Fourier Analysis

Formula

$$s(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f t) + \sum_{n=1}^{\infty} B_n \sin(2\pi n f t)$$

where,

$$\begin{aligned}
A_0 &= \frac{1}{T}\int_0^T s(t)\,\mathrm{d}t \\
A_n &= \frac{2}{T}\int_0^T s(t)\cos(2\pi n f t)\,\mathrm{d}t \\
B_n &= \frac{2}{T}\int_0^T s(t)\sin(2\pi n f t)\,\mathrm{d}t
\end{aligned}$$

1 Fourier Analysis

Square Wave

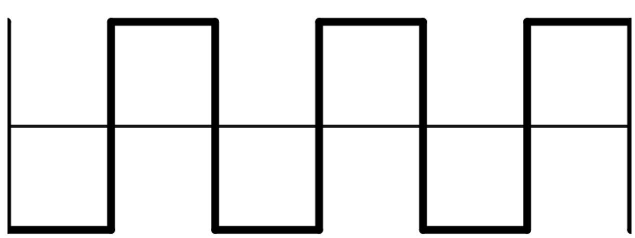

Figure 1: Square Wave

For a square wave, in one cycle, that is, from 0 to T, $s(t)$ can be expressed as A from 0 to $T/2$ and as $-A$ from $T/2$ to T, where A is the amplitude. Now,

$$
\begin{aligned}
A_0 &= \frac{1}{T}\int_0^T s(t)\,\mathrm{d}t \\
&= \frac{1}{T}\left(\int_0^{T/2} A\,\mathrm{d}t + \int_{T/2}^{T} -A\,\mathrm{d}t\right) \\
&= \frac{1}{T}\left(At\big|_0^{T/2} + (-At)\big|_0^{T/2}\right) \\
&= \frac{1}{T}\left(\frac{AT}{2} - \frac{AT}{2}\right) \\
&= 0
\end{aligned}
$$

And,

$$
\begin{aligned}
A_n &= \frac{2}{T}\int_0^T s(t)\cos(2\pi nft)\,\mathrm{d}t \\
&= \frac{2}{T}\left(\int_0^{T/2} A\cos(2\pi nft)\,\mathrm{d}t - \int_{T/2}^{T} A\cos(2\pi nft)\,\mathrm{d}t\right) \\
&= \frac{2A}{T}\left(\frac{1}{2\pi nf}\sin(2\pi nft)\bigg|_0^{T/2} - \frac{1}{2\pi nf}\sin(2\pi nft)\bigg|_{T/2}^{T}\right) \\
&= \frac{A}{\pi n}\left(\sin\left(\frac{2\pi n}{T}\cdot\frac{T}{2}\right) - \sin(0) - \sin\left(\frac{2\pi n}{T}\cdot T\right) + \sin\left(\frac{2\pi n}{T}\cdot\frac{T}{2}\right)\right) \\
&= \frac{A}{\pi n}\cdot 2\sin(n\pi) \\
&= 0
\end{aligned}
$$

And,

$$
\begin{aligned}
B_n &= \frac{2}{T}\int_0^T s(t)\sin(2\pi nft)\,\mathrm{d}t \\
&= \frac{2}{T}\left(\int_0^{T/2} A\sin(2\pi nft)\,\mathrm{d}t - \int_{T/2}^{T} A\sin(2\pi nft)\,\mathrm{d}t\right) \\
&= \frac{2A}{T}\left(-\frac{1}{2\pi nf}\cos(2\pi nft)\bigg|_0^{T/2} + \frac{1}{2\pi nf}\cos(2\pi nft)\bigg|_{T/2}^{T}\right) \\
&= \frac{A}{n\pi}\left(\cos(2n\pi) - \cos(0) - \cos(n\pi) - 2\cos(n\pi)\right) \\
&= \frac{A}{n\pi}\left(2 - 2\cos(n\pi)\right)
\end{aligned}
$$

If n is even,

$$B_n = \frac{A}{n\pi}(2-2) = 0$$

If n is odd,

$$B_n = \frac{A}{n\pi}(2+2) = \frac{4A}{n\pi}$$

Therefore,

$$B_n = \begin{cases} 0 & ; n \text{ is even} \\ \frac{4A}{n\pi} & ; n \text{ is odd} \end{cases}$$

Therefore,

$$s(t) = \frac{4A}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin(2\pi n f t)$$

Sawtooth Wave

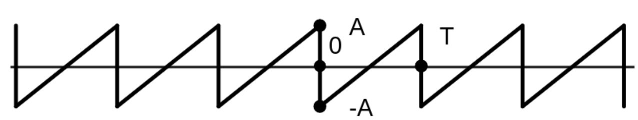

Figure 2: Sawtooth Wave

For a sawtooth wave, in one cycle, that is from O to T, $s(t)$ can be expressed $2At/T - A$.

Some required formulas:

$$\int x \cos(ax) \mathrm{d}x$$
$$= \frac{x}{a} \sin(ax) + \frac{1}{a^2} \cos(ax) + c$$

And,

$$\int x \sin(ax) \mathrm{d}x$$
$$= -\frac{x}{a} \cos(ax) + \frac{1}{a^2} \sin(ax) + c$$

Now,

$$\begin{aligned}
A_0 &= \frac{1}{T}\int_0^T \left(\frac{2A}{T}t - A\right) \mathrm{d}t \\
&= \frac{1}{T}\left(\left(\frac{2A}{T}\cdot\frac{t^2}{2}\right)_0^T - At|_0^T\right) \\
&= \frac{1}{T}\left(AT - 0 - AT + 0\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{2}{T}\int_0^T \left(\frac{2A}{T}t - A\right)\cos(2\pi nft)\mathrm{d}t \\
&= \frac{2}{T}\cdot\frac{2A}{T}\int_0^T t\cos(2\pi nft)\mathrm{d}t - \frac{2A}{T}\int_0^T \cos(2\pi nft)\mathrm{d}t \\
&= \frac{4A}{T^2}\left[\frac{t}{2\pi nf}\sin(2\pi nft) + \frac{1}{4\pi^2n^2f^2}\cos(2\pi nft)\right]_0^T - \frac{2A}{T}\left[\frac{1}{2\pi nf}\sin(2\pi nft)\right]_0^T \\
&= \frac{4A}{T^2}\left(\frac{T}{2\pi nf}\sin(2n\pi) + \frac{1}{4\pi^2n^2f^2}\cos(2n\pi) - 0 - \frac{1}{4\pi^2n^2f^2}\cos(0)\right) - 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{2}{T}\int_0^T \left(\frac{2A}{T} - A\right)\sin(2\pi nft)\mathrm{d}t \\
&= \frac{4A}{T^2}\int_0^T t\sin(2\pi nft)\mathrm{d}t - \frac{2A}{T}\int_0^T \sin(2\pi nft)\mathrm{d}t \\
&= \frac{4A}{T^2}\left[-\frac{t}{2\pi nf}\cos(2\pi nft) + \frac{1}{4\pi^2n^2f^2}\sin(2\pi nft)\right]_0^T + \frac{2A}{T}\left[\frac{1}{2\pi nf}\cos(2\pi nft)\right]_0^T \\
&= \frac{4A}{T^2}\left(\frac{1}{4\pi^2n^2f^2}\sin(2n\pi) - \frac{T^2}{2n\pi}\cos(2n\pi) - \frac{1}{4\pi^2n^2f^2}\sin(0) + 0\right) + \frac{A}{n\pi}\left(\cos(2n\pi) - \cos(0)\right) \\
&= \frac{4A}{T^2}\cdot\frac{T^2}{2n\pi}\left(-\cos(2n\pi)\right) \\
&= -\frac{2A}{n\pi}
\end{aligned}$$

Therefore,

$$s(t) = -\frac{2A}{\pi}\sum_1^\infty \sin(2\pi nft)$$

Triangular Wave

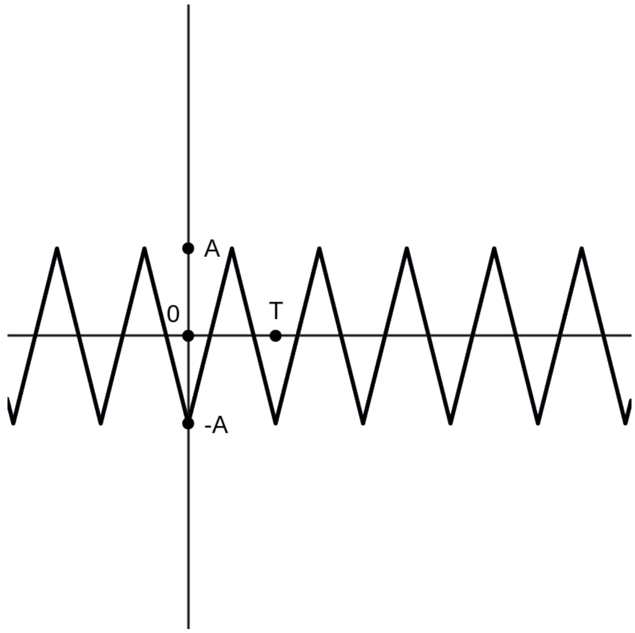

Figure 3: Triangular Wave

In one cycle, $s(t)$ can be expressed as $4At/T - A$ from 0 to $T/2$ and as $-4At/T + 3A$ from $T/2$ to T.

$$
\begin{aligned}
A_0 &= \frac{1}{T}\int_0^T s(t)\mathrm{d}t \\
&= \frac{1}{T}\left(\int_0^{T/2}\left(\frac{4A}{T}t - A\right)\mathrm{d}t + \int_{T/2}^{T}\left(-\frac{4A}{T}t + 3A\right)\mathrm{d}t\right) \\
&= \frac{1}{T}\left(\left[\frac{4A}{T}\cdot\frac{t^2}{2} - At\right]_0^{T/2} + \left[-\frac{4A}{T}\cdot\frac{t^2}{2} + 3At\right]_{T/2}^{T}\right) \\
&= \frac{1}{T}\left(\left(\frac{AT}{2} - \frac{AT}{2}\right) + \left(-2AT + 3AT + \frac{AT}{2} - \frac{3AT}{2}\right)\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
A_n &= \frac{2}{T}\int_0^T s(t)\cos(2\pi n f t)\mathrm{d}t \\
&= \frac{2}{T}\left(\int_0^{T/2}\left(\frac{4A}{T}t - A\right)\cos(2\pi n f t)\mathrm{d}t + \int_{T/2}^{T}\left(-\frac{4A}{T}t + 3A\right)\cos(2\pi n f t)\mathrm{d}t\right) \\
&= \frac{8A}{T^2}\int_0^{T/2} t\cos(2\pi n f t)\mathrm{d}t - \frac{2A}{T}\int_0^{T/2}\cos(2\pi n f t)\mathrm{d}t - \frac{8A}{T^2}\int_{T/2}^{T} t\cos(2\pi n f t)\mathrm{d}t + \frac{6A}{T}\int_{T/2}^{T}\cos(2\pi n f t)\mathrm{d}t \\
&= \frac{8A}{T^2}\left(\left[\frac{t}{2\pi n f}\sin(2\pi n f t) + \frac{1}{4\pi^2 n^2 f^2}\cos(2\pi n f t)\right]_0^{T/2} - \left[\frac{t}{2\pi n f}\sin(2\pi n f t) + \frac{1}{4\pi^2 n^2 f^2}\cos(2\pi n f t)\right]_{T/2}^{T}\right) \\
&- \frac{2A}{T}\left(\left[\frac{1}{2\pi n f}\sin(2\pi n f t)\right]_0^{T/2} - \left[\frac{3}{2\pi n f}\sin(2\pi n f t)\right]_{T/2}^{T}\right) \\
&= \frac{8A}{T^2}\left(\frac{T^2}{4\pi^2 n^2}\cos(n\pi) - \frac{T^2}{4\pi^2 n^2} - \frac{T^2}{4n^2\pi^2} + \frac{T^2}{4n^2\pi^2}\cos(n\pi)\right) - \frac{2A}{T}(0) \\
&= \frac{4A}{n^2\pi^2}\left(\cos(n\pi) - 2)\right)
\end{aligned}
$$

If n is even,

$$A_n = 0$$

If n is odd,

$$A_n = -\frac{8A}{n^2\pi^2}$$

$$
\begin{aligned}
B_n &= \frac{2}{T}\int_0^T s(t)\sin(2\pi nft)\mathrm{d}t \\
&= \frac{2}{T}\left(\int_0^{T/2}\left(\frac{4A}{T}t - A\right)\sin(2\pi nft)\mathrm{d}t + \int_{T/2}^{T}\left(-\frac{4A}{T}t + 3A\right)\sin(2\pi nft)\mathrm{d}t\right) \\
&= \frac{8A}{T^2}\int_0^{T/2} t\sin(2\pi nft)\mathrm{d}t - \frac{2A}{T}\int_0^{T/2}\sin(2\pi nft)\mathrm{d}t - \frac{8A}{T^2}\int_{T/2}^{T} t\sin(2\pi nft)\mathrm{d}t + \frac{6A}{T}\int_{T/2}^{T}\sin(2\pi nft)\mathrm{d}t \\
&= \frac{8A}{T^2}\left(\left[-\frac{t}{2\pi nf}\cos(2\pi nft) + \frac{1}{4\pi^2n^2f^2}\sin(2\pi nft)\right]_0^{T/2} - \left[-\frac{t}{2\pi nf}\cos(2\pi nft) + \frac{1}{4\pi^2n^2f^2}\sin(2\pi nft)\right]_{T/2}^{T}\right) \\
&- \frac{2A}{T}\left(\left[-\frac{1}{2\pi nf}\cos(2\pi nft)\right]_0^{T/2} - \left[-\frac{3}{2\pi nf}\cos(2\pi nft)\right]_{T/2}^{T}\right) \\
&= \frac{8A}{T^2}\left(-\frac{T^2}{4n\pi}\cos(n\pi) + \frac{T^2}{2n\pi} - \frac{T^2}{4n\pi}\cos(n\pi)\right) - \frac{2A}{T}\left(-\frac{T}{2n\pi}\cos(n\pi) + \frac{T}{2n\pi} + \frac{3T}{2n\pi} - \frac{3T}{2n\pi}\cos(n\pi)\right) \\
&= \frac{4A}{n\pi}\left(1 - \cos(n\pi)\right) - \frac{4A}{n\pi}(1 - \cos(n\pi)) \\
&= 0
\end{aligned}
$$

Therefore,

$$
s(t) = -\frac{8A}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\cos(2\pi nft)
$$

2 Fourier Transform

Fourier Transform gives the continuous frequency domain of a nonperiodic signal

$$
S(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi ft}\mathrm{d}t \qquad\qquad s(t) = \int_{-\infty}^{\infty} S(f)e^{-i2\pi ft}\mathrm{d}t
$$

Rectangular Pulse

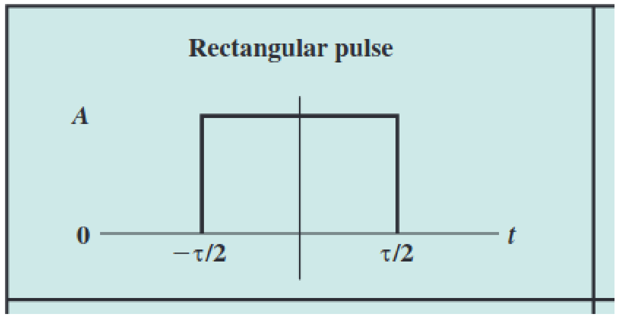

Figure 4: Rectangular Pulse

$$s(t) = \begin{cases} A & \text{if } |t| \leq t/2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} S(f) &= \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} \mathrm{d}t \\ &= \int_{-t/2}^{t/2} A e^{-i2\pi f t} \mathrm{d}t \\ &= -\frac{A}{i2\pi f} \left[e^{-i2\pi f t} \right]_{-t/2}^{t/2} \\ &= -\frac{A}{i2\pi f} \left(e^{-i\pi f t} - e^{i\pi f t} \right) \\ &= \frac{A}{\pi f} \left(\frac{e^{i\pi f t} - e^{-i\pi f t}}{2i} \right) \\ &= \frac{A \sin(\pi f t)}{\pi f} \end{aligned}$$

2.1 Triangular Pulse

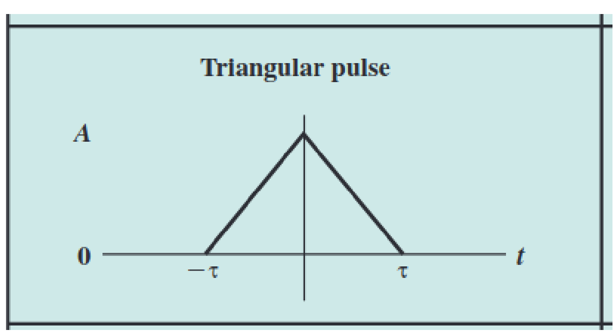

Figure 5: Triangular Pulse

$$s(t) = \begin{cases} \frac{A}{\tau}t + A & \text{if } \tau \leq t \leq 0 \\ -\frac{A}{\tau}t + A & \text{if } 0 \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

$$S(f) = A\tau \left(\frac{\sin(\pi f \tau)}{\pi f \tau} \right)^2$$

Sawtooth Pulse

$$S(f) = \frac{iA}{2\pi f} \left(\frac{\sin(\pi f \tau)}{\pi f \tau} e^{-i\pi f \tau} - 1 \right)$$