TSA02: Exponential Smoothing Models

Jakey BLUE

Decomposition of Time Series

- Level; Trend; Seasonality
  - Additive Model
    - systematic component = level + trend + seasonality
  - Multiplicative
    - systematic component = level × trend × seasonality
  - Mixed
    - (level + trend) × seasonality

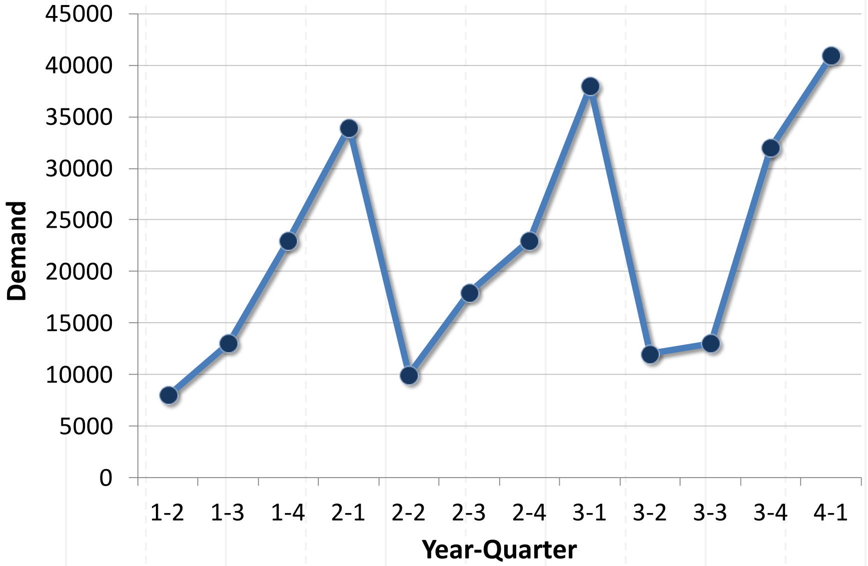
Which Model is Making More Senses?

Let's start with an easy case

1

At time t, we want to forecast $t + k$

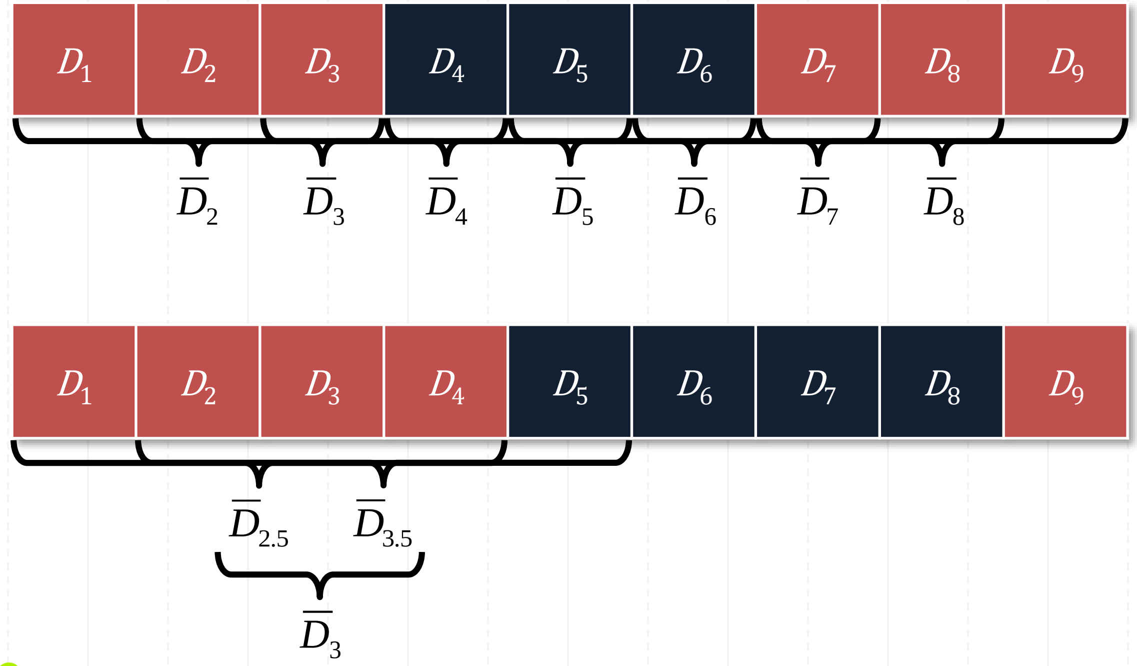
- Additive Model
  $$F_{t+k} = L + (t+k)T + S_{t+k}$$
- Multiplicative Model
  $$F_{t+k} = L \times (t+k)T \times S_{t+k}$$
- Mixed
  $$F_{t+k} = [L + (t+k)T] \times S_{t+k}$$



Procedure to Build a Static Model

1. Identify the # of periods in a season.
2. De-seasonalize the series.
3. Estimate the level and trend.
4. Recover the seasonality factors.
5. Finalize the static model.

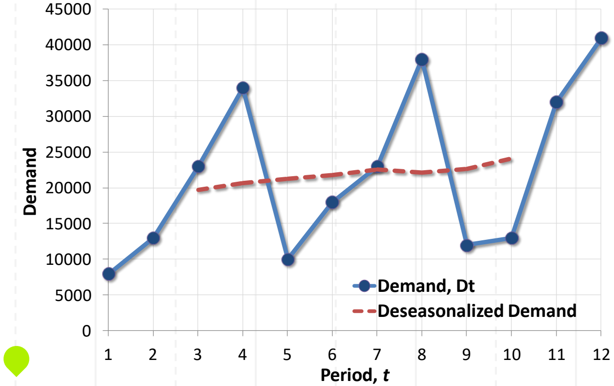
De-seasonalize the series



Formulation of Deseasonalization

$$\overline{D}_t = \begin{cases} \dfrac{\left[D_{t-\left(\frac{p}{2}\right)} + D_{t+\left(\frac{p}{2}\right)} + \sum_{i=t+1-\left(\frac{p}{2}\right)}^{t-1+\left(\frac{p}{2}\right)} 2D_i\right]}{2p} & p \text{ is even} \\ \dfrac{\sum_{i=t-\left(\frac{p-1}{2}\right)}^{t+\left(\frac{p-1}{2}\right)} 2D_i}{p} & p \text{ is odd} \end{cases}$$

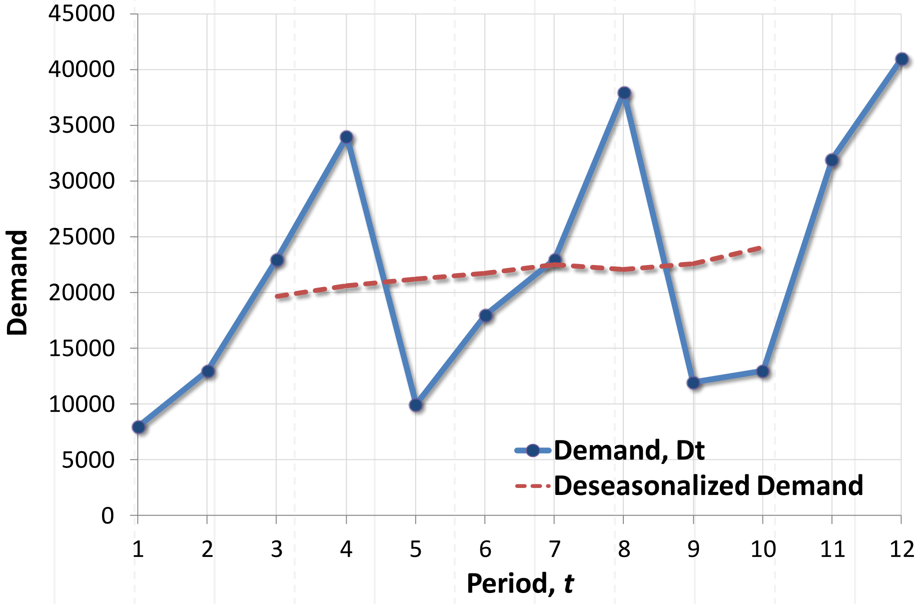
Year	Quarter	Period, t	Demand, D_t	Deseasonalized Demand
1	2	1	8000	
1	3	2	13000	
1	4	3	23000	19750
2	1	4	34000	20625
2	2	5	10000	21250
2	3	6	18000	21750
2	4	7	23000	22500
3	1	8	38000	22125
3	2	9	12000	22625
3	3	10	13000	24125
3	4	11	32000	
4	1	12	41000	



The Trend of Deseasonalized Series

- $\overline{D}_t = L + T_t$

SUMMARY OUTPUT	
Regression Statistics	
Multiple R	0.958065237
R Square	0.917888998
Adjusted R Square	0.90420383
Standard Error	414.5033124
Observations	8
ANOVA	
	df
Regression	1
Residual	6
Total	7
	Coefficients
L	18438.9881
T	523.8095238



Level and Trend are Known. Seasonality?

$$\overline{D}_t = 18439 + 524t \qquad S_t = \frac{D_t}{\overline{D}_t}$$

Year	Quarter	Period, t	Demand, D_t	Deseasonalized Demand	Seasonality, S_t
1	2	1	8000	18963	0.42
1	3	2	13000	19487	0.67
1	4	3	23000	20010	1.15
2	1	4	34000	20534	1.66
2	2	5	10000	21058	0.47
2	3	6	18000	21582	0.83
2	4	7	23000	22106	1.04
3	1	8	38000	22629	1.68
3	2	9	12000	23153	0.52
3	3	10	13000	23677	0.55
3	4	11	32000	24201	1.32
4	1	12	41000	24725	1.66

$\overline{S}_2 = 0.47$

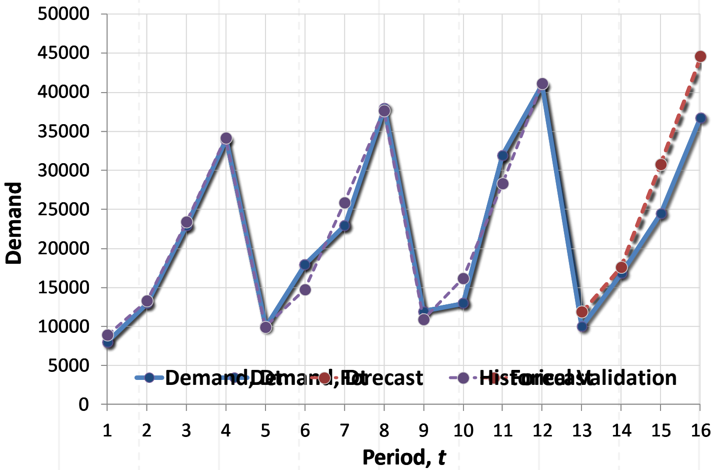
$\overline{S}_3 = 0.68$

$\overline{S}_4 = 1.17$

$\overline{S}_1 = 1.67$

Predict the Next 4 Periods of Demand

$$F_{4-2} = F_{13} = (L + 13T)\overline{S}_2 = (18439 + 13 \times 524) \times 0.47 = 11868$$
$$F_{4-3} = F_{14} = (L + 14T)\overline{S}_3 = (18439 + 14 \times 524) \times 0.68 = 17527$$
$$F_{4-4} = F_{15} = (L + 15T)\overline{S}_4 = (18439 + 15 \times 524) \times 1.17 = 30770$$
$$F_{5-1} = F_{16} = (L + 16T)\overline{S}_1 = (18439 + 16 \times 524) \times 1.67 = 44794$$



Evaluating the Adequacy of Models

- How do we measure the errors? $E_t = F_t - D_t$
  - Mean Squared Error (MSE) ………………………………… $\text{MSE}_n = \frac{1}{n}\sum_{t=1}^{n}(E_t)^2$

    variance of forecasting error
  - Mean Absolute Deviation (MAD) ……………………………. $\text{MAD}_n = \frac{1}{n}\sum_{t=1}^{n}|E_t|$

    estimate the standard deviation $\sigma = 1.25\text{MAD}$
  - Mean Absolute Percentage Error (MAPE) …………… $\text{MAPE}_n = \frac{100\%}{n}\sum_{t=1}^{n}\frac{|E_t|}{|D_t|}$

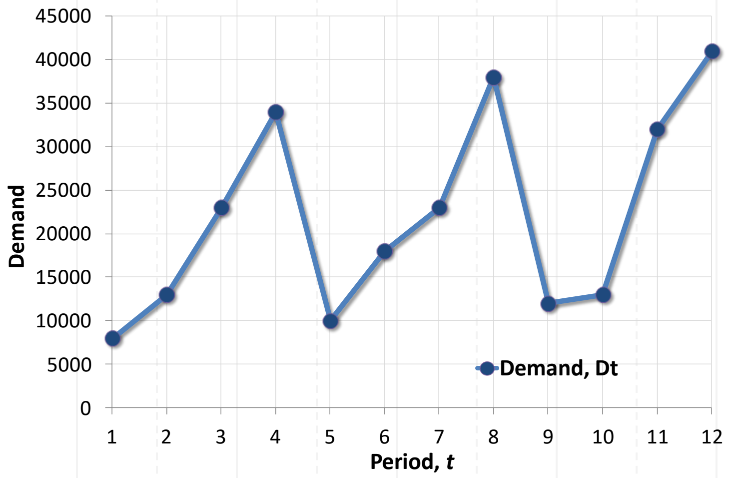
    accuracy in percentage
  - Bias……………………………………………………………… $\text{Bias}_n = \sum_{t=1}^{n} E_t$

    if the model constantly over- or under-estimates demand
  - Tracking Signal (TS)………………………………………… $\text{TS}_t = \frac{\text{Bias}_t}{\text{MAD}_t}$

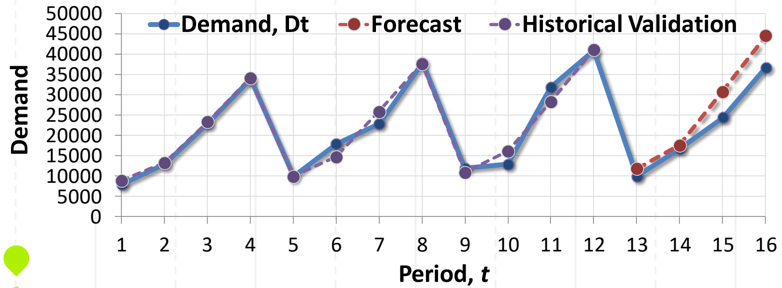
    $\text{TS}_t < -6$ → underforecasting; $\text{TS}_t > 6$ → overforecasting

Homework Revisited

Year	Quarter	Period, t	Demand, D_t
1	2	1	8000
1	3	2	13000
1	4	3	23000
2	1	4	34000
2	2	5	10000
2	3	6	18000
2	4	7	23000
3	1	8	38000
3	2	9	12000
3	3	10	13000
3	4	11	32000
4	1	12	41000



Year	Q	t	Demand, D_t	Forecast, F_t	Error, E_t	E_t^2	MSE_t	$\lvert E_t \rvert$	MAD_t	$\left\lvert \frac{E_t}{D_t} \right\rvert$ %	MAPE_t	Bias_t	TS_t
1	2	1	8000	8944	-944	891863	891863	944	944	12%	11.80%	-944	-1
1	3	2	13000	13317	-317	100637	496250	317	631	2%	7.12%	-1262	-2
1	4	3	23000	23426	-426	181781	391427	426	563	2%	5.37%	-1688	-3
2	1	4	34000	34178	-178	31532	301453	178	466	1%	4.16%	-1866	-4
2	2	5	10000	9933	67	4534	242069	67	387	1%	3.46%	-1798	-5
2	3	6	18000	14749	3251	10568164	1963085	3251	864	18%	5.89%	1453	2
2	4	7	23000	25879	-2879	8290194	2866958	2879	1152	13%	6.84%	-1427	-1
3	1	8	38000	37665	335	112273	2522622	335	1050	1%	6.09%	-1092	-1
3	2	9	12000	10921	1079	1164345	2371702	1079	1053	9%	6.42%	-12	0
3	3	10	13000	16181	-3181	10118912	3146423	3181	1266	24%	8.22%	-3194	-3
3	4	11	32000	28332	3668	13452890	4083375	3668	1484	11%	8.52%	474	0
4	1	12	41000	41152	-152	23191	3745026	152	1373	0%	7.84%	322	0



Model Performance:

$\text{MSE}_{12} = 3745026$
$\text{MAD}_{12} = 1375$
$\text{MAPE}_{12} = 8\%$

Making the Model More Dynamic

What were the STATIC parts?

2

An Adaptive Forecasting Model

- From $F_{t+k} = [L + (t + k)T]S_{t+k}$ to $F_{t+k} = [L_t + (t + k)T_t]S_{t+k}$

Notation Definition:
D_t = actual demand observed in period t
L_t = estimate of level at the end of period t
T_t = estimate of trend at the end of period t
S_t = estimate of seasonal factor for period t
F_{t+k} = demand forecast for period $t+k$
E_t = forecast error in period t

1. Moving Average
2. Single (Simple) Exponential Smoothing
3. Trend-Corrected Exponential Smoothing
4. Trend- and Seasonality-Corrected Exponential Smoothing

Moving Average Model

- What is "MOVING AVERAGE"?

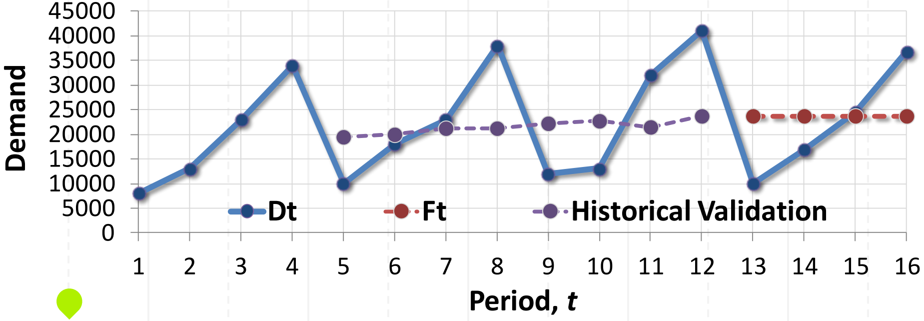
$$F_{t+1} = L_t = \frac{(D_t + D_{t-1} + \cdots + D_{t-N+1})}{N}$$

  - Is it really useful? When?
  - How to predict?
    $F_{t+k} = ?$

- When D_{t+1} arrived, $F_{t+k} = L_{t+1} = \frac{(D_t + D_{t-1} + \cdots + D_{t-N+2})}{N}$

Year	Q	t	D_t	L_t	F_t	E_t	$\lvert Et \rvert$	MSE_t	MAD_t	Error %	$MAPE_t$	$Bias_t$	TS_t
1	2	1	8000										
1	3	2	13000										
1	4	3	23000										
2	1	4	34000	19500									
2	2	5	10000	20000	19500	9500	9500	18050000	9500	95.00%	95.00%	9500	1.00
2	3	6	18000	21250	20000	2000	2000	15708333	5750	11.11%	53.06%	11500	2.00
2	4	7	23000	21250	21250	-1750	1750	13901786	4417	7.61%	37.91%	9750	2.21
3	1	8	38000	22250	21250	-16750	16750	47234375	7500	44.08%	39.45%	-7000	-0.93
3	2	9	12000	22750	22250	10250	10250	53659722	8050	85.42%	48.64%	3250	0.40
3	3	10	13000	21500	22750	9750	9750	57800000	8333	75.00%	53.04%	13000	1.56
3	4	11	32000	23750	21500	-10500	10500	62568182	8643	32.81%	50.15%	2500	0.29
4	1	12	41000	24500	23750	-17250	17250	82151042	9719	42.07%	49.14%	-14750	-1.52
4	2	13	10000		24500	14500	14500	92004808	10250	145.00%	59.79%	-250	-0.02
4	3	14	16800		24500	7700	7700	89668036	9995	45.83%	58.39%	7450	0.75
4	4	15	24500		24500	0	0	83690167	9086	0.00%	53.08%	7450	0.82
5	1	16	36800		24500	-12300	12300	87915156	9354	33.42%	51.45%	-4850	-0.52

$N = 4$

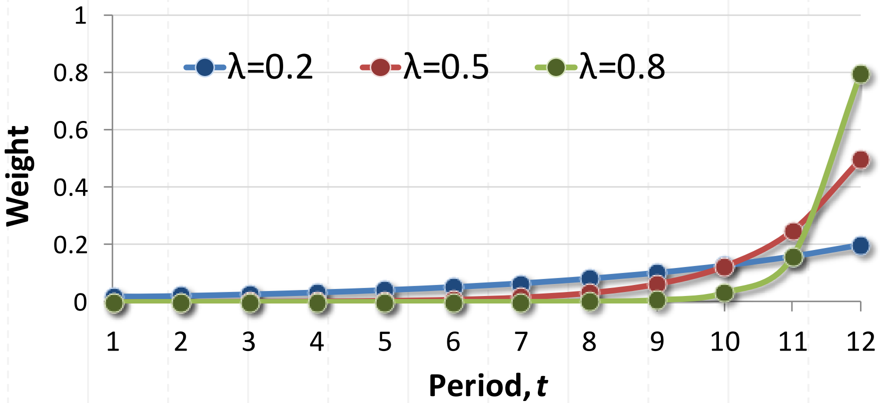


Model Performance:

$MSE_{12} = 82151042$
$MAD_{12} = 9719$
$MAPE_{12} = 49\%$

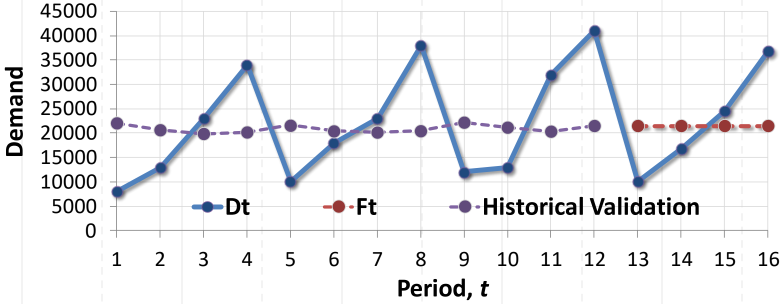
Single Exponential Smoothing

- $F_{t+1} = L_t = \lambda D_t + (1-\lambda)L_{t-1} = \underbrace{(1-\lambda)^{t-1}L_0 + \sum_{k=0}^{t-1}\lambda(1-\lambda)^k D_{t-k}}_{\text{Exponentially Weighted Average}}$
  - $0 < \lambda < 1$: smoothing constant
  - $L_0 = \frac{1}{t}\sum_{i=1}^{t} D_i$



Year	Q	t	D_t	L_t	F_t	E_t	$\lvert Et \rvert$	MSE_t	MAD_t	Error %	$MAPE_t$	$Bias_t$	TS_t
		0		22083									
1	2	1	8000	20675	22083	14083	14083	198340278	14083	176.04%	176.04%	14083	1.00
1	3	2	13000	19908	20675	7675	7675	128622951	10879	59.04%	117.54%	21758	2.00
1	4	3	23000	20217	19908	-3093	3093	88936486	8284	13.45%	82.84%	18666	2.25
2	1	4	34000	21595	20217	-13783	13783	114196860	9659	40.54%	72.27%	4883	0.51
2	2	5	10000	20436	21595	11595	11595	118246641	10046	115.95%	81.00%	16478	1.64
2	3	6	18000	20192	20436	2436	2436	99527532	8777	13.53%	69.76%	18913	2.15
2	4	7	23000	20473	20192	-2808	2808	86435714	7925	12.21%	61.54%	16105	2.03
3	1	8	38000	22226	20473	-17527	17527	114031550	9125	46.12%	59.61%	-1422	-0.16
3	2	9	12000	21203	22226	10226	10226	112979315	9247	85.21%	62.45%	8804	0.95
3	3	10	13000	20383	21203	8203	8203	108410265	9143	63.10%	62.52%	17007	1.86
3	4	11	32000	21544	20383	-11617	11617	110824074	9368	36.30%	60.14%	5389	0.58
4	1	12	41000	23490	21544	-19456	19456	133132065	10208	47.45%	59.08%	-14066	-1.38
4	2	13	10000		23490	13490	13490	136889542	10461	134.90%	64.91%	-576	-0.06
4	3	14	16800		23490	6690	6690	130308553	10192	39.82%	63.12%	6114	0.60
4	4	15	24500		23490	-1010	1010	121689327	9579	4.12%	59.19%	5104	0.53
5	1	16	36800		23490	-13310	13310	125156051	9813	36.17%	57.75%	-8206	-0.84

$\lambda = 0.1$



Model Performance:

$MSE_{12} = 133132065$
$MAD_{12} = 10208$
$MAPE_{12} = 59\%$

Double Exponential Smoothing

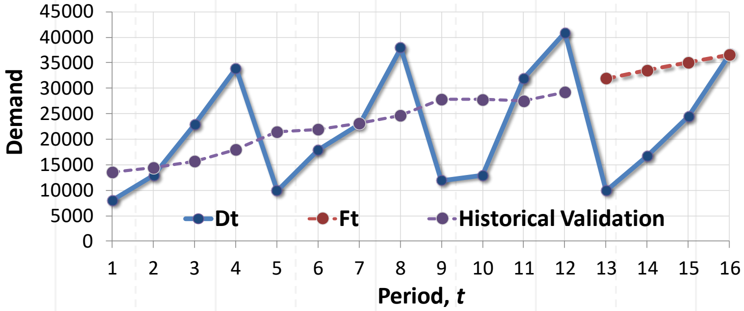
- a.k.a. Trend-Corrected Exponential Smoothing
- a.k.a. Holt's Model

$F_{t+1} = L_t + T_t$ and $F_{t+k} = L_t + kT_t$

- $L_t = \alpha D_t + (1 - \alpha)(L_{t-1} + T_{t-1})$
  - $0 < \alpha < 1$: smoothing constant for LEVEL
- $T_t = \beta(L_t - L_{t-1}) + (1 - \beta)T_{t-1}$
  - $0 < \beta < 1$: smoothing constant for TREND

Year	Q	t	D_t	L_t	F_t	E_t	$\|Et\|$	MSE_t	MAD_t	Error %	$MAPE_t$	$Bias_t$	TS_t	Year
		0		12015	1549									
1	2	1	8000	13008	1438	13564	5564	5564	30959237.34	5564	69.55%	69.55%	5564	1.00
1	3	2	13000	14301	1409	14445	1445	1445	16524153.32	3505	11.12%	40.33%	7009	2.00
1	4	3	23000	16439	1555	15710	-7290	7290	28732809.71	4767	31.70%	37.46%	-281	-0.06
2	1	4	34000	19594	1875	17993	-16007	16007	85604032.58	7577	47.08%	39.86%	-16288	-2.15
2	2	5	10000	20322	1645	21469	11469	11469	94788912.25	8355	114.69%	54.83%	-4819	-0.58
2	3	6	18000	21570	1566	21967	3967	3967	81613688.31	7624	22.04%	49.36%	-852	-0.11
2	4	7	23000	23123	1563	23136	136	136	69957245.79	6554	0.59%	42.39%	-716	-0.11
3	1	8	38000	26017	1830	24686	-13314	13314	83370484.37	7399	35.04%	41.48%	-14030	-1.90
3	2	9	12000	26262	1513	27847	15847	15847	102009888.3	8338	132.06%	51.54%	1817	0.22
3	3	10	13000	26297	1217	27775	14775	14775	113638498.2	8981	113.65%	57.75%	16592	1.85
3	4	11	32000	27963	1307	27514	-4486	4486	105136814.7	8573	14.02%	53.78%	12106	1.41
4	1	12	41000	30443	1541	29270	-11730	11730	107841791.9	8836	28.61%	51.68%	376	0.04
4	2	13	10000			31984	21984	21984	136723869.3	9847	219.84%	64.61%	22361	2.27
4	3	14	16800			33526	16726	16726	146939977.2	10339	99.56%	67.11%	39086	3.78
4	4	15	24500			35067	10567	10567	144588268.3	10354	43.13%	65.51%	49653	4.80
5	1	16	36800			36609	-191	191	135553792.1	9719	0.52%	61.45%	49462	5.09

$\alpha = 0.1$
$\beta = 0.2$



Model Performance:

$MSE_{12} = 107841792$
$MAD_{12} = 8836$
$MAPE_{12} = 52\%$

Triple Exponential Smoothing

- a.k.a. Trend- and Seasonality-Corrected Exponential Smoothing
- a.k.a. Holt-Winter's Model

$F_{t+1} = (L_t + T_t)S_{t+1}$ and $F_{t+k} = (L_t + T_t)S_{t+k}$

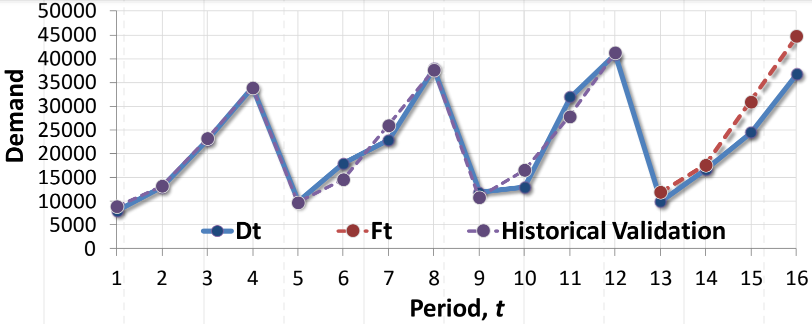
- $L_t = \alpha\left(\frac{D_t}{S_t}\right) + (1-\alpha)(L_{t-1} + T_{t-1})$
  - $0 < \alpha < 1$: smoothing constant for LEVEL
- $T_t = \beta(L_t - L_{t-1}) + (1-\beta)T_{t-1}$
  - $0 < \beta < 1$: smoothing constant for TREND
- $S_t = \gamma\left(\frac{D_{t-p}}{L_{t-p}}\right) + (1-\gamma)S_{t-p}$
  - $0 < \gamma < 1$: smoothing constant for SEASONALITY

Year	Q	t	D_t	L_t	T_t	S_t	F_t	E_t	$\lvert E_t \rvert$	MSE_t	MAD_t	Error%	$MAPE_t$	$Bias_t$	TS_t
		0		18439	524										
1	2	1	8000	18863	514	0.47	8944	944	944	891863.2602	944	11.80%	11.80%	944	1.00
1	3	2	13000	19359	512	0.68	13242	242	242	475208.2322	593	1.86%	6.83%	1186	2.00
1	4	3	23000	19860	511	1.17	23263	263	263	339848.0274	483	1.14%	4.94%	1449	3.00
2	1	4	34000	20373	511	1.66	33905	-95	95	257140.1562	386	0.28%	3.77%	1354	3.51
2	2	5	10000	20911	514	0.47	9751	-249	249	218062.8927	359	2.49%	3.51%	1106	3.08
2	3	6	18000	21673	539	0.68	14616	-3384	3384	2089737.397	863	18.80%	6.06%	-2278	-2.64
2	4	7	23000	22084	526	1.17	25975	2975	2975	3055910.711	1165	12.94%	7.04%	698	0.60
3	1	8	38000	22621	527	1.66	37643	-357	357	2689843.68	1064	0.94%	6.28%	341	0.32
3	2	9	12000	23273	539	0.47	10835	-1165	1165	2541898.04	1075	9.71%	6.66%	-825	-0.77
3	3	10	13000	23554	514	0.70	16598	3598	3598	3582350.293	1327	27.68%	8.76%	2773	2.09
3	4	11	32000	24247	532	1.16	27838	-4162	4162	4831609.551	1585	13.01%	9.15%	-1389	-0.88
4	1	12	41000	24770	531	1.67	41291	291	291	4436030.044	1477	0.71%	8.45%	-1098	-0.74
4	2	13	10000			0.47	11963	1963	1963	4391105.79	1514	19.63%	9.31%	865	0.57
4	3	14	16800			0.68	17631	831	831	4126804.533	1466	4.95%	8.99%	1696	1.16
4	4	15	24500			1.17	30922	6422	6422	6601423.556	1796	26.21%	10.14%	8118	4.52
5	1	16	36800			1.67	44784	7984	7984	10173002.34	2183	21.70%	10.86%	16102	7.38

$\alpha = 0.05$

$\beta = 0.1$

$\gamma = 0.1$



Model Performance:

$MSE_{12} = 4436030$

$MAD_{12} = 1477$

$MAPE_{12} = 8\%$

Comparing the FIVE Models

Forecasting Method	MSE_{12}	MAD_{12}	$MAPE_{12}$	$TS_{1\sim12}$ Range
Static Model	3745026	1373	7.84%	-1.68 to 4.65
Moving Average ($N = 4$)	82151042	9719	49.14%	-1.52 to 2.21
Single Exponential Smoothing ($\lambda = 0.1$)	133132065	10208	59.08%	-1.38 to 2.25
Holt's Model ($\alpha = 0.1, \beta = 0.2$)	107841792	8836	51.68%	-2.15 to 1.85
Holt-Winter's Model ($\alpha = 0.05, \beta = 0.1, \gamma = 0.1$)	4436030	1477	8.45%	-2.74 to 4.00

Everything should be made as simple as possible, but no simpler.

Summary

- Mathematically, a time series is decomposed into:
  - Level; Trend; Seasonality
- Static Model: $F_{t+k} = [L + (t + k)T] \times S_{t+k}$
- Moving Average: $F_{t+1} = L_t = \frac{(D_t + D_{t-1} + \cdots + D_{t-N+1})}{N}$
- Single Exponential Smoothing: $F_{t+1} = L_t = \lambda D_t + (1 - \lambda)L_{t-1}$
- Trend-Corrected Exponential Smoothing: $F_{t+1} = L_t + T_t$
  - $L_t = \alpha D_t + (1 - \alpha)(L_{t-1} + T_{t-1})$
  - $T_t = \beta(L_t - L_{t-1}) + (1 - \beta)T_{t-1}$
- Trend- and Seasonality-Corrected Exponential Smoothing: $F_{t+1} = (L_t + T_t)S_{t+1}$
  - $L_t = \alpha\left(\frac{D_t}{S_t}\right) + (1 - \alpha)(L_{t-1} + T_{t-1})$
  - $T_t = \beta(L_t - L_{t-1}) + (1 - \beta)T_{t-1}$
  - $S_t = \gamma\left(\frac{D_{t-p}}{L_{t-p}}\right) + (1 - \gamma)S_{t-p}$