Proof of variance of stationary time series

Asked 3 years, 11 months ago Active 6 months ago Viewed 1k times

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Suppose that $\{X_t\}$ is a weakly stationary time series with mean $\mu = 0$ and a covariance function $\gamma(h)$, $h \geq 0$, $\mathrm{E}[X_t] = \mu = 0$ and $\gamma(h) = \mathrm{Cov}(X_t, X_{t+h}) = \mathrm{E}\left[X_t X_{t+h}\right]$

Show that:

$$\mathrm{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = \frac{\gamma(0)}{n} + \frac{2}{n}\sum_{u=1}^{n-1}\left(1 - \frac{m}{n}\right)\gamma(m).$$

So far, I've gotten this:

$$\begin{aligned}\mathrm{Var}(\bar{X}) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_i, X_j)\\ &= \frac{1}{n^2}\sum_{i-j=-n}^{n}(n - |i-j|)\gamma(i-j)\\ &= \frac{1}{n}\sum_{m=-n}^{n}\left(1 - \frac{|m|}{n}\right)\gamma(m)\end{aligned}$$

How am I supposed to come up with the $\frac{\gamma(0)}{n} + \frac{2}{n}$?

time-series self-study variance stationarity

edited May 11 '18 at 13:36
gung - Reinstate Monica
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asked Oct 13 '16 at 13:38
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https://stats.stackexchange.com/questions/240010/proof-of-variance-of-stationary-time-series

3 Hint: under stationarity, only the distance of two elements of the proces
– Christoph Hanck Oct 13 '16 at 15:13

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You are almost there! Now you just need to recognise that auto-correlation only depends on the lag, so you have $\gamma(m) = \gamma|m|$, which means that the entire summand depends on m only through $|m|$ (i.e., it is symmetric around $m = 0$). This allows you to split the sum into the middle element ($m = 0$) and two lots of the symmetric part ($|m| = 1, \ldots, n-1$), which gives you:

$$\begin{aligned}
\text{Var}(\bar{X}) &= \frac{1}{n} \sum_{m=-n}^{n} \Big(1 - \frac{|m|}{n}\Big) \gamma(m) \\
&= \frac{1}{n} \sum_{m=-n}^{n} \Big(1 - \frac{|m|}{n}\Big) \gamma|m| \\
&= \frac{1}{n} \Bigg[\gamma(0) + 2 \sum_{|m|=1}^{n} \Big(1 - \frac{|m|}{n}\Big) \gamma|m| \Bigg] \\
&= \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{m=1}^{n} \Big(1 - \frac{m}{n}\Big) \gamma(m) \\
&= \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{m=1}^{n-1} \Big(1 - \frac{m}{n}\Big) \gamma(m).
\end{aligned}$$

(The last step follows from the fact that $1 - \frac{m}{n} = 0$ for $m = n$.) This method of splitting symmetric sums around their mid-point is a common trick used in these kinds of cases to simplify the sum by taking it only over positive arguments. It is a worthwhile trick to learn in general.

edited May 7 '19 at 22:45

answered Jul 18 '18 at 2:28
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0

first, fixing the definition of the problem, the index is m instead of u, to make simpler I will use only the index i and j.

We want to prove that

$\text{Var}\left(\frac{X_1+X_2+\ldots+X_n}{n}\right) = \frac{\gamma(0)}{n} + \frac{2}{n}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\gamma(i).$

The begin is correct,

$$\text{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}(X_i, X_j)$$

We can notice that $\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$ and, from our assumptions about the problem, that $\text{Cov}(X_i, X_i + h) = \text{Cov}(X_i, X_i - h) = \gamma(h)$ for any i and h.

We can visualize the sum of covariances in i and j as follows

$$\begin{vmatrix} \text{Cov}(1,1) & \text{Cov}(1,2) & \cdots & \text{Cov}(1,n-1) & \text{Cov}(1,n) \\ \text{Cov}(2,1) & \text{Cov}(2,2) & \cdots & \text{Cov}(2,n-1) & Cov(2,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \text{Cov}(n-1,1) & \text{Cov}(1,2) & \cdots & \text{Cov}(n-1,n-1) & \text{Cov}(n-1,n) \\ \text{Cov}(n,1) & \text{Cov}(n,2) & \cdots & \text{Cov}(n,n-1) & \text{Cov}(n,n) \end{vmatrix}$$

What is equal to

$$\begin{vmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{vmatrix}$$

To sum all the elements we can first sum the main diagonal, and as it is symmetric sum twice the other diagonals

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}(X_i, X_j) = n\gamma(0) + 2\sum_{i=1}^{n-1}(n-i)\gamma(i)$$

.

Back to the main equation

$$\text{Var}\left(\frac{X_1+X_2+\ldots+X_n}{n}\right) = \frac{\gamma(0)}{n} + \frac{2}{n^2}\sum_{i=1}^{n-1}(n-i)\gamma(i) = \frac{\gamma(0)}{n} + \frac{2}{n}\sum_{i=1}^{n-1}(1-\frac{i}{n})\gamma(i).$$

edited Mar 31 at 7:46 answered Oct 14 '16 at 17:07