ENSAE
Numerical Methods in Financial Engineering

Tutorial on Simulation Methods

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Exercise 1 (Black Scholes Model). *In the Black-Scholes model, the stock price S is solution on $[0, T]$ to*

$$S_t = x + \int_0^t rS_s \mathrm{d}s + \int_0^t \sigma S_s \mathrm{d}W_s$$

where W is a one dimensional Brownian motion, $r \geqslant 0$ is the risk free interest rate and $\sigma > 0$ is the volatility parameter. The price of an European option with maturity T and payoff function g is given by $p := e^{-rT}\mathbb{E}[g(S_T)]$.

1. *Recall the expression of S as a function of the Brownian Motion.*
2. *Implement a function returning the price of a call option and a function returning the price of a put option in the Black & Scholes Model (using the celebrated Black&Scholes Formula). Check the correctness of your implementation on various examples.*
3. *Recall the Monte-Carlo estimator of the Call price in the Black & Scholes Model and implement it. Test the estimator for various value of M (number of samples) indicating standard deviation, confidence interval (95% say) and computational time. Compute each ($r = 3\%$, $\sigma = 20\%$, $T = 1.1$, $K = 95$ $S_0 = 100$)*
4. Antithetic estimator. *Let S^- be the solution to*

$$\mathrm{d}S_t^- = rS_t^-\mathrm{d}t - \sigma S_t^-\mathrm{d}W_t \text{ and } S_0^- = x. \quad (0.1)$$

   (a) *Justify why $p = \frac{e^{-rT}}{2}\mathbb{E}\big[\big(g(S_T) + g(S_T^-)\big)\big]$*

   (b) *Deduce from the previous question a new estimator of p and implement it (we shall call it the "antithetic estimator").*

   (c) *Set $r = 0$, $\sigma = 0.3$, $T = 1$ and $S_0 = 100$, test the estimator for*

i. *the price of a put with strike* $K = 100$

ii. *the payoff of a straddle:* $x \mapsto [x-K]_+ + [K-x]_+$

using various number of samples (say $M = 10000$*,* 40000*,* 100000*).*
Comment the empirical results.

(d) *We assume for this question that g is monotonic.*

i. *Show that* $g(S_T) = \phi(W_T)$ *and* $g(S_T^-) = \psi(W_T)$ *for two functions* ϕ *and* ψ *satisfying*

$$(\phi(z) - \phi(z'))(\psi(z) - \psi(z')) \leqslant 0\,, \quad \forall z, z' \in \mathbb{R}.$$

ii. *Let* $\tilde{W}_T$ *a random variable independent from* W_T *but with same law. Show that*

$$\mathbb{E}\Big[(\phi(W_T) - \phi(\tilde{W}_T))(\psi(W_T) - \psi(\tilde{W}_T))\Big] = 2\mathbb{C}\mathrm{ov}[\phi(W_T)\psi(W_T)]$$

iii. *Compare the efficiency of the antithetic estimator and the classical estimator.*

Exercise 2. *We study the Euler Scheme for the Black Scholes model given by :* $\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$ *on* $[0,T]$*,* $T > 0$*.*

1. *Write down the Euler scheme (denoted* S^π*) for an equidistant time grid* π_N *given by*

$$\pi_N = \{0 =: t_0 < \cdots < t_n < \cdots < T\} \text{ and } h = \frac{T}{N}$$

where N is a positive integer.

2. *Study of the strong error :* $\epsilon_S(N) := \mathbb{E}\big[|S_T - S_T^{\pi_N}|\big]$*.*

(a) *Propose an estimator* $\hat{\epsilon}_S$ *of the error* ϵ_S*.*

(b) *We would to like to understand the behavior of* $\epsilon_S(N)$ *with respect to N. Implement the estimator found in the previous question for different value of N. (What should be the number of samples?)*

(c) *Comment your result.*

Exercise 3 (Biased Monte-Carlo Simulation)**.** *We consider X solution of a Lipschitz SDE on* $[0,T]$ *and its associated Euler scheme* X^{π_N} *where the grid* π_N *is given by*

$$\pi_N = \{0 =: t_0 < \cdots < t_n < \cdots < T\} \text{ and } h = \frac{T}{N}$$

with N a positive integer.
The goal is to estimate numerically $p = \mathbb{E}[g(X)]$ *for a given measurable function g such that* $g(X)$ *is integrable using the following estimator:*

$$p_{M,N} := \frac{1}{M}\sum_{m=1}^{M} g((X_T^{\pi_N})^m) \tag{0.2}$$

where $((X_T^{\pi_N})^m)_{1\leqslant m\leqslant M}$ *are i.i.d. sample of* $X_T^{\pi_N}$*.*

To measure the error (and understand the property of the estimator) we use the so called Mean Square Error *(MSE) defined by:*

$$\mathrm{MSE} = \mathbb{E}\big[|p_{M,N} - p|^2\big]. \tag{0.3}$$

The root MSE (rMSE) is the square root of the above quantity: it will be also useful to set the precision of our method.

1. *Show that the MSE is the sum of the variance of the estimator and its bias squarred.*
2. *Observing that the bias is the weak error associated to the Euler scheme, give an estimate of the MSE in terms in* M *and* N*.*
3. *We want to implement the above estimator: what is the optimal choice of the parameters* M *and* N*?*
4. *What is the associated numerical complexity? Compare with the unbiased case.*
5. *Implement and test the method for the Black Scholes Model.*

Exercise 4 (Importance sampling by translation)**.** 1. *Prove the following relation*

$$\forall \theta \in \mathbb{R}^d, \quad \mathbb{E}[f(G+\theta)e^{-\theta.G-\frac{|\theta|^2}{2}}] = \mathbb{E}[f(G)],$$

where G *is a* $\mathcal{N}(0,1)$ *random variable.*

2. *Prove that*

$$\forall \theta \in \mathbb{R}^d, \quad \mathbb{V}\mathrm{ar}\Big[f(G+\theta)e^{-\theta.G-\theta^2/2}\Big] = v(\theta) - \mathbb{E}[f(G)]^2$$

where

$$v(\theta) := \mathbb{E}\left[f^2(G)e^{-\theta.G+|\theta|^2/2}\right].$$

3. *We assume that the two following conditions hold*

$$\mathbb{P}(f^2(G) > 0) > 0, \tag{0.4}$$

$$\forall R > 0, \ \mathbb{E}\left[|G|^2 f^2(G)e^{R|G|}\right] < +\infty. \tag{0.5}$$

   (a) *Prove that* v *is two times continuously differentiable and give the explicit expressions for* Dv *and* D^2v*.*

   (b) *Prove that* v *is strongly convex.* Hint: observe that (0.4) implies the existence of $\varepsilon > 0$ such that $\mathbb{P}(f^2(G) > \varepsilon, |G| \leqslant \frac{1}{\varepsilon}) > 0$.

   (c) *Prove that* $\lim_{|\theta|\to+\infty} v(\theta) = +\infty$.

4. *Using the above questions, we introduce*

$$\mathfrak{m}_M^\theta = \frac{1}{M}\sum_{m=1}^{M} f(G^m+\theta)e^{-\theta.G^m-\frac{|\theta|^2}{2}}. \tag{0.6}$$

Comment on the choice of θ *to estimate* $\mathbb{E}[f(G)]$.

5. *Implementation in the Black-Scholes model:* $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma \mathrm{d}W_t$.

   (a) *We consider a call option with strike* K *and maturity* T. *Give the expression of* f *associated.*

   (b) *We set* $r = 0, \sigma = 0.25, S_0 = 100$ *and* $T = 1$ *and* $K = 150$. *Implement the IS estimator with* $\theta = 2$ *(recall Example 3.5 in the handouts) Comment.*

6. *Optimising* θ *in* $\mathfrak{m}_M^\theta$: *a stochastic Newton-Raphson approach. Set*

$$\begin{aligned}
v_M(\theta) &= M^{-1}\sum_{m=1}^{M} f^2(G^m)e^{-\theta.G^m+|\theta|^2/2}\\
\nabla v_M(\theta) &= \frac{1}{M}\sum_{m=1}^{M}(\theta-G^m)f^2(G^m)e^{-\theta.G^m+\frac{|\theta|^2}{2}},\\
\nabla^2 v_M(\theta) &= \frac{1}{M}\sum_{m=1}^{M}(I_d+(\theta-G^m)(\theta-G^m)^*)f^2(G^m)e^{-\theta.G^m+\frac{|\theta|^2}{2}}.
\end{aligned}$$

   (a) *Explain rapidly how to approximate the optimal* θ *using* v_M.

   (b) *We shall use the following algorithm:*
   Choose the starting point $\theta_0 \in \mathbb{R}^d$. *Set* $p = 0$.
   while $|\nabla v_M(\theta_p)| > \varepsilon$ *do*

$$\begin{aligned}
\theta_{p+1} &= \theta_p - (\nabla^2 v_M(\theta_p))^{-1}\nabla v_M(\theta_p).\\
p &= p+1.
\end{aligned}$$

   end while

   (c) *Implement this procedure in the setting of question 5.b. What is the (approximated) optimal* θ*?* (remark: I would compute a rough estimate of the optimal θ say using 100-1000 samples.) *Try also different value of* K, *comment.*

   (d) *Two-dimensional application: We consider a Best-of-Call option with payoff* $(\max(S_T^1, S_T^2)-K)_+$, $K > 0$. *Both dynamics are given by the Black-Scholes dynamics*

$$\begin{aligned}
S_T^1 &= S_0^1 e^{(r-\frac{\sigma_1^2}{2})T+\sigma W_T^1},\\
S_T^2 &= S_0^2 e^{(r-\frac{\sigma_2^2}{2})T+\sigma W_T^2}.
\end{aligned}$$

   Here W^1 *et* W^2 *are two correlated B.M., that is* $d < W^1, W^2 >_t = \rho\, dt$. *We choose the following parameters:* $T = 1$, $S_0^1 = S_0^2 = 100$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$ $r = 0.05$, $K = 140$. *Redo the questions of the previous section with this new payoff.*

Exercise 5 (Probabilistic numerical method for the Delta). *Let* $(\Omega, \mathcal{A}, \mathbb{P})$ *be a probability space supporting a one dimensional Brownian motion* W. *We denote by* $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$ *the (augmented) natural filtration of Brownian motion. We work in the Black-Scholes model where the stock price* $S^{t,x}$ *is given, for* $(t,x) \in [0,T] \times (0,\infty)$, *by*

$$S^{t,x}_s = x e^{(r - \frac{\sigma^2}{2})(s-t) + \sigma(W_s - W_t)}, \quad s \in [t,T]. \tag{0.7}$$

By convention, we fix $S^{t,x}_s = x$, *for* $s \in [0,t)$. *In the expression above,* $r \geqslant 0$ *is the risk free interest rate,* $\sigma > 0$ *is the volatility and* T *is the maturity time for some European option with measurable payoff function* g. *We then define* $p(t,x) = \mathbb{E}\Big[e^{-r(T-t)} g(S^{t,x}_T)\Big]$, *for* $(t,x) \in [0,T] \times (0,\infty)$, *namely:* p *is the price functional.*

1. *Assume* p *is smooth. Recall the PDE satisfied by* p *and show that*

$$e^{-rT} g(S^{0,x}_T) = p(0,x) + \int_0^T e^{-rs} \partial_x p(s, S^{0,x}_s) \sigma S^{0,x}_s \mathrm{d}W_s. \tag{0.8}$$

2. *Show that* $\partial_x p(0,x) = \mathbb{E}\Big[e^{-rt} \partial_x p(t, S^{0,x}_t) S^{0,1}_t\Big]$, *for* $x \in (0,\infty)$, $t \in [0,T]$.

3. *Setting* $t = T$, *we obtain from the previous question*

$$\partial_x p(0,x) = \mathbb{E}[\Delta^x_T] \quad \textit{with} \quad \Delta^x_T := e^{-rT} g'(S^{0,x}_T) S^{0,1}_T, \tag{0.9}$$

and where g' *is the derivative of* g.

   (a) *Suggest a Monte-Carlo estimator of the Delta using the previous formula.*

   (b) *We consider a Call option with strike* K. *Give a closed-form expression of its Delta: this will serve as a reference value to verify the numerical algorithms.*

   (c) *Implement the MC estimator suggested by* (0.9). *(Justify rapidly why it is OK to use this formula though* g *is not differentiable). Test it with* $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$ *and various values of* $T \in \mathfrak{T} = \{\tau/10, \tau \in \{1,\dots,10\}\}$.

   (d) *Taking the number of samples as large as possible, estimate the variance of* Δ^x_T *for* $T \in \mathfrak{T}$ *(consider* $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$*). Comment.*

4. *In this question, we will prove another probabilistic representation of* $\partial_x p(0,x)$ *assuming that* p *is smooth.*

   (a) *Show that*

$$\mathbb{E}\left[e^{-rT} g(S^{0,x}_T) \frac{W_T}{x\sigma}\right] = \mathbb{E}\left[\int_0^T e^{-rt} \partial_x p(t, S^{0,x}_t) S^{0,1}_t \mathrm{d}t\right].$$

   (b) *Deduce that*

$$\partial_x p(0,x) = \mathbb{E}[H^x_T] \quad \textit{with} \quad H^x_T := e^{-rT} g(S^{0,x}_T) \frac{W_T}{x\sigma T}. \tag{0.10}$$

5. *We admit that* (0.10) *holds true when* g *is Lipschitz continuous only.*

   (a) *Suggest a Monte-Carlo estimator of the Delta using* (0.10). *Test it for a Call with parameters* $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$ *and various values of* $T \in \mathfrak{T}$.

   (b) *Taking the number of samples as large as possible, estimate the variance of* H_T^x *for* $T \in \mathfrak{T}$ *(consider* $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$*). Comment.*

   (c) *Define*

$$\tilde{H}_T^x := e^{-rT}(g(S_T^{0,x}) - g(x))\frac{W_T}{x\sigma T}. \tag{0.11}$$

      i. *Suggest a new Monte-Carlo estimator for the Delta using* (0.11).

      ii. *Test it for a Call with parameters* $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$ *and various values of* $T \in \mathfrak{T}$.

      iii. *Taking the number of samples as large as possible, estimate the variance of* $\tilde{H}_T^x$ *for* $T \in \mathfrak{T}$ *(consider* $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$*). Comment.*