National Taiwan University
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Time Series Analytics

110-1 Homework #03

Due at 23h59, October 31, 2021; files uploaded to NTU-COOL

1. (10%) y_t is a stationary process with the autocovariance function γ_k. Define $\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$. Show that
$$\mathrm{V}[\bar{y}] = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1 - \frac{|k|}{n}\right)\gamma_k.$$

2. (10%) Assume x_t is a stationary process, and define $y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$
   (a) Show that $\mathrm{COV}[y_t,\ y_{t-k}]$ is independent of t for all lags k.
   (b) Is y_t stationary?

3. (10%) Let y_t be a stationary process with autocovariance function γ_k.
   (a) Show that $z_t = \nabla y_t = y_t - y_{t-1}$ is stationary by finding the mean and autocovariance function for z_t.
   (b) Show that $w_t = \nabla^2 y_t = z_t - z_{t-1} = y_t - 2y_{t-1} + y_{t-2}$ is stationary.

4. (15%) Let x_t be stationary with $\mathrm{E}[x_t] = 0, \mathrm{V}[x_t] = 1$, autocorrelation function ρ_k. Define μ_t is a nonconstant function and σ_t is a positively nonconstant **function** (that is to say: μ_t and σ_t are deterministic and in function of t). Now we observe a time series formed as
$$y_t = \mu_t + \sigma_t x_t.$$
   (a) Find the mean and autocovariance function of y_t.
   (b) Show that the autocorrelation of y_t depends only on the lag k. Is y_t stationary?
   (c) Find a scenario that y_t is nonstationary but with a constant mean, i.e., $\mu_t = \mu_0$, and with ρ_k independent of t?

5. (5%) Let x_t be the series of the "expected" measurements during the production process. Because the measuring tool itself won't be perfect, we actually observe $y_t = x_t + e_t$, assuming x_t and e_t are independent. In general, we call x_t the **signal** and e_t the **measurement noise**.

   If x_t is stationary with the autocorrelation function ρ_k, show that y_t is also a stationary process with
$$\mathrm{corr}(y_t, y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}, \text{for } k \geq 1.$$
   $\frac{\sigma_x^2}{\sigma_e^2}$ is usually referred to as the **signal-to-noise ratio**, or **SNR**.
   The larger the SNR, the closer the autocorrelation function of the observed series y_t is to the autocorrelation function of the desired signal x_t.

6. (10%) Suppose $y_t = \alpha_0 + \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)]$, where $\alpha_0, f_1, f_2, \ldots, f_q$ are constants and $\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_q$ are independent random variables with zero means and variances $\mathrm{V}[\alpha_i] = V[\beta_i] = \sigma_i^2$. Show that y_t is stationary and find its autocovariance function.

   (Hint: try to show $\mathrm{COV}[y_t, y_s]$ depends only on $t - s$.)