R09546042 楊雲終 HW3.

1. $\begin{cases} Y_t & \text{stationary process}, \quad \bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t, \\ \gamma_k & \text{autocovariance} \end{cases}$

① ②

→ show that $V[\bar{Y}] = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k$

①

$V[\bar{Y}] = V\left[\frac{1}{n}\sum_{t=1}^{n} Y_t\right] = \frac{1}{n^2}V\left[\sum_{t=1}^{n} Y_t\right] = \frac{1}{n^2}V[Y_{t+1}, Y_{t+2} \cdots Y_n]$

$\Rightarrow \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(Y_i, Y_j)$ ← $\begin{cases} \gamma_k = Cov(Y_t, Y_{t+k}) = Cov(Y_{t+k}, Y_t) \\ \gamma_0 = Cov(Y_t, Y_t) \end{cases}$

$\Rightarrow \frac{1}{n^2}\left[n\gamma_0 + 2\sum_{k=1}^{n-1}(n-k)\gamma_k\right] = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k$ 得

②

$\frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k \Rightarrow \frac{2}{n}\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k$

$\Rightarrow \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k$

2.

x_t is stationary process. $Y_t = \begin{cases} X_t & \text{for odd } t \\ X_t + 3 & \text{for even } t \end{cases}$

ⓐ show $Cov(Y_t, Y_{t-k})$ is independent of t for all lags k.

$\because$ $cov(Y_t, Y_{t-k}) = Cov(X_t + 3,\ X_{t-k} + 3)$

$\therefore$ $Cov(Y_t, Y_{t-k})$ is independent of t for all lags k.

ⓑ is Y_t stationary ?

NO. because $\begin{cases} E(Y_{t,odd}) = E(Y_t) \\ E(Y_{t,even}) = E(Y_t) + 3 \end{cases}$ not equal.

Thus. Y_t is not a stationary series.

3. Y_t stationary

γ_k autocovariance

(a) show $Z_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary

$E(Z_t) = E(\nabla Y_t) = E(Y_t - Y_{t-1}) = \underbrace{E(Y_t) - E(Y_{t-1})}_{=0}$

$\because Y_t$ is stationary

$\therefore E(Z_t) = 0$ (i)

$Cov(Z_t, Z_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1})$

$\Rightarrow Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-k-1}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-k-1})$

$= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k = \underbrace{2\gamma_k - \gamma_{k+1} - \gamma_{k-1}}_{\text{It is free from } t}$ (ii)

by (i) & (ii), Z_t is stationary series

(b) Assume $X_t = \nabla^2 Y_t = Z_t - Z_{t-1}$, known $\begin{cases} Y_t \\ \nabla(Y_t) = Z_t \end{cases}$ is stationary. Doing (a) again, we will get:

$\begin{cases} E(X_t) = E(Z_t) - E(Z_{t-1}) = 0 \\ Cov(X_t, X_{t-k}) = 2\gamma_k - \gamma_{k+1} - \gamma_{k-1} \Rightarrow \text{free from } t. \end{cases}$

thus, X_t will also be stationary series. #

4. X_t stationary $\Rightarrow \begin{cases} E(X_t) = 0 \\ V[X_t] = 1 \end{cases}$, ρ_k

$\begin{cases} \mu_t \rightarrow \text{nonconstant} \\ \sigma_t \rightarrow \text{positive noconstant} \end{cases}$ $Y_t = \mu_t + \sigma_t X_t$

(a) $E(Y_t) = E(\mu_t + \sigma_t X_t) = E(\mu_t) + \sigma_t \underbrace{E(X_t)}_{0} = \overset{(i)}{E(\mu_t)}$

$Cov(Y_t, Y_{t-k}) = Cov(\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k})$ μ_t is Constant

$\Rightarrow Cov.(\sigma_t X_t, \sigma_{t-k} X_{t-k}) = \sigma_t \sigma_{t-k} \; Cov(X_t, X_{t-k})$

$\Rightarrow \overset{(ii)}{\sigma_t \sigma_{t-k} \cdot \rho_k}$

(b) $Corr.(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{V[Y_t] \, V[Y_{t-k}]}} \rightarrow \frac{\sigma_t \sigma_{t-k} \cdot \rho_k}{\sqrt{\sigma_t^2 \sigma_{t-k}^2}} = \underbrace{\rho_k}_{\text{depends on } k}$

$\because E(Y_t) = E(\mu_t) = \mu_t$.

$\therefore Y_t$ is not stationary, with changing mean.

(c) Y_t is non stationary, but constant mean. $\Rightarrow$ weak stationary

or Gaussian Function

$V[Y_t] = V[\underbrace{\mu_t}_{\text{Constant.}} + \sigma_t X_t] \Rightarrow \underbrace{\sigma_t^2 V[X_t]}_{\text{depends on } t}$.

$\therefore Y_t$ is nonstationary.

5.

$$y_t = x_t + e_t \quad \begin{cases} x & \text{signal} \\ e & \text{measurement noise} \end{cases} \text{ independant}$$

x_t is stationary, ρ_k => y_t is also stationary

$$E(y_t) = E(x_t + e_t) = E(x_t) + \underbrace{E(e_t)}_{\text{const}} \overset{\text{i}}{=} E(x_t)$$

$$Cov(y_t, y_{t-k}) = Cov(x_t + e_t,\ x_{t-k} + e_{t-k})$$

$$= Cov(x_t, x_{t-k}) + \cancel{Cov(x_t, e_{t-k})} + \cancel{Cov(e_t, x_{t-k})} + Cov(e_t, e_{t-k})$$

$\because e$ & x indep. => $Cov(x_t, x_{t-k}) + \underbrace{Cov(e_t, e_{t-k})}_{\text{const}}$

$$\overset{\text{ii}}{=} Cov(x_t, x_{t-k})$$

Thus y_t is stationary (if e_t is iid.) by i, ii

$$Corr(y_t, y_{t-k}) = \frac{Cov(y_t, y_{t-k})}{V[y_t]} = \frac{Cov(x_t, x_{t-k})}{V[x_t + e_t]} = \frac{V[x_t]\rho_k}{V[x_t] + V[e_t]}$$

$$\Rightarrow \frac{\rho_k}{1 + \frac{V[e_t]}{V[x_t]}} = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}},$$

6.

$$Y_t = \alpha_0 + \sum_{i=1}^{q} [\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)] .$$

$\alpha_0, f_1, f_2 \cdots f_q \Rightarrow$ const.

$\left.\begin{matrix}\alpha_1, \alpha_2 \cdots \alpha_q \\ \beta_1, \beta_2 \cdots \beta_q\end{matrix}\right\} \Rightarrow$ independent $\quad E(\alpha) = E(\beta) = 0$, $V(\alpha_i) = E(\beta_i) = \sigma_i^2$

(i)

$$E(Y_t) = E(\alpha_0) + E\left(\sum_{i=1}^{q} \alpha_i \cos(2\pi f_i t)\right) + E\left(\sum_{i=1}^{q} \beta_i \sin(2\pi f_i t)\right)$$

$$\Rightarrow E(\alpha_0) + \underbrace{\sum_{i=1}^{q} E(\alpha_i) E(\cos(2\pi f_i t))}_{\text{by } E(\alpha_i) = 0} + \underbrace{\sum_{i=1}^{q} E(\beta_i) E(\sin(2\pi f_i t))}_{\text{by } E(\beta_i) = 0}$$

$= E(\alpha_0)$ is const

(ii)

$$Cov(Y_t, Y_s) = Cov\Big[\underbrace{\alpha_0}_{\text{const}} + \sum_{i=1}^{q} [\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)],\ \cancel{\alpha_0} + \sum_{i=1}^{q} [\alpha_i \cos(2\pi f_i s) + \beta_i \sin(2\pi f_i s)]\Big]$$

$$\longrightarrow \sum_{i=1}^{q} Cov(\alpha_i \cos(2\pi f_i t), \alpha_i \cos(2\pi f_i s)) + \sum_{i=1}^{q} Cov(\beta_i \sin(2\pi f_i t), \beta_i \sin(2\pi f_i s))$$

α_i, β_i indenp.

$$\rightarrow \sum_{i=1}^{q} \{V[\alpha_i] \cos(2\pi f_i t) \cos(2\pi f_i s)\} + \sum_{i=1}^{q} \{V[\beta_i] \sin(2\pi f_i t) \sin(2\pi f_i s)\}$$

$$\rightarrow \sum_{i=1}^{q} \{\sigma_i^2 [\cos(2\pi f_i t) \cos(2\pi f_i s) + \sin(2\pi f_i t) \sin(2\pi f_i s)]\}$$

$\cos(A)\cos(B) + \sin(A)\sin(B) \Rightarrow \cos(A-B)$

$$\Rightarrow \cos(A-B) = \underbrace{\sum_{i=1}^{q} \{\sigma_i^2 \cos[2\pi f_i (t-s)]\}}_{\text{depends on } (t-s)} \rightarrow \text{auto covariance: ACF}$$

$A = 2\pi f_i t$, $B = 2\pi f_i s$

Y is Staionary.