An extension of PROMETHEE II to temporal evaluations

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Abstract: Conventional PROMETHEE methods have been designed to help decision makers to manage multicriteria problems without considering temporal evaluations. The aim of this paper is to introduce a temporal extension of PROMETHEE II. A new approach to set dynamic thresholds is introduced. The weighted arithmetic mean is used to aggregate net flow scores over time. In addition, an interactive query method is developed in order to determine decision maker's instances importance. Simulation results are used to show the effectiveness of this approach in terms of convergence and speed. Finally, an example is given to illustrate the model.

Keywords: multicriteria decision aid; temporal evaluations; outranking methods; PROMETHEE; GAIA.

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1 Introduction

In this paper we focus ourselves on decision problems that involve multiple conflicting criteria. Different methods have been developed in this context. Among them we can cite Zionts-Wallenius algorithm, STEM, AHP, MAUT, ELECTRE, PROMETHEE, etc. (Brans and De Smet, 2016; Brans, 1982; Kaliszewski and Zionts, 2004; Benayoun et al., 1971; de Freitas et al., 2013).

Most of these approaches belong to three main categories:

- Multi-attribute methods: turning multicriteria problems into a single criterion optimisation problem [ex. AHP (Vaidya and Kumar, 2006)].
- Interactive methods: alternating calculation phases and dialogue with the decision maker [ex. Zionts-Wallenius algorithm (Kaliszewski and Zionts, 2004), STEM (Benayoun et al., 1971)].
- Outranking methods: computing pairwise comparisons between alternatives which allows an enrichment of the traditional dominance relation [ex. ELECTRE (Figueira et al., 2010) and PROMETHEE (Brans and De Smet, 2016)].

In this contribution we focus on PROMETHEE II. This method has been widely used thanks to its simplicity and the existence of user-friendly software products [D-Sight (Hayez et al., 2012), Visual PROMETHEE (Mareschal, 2012–2013)]. Furthermore, without being exhaustive, Behzadian et al. (2010) have listed more than 200 applications based on PROMETHEE published in more than 100 different journals covering various fields such as sport, health care, finance, environmental management, agriculture, etc.

The aim of this paper is to propose an extension of PROMETHEE II to temporal evaluations. In order to illustrate the importance of managing temporal multicriteria problems let us consider the following examples:

1. *Human Development Index (HDI):* the assessment of the global ranking of ten emerging economies with respect to the HDI (UNDP, http://www.undp.org/content/undp/en/home/) during the last three decades. The HDI consists of four main criteria:

   - long and healthy life: a measure of the life expectancy at birth
   - expected years of schooling: measured for adults aged of 25 years and older
   - access to knowledge: a measure of the enrolment by age at all level of education and the number of school age children in the population
   - standard of living: measured by the gross national income per capita.

   This example will be further studied in Section 6.

2. *Football player's assessment:* the coach of a junior football club aims to have a global ranking of dozens of players who have been regularly controlled during one year of training. The assessment is made with respect to three criteria:

   - speed test: time needed for a 30 m sprint
   - anaerobic capacity: capacity to maintain an intense effort (through ergometer)

- peak power: through Sargent jump test.

The coach would also be interested in monitoring the player's evolution over one or more criteria during the year to better advise them afterwards. Hence, the method to be used should reflect the temporal aspect of the problem.

Among recent contributions in the temporal MCDA literature, one can cite (Onder et al., 2013) where the authors aim to assess the financial performance of Turkish banks during the period 2002–2011. This work proceeds in two steps:

1 finding the criteria weights by using AHP then ranking the banks by using TOPSIS

2 based on the final score of each bank, the arithmetic mean score is computed in order to rank the banks over a subset S of the assessment period T.

Besides, the authors stress out the interest of taking time into account in MCDA financial problems. Still, the temporal aspect within has not been further developed. On the contrary, another contribution (Kou et al., 2011) goes a step further. Indeed, in order to assess the urban eco-environmental quality during several years, Kou et al. (2011) develop a temporal extension of TOPSIS where they base the time weighting vector determination on a 'time-degree' (or orness) value $\lambda \in [0, 1]$ which is supposed to be given by the decision maker (DM). Then, in order to find the weights that evolve smoothly, the entropy of these weights is maximised. One can also cite another work (Chunwei and Gang, 2012) whom authors evaluate the sovereign risk of 32 countries with respect to six macroeconomic indicators (criteria) over the time period 1990-2006. It also proposes a prediction up to 2009. This research uses the PROMETHEE II method to find the score of each country on every year. With regard to the time weighting issue, such as in Kou et al. (2011), the time weighting vector is determined based on an artificial value λ. Yet, we think that the nature of λ would remain abstract for many DMs (see Subsection 4.6).

The next sections are organised as follows: first an overview of the PROMETHEE II method and GAIA plan is proposed. Then some assumptions about the new temporal model are stated. In the following section we explore the new model step by step with an emphasis on the dynamic thresholds setting issue. Then we introduce a new approach to determine the time weighting vector followed by simulations. Finally, an example is proposed to illustrate the application of the new model followed by conclusions.

2 PROMETHEE II and GAIA

PROMETHEE II is a multicriteria decision aid method that allows to compute a complete ranking of the alternatives. PROMETHEE I is dedicated to partial rankings. We consider a set of n alternatives (or actions) denoted by $A = \{a_1, a_2, ..., a_n\}$ which are evaluated according to a set of k criteria denoted by $F = \{f_1, f_2, ..., f_k\}$. Each criterion f_h is characterised by a weight $v_h \in V = \{v_1, v_2, ..., v_k\}$. We assume that:

$$\sum_{h=1}^{k} v_h = 1 \ and \ v_h > 0 \qquad \forall h \in H = \{1, 2, ..., k\}.$$

The first step of PROMETHEE II is a pairwise comparison between each couple of alternatives $a_i, a_j \in A$. This is based on the computation of the difference between the evaluations of a_i over a_j on each criterion $f_h \in F$:

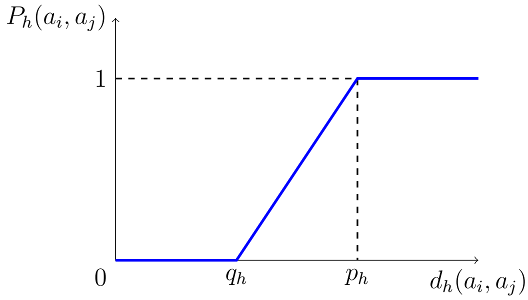
$$d_h(a_i, a_j) = f_h(a_i) - f_h(a_j) \tag{1}$$

Then a preference degree $0 \leq P_h(a_i, a_j) \leq 1$ is computed based on the preference function G_h associated with criterion h such as:

$$P_h(a_i, a_j) = G_h(d_h(a_i, a_j)) \tag{2}$$

Six general preference functions are proposed in most software implementing the PROMETHEE methods. Figure 1 shows one example of these functions:

Figure 1 Example of a linear preference function (see online version for colours)



Practically, assuming that criterion f_h has to be maximised, we set a preference function and two parameters q_h and p_h which are, respectively, the indifference and preference thresholds (with $q_h \leq p_h$) such as:

- if $d_h(a_i, a_j) \leq q_h$ then $P_h(a_i, a_j) = 0$, a_j is preferred or indifferent to a_i with respect to f_h
- if $q_h < d_h(a_i, a_j) < p_h$ then $0 < P_h(a_i, a_j) < 1$, the preference of a_i over a_j with respect to f_h is as strong as $d_h(a_i, a_j)$ is close to p_h
- if $p_h \leq d_h(a_i, a_j)$ then $P_h(a_i, a_j) = 1$, a_i is strictly preferred to a_j with respect to f_h.

A global preference index of a_i over a_j with respect to all criteria is defined in the following way:

$$\pi(a_i, a_j) = \sum_{h=1}^{k} v_h.P_h(a_i, a_j) \tag{3}$$

We have $\pi(a_i, a_j) \geq 0$ and $\pi(a_i, a_j) + \pi(a_j, a_i) \leq 1$. Finally, the alternative a_i is compared to the $n - 1$ other ones. The PROMETHEE net flow score is computed as a difference between the outgoing flow score $\phi^+(a_i)$ and the ingoing flow score $\phi^-(a_i)$ of a_i:

$$\phi(a_i) = \phi^+(a_i) - \phi^-(a_i) = \frac{1}{n-1}\sum_{a\in A}\pi(a_i, a) - \frac{1}{n-1}\sum_{a\in A}\pi(a, a_i) \tag{4}$$

$\phi^+(a_i)$ quantifies the mean preference degree of a_i over all the other alternatives. In the same way $\phi^-(a_i)$ quantifies the mean preference degree of all the alternatives over a_i. The value $\phi(a_i) \in [-1, 1]$ is such that the higher this score, the better the alternative a_i is ranked. For further details, we refer the interested reader to Brans (1982), Brans et al. (1984) and Brans and Vincke (1985). In addition let us note the possibility to calculate unicriterion net flow scores as follows:

$$\phi_h(a_i) = \phi_h^+(a_i) - \phi_h^-(a_i) = \frac{1}{n-1}\sum_{a\in A}P_h(a_i, a) - \frac{1}{n-1}\sum_{a\in A}P_h(a, a_i) \tag{5}$$

Obviously, we have:

$$\phi(a_i) = \sum_{h=1}^{k} v_h \cdot \phi_h(a_i) \tag{6}$$

The PROMETHEE methodology includes an interactive visualisation tool to represent alternatives and criteria on a 2-dimensional plan. The GAIA plan construction turns the k-dimensional unicriterion net flow scores space into a 2-dimensional plan that is called GAIA. For this purpose, a Principal Component Analysis (PCA) technique is applied on the $n \times k$ unicriterion net flow scores matrix. In order to evaluate the quality of the GAIA plan construction one uses the δ ratio. For additional information, the interested reader is referred to Brans and Mareschal (2002a, 2002b) and De Smet and Lidouh (2013). In practice, values of $\delta > 70\%$ are supposed to be associated with good projections. However, the DM should always keep in mind the approximative nature of this representation. It allows to identify similar and contrasted alternatives profiles, redundant and conflicting criteria, best and worst performing alternatives, etc.

3 Model assumptions

The aim of the temporal model we propose is to give a global ranking of the set of alternatives A with respect to the criteria set F and after being evaluated during the period T. Let us stress that forecasting is not the primary aim of this model. All evaluations are considered to have been conducted in the past. Let $T = \{t_1, t_2, ..., t_s\}$ denotes the set of evaluation instants. Each alternative $a_i \in A$ is assumed to be evaluated over each criterion f_h at each instant $t_\ell \in T$. Based on this assumption, one will compute s instantaneous net flow scores for every alternative. We also assume that there is no missing evaluation. In addition, we suppose that the evaluations are crisp and accurate. Consequently, this model is assumed to be synchronous.

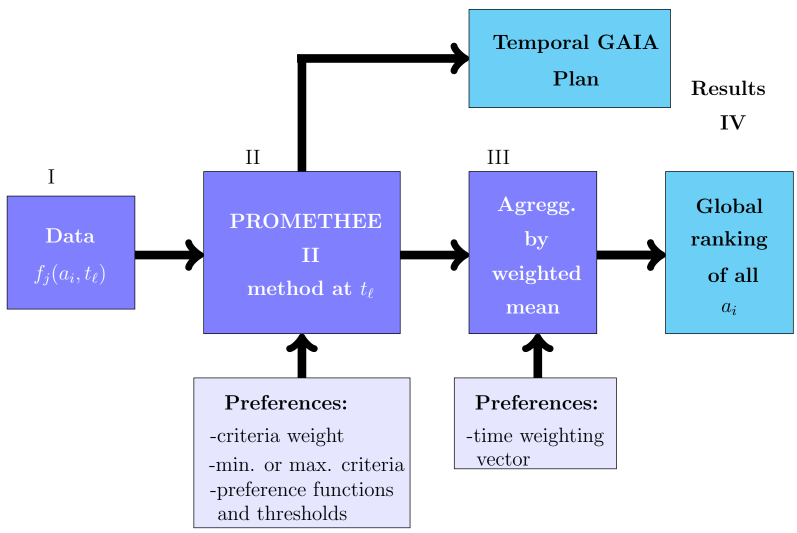
The proposed approach supports both static and temporal preferences of the decision maker. Indeed, we suppose that the DM's preferences (indifference, preference thresholds and instants weights) may change over time. This will be discussed in

Subsections 4.2 and 4.3. We also suppose that the weight v_h assigned to each criterion f_h is constant during the whole period T. However, as we will see in Subsection 4.4, during the aggregation of instantaneous net flow scores the weight v_h will become implicitly dependent on the instant t_ℓ. The global ranking is obtained by aggregating the s net flow scores of every alternative (see Subsection 4.4). For this purpose, we use the weighted arithmetic mean. We assume that each instant t_ℓ is characterised by a relative weight denoted $w_\ell \in W$ such as $W = \{w_1, w_2, ..., w_s\}$. The set W represents the time weighting set that satisfies the starting constraints $\mathcal{C}_0$ such as: $\sum_{\ell=1}^{s} w_\ell = 1$, $w_\ell \geq 0 \quad \forall \ell \in L = \{1, 2, ..., s\}$ and $w_1 \leq w_2 \leq ... \leq w_\ell \leq ... \leq w_s$ where T, W and L are considered to be fixed. The weights $w_1, w_2, ..., w_s$ are increasing because we assume that DMs pay more attention to recent evaluations than to older ones (of course, this assumption can easily be relaxed).

4 General overview of the model

Figure 2 represents the main components of the model that we propose. In the next paragraphs we explore the components of the model one by one.

Figure 2 Overview of the temporal extension of PROMETHEE II (see online version for colours)



4.1 Data

As already stressed, we consider: the alternatives set A, the criteria set F and the instants set T as they have been defined in Sections 2 and 3. These sets are assumed to be finite. Let us stress that, since we have n^2 pairwise comparisons to perform at each instant t_ℓ, the complexity of the procedure is $O(n^2)$. Thus, the size of A has an impact on the computation.

However, it is worth noting that in a recent paper, Calders and Van Assche (2017) have introduced an exact algorithm that computes PROMETHEE's net flow scores with a complexity of $O(k \cdot n \cdot log(n))$.

4.2 Preferences

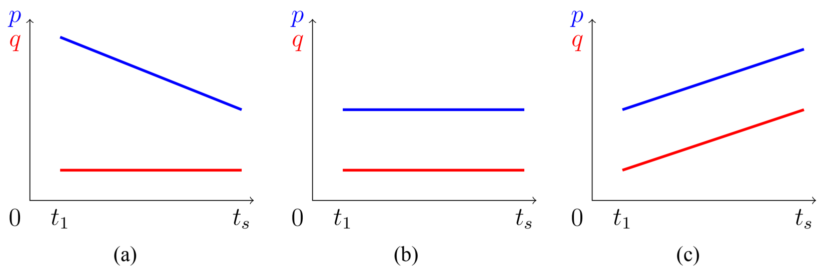
Practically, we ask the DM to give his preferences in five steps:

1. whether a given criterion has to be maximised or minimised
2. the relative importance of each criterion
3. the type of preference function associated with each criterion
4. temporal preference and indifference thresholds for every criterion
5. the relative importance of each instant.

The three first steps are set as in the classical PROMETHEE II method. The last step will be detailed in Subsection 4.6. For Step 4, we consider that indifference and preference thresholds are likely to evolve during the assessment period T.

Indeed, if we refer to the football players example, one can imagine that in the first control, speed test results could be very disparate due to the fact that players were not prepared in the same way. On the contrary, in the last control, results are likely to be less disparate due to the common training. In this context, the coach should pay more attention to the divergence of results in the last control than in the first one. Accordingly, in order to have a realistic approach, the preference threshold of the speed test criterion should have a decreasing trend during T while the indifference threshold could remain static. Figure 3(a) shows thresholds evolution corresponding to this example. For convenience we write p and q instead of p_h and q_h:

Figure 3 Examples of the possible evolution of indifference and preference thresholds during T (see online version for colours)



In Figure 3(b), we have static indifference and preference thresholds over time. Finally, in Figure 3(c) both q and p are increasing. To illustrate this, let us imagine a team of researchers planning to evaluate, annually, the fitness of hundreds of children from their birth until they reach 20 years old. Among the criteria we consider the height of children. Obviously, the height values will be more and more disparate as children grow up. In this context, researchers are likely to pay more attention to the divergence of results in the first years than in the last ones. Consequently, p has to be increasing. Moreover, if the researchers are only interested in the children with the largest heights, then indifference threshold q has to be increasing too. The following subsection investigates the question of setting dynamic thresholds.

4.3 Setting dynamic thresholds

Asking the DM to give his indifference and/or preference instantaneous thresholds over the whole period T for a given criterion may be a tedious task. In this subsection, we introduce an interpolation technique allowing to set these thresholds based on a few questions. For the sake of simplicity, we choose to use the preference function of type III (Brans and Mareschal, 2002a) and we maintain the monotonicity assumption of the dynamic thresholds [Figures 3(a), 3(b), 3(c)]. However, we also assume that these values can eventually be constant during some instants. In the next paragraphs, we treat the case of increasing preference thresholds. Of course, these results can be easily generalised, under assumptions just mentioned above, to the other types of thresholds seen in Figure 3.

Let us assume $P_{real} = \{p_1, p_2, ..., p_s\}$ denote the preference thresholds associated with a given criterion. These are assigned to the instants $t_1, t_2, ..., t_s$. Thus, based on a few known values, we shall find the best approximation to the set P_{real}. For this purpose, four different interpolation techniques are compared based on simulations:

1. Only two values are known, p_s and another $p_i \in P_{real}$. The other values are assumed to be interpolated by the straight line joining p_i to p_s [Figure 4(a)].
2. Three values are known: p_1, p_i and p_s such that $1 < i < s$. The interpolation is done by two segments, the first one joins p_1 to p_i and the second joins p_i to p_s [Figure 4(b)].
3. $p_1 < p_i < p_s$ are known, the interpolation is done by the least squares straight line fit [Figure 5(a)].
4. $p_1 < p_i < p_s$ are known, the interpolation is done by the second order-polynomial line fit [Figure 5(b)].

Figure 4 Example of (a) 1-segment and (b) 2-segment interpolation techniques (see online version for colours)

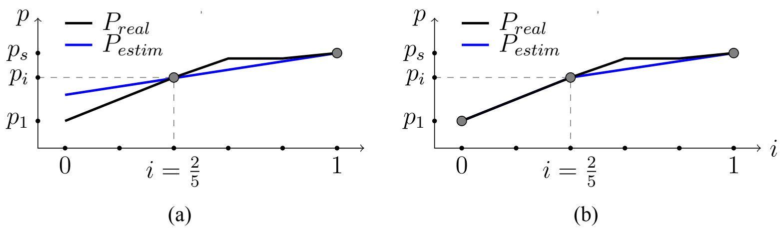
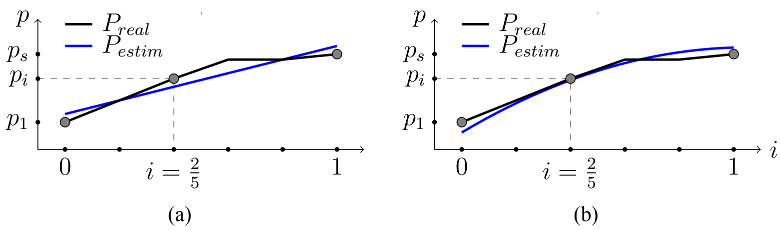


Figure 5 Example of interpolation by (a) linear regression and (b) second-order polynomial fit (see online version for colours)

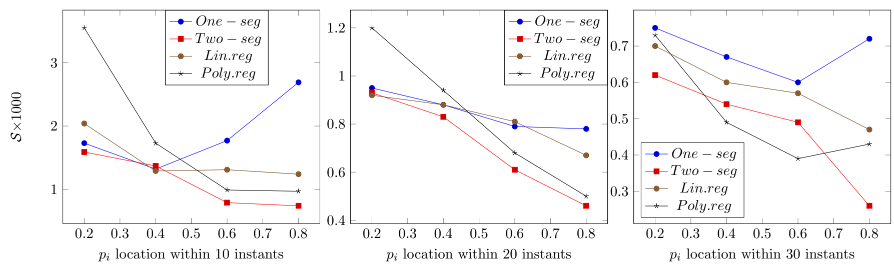


For convenience, i is assumed to be located at either $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ or $\frac{4}{5}$ of the time range. These techniques are compared with each other at each location i. Since the preference threshold of the last instant t_s has the largest impact on the global ranking, p_s has to be known rather than approximated. To perform these comparisons we minimise the sum of squared deviations S between the two obtained unicriterion scores. Every simulation is repeated 100 times. Each time, a unicriterion temporal MCDA problem is created where the characteristic parameters are generated randomly by using the function $randint()$ on $MATLAB\ R2007b$. This function will create:

- Instantaneous evaluations over T for each $a_i \in A$ with an increasing standard deviation as shown in Algorithm 3 in Appendix A.
- Instantaneous preference thresholds. Also, $0 \leq p_\ell \leq p_{max}$ such as $\ell = 1, 2, ..., s$ and p_{max} is the maximum difference between two alternatives evaluations at the last instant t_s (Algorithm 4 in Appendix A).
- Time weighting vector W whose generation will be detailed in Section 5.

The simulations are made for three sizes of T (10, 20 and 30 instants) whereas alternatives set size A has been maintained at 30. The simulations based on the assumptions seen from the beginning of this subsection will be referred to as the dynamic threshold simulation (DTS). Hereafter, Figure 6 shows the results of the DTS.

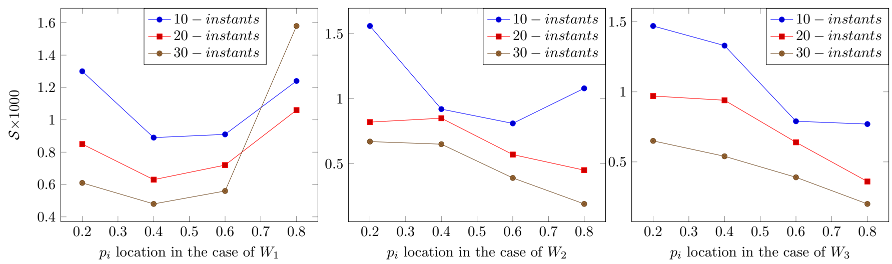
Figure 6 Average fit quality for four interpolation techniques (uniform distribution) S is the sum of squared deviations between the two unicriterion sets (see online version for colours)



According to Figure 6, the 2-segment interpolation provides the best approximation of unicriterion net flow scores based on P_{real}. Furthermore, the approximation is better when the intermediate threshold p_i is located at $\frac{4}{5}$ of the time range. In order to confirm these results, we run the DTS by replacing the uniform distribution with a negative exponential distribution where λ varies randomly at each simulation such that $0.1 \leq \lambda \leq 2$. The obtained results confirm those seen in Figure 6. Additionally, we have applied the DTS in the case where the alternatives size has been changed from 30 to 10, once again, the 2-segment interpolation holds its advantage. The related results are not plotted in this paper.

To continue, we investigate the best location of p_i according to different time weighting vectors W. We shall run the DTS with two changes: W will no longer be randomly generated and only the 2-segment technique will be considered. In fact, we apply these simulations to three different time weighting vectors W_1, W_2 and W_3 whose slopes are, respectively, $0, 0.01$ and 0.02. The slope only relates to the non-zero part of the components of each time weighting vector. Figure 7 shows the behaviour of the 2-segment technique for each time weighting vector:

Figure 7 Average fit quality of 2-segment technique for three different time weighting vectors (uniform distribution) (see online version for colours)



According to Figure 7, for W_1 (equal time weights) the best location of p_i is $\frac{2}{5}$. For W_2 and W_3 the best location is $\frac{4}{5}$. We also note that $\frac{3}{5}$ would be better while the size of instants becomes smaller (10 instants). Indeed, as the instants size gets larger the first instants get less important within a time weighting vector W and vice-versa. Results based on simulations with a negative exponential are concordant with those of Figure 7.

As a conclusion to this subsection, we propose to select the 2-segment interpolation technique for the PROMETHEE temporal model. The intermediate threshold p_i will be set around $\frac{4}{5}$.

4.4 The temporal extension of PROMETHEE II

The PROMETHEE II method (Brans and Mareschal, 2002a, 2002b) is applied at each instant t_ℓ. As a result we have:

$$\begin{aligned} f_h : A \times T &\to \Re \\ (a_i, t_\ell) &\to f_h(a_i, t_\ell) \end{aligned}$$

Given a $f_h \in F$, the evaluation difference between $a_i, a_j \in A$ at instant $t_\ell \in T$ is defined as follows:

$$d_h(a_i, a_j, t_\ell) = f_h(a_i, t_\ell) - f_h(a_j, t_\ell) \tag{7}$$

Figure 8 Succession of net flows during T for each alternative (see online version for colours)

a_1	$\phi(a_1, t_1), \phi(a_1, t_2), ..., \phi(a_1, t_s)$
a_2	$\phi(a_2, t_1), \phi(a_2, t_2), ..., \phi(a_2, t_s)$
$\vdots$	$\vdots \quad \vdots \quad \vdots$
a_n	$\phi(a_n, t_1), \phi(a_n, t_2), ..., \phi(a_n, t_s)$

In the same way as seen in Section 2, the instantaneous net flow score of $a_i \in A$ for each $t_\ell \in T$ can be written as:

$$\phi(a_i, t_\ell) \;=\; \phi^+(a_i, t_\ell) \;-\; \phi^-(a_i, t_\ell) \tag{8}$$

Each alternative from A is characterised by s instantaneous net flows scores at the output of the bloc II as illustrated in Figure 8.
Since we consider the temporal problem as a succession of s individual multicriteria problems, the weighted arithmetic mean is used in order to compute the global ranking of all alternatives over T. One motivation for this choice is its additive property. Indeed, the weighted mean uses an additive aggregation as well as PROMETHEE II method. This distinctive feature will allow to decompose the global score at the criterion level. Hence, the additive property makes the weighted mean operator consistent with PROMETHEE II.

The model aggregates instantaneous net flow scores as follows:

$$\phi_{A,T}(a_i) \;=\; \sum_{\ell=1}^{s} w_\ell . \phi(a_i, t_\ell) \tag{9}$$

where $w_1, w_2, ..., w_s$ denote the time weighting vector.

As already stated in Section 3, v_h is supposed to remain static over T. However v_h is used to find $\phi(a_i, t_\ell)$, the latter depends on t_ℓ. In addition, the global net flow score $\phi_{A,T}(a_i)$ requires both v_h and w_ℓ, such that :

$$\begin{aligned}
\phi_{A,T}(a_i) &= \sum_{\ell=1}^{s} w_\ell \cdot \phi(a_i, t_\ell) \\
&= \sum_{\ell=1}^{s} w_\ell \cdot \frac{1}{n-1} \left(\sum_{x \in A} \pi(a_i, x, t_\ell) - \sum_{x \in A} \pi(x, a_i, t_\ell) \right) \\
&= \frac{1}{n-1} \sum_{\ell=1}^{s} w_\ell \sum_{x \in A} \left(\sum_{h=1}^{k} P_h(a_i, x, t_\ell) \cdot v_h - \sum_{h=1}^{k} P_h(x, a_i, t_\ell) \cdot v_h \right) \\
&= \frac{1}{n-1} \sum_{\ell=1}^{s} w_\ell \sum_{x \in A} \cdot \sum_{h=1}^{k} v_h \left(P_h(a_i, x, t_\ell) - P_h(x, a_i, t_\ell) \right)
\end{aligned}$$

Finally, we have:

$$= \frac{1}{n-1} \; \sum_{\ell=1}^{s} \sum_{x \in A} \sum_{h=1}^{k} w_\ell \cdot v_h \; \left(P_h(a_i, x, t_\ell) - P_h(x, a_i, t_\ell) \right) \tag{10}$$

Since w_ℓ depends on t_ℓ, the product $w_\ell.v_h$ illustrates that the importance of criterion f_h is varying over T. At this stage, one may wonder how the GAIA plan will look like if extended to temporal evaluations too. The following part is dedicated to this question.

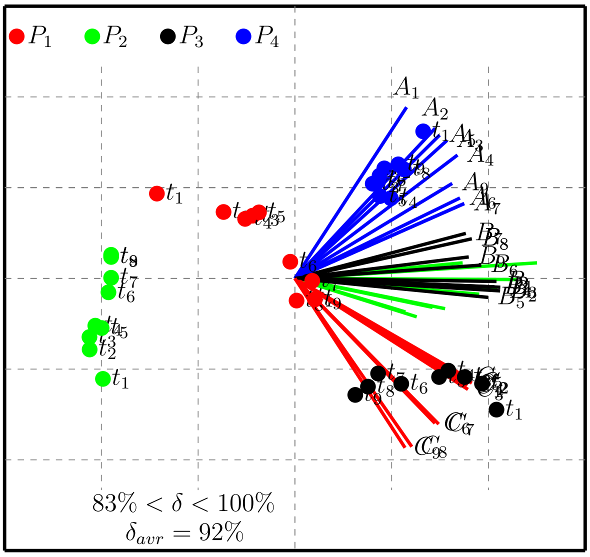
4.5 The temporal GAIA plan

Once bloc II has generated the successive unicriterion net flow scores corresponding to each alternative, we are able to build a temporal GAIA plan in the same way as in the classic version (Brans and Mareschal, 2002a) of PROMETHEE. Since we have s instantaneous net flow scores, the GAIA plan construction is repeated s times. Then these plans are simply overlaid.

Figure 9 shows the temporal GAIA plan based on the footballer example where the aim is to find a global ranking of players after a training period and with respect to three criteria A, B and C as seen in the introduction.

The corresponding evaluation table can be found in Appendix B (Table 5). Let us consider four players: P_1, P_2, P_3 and P_4 evaluated during a period of nine sessions of tests.

Figure 9 Temporal GAIA plan for the football players example (see online version for colours)

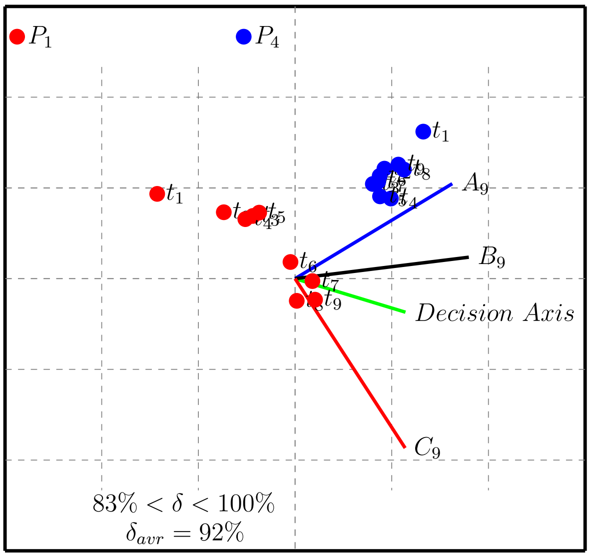


In Figure 9, we observe that each criterion axis moves around a limited area during the nine instants (sessions). The instantaneous decision axes are green, they share almost the same area as the axes of B. To facilitate the interpretation of this plan we only keep the criteria axes belonging to the last instant as shown in Figure 10. Likewise, only positions corresponding to P_1 and P_4 have been kept in order to stress out two different behaviours:

In Figure 10, we observe that:

- P_4, on the right part of the figure, moves in a limited area. He exhibits stable performances during the assessment period.

Figure 10 Temporal GAIA plan with last instant axes (see online version for colours)



- P_1, on the left part of the figure, improves his performances toward the decision axis, he also gets better scores with respect to criterion C.

Nonetheless, this example does not attempt to investigate all facets of the extension of the GAIA plan to the temporal evaluations. A more detailed study will be the topic of a future research.

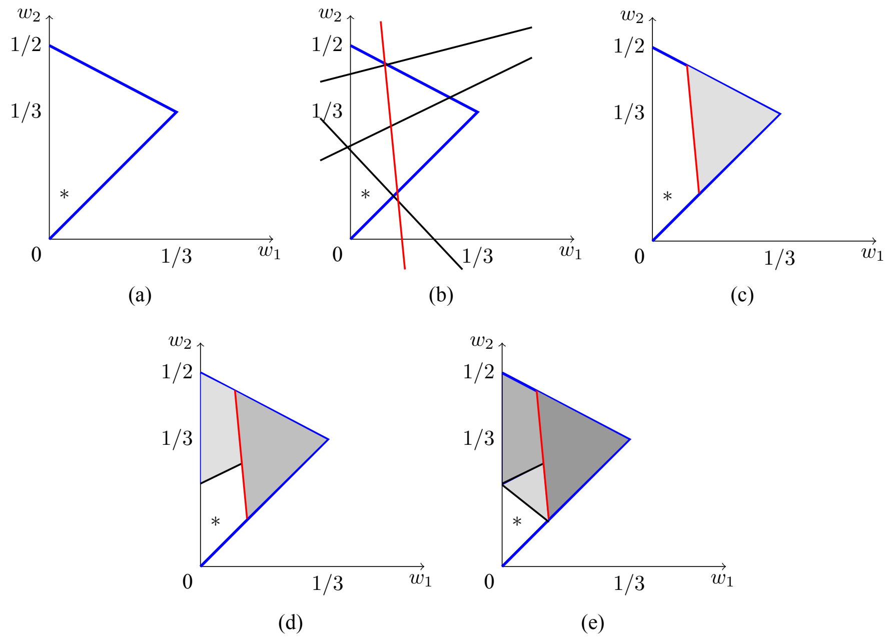
4.6 Time weighting vector determination

In the context of additive temporal aggregation, several methods can be used to find the time weighting vector. For instance one can cite the maximum entropy method (Chunwei and Gang, 2012; Fullér and Majlender, 2001), minimal variability method (Fullér and Majlender, 2003) or minimax disparity approach (Wang and Parkan, 2005). In all these methods the DM is asked to give an orness degree $\lambda \in [0, 1]$ which reflects his preferences over the period T. Instead of asking a technical value we would like to use here a preference elicitation method introduced in 2001 by Iyengar et al. A similar approach has been used by Eppe and De Smet (2014) in order to find the criteria weights regarding the PROMETHEE II method.

The basic idea of this approach is based on the transformation of the weighting vector problem into a s-dimensional constrained polyhedron problem. The weighting vector W that we are looking for is assumed to be an unknown s-dimensional point $W = \{w_1, w_2, ..., w_s\}$ in this polyhedron. So, the aim is to approach the coordinates (weights) of W. In practice, iteratively querying the DM through pairwise alternative comparisons leads to approach these weight values. So, at each query the DM gives his preference over two time series corresponding to two alternatives. We suppose that the DM's current answer is consistent with the previous ones. The choice of this method is motivated by its ability to get closer to the DM's real preferences in an interactive way. In Figure 11, we illustrate the time weighting vector $W = \{w_1, w_2, w_3\}$

determination. We consider the determination of W in the context of a problem based on three dimensions as shown in Figure 11. We draw only w_1 and w_2. w_3 is easily calculated since $\sum_{\ell=1}^{3} w_\ell = 1$.

Figure 11 Reduction of the area of possible time weighting vectors by querying (see online version for colours)



There are five triangles in Figure 11, each of them represents one iteration. A triangle is the projection of the polyhedron on two dimensions (prior to querying the DM). The asterisk in each triangle represents the location of W to approach by the end of the querying process. At each step, the white zone is the area of compatible weights regarding the previous queries. In Figures 11(a), 11(b), 11(c), 11(d) and 11(e), the white area shrinks based on the DM's preference between two given alternatives. Each pair of alternatives is represented by a line in these figures.

In the next paragraphs, we describe the main steps of this approach which is based on the computation of the analytic centre y^a ('a' for analytic centre) of the current feasible region, the selection of the closest hyperplane to y^a, and the querying process. In order to better understand the succession of these steps, we are going, first, to see how to generate an optimum query.

Let us consider Figure 11. Each line is the geometric equivalent of a query addressed to the DM. His answer (preference) determines the side of the line where W lies (white area). Now, consider Figure 11(a). Prior to the DM's answer, we are not able to know the side of the triangle where W lies. To our point of view, the best way to deal with this uncertainty is to select the closest line to the centre of the feasible region. This yields the minimisation of the feasible region whatever the answer will be. In the case of a s-dimensional polyhedron, we would select the closest hyperplane to this analytic centre.

As stated by Bous et al. (2010), a bounded polyhedron $\mathcal{P}$ in $\mathbb{R}^s$ defined by m linear inequalities can be defined such as:

$$\mathcal{P} = \{y \in \mathbb{R}^s : c - G \cdot y \geq 0\} \tag{11}$$

where $G \in \mathbb{R}^{m \times s}$, $m > s$ and $c \in \mathbb{R}^s$.

Ye (1997) has introduced the analytic centre of a polyhedron $\mathcal{P}$ as the point y^a of $\mathcal{P}^0$ that maximises the value of the function $B(y^a)$:

$$\begin{aligned} &B : \mathcal{P}^0 \to \mathcal{R} : \\ &\quad B(y^a) = \max \textstyle\sum_{j=1}^{m} \ln(c_j - g_j \cdot y^a) \end{aligned} \tag{12}$$

where $\mathcal{P}^0 = \{y \in \mathbb{R}^s : c - G.y > 0\}$ is the interior of $\mathcal{P}$.

The difference $c_j - g_j.y$ such as $j \leq m$ is a measure of the distance of y towards the boundary defined by the constraint $c_j > g_j \cdot y$.

Naturally, the analytic centre of $\mathcal{P}$ corresponds to the point y of $\mathcal{P}^0$ that maximises the sum of distances between y and the boundaries of $\mathcal{P}$. In addition, the choice of a logarithmic function results from the high penalty around the boundaries. At this stage, using an appropriate algorithm (next section), we are able to find the analytic centre y^a of the feasible region at iteration k. Before determining the closest hyperplane to y^a, the possible hyperplanes have to be listed. As already stressed, a time series $\Phi(a_i)$ of an alternative a_i is composed of s instantaneous net flow scores $\phi(a_i, t_1), ..., \phi(a_i, t_s)$. In fact, we assume that the DM does not know precisely the importance (weight) of each instant of the period T, but is able to evaluate a time series as a whole. For example, if the DM states that $\Phi(a_i)$ is preferred to $\Phi(a_j)$ (we can write for convenience, respectively, a_i and a_j), it means that:

$$\phi(a_i, t_1) \cdot w_1 + \ldots + \phi(a_i, t_s) \cdot w_s > \phi(a_j, t_1) \cdot w_1 + \ldots + \phi(a_j, t_s) \cdot w_s \tag{13}$$

we have:

$$(\phi(a_i, t_1) - \phi(a_j, t_1)) \cdot w_1 \; + \ldots + (\phi(a_i, t_s) - \phi(a_j, t_s)) \cdot w_s > 0 \tag{14}$$

where $w_1, w_2, ..., w_s$ represent the weights of the different instants. These weights are unknown, and the terms $(\phi(a_i, t_\ell) - \phi(a_j, t_\ell))$ are constant and known. Geometrically, inequation (13) corresponds to the side of the hyperplane [whose equation is (15)] where the point $W = \{w_1, w_2, ..., w_s\}$ lies.

$$(\phi(a_i, t_1) - \phi(a_j, t_1)) \cdot w_1 + \ldots + (\phi(a_i, t_s) - \phi(a_j, t_s)) \cdot w_s = 0 \tag{15}$$

Equation (15) is the equation of the hyperplane denoted by H_{a_i, a_j}, formed by a_i and a_j and passing through the origin. Let us note that this equation also represents the situation where the DM is indifferent between a_i and a_j.

At this level, it is easy to find all possible hyperplanes by computing the differences within each pair of time series a_i and a_j. By doing so, we have $\frac{n^2 - n}{2}$ different hyperplanes. After having listed all possible hyperplanes, we shall compute the distance

$\mathcal{D}$ between each H_{a_i,a_j} and the analytic centre y^a whose coordinates are $y_1^a, y_2^a, ..., y_s^a$ at iteration k such as:

$$\mathcal{D} = \frac{|[\phi(a_i,t_1) - \phi(a_j,t_1)] \cdot y_1^a + \ldots + [\phi(a_i,t_s) - \phi(a_j,t_s)] \cdot y_s^a|}{\sqrt{[\phi(a_i,t_1) - \phi(a_j,t_1)]^2 + \ldots + [\phi(a_i,t_s) - \phi(a_j,t_s))]^2}} \tag{16}$$

Equation (16) is a generalisation of the distance between a point and a plan passing by the origin. Thus, the selected hyperplane will be the one that minimises $\mathcal{D}$. Finally, at each iteration, we ask the DM to give his preference over the two time series a_i and a_j whose hyperplane is the closest one to y^a. The DM has, in fact, three possible answers:

$a_i \succ a_j$ means that a_i is preferred to a_j

$a_i \prec a_j$ means that a_j is preferred to a_i

$a_i \equiv a_j$ means that a_i and a_j are equivalent

From a geometrical point of view, adding a constraint in this way tends to cut the polyhedron in two hopefully equal parts. Then another iteration is started with the same operations but only based on the remaining region. The stopping condition of the iterative process has been set to a maximum number of iterations. Once this number is reached, the coordinates of the last computed analytic centre are considered to be the best approximations of $w_1, w_2, ..., w_s$. However, if we find out at this stage that the DM associates almost equal weights with W components, then we go back to the dynamic threshold preference elicitation step and set p_i to $\frac{2}{5}$ instead of $\frac{4}{5}$ to obtain a better approximation of the dynamic thresholds (see Subsection 4.3 – Figure 7).

5 Simulations about W determination

The W determination process by querying the DM should remain feasible. This means getting the desired weights in a reasonable number of iterations. To investigate this question, we proceed through simulations (Algorithm 1).
Hereafter, we give more details about the algorithm:

Step 1, we begin by creating a temporal MCDA problem such that we consider, among others:

- k criteria to be maximised and having equal weights. This choice is arbitrary.
- For each alternative and for every criterion, s evaluations $f_h(a_i, t_\ell)$ between 1 and 10 are randomly generated (based on a uniform distribution). Then, these evaluations are multiplied by a random value between 0.1 and 1. This multiplication aims to better differentiate the alternatives.
- A static preference threshold has been chosen for this problem such as $p = 7$. Of course p can be lower or higher than 7, but the higher p the lower discrimination of alternatives. Since evaluations are between 1 and 10, $p = 7$ seems to be an acceptable choice.
- W^* is a time weighting vector satisfying the time weighting constraints C_0 (see Section 3). The weights $w_1^*, w_2^*, ..., w_s^*$ can be the ordinates, whose sum is 1, of any non-decreasing segment or up to three joined segments.

Algorithm 1 Iterative querying process

1 **Input:** $n,\ k,\ s,\ q_h,\ p_h,\ f_h(a_i, t_\ell), W^*, iter_{max}$
2 **For** $i = 1...s$ **do**
  Compute the i^{th} instantaneous net flow scores for n alternatives;
end
3 Initialise the time weighting constraints C_0;
4 Compute all possible hyperplanes;
5 Compute the analytic centre y_0 of the polyhedron constrained by C_0;
6 $C_{it} \leftarrow C_0$;
7 $y_{it} \leftarrow y_0$;
8 **For** $j = 1...iter_{max}$ **do**
  a Find the closest hyperplane hyp to the last y_{it};
  b Ask the DM's preference C_{pr} about hyp;
  c Update the constraints set $C_{it} = C_{it} \cup C_{pr}$;
  d Compute the analytic centre y_{it} of the new constrained polyhedron;
end
9 $W_{estim} \leftarrow y_{it}^T$;

Steps 2, 3 and 4 are already cited in the previous section.

Step 5, as introduced in the previous section, we compute the first analytic centre y_0 of the s-dimensional polyhedron constrained by $\mathcal{C}_0$. For this purpose, we use Frank-Wolfe algorithm (Jaggi, 2015) which finds the minimum of a nonlinear function $f(x)$ under the constraints of a domain D. Algorithm 2 shows the main steps of Frank-Wolfe algorithm.

Algorithm 2 Frank-Wolfe algorithm

A **Input**: A starting time weighting vector y_0, $k = 0$ and $iter_{max} = 350$
B **While** $||\nabla f(y_k)||_2 \geq \epsilon$ and $k \leq iter_{max}$ **do**
  a Find z_k that minimises $z^T \nabla f(y_k)\ \forall\ z \in D$;
  b Let $\gamma = \frac{2}{k+2}$ or alternatively find γ that minimises $f(y_k + \gamma(z_k - y_k))$ subject to $0 \leq \gamma \leq 1$;
  c Update $y_{(k+1)} = y_k + \gamma(z_k - y_k)$;
  d $k = k + 1$;
end
C **Output**: y_k

Source: Jaggi (2015)

In Step A, the starting time weighting vector is a point y_0 in the s-dimensional polyhedron $\mathcal{P}$ such as y_0 is far from the boundaries because of the logarithmic function penalty (Subsection 4.6). One method we can use to find y_0 is the centre of Chebyshev (Boyd and Vandenberghe, 2009). It is the centre of the largest inscribed ball in a polyhedron (Appendix C). This method requires simply to solve a linear programming.

Step B.a of Algorithm 2 is a minimisation of the product of the variable z and the gradient of $f(y) = -\sum_{j=1}^{m}\ ln(c_j - g_j.y)$ in the neighbourhood of y_k. This product is nothing else than the product of $(w_1, w_2, ..., w_s)^T$ and constants (result of $\nabla f(y)$ at y_k). In this case we use the simplex algorithm.

After having found a suitable value of γ by linear search in step $B.b$, the value $(z_k - y_k)$ in the following step acts as the minimisation direction. We have considered $\epsilon = 0.1$ and $iter_{max} = 350$.

Steps 6, 7, 8 and 9 of Algorithm 1 are supposed to be clear.

In order to compute statistics for Algorithm 1, simulations have been repeated 100 times for each pair of (A, T) (alternatives and instants sets). The value of $iter_{max}$ can be adapted to the size of both alternatives and instants. According to Table 1, a few queries ($iter_{max} < 10$) are, most of the time, sufficient to reach a Kendall correlation rate greater than 95%. The higher $iter_{max}$ the better the convergence rate.

Table 1 Average number of queries needed to get 95% convergence rate and corresponding standard deviations (in grey)

*Alt.**	*Instants*					
	5	*10*	*15*	*20*	*25*	*30*
5	2.3 +– 1.8	2.3 +– 1.9	2.4 +– 2.1	2.7 +– 2.2	3.2 +– 2.7	2.9 +– 2.3
10	2.4 +– 1.8	2.7 +– 2.2	3.4 +– 3.2	3.7 +– 3.2	4.2 +– 3.7	4.3 +– 4.1
15	2.4 +– 1.9	3.5 +– 2.8	4.5 +– 3.5	5.2 +– 4.5	5.3 +– 4.0	5.3 +– 3.9
20	2.7 +– 1.9	3.9 +– 3.2	4.3 +– 3.6	5.2 +– 3.8	6.4+– 4.3	7.0 +– 4.7
25	2.7 +– 2.0	3.9 +– 2.9	5.6 +– 4.1	6.7 +– 5.1	6.9 +– 5.2	7.0 +– 5.0
30	3.0 +– 1.9	4.2 +– 2.9	5.9 +– 4.0	6.6 +– 4.8	7.4 +– 4.5	7.7 +– 5.3

Note: *Alternatives

Besides, the computation of the analytic centre is the task which is the most time consuming among all other steps of Algorithm 1. To give an idea, the average duration of the entire querying process based on $(A, T) = (100, 10)$, $(A, T) = (100, 20)$ and $(A, T) = (100, 30)$ are, respectively, $< 5s$, $21s$ and $55s$. These simulations are obtained using MATLAB, version 7.5.0.342 (R2007b) on a Pentium Dual-Core T2310-1.47 GHz-32 bits computer with a 1014 MB RAM.

At this stage, one may wonder whether the approach presented in this paper is the same as the one introduced by Eppe and De Smet (2014). Indeed, Eppe and De Smet (2014) have developed an iterative querying approach close to the one of Iyengar et al. (2001) in order to find the criteria weights in the case of PROMETHEE II. The added value of their method relies on three different types of queries. The best one is identified according to its expected impact on the reduction of the admissible weights area (Eppe and De Smet, 2014). The model we propose involves a methodological difference with the one of Eppe and De Smet. This concerns the increasing weights constraint where $w_1 \leq w_2 \leq \ldots \leq w_\ell \leq \ldots \leq w_s$.

Moreover, we have used Frank-Wolfe algorithm instead of Newton's algorithm used by Eppe and De Smet. Frank-Wolfe replaces a differentiable function $f(y)$ with its first-order Taylor expansion. However, Newton's method requires the second-order Taylor expansion. This implies the computation of the inverse of the Hessian matrix (second derivative) at each iteration which can be prohibitively heavy in high dimensions (Hyvarinen et al., 2004). In fact, the Hessian may become close to a singular matrix (non-invertible) which induces computation errors (Hyvarinen et al., 2004; Gill and King, 2004). As a result, since the PROMETHEE temporal model involves working in high dimensions ($s = 10, 20, 30, \ldots$) the Hessian computation may be unstable. In addition, for the sake of simplicity, the query in the model we propose is restricted to two alternatives.

6 Illustration

We consider the global ranking of ten emerging economies with respect to the HDI. This index assumes that economic growth should not be the unique criterion while evaluating people well-being in a given country. Indeed, the HDI is an aggregated measure related to three basic aspects: health, education and standard of living (http://hdr.undp.org/en/content/human-development-index-hdi). They are assessed as follows:

6.1 Life expectancy at birth

This criterion is defined in the United Nations Development Programme (UNDP) 2010 Report as "the number of years a new-born infant could expect to live if prevailing patterns of age-specific mortality rates at the time of birth were to stay the same throughout the child's life".

6.2 Education

Two indicators are used to measure this criterion, they are defined in the UNDP website (http://hdr.undp.org/fr/content/mean-years-schooling-adults-years; http://hdr.undp.org/fr/content/expected-years-schooling-children-years) as:

- "Average number of years of education received by people aged of 25 and older, converted from education attainment levels using official durations of each level".
- "Number of years of schooling that a child of school entrance age can expect to receive if prevailing patterns of age-specific enrolment rates persist throughout the child's life".

The arithmetic mean of these two indicators gives the score on education criterion.

6.3 Gross national income per capita

The UNDP 2010 Report defines this criterion as "the sum of value added by all resident producers in the economy plus any product taxes (less subsidies) not included in the valuation of output plus net receipts of primary income (compensation of employees and property income) from abroad, divided by midyear population".

The global scores of the HDI are annually published by UNDP Human Development Reports (http://hdr.undp.org/en/content/human-development-index-hdi). We draw the attention of the reader to the fact that UNDP uses the geometric mean aggregation to compute those scores. The data we use have been extracted from the Human Development Reports of 2015. The evaluations frequency is five years and covers the period between 1990 and 2015. Based on the evaluations of 2010, 2011, 2013, 2014, values corresponding to 2015 have been extrapolated by linear regression.

The instantaneous evaluations of all countries over the set F are given in Appendix B (Table 6). Each country is evaluated six times over each criterion. We consider:

$$A = \{\text{Brazil, China, India, Indonesia, Malaysia, Mexico, Philippines, Russia, South Africa, Turkey}\}$$
$$F = \{\text{Life expectancy at birth, Education, Gross national income per capita}\}$$
$$T = \{t_{1990}, t_{1995}, \ldots, t_{2015}\}$$

For convenience, we denote by C_1, C_2 and C_3, respectively, the criteria of the set F. They have to be maximised. The weights are chosen to be equal. The evaluations over C_1 and C_2 are expressed in years whereas those over C_3 are expressed in constant 2011 international dollars converted using purchasing power parity (PPP) rates (Human Development Report, 2015).

A preference function of type III has been associated with each criterion. Tables 2, 3 and 4 represent the evaluations disparity variation over T for, respectively, C_1, C_2 and C_3. This helps us to answer the question whether p will be dynamic or static:

6.4 Life expectancy at birth C_1

For this criterion we can propose that p has to be an increasing threshold. This is justified by the fact that it is easier to live longer in 2015 than in 1990, furthermore, Table 2 shows almost a growing disparity in average during T.

Table 2 Standard deviation of evaluations for C_1

	t_{1990}	t_{1995}	t_{2000}	t_{2005}	t_{2010}	t_{2015}
Evaluations Sd	3.860	3.818	5.222	6.624	6.087	5.480

According to the 2-segment technique, we need three values of dynamic thresholds, the intermediate one p_i should be around $3/5$ of T as seen in Subsection 4.3 (case of ten instants). Let us consider then: $p_1 = 7$, $p_i = 8$ and $p_s = 10$. In fact, we consider that seven years are sufficient to make a significant difference between two life expectancies in 1990. We raise this threshold to 10 for 2015. Of course, other choices are possible.

6.5 Education C_2

We propose a decreasing threshold. In fact, since the disparity of the evaluations is slightly declining, a decreasing p will tend to make countries discrimination more perceptible by the last instants:

Table 3 Standard deviation of evaluations for C_2

	t_{1990}	t_{1995}	t_{2000}	t_{2005}	t_{2010}	t_{2015}
Evaluations Sd	1.468	1.496	1.495	1.341	1.231	1.190

Therefore, we need three values of dynamic thresholds. We propose: $p_1 = 3$, $p_i = 2.6$ (around $3/5$ of T) and $p_s = 2.2$. Here, we consider 3 as sufficient to make a significant difference between two countries in 1990 on education criterion. Since the countries become less disparate on this criterion, a decreasing threshold such as $p_s = 2.2$ seems to be suitable for 2015 (Table 6 Appendix B).

6.6 *Gross national income per capita C_3*

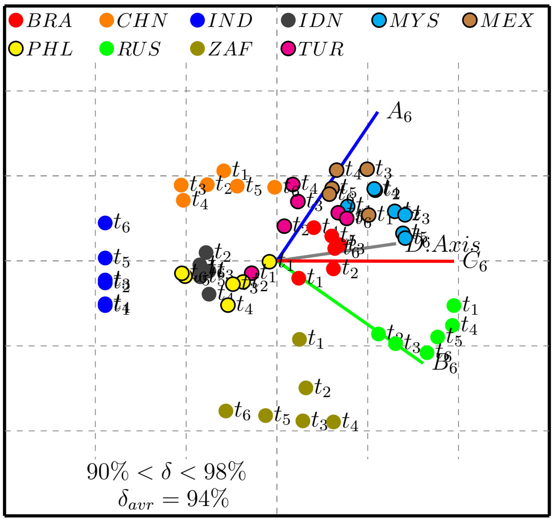
As stated previously, evaluations over C_3 are expressed in constant 2011 international dollars. This means no need to consider inflation rate during T. Table 4 shows dropping evaluations disparity from t_{1995} followed by a slow recovery. Hence, there is no specific trend over T.

Table 4 Standard deviation of evaluations C_3

	t_{1990}	t_{1995}	t_{2000}	t_{2005}	t_{2010}	t_{2015}
Evaluations Sd	5,244.8	4,093.8	4,410.2	4,979.2	5,194.7	5,538.6

For this criterion, we shall rather keep a static preference threshold. Let us consider $p =$ 10,000. According to the corresponding evaluations in Table 6 Appendix B, a preference threshold of 10,000 is a reasonable value. Once again, other choices are possible.

Figure 12 Temporal GAIA Plan of HDI problem (19 countries) (see online version for colours)



Taking into account the corresponding evaluations (Table 6 in Appendix B) and given a time weighting vector $W = \{0.1168, 0.1368, 0.1568, 0.1768, 0.1968, 0.2168\}$. In fact, W is a segment whose slope is 0.02 (this is an arbitrary choice) and whose ordinates sum is 1. Hereafter the results of the global ranking of the countries of A with respect to the HDI after assessment during the period 1990–2015:

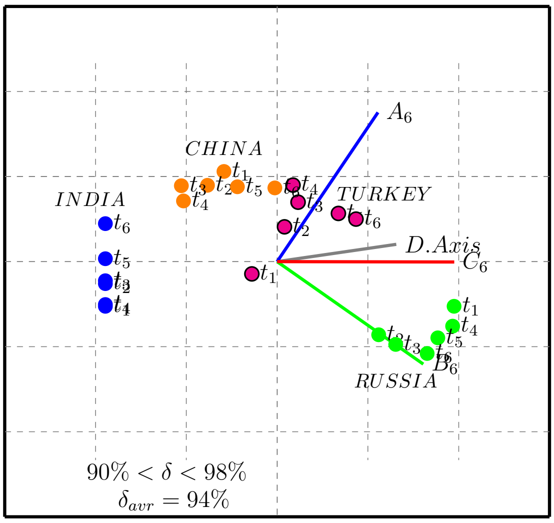
1 – Russia; 2 – Malaysia; 3 – Mexico; 4 – Brazil; 5 – Turkey; 6 – China;
7 – South Africa; 8 – Philippines; 9 – Indonesia; 10 – India

Due to the small size of this example (ten alternatives over six instants), one query has been sufficient to reach 95% of Kendall correlation tau (Section 5). This is consistent with simulation results in Table 1. Yet, to reach 99% of correlation tau, three iterations have been required.

Since this overall ranking represents an aggregated information over T, one may need to visualise the evolution of these countries during the same period. Figure 12 shows the temporal GAIA plan corresponding to this problem, only the criteria axes belonging to the last instant (t_{2015}) are kept.

The shortest axis (D.A) in this figure represents the decision axis. In Figure 13, we stress out the evolution of China and Turkey over T. In addition, we observe the stable behaviour of India during the same period. This plan also confirms the leading rank of Russia.

Figure 13 Temporal GAIA plan of HDI problem (four countries) (see online version for colours)



According to this figure, Turkey and China are also likely to evolve in the future in order to improve their respective ranks with respect to HDI. Based on these results, the temporal model proposed in this paper provides consistent outputs. However, we shall not discuss modelling issues into details as the aim of this section is to illustrate the application of the method.

7 Conclusions and directions for future research

In this paper we have introduced an extension of PROMETHEE II to temporal evaluations. This includes two contributions related to the setting of dynamic thresholds and the time weighting vector elicitation. In addition to the effectiveness of the 2-segment technique, simulations have shown that, in most cases, a few queries addressed to the DM were sufficient to set instants weights.

Moreover, the temporal extension of GAIA allows to monitor the instantaneous behaviour of all alternatives during the evaluation period, this topic deserves a more detailed research.

With regard to whether the temporal extension of PROMETHEE II can be applied to other MCDA methods, the answer would be nuanced. Indeed, except for the time weighting elicitation, we note that the issue of setting (as proposed in this paper) dynamic preference (and/or indifference) thresholds for the methods allowing this type of parameters (ex. ELECTRE methods) deserves a separate research. Moreover, since GAIA is based on the Principle Components Analysis technique, a temporal extension of GAIA construction to other MCDA methods is worth investigation.

Let us note that the time weighting elicitation can be alleviated by asking the DM to choose the best instants weights evolution among a couple of possible shapes. Besides, one may investigate the relevance of approximating these weights by an interpolation technique such as the 2-segment technique.

The proposed model is intended to be suitable for a wide range of temporal MCDA problems. However, further researches are needed to adapt it to particular cases such as evaluations under uncertainty, asynchronous evaluations, dynamic size of alternatives set, etc.

Finally, it is worth noting that forecasting the alternatives behaviours is an interesting future research direction.

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Appendix A

Algorithm 3 Generating random uniformly distributed evaluations with an increasing standard deviation

1	**Input:** n, s
2	**For:** $\ell = 1 : s$
	$Eval(\ell, 1 : n) = randint(1, n, [1, 10 + \ell])$;
	end

We use $randint()$ MATLAB function (uniform distribution). The higher ℓ, the higher the evaluations disparity.

Algorithm 4 Generating a random set of P_{real} by a uniform distribution

1	**Input:** $n, s, Eval$
2	$maxEval$= the largest evaluation at the last instant;
3	$minEval$= the smallest evaluation at the last instant;
4	$slack = maxEval - minEval$;
5	$P_{real}(1, 1) = 3$;
6	**For:** $\ell = 2 : s$
7	**If:** $P_{real}(1, \ell - 1) + 2 \quad < \quad slack$
a	$P_{real}(1, \ell) = randint(1, 1, [P_{real}(1, \ell - 1), P_{real}(1, \ell - 1) + 2])$;
b	**else** $P_{real}(1, \ell) = slack$;
	end

In Steps 2 and 3 evaluations are considered to be the most disparate at the last instant. In Step 5, we have $p_1 = 3$, this choice is random. Since the preference threshold is assumed to increase over T, p_1 should be set as smaller (with respect to $slack$) as possible. A consequence of Step 7a is that $P_{real}(1, \ell - 1) \leq P_{real}(1, \ell) \leq P_{real}(1, \ell - 1) + 2$.

To generate a random value within $[a, b]$ by a negative exponential distribution, we consider the cumulative distribution function according to a truncated negative exponential distribution:

$$F(x) = \frac{\mathrm{e}^{-\lambda a} - \mathrm{e}^{-\lambda x}}{\mathrm{e}^{-\lambda a} - \mathrm{e}^{-\lambda b}} \tag{17}$$

Then, we can find the distribution x by setting $F(x)$ equal to a random uniformly distributed value $U \in [0, 1]$.

Appendix B

Table 5 Evaluations table for football example cited in the introduction

Criteria					*30 m sprint*									*Anaerobic capacity*									*Peak power*				
Weights					0.3									0.3									0.4				
Instants	t1	t2	t3	t4	t5	t6	t7	t8	t9	t1	t2	t3	t4	t5	t6	t7	t8	t9	t1	t2	t3	t4	t5	t6	t7	t8	t9
Max +1 | Min −1					−1									1									1				
Pref. function					III									III									III				
p threshold	1.14	1.13	1.15	1.19	1.11	1.02	0.91	0.83	0.7	0.83	0.55	0.67	0.68	0.71	0.68	0.65	0.61	0.60	0.67	0.60	0.63	0.68	0.75	0.85	0.9	0.92	1
q threshold	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P1	5.5	5.4	5.2	5	4.8	4.7	4.5	4.5	4.4	38	44	45	45	46	48	48	49	50	13.6	13.9	14.1	14.4	14.7	15.1	15.4	15.7	16.3
P2	6.3	6.2	6.1	6	5.8	5.6	5.4	5.2	5	39	41	40	42	44	45	46	48	48	13.7	13.9	14	14.1	14.2	14.3	14.3	14.4	14.6
P3	5.7	5.5	5.3	5.1	5.1	5	4.9	4.8	4.8	49	50	51	52	54	54	53	54	55	15.2	15.3	15.5	15.8	16.1	16.4	16.6	16.7	17
P4	5	4.9	4.8	4.7	4.6	4.5	4.5	4.3	4.3	48	47	48	51	53	55	56	57	57	14	14.3	14.4	14.6	14.7	14.6	14.8	14.9	15.1
Std. deviation	4.66	4.64	4.72	4.85	4.55	4.15	3.7	3.39	2.86	5.02	3.35	4.06	4.15	4.32	4.15	3.96	3.67	3.64	6.38	5.72	5.96	6.46	7.08	8.03	8.58	8.7	9.5

Table 6 Evaluations table for emergent economy problem in illustration section

Criteria	*Life expectancy at birth*						*Education*						*Gross national income per capita*					
Weights	0.333						0.333						0.333					
Instants	1990	1995	2000	2005	2010	2015	1990	1995	2000	2005	2010	2015	1990	1995	2000	2005	2010	2015
Max +1 | Min −1	1						1						1					
Pref. function	III						III						III					
p threshold	7	7.333	7.66	8	9	10	3	2.87	2.74	2.6	2.4	2.2	10,000	10,000	10,000	10,000	10,000	10,000
q threshold	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Brazil	65.3	67.6	70.1	71.9	73.3	74.8	8	8.95	9.95	10.15	11.05	11.6	10,065	10,959	11,161	12,032	14,420	15,062
China	69	69.9	71.7	73.7	75	76	6.8	7.25	7.85	8.75	9.85	10.3	1,520	2,508	3,632	5,632	9,387	13,347
India	57.9	60.4	62.6	64.5	66.5	68.4	5.35	5.9	6.45	7.35	8.25	8.55	1,754	2,046	2,522	3,239	4,499	5,814
Indonesia	63.3	65	66.3	67.2	68.1	69.1	6.75	7.2	8.7	9.3	9.95	10.3	4,337	5,930	5,308	6,547	8,267	10,130
Malaysia	70.7	71.8	72.8	73.6	74.1	74.8	8.1	8.9	10.25	10.15	11.35	11.35	9,772	13,439	14,500	17,157	19,725	23,712
Mexique	70.8	72.8	74.4	75.3	76.1	77	8.05	8.55	9.15	9.85	10.5	10.8	12,074	12,028	14,388	14,693	15,395	16,249
Philippines	65.3	66.1	66.7	67.2	67.7	68.3	8.7	8.95	9.5	9.75	9.75	10.2	3,962	4,111	4,994	6,058	7,478	8,232
Russia	68	66	65.1	65.8	68.6	70.3	10.95	10.85	11.85	12.6	13.1	13.35	19,461	12,011	12,933	17,797	21,075	22,094
South Africa	62.1	61.4	55.9	51.6	54.5	57.9	8.95	10.65	11	11.15	11.55	11.75	9,987	9,566	9,719	10,935	11,833	12,110
Turkey	64.3	67	70	72.5	74.2	75.6	6.7	7.2	8.3	8.95	10.55	11.05	10,494	11,317	12,807	14,987	16,506	18,976
Std. deviation	3.86	3.81	5.22	6.62	6.08	5.48	1.46	1.49	1.49	1.34	1.23	1.19	5,245	4,094	4,410	4,979	5,195	5,539

Appendix C

Let us consider A as a $m \times s$ matrix and b as a m-dimensional vector. $\mathcal{P}$ is a polyhedron subject to $A.x < b$ and whose largest inscribed ball is: $\mathcal{B}(x_c, r)$.

The linear programming related to Chebyshev centre of a polyhedron

max r
Such that $a_i^T x_c + r \|\|a_i\|\|_2 \leq b_i$ where $i = 1, ..., m;$

Source: Boyd and Vandenberghe (2009)

Boyd and Vandenberghe (2009) gives more details about this linear programming.

One may wonder if the Chebyshev centre can act as an analytic centre of the polyhedron $\mathcal{P}$. In fact, if we consider a 3-dimensional polyhedron, the larger w_3 with respect to w_1 the smaller inscribed ball in this polyhedron. It results that the Chebyshev centre cannot match with the analytic centre as defined by equation (16).