ANGULAR INTEGRALS

First, we consider integrals of the type

$$I_{\boldsymbol{e}}^{k,m,p}(\boldsymbol{v},\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m(\boldsymbol{u}\cdot\boldsymbol{e})^p\ , \tag{1}$$

where $\boldsymbol{e}$ is a unit vector, and $\boldsymbol{v},\boldsymbol{u}$ are free vectors. $\Theta(x)$ is the Heaviside step function. The system is three-dimensional, and $k,m,p\geqslant 0$ integers.

The case with $p=0$ and $m=0$.

If $p=m=0$, we have

$$I_{\boldsymbol{e}}^{k,0,0}(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k\ , \tag{2}$$

we can fix $\boldsymbol{v}$, and rotate $\boldsymbol{e}$ around it. In spherical coordinates, we denote the angle between $\boldsymbol{v}$ and $\boldsymbol{e}$ as $0\leqslant\theta\leqslant\pi$. Hence, we can write $\mathrm{d}\boldsymbol{e}=\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, where ϕ is the polar angle. Since $\boldsymbol{v}\cdot\boldsymbol{e}=v\cos\theta$, we have

$$\begin{aligned}
I_{\boldsymbol{e}}^{k,0,0}(\boldsymbol{v}) &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\ \sin\theta\,\Theta(-v\cos\theta)v^k|\cos\theta|^k = \\
&= 2\pi v^k\int_0^{\pi}\mathrm{d}\theta\ \sin\theta\,\Theta(-\cos\theta)|\cos\theta|^k = \\
&= 2\pi v^k\int_{\pi/2}^{\pi}\mathrm{d}\theta\ \sin\theta|\cos\theta|^k = -2\pi v^k\int_{\pi/2}^{\pi}\mathrm{d}(\cos\theta)\ |\cos\theta|^k\ .
\end{aligned} \tag{3}$$

Now, if we check both odd and even cases of k, we arrive at the result

$$I_{\boldsymbol{e}}^{k,0,0}(\boldsymbol{v}) = \frac{2\pi v^k}{k+1}\ . \tag{4}$$

The case with only $p=0$.

If only $p=0$, then we have

$$I_{\boldsymbol{e}}^{k,m,0}(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m\ . \tag{5}$$

Following the same procedure as before, we have

$$I_{\boldsymbol{e}}^{k,m,0}(\boldsymbol{v}) = -2\pi v^{k+m}\int_{\pi/2}^{\pi}\mathrm{d}(\cos\theta)\,|\cos\theta|^k(\cos\theta)^m\ . \tag{6}$$

Since in the region from $\pi/2$ to π, the cosine function is negative, we can rewrite its power as $(\cos\theta)^m = (-1)^m \cdot |\cos\theta|^m$, which gives us

$$I_{\boldsymbol{e}}^{k,m,0}(\boldsymbol{v}) = -2\pi v^{k+m} \cdot (-1)^m \int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\, |\cos\theta|^{k+m} = (-1)^m \cdot I_{\boldsymbol{e}}^{k+m,0,0}(\boldsymbol{v}) \ , \tag{7}$$

or using (4) we write

$$I_{\boldsymbol{e}}^{k,m,0}(\boldsymbol{v}) = (-1)^m \cdot \frac{2\pi v^{k+m}}{k+m+1} \ . \tag{8}$$

The case with $p = 1$.

In the case of $p = 1$, our integral depends on two free vectors, and we cannot simply integrate through the azimuthal angle:

$$I_{\boldsymbol{e}}^{k,m,1}(\boldsymbol{v},\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m(\boldsymbol{u}\cdot\boldsymbol{e}) \ . \tag{9}$$

Let us rewrite this integral in the next form:

$$I_{\boldsymbol{e}}^{k,m,1}(\boldsymbol{v},\boldsymbol{u}) = \boldsymbol{u}\cdot\int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m\,\boldsymbol{e} = \boldsymbol{u}\cdot\boldsymbol{V}^{k,m} \ , \tag{10}$$

where

$$\boldsymbol{V}^{k,m} = \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m\,\boldsymbol{e} \ , \tag{11}$$

is a vector, and since it is being integrated over all possible values of $\boldsymbol{e}$, the only possible direction for vector $\boldsymbol{V}^{k,m}$ is to be along $\boldsymbol{v}$. Hence, we can write

$$\boldsymbol{V}^{k,m} = \boldsymbol{v}\cdot I_{\boldsymbol{v}}^{k,m} \ , \tag{12}$$

where the scalar $I_{\boldsymbol{v}}^{k,m}$ is of an integral form and needs to be calculated. Taking a dot product by $\boldsymbol{v}$ from both sides, we have

$$v^2\cdot I_{\boldsymbol{v}}^{k,m} = \boldsymbol{v}\cdot\boldsymbol{V}^{k,m} = \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^{m+1} = I_{\boldsymbol{e}}^{k,m+1,0}(\boldsymbol{v}) \ , \tag{13}$$

and using (8), we obtain

$$v^2\cdot I_{\boldsymbol{v}}^{k,m} = (-1)^{m+1}\cdot\frac{2\pi v^{k+m+1}}{k+m+2} \ , \tag{14}$$

or

$$I_{\boldsymbol{v}}^{k,m} = (-1)^{m+1}\cdot\frac{2\pi v^{k+m-1}}{k+m+2} \ . \tag{15}$$

Now, the vector integral $\boldsymbol{V}^{k,m}$ is written as

$$\boldsymbol{V}^{k,m} = (-1)^{m+1} \cdot \frac{2\pi v^{k+m-1}}{k+m+2}\boldsymbol{v} \ , \tag{16}$$

and finally, the angular integral is found to be

$$I_{\boldsymbol{e}}^{k,m,1}(\boldsymbol{v},\boldsymbol{u}) = (-1)^{m+1} \cdot \frac{2\pi v^{k+m-1}}{k+m+2}(\boldsymbol{v}\cdot\boldsymbol{u}) \ . \tag{17}$$

The general form of an angular integral is now can be written as

$$I_{\boldsymbol{e}}^{k,m,p}(\boldsymbol{v},\boldsymbol{u}) = (-1)^{m+p} \cdot \frac{2\pi v^{k+m-p}}{k+m+p+1}(\boldsymbol{v}\cdot\boldsymbol{u})^p \ , \quad p=0,1 \ . \tag{18}$$

ANGULAR INTEGRALS WITH A DOMAIN RESTRICTION

We consider domain restrictions in the next form

$$\Theta\big(\lambda^2 - (\boldsymbol{v}\cdot\boldsymbol{e})^2\big) \ , \quad \text{or} \quad \Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2 - \lambda^2\big) \ ,$$

hence the angular integrals in the form

$$I_{\boldsymbol{e},\lambda+}^{k,m,p}(\boldsymbol{v},\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{e} \ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m(\boldsymbol{u}\cdot\boldsymbol{e})^p\Theta\big(\lambda^2 - (\boldsymbol{v}\cdot\boldsymbol{e})^2\big) \ , \tag{19}$$

and

$$I_{\boldsymbol{e},\lambda-}^{k,m,p}(\boldsymbol{v},\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{e} \ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m(\boldsymbol{u}\cdot\boldsymbol{e})^p\Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2 - \lambda^2\big) \ . \tag{20}$$

Following the procedures we did above, we can rewrite these integrals in the next form:

$$\begin{aligned}
I_{\boldsymbol{e},\lambda+}^{k,m,p}(\boldsymbol{v},\boldsymbol{u}) &= (-1)^m \int \mathrm{d}\boldsymbol{e} \ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^{k+m}(\boldsymbol{u}\cdot\boldsymbol{e})^p\Theta\big(\lambda^2 - (\boldsymbol{v}\cdot\boldsymbol{e})^2\big) \ , \\
I_{\boldsymbol{e},\lambda-}^{k,m,p}(\boldsymbol{v},\boldsymbol{u}) &= (-1)^m \int \mathrm{d}\boldsymbol{e} \ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^{k+m}(\boldsymbol{u}\cdot\boldsymbol{e})^p\Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2 - \lambda^2\big) \ ,
\end{aligned} \tag{21}$$

The case with $p=0$.

Again, concentrating first on the case with $p=0$, we write

$$\begin{aligned}
I_{\boldsymbol{e},\lambda+}^{k,m,0}(\boldsymbol{v}) &= (-1)^m \int \mathrm{d}\boldsymbol{e} \ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^{k+m}\Theta\big(\lambda^2 - (\boldsymbol{v}\cdot\boldsymbol{e})^2\big) \ , \\
I_{\boldsymbol{e},\lambda-}^{k,m,0}(\boldsymbol{v}) &= (-1)^m \int \mathrm{d}\boldsymbol{e} \ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^{k+m}\Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2 - \lambda^2\big) \ ,
\end{aligned} \tag{22}$$

and the integrals depend only on the vector $\boldsymbol{v}$. Again, fixing $\boldsymbol{v}$ and denoting the angle between $\boldsymbol{e}$ and $\boldsymbol{v}$ as θ, we write

$$\begin{aligned} I^{k,m,0}_{\boldsymbol{e},\lambda+}(\boldsymbol{v}) &= (-1)^m \cdot 2\pi v^{k+m} \int_{\pi/2}^{\pi} \mathrm{d}\theta\, \sin\theta |\cos\theta|^{k+m} \Theta\big(\lambda^2 - v^2\cos^2\theta\big)\ , \\ I^{k,m,0}_{\boldsymbol{e},\lambda-}(\boldsymbol{v}) &= (-1)^m \cdot 2\pi v^{k+m} \int_{\pi/2}^{\pi} \mathrm{d}\theta\, \sin\theta |\cos\theta|^{k+m} \Theta\big(v^2\cos^2\theta - \lambda^2\big)\ , \end{aligned} \tag{23}$$

or

$$\begin{aligned} I^{k,m,0}_{\boldsymbol{e},\lambda+}(\boldsymbol{v}) &= (-1)^{m+1} \cdot 2\pi v^{k+m} \int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\ |\cos\theta|^{k+m} \Theta\big(\lambda^2 - v^2\cos^2\theta\big)\ , \\ I^{k,m,0}_{\boldsymbol{e},\lambda-}(\boldsymbol{v}) &= (-1)^{m+1} \cdot 2\pi v^{k+m} \int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\ |\cos\theta|^{k+m} \Theta\big(v^2\cos^2\theta - \lambda^2\big)\ . \end{aligned} \tag{24}$$

The restrictions to the angle θ can be written in a more explicit form as

$$\lambda_+ \colon \lambda^2 - v^2\cos^2\theta \geqslant 0\ , \quad \text{and} \quad \lambda_- \colon \lambda^2 - v^2\cos^2\theta \leqslant 0\ , \tag{25}$$

or

$$\lambda_+ \colon\ \cos^2\theta \leqslant \left(\frac{\lambda}{v}\right)^2\ , \quad \text{and} \quad \lambda_- \colon\ \cos^2\theta \geqslant \left(\frac{\lambda}{v}\right)^2\ . \tag{26}$$

The first condition is always true, for $\lambda \geqslant v$, while the second condition is always false for $\lambda \geqslant v$. Hence, a non-trivial case can be found only if $\lambda \leqslant v$. Note, that both λ and v are positive variables. Hence, we can rewrite these conditions as

$$\lambda_+ \colon |\cos\theta| \leqslant \frac{\lambda}{v}\ , \quad \text{and} \quad \lambda_- \colon |\cos\theta| \geqslant \frac{\lambda}{v}\ . \tag{27}$$

Since the angle θ is restricted to be in range from $\pi/2$ to π, the above conditions will restrict θ as

$$\lambda_+ \colon \frac{\pi}{2} \leqslant \theta \leqslant \pi - \arccos\left(\frac{\lambda}{v}\right)\ , \quad \text{and} \quad \lambda_- \colon \pi - \arccos\left(\frac{\lambda}{v}\right) \leqslant \theta \leqslant \pi\ , \tag{28}$$

and now, our integrals read

$$\begin{aligned} I^{k,m,0}_{\boldsymbol{e},\lambda+}(\boldsymbol{v}) &= (-1)^{m+1} \cdot 2\pi v^{k+m} \Theta(v-\lambda) \int_{\pi/2}^{\pi-\arccos(\lambda/v)} \mathrm{d}(\cos\theta)\ |\cos\theta|^{k+m} + \\ &\quad + (-1)^{m+1} \cdot 2\pi v^{k+m} \Theta(\lambda - v) \int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\ |\cos\theta|^{k+m}\ , \\ I^{k,m,0}_{\boldsymbol{e},\lambda-}(\boldsymbol{v}) &= (-1)^{m+1} \cdot 2\pi v^{k+m} \Theta(v-\lambda) \int_{\pi-\arccos(\lambda/v)}^{\pi} \mathrm{d}(\cos\theta)\ |\cos\theta|^{k+m}\ , \end{aligned} \tag{29}$$

and solving them we get

$$\begin{aligned} I^{k,m,0}_{\boldsymbol{e},\lambda+}(\boldsymbol{v}) &= (-1)^m \cdot \frac{2\pi v^{k+m}}{k+m+1} \left[\left(-\frac{\lambda}{v}\right)^{k+m+1} \Theta(v-\lambda) + \Theta(\lambda - v)\right]\ , \\ I^{k,m,0}_{\boldsymbol{e},\lambda-}(\boldsymbol{v}) &= (-1)^m \cdot \frac{2\pi v^{k+m}}{k+m+1} \left((-1)^{k+m+1} - \left(-\frac{\lambda}{v}\right)^{k+m+1}\right)\ . \end{aligned} \tag{30}$$

The case with $p = 1$.

The integrals in this case read

$$
\begin{aligned}
I^{k,m,1}_{\boldsymbol{e},\lambda+}(\boldsymbol{v},\boldsymbol{u}) &= \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m(\boldsymbol{u}\cdot\boldsymbol{e})\Theta\big(\lambda^2-(\boldsymbol{v}\cdot\boldsymbol{e})^2\big)\ , \\
I^{k,m,1}_{\boldsymbol{e},\lambda-}(\boldsymbol{v},\boldsymbol{u}) &= \int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m(\boldsymbol{u}\cdot\boldsymbol{e})\Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2-\lambda^2\big)\ .
\end{aligned}
\tag{31}
$$

Again, writing it as a dot product between vector $\boldsymbol{u}$ and another integrals vector

$$
\begin{aligned}
I^{k,m,1}_{\boldsymbol{e},\lambda+}(\boldsymbol{v},\boldsymbol{u}) &= \boldsymbol{u}\cdot\int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m\Theta\big(\lambda^2-(\boldsymbol{v}\cdot\boldsymbol{e})^2\big)\,\boldsymbol{e} = \boldsymbol{u}\cdot\boldsymbol{V}^{k,m}_{\lambda+}\ , \\
I^{k,m,1}_{\boldsymbol{e},\lambda-}(\boldsymbol{v},\boldsymbol{u}) &= \boldsymbol{u}\cdot\int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^m\Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2-\lambda^2\big)\,\boldsymbol{e} = \boldsymbol{u}\cdot\boldsymbol{V}^{k,m}_{\lambda-}\ .
\end{aligned}
\tag{32}
$$

Since $\boldsymbol{V}$ vectors can only be in the $\boldsymbol{v}$ direction, it can be written as

$$
\boldsymbol{V}^{k,m}_{\lambda+} = \boldsymbol{v}\cdot I^{k,m}_{\lambda+}\ , \quad \boldsymbol{V}^{k,m}_{\lambda-} = \boldsymbol{v}\cdot I^{k,m}_{\lambda-}\ , \tag{33}
$$

hence

$$
\boldsymbol{v}\cdot\boldsymbol{V}^{k,m}_{\lambda+} = v^2\cdot I^{k,m}_{\lambda+}\ , \quad \boldsymbol{v}\cdot\boldsymbol{V}^{k,m}_{\lambda-} = v^2\cdot I^{k,m}_{\lambda-}\ , \tag{34}
$$

which gives us

$$
\begin{aligned}
I^{k,m}_{\lambda+} &= v^{-2}\int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^{m+1}\Theta\big(\lambda^2-(\boldsymbol{v}\cdot\boldsymbol{e})^2\big) = v^{-2}\cdot I^{k,m+1,0}_{\boldsymbol{e},\lambda_+}(\boldsymbol{v})\ , \\
I^{k,m}_{\lambda-} &= v^{-2}\int \mathrm{d}\boldsymbol{e}\ \Theta(-\boldsymbol{v}\cdot\boldsymbol{e})|\boldsymbol{v}\cdot\boldsymbol{e}|^k(\boldsymbol{v}\cdot\boldsymbol{e})^{m+1}\Theta\big((\boldsymbol{v}\cdot\boldsymbol{e})^2-\lambda^2\big) = v^{-2}\cdot I^{k,m+1,0}_{\boldsymbol{e},\lambda_-}(\boldsymbol{v})\ ,
\end{aligned}
\tag{35}
$$

and the integral under consideration can be written in the form of

$$
\begin{aligned}
I^{k,m,1}_{\boldsymbol{e},\lambda+}(\boldsymbol{v},\boldsymbol{u}) &= \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{v^2}\cdot I^{k,m+1,0}_{\boldsymbol{e},\lambda_+}(\boldsymbol{v})\ , \\
I^{k,m,1}_{\boldsymbol{e},\lambda-}(\boldsymbol{v},\boldsymbol{u}) &= \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{v^2}\cdot I^{k,m+1,0}_{\boldsymbol{e},\lambda_-}(\boldsymbol{v})\ .
\end{aligned}
\tag{36}
$$

CENTER OF MASS VELOCITY INTEGRALS

The next type of integral is of the form

$$
I^{q,s}_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{u}\exp\left(-\beta u^2\right)\exp\left(r\boldsymbol{v}\cdot\boldsymbol{u}\right)(\boldsymbol{v}\cdot\boldsymbol{u})^s\,u^q\ , \tag{37}
$$

where $s = \{0, 1\}$. Fixing the vector $\boldsymbol{v}$ and switching to spherical coordinates, with θ being an angle between $\boldsymbol{v}$ and $\boldsymbol{u}$, we write

$$
I^{q,s}_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{u}) = 2\pi v^s\int_0^\infty \mathrm{d}u\ u^{q+s+2}\exp\left(-\beta u^2\right)\int_0^\pi \mathrm{d}\theta\sin\theta\cos^s\theta\exp\left(rvu\cos\theta\right)\ , \tag{38}
$$

where the angular integral for each s are easy to solve:

$$\begin{aligned}\int_0^\pi \mathrm{d}\theta\, \sin\theta\, \exp\left(rvu\cos\theta\right) &= \frac{1}{rvu}\left(e^{rvu} - e^{-rvu}\right)\,, \\ \int_0^\pi \mathrm{d}\theta\, \sin\theta\, \cos\theta\, \exp\left(rvu\cos\theta\right) &= \frac{1}{rvu}\left(e^{rvu} + e^{-rvu}\right) - \frac{1}{r^2v^2u^2}\left(e^{rvu} - e^{-rvu}\right)\,.\end{aligned} \tag{39}$$

Now we have

$$\begin{aligned}I_{\boldsymbol{u}}^{q,0}(\boldsymbol{v},\boldsymbol{u}) &= \frac{2\pi}{rv}\int_0^\infty \mathrm{d}u\; u^{q+1}\exp\left(-\beta u^2\right)\left(e^{rvu} - e^{-rvu}\right)\,, \\ I_{\boldsymbol{u}}^{q,1}(\boldsymbol{v},\boldsymbol{u}) &= \frac{2\pi}{r}\int_0^\infty \mathrm{d}u\; u^{q+2}\exp\left(-\beta u^2\right)\left(e^{rvu} + e^{-rvu}\right) - \,, \\ &- \frac{2\pi}{r^2v}\int_0^\infty \mathrm{d}u\; u^{q+1}\exp\left(-\beta u^2\right)\left(e^{rvu} - e^{-rvu}\right)\,.\end{aligned} \tag{40}$$

We can see that the center of mass velocity integral boiled down to an integral of the form:

$$I_u^{\kappa,\pm}(v,u) = \int_0^\infty \mathrm{d}u\; u^\kappa \exp\left(-\beta u^2\right)\left(e^{rvu} \pm e^{-rvu}\right)\,. \tag{41}$$

Unfortunately, this integral does not have a simple analytic form for a general value of κ. However, the integration with a minus sign, has analytic forms for odd κ, while the version with a plus sign, has analytic forms for even κ.

For few odd values of $\kappa = 1, 3, 5$, the integrals with negative sign are:

$$\begin{aligned}I_u^{1,-}(v,u) &= \sqrt{\frac{\pi}{\beta}}\frac{rv}{2\beta}\exp\left(\frac{r^2v^2}{4\beta}\right)\,, \\ I_u^{3,-}(v,u) &= \sqrt{\frac{\pi}{\beta}}\frac{rv}{(2\beta)^3}\left(6\beta + r^2v^2\right)\exp\left(\frac{r^2v^2}{4\beta}\right)\,, \\ I_u^{5,-}(v,u) &= \sqrt{\frac{\pi}{\beta}}\frac{rv}{(2\beta)^5}\left(60\beta^2 + 20\beta r^2v^2 + r^4v^4\right)\exp\left(\frac{r^2v^2}{4\beta}\right)\,.\end{aligned} \tag{42}$$

For few even values of $\kappa = 0, 2, 4$, the integrals with positive sign are:

$$\begin{aligned}I_u^{0,+}(v,u) &= \sqrt{\frac{\pi}{\beta}}\exp\left(\frac{r^2v^2}{4\beta}\right)\,, \\ I_u^{2,+}(v,u) &= \sqrt{\frac{\pi}{\beta}}\frac{2\beta + r^2v^2}{(2\beta)^2}\exp\left(\frac{r^2v^2}{4\beta}\right)\,, \\ I_u^{4,+}(v,u) &= \sqrt{\frac{\pi}{\beta}}\frac{12\beta^2 + 12\beta r^2v^2 + r^4v^4}{(2\beta)^4}\exp\left(\frac{r^2v^2}{4\beta}\right)\,, \\ I_u^{6,+}(v,u) &= \sqrt{\frac{\pi}{\beta}}\frac{120\beta^3 + 180\beta^2r^2v^2 + 30\beta r^4v^4 + r^6v^6}{(2\beta)^6}\exp\left(\frac{r^2v^2}{4\beta}\right)\,.\end{aligned} \tag{43}$$

Now, we can write our initial integrals for specific values of s and q. Starting with the case $s = 0$:

$$
\begin{aligned}
I_{\boldsymbol{u}}^{0,0}(\boldsymbol{v},\boldsymbol{u}) &= \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{r^2v^2}{4\beta}\right) ,\\
I_{\boldsymbol{u}}^{2,0}(\boldsymbol{v},\boldsymbol{u}) &= \frac{6\beta + r^2v^2}{(2\beta)^2} \cdot \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{r^2v^2}{4\beta}\right) ,\\
I_{\boldsymbol{u}}^{4,0}(\boldsymbol{v},\boldsymbol{u}) &= \frac{60\beta^2 + 20\beta r^2v^2 + r^4v^4}{(2\beta)^4} \cdot \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{r^2v^2}{4\beta}\right) .
\end{aligned}
\tag{44}
$$

For the case of $s = 1$:

$$
\begin{aligned}
I_{\boldsymbol{u}}^{0,1}(\boldsymbol{v},\boldsymbol{u}) &= \frac{rv^2}{2\beta}\left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{r^2v^2}{4\beta}\right) ,\\
I_{\boldsymbol{u}}^{2,1}(\boldsymbol{v},\boldsymbol{u}) &= \frac{rv^2(10\beta + r^2v^2)}{(2\beta)^3}\left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{r^2v^2}{4\beta}\right) ,\\
I_{\boldsymbol{u}}^{4,1}(\boldsymbol{v},\boldsymbol{u}) &= \frac{rv^2(140\beta^2 + 28\beta r^2v^2 + r^4 + v^4)}{(2\beta)^5}\left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{r^2v^2}{4\beta}\right) .
\end{aligned}
\tag{45}
$$

SOLVING KINETIC INTEGRALS

Aggregation integrals

The collision integrals describing the aggregation process, have the next form:

$$
\begin{aligned}
I_k^{\mathrm{agg}}(\boldsymbol{v}_k) = \frac{1}{2}\sum_{i+j=k}\sigma_{ij}^2 &\int \mathrm{d}\boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e}\,\Theta(-\boldsymbol{g}_{ij}\cdot\boldsymbol{e})|\boldsymbol{g}_{ij}\cdot\boldsymbol{e}| \\
&\times f_{ij}(\boldsymbol{v}_i,\boldsymbol{v}_j)\Theta(E_{\mathrm{agg}} - E_{ij})\delta(m_k\boldsymbol{v}_k - m_i\boldsymbol{v}_i - m_j\boldsymbol{v}_j) \\
&-\sum_j \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e}\,\Theta(-\boldsymbol{g}_{kj}\cdot\boldsymbol{e})|\boldsymbol{g}_{kj}\cdot\boldsymbol{e}| \\
&\times f_{kj}(\boldsymbol{v}_k,\boldsymbol{v}_j)\Theta(E_{\mathrm{agg}} - E_{kj}) .
\end{aligned}
\tag{46}
$$

The collision integrals are usually multiplied by a certain velocity function $\psi_k(\boldsymbol{v}_k)$ and then integrated over the whole velocity domain

$$
I_k^{\mathrm{agg}} = \int \mathrm{d}\boldsymbol{v}_k\,\psi_k(\boldsymbol{v}_k) I_k^{\mathrm{agg}}(\boldsymbol{v}_k) , \tag{47}
$$

hence all kinetic parameters involving the aggregation process can be obtained by solving the next integrals:

$$
\begin{aligned}
I_k^{\mathrm{agg}} = & \frac{1}{2} \sum_{i+j=k} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_k \int \mathrm{d}\boldsymbol{v}_i \, \mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g}_{ij} \cdot \boldsymbol{e}) |\boldsymbol{g}_{ij} \cdot \boldsymbol{e}| \psi_k(\boldsymbol{v}_k) \\
& \times f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j) \Theta(E_{\mathrm{agg}} - E_{ij}) \delta(m_k \boldsymbol{v}_k - m_i \boldsymbol{v}_i - m_j \boldsymbol{v}_j) \\
& - \sum_j \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{v}_k \int \mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g}_{kj} \cdot \boldsymbol{e}) |\boldsymbol{g}_{kj} \cdot \boldsymbol{e}| \psi_k(\boldsymbol{v}_k) \\
& \times f_{kj}(\boldsymbol{v}_k, \boldsymbol{v}_j) \Theta(E_{\mathrm{agg}} - E_{kj}) \; .
\end{aligned} \tag{48}
$$

Changing the variables $\boldsymbol{v}_i, \boldsymbol{v}_j \to \boldsymbol{g}_{ij}, \boldsymbol{V}$ and $\boldsymbol{v}_k, \boldsymbol{v}_j \to \boldsymbol{g}_{kj}, \boldsymbol{V}$, and since these transformations have unit jacobian, we can rewrite the aggregation integrals as

$$
\begin{aligned}
I_k^{\mathrm{agg}} = & \frac{1}{2} \sum_{i+j=k} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g} \, \mathrm{d}\boldsymbol{V} \, \psi_k(M\boldsymbol{V}/m_k) \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \\
& \times f_{ij}(\boldsymbol{g}, \boldsymbol{V}) \Theta(E_{\mathrm{agg}} - E_{ij}) \\
& - \sum_j \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g} \, \mathrm{d}\boldsymbol{V} \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \psi_k(\boldsymbol{g}, \boldsymbol{V}) \\
& \times f_{kj}(\boldsymbol{g}, \boldsymbol{V}) \Theta(E_{\mathrm{agg}} - E_{kj}) \; .
\end{aligned} \tag{49}
$$

The two-particle DF has a Maxwellian form:

$$
f_{ij}(\boldsymbol{g}, \boldsymbol{V}) = \alpha_{ij} \exp\left(-A_{ij} V^2\right) \exp\left(-B_{ij} g^2\right) \exp\left(R_{ij} \boldsymbol{g} \cdot \boldsymbol{V}\right) \; , \tag{50}
$$

where the indices ij will be omitted when it is clear exactly what indices we are working on. Now, our aggregation integrals are written as:

$$
\begin{aligned}
I_k^{\mathrm{agg}} = & \frac{1}{2} \sum_{i+j=k} \alpha \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g} \exp\left(-B g^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-A V^2\right) \exp\left(R \boldsymbol{g} \cdot \boldsymbol{V}\right) \psi_k(M\boldsymbol{V}/m_k) \\
& \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \Theta\big(E_{\mathrm{agg}} - \mu_{ij} g^2/2\big) \\
& - \sum_j \alpha \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g} \exp\left(-B g^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-A V^2\right) \exp\left(R \boldsymbol{g} \cdot \boldsymbol{V}\right) \\
& \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \psi_k(\boldsymbol{g}, \boldsymbol{V}) \Theta\big(E_{\mathrm{agg}} - \mu_{kj} g^2/2\big) \; .
\end{aligned} \tag{51}
$$

The remaining step functions restrict the impact speed domain to be less than a certain threshold $g \leqslant \sqrt{2E_{\mathrm{agg}}/\mu} = g_{\mathrm{agg}}$, hence we can now write

$$\begin{aligned} I_k^{\mathrm{agg}} = &\frac{1}{2} \sum_{i+j=k} \alpha\sigma_{ij}^2 \int_{g \leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g} \exp\left(-Bg^2\right) \\ &\int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \psi_k(M\boldsymbol{V}/m_k) \int \mathrm{d}\boldsymbol{e}\, \Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}| \\ &- \sum_j \alpha\sigma_{kj}^2 \int_{g \leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g} \exp\left(-Bg^2\right) \\ &\int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \int \mathrm{d}\boldsymbol{e}\, \Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}| \psi_k(\boldsymbol{g},\boldsymbol{V}) \ . \end{aligned} \tag{52}$$

The first angular integral can be immediately solved,

$$\int \mathrm{d}\boldsymbol{e}\, \Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}| = I_{\boldsymbol{e}}^{1,0,0}(\boldsymbol{g}) = \pi g \ , \tag{53}$$

hence

$$\begin{aligned} I_k^{\mathrm{agg}} = &\frac{\pi}{2} \sum_{i+j=k} \alpha\sigma_{ij}^2 \int_{g \leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\ g \exp\left(-Bg^2\right) \\ &\int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \psi_k(M\boldsymbol{V}/m_k) \\ &- \sum_j \alpha\sigma_{kj}^2 \int_{g \leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g} \exp\left(-Bg^2\right) \\ &\int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \int \mathrm{d}\boldsymbol{e}\, \Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}| \psi_k(\boldsymbol{g},\boldsymbol{V}) \ . \end{aligned} \tag{54}$$

To proceed further, we need to know the exact values of the function $\psi_k(\boldsymbol{v})$.

Number density change during aggregation

Putting $\psi_k(\boldsymbol{v}) = 1$, we obtain the rate of change of the number density of particles in the system

$$\psi_k(\boldsymbol{v}) = 1 \ , \quad \Rightarrow \quad \left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = I_k^{\mathrm{agg}}(\psi_k = 1) \ , \tag{55}$$

or

$$\left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = \frac{\pi}{2} \sum_{i+j=k} \alpha\sigma_{ij}^2 \int_{g\leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\; g \exp\left(-Bg^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) - \sum_j \alpha\sigma_{kj}^2 \int_{g\leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g} \exp\left(-Bg^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \int \mathrm{d}\boldsymbol{e}\, \Theta(-\boldsymbol{g}\cdot\boldsymbol{e}) |\boldsymbol{g}\cdot\boldsymbol{e}| \;, \tag{56}$$

and since the last angular integral is equal to πg, we have

$$\left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = \frac{\pi}{2} \sum_{i+j=k} \alpha\sigma_{ij}^2 \int_{g\leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\; g \exp\left(-Bg^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) - \pi \sum_j \alpha\sigma_{kj}^2 \int_{g\leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\; g \exp\left(-Bg^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) . \tag{57}$$

The center of mass velocity integral $I_{\boldsymbol{V}}^{q,s}(\boldsymbol{g},\boldsymbol{V})$ has already been calculated above. In our case, we need specific values of the integral for $q = s = 0$, hence

$$I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g},\boldsymbol{V}) = \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) = \left(\frac{\pi}{A}\right)^{3/2} \exp\left(\frac{R^2 g^2}{4A}\right) . \tag{58}$$

Now, we are left with standard gaussian integrals

$$\left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = 2\pi^2 \sum_{i+j=k} \left(\frac{\pi}{A}\right)^{3/2} \alpha\sigma_{ij}^2 \int_{g\leqslant g_{\mathrm{agg}}} \mathrm{d}g\; g^3 \exp\left(-Bg^2\right) \exp\left(\frac{R^2 g^2}{4A}\right) - 4\pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \alpha\sigma_{kj}^2 \int_{g\leqslant g_{\mathrm{agg}}} \mathrm{d}g\; g^3 \exp\left(-Bg^2\right) \exp\left(\frac{R^2 g^2}{4A}\right) , \tag{59}$$

or

$$\left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = 2\pi^2 \sum_{i+j=k} \left(\frac{\pi}{A}\right)^{3/2} \alpha\sigma_{ij}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\; g^3 \exp\left(-\lambda g^2\right) - 4\pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \alpha\sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\; g^3 \exp\left(-\lambda g^2\right) , \tag{60}$$

where $\lambda = (4AB - R^2)/(4A)$. Solving the last gaussian integral, we obtain

$$\left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = \pi^2 \sum_{i+j=k} \left(\frac{\pi}{A}\right)^{3/2} \frac{\alpha\sigma_{ij}^2}{\lambda^2} \left[1 - \left(1 + \lambda^2 g_{\mathrm{agg}}^2\right) \exp\left(-\lambda g_{\mathrm{agg}}^2\right)\right] - 2\pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \frac{\alpha\sigma_{kj}^2}{\lambda^2} \left[1 - \left(1 + \lambda^2 g_{\mathrm{agg}}^2\right) \exp\left(-\lambda g_{\mathrm{agg}}^2\right)\right] . \tag{61}$$

Finally, substituting the values of the parameters

$$\begin{aligned}
\alpha_{ij} &= \frac{n_i n_j}{(\pi u_i u_j)^3} \ , \\
A_{ij} &= \frac{1}{u_i^2} + \frac{1}{u_j^2} \ , \\
B_{ij} &= \frac{\mu_j^2}{u_i^2} + \frac{\mu_i^2}{u_j^2} \ , \\
R_{ij} &= \frac{2\mu_j}{u_i^2} - \frac{2\mu_i}{u_j^2} \ ,
\end{aligned} \tag{62}$$

we obtain

$$\lambda_{ij} = \frac{1}{u_i^2 + u_j^2} \ , \tag{63}$$

and

$$\pi^2 \left(\frac{\pi}{A}\right)^{3/2} \frac{\alpha \sigma_{ij}^2}{\lambda^2} = n_i n_j \sigma_{ij}^2 \sqrt{\pi \left(u_i^2 + u_j^2\right)} = \frac{1}{2} n_i n_j \nu_{ij} \ , \tag{64}$$

where ν_{ij} is the collision frequency of particles with sizes i and j. Now we have

$$\left\langle \frac{\mathrm{d}n_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = \frac{1}{2} \sum_{i+j=k} C_{ij} n_i n_j - \sum_j C_{kj} n_k n_j \ , \tag{65}$$

where

$$C_{ij} = \nu_{ij} \lambda^2 I_g^3 = \nu_{ij} \left[1 - \left(1 + \lambda^2 g_{\mathrm{agg}}^2\right) \exp\left(-\lambda g_{\mathrm{agg}}^2\right)\right] \ . \tag{66}$$

Energy change during aggregation

If we take $\psi_k(\boldsymbol{v}_k) = m_k v_k^2/2$, we can obtain the rate of change in the average kinetic energy of the system. In this case, our kinetic integrals are

$$\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = \frac{\pi}{2} \sum_{i+j=k} \alpha \sigma_{ij}^2 & \int_{g \leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g} \ g \exp\left(-Bg^2\right) \\
& \int \mathrm{d}\boldsymbol{V} \, \frac{MV^2}{2} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g} \cdot \boldsymbol{V}\right) \\
& - \sum_j \alpha \sigma_{kj}^2 \int_{g \leqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g} \exp\left(-Bg^2\right) \\
& \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g} \cdot \boldsymbol{V}\right) \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \frac{m_k v_k^2}{2} \ ,
\end{aligned} \tag{67}$$

and since $\boldsymbol{v}_k = \boldsymbol{V} - \mu_j \boldsymbol{g}$, we have

$$v_k^2 = V^2 - 2\mu_j \boldsymbol{g} \cdot \boldsymbol{V} + \mu_j^2 g^2 \ , \tag{68}$$

and the integrals are written as

$$
\begin{aligned}
\left\langle\frac{\mathrm{d}E_k}{\mathrm{d}t}\right\rangle_{\mathrm{agg}} = \frac{\pi}{4}\sum_{i+j=k}\alpha m_k\sigma_{ij}^2 &\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)\\
&\int\mathrm{d}\boldsymbol{V}\;V^2\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)\\
&-\frac{\pi}{2}\sum_j\alpha m_k\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)\\
&\int\mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)\left(V^2-2\mu_j\boldsymbol{g}\cdot\boldsymbol{V}+\mu_j^2g^2\right)\,,
\end{aligned}
\tag{69}
$$

or

$$
\begin{aligned}
\left\langle\frac{\mathrm{d}E_k}{\mathrm{d}t}\right\rangle_{\mathrm{agg}} = \frac{\pi}{4}\sum_{i+j=k}\alpha m_k\sigma_{ij}^2 &\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)\\
&\int\mathrm{d}\boldsymbol{V}\;V^2\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)\\
&-\frac{\pi}{2}\sum_j\alpha m_k\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)\\
&\int\mathrm{d}\boldsymbol{V}\;V^2\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)+\\
&+\pi\sum_j\alpha m_k\mu_j\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)\\
&\int\mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)(\boldsymbol{g}\cdot\boldsymbol{V})-\\
&-\frac{\pi}{2}\sum_j\alpha m_k\mu_j^2\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g^3\exp\left(-Bg^2\right)\\
&\int\mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)\,,
\end{aligned}
\tag{70}
$$

and using the center of mass velocity integral forms we calculated above, we can write

$$
\begin{aligned}
\left\langle\frac{\mathrm{d}E_k}{\mathrm{d}t}\right\rangle_{\mathrm{agg}} = \frac{\pi}{4}\sum_{i+j=k}\alpha m_k\sigma_{ij}^2 &\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)I_{\boldsymbol{V}}^{2,0}(\boldsymbol{g},\boldsymbol{V})\\
&-\frac{\pi}{2}\sum_j\alpha m_k\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)I_{\boldsymbol{V}}^{2,0}(\boldsymbol{g},\boldsymbol{V})+\\
&+\pi\sum_j\alpha m_k\mu_j\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g\exp\left(-Bg^2\right)I_{\boldsymbol{V}}^{0,1}(\boldsymbol{g},\boldsymbol{V})-\\
&-\frac{\pi}{2}\sum_j\alpha m_k\mu_j^2\sigma_{kj}^2\int_{g\leqslant g_{\mathrm{agg}}}\mathrm{d}\boldsymbol{g}\;g^3\exp\left(-Bg^2\right)I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g},\boldsymbol{V})\,.
\end{aligned}
\tag{71}
$$

Writing the values of the integrals

$$
\begin{aligned}
I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g},\boldsymbol{V}) &= \left(\frac{\pi}{A}\right)^{3/2} \exp\left(\frac{R^2 g^2}{4A}\right) , \\
I_{\boldsymbol{V}}^{2,0}(\boldsymbol{g},\boldsymbol{V}) &= \frac{6A + R^2 g^2}{(2A)^2} \left(\frac{\pi}{A}\right)^{3/2} \exp\left(\frac{R^2 g^2}{4A}\right) , \\
I_{\boldsymbol{V}}^{0,1}(\boldsymbol{g},\boldsymbol{V}) &= \frac{R g^2}{2A} \left(\frac{\pi}{A}\right)^{3/2} \exp\left(\frac{R^2 g^2}{4A}\right) ,
\end{aligned}
\tag{72}
$$

we obtain the gaussian integrals

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} &= 3\pi^2 \sum_{i+j=k} \left(\frac{\pi}{A}\right)^{3/2} \frac{1}{A} \alpha m_k \sigma_{ij}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g \; g^3 \exp\left(-\lambda g^2\right) \\
&+ \frac{\pi^2}{4} \sum_{i+j=k} \left(\frac{\pi}{A}\right)^{3/2} \left(\frac{R}{A}\right)^2 \alpha m_k \sigma_{ij}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \\
&- 6\pi^2 \sum_{j} \left(\frac{\pi}{A}\right)^{3/2} \frac{1}{A} \alpha m_k \sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g \; g^3 \exp\left(-\lambda g^2\right) \\
&- \frac{\pi^2}{4} \sum_{j} \left(\frac{\pi}{A}\right)^{3/2} \left(\frac{R}{A}\right)^2 \alpha m_k \sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \\
&+ 2\pi^2 \sum_{j} \left(\frac{\pi}{A}\right)^{3/2} \frac{R}{A} \alpha m_k \mu_j \sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \\
&- 2\pi^2 \sum_{j} \left(\frac{\pi}{A}\right)^{3/2} \alpha m_k \mu_j^2 \sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) .
\end{aligned}
\tag{73}
$$

Denoting the gaussian integrals

$$
I_g^l(b,t) = \int_b^t \mathrm{d}g \; g^l \exp\left(-\lambda g^2\right) ,
\tag{74}
$$

we write

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} &= \sum_{i+j=k} \pi^2 \left(\frac{\pi}{A}\right)^{3/2} \alpha m_k \sigma_{ij}^2 \left(\frac{3}{A} I_g^3(0, g_{\mathrm{agg}}) + \left(\frac{R}{2A}\right)^2 I_g^5(0, g_{\mathrm{agg}}) \right) \\
&- \sum_{j} \pi^2 \left(\frac{\pi}{A}\right)^{3/2} \alpha m_k \sigma_{kj}^2 \left(\frac{6}{A} I_g^3(0, g_{\mathrm{agg}}) + \left[\left(\frac{R}{2A}\right)^2 - \frac{2\mu_j R}{A} + 2\mu_j^2 \right] I_g^5(0, g_{\mathrm{agg}}) \right) .
\end{aligned}
\tag{75}
$$

Solving the gaussian integrals, we obtain

$$
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{agg}} = \frac{1}{2} \sum_{i+j=k} \frac{m_k}{A_{ij}} n_i n_j F_1 - \sum_{j} \frac{m_k}{A_{kj}} n_j n_k F_2 ,
\tag{76}
$$

where

$$
\begin{aligned}
F_1 &= \nu_{ij}\lambda^2\left(6I_g^3(0,g_{\mathrm{agg}}) + \frac{R^2}{2A}I_g^5(0,g_{\mathrm{agg}})\right) , \\
F_2 &= \nu_{kj}\lambda^2\left(6I_g^3(0,g_{\mathrm{agg}}) + \left[\frac{R^2}{4A} - 2\mu_j R + 2A\mu_j^2\right]I_g^5(0,g_{\mathrm{agg}})\right) .
\end{aligned}
\tag{77}
$$

Restitution integrals

The restitutive collisions are considered by the following integrals

$$
I_k^{\mathrm{res}}(\boldsymbol{v}_k) = \sum_j \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{v}_k \,\mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e}\,\Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}| f_{kj}(\boldsymbol{v}_k,\boldsymbol{v}_j)\cdot\Delta\psi_k\Omega(\boldsymbol{v}_k,\boldsymbol{v}_j) , \tag{78}
$$

where $\Delta\psi_k$ is the change in function ψ_k due to a collision between particles of size k and j. $\Omega(\boldsymbol{v}_k,\boldsymbol{v}_j)$ is the domain restriction function. We can immediately see, that if $\psi_k = 1$, which describes the number density change rate, the integral vanishes due to $\Delta\psi_k = 0$, since the number of particles of any size does not change due to restitutive collisions. Now, we can write the change in kinetic energy setting $\psi_k = m_k v_k^2/2$, which gives us

$$
\Delta\psi_k = -\mu_{kj}(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{e})(\boldsymbol{V}\cdot\boldsymbol{e}) - \frac{1-\varepsilon^2}{2}\frac{\mu^2}{m_k}(\boldsymbol{g}\cdot\boldsymbol{e})^2 , \tag{79}
$$

and the domain restriction function is given as

$$
\Omega(\boldsymbol{v}_k,\boldsymbol{v}_j) = \Theta(E_{kj} - E_{\mathrm{agg}})\Theta\big(E_{\mathrm{frag}} - E_{kj}^{\mathrm{n}}\big) , \tag{80}
$$

or

$$
\Omega(\boldsymbol{g}) = \Theta(g - g_{\mathrm{agg}})\Theta\big(g_{\mathrm{frag}}^2 - (\boldsymbol{g}\cdot\boldsymbol{e})^2\big) , \tag{81}
$$

and the energy change rate can be written as

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} &= \sum_j \alpha\sigma_{kj}^2 \int_{g\geqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right) \int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \\
&\int \mathrm{d}\boldsymbol{e}\,\Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}|\cdot\Delta\psi_k\Theta\big(g_{\mathrm{frag}}^2 - (\boldsymbol{g}\cdot\boldsymbol{e})^2\big) .
\end{aligned}
\tag{82}
$$

Substituting the value of $\Delta\psi_k$, we write

$$\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = &-\sum_j \alpha\mu_{kj}(1+\varepsilon)\sigma_{kj}^2 \int_{g\geqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right)\int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \\
&\int \mathrm{d}\boldsymbol{e}\,\Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}|(\boldsymbol{g}\cdot\boldsymbol{e})(\boldsymbol{V}\cdot\boldsymbol{e})\Theta\big(g_{\mathrm{frag}}^2-(\boldsymbol{g}\cdot\boldsymbol{e})^2\big) \\
&-\frac{1}{2}\sum_j \alpha\frac{\mu_{kj}^2}{m_k}(1-\varepsilon^2)\sigma_{kj}^2 \int_{g\geqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right)\int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \\
&\int \mathrm{d}\boldsymbol{e}\,\Theta(-\boldsymbol{g}\cdot\boldsymbol{e})|\boldsymbol{g}\cdot\boldsymbol{e}|(\boldsymbol{g}\cdot\boldsymbol{e})^2\Theta\big(g_{\mathrm{frag}}^2-(\boldsymbol{g}\cdot\boldsymbol{e})^2\big)\ .
\end{aligned} \tag{83}$$

The angular integrals with domain restrictions can be written in a short form as

$$\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = &-\sum_j \alpha\mu_{kj}(1+\varepsilon)\sigma_{kj}^2 \int_{g\geqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right) \\
&\int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)I_{\boldsymbol{e},\lambda+}^{1,1,1}(\boldsymbol{g},\boldsymbol{V}) \\
&-\frac{1}{2}\sum_j \alpha\frac{\mu_{kj}^2}{m_k}(1-\varepsilon^2)\sigma_{kj}^2 \int_{g\geqslant g_{\mathrm{agg}}} \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right) \\
&\int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)I_{\boldsymbol{e},\lambda+}^{1,2,0}(\boldsymbol{g},\boldsymbol{V})\ .
\end{aligned} \tag{84}$$

Writing the angular integrals in explicit forms, we have

$$\begin{aligned}
I_{\boldsymbol{e},\lambda+}^{1,1,1}(\boldsymbol{g},\boldsymbol{V}) &= \frac{\pi}{2}\cdot(\boldsymbol{g}\cdot\boldsymbol{V})\left[\frac{g_{\mathrm{frag}}^4}{g^3}\Theta(g-g_{\mathrm{frag}})+g\Theta(g_{\mathrm{frag}}-g)\right], \\
I_{\boldsymbol{e},\lambda+}^{1,2,0}(\boldsymbol{g},\boldsymbol{V}) &= \frac{\pi}{2}\left[\frac{g_{\mathrm{frag}}^4}{g}\Theta(g-g_{\mathrm{frag}})+g^3\Theta(g_{\mathrm{frag}}-g)\right].
\end{aligned} \tag{85}$$

Now, the energy change rate becomes

$$\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = &-\frac{\pi}{2}\sum_j \alpha\mu_{kj}(1+\varepsilon)\sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right)\Theta(g-g_{\mathrm{agg}}) \\
&\int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)(\boldsymbol{g}\cdot\boldsymbol{V})\left[\frac{g_{\mathrm{frag}}^4}{g^3}\Theta(g-g_{\mathrm{frag}})+g\Theta(g_{\mathrm{frag}}-g)\right], \\
&-\frac{\pi}{4}\sum_j \alpha\frac{\mu_{kj}^2}{m_k}(1-\varepsilon^2)\sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\exp\left(-Bg^2\right)\Theta(g-g_{\mathrm{agg}}) \\
&\int \mathrm{d}\boldsymbol{V}\exp\left(-AV^2\right)\exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right)\left[\frac{g_{\mathrm{frag}}^4}{g}\Theta(g-g_{\mathrm{frag}})+g^3\Theta(g_{\mathrm{frag}}-g)\right],
\end{aligned} \tag{86}$$

or

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = & -\frac{\pi}{2} \sum_j \alpha \mu_{kj} (1+\varepsilon) g_{\mathrm{frag}}^4 \sigma_{kj}^2 \int_{g \geqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g^{-3} \exp\left(-Bg^2\right) \Theta(g - g_{\mathrm{agg}}) \\
& \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) (\boldsymbol{g}\cdot\boldsymbol{V}) \\
& -\frac{\pi}{2} \sum_j \alpha \mu_{kj} (1+\varepsilon) \sigma_{kj}^2 \int_{g \leqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g \exp\left(-Bg^2\right) \Theta(g - g_{\mathrm{agg}}) \\
& \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) (\boldsymbol{g}\cdot\boldsymbol{V}) \\
& -\frac{\pi}{4} \sum_j \alpha \frac{g_{\mathrm{frag}}^4 \mu_{kj}^2}{m_k} (1-\varepsilon^2) \sigma_{kj}^2 \int_{g \geqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g^{-1} \exp\left(-Bg^2\right) \Theta(g - g_{\mathrm{agg}}) \\
& \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) \\
& -\frac{\pi}{4} \sum_j \alpha \frac{\mu_{kj}^2}{m_k} (1-\varepsilon^2) \sigma_{kj}^2 \int_{g \leqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g^{3} \exp\left(-Bg^2\right) \Theta(g - g_{\mathrm{agg}}) \\
& \int \mathrm{d}\boldsymbol{V} \exp\left(-AV^2\right) \exp\left(R\boldsymbol{g}\cdot\boldsymbol{V}\right) ,
\end{aligned}
\tag{87}
$$

and using the short forms of the center of mass velocity integrals, we write

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = & -\frac{\pi}{2} \sum_j \alpha \mu_{kj} (1+\varepsilon) g_{\mathrm{frag}}^4 \sigma_{kj}^2 \int_{g \geqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g^{-3} \exp\left(-Bg^2\right) I_{\boldsymbol{V}}^{0,1}(\boldsymbol{g}, \boldsymbol{V}) \\
& -\frac{\pi}{2} \sum_j \alpha \mu_{kj} (1+\varepsilon) \sigma_{kj}^2 \int_{g_{\mathrm{agg}} \leqslant g \leqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g \exp\left(-Bg^2\right) I_{\boldsymbol{V}}^{0,1}(\boldsymbol{g}, \boldsymbol{V}) \\
& -\frac{\pi}{4} \sum_j \alpha \frac{g_{\mathrm{frag}}^4 \mu_{kj}^2}{m_k} (1-\varepsilon^2) \sigma_{kj}^2 \int_{g \geqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g^{-1} \exp\left(-Bg^2\right) I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g}, \boldsymbol{V}) \\
& -\frac{\pi}{4} \sum_j \alpha \frac{\mu_{kj}^2}{m_k} (1-\varepsilon^2) \sigma_{kj}^2 \int_{g_{\mathrm{agg}} \leqslant g \leqslant g_{\mathrm{frag}}} \mathrm{d}\boldsymbol{g}\ g^{3} \exp\left(-Bg^2\right) I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g}, \boldsymbol{V}) .
\end{aligned}
\tag{88}
$$

Again, by writing the center of mass velocity integrals in explicit forms, we have

$$
\begin{aligned}
I_{\boldsymbol{V}}^{0,1}(\boldsymbol{g}, \boldsymbol{V}) &= \frac{Rg^2}{2A} \left(\frac{\pi}{A}\right)^{3/2} \exp\left(\frac{R^2 g^2}{4A}\right) , \\
I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g}, \boldsymbol{V}) &= \left(\frac{\pi}{A}\right)^{3/2} \exp\left(\frac{R^2 g^2}{4A}\right) ,
\end{aligned}
\tag{89}
$$

and the energy change rate integrals become

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = &- \pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \frac{R}{A} \alpha \mu_{kj} (1+\varepsilon) g_{\mathrm{frag}}^4 \sigma_{kj}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g \; g \exp\left(-\lambda g^2\right) \\
&- \pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \frac{R}{A} \alpha \mu_{kj} (1+\varepsilon) \sigma_{kj}^2 \int_{g_{\mathrm{agg}}}^{g_{\mathrm{frag}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \\
&- \pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \alpha \frac{g_{\mathrm{frag}}^4 \mu_{kj}^2}{m_k} (1-\varepsilon^2) \sigma_{kj}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g \; g \exp\left(-\lambda g^2\right) \\
&- \pi^2 \sum_j \left(\frac{\pi}{A}\right)^{3/2} \alpha \frac{\mu_{kj}^2}{m_k} (1-\varepsilon^2) \sigma_{kj}^2 \int_{g_{\mathrm{agg}}}^{g_{\mathrm{frag}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \; .
\end{aligned}
\tag{90}
$$

where $\lambda = (4AB - R^2)/(4A)$. Now we can write

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = &- \sum_j n_k n_j \nu_{jk} \lambda^2 \cdot \frac{R}{A} \frac{1+\varepsilon}{2} \mu_{kj} \cdot g_{\mathrm{frag}}^4 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g \; g \exp\left(-\lambda g^2\right) \\
&- \sum_j n_k n_j \nu_{jk} \lambda^2 \cdot \frac{R}{A} \frac{1+\varepsilon}{2} \mu_{kj} \int_{g_{\mathrm{agg}}}^{g_{\mathrm{frag}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \\
&- \sum_j n_k n_j \nu_{jk} \lambda^2 \cdot \frac{\mu_{kj}^2}{m_k} \frac{1-\varepsilon^2}{2} \cdot g_{\mathrm{frag}}^4 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g \; g \exp\left(-\lambda g^2\right) \\
&- \sum_j n_k n_j \nu_{jk} \lambda^2 \cdot \frac{\mu_{kj}^2}{m_k} \frac{1-\varepsilon^2}{2} \int_{g_{\mathrm{agg}}}^{g_{\mathrm{frag}}} \mathrm{d}g \; g^5 \exp\left(-\lambda g^2\right) \; ,
\end{aligned}
\tag{91}
$$

or

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = &- \sum_j \frac{\mu_{kj} R}{A^2} \frac{1+\varepsilon}{2} \cdot n_k n_j \nu_{jk} \lambda^2 \cdot \left(A g_{\mathrm{frag}}^4 I_g^1(g_{\mathrm{frag}}, \infty) + A I_g^5(g_{\mathrm{agg}}, g_{\mathrm{frag}})\right) \\
&- \sum_j \frac{\mu_{kj}^2}{m_k A} \frac{1-\varepsilon^2}{2} \cdot n_k n_j \nu_{jk} \lambda^2 \cdot \left(A g_{\mathrm{frag}}^4 I_g^1(g_{\mathrm{frag}}, \infty) + A I_g^5(g_{\mathrm{agg}}, g_{\mathrm{frag}})\right) \; ,
\end{aligned}
\tag{92}
$$

or

$$
\left\langle \frac{\mathrm{d}E_k}{\mathrm{d}t} \right\rangle_{\mathrm{res}} = -\frac{1+\varepsilon}{2} \sum_j \frac{\mu_{kj} R}{A_{kj}^2} n_k n_j F_3 - \frac{1-\varepsilon^2}{2} \sum_j \frac{\mu_{kj}^2}{m_k A_{kj}} n_k n_j F_3 \; , \tag{93}
$$

where

$$
F_3 = \nu_{kj} \lambda^2 \left(A g_{\mathrm{frag}}^4 I_g^1(g_{\mathrm{frag}}, \infty) + A I_g^5(g_{\mathrm{agg}}, g_{\mathrm{frag}})\right) \; . \tag{94}
$$

Fragmentation integrals

The fragmentation process is described by the next kinetic integrals

$$
\begin{aligned}
I_k^{\mathrm{frag}}(\boldsymbol{v}_k) = \frac{1}{2} \sum_{i,j \geqslant k+1} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_i \, \mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \\
\times f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j) \Theta\big(E_{ij}^{\mathrm{n}} - E_{\mathrm{frag}}\big) \big(q_{ki}(\boldsymbol{v}_k, \boldsymbol{v}_i, \boldsymbol{v}_j) + q_{kj}(\boldsymbol{v}_k, \boldsymbol{v}_i, \boldsymbol{v}_j)\big) \\
+ \sum_{i=1}^{k} \sum_{j \geqslant k+1} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_i \, \mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \\
\times f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j) \Theta\big(E_{ij}^{\mathrm{n}} - E_{\mathrm{frag}}\big) q_{kj}(\boldsymbol{v}_k, \boldsymbol{v}_i, \boldsymbol{v}_j) \\
- \sum_{i} (1 - \delta_{k,1}) \sigma_{ki}^2 \int \mathrm{d}\boldsymbol{v}_i \int \mathrm{d}\boldsymbol{e} \, \Theta(-\boldsymbol{g} \cdot \boldsymbol{e}) |\boldsymbol{g} \cdot \boldsymbol{e}| \\
\times f_{ki}(\boldsymbol{v}_k, \boldsymbol{v}_i) \Theta(E_{ki}^{\mathrm{n}} - E_{\mathrm{frag}}) \, .
\end{aligned}
\tag{95}
$$