HW04

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Q1

For $\rho_1 = \frac{\theta}{1+\theta^2} \Rightarrow \frac{\partial}{\partial\theta}\rho_1 = \frac{-1(1+\theta^2)-(-\theta)(2\theta)}{(1+\theta^2)^2}$

$$\Rightarrow \frac{\theta^2-1}{(1+\theta^2)^2} = 0 \Rightarrow \begin{cases} \theta = -1 \\ \theta = 1 \end{cases} \Rightarrow \max\rho_1 = \frac{1}{2} = 0.5$$

$$\min\rho_1 = \frac{-1}{2} = (-0.5)$$

Q2 (a)

$\phi_1 = 1$

$\phi_2 = (-0.5)$

$$\Rightarrow \rho_1 = \frac{1}{1-(-0.5)} = \frac{2}{3}$$

(b)

$$\rho_0 = \frac{\rho_2 - \phi_1\rho_1}{\phi_2} = \frac{(\frac{-1}{3})-(1)\frac{2}{3}}{(-0.5)} = 2$$

$$\rho_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2 = \frac{(-0.5)^2}{1-(-0.5)} + (-0.5) = \frac{-1}{3}$$

$\rho_3 = (-0.17)$

$\rho_4 = 0$

$\rho_5 = 0.083$

$\rho_6 = 0.083$

$\rho_7 = 0.417$

$\rho_8 = 0$

$\rho_9 = -0.021$

$\rho_{10} = -0.021$

$\rho_{11} = -0.01$

$\rho_{12} = 0$

$\rho_{13} = 0.005$

$\rho_{14} = 0.005$

$\rho_{15} = 0.003$

Q3(1)

$y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2} = (1-1.3B+0.4B^2)a_t = \theta(B)a_t$

$\Rightarrow$ roots of $(1-1.3B+0.4B^2)=0 \Rightarrow (4B-5)(B-2)=0$

$\Rightarrow B = \frac{5}{4}, 2 \Rightarrow$ stationary & invertible #

Q3(2)

$\Rightarrow (1-0.5B)y_t = (1-1.3B+0.4B^2)a_t$

$\Downarrow$ $\Downarrow$

$\phi(B)=0$ $\quad\quad$ $\phi(B)=0$

$\Rightarrow B=2.$ $\quad\quad$ $\Rightarrow B = 2, \frac{5}{4}$

$\Rightarrow$ stationary & invertible #

Q3(3)

$\Rightarrow (1-1.5B+0.6B^2) = a_t \Rightarrow$ stationary & invertible #

Q3(4)

$\Rightarrow (1-B)y_t = (1-0.5B)a_t$

$\Rightarrow B = 1, 2 \Rightarrow$ stationary & invertible #

Q4(a)

(1) $(1-\beta)y_t = (1-0.5\beta)a_t$

$$y_t = (1+\beta+\beta^2+\beta^3+\dots)(1-0.5\beta)a_t$$

$$y_t = (1+0.5\beta+0.5\beta^2+0.5\beta^3+\dots+0.5\beta^n)a_t$$

$$= a_t + 0.5a_{t-1} + 0.5a_{t-2} + 0.5a_{t-3}$$

$$\Rightarrow \psi_1 = \psi_2 = \psi_3 = 0.5$$

Q4(b)

$(1-\beta)y_t = (1-0.5\beta)a_t$

$$a_t = (1-\beta)(1+0.5\beta+0.25\beta^2+0.125\beta^3+\dots+0.5^n\beta^n)y_t$$

$$\Rightarrow (1-0.5\beta-0.25\beta^2-0.125\beta^3+\dots-0.5^n\beta^n)y_t$$

$$\Rightarrow y_t = a_t + 0.5y_{t-1} + 0.25y_{t-2} + \dots \sum_{j=4}^{n} 0.5^j y_{t-j}$$

$$\Rightarrow \pi_1 = 0.5, \ \pi_2 = 0.05, \ \pi_3 = 0.125.$$

Q4(c)

$$Var(y_t) = Var[(1+0.5\sum_{j=1}^{n}\beta^j)a_t]$$

$$= Var(a_t) + 0.25 \times n \times Var(a_t)$$

$$= 1 + 0.25n \Rightarrow n \to \infty$$

$$\Rightarrow Var(y_t) = \infty \ \#$$

Q5

$\Rightarrow Var(\nabla y_t) = Var(y_t - y_{t-1}) = Var(y_t) + Var(y_{t-1}) - 2Cov(y_t, y_{t-1})$

$\Rightarrow 2(1-\rho_1)Var(y_t) \Rightarrow$ When $\rho_1 < \frac{1}{2} \Rightarrow 2(1-\rho_1) > 1$

$\Rightarrow$ Still follows a stationary process #

Q6.

(a) $\Rightarrow y_t = \phi y_{t-1} + e_t = \phi(\phi y_{t-2} + e_{t-1}) + e_t$

$= \phi^2 (\phi y_{t-3} + e_{t-2}) + \phi e_{t-1} + e_t$

$\Rightarrow$ finally $y_t = \phi^t y_0 + e_t + \phi e_{t-1} + \dots + \phi^{t-1} e_1$ #

(b)

$E(y_t) = E(\phi^t y_0 + e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots \phi^{t-1} e_1)$

$= \phi^t E(y_0) = \phi^t \cdot \mu_0$ #

Q6(c)

$$Var(y_t) = Var(\phi^t y_0 + e_t + \phi e_{t-1} + \dots \phi^{t-1} e_1)$$

$$= \phi^{2t} \sigma_0^2 + \sigma_e^2 \sum_{k=0}^{t-1} \phi^{2k}$$

$\Rightarrow$ when $\phi \neq (-1) \Rightarrow \dfrac{1-\phi^{2t}}{1-\phi^2}$

$\phi = 1 \Rightarrow t\sigma_e^2 + \sigma_0^2$ #

Q6(d).

if $\mu_0 = 0$ and $\phi = 1 \Rightarrow V(y_t) = V(y_{t-1}) + \sigma_e^2$

$\Rightarrow$ Variance becomes larger and larger

$\Rightarrow$ it's impossible. #

Q6(e)

Since y_t is stationary $\Rightarrow V(y_t) = \dfrac{1-\phi^{2t}}{1-\phi^2} = \dfrac{\sigma_e^2}{(1-\phi^2)}$

While $Var(y_t) \geq 0 \Rightarrow |\phi|$ should ≤ 1 #