4. Public-key techniques

Reminder - Symmetric vs Asymmetric

Encryption #encryption

- *plaintext* $\implies$ ciphertext
- under key $k_\epsilon \in K$

Decryption #decryption

- ciphertext $\implies$ *plaintext*
- Under key $k_d \in K$

In #symmetric cryptography: $k_E = k_D$ is the **secret key**.
In #asymmetric cryptography: k_E is public and k_D is private.

Authentication #authentication

- message $=>$ (message, **tag**)
- Under key $k_A \in K$

Verification #verification

- (message, **tag**) $=>$ message
- Under key $k_V \in K$

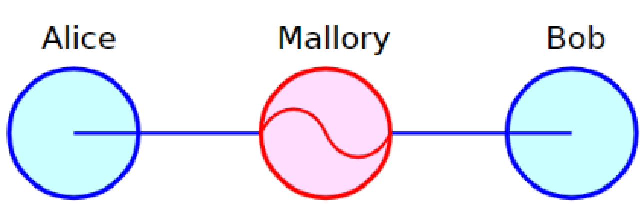
#Symmetric cryptography: $k_A = k_V$ is the secret key. The tag is called a *message authentication code* #MAC .

#Asymmetric cryptography: k_A is private and k_V is public. The tag is called a #signature .

#Symmetric is less versatile but faster. Indeed we can do much more with asymmetric.

4.1 Going public

When going public, several problems arrive. The world is not a goof place and people will try everything to go around the solutions imagined. An example of problem is the #man-in-the-middel-attack . Imagine That *Alice gets Mallory's public key*, thinking it's Bob's and *Bob gets Mallory's public key* thinking it's Alice's. Then **Mallory can decrypt and re-encrypt the traffic at will**!



[By Miraceti on Wikipedia]

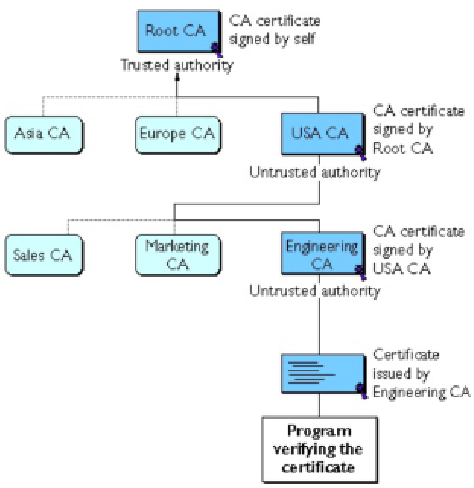
The **binding of a public key to an identify** can be done through #certificates and **public key infrastructure** #PKI or via web of trust (GPG, GnuPG). #TP4-2-ex3a

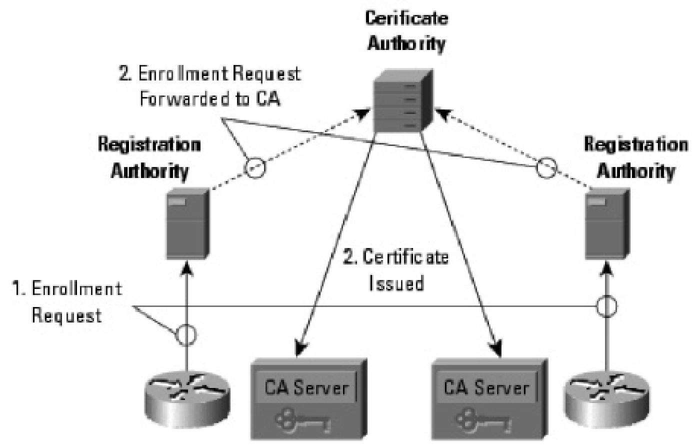
Public key certificate

A #public-key **certificate**:

- *Allows to bind a public key with the identity of its owner*: this is good because it can prevent attacks such as #man-in-the-middel-attack .
- *Contains at least the public key*, the information *allowing to identify its owner* and a *digital signature on the key* and the information.
- Is signed by a **certification authority**.
  Example Slide 7-8. #TP4-2-ex3d

Certificate hierarchy





Web of trust

Manual key verification over an authentic channel

- compare the fingerprint (hash function)
- if correct, sign it
  Public keys are distributed with their signatures.

A key is trusted

- if the key is signed by me, or
- if the key is signed by someone I trust.

Different public-key algorithms

There are multiple #public-key algorithms depending on the problem on which they are based:

- Factorisation
- #discrete-logarithm-problem : **modular exponentiation** or **elliptic curves** (see 4.4).
- Many more...

In practice: Hybrid encryption

In **practice**, to improve both **efficiency and bandwidth**, we can combine #asymmetric with #symmetric #encryption .

Alice computes the encryption c of the message m intended for Bob in the following way:

1. Alice chooses randomly the symmetric key k;
2. Alice computes $(c_k, c_m) = (\mathrm{Enc}_{\text{Bob's public key}}(k), \mathrm{Enc}_k(m))$.

Bob decrypts as follows:

1. Bob recovers $k = \mathrm{Dec}_{\text{Bob's private key}}(c_k)$
2. Bob recovers $m = \mathrm{Dec}_k(c_m)$

Alice sends a **secret key** to Bob encrypted with **his public key**, and she **encrypts the plaintext with her secret key**.

4.2 Mathematical background

4.2.1 Primes

There are **infinitely many** #primes

- Suppose that $p_1 = 2 < p_2 = 3 < \ldots < p_r$ are all the existing primes.
- Let $P = p_1 p_2 \ldots p_r + 1 \implies P$ **cannot be a prime** and then let $p \neq 1$ be one of the existing primes that divides P.
- But p cannot be any of $p_1, p_2, \ldots, p_r$, because otherwise p would divide the difference $P - p_1 p_2 \ldots p_r = 1$, and p would be equal 1.
  $\rightarrow$ So this prime p is still another prime, and $p_1, p_2, \ldots, p_r$ would not be all of the existing prime.

4.2.2 Groups

A #group is a **set** G along with a **binary operation** $\circ$ that *satisfy the following **properties***:

- #closure : $\forall g, h \in G, g \circ h \in G$
- #associativity : $\forall g_1, g_2, g_3 \in G, (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
- #identity (neutre): there exists an identity $e \in G$ such that $\forall g \in G, e \circ g = g \circ e = g$
- #inverse : $\forall g \in G$, there exists $h \in G$ such that $g \circ h = h \circ g = e$
  $\rightarrow$ In easier terms, it is a set in which we can combine elements in a precise way.

Some **examples** of group:

- $(\mathbb{Z}_n, +) \rightarrow$ the set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$
  The law is *addition mod n*
  The neutral/identity is 0
  The inverse of x is $-x \bmod n = n - x$
- $(\mathbb{Z}_p^*, \times) \rightarrow$ the set $\mathbb{Z}_p^* = \{0, 1, 2, \ldots, p-1\}$
  The law is *multiplication mod p*
  Its neutral is 1
  The inverse of x is $x^{-1} \bmod p$
- $(\{0,1\}, \oplus)$ is a group $= (\mathbb{Z}_2^*, +)$
- $(\mathbb{N}, +)$ is **not a group**

Order of a group

Let G be a group, the #order of G, denoted $|G|$ is **the number of elements in** G.
$\rightarrow |\mathbb{Z}_n^*| = n$ and $|\mathbb{Z}_p^*| = p - 1$

Exponentiation

Let $(G, \circ)$ be a group, we define the #exponentiation of an element g as

$$[x]g = g \circ \cdots \circ g$$

with x times g.
Example: In $(\mathbb{Z}_5^*, +) \rightarrow 3[g] = g + g + g$
With $g \in \{1, 2, 3, 4\}$

Order of an element in a finite group

The **order of an element a of a group G** is the smallest positive integer $m = ord(a)$ such that $[m]a = e$, where e denotes the **identity** element of the group.
Example: Let $G = \mathbb{Z}_{10}$ and $a = 2$. The condition is $m \circ a = 0 \bmod 10$. Thus $m = 5 = ord(2)$
#TP4-1-ex7

#Theorem : **For any a its order $ord(a)$ divides the size $|G|$ of the group**.
$\rightarrow$ This tells us that the candidates for i must be dividers of $|G|$. The $ord(a) = i$ such that $a^i \bmod p = 1 \bmod p$ such that i divides $|G|$.

#Fact2 : $ord(a^k) = \frac{ord(a)}{gcd(ord(a),k)}$
$\rightarrow ord(2) = 18 \implies ord(4) = ord(2^2) = \frac{18}{2} = 9$

An element g is said to be #generator of G if $ord(g) = |G|$. Thus (by using the theorem), it can be used to generate all the elements of the group.
Example: In $G = (\mathbb{Z}_5^*, +)$, the generator can be 2 as we can generate all elements of the group with it:
- $1[2] = 2$
- $2[2] = 4$
- $3[2] = 2 + 2 + 2 = 6 \bmod 5 = 1$
- $4[2] = 2 + 2 + 2 + 2 = 8 \bmod 5 = 3$

$\rightarrow$ All numbers of $\mathbb{Z}_5^*$ are generated by 2. So 2 is a #generator .

4.2.3 Operations

4.2.3.1 Modulo

A #modulo is :

- A **binary operation**: $a \bmod n = r$ iff $a = qn + r$ and $0 \leq r < n$
- An **equivalence relation**: $a \equiv b (\bmod n)$
  - iff there exists an integer k such that $(a - b) = kn$
  - iff $(a \bmod n) = (b \bmod n)$

Modular additions and subtractions

$\mathbb{Z}_n, +$ is the set $\{0, 1, \ldots, n-1\}$ together with addition modulo n.
$\mathbb{Z}_n, +$ is a #group with

- #identity is 0
- #inverse of x is $-x \bmod n$

1 is a #generator of the group because all the elements can be represented by adding 1 to itself an appropriate number of times.

4.2.3.2 Greatest common divisor

#gcd or **greatest common divisor** is noted $gcd(a; b)$

Bézout: For any positive integers a and b, there exist integers x and y such that $ax + by = \gcd(a, b)$.

Let $S = \{ax + by \; : \; x, y \in \mathbb{Z}\}$ and $d = \min(S \cap \mathbb{N}_{>0})$
Divide a by d: $a = qd + r$ with $0 \leq r < d$
But $r = a - qd = a - q(ax + by) = (1 - qx)a - (qy)b$ so $r \in S$ and $r < d$ hence $r = 0$ and d divides a
Similarly, d divides b and therefore d divides $\gcd(a, b)$
However $\gcd(a, b)$ divides all elements of S, so $d = \gcd(a, b)$

#corollary : a and b are relatively #primes (or equivalently $gcd(a, b) = 1$) *iff there exist integers x and y such that $ax + by = 1$*

Modular multiplications and multiplicative inversion

$\mathbb{Z}_n^*, \times$ is the set $\{x : 0 < x < n \text{ and } gcd(x, n) = 1\}$ together with multiplication #modulo n.

$\mathbb{Z}_n^*, \times$ is a #group with

- #identity is 1
- #inverse of x is denoted $x^{-1} \bmod n$ and is such that $x^{-1}x = 1 (\text{mod } n)$

The inverse x^{-1} mod n exists and is unique iff $\gcd(x, n) = 1$

- $xy \equiv 1 \pmod n$ means that $xy = 1 + kn$ for some integer k, so $xy + kn = 1$ and $\gcd(x, n) = 1$
- $\gcd(x, n) = 1$ implies that there exist integers y and z such that $yx + zn = 1$, so $xy = 1 - zn$ and $xy \equiv 1 \pmod n$

It is possible to show that the **inverse is unique** under certain conditions (if the number is coprime with the n).

4.2.4 Euler's $\Phi(n)$ function

We note $\Phi(n)$ the **number of integers smaller than n and that are relatively prime/coprime with n**. So $|\mathbb{Z}_n^*| = \Phi(n)$.
If $n = \Pi_{i=1}^r p_i^{e_i}$ for distinct #primes $p_1, p_2, \ldots, p_r$, then

$$\Phi(n) = n * \Pi_{i=1}^r \left(1 - \frac{1}{p_i}\right)$$

Some particular cases
- Prime p: $\Phi(p) = p - 1$
- Product of two distinct primes p and q : $\Phi(pq) = (p - 1)(q - 1)$ #TP4-1-ex5

#Fermats-little-theorem : let p be a prime and a an integer not a multiple of p, then $a^{p-1} = 1 (\text{mod } p)$

#Eulers-theorem : Let a and n two relatively prime integers, then $a^{\Phi(n)} \bmod n = 1 \bmod n$

As a consequence: the **exponent can be reduced** #modulo $\Phi(n)$:

$$a^e \bmod n = a^{e \bmod \Phi(n)} (\text{mod } n) = a^{x \bmod n-1} \bmod n$$

4.2.5 Generator in $\mathbb{Z}_n^*$

The group $\mathbb{Z}_n^*$ has size $\Phi(n)$, hence a #generator g is an element with $ord(g) = \Phi(n)$. Such a generator exists when n is $2, 4, p^a$ or $2p^a$, with p an odd prime.

The **number of generators in** $\mathbb{Z}_p^* = |\mathbb{Z}_{\phi(p)}^*|$ as p is a prime and thus the group is **cyclic**.

Example: consider $\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$.

- $\text{ord}(1) = 1 : 1^1 = 1$
- $\text{ord}(2) = 3 : 2^1 = 2, 2^2 = 4, 2^3 = 1$
- $\text{ord}(\mathbf{3}) = 6 : 3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5, 3^6 = 1$
- $\text{ord}(4) = 3 : 4^1 = 4, 4^2 = 2, 4^3 = 1$
- $\text{ord}(\mathbf{5}) = 6 : 5^1 = 5, 5^2 = 4, 5^3 = 6, 5^4 = 2, 5^5 = 3, 5^6 = 1$
- $\text{ord}(6) = 2 : 6^1 = 6, 6^2 = 1$

4.2.6 Chinese remainder theorem #CRT

The #CRT or **Chinese remainder theorem**:
Let $\{m_1, \ldots, m_k\}$ be a set of relatively prime integers, i.e., $\forall 1 \leq i \neq j \leq k, gcd(m_i, m_j) = 1$ and let m be their product $m = m_1 \times \cdots \times m_k$.
Then the following system of equations:

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ \vdots \\ x \equiv a_k \pmod{m_k} \end{cases}$$

has one and only one solution #modulo m.

To **solve a** #CRT **problem**:
Compute $M_i = \frac{m}{m_i}$, and notice that
- $\gcd(M_i, m_i) = 1$.
- $M_i \equiv 0 \pmod{m_j}$ for any $j \neq i$.
Compute $c_i = M_i^{-1} \pmod{m_i}$.
Let $x = \sum_{i=1}^k a_i c_i M_i$.

Indeed, $x \equiv a_i c_i M_i \equiv a_i \pmod{m_i}$.

4.3 Rivest-Shamir-Adleman #RSA

4.3.1 RSA key generation

In #RSA #key-generation , the user **generates a public-private** key pair as follows:

- **Privately** generate two large distinct primes p and q
- Choose a **public** exponent $3 \leq e \leq (p-1)(q-1) - 3$
  - It must satisfy $gcd(e, (p-1)(q-1)) = 1$
  - often, one chooses $e \in \{2, 17, 2^{16} + 1\}$ then generates the primes
- Compute the **private** exponent $d = e^{-1} \bmod (p-1)(q-1)$
- Compute the **public modulus** $n = pq$ and discard p and q

The #public-key is: (n, e).
The #private-key is (n, d).

Keygen is composed of 5 steps and takes a random value λ

1) Find p, q, two $\neq$ primes in $[2^\lambda, 2^{\lambda+1}-1]$

For $2 \leq i \leq \lceil \sqrt{n} \rceil$

if $n = 0 \bmod i$ → return False

Return true

↳ There are other algorithms used to find them (Miller-Rabin, ...)

2) Set $n = pq$ and $\phi(n) = (p-1)(q-1)$

use Euler's totient function $\phi(N) =$ "# integers $\in [1, N-1]$ that are coprime with N"

↳ a, b are coprime $\iff \gcd(a,b) = 1 \iff$ no shared prime factor

↳ $\phi(10) = 4$ $\quad 1, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}, 7, \cancel{8}, 9$

3) Pick a value for e s.t. $\gcd(e, \phi(n)) = 1$

↳ We use Euclid's algo $\begin{cases} \gcd(a,0) = a \\ \gcd(a,b) = \gcd(b, a \bmod b) \end{cases}$

$$\gcd(140, 13) \iff \gcd(13, 140 \bmod 13) = \gcd(13, 10)$$
$$\iff \gcd(10, 13 \bmod 10) = \gcd(10, 3)$$
$$\iff \gcd(3, 10 \bmod 3) = \gcd(3, 1)$$
$$\iff \gcd(1, 3 \bmod 1) = \gcd(1, 0) = 1$$

↳ if $\phi(n) = 13$ → e could be 140 as they are coprime

4) Compute $d = e^{-1} \bmod \phi(n)$

↳ We use Euclid's extended algorithm

$ed = 1 \bmod \phi(n)$

$d = 29^{-1} \bmod 139$

$\Rightarrow 139.(-5) + 29.(24) = 1$

$\Rightarrow (139.(-5) + 29.(24)) \bmod 139 = 1 \bmod 139$

$\Rightarrow 29.24 = 1 \bmod 139$

$\Rightarrow 24 = 29^{-1} \bmod 139$

$\Rightarrow \boxed{d = 24}$

139 U + 29 V	U	V	
139	1	0	L1
29	0	1	L2
139−4.29 = 23	1	−4	$L3 = L_1 - 4L_2$
29 − 23 = 6	−1	5	L4 = L2 − L3
23 − 3.6 = 5	4	−19	$L5 = L_3 - 3L_4$
1	−5	24	L6 = L4 − L3

5) Return (secret, public) key pair → ((N, d), (N, e))

4.3.2 RSA textbook encryption - DONT USE

#RSA From **plaintext** $m \in \mathbb{Z}_n$ to **ciphertext** $c \in \mathbb{Z}_n$ and back:

- #encryption $Enc(m, (n, e)) \to c = m^e \bmod n$, return c
- #decryption $Dec(c, (n, d)) \to m' = c^d \bmod n$, return m'

Verification

Thanks to #Eulers-theorem , the fact that p and q are distinct #primes and by definition of e and d:

$$\begin{aligned} c^d &\equiv (m^e)^d \equiv m^{ed} \\ &\equiv m^{ed \bmod \Phi(n)} && \text{by Euler's theorem} \\ &\equiv m^{ed \bmod (p-1)(q-1)} && \text{since } p \text{ and } q \text{ are distinct primes} \\ &\equiv m^1 && \text{by definition of } e \text{ and } d \\ &\equiv m \pmod n \end{aligned}$$

RSA textbook signature

#RSA From **plaintext** $m \in \mathbb{Z}_n$ to **signatures** $s \in \mathbb{Z}_n$ and back:

- #signature Sends (m, s), with $s = m^d \bmod n$
- #verification Check whether $m = s^e \bmod n$

  $\to$ Not secure because if another pair $(m', s') = (m \cdot c^e, s \cdot c)$ is also accepted as is uses the fact that $ed = 1 \to (s')^e = (s \times c)^e = s^e \times c^e = m^{ed} \times c^e = m \times c^e = m'$

  #TP4-2-ex6

4.3.3 RSA Attacks

4.3.3.1 RSA and factorisation

#factorisation is an #attack on #RSA , and RSA textbook is not really resistant to it:
If one can factor n into $p \times q$, the *private exponent* d **follows immediately**.

Conversely, **from the knowledge of** d, it is easy to factor n.

$\implies$ This is one reason why we should NOT use textbook #RSA .

Proof: $ed \equiv 1 \pmod{\Phi(n)}$, so for any $a \in \mathbb{Z}_n^*$ we have $a^{ed-1} \equiv 1 \pmod{n}$. As $\Phi(n)$ is even, so is $ed - 1 = 2t$ and $a^{2t} \equiv 1 \pmod{n}$.

Define $z = a^t \bmod n$, so that $z^2 \equiv 1 \pmod{n}$. Assume $z \bmod n \neq \pm 1$ (otherwise change a). Hence n divides $z^2 - 1 = (z-1)(z+1)$.

Let $g_1 = \gcd(z-1, n)$ and $g_2 = \gcd(z+1, n)$. $g_1 \neq n \neq g_2$, as we assumed $z \bmod n \neq \pm 1$. Both g_1 and g_2 cannot be equal to 1 since $z^2 - 1$ is a multiple of n.

Hence g_1 or g_2 is p or q.

There is **no known polynomial time algorithm to factor an integer** (polynomial in the size of the integer).

But **there exist sub exponential algorithms**. The currently best known algorithm (general number field sieve) factors an integer n asymptotically in time:

$$\exp\left(\left(\sqrt[3]{\frac{64}{9}} + o(1)\right)(\ln n)^{\frac{1}{3}}(\ln \ln n)^{\frac{2}{3}}\right).$$

For 128-bit security, NIST recommends n to be at least 3072-bit long (hence p and q are at least 1536-bit long).
[NIST SP 800-57, see also `https://www.keylength.com/`]

4.3.3.2 Other attacks on RSA

There are other #attack such as **cyclic attack**, **message factorisation** or **short message attack**.

#short-message-attack consists in the following:
If e is small (e.g., $e = 3$), and if $m < n^{1/e}$ then $m^e \bmod n = m^e$ *without any modular reduction*. Therefore m **is retrieved** by simply computing $c^{1/e}$ over the integers.

Example: with e=3 and n on 3072 bits, the attack works until m is a 1024-bit integer.

4.3.4 RSA for confidentiality

4.3.4.1 RSA-OAEP

#RSA #OAEP or **Optimal Asymmetric Encryption Padding** allows to **counter** #factorisation and the #short-message-attack and to make RSA #IND-CPA-secure .

#OAEP works by **encrypting a message of small size**, **use randomness** in the encryption and then apply the output of #OAEP as #RSA input.

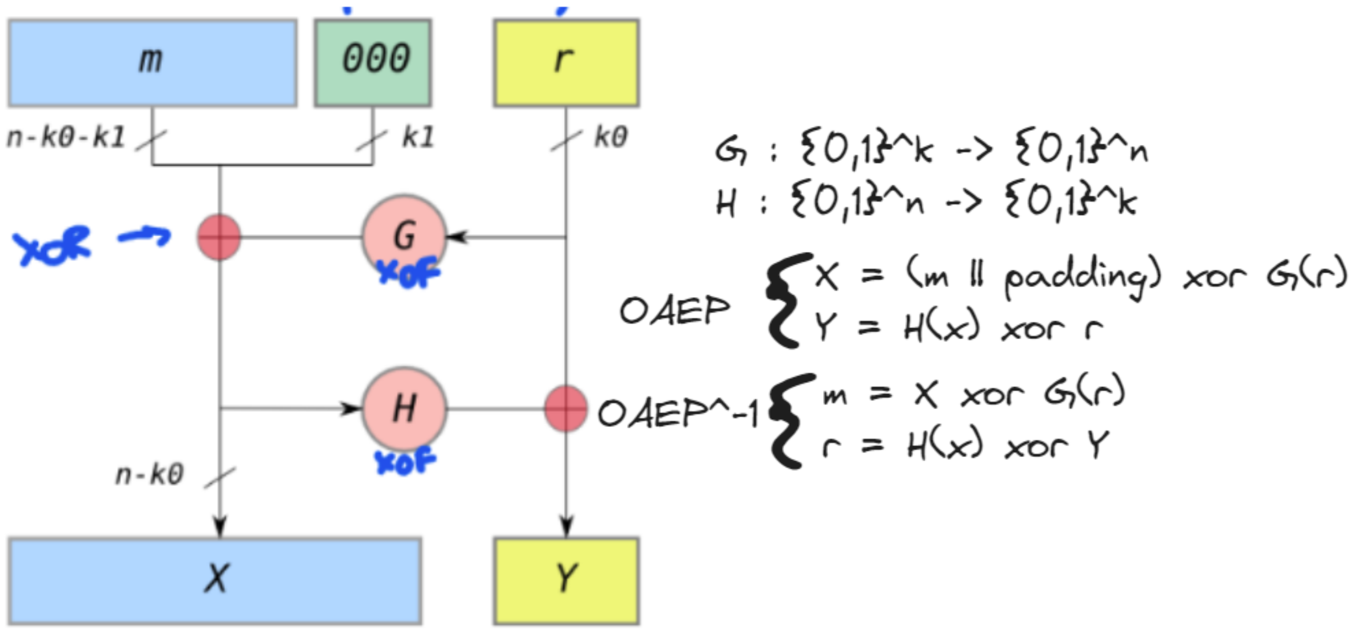
It allows to make the scheme **probabilistic**, **prevents** that **an adversary recovers any portion of the plaintext without being able to invert RSA** (the adversary can either have everything or nothing).

It is shown to be #IND-CPA-secure assuming that G and H behave as #random-oracle .

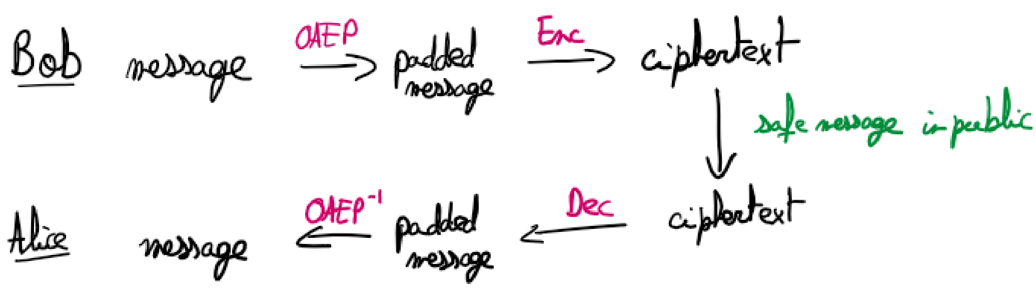
#encryption : $Enc(m, r, (N, e)) = (X||Y)^e \bmod n$ with $(X||Y) = OAEP(m, r)$
- Add a padding full of 0 to the message to have a fix length $m||padding$
- Message m of size n max

- #nonce r of size k

#decryption : $(X||Y) = c^d \bmod n$, then $\ldots m$



Note that it is similar to #feistel-network .



It is applied like this to RSA.

4.3.4.2 RSA-KEM

#RSA-KEM

In hybrid encryption, **Alice does not have to choose the secret key**, but **she can let it be derived from some random bits** by encapsulating the symmetric key using RSA. It enables **shared key creation**.

To #encapsulate , Alice does the following

1. Alice chooses a random m of the same bit size as n
2. Alice encrypts $c = m^e \bmod n$, and she sends c to Bob
3. Alice computes $k = hash(m)$

To #decapsulate , Bob does the following

1. Bob recovers $m = c^d \bmod n$
2. Bob computes $k = hash(m)$

4.3.5 RSA for authenticity

The different options always uses #RSA but there are some modifications made on the message m that is encrypted/signed by RSA.

RSA: How NOT to sign

For #authentication , let's recall the #RSA textbook signature.

- **Signature**: Send (m, s), with $s = m^d \bmod n$
- **Verification**: Check whether $m = s^e \bmod n$

But as said earlier, the textbook signature is **not efficient** as a #forgery attack that exploit the multiplicative structure allows to crack this:

Forgery attack: (exploiting the multiplicative structure)

- Ask for the signature of m_1: $s_1 = m_1^d \bmod n$
- Ask for the signature of m_2: $s_2 = m_2^d \bmod n$
- Compute $m_3 = m_1 \times m_2 \bmod n$ and $s_3 = s_1 \times s_2 \bmod n$
- Submit (m_3, s_3) for verification:

$$\begin{aligned} m_3 &\equiv m_1 m_2 \\ &\equiv s_1^e s_2^e \\ &\equiv s_3^e \pmod n \end{aligned}$$

RSA with message recovery

If the **message** m is **short enough**, we can **embed it in the signature**.
Let $R(m) \to m'$ be a redundancy function from message space M to range $\mathcal{R} \subset \mathbb{Z}_n$.

- **Signature**: Compute $m' = R(m)$ then send $s = (m')^d \bmod n$
- **Verification**:
  - Recover $m' = s^e \bmod n$
  - If $m' \in \mathcal{R}$, return $m = R^{-1}(m')$ and accept it.
  - Otherwise, reject it.

But this is not used much in practice.

RSA with full-domain hashing

Let H be an extendable output function (or take an old-style hash function and use MGF1).

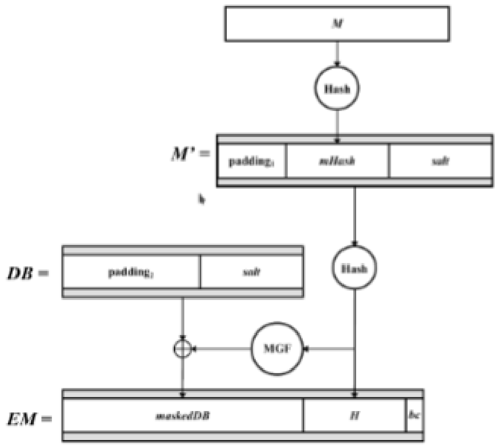
- **Signature**:
  - Compute $h = H(m)$ so that h has the same bit size as n
  - Send (m, s), with $s = h^d \bmod n$
- **Verification**:
  - Compute $h = H(m)$ like above
  - Check whether $h = s^e \bmod n$

It is also not used a lot but it is real nice and easy to achieve.

RSA with probabilistic signature scheme

#RSA-PSS is a #signature scheme used with #RSA , it utilises a probabilistic padding scheme to provide a better resistance against certain types of attacks.

It allows the use of variable *salt* length, and it is very much used (standardised for PKCS #1 v2.1).



[PKCS #1 v2.2] [Bellare and Rogaway]

Signature:

- From m and random $salt$, compute EM
- Send (m, s), with $s = (EM)^d \bmod n$

Verification:

- Recover $EM = s^e \bmod n$
- Recover $salt$
- Check $H \stackrel{?}{=} \text{hash}(M')$

4.3.6 RSA implementation

This section explains how to compute the RSA (manually or with a computer).

How to find large primes

#RSA needs **large primes** to be efficient. Indeed, the security of RSA relies on the difficulty of factoring the product of two large prime numbers.

The number of primes up to n is given by $\pi(n) \sim \frac{n}{\ln n}$ for large n.

$\longrightarrow$The average prime gap for numbers of b bits is about $b \ln 2$.

The **recipe**:

- Draw a random number n
- Test co-primality with the first few primes $2, 3, 5, \ldots$
- Test **pseudo**-primality with (e.g. Miller-Rabin)
- Otherwise, increment n and repeat
  This method has been done and used in the exercise sessions.

Exponentiation using square and multiply

To compute $a^e \bmod n$, write the exponent in binary and apply the **square and multiply** *algorithm*. Reduce the numbers modulo n as you compute.
Here is an example for $e = 26 = 11010_2$

a	initialization 1
$a \rightarrow a^2 \times a = a^3$	square and multiply 1
$a^3 \rightarrow (a^3)^2 = a^6$	square 0
$a^6 \rightarrow (a^6)^2 \times a = a^{13}$	square and multiply 1
$a^{13} \rightarrow (a^{13})^2 = a^{26}$	square 0

The 1's $\longrightarrow$ square and multiply
The 0's $\longrightarrow$ just square.

Optimisation using the CRT

To speed up the decryption or signature generation, keep p and q and use the #CRT Chinese Remainder Theorem. It allows computing to a smaller power, and thus faster.

Instead of computing $m = c^d \bmod n$, compute:

- $m_p = c^d \bmod p = c^{d \bmod (p-1)} \bmod p$
- $m_q = c^d \bmod q = c^{d \bmod (q-1)} \bmod q$

Then recombine:

$$m = (m_p - m_q)(p^{-1} \bmod q)p + m_p$$

4.4 Discrete logarithm problem in $\mathbb{Z}_p^*$

#discrete-logarithm-problem #DLP

The #DLP is a problem that has currently **no known polynomial time algorithm to solve it**. It is therefore used for encryption and signatures as we will see here bellow.

First, let's **fix two domain parameters**:

- Let p be a *large* #prime
- Let g be a #generator of $\mathbb{Z}_p^*$ (for example: $\{g^i \bmod p : i \in \mathbb{N}\} = \mathbb{Z}_p^*$)

> **Discrete logarithm problem (DLP)**
>
> Given $A = g^a \bmod p$, find a.

For 128-bit security, NIST recommends p to be at least 3072-bit long.

4.4.1 Key generation

#key-generation #DLP

The **domain parameters**

- Let p be a *large* #prime
- Let g be a #generator of $\mathbb{Z}_p^*$ (for example: $\{g^i \bmod p : i \in \mathbb{N}\} = \mathbb{Z}_p^*$)

The user generates a #public #private #key-pair as follows:

- The **private key** a is made by privately choose a random integer $a \in [1, p-2]$
- Compute the **public key** $A = g^a \bmod p$

In group notation

The **domain parameters**

- Let G be #group
- Let $g \in G$ be a #generator of G of order $q = |G|$

The user generates a #public #private #key-pair as follows:

- The **private key** a is made by privately choose a random integer $a \in [1, q-1]$
- Compute the **public key** $A = [a]g$

The **public key** is A and the **private key** is a.

4.4.2 El Gamal encryption

Taher ElGamal

The objective of #ElGamal is to use the #DLP to secure its methods.

#encryption of $m \in \mathbb{Z}_p^*$ with Alice's public key A.

- Choose randomly an integer $k \in [1, p-2]$
- Compute:
  - $K = g^k \bmod p$
  - $c = mA^k \bmod p$

#decryption of (K, c) by Alice with her private key $a \rightarrow m = K^{-a} c \bmod p$

A, B and K are all *public*, the **rest** is *private*.

This scheme is #correct because we can recover m by applying encryption and decryption when we fix the parameters.

$$K^{-a} \equiv (g^k)^{-a} \equiv (g^a)^{-k} \equiv (A^k)^{-1} \pmod p$$
$$(A^k)^{-1} c \equiv (A^k)^{-1} m (A^k) \equiv m \pmod p$$

In group notation

Key generation

1) $p \xleftarrow{\$} [2^\lambda, 2^{\lambda+1}]$
2) $a \xleftarrow{\$} \mathbb{Z}_p^*$
3) $g \xleftarrow{\$} \mathbb{Z}_p^*$
4) $A \leftarrow g^a \bmod p$
5) return $(sk, pk) = (a, A)$

Key generation (group)

1) $a \xleftarrow{\$} \mathbb{Z}_q \setminus \{0\}$ with $q \in \mathbb{G}$
2) $A \leftarrow [a]g$
3) return $(sk, pk) = (a, A)$

El Gammal encryption

1) $k \xleftarrow{\$} \mathbb{Z}_p^*$
2) $K \leftarrow g^k$
3) $c \leftarrow mA^k \bmod p$
4) return (c, K)

Decryption $m = cK^{-a} \bmod p$

ElGammal encryption (group)

1) $k \xleftarrow{\$} \mathbb{Z}_p^*$
2) $K \leftarrow [k]g$
3) $C \leftarrow m + [k]A$
4) return (c, K)

$\hookrightarrow m = c + [-a]K$

Security of El Gamal encryption

(K, k) can be seen as an #ephemeral #key-pair, created by the sender.

- K is part of the ciphertext;
- k is protected by the #DLP.
  In order to have a **secure ElGamal** encryption, k must be **secret** and **randomly drawn independently at each encryption**!
  Indeed, if
- k is known, one can compute A^k and recover m from c

- k is repeated to encrypt, say m_1 and m_2 then we have

$$c_1 = m_1 A^k \bmod p \text{ and } c_2 = m_2 A^k \bmod p$$

And thus

$$c_1 c_2^{-1} = m_1 m_2^{-1} \pmod{p}$$

The Diffie-Hellman problem

The Diffie-Hellman Problem #DHP is easier than the #DLP (breaking #DHP does not give us the exponents). However, there are currently **no known polynomial time algorithm** to solve the #DHP either.

First, let's **fix two domain parameters**:

- Let p be a *large* #prime
- Let g be a #generator of $\mathbb{Z}_p^*$ (for example: $\{g^i \bmod p : i \in \mathbb{N}\} = \mathbb{Z}_p^*$)

> **Diffie-Hellman problem**
>
> Given $X = g^x \bmod p$ and $Y = g^y \bmod p$, find $X^y = Y^x = g^{xy} \bmod p$.

To **recover** k or a from K or A $\implies$ #DLP

But to break ElGamal, it is sufficient to recover $K^a = A^k = g^{ak} \bmod p$. Hence, #ElGamal #encryption relies on the #DHP .

4.4.3 Diffie-Hellman key agreement

#Diffie-Hellman
This is **hybrid encryption**, we *do not need to choose the secret key*.
First, let's **fix two domain parameters** p and g and:

- Alice's key pair $A = g^a \bmod p$
- Bob's key pair $B = g^b \bmod p$

$$\begin{array}{cc} \text{Alice computes} & \text{Bob computes} \\ K_{AB} = B^a \bmod p & K_{AB} = A^b \bmod p \\ k_{AB} = \text{hash}(K_{AB}) & k_{AB} = \text{hash}(K_{AB}) \end{array}$$

$$\leftrightarrow$$

$$\text{Secure channel using } k_{AB} \text{ in a symmetric cipher}$$

This is #correct because: $A^b = g^{ab} = B^a \pmod{p}$.

Note that the hashing is to have the same size and to avoid it to be uniformly distributed.

In group notation

#Diffie-Hellman in #group notation.
First, let's **fix two domain parameters**: a #group G and a #generator $g \in G$

- Alice's key pair $A = [a]g$
- Bob's key pair $B = [b]g$

$$\begin{array}{cc} \text{Alice computes} & \text{Bob computes} \\ K_{AB} = [a]B & K_{AB} = [b]A \\ k_{AB} = \text{hash}(K_{AB}) & k_{AB} = \text{hash}(K_{AB}) \\ & \leftrightarrow \\ & \text{Secure channel using } k_{AB} \text{ in a symmetric cipher} \end{array}$$

This is #correct because $[b]A = [ab]g = [a]B$

The **problem** with this is that **when they communicate once**, both **Alice and Bob have the same secret key**. Therefore, a more secure encryption scheme would counter that.

Ephemeral Diffie-Hellman key agreement

#ephemeral #Diffie-Hellman key agreement has for goal to **avoid using the same long-term keys** for all communications.

First, let's **fix two domain parameters** p and g and:

- Alice's key pair $A = g^a \bmod p$
- Bob's key pair $B = g^b \bmod p$

$$\begin{array}{ccc} \text{Alice} & & \text{Bob} \\ \text{randomly chooses } e & & \text{randomly chooses } f \\ \text{sends } E = g^e \bmod p & \leftrightarrow & \text{sends } F = g^f \bmod p \\ \text{along with sign}_a(E) & \leftrightarrow & \text{along with sign}_b(F) \\ \text{checks Bob's signature with } B & & \text{checks Alice's signature with } A \\ K_{AB} = F^e \bmod p & & K_{AB} = E^f \bmod p \\ k_{AB} = \text{hash}(K_{AB}) & & k_{AB} = \text{hash}(K_{AB}) \\ & \leftrightarrow & \\ & \text{Secure channel using } k_{AB} \text{ in a symmetric cipher} & \end{array}$$

Note that the signature exists to make sure that the keys E and F that are sent (public) comes from Alice and Bob. Otherwise anyone could send a public key and this would ruin the key agreement.

4.4.4 El Gamal signature

The #ElGamal #signature of message $m \in \mathbb{Z}_2^*$ by Alice with her private key a:

- Compute $h = hash(m)$
- Choose randomly an integer $k \in [1, p-2]$
- Compute $r = g^k \bmod p$
- Compute $s = k^{-1}(h - ar) \bmod (p-1)$
  - If $s = 0$, restart with a new k
- Send (r, s) along with m.

The #verification of signature (r, s) on m with Alice's public key A.

- Compute $h = hash(m)$
- Check whether $A^r r^s = g^h \pmod{\text{p}}$

This is #correct because:

$$
\begin{aligned}
A^r r^s &\equiv (g^a)^r (g^k)^s \\
&\equiv g^{ar+ks \bmod (p-1)} \\
&\equiv g^{ar+kk^{-1}(h-ar) \bmod (p-1)} \\
&\equiv g^{ar+h-ar \bmod (p-1)} \\
&\equiv g^h \pmod p
\end{aligned}
$$

use

This is done thanks to #Eulers-theorem .

Security of ElGamal signature

r and k can be seen as an #ephemeral #key-pair , created by the signer.

- r is part of the signature
- k is protected by the #DLP

But we must be careful because k **must be secret and randomly drawn independently at each encryption** (Even more than with ElGamal Encryption):

- If k is known, one can recover a from s.
- If k is repeated to sign messages with hashes $h_1 \neq h_2$ then

  $$s_1 - s_2 = k^{-1}(h_1 - ar) - k^{-1}(h_2 - ar) = k^{-1}(h_1 - h_2) \bmod \text{(p-1)}$$

  and we can recover k, then a from s_1 or s_2!

A famous case is SONY that used the same k to sign all their games, so one user recovered k and K and sold a lot of games with it.

To test it in #EC $\rightarrow$ #TP4-2-ex14

4.4.5 Schnorr signature

#Schnorr #signature of a message $m \in \mathbb{Z}_2^*$ by Alice with her private key a:

- Choose randomly an integer $k \in [1, p-2]$
- Compute $r = g^k \bmod p$
- Compute $e = hash(r||m)$
- Compute $s = k - ea \bmod (p-1)$
- Send (s, e) along with m.

#verification of signature (s, e) on m with Alice's public key A:

- Compute $r' = g^s A^e \bmod p$
- Compute $e' = hash(r'||m)$
- Check whether $e' = e$

It is different than #ElGamal as here, e is computed with m and r.

Schnorr signature

Sign(m, a)
1) $k \xleftarrow{\$} \mathbb{Z}_p^* \mid k \in [1, p-2]$
2) $K \leftarrow g^k \bmod p$
3) $e \leftarrow H(K \| m)$
4) $S \leftarrow K - ea \bmod (p-1)$
5) return (S, e)

Verify(m, S, e, A)
1) $K' \leftarrow g^S A^e \bmod p$
2) $e' \leftarrow H(K' \| m)$
3) return $e == e'$

El Gammal Signature

Sign(m, a)
1) $h = H(m)$ in $\mathbb{Z}_p^*$
2) $k \xleftarrow{\$} \mathbb{Z}_p^*$
3) $K \leftarrow g^k \bmod p$
4) $S \leftarrow K^{-1}(h - aK) \bmod (p-1)$
5) return (s, K)

Verify(m, K, S, A)
1) $x \leftarrow A^K K^S$
2) $h' \leftarrow H(m)$
3) return $x == g^{h'} \bmod p$

Security of Schnorr Signature

k **must be different for each signature**.
If the k is reused for two different signatures, then it is possible to retrieve the secret key a.
$s_1 = k_1 - e_1 a$ and $s_2 = k_2 - e_2 a$ with $k_1 = k_2 = k$
$\rightarrow s_1 - s_2 = a(e_2 - e_1)$
$\rightarrow a = (e_1 - e_2)^{-1}(s_2 - s_1) \bmod q$
And thus we retrieve the private key a.
Note that the annoying case where $e_1 = e_2$ which is really rare as the hashing scheme is supposed to **collision resistant**.

4.5 Security of the discrete logarithm problem

Discrete logarithm problem (DLP) in $\mathbb{Z}_p^*$

Given $A = g^a \bmod p$, find a.

The #DLP is not easy to solve but let's take a look at how to solve it.

4.5.1 Solving DLP generically: baby-step giant-step

Let $N = |\mathbb{Z}_p^*|$ and g be a generator of $\mathbb{Z}_p^*$.

- Let $m \approx \sqrt{N}$ and suppose that $a = a_0 + a_1 m$ with $a_0, a_1 < m$
- $A \equiv g^{a_0 + a_1 m} \pmod p \quad \Leftrightarrow \quad Ag^{-a_1 m} \equiv g^{a_0} \pmod p$
- For $i = 0$ to $m - 1$ sequentially (baby steps)
  - Compute and store (i, g^i), i.e., multiply by g at each step
- For $j = 0$ to $m - 1$ sequentially (giant steps)
  - Compute Ag^{-jm}, i.e., multiply by g^{-m} at each step
  - If $Ag^{-jm} = g^i$ then $a = i + jm$ and **exit**

The **baby steps** is to compute and store all exponential values of g.

The **giant steps** is to compute Ag^{-jm} and compare it with the previously stored values.

$\implies$ This takes a complexity of $O(\sqrt{N})$ in **time and memory**

In group notation

Discrete logarithm problem (DLP) for any group G

Given $A = [a]g$, find a.

Let $N = |G|$ and g be a generator of G.

- Let $m \approx \sqrt{N}$ and suppose that $a = a_0 + a_1 m$ with $a_0, a_1 < m$
- $A = [a_0 + a_1 m]g \quad \Leftrightarrow \quad A \circ [-a_1 m]g = [a_0]g$
- For $i = 0$ to $m - 1$ sequentially (baby steps)
  - Compute and store $(i, [i]g)$, i.e., apply $\circ g$ at each step
- For $j = 0$ to $m - 1$ sequentially (giant steps)
  - Compute $A \circ [-jm]g$, i.e., apply $\circ[-m]g$ at each step
  - If $A \circ [-jm]g = [i]g$ then $a = i + jm$ and **exit**

$\implies$ This takes a complexity of $O(\sqrt{N})$ in **time and memory**

4.5.2 Pohlig-Hellman

Let the size of the group be decomposed in prime factors:

$$N = |G| = \Pi_i p_i^{e_i}$$

Then we can solve the #DLP in time

$$O\left(\Sigma_i e_i (\log N + \sqrt{p_i})\right)$$

4.5.3 Conclusions

In conclusion, for the #DLP to be hard to break. The following conditions are **necessary** (but **not sufficient**):

- The **size of the group should be** $\approx 2^{2s}$ for security strength s.
- The size of the group should be prime.

Note: we focus on the actual group used. In the case of DSA, it is the sub-group of size q generated by f.

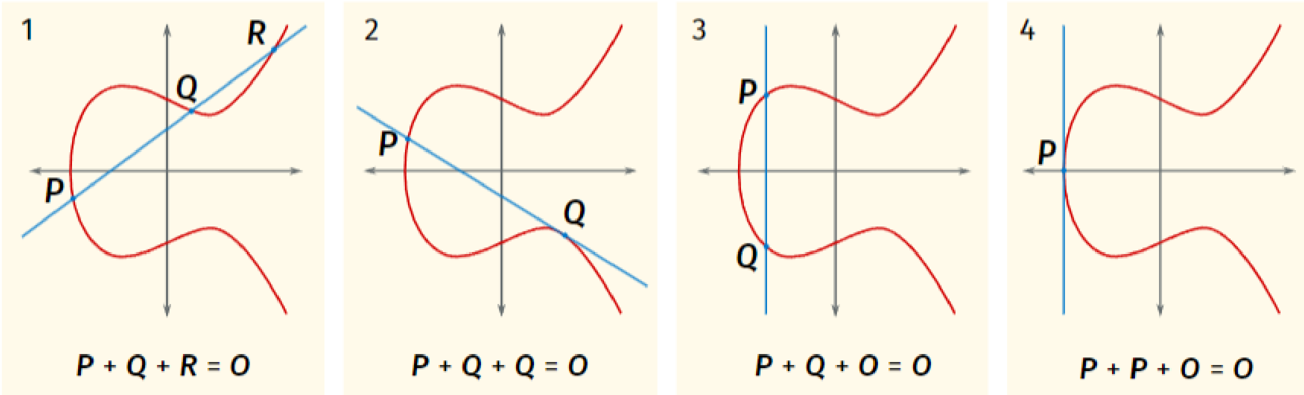
4.6 Elliptic curve cryptography

Given constants a and b, an #elliptic-curve #EC is the **set of points** $(x, y) \in \mathbb{R}^2$ that **satisfy the Weierstrass equation**:

$$y^2 = x^3 + ax + b$$

together with the **point at infinity** denoted O (one can view the point at infinity as $()0, \pm\infty)$).

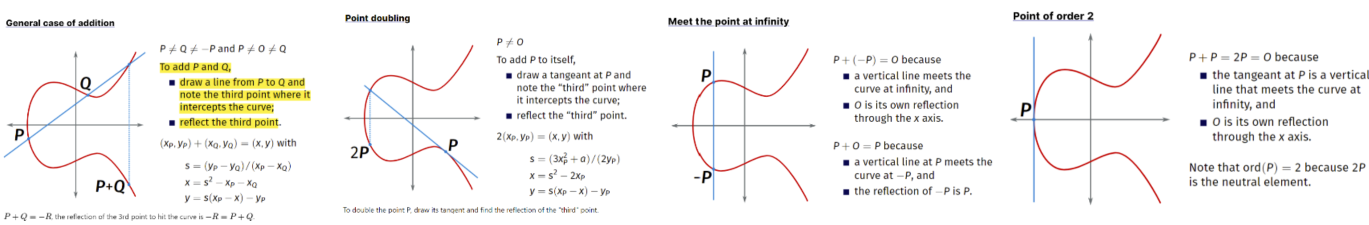
An important #property of the #EC :
If a straight line meets an elliptic curve in two points, then it must cross a third point.
A point on a tangent counts for 2
Don't forget O.



4.6.1 Group law

In #group law, let's build a **rule to add points on the curve**.

- Three aligned points must sum to O
- The point at infinity O is the neutral element.
  So:
- If P and P' are each other's reflections over the x axis, then P, P', O are aligned. So $P + P\prime + O = O$ and $-P = P'$.
- If P, Q, R are aligned, then $P + Q + R = O$ and $P + Q = -R$.



We talk about **order** of a point:

- $ord(P) = 1 \iff$ It is the neutral element (infinity)
- $ord(P) = 2 \iff$ Has a vertical tangent
  #TP4-2-ex13

4.6.2 Elliptic curves over (prime) finite fields

Fix a #prime $p \geq 3$ and constants a and b. An #EC is the set of points $(x, y) \in \mathbb{Z}_p^2$ that satisfy the Weierstrass equation

$$y^2 = x^3 + ax + b$$

together with the point at infinity denoted O.

Fact: the *formulas for point addition and doubling also work here*.
We can thus define a finite group comprising the points on the curve (including O, the neutral element), together with the point addition as operation.

4.6.2.1 Number of points in a curve over a finite field

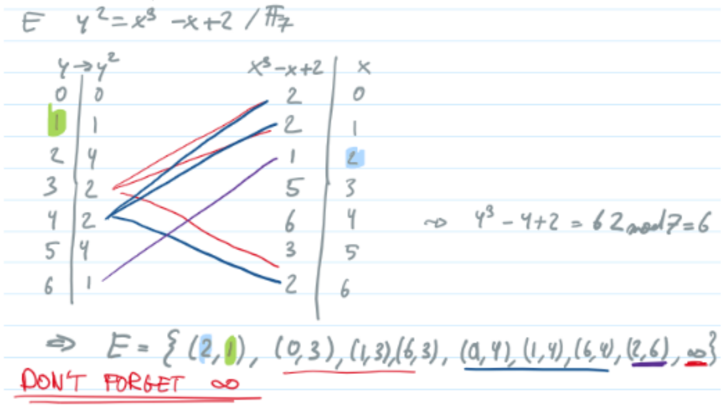
The **number of points on a curve** E is denoted $\#E$. This number depends on the parameters a and b.

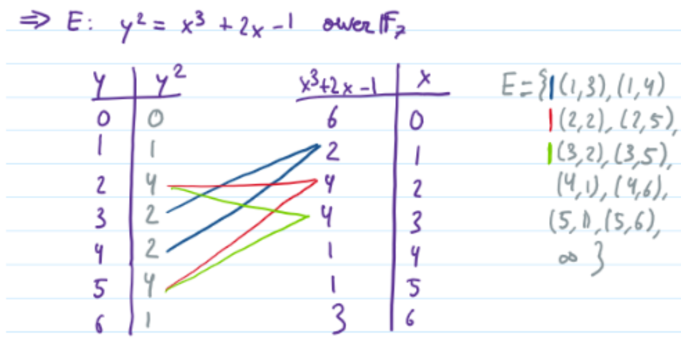
Hasse's theorem

In GF(p), the number of points of an elliptic curve E satisfies

$$p+1-2\sqrt{p} \leq \#E \leq p+1+2\sqrt{p}.$$

4.6.2.2 To find points in a curve over a finite field





4.6.3 Projective coordinates

Instead of representing points on the curve with (x, y) called *affine coordinates*, one can use an alternative representation using **three coordinates** called **projective coordinates**: $(X : Y : Z)$. They are mostly used for their **efficiency** as the operations on the points can be expressed without inversions. #TP4-2-ex16

When $Z \neq 0$, a **point in projective coordinates** $(X : Y : Z)$ **represent** (x, y) **in affine coordinates** with:

- $x = XZ^{-1}$
- $y = YZ^{-1}$

For any $\lambda \neq 0$, the projective coordinates $(X : Y : Z)$ and $(\lambda X : \lambda Y : \lambda Z)$ represents the same point.

The #Weierstrass equation becomes

$$Y^2 Z = X^3 + aXZ^2 + bZ^3$$

The **point at infinity** O in **projective coordinates** can be represented by all points where $Z = 0$ and satisfies the Weierstrass equation $\rightarrow (0 : Y : 0)$ with $Y \neq 0$.

4.7 Protocols

#skip 4-Public

That's it folks!