

## THE MECHANICS OF BINARY INTERACTIONS

Our system comprises an infinite number of mechanically identical, spherical particles known as monomers. These particles have masses  $m_1$  and radii  $R_1$ . When two monomers collide, they lose a certain amount of impact energy and rebound with a coefficient of restitution  $\varepsilon$ . If the impact energy is below a specific threshold value  $E_{\text{imp}} \leq E_{\text{agg}}$ , the monomers stick together due to surface forces like van der Waals forces, forming a larger aggregate particle with mass  $m_2$  and radius  $R_2$ . This process is known as *aggregation* and allows the creation of larger particles from individual monomers. A particle of mass  $m_k$  is an aggregate of  $k$  monomers, hence  $m_k = k \cdot m_1$ . We assume that the aggregates remain spherically shaped, and their radii scale as  $R_k \sim m_k^{1/3} \sim k^{1/3}$ .

Conversely, there is another mechanism called *fragmentation* that decreases the sizes of aggregates. If the impact energy is higher than a certain threshold value  $E_{\text{imp}} \geq E_{\text{frag}}$ , the colliding aggregates break into smaller pieces. The size distribution of the fragmented pieces are difficult to model analytically, and in this work we assume a simplistic model of fragmentation, called *shattering*. When two aggregates of masses  $m_i$  and  $m_j$  collide with a sufficient energy, both of them shatter into singular monomers,  $m_i \rightarrow i \cdot m_1$  and  $m_j \rightarrow j \cdot m_1$ .

Let us consider a collision of two particles of masses  $m_i$ ,  $m_j$ , and velocities  $\mathbf{v}_i$ ,  $\mathbf{v}_j$  and radii  $R_i$ ,  $R_j$  in the center of mass (CoM) frame, which moves with velocity  $\mathbf{V} = \mu_i \mathbf{v}_i + \mu_j \mathbf{v}_j$ , where  $\mu_i = m_i/M$ ,  $\mu_j = m_j/M$  and  $M = m_i + m_j$ . The velocities of particles in the CoM are

$$\mathbf{v}_{ci} = \mu_j \mathbf{g}, \quad \mathbf{v}_{cj} = -\mu_i \mathbf{g}, \quad (1)$$

where  $\mathbf{g} = \mathbf{v}_i - \mathbf{v}_j$  is the relative velocity. Obviously, both particles in the CoM frame have equal amounts of momenta, but oppositely directed

$$\mathbf{p}_{ci} = \mu \mathbf{g} = \mathbf{p}, \quad \mathbf{p}_{cj} = -\mu \mathbf{g} = -\mathbf{p}, \quad (2)$$

where  $\mu = m_j \mu_i = m_i \mu_j$  is the reduced mass. In the CoM frame, the relative velocity vector is

The impact energy is then

$$E_{\text{imp}} = \frac{p^2}{2\mu} = \frac{\mu g^2}{2}. \quad (3)$$

The outcome of the collision depends on the value of  $E_{\text{imp}}$ . Let us discuss each case in more details.

### Generalized collisions

First, let us consider a collision of particles of masses  $m_i$ ,  $m_j$  and velocities  $\mathbf{v}_i$ ,  $\mathbf{v}_j$ . In the most general case, we assume that the outcome of the collision is a collection of particles with various masses and velocities. Introducing the function  $P_k(\mathbf{v})$ , which is the number of particles of mass  $m_k$  and velocity  $\mathbf{v}$  created after the collision, or in other words, introducing the velocity distribution function of the particles of mass  $m_k$ . Using this distribution function, we can write the total mass, momentum and energy of particles in the outcome of the generalized collision

$$\begin{aligned} M &= \sum_{k=1}^{i+j} \int d\mathbf{v} m_k P_k(\mathbf{v}), \\ \mathbf{p} &= \sum_{k=1}^{i+j} \int d\mathbf{v} m_k \mathbf{v} P_k(\mathbf{v}), \\ E &= \sum_{k=1}^{i+j} \int d\mathbf{v} \frac{m_k v^2}{2} P_k(\mathbf{v}). \end{aligned} \quad (4)$$

Here we implied that  $k \leq i + j$ . The total mass and total momentum obey the conservation laws, hence we can write

$$\begin{aligned} i + j &= \sum_{k=1}^{i+j} k \int d\mathbf{v} P_k(\mathbf{v}), \\ i\mathbf{v}_i + j\mathbf{v}_j &= \sum_{k=1}^{i+j} k \int d\mathbf{v} \mathbf{v} P_k(\mathbf{v}), \\ \frac{i^2 v_i^2 + (2\mathbf{v}_i \cdot \mathbf{v}_j + \varepsilon^2 g^2)ij + j^2 v_j^2}{i + j} &= \sum_{k=1}^{i+j} k \int d\mathbf{v} v^2 P_k(\mathbf{v}). \end{aligned} \quad (5)$$

Although the total energy does not conserve in a collision, we can estimate the energy loss using the restitution coefficient  $\varepsilon$ . So, remaining total energy after the collision is

$$E = E_{\text{trans}} + \varepsilon^2 E_{\text{imp}} = \frac{MV^2}{2} + \frac{\mu}{2} \varepsilon^2 g^2, \quad (6)$$

which is distributed among the debris.

### Restitution

For a restitutive rebound of particles, the outcome velocities are analytic and given by

$$\begin{aligned} \mathbf{v}'_i &= \mathbf{v}_i - \mu_j(1 + \varepsilon)(\mathbf{g} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}}, \\ \mathbf{v}'_j &= \mathbf{v}_j + \mu_i(1 + \varepsilon)(\mathbf{g} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}}, \end{aligned} \quad (7)$$

where  $\hat{\mathbf{e}}$  is the unit vector of the impact geometry, e.g. the unit vector pointing from the center of the first particle to the center of the second particle at the moment of the collision

$$\hat{\mathbf{e}} = \frac{\mathbf{r}_i - \mathbf{r}_j}{R_i + R_j}. \quad (8)$$

Let us write the distribution function  $P_k(\mathbf{v})$  for the restitutive collision. First of all, the masses of impacting particles do not change, hence the distribution function should contain  $\delta$ -functions to control this. Together with the analytic expression for the outcome velocities, we can write

$$P_k^{\text{res}}(\mathbf{v}) = \delta_{ik}\delta[\mathbf{v} - \mathbf{v}_i + \mu_j(1 + \varepsilon)(\mathbf{g} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}}] + \delta_{jk}\delta[\mathbf{v} - \mathbf{v}_j - \mu_i(1 + \varepsilon)(\mathbf{g} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}}]. \quad (9)$$

### Aggregation

If the impact energy is smaller than a certain threshold  $E_{\text{imp}} \leq E_{\text{agg}}$ , the outcome of the collision is merging of two particles. From the momentum conservation we can write the outcome of the aggregative collision, which is a single particle with a mass and velocity

$$m' = m_i + m_j \quad \mathbf{v}' = \mathbf{V} = \frac{m_i \mathbf{v}_i + m_j \mathbf{v}_j}{m_i + m_j}. \quad (10)$$

The total energy loss is

$$\Delta E = \frac{MV^2}{2} - \frac{m_i v_i^2}{2} - \frac{m_j v_j^2}{2} = -\frac{\mu g^2}{2} = -E_{\text{imp}}, \quad (11)$$

so, all the impact energy is lost during the aggregative collision. The threshold energy value  $E_{\text{agg}}$  is in general a function of the sizes of particles.

Let us write the debris velocity distribution function for the aggregation process. Since the outcome is a single particle of mass  $m_i + m_j$ , with velocity  $\mathbf{V}$ , we have

$$P_k^{\text{agg}}(\mathbf{v}) = \delta_{k,i+j}\delta[\mathbf{v} - \mu_i \mathbf{v}_i - \mu_j \mathbf{v}_j]. \quad (12)$$

The restitution coefficient for the aggregation process is  $\varepsilon = 0$ .

## Fragmentation

If the impact energy exceeds the certain threshold value  $E_{\text{imp}} \geq E_{\text{frag}}$ , the two impactors break into into smaller particles in the collision. We cannot obtain the velocities of the monomers from only conservation laws, hence we have to assume that certain constraints are valid. Namely, we assume two constraints:

1. Both particles shatter into their constituent monomers;
2. Complete isotropy of the momenta of the monomers in CoM frame;

These two constraints allow us to write the outcome velocities of the fragmented pieces. Let us write the energy needed to release a single monomer from a particle as  $\gamma$ . Hence, the total energy needed for a complete decomposition of an aggregate of mass  $m_k$  can be estimated as

$$E_k = \gamma \cdot k. \quad (13)$$

The fragmentation process of two particles of masses  $m_i$  and  $m_j$ , with velocities  $\mathbf{v}_i$  and  $\mathbf{v}_j$  can be then described as a decay of a single particle of mass  $m_k = m_i + m_j$  with velocity  $\mathbf{v}_k = \mathbf{V} = \mu_i \mathbf{v}_i + \mu_j \mathbf{v}_j$ . The decay energy can be estimated as

$$E_{\text{decay}} = E_{\text{imp}} - \gamma \cdot k, \quad (14)$$

which is the amount of energy which is equally distributed among all the shattered monomers. From this, we can see that the impact energy should be larger than  $\gamma \cdot k$ , which can be treated as the threshold energy. In the frame, each released monomer has an energy

$$E'_c = \frac{m_1 v_c'^2}{2} = \frac{\mu g^2}{2k} - \gamma, \quad (15)$$

where  $v'_c$  is the speed of a monomer in CoM frame

$$v'_c = \sqrt{\frac{ij}{(i+j)^2} \cdot g^2 - \frac{2\gamma}{m_1}}. \quad (16)$$

Let us estimate the number of monomers  $dN$  in a small solid angle  $d\Omega$ . From the second constraint, we deduce that this number has to be proportional to the angle itself, hence

$$dN = \frac{k}{4\pi} d\Omega, \quad k = i + j. \quad (17)$$

In the Lab frame, the speeds of monomers are not equal, but rather uniformly distribution in the range

$$v'_{\min} = |\mu_i \mathbf{v}_i + \mu_j \mathbf{v}_j| - \sqrt{\frac{ij}{i+j} \cdot g^2 - \frac{2\gamma}{m_1}}, \quad v'_{\max} = |\mu_i \mathbf{v}_i + \mu_j \mathbf{v}_j| + \sqrt{\frac{ij}{i+j} \cdot g^2 - \frac{2\gamma}{m_1}}. \quad (18)$$

## DISTRIBUTION FUNCTION

The statistical description of the system is fully described by a set of distribution functions  $f_k(\mathbf{r}, \mathbf{v}, t)$ . It is normalized, such that  $f_k(\mathbf{r}, \mathbf{v}, t) d\mathbf{r} d\mathbf{v}$  gives the number of particles of size  $k$  in the phase space volume  $d\Gamma = d\mathbf{r} d\mathbf{v}$ , around the point  $(\mathbf{r}, \mathbf{v})$ . Hence, integrating over the whole phase space gives us the total number of particles of size  $k$

$$N_k = \int d\mathbf{r} d\mathbf{v} f_k(\mathbf{r}, \mathbf{v}, t). \quad (19)$$

The spacial distribution of particles is not very important for us, hence in the following we assume that the system is spatially homogeneous, and we use only the velocity distribution function  $f_k(\mathbf{v}, t)$

$$N_k = \int d\mathbf{r} \int d\mathbf{v} f_k(\mathbf{v}, t), \quad (20)$$

hence

$$n_k \equiv \frac{N_k}{V} = \int d\mathbf{v} f_k(\mathbf{v}, t), \quad (21)$$

is the number density of the subsystem of particles with size  $k$ . The other field functions, such as the mean flow velocity  $\mathbf{u}_k$  or granular temperature  $T_k$  can be defined as velocity moments of the distribution function

$$\begin{aligned} n_k \mathbf{u}_k &= \int d\mathbf{v} \mathbf{v} f_k(\mathbf{v}, t), \\ \frac{3}{2} n_k T_k &= \int d\mathbf{v} \frac{m_k c_k^2}{2} f_k(\mathbf{v}, t), \\ \mathbf{c}_k &= \mathbf{v} - \mathbf{u}_k. \end{aligned} \quad (22)$$

## KINETIC EQUATIONS

The time evolution of the distribution functions obey the Boltzmann equations

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} - \frac{1}{m_k} \frac{\partial U(r)}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{v}} \right) f_k(\mathbf{r}, \mathbf{v}, t) = \sum_j \mathcal{I}(f_k, f_j), \quad (23)$$

where  $U(r)$  is the potential of the external gravitational field. The LHS of the Boltzmann equation describes the change over time in the function  $f_k$  due to the local flow of the particles, subject to external driving. The function  $\mathcal{I}(f_k, f_j)$  on the RHS is the *collision integral*, which describes the change over time in the function  $f_k$  due to collisions of particles  $k$  with particles of size  $j$ . Since we have three types of collisional outcomes, the collision integral  $\mathcal{I}$  has to take into account all these types of outcomes. Without the loss of generality, we can write the collision integral as a sum of three functions

$$\mathcal{I}(f_k, f_j) = \mathcal{I}^{\text{agg}}(f_k, f_j) + \mathcal{I}^{\text{res}}(f_k, f_j) + \mathcal{I}^{\text{frag}}(f_k, f_j), \quad (24)$$

each corresponding to the specific type of collision.

### General structure of collision integrals

Let us consider a collision integral  $\mathcal{J}(f_k, f_j)$  for a generalized collision.