_{i+1})$, and $\bar{f}_{\alpha l}$, $\bar{f}_{\alpha r}$ be the averages of f over (x_j,α) and (α,x_{j+1}), respectively.

We define for $i\neq j$

$$
u_h(x) := -\bar{\beta}_{i+1/2}p_h'(x_{i+1/2})+\bar{f}_{i+1/2}(x-x_{i+1/2})+C_{i+1/2}, \quad x\in[x_i,x_{i+1}]
$$

where

$$
C_{i+1/2} = \frac{1}{2}\int_{x_i}^{x_{i+1}}(\phi_{i+1}-\phi_i)f\,dx.
$$

Now let's motivate the above definition. Multiply (1.1) by ϕ_i and integrate by parts over $[x_{i-1}, x_i], i \neq j+1$, $1 \leq i \leq n$, to get

$$u(x_i^-) = -\beta(x_i)p'(x_i) = -\int_{x_{i-1}}^{x_i} \beta p' \phi_i' \, dx + \int_{x_{i-1}}^{x_i} f\phi_i \, dx$$

and do the same over $[x_i, x_{i+1}]$, $i \neq j$, $0 \leq i \leq n-1$ to get

$$u(x_i^+) = -\beta(x_i)p'(x_i) = \int_{x_i}^{x_{i+1}} \beta p' \phi_i' \, dx - \int_{x_i}^{x_{i+1}} f\phi_i \, dx.$$

By a slight adjustment of argument, the above two relations also hold when $i = j+1, j$, respectively. For example, to get the expression for $u(x_j^+)$, we integrate (1.1) against ϕ_j over $[x_j, x_{j+1}]$ to get

$$-\int_{x_j}^{\alpha} (\beta p')' \phi_j \, dx - \int_{\alpha}^{x_{j+1}} (\beta p')' \phi_j dx = \int_{x_j}^{x_{j+1}} f\phi_j dx.$$

By integration by parts on the two integrals on the left side and the continuity of the flux at α, the left side becomes

$$\int_{x_j}^{\alpha} \beta p' \phi_j' dx - (\beta p')(\alpha^-)\phi_j(\alpha) - u(x_j^+) + \int_{\alpha}^{x_{j+1}} \beta p' \phi_j' dx + (\beta p')(\alpha^+)\phi_j(\alpha)$$

or

$$\int_{x_j}^{x_{j+1}} \beta p' \phi_j' dx - u(x_j^+).$$

The remaining case can be derived similarly.

Hence

$$u(x_i^-) = -\beta(x_i)p'(x_i) = -\int_{x_{i-1}}^{x_i} \beta p' \phi_i' \, dx + \int_{x_{i-1}}^{x_i} f\phi_i \, dx, \quad 1 \leq i \leq n, \tag{4.1}$$

and

$$u(x_i^+) = -\beta(x_i)p'(x_i) = \int_{x_i}^{x_{i+1}} \beta p' \phi_i' \, dx - \int_{x_i}^{x_{i+1}} f\phi_i \, dx, \quad 0 \leq i \leq n-1. \tag{4.2}$$

Thus if p_h is a good approximate of p, it is natural to define

$$\begin{aligned}
u_h(x_i^-) &:= -\int_{x_{i-1}}^{x_i} \beta p_h' \phi_i' \, dx + \int_{x_{i-1}}^{x_i} f\phi_i \, dx && (4.3)\\
&= -\bar{\beta}_{i-1/2} p_h' + \int_{x_{i-1}}^{x_i} f\phi_i \, dx \\
&= -\bar{\beta}_{i-1/2} p_h' + \int_{x_{i-1}}^{x_i} (\phi_i + \phi_{i-1}) f \, dx - \int_{x_{i-1}}^{x_i} f\phi_{i-1} \, dx \\
&= -\bar{\beta}_{i-1/2} p_h' + \int_{x_{i-1}}^{x_i} f \, dx - \int_{x_{i-1}}^{x_i} f\phi_{i-1} \, dx \\
&= -\bar{\beta}_{i-1/2} p_h' + \bar{f}_{i-1/2}(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f\phi_{i-1} \, dx && (4.4)
\end{aligned}$$

and similarly

$$\begin{aligned} u_h(x_i^+) &:= \int_{x_i}^{x_{i+1}} \beta p_h' \phi_i' \, dx - \int_{x_i}^{x_{i+1}} f \phi_i \, dx && (4.5) \\ &= -\bar{\beta}_{i+1/2} p_h' - \int_{x_i}^{x_{i+1}} (\phi_i + \phi_{i+1}) f \, dx + \int_{x_i}^{x_{i+1}} f \phi_{i+1} \, dx \\ &= -\bar{\beta}_{i+1/2} p_h' - \bar{f}_{i+1/2}(x_{i+1} - x_i) + \int_{x_i}^{x_{i+1}} f \phi_{i+1} \, dx. && (4.6) \end{aligned}$$

Test (2.6) with $q = \phi_i$ to get

$$-\int_{x_{i-1}}^{x_i} \beta p_h' \phi_i' \, dx + \int_{x_{i-1}}^{x_i} f \phi_i \, dx = \int_{x_i}^{x_{i+1}} \beta p_h' \phi_i' \, dx - \int_{x_i}^{x_{i+1}} f \phi_i \, dx. \qquad (4.7)$$

Thus we have by (4.3) and (4.5) that $u_h(x_i^-) = u_h(x_i^+)$, and there is no ambiguity in defining $u_h(x_i) = u_h(x_i^-) = u_h(x_i^+)$.

The idea of the above construction for the standard finite element method is contained in [1]. We simply interpreted (4.6) as an analogy of the Taylor expansion at the midpoint of the interval with an extra term $C_{i+1/2}$ below used to connect to information on other adjacent elements. Therefore for each (x_i, x_{i+1}) which does not include α, $u_h(x)$ is of the form:

$$u_h(x) = -\bar{\beta}_{i+1/2} p_h'(x_{i+1/2}) + \bar{f}_{i+1/2}(x - x_{i+1/2}) + C_{i+1/2}, \quad x \in [x_i, x_{i+1}]. \qquad (4.8)$$

where

$$C_{i+1/2} = \frac{1}{2} \int_{x_i}^{x_{i+1}} (\phi_{i+1} - \phi_i) f \, dx$$

is obtained so that (4.8) evaluated at x_i is the same as (4.6). The reason for using this form is that when we generalize the scheme to higher dimensions, this formula still makes sense, the midpoint base point becomes the barycenter. For (x_j, α) and (α, x_{j+1}), we use the following form

$$\begin{aligned} u_h(x) &= -\bar{\beta}_{\alpha l} p_h'(\alpha^-) + \bar{f}_{\alpha l}(x - \alpha) + C_{\alpha l}, \quad x \in (x_j, \alpha], && (4.9) \\ u_h(x) &= -\bar{\beta}_{\alpha r} p_h'(\alpha^+) + \bar{f}_{\alpha r}(x - \alpha) + C_{\alpha r}, \quad x \in [\alpha, x_{j+1}), && (4.10) \end{aligned}$$

where

$$\begin{aligned} C_{\alpha l} &= \bar{f}_{\alpha l}(\alpha - x_j) - \int_{x_j}^{x_{j+1}} f \phi_j dx, && (4.11) \\ C_{\alpha r} &= \bar{f}_{\alpha r}(\alpha - x_{j+1}) + \int_{x_j}^{x_{j+1}} f \phi_{j+1} dx. && (4.12) \end{aligned}$$

The above formulas are motivated as follows. Again we start with the fundamental relation (4.7) with $i = j$, i.e.,

$$-\int_{x_{j-1}}^{x_j} \beta p_h' \phi_j' \, dx + \int_{x_{j-1}}^{x_j} f \phi_j \, dx = \int_{x_j}^{x_{j+1}} \beta p_h' \phi_j' \, dx - \int_{x_j}^{x_{j+1}} f \phi_j \, dx. \qquad (4.13)$$

Note that from (4.3), the left side is $u_h(x_j^-)$ and thus

$$u_h(x_j^-) = \int_{x_j}^{x_{j+1}} \beta p_h' \phi_j' \, dx - \int_{x_j}^{x_{j+1}} f \phi_j \, dx.$$

The first term on the right–hand side is

$$\begin{aligned}&\int_{x_j}^{\alpha}\beta p_h'\phi_j'\,dx+\int_{\alpha}^{x_{j+1}}\beta p_h'\phi_j'\,dx\\&=\bar\beta_{\alpha l}p_h'(\alpha^-)(\alpha-x_j)\phi_j'(\alpha^-)+\bar\beta_{\alpha r}p_h'(\alpha^+)(x_{j+1}-\alpha)\phi_j'(\alpha^+)\\&=-\bar\beta_{\alpha l}p_h'(\alpha^-),\end{aligned}$$

where we have used the facts that $\bar\beta_{\alpha l}p_h'(\alpha^-)=\bar\beta_{\alpha r}p_h'(\alpha^+)$ and that

$$\phi_j'(\alpha^-)=\frac{-1}{D},\quad \phi_j'(\alpha^+)=\frac{-\rho}{D},$$

and (2.5). Consequently, we have

$$u_h(x_j^-)=-\bar\beta_{\alpha l}p_h'(\alpha^-)-\int_{x_j}^{x_{j+1}}f\phi_j\,dx. \tag{4.14}$$

Comparing this with (4.9), we see that again

$$u_h(x_j^-)=u_h(x_j^+). \tag{4.15}$$

Similarly we can show that $u_h(x_{j+1}^-)=u_h(x_{j+1}^+)$. Also from (4.9), (4.14), and (4.15), we see that

$$u_h(\alpha)=u_h(x_j^+)-\int_{\alpha}^{x_j}f(x)dx. \tag{4.16}$$

This conservation law will force $u_h(\alpha)=u(\alpha)$ in a certain case, as we shall see below in Theorem 4.2. Now let's look at within the interval $[x_j,x_{j+1}]$. The fact that u_h is also continuous at α can be easily checked by noting that $\phi_j+\phi_{j+1}=1$ over $[x_j,x_{j+1}]$ and

$$\bar f_{\alpha l}(\alpha-x_j)=\int_{x_j}^{\alpha}f(x)dx\qquad\text{and}\qquad \bar f_{\alpha r}(\alpha-x_{j+1})=-\int_{\alpha}^{x_{j+1}}f(x)dx.$$

Finally, by taking the derivative of u_h we see that the conservation law holds over each element. We collect our findings in the following theorem.

Theorem 4.1. *The approximated flux u_h defined by (4.8), (4.9) and (4.10) are continuous at all nodes x_i, $i=1,\dots,n$ and α. Furthermore, over each $[x_{i-1},x_i], i=1,\dots,j,j+2,\dots,n$ the conservation law holds:*

$$\mathrm{div}\,\mathrm{u_h}=\bar{\mathrm{f}}_{\mathrm{i-1/2}},$$

where div$=\frac{d}{dx}$. *Similar statement holds for $[x_j,\alpha]$ and $[\alpha,x_{j+1}]$.*

There is another choice in constructing an approximation of u over $[x_j,x_{j+1}]$: We simply define u_h to be the linear interpolant using the two endpoints values:

$$u_h(x)=\frac{x-x_j}{x_{j+1}-x_j}u_h(x_{j+1})+\frac{x_{j+1}-x}{x_{j+1}-x_j}u_h(x_j). \tag{4.17}$$

4.1. **Superconvergence of flux approximation.** If

$$\beta(x)=\begin{cases}\beta^-, & x\in[a,\alpha),\\ \beta^+, & x\in[\alpha,b],\end{cases} \tag{4.18}$$

then it is shown on p. 260 of [3] that the immersed finite element solution p_h is nothing but the interpolant $\hat I_hp\in V_h$ of the exact solution p. This is no surprise, since the standard linear finite element solution of the positive constant coefficient $\beta_-=\beta_+$ case has this well known superconvergence behavior at the nodes as well.

Furthermore, in this smooth case, Chou and Tang [1] gave an example in which their approximate flux formula reproduced the exact flux, see Example 2.1 on p. 667 of [1]. These facts motivate us to derive the following theorem.

Theorem 4.2. (i) Superconvergence at the nodes and interface point. *Let the coefficient β be piecewise constant with respect to the whole mesh for which (4.18) is a special case. Then the approximated flux u_h defined by (4.8), (4.9) and (4.10) coincides with the exact flux at all nodes $x_i, i = 0, \ldots, n$. Furthermore, the approximate flux reproduces the exact flux at the interface point α.*

(ii) Uniform error at the nodes and interface point. *Suppose that the coefficient β is a general function, not necessarily piecewise constant, then the errors at the nodes and the interface point are identical, i.e.,*

$$E(x) := u(x) - u_h(x) = C \qquad \text{for all} \quad x = x_i, i = 0, \ldots, n \text{ and } \alpha, \tag{4.19}$$

where C is a constant.

(iii) First order flux error at the nodes and interface point.
Furthermore, if $\beta \in C^1[a, \alpha] \cap C^1(\alpha, b)$, then the constant error in (4.19) satisfies the following property: there exists a constant $\tilde{C}$ such that

$$|u(x) - u_h(x)| \leq \tilde{C}h \tag{4.20}$$

for all $x = x_i, i = 0, \ldots, n$ and α.

Proof. We prove (i) first. From [3] or p. 167 of [4], we know that in this case p_h is the interpolant of the exact pressure p. Using (4.1) and (4.3) we have for $i = 1, \ldots, j, j+2, \ldots, n$,

$$\begin{aligned} u_h(x_i) - u(x_i) &= \int_{x_{i-1}}^{x_i} \beta(p' - p_h')\phi_i'\,dx \\ &= \frac{\beta}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (p' - p_h')dx = \frac{\beta}{x_i - x_{i-1}} (p - p_h)|_{x_{i-1}}^{x_i} = 0. \end{aligned}$$

Similarly for $i = 0, j+1$ we can show $u_h(x_i) = u(x_i)$ by (4.2) and (4.5). As a consequence of the nodal exactness, (4.16) becomes

$$u_h(\alpha) = u(x_j^+) - \int_\alpha^{x_j} f(x)dx = u(\alpha). \tag{4.21}$$

This completes the proof of (i).

As for (ii), we will prove the assertion for $i = 1, \ldots, n-1$, as the endpoint cases can be handled similarly. By (4.1) and (4.3),

$$E(x_i) = -\int_{x_{i-1}}^{x_i} \beta(p' - p_h')\phi_i'dx,$$

and similarly

$$\begin{aligned} E(x_{i+1}) &= -\int_{x_i}^{x_{i+1}} \beta(p' - p_h')\phi_{i+1}'dx \\ &= \int_{x_i}^{x_{i+1}} \beta(p' - p_h')\phi_i'dx, \end{aligned}$$

where we have used the fact that $\phi_i' = -\phi_{i+1}'$ over the interval $[x_i, x_{i+1}]$. Thus

$$\begin{aligned} E(x_i) - E(x_{i+1}) &= -\int_{x_{i-1}}^{x_{i+1}} \beta(p' - p_h')\phi_i'dx \\ &= -a(p - p_h, \phi_i) = 0. \end{aligned}$$

As for the interface point, we use (4.16) and subtract from it the corresponding equation for the exact flux u to get $E(\alpha) = E(x_j)$. This completes the proof of (ii).

We now prove assertion (iii). Let $\bar{\beta}$ be a fixed value $\beta(\eta), \eta \in (a, x_1)$. Setting $i = 0$ in (4.2) and (4.5), we have

$$\begin{aligned} u(a) - u_h(a) &= \int_a^{x_1} \beta(p' - p_h')\phi_0' dx \\ &= \int_a^{x_1} (\beta - \bar{\beta})(p' - p_h')\phi_0' dx + \int_a^{x_1} \bar{\beta}(p' - p_h')\phi_0' dx \\ &= \int_a^{x_1} (\beta - \bar{\beta})(p' - p_h')\phi_0' dx - h^{-1} \int_a^{x_1} \bar{\beta}(p' - p_h') dx \\ &= -h^{-1} \int_a^{x_1} (\beta - \bar{\beta})(p' - p_h') dx - \bar{\beta} h^{-1}(p(x_1) - p_h(x_1)) \\ &= I_1 + I_2. \end{aligned}$$

Here

$$|I_1| = |h^{-1} \int_a^{x_1} (\beta - \bar{\beta})(p_h' - p') dx| \le h^{-1}(||\beta'||_{\infty,(a,\alpha)} h)(C_1 h ||p||_{2,I}) \le Ch,$$

where we have used Theorem 5.1 below to estimate $|p' - p_h'|_{1,(a,x_1)}$.

$$|I_2| = |\bar{\beta} h^{-1}(p_h(x_1) - p(x_1))| \le (||\beta||_{\infty,I} h^{-1})(C_2 h^2 ||\beta||_{\infty,I} ||p||_{2,I}) \le Ch,$$

where we have used (4.22) in Theorem 4.3 below to estimate $|p(x_1) - p_h(x_1)|$. This completes the proof of (iii). □

It remains to prove the second order pressure error at the nodes used in the last proof.

Theorem 4.3. Second order pressure error at nodes. *Let $\beta \in C^1(a, \alpha) \cap C^1(\alpha, b)$. Then there exists a constant $C > 0$ such that*

$$|p(x_i) - p_h(x_i)| \le Ch^2 ||\beta||_{\infty,I} ||p||_{2,I}, \quad 1 \le i \le n - 1. \tag{4.22}$$

where C depends on certain norms of the Green's function at x_i.

Proof. Let $g(x, \xi)$ be the Green's function satisfying

$$a(g, v) =< \delta(x - \xi), v >, \quad v \in H_0^1(a, b).$$

By working out the closed form of g satisfying the classical formulation

$$-(\beta g')' = \delta(x - \xi), \quad [g]_\alpha = 0, \quad [\beta g']_\alpha = 0, \quad g(a, \xi) = g(b, \xi) = 0,$$

we see that g can be expressed in terms of $\int_d^x \frac{1}{\beta(t)} dt$ for different d. For instance, the Green's function for $(a, b) = (0, 1)$ and $\xi < \alpha$ takes the form

$$g(x, \xi) = \begin{cases} A \displaystyle\int_0^x \frac{1}{\beta(t)} dt, & 0 < x \le \xi, \\ (A - 1) \displaystyle\int_\xi^x \frac{1}{\beta(t)} dt + A \int_0^\xi \frac{1}{\beta(t)} dt, & \xi \le x \le \alpha, \\ (1 - A) \displaystyle\int_x^1 \frac{1}{\beta(t)} dt, & \alpha \le x \le 1, \end{cases}$$

where

$$A = \frac{\int_\xi^1 \frac{1}{\beta(t)} dt}{\int_0^1 \frac{1}{\beta(t)} dt}.$$

Thus $g(x, x_i) \in H^2(\Omega)$, for $\Omega = (x_k, x_{k+1}), k \neq j$ and $\Omega = (x_j, \alpha), (\alpha, x_{j+1})$. By the local approximation estimates in Section 2, we see that there exists $\hat{I}_h g \in V_h$, an interpolant of g, such that

$$|g - \hat{I}_h g|_{1,\Omega} \leq Ch||g''||_{0,\Omega} \tag{4.23}$$

for all the Ω's listed above. Now let $g = g(x, x_i)$ and use Galerkin orthogonality property, then

$$e(x_i) = a(g, e) = a(g - \hat{I}_h g, e) = (\beta(g - \hat{I}_h g)', e'),$$

which implies by (4.23)

$$\begin{aligned} |e(x_i)| &\leq ||\beta||_{\infty,I}||g' - (\hat{I}_h g)'||_{0,I}||e'||_{0,I} \\ &\leq ||\beta||_{\infty,I}(C_1 h||g||_{2,*})(C_2 h||p||_{2,I}) \\ &\leq C_3 h^2||g||_{2,*}||p||_{2,I}, \end{aligned}$$

where the result

$$||e'||_{0,I} \leq C_2 h||p||_{2,I}$$

will be shown in Theorem 5.1 below and where $||g||^2_{2,*} := \sum ||g||^2_{2,\Omega}$, the summation being over all Ω's listed above. $\square$

5. **Convergence of the pressure and the flux.**

The following consequence is standard since we have a conforming method.

Theorem 5.1.

$$||p - p_h||_{0,I} + h|p - p_h|_{1,I} \leq Ch^2||p||_{2,I},$$

where $||p||^2_{2,I} = ||p||^2_{2,(a,\alpha)} + ||p||^2_{2,(\alpha,b)}$.

Proof. Let $\beta^* = \sup_{x\in I}\beta(x), \beta_* = inf_{x\in I}\beta(x)$. Then $0 < \beta_* < \beta^* < \infty$. It is easy to see that

$$a(p - p_h, q_h) = 0 \qquad \forall q_h \in V_h,$$

and hence,

$$\begin{aligned} \beta_*|p - p_h|^2_{1,I} &\leq a(p - p_h, p - p_h) = a(p - p_h, p - q_h) \\ &\leq \beta^*|p - p_h|_{1,I}|p - q_h|_{1,I}. \end{aligned}$$

Thus

$$\begin{aligned} |p - p_h|_{1,I} &\leq \frac{\beta^*}{\beta_*}\inf_{q\in V_h}|p - q_h|_{1,I} \\ &\leq \frac{\beta^*}{\beta_*}|p - \hat{I}_h p|_{1,I} \leq Ch||p||_{2,I}. \end{aligned}$$

If we assume that $\beta \in C^1[a, \alpha] \cap C^1[\alpha, b]$, then the usual duality argument gives

$$||p - p_h||_{0,I} \leq Ch^2||p||_{2,I}.$$

$\square$

Now we begin to estimate $||u - u_h||_{0,I}$. For each (x_i, x_{i+1}) not containing the point α we have

$$\begin{aligned}||u-u_h|| &\leq ||\bar{\beta}_{i+1/2}p_h'(x_{i+1/2})-\beta p'||_{0,I_{i+1}}+||\bar{f}_{i+1/2}(x-x_{i+1/2})||_{0,I_{i+1}}\\ &\quad +||\int_{x_i}^{x_{i+1}}\left(\frac{\phi_{i+1}-\phi_i}{2}\right)f\,dx||_{0,I_{i+1}}\\ &\leq ||(\beta-\bar{\beta}_{i+1/2})p'||+|\bar{\beta}_{i+1/2}|\,||p'-p_h'||_{0,I_{i+1}}+Ch||f||_{0,I_{i+1}}\\ &\leq ||\beta-\bar{\beta}_{i+1/2}||\,|p'|_{\infty,I_{i+1}}+|\beta|_{\infty,I_{i+1}}\,||p'-p_h'||_{0,I_{i+1}}+Ch||f||_{0,I_{i+1}}\\ &\leq Ch|\beta|_{1,I_{i+1}}\,||p||_{2,I_{i+1}}+|\beta|_{\infty,I_{i+1}}\,||p'-p_h'||_{0,I_{i+1}}\\ &\quad +Ch||f||_{0,I_{i+1}} \end{aligned} \tag{5.1}$$

Similarly, for (x_j,α) we have

$$\begin{aligned}||u-u_h||_{0,I_{\alpha l}} &\leq ||\bar{\beta}_{\alpha l}p_h'(\alpha^-)-\beta p'||_{0,I_{\alpha l}}+||\bar{f}_{\alpha l}(x-\alpha)||_{0,I_{\alpha l}}\\ &\quad +||\int_{x_j}^{x_{j+1}}f\phi_j\,dx||_{0,I_{\alpha l}}\\ &\leq ||(\beta-\bar{\beta}_{\alpha l})p'||_{0,I_{\alpha l}}+|\bar{\beta}_{\alpha l}|\,||p'-p_h'||_{0,I_{\alpha l}}\\ &\quad +Ch(||f||_{0,I_{\alpha l}}+||f||_{0,I_{\alpha r}})\\ &\leq Ch||p||_{2,I_{\alpha l}}\,|\beta|_{1,I_{\alpha l}}\,+|\beta|_{\infty,I_{\alpha l}}||p'-p_h'||_{0,I_{\alpha l}}\\ &\quad +Ch(||f||_{0,I_{\alpha l}}+||f||_{0,I_{\alpha r}}) \end{aligned} \tag{5.2}$$

and for (α,x_{j+1})

$$\begin{aligned}||u-u_h||_{0,I_{\alpha r}} &\leq Ch||p||_{2,I_{\alpha r}}\,|\beta|_{1,I_{\alpha r}}\,+|\beta|_{\infty,I_{\alpha r}}||p'-p_h'||_{0,I_{\alpha r}}\\ &\quad +Ch(||f||_{0,I_{\alpha l}}+||f||_{0,I_{\alpha r}}). \end{aligned} \tag{5.3}$$

From (5.1)–(5.3), we have

$$\begin{aligned}||u-u_h||_{0,I} &\leq Ch(||p||_{2,I}+|\beta|_{1,I}+|\beta|_{\infty,I}+||f||_{0,I})\\ &\leq Ch(||p||_{2,I}+||\beta||_{1,I}+||f||_{0,I}),\end{aligned}$$

where $||\beta||_{1,I}^2=||\beta||_{1,(a,\alpha)}^2+||\beta||_{1,(\alpha,b)}$. Thus

Theorem 5.2.

$$||u-u_h||_{0,I}\leq Ch(||p||_{2,I}+||\beta||_{1,I}+||f||_{0,I}).$$

The above theorem obviously also holds when we replace (4.9) and (4.10) over the interval (x_j,x_{j+1}) by the interpolation formula (4.17).

6. **Numerical examples.** In this section we present two numerical examples to confirm our theory.

Problem 1. Consider

$$-(\beta p')'=f(x)=sx^m,\qquad p(0)=p(1)=0,$$

where s is a constant and m is a nonnegative integer. The interface point is located at α and

$$\beta(x)=\begin{cases}\beta^-, & x\in[0,\alpha),\\ \beta^+, & x\in[\alpha,1].\end{cases} \tag{6.1}$$

The exact solution is

$$p(x) = \begin{cases} \dfrac{-s}{(m+1)(m+2)\beta^-} x^{m+2} + \dfrac{t_-}{\beta^-} x, & x \le \alpha, \\ \dfrac{-s}{(m+1)(m+2)\beta^+} x^{m+2} + \dfrac{t_+}{\beta^+} x - \dfrac{t_+}{\beta^+} + \dfrac{s}{(m+1)(m+2)\beta^+}, & x \ge \alpha, \end{cases} \tag{6.2}$$

where

$$\begin{aligned} t_+ &= t_- \\ &= \left(\frac{\alpha - 1}{\beta^+} - \frac{\alpha}{\beta^-} \right)^{-1} \\ &\quad \times \left[\frac{-\alpha^{m+2}}{(m+1)(m+2)\beta^-} + \frac{\alpha^{m+2}}{(m+1)(m+2)\beta^+} - \frac{1}{(m+1)(m+2)\beta^+} \right] s. \end{aligned}$$

The flux

$$u(x) = -\beta p'(x) = \frac{s}{m+1} x^{m+1} - t_-$$

is continuous over $[0, 1]$.

We now use this problem to confirm exactness of the approximate flux u_h at the nodes and the interface point, as predicted in Theorem 4.2. Let $\beta^+ = 1000, \beta^- = 1$, $f(x) = x^m$ and the interface point $\alpha = 0.3$. Below we list in Table 1

$$uErrNodes = \max_{1 \le i \le n-1} |u(x_i) - u_h(x_i)|,$$

the maximum error over all the nodes $x_i, 1 \le i \le n-1$ for $m = 2, 5, 10$. In Table 2, we list the error at the interface point α for different mesh sizes. The last column in both tables lists the values of m, the exponent of the right hand side function. We used the Gaussian quadrature of order four to evaluate all integrals involved. We use $m = 10$ to show the influence of quadrature errors on the exactness. We can see a slight but not significant deterioration of the point errors at the nodes in Table 1 for $h = 1/16, 1/32$. In Table 2, uErr@alp denotes the error of the approximate flux at the interface point α using formula (4.9). The quantity uErr@alp$_I$ is the error in the approximate flux using the interpolation at the two endpoints of (x_j, x_{j+1}).

Table 1. Maximum Errors at Nodes of Approximate Flux

Problem 1	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Exponent m
uErrNodes	2.0428e-14	4.9405e-15	2.8449e-15	3.1225e-15	2
uErrNodes	3.3973e-14	9.2842e-15	1.6433e-15	1.8596e-15	5
uErrNodes	3.0552e-13	1.5363e-14	3.5943e-15	1.6653e-15	10

Problem 2. Consider

$$-(\beta p')' = f(x) = 2x, \qquad p(0) = p(1) = 0,$$

where

$$\beta(x) = \begin{cases} x^2 + 1, & x \in [0, \alpha), \\ x^2, & x \in [\alpha, 1]. \end{cases}$$

TABLE 2. Errors at Interface Point of Approximate Flux

Problem 1	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Exponent m
uErr@alp	1.2185e-14	3.2058e-15	6.9389e-16	3.8858e-16	2
uErr@alp	9.3606e-15	2.4217e-15	6.7307e-16	1.3878e-16	5
uErr@alp	2.0696e-13	3.0809e-15	6.2624e-16	1.1102e-16	10
uErr@alp$_I$	1.7969e-4	6.9824e-5	1.1841e-5	4.4022e-6	2
uErr@alp$_I$	1.0814e-5	4.6285e-6	8.2511e-7	2.9875e-7	5
uErr@alp$_I$	4.5151e-8	2.2024e-8	4.2392e-9	1.4658e-9	10

The exact solution is

$$p(x) = \begin{cases} -x + (1-d)\tan^{-1} x, & x \le \alpha, \\ -x + \dfrac{d}{x} + (1-d), & x \ge \alpha, \end{cases}$$

where

$$d = \frac{\alpha \tan^{-1}\alpha - \alpha}{1 - \alpha + \alpha\tan^{-1}\alpha}.$$

The flux

$$u(x) = -\beta p' = x^2 + d$$

is continuous over $[0,1]$. We take $\alpha = 0.3$. Again we list in Table 3

$$uErrNodes = \max_{1\le i\le n-1} |u(x_i) - u_h(x_i)|,$$

$$pErrNodes = \max_{1\le i\le n-1} |p(x_i) - p_h(x_i)|,$$

and $uErrL2Norm$, the L^2 norm error in the flux, which is one order higher than predicted by Theorem 5.2. First note that the second order error for pressure at nodes confirms the assertion of Theorem 4.3. We can interpret the second order of the L^2 error of u_h via the second order L^∞ norm error, which in turn is implied by the second order of uErrNodes shown in Table 3. The max norm and uErrNodes have the same order accuracy, since u_h is linear over each element, local linear interpolation error is second order, and uErrNodes is second order. It remains to explain the second order of uErrNodes, which is better than predicted in (iii) of Theorem 4.2. In view of (ii) of Theorem 4.2, it suffices to look at the error $E(0) = |u(0) - u_h(0)|$, but we created the coefficient β as $1 + x^2$ so that it is an $O(h^2)$ perturbation of a constant near the left endpoint 0. In view of (i) of Theorem 4.2, it is reasonable to expect that $E(0)$ is second order, and our numerical results do confirm that. It may take some effort to prove superconvergence results for general β.

The flux at the interface point is second order, as shown in Table 4. Finally we remark that we have extended most of the results in this paper to a quadratic immersed finite element method.

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Table 3. Maximum Errors at Nodes of Approximate Flux

Problem 2	$h = 1/32$	$h = 1/64$	$h = 1/128$	h=1/256	Order
pErrNodes	1.5729e-4	4.5597e-5	1.1775e-5	3.1019e-6	≈2
uErrNodes	3.6224e-4	9.6479e-5	2.4453e-5	6.2538e-6	≈2
uErrL2Norm	1.2990e-4	3.3919e-5	8.4569e-6	2.1467e-6	≈2

Table 4. Errors at Interface Point of Approximate Flux

Problem 2	$h = 1/32$	$h = 1/64$	$h = 1/128$	$h = 1/256$	Order
uErr@alp	3.6225e-4	9.64795e-5	2.44526e-5	6.2538e-6	≈ 2
uErr@alp$_I$	1.2787e-4	5.7417e-5	9.8041e-6	3.8124e-6	≈ 2

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