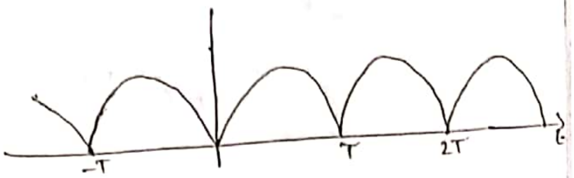
4th chapter problems

1) consider the full wave rectified sinusoid

a) Determine its spectrum $x_a(f)$

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{j2\pi k F_0 t}$$



let $F_0 = \frac{1}{T}$

$$= \sum_{k=-\infty}^{\infty} C_k\, e^{j2\pi k \frac{t}{T}}$$

$$C_k = \frac{1}{T}\int_0^T A\sin(\pi \tfrac{t}{T})\, e^{-j2\pi k t/T}\, dt$$

$$= \frac{1}{T}\int_0^T A\cdot \frac{e^{\frac{j\pi t}{T}} - e^{\frac{-j\pi t}{T}}}{2j}\, e^{-j2\pi k\frac{t}{T}}\, dt$$

$$= \frac{A}{2j\cdot T}\int_0^T \left(e^{\frac{j\pi t}{T}} - e^{\frac{-j\pi t}{T}}\right) e^{-j2\pi k\frac{t}{T}}\, dt$$

$$= \frac{A}{2j\cdot T}\int_0^T \left(e^{j\pi\frac{t}{T}}\cdot e^{-j2\pi k\frac{t}{T}}\right) - \left(e^{-j\pi\frac{t}{T}}\cdot e^{-j2\pi k\frac{t}{T}}\right) dt$$

$$= \frac{A}{2j\cdot T}\int_0^T e^{j\pi(1-2k)\frac{t}{T}} - e^{-j\pi(1+2k)\frac{t}{T}}\, dt$$

$$= \frac{A}{2j\,T}\left[\int_0^T e^{j\pi(1-2k)\frac{t}{T}}\, dt - \int_0^T e^{-j\pi(1+2k)\frac{t}{T}}\, dt\right]$$

$$= \frac{A}{2j\cdot T}\left[\left[\frac{e^{j\pi(1-2k)\frac{t}{T}}}{j\pi(1-2k)\frac{1}{T}}\right]_0^T - \left[\frac{e^{-j\pi(1+2k)\frac{t}{T}}}{-j\pi(1+2k)\frac{1}{T}}\right]_0^T\right]$$

$$= \frac{A}{2j\cdot T}\left[\frac{e^{j\pi(1-2k)} - e^{0}}{j\pi(1-2k)\frac{1}{T}} - \frac{e^{-j\pi(1+2k)} - e^{-0}}{-j\pi(1+2k)\cdot\frac{1}{T}}\right]$$

$$= \frac{A}{2j\cancel{T}}\cdot\frac{\cancel{T}}{j\pi}\left[\frac{e^{j\pi(1-2k)}-1}{(1-2k)} + \frac{e^{-j\pi(1+2k)}-1}{(1+2k)}\right]$$

$$= \frac{-A}{2\pi}\left[\frac{-1-1}{(1-2k)} + \frac{-1-1}{(1+2k)}\right]$$

$$= \frac{\cancel{-}A}{\cancel{2}\pi}\left[\cancel{-2}\left(\frac{1}{1-2k} + \frac{1}{1+2k}\right)\right]$$

$$= \frac{A}{\pi}\left(\frac{1}{1-2k} + \frac{1}{1+2k}\right)$$

$$= \frac{A}{\pi}\left(\frac{1+\cancel{2k}+1-\cancel{2k}}{1-4k^2}\right)$$

$$C_k = \frac{2A}{\pi(1-4k^2)}$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{j\pi(1-2k)}$

$= \cos(\pi(1-2k)) + j\sin(\pi(1-2k$

$= -1+0 \Rightarrow -1$

$\text{llly} \Rightarrow e^{-j\pi(1+2k)} = -1$

$$X(F) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j2\pi F t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi F_0 k t} \cdot e^{-j2\pi F t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} c_k\, e^{-j2\pi (F-kF_0)t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} c_k\, e^{-j2\pi (F-\frac{k}{T})t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} e^{-j2\pi (F-\frac{k}{T})t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k\, \delta(F-\frac{k}{T})$$

b) compute the power of the signal.

$$P_x = \frac{1}{T}\int_0^T x_a^2(t)\, dt$$

$$= \frac{1}{T}\int_0^T (A \sin\frac{\pi t}{T})^2\, dt$$

$$= \frac{A^2}{T}\int_0^T \sin^2\frac{\pi t}{T}\, dt$$

$$= \frac{A^2}{T}\int_0^T \frac{1-\cos 2(\frac{\pi}{T})t}{2}$$

$$= \frac{A^2}{T}\int_0^T \frac{1}{2} - \frac{\cos 2(\frac{\pi}{T})t}{2}$$

$$= \frac{A^2}{T}\left(\frac{T}{2} - \left(\frac{\cos 2\pi}{2} - 0\right]\right]$$

$$= \frac{A^2}{T}\times\frac{T}{2} - 0$$

$$= \frac{A^2}{2}$$

c) plot the power spectral density.

→ power spectral density $= |c_k|^2$, $k = 0, \pm 1, \pm 2, \pm 3$ - - -

1) check the validity of parseval's relation for the signal.

$$P_x = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{2A}{\pi(1-4k^2)} \right)^2$$

$$= \frac{4A^2}{\pi^2} \sum_{k=-\infty}^{\infty} \left(\frac{1}{(4k^2-1)^2} \right)$$

$$= \frac{4A^2}{\pi^2} \left[\frac{1}{(4k^2-1)^2}\bigg|_{k=0} + 2 \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2} \right]$$

$$= \frac{4A^2}{\pi^2} \left[1 + \frac{2}{3^2} + \frac{2}{15^2} + \cdots \right]$$

$$= \frac{4A^2}{\pi^2} (1.231)$$

$$= 0.498A^2 \simeq 0.5A^2 = \frac{A^2}{2}$$

2) compute and sketch the magnitude & phase spectra for the following signals (a>0)

1) $x_a(t) = \begin{cases} Ae^{-at} , & t \geq 0 \\ 0 , & t < 0 \end{cases}$

$$x_a(F) = \int_0^{\infty} Ae^{-at} e^{-j2\pi Ft} dt$$

$$= A \int_0^{\infty} e^{-(a+2\pi Fj)t} dt$$

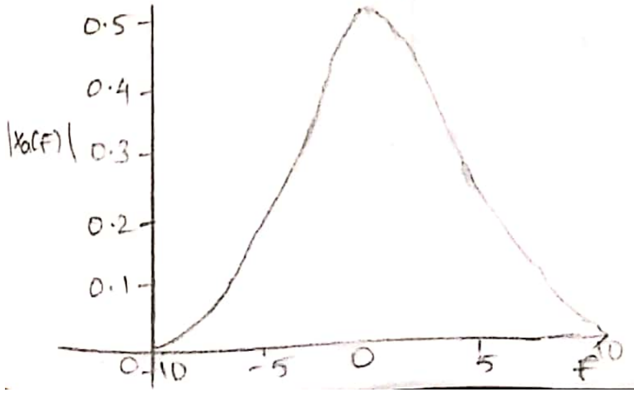
$$= A \left(\frac{e^{-(a+2\pi fj)t}}{-(a+2\pi fj)} \right]_0^{\infty}$$

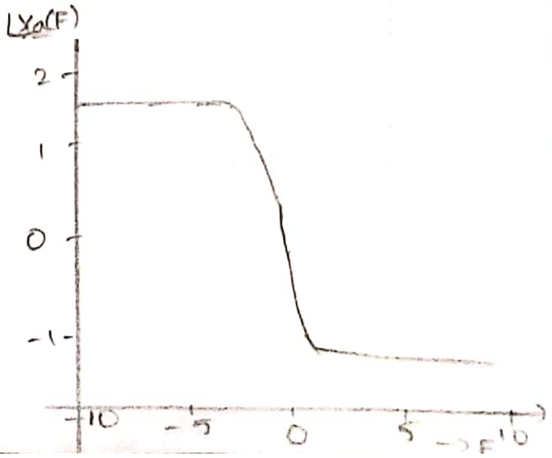
$$= A \cdot \frac{1}{a+j2\pi f}$$

$A = 2$, $a = 4$ $\Rightarrow |X_a(0)| = \frac{2}{4} = 1/2$

$$|x_a(F)| = \frac{A}{\sqrt{a^2+(2\pi F)^2}}$$

$$\angle X_a(F) = -\tan^{-1}\left(\frac{2\pi f}{a} \right)$$





b) $x_a(t) = Ae^{-a|t|}$

$$X_a(F) = \int_{-\infty}^{\infty} Ae^{-a|t|} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0} Ae^{at} e^{-j2\pi ft} dt + \int_{0}^{\infty} Ae^{-at} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0} A e^{(a-j2\pi f)t} + \int_{0}^{\infty} Ae^{-(a+j2\pi f)t} dt$$

$$= \left[\frac{A e^{(a-j2\pi f)t}}{a-j2\pi f}\right]_{-\infty}^{0} + \left[\frac{A e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\right]_{0}^{\infty}$$

$$= \frac{A}{a-j2\pi f} - 0 + 0 - \left[\frac{-A}{(a+j2\pi f)}\right]$$

$$= \frac{A}{a-j2\pi f} + \frac{A}{a+j2\pi f}$$

$$= \frac{Aa + Aj\cancel{2\pi} f + Aa - Aj\cancel{2\pi} f}{a^2+(2\pi f)^2}$$

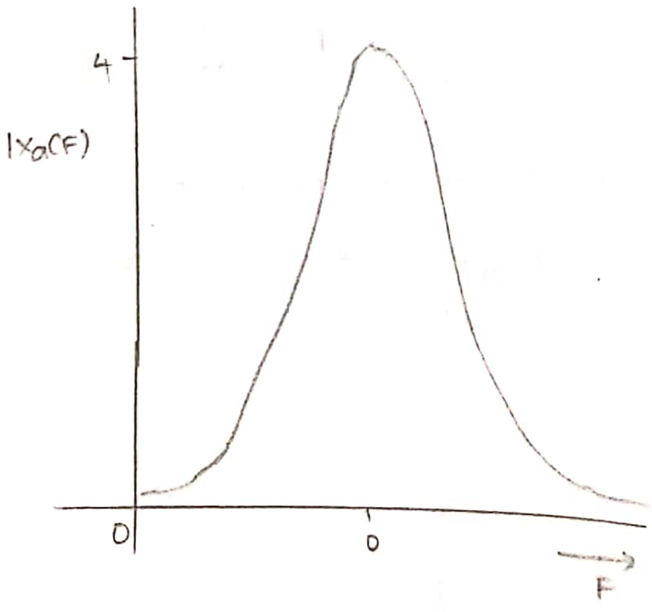
$$= \frac{2aA}{a^2+(2\pi f)^2}$$

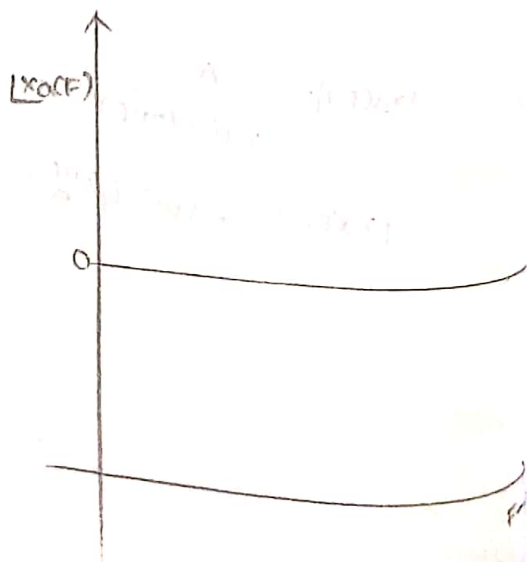
$|X_a(F)| = X_a(f)$

$\angle X_a(f) = \tan^{-1}\left(\frac{0}{\frac{2aA}{a^2+(2\pi f)^2}}\right) = 0$

$|X_a(f)|$

$F=0 \Rightarrow a=2, \quad A=4 \Rightarrow \frac{16}{4} = 4$





3) consider the signal

$$x(t) = \begin{cases} 1 - \frac{|t|}{T} , & |t| < T \\ 0 , & \text{elsewhere} \end{cases}$$

1) Determine & sketch its magnitude & phase spectra. $|X_a(F)|$ and $\angle X_a(F)$ respectively.

a) $$X_a(F) = \int_{-T}^{0} \left(1+\frac{t}{T}\right) e^{-j2\pi Ft} dt + \int_{0}^{T} \left(1-\frac{t}{T}\right) e^{-j2\pi Ft} dt$$

first we find

Let $y(t) = \left\{ \frac{1}{T} \right. ; -T \leq t \leq T$

$$Y(F) = \int_{-T}^{0} \frac{1}{T} e^{-j2\pi ft} dt + \int_{0}^{T} -\frac{1}{T} e^{-j2\pi ft} dt$$

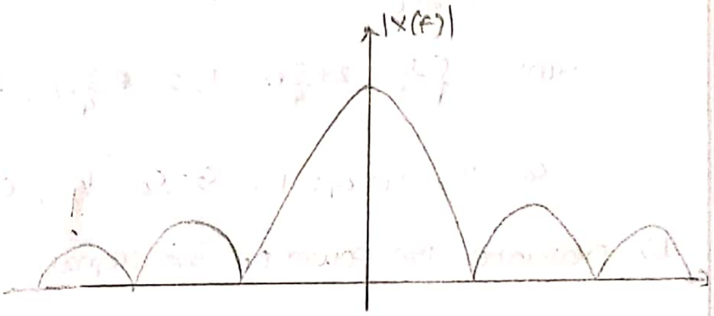
$$= -2 \frac{\sin^2 \pi fT}{j\pi fT}$$

$$X(F) = \frac{1}{j2\pi f} Y(F)$$

$$= T \left(\frac{\sin \pi fT}{\pi fT} \right)^2$$

$$|X(F)| = T \left(\frac{\sin \pi fT}{\pi fT} \right)^2 \Rightarrow \text{Let } ~~fct~~ f = \frac{t}{T} \Rightarrow \left(\frac{\sin \pi t}{\pi t} \right)^2 = \text{sinc}^2 t$$

$\angle X(F) = 0$



b) Create a periodic signal $x_p(t)$ with fundamental period $T_p \geq 2T$. so that $x(t) = x_p(t)$ for $|t| < T_p/2$. What are the fourier coefficients c_k for the signal $x_p(t)$?

a) $$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{\frac{-j2\pi kt}{T_p}} dt$$

$$= \frac{1}{T_p} \int_{-T}^{0} \left(1+\frac{t}{T}\right) e^{\frac{-j2\pi kt}{T_p}} dt + \int_{0}^{T} \left(1-\frac{t}{T}\right) e^{-j2\pi kt/T_p} dt$$

$$= \frac{T}{T_p} \left[\frac{\sin \pi kT/T_p}{\pi kT/T_p} \right]^2$$

c) Using the results in parts (a) & (b), show that $c_k = \frac{1}{T_p} X_a(k/T_p)$

Sol) $\frac{1}{T_p} X_a(k/T_p)$

$$= \frac{1}{T_p} \tau \left(\frac{\sin \frac{\pi k}{T_p} \tau}{\pi \frac{k}{T_p} \tau} \right)^2$$

$$= \frac{\tau}{T_p} \left(\frac{\sin \frac{\pi k \tau}{T_p}}{\pi k \tau / T_p} \right)^2 = c_k$$

$\therefore c_k = \frac{1}{T_p} X_a\left(\frac{k}{T_p}\right)$ hence proved.

4.4) consider the signal

$$x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$$

a) Determine and sketch its power density spectrum.

Sol) $x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$

$$= 2 + 2\left[\frac{e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}}{2}\right] + \frac{e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}}{2} + \frac{1}{2}\left[\frac{e^{j\frac{3\pi n}{4}} + e^{-j\frac{3\pi n}{4}}}{2}\right]$$

$$= 2 + e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}} + \frac{1}{2}e^{j\frac{\pi n}{2}} + \frac{1}{2}e^{-j\frac{\pi n}{2}} + \frac{1}{4}e^{j\frac{3\pi n}{4}} + \frac{1}{4}e^{-j\frac{3\pi n}{4}}$$

$N = 8$

$$c_k = \frac{1}{8}\sum_{n=0}^{7} x(n)\, e^{-j\frac{\pi k n}{4}}$$

$$x(n) = \left\{\frac{11}{2},\ 2+\frac{3}{4}\sqrt{2},\ 1,\ 2-\frac{3}{4}\sqrt{2},\ \frac{1}{2},\ 2-\frac{3}{4}\sqrt{2},\ 1,\ 2+\frac{3}{4}\sqrt{2}\right\}$$

$c_0 = 2$, $c_1 = c_7 = 1$, $c_2 = c_6 = 1/2$, $c_3 = c_5 = \frac{1}{4}$, $c_4 = 0$.

b) Evaluate the power of the signal.

$$\sum_{n=0}^{7} |c_k|^2$$

$$= \left(2^2 + 1^2 + 1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2\right)$$

$$= \left(4 + 2 + \frac{1}{2} + \frac{1}{8}\right)$$

$$= \frac{32 + 16 + 4 + 1}{8}$$

$$= \frac{53}{8}$$

5) consider the following periodic signal.

$$x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, \ldots\}$$

a) sketch the signal x(n) and its magnitude & phase spectra.

$$\underbrace{1, 0, 1, 2, 3, 2,}_{N=6} 1\ 0\ 1$$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi nk}{N}}$$

$$= \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-\frac{j2\pi kn}{6}}$$

for $n=0 \Rightarrow x(0) e^{-\frac{j2\pi k}{6}(0)} = 3$

$n=1 \Rightarrow x(1) e^{-\frac{j2\pi k}{6}} = 2e^{-\frac{j2\pi k}{6}}$

$n=2 \Rightarrow x(2) e^{-\frac{j2\pi k(2)}{6}} = e^{-\frac{j2\pi k}{3}}$

$n=3 \Rightarrow x(3) e^{-\frac{j2\pi k(3)}{6}} = 0$

$n=4 \Rightarrow x(4) e^{\frac{j2\pi k \cdot 4}{6}} = e^{-\frac{j4\pi k}{6}}$

$n=5 \Rightarrow x(5) e^{-\frac{j2\pi k 5}{6}} = 2e^{-\frac{j10\pi k}{6}}$

$$= \frac{1}{6}\left[3 + 2e^{-\frac{j\pi 2k}{6}} + e^{-\frac{j2\pi k}{3}} + 0 + e^{-\frac{j4\pi k}{3}} + 2e^{-\frac{j10\pi k}{6}}\right]$$

For $k=0$

$$= \frac{1}{6}[3+2+1+0+1+2]$$

$$= \frac{1}{6}(9) = \frac{9}{6}$$

for $k=1$

$$= \frac{1}{6}\left[3 + 2e^{-\frac{j\pi}{3}} + e^{-\frac{j2\pi}{3}} + 0 + e^{-\frac{j4\pi}{3}} + 2e^{-\frac{j5\pi}{3}}\right]$$

$$= \frac{1}{6}\left[3 + 2(\cos(\tfrac{\pi}{3}) - j\sin(\tfrac{\pi}{3}) + \cos(\tfrac{2\pi}{3}) - j\sin(\tfrac{2\pi}{3}) + \cos(\tfrac{4\pi}{3}) - j\sin\tfrac{4\pi}{3} + 2(\cos\tfrac{5\pi}{3} - j\sin\tfrac{5\pi}{3})\right]$$

$$= \frac{4}{6}$$

4

for $k=2$; $C_2 = 0$

$k=3$; $C_3 = 1/6$

$k=4$, $C_4 = 0$

$k=5$; $C_5 = \frac{4}{6}$

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b) Using the results in part (a) verify parsevals relation by comp

the power in the time & frequency domains.

Sol) $P_t = \frac{1}{6} \sum_{n=0}^{5} |x(n)|^2$

$= \frac{1}{6} \left(1^2 + 0^2 + 1^2 + 2^2 + 3^2 + 2^2 \right] = \frac{1}{6} \left(1 + \cancel{1} + 4 + 9 + 4 \right] = \frac{19}{6}$

$P_f = \sum_{n=0}^{5} |C_n|^2$

$= \left(\frac{9}{6}\right)^2 + \left(\frac{4}{6}\right)^2 + 0^2 + \left(\frac{1}{6}\right)^2 + 6^2 + \left(\frac{4}{6}\right)^2$

$= \frac{114}{36} = \frac{19}{6}$.

4.6) Determine & sketch the magnitude and phase spectra of the follow

periodic signal.

a) $x(n) = 4 \sin\left(\frac{\pi(n-2)}{3}\right)$

$= 4 \left[\frac{e^{j\pi\frac{(n-2)}{3}} - e^{-j\pi\frac{(n-2)}{3}}}{2j} \right]$

$= 4 \left[e^{3j\pi(n-2)} - e^{-3j\pi(n-2)} \right]$

$\therefore N = 6$

$C_k = \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-\frac{j2\pi kn}{6}}$.

$= \frac{4}{6} \sum_{n=0}^{5} \left(\frac{e^{j\pi\left(\frac{n-2}{3}\right)} - e^{-j\pi\left(\frac{n-2}{3}\right)}}{2j} \right] e^{-\frac{j2\pi kn}{\cancel{6}3}}$

$= \frac{1}{\sqrt{3}} \left(-e^{-j2\pi k/3} - e^{-j\pi k/3} + e^{-j\pi k/3} + e^{-j2\pi k/3} \right]$

$= \frac{1}{\sqrt{3}} (-j2) \left(\sin\frac{2\pi k}{6} + \sin\frac{\pi k}{3} \right] e^{-j2\pi k/3}$

$C_0 = 0,\ C_1 = -j2 e^{-j25/3},\ C_2 = C_3 = C_4 = 0,\ C_5 = 9$

$\angle C_1 = \frac{5\pi}{6},\ \angle C_5 = -\frac{5\pi}{6},\ \angle C_0 = \angle C_1 = \angle C_3 = \angle C_4 = 0^\circ$

b) $x(n) = \cos\frac{2\pi}{3}n + \sin\frac{2\pi}{5}n$

Sol) $N = 15$

$\cos\frac{2\pi}{3}n$

$= \frac{1}{2} \left(e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n} \right]$

$$\Rightarrow e^{\frac{j2\pi}{3}n} = e^{\frac{-2j\pi kn}{N}}$$

$$k = \frac{N}{3} = 5$$

$$15 - 5 = 10$$

$$c_{1k} = \begin{cases} \frac{1}{2} & ; k = 5, 10 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\sin \frac{2\pi}{5} n$$

$$\Rightarrow \frac{1}{2j}\left(e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n} \right]$$

$$e^{-j\frac{2\pi}{5}n} = e^{-\frac{j2\pi kn}{N}}$$

$$k = \frac{N}{5} = \frac{15}{5} = 3$$

$$15 - 3 = 12$$

$$c_{2k} = \begin{cases} \frac{1}{2j} & ; k = 3. \\ \frac{-1}{2j} & ; k = 12 \\ 0 & ; \text{otherwise} \end{cases}$$

$$c_k = c_{1k} + c_{2k} = \begin{cases} \frac{1}{2j} & , k = 3 \\ \frac{1}{2} & , k = 5 \\ \frac{1}{2} & , k = 10 \\ \frac{-1}{2j} & , k = 12 \\ 0 & , \text{otherwise} \end{cases}$$

c) $x(n) = \cos \frac{2\pi}{3} n \cdot \sin \frac{2\pi}{5} n.$

$$\cos a \cdot \cos b = \frac{\sin(a+b)}{2} - \frac{\sin(a-b)}{2}$$

$$= \frac{1}{2}\left\{ \sin \frac{10\pi n + 6\pi n}{15} - \sin \frac{10\pi n - 6\pi n}{15} \right]$$

$$= \frac{1}{2} \sin \frac{16\pi n}{15} - \frac{1}{2} \sin \frac{4\pi n}{15}$$

$$= \frac{1}{2}\left\{ \frac{e^{j\frac{16\pi n}{15}} - e^{-j\frac{16\pi n}{15}}}{2j} \right] - \frac{1}{2}\left(\frac{e^{\frac{j4\pi n}{15}} - e^{\frac{-j4\pi n}{15}}}{2j} \right)$$

$$= \frac{1}{4j}\left(e^{j\frac{16\pi n}{15}} - e^{\frac{-j16\pi n}{15}} \right] - \frac{1}{4j}\left(e^{j\frac{4\pi n}{15}} - e^{-j\frac{4\pi n}{15}} \right]$$

$$\frac{8}{15} = \frac{k}{N}$$

$$k = 8 \Rightarrow \frac{1}{4j}$$

$$15 - 8 = 7 \Rightarrow -\frac{1}{4j}$$

$$\frac{2}{15} = \frac{k}{N}$$

$$k = 2$$

$$-\frac{1}{4j} \Rightarrow k = 2$$

$$15 - 2 \Rightarrow 13 \rightarrow \frac{1}{4j}$$

$$c_k = \begin{cases} \frac{1}{49} & ; 8,13 \\ \frac{-1}{49} & ; 7,2 \\ 0 & ; \text{otherwise} \end{cases}$$

d) $x(n) = \{ \cdots -2, \underbrace{-1, 0, 1, 2, -2}_{N=5}, -1, 0, 1, 2 \cdots \}$

$$c_k = \frac{1}{5} \sum_{n=0}^{4} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{5} \sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi k n}{5}}$$

$$= \frac{1}{5} \left[0 + e^{-\frac{j2\pi k}{5}} + 2e^{\frac{-j4\pi k}{5}} - 2e^{\frac{-j6\pi k}{5}} - e^{\frac{-j8\pi k}{5}} \right]$$

$$= \frac{2j}{5} \left[-\sin\left(\frac{2\pi k}{5}\right) - 2\sin\left(\frac{4\pi k}{5}\right) \right]$$

For plotting k values.

$k=0$; $c_0 = 0$

$k=1$, $c_1 = \frac{2j}{5}\left[-\sin\frac{2\pi}{5} - 2\sin\frac{4\pi}{5}\right]$

$k=2$; $c_2 = \frac{2j}{5}\left[-\sin\frac{4\pi}{5} - 2\sin\frac{8\pi}{5}\right]$

$c_3 = -c_2$

$c_4 = -c_1$

e) $x(n) = \{ \cdots \underbrace{-1, 2, \underset{\uparrow}{1}, 2, -1, 0}_{N=6}, -1, 2, 1, 2 \cdots \}$

$$c_k = \frac{1}{6} \sum_{n=0}^{5} x(n)\, e^{\frac{-j2\pi k n}{N}}$$

Similarly by substituting from 0 to 5 we get

$$= \frac{1}{6} \left[1 + 2e^{-\frac{j\pi k}{3}} - e^{\frac{-j2\pi k}{3}} - e^{\frac{-j4\pi k}{3}} + 2e^{\frac{-5\pi k}{3}} \right]$$

$$= \frac{1}{6} \left[1 + \cos\frac{4\pi k}{3} - 2\cos\frac{2\pi k}{3} \right]$$

$c_0 = 1/2$; $c_1 = 2/3$; $c_2 = 0$; $c_3 = -5/6$, $c_4 = 0$, $c_5 = 2/3$

f) $x(n) = \{ \cdots 0, 0, \underbrace{\underset{\uparrow n=0}{1}, 1, 0, 0, 0}_{5}, 1, 1, 0, 0 \cdots \}$

$N = 5$

$$c_k = \frac{1}{5} \sum_{n=0}^{4} x(n)\, e^{\frac{-j2\pi k n}{5}}$$

$$= \frac{1}{5} \left[1 + e^{-\frac{j2\pi k}{5}} \right]$$

$= \frac{2}{5} \cos(\frac{\pi k}{5}) e^{-\frac{j\pi k}{5}}$

$\therefore c_0 = \frac{2}{5}$

$c_1 = \frac{2}{5} \cos(\frac{\pi}{5}) e^{-\frac{j\pi}{5}}$

$c_2 = \frac{2}{5} \cos(\frac{2\pi}{5}) e^{-\frac{j2\pi}{5}}$

$c_3 = \frac{2}{5} \cos(\frac{3\pi}{5}) e^{-\frac{j3\pi}{5}}$

$c_4 = \frac{2}{5} \cos(\frac{4\pi}{5}) e^{-\frac{j4\pi}{5}}$

a) $x(n) = 1$, $-\infty < n < \infty$.

$N = 1$

$c_k = x(0) = 1$

(or)

$c_0 = 1$

b) $x(n) = (-1)^n$, $-\infty < n < \infty$

$N = 2$

$c_k = \frac{1}{2} \sum_{n=0}^{1} x(n) e^{-j\pi n k}$

$= \frac{1}{2}(1 - e^{-j\pi k})$

$\therefore c_0 = 0$; $c_1 = 1$

4.7) Determine the periodic signals $x(n)$ with fundamental period $N=8$. If their fourier coefficients are given by.

a) $c_k = \cos(\frac{k\pi}{4}) + \sin(\frac{3k\pi}{4})$

Sol) $x(n) = \sum_{n=0}^{7} c_k e^{\frac{j2\pi nk}{N}}$

Let

$c_k = e^{\frac{j2\pi pk}{N}}$

$\therefore \sum_{n=0}^{7} e^{\frac{j2\pi pk}{N}} \cdot e^{\frac{j2\pi nk}{N}}$

$= \sum_{n=0}^{7} e^{\frac{j2\pi(p+n)k}{N}}$

It gives 8 ; when $p = -n$

0 ; when $p \neq n$

$\therefore c_k = \frac{1}{2}\left(e^{\frac{j2\pi k}{8}} + e^{-\frac{j2\pi k}{8}}\right) - \frac{1}{2j}\left[e^{\frac{j6\pi k}{8}} - e^{-\frac{6\pi k}{8}}\right]$

$\therefore x(n) = 4\delta(n+1) + 4\delta(n-1) - 4j\delta(n+3) + 4j\delta(n+3) + 4j\delta(n-3)$ $\quad -3 \leq n \leq 5$

b) $$c_k = \begin{cases} \sin\frac{k\pi}{3} , & 0 \le k \le 6 \\ 0 ; & k=7 \end{cases}$$

$c_0 = 0 ;\quad c_1 = \frac{\sqrt{3}}{2} ;\quad c_2 = \frac{\sqrt{3}}{2} ;\quad c_3 = 0 ;\quad c_4 = -\frac{\sqrt{3}}{2} ;\quad c_5 = -\frac{\sqrt{3}}{2} ,\quad c_6 = c_7 = 0$

$$x(n) = \sum_{k=0}^{7} c_k \, e^{\frac{j2\pi nk}{8}}$$

$$= \frac{\sqrt{3}}{2}\left[e^{\frac{j\pi n}{4}} + e^{\frac{j\pi n 2}{4}} - e^{-\frac{j4\pi n}{4}} - e^{-\frac{j5\pi n}{4}} \right]$$

$$= \sqrt{3}\left(\frac{\sin\pi n}{4} + \sin\frac{\pi n}{4} \right] e^{\frac{j\pi(3n-2)}{4}}$$

c) $c_k = \{ \cdots 0, \frac{1}{4}, \frac{1}{2}, 1, \underset{\uparrow}{2}, 1, \frac{1}{2}, \frac{1}{4}, 0 \cdots \}$

$$x(n) = \sum_{k=-3}^{4} c_k \, e^{\frac{j2\pi nk}{8}}$$

$$= 2 + e^{\frac{j\pi n}{4}} + e^{-\frac{j\pi n}{4}} + \frac{1}{2} e^{\frac{j\pi n}{2}} + \frac{1}{2} e^{-\frac{j\pi n}{2}} + \frac{1}{4} e^{\frac{j3\pi n}{4}} + \frac{1}{4} e^{-\frac{j3\pi n}{4}}$$

$$= 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}.$$

4.10) Determine the signals having the following fourier transforms

a) $$x(\omega) = \begin{cases} 0 , & 0 \le |\omega| \le \omega_0 \\ 1 , & \omega_0 < |\omega| \le \pi \end{cases}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\omega) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\omega_0} x(\omega) e^{j\omega n} d\omega + \int_{\omega_0}^{\pi} x(\omega) e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\omega_0} 1 . e^{j\omega n} d\omega + \int_{\omega_0}^{\pi} e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{-j\omega n}}{jn} \right]_{-\pi}^{\omega_0} + \left(\frac{e^{j\omega n}}{jn} \right]_{\omega_0}^{\pi} \right]$$

$$= \frac{1}{2\pi}\left(\frac{e^{-j\omega_0 n} - e^{+j\pi n}}{jn} + \frac{e^{j\pi n} - e^{j\omega_0 n}}{jn} \right]$$

$$= \frac{1}{2\pi}\left[2 . \frac{e^{-j\omega_0 n} - e^{j\omega_0 n}}{2jn} + 2 \frac{e^{j\pi n} - e^{+j\pi n}}{2jn} \right)$$

$$= \frac{1}{\cancel{2}\pi}\left(\frac{\cancel{2}}{n} \left(- \frac{\sin\omega_0 n}{n\pi} \right) \right] = \frac{-\sin\omega_0 n}{n\pi} \quad ; n \ne 0$$

for $n=0$

from eqn ①

$$= \frac{1}{2\pi}(\pi-\omega_0)+\frac{1}{2\pi}(\pi-\omega_0)$$

$$= \frac{\cancel{2}(\pi-\omega_0)}{\cancel{2}\pi}$$

$$= \frac{(\pi-\omega_0)}{\pi} \quad ; \text{ when } n=0$$

b) $X(\omega)= \cos^2\omega$

$$= \left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)^2$$

$$= \frac{1}{4}\left[(e^{j\omega})^2+2e^{j\omega}\cdot e^{-j\omega}+(e^{-j\omega})^2\right]$$

$$= \frac{1}{4}\left(e^{j2\omega}+2+e^{-j2\omega}\right)$$

$$= \frac{1}{4}e^{j2\omega}+\frac{1}{2}+\frac{1}{4}e^{-j2\omega}$$

Inverse Fourier transform↓

$$= \frac{1}{4}\delta(n+2)+\delta(n)\frac{1}{2}+\frac{1}{4}\delta(n-2)$$

$$= \frac{1}{4}\left[\delta(n+2)+2\delta(n)+\delta(n-2)\right]$$

c) $X(\omega) = \begin{cases} 1 \ ; & \omega_0-\frac{\delta\omega}{2} \le |\omega| \le \omega_0+\frac{\delta\omega}{2} \\ 0 \ ; & \text{elsewhere} \end{cases}$

sol) $\omega_0-\frac{\delta\omega}{2} \le \omega \le \omega_0+\frac{\delta\omega}{2}$

$$\omega_0-\frac{\delta\omega}{2} \le -\omega \le \omega_0+\frac{\delta\omega}{2}$$

$$-\omega_0+\frac{\delta\omega}{2} \le \omega \le -\omega_0-\frac{\delta\omega}{2}$$

consider limits $\omega_0-\frac{\delta\omega}{2} \le \omega \le \omega_0+\frac{\delta\omega}{2}$

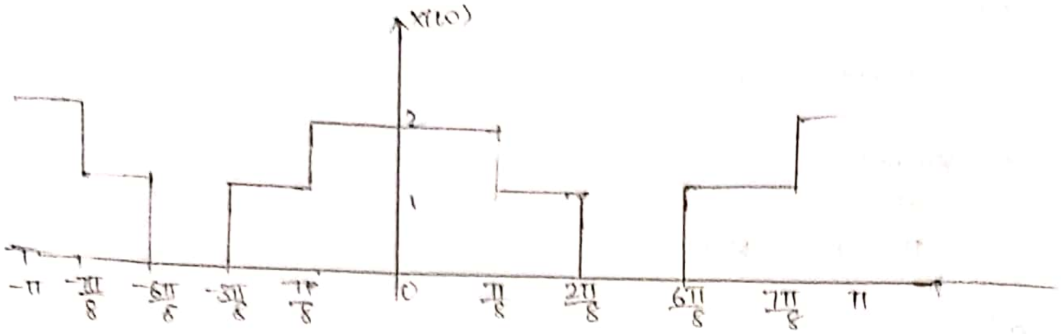
$$= \frac{1}{2\pi}\int_{\omega_0-\frac{\delta\omega}{2}}^{\omega_0+\frac{\delta\omega}{2}} 1\cdot e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_0-\frac{\delta\omega}{2}}^{\omega_0+\frac{\delta\omega}{2}}$$

$$= \frac{2}{2\pi}\cdot\frac{e^{j(\omega_0+\frac{\delta\omega}{2})n}-e^{j(\omega_0-\frac{\delta\omega}{2})n}}{2jn}$$

$$= \cancel{\delta\omega}\ \delta\omega\ \frac{2}{\pi}\left(\frac{\sin(\frac{\delta\omega}{2}n)}{\frac{n\delta\omega}{2}}\right]e^{jn\omega_0}$$

d) The signal shown in fig X(ω)



sol) $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

Let consider limits 0 to π

$$\Rightarrow 2 \times \frac{1}{2\pi}\left[\int_{0}^{\frac{\pi}{8}} 2e^{j\omega n} d\omega + \int_{\frac{\pi}{8}}^{\frac{2\pi}{8}} e^{j\omega n} d\omega + \int_{\frac{6\pi}{8}}^{\frac{7\pi}{8}} e^{j\omega n} d\omega + \int_{\frac{7\pi}{8}}^{\pi} 2e^{j\omega n} d\omega\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\frac{\pi}{8}} 2\cos\omega n \, d\omega + \int_{\frac{\pi}{8}}^{\frac{2\pi}{8}} \cos\omega n \, d\omega + \int_{\frac{6\pi}{8}}^{\frac{7\pi}{8}} \cos\omega n \, d\omega + \int_{\frac{7\pi}{8}}^{\pi} 2\cos\omega n \, d\right.$$

[neglected $j\sin\omega n$ because $\sin(\pm n\pi) = 0$

$$= \frac{1}{\pi}\left(\left[-2\sin\omega n\right]_0^{\frac{\pi}{8}} + \left(-\sin\omega n\right]_{\frac{\pi}{8}}^{\frac{2\pi}{8}} + \left(-\sin\omega n\right]_{\frac{6\pi}{8}}^{\frac{7\pi}{8}} + \left(-2\sin\omega n\right]_{\frac{7\pi}{8}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(-2\sin\frac{\pi}{8}n - \sin\frac{3\pi}{8}n + \sin\frac{\pi}{8}n - \sin\frac{7\pi}{8}n - 1\sin\frac{6\pi}{8}n - 2\sin\pi n + 2\sin\frac{7\pi}{8}n\right]$$

$$= \frac{1}{\pi}\left(\sin\frac{7\pi}{8}n - \sin\frac{\pi}{8}n + \sin\frac{6\pi}{8}n - \sin\frac{3\pi}{8}n\right]$$

4.11) consider the signal

$$x(n) = \{1, 0, -1, \underset{\uparrow}{2}, 3\}$$
-3 -2 -1 0 1

with Fourier transform $X(\omega) = X_R(\omega) + j(X_I(\omega))$. Determine & sketch the signal $y(n)$ with fourier transform.

$$Y(\omega) = X_I(\omega) + X_R(\omega)$$

sol) $x_e(n) = \frac{x(n) + x(-n)}{2}$

$n=0$; $x_e(0) = \frac{x(0) + x(-0)}{2} = \frac{2+2}{2} = 2$

$n=1$; $\frac{x(1) + x(-1)}{2} = \frac{3 + (-1)}{2} = 1$

$n=1;\ \frac{x(-1)+x(1)}{2} = \frac{-1+3}{2} = 1$

$n=2;\ \frac{x(2)+x(-2)}{2} = 0$

$n=3;\ \frac{x(3)+x(-3)}{2} = \frac{0+1}{2} = 1/2$

$n=-2;\ \frac{x(-2)+x(2)}{2} = 0$

$n=-3;\ \frac{x(-3)+x(3)}{2} = \frac{1+0}{2} = 1/2$

$x_e(n) = \{\frac{1}{2}, 0, 1, \underset{\uparrow}{2}, 1, 0, 1/2\}$

II we get

$x_o(n) = \{\frac{1}{2}, 0, -2, \underset{\uparrow}{0}, 2, 0, 1/2\}$

From $x_o(n) = \frac{x(n)-x(-n)}{2}$

$X_R(\omega) = \sum_{n=-3}^{3} x_e(n)\, e^{-jn\omega}$

$j X_I(\omega) = \sum_{n=-3}^{3} x_o(n)\, e^{-jn\omega}$

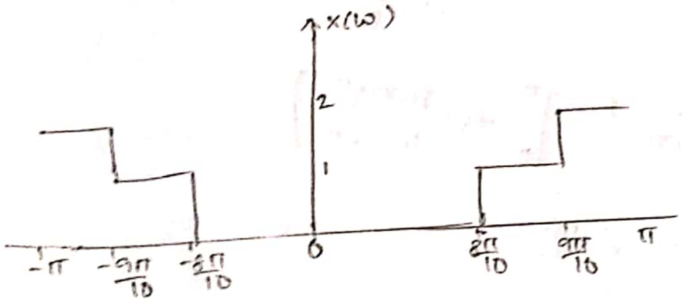
$Y(\omega) = X_I(\omega) + X_R(\omega)\, e^{j2\omega}$

$= \frac{x_o(n)}{j} + x_e(n+2) \longrightarrow$ From IFT of $Y(\omega)$

$= -j\, x_o(n) + x_e(n+2)$

$= \{\frac{1}{2}, 0, 1+\frac{j}{2}, 2, 1+\frac{j}{2}, 0\frac{1}{2} - j2, 0, \frac{j}{2}\}$

Determine the signal $x(n)$ if its fourier transform is as given in



$$x(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{9\pi}{10}} 2e^{jn\omega}d\omega + \int_{-\frac{9\pi}{10}}^{-\frac{8\pi}{10}} e^{jn\omega}d\omega + \int_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} e^{jn\omega}d\omega + 2\int_{\frac{9\pi}{10}}^{\pi} e^{jn\omega}d\omega\right]$$

$$= \frac{1}{2\pi}\left\{2\left(\frac{e^{jn\omega}}{jn}\right)_{-\pi}^{-\frac{9\pi}{10}} + \left[\frac{e^{jn\omega}}{jn}\right]_{-\frac{9\pi}{10}}^{-\frac{8\pi}{10}} + \left[\frac{e^{jn\omega}}{jn}\right]_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} + 2\left(\frac{e^{jn\omega}}{jn}\right)_{\frac{9\pi}{10}}^{\pi}\right)$$

$$= \frac{1}{2\pi jn}\left[2\left(e^{jn\frac{9\pi}{10}} - e^{-j\pi n}\right] + e^{-jn\frac{8\pi}{10}} - e^{-jn\frac{9\pi}{10}} + e^{jn\frac{9\pi}{10}} - e^{jn\frac{8\pi}{10}} + 2e^{jn\pi} - 2e^{jn\frac{9\pi}{10}}\right]$$

$$= \frac{1}{jn\cancel{\pi}}\left[e^{jn(-\omega_c+\frac{\omega}{2})} - e^{jn(-\omega_c-\frac{\omega}{2})} + e^{jn(\omega_c+\frac{\omega}{2})} \right.$$

$$= \frac{1}{2\pi jn}\left[2e^{-jn\frac{9\pi}{10}} - 2e^{-jn\pi} + e^{-jn\frac{8\pi}{10}} - e^{-jn\frac{9\pi}{10}} + e^{jn\frac{9\pi}{10}} - e^{jn\frac{8\pi}{10}} \right.$$

$$\left. + 2e^{jn\pi} - 2e^{jn\frac{9\pi}{10}} \right]$$

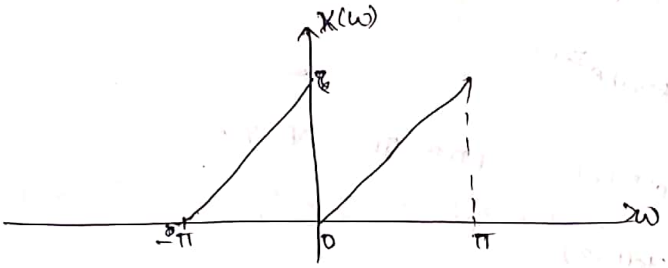
$$= \frac{1}{2\pi jn}\left[e^{-jn\frac{9\pi}{10}} - 2e^{-jn\pi} + 2e^{jn\pi} + e^{-jn\frac{8\pi}{10}} - e^{jn\frac{8\pi}{10}} + e^{jn\frac{9\pi}{10}} - 2e^{jn\frac{9\pi}{10}} \right.$$

$$= \frac{1}{2\pi jn}\left(e^{-jn\frac{9\pi}{10}} - e^{jn\frac{9\pi}{10}} - 2e^{-jn\pi} + 2e^{jn\pi} + e^{-jn\frac{8\pi}{10}} - e^{jn\frac{8\pi}{10}} \right)$$

$$= \frac{1}{jn\pi}\left[-\sin\left(\frac{9\pi n}{10}\right) - \sin\left(\frac{\overset{4}{8}n\pi}{\underset{5}{10}}\right) + \sin \overset{\pi 0}{n\pi} \right]$$

$$= \frac{-1}{jn\pi}\left[\sin\left(\frac{9\pi n}{10}\right) + \sin\left(\frac{4n\pi}{5}\right) \right]$$

b)



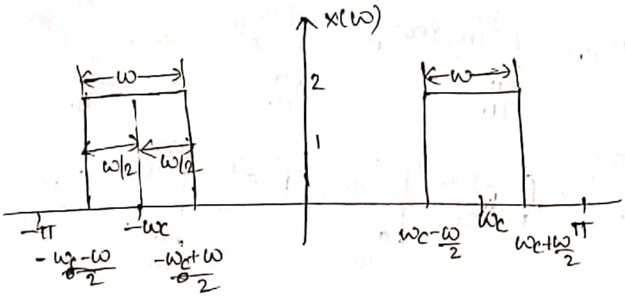
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}\left(\frac{\omega}{\pi}+1\right) e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{0}^{\pi} \frac{\omega}{\pi}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\left(\frac{\omega}{jn\pi} e^{j\omega n}\right]_{-\pi}^{\pi} + \left(\frac{e^{j\omega n}}{jn}\right)_{-\pi}^{0} \right]$$

$$= \frac{1}{\pi n} \sin\frac{\pi n}{2}\; e^{-j\frac{n\pi}{2}}$$

c)



sol) $x(n) = \frac{1}{2\pi}\left[\int_{-\omega_c-\frac{\omega}{2}}^{-\omega_c+\frac{\omega}{2}} 2e^{jn\omega}\, d\omega + \int_{\omega_c-\frac{\omega}{2}}^{\omega_c+\frac{\omega}{2}} 2e^{jn\omega}\, d\omega \right]$

$$= \frac{1}{\pi}\left[\left[\frac{e^{jn\omega}}{jn}\right]_{-\omega_c-\frac{\omega}{2}}^{-\omega_c+\frac{\omega}{2}} + \left[\frac{e^{jn\omega}}{jn}\right]_{\omega_c-\frac{\omega}{2}}^{\omega_c+\frac{\omega}{2}}\right]$$

$$= \frac{1}{jn\pi}\left[e^{jn(-\omega_c+\frac{\omega}{2})} - e^{jn(-\omega_c-\frac{\omega}{2})} + e^{jn(\omega_c+\frac{\omega}{2})} - e^{jn(\omega_c-\frac{\omega}{2})}\right]$$

$$= \frac{1}{jn\pi}\left(e^{jn(-\omega_c+\frac{\omega}{2})} - e^{jn(-\omega_c-\frac{\omega}{2})} + e^{-jn(-\omega_c-\frac{\omega}{2})} - e^{-jn(\frac{\omega}{2}-\omega_c)}\right]$$

$$= \frac{2}{jn\pi}\left[\sin(\tfrac{\omega}{2}-\omega_c)n - \sin(-\omega_c-\tfrac{\omega}{2})n\right]$$

$$= \frac{2}{n\pi}\left(-\sin(\omega_c-\tfrac{\omega}{2})n + \sin(\omega_c+\tfrac{\omega}{2})n\right]$$

$$= \frac{2}{n\pi}\left(\sin(\omega_c+\tfrac{\omega}{2})n - \sin(\omega_c-\tfrac{\omega}{2})n\right]$$

4.13)

$$x_1(\omega) = \sum_{n=0}^{m} 1.e^{-j\omega n}$$

$$1+e^{-j\omega}+e^{-j\omega 2}+e^{-j\omega 3}+ - - - -\left(e^{-j\omega(m+1)}+e^{-j\omega(m+2)} - -\right)$$

$$\longleftrightarrow$$

$$\frac{1}{1-e^{-j\omega}} - \frac{e^{-j\omega(m+1)}}{1-e^{-j\omega}} \qquad \frac{e^{-j\omega(m+1)}}{1-e^{-j\omega}}$$

$$= \frac{1-e^{-j\omega(m+1)}}{1-e^{-j\omega}}$$

$$x_2(\omega) = \sum_{n=-m}^{-1} e^{-j\omega n}$$

$$= \sum_{n=1}^{m} e^{j\omega n}$$

$$= \frac{1-e^{j\omega m}}{1-e^{j\omega}}\cdot e^{j\omega}$$

$$X(\omega) = x_1(\omega)+x_2(\omega)$$

$$= \frac{1-e^{-j\omega(m+1)}}{1-e^{-j\omega}} + \frac{1-e^{j\omega m}}{1-e^{j\omega}}\cdot e^{j\omega}$$

$$= \frac{1+e^{j\omega}-e^{j\omega}-1-e^{-j\omega(m+1)}-e^{j\omega(m+1)}+e^{j\omega m}+e^{-j\omega m}}{2-e^{-j\omega}-e^{j\omega}}$$

$$= \frac{2\cos \omega m - 2\cos \omega(m+1)}{2 - 2\cos\omega}$$

= ~~2 ceo~~ $\dfrac{2\sin(\omega m + \frac{\omega}{2})\cdot\cos(\frac{\omega}{2})}{2\sin^2\frac{\omega}{2}}$

$$= \frac{\sin(m+1/2)\omega}{\sin(\frac{\omega}{2})}$$

proved that $1 + 2\sum_{n=1}^{m} \cos \omega n = \dfrac{\sin(m+\frac{1}{2})\omega}{\sin(\frac{\omega}{2})}$

4.14) consider the signal

$$x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$$

with fourier transform $X(\omega)$. compute the following quantities, without explicitly computing $X(\omega)$:

a) $X(0)$

$$X(\omega)\Big|_{\omega=0} = \sum_{\omega=0} x(n) e^{-j\omega n}$$

$$X(0) = -3e^0 = -3$$

b) $\angle X(\omega) = \pi$ for all 'ω'

c) $\int_{-\pi}^{\pi} X(\omega)\, d\omega$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, d\omega$$

$$\int_{-\pi}^{\pi} X(\omega)\, d\omega = 2\pi\, x(0)$$

$$= 2\pi(-3)$$

$$= -6\pi$$

d) $X(\pi) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega}$

$$= \sum_n e^{-jn\pi} \cdot x(n)$$

$$= \sum_n \left[\cos(n\pi) - j\overset{\to 0}{\sin(n\pi)}\right] x(n)$$

$$= \sum_n (-1)^n x(n)$$

For $n=0$; $(-1)^0 x(n) = 1\cdot(-3) = -3$

$n=1$, $(-1)^1 x(n) = -1\cdot 2 = -2$

$n=2$, $(-1)^2 x(2) = -1 = -1$

$n=-1$ $(-1)^{-1}$ $x(-1) \Rightarrow -2$

$n=-2$ $(-1)^{-2}$ $x(-2) = -1$

$\Rightarrow -3-2-1-2-1$ $\Rightarrow -9$

e) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

we know

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 = \sum_{n} |x(n)|^2$$

$$= (-1)^2+(2)^2+(-3)^2+(2)^2+(-1)^2$$

$$= 19$$

$$\int_{-\pi}^{\pi} |X(\omega)|^2 = 2\pi \times 19 = 38\pi$$

4.15) The center of gravity of a signal $x(n)$ is defined as

$$C = \frac{\sum_{n=-\infty}^{\infty} n x(n)}{\sum_{n=-\infty}^{\infty} x(n)}$$

and provide a measure of the "time delay" of the signal.

a) Express c in terms of $X(\omega)$.

We know

$$x(n) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(0) = \sum_{n=-\infty}^{\infty} x(n) e^{0}$$

$$x(0) = \sum_{n=-\infty}^{\infty} x(n)$$

We know from differentiation in 'ω' domain multiplies 'n' with $x(n)$

$$n x(n) \xleftrightarrow{F.T} j \frac{dX(\omega)}{d\omega}$$

$$-j n x(n) \xleftrightarrow{F.T} \frac{dX(\omega)}{d\omega}$$

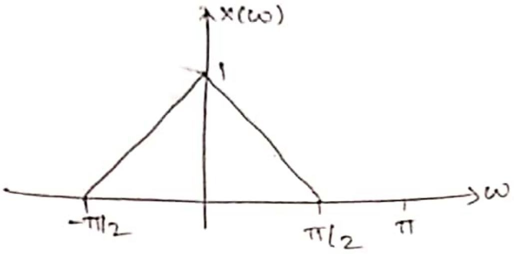
$$\frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} -j\, n x(n) e^{-jn\omega} d\omega$$

$$= -j \sum_{n=-\infty}^{\infty} n x(n) e^{-jn\omega} d\omega$$

$$\frac{j\, dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} n x(n) e^{-jn\omega} d\omega$$

$$\therefore C = \frac{j \frac{dX(\omega)}{d\omega}\Big|_{\omega=0}}{X(0)}$$

b) compute c for the signal $x(n)$ whose fourier transform is shown



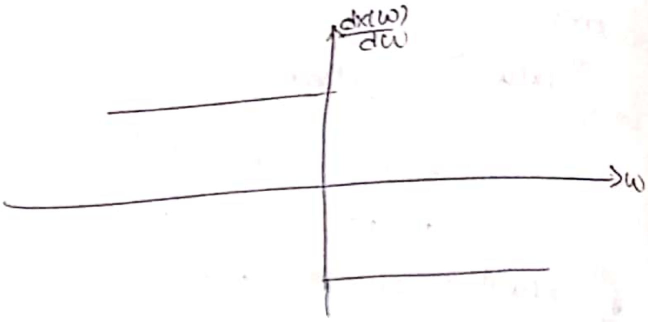
$A\left(1-\frac{|t|}{T}\right)$

$1\left(1-\frac{|t|}{\pi/2}\right)$

From given figure $x(0)=1$

$$c = \frac{j\frac{dx(0)}{d\omega}}{x(0)}$$

$$= \frac{0}{1} = 0$$



4.16) consider the fourier transform pair.

$$a^n u(n) \xleftrightarrow{F.T} \frac{1}{1-ae^{-j\omega}} \qquad |a|<1$$

Use the differentiation in frequency theorem & induction to those

$$x(n) = \frac{(n+l-1)!}{n!\,(l-1)!} a^n u(n) \xleftrightarrow{F.T} X(\omega) = \frac{1}{(1-ae^{-j\omega})^l}$$

Let $l = k+1$

$$x(n) = \frac{(n+k+1-1)!}{n!\,(k+1-1)!} a^n u(n)$$

$$= \frac{(n+k)!}{n!\,k!} a^n u(n)$$

$$= \frac{(n+k)(n+k-1)!}{k\; n!\,(k-1)!} a^n u(n)$$

Let $x_k(n) = \frac{(n+k-1)!}{n!\,(k-1)!} a^n u(n)$

$$x_{k+1}(n) = \frac{n+k}{k!} x_k(n)$$

$$X_{k+1}(\omega) = \sum_{n=-\infty}^{\infty} \frac{n+k}{k} x_k(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{n}{k} x_k(n) + x_k(n)\right) e^{-j\omega n}$$

$$= \frac{1}{k} \sum_{n=-\infty}^{\infty} n\, x_k(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x_k(n) e^{-j\omega n}$$

$$= \frac{1}{k} \sum_{n=-\infty}^{\infty} n\, x_k(n) e^{-j\omega n} + X_k(\omega)$$

$$= \frac{1}{k} j \frac{dX_k(\omega)}{d\omega} + X_k(\omega)$$

$$= \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^{k+1}} + \frac{1}{(1-ae^{-j\omega})^{k}}$$

4.17) Let $x(n)$ be a arbitrary signal, not necessarily real valued with F.T $X(\omega)$. Express the fourier tansform of the following signals in terms of $X(\omega)$.

a) $x^*(n)$

$$\sum_{n=-\infty}^{\infty} x^*(n)e^{-j\omega n}$$

$$\sum_{n=-\infty}^{\infty} \left(x(n)e^{-j\omega n}\right]^*$$

$$X(-\omega)^*$$

b) $x^*(-n)$

$$\sum_{n=-\infty}^{\infty} x^*(-n)e^{-j\omega n}$$

replace $-n$ with 'n'

$$\sum_{n=-\infty}^{\infty} x^*(n)e^{j\omega n}$$

$$\sum_{n=-\infty}^{\infty} (x(n)e^{-j\omega n})^*$$

$$X^*(\omega)$$

c) $y(n) = x(n) - x(n-1)$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} - \sum_{n=-\infty}^{\infty} x(n-1)e^{-j\omega n} d\omega$$

$$= X(\omega) - \sum_{n=-\infty}^{\infty} x(n-1)e^{-j\omega n} d\omega$$

Let $l = n-1$ (dummy variable)

$$= X(\omega) - \sum_{l=-\infty}^{\infty} x(l)\, e^{-j\omega(l+1)}$$

$$= X(\omega) - \sum_{l=-\infty}^{\infty} x(l)e^{-j\omega l} \cdot e^{-j\omega}$$

$$= X(\omega) - e^{-j\omega}\sum_{l=-\infty}^{\infty} x(l)e^{-j\omega l}$$

Replace l by n

$$= X(\omega) - e^{-j\omega}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$= X(\omega) - e^{-j\omega}X(\omega)$$

$$= X(\omega)\left[1 - e^{-j\omega}\right]$$

d) $y(n) = \sum_{n=-\infty}^{n} x(k)$

$$= y(n) - y(n-1)$$

$$= x(n)$$

$X(\omega) = Y(\omega)\left[1 - e^{-j\omega}\right]$ → from ②

$$Y(\omega) = \frac{X(\omega)}{1 - e^{-j\omega}}$$

e) $y(n) = x(2n)$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} x(2n)e^{-jn\omega}$$

Let $l = 2n$

$$= \sum_{l=-\infty}^{\infty} x(l)\, e^{-j\frac{l}{2}\omega}$$

$$= X(\omega/2)$$

f) $y(n) = \begin{cases} x(n/2) ; & \text{'n' even} \\ 0 ; & \text{'n' odd} \end{cases}$

$$Y(\omega) = \sum_{n} x(n/2)e^{jn\omega}$$

Let $n = 2l$

$$= \sum_{n} x\left(\frac{2l}{2}\right)e^{j2l\omega}$$

$$= \sum_{n} x(l)e^{j2l\omega}$$

$$= X(2\omega)$$

4.18) Determine & sketch the fourier transforms $x_1(\omega)$, $x_2(\omega)$ & $x_3(\omega)$ the following signals.

a) $x_1(n) = \{\underset{-2}{1}, 1, \underset{\uparrow}{1}, 1, \underset{2}{1}\}$

$$\sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n}$$

$$\sum_{n=-2}^{2} x_1(n)\, e^{-j\omega n}$$

for $n=-2$; $1.e^{j2\omega} = e^{j2\omega}$

$n=-1$; $1.e^{j\omega} = e^{j\omega}$

$n=0$; $e^{0} = 1$

$n=1$; $1.e^{-j\omega} = e^{-j\omega}$

$n=2$; $1.e^{-j2\omega} = e^{-j2\omega}$

$\Rightarrow e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} = 2.\cos(2\omega) + 2\cos(\omega) + 1.$

b) $x_2(n) = \{\underset{-4}{1}, 0, \underset{-2}{1}, 0, \underset{\underset{0}{\uparrow}}{1}, 0, 1, 0, \underset{4}{1}\}$

for $n=-4 \Rightarrow 1.e^{+j4\omega}$

$n=-2 \Rightarrow 1.e^{j2\omega}$

$n=0 \Rightarrow 1e^{0} = 1$

$n=2 \Rightarrow 1.e^{-j2\omega}$

$n=4 \Rightarrow 1.e^{-j4\omega}$

$= e^{j2\omega} + e^{-j2\omega} + 1 + e^{+j4\omega} + e^{-j4\omega}$

$= 2\cos(2\omega) + 2\cos(4\omega) + 1$

c) $x_3(n) = \{\underset{-6}{1}, 0, 0, 1, 0, 0, \underset{\underset{0}{\uparrow}}{1}, 0, 0, 1, 0, 0, \underset{6}{1}\}$

sol) for $n=-6$; $1.e^{j6\omega}$

$n=-3$; $1.e^{j3\omega}$

$n=0$; 1

$n=3$; $1e^{-j3\omega}$

$n=6$; $1e^{-j6\omega}$

$= e^{j3\omega} + e^{-j3\omega} + e^{+j6\omega} + e^{-j6\omega} + 1$

$= 2\cos(3\omega) + 2\cos(6\omega) + 1$

d) Is there any relationship b/w $x_1(\omega)$, $x_2(\omega)$ & $x_3(\omega)$? What is its physical meaning.

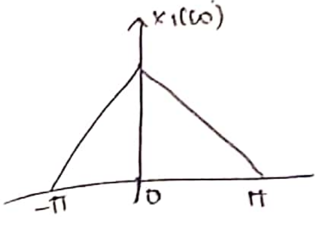
sol) $x_1(\omega) = 2\cos(2\omega) + 2\cos(\omega) + 1$

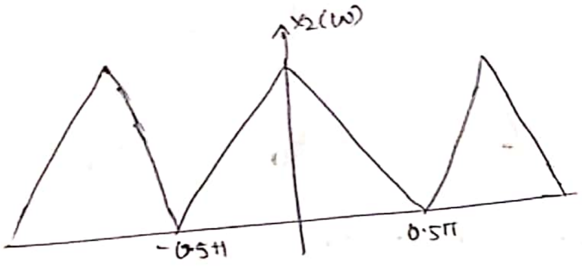
$x_2(\omega) = 2\cos(2\omega) + 2\cos(4\omega) + 1$

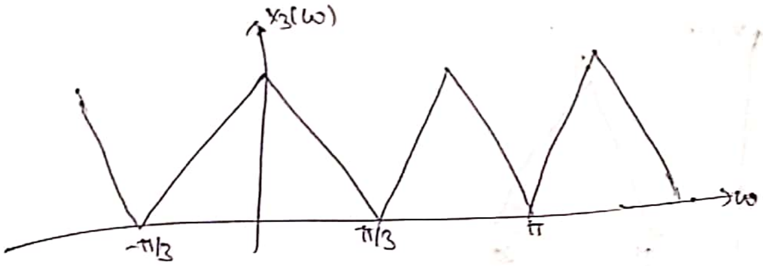
$$\boxed{x_2(\omega) = x_1(2\omega)}$$

$$x_3(\omega) = 2\cos(6\omega)+2\cos(\varepsilon\omega)+1$$

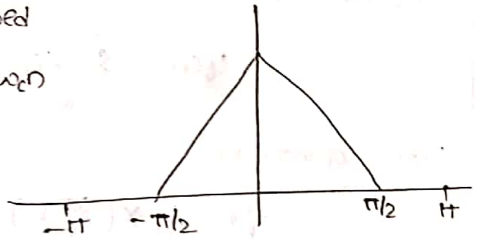
$$\boxed{x_3(\omega) = x_1(3\omega)}$$







4.9) let $x(n)$ be a signal with fourier transform as shown. Determine and sketch the fourier transforms of the following signals. note that these signal sequences are obtained by amplitude modulation of a carrier $\cos\omega_c n$ or $\sin\omega_c n$ by the sequence $x(n)$.

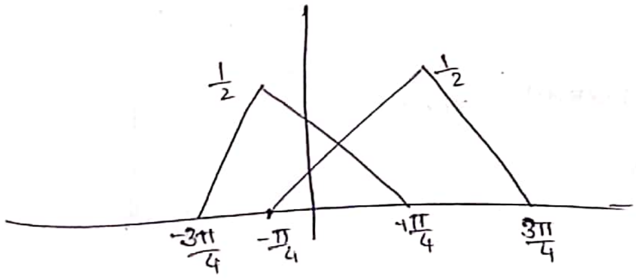


a) $x_1(n) = x(n)\cdot\cos(\frac{\pi n}{4})$.

we know

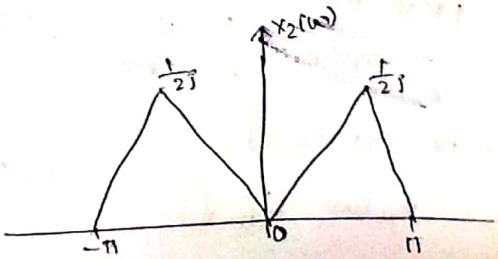
$$x(n)\cos(\omega_0 n) \xleftrightarrow{F.T} \frac{1}{2}\left(x(\omega-\omega_0)+x(\omega+\omega_0)\right]$$

$$x_1(\omega) = \frac{1}{2}\left[x(\omega+\frac{\pi}{4}) + x(\omega-\frac{\pi}{4})\right]$$



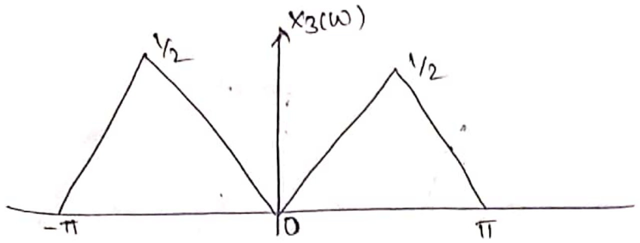
b) $x_2(n) = x(n)\sin(\frac{\pi n}{2})$

$$x_2(\omega) = \frac{1}{2j}\left(x(\omega+\frac{\pi}{2}) + x(\omega-\frac{\pi}{2})\right]$$



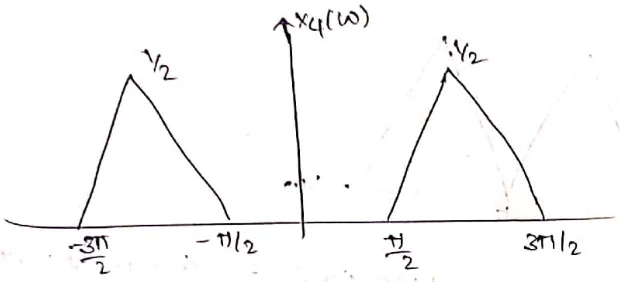
c) $x_3(n) = x(n)\cos(\frac{\pi}{2}n)$

$$X_3(\omega) = \frac{1}{2}\left(X(\omega-\frac{\pi}{2}) + X(\omega+\frac{\pi}{2})\right)$$



d) $x_4(n) = x(n)\cos(\pi n)$

$$X_4(\omega) = \frac{1}{2}\left(X(\omega-\pi) + X(\omega+\pi)\right)$$



4.20) consider all aperiodic signal $x(n)$ with F.T $X(\omega)$. show that the fourier series coefficients c_k^y of the periodic signal.

$$y(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

are given by

$$c_k^y = \frac{1}{N} X\left(\frac{2\pi}{N}k\right), \quad k = 0, 1, \ldots N-1$$

$$c_k^y = \frac{1}{N}\sum_{n=0}^{N-1} y(n)\, e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{l=-\infty}^{\infty}\sum_{m=-lN}^{N-l-lN} x(m)\, e^{-j2\pi k(m+lN)/N}$$

But $$\sum_{l=-\infty}^{\infty}\sum_{m=-lN}^{N-l-lN} x(m)\, e^{-j\omega(m+N)} = X(\omega)$$

$$c_k^y = \frac{1}{N} X\left(\frac{2\pi k}{N}\right)$$

4.21) prove that

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin\omega_c n}{n\pi} e^{-j\omega n}.$$

May be expressed as

$$X_N(\omega) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{\sin[(2N+1)(\omega-\theta)/2)]}{\sin[(\omega-\theta)/2]}\, d\theta.$$

Let

$$x_N(n) = \frac{\sin\omega_c n}{\pi n} \quad ; \quad -N \le n \le N$$

$$= x(n)\,\omega(n)$$

where $x(n) = \frac{\sin\omega_c n}{\pi n}$; $-N \le n \le N$

$$w(n) = 1 \quad ; \quad -N \le n \le N$$

$$= 0 \quad ; \quad \text{otherwise}$$

$$\frac{\sin\omega_c n}{\pi n} \xleftrightarrow{F} X(\omega)$$

$$= 1 \quad ; \quad |\omega| \le \omega_c$$

$$0 \quad ; \quad \text{otherwise}$$

$$X_N(\omega) = X(\omega) * W(\omega)$$

$$= \int_{-\pi}^{\pi} X(\theta) \cdot W(\omega-\theta)\, d\theta$$

$$= \int_{-\omega_c}^{\omega_c} \frac{\sin(2N+1)(\omega-\theta)/2}{\sin(\omega-\theta)/2}\, d\theta$$

422) A signal $x(n)$ has the following fourier transform

$$X(\omega) = \frac{1}{1-ae^{-j\omega}}$$

Determine the fourier transforms of the following signals.

a) $x(2n+1)$

$$\sum_{n=-\infty}^{\infty} x(2n+1)e^{-j\omega n}$$

Let $2n+1 = l$

$$\sum_{l=-\infty}^{\infty} x(l)\, e^{-j\omega\left(\frac{l-1}{2}\right)}$$

$$\sum_{l=-\infty}^{\infty} x(l)\, e^{\frac{-j\omega l}{2}} \cdot e^{\frac{j\omega}{2}}$$

$$= e^{\frac{j\omega}{2}} \sum_{l=-\infty}^{\infty} x(l)\, e^{\frac{-j\omega l}{2}}$$

$$= e^{\frac{j\omega}{2}} \sum_{l=-\infty}^{\infty} x(n)\, e^{\frac{-j\omega}{2} n}$$ 'Replacing 'l' by 'n'.

$$= e^{\frac{j\omega}{2}}\, X\left(\frac{\omega}{2}\right)$$

$$= e^{\frac{j\omega}{2}} \cdot \frac{1}{1-a e^{-j\left(\frac{\omega}{2}\right)}}$$

$$= \frac{e^{\frac{j\omega}{2}}}{1-ae^{-j\left(\frac{\omega}{2}\right)}}$$

b) $e^{\frac{j\pi}{2}} x(n+2)$.

$e^{j2\omega} \cdot X(\omega - \frac{\pi j}{2})$

$x(n) \longleftrightarrow X(\omega)$

$x(n+2) \longleftrightarrow e^{j2\omega} X(\omega)$

$e^{\frac{\pi j}{2}} x(n+2) \longleftrightarrow e^{j2\omega} X(\omega - \frac{\pi j}{2})$

$e^{j2\omega} \cdot X(\omega - \frac{\pi j}{2})$

c) $x(-2n)$

$x(n) \longleftrightarrow X(\omega)$

$x(2n) \longleftrightarrow X(\frac{\omega}{2})$

$x(-2n) \longleftrightarrow X(-\frac{\omega}{2})$

d) $x(n) \cos(0.3\pi n)$

$x(n) \cos \omega_0 n \longleftrightarrow \frac{1}{2} \left(X(\omega + \omega_0) + X(\omega - \omega_0) \right]$

$x(n) \cos(0.3\pi n) \longleftrightarrow \frac{1}{2} \left[X(\omega + 0.3\pi) + X(\omega - 0.3\pi) \right]$

e) $x(n) * x(n-1)$

$X(\omega)\, e^{-j\omega} X(\omega)$

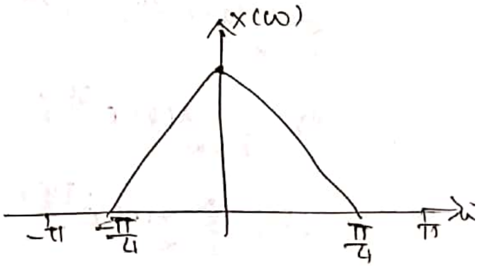
$X^2(\omega)\, e^{-j\omega}$

f) $x(n) * x(-n)$

$X(\omega) \cdot X^*(\omega)$

$$\frac{1}{1-ae^{-j\omega}} \cdot \frac{1}{1-ae^{j\omega}} = \frac{1}{1-ae^{j\omega}-ae^{-j\omega}+a^2e^0} = \frac{1}{1+a^2-2\cos\omega}$$

4.23) From a discrete time signal $x(n)$ with fourier transform X(ω shown in figure determine & sketch the fourier transform of the following signals.



Note that $y_1(n) = x(n) s(n)$

where $s(n) = \{ \cdots 0, 1, 0, 1, 0, 1, 0, 1 \cdots \}$

a) $y_1(n) = \begin{cases} x(n), & 'n' \text{ even} \\ 0, & 'n' \text{ odd} \end{cases}$

b) $y_2(n) = x(2n)$

$Y_2(\omega) = \sum_n y_2(n) e^{-j\omega n}$

$= \sum_n x(2n) e^{-j\omega n}$

$= X(\frac{\omega}{2})$

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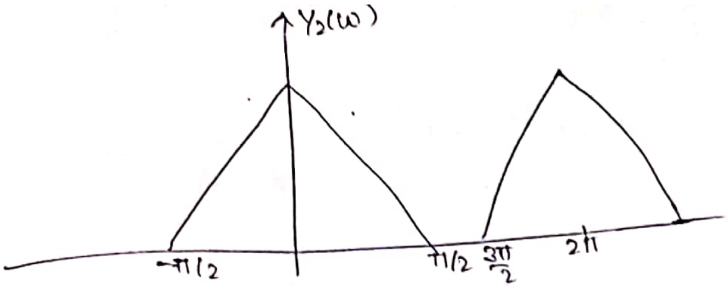


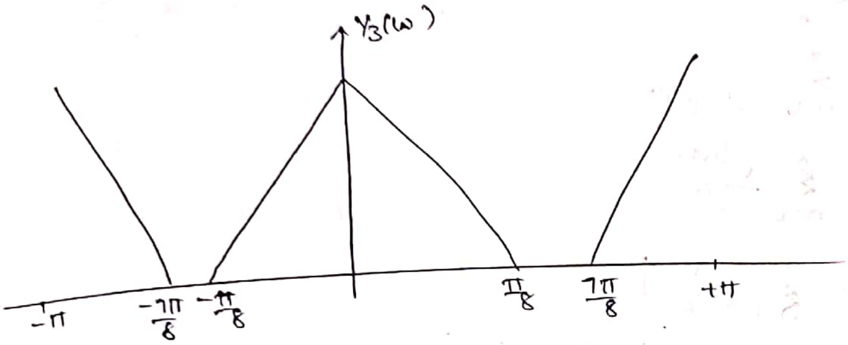
c) $y_3(n) = \begin{cases} x(n/2), & 'n' \text{ even} \\ 0, & 'n' \text{ odd} \end{cases}$

$$Y_3(\omega) = \sum_n y_3(n)\, e^{-j\omega n}$$

$$= \sum_{n \text{ even}} x(\tfrac{n}{2})\, e^{-j\omega n}$$

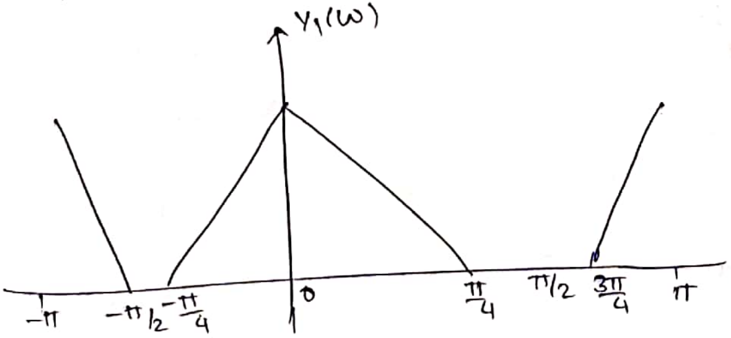
$$= \sum_m x(m)\, e^{-j2\omega m}$$

$$= X(2\omega)$$



e) $y_1(n) = \begin{cases} y_2(n/2); & 'n' \text{ even} \\ 0; & 'n' \text{ odd} \end{cases}$

$$Y_1(\omega) = Y_2(2\omega)$$



7a) Find FT of the signals.

a) $x(n) = u(n) - u(n-6)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n} - \sum_{n=-\infty}^{\infty} u(n-6)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=6}^{\infty} e^{-j\omega n}$$

$$\sum_{n=0}^{\infty} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + \cdots$$

$$= \frac{1}{1-e^{-j\omega}}$$

$$\therefore \; x(\omega) = \frac{1}{1-e^{-j\omega}} = \frac{e^{-j6\omega}}{1-e^{-j\omega}}$$

$$= \frac{1-e^{-j6\omega}}{1-e^{-j\omega}}$$

b) $x(n) = 2^n u(-n)$

$$x(\omega) = \sum_{n=-\infty}^{\infty} 2^n u(-n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n e^{-j\omega n}}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n}}{2^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega}}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{e^{j\omega}}{2}\right)^n = 1 + \frac{e^{j\omega}}{2} + \left(\frac{e^{j\omega}}{2}\right)^2 + \left(\frac{e^{j\omega}}{2}\right)^3 + \cdots$$

$$= \cancel{da} \; \frac{1}{1-\frac{e^{j\omega}}{2}}$$

$$= \frac{1}{\frac{2-e^{j\omega}}{2}}$$

$$= \frac{2}{2-e^{j\omega}}$$

c) $x(n) = \left(\frac{1}{4}\right)^n u(n+4)$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n+4) e^{-j\omega n}$$

$$= \sum_{n=-4}^{\infty} \frac{1}{4^n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4} e^{-j\omega n} \left(\frac{1}{4^{-4}} e^{-j\omega(-4)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}^n e^{-j\omega n} \left[4^4 e^{j\omega 4} \right]$$

$$= 4^4 e^{j\omega 4} \sum_{n=0}^{\infty} \frac{1}{4}^n e^{-j\omega n}$$

$$= 4^4 e^{j\omega 4} \left(1 + \frac{1}{4} e^{-j\omega} + \left(\frac{1}{4} e^{-j\omega}\right)^2 + \cdots \right]$$

$$= 4^4 e^{j\omega 4} \cdot \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

$$X(\omega) = \frac{4^4 e^{j\omega 4}}{1 - \frac{1}{4} e^{-j\omega}}$$

d) $x(n) = |\alpha|^n \sin\omega_0 n$, $|\alpha| < 1$

checking for stability

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |\alpha|^n \sin\omega_0 n$$

$$\sum_{n=-\infty}^{\infty} |\alpha|^n |\sin\omega_0 n|$$

if $\omega_0 = \pi/2$, $\sin\omega_0 n = 1$

$$\sum_{n=-\infty}^{\infty} |\alpha|^n = \infty$$

∴ It not satisfies the condition of stability

∴ It has no Fourier transform (doesn't exist]

e) $x(n) = \{-2, -1, 0, 1, 2\}$

sol) $\sum_{n=-2}^{2} x(n) e^{-j\omega n}$

$\Rightarrow x(-2) e^{j2\omega} + x(-1) e^{j\omega} + x(0) e^{0} + x(1) e^{-j\omega} + x(2) e^{-j2\omega}$

$\Rightarrow -2 e^{j2\omega} - e^{j\omega} + 0 + e^{-j\omega} + 2 e^{-j2\omega}$

$\Rightarrow -2 \left(e^{j2\omega} - e^{-j2\omega} \right] - \left(e^{j\omega} - e^{-j\omega} \right]$

$\Rightarrow -4j \left(\frac{e^{j2\omega} - e^{-j2\omega}}{2j} \right] - 2j \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right)$

$\Rightarrow -4j \left(\sin(2\omega) \right] - 2j \left(\sin(\omega) \right]$

F) $x(n) = \begin{cases} A\ (2M+1-|n|) & ; \quad |n| \leq m \\ 0 & ; \quad |n| > m \end{cases}$

$$X(\omega) = \sum_{n=-m}^{m} x(n)\, e^{-j\omega n}$$

$$= A \sum_{n=-m}^{m} (2m+1-|n|)\, e^{-j\omega n}$$

$$= (2m+1)A + A \sum_{k=1}^{N} (2m+1-k)(e^{-j\omega k} + e^{j\omega k})$$

$$= (2m+1) + 2A \sum_{k=1}^{m} (2m+1-k) \cos\omega k$$