Q1.

$$\text{P(guessing the right answer)} = \frac{1}{4}$$

$$\text{P(get at least four questions correct)} = \text{P(five correct)} + \text{P(four correct)}$$

$$\text{P(get at least four questions correct)} = \left(\frac{1}{4}\right)^5 + \ C_1^5 \times \left(\frac{3}{4}\right) \times \left(\frac{1}{4}\right)^4 = 0.015625$$

Q2.

$$\text{P(E team win)} = \ \text{p}$$

$$P(4\ games) = (1-p)^0 p^4 + (1-p)^4 p^0$$

$$P(5\ games) = C_1^4 (1-p)^1 p^4 + C_1^4 (1-p)^4 p^1$$

$$P(6\ games) = C_2^5 (1-p)^2 p^4 + C_2^5 (1-p)^4 p^2$$

$$P(7\ games) = C_3^6 (1-p)^3 p^4 + C_3^6 (1-p)^4 p^3$$

$$\text{E(games)} = 4(p^4 + (1-\text{p})^4) + \ 5([C_1^4(1-p)^1p^4 + C_1^4(1-p)^4p^1]) + 6\left(\left[C_2^5(1-p)^2p^4 + C_2^5(1-p)^4p^2\right]\right)$$

$$+ 7(C_3^6(1-p)^3p^4 + C_3^6(1-p)^4p^3)$$

$$\text{E(games|p} = \frac{1}{2}) \ = \ 5.8125$$

Q3.

$$\text{Define X} \ = \ \text{wait time till the show}, \qquad \text{X}\sim\text{Uniform(0,80)}, \qquad \int(x) = \frac{1}{80}, \qquad 0 < \text{x} < 80$$

$$\text{P(X} \le 20) = \int_0^{20} \frac{1}{80}\text{dx} \ = \frac{20}{80} = \frac{1}{4}$$

Q4.

$$\text{Define clerk1's service time} \ = \ \text{X}, \qquad \text{X}\sim\text{Exponential}\left(\frac{1}{\mu_1}\right), \qquad \text{E(X)} = \mu_1$$

$$\text{Define clerk2's service time} \ = \ \text{Y}, \qquad \text{Y}\sim\text{Exponential}\left(\frac{1}{\mu_2}\right), \qquad \text{E(Y)} = \mu_2$$

$$\text{a.}\,P(\text{Jhon still being served}) = P(\text{X} > 10) = \int_{10}^{\infty} \mu_1 e^{-\mu_1 x}\,\text{d}x = \ [-e^{-\mu_1 \text{x}}]_{10}^{\infty} = e^{-10\mu_1}$$

$$b.\,P(\text{Mary finish first}) = P(X > Y)$$

$$f_{xy}(\text{x,y}) = f_x(\text{x}) \times f_y(\text{y}) = \ \mu_1\mu_2 \times e^{-\mu_1\mu_2 xy}, \qquad x > 0, y > 0$$

$$P(X > Y) = \int_0^{\infty}\int_0^{x} \mu_1\mu_2 e^{-\mu_1 x - \mu_2 y}\,\text{d}xdy = \ [-e^{-\mu_2 \text{X}_2}]_0^{\mu_1} = \mu_1\mu_2 \int_0^{\infty} e^{-\mu_1 x} \int_0^{x} e^{-\mu_2 y}\,\text{d}xdy$$

$$= \mu_1\mu_2 \int_0^\infty e^{-\mu_1 x}\left[\frac{-1}{\mu_2}e^{-\mu_1 y}\right]_0^x dx = -\mu_1 \int_0^\infty e^{-\mu_2 y}\left[e^{-\mu_1 x}\right]_0^x dx = -\mu_1\left[\frac{-1}{\mu_1+\mu_2}e^{-(\mu_1+\mu_2)x} + \frac{1}{\mu_1}e^{-\mu_1 x}\right]_0^\infty$$

$$= \frac{-\mu_1}{\mu_1+\mu_2} + 1 = \frac{\mu_2}{\mu_1+\mu_2}$$

Q5.

Define A is the time of train arrive at the train station(minutes)

Define B is the time of bus departure from the train station (minutes), B~Normal($\mu = 8{:}20, \sigma = 2$)

$$P(late\ for\ work) = P(A > B) + P(B > 8{:}20)$$

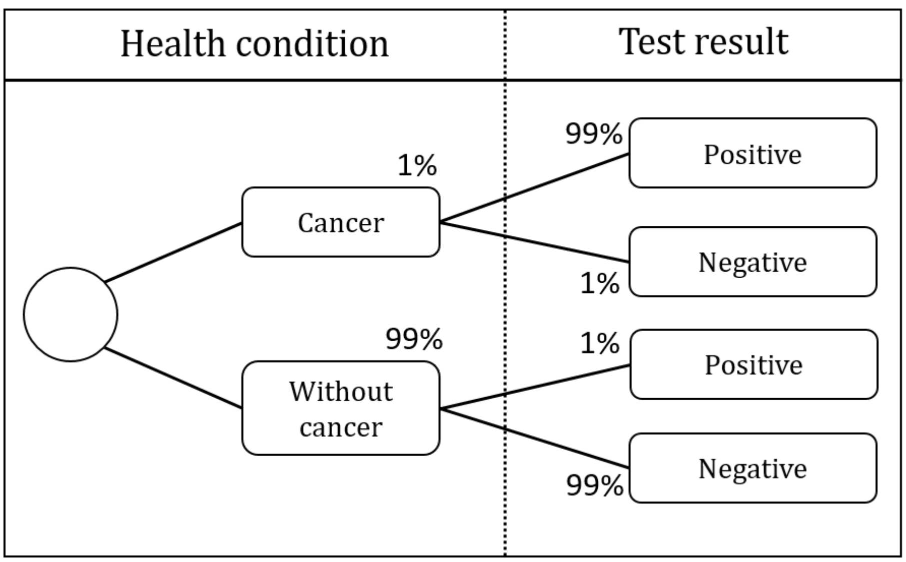
$$= P(A = 8{:}14) \times P(B < 8{:}14) + P(A = 8{:}16) \times P(B < 8{:}16) + P(A = 8{:}18) \times P(B < 8{:}18)$$

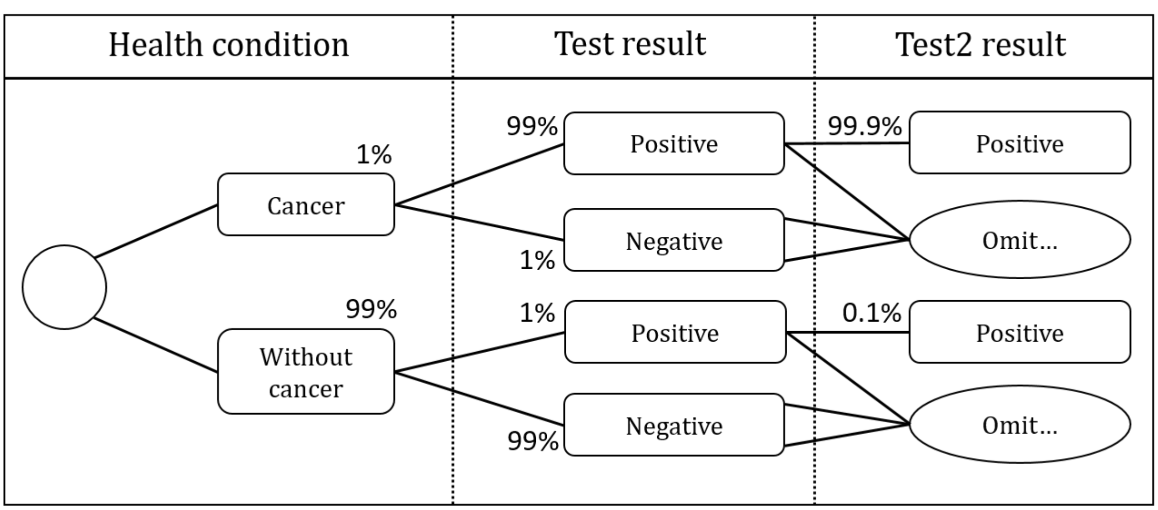
$$+ P(A = 8{:}20) \times P(B < 8{:}20) + P(A = 8{:}22) + P(B > 8{:}20)$$

$$= \frac{1}{16} \times P(Z > 3) + \frac{1}{8} \times P(Z > 2) + \frac{1}{2} \times P(Z > 1) + \frac{1}{4} \times P(Z > 0) + \frac{1}{16} + P(Z > 0)$$

$$= \frac{0.0013}{16} + \frac{0.0228}{8} + \frac{0.1587}{2} + \frac{0.5}{4} + \frac{1}{16} + \frac{1}{2} = 0.76978125$$

Q6.





$$Known:\ \ P(cancer) = \frac{1}{100}\ \ and,\qquad P(positive|\ cancer) = \frac{99}{100}$$

$$P(positive \mid cancer) = \frac{P(positive\ \cap\ cancer)}{P(cancer)}$$

$$P(positive\ \cap cancer) = \frac{99}{100} \times \frac{1}{100} = \frac{99}{10000}$$

$$a. P(Jhon\ got\ cancer) = P(cancer \mid positive) = \frac{P(positive\ \cap\ cancer)}{P(positive)}$$

$$= \frac{P(positive\ \cap\ cancer)}{P(cancer \mid positive) \times P(positive) + P(without\ cancer \mid positive) \times P(positive)}$$

$$= \frac{\frac{99}{10000}}{\frac{99}{10000} + \frac{99}{10000}} = \frac{1}{2}$$

$$b. P(Jhon\ still\ got\ cancer) = \frac{\frac{99}{10000} \times \frac{999}{1000}}{\frac{99}{10000} \times \frac{999}{1000} + \frac{99}{10000} \times \frac{1}{1000}} = 0.999$$

Q7.

$$\text{Define X} = \text{customer waiting time,}\qquad \overline{\mathrm{X}} \overset{a}{\underset{CLT}{\rightarrow}} \mathrm{Normal}(\mu = 8.5, \sigma = 3.5)$$

while sample size = 49 (By Central Limit Theorem)

$$a. \mathrm{P}(\overline{\mathrm{X}} < 10) = \mathrm{P}\left(\frac{\overline{\mathrm{X}} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{10 - \mu}{\frac{\sigma}{\sqrt{n}}}\right) = \mathrm{p}\left(\mathrm{Z} < \frac{10 - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= \mathrm{P}\left(Z < \frac{10 - 8.5}{\frac{3.5}{\sqrt{49}}}\right) = \mathrm{P}(\mathrm{Z} < 3) = 0.9987$$

$$b. \mathrm{P}(7 \le \overline{\mathrm{X}} \le 10) = \mathrm{P}\left(\frac{7 - \mu}{\frac{\sigma}{\sqrt{n}}} \le \frac{\overline{\mathrm{X}} - \mu}{\frac{\sigma}{\sqrt{n}}} \le \frac{10 - \mu}{\frac{\sigma}{\sqrt{n}}}\right) = \mathrm{p}\left(\frac{7 - \mu}{\frac{\sigma}{\sqrt{n}}} \le \mathrm{Z} \le \frac{10 - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= \mathrm{P}\left(\frac{7 - 8.5}{\frac{3.5}{\sqrt{49}}} \le Z \le \frac{10 - 8.5}{\frac{3.5}{\sqrt{49}}}\right) = \mathrm{P}(-3 \le \mathrm{Z} \le 3) = 0.9974$$

$$c. \mathrm{P}(7.5 < \overline{\mathrm{X}}) = \mathrm{P}\left(\frac{7.5 - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\overline{\mathrm{X}} - \mu}{\frac{\sigma}{\sqrt{n}}}\right) = \mathrm{p}\left(\frac{7.5 - \mu}{\frac{\sigma}{\sqrt{n}}} < \mathrm{Z}\right)$$

$$= \mathrm{P}\left(\frac{7.5-8.5}{\frac{3.5}{\sqrt{49}}} < Z\right) = \mathrm{P}(-2 < \mathrm{Z}) = 0.0228$$

Q8.

$$\begin{cases} \mathrm{H_0}: \ \mathrm{p} = 0.4 \\ \mathrm{H_1}: \ \mathrm{p} \neq 0.4 \end{cases}$$

Define X = use iPhone or not, $R_x = \{0,1\}$, X~Bernoulli(p)

np = iPhone users of the 600 people.

$$\mathrm{np} \rightarrow \mathrm{Binomial}(\mathrm{n} = 600, \mathrm{p}), \qquad \mathrm{np} \underset{CLT}{\overset{a}{\rightarrow}} \mathrm{Normal}\left(\mu = \mathrm{np}, \sigma = \sqrt{np(1-p)}\right)$$

$$\alpha = \mathrm{P(Type\ 1\ error)} = \mathrm{P(reject\ H_0 \mid H_0\ is\ true)} = \mathrm{P(264 < X\ or\ X < 216 \mid p = 0.4)}$$

$$= \mathrm{P}\left(\frac{264-np}{\sqrt{np(1-p)}} < \frac{X-np}{\sqrt{np(1-p)}} \ or \ \frac{X-np}{\sqrt{np(1-p)}} < \frac{216-np}{\sqrt{np(1-p)}} \mid p = 0.4\right)$$

$$= \mathrm{P}\left(\frac{264-240}{\sqrt{600\times0.4\times0.6}} < \mathrm{Z\ or\ Z} < \frac{216-240}{\sqrt{600\times0.4\times0.6}}\right) = P(2 < Z) or\ P(P > 2) = 0.0228 \times 2 = 0.0456$$

Q9.

$$\begin{cases} \mathrm{H_0}: \ \mathrm{X\ follow\ Binomial\ distribution} \\ \mathrm{H_1}: \ \mathrm{X\ not\ follow\ Binomial\ distribution} \end{cases}$$

Significant level $\alpha = 0.05$

Assume $\mathrm{X} \sim \mathrm{Binomial}(\mathrm{n} = 24, \mathrm{p})$

Define $\mathrm{N_b}$ as numbers of the Binomial trial, $\mathrm{N_b} = 75$

Define N Total bottles, $\mathrm{N} = \mathrm{n} \times \mathrm{N_b} = 24 \times 75 = 1800$

$$\text{Total under} - \text{filled beer bottles} = \mathrm{N}\hat{\mathrm{p}} = \sum_{\mathrm{i}=1}^{4} \mathrm{frequency_i} \times \mathrm{X_i} = 50$$

$$\widehat{\mathrm{P}} = \frac{\mathrm{N}\widehat{\mathrm{P}}}{\mathrm{N}} = \frac{1}{36} \ \overset{estimate}{\rightarrow} \ \mathrm{P}$$

X	0	1	2	3
O_i	39	23	12	1
P_i	$C_0^{24} \times \left(\frac{1}{36}\right)^0 \times \left(\frac{35}{36}\right)^{24} = 0.5086$	$C_1^{24} \times \left(\frac{1}{36}\right)^1 \times \left(\frac{35}{36}\right)^{23} = 0.3488$	$C_2^{24} \times \left(\frac{1}{36}\right)^2 \times \left(\frac{35}{36}\right)^{22} = 0.1146$	$C_3^{24} \times \left(\frac{1}{36}\right)^3 \times \left(\frac{35}{36}\right)^{21} = 0.0240$
E_i	38.1447	26.1564	8.5942	1.8007

$(E_i = \mathrm{N_b} \times P_i)$

$$\text{Teststatistic}: \sum_{i=1}^{4} \frac{(\mathrm{O_i}-\mathrm{E_i})^2}{\mathrm{E}_i} \overset{\mathrm{H_0}}{\sim} \mathrm{X}^2(\mathrm{i}-1-1) = \mathrm{X}^2(4-1-1) = \mathrm{X}^2(2)$$

$$Reject\ region = \{\mathrm{X}^2 \geq X^2_{0.05}(2)\}$$

$$X_0^2 = \sum_{i=0}^{3} \frac{(\mathrm{O_i}-\mathrm{E_i})^2}{\mathrm{Ei}} = \sum_{i=0}^{3} \frac{\mathrm{O_i}^2}{\mathrm{Ei}} - 75$$

$$= \frac{39^2}{38.1147} + \frac{23^2}{26.1564} + \frac{12^2}{8.5942} + \frac{1^2}{1.8007} - 75$$

$$= 2.4412$$

$\because X_0^2 = 2.4412 < X^2_{0.05}(2) = 5.991,\quad \therefore$ do not reject $\mathrm{H_0}$,

$\gg$ can not reject that x follow Binomial distribution, at $\alpha = 0.05$ singnificant level.

Q10.

$$\begin{cases} \mathrm{H_0}: \text{ grades in Prob and OR are related} \\ \mathrm{H_1}: \text{ grades in Prob and OR are not related} \end{cases}$$

Significant level $\alpha = 0.01$

PRO \ OR	A	B	C	
A	24 / 15.75	11 / 13.5	10 / 15.75	45
B	7 / 8.75	13 / 7.5	5 / 8.75	25
C	4 / 10.5	6 / 9	20 / 10.5	30
	35	30	35	100

$$(E_{ij} = \mathrm{n} \times p_i = \frac{R_i \times C_i}{n})$$

$$Teststatistic: \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}} \overset{H_0}{\sim} X^2\big((3-1)\times(3-1)\big) = X^2(4)$$

$$Reject\ region = \{X^2 \geq X^2_{0.01}(4)\}$$

$$X_0^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{{O_i}^2}{Ei} - 100$$

$$= \frac{24^2}{15.75} + \frac{11^2}{13.5} + \frac{10^2}{15.75} + \frac{7^2}{8.75} + \frac{13^2}{7.5} + \frac{5^2}{8.75} + \frac{4^2}{10.5} + \frac{6^2}{9} + \frac{20^2}{10.5} - 100$$

$$= 26.4931$$

$\because X_0^2 = 26.4931 > X^2_{0.01}(4) = 13.277,\quad \therefore$ reject $\mathrm{H_0}$,

$\gg$ reject that grades in Prob &and OR are related, at $\alpha = 0.01$ singnificant level.