Supporting Information

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Collisional integrals

The most important for our study are the collision integrals for the aggregative and fragmentative collisions. The former describes all possible collisions which change the concentration of particles of size k:

$$I_k^{\rm agg}(\vec{v}_k) = \frac{1}{2}\sum_{i+j=k}\sigma_{ij}^2\int d\vec{v}_i\int d\vec{v}_j\int d\vec{e}\,\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)|\vec{v}_{ij}\cdot\vec{e}|$$
$$\times f_i\left(\vec{v}_i\right) f_j\left(\vec{v}_j\right)\Theta\left(E_{\rm agg}-E_{ij}\right)\delta(m_k\vec{v}_k-m_i\vec{v}_i-m_j\vec{v}_j)$$
$$-\sum_j\sigma_{kj}^2\int d\vec{v}_j\int d\vec{e}\,\Theta\left(-\vec{v}_{kj}\cdot\vec{e}\right)|\vec{v}_{kj}\cdot\vec{e}|$$
$$\times f_k\left(\vec{v}_k\right) f_j\left(\vec{v}_j\right)\Theta\left(E_{\rm agg}-E_{kj}\right)\,. \qquad [1]$$

Here the first sum in the right-hand side of Eq. (1) refers to collisions where an aggregate of mass k is formed from smaller aggregates of masses i and j, while the second sum describes the collisions of k-aggregates with all other particles. In the first sum $m_k = m_i + m_j$ and $m_k\vec{v}_k = m_i\vec{v}_i + m_j\vec{v}_j$ due to mass and momentum conservation. The rest of the notation is standard, see e.g. [1]: $\sigma_{ij}^2 = r_1\left(i^{1/3}+j^{1/3}\right)^2$ quantifies the collision cross-section and $|\vec{v}_{ij}\cdot\vec{e}|$ is the length of the collision cylinder, where the unit vector $\vec{e}$ specifies the direction of the inter-center vector at the collision instant; $\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)$ selects only approaching particles. The factor $\Theta\left(E_{\rm agg}-E_{ij}\right)$ in the integrands guarantees that the relative kinetic energy does not exceed $E_{\rm agg}$ to cause the aggregation. Here the kinetic energy of the relative motion reads, $E_{ij} = \frac{1}{2}\mu_{ij}\vec{v}_{ij}^{\,2}$, with the relative velocity, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$ and the respective reduced mass, $\mu_{ij} = m_i m_j/(m_i+m_j)$. The notations in the second sum in the right-hand side of Eq. (1) have the similar meaning.

For the collisions leading to fragmentation we have

$$I_k^{\rm frag}(\vec{v}_k) = \frac{1}{2}\sum_{i,j\geq k+1}\sigma_{ij}^2\int d\vec{v}_j\int d\vec{v}_i\int d\vec{e}\,\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)$$
$$\times|\vec{v}_{ij}\cdot\vec{e}|\,f_j\left(\vec{v}_j\right) f_i\left(\vec{v}_i\right)\Theta\left(E_{ij}^{\rm n}-E_{\rm frag}\right)$$
$$\times\left(q_{ki}(\vec{v}_k,\,\vec{v}_i,\,\vec{v}_j)+q_{kj}(\vec{v}_k,\,\vec{v}_i,\,\vec{v}_j)\right)$$
$$+\sum_{i=1}^{k}\sum_{j\geq k+1}\sigma_{ij}^2\int d\vec{v}_j\int d\vec{v}_i\int d\vec{e}\,\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)$$
$$\times|\vec{v}_{ij}\cdot\vec{e}|\,f_j\left(\vec{v}_j\right) f_i\left(\vec{v}_i\right)\Theta\left(E_{ij}^{\rm n}-E_{\rm frag}\right)q_{kj}\left(\vec{v}_k,\,\vec{v}_i,\,\vec{v}_j\right))$$
$$-\sum_i\left(1-\delta_{k,1}\right)\sigma_{ki}^2\int d\vec{v}_i\int d\vec{e}\,\Theta\left(-\vec{v}_{ki}\cdot\vec{e}\right)|\vec{v}_{ki}\cdot\vec{e}|$$
$$\times f_k\left(\vec{v}_k\right) f_i\left(\vec{v}_i\right)\Theta\left(E_{ki}^{\rm n}-E_{\rm frag}\right) \qquad [2]$$

where we define the kinetic energy of the relative *normal* motion, $E_{ij}^{\rm n} = \frac{1}{2}\mu_{ij}(\vec{v}_{ij}\cdot\vec{e})^2$, with $(\vec{v}_{ij}\cdot\vec{e})$ being the normal relative velocity. Note that in contrast to the case of aggregation, where both normal and tangential components must be small, so that the total energy of the relative motion E_{ij} matters, for a fragmentation process only the relative normal motion is important: Only normal motion causes a compression and the subsequent breakage of particles' material. Hence the kinetic energy of the relative *normal* motion, $E_{ij}^{\rm n}$ must exceed some threshold. The first sum in Eq. (2) describes the collision of particles $i > k$ and $j > k$ with the relative kinetic energy of the normal motion above the fragmentation threshold $E_{\rm frag}$; both particles give rise to fragments of size k. $q_{ki}(\vec{v}_k,\,\vec{v}_i,\,\vec{v}_j)$ indicates the number of debris of mass $m_k = m_1\,k$ with the velocity $\vec{v}_k$, when a particle of mass $m_i = m_1\,i$ desintegrates at a collision with a particle of mass $m_j = m_1\,j$, provided that the according pre-collision velocities were $\vec{v}_i$ and $\vec{v}_j$. Obviously, $q_{ki} = 0$ if $k \geq i$. The function $q_{ki}(\vec{v}_k,\,\vec{v}_i,\,\vec{v}_j)$ depends on a particular collision model. The second sum describes the process, when only one particle with $j > k$ (but not with $i < k$) gives rise to debris of size k. Finally, the third term accounts for the breakage of particles of size $k \neq 1$ in collisions with all other particles.

In the present study only evolution of particle concentrations n_k is addressed, therefore: (i) the particular form of two other collision integrals $I_k^{\rm res}$ and $I_k^{\rm heat}$ is not important, since these terms do not change concentrations of the species and (ii) it is sufficient to exploit a more simple distribution, $x_k(i)$, defined as,

$$x_k(i) = \left(A_{ij}\right)^{-1}\sigma_{ij}^2\int d\vec{v}_k\int d\vec{v}_i\int d\vec{v}_j\int d\vec{e}\,\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)$$
$$\times|\vec{v}_{ij}\cdot\vec{e}|\,f_i\left(\vec{v}_i\right) f_j\left(\vec{v}_j\right)\Theta\left(E_{ij}^{\rm n}-E_{\rm frag}\right)q_{ki}(\vec{v}_k,\,\vec{v}_i,\,\vec{v}_j)$$
$$A_{ij} = \left(n_i\,n_j\right)^{-1}\sigma_{ij}^2\int d\vec{v}_i\,d\vec{v}_j\int d\vec{e}\,\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)|\vec{v}_{ij}\cdot\vec{e}|$$
$$\times\, f_i\left(\vec{v}_i\right) f_j\left(\vec{v}_j\right)\Theta\left(E_{ij}^{\rm n}-E_{\rm frag}\right)\,. \qquad [3]$$

$x_k(i)$ gives the *average* number of debris of size k of *any velocity*, obtained in fragmentation of particle of size i in all possible disruptive collisions. For simplicity we assume that this averaged distribution depends neither on the size of the colliding partner j, nor on its velocity $\vec{v}_j$. Also note, that while $I_k^{\rm res}$ describes the loss of energy in dissipative collisions, $I_k^{\rm heat}$ characterizes the energy input, so that the average kinetic energy of all species $\langle E_k\rangle$, $k = 1, 2, \ldots$ is kept in a steady-state.

Self-gravity wakes and averaged kinetic equations

Saturn's rings are not uniform but exhibit a large variety of structures [2]. One example are the self-gravity wakes

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[3, 4, 5, 6], arising from self-gravitational instability, forming a transient and fluctuating pattern in the surface mass density of the rings. These are canted relative to the azimuthal direction, with a typical length scale of about $L_w \sim 10^2$ m, one Toomre critical wavenlength [7]. Thus, to describe adequately particle kinetics one needs, in principle, to take into account effects of dense packing and non-homogeneity.

Two important comments are to be done in this respect. First, due to the low velocity dispersion of particles in the dense parts of the wakes, the collision duration is still significantly smaller than the time between particle collisions. This implies the validity of the assumption of binary collisions, as the dominant mechanism of particles' kinetics. Therefore a kinetic description in terms of the Enskog-Boltzmann equation is possible [8]. Although this Markovian equation ignores memory effects in particle kinetics, it may be still applicable, when the mean free path is comparable to, or even smaller then the particle size [9]. Second, the characteristic length scale of the density wakes, L_w, and the upper cut-off radius (less then 10m) are well separated. This allows to neglect variations of density and distribution functions on the latter length scale and use a local approximation for the distribution function of two particles at contact:

$$f_2(\vec{v}_k, \vec{r}-\vec{e}r_k, \vec{v}_l, \vec{r}+\vec{e}r_l, t) \simeq g_2(\sigma_{lk}) f_k(\vec{v}_k, \vec{r}, t) f_l(\vec{v}_l, \vec{r}, t) \quad [4]$$

where f_2 is the two-particle distribution function, for particles of radius r_k, and r_l, which have a contact at point $\vec{r}$, the unit vector $\vec{e}$ joins the centers of particles and $g_2(\sigma_{lk})$, with $\sigma_{lk} = r_l + r_k$, is the contact value of the pair distribution function, which may be well approximated by its equilibrium value for the hard sphere fluid; explicit expressions for $g_2(\sigma_{lk})$ may be found, e.g. in [1]. In the local approximation, Eq. (4), the collision integrals depend only on local values at a particular space point $\vec{r}$. This significantly simplifies the kinetic description of a high-density gas, since the density effects are taken into account in this approach by the multiplicative Enskog factor, $g_2(\sigma_{lk})$ only, leaving the structure of the collision integrals unchanged. Therefore the Enskog-Boltzmann equation valid for the case of wakes reads:

$$\frac{\partial}{\partial t} f_k(\vec{v}_k, \vec{r}, t) + \vec{v}_k \cdot \vec{\nabla} f_k(\vec{v}_k, \vec{r}, t) + \vec{F}_k(\vec{r}) \cdot \frac{\partial}{\partial \vec{v}_k} f_k(\vec{v}_k, \vec{r}, t) = I_k^{\rm agg}(\vec{r}) + I_k^{\rm res}(\vec{r}) + I_k^{\rm frag}(\vec{r}), \quad [5]$$

where $f_k(\vec{v}_k, \vec{r}, t)$ is the velocity distribution function of particles of size k (comprised of k-monomers), which depends on the space coordinate $\vec{r}$ and $\vec{F}_k(\vec{r})$ is the total gravitational force, acting on the particle of size k, which includes both the gravitational force from the central planet as well as self-gravitation of the ring particles. In what follows we do not need an explicit expression for this term. The collision integrals in the r.h.s. of Eq. (5) have the same form as in the previous case of a uniform system, with the only difference, that they depend on local parameters taken at a point $\vec{r}$, and that the collision cross-sections are re-normalized according to the rule,

$$\sigma_{ij}^2 \longrightarrow \sigma_{ij}^2 \, g_2(\sigma_{ij}) \quad [6]$$

which accounts for the high-density effects in the local approximation, see e.g. [1]. Note that we do not need here the term $I_k^{\rm heat}(\vec{r})$, which mimicked heating for the model of a uniform gas, since the Eq. (5) implicitly contains the spacial gradients and fluxes, responsible for the heating.

If we now integrate (as in the case of a uniform gas) the kinetic equation (5) over $\vec{v}_k$, we find the equations for the concentrations $n_k(\vec{r})$:

$$\frac{\partial}{\partial t} n_k(\vec{r}) + \vec{\nabla} \cdot \vec{j}_k(\vec{r}) = \frac{1}{2} \sum_{i+j=k} C_{ij}(\vec{r}) n_i(\vec{r}) n_j(\vec{r}) - n_k(\vec{r}) \sum_{i \geq 1} C_{ki}(\vec{r}) n_i(\vec{r}) - \sum_{i \geq 1} A_{ki}(\vec{r}) n_k(\vec{r}) n_i(\vec{r}) (1 - \delta_{k1}) + \sum_{i=1}^{k} n_i(\vec{r}) \sum_{j \geq k+1} A_{ij}(\vec{r}) n_j(\vec{r}) x_k(j) + \frac{1}{2} \sum_{i,j \geq k+1} A_{ij}(\vec{r}) n_i(\vec{r}) n_j(\vec{r}) \left(x_k(i) + x_k(j)\right) \quad [7]$$

where the partial flux $\vec{j}_k(\vec{r})$ is defined as

$$\vec{j}_k(\vec{r}) = \int \vec{v}_k \, f_k(\vec{v}_k, \vec{r}, t) \, d\vec{v}_k \, .$$

It describes the macroscopic (hydrodynamic flux), associated with particles of size k. The important feature of Eq. (7) is the spatial dependence of the kinetic coefficients A_{ij} and C_{ij}. Although the structure of these coefficients coincides with that of kinetic coefficients in the uniform system (apart from the trivial re-normalization, Eq. (6)), all quantities here are local. Naturally, the local velocity dispersion $\langle E_i \rangle(\vec{r})$ in the dense parts of the wakes significantly differs from that of the dilute regions inbetween.

Now we average Eqs. (7) over a suitable control volume V, which contains a large number of wakes. Applying then Green's theorem and taking into account that the surface integral of the flux $\vec{j}_k$ over the boundary of the control volume vanishes as $1/V$,

$$\int \vec{\nabla} \cdot \vec{j}_k \, d\vec{r} = \int_S \vec{j}_k \cdot d\vec{s} = 0 \, ,$$

we finally arrive at a set of equations with the space-averaged quantities:

$$\frac{d}{dt} \bar{n}_k = \frac{1}{2} \sum_{i+j=k} \bar{C}_{ij} \bar{n}_i \bar{n}_j - \bar{n}_k \sum_i \bar{C}_{ki} \bar{n}_i - \sum_i \bar{A}_{ki} \bar{n}_k \bar{n}_i (1 - \delta_{k1}) + \sum_{i=1}^{k} \bar{n}_i \sum_{j=k+1}^{\infty} \bar{A}_{ij} \bar{n}_j x_k(j) + \frac{1}{2} \sum_{i,j \geq k+1} \bar{A}_{ij} \bar{n}_i \bar{n}_j \left(x_k(i) + x_k(j)\right) \quad [8]$$

Here, by the definition,

$$\bar{n}_k = \frac{1}{V} \int n_k(\vec{r}) d\vec{r}.$$

and

$$\bar{C}_{ij} = \frac{1}{\bar{n}_i \bar{n}_j} \frac{1}{V} \int n_i(\vec{r}) n_j(\vec{r}) C_{ij}(\vec{r}) d\vec{r},$$

where $C_{ij}(\vec{r})$ are defined by Eqs. (3) with the local velocity dispersions $\langle E_i \rangle(\vec{r})$. Similar expression holds true for the coefficients $\bar{A}_{ij}$.

It is important to note that the coefficients $\bar{C}_{ij}$ and $\bar{A}_{ij}$ are density-weighted quantities. Therefore the contribution to the average value is proportional to the local density. This in turn implies that the values of these coefficients practically coincide with these for the dense part of the wakes,

$$\bar{C}_{ij} = C_{ij}^{(\text{dense part})} \qquad \bar{A}_{ij} = A_{ij}^{(\text{dense part})} \, .$$

Hence we conclude that the kinetic equations for the average concentrations of particles $\bar{n}_k$ coincide with the previously derived equations for n_k for the case of a uniform system. In what follows we will use n_k, C_{ij} and A_{ij} for the notation brevity, keeping although in mind that they correspond to the average values, that are almost equal to these values in the dense part of the wakes.

Maxwell approximation for the velocity distribution functions

Owing to the permanent aggregation and fragmentation we have a dynamically sustained mixture of particles of different mass. Mixtures of dissipative particles, generally, have different velocity dispersion, or mean kinetic energy ("granular temperature") of each species. The partial number density (concentration) n_i and the respective mean kinetic energy $\langle E_i\rangle$ of the species read, e.g. [10],

$$n_i = \int d\vec{v}_i f_i(\vec{v}_i)\,, \qquad n_i\langle E_i\rangle = \int \frac{m_i\vec{v}_i^{\,2}}{2} f(m_i,\vec{v}_i)d\vec{v}_i\,. \tag{9}$$

We assume that the distribution function $f_i(\vec{v}_i,t)$ may be written as [11, 12, 10]

$$f_i(\vec{v}_i,t) = \frac{n_i(t)}{v_{0,\,i}^3(t)}\phi_M(\vec{c}_i)\,, \qquad \vec{c}_i \equiv \frac{\vec{v}_i}{v_{0,\,i}}\,, \tag{10}$$

where $v_{0,\,i}^2(t) = 2\langle E_i\rangle(t)/m_i$ is the thermal velocity and $\phi(c_i)$ the reduced distribution function. For force-free granular mixtures [13] and interacting particles (which suffer ballistic annihilation) [14] the reduced distribution function is represented in the form of the Sonine polynomial expansion,

$$\phi_i(\vec{c}) = \phi_M(\vec{c})\left[1+\sum_{a=1}^{\infty} a_k S_k(c^2)\right]$$

where $\phi_M(\vec{c})$ is the Maxwellian distribution function,

$$\phi_M(\vec{c}) = \pi^{-3/2}\exp(-c^2)\,, \tag{11}$$

$S_k(c^2)$ are Sonine the polynomials and a_k the respective Sonine coefficients. It has been shown, however, that these coefficients are rather small in both cases, of dissipative collisions [13], as well as for the case of reacting particles [14]. Therefore, with a high accuracy we can use the Maxvellian distribution function (11) in all further calculations. Integration of Eqs. (5) over $\vec{v}_k$ with the use of the Maxwellian distribution functions (10) is rather straightforward, since all the arising integrals are Gaussian.

Estimates of the aggregation energy

The threshold energy of aggregation for two particles of radii R_i and R_j has the following form [15]:

$$E_{\rm agg} = q_0\left(\pi^5\gamma^5 R_{\rm eff}^4 D^2\right)^{1/3}\,. \tag{12}$$

Here $q_0 = 1.457$ is a constant, γ is the adhesion coefficient, $D = (3/2)(1-\nu^2)/Y$, where Y and ν are respectively the Young modulus and Poisson ratio and $R_{\rm eff} = R_iR_j/(R_i+R_j)$.

The corresponding coefficients B_{ij} read for the case of equal velocity dispersion $\mathbf{v}_0^2$ and equal radii of particles,

$$B_{ij} = \frac{3}{2}\frac{m_i+m_j}{\left(\frac{1}{2}m_im_j\mathbf{v}_0^2+\frac{1}{2}m_im_j\mathbf{v}_0^2\right)} = \frac{9}{4\pi\rho\phi\mathbf{v}_0^2}R^{-3}\,, \tag{13}$$

where $m_i = (4\pi/3)\rho\phi R^3$, ρ is the material density of ice and $\phi = 0.3$ is the approximate packing fraction of aggregates. Let us estimate the average value of $B_{ij}E_{\rm agg}$, which we define as,

$$\langle B_{ij}E_{\rm agg}(R)\rangle = \frac{\int_{r_1}^{R_c} B_{ij}E_{\rm agg}(R)F(R)dR}{\int_{r_1}^{R_c}F(R)dR}\,,$$

Here $F(R)\simeq R^{-q}$ is the radii distribution function, which behaves as a power-law with $q=3$ for $R<R_c$, where $R_c\gg r_1$ is the cutoff radius for this distribution. Performing the averaging, we obtain,

$$\langle B_{ij}E_{\rm agg}(R)\rangle = \frac{9\pi^{2/3}\gamma^{5/3}D^{2/3}\,(q-1)\,q_0}{8\cdot 2^{1/3}\rho\phi\mathbf{v}_0^2\,(q+2/3)}r_1^{-5/3}\,.$$

Using material parameters for ice, $Y = 7\times10^9$ Pa, $\nu = 0.25$, $\gamma = 0.74\,{\rm N/m}$ [16], $\rho = 900\,{\rm kg/m^3}$, the primary grain size $r_1 = 1\,{\rm cm}$ and the velocity dispersion $(\mathbf{v}_0^2)^{1/2} = 0.01\,{\rm cm/s}$, [17] we obtain, $\langle B_{ij}E_{\rm agg}(R)\rangle = 251 \gg 1$. If we use $(\mathbf{v}_0^2)^{1/2} = 0.1\,{\rm cm/s}$, which corresponds the maximal value for the dispersion velocity diapason [17], we still have $\langle B_{ij}E_{\rm agg}(R)\rangle = 2.51 > 1$. Hence, we conclude that for the most of collisions in the rings, the condition $\langle B_{ij}E_{\rm agg}(R)\rangle \gg 1$ is fulfilled. One should note, however, that recent data for the adhesion coefficient for ice suggests smaller values of γ, namely $\gamma = 0.19\,N/m$ [18]; in this case the condition $\langle B_{ij}E_{\rm agg}(R)\rangle \gg 1$ would be fulfilled only for the smaller part of the velocity dispersion diapason $(\mathbf{v}_0^2)^{1/2}\sim 0.01\,{\rm cm/s}$.

Universality of particles size distribution for steep distribution of debris

As it has been already mentioned in the main text, distribution of debris in a collision obeys in its main part a power-law. That is, if an aggregate of size i suffers a disruption in an impact a plenty of fragments of size $k<i$ appear. Let $x_k(i)$ denote the number of fragments of size k; the power-law fragment distribution implies that $x_k(i)\sim k^{-\alpha}$ in the main part of the distribution. This allows to quantify the pre-factor of the distribution, $x_k(i) = B(i)k^{-\alpha}$ from the normalization condition, that is, from the condition that the total mass of all debris is equal to the mass of the parent body. Although we have a discrete mass spectrum of debris, $m_k = m_1k$, $k = 1,2,\ldots$, for $i\gg1$ one can approximate summation by integration to obtain,

$$i \simeq \int_1^{(i-1)} Bk^{-\alpha}kdk = \begin{cases} \beta\frac{B}{(2-\alpha)}\left[(i-1)^{2-\alpha}-1\right] & \text{if } \alpha\neq 2 \\ \beta_1 B\log(i-1) & \text{if } \alpha = 2, \end{cases} \tag{14}$$

where the factors β and β_1 stand for an approximate correction when the summation is approximated by the integration (see the next section for more detail). This yields for $i\gg1$

$$B(i)\sim\begin{cases} i^{\alpha-1} & \text{if } \alpha<2 \\ i(\log i)^{-1} & \text{if } \alpha = 2 \\ i & \text{if } \alpha>2\,. \end{cases} \tag{15}$$

Now we perform analysis of the general system of equations (2) in the main text to show that under certain conditions the solution to Eqs. (2) (fragmentation with a particular debris-size distribution) coincides with the solution to Eqs. (5)–(6) (complete fragmentation into monomers). First, we notice that if $x_k(i)\sim i/k^\alpha$, which holds true for $\alpha>2$, the equations for monomers are identical (up to a factor at the coefficients A_{ij}) for the both models. Next, we write Eqs. (2) as

$$\frac{dn_k}{dt} = K_1 - K_2 - K_3 + K_4 + K_5\,, \tag{16}$$

where

$$K_1 = \frac{1}{2}\sum_{i+j=k} C_{ij}n_in_j$$

$$K_2 = \sum_{i=1}^{\infty} C_{ik} n_i n_k$$

$$K_3 = \sum_{i=1}^{\infty} A_{ki} n_i n_k \left(1 - \delta_{k1}\right)$$

$$K_4 = \sum_{i=1}^{k} \sum_{j=k+1}^{\infty} A_{ij} n_i n_j x_k(j)$$

$$K_5 = \frac{1}{2} \sum_{i,j \geq k+1} A_{ij} n_i n_j \left[x_k(i) + x_k(j)\right] .$$

In these notations Eqs. (5) for the case of decomposition into monomers read,

$$\frac{dn_k}{dt} = K_1 - K_2 - K_3 \,, \qquad [17]$$

that is, the two models differ by the two terms K_4 and K_5 only. Now we estimate the relative magnitude of the terms $(K_1 - K_2 - K_3)$ and K_4 and K_5 using the scaling approach. We assume that under certain conditions, the distribution of aggregate concentrations in a steady state (when $\dot{n}_k = 0$) has the form,

$$n_k \sim k^{-\gamma} e^{-ak} \,,$$

the same as for the case of decomposition into monomers. We perform the analysis for the simplified kernels $C_{ij} = (ij)^{\mu}$ and $A_{ij} = \lambda C_{ij}$ for the scaling regime, $k \gg 1$ and focus on the power-law domain, where $ka \ll 1$. Additionally, we assume that $1 < \gamma - \mu < 2$ and check all the assumptions a'posteriori. Approximating again the summation by the integration we obtain for the first term K_1:

$$K_1 \sim \sum_{i+j=k} (ij)^{\mu-\gamma} e^{-ak} \sim \int_1^{k-1} i^{\mu-\gamma} (k-i)^{\mu-\gamma} di$$

$$\sim k^{\mu-\gamma} \int_1^{k/2} i^{\mu-\gamma} \left(1 - \frac{i}{k}\right)^{\mu-\gamma} di$$

$$\sim k^{\mu-\gamma} \left[i^{\mu-\gamma+1} - b\, \frac{i^{\mu-\gamma+2}}{k} \ldots \right]_1^{k/2} \sim k^{\mu-\gamma}, \qquad [18]$$

where $b = (\mu-\gamma)(\mu-\gamma+1)(\mu-\gamma+2)^{-1}$. Here we take into account the symmetry of the integrand around $k/2$, make an expansion of the factor $(1-i/k)^{\mu-\gamma}$ and keep only the leading term in the obtained series. Now we evaluate the second and the third terms:

$$K_2 + K_3 \sim (1+\lambda) k^{\mu-\gamma} e^{-ak} \int_1^{\infty} i^{\mu-\gamma} e^{-ai} di \sim k^{\mu-\gamma}. \quad [19]$$

Similarly, we find for the fourth term:

$$K_4 \sim \lambda \sum_{i=1}^{k} i^{-\gamma} e^{-ai} \sum_{j=k+1}^{\infty} (ij)^{\mu} j^{-\gamma} e^{-aj} B(j) k^{-\alpha}$$

$$\sim k^{-\alpha} \int_1^{k} i^{\mu-\gamma} e^{-ai} di \int_k^{\infty} j^{\mu-\gamma} e^{-aj} B(j) dj.$$

Using $B(j)$ from Eq. (14), it is straightforward to show, that the forth term scales as

$$K_4 \sim \begin{cases} k^{-\alpha} & \text{if } \alpha > \gamma - \mu \,,\ \alpha \neq 2 \\ k^{-\alpha} \left(\log k\right)^{-1} & \text{if } \alpha = 2 \\ k^{-\alpha} \left|\log ka\right| & \text{if } \alpha = \gamma - \mu \\ k^{\mu-\gamma} & \text{if } \alpha < \gamma - \mu \end{cases} \qquad [20]$$

Analogously, the fifth term may be estimated as

$$K_5 \sim \sum_{i,j \geq k} (ij)^{\mu} (ij)^{-\gamma} e^{-a(i+j)} \left[B(i) + B(j)\right] k^{-\alpha}$$

$$\sim k^{-\alpha} \int_k^{\infty} dii^{\mu-\gamma} e^{-ai} \int_k^{\infty} j^{\mu-\gamma} e^{-aj} B(j) dj.$$

Finally, we get:

$$K_5 \sim \begin{cases} k^{\mu+1-\alpha-\gamma} & \text{if } \alpha > \gamma - \mu \,,\ \alpha \neq 2 \\ k^{\mu+1-\alpha-\gamma} \left(\log k\right)^{-1} & \text{if } \alpha = 2 \\ k^{\mu+1-\alpha-\gamma} \left|\log ka\right| & \text{if } \alpha = \gamma - \mu \\ k^{1-2(\gamma-\mu)} & \text{if } \alpha < \gamma - \mu \end{cases} \qquad [21]$$

Comparing Eqs. (19) and (18) with Eqs. (20) and (21) we conclude that the forth and fifth terms of the basic Eq. (2) are negligibly small for $k \gg 1$ as compared with the first, second and third terms of this equation, provided $\alpha > (\gamma - \mu)$ under the condition $1 < \gamma - \mu < 2$. If we additionally take into account that for $\alpha > 2$ the equations for the monomer concentration coincide for the two models, we conclude that if $\alpha > \max\{\gamma - \mu, 2\}$, the steady-state size distribution of aggregates for the case of complete decomposition into monomers and for the power-law decomposition would be the same in the domain $k \gg 1$ and $ka \ll 1$. Since for the case of monomer decomposition $\gamma = 3/2 + \mu$ and $a = \lambda^2$, the condition $1 < \gamma - \mu < 2$ holds true, and $ka \ll 1$ is fulfilled for $k \gg 1$, if $\lambda \ll 1$. Hence it is expected that for a steep size distribution of debris with $\alpha > \alpha_0 = 2$ the steady-state distribution

$$n_k \sim k^{3/2+\mu} \qquad [22]$$

is *universal*, that is, it does not depend on the particular value of α.

Moreover, the same conclusion of the universality of the distribution (22) holds true for any functional form of a steep distribution of debris size. If we write it as $x_k(i) = B(i)\phi(k)$, where the function $\phi(k)$ is steep enough, so that

$$\int_1^{i-1} \phi(k) k dk \simeq \int_1^{\infty} \phi(k) k dk = C^{-1},$$

the pre-factor $B(i)$ reads, $B(i) = Ci$. Then for any function $\phi(k)$ satisfying the condition, $\phi(k) \ll k^{-3/2}$ for $k \gg 1$ the resultant distribution of aggregates will have the universal form (22). This has been confirmed numerically for the exponential distribution of debris, see Fig. 2 in the main text.

Efficient numerical solution of the kinetic equations

We use the Euler's method for the numerical solution of the kinetic equations. This method is rather suitable for the case of interest, since we search stationary, continuous and smooth solutions. The first problem in the numerical analysis of the infinite number of rate equations is the conservation of mass. Indeed, in any real simulation one can handle only a *finite* number of equations say N, which describe evolution of particles of size 1, 2, ... N (a particle of size k has mass $m_k = m_1 k$). These equations have both aggregation and fragmentation terms. In particular, they have a term which describes aggregation of particles of size $i < N$ and $j < N$, resulting in an aggregate of size $i + j > N$. Since the system of N equations does not account for particles larger that N, such processes would effectively lead to the leak of particles' mass and violation of the mass conservation. To preserve the mass conservation we assume that all collisions of particles of mass i and j are *fragmentative* if $i + j \geq N$. We have checked

that this assumption does not lead to any noticeable distortion of the numerical solution of the rate equations n_k, if k is smaller than some fraction of N.

Another problem is to handle efficiently a large number of equation, say up to $\sim 10^6$. One possible way is an application of the coarse-graining, that is, grouping concentrations $n_k - n_{k+l}$ into coarse-grained variables $\tilde{n}_K$ with increasing l as k grows. In the case of interest, however, we have a drastic variation of the functional dependence of $n_k(k)$, which changes from a power-law to the exponential decay. This hinders an effective application of the coarse-graining and we need to keep explicitly all individual concentrations. Hence we have to work with a large number of equations, which is computationally costly. To speed up the computations we have developed a recursive procedure.

In the case of fragmentation into monomers the system of kinetic equations has the following form:

$$\begin{aligned}
&\frac{dn_1}{dt} = -n_1 \sum_{j=1}^{N} n_j + \lambda(1-n_1) \sum_{j=1}^{N} n_j \\
&... \\
&\frac{dn_k}{dt} = \frac{1}{2} \sum_{i+j=k} C_{i,j} n_i n_j - (1+\lambda) \sum_{i=1}^{N} C_{i,k} n_i n_k \\
&\frac{dn_{k+1}}{dt} = \frac{1}{2} \sum_{i+j=k+1} C_{i,j} n_i n_j - (1+\lambda) \sum_{i=1}^{N} C_{i,k+1} n_i n_{k+1} \\
&...
\end{aligned} \qquad [23]$$

Taking into account, that we search for a stationary solution, $dn_{k+1}/dt = 0$, we obtain for the number density n_{k+1}:

$$\frac{1}{2(1+\lambda)} \sum_{i+j=k+1} C_{i,j} n_i n_j - \sum_{i=1}^{N} C_{i,k+1} n_i n_{k+1} = 0\,. \qquad [24]$$

The first sum in Eq. (24) contains only n_i with $i \leq k$, while we write the second sum as

$$\begin{aligned}
\sum_{i=1}^{N} C_{i,k+1} n_i n_{k+1} &= n_{k+1} \sum_{i=1}^{k} C_{i,k+1} n_i + C_{k+1,k+1} n_{k+1}^2 \\
&+ n_{k+1} \sum_{i=k+1}^{N} C_{i,k+1} n_i\,.
\end{aligned} \qquad [25]$$

Now we use the properties of the kinetic kernel C_{ij} and the steady-state distribution $n_k = n_k(m_k)$, which we assume to be decreasing function of k. Namely, we assume that the coefficients C_{ij} increase with i and j at a smaller rate than the rate at which n_k decreases with k. That is, we assume that for $k \gg 1$ the following condition holds true:

$$\sum_{i=1}^{k} C_{i,k+1} n_i >> \sum_{i=k+1}^{N} C_{i,k+1} n_i\,. \qquad [26]$$

This allows to neglect the last sum in Eq. (25) and obtain the quadratic equation for n_{k+1}:

$$C_{k+1,k+1} n_{k+1}^2 + n_{k+1} \sum_{i=1}^{k} C_{i,k+1} n_i - \sum_{i+j=k+1} \frac{C_{i,j} n_i n_j}{2(1+\lambda)} = 0\,. \qquad [27]$$

Solving the above equation and choosing the positive root, we arrive at the recurrent relation for the concentrations n_k:

$$\begin{aligned}
n_{k+1} &= \frac{\sqrt{D} - \sum_{i=1}^{k} C_{i,k+1} n_i}{2C_{k+1,k+1}} \qquad [28] \\
D &= \frac{2C_{k+1,k+1}}{(1+\lambda)} \sum_{i+j=k+1} C_{i,j} n_i n_j + \left(\sum_{i=1}^{k} C_{i,k+1} n_i \right)^2 .
\end{aligned}$$

Using the recurrent relation (28) one can significantly accelerate computations. This may be done as follows: First, one solves explicitly the system of rate equations (24) for $k \ll N$, choosing the value of k to fulfil the condition (26). Then the concentrations n_i with $k < i \leq N$ may be straightforwardly obtained from the recurrence (28). Performing numerical solution of the rate equations with different kernels directly, and with the use of the recurrence (28), we proved the efficiency and accuracy of the above accelerating approach.

Numerical calculation of the distribution of fragments

In the numerical solution we calculate $x_k\,(i)$ using the mass conservation and taking into account the discreteness of the distribution of particles:

$$k x_k\,(i) = B\,(i) \int_{k-1/2}^{k+1/2} k_1 k_1^{-\alpha} dk_1 \qquad [29]$$

Computing the integral, we find:

if $\alpha \neq 2$:

$$x_k\,(i) = \frac{B\,(i)}{k} \frac{1}{2-\alpha} \left[(k+1/2)^{2-\alpha} - (k-1/2)^{2-\alpha} \right] \qquad [30]$$

if $\alpha = 2$:

$$x_k\,(i) = \frac{B\,(i)}{k} \left[\ln(k+1/2) - \ln(k-1/2) \right]\,. \qquad [31]$$

Note, that $x_k\,(i) \to B\,(i)\, k^{-\alpha}$ for $k >> 1$, when the discreteness of the system becomes insignificant.

Here $B\,(i)$ represents a normalization constant, which can be computed from:

$$B\,(i) \int_{1/2}^{i-1/2} k k^{-\alpha} dk = i\,. \qquad [32]$$

Thus we obtain:

if $\alpha \neq 2$:

$$B\,(i) = \frac{i\,(2-\alpha)}{(i-1/2)^{2-\alpha} - (1/2)^{2-\alpha}}\,, \qquad [33]$$

so that $B(i) = 2^{2-\alpha}(\alpha-2)i$ for $i \gg 1$;

if $\alpha = 2$:

$$B\,(i) = \frac{i}{\ln(i-1/2) - \ln(1/2)}\,. \qquad [34]$$

From the last equations we see that the correction factor β, introduced above reads, e.g. for the case of $\alpha \neq 2$, $\beta = 2^{2-\alpha}$. In the case of exponential distribution we get analogously:

$$\begin{aligned}
x_k(i) &= \frac{B\,(i)}{k} \int_{k-1/2}^{k+1/2} k_1 \exp(-k_1)\, dk_1 \qquad [35] \\
&= \frac{B\,(i)}{k} \left[(k+1/2) e^{1/2-k} - (k+3/2) e^{-1/2-k} \right] .
\end{aligned}$$

Here $B(i) = i/I_0$, with

$$I_0(i) \quad = \quad \int_{1/2}^{i-1/2} ke^{-k}dk = (3/2)e^{-1/2} - (i+1/2)e^{1/2-i}.$$

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