ENSAE
Numerical Methods in Financial Engineering

Tutorial on Finite Difference Methods

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Exercise 1. *We consider the Black-Scholes PDE forward in time and in log stock:*

$$\begin{cases} \partial_t u(t,x) - \frac{\sigma^2}{2}\partial^2_{xx}u(t,x) - \left(r - \frac{\sigma^2}{2}\right)\partial_x u(t,x) + ru(t,x) = 0, & \\ & x \in \mathbb{R}, \quad t \in (0,T] \\ u(0,x) = \Psi(x) \qquad \textit{(initial condition)} & \end{cases}$$

1. *Explicit Euler scheme for a put option:*

   (a) *Give the expression of* Ψ. *The domain is truncated to* $[x_{min}, x_{max}]$. *Give the boundary condition* $t \mapsto \underline{g}(t)$ *at* x_{min} *and* $t \mapsto \overline{g}(t)$ *at* x_{max} *(cf. lecture slides)*

   (b) *Give the expression of the explicit Euler scheme, using centered finite difference to approximate the space derivatives.*

   (c) *Take* $x_{min} = \log(0.6)$, $x_{max} = \log(1.65)$. *Use a discretization with* 100 *intervals in space and* 100 *intervals in time (i.e.* $N = 100$ *and* $M = 100$ *in the lecture slides). Write an implementation of the explicit Euler method in order to calculate. Plot the solution. What do you observe?*

   (d) *Try the previous implementation with* $M = 1000$. *Comment.*

   (e) *Plot the final values* U^M *versus the price of the underlying asset (not the log-price), and compare them with the true values* $(u(T, x_i))$, $i = 1, \ldots, N-1$, *obtained by the Black-Scholes formula. Are the two solutions comparable? Compute the maximum error.*

2. *We repeat the previous questions but this time we use an implementation of the* θ*-method.*

(a) *Test implicit Euler scheme ($\theta = 1$). Compare with the results of the previous part and explain the difference.*

(b) *Test the Crank-Nicolson scheme ($\theta = 0.5$). What are the theoretical differences between the CN scheme and explicit/implicit Euler. Can you observe them numerically (e.g. set $N = 25$ and try different value of M).*

Exercise 2. *We consider the Black-Scholes PDE in the primitive variables, namely*

$$\partial_t u + rx\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} u = ru \text{ on } [0,T) \times \mathbb{R} \tag{0.1}$$

and $u(T,x) = g(x)$.

1. *Write down an implicit Euler scheme, using centered finite difference for the space derivatives.*

2. *Implement the (backward) scheme obtained from the previous question for the put option on $[0, S_{max}]$ where $S_{max} \gg K$.*

Exercise 3. 1. *Show that a tridiagonal matrix $B \in \mathbb{R}^{(N-1)\times(N-1)}$ can be decomposed as $B = \mathcal{L}\mathcal{U}$ with $\mathcal{L}$ lower bi-diagonal and $\mathcal{U}$ upper bi-diagonal.*

$$\underbrace{\begin{bmatrix} a_1 & c_1 & & \\ b_2 & a_2 & c_2 & \\ & b_3 & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & & & \\ \beta_2 & 1 & & \\ & \beta_3 & \ddots & \\ & & \ddots & \ddots \end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} \alpha_1 & c_1 & & \\ & \alpha_2 & c_2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{bmatrix}}_{\mathcal{U}}$$

2. *Suggest an algorithm using the previous decomposition to solve system $Bx = f$.*

3. *Implement the previous algorithm for an implicit Euler scheme.*

Exercise 4. *The goal of this exercise is to approximate an American put option in the Black-Scholes model. From the lectures, we know that its pricing function satisfies the following PDE:*

$$\min\{-\partial_t u - rx\partial_x u - \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} u + ru \ , \ u - g\} = 0 \text{ on } [0,T) \times \mathbb{R}$$
$$u(T,.) = g(.)$$

with $x \mapsto g(x) = (K - x)_+$. We will consider approximation on the truncated domain $[0,T] \times [0, S_{max}]$ for $S_{max} \gg K$. There is M steps in time and N intervals in space. Namely, $\Delta t := \frac{T}{M}$, $h := \frac{S_{max}}{N}$, the time grid is $(t_m = m * \Delta t)_{0 \leqslant m \leqslant M}$ and space grid is $(S_n = n * h)_{0 \leqslant n \leqslant N}$.

1. *We consider first the case of an European option.*

   (a) *Write down an explicit Euler scheme using forward difference for the first order spatial derivative. This is a backward scheme that should read: for* $0 \leqslant m < M$,

   $$U^m = (1 - \Delta t A) U^{m+1},$$

   where $U^m \in \mathbb{R}^N$, $A \in \mathbb{R}^{N \times N}$ *(tridiagonal matrix that you have to find). The terminal condition is* $U_n^M = g(S_{n-1})$ *for* $1 \leqslant n \leqslant N$.

   (b) *Assuming the solution* u *of the pricing PDE is smooth, study the order of convergence of the scheme.*

   (c) *Show that if*

   $$\Delta t \leqslant \frac{h^2}{r + r S_{max} h + \sigma^2 S_{\max}^2}$$

   then the scheme is stable for the norm $|x|_\infty = \max_{1 \leqslant n \leqslant N} |x_n|$.

   (d) *Implement the previous scheme for the parameter value:*

   $$r = 0.015, \quad \sigma = 0.21, \quad K = 1, \quad T = 1 \text{ and } \quad S_{max} = 2.$$

   Set $N = 20$ *and the parameter* M *in order to guarantee stability. Compute the maximum error made at time* 0. *Set* $N = 200$ *and* M *accordingly. Comment on the rate of convergence. Illustrate what happens for a bad choice of* M.

2. *We now study the approximation of the American put.*

   (a) *Write down an explicit Euler scheme for the American option (using the setting of the previous question).*

   (b) *Give a sufficient condition for stability.*

   (c) *Test numerically the scheme.*

   (d) *Write down and implement a splitting method combined with an implicit Euler scheme (using again forward difference). Comment on the stability of the scheme (no proof required)*

   (e) *For both scheme, explain what is the best expected order of convergence (no proof required)*