Chapter 1 - Probability

Q1: Fill in details of proof Theorem 1.8

prove monotone decreasing case

part 1: $B_i \cap B_j = \Phi \quad i \neq j \Rightarrow$ disjoint

$A_1 \subset A_2 \subset \cdots A_n \Rightarrow A_n = \bigcup_{i=1}^{n} A_i$

$\bigcup_{i=1}^{n} B_i = \{\omega \in \Omega : \omega \in A_1 \,\|\, (\omega \in A_2 \,\&\, \omega \notin A_1)$
$\|\, \omega \in A_3 \,\&\, \omega \notin A_2 \,\&\, \omega \notin A_1), \cdots\}$

$= \{\omega \in \Omega : \omega \in A_1 \,\|\, \omega \in A_2 \cdots \|\, \omega \in A_n\}$

$A_1 \subset A_2 \subset A_n \Rightarrow \bigcup_{i=1}^{n} B_i = \{\omega \in \Omega : \omega \in A_n\} = A_n$

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i = \lim_{n \to \infty} A_n = A$

Part 2: monotone decreasing:

$A_1 \supset A_2 \supset \cdots \cdots \supset A_n$

$A = \lim_{n \to \infty} A_n = \bigcap_{i=1}^{\infty} A_i \qquad \lim_{n \to \infty} P(A_n) = P(A)$

$$P(A_n) = P\left(\bigcap_{i=1}^{n} A_i\right) \Rightarrow$$

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} A_i\right) = P(A)$$

Q2: prove statements in equation 1.1

1) $P(\emptyset) = 0$

Using axioms 2 & 3:

$P(\Omega) = 1 \Rightarrow P(\emptyset) = 0$

2) $A \subset B \Rightarrow P(A) \leq P(B)$

define $C = B - A \Rightarrow C$ and A are disjoint

Axiom 3 $\Rightarrow P(B) = P(C) + P(A)$

Axiom 1 $\Rightarrow P(C) \geq 0$

$\Rightarrow P(B) \geq P(A)$

3) $0 \leq P(A) \leq 1$

Axiom 1 $\Rightarrow$ $P(A) \geq 0$

Axiom 2 $\Rightarrow$ $P(\Omega) = 1$

$A \subset \Omega$, using prev part $\Rightarrow$ $P(A) \leq P(\Omega)$

$\Rightarrow$ $P(A) \leq 1$

4) $P(A^c) = 1 - P(A)$

Axiom 1: $P(\Omega) = 1$ $\Rightarrow$ $P(A) + P(A^c) = 1$

5) $A \cap B = \emptyset$ $\Rightarrow$ $P(A \cup B) = P(A) + P(B)$

A and B are disjoint

Axiom 3 $\Rightarrow$ $P(A \cup B) = P(A) + P(B)$

Q3: events $A_1, A_2, \dots$ $\quad B_n = \bigcup_{i=n}^{\infty} A_i \quad C_n = \bigcap_{i=n}^{\infty} A_i$

a) show $B_1 \supset B_2 \supset \dots \quad C_1 \subset C_2 \subset \dots$

$$B_{k+1} = \bigcup_{i=k+1}^{\infty} A_i \qquad B_k = \bigcup_{i=k}^{\infty} A_i \Rightarrow B_k = B_{k+1} \cup A_k \Rightarrow B_k \supset B_{k+1}$$

$C_{k+1} = \bigcap_{i=k+1}^{\infty} A_i$

$C_k = \bigcap_{i=k}^{\infty} A_i \Rightarrow C_k = C_{k+1} \cap A_k \Rightarrow C_k \subset C_{k+1}$

b) $\omega \in \bigcap_{n=1}^{\infty} B_n \iff \omega$ belongs to infinite number of events $A_1, A_2, \ldots$

if $\omega \in \bigcap_{n=1}^{\infty} B_n \Rightarrow \omega \in B_1 \;\&\; B_2 \;\&\; \cdots \;\&\; B_n \;\&\; B_\infty$

$B_n = \bigcup_{i=n}^{\infty} A_i \Rightarrow$ if $\omega \in A_1, \ldots, A_k$ i.e. finite number of A_i

$\Rightarrow \exists\, j$ s.t. $\omega \notin B_j$

$\Rightarrow \omega \notin \bigcap_{n=1}^{\infty} B_n$ contradiction!

$\Rightarrow \omega$ belongs to infinite number of events A_i

if ω belongs to infinte number of events A_i

$\Rightarrow \omega \in B_1 \;\&\; B_2 \;\&\; \cdots B_\infty \Rightarrow \omega \in \bigcap_{n=1}^{\infty} B_n$

c) $\omega \in \bigcup_{n=1}^{\infty} C_n \iff \omega \in A_1, A_2 \ldots$ except finite number

$w \in \bigcup_{n=1}^{\infty} C_n$

$\iff \exists k$ where $w \in C_k$

$\iff w \in \bigcap_{i=k}^{\infty} A_k$

$\iff w \in A_k, A_{k+1}, \dots, A_{\infty}$

Q4: Show $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

$B = \bigcup_{i=1}^{n} A_i$

$B^c = \{w \in \Omega, w \notin B\}$

$\Rightarrow w \notin A_1$ & $w \notin A_2$ ... & $w \notin A_n$

$\Rightarrow B^c = \bigcap_{i=1}^{n} A_i$

Show $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$

$\left(\bigcap_{i=1}^{n} A_i\right)^C = \{w \in \Omega : w \notin A_1 \text{ or } w \notin A_2 \text{ or} \dots \text{ or } w \notin A_n\}$

$= \bigcup_{i=1}^{n} A_i^C$

Index $= I = \{i, j, k, \dots\}$

define $B_j \quad j = 1, \dots, n \qquad$ where $n = \text{size}(I)$

$= A_i : i \in I$

Q5: toss coin until get 2 heads

sample space?

probability that k tosses are required?

$$\Omega = \{(H, H), (T, T, \dots, H, \dots, H), \dots, (\dots, H, \dots, H)\}$$

$$P(n=k) = P(\text{one } H \text{ between } k-1 \text{ tosses}) \times P(\omega_k = H)$$

$$P(\omega_k = H) = 0.5$$

$$P(H, \underbrace{T, T, \dots, T}_{k-2}) = 0.5\,(0.5)^{k-2}$$

$$\Rightarrow P(\text{one } H \text{ in } k-1 \text{ tosses}) = (k-1) \times 0.5^{k-1}$$

$$\Rightarrow P(n=k) = (k-1)\, 0.5^{k-1} \times 0.5$$

$$= (k-1)\, 0.5^{k}$$

Q6: $\Omega = \{0, 1, \dots\}$

prove does not exis a unif. dist. on Ω

let $A_0=\{0\}$, $A_1=\{1\}$, ...

if uniform dist. $\Rightarrow$ $P(A_i)=c$ $\quad i=0,1,\cdots,\infty$

$\Rightarrow P(\Omega)=c\times\infty=\infty\neq 1$ $\Rightarrow$ contradiction

Q7: events $A_1, A_2, \ldots$

show $P\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} P(A_n)$

hints: define $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$

then show $\begin{cases} B_n \text{ are disjoint} \\ \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \end{cases}$

$B_n = A_n - \bigcup_{i=1}^{n-1} A_i = \{\omega\in\Omega : \omega\in A_n, \omega\notin A_{1,\ldots,n-1}\}$

$$B_i \cap B_j = \underbrace{\left(A_i - \bigcup_{k=1}^{i-1} A_k\right)}_{\substack{\omega\in A_i \\ \notin A_1,\ldots,A_{i-1}}} \cap \underbrace{\left(A_j - \bigcup_{k=1}^{j-1} A_k\right)}_{\substack{\omega\in A_j \\ \notin A_1,\ldots,A_{j-1}}}$$

$\Rightarrow B_i \cap B_j = \emptyset$

$\Rightarrow B_n$ are disjoint

$$\begin{cases} B_1 = A_1 \qquad B_2 = A_2 - A_1 \qquad B_3 = A_3 - (A_1 \cup A_2) \\ B_1 \cup B_2 = A_1 \cup A_2 \\ B_1 \cup B_2 \cup B_3 = B_1 \cup B_2 \cup \left[A_3 - (B_1 \cup B_2) \right] \end{cases}$$

proved for $k = 1, 2, 3$

assume it is true for k, prove for $k+1$

assume $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$

$$\Rightarrow \bigcup_{i=1}^{k+1} B_i = \left(\bigcup_{i=1}^{k} B_i \right) \cup B_{k+1}$$
$$= \left(\bigcup_{i=1}^{k} A_i \right) \cup \left(A_{k+1} - \bigcup_{i=1}^{k} A_i \right)$$
$$= \bigcup_{i=1}^{k+1} A_i$$

$\Rightarrow$ proved for case $k+1 \Rightarrow \bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n \right) = P\left(\bigcup_{n=1}^{\infty} B_n \right)$$

$$\left. \begin{array}{l} B_i \text{ disjoint} \Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P(B_n) \\ B_n = A_n - \bigcup_{i=1}^{n-1} A_i \Rightarrow P(B_n) \leq P(A_n) \end{array} \right\} \Rightarrow$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

Q8: if $P(A_i)=1 \quad \forall i$ prove $P(\bigcap_{i=1}^{\infty} A_i)=1$

$$\left.\begin{array}{l} A_i \subseteq \Omega \\ |A_i| = |\Omega| \end{array}\right\} \Rightarrow A_i = \Omega$$

$$\Rightarrow P(\bigcap_{i=1}^{\infty} A_i) = P(\Omega) = 1$$

Q9: $P(B>0)$

Show $P(\cdot|B)$ satisfies prob. axioms

Axiom 1: $P(\cdot|B) = \dfrac{P(\cdot B)}{P(B)} \geq 0$

Axiom 2: $P(\Omega|B) = \dfrac{P(\Omega B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

Axiom 3: if $A_1, A_2, \dots$ disjoint $\rightarrow P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$$P(\bigcup_i A_i|B) \stackrel{?}{=} \sum_i P(A_i|B)$$

$$P(\bigcup_i A_i | B) = P[(\bigcup_i A_i) \cap B] / P(B)$$

$$= P[\bigcup_i (A_i \cap B)] / P(B)$$

$$\overset{A_i \text{ disjoint}}{=} \sum_{i=1}^{\infty} P(A_i \cap B)/P(B)$$

$$= \sum_{i=1}^{\infty} P(A_i | B)$$

Q10: Monty Hall problem

you pick door 1 → monty shows 2 or 3 is empty.

prove that you should switch afte Monty

$\Omega = \{(\omega_1, \omega_2): \omega_i \in \{1,2,3\}\}$ ω_1 prize
ω_2 Monty

$\Omega = \{(1,2),(1,3),(2,3),(3,2)\}$

assumption: we always choose door 1 first if $\omega_1 = 1 \Rightarrow$ Monty chooses 2 or 3 with same prob

$$\Rightarrow P(1,2) = P(1,3)$$

if $\omega_1 = 2$ or $\omega_1 = 3 \Rightarrow$ Monty chooses other door

$$\Rightarrow P(3,2) = P(2,3)$$

lets compare $P(1,2)$ and $P(3,2)$

if $P(3,2) > P(1,2) \Rightarrow$ we should switch

$$P(3,2) = P(\omega_1 = 3 \mid \omega_2 = 2) = \frac{P(\omega_2 = 2 \mid \omega_1 = 3)\, P(\omega_1 = 3)}{P(\omega_2 = 2)}$$

$$= \frac{1 \times 1/3}{1/2} = 2/3$$

$$P(1,2) = P(\omega_1 = 1 \mid \omega_2 = 2) = \frac{P(\omega_2 = 2 \mid \omega_1 = 1)\, P(\omega_1 = 1)}{P(\omega_2 = 2)}$$

$$= \frac{1/2 \times 1/3}{1/2} = 1/3$$

$\Rightarrow$ if $\omega_2 = 2$ $\begin{cases} P(\omega_1 = 3 \mid \omega_2 = 2) = 2/3 \\ P(\omega_1 = 1 \mid \omega_2 = 2) = 1/3 \end{cases}$

if $\omega_2 = 3$ $\begin{cases} P(\omega_1 = 2 \mid \omega_2 = 3) = 2/3 \\ P(\omega_1 = 1 \mid \omega_2 = 3) = 1/3 \end{cases}$

$\Rightarrow$ better to swicth

Q11: show if A & B indep. $\rightarrow$ A^c & B^c are indep.

$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$

also using Venn diagram one can prove that:

$P(A^c \cup B^c) = 1 - P(A \cap B) = 1 - P(A) \cdot P(B)$

(since A & B indep)

$\Rightarrow P(A^c) + P(B^c) - P(A^c \cap B^c) = 1 - P(A) P(B)$

$\Rightarrow P(A^c \cap B^c) = 1 - P(A) + 1 - P(B) - 1 + P(A) P(B)$

$= (1 - P(A))(1 - P(B))$

$= P(A^c) \, P(B^c)$

$\Rightarrow A^c$ & B^c are independent.

Q_{12}: 3 cards : red-red , green-green, red-green

- choos a card at random

see one side at random

$$P(\underbrace{\text{second side} = \text{green}}_{\omega_2} \mid \underbrace{\text{first side green}}_{\omega_1}) = ?$$

$$\Omega = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \{r, g\}\}$$

$$\Omega = \{(r,r), (r,r), (g,g), (g,g), (r,g), (g,r)\}$$

$$P(\omega_2 = g \mid \omega_1 = g) = \frac{P(\omega_2 = g, \omega_1 = g)}{P(\omega_1 = g)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Q_{13}: toss fair coin until one Head one Tail

a) Sample space?

b) probability that 3 tosses required?

a) $$\Omega = \left\{ (\omega_1, \dots, \omega_n) : \omega_i \in \{H, T\},\ n = 2, 3, \dots, \infty \right.$$
$$\left. \omega_n \neq \omega_i \quad i = 1, 2, \dots, n-1 \right\}$$

b) $p(n=k) = p(\omega_{i=1,\dots,k-1} = H, \omega_k = T) + P(\omega_{i=1,\dots,k-1} = T, \omega_k = H)\}$

$$= 2 \times \tfrac{1}{2} \times \left(\tfrac{1}{2}\right)^{k-1} = 2^{1-k}$$

$\Rightarrow p(n=3) = 1/4$

Q14: show if $p(A)=0$ or $1 \rightarrow A$ indep of all events

show if A indep. of itself $\rightarrow P(A)=0$ or 1

- if $p(A)=0$ or $1 \Rightarrow$

assume general event B

$$p(A \cap B) = \begin{cases} P(B) & \text{if } p(A)=1 \\ 0 & \text{if } p(A)=0 \end{cases}$$

$\Rightarrow p(A \cap B) = P(A)\,P(B) \Rightarrow$ A & B indep.

- A indep of A $\Rightarrow P(A) = P(A \cap A) = P(A) \cdot P(A)$

$\Rightarrow P(A) = P(A)^2 \Rightarrow P(A) = 1$ or 0

Q_{15}: prob blue eye = 1/4

family with 3 children

assume indep between children

a) if at least one child has blue eye

prob of at least 2 child blue eyes?

b) if youngest has blue eyes

prob of at least 2 child blue eyes?

$\Omega = \{ (\omega_1, \omega_2, \omega_3) : \omega_i \in \{1, 0\} \}$ 1: blue, 0: not blue

$n = \omega_1 + \omega_2 + \omega_3 \quad , \quad 0 \leqslant n \leqslant 3$

$P(\omega_i = 1) = 1/4 = p$

$P(\omega_i = 0) = 3/4 = q$

a) $P(n \geqslant 2 \mid n \geqslant 1) = \dfrac{P(n \geqslant 2)}{P(n \geqslant 1)} = \dfrac{1 - P(n=0) - P(n=1)}{1 - P(n=0)}$

$P(n=0) = q^3 = 27/64$

$$P(n=1) = 3pq^2 = 27/64$$

$$\Rightarrow P(n \geq 2 \mid n \geq 1) = \frac{1 - \frac{27}{64} - \frac{27}{64}}{1 - \frac{27}{64}} = \frac{10}{37}$$

b) $$P(n \geq 2 \mid w_1 = 1) = \frac{P(w_1 = 1, w_2 + w_3 \geq 1)}{P(w_1 = 1)}$$

$$= P(w_2 + w_3 \geq 1) = 1 - P(w_2 + w_3 = 0)$$

$$= 1 - q^2 = 7/16$$

Q16: prove if A & B indep $\rightarrow P(A \mid B) = P(A)$

prove $\forall A, B$: $P(AB) = P(A \mid B) P(B) = P(B \mid A) P(A)$

- if A & B indep

$$\Rightarrow P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$- \quad P(A|B) = \frac{P(AB)}{P(B)} \Rightarrow P(AB) = P(A|B)\,P(B)$$

$$P(B|A) = \frac{P(AB)}{P(A)} \Rightarrow P(AB) = P(B|A)\,P(A)$$

Q_{17}: Show $P(ABC) = P(A|BC)\,P(B|C)\,P(C)$

$$P(ABC) = P(A|BC)\,P(BC)$$

$$P(BC) = P(B|C)\,P(C)$$

$$\Rightarrow P(ABC) = P(A|BC)\,P(B|C)\,P(C)$$

Q_{18}: A_i are disjoint, $\bigcup_{i=1}^{k} A_i = \Omega$

prove if $P(A_1|B) < P(A_1)$ $\quad P(B) > 0$

$\Rightarrow \quad P(A_i|B) > P(A_i)$ for some $i = 2, \ldots, k$

$$\sum_{i=1}^{K} P(A_i) = \sum_{i=1}^{k} P(A_i|B) = 1$$

$$\Rightarrow P(A_1) + \sum_{i=2}^{k} P(A_i) = P(A_1|B) + \sum_{i=2}^{k} P(A_i|B)$$

$$P(A_1|B) < P(A_1) \Rightarrow \sum_{i=2}^{k} P(A_i|B) > \sum_{i=2}^{k} P(A_i)$$

$$\Rightarrow \exists i \in \{2, \dots k\} \text{ s.t. } P(A_i|B) > P(A_i)$$

Q19: $P(Mac) = 0.3 \quad P(Win) = 0.5 \quad P(Lin) = 0.2$

$P(V|Mac) = 0.65 \quad P(V|Win) = 0.82 \quad P(V|Lin) = 0.5$

$P(Win|V) = ?$

$$P(Win|V) = \frac{P(V|Win)\,P(Win)}{P(V)} = \frac{0.82 \times 0.5}{\sum_x P(V|x)\,P(x)}$$

$$x \in \{Mac, Win, Lin\} = \frac{0.82 \times 0.5}{0.65 \times 0.3 + 0.82 \times 0.5 + 0.5 \times 0.2} = 0.58$$

Q20: $p_1, \dots, p_5$ prob of head for each coin

$p_1 = 0 \quad p_2 = 1/4 \quad p_3 = 1/2 \quad p_4 = 3/4 \quad p_5 = 1$

C_i event that coin i is selected

a) $P(C_i \mid H) = ?$ $\quad i = 1, \dots, 5$

b) $P(H_2 \mid H_1) = ?$ $\quad H_j$ = heads on toss j

c) $P(C_i \mid B_4) = ?$ $\quad B_4$ = first head on toss 4

$P(H \mid C_i) = P_i \quad , \quad \sum_i P_i = \frac{5}{2} \quad , \quad P(C_i) = 1/5$

a) $$P(C_i \mid H) = \frac{P(H \mid C_i)\, P(C_i)}{P(H)} = \frac{P_i\, \cancel{P(C_i)}}{\sum P_i\, \cancel{P(C_i)}} = \frac{P_i}{\sum P_i}$$

$$= \frac{2}{5} P_i$$

b) $$P(H_2 \mid H_1) = \frac{P(H_2 \cap H_1)}{P(H_1)} = \frac{\sum P_i^2\, \cancel{P(C_i)}}{\sum P_i\, \cancel{P(C_i)}}$$

$$= \frac{2}{5} \sum_i P_i^2$$

c) $$P(C_i \mid B_4) = \frac{P(B_4 \mid C_i)\, \cancel{P(C_i)}}{\sum P(B_4 \mid C_i)\, \cancel{P(C_i)}}$$

$$P(B_4 \mid C_i) = (1 - P_i)^3 P_i$$

$$\Rightarrow P(C_i|B_4) = \frac{(1-p_i)^3 p_i}{\sum_j (1-p_j)^3 P(j)}$$

Q21: Computer experiment

find solution in Jupyter notebook.

Q22: Computer experiment

find solution in Jupyter notebook.

Q23: Computer experiment

find solution in Jupyter notebook.