GENERAL FORM OF KINETIC COLLISION INTEGRALS

In our model, we face three types of collision integrals, for each type of collision. The aggregative integrals, the restitutive integrals and the fragmentative integrals. We can solve the typical forms of each type of collision integrals in the most general form

$$
\begin{aligned}
I_a^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^l e^{-Au^2-Bw^2+R\boldsymbol{u}\cdot\boldsymbol{w}}\Theta(\lambda_a - u)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q,\\
I_r^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^l e^{-Au^2-Bw^2+R\boldsymbol{u}\cdot\boldsymbol{w}}\Theta(u-\lambda_a)\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[\lambda_f^2-(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big],\\
I_f^{k,l,m,p,q} &= \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{w}\, u^k w^l e^{-Au^2-Bw^2+R\boldsymbol{u}\cdot\boldsymbol{w}}\int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2-\lambda_f^2\Big],
\end{aligned}
\tag{1}
$$

where k, l, m, p, q, s are integers, and $q = \{0, 1\}$. The difference in each type of integrals is in the domains of the vector $\boldsymbol{u}$. In the aggregative case, the values of u have to be less than a certain threshold λ_a, in the restitutive case, the values of u have to be larger than λ_a, but restricted by the parameter λ_f from above. Finally, in the fragmentative case, the values of u are restricted by the parameter λ_f from below.

Angular integrals

We start by first solving the inner integrals over $\hat{\boldsymbol{n}}$. By its physical meaning, we can call them angular integrals. Note, that q can be either 0 or 1, meaning that the corresponding term either do exist or is absent

$$
\begin{aligned}
I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q,\\
I_{\hat{\boldsymbol{n}},r}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[\lambda^2-(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2\Big],\\
I_{\hat{\boldsymbol{n}},f}^{m,p,q}(\boldsymbol{u},\boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})^q\Theta\Big[(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^2-\lambda^2\Big].
\end{aligned}
\tag{2}
$$

If $q = 0$, then the angular integral is a function of only the vector $\boldsymbol{u}$, otherwise it is a function of both vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

Let us first solve the aggregative angular integrals

Aggregative angular integrals

We start with a simpler case when $q = 0$ and the angular integral is a function of only $\boldsymbol{u}$

$$
I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = \int \mathrm{d}\hat{\boldsymbol{n}}\,\Theta(-\boldsymbol{u}\cdot\hat{\boldsymbol{n}})|\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|^m(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})^p. \tag{3}
$$

To solve this integral, we fix the vector $\boldsymbol{u}$, and denote by θ the angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{n}}$. In the spherical coordinates we have $\mathrm{d}\hat{\boldsymbol{n}} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, and the integral can be written as

$$
\begin{aligned}
I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) &= 2\pi u^{m+p}\int_0^\pi \mathrm{d}\theta\sin\theta\Theta(-\cos\theta)|\cos\theta|^m(\cos\theta)^p =\\
&= 2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}\theta\sin\theta|\cos\theta|^m(\cos\theta)^p =\\
&= -2\pi u^{m+p}\int_{\pi/2}^\pi \mathrm{d}(\cos\theta)\,|\cos\theta|^m(\cos\theta)^p,
\end{aligned}
\tag{4}
$$

where we have integrated out over φ to give us the 2π factor. Now, substituting $\cos\theta = z$, we write

$$
I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = 2\pi u^{m+p}\int_{-1}^0 \mathrm{d}z\,|z|^m z^p. \tag{5}
$$

Since in the integration domain z is always negative, we now that $z^p < 0$ for odd values of p, and $z^p > 0$ for even values of p, hence we can write $z^p = (-1)^p |z|^p$, and

$$I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = 2\pi u^{m+p} \cdot (-1)^p \int_{-1}^{0} \mathrm{d}z\, |z|^{m+p}. \tag{6}$$

We can see that for both odd and even values of $m+p$, the integral gives the same result, and finally we have

$$I_{\hat{\boldsymbol{n}},a}^{m,p,0}(\boldsymbol{u}) = (-1)^p \cdot \frac{2\pi u^{m+p}}{m+p+1}. \tag{7}$$

The case with $q = 1$ is trickier, since we have two arbitrary angles $\angle(\hat{\boldsymbol{n}}, \boldsymbol{u})$ and $\angle(\hat{\boldsymbol{n}}, \boldsymbol{w})$. However, we can write it as a dot product of $\boldsymbol{w}$ and another vector $\boldsymbol{F}$ as

$$\begin{aligned} I_{\hat{\boldsymbol{n}},a}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) |\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^p (\boldsymbol{w} \cdot \hat{\boldsymbol{n}}) = \\ &= \boldsymbol{w} \cdot \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) |\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}} = \boldsymbol{w} \cdot \boldsymbol{F}, \end{aligned} \tag{8}$$

where the vector $\boldsymbol{F}$ is constructed by vectors $\hat{\boldsymbol{n}}$ and $\boldsymbol{u}$

$$\boldsymbol{F} = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) |\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}}. \tag{9}$$

Since it is being integrated over $\hat{\boldsymbol{n}}$, it cannot depend on $\hat{\boldsymbol{n}}$. This means that it can be oriented only along the vector $\boldsymbol{u}$, or $\boldsymbol{F} = f\boldsymbol{u}$. Now we can write

$$\begin{aligned} u^2 f = \boldsymbol{u} \cdot \boldsymbol{F} &= \boldsymbol{u} \cdot \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) |\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^p \hat{\boldsymbol{n}} = \\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) |\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^{p+1} = I_{\hat{\boldsymbol{n}},a}^{m,p+1,0}(\boldsymbol{u}), \end{aligned} \tag{10}$$

or

$$I_{\hat{\boldsymbol{n}},a}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) = f \boldsymbol{w} \cdot \boldsymbol{u} = \frac{\boldsymbol{w} \cdot \boldsymbol{u}}{u^2} \cdot I_{\hat{\boldsymbol{n}},a}^{m,p+1,0}(\boldsymbol{u}), \tag{11}$$

which gives us the value of the integral

$$I_{\hat{\boldsymbol{n}},a}^{m,p,1}(\boldsymbol{u}, \boldsymbol{w}) = (-1)^{p+1} \cdot \frac{2\pi u^{m+p-1}}{m+p+2} (\boldsymbol{w} \cdot \boldsymbol{u}). \tag{12}$$

Now, we can combine both cases of $q = 0$ and $q = 1$, and write

$$I_{\hat{\boldsymbol{n}},a}^{m,p,q}(\boldsymbol{u}, \boldsymbol{w}) = \int \mathrm{d}\hat{\boldsymbol{n}}\, \Theta(-\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) |\boldsymbol{u} \cdot \hat{\boldsymbol{n}}|^m (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})^p (\boldsymbol{w} \cdot \hat{\boldsymbol{n}})^q = (-1)^{p+q} \cdot \frac{2\pi u^{m+p-q}}{m+p+q+1} \cdot (\boldsymbol{w} \cdot \boldsymbol{u})^q, \quad q = \{0, 1\}. \tag{13}$$

Restitutive angular integrals