12.03.2024 ДИС 2

0. Th на Нютон - Лайбниц

$f: \Delta \to \mathbb{R}$, Δ - интервал, $a \in \Delta$

f е интегруема в $[a, c]$ $\forall c \in \Delta$ } предположение

$F(x) = \int_a^x f(t)\, dt \quad \forall x \in \Delta$, $F: \Delta \to \mathbb{R}$

Ако f е непр. в $x_0 \in \Delta$, то F е диференцируема в x_0 и $F'(x_0) = f(x_0)$

F' е непр в $[a, b] \Rightarrow \int_a^b F'(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)$

1. Th за смяна на променливите

$f: \Delta \to \mathbb{R}$, Δ - интервал, а f е непрекъсната, $a, b \in \Delta$ (търсим $\int_a^b f(x)\, dx$)

$\varphi: [\alpha, \beta] \to \Delta$ (стойностите не трябва да излизат от Δ, т.е. там, където е непр.)

$\varphi(\alpha) = a$, $\varphi(\beta) = b$ и φ е гладка ($\exists \varphi'$ и е непр.)

Тогава $\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt = \int_\alpha^\beta f(\varphi(t))\, d\varphi(t)$

NB!! Не е нужно φ да е биекция

1.1 Доказателство:

$\int_a^b f(x)\, dx = F(b) - F(a)$, където F е примитивна за f в Δ

$\Phi(t) = F(\varphi(t))$ е примитивна за $f(\varphi).\varphi'$ в $[\alpha, \beta]$

$\Phi' = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) . \varphi'(t)$

$\Rightarrow \int_a^b f(x)\, dx = F(b) - F(a) = F(\varphi(\beta)) - F(\varphi(\alpha)) =$

$= \Phi(\beta) - \Phi(\alpha) = \int_\alpha^\beta \Phi'(t)\, dt = \int_\alpha^\beta f(\varphi(t)) . \varphi'(t)\, dt$

пр: $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx =$ $\left| \begin{array}{l} \text{пол. } x = a\sin t \\ t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array} \right.$ $a \geq 0$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\; d(a\sin t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{\geq 0}{a\cos t} \cdot a\cos t \, dt =$

$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\, dt = \frac{a^2}{2} \cdot t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{a^2}{2} \cdot \frac{1}{2} \sin 2t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$

$= \frac{a^2 \pi}{2}$

пр: $\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{(n-1)!!}{n!!} \begin{cases} \frac{\pi}{2}, & \text{ако } n \text{ е четно} \\ 1, & \text{ако } n \text{ е нечетно} \end{cases}$

$\int_0^{\frac{\pi}{2}} \sin^n x\, dx =>$ пол. $x = \frac{\pi}{2} - t$

$t = \frac{\pi}{2} - x$

$\int_{\frac{\pi}{2}}^{0} \sin^n(\frac{\pi}{2} - t)\, d(\frac{\pi}{2} - t) =$ за $x = 0$ $t = \frac{\pi}{2}$

за $x = \frac{\pi}{2}$ $t = 0$

$= \int_0^{\frac{\pi}{2}} \cos^n t\, dt$

пр: Формула на Уолис

$a_n = \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{1}{2n+1}$

$0 \leq x \leq \frac{\pi}{2}$

$\sin^{2n-1} x \geq \sin^{2n} x \geq \sin^{2n+1} x$, т.к. $\sin x \in [0, 1]$

интегрираме:

$\frac{(2n-2)!!}{(2n-1)!!} \cdot 1 \geq \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \geq \frac{(2n)!!}{(2n+1)!!}$

$$\frac{(2n-2)!!}{(2n-1)!!} \geq \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \geq$$

$$\iff \frac{(2n)!! \cdot (2n-2)!!}{((2n-1)!!)^2} \geq \frac{\pi}{2} \Rightarrow$$

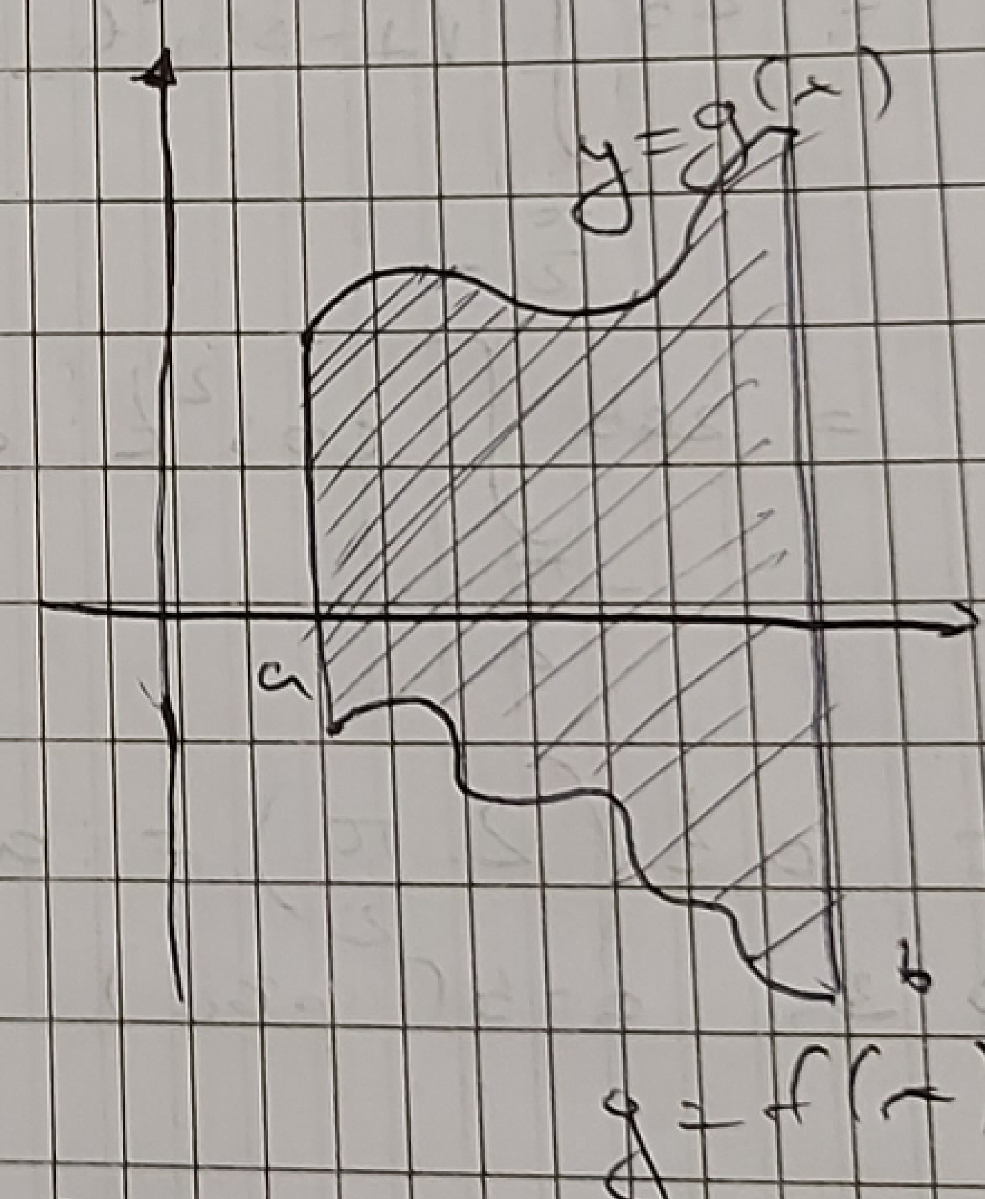
$$a_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n+1} \geq \frac{\pi}{2} \cdot \frac{2n}{2n+1}$$

$$\frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \geq \frac{(2n)!!}{(2n+1)!!} \Rightarrow a_n = \frac{((2n)!!)^2}{(2n-1)!! \cdot (2n+1)!!} \leq \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right) \leq a_n \leq \frac{\pi}{2} \Rightarrow \lim_{n\to\infty} a_n = \frac{\pi}{2}$$

2. Геометрични приложения на опр. интеграл

2.1 Лице на криволинеен трапец



$f, g : [a,b] \to \mathbb{R}$ непр.

$f(x) \leq g(x) \quad \forall x \in [a,b]$

$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \leq x \leq b \\ f(x) \leq y \leq g(x) \end{array} \right\}$ е

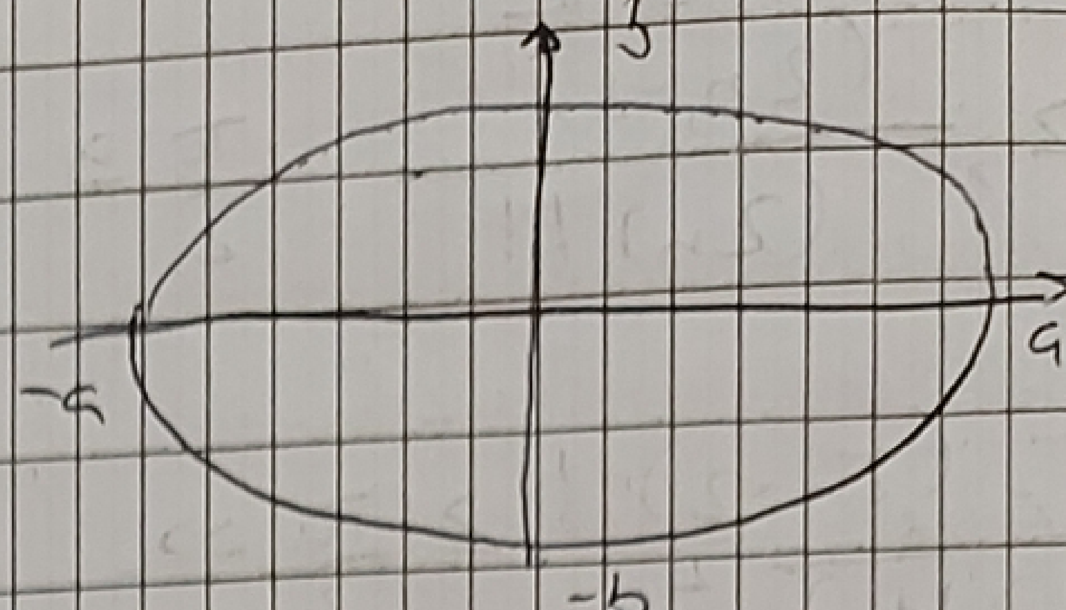
криволинеен трапец

f непр. $\xRightarrow{\text{T Вайерщрас}} \exists\, C \in \mathbb{R} \;\; \forall x \in [a,b] : f(x) \geq C$

$f(x) - c \geq 0, \quad g(x) - c \geq 0$

$$S_D = \int_a^b (g(x)-c)\,dx - \int_a^b (f(x)-c)\,dx = \int_a^b (g(x)-f(x))\,dx$$

2.2 Лицето на елипсата



$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$$

$$\frac{y^2}{b^2} \le 1-\frac{x^2}{a^2}$$

$$y^2 \le b^2-\frac{b^2x^2}{a^2} \Rightarrow |y| \le \sqrt{b^2-\frac{b^2}{a^2}x^2}$$

$D := \left\{(x,y)\in \mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1\right\}$ $a>0,\ b>0$

$$-\sqrt{b^2-\frac{b^2}{a^2}x^2} \le y \le \sqrt{b^2-\frac{b^2}{a^2}x^2}$$

$$D := \left\{(x,y)\in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -a \le x \le a \\ -\sqrt{b^2-\frac{b^2}{a^2}x^2} \le y \le \sqrt{b^2-\frac{b^2}{a^2}x^2} \end{array}\right\}$$

$$2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx =$$

пол. $\frac{x}{a}=\sin t,\ t\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$x = a\sin t$

$dx = a\cos t\, dt$

за $x=-a$ $t=-\frac{\pi}{2}$

за $x=a$ $t=\frac{\pi}{2}$

$$= 2b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 t}\; d(a\sin t) =$$

$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\, dt = \frac{2ab}{\cancel{2}}\, t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{2ab}{2}\cdot\frac{1}{2}\sin 2t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

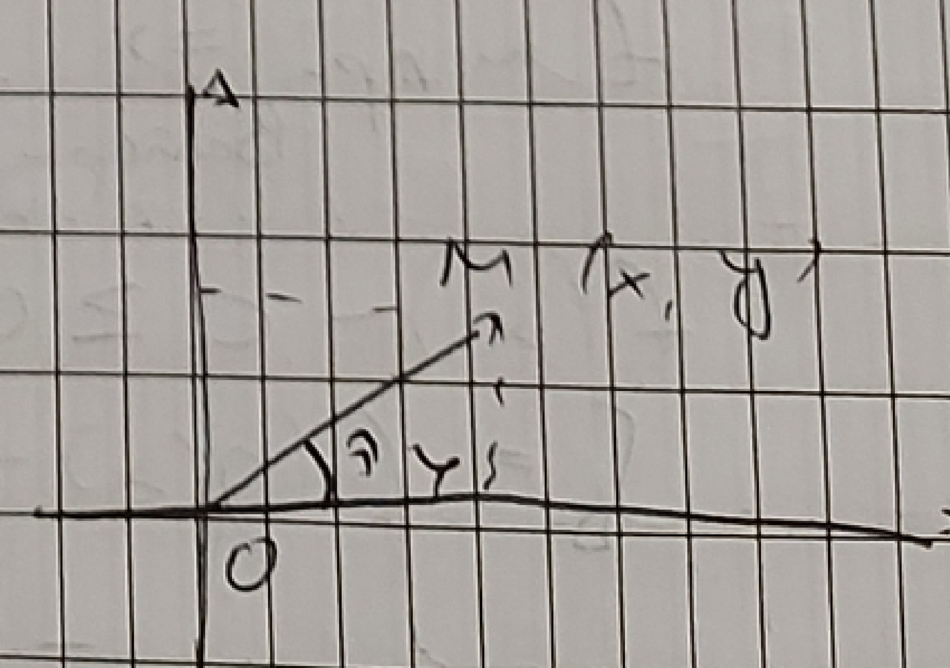
$$2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt$$

$$= ab\left(2\cdot\frac{\pi}{2}\right) = ab\pi$$

$\Rightarrow$ за $a=b$ (кръг) $S=\pi r^2$

2.3 Лица с полярни координати

$\rho: |\overrightarrow{OM}|$ и $\varphi = \measuredangle(\overrightarrow{Ox}, \overrightarrow{OM})$



$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

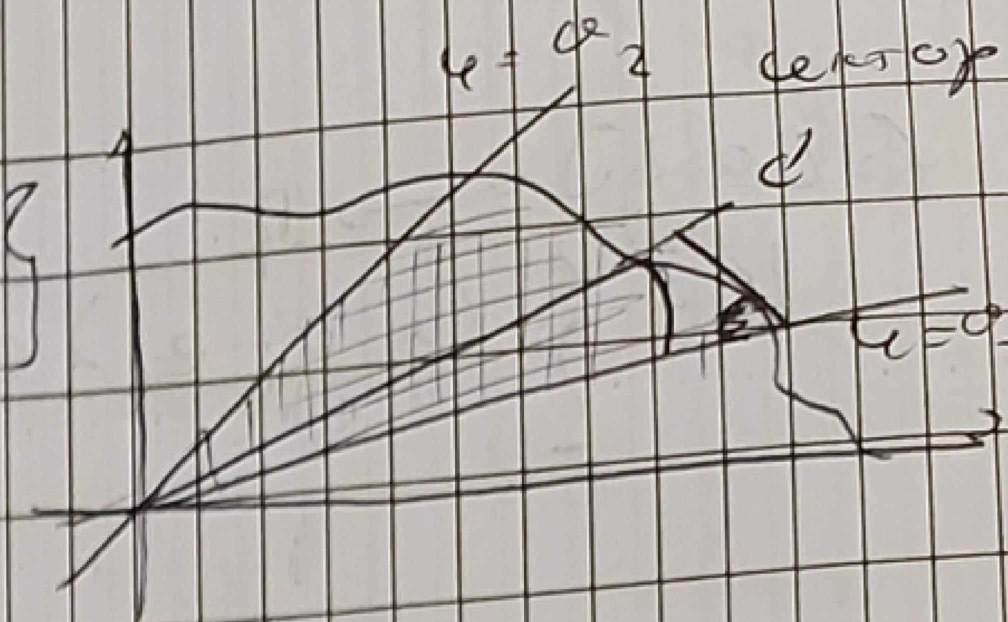
$\rho = \sqrt{x^2+y^2}$

$\varphi =$

$\sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$, $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$

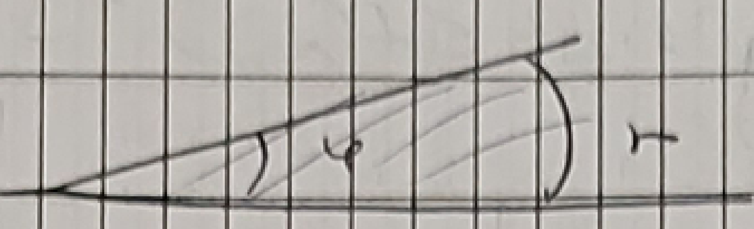
$\theta_1 \le \varphi \le \theta_2$ ($\varphi_0 \quad \varphi_1$)

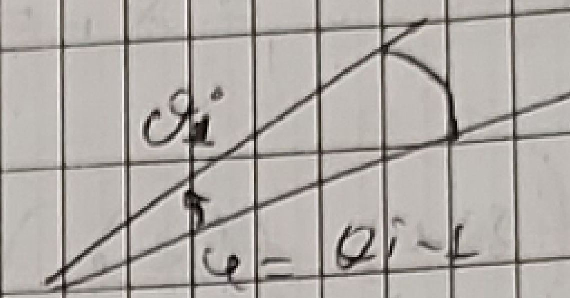
$0 \le \rho \le f(\varphi)$ $\Rightarrow$ $\left\{ (\rho, \varphi) : \begin{array}{l} \theta_1 \le \varphi \le \theta_2 \\ 0 \le \rho \le f(\varphi) \end{array} \right\}$



Лице на сектор

$S = \frac{\varphi}{2} r^2$





$\Rightarrow S = \frac{\theta_i - \theta_{i-1}}{2} r^2$

$\varphi_0 = \theta_0 < \theta_1 < \dots < \theta_n = \varphi_1$

$$S_D \ge \sum_{i=1}^{n} \frac{\theta_i - \theta_{i-1}}{2} \left(\inf_{[\theta_{i-1}, \theta_i]} f \right)^2 \le S_D \le \sum_{i=1}^{n} \frac{\theta_i - \theta_{i-1}}{2} \cdot \sup_{\varphi \in [\theta_{i-1}, \theta_i]} f(\varphi)^2$$

$f(\varphi) \ge 0$

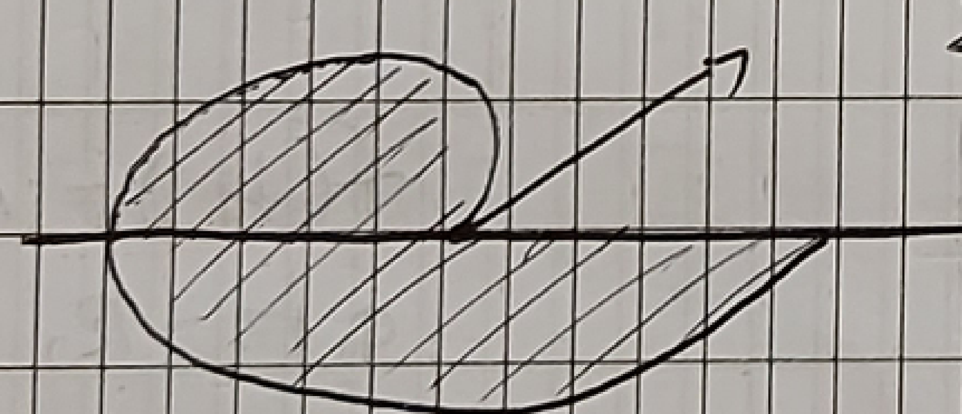
$$\sum_{i=1}^{n} \left(\inf_{\varphi \in [\theta_{i-1}, \theta_i]} \frac{f^2(\varphi)}{2} \right) \cdot \frac{\theta_i - \theta_{i-1}}{} \le S_D \le \sum_{i=1}^{n} \sup_{\varphi \in [\theta_{i-1}, \theta_i]} \left(\frac{f^2(\varphi)}{2} \right) (\theta_i - \theta_{i-1})$$

$$\boxed{f \text{ интегруеме} \Rightarrow S_D = \int_{\varphi_0}^{\varphi_1} \frac{f^2(\varphi)}{2} d\varphi = \frac{1}{2} \int_{\varphi_0}^{\varphi_1} f^2(\varphi) d\varphi}$$

2.3.1 Архимедова спирала

$\rho = k\varphi, \ k > 0$

$$\frac{1}{2} \int_0^{2\pi} (k\varphi)^2 d\varphi = \frac{1}{2} k^2 \frac{\varphi^3}{3} \Big|_0^{2\pi} =$$



$$= \frac{1}{6} k^2 8\pi^3 = \frac{4}{3} k^2 \pi^3$$

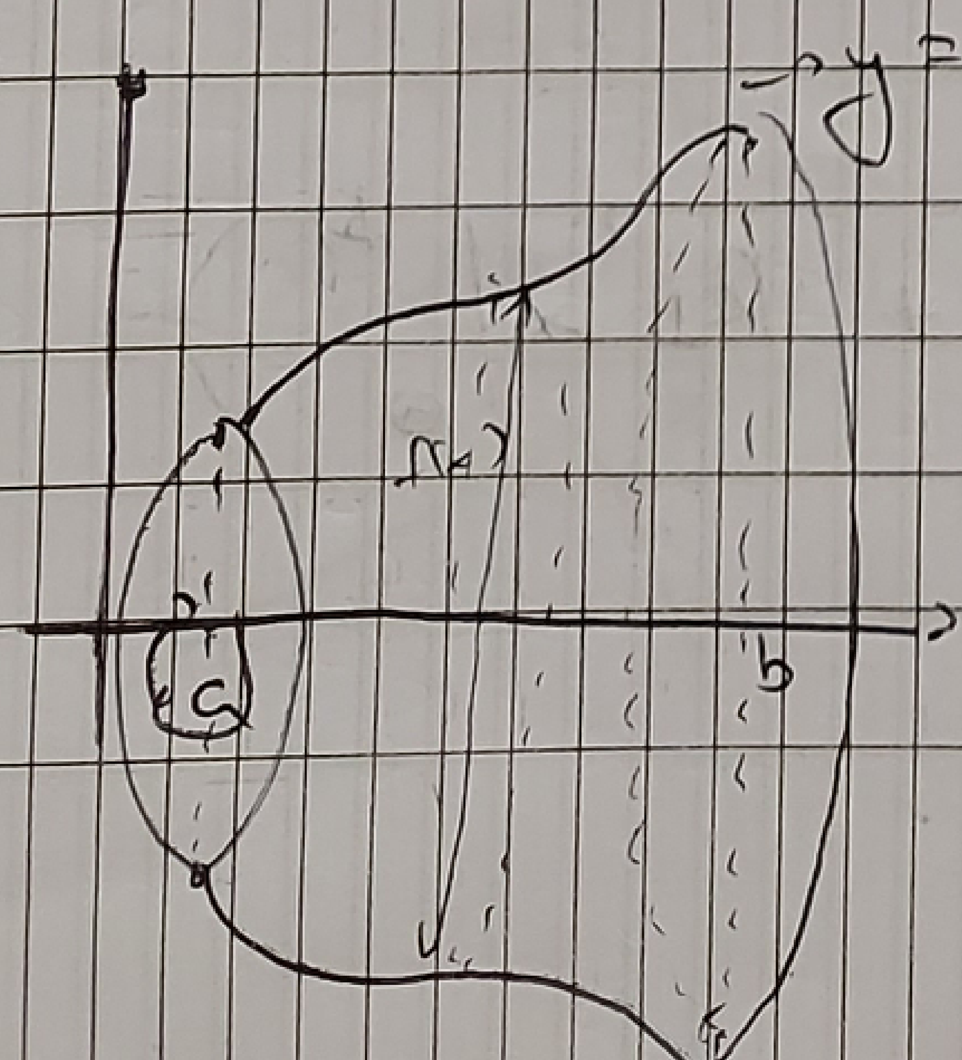
2.3.2 Обем на ротационно тяло

$y = f(x)$

$f : [a, b] \to \mathbb{R}$

$f(x) \ge 0 \ \forall x \in [a, b]$

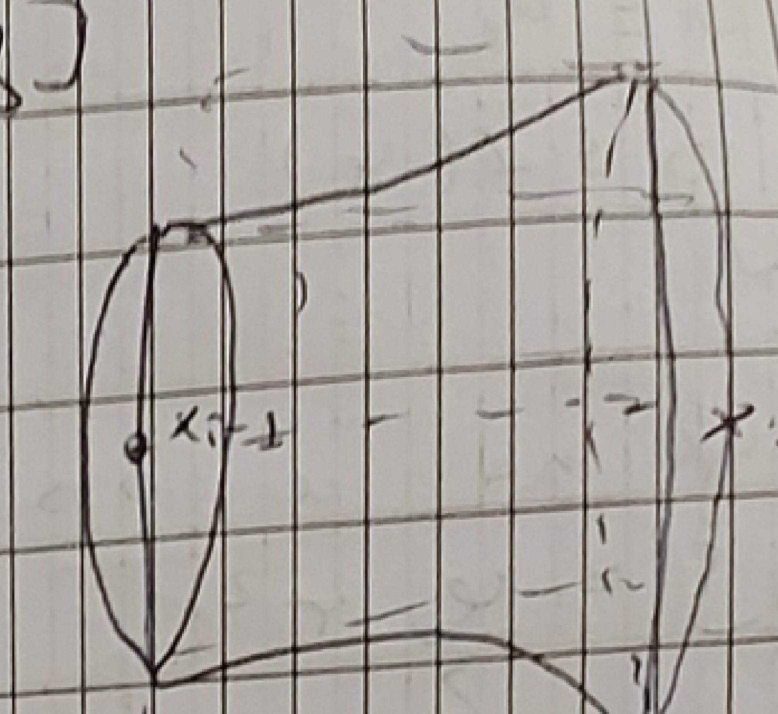
$K = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{array}{l} a \le x \le b \\ y^2 + z^2 = (f(x))^2 \end{array} \right\} \subset$



ротационно тяло

$f:[a,b]\to\mathbb{R}$ е непр., $f(x)\ge 0$ в/у $[a,b]$

$$K=\left\{(x,y,z)\in\mathbb{R}^3 : \begin{array}{l} a\le x\le b \\ y^2+z^2\le (f(x))^2\end{array}\right\}$$



$a=x_0<x_1<\dots<x_n=b$

$[x_{i-1},x_i]$

оценяваме с цилиндър

$$V_K\le\sum_{i=1}^{n}\pi\Big(\sup_{x\in[x_{i-1},x_i]} f(x)\Big)^2(x_i-x_{i-1})$$

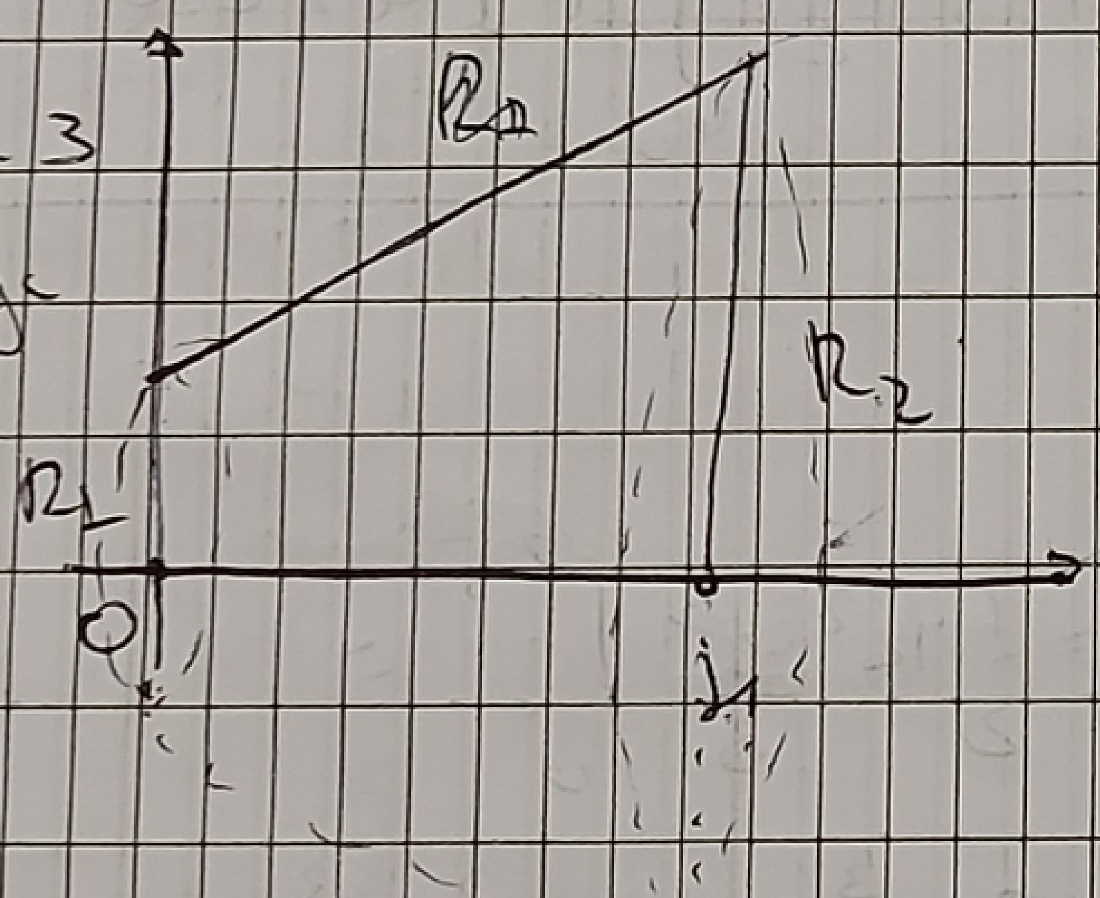
$$V_K\ge\sum_{i=1}^{n}\pi\Big(\inf_{x\in[x_{i-1},x_i]} f(x)\Big)^2(x_i-x_{i-1})$$

забележка: $\left(\sup_A f\right)^2=\sup_A(f^2)$, $f\ge 0$

$$\Rightarrow \sum_{i=1}^{n}\inf_{[x_{i-1},x_i]}(\pi f^2)(x_i-x_{i-1})\le V_K\le\sum_{i=1}^{n}\sup_{[x_{i-1},x_i]}(\pi f^2)(x_i-x_{i-1})$$

f интегр. $\Rightarrow$ $V_K=\pi\int_a^b f^2(x)\,dx$

2.3.3 конус



$$\pi\int_0^H\left(x\cdot\frac{R_2-R_1}{H}+R_1\right)^2dx=$$

$f(x)=ax+b$ $\quad f(0)=b=R_1$

$a=f(x)$

зс $x=$

$f(x)=x\,\frac{R_2-R_1}{H}+R_1$

$f(H)=R_2$

$aH+R_1=R_2$

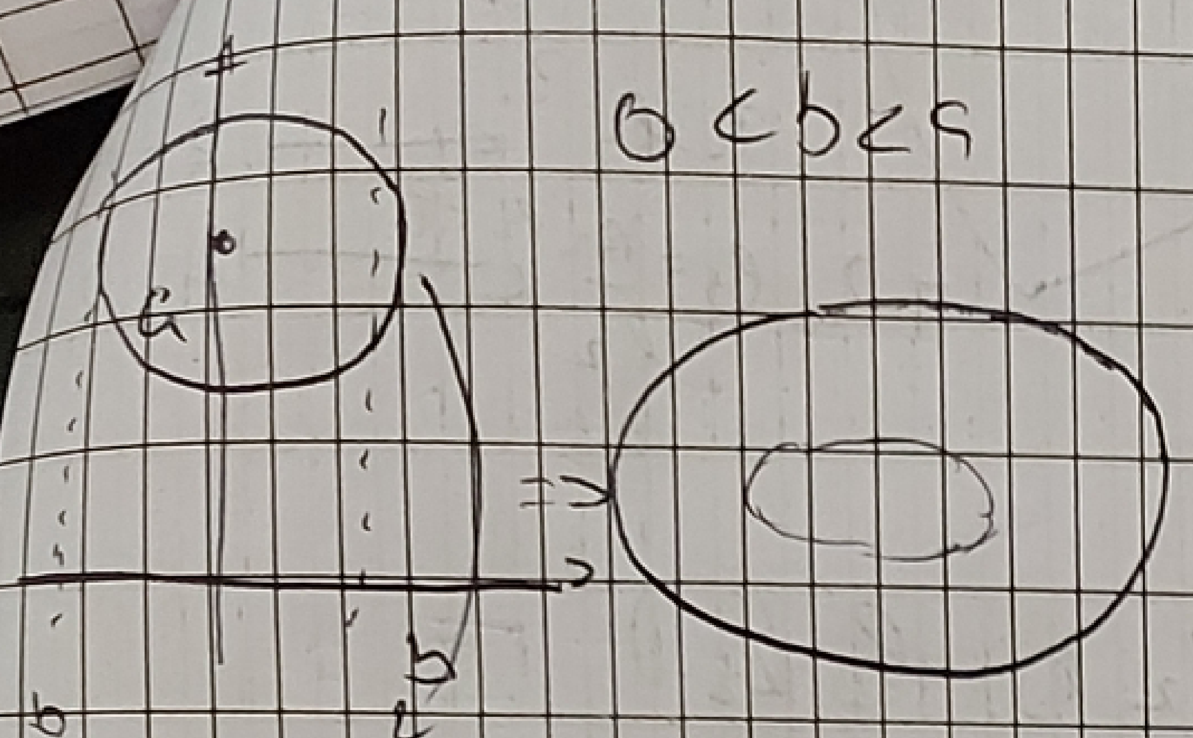
$$=R_1\pi x\Big|_a^b+\pi\frac{R_2-R_1}{H}\cdot\frac{x^2}{2}\Big|_a^b$$

$$=\pi\int_0^H x^2\left(\frac{R_2-R_1}{H}\right)^2+2x\frac{R_2-R_1}{H}\cdot R_1+R_1^2\,dx=$$

$$=\pi\left(\left(\frac{R_2-R_1}{H}\right)^2\frac{x^3}{3}\Big|_0^H+\frac{2x^2}{2}\,\frac{R_2-R_1}{H}\,R_1\Big|_0^H+R_1^2x\Big|_0^H\right)=$$

$$=\dots=\frac{\pi H}{3}\left(R_1^2+R_1R_2+R_2^2\right)$$

3.4 Обем на тор



$0 < b < a$ Кръг с център $(0, a)$ и радиус b

$$x^2+(y-a)^2 \le b^2$$

$$(y-a)^2 \le b^2-x^2$$

$$a-\sqrt{b^2-x^2} \le y \le a+\sqrt{b^2-x^2}$$

$$\pi\int_{-b}^{b}\left(a+\sqrt{b^2-x^2}\right)^2 dx - \pi\int_{-b}^{b}\left(a-\sqrt{b^2-x^2}\right)^2 dx =$$

$$=\pi\int_{-b}^{b}\left(a+\cancel{\sqrt{b^2-x^2}}+a-\cancel{\sqrt{b^2-x^2}}\right)\left(\cancel{a}+\sqrt{b^2-x^2}-\cancel{a}+\sqrt{b^2-x^2}\right)dx=$$

$$=4\pi\int_{-b}^{b} a\sqrt{b^2-x^2}\,dx = 8a\pi\int_{0}^{b}\sqrt{b^2-x^2}\,dx = \quad \left|\begin{array}{l} \text{пол. } x = b\sin t \\ t\in[0,\frac{\pi}{2}] \\ \text{за } x=0 \quad t=0 \\ \text{за } x=b \quad t=\frac{\pi}{2} \\ dx = b\cos t\,dt \end{array}\right.$$

$$= 8a\pi b^2\int_{0}^{\frac{\pi}{2}}\cos^2 t\,dt =$$

$$=\frac{8ab^2\pi}{2}\left(t\Big|_0^{\frac{\pi}{2}}+\frac{1}{2}\cdot\sin 2t\Big|_0^{\frac{\pi}{2}}\right)=$$

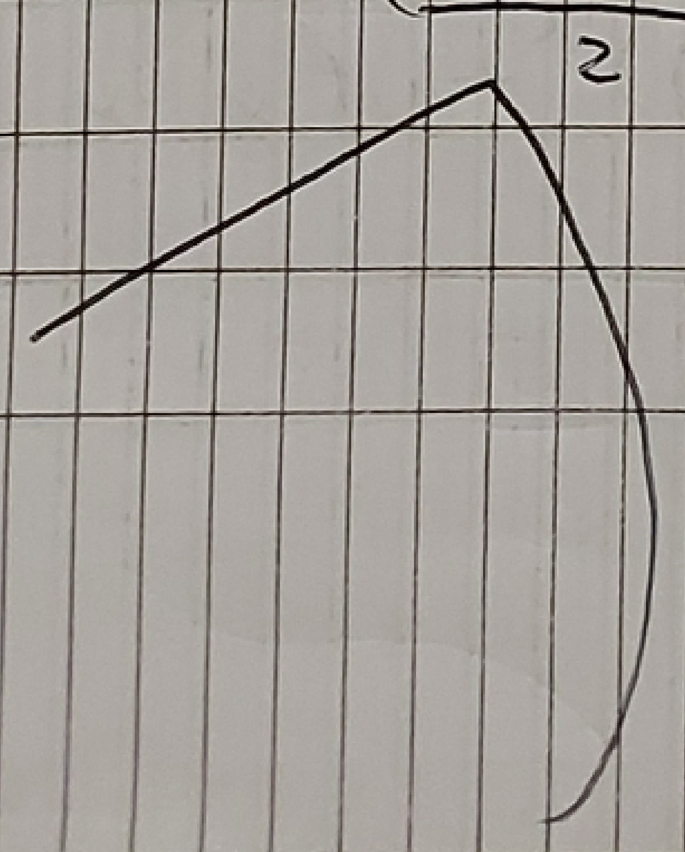
$$=4ab^2\pi\left(\frac{\pi}{2}+\cancel{\frac{1}{2}\cdot 1-0}\right)=2ab^2\pi^2\cancel{(\pi-1)}$$

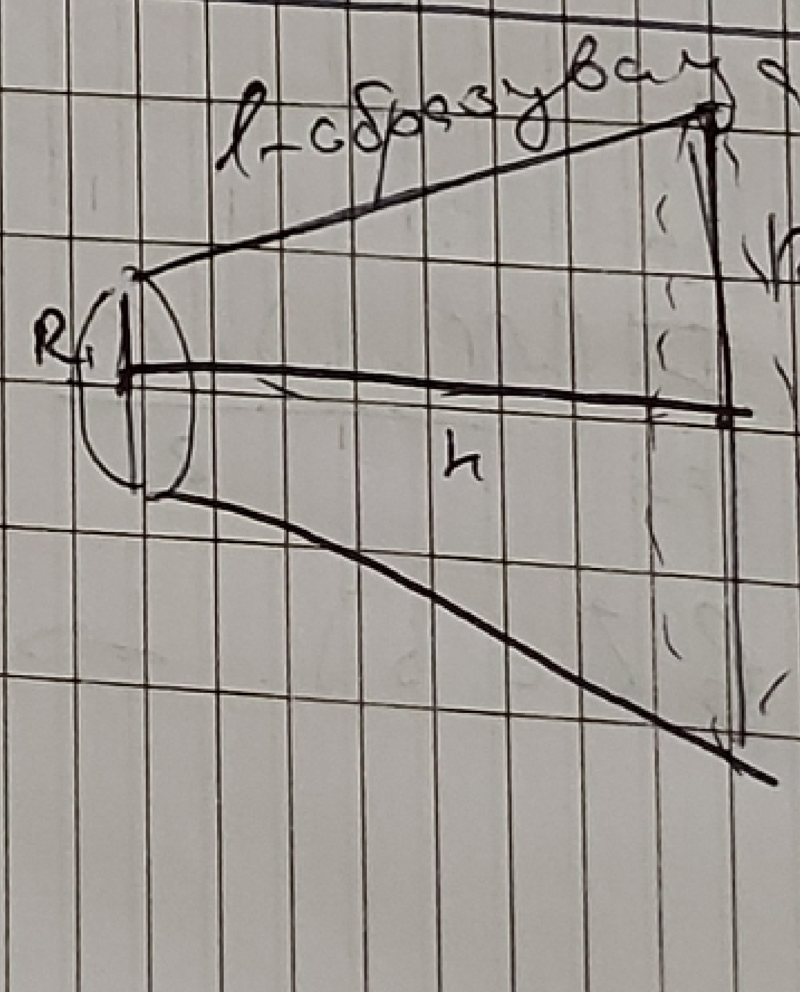
2.4 Лице на ротационно тяло (повърхнина)

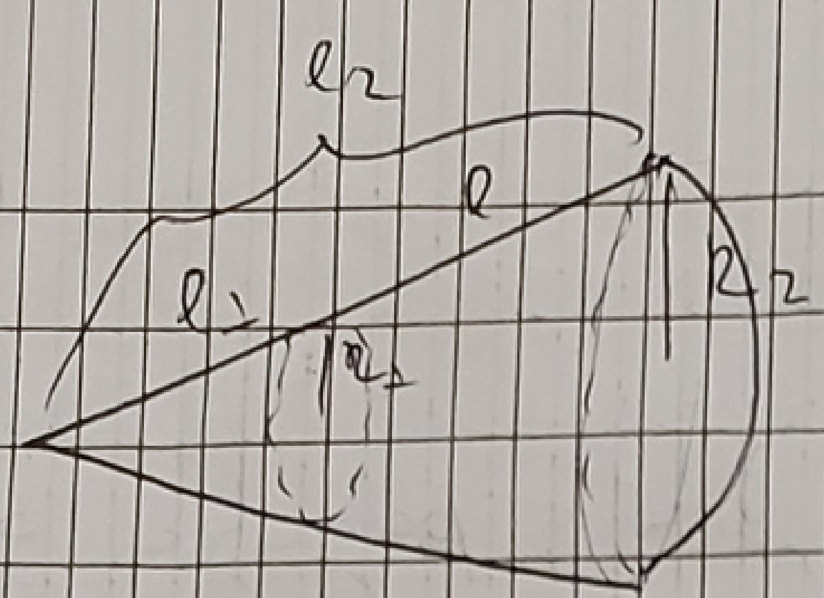
апроксимираме лице на ос. повърхнина с конус

(за гладки функции, а не само непр.)

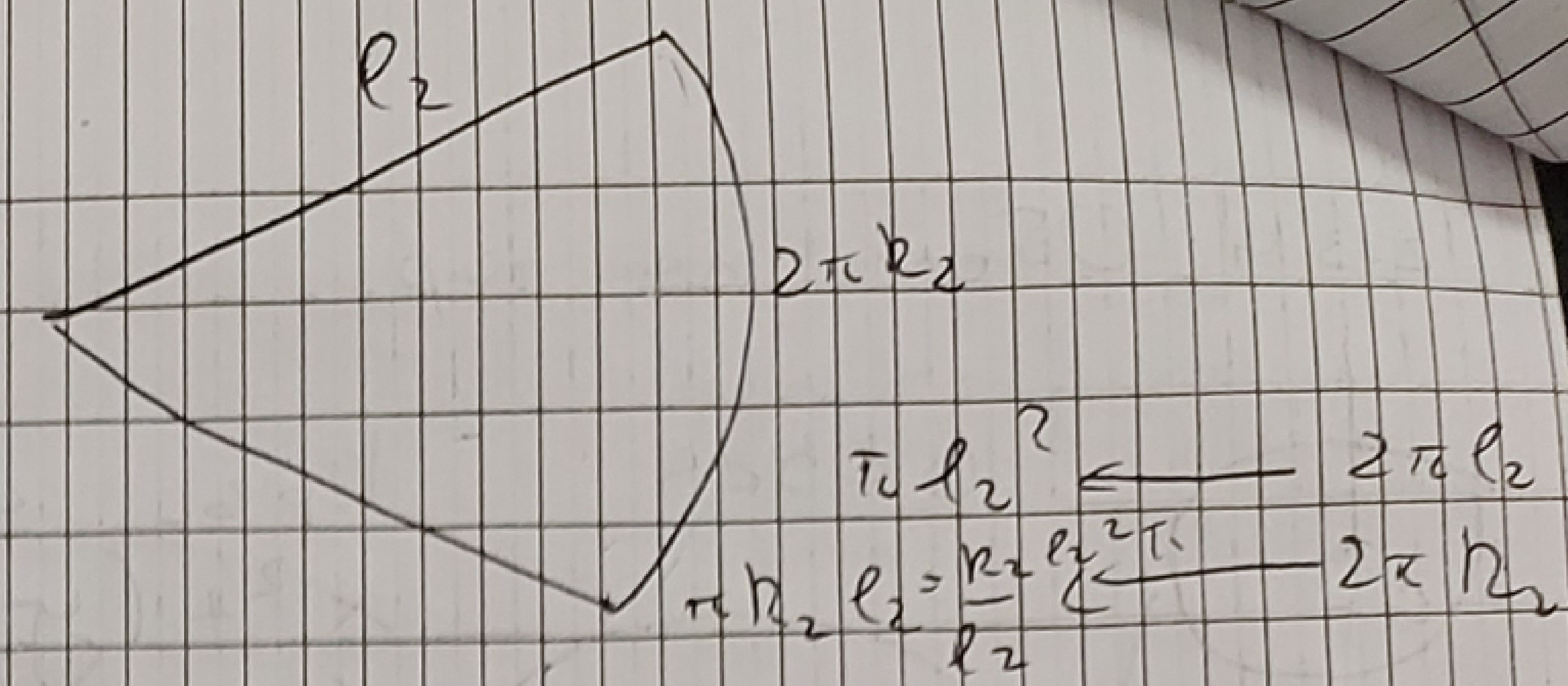
$$S=2\pi l\left(\frac{R_1+R_2}{2}\right)$$







$l = l_2 - l_1$



$\pi r_2 l_2 - \pi r_1 l_1 = \pi l (r_1 + r_2)$

$\pi (l_2 - l_1)(r_1 + r_2) = \pi (l_2 r_2 - l_1 r_1 + l_2 r_1 - l_1 r_2)$

$\frac{l_2}{l_1} = \frac{r_2}{r_1} \to l_2 r_1 = l_1 r_2$

$\Rightarrow S = 2\pi l \cdot \frac{r_1 + r_2}{2}$

$\Pi: a = x_0 < x_1 < \dots < x_n = b$

$S_{ок} \sim \sum_{i=1}^{n} 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2} =$

$= 2\pi \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}\right)^2 + 1} \cdot (x_i - x_{i-1}) =$

$\longrightarrow \exists \xi_i \in (x_{i-1}, x_i)$ средна стойност

$= 2\pi \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{(f'(\xi_i))^2 + 1}\, (x_i - x_{i-1}) =$

$= 2\pi \sum_{i=1}^{n} f(\xi_i) \sqrt{(f'(\xi_i))^2 + 1}\, (x_i - x_{i-1}) + 2\pi \sum_{i=1}^{n} \left(\frac{f(x_{i-1}) + f(x_i)}{2} - f(\xi_i)\right) \sqrt{(f'(\xi_i))^2 + 1}\, (x_i - x_{i-1})$

$=$

$= 2\pi \sum_{i=1}^{n} f(\xi_i) \sqrt{(f'(\xi_i))^2 + 1}\, (x_i - x_{i-1}) = 2\pi\, \sigma_{f\sqrt{f'^2+1}} (\Pi, \xi)$

f е гладка $\Rightarrow f\sqrt{(f')^2 + 1}$ е непр $\Rightarrow$ интегруема

$\xrightarrow[d(\Pi) \to 0]{} 2\pi \int_a^b f(x) \sqrt{(f'(x))^2 + 1}\, dx$

а пък $\left| 2\pi \sum_{i=1}^{n} \left(\frac{f(x_{i-1}) + f(x_i)}{2} - f(\xi_i)\right) \sqrt{(f'(\xi_i))^2 + 1}\, (x_i - x_{i-1}) \right|$

f' непр в $[a, b] \Rightarrow$ Вайерщрас

$|f'(x)| \leq M \quad \forall x \in [a, b]$

$\Rightarrow \ldots \leq 2\pi \sqrt{M^2+1} \sum_{i=1}^{n} \left| \frac{f(x_i) + f(x_{i-1})}{2} - f(\xi_i) \right| (x_i - x_{i-1})$

f непр в $[a, b] \xRightarrow{\text{Кантор}} f$ е равн. непр. в $[a, b]$, $\varepsilon > 0$

$\Rightarrow \exists \delta > 0 \quad \forall x', x'' \in [a, b], \; |x' - x''| < \delta : |f(x') - f(x'')| < \varepsilon$

$T_1, \; d(T_1) < \delta \quad \xi_i \in (x_{i-1}, x_i) \rightarrow |x_{i-1} - \xi_i| < \delta$

$\Rightarrow |f(x_{i-1}) - f(\xi_i)| < \varepsilon \qquad |x_i - \xi_i| < \delta$

$|f(x_i) - f(\xi_i)| < \varepsilon$

$\Rightarrow \left| \frac{f(x_i) + f(x_{i-1})}{2} - f(\xi_i) \right| \leq \frac{1}{2} |f(x_i) - f(\xi_i)| + \frac{1}{2} |f(x_{i-1}) - f(\xi_i)|$

$< \varepsilon$

$\Rightarrow \ldots < 2\pi \sqrt{M^2+1} \sum_{i=1}^{n} \varepsilon (x_i - x_{i-1}) = \varepsilon \pi \sqrt{M^2+1} (b-a)$

$\Rightarrow$ клони към 0

$\Rightarrow S_{\text{ок}} = 2\pi \int_a^b f(x) \sqrt{(f'(x))^2 + 1} \, dx$

ок. на тор

$f_1(x) = a + \sqrt{b^2 - x^2}$

$f_2(x) = a - \sqrt{b^2 - x^2}$

$S_{\text{ок}} = 2\pi \int_{-b}^{b} f_1(x) \sqrt{(f_1'(x))^2 + 1} + f_2(x) \sqrt{(f_2'(x))^2 + 1} \, dx =$

$= 2\pi \int_{-b}^{b} (a + \sqrt{b^2 - x^2}) \sqrt{\left(\frac{-2x}{\sqrt{b^2 - x^2}} \right)^2 + 1} + (a - \sqrt{b^2 - x^2}) \sqrt{\left(\frac{-2x}{\sqrt{b^2 - x^2}} \right)^2 + 1} \, dx =$