Mathematics

-Presented by

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CHAPTER 1

SETS

Mathematics, a set is defined as a collection of distinct, well-defined objects forming a group. There can be any number of items, be it a collection of whole numbers, months of a year, types of birds, and so on. Each item in the set is known as an element of the set. We use curly brackets while writing a set.

- Suppose two sets $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Find $A \times B$ and $B \times A$. Here, $A = \{a, b\}$ $B = \{1, 2, 3\}$ Now, $A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$ $B \times A = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

Roaster Form

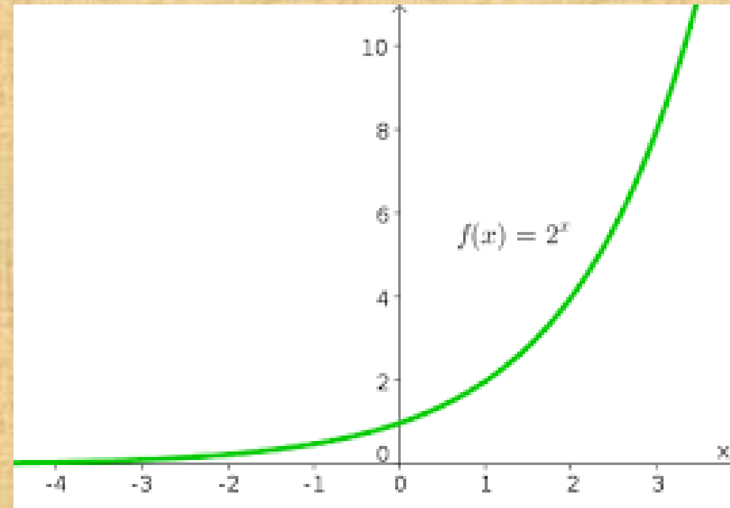
- Roster FormIn Roster form, all the elements of a set are listed.For example, the set of natural numbers less than 5.Natural Number = 1, 2, 3, 4, 5, 6, 7, 8,……….Natural Number less than 5 = 1, 2, 3, 4Therefore, the set is N = { 1, 2, 3, 4 }Set Builder FormThe general form is, A = { x : property }
- Example: Write the following sets in set builder form: A={2, 4, 6, 8}Solution:2 = 2 x 14 = 2 x 26 = 2 x 38 = 2 x 4So, the set builder form is A = {x: x=2n, n ∈ N and 1 ≤ n ≤ 4}Also, Venn Diagrams are the simple and best way for visualized representation of sets

- Example 2: Find A ∪ B and A ∩ B and A – B.If A = {a, b, c, d} and B = {c, d} .Solution: A = {a, b, c, d} and B = {c, d}A ∪ B = {a, b, c, d} A ∩ B = {c, d} and A – B = {a, b}
- Sets FormulasSome of the most important set formulas are:For any three sets A, B and Cn (A ∪ B) = n(A) + n(B) – n (A ∩ B)If A ∩ B = ∅, then n (A ∪ B) = n(A) + n(B)n(A – B) + n(A ∩ B) = n(A)n(B – A) + n(A ∩ B) = n(B)n(A – B) + n (A ∩ B) + n(B – A) = n (A ∪ B)n (A ∪ B ∪ C) = n(A) + n(B) + n(C) – n (A ∩ B) – n (B ∩ C) – n (C ∩ A) + n (A ∩ B ∩ C)

CHAPTER 2
Exponential and logarithm functions

- specify for which values of a the exponential function f(x) = a x may be defined,
- • recognize the domain and range of an exponential function,
- • identify a particular point which is on the graph of every exponential function,
- • specify for which values of a the logarithm function f(x) = loga x may be defined,
- • recognize the domain and range of a logarithm function,
- • identify a particular point which is on the graph of every logarithm function,
- • understand the relationship between the exponential function f(x) = e x and t

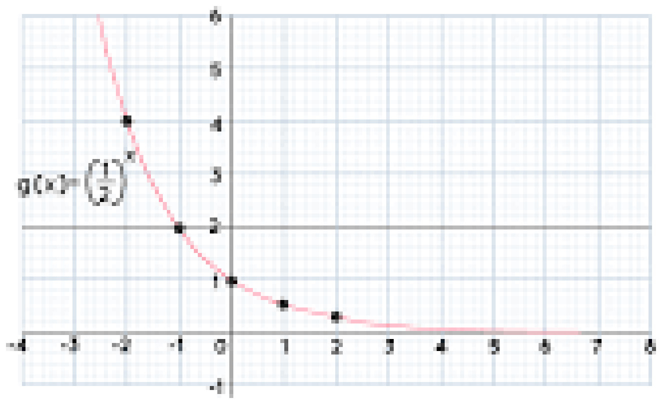
- 1. Exponential functions
- Consider a function of the form f(x) = a x , where a > 0. Such a function is called an exponential function. We can take three different cases, where a = 1, 0 < a < 1 and a > 1. If a = 1 then f(x) = 1x = 1. So this just gives us the constant function f(x) = 1.
- This example demonstrates the general shape for graphs of functions of the form f(x) = a x when a > 1. What is the effect of varying a? We can see this by looking at sketches of a few graphs of similar functions. f(x) x



Graphing the function $g(x) = (1/2)^x$

$$g(x) = \left(\frac{1}{2}\right)^x$$

x	g(x)	
-2	4	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	2	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$
0	1	$\left(\frac{1}{2}\right)^{0} = 1$
1	0.5	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2} = 0.5$
2	0.25	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4} = 0.25$

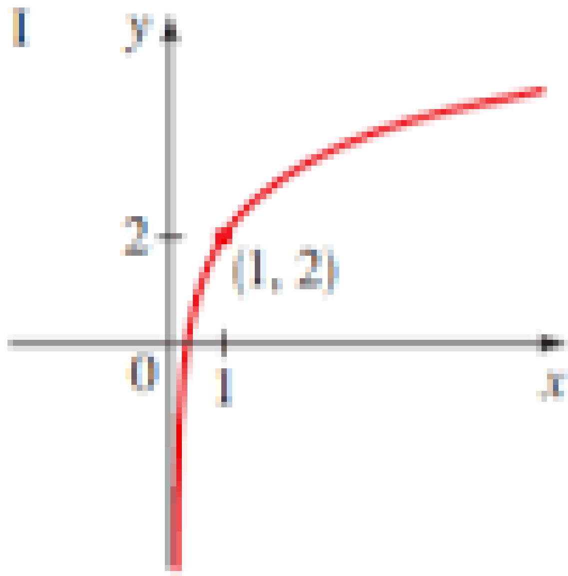


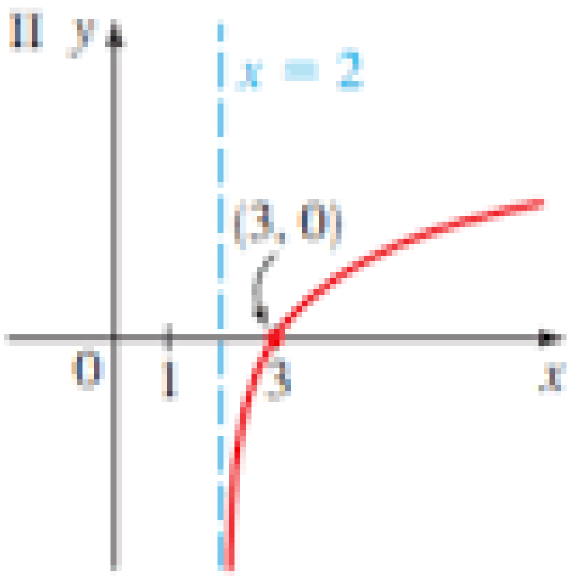
Logarithm functions

- We shall now look at logarithm functions. These are functions of the form f(x) = loga x where a > 0. We do not consider the case a = 1, as this will not give us a valid function.
- What happens if a > 1? To examine this case, take a numerical example. Suppose that a = 2. Then
- f(x) = log2 x means 2 f(x) = x.
- An important point to note here is that, regardless of the argument, 2 f(x) > 0. So we shall consider only positive arguments.
- f(1) = log2 1 means 2 f(1) = 1 so f(1) = 0
- f(2) = log2 2 means 2 f(2) = 2 so f(2) = 1
- f(4) = log2 4 means 2 f(4) = 4 so f(4) = 2
- f(1 2) = log2 (1 2) means 2 f(1 2) = 1 2 = 2−1 so f(1 2) = −1
- f(1 4) = log2 (1 4) means 2 f(1 4) = 1 4 = 2−2 so f(1 4) = −2
- We can put these results into a table, and plot a graph of the function.

57–58 ■ Graphing Logarithmic Functions Match the logarithmic function with one of the graphs labeled I or II.

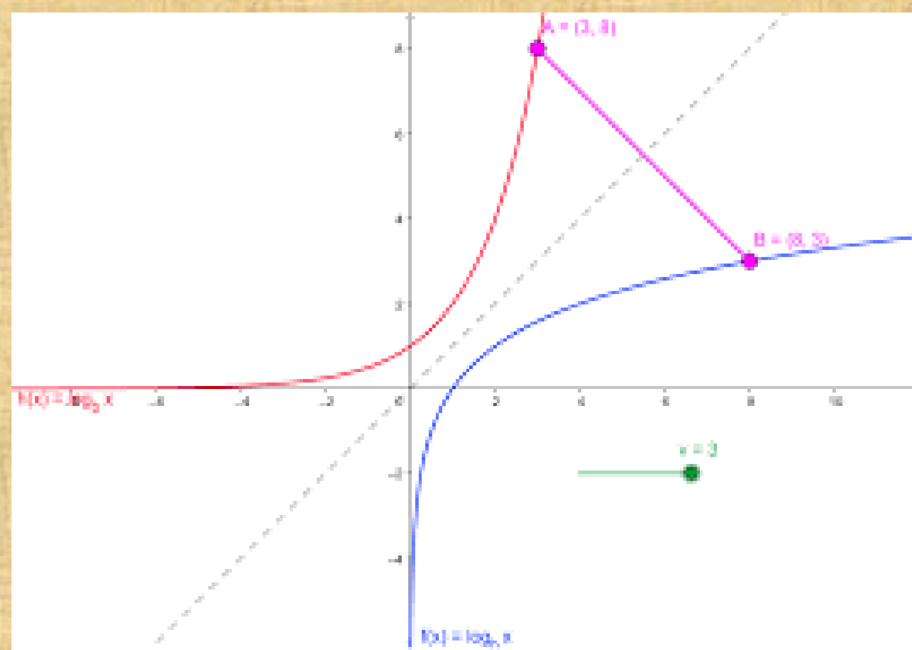
57. $f(x) = 2 + \ln x$ **58.** $f(x) = \ln(x - 2)$





The relationship between exponential functions and logarithm functions

- We can see the relationship between the exponential function f(x) = e x and the logarithm function f(x) = ln x by looking at their graphs.
- f(You can see straight away that the logarithm function is a reflection of the exponential function in the line represented by f(x) = x



CHAPTER 3
Euler's Theorem

- 1.Homogeneous Function
- A function f of two independent variables x, y is said to be a homogeneous function of degree n if it can be put in either of the following two forms :
-

$$\frac{\partial f}{\partial z} = 3x^2y + 5xy^2 + 16z^3$$

Now

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 6xyz + 5y^2z + 3x^2z + 10xyz + 3x^2y + 5xy^2 + 16z^3$$

$$= 4(3x^2yz + 5xy^2z + 4z^4)$$

$$= 4\ f(x, y, z)$$

This verify Euler's theorem.

Generalization of Euler's theorem-

If $f(x_1, x_2, x_3, \ldots\ldots, x_m)$ be a homogeneous function of variables $x_1, x_2, x_3, \ldots\ldots, x_m$ of degree n, than

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_m\frac{\partial f}{\partial x_m} = nf$$

Key Words: Homogeneous function, degree of function.

Contents

1. Homogeneous Function

Example : The function

$$f(x,y)=\frac{x^4+y^4}{x-y}$$

is a homogeneous function of degree 3, since

$$f(x,y)=\frac{x^4+y^4}{x-y}=\frac{x^4\left[1+\left(\frac{y}{x}\right)^4\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}=x^3\frac{\left[1+\left(\frac{y}{x}\right)^4\right]}{\left[1-\left(\frac{y}{x}\right)\right]}=x^3\phi\left(\frac{y}{x}\right)$$

or alternatively,

$$f(t\,x,t\,y)=\frac{(t\,x)^4+(t\,y^4)}{t\,x-t\,y}=\frac{t^4\left(x^4+y^4\right)}{t\,(x-y)}=t^3\,\frac{\left(x^4+y^4\right)}{(x-y)}=t^3 f\,(x,y).$$

Similarly, the function

$$f(x,y,z)=\frac{x^3+y^3+z^3}{x+y+z}$$

is a homogeneous function of degree 2, since

$$f(x,y,z)=\frac{x^3+y^3+z^3}{x+y+z}=\frac{x^3\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{x\left[1+\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\right]}=x^2\frac{\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{\left[1+\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\right]}=x^2\phi\left(\frac{y}{x},\frac{z}{x}\right)$$

$$f(t\,x,t\,y,t\,z)=\frac{(t\,x)^3+(t\,y^3)+(t\,z)^3}{t\,x+t\,y+t\,z}=\frac{t^3\left(x^3+y^3+z^3\right)}{t\left(x+y+z\right)}=t^2\,\frac{\left(x^3+y^3+z^3\right)}{\left(x+y+z\right)}=t^2 f\left(x,y,z\right).$$

2. Euler's Theorem on Homogeneous Function of Two Variables

Statement : If u be a homogeneous function of degree n in two independent variables x, y, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Proof : Let

$$u=A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}+A_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots+A_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(1)$$

where $\alpha_1+\beta_1=\alpha_2+\beta_2=\alpha_3+\beta_3=\ldots\ldots\ldots\ldots=\alpha_n+\beta_n=n$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=A_1\left(\alpha_1x^{\alpha_1-1}\right)y^{\beta_1}+A_2\left(\alpha_2x^{\alpha_2-1}\right)y^{\beta_2}+A_3\left(\alpha_3x^{\alpha_3-1}\right)y^{\beta_3}+\ldots\ldots+A_n\left(\alpha_nx^{\alpha_n-1}\right)y^{\beta_n}$$

This $\Rightarrow x\frac{\partial u}{\partial x}=A_1\alpha_1x^{\alpha_1}y^{\beta_1}+A_2\alpha_2x^{\alpha_2}y^{\beta_2}+A_3\alpha_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots+A_n\alpha_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(2)$

Now, differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=A_1x^{\alpha_1}\left(\beta_1y^{\beta_1-1}\right)+A_2x^{\alpha_2}\left(\beta_2y^{\beta_2-1}\right)+A_3x^{\alpha_3}\left(\beta_3y^{\beta_3-1}\right)+\ldots\ldots+A_nx^{\alpha_n}\left(\beta_ny^{\beta_n-1}\right)$$

This $\Rightarrow y\frac{\partial u}{\partial y}=A_1\beta_1x^{\alpha_1}y^{\beta_1}+A_2\beta_2x^{\alpha_2}y^{\beta_2}+A_3\beta_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots+A_n\beta_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(3)$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(\alpha_1+\beta_1)A_1x^{\alpha_1}y^{\beta_1}+(\alpha_2+\beta_2)A_2\,x^{\alpha_2}y^{\beta_2}+(\alpha_3+\beta_3)A_3\,x^{\alpha_3}y^{\beta_3}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots+(\alpha_n+\beta_n)A_n\,x^{\alpha_n}y^{\beta_n}$$

$$=n\,A_1x^{\alpha_1}y^{\beta_1}+n\,A_2\,x^{\alpha_2}y^{\beta_2}+n\,A_3\,x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots\ldots\ldots+n\,A_n\,x^{\alpha_n}y^{\beta_n}$$

$$(\because\ \alpha_1+\beta_1=\alpha_2+\beta_2=\alpha_3+\beta_3=\ldots\ldots\ldots\ldots=\alpha_n+\beta_n=n)$$

$$=n\left(A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}+A_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots+A_nx^{\alpha_n}y^{\beta_n}\right)$$

$=nu$ (using equation (1))

i.e., $$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Example

Example 1 : Verify Euler's Theorem when $u = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$.

Solution : According to Euler's Theorem, if u be a homogeneous function of degree n in two independent variables x, y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Given that

$$u = \frac{x(x^3 - y^3)}{x^3 + y^3} \qquad .....(1)$$

i.e., $$u = \frac{x^4\left[1-\left(\frac{y}{x}\right)^3\right]}{x^3\left[1+\left(\frac{y}{x}\right)^3\right]} = x\frac{\left[1-\left(\frac{y}{x}\right)^3\right]}{\left[1+\left(\frac{y}{x}\right)^3\right]} = x\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

This $\Rightarrow$ The given function u is a homogeneous function of degree 1 in two independent variables x, y. Therefore Euler's Theorem will be verified if we can prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u.$$

Taking logarithm of both sides of equation (1), we get

$$\log u=\log x+\log(x^3-y^3)-\log(x^3+y^3) \quad ....(2)$$

Now, differentiating both sides of equation (2) partially w. r. t. x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}+\frac{1}{x^3-y^3}(3x^2)-\frac{1}{x^3+y^3}(3x^2)$$

This $\Rightarrow \dfrac{1}{u}\left(x\dfrac{\partial u}{\partial x}\right)=1+\dfrac{3x^3}{x^3-y^3}-\dfrac{3x^3}{x^3+y^3}$ ....(3)

Similarly, differentiating both sides of equation (2) partially w. r. t. y, we get

$$\frac{1}{u}\frac{\partial u}{\partial y}=0+\frac{1}{x^3-y^3}(-3y^2)-\frac{1}{x^3+y^3}(3y^2)$$

This $\Rightarrow \dfrac{1}{u}\left(y\dfrac{\partial u}{\partial y}\right)=-\dfrac{3y^3}{x^3-y^3}-\dfrac{3y^3}{x^3+y^3}$ ....(4)

Adding equations (3) and (4), we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3-y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{x^3+y^3}$$

$$=1+3-3$$

$$=1$$

This $\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$

$\Rightarrow$ Euler's Theorem is verified for the given function.

Example 4 : If $u = \cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0.$

Solution : Given that

$$u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}.$$

This $\Rightarrow \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left[1++\dfrac{y}{x}\right]}{\sqrt{x}\left[1++\sqrt{\dfrac{y}{x}}\right]} = x^{\frac{1}{2}}\dfrac{\left[1++\dfrac{y}{x}\right]}{\left[1++\sqrt{\dfrac{y}{x}}\right]} = x^{\frac{1}{2}}\varphi\left(\dfrac{y}{x}\right)$, where ϕ is a function of $\dfrac{y}{x}$.

$\Rightarrow \cos u$ is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y.

Let $v = \cos u$ ......(1)

Then v is a homogeneous function of degree $\frac{1}{2}$ in two independent variables x, y. Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v.$$

This $\Rightarrow x\left(-\sin u\frac{\partial u}{\partial x}\right)+y\left(-\sin u\frac{\partial u}{\partial y}\right)=\frac{1}{2}\cos u$

$$\left(\because \frac{\partial v}{\partial x}=-\sin u\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y}=-\sin u\frac{\partial u}{\partial y}\text{ and } v=\cos u, \text{ by equation (1)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\cos u}{\sin u}$$

$$=-\frac{1}{2}\cot u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$$