Nonlocal transport coefficients in Planetary rings

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KINETIC EQUATIONS

Our systems consists of granular particles of various sizes and masses. If the system contains in total N particles, N_α of which are mechanically identical, we can write the number density of species α as

$$n_\alpha = \frac{N_\alpha}{N} \ , \tag{1}$$

where $\alpha = (1, 2, \ldots, s)$. In the limit $N \to \infty$ and $s \to \infty$, we get the size distribution function $n_\alpha \to n(\alpha)$ with an obvious property

$$\int_1^\infty n(\alpha)\, d\alpha = 1 \ . \tag{2}$$

In order to describe a statistical system, we need to introduce also the one particle distribution function $f_\alpha(\mathbf{r}_1, \mathbf{v}_1; t)$, which gives us the probability to find a particle of species α, at position $\mathbf{r}_1$ with velocity $\mathbf{v}_1$. The evolution of this function obeys the kinetic equation

$$\left(\partial_t + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} + m_\alpha^{-1} \mathbf{F}_\alpha(\mathbf{r}_1) \cdot \nabla_{\mathbf{v}_1}\right) f_\alpha(\mathbf{r}_1, \mathbf{v}_1; t) = C_\alpha(\mathbf{r}_1, \mathbf{v}_1; t) \ , \tag{3}$$

where

$$\begin{aligned} C_\alpha(\mathbf{r}_1, \mathbf{v}_1; t) = \sum_{\beta=1}^{s} \sigma_{\alpha\beta}^2 \int d\mathbf{v}_2 \int d\hat{\mathbf{n}} \Theta(\hat{\mathbf{n}} \cdot \mathbf{g}_{12})(\hat{\mathbf{n}} \cdot \mathbf{g}_{12}) \times \\ \times \left[\varepsilon^{-2} f_{\alpha\beta}(\mathbf{r}_1, \mathbf{v}_1'', \mathbf{r}_1 + \boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2''; t) - f_{\alpha\beta}(\mathbf{r}_1 \mathbf{v}_1, \mathbf{r}_1 + \boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t)\right] \ , \end{aligned} \tag{4}$$

where m_α is the mass of a particle of species α, and $\mathbf{F}_\alpha(\mathbf{r}_1)$ is the external force acting on this particle at a position $\mathbf{r}_1$. Here $\sigma_{\alpha\beta}$ is the sum of radii of particles α and β. At the moment of contact, we have $\mathbf{r}_2 = \mathbf{r}_1 + \boldsymbol{\sigma}_{\alpha\beta}$, where $\boldsymbol{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta}\hat{\mathbf{n}}$, and $\hat{\mathbf{n}}$ is the unit vector connecting centers of two particles, from particle 1 to 2. Here, $f_{\alpha\beta}(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2; t)$ is the two particle distribution function, and its relation to the one particle distribution function we address later. The collision integral $C_\alpha(\mathbf{r}_1, \mathbf{v}_1; t)$ has one important property: given a certain dynamic function of the velocity $\psi_\alpha(\mathbf{v})$, we can write

$$\begin{aligned} \int d\mathbf{v}_1 \psi_\alpha C_\alpha = \sum_{\beta=1}^{s} \sigma_{\alpha\beta}^2 \int d\mathbf{v}_1 \int d\mathbf{v}_2 \int d\hat{\mathbf{n}} \Theta(\hat{\mathbf{n}} \cdot \mathbf{g}_{12})(\hat{\mathbf{n}} \cdot \mathbf{g}_{12}) \times \\ \times \left[\psi_\alpha(\mathbf{v}_1') - \psi_\alpha(\mathbf{v}_1)\right] f_{\alpha\beta}(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_1 + \boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t) \ . \end{aligned} \tag{5}$$

Performing summation over all species, we write

$$\begin{aligned}\sum_{\alpha=1}^{s}\int d\mathbf{v}_1\psi_\alpha C_\alpha = \sum_{\alpha,\beta=1}^{s}\sigma_{\alpha\beta}^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\mathbf{n}}\Theta(\hat{\mathbf{n}}\cdot\mathbf{g}_{12})(\hat{\mathbf{n}}\cdot\mathbf{g}_{12})\times \\ \times\left[\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1)\right] f_{\alpha\beta}(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_1+\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_2;t) = \\ = \frac{1}{2}\sum_{\alpha,\beta=1}^{s}\sigma_{\alpha\beta}^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\mathbf{n}}\Theta(\hat{\mathbf{n}}\cdot\mathbf{g}_{12})(\hat{\mathbf{n}}\cdot\mathbf{g}_{12})\times \\ \times\left[\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1)+\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1)\right] f_{\alpha\beta}(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_1+\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_2;t)\ .\end{aligned} \tag{6}$$

Since exchanging of α to β and $\mathbf{v}_1$ to $\mathbf{v}_2$, does not change (6), we can write it in the next form

$$\begin{aligned}\sum_{\alpha=1}^{s}\int d\mathbf{v}_1\psi_\alpha C_\alpha = \frac{1}{2}\sum_{\alpha,\beta=1}^{s}\sigma_{\alpha\beta}^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\boldsymbol{\Omega}\times \\ \times\left([\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1)]f_{\alpha\beta}(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_1+\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_2;t)+\right. \\ \left.+[\psi_\beta(\mathbf{v}_2')-\psi_\beta(\mathbf{v}_2)]f_{\beta\alpha}(\mathbf{r}_1,\mathbf{v}_2,\mathbf{r}_1+\boldsymbol{\sigma}_{\beta\alpha},\mathbf{v}_1;t)\right)\ .\end{aligned} \tag{7}$$

Using the relations $f_{\alpha\beta}(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2;t) = f_{\beta\alpha}(\mathbf{r}_2,\mathbf{v}_2,\mathbf{r}_1,\mathbf{v}_1;t)$ and $\boldsymbol{\sigma}_{\alpha\beta} = -\boldsymbol{\sigma}_{\beta\alpha}$, we have

$$\begin{aligned}\sum_{\alpha=1}^{s}\int d\mathbf{v}_1\psi_\alpha C_\alpha = \frac{1}{2}\sum_{\alpha,\beta=1}^{s}\sigma_{\alpha\beta}^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\boldsymbol{\Omega}\times \\ \times\left([\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1)]f_{\alpha\beta}(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_1+\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_2;t)+\right. \\ \left.+[\psi_\beta(\mathbf{v}_2')-\psi_\beta(\mathbf{v}_2)]f_{\alpha\beta}(\mathbf{r}_1-\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_1,\mathbf{r}_1,\mathbf{v}_2;t)\right)\ .\end{aligned} \tag{8}$$

Here we used a short notation $d\boldsymbol{\Omega} = d\hat{\mathbf{n}}\Theta(\hat{\mathbf{n}}\cdot\mathbf{g}_{12})(\hat{\mathbf{n}}\cdot\mathbf{g}_{12})$. Let us also use the next shorthand notations: $f^+ = f_{\alpha\beta}(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_1+\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_1;t)$, $f^- = f_{\alpha\beta}(\mathbf{r}_1-\boldsymbol{\sigma}_{\alpha\beta},\mathbf{v}_1,\mathbf{r}_1,\mathbf{v}_2;t)$, and rewrite

$$\sum_{\alpha=1}^{s}\int d\mathbf{v}_1\psi_\alpha C_\alpha = \frac{1}{2}\sum_{\alpha,\beta=1}^{s}\sigma_{\alpha\beta}^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\boldsymbol{\Omega}\cdot I_{\alpha\beta}\ , \tag{9}$$

where

$$I_{\alpha\beta} = [\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1)]f^+ + [\psi_\beta(\mathbf{v}_2')-\psi_\beta(\mathbf{v}_2)]f^-\ . \tag{10}$$

After some algebra, we can write

$$\begin{aligned}I_{\alpha\beta} &= \psi_\alpha(\mathbf{v}_1')f^+ - \psi_\alpha(\mathbf{v}_1)f^+ + \psi_\beta(\mathbf{v}_2')f^- - \psi_\beta(\mathbf{v}_2)f^- + \\ &+ \psi_\beta(\mathbf{v}_2')f^+ - \psi_\beta(\mathbf{v}_2')f^+ + \psi_\beta(\mathbf{v}_2)f^+ - \psi_\beta(\mathbf{v}_2)f^+ = \\ &= [\psi_\alpha(\mathbf{v}_1')+\psi_\beta(\mathbf{v}_2')-\psi_\alpha(\mathbf{v}_1)-\psi_\beta(\mathbf{v}_2)]f^+ + [\psi_\beta(\mathbf{v}_2')-\psi_\beta(\mathbf{v}_2)](f^- - f^+) = \\ &= [\psi_\alpha(\mathbf{v}_1')+\psi_\beta(\mathbf{v}_2')-\psi_\alpha(\mathbf{v}_1)-\psi_\beta(\mathbf{v}_2)]f^+ + [\psi_\alpha(\mathbf{v}_1')-\psi_\alpha(\mathbf{v}_1))](f^+ - f^-)\ .\end{aligned} \tag{11}$$

Both, f^+ and f^- can be unified by introducing a scalar parameter x

$$f_{\alpha\beta}(x) = f_{\alpha\beta}(\mathbf{r}_1 - (1-x)\boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_1, \mathbf{r}_1 + x\boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t) \ , \tag{12}$$

where

$$f^+ = f_{\alpha\beta}(x=1) \ , \quad f^- = f_{\alpha\beta}(x=0) \ . \tag{13}$$

Now, the difference $f^+ - f^-$ can be written as

$$\begin{aligned} f_{\alpha\beta}(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_1 + \boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t) - f_{\alpha\beta}(\mathbf{r}_1 - \boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_1, \mathbf{r}_1, \mathbf{v}_2; t) &= \\ = \int_0^1 dx\, \frac{\partial}{\partial x} f_{\alpha\beta}(\mathbf{r}_1 - (1-x)\boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_1, \mathbf{r}_1 + x\boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t) \ , \end{aligned} \tag{14}$$

and finally we can rewrite (6) in the next form

$$\sum_{\alpha=1}^{s} \int d\mathbf{v}_1 \psi_\alpha C_\alpha = \frac{1}{2} \sum_{\alpha,\beta=1}^{s} \sigma_{\alpha\beta}^2 \int d\mathbf{v}_1 \int d\mathbf{v}_2 \int d\hat{\mathbf{n}} \Theta(\hat{\mathbf{n}} \cdot \mathbf{g}_{12})(\hat{\mathbf{n}} \cdot \mathbf{g}_{12}) \cdot I_{\alpha\beta} \ , \tag{15}$$

where

$$\begin{aligned} I_{\alpha\beta} = {} & [\psi_\alpha(\mathbf{v}_1') + \psi_\beta(\mathbf{v}_2') - \psi_\alpha(\mathbf{v}_1) - \psi_\beta(\mathbf{v}_2)]\, f_{\alpha\beta}(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_1 + \boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t) + \\ & + [\psi_\alpha(\mathbf{v}_1') - \psi_\alpha(\mathbf{v}_1)] \int_0^1 dx \frac{\partial}{\partial x} f_{\alpha\beta}(\mathbf{r}_1 - (1-x)\boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_1, \mathbf{r}_1 + x\boldsymbol{\sigma}_{\alpha\beta}, \mathbf{v}_2; t) \ . \end{aligned} \tag{16}$$

The first term corresponds to the total change of $\psi(\mathbf{v})$ during a binary collision at fixed locations of both particles. Hence, it represents the *local* changes. The second term, as we can see, represents the *nonlocal* transport of $\psi(\mathbf{v})$ over the distance $\boldsymbol{\sigma}_{\alpha\beta}$.

DISTRIBUTION FUNCTIONS

The two particle distribution function $f_{\alpha\beta}$ will be considered in the Enskog form

$$f_{\alpha\beta}(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2; t) = \chi(\sigma_{\alpha\beta}; n_\gamma) f_\alpha(\mathbf{r}_1, \mathbf{v}_1; t) f_\beta(\mathbf{r}_2, \mathbf{v}_2; t) \ , \tag{17}$$

where χ is the Enskog factor. We assume that at the kinetic scale of constituent sizes, the number density of species is constant, hence the Enskog factor is approximated as a constant as well. The position dependence of the one particle distribution function appears from the difference in the orbital speeds of colliding pairs of particles. In the Hill's problem, this orbital velocity is given as

$$\mathbf{u}(r) = -\frac{3}{2}\Omega r \cdot \mathbf{e}_y \ , \tag{18}$$

where r is the radial distance from the Hill's box origin, $\mathbf{e}_y$ is the base vector of the azimuthal direction. Now, the one particle distribution function can be written in the next form

$$f_\alpha(\mathbf{r}_1, \mathbf{v}_1; t) = n_\alpha \left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} \exp\left\{-\frac{m_\alpha}{2T_\alpha}\left(\mathbf{v}_1 - \mathbf{u}(r_1)\right)^2\right\} , \tag{19}$$

where T_α is the granular temperature of species α. If the radial position of the first particle is r, then the radial position of the second particle, which is in contact with the first particle at the moment of collision, is written

$$r_2 = r_1 + \boldsymbol{\sigma}_{\alpha\beta} \cdot \mathbf{e}_x = r_1 + \sigma_{\alpha\beta}(\hat{\mathbf{n}} \cdot \mathbf{e}_x) , \tag{20}$$

where $\mathbf{e}_x$ is the base vector of the radial direction. Now, introducing the parameter x, the orbital velocities of colliding pair of particles is written in the next form

$$\begin{aligned} \mathbf{u}(\mathbf{r}) &= -\frac{3}{2}\Omega r \mathbf{e}_y , \\ \mathbf{u}(\mathbf{r}_1 + x\boldsymbol{\sigma}_{\alpha\beta}) &= \mathbf{u}(\mathbf{r})\left(1 + x\frac{\sigma_{\alpha\beta}}{r}(\hat{\mathbf{n}} \cdot \mathbf{e}_x)\right) , \\ \mathbf{u}(\mathbf{r}_1 - (1-x)\boldsymbol{\sigma}_{\alpha\beta}) &= \mathbf{u}(\mathbf{r})\left(1 - (1-x)\frac{\sigma_{\alpha\beta}}{r}(\hat{\mathbf{n}} \cdot \mathbf{e}_x)\right) . \end{aligned} \tag{21}$$

Here, we assume r to be in the scale of the Hill's box, and $\sigma_{\alpha\beta}$ is in the scale of particle sizes, hence $\sigma_{\alpha\beta} \ll r$. Further, we will consider only the linear terms in the parameter $\sigma_{\alpha\beta}/r$. Before writing the two particle distribution function, let us change the velocity space from $\mathbf{v}_1, \mathbf{v}_2$ into $\mathbf{g}, \mathbf{v}_C$, where $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity and $\mathbf{v}_C$ is the center of mass velocity. The transformation reads

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_C + \frac{\mu}{m_\alpha}\mathbf{g} , \\ \mathbf{v}_2 &= \mathbf{v}_C - \frac{\mu}{m_\beta}\mathbf{g} . \end{aligned} \tag{22}$$

Since the Jacobian of this transformation is unity, we can directly replace $\mathbf{v}_1$ and $\mathbf{v}_2$ in distribution functions, and finally write

$$f_{\alpha\beta}(r, \hat{\mathbf{n}}, \mathbf{g}, \mathbf{v}_C) = \chi_{\alpha\beta} n_\alpha n_\beta \left(\frac{m_\alpha m_\beta}{4\pi^2 T_\alpha T_\beta}\right)^{3/2} \times \exp(-E_{\alpha\beta}(r, \hat{\mathbf{n}}, \mathbf{g}, \mathbf{v}_C)) , \tag{23}$$

where

$$\begin{aligned} E_{\alpha\beta}(r, \hat{\mathbf{n}}, \mathbf{g}, \mathbf{v}_C) = &\frac{m_\alpha}{2T_\alpha}\left(\mathbf{v}_C + \frac{\mu}{m_\alpha}\mathbf{g} - \mathbf{u}(r)\left(1 + x\frac{\sigma_{\alpha\beta}}{r}(\hat{\mathbf{n}} \cdot \mathbf{e}_x)\right)\right)^2 \\ &+ \frac{m_\beta}{2T_\beta}\left(\mathbf{v}_C - \frac{\mu}{m_\beta}\mathbf{g} - \mathbf{u}(r)\left(1 - (1-x)\frac{\sigma_{\alpha\beta}}{r}(\hat{\mathbf{n}} \cdot \mathbf{e}_x)\right)\right)^2 . \end{aligned} \tag{24}$$