工業工程所碩士班 R08546036 趙上涵 HW03

(1)

$$\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$$

$$Var(\bar{y}) = Var\left(\frac{y_1+y_2+y_3\ldots+y_n}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j)$$

$$\Rightarrow \frac{1}{n^2}\sum_{i-j=(-n)}^{n} (n-|i-j|)\,r(i-j) = \frac{1}{n}\sum_{k=(-n+1)}^{n-1}\left(1-\frac{|k|}{n}\right) r(k) = \frac{1}{n}\sum_{k=(-n+1)}^{n-1}\left(1-\frac{|k|}{n}\right) r_{|k|} \#$$

$$\Rightarrow \frac{1}{n}\left[r_0 + 2\cdot\sum_{|k|=1}^{n-1}\left(1-\frac{|k|}{n}\right) r(|k|)\right] \Rightarrow \frac{r_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right) r(k) \#$$

$$\Rightarrow \frac{r_0}{n} + \frac{2}{n}\sum_{m=1}^{n-1}\left(1-\frac{k}{n}\right) r(k) \#$$

(2)(a) $y_t = \begin{cases} X_t & (\text{Odd } t) \\ X_t + 3 & (\text{Even } t) \end{cases}$; while X_t is stationary process.

$Cov(y_t, y_{t-k}) = Cov(X_t + 3, X_{t-k} + 3) = Cov(X_t, X_{t-k})$

is free / independent since X_t is stationary. #

(2)(b)

$\{Y_t\}$ is not stationary, since $E(y_t) = E(X_t) = \mu$ for odd t

while also $E(y_t) = E(X_t + 3) = (\mu + 3)$ for even t,

so y_t is not independent. #

(3)(a) Y_t is stationary process with ACF r_k

$Z_t = \nabla Y_t = Y_t - Y_{t-1}$

$E(Z_t) = E(Y_t - Y_{t-1}) = E(Y_t) - E(Y_{t-1}) = 0$ since Y_t is stationary

$Cov(Y_t, Y_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1})$

$= Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-k-1}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-k-1})$

$= r_k - r_{k+1} - r_{k-1} + r_k = 2r_k - r_{k+1} - r_{k-1} \Rightarrow$ free for t

$\Rightarrow Z_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary. #

3(b) $W_t = \nabla^2 Y_t = Z_t - Z_{t-1}$

Since Y_t, $\nabla Y_t (Z_t)$ are stationary.

and W_t is first difference of process Y_t.

Hence we could say that W_t is also stationary. #

Q4(a)

X_t with $E(X_t)=0$; $V(X_t)=1$ ACF : ρ_k.

μ_t nonconstant function σ_t is postive nonconstant function.

$Y_t = \mu_t + \sigma_t X_t$

$Cov(X, X_{t-k}) = Cor(X, X_{t-k})$ since $V(X_t)=1$

$E(Y_t) = E(\mu_t + \sigma_t X_t) = \mu_t + \sigma_t E(X_t) = \mu_t$ = Mean of Y_t #

$Cov(Y_t, Y_{t-k}) = Cov(\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k})$

$\Rightarrow \sigma_t \sigma_{t-k} Cov(X_t, X_{t-k}) = \sigma_t \sigma_{t-k} \cdot \rho_k$ = ACF of Y_t #

Q4(b)

$Corr(Y_t, Y_{t-k}) = \dfrac{\cancel{\sigma_t} \cdot \cancel{\sigma_{t-k}} \cdot \rho_k}{\cancel{\sigma_t} \cancel{\sigma_{t-k}}} = \rho_k$,

but Y_t is not necessary stationary, since $E(Y_t) = \mu_t$ #

Q4(c)

If μ_t remains constant, and σ_t varies within time.

then this time series would fit the requirement. #

Q5

$y_t = X_t + e_t$, with X: signal, e_t: measurement noise.

$E(y_t) = E(X_t) + E(e_t)$

when $k \geq 1$, $Cov(y_t, y_{t-k}) = Cov(X_t + e_t, X_{t-k} + e_{t-k})$

$\Rightarrow Cov(X_t, X_{t-k}) + Cov(e_t, e_{t-k})$ ($\because X_t, e_t$ independent)

$\Rightarrow Cov(X_t, X_{t-k}) = Var(X_t)\rho_k$ free for t.

$$Corr(y_t, y_{t-k}) = \frac{Cov(y_t, y_{t-k})}{Var(y_t)} = \frac{Var(X_t)\rho_k}{Var(X_t)+\sigma_e^2} = \frac{\sigma_X^2 \rho_k}{\sigma_X^2+\sigma_e^2}$$

$$= \frac{\rho_k}{1+\frac{\sigma_e^2}{\sigma_X^2}}$$ while $k \geq 1$ #

Q6.

$$E(y_t) = a_0 + \sum_{i=1}^{q} \left[E(a_i)\cos(2\pi f_i t) + E(\beta_i)\sin(2\pi f_i s) \right]$$

$$= a_0 \quad (\text{since } E(a_i) \text{ \& } E(\beta_i) \text{ are } 0).$$

$$Cov(y_t, y_s) = Cov\left[\sum_{i=1}^{q} [a_i \cos(2\pi f_i t) + (\beta_i)\sin(2\pi f_i t)), \right.$$

$$\left. \sum_{j=1}^{q} [a_j \cos(2\pi f_j s) + (\beta_i)\sin(2\pi f_j s)) \right]$$

$$\Rightarrow \sum_{i=1}^{q} Cov\{a_i \cos(2\pi f_i t), a_i \cos(2\pi f_i s)\} + \sum_{i=1}^{q} Cov(\beta_i \sin(2\pi f_i t), \beta_i \sin(2\pi f_i s))$$

$$\Rightarrow \sum_{i=1}^{q} \{V(a_i) \cdot \cos(2\pi f_i t) \cdot \cos(2\pi f_i s)\} + \sum_{i=1}^{q} Cov(Var(\beta_i)\sin(2\pi f_i t)(2\pi f_i s))$$

$$\Rightarrow \sum_{i=1}^{q} \{\cos(2\pi f_i(t-s)) + \cos(2\pi f_i(t+s))\} \frac{\sigma_i^2}{2} + \sum_{i=1}^{q} \{\cos(2\pi f_i(t-s)) - \cos(2\pi f_i(t+s))\} \frac{\sigma_i^2}{2}$$

$$\Rightarrow \sum_{i=1}^{k} \cos(2\pi f_i(t-s))\, \sigma_i^2 = ACF_{y_t}$$

$\Rightarrow y_t$ is stationary #