GENERAL FORM OF KINETIC COLLISION INTEGRALS

In our model, we face three types of collision integrals, for each type of collision. The aggregative integrals, the restitutive integrals and the fragmentative integrals. We can solve the typical forms of each type of collision integrals in the most general form

$$\begin{aligned}
I_a^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^{2l} e^{-Aw^2-Bu^2+R\boldsymbol{w}\cdot\boldsymbol{u}}\Theta(v_a-u)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q,\\
I_r^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^{2l} e^{-Aw^2-Bu^2+R\boldsymbol{w}\cdot\boldsymbol{u}}\Theta(u-v_a)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[v_f^2-(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big],\\
I_f^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^{2l} e^{-Aw^2-Bu^2+R\boldsymbol{w}\cdot\boldsymbol{u}}\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2-v_f^2\Big],
\end{aligned} \tag{1}$$

where k, l, m, p, q are integers, and $q = \{0, 1\}$. The difference in each type of integrals is in the domains of the vector $\boldsymbol{u}$. In the aggregative case, the values of u have to be less than a certain threshold v_a, in the restitutive case, the values of u have to be larger than v_a, but restricted by the parameter v_f from above. Finally, in the fragmentative case, the values of u are restricted by the parameter v_f from below.

Angular integrals

We start by first solving the inner integrals over $\hat{\boldsymbol{n}}$. By its physical meaning, we can call them angular integrals. Note, that q can be either 0 or 1, meaning that the corresponding term either do exist or is absent

$$\begin{aligned}
I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q,\\
I_{\hat{\boldsymbol{n}},r}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[v_f^2-(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big],\\
I_{\hat{\boldsymbol{n}},f}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2-v_f^2\Big].
\end{aligned} \tag{2}$$

If $q = 0$, then the angular integral is a function of only the vector $\boldsymbol{u}$, otherwise it is a function of both vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

Let us first solve the aggregative angular integrals

Aggregative angular integrals

We start with a simpler case when $q = 0$ and the angular integral is a function of only $\boldsymbol{u}$

$$I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p. \tag{3}$$

To solve this integral, we fix the vector $\boldsymbol{u}$, and denote by θ the angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{n}}$. In the spherical coordinates we have $\mathrm{d}\hat{\boldsymbol{n}} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, and the integral can be written as

$$\begin{aligned}
I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) &= 2\pi u^{m+p}\int_0^\pi \mathrm{d}\theta\sin\theta\Theta(-\cos\theta)|\cos\theta|^m(\cos\theta)^p =\\
&= 2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}\theta\sin\theta|\cos\theta|^m(\cos\theta)^p =\\
&= -2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}(\cos\theta)\,|\cos\theta|^m(\cos\theta)^p,
\end{aligned} \tag{4}$$

where we have integrated out over φ to give us the 2π factor. Now, substituting $\cos\theta = z$, we write

$$I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = 2\pi u^{m+p}\int_{-1}^0 \mathrm{d}z\,|z|^m z^p. \tag{5}$$

Since in the integration domain z is always negative, we now that $z^p < 0$ for odd values of p, and $z^p > 0$ for even values of p, hence we can write $z^p = (-1)^p |z|^p$, and

$$I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = 2\pi u^{m+p} \cdot (-1)^p \int_{-1}^{0} \mathrm{d}z\, |z|^{m+p} = -(-1)^p \cdot 2\pi u^{m+p} \int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\, |\cos\theta|^{m+p}. \tag{6}$$

We can see that for both odd and even values of $m+p$, the integral gives the same result, and finally we have

$$I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = (-1)^p \cdot \frac{2\pi u^{m+p}}{m+p+1}. \tag{7}$$

The case with $q = 1$ is trickier, since we have two arbitrary angles $\angle(\hat{\boldsymbol{n}}, \boldsymbol{u})$ and $\angle(\hat{\boldsymbol{n}}, \boldsymbol{w})$. However, we can write it as a dot product of $\boldsymbol{w}$ and another vector $\boldsymbol{F}$ as

$$\begin{aligned} I_{\hat{\boldsymbol{n}},a}^{m,p,1}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p (\boldsymbol{w}\cdot\hat{\boldsymbol{n}}) = \\ &= \boldsymbol{w} \cdot \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}} = \boldsymbol{w}\cdot\boldsymbol{F}, \end{aligned} \tag{8}$$

where the vector $\boldsymbol{F}$ is constructed by vectors $\hat{\boldsymbol{n}}$ and $\boldsymbol{u}$

$$\boldsymbol{F} = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}}. \tag{9}$$

Since it is being integrated over $\hat{\boldsymbol{n}}$, it cannot depend on $\hat{\boldsymbol{n}}$. This means that it can be oriented only along the vector $\boldsymbol{u}$, or $\boldsymbol{F} = f\boldsymbol{u}$. Now we can write

$$\begin{aligned} u^2 f = \boldsymbol{u}\cdot\boldsymbol{F} = \boldsymbol{u} \cdot &\int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}} = \\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^{p+1} = I_{\hat{\boldsymbol{n}},a}^{m,p+1,0}(\boldsymbol{u}), \end{aligned} \tag{10}$$

or

$$I_{\hat{\boldsymbol{n}},a}^{m,p,1}(\boldsymbol{u},\boldsymbol{w}) = f\boldsymbol{w}\cdot\boldsymbol{u} = \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{u^2} \cdot I_{\hat{\boldsymbol{n}},a}^{m,p+1,0}(\boldsymbol{u}), \tag{11}$$

which gives us the value of the integral

$$I_{\hat{\boldsymbol{n}},a}^{m,p,1}(\boldsymbol{u},\boldsymbol{w}) = (-1)^{p+1} \cdot \frac{2\pi u^{m+p-1}}{m+p+2} (\boldsymbol{w}\cdot\boldsymbol{u}). \tag{12}$$

Now, we can combine both cases of $q = 0$ and $q = 1$, and write

$$I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p (\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q = (-1)^{p+q} \cdot \frac{2\pi u^{m+p-q}}{m+p+q+1} \cdot (\boldsymbol{w}\cdot\boldsymbol{u})^q, \quad q = \{0,1\}. \tag{13}$$

Restitutive angular integrals

These type of integrals have a domain restriction terms given by the parameter v_f. We can start with a simpler case when $q = 0$, and write

$$\begin{aligned} I_{\hat{\boldsymbol{n}},r}^{m,p,0}(\boldsymbol{u}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p \Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big] = \\ &= (-1)^p \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^{m+p} \Theta\Big[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big]. \end{aligned} \tag{14}$$

Again, switching to spherical coordinates, and denoting the angle $\angle(\hat{\boldsymbol{n}}, \boldsymbol{u})$ by θ, we write

$$\begin{aligned} I_{\hat{\boldsymbol{n}},r}^{m,p,0}(\boldsymbol{u}) &= (-1)^p \cdot 2\pi u^{m+p} \int_0^{\pi} \mathrm{d}\theta \sin\theta \Theta(-\cos\theta) |\cos\theta|^{m+p} \Theta\Big[v_f^2 - (u\cos\theta)^2\Big] = \\ &= (-1)^p \cdot 2\pi u^{m+p} \int_{\pi/2}^{\pi} \mathrm{d}\theta \sin\theta |\cos\theta|^{m+p} \Theta\left[\frac{v_f^2}{u^2} - \cos^2\theta\right]. \end{aligned} \tag{15}$$

The domain restriction implies

$$|\cos\theta| \leqslant \frac{v_f}{u}. \tag{16}$$

This constraint restricts two variable, both θ and u, although we do not perform integration over u at this moment. Since the variable u changes from 0 to ∞, the restriction can be split into two cases, (i) when $u \leqslant v_f$, (ii) when $u > v_f$. In the first case, when $u \leqslant v_f$, the restriction holds true for any values of $\theta \in [\pi/2, \pi]$, e.g. no constraint in the angle θ. In the second case, when $u > v_f$, the restriction holds true only within a certain range of values of θ, namely $\theta \in [\pi/2, \pi - \arccos(v_f/u)]$. Now, we can rewrite the domain restriction term as

$$\Theta\left[\frac{v_f^2}{u^2} - \cos^2\theta\right] = \Theta(v_f - u) + \Theta(u - v_f)\Theta\left[\pi - \arccos\left(\frac{v_f}{u}\right) - \theta\right]. \tag{17}$$

Using this form of the restriction allows us to solve the restitutive angular integrals

$$\begin{aligned} I_{\hat{\boldsymbol{n}},r}^{m,p,0}(\boldsymbol{u}) = &- (-1)^p \cdot 2\pi u^{m+p}\Theta(v_f - u)\int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p} - \\ &- (-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f)\int_{\pi/2}^{\pi-\arccos(v_f/u)} \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p}. \end{aligned} \tag{18}$$

The first integral is already solved for the aggregative angular case, and in the second integral we substitute $z = \cos\theta$, and write

$$\begin{aligned} I_{\hat{\boldsymbol{n}},r}^{m,p,0}(\boldsymbol{u}) &= \Theta(v_f - u) \cdot I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) - (-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f)\int_0^{-v_f/u} \mathrm{d}z\,|z|^{m+p} = \\ &= (-1)^p \cdot \Theta(v_f - u) \cdot \frac{2\pi u^{m+p}}{m+p+1} + (-1)^p \cdot \Theta(u - v_f) \cdot \frac{2\pi u^{m+p}}{m+p+1}\left(\frac{v_f}{u}\right)^{m+p+1} = \\ &= (-1)^p \cdot \frac{2\pi u^{m+p}}{m+p+1}\left[\Theta(v_f - u) + \Theta(u - v_f)\left(\frac{v_f}{u}\right)^{m+p+1}\right], \end{aligned} \tag{19}$$

or

$$I_{\hat{\boldsymbol{n}},r}^{m,p,0}(\boldsymbol{u}) = I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) \cdot \left[\Theta(v_f - u) + \Theta(u - v_f)\left(\frac{v_f}{u}\right)^{m+p+1}\right]. \tag{20}$$

For the case $q = 1$, we can perform the same procedure as before, and write

$$I_{\hat{\boldsymbol{n}},r}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{w} \cdot \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\Theta\left[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\right]\hat{\boldsymbol{n}} = \boldsymbol{w}\cdot\boldsymbol{F}, \tag{21}$$

where

$$\boldsymbol{F} = \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\Theta\left[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\right]\hat{\boldsymbol{n}}. \tag{22}$$

Again, we see that $\boldsymbol{F}$ vector cannot depend on $\hat{\boldsymbol{n}}$, and depends only on the vector $\boldsymbol{u}$. This implies that $\boldsymbol{F} = f\boldsymbol{u}$, or

$$f = \frac{\boldsymbol{F}\cdot\boldsymbol{u}}{u^2} = \frac{1}{u^2}\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^{p+1}\Theta\left[v_f^2 - (\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\right] = u^{-2}\cdot I_{\hat{\boldsymbol{n}},r}^{m,p+1,0}(\boldsymbol{u}). \tag{23}$$

Since,

$$I_{\hat{\boldsymbol{n}},r}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{w}\cdot\boldsymbol{F} = (\boldsymbol{w}\cdot\boldsymbol{u})f = \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{u^2}\cdot I_{\hat{\boldsymbol{n}},r}^{m,p+1,0}(\boldsymbol{u}), \tag{24}$$

or writing explicitly, we have

$$I_{\hat{\boldsymbol{n}},r}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) = (-1)^{p+1}\cdot\frac{2\pi u^{m+p-1}}{m+p+2}\left[\Theta(v_f - u) + \Theta(u - v_f)\left(\frac{v_f}{u}\right)^{m+p+2}\right](\boldsymbol{w}\cdot\boldsymbol{u}). \tag{25}$$

By combining both cases $q = 0$ and $q = 1$, we write the final solution of the restitutive angular integrals as

$$I_{\hat{\boldsymbol{n}},r}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) = (-1)^{p+q}\cdot\frac{2\pi u^{m+p-q}}{m+p+q+1}\left[\Theta(v_f - u) + \Theta(u - v_f)\left(\frac{v_f}{u}\right)^{m+p+q+1}\right](\boldsymbol{w}\cdot\boldsymbol{u})^q. \tag{26}$$

Fragmentative angular integrals

The last type of angular integrals is the fragmentative type, which is very similar to the restitutive angular case. Again, we start with the simpler case of $q = 0$

$$I^{m,p,0}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2 - v_f^2\Big]. \tag{27}$$

The difference of this type of angular integrals is in the inverse domain restriction function. Switching into spherical coordinates, we write

$$I^{m,p,0}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u}) = -(-1)^p \cdot 2\pi u^{m+p}\int_{\pi/2}^{\pi} \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p}\Theta\left[\cos^2\theta - \frac{v_f^2}{u^2}\right]. \tag{28}$$

The domain restriction is now given as

$$|\cos\theta| \geqslant \frac{v_f}{u}. \tag{29}$$

This condition can be satisfied only for $u \geqslant v_f$, and we can rewrite the domain restriction as

$$\Theta\left[\cos^2\theta - \frac{v_f^2}{u^2}\right] = \Theta(u - v_f)\Theta\Big[\theta - \pi - \arccos\Big(\frac{v_f}{u}\Big)\Big], \tag{30}$$

and our fragmentative angular integral becomes

$$\begin{aligned} I^{m,p,0}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u}) &= -(-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f)\int_{\pi-\arccos(v_f/u)}^{\pi} \mathrm{d}(\cos\theta)\,|\cos\theta|^{m+p} = \\ &= -(-1)^p \cdot 2\pi u^{m+p}\Theta(u - v_f)\int_{-v_f/u}^{-1} \mathrm{d}z\,|z|^{m+p} = \\ &= (-1)^p \cdot \frac{2\pi u^{m+p}}{m+p+1}\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+1}\right]\Theta(u - v_f). \end{aligned} \tag{31}$$

The case with $q = 1$ can be solved exactly as the previous cases, and we can immediately write

$$I^{m,p,1}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u},\boldsymbol{w}) = \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{u^2}\cdot I^{m,p+1,0}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u}) = (-1)^{p+1}\cdot\frac{2\pi u^{m+p-1}}{m+p+2}\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+2}\right]\Theta(u - v_f)(\boldsymbol{w}\cdot\boldsymbol{u}), \tag{32}$$

and combining both cases $q = 0$ and $q = 1$, we have

$$I^{m,p,q}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u},\boldsymbol{w}) = (-1)^{p+q}\cdot\frac{2\pi u^{m+p-q}}{m+p+q+1}\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+q+1}\right]\Theta(u - v_f)(\boldsymbol{w}\cdot\boldsymbol{u})^q. \tag{33}$$

Final results of angular integrals

Let us write the final results of solutions of the angular integrals for all three types of collision integrals. First, the solution of the aggregative angular integrals is

$$I^{m,p,q}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q = (-1)^{p+q}\cdot\frac{2\pi u^{m+p-q}}{m+p+q+1}\cdot(\boldsymbol{w}\cdot\boldsymbol{u})^q, \quad q = \{0,1\}. \tag{34}$$

The restitutive and fragmentative angular integrals are solved to give us

$$\begin{aligned} I^{m,p,q}_{\hat{\boldsymbol{n}},r}(\boldsymbol{u},\boldsymbol{w}) &= I^{m,p,q}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w})\cdot\left[\Theta(v_f - u) + \Theta(u - v_f)\Big(\frac{v_f}{u}\Big)^{m+p+q+1}\right], \\ I^{m,p,q}_{\hat{\boldsymbol{n}},f}(\boldsymbol{u},\boldsymbol{w}) &= I^{m,p,q}_{\hat{\boldsymbol{n}},a}(\boldsymbol{u},\boldsymbol{w})\cdot\left[1 - \Big(\frac{v_f}{u}\Big)^{m+p+q+1}\right]\Theta(u - v_f). \end{aligned} \tag{35}$$

Center of mass velocity integrals

We refer to the integrals over the vector $\boldsymbol{w}$ as the center of mass velocity integrals. All three types of collision integrals contain similar forms of the center of mass velocity integrals, and we can write a generic form of such integrals as

$$I_{\boldsymbol{w}}^{l,q}(\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{w}\, w^{2l} (\boldsymbol{w}\cdot\boldsymbol{u})^q \exp\left(-Aw^2\right) \exp\left(R\boldsymbol{w}\cdot\boldsymbol{u}\right), \quad q = \{0,1\}. \tag{36}$$

Switching into spherical coordinates, and denoting by θ the angle between vectors $\boldsymbol{w}$ and $\boldsymbol{u}$, we have $\mathrm{d}\boldsymbol{w} = w^2 \sin\theta\, \mathrm{d}w\, \mathrm{d}\theta\, \mathrm{d}\varphi$,

$$I_{\boldsymbol{w}}^{l,q}(\boldsymbol{u}) = -2\pi u^q \int_0^\infty \mathrm{d}w\, w^{2l+q+2} \exp\left(-Aw^2\right) \int_0^\pi \mathrm{d}(\cos\theta)\, (\cos\theta)^q \exp\left(Rwu\cdot\cos\theta\right), \quad q = \{0,1\}. \tag{37}$$

Again, we solve these integrals for two different cases of q, starting with the simpler case

The case with $q = 0$

In this case we write

$$I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) = -2\pi \int_0^\infty \mathrm{d}w\, w^{2l+2} \exp\left(-Aw^2\right) \int_0^\pi \mathrm{d}(\cos\theta) \exp\left(Rwu\cdot\cos\theta\right). \tag{38}$$

The inner angular integral is solved to give us

$$I_R^0(\boldsymbol{u},\boldsymbol{w}) = -\int_0^\pi \mathrm{d}(\cos\theta) \exp\left(Rwu\cdot\cos\theta\right) = \frac{1}{Rwu}\left(e^{Rwu} - e^{-Rwu}\right), \tag{39}$$

and substituting into the center of mass velocity integral, we have

$$I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) = \frac{2\pi}{Ru} \int_0^\infty \mathrm{d}w\, w^{2l+1} \exp\left(-Aw^2\right)\left(e^{Ru\cdot w} - e^{-Ru\cdot w}\right). \tag{40}$$

The case with $q = 1$

The case with $q = 1$ is

$$I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) = -2\pi u \int_0^\infty \mathrm{d}w\, w^{2l+3} \exp\left(-Aw^2\right) \int_0^\pi \mathrm{d}(\cos\theta)\, (\cos\theta) \exp\left(Rwu\cdot\cos\theta\right), \tag{41}$$

where the inner angular integral is solved to give us

$$I_R^1(\boldsymbol{u},\boldsymbol{w}) = -\int_0^\pi \mathrm{d}(\cos\theta)\, (\cos\theta) \exp\left(Rwu\cdot\cos\theta\right) = \frac{1}{Rwu}\left(e^{Rwu} + e^{-Rwu}\right) - \frac{1}{R^2w^2u^2}\left(e^{Rwu} - e^{-Rwu}\right). \tag{42}$$

Now, the center of mass velocity integral reads

$$I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) = \frac{2\pi}{R} \int_0^\infty \mathrm{d}w\, w^{2l+2} \exp\left(-Aw^2\right)\left(e^{Ru\cdot w} + e^{-Ru\cdot w}\right) - \frac{2\pi}{R^2u} \int_0^\infty \mathrm{d}w\, w^{2l+1} \exp\left(-Aw^2\right)\left(e^{Ru\cdot w} - e^{Ru\cdot w}\right). \tag{43}$$

Shifted Gaussian integrals

To proceed further, let us analyze the specific types of shifted Gaussian integrals, namely

$$I_{G,\pm}^n = \int_0^\infty \mathrm{d}x\, x^n \exp\left(-ax^2 \pm bx\right), \qquad n \in \{0,1,2,\ldots\}. \tag{44}$$

To get a general solution for these types of integrals, let us write them in a more canonical form first. To do so, let us introduce a variable transformation

$$t := \sqrt{a}x \pm \lambda, \qquad \lambda = \frac{b}{2\sqrt{a}}, \tag{45}$$

this implies

$$\begin{aligned} -ax^2 \pm bx &= -t^2 + \lambda^2, \\ \mathrm{d}x &= \frac{\mathrm{d}t}{\sqrt{a}}, \\ x^n &= a^{-n/2} \cdot (t \pm \lambda)^n = a^{-n/2} \cdot \sum_{k=0}^{n} \binom{n}{k} (\pm\lambda)^{n-k} t^k, \\ \binom{n}{k} &= \frac{n!}{k!(n-k)!}. \end{aligned} \tag{46}$$

Now, our shifted Gaussian integrals become

$$I^n_{G,\pm} = \frac{\exp\left(\lambda^2\right)}{\sqrt{a^{n+1}}} \sum_{k=0}^{n} \binom{n}{k} (\pm\lambda)^{n-k} \int_{\pm\lambda}^{\infty} \mathrm{d}t\, t^k e^{-t^2}. \tag{47}$$

The $\pm$ sign is now hidden in the parameter λ. Let us concentrate on the canonical Gaussian integral

$$G^k_\pm = \int_{\pm\lambda}^{\infty} \mathrm{d}t\, t^k e^{-t^2}. \tag{48}$$

Let us start with integration by parts and put $u = e^{-t^2}$ and $\mathrm{d}v = t^k\,\mathrm{d}t$. This gives us $\mathrm{d}u = -2te^{-t^2}\,\mathrm{d}t$, and $v = t^{k+1}/(k+1)$. Now we have

$$G^k_\pm = \frac{t^{k+1}}{k+1}\, e^{-t^2}\Big|_{\pm\lambda}^{\infty} + \frac{2}{k+1}\int_{\pm\lambda}^{\infty} \mathrm{d}t\, t^{k+2} e^{-t^2} = -\frac{(\pm\lambda)^{k+1}}{k+1} e^{-\lambda^2} + \frac{2}{k+1} G^{k+2}_\pm, \tag{49}$$

a recurrent relation for the integral

$$G^{k+2}_\pm = \frac{k+1}{2} G^k_\pm + (\pm\lambda)^{k+1} \cdot \frac{1}{2} e^{-\lambda^2}. \tag{50}$$

In order to get a full solution, let us calculate the first two cases $k = 0$ and $k = 1$. We have

$$\begin{aligned} G^0_\pm &= \int_{\pm\lambda}^{\infty} \mathrm{d}t\, e^{-t^2} = \frac{\sqrt{\pi}}{2}(1 \pm \operatorname{erf}(\lambda)), \\ G^1_\pm &= G^1 = \int_{\pm\lambda}^{\infty} \mathrm{d}t\, te^{-t^2} = \frac{1}{2} e^{-\lambda^2}. \end{aligned} \tag{51}$$

Given this two functions, we can obtain the solution for any order k

$$G^k_\pm = \frac{k-1}{2} G^{k-2}_\pm + (\pm\lambda)^{k-1} G^1, \tag{52}$$

where we have rewritten our recurrent relation with the help of G^1. Extending the the recurrent relation, we obtain two different results, for odd and even k. For an even $k = 2p$, then we have

$$G^{2p}_\pm = \frac{(2p-1)!!}{2^p} G^0_\pm \pm G^1 \sum_{j=1}^{p} \frac{1}{2^{j-1}} \frac{(2p-1)!!}{(2p-2j+1)!!} \lambda^{2p-2j+1}. \tag{53}$$

Here we used the fact that $2p - 2j + 1$ is always an odd number, hence $(\pm\lambda)^{2p-2j+1} = \pm\lambda^{2p-2j+1}$. For odd valued $k = 2p+1$, we have

$$G^{2p+1}_\pm = G^{2p+1} = G^1 \sum_{j=1}^{p+1} \frac{1}{2^{j-1}} \frac{(2p)!!}{(2p-2j+2)!!} \lambda^{2p-2j+2}. \tag{54}$$

Here we used the fact that $2p-2j+2$ is always an even number, hence $(\pm\lambda)^{2p-2j+2} = \lambda^{2p-2j+2}$. Now, we can write the solution of the original shifted Gaussian integrals as

$$I^n_{G,\pm} = \frac{\exp\left(\lambda^2\right)}{\sqrt{a^{n+1}}} \sum_{k=0}^{n} \binom{n}{k} (\pm\lambda)^{n-k} G^k_{\pm}. \tag{55}$$

Symmetric Gaussian integrals

In the context of our problem, the shifted Gaussian integrals appear in symmetric pairs, such as

$$I^n_{\pm} = I^n_{G,+} \pm I^n_{G,-} = \int_0^{\infty} \mathrm{d}x\, x^n e^{-ax^2} \left(e^{bx} \pm e^{-bx}\right). \tag{56}$$

From (55), we have

$$I^n_{\pm} = \frac{\exp\left(\lambda^2\right)}{\sqrt{a^{n+1}}} \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} \left[G^k_+ \pm (-1)^{n-k} G^k_-\right]. \tag{57}$$

If we notice, that $G^0_+ + G^0_- = \sqrt{\pi}$, meaning that the error functions cancel each other, we can look for specific solution for odd or even values of n. Let us look at $n = 2p$ for a positive version of the symmetric integral, namely

$$I^{2p}_{+} = \frac{\exp\left(\lambda^2\right)}{a^p\sqrt{a}} \sum_{k=0}^{2p} \binom{2p}{k} \lambda^{2p-k} \left[G^k_+ + (-1)^{2p-k} G^k_-\right]. \tag{58}$$

We can split the sum into two, summing up only the odd indices and only the even indices separately. This gives us

$$I^{2p}_{+} = \frac{\exp\left(\lambda^2\right)}{a^p\sqrt{a}} \left[\sum_{k=0}^{p} \binom{2p}{2k} \lambda^{2p-2k} \left(G^{2k}_+ + G^{2k}_-\right) + \sum_{k=0}^{p-1} \binom{2p}{2k+1} \lambda^{2p-2k-1} \left(G^{2k+1}_+ - G^{2k+1}_-\right)\right], \tag{59}$$

where we have used $(-1)^{2p-2k} = 1$ and $(-1)^{2p-2k-1} = -1$. From (53) and (54) we have

$$G^{2k}_+ + G^{2k}_- = \frac{(2k-1)!!}{2^k}\sqrt{\pi}, \qquad G^{2k+1}_+ - G^{2k+1}_- = 0. \tag{60}$$

Now, we have

$$I^{2p}_{+} = \sqrt{\frac{\pi}{a}} \frac{\exp\left(\lambda^2\right)}{a^p} \sum_{k=0}^{p} \binom{2p}{2k} \frac{(2k-1)!!}{2^k} \cdot \lambda^{2p-2k}. \tag{61}$$

Next, let us look at $n = 2p+1$ for the negative symmetric integral, namely

$$I^{2p+1}_{-} = \frac{\exp\left(\lambda^2\right)}{a^p\sqrt{a}} \sum_{k=0}^{2p+1} \binom{2p+1}{k} \lambda^{2p+1-k} \left[G^k_+ - (-1)^{2p-k} G^k_-\right]. \tag{62}$$

Again, splitting the sum into two, we write

$$I^{2p+1}_{-} = \frac{\exp\left(\lambda^2\right)}{a^{p+1}} \left[\sum_{k=0}^{p} \binom{2p+1}{2k} \lambda^{2p-2k+1} \left(G^{2k}_+ + G^{2k}_-\right) + \sum_{k=0}^{p} \binom{2p+1}{2k+1} \lambda^{2p-2k} \left(G^{2k+1}_+ - G^{2k+1}_-\right)\right], \tag{63}$$

which results in

$$I^{2p+1}_{-} = \frac{\sqrt{\pi}\exp\left(\lambda^2\right)}{a^{p+1}} \sum_{k=0}^{p} \binom{2p+1}{2k} \frac{(2k-1)!!}{2^k} \cdot \lambda^{2p-2k+1}. \tag{64}$$

As we can see, the odd and negative symmetric integral, together with the even and positive symmetric integrals, do not contain the error function, and can be written purely in algebraic terms. Luckily, in our problem, for only even powers of w, in the original kinetic integrals, we will deal with only the analytic versions of the center of mass velocity integrals. Also, the main physical parameters, are described by the analytic versions of these integrals as we will see later.

Solutions of the center of mass velocity integrals

Let us write the center of mass velocity integrals for $q = 0$ and $q = 1$ once again.

$$I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) = \frac{2\pi}{Ru}\int_0^\infty \mathrm{d}w\, w^{2l+1}\exp\left(-Aw^2\right)\left(e^{Ru\cdot w} - e^{-Ru\cdot w}\right), \tag{65}$$

for $q = 0$ and

$$I_{\boldsymbol{w}}^{1}(\boldsymbol{u}) = \frac{2\pi}{R}\int_0^\infty \mathrm{d}w\, w^{2l+2}\exp\left(-Aw^2\right)\left(e^{Ru\cdot w} + e^{-Ru\cdot w}\right) - \frac{2\pi}{R^2u}\int_0^\infty \mathrm{d}w\, w^{2l+1}\exp\left(-Aw^2\right)\left(e^{Ru\cdot w} - e^{Ru\cdot w}\right), \tag{66}$$

for $q = 1$. They can be written in terms of the symmetric Gaussian integrals as

$$\begin{aligned} I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) &= \frac{2\pi}{Ru}\cdot I_{-}^{2l+1}, \\ I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) &= \frac{2\pi}{R}\cdot I_{+}^{2l+2} - \frac{1}{R}\cdot I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}), \\ a = A, \quad b &= Ru, \quad \Longrightarrow \quad \lambda = \frac{Ru}{2\sqrt{A}}. \end{aligned} \tag{67}$$

Using (61) and (64), we have

$$\begin{aligned} I_{\boldsymbol{w}}^{l,0}(\boldsymbol{u}) &= \sqrt{\frac{\pi}{A}}\cdot\frac{\pi\exp\left(\lambda^2\right)}{A^{l+1}}\sum_{k=0}^{l}\binom{2l+1}{2k}\frac{(2k-1)!!}{2^k}\cdot\lambda^{2l-2k}, \\ I_{\boldsymbol{w}}^{l,1}(\boldsymbol{u}) &= \frac{2\pi}{R}\sqrt{\frac{\pi}{A}}\frac{\exp\left(\lambda^2\right)}{A^{l+1}}\left[\frac{(2l+1)!!}{2^{l+1}} + \sum_{k=0}^{l}\binom{2l+1}{2k}\frac{(2k-1)!!}{2^k}\cdot\lambda^{2l-2k}\left(\frac{2l+2}{2l+2-2k}\lambda^2 - \frac{1}{2}\right)\right]. \end{aligned} \tag{68}$$

Both cases of $q = 0$ and $q = 1$ can now be combined simply as

$$I_{\boldsymbol{w}}^{l,q}(\boldsymbol{u}) = \left(\frac{2}{R}\right)^q\sqrt{\frac{\pi}{A}}\frac{\pi\exp\left(\lambda^2\right)}{A^{l+1}}\left[\left(\frac{(2l+1)!!}{2^{l+1}}\right)^q + \sum_{j=0}^{l}\binom{2l+1}{2j}\frac{(2j-1)!!}{2^j}\cdot\lambda^{2l-2j}\cdot\left(\frac{2l+2}{2l+2-2j}\lambda^2 - \frac{1}{2}\right)^q\right]. \tag{69}$$

We have changed the dummy index k, not to confuse with the index k in the original kinetic integrals.

Relative velocity integrals

The final type of integrals over $\boldsymbol{u}$, is what we call the relative velocity integrals. The general form of these integrals are simple Gaussian integrals. Depending on the type of the kinetic integral, they can be written as

$$\begin{aligned} I_{\boldsymbol{u},a}^{k,i} &= \int \mathrm{d}\boldsymbol{u}\, u^{k+i}\exp\left(-Cu^2\right)\Theta(v_a - u), \\ I_{\boldsymbol{u},f}^{k,i} &= \int \mathrm{d}\boldsymbol{u}\, u^{k+i}\exp\left(-Cu^2\right)\Theta(u - v_f), \\ I_{\boldsymbol{u},r}^{k,i} &= \int \mathrm{d}\boldsymbol{u}\, u^{k+i}\exp\left(-Cu^2\right)\Theta(u - v_a)\Theta(v_f - u), \end{aligned} \tag{70}$$

where we C is constant and i is a combined power of all u vectors appearing in the inner integrals. The index k is the original power of u in the kinetic integral. Changing into spherical coordinates, we have

$$\begin{aligned} I_{\boldsymbol{u},a}^{k,i} &= 4\pi\int_0^{v_a}\mathrm{d}u\, u^{k+i+2}\exp\left(-Cu^2\right), \\ I_{\boldsymbol{u},r}^{k,i} &= 4\pi\int_{v_a}^{v_f}\mathrm{d}u\, u^{k+i+2}\exp\left(-Cu^2\right), \\ I_{\boldsymbol{u},f}^{k,i} &= 4\pi\int_{v_f}^{\infty}\mathrm{d}u\, u^{k+i+2}\exp\left(-Cu^2\right). \end{aligned} \tag{71}$$

We can write these integrals in canonical forms by setting $t = \sqrt{C}u$. This implies

$$
\begin{aligned}
u &= \frac{t}{\sqrt{C}}, \qquad \mathrm{d}u = \frac{\mathrm{d}t}{\sqrt{C}}, \\
u^{k+i+2} &= \frac{t^{k+i+2}}{\sqrt{C^{k+i+2}}}, \\
u &= 0, \quad \Rightarrow \quad t = 0, \\
u &= \infty, \quad \Rightarrow \quad t = \infty, \\
u &= v_{a/f}, \quad \Rightarrow \quad t = \sqrt{C}v_{a/f}.
\end{aligned}
\tag{72}
$$

Now, our relative velocity integrals become

$$
\begin{aligned}
I^{k,i}_{\boldsymbol{u},a} &= \frac{4\pi}{\sqrt{C^{k+i+3}}} \int_0^{\sqrt{C}v_a} \mathrm{d}t\, t^{k+i+2} e^{-t^2}, \\
I^{k,i}_{\boldsymbol{u},r} &= \frac{4\pi}{\sqrt{C^{k+i+3}}} \int_{\sqrt{C}v_a}^{\sqrt{C}v_f} \mathrm{d}t\, t^{k+i+2} e^{-t^2}, \\
I^{k,i}_{\boldsymbol{u},f} &= \frac{4\pi}{\sqrt{C^{k+i+3}}} \int_{\sqrt{C}v_f}^{\infty} \mathrm{d}t\, t^{k+i+2} e^{-t^2}.
\end{aligned}
\tag{73}
$$

As we did with previously, we can integrate by parts, writing $u = e^{-t^2}$, $\mathrm{d}v = t^{k+i+2}\,\mathrm{d}t$, which gives us $\mathrm{d}u = -2te^{-t^2}\,\mathrm{d}t$ and $v = t^{k+i+3}/(k+i+3)$. Hence,

$$
I^k = \int_{L_1}^{L_2} \mathrm{d}t\, t^{k+i+2} e^{-t^2} = \frac{t^{k+i+3}}{k+i+3}\, e^{-t^2}\bigg|_{L_1}^{L_2} + \frac{2}{k+i+3} \int_{L_1}^{L_2} \mathrm{d}t\, t^{k+i+4} e^{-t^2},
\tag{74}
$$

and we write the recurrent relation

$$
I^k = \frac{k+i+1}{2} I^{k-2} + \frac{1}{2}\Big(L_1^{k+i+1} e^{-L_1^2} - L_2^{k+i+1} e^{-L_2^2}\Big).
\tag{75}
$$

SOLUTION OF THE KINETIC INTEGRALS