21.05.2024 ДИС 2

0. От предната лекция

$U \underset{\text{отв.}}{\subset} \mathbb{R}^n$, $f: U \to \mathbb{R}$, $x_0 \in U$

f е диф в x_0, ако $\exists$ лин. оператор $df(x_0): \mathbb{R}^n \to \mathbb{R}$

$$f(x) = f(x_0) + df(x_0)(x - x_0) + o(\|x - x_0\|)$$

или

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - df(x_0)(h)}{\|h\|} = 0$$

0.1 От диф $\Rightarrow$ непр.

0.2 $$\frac{\partial f}{\partial x_i}(x_0) = \lim_{\lambda \to 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda}$$

f е диф в $x_0 \Rightarrow \frac{\partial f}{\partial x_i}(x_0)$ същ. за $i \in \{1, \dots, n\}$

и $df(x_0)(h) = \langle \operatorname{grad} f(x_0), h \rangle$, където

$$\operatorname{grad} f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right)$$

0.3 Ако $\frac{\partial f}{\partial x_i}$ същ. в U $\forall i \in \{1, \dots, n\}$ и са непр. в $x_0 \in U$, то f е диф. в $x_0 \in U$

1. Th: Диференциране на композиции

$x_0 \in U \underset{\text{отв.}}{\subset} \mathbb{R}^n$

$$U \xrightarrow{f} V \xrightarrow{g} W$$

$V \underset{\text{отв}}{\subset} \mathbb{R}^m$

$f: U \to V$, f е диф. в $x_0 \in U$
$g: V \to \mathbb{R}^k$, g е диф. в $y_0 = f(x_0) \in V$

Тогава: $g \circ f: U \to \mathbb{R}^k$ е диф. в $x_0 \in U$ и $d(g \circ f)(x_0)$ е композиция на $df(x_0)$ и $dg(f(x_0))$

$$f(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$$

$df(x_0)$ съществува $\Leftrightarrow$ $df_1(x_0), \dots, df_m(x_0)$ същ.

$$df(x_0)(h) = \begin{pmatrix} \operatorname{grad} f_1(x_0) \\ \vdots \\ \operatorname{grad} f_m(x_0) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0), \dots, \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0), \dots, \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

$$= \underbrace{f'(x_0)}_{\text{якобиан (Jacobian)}} h$$

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

$$g'(y_0) = \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(y_0), \dots, \frac{\partial g_1}{\partial y_m}(y_0) \\ \vdots \\ \frac{\partial g_k}{\partial y_1}(y_0), \dots, \frac{\partial g_k}{\partial y_m}(y_0) \end{pmatrix}$$

$$\frac{\partial (g \circ f)_s}{\partial x_i}(x_0) = \sum_{j=1}^{m} \frac{\partial g_s}{\partial y_j}(f(x_0)) \cdot \frac{\partial f_j}{\partial x_i}(x_0) \quad \forall i \in \{1, \dots, n\},\ \forall s \in \{1, \dots, k\}$$

1.1 Доказателство:

* за простота нека $k=1$: бОО. $k=1$

• g е диференцируема в $y_0 = f(x_0)$, т.е.: $g(y) =$

$$g(y) = g(y_0) + dg(y_0)(y-y_0) + \psi(y, y_0) \text{ и } \psi(y,y_0) = o(\|y-y_0\|)$$

• $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \Rightarrow f_j(x) = f_j(x_0) + df_j(x_0)(x-x_0) + \varphi_j(x,x_0)$ и $\varphi_j(x,x_0) = o(\|x-x_0\|)$

• $F(x) = g(f(x)) = g(f_1(x_1,\dots,x_n), \dots, f_m(x_1,\dots,x_n))$

разгл.: $F(x) - F(x_0) = g(f(x)) - g(f(x_0)) \overset{\text{диф. } g \text{ в } y_0}{=}$

$$= \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(y_0) \cdot \overbrace{\left(f_j(x) - f_j(x_0)\right)}^{\text{отместване}} + \psi(f(x), f(x_0)) \overset{\text{диф. } f_j}{=}$$

$$= \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(y_0) \left[\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x_0)(x-x_0)_i + \varphi_j(x, x_0) \right] + \psi(f(x), f(x_0))$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial g}{\partial y_j}(y_0) \cdot \frac{\partial f_j}{\partial x_i}(x_0) \underbrace{(x-x_0)_i}_{i\text{-та коорд. на отмест.}} + \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(y_0)\, \varphi_j(x, x_0) + \psi(f(x), f(x_0))$$

$$= \sum_{i=1}^{n} \underbrace{\left[\sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(f(x_0)) \cdot \frac{\partial f_j}{\partial x_i}(x_0) \right]}_{\frac{\partial F}{\partial x_i}(x_0)} (x-x_0)_i + \boxed{\sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(y_0)\, \varphi_j(x, x_0) + \psi(f(x), f(x_0))}$$

(*) →

Искаме да го докараме до 0:

$$\underbrace{\frac{\left\|\sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(y_0)\cdot \varphi_j(x,x_0)\right\|}{\|x-x_0\|}}_{\xrightarrow[x\to x_0]{} 0} \ \underset{\text{нер. }\Delta}{\le}\ \sum_{j=1}^{m} \underbrace{\left|\frac{\partial g}{\partial y_j}(y_0)\right|}_{\text{const}} \cdot \underbrace{\frac{|\varphi_j(x,x_0)|}{\|x-x_0\|}}_{\to 0 \text{ при } x\to x_0}$$

Знаем:

$$\varphi(y, y_0) = o(\|y - y_0\|), \text{ тогава } \alpha(y,y_0) = \begin{cases} \dfrac{\varphi(y,y_0)}{\|y-y_0\|}, & \text{ако } y \neq y_0 \\ 0, & \text{ако } y = y_0 \end{cases}$$

(y_0 фиксирано)

$$\alpha(y, y_0) \xrightarrow[y\to y_0]{} 0 \quad \left(\alpha(\cdot, y_0) \text{ е непр. в } y_0\right)$$

като ф-я на y, т.к. y_0 е фиксирано

$$\frac{|\varphi(f(x), f(x_0))|}{\|x-x_0\|} = \frac{|\alpha(f(x), f(x_0))| \cdot \|f(x) - f(x_0)\|}{\|x-x_0\|}$$

$$x \to x_0 \Rightarrow f(x) \xrightarrow[x\to x_0]{\text{т.к. диф} \Rightarrow \text{непр.}} f(x_0) \ \text{ и } \ \alpha(f(x), f(x_0)) \xrightarrow[x\to x_0]{} 0$$

$$\Rightarrow \frac{\|f(x) - f(x_0)\|}{\|x-x_0\|} = \frac{\|df(x_0)(x-x_0) + \varphi(x,x_0)\|}{\|x-x_0\|} \le$$

$$\le \underbrace{\left\| df(x_0)\left(\frac{x-x_0}{\|x-x_0\|}\right)\right\|}_{\text{огр.}} + \underbrace{\sqrt{\sum_{j=1}^{m}\left(\frac{\varphi_j(x,x_0)}{\|x-x_0\|}\right)^2}}_{\xrightarrow[x\to x_0]{} 0}$$

$$\left\| df(x_0)\left(\frac{x-x_0}{\|x-x_0\|}\right)\right\| = \sqrt{\sum_{j=1}^{m}\left(\left\langle \operatorname{grad} f_j(x_0), \frac{x-x_0}{\|x-x_0\|}\right\rangle\right)^2} \ \underset{}{\overset{\text{Коши-Буняковски}}{\le}}$$

$$\le \sqrt{\sum_{j=1}^{m} \|\operatorname{grad} f_j(x_0)\|^2 \cdot \underbrace{\left\|\frac{x-x_0}{\|x-x_0\|}\right\|^2}_{=1}} \le \sqrt{\sum_{j=1}^{m} \|\operatorname{grad} f_j(x_0)\|^2}$$

$\Rightarrow$ (*) е $O \xrightarrow{\text{клони към } 0}$ отворено $\square$

$$F(x_1, \dots, x_n) = g\left(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\right)$$

$$\frac{\partial F}{\partial x_i}(x_0) = \frac{\partial g}{\partial y_1}(f(x_0)) \cdot \frac{\partial f_1}{\partial x_i}(x_0) + \dots + \frac{\partial g}{\partial y_m}(f(x_0)) \cdot \frac{\partial f_m}{\partial x_i}(x_0)$$

(1) $f(x, y) = x^y$

$U = (0, +\infty) \times (-\infty, +\infty)$

$$\frac{\partial f}{\partial x}(x, y) = y x^{y-1}$$

$$\frac{\partial f}{\partial y}(x, y) = \ln x \cdot x^y$$

$$g(t) = f(t, t) = t^t$$

$$g'(t) = \frac{\partial f}{\partial x}(t, t) \cdot 1 + \frac{\partial}{\partial y} f(t, t) =$$

$$= t \cdot t^{t-1} + t^t \cdot \ln t =$$

$$= t^t (1 + \ln t)$$

(2) $z(x, y)$ $\quad (x, y) \in \mathbb{R}^2$

$z'_x = z'_y$ $\quad$ пол. $\quad u = x + y$, $v = x - y$ $\Rightarrow$ $x = x(u, v)$, $y = y(u, v)$

нека $w(u, v) = z(x(u, v), y(u, v))$

$z(x, y) = w(u(x, y), v(x, y))$

$$z'_x = w'_u(u(x, y), v(x, y)) \cdot u'_x(x, y) + w'_v(u(x, y), v(x, y)) \cdot v'_x(x, y)$$

$$z'_x = w'_u(x + y, x - y) \cdot 1 + w'_v(x + y, x - y) \cdot 1 = w'_u + w'_v$$

$z'_y(x,y) = w'_u(x+y, x-y) \cdot \frac{\partial}{\partial y}(x+y) +$

$w'_v(x+y, x-y) \cdot \frac{\partial}{\partial y}(x-y) = w'_u - w'_v$

$\Rightarrow \cancel{w'_u} + w'_v = \cancel{w'_u} - w'_v \Rightarrow w'_v \equiv 0$

$w(u,v) = \varphi(u)$ ↳ w не завис от v

$z(x,y) = \varphi(x+y)$

(3) $y z'_x - x z'_y = 0$

смена $u = x$

$v = x^2 + y^2$

$z(x,y) = w(u(x,y), v(x,y))$

$z'_x = w'_u(\overbrace{u(x,y)}^{x}, \overbrace{v(x,y)}^{x^2+y^2}) \frac{\partial}{\partial x}(x) +$

$w'_v(x, x^2+y^2) \frac{\partial}{\partial x}(x^2+y^2) =$

$= w'_u(x, x^2+y^2) \cdot 1 + w'_v(x, x^2+y^2) \cdot 2x$

$z'_y = w'_u(x, x^2+y^2) \underbrace{\frac{\partial}{\partial y}(x)}_{=0} + w'_v(x, x^2+y^2) \frac{\partial}{\partial y}(x^2+y^2)$

$= w'_v(x, x^2+y^2) \cdot 2y$

$\Rightarrow y[w'_u(x, x^2+y^2) + \cancel{w'_v(x, x^2+y^2) \cdot 2x}] =$

$x[\cancel{w'_v(x, x^2+y^2) \cdot 2y}]$

$\Rightarrow y\, w'_u(x, x^2+y^2) = 0 \quad w'_u = 0$

$w = \varphi(v)$

$f(x,y) = \varphi(x^2+y^2)$

2. Инвариантност на формата на диференциала

$h = (h_1, \dots, h_n) \in \mathbb{R}^n$

$x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$x_i(h) = h_i$

$dx_i(x_0)(h) = x_i(h) = h_i$

знаем $df(x_0)(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) \cdot h_i = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) \cdot dx_i(x_0)(h)$

$df(x_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)\, dx_i(x_0)$ или $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$

$F(x_1, \dots, x_n) = g(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$

$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i = \sum_{i=1}^{n} \left[\sum_{j=1}^{m} \frac{\partial g}{\partial y_j} \cdot \frac{\partial f_j}{\partial x_i} \right] dx_i =$

$= \sum_{j=1}^{m} \frac{\partial g}{\partial y_j} \underbrace{\left(\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} dx_i \right)}$

$= \sum_{j=1}^{m} \frac{\partial g}{\partial y_j} \cdot df_j = \boxed{\sum_{j=1}^{m} \frac{\partial g}{\partial f_j} dy_j}$

пр. $g(u,v) = u+v$

$d(u+v) = \frac{\partial}{\partial u}(u+v)\,du + \frac{\partial}{\partial v}(u+v)\,dv = du + dv$

$\Rightarrow d(u+v) = du + dv$

пр. $d(u \cdot v) = \frac{\partial}{\partial u}(u \cdot v)\,du + \frac{\partial}{\partial v}(u \cdot v)\,dv =$

$= v\,du + u\,dv \Rightarrow d(u \cdot v) = v \cdot du + u \cdot dv$

$$d\left(\frac{u}{v}\right) = \frac{\partial}{\partial u}\left(\frac{u}{v}\right)du + \frac{\partial}{\partial v}\left(\frac{u}{v}\right)dv = \frac{du}{v} - \frac{u}{v^2}\, dv \Rightarrow$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

3. Производни от по-висок ред

$f: U \to \mathbb{R}$

$\frac{\partial f}{\partial x_i} : U \to \mathbb{R}$, тогава $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$

е пр. от втори ред — първо по x_i!

4. Th на Шварц за равенство на смесените производни

$f: U \to \mathbb{R}$, $\underset{\text{отв.}}{U \subset \mathbb{R}^n}$, $x_0 \in \mathbb{R}^n$

$\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \underbrace{\frac{\partial^2 f}{\partial x_i x_j}, \frac{\partial^2 f}{\partial x_j x_i}}_{\text{са непр. в } x_0}$ съществуват в U

Тогава $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial f}{\partial x_j \partial x_i}(x_0)$

4.1 Следствие:

$f \in C^2(U, \mathbb{R})$

$\frac{\partial^2 f}{\partial x_i \partial x_j}$, $i, j \in \{1, \dots, n\}$ същ. и непр. в U

f двукратно гладка в U

$\Rightarrow f \in C^2(U, \mathbb{R})$

$\Rightarrow \dfrac{\partial^2 f}{\partial x_i \partial x_j}(x) = \dfrac{\partial^2 f}{\partial x_j \partial x_i}(x) \quad \forall x \in U \quad \forall i,j \in \{1, \dots, n\}$

ч.2 Доказателство:

Б.о.о. нека $n = 2$

$f(x_1, x_2) \qquad (x_1, x_2) \in U \subset \mathbb{R}^2$

$x_0 = (x_1^0, x_2^0), \quad B_\delta(x_0) \subset U$

$\|h\| < \delta$

нека $\varphi(h_1, h_2) = f(x_1^0 + h_1, x_2^0 + h_2) + f(x_1^0, x_2^0)$
$\quad - f(x_1^0 + h_1, x_2^0) - f(x_1^0, x_2^0 + h_2)$

$a(t) = f(x_1^0 + h_1, t) - f(x_1^0, t)$

$\varphi(h_1, h_2) = a(x_2^0 + h_2) - a(x_2^0)$

$t \in (x_2^0 - \delta, x_2^0 + \delta)$

$a' = \dfrac{\partial f}{\partial x_2}(x_1^0 + h_1, t) - \dfrac{\partial f}{\partial x_2}(x_1^0, t)$

$0 < \theta < 1$

$\varphi(h_1, h_2) = a(x_2^0 + h_2) - a(x_2^0) \overset{\text{Лагранж}}{=} h_2 \cdot a'(x_2^0 + \theta_1 h_2)$

$\varphi(h_1, h_2) = h_2 \cdot \left(\dfrac{\partial f}{\partial x_2}(x_1^0 + h_1, x_2^0 + \theta_1 h_2) - \dfrac{\partial f}{\partial x_2}(x_1^0, x_2^0 + \theta_1 h) \right)$

$= h_2 \left(b(x_1^0 + h_1) - b(x_1^0) \right) = h_2 \cdot h_1 \cdot b'(x_1^0 + \theta_2 h_1) =$

$0 < \theta_2 < 1$

$= h_2 \cdot h_1 \cdot \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(x_1^0 + \theta_2 h_1, x_2^0 + \theta_1 h_2)$

$b(t) = \dfrac{\partial f}{\partial x_2}(t, x_2^0 + \theta_1 h_2)$

$\varphi(t) = f(t, x_2^0 + h_2) - f(t, x_2)$

$\psi(h_1, h_2) = \varphi(x_1^0 + h_1) - \varphi(x_1^0) =$

$= h_1 \left(\frac{\partial f}{\partial x_1}(x_1^0 + \theta_3 h_1, x_2^0 + h_2) - \frac{\partial f}{\partial x_1}(x_1^0 + \theta_3 h_1, x_2 \right.$

$= h_1 h_2 \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1^0 + \theta_3 h, x_2^0 + \theta_4 h_2)$

приравняваме и съкр. $h_1 \cdot h_2$

$\Rightarrow \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1^0 + \theta_2 h_1, x_2^0 + \theta_1 h_2) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1^0 + \theta_3 h_1, x_2^0 + \theta_4 h_2)$

при $h \to \theta \Rightarrow \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0)$

т.к. + непрекъснатост

$\| (x_1^0 + \theta_2 h_1, x_2^0 + \theta_1 h_2) - (x_1^0, x_2^0) \| = \sqrt{\theta_2^2 h_1^2 + \theta_1^2 h_2^2}$

$\leq \|h\| \xrightarrow[h \to \theta]{} 0$

$\square$

<u>пр:</u>

$f(x,y) = \begin{cases} xy \cdot \frac{x^2 - y^2}{x^2 + y^2} & \text{ако } (x,y) \neq (x,y) \\ 0 & \text{ако } (x,y) = (0,0) \end{cases}$

$f_x', f_y', f_{xy}'', f_{yx}''$ съм. в $\mathbb{R}^2$

$f_{xy}''(0,0) = -1$

$f_{yx}''(0,0) = 1$