Orbital Mechanics: Keplerian model

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Preface

This book aims to provide a comprehensive guide to the Keplerian model of orbital mechanics, covering the fundamental principles and equations governing the motion of celestial bodies. We will explore the historical context of orbital mechanics, starting with the work of Johannes Kepler and Isaac Newton, and progressing to modern applications in astrodynamics and space exploration.

1 Newton's law of gravitation

Newton formulated that the gravitational force between two point masses is directly proportional to the product of their masses and inversely proportional to the square of the distance between them (inverse square law):

$$\vec{F} \propto \frac{m_1 m_2}{r^2}$$

1.1 How did Newton arrive at the inverse square law?

Newton deduced the inverse square law through a combination of Kepler's empirical laws, geometrical analysis, and physical intuition:

- Kepler, building on **Tycho Brahe**'s detailed astronomical data, established these laws:
  - **1st Law**: Planets move in elliptical orbits with the Sun at one focus.
  - **2nd Law**: The line joining a planet and the Sun sweeps out equal areas in equal time.
  - **3rd Law**: The square of a planet's orbital period is proportional to the cube of its semi-major axis: $T^2 \propto r^3$
- Newton inferred that the force must be a **central force**—pointing toward the Sun—because Kepler's Second Law (the area law) states that a planet sweeps out equal areas in equal time. This only happens if the torque about the Sun is zero, which in turn implies that the force has no component perpendicular to the radius vector. Therefore, the force must always point along the line joining the planet and the Sun—making it a central force.
- **Deriving the inverse-square relationship:** Starting with Kepler's Third law: $T^2 \propto r^3 \Rightarrow \left(\frac{2\pi r}{v}\right)^2 \propto r^3 \Rightarrow v^2 \propto \frac{1}{r} \Rightarrow a = \frac{v^2}{r} \propto \frac{1}{r^2}$ This showed Newton that the acceleration required to keep a planet in orbit must follow an inverse-square law
- To test the universality of this law, Newton compared the gravitational acceleration near Earth's surface ($a_E \approx 9.81 m/s^2$) with the centripetal acceleration needed to keep the Moon in its orbit using the following measurements to calculate the velocity in its orbit:

- The Moon's distance from the Earth: about 60 Earth radii (or 384,400 km).
- And the orbital period was around 27.3 days.

And he found: $\frac{a_E}{a_M} \approx \left(\frac{r_M}{r_E}\right)^2$. This confirmed that the same law governed both falling of objects near Earth as well as orbiting moons.

2 Two Body Problem

2.1 Equations of Motion for the Two body problem

Let $\vec{R}_1$ and $\vec{R}_2$ denote the position vectors of masses m_1 and m_2 in an inertial reference frame. According to Newton's Second Law and the law of gravitation:

$$m_1\ddot{\vec{R}}_1 = -G\frac{m_1 m_2}{r^3}\vec{r} \tag{2.1}$$

$$m_2\ddot{\vec{R}}_2 = G\frac{m_1 m_2}{r^3}\vec{r} \tag{2.2}$$

where $\vec{r} = \vec{R}_2 - \vec{R}_1$.

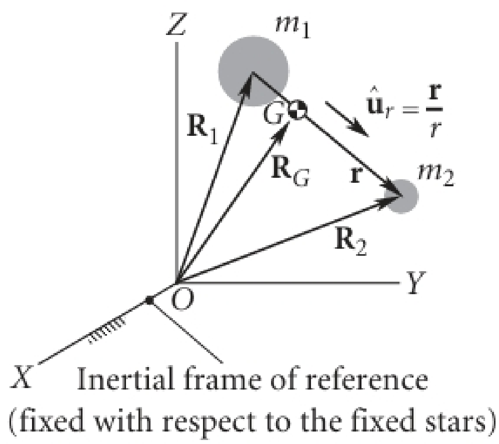

Figure 2.1: Diagram of two masses in an inertial frame $OXYZ$ (Curtis (2005))

2.1.1 Deriving the relative equation of motion

Substituting Equation 2.1 and Equation 2.2 to $\vec{r} = \vec{R}_2 - \vec{R}_1$:

$\ddot{\vec{r}} = -G\frac{m_1 m_2}{r^3}\vec{r}\left(\frac{1}{m_2} + \frac{1}{m_1}\right) = -G\frac{m_1+m_2}{r^3}\vec{r}$

Let the gravitational parameter be defined as:

$$\mu = G(m_1 + m_2)$$

Hence the final equation that we use turns out to be:

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}} \tag{2.3}$$

2.1.2 Motion of Center of Mass

The center of mass vector is: $\vec{R}_G = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1+m_2}$

Using Equation 2.1 and Equation 2.2 :

$$\ddot{\vec{R}}_G = \frac{m_1\ddot{\vec{R}}_1 + m_2\ddot{\vec{R}}_2}{m_1 + m_2} \tag{2.4}$$

$$\ddot{\vec{R}}_G = \frac{-G\frac{m_1 m_2}{r^3}\vec{r} + G\frac{m_1 m_2}{r^3}\vec{r}}{m_1 + m_2} = 0$$

Hence proved that the acceleration of center of mass is zero.

2.2 Angular momentum in two body problem

The angular momentum of body m_2 relative to m_1 is moment of m_2's relative linear momentum $m_2\dot{\vec{r}}$:

$$\vec{H}_{2/1} = \vec{r} \times m_2\dot{\vec{r}}$$

where $\dot{\vec{r}}$ is the velocity of m_2 relative to m_1. We divide this equation of the mass term and get the specific relative angular momentum:

$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$

On calculating the time derivative:

$$\frac{d\vec{h}}{dt} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}$$

From the Equation 2.3, we see that $\ddot{\vec{r}} \parallel \vec{r}$. Hence $\frac{d\vec{h}}{dt} = 0 \Rightarrow \vec{h}$ is conserved.

2.2.1 Eccentricity of orbit from Laplace Runge Lenz vector

Differentiating $\dot{\vec{r}} \times \vec{h}$:

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \ddot{\vec{r}} \times \vec{h} + \dot{\vec{r}} \times \underbrace{\dot{\vec{h}}}_{0} = -\frac{\mu}{r^3}\vec{r} \times (\vec{r} \times \dot{\vec{r}}) \tag{2.5}$$

Using the triple product identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$:

$$\vec{r} \times (\vec{r} \times \dot{\vec{r}}) = (\vec{r} \cdot \dot{\vec{r}})\vec{r} - r^2\dot{\vec{r}}$$

And using the identity $\vec{r} \cdot \dot{\vec{r}} = r\dot{r}$(See **?@sec-rrdot**), we get:

$$\vec{r} \times (\vec{r} \times \dot{\vec{r}}) = (r\dot{r})\vec{r} - r^2\dot{\vec{r}} \tag{2.6}$$

Inserting Equation 2.6 into Equation 2.5 :

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = -\frac{\mu}{r^3}\left[(r\dot{r})\vec{r} - r^2\dot{\vec{r}}\right] = -\mu\left[\frac{\dot{r}\vec{r} - r\dot{\vec{r}}}{r^2}\right] \tag{2.7}$$

But ,

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{r\dot{\vec{r}} - \dot{r}\vec{r}}{r^2} = -\frac{\dot{r}\vec{r} - r\dot{\vec{r}}}{r^2}$$

substituting the above into Equation 2.7;

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \frac{d}{dt}\left(\mu\frac{\vec{r}}{r}\right)$$

Which on integration, gives this solution:

$$\frac{1}{\mu}(\dot{\vec{r}} \times \vec{h}) - \frac{\vec{r}}{r} = \vec{e} \tag{2.8}$$

where $\vec{e}$ is the constant of integration and called as the Laplace-Runge-Lenz Vector. The significance of this vector is that its magnitude $|\vec{e}|$ gives the eccentricity e and it faces in the direction of the periapsis of the orbit.

2.3 Orbit Equation (Trajectory Under Gravity)

Equation 2.8 is the vector equation that represents the relative motion of one body wrt the other in two body problem. In order to obtain the scalar form, we take a dot product with $\vec{r}$:

$$\frac{1}{\mu}(\dot{\vec{r}} \times \vec{h}) \cdot \vec{r} - \frac{\vec{r} \cdot \vec{r}}{r} = \vec{e} \cdot \vec{r}$$

Using the identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$;

$$\frac{1}{\mu}\underbrace{(\vec{r} \times \dot{\vec{r}})}_{\vec{h}} \cdot \vec{h} - \frac{\vec{r} \cdot \vec{r}}{r} = \vec{e} \cdot \vec{r}$$

$$\frac{h^2}{\mu} - r = re\cos\theta \text{ where } \theta \text{ is the true anomaly angle}$$

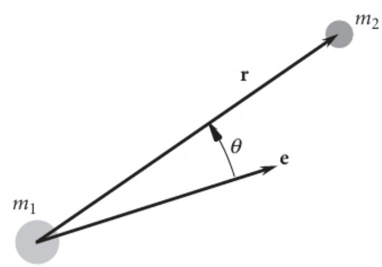

Figure 2.2: The true anomaly θ is the angle between the eccentricity vector $\vec{e}$ and the position vector $\vec{r}$.

Hence the final equation of orbit will turn out to be as follows:

$$\boxed{r = \frac{\frac{h^2}{\mu}}{1 + e\cos\theta}}$$

2.4 Two body problem simulation

Appendix

Table of equations

Quantity	Equation
Ellipse	$r = \dfrac{p}{1 + e\cos\theta}$
Area law	$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{h}{2} = \text{const}$
Harmonic law	$T^2 \propto a^3$
Gravity	$\vec{F}_{21} = -\dfrac{G m_1 m_2}{r^2} \hat{u_r}$
Reduced EoM	$\ddot{\vec{r}} = -\dfrac{\mu}{r^3} \vec{r}$
Specific angular momentum	$\vec{h} = \vec{r} \times \dot{\vec{r}}$
Laplace–Runge–Lenz vector	$\vec{e} = \dfrac{\dot{\vec{r}} \times \vec{h}}{\mu} - \dfrac{\vec{r}}{r}$

Identities derivation

1.

We know that: $\vec{r} \cdot \vec{r} = r^2$. Then :

$$\frac{d}{dt}(\vec{r} \cdot \vec{r}) = 2r \frac{dr}{dt} = 2r\dot{r}$$

But,

$$\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \vec{r} \cdot \frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{r} = 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 2\vec{r} \cdot \dot{\vec{r}}$$

Hence:

$$\boxed{\vec{r} \cdot \dot{\vec{r}} = r\dot{r}}$$

References

Curtis, H. D. 2005. *Orbital Mechanics: For Engineering Students*. Aerospace Engineering. Elsevier Science. https://books.google.co.in/books?id=nEO7lAEACAAJ.