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The Vertical Structure and Thickness of Saturn's Rings

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We have considered the steady state vertical structure of Saturn's rings with regard to whether collapse to a monolayer due to collisions between particles, the end state predicted by Jeffreys (1947a), may be prevented by any of a variety of mechanisms. Given a broad distribution of particle sizes such as a typical power law $n(R) = n_0R^{-3}$, it is found that gravitational scattering of small particles by large particles maintains a true ring thickness of several times the radius of the largest particles, or many times the radius of the smallest particles. Thus the "many-particle-thick" condition which best satisfies optical observations, such as the opposition effect, may be reconciled with ongoing particle collisions. If we consider the obvious sources of energy available for such a process, we find that a ring thickness of only tens of meters may be sustained over the lifetime of the solar system. This implies a maximum particle size on the order of a few meters.

I. INTRODUCTION

There are two distinctly different hypotheses for the origin of Saturn's rings: either they resulted from the tidal breakup of a satellite or a comet, or they condensed from the protoplanetary nebula in the region where they are seen today. It has not as yet been possible to decide between these hypotheses. In fact, there is as yet no general agreement even as to the physical structure of the rings. In this paper we will discuss the thickness and vertical structure of the rings, and note some possible restrictions on ring origins which may be derived from our consideration of the ring properties.

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There are no direct observations of the physical thickness of the rings. However, at the time of the last crossing of the Earth through the ring plane, the minimum brightness of the rings was obtained in an extremely difficult measurement by Kiladze (1969) and Focas and Dollfus (1969). Assuming the same surface brightness for the edge-on rings as for the inclined rings, these authors inferred apparent edge-on full thicknesses of 2.8 ± 1.5 and 1.4 ± 0.5 km, respectively (see, e.g., Bobrov, 1970, and Fountain and Larson 1978). A reanalysis of this data by Lumme and Irvine (1978) suggests that an upper limit of 3 km for the thickness is all that can be inferred from the data. Nevertheless, for simplicity, throughout this paper we will refer to these measurements as giving an "observed" thickness.

The vertical structure of the rings must also be inferred from observations. Optical observations, in particular the opposition effect, favor a many-particle-thick ring sys-

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tem with a very small ratio of particle volume to total volume (10^{-2} to 10^{-3}). As proposed originally by Seeliger nearly a century ago, this hypothesis explains the opposition effect by mutual shadowing of the particles in the many-particle-thick ring. More recent analyses of the opposition effect as well as other optical phenomena continue to be readily understandable in the context of this *classical model* (see, e.g., Bobrov, 1970; Kawata and Irvine, 1974; Pollack, 1975). Such a many-particle-thick model implies random velocities of the particles with respect to the average local motion. On the other hand, dynamical arguments show that inelastic collisions between particles will decrease the energy contained in random particle motions. In the absence of sources of random motion, the system will thus flatten and spread to a layer that is at most only a few particles thick in a time short compared to the age of the solar system (Jeffreys, 1947a; c.f., however, Brahic, 1977a and Goldreich and Tremaine, 1978). We refer to this as the *monolayer model.* Under such conditions, the opposition effect would have to be produced by intricate surface microstructure, as is the less dramatic opposition effect for the Moon. However, the extremely low volume densities implied by the opposition effect of the rings, when coupled with particle size and separation constraints imposed by geometrical optics, make the surface microstructure hypothesis an unlikely solution (see, e.g., Hämeen-Anttila and Vaaraniemi, 1975).

Each of these alternative models of vertical ring structure offers a different explanation for the apparent ring thickness. In the case of the classical model, it may be understood as a distance equal to several scale heights of a vertical particle density distribution (Pollack, 1975). In the case of the monolayer model, it is a rough measure of the particle size. The difficulty in the latter case is that particles as large as a kilometer—if composed of materials which are expected to be common in the outer solar system—would exhibit much stronger radio emission than observed (Pollack, 1975).

We have sought mechanisms which could be replenishing the energy in the dispersed component of particle velocities, and thereby preventing the ring system from collapsing to a monolayer. This would resolve the apparent conflict between observations and dynamical arguments discussed above. In Section II we describe in more detail the characteristics which distinguish rings which are truly many particles thick from rings which are not. In Section III we review perturbing mechanisms which have been invoked to affect the vertical ring structure. These mechanisms may be divided into two classes according to the nature of the ring structure that they produce: "coherent" (satellite and solar gravitational perturbations) and dispersive (meteoroid impact; radiation pressure; differential rotation or Kepler shear). In Section IV we show how gravitational scattering within a broad particle size distribution could be responsible for the present existence of a ring that is many particles thick. We find, however, that even then it is not easy to understand a thickness as great as a kilometer without changing at least one of our current ideas about the rings.

II. CHARACTERISTICS OF "THICK" RINGS

The thickening mechanisms to be discussed in Section III can be classified according to whether they are "coherent" or "dispersive." The former class of mechanisms lead to what we will call "warped" or "wavy" rings; the latter lead to "thick" rings. In this section we describe some of the important properties of "thick" rings.

a. *Thick vs Wavy Rings*

The distinction between a "thick" ring and a "wavy" monolayer should be clearly stated. Consider a system of orbiting particles having some mean inclination about the midplane of the system. If the nodes of the particle orbit planes are similar at a given radial distance and they vary systematically with distance to the planet, a warp or wave exists. On the other hand, if the nodes of the particle orbit planes are randomly distributed with radial distance and/or uniformly distributed with azimuth, a "thick" ring results in which a given vertical section may pass through several particles.

In a "wavy" ring, adjacent particles move coherently through their orbital excursions and do not collide. However, regarded in a local frame rotating with the mean orbital rate, the particles in any local region of a "thick" ring exhibit random relative velocities, denoted by V. These random velocities produce collisions; during such impacts the relative velocity $\Delta V \sim V/3^{1/2}$ decreases to $\Delta V' = e\Delta V$, where e is the elasticity. We discuss this further below.

b. Flattening of a Thick Ring

The collisions discussed above which characterize thick rings decrease the energy contained in random particle motions. In a "thick" system of normal optical depth τ_0, the mean time between collisions is on the order of $P/2\tau_0$, where P is the orbital period. In each collision, the relative velocity ΔV is decreased by $(1 - e)$. Hence, the energy in dispersive motion is decreased during a collision by $(1 - e^2)\Delta V^2 \approx (1-e^2) V^2/3$, where $V^2 = V_r^2 + V_\theta^2 + V_z^2$. The time scale on which the energy (per unit mass) in random motions is damped out is thus

$$T_f \approx \frac{E_{\text{random}}}{\Delta E/\Delta t} \approx \frac{3V^2P}{2(1-e^2)V^2\tau_0} \approx \frac{3P}{2\tau_0(1-e^2)}.$$

T_f is much, much less than the age of the solar system for any reasonable value of elasticity e. Therefore, to maintain a "thick" ring, an energy source must continually impart random velocity components, which in turn implies that particles in the same region of space must suffer different orbital perturbations in order to produce the dispersion.

c. Vertical Distribution of Particles in a Thick Ring

The distribution of random velocities in an ensemble of particles undergoing collisional distribution of energy is expected to be of the form $f(V_k)\alpha e^{-V_k^2/2V_{0k}^2}$, where the subscript $k = (r,\ \theta,\ z)$. Thus, we expect the spatial density of particles to decrease with vertical distance z as e^{-z^2/z_0^2}. The scale factor z_0 is given by $z_0 = 2^{\frac{1}{2}}V_{0z}a/V_c$, where a is the orbital semimajor axis and V_c is the corresponding circular velocity. The region bounded by $z = \pm z_0$ contains about 85% of the particles. We will thus refer to $2z_0$ as the "full" thickness of the ring.

Given the above spatial density distribution, the apparent thickness inferred from observations will be significantly greater than the characteristic vertical dimension $2z_0$ for, when the rings are viewed edge-on, an extremely long path length ($\sim 2R_s$) results. The apparent thickness z_{app} will be that value of z giving $\tau \approx 1$ along this very long line of sight, a region characterized by much lower spatial density $N(z)$ than is z_0. That is,

$$\tau(z_{\text{app}}) \approx 1 \approx N(z_{\text{app}})\cdot\pi R^2\cdot 2R_s, \quad (1)$$

where R is particle radius, and R_s is Saturn's radius. We determine $N(z)$ using the normal ring optical depth τ_0:

$$\tau_0 = \int_{-\infty}^{\infty} N(z)\pi R^2 dz \approx \pi R^2 N_0 \times \int_{-\infty}^{\infty} e^{-(z/z_0)^2} dz = \pi^{3/2}R^2 z_0 N_0. \quad (2)$$

Because $\tau_0 \approx 1$, we find $(z_{\mathrm{app}}/z_0) \approx 4$. This implies that a "thick" ring of $\tau_0 \approx 1$ with a characteristic full thickness $2z_0 \sim 250$ to 500 m due to particle velocity dispersion will appear to be 1 to 2 km thick when viewed edge-on. This would not be the case with a "wavy" or warped monolayer; its apparent thickness viewed edge-on would be equal to the amplitude of the wave.

III. PERTURBATIONS

a. *Coherent Perturbations*

Satellite perturbations have been proposed as a mechanism that could produce a thick ring (Bobrov, 1970; 1978). Such perturbations are addressed in a companion paper (Burns *et al.*, 1979) where we analyze in detail the effects due to satellite and solar perturbations of long period (i.e., averaged over a particle orbit), including the planetary oblateness and precession, to obtain solutions in closed form for the perturbed inclination of the particle orbits.

The computed peak-to-peak vertical amplitudes of these perturbations (measured at 10^5 km from Saturn) about the mean ring plane, taken to be the Laplacian plane, are 9, 1.8, 0.3, and 0.1 m for the Sun, Titan, Mimas, and Tethys, respectively; other satellites have little effect. In each case the wave magnitudes vary by about an order of magnitude from the inner to the outer edge of the ring. Because the perturbations experienced by adjacent particles are essentially the same, these forced particle motions are "coherent," or nondispersive in nature (Ward, 1977) and produce a monolayer, not a many-particle-thick ring system. The satellite perturbations do, nevertheless, warp the mean plane so that an about 400-m "thickness" may be apparent during ring plane passage. In addition, Burns *et al.* (1979) point out that reflected light from an exterior E ring can also appreciably contribute to the measured "thickness." Vaaraniemi (1973) has treated short-period perturbations (those having a period similar to the period of a particle orbit) due to the satellites, which are also "coherent," and found their amplitudes to be on the same order as those of long period. Numerical calculations of satellite perturbations by other authors are discussed by Burns *et al.* (1979).

b. *Dispersive Perturbations*

A number of physical mechanisms can generate random motions of the ring particles. We first describe mechanisms which do not involve the differential rotation (*Kepler shear*) of the disk. We then consider mechanisms in which the shear motion of the disk is important. Initially, we restrict the discussion to identical particles, whether visualized as infinitesimal or of finite size. Later, in Section IV, we investigate the importance of a size distribution for the Kepler shear case.

(1) One mechanism which would clearly provide a source of random particle motions is meteoroid impact. We estimate the ring thickness maintained in a steady state due to ongoing impact by using $\dot{p}_\perp$, the rate of momentum flux into the rings. To produce a generous upper limit we assume the process is completely efficient. Then, to an order of magnitude, the mean thickness will increase as:

$$\dot{z}_0 \approx a \left(\frac{\dot{p}_\perp}{p_c}\right)_{\text{ring particles}} \approx a \left[\frac{\dot{p}_\perp(\text{meteoroids})}{p_c(\text{ring particles})}\right] \approx \frac{\rho V_e^2}{\Sigma V_c} a, \quad (3)$$

where V_e is the escape velocity at the ring distance, Σ is the surface mass density of the rings, ρ is the space mass density of meteoroids, and p_c is the orbital momentum of the ring particles. In a steady state process characterized by flattening time T_f the ring thickness will be about $\dot{z}_0 T_f$.

We use the space density of meteoroids in Saturn's vicinity as crudely extrapolated by Cook and Franklin (1970), $\rho \approx 1.2 \times 10^{-23}$ g cm^{-3}. The surface mass density of the rings is not well known, but is surely greater than 1 g cm^{-2}. Then for $\tau_0 \approx 1$, $z_0 \lesssim 10^{-2}/(1 - e^2)$ cm. Thus meteoroid impact is not likely to have a significant effect on the ring thickness.

(2) Another mechanism that could produce a thick ring is radiation pressure (Vaaraniemi, 1973). As the particles go in and out of Saturn's shadow, perturbations arise which are on the order of 1 to 10 cm for a 1-cm diameter particle, depending on the saturnicentric latitude of the Sun. Because the perturbation varies inversely with particle diameter, adjacent particles will be perturbed differently as long as they are of different sizes. Thus, only extremely small particles ($\lesssim 1$ mm) will, in general, attain vertical displacements many times their own size. Some fraction of such tiny particles undoubtedly exists in the ring system. However, radar observations (Goldstein *et al.*, 1976) require the preponderance of particles to be larger than 1 cm in size (Pollack *et al.*, 1973; Pollack, 1975; Cuzzi and Pollack, 1978); this is not surprising because small particles would be easily "swept up" in collisions with larger particles. Therefore, the total optical depth represented by small particles ($\ll 1$ cm) is insufficient to explain either the observed opposition effect (Kawata and Irvine, 1974) or the apparent thickness.

(3) Several authors have discussed the effects of Kepler shear and radial ring spreading (Cook and Franklin, 1964; Bobrov, 1970; Trulsen, 1972; Goldreich and Tremaine, 1978). In these analyses, identical particles of substantial elasticity (>0.5) have been assumed. Results of these calculations, which merely assume that the energy given up by a radially spreading ring is transformed to velocity dispersions uniformly throughout the age of the solar system, indicate that a steady state full thickness of about 10 to 35 m is possible; this is more than an order of magnitude smaller than indicated by observations. For cases in which the particle interactions are extremely inelastic (Baxter and Thompson, 1963; Hämeen-Anttila, 1975; 1976), quite different results are produced (e.g., radial clumping) than those observed in the ring system. Such behavior will be transient if the elasticity increases with decreasing relative velocity, as seen for most materials (Goldsmith, 1960); radial ring spreading will ultimately result.

The rate at which energy is dissipated increases with dispersion velocity in the above analyses due to an increase in the effective viscosity of the system. For infinitesimal particles, this is independent of particle size. For finite-sized particles, however, the dispersion velocity may be determined in part by particle size. As the centers of finite-sized particles are separated by at least the sum of the particle radii, they each experience a slightly different potential and therefore move with systematically different velocities at the moment of collision. The postcollision components of this incident velocity will in general reappear in randomly different directions, thereby contributing to a true velocity dispersion.

As an example, consider particles moving in a simple central gravity field GM/a^2. The circular velocity of a particle is $V_c = (GM/a)^{1/2}$. Due to the existence of this differential rotation, or Kepler velocity shear, two particles of radii R_1 and R_2 will have a relative velocity $\mathbf{\Delta V}$ at the moment of collision of amplitude

$$|\mathbf{\Delta V}| \approx (R_1 + R_2)(dV_c/da) \approx -(R_1 + R_2)V_c/2a. \quad (4)$$

The postcollision relative velocity $\Delta V'$ will be of amplitude $\Delta V' \approx e|\mathbf{\Delta V}| \approx \Delta V$, but randomly oriented in direction. The particles will typically attain inclinations

$i \approx \Delta V/V_c$, and will thus constitute a "thick" layer of full thickness $2z_0 \approx 2ai \approx 2a(\Delta V/V_c) \approx R_1 + R_2$. Thus we would expect a layer of identical particles to be about a few particle radii thick just due to Kepler shear, as long as e is on the order of unity. This agrees well with the results of numerical calculations for identical particles by Brahic (1977a).

Brahic (1977a) converts his direct result of a few-particle-thicknesses to a typical particle size of 2.5 m, or less, using the energy agrument mentioned above. This value, viewed as a "scale height," is also about an order of magnitude smaller than indicated by the "observations." In addition, if regarded as characterizing a layer of particles of finite, identical size, this result does not explain how the rings may be many particles thick in order to explain the opposition effect. We shall return to the question of the dispersive energy budget of the rings in the next section.

IV. SCATTERING OF SMALL PARTICLES BY LARGE PARTICLES WITHIN THE RINGS

a. A Likely Size Distribution

It is unlikely that the ring particles are of only one size, since collisional and accretional processes must, at least at some time, have been important. Such processes are probably operating even at present. We now investigate the effect of a particle size distribution on scattering processes. In addition to its relevance to the question of the vertical structure of the rings, the existence of a broad distribution of particle sizes in the rings provides a natural explanation of several other recent observations. The observed nonaxisymmetric brightness of ring A (Reitsema *et al.*, 1976; Lumme and 1976) Irvine, has been ascribed to large particles either rotating synchronously (Reitsema *et al.*, 1976; Lumme and Irvine, 1976) or acting as sources of gravitational "wakes" by perturbing the orbits of nearby, smaller particles (Colombo *et al.*, 1977; Franklin and Colombo, 1978).

The high radar reflectivity and low microwave emissivity of the rings is naturally explained in the context of the classical model if most of the ring particles by number are centimeters to meters in size (Pollack *et al.*, 1973; Pollack, 1975; Cuzzi and Pollack, 1978). The wavelength independence of the radar reflectivity requires in this case a broad size distribution, consistent with the distribution

$$n(R) = n_0 R^{-3}, \quad (5)$$

with a minimum radius of about 1 cm, where $n(R)dR$ is the number of particles per unit volume with radii between R and $R + dR$, and n_0 is a constant. Such a size distribution is quite commonly found to characterize processes dominated by comminution (mutual fragmentation) such as might be taking place in the ring system (Anders, 1965; Hartmann, 1969; Greenberg *et al.*, 1976). The upper limit on the size distribution (5) is not well established by radar observations alone, but is larger than a meter. In such a size distribution, most of the particles are at the small-size end of the distribution, whereas equal cross-sectional area is presented by all particle sizes for equal fractional radius increment. The bulk of the mass and therefore of the gravitational (orbital) energy is contained in the large particles.

We have attempted to understand the effect of such a distribution of particle sizes on the vertical structure of the rings. We will show that the less numerous large particles do most of the perturbing, and the many small particles which are so affected then delineate the structural form of the ring system. We shall ignore physical collisions of small particles with large particles as a source of random velocities, because the nature of the surfaces (regoliths) of the large particles is unknown and such collisions may be quite inelastic.

However, the gravitational perturbations of the large particles are at least as effective as physical collisions in maintaining the random velocities of the small particles, and gravitational scattering is completely elastic.

b. Gravitational Scattering, Another Dispersive Mechanism

Our picture is then that small particles are continually scattered out of their mean orbits by close encounters with large particles, and then distribute and decrease their random velocities by physical collisions with other small particles. In steady state, these processes will be in balance with some mean random velocity (vertical thickness) determined by the number and size of the large particles and the frequency and inelasticity of the collisions between small particles. We will obtain a steady-state value for ring thickness by setting the rate of increase of random energy due to a combination of Kepler shear and gravitational scattering equal to the rate of decrease due to energy lost in collisions. The same approach was adopted by Goldreich and Tremaine (1978) but in our case an explicit result for ring thickness will result, as our rate of increase depends on dispersion velocity in a different way than does the rate of decrease.

The rate of increase of random energy of the small particles per unit mass and volume is

$$\frac{d\langle V^2\rangle^+}{dt} = \left.\frac{d\langle V^2\rangle^+}{dt}\right|_{\text{collisional}} + \left.\frac{d\langle V^2\rangle}{dt}\right|_{\text{gravitational}} = \nu\left(a\frac{d\Omega}{da}\right)^2 + \int_M d\,\frac{\langle V^2\rangle}{t_{\text{enc}}(M)}, \quad (6)$$

where $\langle V^2\rangle$ is the sum of the one-dimensional velocity dispersions or twice the average "random" energy per unit mass; ν is viscosity and Ω is the orbital angular velocity. The typical ring half-thickness will be $z_0 \approx 2^{1/2}\sigma/\Omega$ as shown in Section IIc, where $\sigma^2 = \langle V^2\rangle/3$ is the one-dimensional velocity dispersion. For simplicity we assume an isotropic distribution of particle random velocities. In (6), both Kepler shear and gravitational scattering, the first and second terms on the right-hand side, are represented. Because there is significant variation of the gravitational scattering term with $\langle V^2\rangle$ and particle mass M, the net contribution is obtained as an integral over M. The term $t_{\text{enc}}(M)$ is the mean time it takes for encounters of a "test" particle with particles of mass M to cause a large change in the test particle's velocity. Thus, the term $\langle V^2\rangle/t_{\text{enc}}(M)$ is a crude measure of $d\langle V^2\rangle/dt|_{\text{grav}}$ due to the particles in the mass range $(M, M + dM)$. We evaluate the contribution from each mass range $(M, M + dM)$ to the energy per unit mass of a given "test" particle as:

$$d[\langle V^2\rangle/t_{\text{enc}}(M)] \approx \langle V^2\rangle n(M)\pi b^2(M) V_\infty dM. \quad (7)$$

In (7), $\pi b^2(M)$ is the cross section for gravitational scattering by a particle of mass M, V_∞ is the encounter velocity at "infinity," and $n(M)$ is the mass distribution. The impact parameter for large-angle scattering, b_{la}, is given by

$$\tfrac{1}{2}mV_\infty^2 \cong GMm/b_{\text{la}} \Rightarrow b_{\text{la}}(M) \approx 2GM/V_\infty^2. \quad (8)$$

In the Appendix we show that use of (8) in (7) will take adequate account of both small- and large-angle scattering when the scatterers are confined to a thin disk. We can ignore the cumulative effects of small-angle scattering here because of the thin-disk geometry. Therefore,

$$d[\langle V^2\rangle/t_{\text{enc}}(M)] \approx 4\pi G^2M^2\langle V^2\rangle n(M)dM/V_\infty^3.$$

The encounter velocity V_∞ is approximately equal to the dispersion velocity, which is significantly larger than the relative velocity due to Kepler shear ΔV_c at a radial distance b from the mass M, as will be seen. Thus in the frame of the large particle, small particles are seen to approach with more or less constant velocity $V_\infty \approx \sigma 3^{1/2}$, and, to zeroth order, isotropically.

The contribution to particle dispersions due to Kepler shear and physical collisions is not a strong function of $\langle V^2 \rangle$ or M and depends essentially on total optical depth τ_0 through the viscosity ν. We shall simply adopt for viscosity ν in (6) the expression,

$$\nu = (C_1 \langle V^2 \rangle / 3\Omega)(\tau_0 + 1/\tau_0)^{-1},$$

where C_1 is of order unity (Goldreich and Tremaine, 1978), valid for both large and small optical depths, and use $d\Omega/da \sim \Omega/a$, to obtain

$$\frac{d\langle V^2 \rangle^+}{dt} \approx \frac{C_1 \langle V^2 \rangle \Omega}{3(\tau_0 + 1/\tau_0)} + \int_M \frac{4\pi G^2 M^2 \langle V^2 \rangle n(M) dM}{\langle V^2 \rangle^{3/2}}. \quad (9)$$

In an inelastic collision between particles of elasticity e, the relative velocities after impact are a factor e times their precollision values. If we adopt the one-dimensional velocity dispersion σ as a representative value of the average relative velocity,

$$\frac{d\langle V^2 \rangle^-}{dt} = \left.\frac{d\langle V^2 \rangle^-}{dt}\right|_{\text{collisional}} \approx \frac{\Delta \langle V^2 \rangle}{T_{\text{coll}}} \sim \frac{C_2(\sigma^2 - e^2\sigma^2)}{\pi(\Omega\tau_0)^{-1}} \approx \frac{C_2 \langle V^2 \rangle \tau_0 \Omega (1 - e^2)}{3\pi}, \quad (10)$$

where C_2 is of order unity. Equating (9) and (10) and cancelling terms gives

$$C_2\tau_0(1 - e^2) - \pi C_1(\tau_0 + 1/\tau_0)^{-1} = \frac{12\pi^2 G^2}{\Omega \langle V^2 \rangle^{3/2}} \int_{M_1}^{M_2} M^2 n(M) dM. \quad (11)$$

Note that V remains explicitly in this expression whereas in Goldreich and Tremaine (1978) the right-hand side equals zero. The scale height z_0 enters (11) through

$$z_0 \sim \left(\frac{2\langle V^2 \rangle}{3}\right)^{1/2} \Omega^{-1} \approx 2^{1/2} \frac{\sigma}{\Omega}.$$

We assume that the distribution of particle radii indeed follows a power law distribution with index -3 and that it has lower and upper radius cutoffs of R_1 and R_2, respectively.

$$\left.\begin{aligned} n(R) &= n_0 R^{-3}; \; R_1 \le R \le R_2 \\ &= 0; \qquad R < R_1, \; R > R_2 \end{aligned}\right\}. \quad (12a)$$

This then implies a mass distribution

$$\left.\begin{aligned} n(M)dM &= \tfrac{1}{3} n_0 (\tfrac{4}{3}\pi\rho_p)^{2/3} M^{-5/3} dM; \\ &\qquad M_1 \le M \le M_2 \\ &= 0; \\ &\qquad M < M_1, \; M > M_2 \end{aligned}\right\}, \quad (12b)$$

where ρ_p is the internal mass density of the particles. We obtain n_0 from the observed normal optical depth

$$\tau_0 = \pi^{1/2} z_0 \int_{R_1}^{R_2} \pi R^2 n(R) dR \approx \pi^{3/2} n_0 z_0 \ln (R_2/R_1),$$

where we have integrated over z assuming a Gaussian distribution as in (2). Substituting in (11) and integrating over mass we obtain

$$z_0 \approx \{G^2 \tau_0 (4\pi\rho_p/3)^2 2\pi (\tfrac{2}{3}\pi)^{1/2} / [[C_2\tau_0(1 - e^2) - \pi C_1(\tau_0 + 1/\tau_0)^{-1}] \times \ln (R_2/R_1)]\}^{1/4} (R_2/\Omega). \quad (13)$$

For a reasonably broad size distribution ($R_1 \sim 1$ cm, $R_2 \gtrsim 10$ m) and for $\rho_p \approx 1$ g cm^{-3}, $\tau_0 \approx 1$, $e \approx 0.75$, $C_1 \approx 0.2$ (Goldreich and Tremaine, 1978), and $C_2 \approx 1$, we find at a typical location in the rings that a full thickness $2z_0 \approx 10R_2$ will be attained by the small particles, where R_2 is the radius of the *largest* particle in the R^{-3}

size distribution. With this size distribution, then, the small particles constitute the major portion of the ring system by number and surface area, and produce all the observed optical and microwave phenomena, while being maintained in a layer of thickness many times the typical particle size by a numerically small but gravitationally dominant fraction of much larger particles. This result is in agreement with earlier theoretical work (Safronov, 1972) and recent numerical simulations (Brahic, 1977b) of physical collisions with high elasticity.

It has been suggested (P. Goldreich, private communication, 1978) that collective density fluctuations in a thin ring layer may also act, in some sense, as gravitational scatterers. Stability analysis (see Hunter, 1972) predicts instability to clumping in a Keplerian disk if $C\Omega/\pi G\sigma \leq 1$, where C = local sound speed of about $\langle V \rangle$, and σ is the surface mass density. For reasonable parameter values, clumping is therefore only expected in the outer A ring. In this region, the gravitational effects of such clumps could be nonnegligible, as estimated in the following simple way: We assume that the clumps have scale size on the order of the ring thickness (tens of meters), and the mass density of the rings ($\sim 10^{-2}$ or 10^{-3} g cm^{-3} for ice particles). Such clumps have a net mass comparable to the mass of solid particles with radii of a meter or so. Thus, acting merely as another sort of gravitational scatterer, such clumps may have effects comparable in the outer A ring to our hypothesized mechanism. The real situation is, in fact, sure to be a good deal more complex. However, a successful ring-thickening mechanism must be active throughout the entire ring system. Therefore, we will neglect the gravitational effects of such density fluctuations, while keeping in mind that they may produce some additional thickening in the outer parts of the A ring.

Several aspects of the above development merit further discussion. Goldreich and Tremaine (1978) did not include our scattering term in their energy balance, so the terms $C_2\tau_0(1 - e^2)$ and $C_1\pi(\tau_0 + 1/\tau_0)^{-1}$ become equal as the value of e (a weak function of velocity) adjusted to produce a steady state. Their thickness $z_0(\sigma)$ then depended on the unknown relation $e(\sigma)$ for the material constituting the ring particles. In our case, gravitational scattering is strongly dependent on σ and this dependence dominates for the low values of σ characterizing $R_2 \lesssim 100$ m. Therefore σ and z_0 may be obtained directly. The exact values for e and for other factors of order unity are not critical, because they appear within the bracket in (13) to a small power. Values of e for a variety of materials approach unity rapidly for collision velocities characterizing 10- to 1000-m thick rings (Goldsmith, 1960). Unfortunately, a value for water ice has not been measured.

Also, the assumption that $V_\infty \sim \sigma 3^{1/2} > \Delta V_c(b_{1a})$ proves to be justified. As shown in Section III, $V_c(b_{1a}) \sim b_{1a}V_c/2a$. Also recall $b_{1a} \sim 2GM/V_\infty^2$. Then

$$\begin{aligned}\Delta V_c/V_\infty &\approx (GM/a)(V_c/V_\infty^3)\\ &\approx (GM_s/a)(M/M_s)(V_c/V_\infty^3)\\ &\approx (M/M_s)(V_c/V_\infty)^3\\ &\approx (R_2/R_s)^3(V_c/V_\infty)^3.\end{aligned}$$

Using (13), we find $V_\infty \approx z_0\Omega \approx 5R_2\Omega \approx 5R_2V_c/a$. Substituting into the expression for $\Delta V_c/V_\infty$ we find $\Delta V_c/V_\infty \approx a^3/125R_s^3 \approx 1/16$ for $a \approx 2R_s$. Therefore, our earlier assumption that $V_\infty \gg \Delta V_c$ appears justified, and the particles approach each other more or less isotropically with a velocity of about V_∞. What this means is that gravitational scattering for the R^{-3} size distribution is both an extremely efficient thickening mechanism for rings initially "thinner" than a few times the largest particle radius, and a self-limiting mechanism due to the strong velocity dependence of the scattering cross section.

Observations may be used to obtain upper limits on R_2. The apparent edge-on thickness of 1–2 km implies that $2z_0 \approx 250$–500 m as shown in Section IIc. This then would require a maximum particle radius $R_2 \approx 25$–50 m. We may then calculate the mass in such a particle distribution to be 3–5 $\times 10^{-7}$ M_s if the particles are primarily water ice. Previous estimates of the ring mass are $\gtrsim 6 \times 10^{-6}$ M_s (Franklin *et al.*, 1971) and $\sim 6 \times 10^{-6}$ M_s (McLaughlin and Talbot, 1977), coming, respectively, from an analysis of the position of the Cassini division and from a fit to the orbital perturbations of the inner satellites. However, both estimates—and our own as well—are model dependent: The first accepts a simple resonance model for the formation of the Cassini division, while the second relies on a particular planetary interior model for Saturn.

c. The Energy Balance of the Rings

Energy considerations combined with the assumption that the rings are as old as the solar system lead to a fairly stringent constraint on ring thickness, and, consequently, maximum particle size. The random velocities imparted to small particles and eventually damped by collisions represent an energy that must be drawn from the orbital energy of the ring system. We may evaluate the energy dispersed and lost by collisions (E_{coll}) in rings of given full thickness $2z_0$ over the lifetime of the solar system as a fraction of the present gravitational energy E_G in the rings, based on our assumed R^{-3} size distribution. The available energy is

$$E_G = 2\pi^{3/2} z_0 \int_{a_{\text{in}}}^{a_{\text{out}}} \int_{M_1}^{M_2} (GM_s/a) \times Mn(M)dMada$$
$$= 2\pi^{1/2} GM_s(a_{\text{out}} - a_{\text{in}})\tau_0(\tfrac{4}{3}\pi\rho_p)R_2/ \times \ln (R_2/R_1). \quad (14)$$

The required energy is

$$E_{\text{coll}} = 2\pi^{3/2} z_0 (T_{ss}/T_f) \times \int_{a_{\text{in}}}^{a_{\text{out}}} \int_{M_1}^{M^*} \tfrac{1}{2} M \langle V^2 \rangle n(M) dMada, \quad (15)$$

where T_{ss} is the age of the rings, assumed to be equal to the age of the solar system. As T_f is the "flattening time" discussed in Section IIb, (T_{ss}/T_f) is the number of times the energy of dispersion (under the integral) must be supplied. When evaluating the integral in (15), we assume that $\langle V^2 \rangle$ is independent of mass. This is justified because Kepler shear maintains a $\langle V^2 \rangle$ for large particles which is roughly equal to the $\langle V^2 \rangle$ for small particles obtained from (11). Exchange of energy among particles of different mass is inefficient on a flattening time, and so equipartition, with $\langle V^2 \rangle \sim 1/M$, cannot be achieved. The upper limit (M^*) to the integration over mass is less than M_2 because the energy loss rate is a decreasing function of mass. This is due to the decreasing physical collision cross section per unit mass ($\pi R^2/M \propto R^{-1}$). In fact particles of mass greater than $\tfrac{1}{3}M_2$, constituting one-third of the total mass and energy, produce less than 10% of the total energy loss. Then, from (14) and (15),

$$E_{\text{coll}}/E_G \approx (T_{ss}/2\pi)\Omega\tau_0(1 - e^2) \times (R^*/R_2)(z_0/\bar{a})^2, \quad (16)$$

where $\bar{a}$ is the mean semimajor axis of the rings. Or, $z_0/\bar{a} \approx 10^{-6}(E_{\text{coll}}/E_G)^{1/2}$ for $e^2 \approx 0.5$ (a conservative value), $M^*/M_2 \approx \tfrac{1}{3}$, $\tau_0 \approx 1$, and $\Omega \approx 1.7 \times 10^{-4}$ sec^{-1}.

When particles collide, orbital energy is lost but the total angular momentum about Saturn is conserved. Thus, one particle moves out while the other moves in, with a net decrease in the sum of the orbital energies. Brahic (1977a) shows that the rings could have lost no more than 1% of their original ($\approx$present) energy E_G in spreading radially from a single central location to their present boundaries. For

$$E_{\mathrm{coll}}/E_{\mathrm{G}} \lesssim 0.01, \quad z_0/\bar{a} \lesssim 10^{-7},$$

which implies a thickness $2z_0 \lesssim 20$ m, consistent with the results of previous studies (Bobrov, 1970; Goldreich and Tremaine, 1978). For $e^2 \approx 0.8$ (a reasonable value), $2z_0 \lesssim 40$ m. In the present context this would imply that, in the absence of other energy sources, the largest particle size R_2 would have to be less than or about 2 to 4 m, as Brahic (1977a) also concludes.

Other energy sources are not difficult to suggest, but are difficult to evaluate. For instance, perturbation of ring particles by Mimas at the resonances which bound the rings will surely induce some high-velocity collisions, thereby imparting random energy to the system. The orbital energy of Mimas is comparable to the source E_{G} discussed above. One possible way of coupling the satellite orbital energy into the ring is through the satellite perturbations of the individual ring particles. We (Burns *et al.*, 1979) have argued elsewhere that satellite perturbations may be most significant for particles of large mass since their perturbed motion can develop most fully as they are less affected by interparticle collisions which damp the forced motion. These particles may then stir the rest of the system by their up and down motion.

Another energy source is the orbital decay of the large particles. Because the small particles have random "thermal" velocities about $\langle V^2 \rangle$, their mean circular velocity at a given radial distance from Saturn will be less than the orbital speed of a massive particle at the same distance by a fractional amount on the order of $\langle V^2 \rangle / V_c^2$. Therefore, the massive particle will feel a "drag" force due to a net momentum transfer in the backwards direction and its orbit will slowly decay. Of course, angular momentum would have to be conserved and the fate of the outwardly moving *small* particles is not clear. It appears, however, that this energy source is probably orders of magnitude smaller than those discussed above, and that the process proceeds very slowly.

Therefore, considerations of the obvious energy sources indicate that a full thickness of only 20 m $\lesssim 2z_0 \lesssim$ 50 m or so is maintainable over the age of the solar system. This implies that the maximum particle size characterizing the hypothetical R^{-3} size distribution is 2 m $\lesssim R_2 \lesssim$ 5 m. This would not be consistent with the existing inferences of apparent ring thickness (see, however, Burns *et al.*, 1979) from the observed brightness at ring crossing. It has, in fact, been suggested by Lumme and Irvine (1977) that the observations at ring crossing are best regarded as an upper limit. However, if many of the ring particles are smaller than a meter in radius, such a ring *would* be many particles thick, thus satisfying the conditions for the classical ring model.

IV. DISCUSSION AND CONCLUSIONS

Our hypothesis of a broad particle size distribution explains how the rings of Saturn may be many particles thick at the present time due to dispersion of the small particle component by the large particles. This reconciles dynamical arguments with observed optical behavior of the rings and in particular with the opposition effect. The R^{-3} size distribution we assume is consistent with microwave radar observations (Cuzzi and Pollack, 1978) and is theoretically expected for an ensemble of particles which, perhaps only very early in their history, although possibly even at present, undergo comminution and mutual fragmentation (Anders, 1965; Hartmann, 1969; Greenberg *et al.*, 1976). However, the apparent thickness of the rings inferred from brightness at ring crossing is not consistent with the thickness which can be supported by known energy sources over the age of the solar system. Therefore, one or more of the following must be true:

(a) the rings are significantly younger than the solar system; (b) there is another energy source which has as yet not been considered; or (c) the observations do not actually measure the top-to-bottom ring thickness but instead a warping of the ring plane and even then represent only an upper limit to the "true" ring thickness. Cases (a) and (b) accept that the observations measured a true thickness and imply a maximum particle radius of about 25–50 m. On the other hand, the requirement that known energy sources support the rings over the age of the solar system would imply a maximum particle size for the R^{-3} distribution of 2 m $\lesssim R_2 \lesssim$ 5 m or so, and thus would imply case (c).

The general result that the "scale height" of the rings will be a few times the maximum particle size, given a roughly R^{-3} size distribution, has other implications for the origin of the rings. Jeffreys (1947b), Harris (1975), and Greenberg *et al.* (1976) have discussed the maximum particle sizes expected to result from Roche or collisional breakup of a large satellite followed by mutual collisions. It is their conclusion that ice particles at least 1 km and perhaps tens of km in radius would survive and could in fact be enhanced relative to the R^{-3} distribution characterizing the smaller particles. Even if the largest bodies should be removed from the ring system by tidal evolution, particles several km in size would remain (Harris, 1975; Greenberg *et al.*, 1976). An apparent ring thickness of about 10 km would thus result from gravitational scattering, a value significantly larger than even the observational upper limit on the ring thickness. Of course, a few much larger particles could be present in the rings without appreciably affecting the average ring thickness. However, there must be far fewer of such particles than given by a continuation of the R^{-3} power law beyond the above values of R_2. This implies that such large particles are not the ultimate source of the centimeter-to-meter-sized particles, and would appear to be a serious criticism of the hypothesis that the rings originated through breakup of a large satellite.

APPENDIX A

Small-Angle Scattering

A "small-angle scattering" here means a two-body gravitational encounter which produces only a small deflection in the relative velocity. Ordinarily, in systems of gravitating masses, the cumulative effect of small-angle scatterings can be shown to dominate over large-angle scattering. Since this result is usually taken for granted, our use of the large-angle scattering cross section in Section IVb warrants some discussion. This Appendix demonstrates that use of the large-angle scattering cross section takes adequate account of both small- and large-angle scattering when the scatterers are confined to a thin disk.

First, it will be useful to review the standard case where small-angle scattering dominates. We will follow the approximate treatment described in Ogorodnikov (1965, Chap. IV). Let us consider the scattering of small-mass particles M_1 by large-mass particles M_2 where $M_1 \ll M_2$. Let us assume that the scatterers M_2 are essentially stationary but randomly distributed. In the standard case, the scatterers are envisioned as filling 3-space with uniform number density n out to some great distance L in all directions. When a small particle with velocity V encounters a scatterer at a large impact parameter b, its trajectory is approximately a straight line. The net impulse experienced by the small particle can then readily be shown to have a magnitude $2GM_1M_2/bV$. This exact result can be obtained crudely by multiplying the force at closest approach, GM_1M_2/b^2, by the time it takes M_1 to travel $2b$, namely $2b/V$. The impulse produces a velocity deflection $\delta V = 2GM_2/bV$. The cumulative effect of small-angle scatterings is equivalent to one large-angle scattering when the accumulated velocity dispersion $\Sigma(\delta V)^2$ is of order V^2. The rate of change of $\Sigma(\delta V)^2$ is

$$\frac{d}{dt}\Sigma(\delta V)^2 \approx nV\int_{b_{1a}}^{L}(\delta V)^2 2\pi b db$$
$$= 8\pi G^2 M_2{}^2 n \ln(L/b_{1a})/V, \quad \text{(A1)}$$

where b_{1a} is the impact parameter for large-angle scattering. The time t_{sa} for small-angle scatterings to fully deflect V is then

$$t_{sa} \equiv \left[\frac{1}{V^2}\frac{d}{dt}\Sigma(\delta V)^2\right]^{-1}$$
$$\approx \left[\frac{V^3}{8\pi G^2 M_2{}^2 n \ln(L/b_{1a})}\right]$$
$$\approx \left[\frac{t_{1a}}{2\ln(L/b_{1a})}\right] < t_{1a}, \quad \text{(A2)}$$

where t_{1a} is the time between large-angle scatterings as given by $t_{enc}(M)$ in Eqs. (7) and (8) of Section IVb when $n(M)$ is a δ-function at M_2. Since $\ln(L/b_{1a})$ is typically about 10 (see Ogorodnikov, 1965, p. 113), small-angle scattering dominates.

In our model for Saturn's rings, there is a continuous distribution of particle masses. For simplicity here, we use a bimodal distribution containing only the extreme masses M_1 and M_2 of Eq. (12). The large particles M_2 which do the scattering are in a monolayer. So we approximate the scatterer distribution by a disk of uniform number density n with half-thickness R_2. A small particle no longer follows a straight-line trajectory during small-angle scattering because its z-motion is an oscillation about the plane of the disk. The motion parallel to the disk is locally linear. The effect of the combined sinusoidal trajectory is to prolong the time an effective impulse is delivered during an encounter. Instead of $2b/V$, this time is roughly $2b/V\sin\alpha$, where α is defined to be the angle between V and the normal to the disk at a time when V_z is passing through its maximum. The impulse resulting from an encounter with large b is then $2GM_1M_2/bV\sin\alpha$, which gives a velocity deflection $\delta V = 2GM_2/bV\sin\alpha$. For the assumed distribution of scatterers,

$$\frac{d}{dt}\Sigma(\delta V)^2 \approx nV\sin\alpha\int_{\mathcal{L}}^{L} 2\cdot(\delta V)^2\cdot 2R_2 db$$
$$\approx \frac{16G^2M_2{}^2n}{V\sin\alpha}\frac{R_2}{\mathcal{L}}, \quad \text{(A3)}$$

where $\mathcal{L}$ is the minimum of b_{1a} and R_2, and L is the radial width of the rings ($\gg\mathcal{L}$). The choice of lower limit on the b-integral is justified as follows. If $b_{1a} > R_2$, the lower limit is b_{1a} for the same reason as in (A1), that is, by definition of small-angle scattering. If $b_{1a} < R_2$, the lowest impact parameter that leads to gravitational scattering rather than a physical collision is $\approx R_2$, and such scatterings will be small-angle. Equation (A3) is somewhat cruder than (A1) because the result is more sensitive to the integration limits. The scatterings which dominate (A3) are those for $b \gtrsim \mathcal{L}$, which should perhaps be called "intermediate"- rather than "small"-angle scatterings. The method for calculating the deflection is also cruder when b is only somewhat larger than b_{1a}. However, (A3) should be good to within about a factor of two. Averaging (A3) over α for an isotropic velocity distribution, we find

$$\frac{d}{dt}\Sigma(\delta V)^2 \approx \frac{8\pi G^2M_2{}^2n}{V}\frac{R_2}{\mathcal{L}}, \quad \text{(A4)}$$

and so

$$t_{sa} \approx (2\mathcal{L}/R_2)t_{1a} \gtrsim 2t_{1a}. \quad \text{(A5)}$$

The difference between this result and the standard case can be understood by comparing (A1) and (A3). For a monolayer of scatterers, there are far fewer encounters at large impact parameters, and so the total effect of small-angle scatterings is considerably lessened.

From (A5), we see that there are two cases for our multicomponent ring model. When $b_{1a} > R_2$, then $t_{sa} > t_{1a}$ and large-angle scatterings dominate. When $b_{1a} < R_2$, so that $\mathcal{L} = R_2$, then only small- or intermediate-angle gravitational scatterings can occur and (A5) tells us that $t_{sa} \approx 2t_{1a}$, where t_{1a} is computed as if large-angle scatterings could occur. In *either* regime then, the appropriate $t_{enc}(M)$ to use in Eqs. (6) and (7) is t_{1a} to within a factor of order unity. In view of the $\frac{1}{4}$ power to which this factor is raised in Eq. (13), we use $t_{enc}(M) = t_{1a}$ for all cases. In our final ring model, the small particles have a half-thickness $z_0 \approx 5R_2$ which corresponds to $\langle V^2 \rangle \approx (6R_2\Omega)^2$. Plugging this $\langle V^2 \rangle$ into (8), gives $b_{1a} \approx R_2/3$. If the collisions with large particles are inelastic, the dispersion will adjust itself such that $b_{1a} \gtrsim R_2$. This will occur after a reduction in z by a factor on the order of unity. We will not make this change, due to our complete lack of knowledge as to the surface nature of the large particles. So, in our final ring model, the gravitational scattering of small particles occurs via small- or intermediate-angle scattering, but the use of the large-angle scattering cross section is still appropriate.

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