1.

$$MA(1)\ process: y_t = a_t - \theta_1 a_{t-1}$$

$$\gamma_0 = E[y_t y_t] = E[(a_t - \theta_1 a_{t-1})(a_t - \theta_1 a_{t-1})]$$
$$= E[a_t^2 - 2\theta_1 a_t a_{t-1} + \theta_1^2 a_{t-1}^2] = E[a_t^2] + E[\theta_1^2 a_{t-1}^2]$$
$$= \sigma_a^2 + \theta_1^2 \sigma_a^2 = \sigma_a^2(1 + \theta_1^2)$$

$$\gamma_1 = E[y_t y_{t-1}] = E[(a_t - \theta_1 a_{t-1})(a_{t-1} - \theta_1 a_{t-2})]$$
$$= E[a_t a_{t-1} - \theta_1 a_t a_{t-2} - \theta_1 a_{t-1}^2 + \theta_1^2 a_{t-1} a_{t-2}] = -\theta_1 \sigma_a^2$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1 \sigma_a^2}{\sigma_a^2(1 + \theta_1^2)} = \frac{-\theta_1}{1 + \theta_1^2}$$

$$The\ condition\ for\ y_t\ to\ be\ stationary\ is\ |\theta_1| < 1.$$

$$When\ \theta_1 = 1, \rho_1 = \frac{-1}{1+1} = -0.5, and\ when\ \theta_1 = -1, \rho_1 = \frac{1}{1+1} = 0.5.$$

$$Therefore,\ \max_{-\infty \le x \le \infty} \rho_1 = 0.5\ and\ \min_{-\infty \le x \le \infty} \rho_1 = -0.5.$$

2.

(a)

$$\gamma_0 = E[y_t y_t] = E[y_t(y_{t-1} - 0.5 y_{t-2} + a_t)]$$
$$= E[y_t y_{t-1} - 0.5 y_t y_{t-2} + y_t a_t] = \gamma_1 - 0.5\gamma_2 + \sigma_a^2$$
$$Note: E[y_t a_t] = E\big[[(y_{t-1} - 0.5 y_{t-2} + a_t) a_t]\big] = \sigma_a^2$$

$$\gamma_1 = E[y_t y_{t-1}] = E[(y_{t-1} - 0.5 y_{t-2} + a_t) y_{t-1}]$$
$$= E[(y_{t-1} - 0.5 y_{t-2} + a_t) y_{t-1}] = E[y_{t-1} y_{t-1} - 0.5 y_{t-2} y_{t-1} + a_t y_{t-1}]$$
$$= E[y_{t-1} y_{t-1}] - 0.5 E[y_{t-2} y_{t-1}] = \gamma_0 - 0.5\gamma_1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\gamma_0 - 0.5\gamma_2}{\gamma_0} = \rho_0 - 0.5\rho_1$$

$$\Rightarrow 1.5\rho_1 = \rho_0 = 1 \quad \Rightarrow \rho_1 = \frac{1}{1.5}$$

(b)

According to 2.(a), we get $\gamma_k = \gamma_{k-1} - 0.5\gamma_{k-2}$
divide $\gamma_0$, we get $\rho_k = \rho_{k-1} - 0.5\rho_{k-2}$
and $\rho_0 = 1$, $\rho_1 = \frac{2}{3}$
we get:

$\rho_2$ 0.166666667
$\rho_3$ -0.166666667
$\rho_4$ -0.25
$\rho_5$ -0.166666667
$\rho_6$ -0.041666667
$\rho_7$ 0.041666667
$\rho_8$ 0.0625
$\rho_9$ 0.041666667
$\rho_{10}$ 0.010416667
$\rho_{11}$ -0.010416667
$\rho_{12}$ -0.015625
$\rho_{13}$ -0.010416667
$\rho_{14}$ -0.002604167
$\rho_{15}$ 0.002604167

3.

(1)

$$y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2} = \theta(B)a_t \text{ where } \theta(B) = (1 - 1.3B + 0.4B^2)$$

$$\text{The condition for } y_t \text{ to be stationary is that}$$

$$\text{the roots of } \theta(B) = 0 \text{ lie outside the unit circle.}$$

$$1 - 1.3B + 0.4B^2 = 0 \quad \Rightarrow B = 1.25 \text{ or } 2$$

$$\text{Therefore } y_t \text{ is stationary and invertible.}$$

(2)

$$y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2}$$

$$\Rightarrow \phi(B)y_t = \theta(B)a_t \text{ where } \phi(B) = (1 - 0.5B) \text{ and } \theta(B) = (1 - 1.3B + 0.4B^2)$$

$$\text{The condition for } y_t \text{ to be stationary is that}$$

$$\text{the roots of } \phi(B) = 0 \text{ and } \theta(B) = 0 \text{ lie outside the unit circle.}$$

$$1 - 0.5B = 0 \quad \Rightarrow B = 2$$

$$1 - 1.3B + 0.4B^2 = 0 \quad \Rightarrow B = 1.25 \text{ or } 2$$

$$\text{Therefore } y_t \text{ is stationary and invertible.}$$

(3)

$$y_t - 1.5y_{t-1} + 0.6y_{t-2} = a_t$$

$$\Rightarrow \phi(B)y_t = a_t \text{ where } \phi(B) = (1 - 1.5B + 0.6B^2)$$

$$\text{The condition for } y_t \text{ to be stationary is that}$$

$$\text{the roots of } \phi(B) = 0 \text{ lie outside the unit circle.}$$

$$1 - 1.5B + 0.6B^2 = 0 \quad \Rightarrow B = 1.25 \pm 0.3227i$$

$$\text{Therefore } y_t \text{ is stationary and invertible.}$$

(4)

$$y_t - y_{t-1} = a_t - 0.5a_{t-1} \quad \Rightarrow \phi(B)y_t = \theta(B)a_t$$
$$where\ \phi(B) = (1 - B)\ and\ \theta(B) = (1 - 0.5B)$$
$$The\ condition\ for\ y_t\ to\ be\ stationary\ is\ that$$
$$the\ roots\ of\ \phi(B) = 0\ and\ \theta(B) = 0\ lie\ outside\ the\ unit\ circle.$$
$$But\ 1 - B = 0 \quad \Rightarrow B = 1$$
$$Therefore\ y_t\ is\ not\ stationary\ nor\ invertible.$$

4.

(a)

$$\boldsymbol{\psi_j\ for\ (1): y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2}}$$
$$\because \psi(B) = \theta(B)$$
$$\therefore The\ first\ three\ \psi_j\ is \begin{cases} \psi_1 = -1.3 \\ \psi_2 = 0.4 \\ \psi_3 = 0 \end{cases}$$

$$\boldsymbol{\psi_j\ for\ (2): y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2}}$$
$$\because \psi(B) = \phi^{-1}(B)\theta(B) \quad \Rightarrow \psi(B)\phi(B) = \theta(B)$$
$$\therefore (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots)(1 - 0.5B) = 1 - 1.3B + 0.4B^2$$
$$\Rightarrow \begin{cases} -0.5 + \psi_1 = -1.3 \\ -0.5\psi_1 + \psi_2 = 0.4 \\ -0.5\psi_2 + \psi_3 = 0 \end{cases} \Rightarrow The\ first\ three\ \psi_j\ is \begin{cases} \psi_1 = -0.8 \\ \psi_2 = 0 \\ \psi_3 = 0 \end{cases}$$

$$\boldsymbol{\psi_j\ for\ (3): y_t - 1.5y_{t-1} + 0.6y_{t-2} = a_t}$$
$$\because \psi(B)\phi(B) = 1$$
$$\therefore (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots)(1 - 1.5B + 0.6B^2) = 1$$
$$\Rightarrow \begin{cases} -1.5 + \psi_1 = 0 \\ 0.6 - 1.5\psi_1 + \psi_2 = 0 \\ 0.6\psi_1 - 1.5\psi_2 + \psi_3 = 0 \end{cases} \Rightarrow The\ first\ three\ \psi_j\ is \begin{cases} \psi_1 = 1.5 \\ \psi_2 = 1.65 \\ \psi_3 = 1.575 \end{cases}$$

$$\boldsymbol{\psi_j\ for\ (4): y_t - y_{t-1} = a_t - 0.5a_{t-1}}$$
$$\because \psi(B) = \phi^{-1}(B)\theta(B) \quad \Rightarrow \psi(B)\phi(B) = \theta(B)$$
$$\therefore (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots)(1 - B) = 1 - 0.5B$$

$$\Rightarrow \begin{cases} -1+\psi_1=-0.5 \\ -\psi_1+\psi_2=0 \\ -\psi_2+\psi_3=0 \end{cases} \Rightarrow The\ first\ three\ \psi_j\ is \begin{cases} \psi_1=0.5 \\ \psi_2=0.5 \\ \psi_3=0.5 \end{cases}$$

(b)

$$\boldsymbol{\pi_j\ for\ (1): y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2}}$$

$$\because \theta(B)\pi(B)=1$$

$$\therefore (1-1.3B+0.4B^2)(1+\pi_1B+\pi_2B^2+\pi_3B^3+\cdots)=1$$

$$\Rightarrow \begin{cases} \pi_1-1.3=0 \\ \pi_2-1.3\pi_1+0.4=0 \\ \pi_3-1.3\pi_2+0.4\pi_1=0 \end{cases} \Rightarrow The\ first\ three\ \pi_j\ is \begin{cases} \pi_1=1.3 \\ \pi_2=1.29 \\ \pi_3=1.157 \end{cases}$$

$$\boldsymbol{\pi_j\ for\ (2): y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2}}$$

$$\because \pi(B)=\theta^{-1}(B)\phi(B) \quad \Rightarrow \pi(B)\theta(B)=\phi(B)$$

$$\therefore (1+\pi_1B+\pi_2B^2+\pi_3B^3+\cdots)(1-1.3B+0.4B^2)=1-0.5B$$

$$\Rightarrow \begin{cases} -1.3+\pi_1=0.5 \\ 0.4-1.3\pi_1+\pi_2=0 \\ 0.4\pi_1-1.3\pi_2+\pi_3=0 \end{cases} \Rightarrow The\ first\ three\ \pi_j\ is \begin{cases} \pi_1=1.8 \\ \pi_2=1.94 \\ \pi_3=1.802 \end{cases}$$

$$\boldsymbol{\pi_j\ for\ (3): y_t - 1.5y_{t-1} + 0.6y_{t-2} = a_t}$$

$$\because \pi(B)=\phi(B)$$

$$\therefore The\ first\ three\ \pi_j\ is \begin{cases} \pi_1=-1.5 \\ \pi_2=0.6 \\ \pi_3=0 \end{cases}$$

$$\boldsymbol{\pi_j\ for\ (4): y_t - y_{t-1} = a_t - 0.5a_{t-1}}$$

$$\because \pi(B)=\theta^{-1}(B)\phi(B) \quad \Rightarrow \pi(B)\theta(B)=\phi(B)$$

$$\therefore (1+\pi_1B+\pi_2B^2+\pi_3B^3+\cdots)(1-0.5B)=1-B$$

$$\Rightarrow \begin{cases} -0.5+\pi_1=-1 \\ -0.5\pi_1+\pi_2=0 \\ -0.5\pi_2+\pi_3=0 \end{cases} \Rightarrow The\ first\ three\ \pi_j\ is \begin{cases} \pi_1=-0.5 \\ \pi_2=-0.25 \\ \pi_3=-0.125 \end{cases}$$

(c)

$$\boldsymbol{V[y_t]\ for\ (1)\ y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2}}$$

$$\Rightarrow V[y_t]=E[(a_t-1.3a_{t-1}+0.4a_{t-2})(a_t-1.3a_{t-1}+0.4a_{t-2})]$$

$$=E[a_t^2+1.69a_{t-1}^2+0.16a_{t-2}^2]=1+1.69+0.16=2.85$$

$$\Rightarrow V[y_t]=2.85$$

$$\boldsymbol{V[y_t]\ for\ (2)\ y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2}}$$

$$V[y_t] = E[(0.5y_{t-1} + a_t - 1.3a_{t-1} + 0.4a_{t-2})(0.5y_{t-1} + a_t - 1.3a_{t-1} + 0.4a_{t-2})]$$

$$= E[0.25y_{t-1}y_{t-1} - 0.65y_{t-1}a_{t-1} + 0.2y_{t-1}a_{t-2} + a_t^2 - 0.65y_{t-1}a_{t-1} + 1.69a_{t-1}^2 + 0.2y_{t-1}a_{t-2} + 0.16a_{t-2}^2]$$

$$= 0.25\gamma_0 - 1.3\gamma_{ay}(0) + 0.4\gamma_{ay}(-1) + 2.85\sigma_a^2$$

$$Note\ \gamma_{ay}(0) = E[y_t a_t] = E[(0.5y_{t-1} + a_t - 1.3a_{t-1} + 0.4a_{t-2})a_t] = \sigma_a^2$$

$$\gamma_{ay}(-1) = E[y_t a_{t-1}] = E[(0.5y_{t-1} + a_t - 1.3a_{t-1} + 0.4a_{t-2})a_{t-1}]$$

$$= \gamma_{ay}(0) - 1.3\sigma_a^2 = \sigma_a^2 - 1.3\sigma_a^2$$

$$Hence, V[y_t] = 0.25\gamma_0 - 1.3\gamma_{ay}(0) + 0.4\gamma_{ay}(-1) + 2.85\sigma_a^2$$

$$\Rightarrow 0.75\ V[y_t] = -1.3\gamma_{ay}(0) + 0.4\gamma_{ay}(-1) + 2.85\sigma_a^2$$

$$= -1.3\sigma_a^2 + 0.4(\sigma_a^2 - 1.3\sigma_a^2) + 2.85\sigma_a^2 = -1.3 + 0.4 * (-0.3) + 2.85 = 1.43$$

$$\Rightarrow V[y_t] = 1.9067$$

$$\boldsymbol{V[y_t]\ for\ (3)\ y_t - 1.5y_{t-1} + 0.6y_{t-2} = a_t}$$

$$V[y_t] = E[y_t(1.5y_{t-1} - 0.6y_{t-2} + a_t)]$$

$$= E[1.5y_t y_{t-1} - 0.6y_t y_{t-2} + y_t a_t] = 1.5\gamma_1 - 0.6\gamma_2 + \sigma_a^2$$

$$\Rightarrow V[y_t] = \gamma_0 = 1.5\gamma_1 - 0.6\gamma_2 + 1$$

$$\gamma_1 = E[(1.5y_{t-1} - 0.6y_{t-2} + a_t)y_{t-1}] = 1.5\gamma_0 - 0.6\gamma_1$$

$$\Rightarrow 1.6\gamma_1 = 1.5\gamma_0 \quad \Rightarrow \gamma_1 = \frac{1.5\gamma_0}{1.6}$$

$$\gamma_2 = E[(1.5y_{t-1} - 0.6y_{t-2} + a_t)y_{t-2}] = 1.5\gamma_1 - 0.6\gamma_0$$

$$\Rightarrow \gamma_2 = 1.5\gamma_1 - 0.6\gamma_0$$

$$\begin{cases} \gamma_0 = 1.5\gamma_1 - 0.6\gamma_2 + 1 \\ \gamma_1 = \dfrac{1.5\gamma_0}{1.6} \\ \gamma_2 = 1.5\gamma_1 - 0.6\gamma_0 \end{cases} \Rightarrow \begin{cases} \gamma_0 = \dfrac{400}{31} \\ \gamma_1 = \dfrac{375}{31} \\ \gamma_2 = \dfrac{645}{62} \end{cases}$$

$$Hence, V[y_t] = \frac{400}{31}.$$

$$\boldsymbol{V[y_t]\ for\ (4)\ y_t - y_{t-1} = a_t - 0.5a_{t-1}}$$

$$V[y_t] = E[(y_{t-1} + a_t - 0.5a_{t-1})(y_{t-1} + a_t - 0.5a_{t-1})]$$
$$= E[y_{t-1}y_{t-1} - 0.5y_{t-1}a_{t-1} + a_t^2 - 0.5y_{t-1}a_{t-1} + 0.25a_{t-1}^2]$$
$$= \gamma_0 - \gamma_{ay}(0) + 1.25\sigma_a^2 = \gamma_0 - 1 + 1.25$$
$$\Rightarrow V[y_t] = \gamma_0 = \gamma_0 - 1 + 1.25$$
$$\Rightarrow V[y_t]\ does\ not\ exist.$$

5.

$$V[(1-B)y_t] = V[y_t - y_{t-1}] = V[y_t] - V[y_{t-1}] + 2COV[y_t, y_{t-1}] = 2V[y_t] - 2\gamma_1$$

$$\rho_1 < \frac{1}{2} \quad \Rightarrow \gamma_1 < \frac{1}{2}V[y_t] \quad \Rightarrow 2\gamma_1 < V[y_t]$$

$$Hence, 2V[y_t] - 2\gamma_1 > V[y_t] \quad \Rightarrow V[(1-B)y_t] > V[y_t]$$

6.

(a)

$$y_t = \phi y_{t-1} + e_t = \phi[\phi y_{t-2} + e_{t-1}] + e_t = \phi[\phi(\phi y_{t-3} + e_{t-2}) + e_{t-1}] + e_t$$
$$= \cdots = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1}e_1 + \phi^t y_0\ for\ all\ t > 0$$

(b)

$$\because e_t\ is\ white\ noise$$
$$\therefore E[y_t] = E[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1}e_1 + \phi^t y_0] = \phi^t E[y_0] = \phi^t \mu_0$$

(c)

$$\because e_t\ is\ white\ noise\ and\ independent\ of\ past\ y_{t-1}, y_{t-2}, \ldots$$
$$V[y_t] = V[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1}e_1 + \phi^t y_0]$$
$$= V[e_t] + \phi^2 V[e_{t-1}] + \phi^4 V[e_{t-2}] + \cdots + \phi^{2t-2}V[e_1] + \phi^{2t}V[y_0]$$
$$= (1 + \phi^2 + \phi^4 + \cdots + \phi^{2t-2})\sigma_e^2 + \phi^{2t}\sigma_0^2 = \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2$$
$$When\ \phi = 1, (1 + \phi^2 + \phi^4 + \cdots + \phi^{2t-2})\sigma_e^2 + \phi^{2t}\sigma_0^2 = t\sigma_e^2 + \sigma_0^2$$
$$Hence, V[y_t] = \begin{cases} \dfrac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2, & for\ \phi \neq 1 \\ t\sigma_e^2 + \sigma_0^2, & for\ \phi = 1 \end{cases}$$

(d)

$$When\ \phi = 1, V[y_t] = t\sigma_e^2 + \sigma_0^2, then\ y_t\ is\ not\ stationary$$
$$since\ the\ variance\ of\ y_t\ is\ a\ function\ of\ t.$$
$$When\ \phi \neq 1, E[y_t] = 0\ under\ the\ assumption\ \mu_0 = 0.$$

$$V[y_t] = \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2 = \frac{\sigma_e^2}{1-\phi^2} - \frac{\sigma_e^2\phi^{2t}}{1-\phi^2} + \phi^{2t}\sigma_0^2$$

$$= \frac{\sigma_e^2}{1-\phi^2} + \phi^{2t}\left(\sigma_0^2 - \frac{\sigma_e^2}{1-\phi^2}\right) = \frac{\sigma_e^2}{1-\phi^2} + \phi^{2t}(\gamma_0 - \gamma_0) = \frac{\sigma_e^2}{1-\phi^2}$$

$$\left(\because\ V[y_t] = \gamma_0 = \phi^2\gamma_0 + \sigma_e^2 \quad \Rightarrow \gamma_0 = \frac{\sigma_e^2}{1-\phi^2}\right)$$

$$Hence, we\ must\ have\ \phi \neq 1\ to\ make\ y_t\ stationary.$$

(e)

$$From\ the\ previous\ question, we\ have\ shown\ that\ V[y_t] = \frac{\sigma_e^2}{1-\phi^2}$$

$$under\ the\ assumption\ that\ \mu_0 = 0\ and\ y_t\ is\ stationary.$$

$$Since\ 0 < V[y_t] < \infty, |\phi|\ must\ be\ less\ than\ 1\ s.t. V[y_t] = \frac{\sigma_e^2}{1-\phi^2}$$

$$= (1 + \phi^2 + \phi^4 + \cdots)\sigma_e^2\ would\ be\ summable.$$

$$That\ is\ we\ must\ have\ |\phi| < 1\ so\ that\ V[y_t] = \frac{\sigma_e^2}{1-\phi^2}.$$