MA(1)

$\boxed{\text{All : stationary} : |B|>1}$

$y_t=(1-\theta_1 B)a_t$ $\quad \boxed{B>1}$

$E[y_t]=0$

$r_0=V[y_t]=E[y_t y_t]=E[(a_t-\theta a_{t-1})(a_t-\theta a_{t-1})] = (1+\theta_1^2)\sigma_a^2$

$r_1=Cov[y_t,y_{t-1}]=E[-\theta^2 a_{t-1}]=-\theta\sigma_a^2$

$r_{2,\dots}=0$

$\rho_k=\begin{cases}1, & k=0\\ \left(-\frac{\theta_1}{1+\theta_1^2}\right), & k=1\\ 0, & k\ge 2\end{cases}$

$\theta_1<0 \Rightarrow \rho_1>0$

$\theta_1>0 \Rightarrow \rho_1<0$

$(-0.5)\le\rho_1\le(0.5)$

$\boxed{\text{Invertible: } |\theta|<1}$

MA(2)

$y_t=a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2}=(1-\theta_1 B-\theta_2 B^2)a_t$

$E[y_t]=0$

$r_0=V[y_t]=\sigma_a^2(1+\theta_1^2+\theta_2^2)$

$\rho_k\begin{cases}1; & k=0\\ \left(\frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}\right); & k=1\\ \left(\frac{-\theta_2}{1+\theta_1^2+\theta_2^2}\right); & k=2\\ 0; & k\ge 3\end{cases}$

MA(q)

$y_t=(1-\theta_1 B-\theta_2 B^2-\dots\theta_q B^q)a_t.$

$E[y_t]=0$

$r_0=V[y_t]=\sigma_a^2\left(\sum_{0}^{q}\theta_i^2\right);\ \theta_0=1$

$r_k=\boxed{\sigma_a^2\left(-\theta_k+\sum_{j=1}^{q-k}\theta_j\theta_{j+k}\right);}\ 1\le k\le q$

Invertible: $|\theta_1|<1$

$\pi(B)=(1-\theta_1 B)^{-1}=\sum\theta_1^j B^j$ converge unit circle

$B=\theta_1^{-1}$ of $(1-\theta_1 B)=0$ lies out side.

$D_t\doteq(L+T)=S_t \quad S_t\doteq n=\overline{St}$

$V_t=(L+T)_{前}\times S_{t_{now}}$

MSE $\frac{1}{n}(E_t)^2$

MAD

分数微分：

“(分母×分子微－分子×分母微)÷(分母平方)”

AR

AR(P)

$y_t=\phi_1 y_{t-1}+\phi_2 y_{t-2}+\dots\phi_p y_{t-p}+a_t$

$(1-\phi_1 B-\phi_2 B^2-\dots-\phi_p B^p)y_t=a_t$

Stationary:

$|\phi_1|<1 \Rightarrow \boxed{|B|>1}$ (AR1)

$(1-G_i B)^{-1}$

$\phi(B)=(1-G_1 B)(1-G_2 B)\dots(1-G_p B)$

$y_t=\phi^{-1}(B)a_t=\sum_{i=1}^{p}\frac{K_i}{1-G_i B}a_t$

$\Rightarrow |G_i|<1$; $|B_i|>1$

$\boxed{\phi_1-\phi_1\phi_0=0;}$ with $\phi_0=1$

AR(1)

$y_t=\phi_1 y_{t-1}+a_t$

$E[y_t]=0$

$V[y_t]=\frac{\sigma_e^2}{1-\phi_1^2}$

$\phi_1=\rho_1$

$r_k=\phi^k\cdot\frac{\sigma_e^2}{1-\phi^2}$

$\phi>0 \Rightarrow \rho\ge 0.$

AR(2)

$y_t=\phi_1 y_{t-1}+\phi_2 y_{t-2}+a_t$

$E[y_t]=0.$

$V[y_t]=\phi_1 r_1+\phi_2 r_2+\sigma_a^2=r_0$

$r_0=\frac{\sigma_a^2}{1-\phi_1\rho_1-\phi_2\rho_2}$

where $\rho_1=\frac{\phi_1}{1-\phi_2}$ $\rho_2=\left(\frac{\phi_1^2}{1-\phi_2}+\phi_2\right)$

$\infty > V[y_t]=\frac{(1+\phi_2)\sigma_a^2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}>0$

$\Rightarrow 1-\phi_2^2>0$; $|\phi_2|<1$ & $\begin{cases}\phi_1+\phi_2<1\\ \phi_2-\phi_1<1\end{cases}$

☆ AR2 Stationary

$\begin{cases}|\phi_2|<1\\ \phi_1+\phi_2<1\\ \phi_2-\phi_1<1\end{cases}$

$\boxed{r_k=\phi_1 r_{k-1}+\phi_2 r_{k-2}}$

$\rho_k=\phi_1\rho_{k-1}+\phi_2\rho_{k-2}$

$\rho_1=\frac{\phi_1}{1-\phi_2}$

$\rho_2=\frac{\phi_1^2}{1-\phi_2}+\phi_2$

$\phi_1=\frac{\rho_1(1-\rho_2)}{1-\rho_1^2}$

$\phi_2=\frac{\rho_2-\rho_1^2}{1-\rho_1^2}$

$(1+\psi_1 B+\psi_2 B^2+\dots)(1-0.5B)=(\dots\dots)$

$1=(1+\pi_1 B+\pi_2 B+\pi_3 B\dots)$

ARMA(p,q)

$y_t=\phi_1 y_{t-1}+\dots+\phi_p y_{t-p}+a_t-\theta_1 a_{t-1}\dots-\theta_q a_{t-q}$

$\Rightarrow \phi(B)y_t=\phi(B)a_t.$

$r_k=E[y_t,y_{t-k}]=\phi_1 r_{k-1}+\dots+\phi_p r_{k-p}+r_{ay}(k)-\theta_1 r_{ay}(k-1)-\dots-\theta_q r_{ay}(k-q).$

$r_{ay}(j)=Cov[a_t,y_{t-j}]=E[a_t\cdot y_{t-j}]$

ARMA(1,1)

$r_0=\left(\frac{1-2\theta\phi+\theta^2}{1-\phi^2}\right)\sigma_a^2$

$r_1=\left(\frac{(\phi-\theta)(1-\theta\phi)}{1-\phi^2}\right)\sigma_a^2$

$\rho_1=\frac{(\phi-\theta)(1-\phi\theta)}{1-2\theta\phi+\theta^2}$

$\rho_k=\phi^{k-1}\cdot\frac{(\phi-\theta)(1-\theta\phi)}{1-2\theta\phi+\theta^2}$

★Check White Noise

$\frac{1}{\sqrt{n}} \Rightarrow 95\% \Rightarrow \frac{2}{\sqrt{n}}$

$\sigma(\rho_k)$

ARIMA(X,X,X)

ARIMA(1,1,1) $\ne 0$ y_t is not stationary

$(1-\phi B)(1-B)y_t=(1-\theta B)a_t$

$\Rightarrow [1-(\phi+1)B+\phi B^2]y_t=a_t-\theta a_{t-1}$

$\Rightarrow y_t=(\phi+1)y_{t-1}-\phi y_{t-2}+a_t-\theta a_{t-1}$

Invertible but Non Stationary $\Rightarrow \theta(1)\pi(1)=0$

PACF

$\phi_{11}=\rho_1$;

$\phi_{22}=\frac{\begin{vmatrix}1 & \rho_1\\ \rho_1 & \rho_2\end{vmatrix}}{\begin{vmatrix}1 & \rho_1\\ \rho_1 & 1\end{vmatrix}}=\frac{\rho_2-\rho_1^2}{1-\rho_1^2}$

$\phi_{33}=\frac{\begin{vmatrix}1 & \rho_1 & \rho_1\\ \rho_1 & 1 & \rho_2\\ \rho_2 & \rho_1 & \rho_3\end{vmatrix}}{\begin{vmatrix}1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_1\\ \rho_2 & \rho_1 & 1\end{vmatrix}}$

$y_t=3+y_{t-1}+a_t-0.75a_{t-1} \Rightarrow ARIMA(0,1,1) \Rightarrow E=3$

$V(\nabla y_t)=E[(y_t-y_{t-1})^2]-E[y_t-y_{t-1}]^2=E(9+a_t^2+0.5625a_{t-1}^2)-9=1.5625\sigma_a^2$

$y_t=10+1.25y_{t-1}-0.25y_{t-2}+a_t-0.1a_{t-1} \Rightarrow ARIMA(1,1,1)$

$\Rightarrow y_t-y_{t-1}=10+0.25(y_{t-1}-y_{t-2})+a_t-10a_{t-1} \Rightarrow ARMA(1,1)$

$y_t=5+2y_{t-1}-1.7y_{t-2}+0.7y_{t-3}+a_t-0.5a_{t-1}+0.25a_{t-2} \Rightarrow ARIMA(2,1,2)$

$\Rightarrow ARMA(2,2)$

Different Equation Form

$y_t=\phi_1 y_{t-1}+\dots+\phi_{p+d}y_{t-p-d}+a_t-\theta a_{t-1}-\dots-\theta_q a_{t-q}$

Random Shock Form

$y_t=a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\psi_3 a_{t-3}+\dots$

Inverted Form

$y_t=\pi_1 y_{t-1}+\pi_2 y_{t-2}+\pi_3 y_{t-3}+\dots+a_t.$

ARIMA

$\phi(B)\nabla^d y_t=\theta(B)a_t+L \Rightarrow$ where. $\begin{cases}\phi(B)=1-\phi_1 B-\phi_2 B\dots\\ \theta(B)=1-\theta_1 B-\theta_2 B\dots\\ L \text{ is constant}\end{cases}$

☆ ϕB & θB s' roots out of unit circle $\Rightarrow$ stationary & invertible.

Unit Root Test = Identify d.

MA(q) = ACF Cut off

AR(p) = PACF Cut off.

ARMA = EACF NW Corner

Param Estimate = MLE

$Var(r_k) = \frac{1}{n}\left[1+2\sum \rho_j^2\right] \quad k>q$

$\Rightarrow \frac{r_k}{\sqrt{V(r_k)}} \Rightarrow$ see if Reject MA

ϕ	$\sqrt{Var(r_1)}$	$\sqrt{Var(r_2)}$	Corr $r_1 r_2$	$\sqrt{Var\, r_{10}}$
± 0.9	$0.44/\sqrt{n}$	$0.807/\sqrt{n}$	$\pm .97$	$2.44/\sqrt{n}$
$\pm .7$	$0.71/\sqrt{n}$	$1.12/\sqrt{n}$	$\pm .89$	$1.7/\sqrt{n}$
$\pm .4$	$0.92/\sqrt{n}$	$1.11/\sqrt{n}$	$\pm .66$	$1.18/\sqrt{n}$
$\pm .2$	$0.98/\sqrt{n}$	$1.04/\sqrt{n}$	$\pm .38$	$1.04/\sqrt{n}$

S.MA(1)

$Y_t = a_t - \Theta a_{t-s}$

$\rho_k = \begin{cases} 1 & k=0 \\ -\frac{\Theta_s}{1+\Theta_s^2} & k=s \\ 0 & k\neq s;\ k\geq 1 \end{cases}$

SMA(Q): $Y_t = a_t - \Theta_s a_{t-s} - \Theta_{2s} a_{t-2s} - \dots - \Theta_{Qs} a_{t-Qs}$

$\rho_s = \frac{-\Theta_s + \Theta_s\Theta_{2s} + \dots + \Theta_{(Q-1)s}\Theta_{Qs}}{1+\Theta_s^2+\Theta_{2s}^2+\dots+\Theta_{Qs}^2}$

$\vdots$

$\rho_{Qs} = \frac{-\Theta_{Qs}}{1+\Theta_s^2+\Theta_{2s}^2+\dots+\Theta_{Qs}^2}$

$\rho_k = 0,\ k\neq s, 2s, \dots Qs$

S.AR(1)

$Y_t = \Phi_s Y_{t-s} + a_t$.

$\rho_k = \begin{cases} \Phi_s^{k/s} & k=0, s, 2s \\ 0 & k\neq (0, s, \dots) \end{cases}$

SAR(P) $Y_t = \Phi_s Y_{t-s} + \Phi_{2s} Y_{t-2s} + \dots \Phi_{Ps} Y_{t-Ps} + a_t$

$\begin{cases} (1-\Phi_s B^s - \Phi_{2s}B^{2s} - \dots - \Phi_{Ps}B^{Ps})\rho_k = 0 \\ \rho_k = 0. \end{cases}$

ARIMA $(0\,0\,1)\times(0\,0\,1)_s$ (intra, inter) $Y_t = (1-\theta_1 B)(1-\Theta_s B^s)a_t$

$= a_t - \theta_1 a_{t-1} - \Theta_s a_{t-s} + \theta_1\Theta_s a_{t-s-1}$

$\rho_1 = \frac{-\theta_1}{1+\theta_1^2} \qquad \rho_{s-1} = \rho_{s+1} = \rho_1 \times \rho_s = \frac{\theta_1 \cdot \Theta_s}{(1+\theta_1^2)(1+\Theta_s^2)}$

$\rho_s = \frac{-\Theta_s}{1+\Theta_s^2} \qquad \rho_k = 0,\ k\neq(1, s-1, s+1)$

ARIMA $(1\,0\,0)\times(1\,0\,0)_s$

$(1-\phi_1 B)(1-\Phi_s B^s)Y_t = a_t = Y_t - \phi_1 Y_{t-1} - \Phi_s Y_{t-s} + \phi_1\Phi_s Y_{t-1-s}$

ACF: $\rho_k = \phi_1\rho_{k-1} + \Phi_s\rho_{k-s} - \phi_1\Phi_s\rho_{k-s-1}$

ARIMA $(1\,0\,0)\times(0\,0\,1)_s$ $\quad (1-\phi_1 B)Y_t = (1-\Theta_s B^s)a_t$, $Y_t = \phi_1 Y_{t-1} + a_t - \Theta_s a_{t-s}$

$r_k = \phi_1 r_{k-1} - \Theta_s r_{ay}(k-s)$

$r_{ay}(k-s) = \begin{cases} 0 & k>s \\ \sigma_a^2 & k=s \\ \phi_1^{s-k}\sigma_a^2 & ;k<s \end{cases}$

ARIMA $(0\,0\,1)(1\,0\,0)_s$

$(1-\Phi_s B^s)Y_t = (1-\theta_1 B)a_t$, $Y_t = \Phi_s Y_{t-s} + a_t - \theta_1 a_{t-1}$

$\rho_1 = \frac{-\theta_1}{1+\theta_1^2}$

$\rho_2 = \rho_3 = \dots = \rho_{s-2} = 0$

$\rho_{s-1} = \Phi_s\rho_1 = \frac{-\Phi_s\theta_1}{1+\theta_1^2}$

$\rho_s = \Phi_s$

$\rho_{s+1} = \rho_{s-1}$

ARIMA $(0\,1\,1)\times(0\,1\,1)_s$

$\underbrace{Y_t - Y_{t-1}}\ \ Y_t - Y_{t-s}$

$(1-B)(1-B^s)Y_t = (1-\theta_1 B)(1-\Theta_s B^s)a_t$

$(Y_t - Y_{t-1}) - (Y_{t-s} - Y_{t-s-1})$

$= a_t - \theta_1 a_{t-1} - \Theta_s a_{t-s} + \theta_1\Theta_s a_{t-1-s}$

$e_t(l) = a_{t+l} + \psi_1 a_{t+l-1} + \dots + \psi_{l-1} a_{t+1}$

$V(e_t(l)) = \sigma_a^2(1+\psi_1^2 + \dots \psi_{l-1}^2)$

interval: $\hat{Y}_{t+l} \pm Z_{\frac{\alpha}{2}}\sqrt{V(e_t(l))}$

Lag l Prediction Error.

$E[(Y_{t+l} - \hat{Y}_{t+l})^2] = \sigma_a^2(1+\psi_1^2 + \dots \psi_{l-1}^2)$

$+ \sigma_a^2[(\psi_l - \psi_l^*)^2 + (\psi_{l+1} - \psi_{l+1}^*)^2 + \dots]$

$= \sigma_a^2(1+\psi_1^2 + \dots \psi_{l-1}^2) + \sigma_a^2\sum(\psi_{l+j} - \psi_{l+j}^*)^2$

MSE

$\rho_t(j) = \rho(e_t(l), e_t(l+j))$

$= \frac{\sum \psi_i \psi_{i+j}}{\sqrt{V(e_t(l)) \times V(e_t(l+j))}}$

AR(1) Forecast

$\hat{Y}_t(1) = \mu + \phi(Y_t - \mu)$

$\hat{Y}_t(l) = \mu + \phi(\hat{Y}_t(l-1) - \mu);\ l\geq 1$

$= \mu + \phi^l(Y_t - \mu)$

MA(1) Forecast

$\hat{Y}_t(1) = \mu - \theta a_t$

$a_t(1) = Y_{t+1} - \hat{Y}_t(1)$

$= (\mu + a_{t+1} - \theta a_t) - (\mu - \theta a_t)$

$= a_{t+1}$

$\hat{Y}_t(l) = \mu$ for $l>1$.

ARMA (1,1) Forecast

$\hat{Y}_t(1) = \phi Y_t + \theta_0 - \theta a_t$

$\hat{Y}_t(2) = \phi \hat{Y}_t(1) + \theta_0$

$\hat{Y}_t(l) = \phi \hat{Y}_t(l-1) + \theta_0$

ARMA(p,q)

$\hat{Y}_t(l) = \phi_1 \hat{Y}_t(l-1) + \phi_2 \hat{Y}_t(l-2)$

$+ \dots \phi_p \hat{Y}_t(l-p) + \theta_0$

$- \theta_1 E(e_{t+l-1} \mid Y_1, Y_2, \dots Y_t)$

$- \theta_2 E(e_{t+l-2} \mid Y_1, Y_2, \dots Y_t)$

$- \dots - \theta_q E(e_{t+l-q} \mid Y_1, Y_2, \dots Y_t)$

where

$E(e_{t+j} \mid Y_1, Y_2, \dots Y_t) = \begin{cases} 0 & \text{for } j>0 \\ e_{t+j} & \text{for } j\leq 0 \end{cases}$