THE MECHANICS OF BINARY INTERACTIONS

Our system comprises an infinite number of mechanically identical, spherical particles known as monomers. These particles have masses m_1 and radii R_1. When two monomers collide, they lose a certain amount of impact energy and rebound with a coefficient of restitution ε. If the impact energy is below a specific threshold value $E_{\mathrm{imp}} \leqslant E_{\mathrm{agg}}$, the monomers stick together due to surface forces like van der Waals forces, forming a larger aggregate particle with mass m_2 and radius R_2. This process is known as *aggregation* and allows the creation of larger particles from individual monomers. A particle of mass m_k is an aggregate of k monomers, hence $m_k = k \cdot m_1$. We assume that the aggregates remain spherically shaped, and their radii scale as $R_k \sim m_k^{1/3} \sim k^{1/3}$.

Conversely, there is another mechanism called *fragmentation* that decreases the sizes of aggregates. If the impact energy is higher than a certain threshold value $E_{\mathrm{imp}} \geqslant E_{\mathrm{frag}}$, the colliding aggregates break into smaller pieces. The size distribution of the fragmented pieces are difficult to model analytically, and in this work we assume a simplistic model of fragmentation, called *shattering*. When two aggregates of masses m_i and m_j collide with a sufficient energy, both of them shatter into singular monomers, $m_i \to i \cdot m_1$ and $m_j \to j \cdot m_1$.

Generalized collisions

Let us consider a collision of particles of masses m_i, m_j and velocities $\boldsymbol{v}_i$, $\boldsymbol{v}_j$, and radii R_i, R, j. The collision geometry is characterized by the unit vector $\hat{\boldsymbol{n}}$, which is directed from the center of particle j to the center of particle i at the moment of contact of two particles

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{R_i + R_j}, \tag{1}$$

where $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are position vectors of the particles. The next parameter which describes the collision, is the *restitution coefficient* $0 \leqslant \varepsilon \leqslant 1$. This parameter controls the amount of energy dissipated after the collision. The total energy of this binary system, can be split into two parts, the translational energy and the internal energy

$$E = E_{\mathrm{translation}} + E_{\mathrm{internal}} = \frac{MV^2}{2} + \frac{\mu g^2}{2}, \tag{2}$$

where

$$\begin{aligned} &\boldsymbol{V} = \mu_i \boldsymbol{v}_i + \mu_j \boldsymbol{v}_j, \quad \boldsymbol{g} = \boldsymbol{v}_i - \boldsymbol{v}_j, \\ &M = m_i + m_j, \quad \mu = \frac{m_i m_j}{m_i + m_j}, \\ &\mu_i = \frac{m_i}{m_i + m_j}, \quad \mu_j = \frac{m_j}{m_i + m_j}. \end{aligned} \tag{3}$$

The translational energy does not change after the collision, but the internal part dissipates. Using $\hat{\boldsymbol{n}}$, we can split the relative velocity into normal and tangential parts

$$\boldsymbol{g}_n = (\boldsymbol{g} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}, \quad \boldsymbol{g}_t = \boldsymbol{g} - \boldsymbol{g}_n, \tag{4}$$

and write the total energy as

$$E = \frac{MV^2}{2} + \frac{\mu g_t^2}{2} + \frac{\mu g_n^2}{2}. \tag{5}$$

Now, we can write the post-collision total energy E' as

$$E' = \frac{MV^2}{2} + \frac{\mu g_t^2}{2} + \varepsilon^2 \frac{\mu g_n^2}{2}, \tag{6}$$

where only the normal part of the internal energy dissipates.

In the most general case, we assume that the outcome of the collision is a collection of particles with various masses and velocities. Introducing the function $P_k(\boldsymbol{v}_k | \boldsymbol{v}_i, \boldsymbol{v}_j)$, which is the number of particles of mass m_k and velocity $\boldsymbol{v}_k$ created after the collision of particles with velocities $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ and sizes i, j, or in other words, introducing the velocity distribution function of the particles of mass m_k. Using this distribution function, we can write the total mass, momentum and energy of particles in the outcome of the generalized collision

$$\begin{aligned} M &= \sum_{k=1}^{i+j} \int \mathrm{d}\boldsymbol{v}\, m_k P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j), \\ M\boldsymbol{V} &= \sum_{k=1}^{i+j} \int \mathrm{d}\boldsymbol{v}\, m_k \boldsymbol{v} P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j), \\ \frac{MV^2}{2} + \frac{\mu g_t^2}{2} + \varepsilon^2 \frac{\mu g_n^2}{2} &= \sum_{k=1}^{i+j} \int \mathrm{d}\boldsymbol{v}\, \frac{m_k v^2}{2} P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j). \end{aligned} \tag{7}$$

Restitution

For a restitutive rebound of particles, the outcome velocities are analytic and given by

$$\begin{aligned} \boldsymbol{v}_i' &= \boldsymbol{v}_i - \mu_j(1+\varepsilon)\boldsymbol{g}_n, \\ \boldsymbol{v}_j' &= \boldsymbol{v}_j + \mu_i(1+\varepsilon)\boldsymbol{g}_n, \end{aligned} \tag{8}$$

Let us write the distribution function $P_k(\boldsymbol{v})$ for the restitutive collision. First of all, the masses of impacting particles do not change, hence the distribution function should contain δ-functions to control this. Together with the analytic expression for the outcome velocities, we can write

$$P_k^{\mathrm{res}}(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{k,i}\delta[\boldsymbol{v}_k - \boldsymbol{v}_i + \mu_j(1+\varepsilon)\boldsymbol{g}_n] + \delta_{k,j}\delta[\boldsymbol{v}_k - \boldsymbol{v}_j - \mu_i(1+\varepsilon)\boldsymbol{g}_n]. \tag{9}$$

The $\delta_{k,x}$ is a Kronecker-delta operator.

Aggregation

If the impact energy is smaller than a certain threshold $E_{\mathrm{imp}} \leqslant E_{\mathrm{agg}}$, the outcome of the collision is merging of two particles. From the momentum conservation we can write the outcome of the aggregative collision, which is a single particle with a mass and velocity

$$m' = m_i + m_j \qquad \boldsymbol{v}' = \boldsymbol{V} = \frac{m_i\boldsymbol{v}_i + m_j\boldsymbol{v}_j}{m_i + m_j}. \tag{10}$$

The total energy loss is

$$\Delta E = \frac{MV^2}{2} - \frac{m_i v_i^2}{2} - \frac{m_j v_j^2}{2} = -\frac{\mu g^2}{2} = -E_{\mathrm{internal}}, \tag{11}$$

so, all the internal energy is lost during the aggregative collision. The threshold energy value E_{agg} is in general a function of the sizes of particles.

Let us write the debris velocity distribution function for the aggregation process. Since the outcome is a single particle of mass $m_i + m_j$, with velocity $\boldsymbol{V}$, we have

$$P_k^{\mathrm{agg}}(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{k,i+j}\delta[\boldsymbol{v}_k - \mu_i\boldsymbol{v}_i - \mu_j\boldsymbol{v}_j]. \tag{12}$$

Fragmentation

If the impact energy exceeds the certain threshold value $E_{\mathrm{imp}} \geqslant E_{\mathrm{frag}}$, the two impactors break into into smaller particles in the collision. We cannot obtain the velocities of the monomers from only conservation laws, hence we have to assume that certain constraints are valid. Namely, we assume two constraints:

1. Both particles shatter into their constituent monomers;

2. Complete isotropy of the momenta of the monomers in CoM frame;

These two constraints allow us to write the outcome velocities of the fragmented pieces. Let us write the energy needed to release a single monomer from a particle as γ. Hence, the total energy needed for a complete decomposition of an aggregate of mass m_k can be estimated as

$$E_k = \gamma \cdot k. \tag{13}$$

The fragmentation process of two particles of masses m_i and m_j, with velocities $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ can be then described as a decay of a single particle of mass $m_k = m_i + m_j$ with velocity $\boldsymbol{v}_k = \boldsymbol{V} = \mu_i \boldsymbol{v}_i + \mu_j \boldsymbol{v}_j$. The decay energy can be estimated as

$$E_{\text{decay}} = E_{\text{imp}} - \gamma \cdot k, \tag{14}$$

which is the amount of energy which is equally distributed among all the shattered monomers. From this, we can see that the impact energy should be larger than $\gamma \cdot k$, which can be treated as the threshold energy. Since the impact energy is the normal part of the internal energy, we can write

$$E_{\text{decay}} = \frac{\mu g_n^2}{2} - \gamma \cdot k = \varepsilon^2 \frac{\mu g_n^2}{2}, \tag{15}$$

and the restitution coefficient for the fragmentation is

$$\varepsilon = \sqrt{1 - \frac{2\gamma k}{\mu g_n^2}}. \tag{16}$$

Since the decay energy has to be positive, we can write the threshold value for the normal relative velocity as

$$g_n \geqslant \sqrt{\frac{2\gamma k}{\mu}} = \sqrt{\frac{2\gamma}{m_1}} \cdot \frac{i+j}{\sqrt{ij}}. \tag{17}$$

In the CoM frame, each released monomer has an energy

$$E_c' = \frac{m_1 v_c'^2}{2} = \frac{E_{\text{decay}}}{k} = \varepsilon^2 \frac{\mu g_n^2}{2k}, \tag{18}$$

where v_c' is the speed of a monomer in CoM frame

$$v_c' = \frac{\sqrt{ij}}{i+j} \cdot \varepsilon g_n. \tag{19}$$

Let us estimate the number of monomers $\mathrm{d}N$ in a small solid angle $\mathrm{d}\Omega$. From the second constraint, we deduce that this number has to be proportional to the angle itself, hence

$$\mathrm{d}N = \frac{k}{4\pi}\,\mathrm{d}\Omega\,, \qquad k = i + j. \tag{20}$$

In the Lab frame, the speeds of monomers are not equal, but rather uniformly distribution in the range

$$v'_{\text{min}} = V - v_c', \qquad v'_{\text{max}} = V + v_c'. \tag{21}$$

Since the fragmented debris consist of only monomers, the distribution function $P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j)$ has to contain the term $\delta_{k,1}$. In the CoM frame, we can write

$$P_k^{\text{frag,CoM}}(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{k,1}\delta(v_k - v_c')\frac{i+j}{4\pi}. \tag{22}$$

In this case, the integral of any velocity function $\varphi(\boldsymbol{v}_k)$ in the form of

$$\int \mathrm{d}\boldsymbol{v}_k\, \varphi(\boldsymbol{v}_k) P_k^{\text{frag,CoM}}(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{k,1}\frac{i+j}{4\pi} \int \mathrm{d}\boldsymbol{v}_k\, \varphi(\boldsymbol{v}_k)\delta(v_k - v_c'), \tag{23}$$

can be written as

$$\delta_{k,1}\frac{i+j}{4\pi}\int \mathrm{d}\hat{\boldsymbol{e}}\int_0^\infty \mathrm{d}v\,\varphi(v,\hat{\boldsymbol{e}})\delta(v-v_c') = \delta_{k,1}\frac{i+j}{4\pi}\int \mathrm{d}\hat{\boldsymbol{e}}\,\varphi(v_c',\hat{\boldsymbol{e}}). \tag{24}$$

If $\varphi(\boldsymbol{v}) \equiv \varphi(v,\hat{\boldsymbol{e}}) = \hat{\boldsymbol{e}}\varphi(v)$, such as $\boldsymbol{v} = v\hat{\boldsymbol{e}}$, then

$$\int \mathrm{d}\hat{\boldsymbol{e}}\,\hat{\boldsymbol{e}}\varphi(v) = \boldsymbol{0}. \tag{25}$$

If $\varphi(\boldsymbol{v}) \equiv \varphi(v,\hat{\boldsymbol{e}}) = \varphi(v)$, then

$$\int \mathrm{d}\hat{\boldsymbol{e}}\,\varphi(v) = 4\pi\varphi(v). \tag{26}$$

To write the debris velocity distribution function in the Lab frame, we have to add the center of mass velocity to all the velocities of the monomers. This can be written as

$$P_k^{\mathrm{frag}}(\boldsymbol{v}_k|\boldsymbol{v}_i,\boldsymbol{v}_j) = \delta_{k,1}\frac{i+j}{4\pi}\int \mathrm{d}\hat{\boldsymbol{e}}\,\delta(\boldsymbol{v}_k-\boldsymbol{V}-v_c'\hat{\boldsymbol{e}}). \tag{27}$$

Now, integrating over a function $\varphi(\boldsymbol{v}_k)$ becomes

$$\delta_{k,1}\frac{i+j}{4\pi}\int \mathrm{d}\hat{\boldsymbol{e}}\int \mathrm{d}\boldsymbol{v}_k\,\varphi(\boldsymbol{v}_k)\delta(\boldsymbol{v}_k-\boldsymbol{V}-v_c'\hat{\boldsymbol{e}}) = \delta_{k,1}\frac{i+j}{4\pi}\int \mathrm{d}\hat{\boldsymbol{e}}\,\varphi(\boldsymbol{V}-v_c'\hat{\boldsymbol{e}}). \tag{28}$$

If $\varphi(\boldsymbol{V}-v_c'\hat{\boldsymbol{e}}) = \varphi(\boldsymbol{V}) - \hat{\boldsymbol{e}}\varphi(v_c')$, then we have

$$\delta_{k,1}\frac{i+j}{4\pi}\int \mathrm{d}\hat{\boldsymbol{e}}\int \mathrm{d}\boldsymbol{v}\,\varphi(\boldsymbol{v})\delta(\boldsymbol{v}-\boldsymbol{V}-v_c'\hat{\boldsymbol{e}}) = \delta_{k,1}(i+j)\varphi(\boldsymbol{V}). \tag{29}$$

DISTRIBUTION FUNCTION

The statistical description of the system is fully described by a set of distribution functions $f_k(\boldsymbol{r},\boldsymbol{v},t)$. It is normalized, such that $f_k(\boldsymbol{r},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{v}$ gives the number of particles of size k in the phase space volume $\mathrm{d}\Gamma = \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{v}$, around the point $(\boldsymbol{r},\boldsymbol{v})$. Hence, integrating over the whole phase space gives us the total number of particles of size k

$$N_k = \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{v}\,f_k(\boldsymbol{r},\boldsymbol{v},t). \tag{30}$$

The spacial distribution of particles is not very important for us, hence in the following we assume that the system is spatially homogeneous, and we use only the velocity distribution function $f_k(\boldsymbol{v},t)$

$$N_k = \int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{v}\,f_k(\boldsymbol{v},t), \tag{31}$$

hence

$$n_k \equiv \frac{N_k}{V} = \int \mathrm{d}\boldsymbol{v}\,f_k(\boldsymbol{v},t), \tag{32}$$

is the number density of the subsystem of particles with size k. The other field functions, such as the mean flow velocity $\boldsymbol{u}_k$ or granular temperature T_k can be defined as velocity moments of the distribution function

$$\begin{aligned} n_k\boldsymbol{u}_k &= \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}f_k(\boldsymbol{v},t), \\ \frac{3}{2}n_kT_k &= \int \mathrm{d}\boldsymbol{v}\,\frac{m_kc_k^2}{2}f_k(\boldsymbol{v},t), \\ \boldsymbol{c}_k &= \boldsymbol{v}-\boldsymbol{u}_k. \end{aligned} \tag{33}$$

KINETIC EQUATIONS

The time evolution of the distribution functions obey the Boltzmann equations

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} - \frac{1}{m_k}\frac{\partial U(r)}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{v}}\right) f_k(\boldsymbol{r}, \boldsymbol{v}, t) = \mathcal{I}_k(f_i, f_j), \tag{34}$$

where $U(r)$ is the potential of the external gravitational field. The LHS of the Boltzmann equation describes the change over time in the function f_k due to the local flow of the particles, subject to external driving. The function $\mathcal{I}(f_i, f_j)$ on the RHS is the *collision integral*, which describes the change over time in the function f_k due to collisions of particles i with particles of size j. Since we have three types of collisional outcomes, the collision integral $\mathcal{I}$ has to take into account all these types of outcomes. Without the loss of generality, we can write the collision integral as a sum of three functions

$$\mathcal{I}_k(f_i, f_j) = \mathcal{I}_k^{\text{agg}}(f_i, f_j) + \mathcal{I}_k^{\text{res}}(f_i, f_j) + \mathcal{I}_k^{\text{frag}}(f_i, f_j), \tag{35}$$

each corresponding to the specific type of collision.

General structure of collision integrals

Let us consider a collision integral $\mathcal{J}(f_i, f_j)$ for a generalized collision. If we consider a small volume in the phase space $\mathrm{d}\Gamma$ around a point $(\boldsymbol{r}, \boldsymbol{v})$, the term $f_i(\boldsymbol{r}, \boldsymbol{v}, t)\,\mathrm{d}\Gamma$ gives us the number of particles of size i in that volume at time t. The collision integral shows how many particles leave and enter this phase space volume per unit time, due to collisions only. So, the collision integral contains two terms, the gain term, which shows the number of particles that enter the phase space volume per unit time, and the loss terms, which shows the number of particles that leave this phase space volume per unit time

$$\mathcal{J}_k(f_i, f_j) = \mathcal{G}_k(f_i, f_j) - \mathcal{L}_k(f_i, f_j). \tag{36}$$

These terms are proportional to the number of collisions happening in unit volume per unit time. Estimation of the number of collisions between the particles of sizes i and j gives us

$$\mathrm{d}N_{ij}^{\text{cols}} = \sigma_{ij}^2\,\mathrm{d}\boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| f_i(\boldsymbol{v}_i, t) f_j(\boldsymbol{v}_j, t). \tag{37}$$

Now, integrating over all possible pairs of velocities and mass combinations, we can write the gain and loss terms as

$$\begin{aligned}
\mathcal{G}_k(f_i, f_j) &= \sum_{i,j} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\Omega\, P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) f_i(\boldsymbol{v}_i, t) f_j(\boldsymbol{v}_j, t), \\
\mathcal{L}_k(f_i, f_j) &= \sum_{i,j} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\Omega\, \delta_{k,i}\delta(\boldsymbol{v}_i - \boldsymbol{v}_k) f_i(\boldsymbol{v}_i, t) f_j(\boldsymbol{v}_j, t).
\end{aligned} \tag{38}$$

The delta functions in the loss term make sure that one of the collision partners is always the considered particle of size k and velocity $\boldsymbol{v}_k$. Now, the collision integral for a general type of collision is written as

$$\mathcal{I}_k(f_i, f_j) = \sum_{i,j} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\Omega\,[P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) - \delta_{k,i}\delta(\boldsymbol{v}_i - \boldsymbol{v}_k)] f_i f_j, \tag{39}$$

where

$$\mathrm{d}\Omega = \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|, \quad \boldsymbol{g} = \boldsymbol{v}_i - \boldsymbol{v}_j. \tag{40}$$

For specific types of collisions, e.g. aggregation, fragmentation, restitution, we have to make sure that the integration domains are specified as well. Usually, this domains are the function of the relative velocity $\mathcal{D}(\boldsymbol{g})$. Hence, the specific collision integrals are written as

$$\mathcal{I}_k^{\text{type}}(f_i, f_j) = \sum_{i,j} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_j\,\mathcal{D}^{\text{type}}(\boldsymbol{g}) \int \mathrm{d}\Omega\,\left[P_k^{\text{type}}(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) - \delta_{k,i}\delta(\boldsymbol{v}_i - \boldsymbol{v}_k)\right] f_i f_j, \tag{41}$$

where *type* can be aggregation, fragmentation or restitution.

HYDRODYNAMIC EQUATIONS

Using the kinetic equations, we can construct the balance equations for macroscopic or hydrodynamic fields. Namely, we focus on three of them, the number density fields $\{n_k(\boldsymbol{r},t)\}$, the mean velocity field $\boldsymbol{u}_k = \boldsymbol{u}(\boldsymbol{r},t)$, and the temperature fields $\{T_k(\boldsymbol{r},t)\}$. All these fields can be defined as certain moments of the distribution function $f_k(\boldsymbol{v},\boldsymbol{r},t)$

$$\begin{aligned} n_k(\boldsymbol{r},t) &= \int \mathrm{d}\boldsymbol{v}\, f_k(\boldsymbol{v},\boldsymbol{r},t), \\ n_k\boldsymbol{u}_k(\boldsymbol{r},t) &= \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_k(\boldsymbol{v},\boldsymbol{r},t), \\ \frac{3}{2}n_k T_k(\boldsymbol{r},t) &= \int \mathrm{d}\boldsymbol{v}\, \frac{m_k c_k^2}{2} f_k(\boldsymbol{v},\boldsymbol{r},t), \\ \boldsymbol{c}_k(\boldsymbol{r},t) &= \boldsymbol{v} - \boldsymbol{u}_k(\boldsymbol{r},t). \end{aligned} \tag{42}$$

Also, we introduce the momentum flux (pressure tensor) and heat flux terms as

$$\begin{aligned} \boldsymbol{P}_k(\boldsymbol{r},t) &= \int \mathrm{d}\boldsymbol{v}\, m_k \boldsymbol{c}_k \boldsymbol{c}_k f_k(\boldsymbol{r},\boldsymbol{v},t), \\ \boldsymbol{q}_k(\boldsymbol{r},t) &= \int \mathrm{d}\boldsymbol{v}\, \frac{m_k c_k^2}{2} \boldsymbol{c}_k f_k(\boldsymbol{r},\boldsymbol{v},t). \end{aligned} \tag{43}$$

Here, the notation with two vectors next to each other $\boldsymbol{cc}$ denote a dyadic tensor, while $\boldsymbol{c}\cdot\boldsymbol{c}$ is a dot product.

We can write the mean field equations, by averaging the parameters over all ensembles k.

$$\begin{aligned} n(\boldsymbol{r},t) &= \sum_k n_k(\boldsymbol{r},t) = const, \\ n\boldsymbol{u}(\boldsymbol{r},t) &= \sum_k n_k \boldsymbol{u}_k(\boldsymbol{r},t), \\ nT(\boldsymbol{r},t) &= \sum_k n_k T_k(\boldsymbol{r},t). \end{aligned} \tag{44}$$

Since the total mass of the entire system is constant $n(\boldsymbol{r},t) = const$, we need a more informative parameter to describe the dynamics of the mean mass of the system. We can use the squared number density as a more informative parameter

$$\bar{n}(\boldsymbol{r},t) = \frac{1}{n(\boldsymbol{r},t)} \sum_k n_k^2(\boldsymbol{r},t). \tag{45}$$

In the following, we normalize the total number density as one $n(\boldsymbol{r},t) = 1$ for the sake of brevity. The hydrodynamic balance equations can be obtained by multiplying the kinetic equation with specific functions and integrating over the velocities.

Balance equations

Let us take a certain function of the velocity $\psi_k(\boldsymbol{v})$, which describe a specific physical characteristics of the system. We can multiply the kinetic equation (34) by this function, and integrate over the velocity

$$\int \mathrm{d}\boldsymbol{v} \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{w}\cdot\frac{\partial}{\partial \boldsymbol{v}} \right) \psi_k(\boldsymbol{v}) f_k(\boldsymbol{r},\boldsymbol{v},t) = \int \mathrm{d}\boldsymbol{v}\, \mathcal{I}_k(f_i,f_j)\psi_k(\boldsymbol{v}), \tag{46}$$

where

$$\boldsymbol{w} = -\frac{1}{m_k}\frac{\partial U(r)}{\partial \boldsymbol{r}}. \tag{47}$$

The RHS of this equation can be written as a difference of gain and loss terms, without a loss of generality

$$\int \mathrm{d}\boldsymbol{v}\, \mathcal{I}_k(f_i,f_j)\psi_k(\boldsymbol{v}) = \mathcal{G}(\psi_k) - \mathcal{L}(\psi_k) = \left\langle \frac{\partial}{\partial t}(n_k\psi_k) \right\rangle_{\text{coll}}, \tag{48}$$

where the exact forms of the gain and loss functions are obtained only after specifying the distribution function $f_k(\boldsymbol{r}, \boldsymbol{v}, t)$. If the physical characteristic specified by the function $\psi_k(\boldsymbol{v})$ is conserved after a collision, the gain and loss terms are identical and the RHS vanish. The piece by piece integration gives us the next generalized form for balance equations of the physical value $\psi_k(\boldsymbol{v})$

$$\frac{\partial}{\partial t}\int \mathrm{d}\boldsymbol{v}\, \psi_k(v_\beta) f_k + \frac{\partial}{\partial r_\alpha}\int \mathrm{d}\boldsymbol{v}\, \psi_k(v_\beta) v_\alpha f_k - w_\beta \int \mathrm{d}\boldsymbol{v}\, \frac{\partial \psi_k(v_\alpha)}{\partial v_\alpha} f_k = \left\langle \frac{\partial}{\partial t}(n_k \psi_k) \right\rangle_{\mathrm{coll}}, \tag{49}$$

where we use the index notation forms, and greek letters α, β, γ denote the coordinate indices. Also, we invoke the summation notation convention for the repeated indices. By plugging in the functions $\psi_k = m_k$, $\psi_k = m_k v_\beta$ and $\psi_k = m_k v^2/2$, we obtain the balance equations for the mass density, momentum density and energy density

$$\begin{aligned}
\frac{\partial \rho_k}{\partial t} + \frac{\partial}{\partial r_\alpha}(\rho_k u_\alpha) &= \left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}, \\
\rho_k \frac{\partial u_\beta}{\partial t} + \rho_k u_\alpha \frac{\partial u_\beta}{\partial r_\alpha} + \frac{\partial P_{\alpha\beta}}{\partial r_\alpha} &= \rho_k w_\beta - u_\beta \left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}, \\
\frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) + \frac{\partial}{\partial r_\alpha}\left(\frac{3}{2} n_k T_k u_\beta\right) + \frac{\partial q_\alpha}{\partial r_\alpha} + P_{\alpha\beta}\frac{\partial u_\beta}{\partial r_\alpha} &= \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_{\mathrm{coll}} + \frac{u^2}{2}\left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}.
\end{aligned} \tag{50}$$

To proceed further, we assume certain condition in our system. First condition is that the mass density and temperature fields are homogeneous in space, hence their gradients vanish. This also implies that the heat flux is absent. Second condition is that the mean velocity field is stationary but has a steady spacial gradient. These conditions give us

$$\frac{\partial \rho_k}{\partial r_\alpha} = \mathbf{0}, \quad \frac{\partial T_k}{\partial r_\alpha} = \mathbf{0}, \quad q_\alpha = \mathbf{0}. \quad \frac{\partial u_\alpha}{\partial t} = 0, \tag{51}$$

Using these conditions, our balance equations simplify into

$$\begin{aligned}
\frac{\partial \rho_k}{\partial t} + \rho_k \frac{\partial u_\alpha}{\partial r_\alpha} &= \left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}, \\
\rho_k u_\alpha \frac{\partial u_\beta}{\partial r_\alpha} + \frac{\partial P_{\alpha\beta}}{\partial r_\alpha} &= \rho_k w_\beta - u_\beta \left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}, \\
\frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) + \frac{3}{2} n_k T_k \frac{\partial u_\alpha}{\partial r_\alpha} + P_{\alpha\beta}\frac{\partial u_\beta}{\partial r_\alpha} &= \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_{\mathrm{coll}} + \frac{u^2}{2}\left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}.
\end{aligned} \tag{52}$$

Next, we can assume a zero divergence mean velocity field, which yields the balance equations in the next form

$$\begin{aligned}
\frac{\partial \rho_k}{\partial t} &= \left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}, \\
\frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) &= -P_{\alpha\beta}\frac{\partial u_\beta}{\partial r_\alpha} + \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_{\mathrm{coll}} + \frac{u^2}{2}\left\langle \frac{\partial \rho_k}{\partial t} \right\rangle_{\mathrm{coll}}.
\end{aligned} \tag{53}$$

To close the balance equations, we have to write the pressure tensor in terms of the macroscopic fields ρ_k, $\boldsymbol{u}$ and T_k. In the simplest form, we can write the empirical dependence

$$P_{\alpha\beta} = p\delta_{\alpha\beta} - \eta\left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial r_\gamma}\right), \tag{54}$$

where p is the hydrostatic pressure and η is the shear viscosity coefficient. It is shown that the hydrostatic pressure depends on the granular temperature through the equation of state

$$p = nT\left(1 + \frac{1+\varepsilon}{3}\pi n \sigma^3 g_2(\sigma)\right). \tag{55}$$

Writing a similar form for te polydisperse system, and assuming the zero divergence mean velocity field, we have for the pressure tensor

$$P_{\alpha\beta} = n_k T_k (1 + \phi_k)\delta_{\alpha\beta} - \eta\left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha}\right), \tag{56}$$

and write the final form of the balance equations as

$$\begin{aligned}
\frac{\partial n_k}{\partial t} &= \left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}, \\
\frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) &= \eta_k \left[\frac{\partial u_\alpha}{\partial r_\beta}\frac{\partial u_\beta}{\partial r_\alpha} + \frac{\partial u_\beta}{\partial r_\alpha}\frac{\partial u_\beta}{\partial r_\alpha}\right] + \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_{\text{coll}} + \frac{m_k u^2}{2}\left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}.
\end{aligned} \tag{57}$$

Here we used

$$\delta_{\alpha\beta}\frac{\partial u_\beta}{\partial r_\alpha} = \frac{\partial u_\alpha}{\partial r_\alpha} = 0. \tag{58}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) = \frac{3}{2} n_k \frac{\partial T_k}{\partial t} + \frac{3}{2} T_k \frac{\partial n_k}{\partial t} = \frac{3}{2} n_k \frac{\partial T_k}{\partial t} + \frac{3}{2} T_k \left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}, \tag{59}$$

we can rewrite the hydrodynamic balance equations as

$$\begin{aligned}
\frac{\partial n_k}{\partial t} &= \left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}, \\
\frac{3}{2} n_k \frac{\partial T_k}{\partial t} &= \eta_k \left[\frac{\partial u_\alpha}{\partial r_\beta}\frac{\partial u_\beta}{\partial r_\alpha} + \frac{\partial u_\beta}{\partial r_\alpha}\frac{\partial u_\beta}{\partial r_\alpha}\right] + \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_{\text{coll}} + \left(\frac{m_k u^2}{2} - \frac{3T_k}{2}\right)\left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}.
\end{aligned} \tag{60}$$

The mean velocity field in the ring environment is expressed as

$$\boldsymbol{u} = -\frac{3}{2}\Omega x \hat{\boldsymbol{e}}_y, \tag{61}$$

in the local Hill's box, Ω is the orbital speed. Hence, the velocity gradients tensor has a single non-zero term

$$\frac{\partial u_y}{\partial x} = -\frac{3}{2}\Omega, \tag{62}$$

which means that

$$\frac{\partial u_\alpha}{\partial r_\beta}\frac{\partial u_\beta}{\partial r_\alpha} + \frac{\partial u_\beta}{\partial r_\alpha}\frac{\partial u_\beta}{\partial r_\alpha} = \frac{9}{2}\Omega^2. \tag{63}$$

Now, we write our balance equations as

$$\begin{aligned}
\frac{\partial n_k}{\partial t} &= \left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}, \\
\frac{3}{2} n_k \frac{\partial T_k}{\partial t} &= \frac{9}{2}\Omega^2 \eta_k + \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_{\text{coll}} + \left(\frac{m_k u^2}{2} - \frac{3T_k}{2}\right)\left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}}.
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
\left\langle \frac{\partial n_k}{\partial t} \right\rangle_{\text{coll}} &= \frac{1}{2}\sum_{i+j=k} K_{ij}^{an} n_i n_j + \delta_{k,1}\sum_{i,j}(i+j) K_{ij}^{fn} n_i n_j - n_k \sum_j \left(K_{kj}^{an} + K_{kj}^{fn}\right) n_j, \\
K_{ij}^{an} &= \nu_{ij}\left[1 - \left(1 + C_{ij} g_{\text{agg}}^2\right)\exp\left(-C_{ij} g_{\text{agg}}^2\right)\right], \\
K_{ij}^{fn} &= \nu_{ij}\exp\left(-C_{ij} g_{\text{frag}}^2\right), \\
C_{ij} &= \left(\frac{2T_i}{m_i} + \frac{2T_j}{m_j}\right)^{-1}, \\
\nu_{ij} &= 2\sigma_{ij}^2 \sqrt{\frac{2\pi T_i}{m_i} + \frac{2\pi T_j}{m_j}},
\end{aligned} \tag{65}$$

and

$$
\begin{aligned}
\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_{\text{coll}} = & \frac{1}{2}m_k \sum_{i+j=k}\left[\frac{1}{2}K_{ij}^{aT}\left[\frac{\mu_i u_i^2 - \mu_j u_j^2}{u_i^2+u_j^2}\right]^2 + \frac{3}{4}K_{ij}^{an}\frac{u_i^2 u_j^2}{u_i^2+u_j^2}\right] n_i n_j - \\
& - m_k n_k \sum_j \left[\frac{1}{2}K_{kj}^{aT}\left(\frac{u_k^2}{u_k^2+u_j^2}\right)^2 + \frac{3}{4}K_{kj}^{an}\frac{u_k^2 u_j^2}{u_k^2+u_j^2}\right] n_j - \\
& - \frac{1+\varepsilon}{2} m_k n_k \sum_j \frac{\mu_j R_{kj}}{A_{kj}^2} F_{kj}\cdot n_j - \frac{1-\varepsilon^2}{2} m_k n_k \sum_j \frac{\mu_j^2}{A_{kj}} F_{kj}\cdot n_j + \\
& + \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{R_{ij}^2}{4A_{ij}} - \frac{\gamma_{ij}A_{ij}}{2}\right]\alpha_{ij}\sigma_{ij}^2\frac{1}{2C_{ij}^3}\left(2+2C_{ij}g_{\text{frag}}^2 + C_{ij}^2 g_{\text{frag}}^4\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right) - \\
& - \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{R_{ij}^2}{4A_{ij}}g_{\text{frag}}^2 - \frac{3}{2}\right]\alpha_{ij}\sigma_{ij}^2\frac{1}{2C_{ij}^2}\left(1+C_{ij}g_{\text{frag}}^2\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right) - \\
& - \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{3}{2}g_{\text{frag}}^2 + \frac{\gamma_{ij}A_{ij}}{2}g_{\text{frag}}^4\right]\alpha_{ij}\sigma_{ij}^2\frac{1}{2C_{ij}}\exp\left(-C_{ij}g_{\text{frag}}^2\right) - \\
& - \frac{m_k}{2}\sum_j 4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\left[\frac{R_{kj}^2}{4A_{kj}} + \mu_j^2 A_{kj} + \mu_j R_{kj}\right]\alpha_{kj}\sigma_{kj}^2\frac{1}{2C_{ij}^3}\left(2+2C_{ij}g_{\text{frag}}^2 + C_{ij}^2 g_{\text{frag}}^4\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right) + \\
& + \frac{m_k}{2}\sum_j 4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\left[\left(\frac{R_{kj}^2}{4A_{kj}} + \mu_j^2 A_{kj} + \mu_j R_{kj}\right)\cdot g_{\text{frag}}^2 - \frac{3}{2}\right]\alpha_{kj}\sigma_{kj}^2\frac{1}{2C_{ij}^2}\left(1+C_{ij}g_{\text{frag}}^2\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right) + \\
& + \frac{m_k}{2}\sum_j 4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\frac{3}{2}g_{\text{frag}}^2\alpha_{kj}\sigma_{kj}^2\frac{1}{2C_{ij}}\exp\left(-C_{ij}g_{\text{frag}}^2\right).
\end{aligned}
\tag{66}
$$

GENERAL FORM OF KINETIC COLLISION INTEGRALS

In our model, we face three types of collision integrals, for each type of collision. The aggregative integrals, the restitutive integrals and the fragmentative integrals. We can solve the typical forms of each type of collision integrals in the most general form

$$
\begin{aligned}
I_a^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^{2l} e^{-Aw^2 - Bu^2 + R\boldsymbol{w}\cdot\boldsymbol{u}}\Theta(v_a - u)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q, \\
I_r^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^{2l} e^{-Aw^2 - Bu^2 + R\boldsymbol{w}\cdot\boldsymbol{u}}\Theta(u - v_a)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big], \\
I_f^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^{2l} e^{-Aw^2 - Bu^2 + R\boldsymbol{w}\cdot\boldsymbol{u}}\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2 - v_f^2\Big],
\end{aligned}
\tag{67}
$$

where k,l,m,p,q are integers, and $q=\{0,1\}$. The difference in each type of integrals is in the domains of the vector $\boldsymbol{u}$. In the aggregative case, the values of u have to be less than a certain threshold v_a, in the restitutive case, the values of u have to be larger than v_a, but restricted by the parameter v_f from above. Finally, in the fragmentative case, the values of u are restricted by the parameter v_f from below.

Angular integrals

We start by first solving the inner integrals over $\hat{\boldsymbol{n}}$. By its physical meaning, we can call them angular integrals. Note, that q can be either 0 or 1, meaning that the corresponding term either do exist or is absent

$$\begin{aligned} I^{m,p,q}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q, \\ I^{m,p,q}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[v_f^2-(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big], \\ I^{m,p,q}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2-v_f^2\Big]. \end{aligned} \tag{68}$$

If $q=0$, then the angular integral is a function of only the vector $\boldsymbol{u}$, otherwise it is a function of both vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

Let us first solve the aggregative angular integrals

Aggregative angular integrals

We start with a simpler case when $q=0$ and the angular integral is a function of only $\boldsymbol{u}$

$$I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p. \tag{69}$$

To solve this integral, we fix the vector $\boldsymbol{u}$, and denote by θ the angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{n}}$. In the spherical coordinates we have $\mathrm{d}\hat{\boldsymbol{n}} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, and the integral can be written as

$$\begin{aligned} I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}) &= 2\pi u^{m+p}\int_0^\pi \mathrm{d}\theta \sin\theta\Theta(-\cos\theta)|\cos\theta|^m(\cos\theta)^p = \\ &= 2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}\theta \sin\theta|\cos\theta|^m(\cos\theta)^p = \\ &= -2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}(\cos\theta)\,|\cos\theta|^m(\cos\theta)^p, \end{aligned} \tag{70}$$

where we have integrated out over φ to give us the 2π factor. Now, substituting $\cos\theta = z$, we write

$$I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}) = 2\pi u^{m+p}\int_{-1}^0 \mathrm{d}z\,|z|^m z^p. \tag{71}$$

Since in the integration domain z is always negative, we now that $z^p<0$ for odd values of p, and $z^p>0$ for even values of p, hence we can write $z^p = (-1)^p|z|^p$, and

$$I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}) = 2\pi u^{m+p}\cdot(-1)^p\int_{-1}^0 \mathrm{d}z\,|z|^{m+p} = -(-1)^p\cdot 2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p}. \tag{72}$$

We can see that for both odd and even values of $m+p$, the integral gives the same result, and finally we have

$$I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}) = (-1)^p\cdot\frac{2\pi u^{m+p}}{m+p+1}. \tag{73}$$

The case with $q=1$ is trickier, since we have two arbitrary angles $\angle(\hat{\boldsymbol{n}},\boldsymbol{u})$ and $\angle(\hat{\boldsymbol{n}},\boldsymbol{w})$. However, we can write it as a dot product of $\boldsymbol{w}$ and another vector $\boldsymbol{F}$ as

$$\begin{aligned} I^{m,p,1}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}}) = \\ &= \boldsymbol{w}\cdot\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\hat{\boldsymbol{n}} = \boldsymbol{w}\cdot\boldsymbol{F}, \end{aligned} \tag{74}$$

where the vector $\boldsymbol{F}$ is constructed by vectors $\hat{\boldsymbol{n}}$ and $\boldsymbol{u}$

$$\boldsymbol{F} = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}}. \tag{75}$$

Since it is being integrated over $\hat{\boldsymbol{n}}$, it cannot depend on $\hat{\boldsymbol{n}}$. This means that it can be oriented only along the vector $\boldsymbol{u}$, or $\boldsymbol{F} = f\boldsymbol{u}$. Now we can write

$$\begin{aligned} u^2 f = \boldsymbol{u}\cdot\boldsymbol{F} = \boldsymbol{u}\cdot \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}} = \\ = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^{p+1} = I^{m,p+1,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}), \end{aligned} \tag{76}$$

or

$$I^{m,p,1}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w}) = f\boldsymbol{w}\cdot\boldsymbol{u} = \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{u^2}\cdot I^{m,p+1,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}), \tag{77}$$

which gives us the value of the integral

$$I^{m,p,1}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w}) = (-1)^{p+1}\cdot\frac{2\pi u^{m+p-1}}{m+p+2}(\boldsymbol{w}\cdot\boldsymbol{u}). \tag{78}$$

Now, we can combine both cases of $q = 0$ and $q = 1$, and write

$$I^{m,p,q}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p (\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q = (-1)^{p+q}\cdot\frac{2\pi u^{m+p-q}}{m+p+q+1}\cdot(\boldsymbol{w}\cdot\boldsymbol{u})^q, \quad q=\{0,1\}. \tag{79}$$

Restitutive angular integrals

These type of integrals have a domain restriction terms given by the parameter v_f. We can start with a simpler case when $q = 0$, and write

$$\begin{aligned} I^{m,p,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big] = \\ = (-1)^p \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^{m+p} \Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big]. \end{aligned} \tag{80}$$

Again, switching to spherical coordinates, and denoting the angle $\angle(\hat{\boldsymbol{n}},\boldsymbol{u})$ by θ, we write

$$\begin{aligned} I^{m,p,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}) = (-1)^p\cdot 2\pi u^{m+p}\int_0^\pi \mathrm{d}\theta \sin\theta\Theta(-\cos\theta)|\cos\theta|^{m+p}\Theta\Big[v_f^2 - (u\cos\theta)^2\Big] = \\ = (-1)^p\cdot 2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}\theta \sin\theta|\cos\theta|^{m+p}\Theta\left[\frac{v_f^2}{u^2} - \cos^2\theta\right]. \end{aligned} \tag{81}$$

The domain restriction implies

$$|\cos\theta| \leqslant \frac{v_f}{u}. \tag{82}$$

This constraint restricts two variable, both θ and u, although we do not perform integration over u at this moment. Since the variable u changes from 0 to ∞, the restriction can be split into two cases, (i) when $u \leqslant v_f$, (ii) when $u > v_f$. In the first case, when $u \leqslant v_f$, the restriction holds true for any values of $\theta \in [\pi/2, \pi]$, e.g. no constraint in the angle θ. In the second case, when $u > v_f$, the restriction holds true only within a certain range of values of θ, namely $\theta \in [\pi/2, \pi - \arccos(v_f/u)]$. Now, we can rewrite the domain restriction term as

$$\Theta\left[\frac{v_f^2}{u^2} - \cos^2\theta\right] = \Theta(v_f - u) + \Theta(u - v_f)\Theta\Big[\pi - \arccos\left(\frac{v_f}{u}\right) - \theta\Big]. \tag{83}$$

Using this form of the restriction allows us to solve the restitutive angular integrals

$$
\begin{aligned}
I^{m,p,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}) = &- (-1)^p \cdot 2\pi u^{m+p}\Theta(v_f - u)\int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p} - \\
&- (-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f)\int_{\pi/2}^{\pi-\arccos(v_f/u)} \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p}.
\end{aligned}
\tag{84}
$$

The first integral is already solved for the aggregative angular case, and in the second integral we substitute $z = \cos\theta$, and write

$$
\begin{aligned}
I^{m,p,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}) &= \Theta(v_f - u)\cdot I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u}) - (-1)^p\cdot 2\pi u^{m+p}\Theta(u - v_f)\int_0^{-v_f/u} \mathrm{d}z\,|z|^{m+p} = \\
&= (-1)^p\cdot\Theta(v_f - u)\cdot\frac{2\pi u^{m+p}}{m+p+1} + (-1)^p\cdot\Theta(u - v_f)\cdot\frac{2\pi u^{m+p}}{m+p+1}\Big(\frac{v_f}{u}\Big)^{m+p+1} = \\
&= (-1)^p\cdot\frac{2\pi u^{m+p}}{m+p+1}\bigg[\Theta(v_f - u) + \Theta(u - v_f)\Big(\frac{v_f}{u}\Big)^{m+p+1}\bigg],
\end{aligned}
\tag{85}
$$

or

$$
I^{m,p,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}) = I^{m,p,0}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u})\cdot\bigg[\Theta(v_f - u) + \Theta(u - v_f)\Big(\frac{v_f}{u}\Big)^{m+p+1}\bigg].
\tag{86}
$$

For the case $q = 1$, we can perform the same procedure as before, and write

$$
I^{m,p,1}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u},\boldsymbol{w}) = \boldsymbol{w}\cdot\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big]\hat{\boldsymbol{n}} = \boldsymbol{w}\cdot\boldsymbol{F},
\tag{87}
$$

where

$$
\boldsymbol{F} = \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big]\hat{\boldsymbol{n}}.
\tag{88}
$$

Again, we see that $\boldsymbol{F}$ vector cannot depend on $\hat{\boldsymbol{n}}$, and depends only on the vector $\boldsymbol{u}$. This implies that $\boldsymbol{F} = f\boldsymbol{u}$, or

$$
f = \frac{\boldsymbol{F}\cdot\boldsymbol{u}}{u^2} = \frac{1}{u^2}\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^{p+1}\Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big] = u^{-2}\cdot I^{m,p+1,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}).
\tag{89}
$$

Since,

$$
I^{m,p,1}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u},\boldsymbol{w}) = \boldsymbol{w}\cdot\boldsymbol{F} = (\boldsymbol{w}\cdot\boldsymbol{u})f = \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{u^2}\cdot I^{m,p+1,0}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u}),
\tag{90}
$$

or writing explicitly, we have

$$
I^{m,p,1}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u},\boldsymbol{w}) = (-1)^{p+1}\cdot\frac{2\pi u^{m+p-1}}{m+p+2}\bigg[\Theta(v_f - u) + \Theta(u - v_f)\Big(\frac{v_f}{u}\Big)^{m+p+2}\bigg](\boldsymbol{w}\cdot\boldsymbol{u}).
\tag{91}
$$

By combining both cases $q = 0$ and $q = 1$, we write the final solution of the restitutive angular integrals as

$$
I^{m,p,q}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u},\boldsymbol{w}) = (-1)^{p+q}\cdot\frac{2\pi u^{m+p-q}}{m+p+q+1}\bigg[\Theta(v_f - u) + \Theta(u - v_f)\Big(\frac{v_f}{u}\Big)^{m+p+q+1}\bigg](\boldsymbol{w}\cdot\boldsymbol{u})^q.
\tag{92}
$$

Fragmentative angular integrals

The last type of angular integrals is the fragmentative type, which is very similar to the restitutive angular case. Again, we start with the simpler case of $q = 0$

$$
I^{m,p,0}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u}) = \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2 - v_f^2\Big].
\tag{93}
$$

The difference of this type of angular integrals is in the inverse domain restriction function. Switching into spherical coordinates, we write

$$I_{\hat{\boldsymbol{n}},f}^{m,p,0}(\boldsymbol{u}) = -(-1)^p \cdot 2\pi u^{m+p} \int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\, |\cos\theta|^{m+p} \Theta\left[\cos^2\theta - \frac{v_f^2}{u^2}\right]. \tag{94}$$

The domain restriction is now given as

$$|\cos\theta| \geqslant \frac{v_f}{u}. \tag{95}$$

This condition can be satisfied only for $u \geqslant v_f$, and we can rewrite the domain restriction as

$$\Theta\left[\cos^2\theta - \frac{v_f^2}{u^2}\right] = \Theta(u - v_f)\Theta\Big[\theta - \pi - \arccos\Big(\frac{v_f}{u}\Big)\Big], \tag{96}$$

and our fragmentative angular integral becomes

$$\begin{aligned} I_{\hat{\boldsymbol{n}},f}^{m,p,0}(\boldsymbol{u}) &= -(-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f) \int_{\pi - \arccos(v_f/u)}^{\pi} \mathrm{d}(\cos\theta)\, |\cos\theta|^{m+p} = \\ &= -(-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f) \int_{-v_f/u}^{-1} \mathrm{d}z\, |z|^{m+p} = \\ &= (-1)^p \cdot \frac{2\pi u^{m+p}}{m+p+1}\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+1}\right]\Theta(u - v_f). \end{aligned} \tag{97}$$

The case with $q = 1$ can be solved exactly as the previous cases, and we can immediately write

$$I_{\hat{\boldsymbol{n}},f}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) = \frac{\boldsymbol{w} \cdot \boldsymbol{u}}{u^2} \cdot I_{\hat{\boldsymbol{n}},f}^{m,p+1,0}(\boldsymbol{u}) = (-1)^{p+1} \cdot \frac{2\pi u^{m+p-1}}{m+p+2}\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+2}\right]\Theta(u - v_f)(\boldsymbol{w} \cdot \boldsymbol{u}), \tag{98}$$

and combining both cases $q = 0$ and $q = 1$, we have

$$I_{\hat{\boldsymbol{n}},f}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) = (-1)^{p+q} \cdot \frac{2\pi u^{m+p-q}}{m+p+q+1}\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+q+1}\right]\Theta(u - v_f)(\boldsymbol{w} \cdot \boldsymbol{u})^q. \tag{99}$$

Final results of angular integrals

Let us write the final results of solutions of the angular integrals for all three types of collision integrals. First, the solution of the aggregative angular integrals is

$$I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}})|\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^p (\boldsymbol{w} \cdot \hat{\boldsymbol{n}})^q = (-1)^{p+q} \cdot \frac{2\pi u^{m+p-q}}{m+p+q+1} \cdot (\boldsymbol{w} \cdot \boldsymbol{u})^q, \quad q = \{0, 1\}. \tag{100}$$

The restitutive and fragmentative angular integrals are solved to give us

$$\begin{aligned} I_{\hat{\boldsymbol{n}},r}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) &= I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) \cdot \left[\Theta(v_f - u) + \Theta(u - v_f)\Big(\frac{v_f}{u}\Big)^{m+p+q+1}\right], \\ I_{\hat{\boldsymbol{n}},f}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) &= I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) \cdot \left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+q+1}\right]\Theta(u - v_f). \end{aligned} \tag{101}$$

Center of mass velocity integrals

We refer to the integrals over the vector $\boldsymbol{w}$ as the center of mass velocity integrals. All three types of collision integrals contain similar forms of the center of mass velocity integrals, and we can write a generic form of such integrals as

$$I_{\boldsymbol{w}}^{l,q}(\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{w}\, w^{2l} (\boldsymbol{w} \cdot \boldsymbol{u})^q \exp\left(-Aw^2\right) \exp\left(R\boldsymbol{w} \cdot \boldsymbol{u}\right), \quad q = \{0, 1\}. \tag{102}$$

Switching into spherical coordinates, and denoting by θ the angle between vectors $\boldsymbol{w}$ and $\boldsymbol{u}$, we have $\mathrm{d}\boldsymbol{w} = w^2 \sin\theta \,\mathrm{d}w \,\mathrm{d}\theta \,\mathrm{d}\varphi$,

$$I_{\boldsymbol{w}}^{l,q}(\boldsymbol{u}) = -2\pi u^q \int_0^\infty \mathrm{d}w \, w^{2l+q+2} \exp\left(-Aw^2\right) \int_0^\pi \mathrm{d}(\cos\theta) \, (\cos\theta)^q \exp\left(Rwu \cdot \cos\theta\right), \quad q = \{0, 1\}. \tag{103}$$

Again, we solve these integrals for two different cases of q, starting with the simpler case

The case with $q = 0$

In this case we write

$$I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) = -2\pi \int_0^\infty \mathrm{d}w \, w^{2l+2} \exp\left(-Aw^2\right) \int_0^\pi \mathrm{d}(\cos\theta) \exp\left(Rwu \cdot \cos\theta\right). \tag{104}$$

The inner angular integral is solved to give us

$$I_R^0(\boldsymbol{u}, \boldsymbol{w}) = -\int_0^\pi \mathrm{d}(\cos\theta) \exp\left(Rwu \cdot \cos\theta\right) = \frac{1}{Rwu}\left(e^{Rwu} - e^{-Rwu}\right), \tag{105}$$

and substituting into the center of mass velocity integral, we have

$$I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) = \frac{2\pi}{Ru} \int_0^\infty \mathrm{d}w \, w^{2l+1} \exp\left(-Aw^2\right)\left(e^{Ru \cdot w} - e^{-Ru \cdot w}\right). \tag{106}$$

The case with $q = 1$

The case with $q = 1$ is

$$I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) = -2\pi u \int_0^\infty \mathrm{d}w \, w^{2l+3} \exp\left(-Aw^2\right) \int_0^\pi \mathrm{d}(\cos\theta) \, (\cos\theta) \exp\left(Rwu \cdot \cos\theta\right), \tag{107}$$

where the inner angular integral is solved to give us

$$I_R^1(\boldsymbol{u}, \boldsymbol{w}) = -\int_0^\pi \mathrm{d}(\cos\theta) \, (\cos\theta) \exp\left(Rwu \cdot \cos\theta\right) = \frac{1}{Rwu}\left(e^{Rwu} + e^{-Rwu}\right) - \frac{1}{R^2 w^2 u^2}\left(e^{Rwu} - e^{-Rwu}\right). \tag{108}$$

Now, the center of mass velocity integral reads

$$I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) = \frac{2\pi}{R} \int_0^\infty \mathrm{d}w \, w^{2l+2} \exp\left(-Aw^2\right)\left(e^{Ru \cdot w} + e^{-Ru \cdot w}\right) - \frac{2\pi}{R^2 u} \int_0^\infty \mathrm{d}w \, w^{2l+1} \exp\left(-Aw^2\right)\left(e^{Ru \cdot w} - e^{Ru \cdot w}\right). \tag{109}$$

Shifted Gaussian integrals

To proceed further, let us analyze the specific types of shifted Gaussian integrals, namely

$$I_{G,\pm}^n = \int_0^\infty \mathrm{d}x \, x^n \exp\left(-ax^2 \pm bx\right), \qquad n \in \{0, 1, 2, \ldots\}. \tag{110}$$

To get a general solution for these types of integrals, let us write them in a more canonical form first. To do so, let us introduce a variable transformation

$$t := \sqrt{a}x \pm \lambda, \qquad \lambda = \frac{b}{2\sqrt{a}}, \tag{111}$$

this implies

$$
\begin{aligned}
-ax^2 \pm bx &= -t^2 + \lambda^2, \\
\mathrm{d}x &= \frac{\mathrm{d}t}{\sqrt{a}}, \\
x^n &= a^{-n/2} \cdot (t \pm \lambda)^n = a^{-n/2} \cdot \sum_{k=0}^{n} \binom{n}{k} (\pm\lambda)^{n-k} t^k, \\
\binom{n}{k} &= \frac{n!}{k!(n-k)!}.
\end{aligned}
\tag{112}
$$

Now, our shifted Gaussian integrals become

$$
I^n_{G,\pm} = \frac{\exp\left(\lambda^2\right)}{\sqrt{a^{n+1}}} \sum_{k=0}^{n} \binom{n}{k} (\pm\lambda)^{n-k} \int_{\pm\lambda}^{\infty} \mathrm{d}t\, t^k e^{-t^2}. \tag{113}
$$

The $\pm$ sign is now hidden in the parameter λ. Let us concentrate on the canonical Gaussian integral

$$
G^k_{\pm} = \int_{\pm\lambda}^{\infty} \mathrm{d}t\, t^k e^{-t^2}. \tag{114}
$$

Let us start with integration by parts and put $u = e^{-t^2}$ and $\mathrm{d}v = t^k\,\mathrm{d}t$. This gives us $\mathrm{d}u = -2te^{-t^2}\,\mathrm{d}t$, and $v = t^{k+1}/(k+1)$. Now we have

$$
G^k_{\pm} = \frac{t^{k+1}}{k+1}\, e^{-t^2}\bigg|_{\pm\lambda}^{\infty} + \frac{2}{k+1} \int_{\pm\lambda}^{\infty} \mathrm{d}t\, t^{k+2} e^{-t^2} = -\frac{(\pm\lambda)^{k+1}}{k+1} e^{-\lambda^2} + \frac{2}{k+1} G^{k+2}_{\pm}, \tag{115}
$$

a recurrent relation for the integral

$$
G^{k+2}_{\pm} = \frac{k+1}{2} G^k_{\pm} + (\pm\lambda)^{k+1} \cdot \frac{1}{2} e^{-\lambda^2}. \tag{116}
$$

In order to get a full solution, let us calculate the first two cases $k = 0$ and $k = 1$. We have

$$
\begin{aligned}
G^0_{\pm} &= \int_{\pm\lambda}^{\infty} \mathrm{d}t\, e^{-t^2} = \frac{\sqrt{\pi}}{2}(1 \pm \mathrm{erf}(\lambda)), \\
G^1_{\pm} = G^1 &= \int_{\pm\lambda}^{\infty} \mathrm{d}t\, te^{-t^2} = \frac{1}{2} e^{-\lambda^2}.
\end{aligned}
\tag{117}
$$

Given this two functions, we can obtain the solution for any order k

$$
G^k_{\pm} = \frac{k-1}{2} G^{k-2}_{\pm} + (\pm\lambda)^{k-1} G^1, \tag{118}
$$

where we have rewritten our recurrent relation with the help of G^1. Extending the the recurrent relation, we obtain two different results, for odd and even k. For an even $k = 2p$, then we have

$$
G^{2p}_{\pm} = \frac{(2p-1)!!}{2^p} G^0_{\pm} \pm G^1 \sum_{j=1}^{p} \frac{1}{2^{j-1}} \frac{(2p-1)!!}{(2p-2j+1)!!} \lambda^{2p-2j+1}. \tag{119}
$$

Here we used the fact that $2p - 2j + 1$ is always an odd number, hence $(\pm\lambda)^{2p-2j+1} = \pm\lambda^{2p-2j+1}$. For odd valued $k = 2p + 1$, we have

$$
G^{2p+1}_{\pm} = G^{2p+1} = G^1 \sum_{j=1}^{p+1} \frac{1}{2^{j-1}} \frac{(2p)!!}{(2p-2j+2)!!} \lambda^{2p-2j+2}. \tag{120}
$$

Here we used the fact that $2p - 2j + 2$ is always an even number, hence $(\pm\lambda)^{2p-2j+2} = \lambda^{2p-2j+2}$. Now, we can write the solution of the original shifted Gaussian integrals as

$$
I^n_{G,\pm} = \frac{\exp\left(\lambda^2\right)}{\sqrt{a^{n+1}}} \sum_{k=0}^{n} \binom{n}{k} (\pm\lambda)^{n-k} G^k_{\pm}. \tag{121}
$$

Symmetric Gaussian integrals

In the context of our problem, the shifted Gaussian integrals appear in symmetric pairs, such as

$$I_{\pm}^{n} = I_{G,+}^{n} \pm I_{G,-}^{n} = \int_0^{\infty} \mathrm{d}x\, x^n e^{-ax^2}\left(e^{bx} \pm e^{-bx}\right). \tag{122}$$

From (121), we have

$$I_{\pm}^{n} = \frac{\exp\left(\lambda^2\right)}{\sqrt{a^{n+1}}} \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k}\left[G_{+}^{k} \pm (-1)^{n-k} G_{-}^{k}\right]. \tag{123}$$

If we notice, that $G_{+}^{0} + G_{-}^{0} = \sqrt{\pi}$, meaning that the error functions cancel each other, we can look for specific solution for odd or even values of n. Let us look at $n = 2p$ for a positive version of the symmetric integral, namely

$$I_{+}^{2p} = \frac{\exp\left(\lambda^2\right)}{a^p\sqrt{a}} \sum_{k=0}^{2p} \binom{2p}{k} \lambda^{2p-k}\left[G_{+}^{k} + (-1)^{2p-k} G_{-}^{k}\right]. \tag{124}$$

We can split the sum into two, summing up only the odd indices and only the even indices separately. This gives us

$$I_{+}^{2p} = \frac{\exp\left(\lambda^2\right)}{a^p\sqrt{a}} \left[\sum_{k=0}^{p} \binom{2p}{2k} \lambda^{2p-2k}\left(G_{+}^{2k} + G_{-}^{2k}\right) + \sum_{k=0}^{p-1} \binom{2p}{2k+1} \lambda^{2p-2k-1}\left(G_{+}^{2k+1} - G_{-}^{2k+1}\right)\right], \tag{125}$$

where we have used $(-1)^{2p-2k} = 1$ and $(-1)^{2p-2k-1} = -1$. From (119) and (120) we have

$$G_{+}^{2k} + G_{-}^{2k} = \frac{(2k-1)!!}{2^k}\sqrt{\pi}, \qquad G_{+}^{2k+1} - G_{-}^{2k+1} = 0. \tag{126}$$

Now, we have

$$I_{+}^{2p} = \sqrt{\frac{\pi}{a}} \frac{\exp\left(\lambda^2\right)}{a^p} \sum_{k=0}^{p} \binom{2p}{2k} \frac{(2k-1)!!}{2^k} \cdot \lambda^{2p-2k}. \tag{127}$$

Next, let us look at $n = 2p+1$ for the negative symmetric integral, namely

$$I_{-}^{2p+1} = \frac{\exp\left(\lambda^2\right)}{a^p\sqrt{a}} \sum_{k=0}^{2p+1} \binom{2p+1}{k} \lambda^{2p+1-k}\left[G_{+}^{k} - (-1)^{2p-k} G_{-}^{k}\right]. \tag{128}$$

Again, splitting the sum into two, we write

$$I_{-}^{2p+1} = \frac{\exp\left(\lambda^2\right)}{a^{p+1}} \left[\sum_{k=0}^{p} \binom{2p+1}{2k} \lambda^{2p-2k+1}\left(G_{+}^{2k} + G_{-}^{2k}\right) + \sum_{k=0}^{p} \binom{2p+1}{2k+1} \lambda^{2p-2k}\left(G_{+}^{2k+1} - G_{-}^{2k+1}\right)\right], \tag{129}$$

which results in

$$I_{-}^{2p+1} = \frac{\sqrt{\pi}\exp\left(\lambda^2\right)}{a^{p+1}} \sum_{k=0}^{p} \binom{2p+1}{2k} \frac{(2k-1)!!}{2^k} \cdot \lambda^{2p-2k+1}. \tag{130}$$

As we can see, the odd and negative symmetric integral, together with the even and positive symmetric integrals, do not contain the error function, and can be written purely in algebraic terms. Luckily, in our problem, for only even powers of w, in the original kinetic integrals, we will deal with only the analytic versions of the center of mass velocity integrals. Also, the main physical parameters, are described by the analytic versions of these integrals as we will see later.

Solutions of the center of mass velocity integrals

Let us write the center of mass velocity integrals for $q = 0$ and $q = 1$ once again.

$$I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) = \frac{2\pi}{Ru} \int_0^\infty \mathrm{d}w\, w^{2l+1} \exp\left(-Aw^2\right)\left(e^{Ru\cdot w} - e^{-Ru\cdot w}\right), \tag{131}$$

for $q = 0$ and

$$I_{\boldsymbol{w}}^{1}(\boldsymbol{u}) = \frac{2\pi}{R} \int_0^\infty \mathrm{d}w\, w^{2l+2} \exp\left(-Aw^2\right)\left(e^{Ru\cdot w} + e^{-Ru\cdot w}\right) - \frac{2\pi}{R^2 u} \int_0^\infty \mathrm{d}w\, w^{2l+1} \exp\left(-Aw^2\right)\left(e^{Ru\cdot w} - e^{Ru\cdot w}\right), \tag{132}$$

for $q = 1$. They can be written in terms of the symmetric Gaussian integrals as

$$\begin{aligned} I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) &= \frac{2\pi}{Ru} \cdot I_-^{2l+1}, \\ I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) &= \frac{2\pi}{R} \cdot I_+^{2l+2} - \frac{1}{R} \cdot I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}), \\ a = A, \quad b &= Ru, \quad \Longrightarrow \quad \lambda = \frac{Ru}{2\sqrt{A}}. \end{aligned} \tag{133}$$

Using (127) and (130), we have

$$\begin{aligned} I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) &= \sqrt{\frac{\pi}{A}} \cdot \frac{\pi \exp\left(\lambda^2\right)}{A^{l+1}} \sum_{k=0}^{l} \binom{2l+1}{2k} \frac{(2k-1)!!}{2^k} \cdot \lambda^{2l-2k}, \\ I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) &= \frac{2\pi}{R} \sqrt{\frac{\pi}{A}} \frac{\exp\left(\lambda^2\right)}{A^{l+1}} \left[\frac{(2l+1)!!}{2^{l+1}} + \sum_{k=0}^{l} \binom{2l+1}{2k} \frac{(2k-1)!!}{2^k} \cdot \lambda^{2l-2k} \left(\frac{2l+2}{2l+2-2k}\lambda^2 - \frac{1}{2}\right)\right]. \end{aligned} \tag{134}$$

Both cases of $q = 0$ and $q = 1$ can now be combined simply as

$$I_{\boldsymbol{w}}^{l,q}(\boldsymbol{u}) = \left(\frac{2}{R}\right)^q \sqrt{\frac{\pi}{A}} \frac{\pi \exp\left(\lambda^2\right)}{A^{l+1}} \left[\left(\frac{(2l+1)!!}{2^{l+1}}\right)^q + \sum_{j=0}^{l} \binom{2l+1}{2j} \frac{(2j-1)!!}{2^j} \cdot \lambda^{2l-2j} \cdot \left(\frac{2l+2}{2l+2-2j}\lambda^2 - \frac{1}{2}\right)^q\right]. \tag{135}$$

We have changed the dummy index k, not to confuse with the index k in the original kinetic integrals.

Relative velocity integrals

The final type of integrals over $\boldsymbol{u}$, is what we call the relative velocity integrals. The general form of these integrals are simple Gaussian integrals. Depending on the type of the kinetic integral, they can be written as

$$\begin{aligned} I_{\boldsymbol{u},a}^{k,i} &= \int \mathrm{d}\boldsymbol{u}\, u^{k+i} \exp\left(-Cu^2\right)\Theta(v_a - u), \\ I_{\boldsymbol{u},f}^{k,i} &= \int \mathrm{d}\boldsymbol{u}\, u^{k+i} \exp\left(-Cu^2\right)\Theta(u - v_f), \\ I_{\boldsymbol{u},r}^{k,i} &= \int \mathrm{d}\boldsymbol{u}\, u^{k+i} \exp\left(-Cu^2\right)\Theta(u - v_a)\Theta(v_f - u), \end{aligned} \tag{136}$$

where we C is constant and i is a combined power of all u vectors appearing in the inner integrals. The index k is the original power of u in the kinetic integral. Changing into spherical coordinates, we have

$$\begin{aligned} I_{\boldsymbol{u},a}^{k,i} &= 4\pi \int_0^{v_a} \mathrm{d}u\, u^{k+i+2} \exp\left(-Cu^2\right), \\ I_{\boldsymbol{u},r}^{k,i} &= 4\pi \int_{v_a}^{v_f} \mathrm{d}u\, u^{k+i+2} \exp\left(-Cu^2\right), \\ I_{\boldsymbol{u},f}^{k,i} &= 4\pi \int_{v_f}^{\infty} \mathrm{d}u\, u^{k+i+2} \exp\left(-Cu^2\right). \end{aligned} \tag{137}$$

We can write these integrals in canonical forms by setting $t = \sqrt{C}u$. This implies

$$\begin{aligned} u &= \frac{t}{\sqrt{C}}, \qquad \mathrm{d}u = \frac{\mathrm{d}t}{\sqrt{C}}, \\ u^{k+i+2} &= \frac{t^{k+i+2}}{\sqrt{C^{k+i+2}}}, \\ u &= 0, \quad \Rightarrow \quad t = 0, \\ u &= \infty, \quad \Rightarrow \quad t = \infty, \\ u &= v_{a/f}, \quad \Rightarrow \quad t = \sqrt{C}v_{a/f}. \end{aligned} \tag{138}$$

Now, our relative velocity integrals become

$$\begin{aligned} I^{k,i}_{\boldsymbol{u},a} &= \frac{4\pi}{\sqrt{C^{k+i+3}}} \int_0^{\sqrt{C}v_a} \mathrm{d}t\, t^{k+i+2} e^{-t^2}, \\ I^{k,i}_{\boldsymbol{u},r} &= \frac{4\pi}{\sqrt{C^{k+i+3}}} \int_{\sqrt{C}v_a}^{\sqrt{C}v_f} \mathrm{d}t\, t^{k+i+2} e^{-t^2}, \\ I^{k,i}_{\boldsymbol{u},f} &= \frac{4\pi}{\sqrt{C^{k+i+3}}} \int_{\sqrt{C}v_f}^{\infty} \mathrm{d}t\, t^{k+i+2} e^{-t^2}. \end{aligned} \tag{139}$$

As we did with previously, we can integrate by parts, writing $u = e^{-t^2}$, $\mathrm{d}v = t^{k+i+2}\,\mathrm{d}t$, which gives us $\mathrm{d}u = -2te^{-t^2}\,\mathrm{d}t$ and $v = t^{k+i+3}/(k+i+3)$. Hence,

$$I^k = \int_{L_1}^{L_2} \mathrm{d}t\, t^{k+i+2} e^{-t^2} = \frac{t^{k+i+3}}{k+i+3}\, e^{-t^2}\Big|_{L_1}^{L_2} + \frac{2}{k+i+3} \int_{L_1}^{L_2} \mathrm{d}t\, t^{k+i+4} e^{-t^2}, \tag{140}$$

and we write the recurrent relation

$$I^k = \frac{k+i+1}{2} I^{k-2} + \frac{1}{2}\Big(L_1^{k+i+1} e^{-L_1^2} - L_2^{k+i+1} e^{-L_2^2}\Big). \tag{141}$$

SOLUTION OF THE KINETIC INTEGRALS

The collision integrals of the Boltzmann equation have the next structure

$$I_k = \sum_{i,j} \sigma_{ij}^2 \int_{\mathcal{D}} \mathrm{d}\boldsymbol{v}_i\, \mathrm{d}\boldsymbol{v}_j\, f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j) \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| (P_k(\boldsymbol{v}_k|\boldsymbol{v}_i, \boldsymbol{v}_j) - \delta_{k,i}\delta(\boldsymbol{v}_i - \boldsymbol{v}_k)). \tag{142}$$

The two-particle distribution function $f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j)$, can be approximated by a product of two one-particle distribution functions $f_i(\boldsymbol{v}_i)$ and $f_j(\boldsymbol{v}_j)$

$$f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j) \approx f_i(\boldsymbol{v}_i) f_j(\boldsymbol{v}_j). \tag{143}$$

In the context of our work, we assume that the one-particle distribution functions have the Maxwellian form

$$f_i(\boldsymbol{v}_i) = n_i \left(\frac{m_i}{2\pi T_i}\right)^{3/2} \cdot \exp\left(-\frac{m_i v_i^2}{2T_i}\right) = \frac{n_i}{\pi^{3/2} u_i^3} \cdot \exp\left(-\frac{v_i^2}{u_i^2}\right), \quad u_i^2 = \frac{2T_i}{m_i}. \tag{144}$$

The two-particle distribution function is then approximated as

$$f_{ij}(\boldsymbol{v}_i, \boldsymbol{v}_j) \approx \frac{n_i n_j}{(\pi u_i u_j)^3} \cdot \exp\left(-\frac{v_i^2}{u_i^2} - \frac{v_j^2}{u_j^2}\right). \tag{145}$$

Changing impact velocities into the center of mass velocity $\boldsymbol{V}$ and the relative velocity $\boldsymbol{g}$ vectors, we have

$$
\begin{aligned}
\frac{v_i^2}{u_i^2} + \frac{v_j^2}{u_j^2} &= AV^2 + Bg^2 - R(\boldsymbol{V}\cdot\boldsymbol{g}),\\
A_{ij} &= \frac{1}{u_i^2} + \frac{1}{u_j^2},\\
B_{ij} &= \frac{\mu_j^2}{u_i^2} + \frac{\mu_i^2}{u_j^2},\\
R_{ij} &= \frac{2\mu_i}{u_j^2} - \frac{2\mu_j}{u_i^2},\\
\alpha_{ij} &= \frac{n_i n_j}{(\pi u_i u_j)^3},
\end{aligned}
\tag{146}
$$

hence the two-particle distribution function in terms of $\boldsymbol{V}$ and $\boldsymbol{g}$ is written as

$$
f_{ij}(\boldsymbol{V},\boldsymbol{g}) = \alpha_{ij}\exp\left(-A_{ij}V^2\right)\exp\left(-B_{ij}g^2\right)\exp\left(R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right),
\tag{147}
$$

and the collision integral becomes

$$
I_k = \sum_{i,j}\alpha_{ij}\sigma_{ij}^2\int_{\mathcal{D}}\mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\,e^{-A_{ij}V^2-B_{ij}g^2+R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}}\int\mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|(P_k(\boldsymbol{v}_k|\boldsymbol{v}_i,\boldsymbol{v}_j)-\delta_{k,i}\delta(\boldsymbol{v}_i-\boldsymbol{v}_k)).
\tag{148}
$$

In general, the collision integral is multiplied by a function of a velocity $\psi_k(\boldsymbol{v}_k)$, and then integrated over $\boldsymbol{v}_k$, hence

$$
\begin{aligned}
\left\langle\frac{\partial}{\partial t}(n_k\psi_k)\right\rangle_c = \sum_{i,j}\alpha_{ij}\sigma_{ij}^2\int\mathrm{d}\boldsymbol{v}_k\int_{\mathcal{D}}\mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\,e^{-A_{ij}V^2-B_{ij}g^2+R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}}\\
\int\mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|\cdot\psi_k(\boldsymbol{v}_k)\cdot(P_k(\boldsymbol{v}_k|\boldsymbol{v}_i(\boldsymbol{g},\boldsymbol{V}),\boldsymbol{v}_j(\boldsymbol{g},\boldsymbol{V}))-\delta_{k,i}\delta(\boldsymbol{v}_k-\boldsymbol{v}_i(\boldsymbol{g},\boldsymbol{V}))).
\end{aligned}
\tag{149}
$$

In our work, we will be dealing with two functions $\psi_k(\boldsymbol{v}_k) = 1$ and $\psi_k(\boldsymbol{v}_k) = m_k v_k^2/2$.

Collision frequency density

The collision frequency density ν_{ij} between the particles of sizes i and j, is given by

$$
\nu_{ij} = \pi\sigma_{ij}^2\langle g_{ij}\rangle,
\tag{150}
$$

where $\langle g_{ij}\rangle$ is the mean impact speed. Given the two-particle distribution function $f_{ij}(\boldsymbol{V},\boldsymbol{g})$, the mean value of the impact velocity is given by

$$
\begin{aligned}
n_i n_j\langle g\rangle &= \int\mathrm{d}\boldsymbol{V}\,\mathrm{d}\boldsymbol{g}\,g f_{ij}(\boldsymbol{V},\boldsymbol{g}) = \alpha_{ij}\int\mathrm{d}\boldsymbol{g}\,g\exp\left(-B_{ij}g^2\right)\int\mathrm{d}\boldsymbol{V}\exp\left(-A_{ij}V^2+R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right) =\\
&= 4\alpha_{ij}\pi\left(\frac{\pi}{A_{ij}}\right)^{3/2}\int_0^\infty\mathrm{d}g\,g^3\exp\left(-\lambda_{ij}g^2\right) = \frac{2\pi\alpha_{ij}}{\lambda_{ij}^2}\left(\frac{\pi}{A_{ij}}\right)^{3/2},
\end{aligned}
\tag{151}
$$

where

$$
\lambda_{ij} = \frac{1}{u_i^2+u_j^2},\qquad A_{ij} = \frac{u_i^2+u_j^2}{u_i^2u_j^2}.
\tag{152}
$$

After simple algebra, we have

$$
\langle g_{ij}\rangle = \frac{2}{\sqrt{\pi}}\sqrt{u_i^2+u_j^2},
\tag{153}
$$

hence the collision frequency is

$$
n_i n_j\nu_{ij} = \frac{2\pi^2\sigma_{ij}^2\alpha_{ij}}{\lambda_{ij}^2}\left(\frac{\pi}{A_{ij}}\right)^{3/2} = 2n_i n_j\sigma_{ij}^2\sqrt{\pi\left(u_i^2+u_j^2\right)}.
\tag{154}
$$

Aggregation terms of the collision integrals

For the aggregative collision integrals, we have

$$P_k(\boldsymbol{v}_k|\boldsymbol{v}_i,\boldsymbol{v}_j) = \delta_{k,i+j}\delta[\boldsymbol{v}_k - \boldsymbol{V}], \qquad \mathcal{D} = \Theta(g_{\text{agg}} - g), \tag{155}$$

and

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}(n_k\psi_k)\right\rangle_a = \sum_{i,j}\alpha_{ij}\sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_k \int \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\, e^{-A_{ij}V^2 - B_{ij}g^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}}\Theta(g_{\text{agg}} - g) \\ \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|\cdot\psi_k(\boldsymbol{v}_k)\cdot(\delta_{k,i+j}\delta[\boldsymbol{v}_k - \boldsymbol{V}] - \delta_{k,i}\delta(\boldsymbol{v}_k - \boldsymbol{v}_i(\boldsymbol{g},\boldsymbol{V}))).\end{aligned} \tag{156}$$

Now, performing the actions of the delta functions, we write

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}(n_k\psi_k)\right\rangle_a = \frac{1}{2}\sum_{i+j=k}\alpha_{ij}\sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\, e^{-A_{ij}V^2 - B_{ij}g^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}}\Theta(g_{\text{agg}} - g)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|\cdot\psi_k(\boldsymbol{V}) - \\ - \sum_j \alpha_{kj}\sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\, e^{-A_{kj}V^2 - B_{kj}g^2 + R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}}\Theta(g_{\text{agg}} - g)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|\cdot\psi_k(\boldsymbol{V} + \mu_j\boldsymbol{g}).\end{aligned} \tag{157}$$

Number density evolution inegrals for aggregation

We put $\psi_k = 1$ and solve the integral

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t}\right\rangle_a = \frac{1}{2}\sum_{i+j=k}\alpha_{ij}\sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\, e^{-A_{ij}V^2 - B_{ij}g^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}}\Theta(g_{\text{agg}} - g)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| - \\ - \sum_j \alpha_{kj}\sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\, e^{-A_{kj}V^2 - B_{kj}g^2 + R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}}\Theta(g_{\text{agg}} - g)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|.\end{aligned} \tag{158}$$

Both angular integrals are immediately solved as

$$\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| = \pi g, \tag{159}$$

and we have

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t}\right\rangle_a = \frac{\pi}{2}\sum_{i+j=k}\alpha_{ij}\sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\, g\exp\left(-B_{ij}g^2\right)\Theta(g_{\text{agg}} - g)\int \mathrm{d}\boldsymbol{V}\exp\left(-A_{ij}V^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right) - \\ - \pi\sum_j \alpha_{kj}\sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\, g\exp\left(-B_{ij}g^2\right)\Theta(g_{\text{agg}} - g)\int \mathrm{d}\boldsymbol{V}\exp\left(-A_{kj}V^2 + R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}\right).\end{aligned} \tag{160}$$

The center of mass velocity integrals are solved to be

$$\int \mathrm{d}\boldsymbol{V}\exp\left(-A_{ij}V^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right) = I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g}) = \left(\frac{\pi}{A_{ij}}\right)^{3/2}\cdot\exp\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2\right). \tag{161}$$

Defining

$$C_{ij} = \frac{4A_{ij}B_{ij} - R_{ij}^2}{4A_{ij}} = \frac{1}{u_i^2 + u_j^2}, \tag{162}$$

we have

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t}\right\rangle_a = 2\pi^2\sum_{i+j=k}\left(\frac{\pi}{A_{ij}}\right)^{3/2}\alpha_{ij}\sigma_{ij}^2\int_0^{g_{\text{agg}}}\mathrm{d}g\, g^3\exp\left(-C_{ij}g^2\right) - \\ - 4\pi^2\sum_j\left(\frac{\pi}{A_{kj}}\right)^{3/2}\alpha_{kj}\sigma_{kj}^2\int_0^{g_{\text{agg}}}\mathrm{d}g\, g^3\exp\left(-C_{kj}g^2\right).\end{aligned} \tag{163}$$

The final incomplete Gaussian integral has an analytic solution

$$\int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-C_{ij} g^2\right) = \frac{1}{2C_{ij}^2}\left[1 - \left(1 + C_{ij} g_{\mathrm{agg}}^2\right) \exp\left(-C_{ij} g_{\mathrm{agg}}^2\right)\right], \tag{164}$$

and the number density evolution due to aggregation is given by

$$\begin{aligned} \left\langle \frac{\partial n_k}{\partial t} \right\rangle_a &= \pi^2 \sum_{i+j=k} \left(\frac{\pi}{A_{ij}}\right)^{3/2} \frac{\alpha_{ij}}{C_{ij}^2} \sigma_{ij}^2 \left[1 - \left(1 + C_{ij} g_{\mathrm{agg}}^2\right) \exp\left(-C_{ij} g_{\mathrm{agg}}^2\right)\right] - \\ &- 2\pi^2 \sum_j \left(\frac{\pi}{A_{kj}}\right)^{3/2} \frac{\alpha_{kj}}{C_{kj}^2} \sigma_{kj}^2 \left[1 - \left(1 + C_{kj} g_{\mathrm{agg}}^2\right) \exp\left(-C_{kj} g_{\mathrm{agg}}^2\right)\right]. \end{aligned} \tag{165}$$

From simple algebra we have

$$\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \frac{\alpha_{ij}}{C_{ij}^2} = n_i n_j \sqrt{\pi\left(u_i^2 + u_j^2\right)}, \tag{166}$$

and we write the final result as

$$\begin{aligned} \left\langle \frac{\partial n_k}{\partial t} \right\rangle_a &= \frac{1}{2} \sum_{i+j=k} K_{ij}^{an} n_i n_j - n_k \sum_j K_{kj}^{an} n_j, \\ K_{ij}^{an} &= \nu_{ij}\left[1 - \left(1 + C_{ij} g_{\mathrm{agg}}^2\right) \exp\left(-C_{ij} g_{\mathrm{agg}}^2\right)\right]. \end{aligned} \tag{167}$$

Temperature evolution inegrals for aggregation

For the temperature evolution due to aggregative collisions, we put $\psi_k(\boldsymbol{v}_k) = m_k v_k^2/2$, hence our collision integrals become

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_a &= \frac{m_k}{4} \sum_{i+j=k} \pi \alpha_{ij} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, g V^2 e^{-A_{ij} V^2 - B_{ij} g^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} \Theta(g_{\mathrm{agg}} - g) - \\ &- \frac{m_k}{2} \sum_j \pi \alpha_{kj} \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V} \left(g V^2 + \mu_j^2 g^3 + 2\mu_j g (\boldsymbol{V}\cdot\boldsymbol{g})\right) e^{-A_{kj} V^2 - B_{kj} g^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}} \Theta(g_{\mathrm{agg}} - g), \end{aligned} \tag{168}$$

where have performed the angular integration. Now, separating integrals over $\boldsymbol{g}$ and $\boldsymbol{V}$, we write

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_a &= m_k \sum_{i+j=k} \pi^2 \alpha_{ij} \sigma_{ij}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-B_{ij} g^2\right) \int \mathrm{d}\boldsymbol{V}\, V^2 \exp\left(-A_{ij} V^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}\right) - \\ &- 2m_k \sum_j \pi^2 \alpha_{kj} \sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-B_{kj} g^2\right) \int \mathrm{d}\boldsymbol{V}\, V^2 \exp\left(-A_{kj} V^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}\right) - \\ &- 2m_k \sum_j \pi^2 \alpha_{kj} \sigma_{kj}^2 \mu_j^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^5 \exp\left(-B_{kj} g^2\right) \int \mathrm{d}\boldsymbol{V} \exp\left(-A_{kj} V^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}\right) - \\ &- 4m_k \sum_j \pi^2 \alpha_{kj} \sigma_{kj}^2 \mu_j \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-B_{kj} g^2\right) \int \mathrm{d}\boldsymbol{V}\, (\boldsymbol{V}\cdot\boldsymbol{g}) \exp\left(-A_{kj} V^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}\right), \end{aligned} \tag{169}$$

or

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_a &= m_k \sum_{i+j=k} \pi^2 \alpha_{ij} \sigma_{ij}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-B_{ij} g^2\right) \cdot I_{\boldsymbol{V}}^{1,0}(\boldsymbol{g}) - \\ &- 2m_k \sum_j \pi^2 \alpha_{kj} \sigma_{kj}^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-B_{kj} g^2\right) \cdot I_{\boldsymbol{V}}^{1,0}(\boldsymbol{g}) - \\ &- 2m_k \sum_j \pi^2 \alpha_{kj} \sigma_{kj}^2 \mu_j^2 \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^5 \exp\left(-B_{kj} g^2\right) \cdot I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g}) - \\ &- 4m_k \sum_j \pi^2 \alpha_{kj} \sigma_{kj}^2 \mu_j \int_0^{g_{\mathrm{agg}}} \mathrm{d}g\, g^3 \exp\left(-B_{kj} g^2\right) \cdot I_{\boldsymbol{V}}^{0,1}(\boldsymbol{g}). \end{aligned} \tag{170}$$

Putting in the solutions of the center of mass velocity integrals, we have

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_a &= m_k \sum_{i+j=k}\left(\frac{\pi}{A_{ij}}\right)^{3/2}\frac{\pi^2\alpha_{ij}}{A_{ij}}\sigma_{ij}^2\int_0^{g_{\rm agg}} \mathrm{d}g\, g^3 \exp\left(-C_{ij}g^2\right)\cdot\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2+\frac{3}{2}\right)- \\ &- 2m_k\sum_j\left(\frac{\pi}{A_{kj}}\right)^{3/2}\frac{\pi^2\alpha_{kj}}{A_{kj}}\sigma_{kj}^2\int_0^{g_{\rm agg}} \mathrm{d}g\, g^3 \exp\left(-C_{kj}g^2\right)\cdot\left(\frac{R_{kj}^2}{4A_{kj}}\cdot g^2+\frac{3}{2}\right)- \\ &- 2m_k\sum_j\left(\frac{\pi}{A_{kj}}\right)^{3/2}\pi^2\alpha_{kj}\sigma_{kj}^2\mu_j^2\int_0^{g_{\rm agg}} \mathrm{d}g\, g^5 \exp\left(-C_{kj}g^2\right)- \\ &- 2m_k\sum_j\left(\frac{\pi}{A_{kj}}\right)^{3/2}\frac{R_{kj}}{A_{kj}}\pi^2\alpha_{kj}\sigma_{kj}^2\mu_j\int_0^{g_{\rm agg}} \mathrm{d}g\, g^5 \exp\left(-C_{kj}g^2\right),\end{aligned} \tag{171}$$

and opening the brackets we write

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_a &= m_k \sum_{i+j=k}\left(\frac{\pi}{A_{ij}}\right)^{3/2}\frac{R_{ij}^2}{4A_{ij}^2}\cdot\pi^2\alpha_{ij}\sigma_{ij}^2 G_{ij}^{(5)}+ \\ &+ m_k \sum_{i+j=k}\left(\frac{\pi}{A_{ij}}\right)^{3/2}\frac{3}{2A_{ij}}\cdot\pi^2\alpha_{ij}\sigma_{ij}^2 G_{ij}^{(3)}- \\ &- 2m_k\sum_j\left(\frac{\pi}{A_{kj}}\right)^{3/2}\left(\frac{R_{kj}}{2A_{kj}}+\mu_j\right)^2\cdot\pi^2\alpha_{kj}\sigma_{kj}^2 G_{kj}^{(5)}- \\ &- 2m_k\sum_j\left(\frac{\pi}{A_{kj}}\right)^{3/2}\frac{3}{2A_{kj}}\cdot\pi^2\alpha_{kj}\sigma_{kj}^2 G_{kj}^{(3)},\end{aligned} \tag{172}$$

where the incomplete Gaussian integrals have the next solutions

$$\begin{aligned} G_{ij}^{(3)}(0,g_{\rm agg}) &= \int_0^{g_{\rm agg}} \mathrm{d}g\, g^3\exp\left(-C_{ij}g^2\right) = \frac{1}{2C_{ij}^2}\left[1-\left(1+C_{ij}g_{\rm agg}^2\right)\exp\left(-C_{ij}g_{\rm agg}^2\right)\right], \\ G_{ij}^{(5)}(0,g_{\rm agg}) &= \int_0^{g_{\rm agg}} \mathrm{d}g\, g^5\exp\left(-C_{ij}g^2\right) = \frac{1}{2C_{ij}^3}\left[1-\left(2+2C_{ij}g_{\rm agg}^2+C_{ij}^2 g_{\rm agg}^4\right)\exp\left(-C_{ij}g_{\rm agg}^2\right)\right].\end{aligned} \tag{173}$$

After some algebraic manipulations, we can rewrite

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_a &= \frac{1}{2}m_k\sum_{i+j=k}\left[\frac{1}{2}K_{ij}^{aT}\left[\frac{\mu_i u_i^2-\mu_j u_j^2}{u_i^2+u_j^2}\right]^2+\frac{3}{4}K_{ij}^{an}\frac{u_i^2 u_j^2}{u_i^2+u_j^2}\right]n_i n_j- \\ &- m_k n_k\sum_j\left[\frac{1}{2}K_{kj}^{aT}\left(\frac{u_k^2}{u_k^2+u_j^2}\right)^2+\frac{3}{4}K_{kj}^{an}\frac{u_k^2 u_j^2}{u_k^2+u_j^2}\right]n_j,\end{aligned} \tag{174}$$

where

$$\begin{aligned} K_{ij}^{an} &= 4\sigma_{ij}^2 C_{ij}^2 G_{ij}^{(3)}\sqrt{\pi\left(u_i^2+u_j^2\right)}, \\ K_{ij}^{aT} &= 4\sigma_{ij}^2 C_{ij}^2 G_{ij}^{(5)}\sqrt{\pi\left(u_i^2+u_j^2\right)}.\end{aligned} \tag{175}$$

Restitution terms of the collision integrals

The collision integrals for the restitutive terms read

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}(n_k\psi_k)\right\rangle_r = \sum_j \alpha_{kj}\sigma_{kj}^2 &\int \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{V}\, e^{-A_{kj}V^2-B_{kj}g^2+R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}}\Theta(g-g_{\rm agg}) \\ &\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}|\cdot\Delta\psi_k(\boldsymbol{V}+\mu_j\boldsymbol{g})\Theta\left(g_{\rm frag}^2-(\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2\right),\end{aligned} \tag{176}$$

where $\Delta\psi_k$ is the change of the physical value ψ_k due to a single restitutive collision. For the restitutive type of collisions, the number densities do not change, since $\Delta(n_k) = 0$. Hence we have only the temperature evolution term

Temperature evolution inegrals for restitution

Putting $\psi_k(\boldsymbol{v}_k) = m_k v_k^2/2$, we have

$$\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right)\right\rangle_r = -\frac{1+\varepsilon}{2} m_k n_k \sum_j \frac{\mu_j R_{kj}}{A_{kj}^2} F_{kj} \cdot n_j - \frac{1-\varepsilon^2}{2} m_k n_k \sum_j \frac{\mu_j^2}{A_{kj}} F_{kj} \cdot n_j, \tag{177}$$

where

$$F_{kj} = 2\sigma_{kj}^2 C_{kj}^2 \sqrt{\pi\left(u_i^2 + u_j^2\right)} \Big(A_{kj} g_{\text{frag}}^4 G_{kj}^{(1)}(g_{\text{frag}}, \infty) + A_{kj} G_{kj}^{(5)}(g_{\text{agg}}, g_{\text{frag}}) \Big), \tag{178}$$

where

$$\begin{aligned} G_{kj}^{(1)}(g_{\text{frag}}, \infty) &= \int_{g_{\text{frag}}}^{\infty} \mathrm{d}g\, g \exp\left(-C_{kj} g^2\right) = \frac{1}{2C_{kj}^2} \exp\left(-C_{kj} g_{\text{frag}}^2\right), \\ G_{kj}^{(5)}(g_{\text{agg}}, g_{\text{frag}}) &= \int_{g_{\text{agg}}}^{g_{\text{frag}}} \mathrm{d}g\, g^5 \exp\left(-C_{kj} g^2\right) = \\ &= \frac{1}{2C_{kj}^3} \left[\left(2 + 2C_{kj} g_{\text{agg}}^2 + C_{kj}^2 g_{\text{agg}}^4\right) \exp\left(-C_{kj} g_{\text{agg}}^2\right) - \left(2 + 2C_{kj} g_{\text{frag}}^2 + C_{kj}^2 g_{\text{frag}}^4\right) \exp\left(-C_{kj} g_{\text{frag}}^2\right) \right]. \end{aligned} \tag{179}$$

Fragmentation terms of the collision integrals

The fragmentative terms of the collision integral reads

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}(n_k \psi_k) \right\rangle_f = \sum_{i,j} \alpha_{ij} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{v}_k \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{ij} V^2 - B_{ij} g^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} \\ \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \psi_k(\boldsymbol{v}_k) \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big) \times \\ \times \left(\delta_{k,1} \frac{i+j}{4\pi} \int \mathrm{d}\hat{\boldsymbol{e}}\, \delta(\boldsymbol{v}_k - \boldsymbol{V} - v_c' \hat{\boldsymbol{e}}) - \delta_{k,i} \delta(\boldsymbol{v}_k - \boldsymbol{v}_i(\boldsymbol{g}, \boldsymbol{V})) \right). \end{aligned} \tag{180}$$

Performing the actions of the delta functions, we have

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}(n_k \psi_k) \right\rangle_f = \frac{\delta_{k,1}}{4\pi} \sum_{i,j} (i+j) \alpha_{ij} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{ij} V^2 - B_{ij} g^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} \\ \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big) \int \mathrm{d}\hat{\boldsymbol{e}}\, \psi_k(\boldsymbol{V} - v_c' \hat{\boldsymbol{e}}) - \\ - \sum_j \alpha_{kj} \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{kj} V^2 - B_{kj} g^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}} \\ \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \psi_k(\boldsymbol{V} + \mu_j \boldsymbol{g}) \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big). \end{aligned} \tag{181}$$

Again, putting $\psi_k = 1$, we have

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t} \right\rangle_f &= \delta_{k,1} \sum_{i,j} (i+j)\alpha_{ij}\sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{ij}V^2 - B_{ij}g^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}} \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\mathrm{frag}}^2\big) - \\ &- \sum_j \alpha_{kj}\sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{kj}V^2 - B_{kj}g^2 + R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}} \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\mathrm{frag}}^2\big).\end{aligned} \tag{182}$$

Solving the angular integrals, we get

$$\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}})|\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\mathrm{frag}}^2\big) = I_{\hat{\boldsymbol{n}},f}^{1,0,0}(\boldsymbol{g}) = \pi g \left[1 - \left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right] \Theta(g - g_{\mathrm{frag}}), \tag{183}$$

hence

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t} \right\rangle_f &= \delta_{k,1} \sum_{i,j} (i+j) 4\pi^2 \alpha_{ij}\sigma_{ij}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right] \exp\left(-B_{ij}g^2\right) \int \mathrm{d}\boldsymbol{V}\, e^{-A_{ij}V^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}} - \\ &- \sum_j 4\pi^2 \alpha_{kj}\sigma_{kj}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right] \exp\left(-B_{ij}g^2\right) \int \mathrm{d}\boldsymbol{V}\, e^{-A_{kj}V^2 + R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}}.\end{aligned} \tag{184}$$

The center of mass velocity integral is solved to give us

$$\int \mathrm{d}\boldsymbol{V} \exp\left(-A_{kj}V^2 + R_{kj}\boldsymbol{V}\cdot\boldsymbol{g}\right) = I_{\boldsymbol{V}}^{0,0}(\boldsymbol{g}) = \left(\frac{\pi}{A_{ij}}\right)^{3/2} \exp\left(\frac{R_{ij}^2}{4A_{ij}} \cdot g^2\right). \tag{185}$$

Now, we write the number density evolution integrals as

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t} \right\rangle_f &= \delta_{k,1} \sum_{i,j} (i+j) 4\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \alpha_{ij}\sigma_{ij}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right] \exp\left(-C_{ij}g^2\right) - \\ &- \sum_j 4\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \alpha_{kj}\sigma_{kj}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right] \exp\left(-C_{ij}g^2\right),\end{aligned} \tag{186}$$

and opening the brackets we have

$$\begin{aligned}\left\langle \frac{\partial n_k}{\partial t} \right\rangle_f &= \delta_{k,1} \sum_{i,j} (i+j) 4\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \alpha_{ij}\sigma_{ij}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \exp\left(-C_{ij}g^2\right) - \\ &- \delta_{k,1} g_{\mathrm{frag}}^2 \sum_{i,j} (i+j) 4\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \alpha_{ij}\sigma_{ij}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g \exp\left(-C_{ij}g^2\right) - \\ &- \sum_j 4\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \alpha_{kj}\sigma_{kj}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \exp\left(-C_{ij}g^2\right) + \\ &+ g_{\mathrm{frag}}^2 \sum_j 4\pi^2 \left(\frac{\pi}{A_{ij}}\right)^{3/2} \alpha_{kj}\sigma_{kj}^2 \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g \exp\left(-C_{ij}g^2\right).\end{aligned} \tag{187}$$

Solving the incomplete Gaussian integrals

$$\begin{aligned}G_{ij}^{(1)}(g_{\mathrm{frag}}, \infty) &= \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g \exp\left(-C_{ij}g^2\right) = \frac{1}{2C_{ij}} \exp\left(-C_{ij}g_{\mathrm{frag}}^2\right), \\ G_{ij}^{(3)}(g_{\mathrm{frag}}, \infty) &= \int_{g_{\mathrm{frag}}}^{\infty} \mathrm{d}g\, g^3 \exp\left(-C_{ij}g^2\right) = \frac{1}{2C_{ij}^2}\left(1 + C_{ij}g_{\mathrm{frag}}^2\right) \exp\left(-C_{ij}g_{\mathrm{frag}}^2\right),\end{aligned} \tag{188}$$

we write

$$\left\langle \frac{\partial n_k}{\partial t} \right\rangle_f = \delta_{k,1} \sum_{i,j} (i+j) K_{ij}^{fn} n_i n_j - \sum_j K_{ij}^{fn} n_i n_j, \tag{189}$$

where

$$K_{ij}^{fn} = 2\sigma_{kj}^2 \sqrt{\pi\big(u_i^2 + u_j^2\big)} \exp\big(-C_{ij} g_{\text{frag}}^2\big). \tag{190}$$

Temperature evolution inegrals for fragmentation

By putting $\psi_k(\boldsymbol{v}_k) = m_k v_k^2/2$, we have

$$\begin{aligned}
\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_f = &\frac{\delta_{k,1}}{4\pi} \frac{m_k}{2} \sum_{i,j} (i+j) \alpha_{ij} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, V^2 e^{-A_{ij} V^2 - B_{ij} g^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} \\
&\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big) + \\
&+ \frac{\delta_{k,1}}{4\pi} \frac{m_k}{2} \sum_{i,j} (i+j) \gamma_{ij} \alpha_{ij} \sigma_{ij}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{ij} V^2 - B_{ij} g^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} \\
&\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| (\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big) - \\
&- \frac{m_k}{2} \sum_j \alpha_{kj} \sigma_{kj}^2 \int \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{V}\, e^{-A_{kj} V^2 - B_{kj} g^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}} \big(V^2 + \mu_j^2 g^2 + 2\mu_j (\boldsymbol{V}\cdot\boldsymbol{g})\big) \\
&\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big),
\end{aligned} \tag{191}$$

where we have used $v_c'^2 = \gamma_{ij} (\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2$. The angular integral parts are

$$\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big) = I_{\hat{\boldsymbol{n}},f}^{1,0,0}(\boldsymbol{g}) = \pi g \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^2\right] \Theta(g - g_{\text{frag}}), \tag{192}$$

and

$$\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{g}\cdot\hat{\boldsymbol{n}}| (\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 \cdot \Theta\big((\boldsymbol{g}\cdot\hat{\boldsymbol{n}})^2 - g_{\text{frag}}^2\big) = I_{\hat{\boldsymbol{n}},f}^{1,2,0} = \frac{\pi g^3}{2} \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^4\right] \Theta(g - g_{\text{frag}}). \tag{193}$$

Now we have

$$\begin{aligned}
\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2} n_k T_k\right) \right\rangle_f = &\delta_{k,1} \frac{m_k}{2} \sum_{i,j} (i+j) \pi \alpha_{ij} \sigma_{ij}^2 \int_{g_{\text{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^2\right] e^{-B_{ij} g^2} \int \mathrm{d}\boldsymbol{V}\, V^2 e^{-A_{ij} V^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} - \\
&- \delta_{k,1} \frac{m_k}{4} \sum_{i,j} (i+j) \gamma_{ij} \pi \alpha_{ij} \sigma_{ij}^2 \int_{g_{\text{frag}}}^{\infty} \mathrm{d}g\, g^5 \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^4\right] e^{-B_{ij} g^2} \int \mathrm{d}\boldsymbol{V}\, e^{-A_{ij} V^2 + R_{ij} \boldsymbol{V}\cdot\boldsymbol{g}} - \\
&- \frac{m_k}{2} \sum_j 4\pi \alpha_{kj} \sigma_{kj}^2 \int_{g_{\text{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^2\right] e^{-B_{kj} g^2} \int \mathrm{d}\boldsymbol{V}\, V^2 e^{-A_{kj} V^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}} - \\
&- \frac{m_k}{2} \sum_j \mu_j^2 \cdot 4\pi \alpha_{kj} \sigma_{kj}^2 \int_{g_{\text{frag}}}^{\infty} \mathrm{d}g\, g^5 \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^2\right] e^{-B_{kj} g^2} \int \mathrm{d}\boldsymbol{V}\, e^{-A_{kj} V^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}} - \\
&- \frac{m_k}{2} \sum_j 2\mu_j \cdot 4\pi \alpha_{kj} \sigma_{kj}^2 \int_{g_{\text{frag}}}^{\infty} \mathrm{d}g\, g^3 \left[1 - \left(\frac{g_{\text{frag}}}{g}\right)^2\right] e^{-B_{kj} g^2} \int \mathrm{d}\boldsymbol{V}\, (\boldsymbol{V}\cdot\boldsymbol{g}) e^{-A_{kj} V^2 + R_{kj} \boldsymbol{V}\cdot\boldsymbol{g}}.
\end{aligned} \tag{194}$$

The center of mass velocity integrals read

$$
\begin{aligned}
I_{\boldsymbol{V}}^{0,0}(\boldsymbol{u}) &= \int \mathrm{d}\boldsymbol{V} \exp\left(-A_{ij}V^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right) = \left(\frac{\pi}{A_{ij}}\right)^{3/2} \exp\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2\right),\\
I_{\boldsymbol{V}}^{1,0}(\boldsymbol{u}) &= \int \mathrm{d}\boldsymbol{V}\, V^2 \exp\left(-A_{ij}V^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right) = \left(\frac{\pi}{A_{ij}}\right)^{3/2} \frac{1}{A_{ij}} \exp\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2\right)\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2 + \frac{3}{2}\right),\\
I_{\boldsymbol{V}}^{0,1}(\boldsymbol{u}) &= \int \mathrm{d}\boldsymbol{V}\, (\boldsymbol{V}\cdot\boldsymbol{g}) \exp\left(-A_{ij}V^2 + R_{ij}\boldsymbol{V}\cdot\boldsymbol{g}\right) = \frac{2}{R}\left(\frac{\pi}{A_{ij}}\right)^{3/2} \frac{R_{ij}^2 g^2}{4A_{ij}} \exp\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2\right).
\end{aligned}
\tag{195}
$$

Using them, we write

$$
\begin{aligned}
\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_f &= \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\alpha_{ij}\sigma_{ij}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^3\left[1-\left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right]\left(\frac{R_{ij}^2}{4A_{ij}}\cdot g^2+\frac{3}{2}\right)e^{-C_{ij}g^2} -\\
&- \delta_{k,1}\frac{m_k}{4}\sum_{i,j}(i+j)\gamma_{ij}\pi\left(\frac{\pi}{A_{ij}}\right)^{3/2}\alpha_{ij}\sigma_{ij}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^5\left[1-\left(\frac{g_{\mathrm{frag}}}{g}\right)^4\right]e^{-C_{ij}g^2} -\\
&- \frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\alpha_{kj}\sigma_{kj}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^3\left[1-\left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right]\left(\frac{R_{kj}^2}{4A_{kj}}\cdot g^2+\frac{3}{2}\right)e^{-C_{kj}g^2} -\\
&- \frac{m_k}{2}\sum_{j}\mu_j^2\cdot 4\pi\left(\frac{\pi}{A_{kj}}\right)^{3/2}\alpha_{kj}\sigma_{kj}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^5\left[1-\left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right]e^{-C_{kj}g^2} -\\
&- \frac{m_k}{2}\sum_{j}2\mu_j\cdot 4\pi\frac{2}{R_{kj}}\left(\frac{\pi}{A_{kj}}\right)^{3/2}\frac{R_{kj}^2}{4A_{kj}}\alpha_{kj}\sigma_{kj}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^5\left[1-\left(\frac{g_{\mathrm{frag}}}{g}\right)^2\right]e^{-C_{kj}g^2}.
\end{aligned}
\tag{196}
$$

Opening the brackets, we have

$$
\begin{aligned}
\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_f &= \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{R_{ij}^2}{4A_{ij}}-\frac{\gamma_{ij}A_{ij}}{2}\right]\alpha_{ij}\sigma_{ij}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^5 e^{-C_{ij}g^2} -\\
&- \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{R_{ij}^2}{4A_{ij}}g_{\mathrm{frag}}^2-\frac{3}{2}\right]\alpha_{ij}\sigma_{ij}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^3 e^{-C_{ij}g^2} -\\
&- \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{3}{2}g_{\mathrm{frag}}^2+\frac{\gamma_{ij}A_{ij}}{2}g_{\mathrm{frag}}^4\right]\alpha_{ij}\sigma_{ij}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g e^{-C_{ij}g^2} -\\
&- \frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\left[\frac{R_{kj}^2}{4A_{kj}}+\mu_j^2A_{kj}+\mu_jR_{kj}\right]\alpha_{kj}\sigma_{kj}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^5 e^{-C_{kj}g^2} +\\
&+ \frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\left[\left(\frac{R_{kj}^2}{4A_{kj}}+\mu_j^2A_{kj}+\mu_jR_{kj}\right)\cdot g_{\mathrm{frag}}^2-\frac{3}{2}\right]\alpha_{kj}\sigma_{kj}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^3 e^{-C_{kj}g^2} +\\
&+ \frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\frac{3}{2}g_{\mathrm{frag}}^2\alpha_{kj}\sigma_{kj}^2\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g e^{-C_{kj}g^2}.
\end{aligned}
\tag{197}
$$

The incomplete Gaussian integrals are

$$
\begin{aligned}
\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g e^{-C_{ij}g^2} &= \frac{1}{2C_{ij}}\exp\left(-C_{ij}g_{\mathrm{frag}}^2\right),\\
\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^3 e^{-C_{ij}g^2} &= \frac{1}{2C_{ij}^2}\left(1+C_{ij}g_{\mathrm{frag}}^2\right)\exp\left(-C_{ij}g_{\mathrm{frag}}^2\right),\\
\int_{g_{\mathrm{frag}}}^{\infty}\mathrm{d}g\, g^5 e^{-C_{ij}g^2} &= \frac{1}{2C_{ij}^3}\left(2+2C_{ij}g_{\mathrm{frag}}^2+C_{ij}^2g_{\mathrm{frag}}^4\right)\exp\left(-C_{ij}g_{\mathrm{frag}}^2\right),
\end{aligned}
\tag{198}
$$

and the fragmentative temperature evolution terms become

$$
\begin{aligned}
\left\langle \frac{\partial}{\partial t}\left(\frac{3}{2}n_k T_k\right)\right\rangle_f = {} & \delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{R_{ij}^2}{4A_{ij}}-\frac{\gamma_{ij}A_{ij}}{2}\right]\alpha_{ij}\sigma_{ij}^2\frac{1}{2C_{ij}^3}\left(2+2C_{ij}g_{\text{frag}}^2+C_{ij}^2g_{\text{frag}}^4\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right)- \\
& -\delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{R_{ij}^2}{4A_{ij}}g_{\text{frag}}^2-\frac{3}{2}\right]\alpha_{ij}\sigma_{ij}^2\frac{1}{2C_{ij}^2}\left(1+C_{ij}g_{\text{frag}}^2\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right)- \\
& -\delta_{k,1}\frac{m_k}{2}\sum_{i,j}(i+j)\left(\frac{\pi}{A_{ij}}\right)^{5/2}\left[\frac{3}{2}g_{\text{frag}}^2+\frac{\gamma_{ij}A_{ij}}{2}g_{\text{frag}}^4\right]\alpha_{ij}\sigma_{ij}^2\frac{1}{2C_{ij}}\exp\left(-C_{ij}g_{\text{frag}}^2\right)- \\
& -\frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\left[\frac{R_{kj}^2}{4A_{kj}}+\mu_j^2A_{kj}+\mu_jR_{kj}\right]\alpha_{kj}\sigma_{kj}^2\frac{1}{2C_{ij}^3}\left(2+2C_{ij}g_{\text{frag}}^2+C_{ij}^2g_{\text{frag}}^4\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right)+ \\
& +\frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\left[\left(\frac{R_{kj}^2}{4A_{kj}}+\mu_j^2A_{kj}+\mu_jR_{kj}\right)\cdot g_{\text{frag}}^2-\frac{3}{2}\right]\alpha_{kj}\sigma_{kj}^2\frac{1}{2C_{ij}^2}\left(1+C_{ij}g_{\text{frag}}^2\right)\exp\left(-C_{ij}g_{\text{frag}}^2\right)+ \\
& +\frac{m_k}{2}\sum_{j}4\left(\frac{\pi}{A_{kj}}\right)^{5/2}\frac{3}{2}g_{\text{frag}}^2\alpha_{kj}\sigma_{kj}^2\frac{1}{2C_{ij}}\exp\left(-C_{ij}g_{\text{frag}}^2\right).
\end{aligned}
\tag{199}
$$