1.

(a)

$$\mathrm{V}[\bar{y}]$$

$$= \mathrm{V}[\frac{y_1 + y_2 + ... + y_n}{n}]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{COV}[y_i, y_j]$$

$$= \frac{1}{n^2}\mathbf{1}_n^T \Gamma_n \mathbf{1}_n$$

$$= \frac{1}{n^2}[\sum_{i=1}^{n}\Gamma_{ii} + 2\sum_{i=1}^{n}\sum_{i<k}\Gamma_{ik}]$$

$$= \frac{1}{n}\gamma_0 + \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{i<k}\Gamma_{ik}$$

$$\because \Gamma_{ij} = \Gamma_{ji}$$

$$\therefore \sum_{i=1}^{n-1}\sum_{i<k}\Gamma_{ik} = \sum_{i=1}^{n-1}(n-2k)\Gamma_{ki}$$

$$\Rightarrow \frac{1}{n}\gamma_0 + \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{i<k}\Gamma_{ik}$$

$$= \frac{1}{n}\gamma_0 + \frac{2}{n}\sum_{k=1}^{n-1}(1-\frac{k}{n})\gamma_k$$

(b)

$$\mathrm{V}[\bar{y}]$$

$$= \frac{1}{n}\gamma_0 + \frac{2}{n}\sum_{i=1}^{n-1}(1-\frac{k}{n})\gamma_k$$

$$= \frac{1}{n}\gamma_0 + \frac{1}{n}\sum_{k=-n+1}^{-1}(1-\frac{|k|}{n})\gamma_k + \frac{1}{n}\sum_{k=1}^{n-1}(1-\frac{k}{n})\gamma_k$$

$$= \frac{1}{n}\sum_{k=-n+1}^{n-1}(1-\frac{|k|}{n})\gamma_k$$

2.

(a)

y_t can be written as $y_t = x_t + 1.5(-1)^t + 1.5$

$\therefore COV[y_t, y_{t-k}]$

$= COV[x_t + 1.5(-1)^t + 1.5, x_{t-k} + 1.5(-1)^{t-k} + 1.5]$

$= COV[x_t + 1.5(-1)^t, x_{t-k} + 1.5(-1)^{t-k}]$

$= COV[x_t, x_{t-k}] + COV[x_t, 1.5(-1)^{t-k}] + COV[1.5(-1)^t, x_{t-k}] + COV[1.5(-1)^t, 1.5(-1)^{t-k}]$

$\because x_t$ is a stationary process $\Rightarrow COV[x_t, x_{t-k}]$ only depends on k

$\because COV[x_t, 1.5(-1)^{t-k}] = -COV[1.5(-1)^t, x_{t-k}] \Rightarrow COV[x_t, 1.5(-1)^{t-k}] + COV[1.5(-1)^t, x_{t-k}] = 0$

$\because COV[1.5(-1)^t, 1.5(-1)^{t-k}] = 1.5^2(-1)^{-k} COV[(-1)^t, (-1)^t]$

$\because COV[(-1)^t, (-1)^t]$ is a constant

$\Rightarrow 1.5^2(-1)^{-k} COV[(-1)^t, (-1)^t]$ only depends on k

$\therefore COV[x_t, x_{t-k}] + COV[x_t, 1.5(-1)^{t-k}] + COV[1.5(-1)^t, x_{t-k}] + COV[1.5(-1)^t, 1.5(-1)^{t-k}]$
only depends on k

$\Rightarrow COV[y_t, y_{t-k}]$ only depends on k

(b)

$\because \mu_y$ is not fixed, it will depends on the ratio between odd numbers and even numbers

$\therefore y_t$ is not stationary

3.

(a)

$E[z_t]$

$= E[y_t - y_{t-1}]$

$= E[y_t] - E[y_{t-1}]$

$= \mu_y - \mu_y$

$= 0$

$COV[z_t, z_{t-k}]$

$= COV[y_t - y_{t-1}, y_{t-k} - y_{t-k-1}]$

$= COV[y_t, y_{t-k}] - COV[y_t, y_{t-k-1}] - COV[y_{t-1}, y_{t-k}] + COV[y_{t-1}, y_{t-k-1}]$

$= \gamma_k - \gamma_{k-1} - \gamma_{k+1} + \gamma_k$

$= 2\gamma_k - \gamma_{k-1} - \gamma_{k+1}$

(b)

$E[w_t]$
$= E[z_t - z_{t-1}]$
$= E[z_t] - E[z_{t-1}]$
$= 0$

$COV[w_t, w_{t-k}]$
$= COV[z_t - z_{t-1}, z_{t-k} - z_{t-k-1}]$
$= COV[z_t, z_{t-k}] - COV[z_t, z_{t-k-1}] - COV[z_{t-1}, z_{t-k}] + COV[z_{t-1}, z_{t-k-1}]$
$= (2\gamma_k - \gamma_{k-1} - \gamma_{k+1}) - (2\gamma_{k-1} - \gamma_{k-2} - \gamma_k) - (2\gamma_{k+1} - \gamma_k - \gamma_{k+2}) + (2\gamma_k - \gamma_{k-1} - \gamma_{k+1})$
$= 6\gamma_k - 4\gamma_{k-1} - 4\gamma_{k+1} + \gamma_{k-2} + \gamma_{k+2}$

$\because$ $E[w_t]$ and $COV[w_t, w_{t-k}]$ are both free of t
$\therefore$ w_t is a stationary process

4.
(a)

$E[y_t]$
$= E[\mu_t + \sigma_t x_t]$
$= \mu_t + \sigma_t E[x_t]$
$= \mu_t$

$COV[y_t, y_{t-k}]$
$= COV[\mu_t + \sigma_t x_t, \mu_{t-k} + \sigma_{t-k} x_{t-k}]$
$= COV[\sigma_t x_t, \sigma_{t-k} x_{t-k}]$
$= \sigma_t \sigma_{t-k} COV[x_t, x_{t-k}]$
$= \sigma_t \sigma_{t-k} \gamma_k$

(b)

$\because V[x_t] = 1$
$\Rightarrow \gamma_k = \rho_k$
$\because V[y_t]$
$= V[\mu_t + \sigma_t x_t]$
$= \sigma_t^2 V[x_t]$
$= \sigma_t^2$

$corr[y_t, y_{t-k}]$

$= \dfrac{COV[y_t, y_{t-k}]}{\sqrt{V[y_t] \times V[y_{t-k}]}}$

$= \dfrac{\sigma_t \sigma_{t-k} \gamma_k}{\sqrt{\sigma_t^2 \times \sigma_{t-k}^2}}$

$= \dfrac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}}$

$= \rho_k$

$\because E[y_t] = \mu_t$, which depends on t

$\therefore y_t$ is not a stationary process

(c)

Let $y_t = \mu_0 + \sigma_t x_t$

$\Rightarrow [y_t] = \mu_0$ (constant)

$\Rightarrow COV[y_t, y_{t-k}] = \sigma_t \sigma_{t-k} \gamma_k$

$\therefore y_t$ is not a stationary process with a constant mean

5.

$COV[y_t, y_{t-k}]$

$= COV[x_t + e_t, x_{t-k} + e_{t-k}]$

$= COV[x_t, x_{t-k}] + COV[x_t, e_{t-k}] + COV[e_t, x_{t-k}] + COV[e_t, e_{t-k}]$

$\because e_t$ is random noise

$\therefore$ every term associated with e_t will be 0

$\therefore COV[y_t, y_{t-k}]$

$= COV[x_t, x_{t-k}]$

$= \gamma_k$

$V[y_t]$

$= V[x_t + e_t]$

$= \sigma_x^2 + \sigma_e^2$

$= V[y_{t-k}]$

$$corr[y_t, y_{t-k}]$$
$$= \frac{COV[y_t, y_{t-k}]}{\sqrt{V[y_t] \times V[y_{t-k}]}}$$
$$= \frac{\gamma_k}{\sqrt{(\sigma_x^2 + \sigma_e^2)(\sigma_x^2 + \sigma_e^2)}}$$
$$= \frac{\gamma_k}{\sigma_x^2 + \sigma_e^2}$$
$$= \frac{V[x_t]\rho_k}{\sigma_x^2 + \sigma_e^2}$$
$$= \frac{\sigma_x^2 \rho_k}{\sigma_x^2 + \sigma_e^2}$$
$$= \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}$$

6.

$$COV[y_t, y_s]$$
$$= COV[\alpha_0 + \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)], \alpha_0 + \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i s) + \beta_i \sin(2\pi f_i s)]]$$
$$= COV[\sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)], \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i s) + \beta_i \sin(2\pi f_i s)]]$$
$$= COV[\sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t)], \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i s)]] + COV[\sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t)], \sum_{i=1}^{q}[\beta_i \sin(2\pi f_i s)]]$$
$$+ COV[\sum_{i=1}^{q}[\beta_i \sin(2\pi f_i t)], \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i s)]] + COV[\sum_{i=1}^{q}[\beta_i \sin(2\pi f_i t)], \sum_{i=1}^{q}[\beta_i \sin(2\pi f_i s)]]$$

$\because$ α_i and β_i are independent

$$\therefore COV[\sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t)], \sum_{i=1}^{q}[\beta_i \sin(2\pi f_i s)]] = COV[\sum_{i=1}^{q}[\beta_i \sin(2\pi f_i t)], \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i s)]] = 0$$
$$\therefore COV[y_t, y_s]$$
$$= COV[\sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t)], \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i s)]] + COV[\sum_{i=1}^{q}[\beta_i \sin(2\pi f_i t)], \sum_{i=1}^{q}[\beta_i \sin(2\pi f_i s)]]$$
$$= \sum_{i=1}^{q}[\cos(2\pi f_i t)\cos(2\pi f_i s)] \times V[\alpha_i] + \sum_{i=1}^{q}[\sin(2\pi f_i t)\sin(2\pi f_i s)] \times V[\beta_i]$$
$$= \sum_{i=1}^{q}[\cos(2\pi f_i t)\cos(2\pi f_i s) + \sin(2\pi f_i t)\sin(2\pi f_i s)] \times \sigma_i^2$$
$$= \frac{1}{2}\sum_{i=1}^{q}[\cos(2\pi f_i t + 2\pi f_i s) + \cos(2\pi f_i t - 2\pi f_i s) - \cos(2\pi f_i t + 2\pi f_i s) + \cos(2\pi f_i t - 2\pi f_i s)] \times \sigma_i^2$$
$$= \frac{1}{2}\sum_{i=1}^{q}[2\cos(2\pi f_i t - 2\pi f_i s)] \times \sigma_i^2$$

$$= \sum_{i=1}^{q} \sigma_i^2 [\cos(2\pi f_i t - 2\pi f_i s)]$$

$$= \sum_{i=1}^{q} \sigma_i^2 [\cos(2\pi f_i (t - s))]$$

$$\therefore COV[y_t, y_s] = \sum_{i=1}^{q} \sigma_i^2 [\cos(2\pi f_i (t - s))]$$

$\therefore COV[y_t, y_s]$ is a fuction of $(t - s) \Rightarrow y_t$ is stationary