TSA08 – Multivariate Time Series Models

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When there are more than one time series

- If h time series: $\{y_{1,t}, y_{2,t}, \ldots, y_{h,t}\}$, are having interactive and dynamic correlation structure, we can use the multivariate stochastic models to characterize these time series.
  - MA, AR, ARMA, ARIMA, SARIMA ➔ VMA, VAR, VARMA, VARIMA, VSARIMA

$$\mathbf{y}_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{h,t} \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,t} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ y_{h,1} & y_{h,2} & \cdots & y_{h,t} \end{bmatrix}, \mathbf{a}_t \sim N_h\left(\mathbf{0}, \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1h} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1h} & \sigma_{2h} & \cdots & \sigma_{hh} \end{bmatrix}\right)$$

VMA(q) Models

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Bivariate VMA(1), i.e., $h = 2$

- Model: $\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{a}_t - \boldsymbol{\Theta}\mathbf{a}_{t-1} = \boldsymbol{\mu} + (\mathbf{I}_2 - \boldsymbol{\Theta}B)\mathbf{a}_t$

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} B \right) \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix}.$$

$$y_{1,t} = \mu_1 + a_{1,t} - \theta_{11}a_{1,t-1} - \theta_{12}a_{2,t-1} = \mu_1 + (1 - \theta_{11}B)a_{1,t} - \theta_{12}a_{2,t-1}$$
$$y_{2,t} = \mu_2 + a_{2,t} - \theta_{21}a_{1,t-1} - \theta_{22}a_{2,t-1} = \mu_2 + (1 - \theta_{22}B)a_{2,t} - \theta_{21}a_{1,t-1}$$

- Without loss of generality, $\mathbf{y}_t$ is centered, i.e., $\boldsymbol{\mu} = \mathbf{0}$.

$$\mathbf{y}_t = (\mathbf{I}_2 - \boldsymbol{\Theta}B)\mathbf{a}_t$$

- Invertible Form:

$$\mathbf{a}_t = (\mathbf{I}_2 - \boldsymbol{\Theta}B)^{-1}\mathbf{y}_t = (\mathbf{I}_2 + \boldsymbol{\Theta}B + \boldsymbol{\Theta}^2B^2 + \boldsymbol{\Theta}^3B^3 + \cdots)\mathbf{y}_t$$

h Variates VMA(1)

- Model: $\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{a}_t - \boldsymbol{\Theta}\mathbf{a}_{t-1} = \boldsymbol{\mu} + (\mathbf{I}_h - \boldsymbol{\Theta}B)\mathbf{a}_t$

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{h,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_h \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1h} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{h1} & \theta_{h2} & \cdots & \theta_{hh} \end{bmatrix} B \right) \begin{bmatrix} a_{1,t} \\ a_{2,t} \\ \vdots \\ a_{h,t} \end{bmatrix}$$

- In fact, each marginal series $y_{i,t}$ can be seen as an univariate MA(1).
- The invertible condition lies in: the roots to $|(\mathbf{I}_h - \boldsymbol{\Theta}B)| = 0$ are outside the unit circle.

h Variates VMA(q)

- Model: $\mathbf{y}_t = \boldsymbol{\mu} + \left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\mathbf{a}_t$

$$\boldsymbol{\Theta}_i \triangleq \begin{bmatrix} \theta_{11}^{(i)} & \theta_{12}^{(i)} & \cdots & \theta_{1h}^{(i)} \\ \theta_{21}^{(i)} & \theta_{22}^{(i)} & \cdots & \theta_{2h}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{h1}^{(i)} & \theta_{h2}^{(i)} & \cdots & \theta_{hh}^{(i)} \end{bmatrix}$$

- Each marginal series $y_{i,t}$ can be seen as an univariate MA(q).
- The invertible condition lies in: the roots to $\left|\left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\right| = 0$ are outside the unit circle.

Inverted Form of VMA(q)

- Let $\mathbf{y}_t$ be centered, i.e, $\mathbf{y}_t = \mathbf{y}_t - \boldsymbol{\mu}$.

$$\mathbf{a}_t = \left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)^{-1} \mathbf{y}_t \triangleq (\mathbf{I}_h - \boldsymbol{\Pi}_1 B - \boldsymbol{\Pi}_2 B^2 - \cdots)\mathbf{y}_t$$

$$\left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)(\mathbf{I}_h - \boldsymbol{\Pi}_1 B - \boldsymbol{\Pi}_2 B^2 - \cdots) = \mathbf{I}_h$$

- By comparing the coefficients in front of B^i

$$\boldsymbol{\Pi}_1 = -\boldsymbol{\Theta}_1$$
$$\boldsymbol{\Pi}_2 = -\boldsymbol{\Theta}_1^2 - \boldsymbol{\Theta}_2$$
$$\boldsymbol{\Pi}_j = \boldsymbol{\Theta}_1 \boldsymbol{\Pi}_{j-1} + \boldsymbol{\Theta}_2 \boldsymbol{\Pi}_{j-2} + \cdots + \boldsymbol{\Theta}_q \boldsymbol{\Pi}_{j-q}$$

- Therefore, the inverted form of VMA(q) :

$$\mathbf{y}_t = \mathbf{a}_t + \boldsymbol{\Pi}_1 \mathbf{y}_{t-1} + \boldsymbol{\Pi}_2 \mathbf{y}_{t-2} + \cdots$$

VAR(p) Models

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Bivariate VAR(1)

- Model: $\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{y}_{t-1} + \mathbf{a}_t$ or $(\mathbf{I}_2 - \boldsymbol{\Phi}B)\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{a}_t$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} B\right)\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix}.$$

$$y_{1,t} = \mu_1 - \phi_{11} y_{1,t-1} - \phi_{12} y_{2,t-1} + a_{1,t}$$
$$y_{2,t} = \mu_2 - \phi_{21} y_{1,t-1} - \phi_{22} y_{2,t-1} + a_{2,t}$$

- With the same fashion, the stationary condition for VAR(1) is to have the roots, i.e., b_1 and b_2, $|(\mathbf{I}_2 - \boldsymbol{\Phi}B)| = 0$ outside the unit circle.
- Each marginal series, $y_{i,t}$, can be viewed as an ARMA(2,1) model.

h Variates VAR(p)

- Model: $\left(\mathbf{I}_h - \mathbf{\Phi}_1 B - \mathbf{\Phi}_2 B^2 - \cdots - \mathbf{\Phi}_p B^p\right)\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{a}_t$

$$\mathbf{\Phi}_i \triangleq \begin{bmatrix} \phi_{11}^{(i)} & \phi_{12}^{(i)} & \cdots & \phi_{1h}^{(i)} \\ \phi_{21}^{(i)} & \phi_{22}^{(i)} & \cdots & \phi_{2h}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{h1}^{(i)} & \phi_{h2}^{(i)} & \cdots & \phi_{hh}^{(i)} \end{bmatrix}.$$

- Stationary conditions of VAR(p): the roots to $\left|\left(\mathbf{I}_h - \mathbf{\Phi}_1 B - \mathbf{\Phi}_2 B^2 - \cdots - \mathbf{\Phi}_p B^p\right)\right| = 0$ are outside the unit circle.
- Each marginal series, $y_{i,t}$, is following an ARMA$(hp, hp - p)$ model.

Random Shock Form of VAR(1)

- Let $\mathbf{y}_t$ be centered, i.e, $\mathbf{y}_t = \mathbf{y}_t - \boldsymbol{\mu}$.

$$\mathbf{y}_t = \left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)^{-1} \mathbf{a}_t \triangleq (\mathbf{I}_h + \boldsymbol{\Psi}_1 B + \boldsymbol{\Psi}_2 B^2 + \cdots)\mathbf{a}_t$$

$$\left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)(\mathbf{I}_h + \boldsymbol{\Psi}_1 B + \boldsymbol{\Psi}_2 B^2 + \cdots) = \mathbf{I}_h$$

- By comparing the coefficients in front of B^i

$$\boldsymbol{\Psi}_1 = \boldsymbol{\Phi}_1$$

$$\boldsymbol{\Psi}_2 = \boldsymbol{\Phi}_1 \boldsymbol{\Psi}_1 + \boldsymbol{\Phi}_2$$

$$\boldsymbol{\Psi}_j = \boldsymbol{\Phi}_1 \boldsymbol{\Psi}_{j-1} + \boldsymbol{\Phi}_2 \boldsymbol{\Psi}_{j-2} + \cdots + \boldsymbol{\Phi}_p \boldsymbol{\Psi}_{j-p}, \text{ for } j \geq 3.$$

VARMA(p, q) and VARIMA(p, d, q) VSARIMA

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h Variates VARMA(p, q)

- Model: $(\mathbf{y}_t = \mathbf{y}_t - \boldsymbol{\mu})$

$$\left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)\mathbf{y}_t = \left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\mathbf{a}_t$$

- The invertible and stationary conditions are the same as those in VMA(q) and VAR(p).
  - Invertibility: $\left|\left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\right| = 0$
  - Stationarity: $\left|\left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)\right| = 0$

Inverted & Random Shock Forms

- Inverted Form

$$\mathbf{a}_t = \left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)^{-1}\left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)\mathbf{y}_t$$
$$\triangleq (\mathbf{I}_h - \boldsymbol{\Pi}_1 B - \boldsymbol{\Pi}_2 B^2 - \cdots)\mathbf{y}_t$$

- Random Shock Form

$$\mathbf{y}_t = \left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)^{-1}\left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\mathbf{a}_t$$
$$= (\mathbf{I}_h + \boldsymbol{\Psi}_1 B + \boldsymbol{\Psi}_2 B^2 + \cdots)\mathbf{a}_t$$

VARMA(1, 1)

- Inverted Form

$$\mathbf{a}_t = (\mathbf{I}_h - \boldsymbol{\Theta} B)^{-1}(\mathbf{I}_h - \boldsymbol{\Phi} B)\mathbf{y}_t = (\mathbf{I}_h + \boldsymbol{\Theta} B + \boldsymbol{\Theta}^2 B^2 + \cdots)(\mathbf{I}_h - \boldsymbol{\Phi} B)\mathbf{y}_t$$
$$\triangleq (\mathbf{I}_h - \boldsymbol{\Pi}_1 B - \boldsymbol{\Pi}_2 B^2 - \cdots)\mathbf{y}_t$$

$$\boldsymbol{\Pi}_1 = \boldsymbol{\Phi} - \boldsymbol{\Theta}$$
$$\boldsymbol{\Pi}_j = \boldsymbol{\Theta}^{j-1}\boldsymbol{\Pi}_1 = \boldsymbol{\Theta}^{j-1}(\boldsymbol{\Phi} - \boldsymbol{\Theta})$$

- Random Shock Form

$$\mathbf{y}_t = (\mathbf{I}_h - \boldsymbol{\Phi} B)^{-1}(\mathbf{I}_h - \boldsymbol{\Theta} B)\mathbf{a}_t = (\mathbf{I}_h + \boldsymbol{\Phi} B + \boldsymbol{\Phi}^2 B^2 + \cdots)(\mathbf{I}_h - \boldsymbol{\Theta} B)\mathbf{a}_t$$
$$\triangleq (\mathbf{I}_h + \boldsymbol{\Psi}_1 B + \boldsymbol{\Psi}_2 B^2 + \cdots)\mathbf{a}_t$$

$$\boldsymbol{\Psi}_1 = \boldsymbol{\Theta} - \boldsymbol{\Phi}$$
$$\boldsymbol{\Psi}_j = \boldsymbol{\Phi}^{j-1}\boldsymbol{\Psi}_1 = \boldsymbol{\Phi}^{j-1}(\boldsymbol{\Theta} - \boldsymbol{\Phi})$$

h Variates VARIMA(p, d, q)

- Model: ($\mathbf{y}_t = \mathbf{y}_t - \boldsymbol{\mu}$)

$$\left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)(\mathbf{I}_h - B)^d \mathbf{y}_t = \left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\mathbf{a}_t$$

- Special Case: VARIMA$(0,1,1) =$ VIMA$(1,1)$,

$$(\mathbf{I}_h - B)\mathbf{y}_t = (\mathbf{I}_h - \boldsymbol{\Theta} B)\mathbf{a}_t$$

  - is also known as the “multivariate exponential smoothing model”.

Seasonal VARIMA ➔ VSARIMA

- h variates VSARIMA$\left(p, \{d_i\}_1^h, q\right) \times \left(P, \{D_i\}_1^h, Q\right)_s$, model is expressed as

$$\boldsymbol{\Phi}(B)\boldsymbol{\Phi}_s(B^s)\mathbf{D}(B)\mathbf{D}(B^s)\mathbf{y}_t = \boldsymbol{\Theta}(B)\boldsymbol{\Theta}_s(B^s)\mathbf{a}_t$$

$$\boldsymbol{\Phi}(B) = \mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p$$
$$\boldsymbol{\Theta}(B) = \mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q$$
$$\boldsymbol{\Phi}_s(B^s) = \mathbf{I}_h - \boldsymbol{\Phi}_s^* B^s - \boldsymbol{\Phi}_{2s}^* B^{2s} - \cdots - \boldsymbol{\Phi}_{Ps}^* B^P$$
$$\boldsymbol{\Theta}_s(B^s) = \mathbf{I}_h - \boldsymbol{\Theta}_s^* B^s - \boldsymbol{\Theta}_{2s}^* B^{2s} - \cdots - \boldsymbol{\Theta}_{Qs}^* B^Q$$

$$\mathbf{D}(B) = \begin{bmatrix} (1-B)^{d_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (1-B)^{d_h} \end{bmatrix}, \mathbf{D}(B^s) = \begin{bmatrix} (1-B^s)^{D_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (1-B^s)^{D_h} \end{bmatrix}$$

An Example: Investment, Consumption, and Income

- Investment usually works as an external factor, while consumption and income interact internally.

$$\begin{cases} y_{1,t} - \beta_1 y_{2,t} - \alpha_{11} y_{1,t-1} - \alpha_{12} y_{2,t-1} - \gamma_{11} x_t - \gamma_{12} x_{t-1} = a_{1,t} \\ y_{2,t} - \beta_2 y_{1,t} - \alpha_{21} y_{1,t-1} - \alpha_{22} y_{2,t-1} - \gamma_{21} x_t - \gamma_{22} x_{t-1} = a_{2,t} \\ x_t - \omega x_{t-1} = a_{3,t} \end{cases}$$

$$\begin{bmatrix} 1 & -\beta_1 & -\gamma_{11} \\ -\beta_2 & 1 & -\gamma_{21} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1,t} \\ y_{2,t} \\ x_t \end{bmatrix} - \begin{bmatrix} \alpha_{11} & \alpha_{12} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \gamma_{22} \\ 0 & 0 & \omega \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \\ a_{3,t} \end{bmatrix}$$

$$\mathbf{A}\mathbf{z}_t - \mathbf{C}\mathbf{z}_{t-1} = \mathbf{a}_t \text{ or } \mathbf{z}_t - \mathbf{A}^{-1}\mathbf{C}\mathbf{z}_{t-1} = \mathbf{A}^{-1}\mathbf{a}_t$$

Constructing a Multivariate Time Series Model

Identification, Estimation, Diagnosis

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Multivariate Time Series Model Construction

Identifying p, d, q → Estimating Φ_i, Θ_j → Checking Residuals

Identifying VAR, VMA, VARMA

- Recall the art of identifying AR(p), MA(q), ARMA(p, q)
  - ACF, PACF, EACF

- To identify VAR, VMA, VARMA, we need
  - CCM (Cross Covariance Matrix) ← ACF
  - PAR (Partial Autoregressive Matrix) ← PACF
  - ECCM (Extended Cross Covariance Matrix) ← EACF

CCM (Cross Covariance Matrix)

- Let $\mathbf{y}_t = \mathbf{y}_t - \boldsymbol{\mu}$, the lag k CCM is defined as:

$$\Gamma(k) = \text{cov}[\mathbf{y}_t, \mathbf{y}_{t-k}] = \text{E}[\mathbf{y}_t \mathbf{y}'_{t-k}] = \text{E}\left(\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{h,t} \end{bmatrix} [y_{1,t-k} \quad y_{2,t-k} \quad \cdots \quad y_{h,t-k}]\right)$$

$$= \begin{bmatrix} \sigma_{11}(k) & \sigma_{12}(k) & \cdots & \sigma_{1h}(k) \\ \sigma_{21}(k) & \sigma_{22}(k) & \cdots & \sigma_{2h}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{h1}(k) & \sigma_{h2}(k) & \cdots & \sigma_{hh}(k) \end{bmatrix}, \text{where } \sigma_{ii}(k) = \text{E}\left[y_{i,t} y_{i,t-k}\right], \text{and } \sigma_{ij}(k) = \text{E}\left[y_{i,t} y_{j,t-k}\right]$$

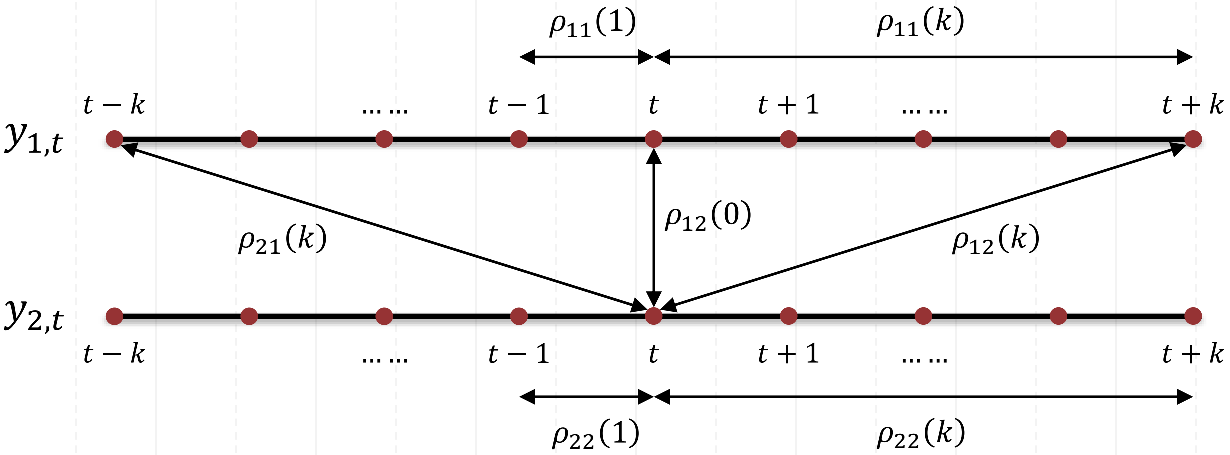
CCM (Cross Correlation Matrix)

$$\rho_{ii}(k) = \frac{\operatorname{cov}[y_{i,t}, y_{i,t-k}]}{\mathrm{V}[y_{i,t}]} = \frac{\sigma_{ii}(k)}{\sigma_{ii}(0)}, \rho_{ij}(k) = \frac{\operatorname{cov}[y_{i,t}, y_{j,t-k}]}{\sqrt{\mathrm{V}[y_{i,t}]\mathrm{V}[y_{j,t}]}} = \frac{\sigma_{ij}(k)}{\sqrt{\sigma_{ii}(0)\sigma_{jj}(0)}}$$

- The Cross Correlation Matrix is expressed as:

$$\mathrm{P}(k) = \begin{bmatrix} \rho_{11}(k) & \rho_{12}(k) & \cdots & \rho_{1h}(k) \\ \rho_{21}(k) & \rho_{22}(k) & \cdots & \rho_{2h}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{h1}(k) & \rho_{h2}(k) & \cdots & \rho_{hh}(k) \end{bmatrix}$$

CCM Properties



$$\mathrm{P}(k) = \Lambda^{-1}\Gamma(k)\Lambda^{-1}, \text{where } \Lambda = \begin{bmatrix} \sqrt{\sigma_{11}(0)} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}(0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{hh}(0)} \end{bmatrix}$$

CCM of VMA(1): $\mathbf{y}_t = \mathbf{a}_t - \boldsymbol{\Theta}\mathbf{a}_{t-1}$

$$\boldsymbol{\Gamma}(k) = \mathrm{E}[\mathbf{y}_t\mathbf{y}'_{t-k}] = \mathrm{E}[\mathbf{a}_t\mathbf{y}'_{t-k}] - \mathrm{E}[\boldsymbol{\Theta}\mathbf{a}_{t-1}\mathbf{y}'_{t-k}] = \mathrm{E}[\mathbf{a}_t\mathbf{y}'_{t-k}] - \boldsymbol{\Theta}\mathrm{E}[\mathbf{a}_{t-1}\mathbf{y}'_{t-k}].$$

- If $k = 0$,

$$\begin{aligned}\boldsymbol{\Gamma}(0) &= \mathrm{E}[\mathbf{y}_t\mathbf{y}'_t] = \mathrm{E}[\mathbf{a}_t(\mathbf{a}'_t - \mathbf{a}'_{t-1}\boldsymbol{\Theta}')] - \boldsymbol{\Theta}\mathrm{E}[\mathbf{a}_{t-1}(\mathbf{a}'_t - \mathbf{a}'_{t-1}\boldsymbol{\Theta}')]\\ &= \mathrm{E}[\mathbf{a}_t\mathbf{a}'_t] - \mathrm{E}[\mathbf{a}_t\mathbf{a}'_{t-1}]\boldsymbol{\Theta}' - \boldsymbol{\Theta}\mathrm{E}[\mathbf{a}_{t-1}\mathbf{a}'_t] + \boldsymbol{\Theta}\mathrm{E}[\mathbf{a}_{t-1}\mathbf{a}'_{t-1}]\boldsymbol{\Theta}'\\ &= \boldsymbol{\Sigma} + \boldsymbol{\Theta}\boldsymbol{\Sigma}\boldsymbol{\Theta}'\end{aligned}$$

- If $k = 1$,

$$\boldsymbol{\Gamma}(1) = \mathrm{E}[\mathbf{y}_t\mathbf{y}'_{t-1}] = \mathrm{E}[\mathbf{a}_t(\mathbf{a}'_{t-1} - \mathbf{a}'_{t-2}\boldsymbol{\Theta}')] - \boldsymbol{\Theta}\mathrm{E}[\mathbf{a}_{t-1}(\mathbf{a}'_{t-1} - \mathbf{a}'_{t-2}\boldsymbol{\Theta}')] = -\boldsymbol{\Theta}\boldsymbol{\Sigma}$$

- For $k \geq 2, \boldsymbol{\Gamma}(k) = \mathbf{0}$.

What do you observe?

CCM of VAR(1): $\mathbf{y}_t = \mathbf{\Phi}\mathbf{y}_{t-1} + \mathbf{a}_t$

$$\mathbf{\Gamma}(k) = \mathrm{E}[\mathbf{y}_t\mathbf{y}_{t-k}'] = \mathbf{\Phi}\mathrm{E}[\mathbf{y}_{t-1}\mathbf{y}_{t-k}'] - \mathrm{E}[\mathbf{a}_t\mathbf{y}_{t-k}'].$$

- If $k = 0$,

$$\mathbf{\Gamma}(0) = \mathbf{\Phi}\mathbf{\Gamma}(-1) + \mathbf{\Sigma} = \mathbf{\Phi}\mathbf{\Gamma}'(1) + \mathbf{\Sigma}$$

- If $k = 1$,

$$\mathbf{\Gamma}(1) = \mathbf{\Phi}\mathbf{\Gamma}(0) + \mathbf{0} = \mathbf{\Phi}\mathbf{\Gamma}(0)$$

- If $k = 2$,

$$\mathbf{\Gamma}(2) = \mathbf{\Phi}\mathbf{\Gamma}(1) = \mathbf{\Phi}^2\mathbf{\Gamma}(0)$$

- For $k \geq 2$,

$$\mathbf{\Gamma}(k) = \mathbf{\Phi}^k\mathbf{\Gamma}(0)$$

What do you observe?

- Furthermore,

$$\mathbf{P}(k) = \mathbf{\Lambda}\mathbf{\Phi}^k\mathbf{\Gamma}(0)\mathbf{\Lambda} = \mathbf{\Lambda}\mathbf{\Phi}^k\mathbf{\Lambda}^{-1}\mathbf{\Lambda}\mathbf{\Gamma}(0)\mathbf{\Lambda} = \left(\mathbf{\Lambda}\mathbf{\Phi}^k\mathbf{\Lambda}^{-1}\right)\mathbf{P}(0) \triangleq \widetilde{\mathbf{\Phi}}^k\mathbf{P}(0)$$

CCM of ARMA(1, 1): $\mathbf{y}_t = \mathbf{\Phi}\mathbf{y}_{t-1} + \mathbf{a}_t - \mathbf{\Theta}\mathbf{a}_{t-1}$

$$\mathbf{\Gamma}(k) = \mathrm{E}[\mathbf{y}_t\mathbf{y}'_{t-k}] = \mathbf{\Phi}\mathrm{E}[\mathbf{y}_{t-1}\mathbf{y}'_{t-k}] + \mathrm{E}[\mathbf{a}_t\mathbf{y}'_{t-k}] - \mathrm{E}[\mathbf{\Theta}\mathbf{a}_{t-1}\mathbf{y}'_{t-k}]$$

- For $k \geq 2$,

$$\mathbf{\Gamma}(k) = \mathbf{\Phi}^k\mathbf{\Gamma}(k-1)$$

- If $k = 1$,

$$\mathbf{\Gamma}(1) = \mathbf{\Phi}\mathbf{\Gamma}(0) - \mathbf{\Theta}\mathbf{\Sigma}$$

- If $k = 0$,

$$\begin{aligned}\mathbf{\Gamma}(0) &= \mathbf{\Phi}\mathbf{\Gamma}(-1) + \mathbf{\Sigma} - \mathbf{\Theta}\mathrm{E}[\mathbf{a}_{t-1}(\mathbf{y}'_{t-1}\mathbf{\Phi}' + \mathbf{a}'_t - \mathbf{a}'_{t-1}\mathbf{\Theta}')] \\ &= \mathbf{\Phi}\mathbf{\Gamma}'(1) + \mathbf{\Sigma} - \mathbf{\Theta}\mathbf{\Sigma}\mathbf{\Phi}' + \mathbf{\Theta}\mathbf{\Sigma}\mathbf{\Theta}' \\ &= \mathbf{\Phi}[\mathbf{\Gamma}'(0)\mathbf{\Phi}' - \mathbf{\Sigma}\mathbf{\Theta}'] + \mathbf{\Sigma} - \mathbf{\Theta}\mathbf{\Sigma}\mathbf{\Phi}' + \mathbf{\Theta}\mathbf{\Sigma}\mathbf{\Theta}' \\ &= \mathbf{\Sigma} + \mathbf{\Phi}\mathbf{\Gamma}'(0)\mathbf{\Phi}' - \mathbf{\Phi}\mathbf{\Sigma}\mathbf{\Theta}' - \mathbf{\Theta}\mathbf{\Sigma}\mathbf{\Phi}' + \mathbf{\Theta}\mathbf{\Sigma}\mathbf{\Theta}' \\ &= \cdots\end{aligned}$$

What do you observe?

The Significance of CCM

- With Bartlett formula, the sample cross correlation is approximately following a centered Normal distribution with the variance $\frac{1}{n}$, i.e.,

$$\hat{\rho}_{ij}(k) \sim \mathrm{N}\left(0, \frac{1}{n}\right).$$

- To reject $H_0\colon \rho_{ij}(k) = 0$, one needs $\left|\hat{\rho}_{ij}(k)\right| > \frac{2}{\sqrt{n}}$, given type I error 5%.
  - In most packages, you see
    - $+$, when $\hat{\rho}_{ij}(k) > \frac{2}{\sqrt{n}}$,
    - $-$, when $\hat{\rho}_{ij}(k) > \frac{2}{\sqrt{n}}$, and
    - $\cdot$, when $\left|\hat{\rho}_{ij}(k)\right| \leq \frac{2}{\sqrt{n}}$.

Bivariate VMA(1) CCM Example

$\mathbf{P}(k), k = 1, \dots, 12$

$$\begin{bmatrix} -0.28 & -.021 \\ 0.37 & -.019 \end{bmatrix} \begin{bmatrix} 0.03 & 0.02 \\ 0.08 & 0.01 \end{bmatrix} \begin{bmatrix} 0.04 & -0.01 \\ -0.03 & -0.08 \end{bmatrix} \begin{bmatrix} -0.11 & -0.03 \\ 0.04 & 0.09 \end{bmatrix} \begin{bmatrix} -0.02 & -0.02 \\ -0.09 & -0.08 \end{bmatrix} \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.00 \end{bmatrix}$$

$$\begin{bmatrix} -0.11 & -0.17 \\ 0.01 & -0.06 \end{bmatrix} \begin{bmatrix} -0.09 & -0.03 \\ -0.12 & -0.16 \end{bmatrix} \begin{bmatrix} 0.01 & 0.08 \\ -0.06 & 0.10 \end{bmatrix} \begin{bmatrix} 0.00 & 0.01 \\ 0.02 & -0.04 \end{bmatrix} \begin{bmatrix} 0.03 & 0.08 \\ 0.00 & 0.08 \end{bmatrix} \begin{bmatrix} 0.06 & -0.01 \\ 0.04 & 0.01 \end{bmatrix}$$

- The significance of $\mathbf{P}(k)$

$$\begin{bmatrix} - & - \\ + & - \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & - \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & - \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

	$y_{1,t}$												$y_{2,t}$											
$y_{1,t}$	−	·	·	·	·	·	·	·	·	·	·	·	−	·	·	·	·	·	−	·	·	·	·	·
$y_{2,t}$	+	·	·	·	·	·	·	·	·	·	·	·	−	·	·	·	·	·	·	−	·	·	·	·

Bivariate VAR(1) CCM Example

$\mathbf{P}(k), k = 1, \dots, 12$

$$\begin{bmatrix} 0.41 & 0.51 \\ 0.13 & 0.88 \end{bmatrix} \begin{bmatrix} 0.14 & 0.46 \\ -0.04 & 0.74 \end{bmatrix} \begin{bmatrix} 0.01 & 0.04 \\ -0.10 & 0.61 \end{bmatrix} \begin{bmatrix} -0.04 & 0.33 \\ -0.12 & 0.50 \end{bmatrix} \begin{bmatrix} -0.06 & 0.27 \\ -0.11 & 0.40 \end{bmatrix} \begin{bmatrix} -0.06 & 0.22 \\ -0.10 & 0.32 \end{bmatrix}$$

$$\begin{bmatrix} 0.39 & 0.54 \\ 0.06 & 0.81 \end{bmatrix} \begin{bmatrix} 0.08 & 0.49 \\ -0.12 & 0.64 \end{bmatrix} \begin{bmatrix} 0.02 & 0.42 \\ -0.16 & 0.47 \end{bmatrix} \begin{bmatrix} -0.08 & 0.29 \\ -0.19 & 0.32 \end{bmatrix} \begin{bmatrix} -0.04 & 0.25 \\ -0.15 & 0.23 \end{bmatrix} \begin{bmatrix} 0.02 & 0.23 \\ -0.11 & 0.16 \end{bmatrix}$$

- The significance of $\mathbf{P}(k)$

$$\begin{bmatrix} + & + \\ \cdot & + \end{bmatrix} \begin{bmatrix} \cdot & + \\ \cdot & + \end{bmatrix} \begin{bmatrix} \cdot & + \\ \cdot & + \end{bmatrix} \begin{bmatrix} \cdot & + \\ - & + \end{bmatrix} \begin{bmatrix} \cdot & + \\ \cdot & + \end{bmatrix} \begin{bmatrix} \cdot & + \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} - & \cdot \\ - & \cdot \end{bmatrix} \begin{bmatrix} - & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

	$y_{1,t}$												$y_{2,t}$											
$y_{1,t}$	+	·	·	·	·	·	−	−	·	·	·	·	+	+	+	+	+	+	·	·	·	·	·	·
$y_{2,t}$	·	·	·	−	·	·	−	·	·	·	·	·	+	+	+	+	+	·	·	·	·	·	·	·

PAR (Partial Autoregressive Matrix)

$$\mathbf{y}_t = \mathbf{\Phi}_{11}\mathbf{y}_{t-1} + \mathbf{a}_t$$
$$\mathbf{y}_t = \mathbf{\Phi}_{21}\mathbf{y}_{t-1} + \mathbf{\Phi}_{22}\mathbf{y}_{t-2} + \mathbf{a}_t$$
$$\mathbf{y}_t = \mathbf{\Phi}_{31}\mathbf{y}_{t-1} + \mathbf{\Phi}_{32}\mathbf{y}_{t-2} + \mathbf{\Phi}_{33}\mathbf{y}_{t-3} + \mathbf{a}_t$$
$$\vdots$$
$$\mathbf{y}_t = \mathbf{\Phi}_{k1}\mathbf{y}_{t-1} + \mathbf{\Phi}_{k2}\mathbf{y}_{t-2} + \cdots + \mathbf{\Phi}_{kk}\mathbf{y}_{t-k} + \mathbf{a}_t$$

- Estimate all the $\{\mathbf{\Phi}_{jj}\}_{j=1}^{k}$ using LSE.
- VAR(p) model has $\mathbf{\Phi}_{jj}$ cut-off after lag p, i.e., $\mathbf{\Phi}_{jj} = \mathbf{0}$ for $j > p$.
- VMA(q) and VARMA(p, q) models have $\mathbf{\Phi}_{jj}$ exponentially decayed.

The Significance of PAR

- After fitting a VAR(k) model, we get the residual error matrix.

$$\mathbf{S}(k) = \sum_{t=k+1}^{n} \left[\mathbf{y}_t - \sum_{j=1}^{k} \widehat{\mathbf{\Phi}}_{kj} \mathbf{y}_{t-j} \right] \left[\mathbf{y}_t - \sum_{j=1}^{k} \widehat{\mathbf{\Phi}}_{kj} \mathbf{y}_{t-j} \right]'$$

  - Using Sequential Likelihood Ratio Test:

$$M(k) = -\left(n - k - \frac{3}{2} - kh \right) \ln \frac{|\mathbf{S}(k)|}{|\mathbf{S}(k-1)|} \sim \chi^2_{h^2}.$$

  - The hypothesis: $H_0: \mathbf{\Phi}_{kk} = 0$, is rejected if $M(k) \geq \chi^2_{h^2}(1-\alpha)$

Bivariate VAR(1) PAR Example

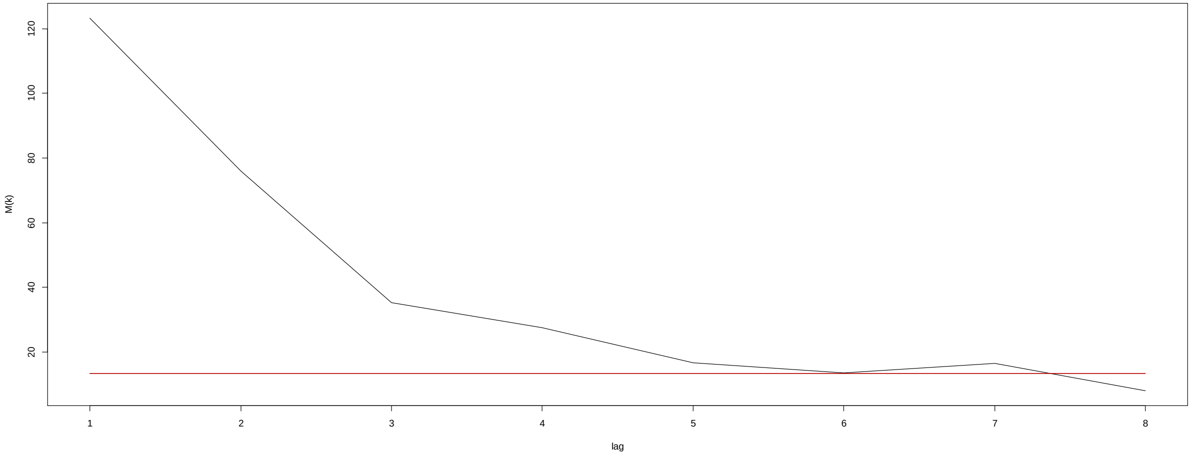
- $\chi^2_{4,0.95} = 9.49$ or $\chi^2_{4,0.99} = 13.28$

k	$\hat{\mathbf{\Phi}}_{kk}$	$M(k) \sim \chi^2_4$	$\hat{\mathbf{\Sigma}}$ diagnoals
1	$\begin{bmatrix} 1.70 & 5.79 \\ -16.28 & 32.90 \end{bmatrix}$	356.96	$\begin{bmatrix} 5.30 & \\ & 1.08 \end{bmatrix}$
2	$\begin{bmatrix} -1.68 & 1.98 \\ -1.64 & 2.39 \end{bmatrix}$	7.04	$\begin{bmatrix} 5.16 & \\ & 1.03 \end{bmatrix}$
3	$\begin{bmatrix} 1.20 & -0.54 \\ 0.30 & 2.39 \end{bmatrix}$	2.63	$\begin{bmatrix} 5.07 & \\ & 1.03 \end{bmatrix}$
4	$\begin{bmatrix} 0.90 & -1.25 \\ -0.85 & 0.76 \end{bmatrix}$	4.38	$\begin{bmatrix} 5.01 & \\ & 1.02 \end{bmatrix}$
5	$\begin{bmatrix} 0.51 & 0.10 \\ 1.11 & -0.56 \end{bmatrix}$	2.42	$\begin{bmatrix} 4.95 & \\ & 1.01 \end{bmatrix}$

Bivariate VMA(1) PAR Example

- $\chi^2_{4,0.95} = 9.49$ or $\chi^2_{4,0.99} = 13.28$

$$M(k) = \{123.2, 75.9, 35.2, 27.5, 16.6, 13.5, 16.5, 8.1\}$$



The Ultimate Way of Identify p, q

- Recall EACF for ECCM (Extended Cross Correlation Matrix).

Tiao, G., & Tsay, R. (1983). [Multiple Time Series Modeling and Extended Sample Cross-Correlations](). *Journal of Business & Economic Statistics, 1*(1), 43-56. doi:10.2307/1391772

  - Look for the NW corner.

Table 9. The ESCC Matrix Table for the Hog Data

a. The ESCC Matrices

AR \ MA	0	1	2	3	4
0	.86 .60 .48 .71 .76 .65 .85 .55 .58 .85 .30 .77 .77 .25 .74 .83 .54 .29 .74 .61 .68 .87 .65 .58 .93	.71 .66 .53 .63 .76 .62 .67 .45 .52 .75 .24 .73 .56 .21 .65 .77 .51 .37 .72 .63 .62 .78 .58 .52 .85	.61 .70 .56 .56 .73 .56 .55 .47 .40 .67 .27 .60 .41 .21 .55 .67 .52 .41 .62 .61 .56 .68 .54 .44 .76	.56 .69 .50 .55 .70 .50 .51 .50 .32 .62 .33 .42 .33 .15 .47 .57 .57 .42 .60 .59 .49 .59 .51 .36 .68	.47 .58 .41 .48 .60 .43 .50 .48 .27 .57 .28 .28 .32 .10 .40 .55 .58 .42 .54 .57 .42 .53 .47 .29 .60
1	.15 .10 .16 .11 .17 −.01 .12 −.04 −.02 −.01 .30 .12 .25 .27 .29 −.28 −.23 −.25 −.31 −.28 .05 .02 .03 .03 .06	−.06 .06 .06 −.06 −.06 .02 −.02 −.02 .02 .02 .02 −.02 −.02 .01 .02 −.03 .04 .04 −.03 −.03 −.03 .03 .03 −.03 −.02	−.00 .03 .02 −.03 −.03 −.07 −.00 .00 .00 .04 .03 .09 .05 .01 −.01 −.02 −.06 −.04 −.03 .05 .09 .06 .02 .06 .05	.10 −.04 .07 −.01 .07 −.11 .05 −.04 .03 −.11 .25 −.07 .14 −.12 .15 −.17 .06 −.11 .16 −.10 .05 .01 .09 −.03 .06	−.10 −.05 .03 −.06 −.04 .09 −.00 .06 .04 .05 .05 −.20 −.05 −.19 −.23 .08 .14 .13 .10 .18 .06 −.02 .04 −.01 −.00
2	.21 .18 .01 .06 .19 .33 −.27 .23 −.27 .11 .12 .05 .39 −.36 −.02 .21 −.00 −.08 .05 .17 .16 .11 .21 −.25 −.21	−.05 −.02 −.05 .06 .05 −.35 .07 −.29 .27 .32 −.02 −.03 −.02 .05 .04 −.06 −.08 −.02 −.05 .03 −.01 −.04 .01 −.02 .00	−.02 .02 −.02 .02 −.02 .01 −.01 .01 −.01 .01 −.01 .01 −.01 .01 −.01 .02 −.02 .02 −.02 .02 −.02 .02 −.01 .02 −.02	.02 .00 .04 −.03 .04 −.04 .04 .02 .01 −.04 .09 −.04 .05 −.16 .06 −.06 .01 −.01 .17 −.05 −.04 .01 .01 −.07 −.06	.00 −.01 .04 −.03 .02 −.06 −.04 −.04 .06 −.03 .09 .00 .03 −.00 −.01 −.03 .00 −.01 .01 −.00 .02 .06 −.03 .02 −.06
3	−.03 .00 .01 −.01 .07 .05 .15 −.05 .07 .03 −.12 −.10 .07 −.09 −.03 −.05 −.03 .07 −.06 .07 −.20 −.17 .16 −.16 −.01	−.06 −.06 .05 −.05 .05 −.06 −.05 .06 −.07 .06 .06 .03 −.05 .05 −.05 −.07 −.06 .07 −.07 .07 .09 .06 −.09 .08 −.11	−.01 .00 .02 −.02 −.01 −.09 −.08 .10 −.10 .11 .11 .14 −.09 .09 −.05 −.15 −.16 .17 −.16 .08 −.08 −.05 .09 −.09 .05	−.02 .01 .02 −.04 .04 .08 .02 −.06 .10 −.07 .06 .21 −.06 −.12 −.06 −.07 −.19 −.06 .15 .06 .01 .03 −.07 .06 −.06	.00 −.00 .00 −.00 −.00 −.01 .00 −.01 .01 .01 .01 −.00 .01 −.01 −.01 −.01 .00 −.01 .01 .01 −.00 .00 −.00 .00 .00

b. Indicator Symbols

AR \ MA	0	1	2	3	4
0	+++++ +++++ +++++ +++++ +++++	+++++ +++++ +++.+ +++++ +++++	+++++ +++++ +++.+ +++++ +++++	+++++ +++++ +++.+ +++++ +++++	+++++ +++++ +++.+ +++++ +++++
1	.... +.+++ −−−−−	.... [illegible]	.... [illegible]	.... [illegible]	.... [illegible] − ....
2	..... +−+−. ..+−. ....−.	..... −.−++	....	....	....
3	....	....	....	....	....

Multivariate Time Series Model Construction

Identifying p, d, q → **Estimating Φ_i, Θ_j** → Checking Residuals

Estimation of the Coefficients VARMA(p, q)

$$\mathbf{a}_t = \mathbf{y}_t - \mathbf{\Phi}_1 \mathbf{y}_{t-1} - \cdots - \mathbf{\Phi}_p \mathbf{y}_{t-p} + \mathbf{\Theta}_1 \mathbf{a}_{t-1} + \mathbf{\Theta}_2 \mathbf{a}_{t-2} + \cdots + \mathbf{\Theta}_q \mathbf{a}_{t-q}$$

$$\mathbf{L}\left(\mathbf{\Phi}_i, \mathbf{\Theta}_j, \mathbf{\Sigma}\right) = (2\pi)^{-\frac{nh}{2}} |\mathbf{\Sigma}|^{\frac{n}{2}} \exp\left(-\sum_{t=1}^{n} \mathbf{a}_t' \mathbf{a}_t\right).$$

- Conditional vs. Exact MLE
  - Exact MLE will be slow (backcasting method to get the initial values).
  - Exact MLE has smaller bias, especially, when $q \neq 0$.

Multivariate Time Series Model Construction

Identifying p, d, q → Estimating Φ_i, Θ_j → **Checking Residuals**

Model Diagnostics

$$\mathbf{e}_t = \mathbf{y}_t - \widehat{\mathbf{\Phi}}_1\mathbf{y}_{t-1} - \cdots - \widehat{\mathbf{\Phi}}_p\mathbf{y}_{t-p} + \widehat{\mathbf{\Theta}}_1\mathbf{a}_{t-1} + \widehat{\mathbf{\Theta}}_2\mathbf{a}_{t-2} + \cdots + \widehat{\mathbf{\Theta}}_q\mathbf{a}_{t-q}$$

$$\because\ \mathbf{a}_t \sim^{iid} N_h(\mathbf{0}, \mathbf{\Sigma})$$

- We need to check the multi-normality on $\mathbf{e}_t$: Normal Probability Plot
- Evaluate SCCM, SPAR, ECCM to see if $\mathbf{e}_t$ is autocorrelated.
  - e.g.,

Lags 1–6	$\begin{matrix} \cdot & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} .26 \\ \ \end{bmatrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} .24 \\ \ \end{bmatrix}$	$\begin{matrix} \cdot & - & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} -.26 \\ .29 \end{bmatrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} .25 \\ \ \end{bmatrix}$	$\begin{matrix} - & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} -.28 \\ .24 \end{bmatrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} .24 \\ \ \end{bmatrix}$
Lags 7–12	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & - & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} .26 \\ -.24 \end{bmatrix}$	$\begin{matrix} + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$ $\begin{bmatrix} .26 \\ .23 \end{bmatrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$

Forecasting Multivariate Time Series

5

VAR(1) Forecasts

- $\mathbf{y}_t = \mathbf{\Phi}\mathbf{y}_{t-1} + \mathbf{a}_t$, and $\mathbf{y}_{t+l}$ to be forecasted.
  - $l = 1, \hat{\mathbf{y}}_t(1) = \mathbf{\Phi}\mathbf{y}_t, \hat{\mathbf{e}}_t(1) = \mathbf{y}_{t+1} - \hat{\mathbf{y}}_t(1) = \mathbf{a}_{t+1}$.
  - $l = 2, \hat{\mathbf{y}}_t(2) = \mathbf{\Phi}\hat{\mathbf{y}}_t(1) = \mathbf{\Phi}^2\mathbf{y}_t, \hat{\mathbf{e}}_t(2) = \mathbf{\Phi}\hat{\mathbf{e}}_t(1) + \mathbf{a}_{t+2}$.
  - $l \geq 3, \hat{\mathbf{y}}_t(l) = \mathbf{\Phi}\hat{\mathbf{y}}_t(l-1), \hat{\mathbf{e}}_t(l) = \mathbf{\Phi}^{l-1}\mathbf{a}_{t+1} + \mathbf{\Phi}^{l-2}\mathbf{a}_{t+2} + \cdots + \mathbf{a}_{t+l}$.
- For a stationary VAR(1)
  - $\lim_{l\to\infty} \hat{\mathbf{y}}_t(l) = 0$
  - $\lim_{l\to\infty} \mathrm{V}[\hat{\mathbf{e}}_t(l)] = \mathbf{\Gamma}(0)$

VARIMA(0, 1, 1) Forecasts

$$(\mathbf{I}_h - B)\mathbf{y}_t = (\mathbf{I}_h - \mathbf{\Theta}B)\mathbf{a}_t$$

VARIMA(p, d, q) Forecasts

$$\left(\mathbf{I}_h - \boldsymbol{\Phi}_1 B - \boldsymbol{\Phi}_2 B^2 - \cdots - \boldsymbol{\Phi}_p B^p\right)(\mathbf{I}_h - B)^d \mathbf{y}_t = \left(\mathbf{I}_h - \boldsymbol{\Theta}_1 B - \boldsymbol{\Theta}_2 B^2 - \cdots - \boldsymbol{\Theta}_q B^q\right)\mathbf{a}_t$$

Updating the Forecasts

- Given the random shock form out of the VARIMA model:

$$\mathbf{y}_t = \mathbf{a}_t + \boldsymbol{\Psi}_1\mathbf{a}_{t-1} + \boldsymbol{\Psi}_2\mathbf{a}_{t-2} + \boldsymbol{\Psi}_3\mathbf{a}_{t-3} \dots$$

- It is the same as what has been proposed in the univariate models. Standing at time T,

$$\hat{\mathbf{y}}_{T+1}(l) = \hat{\mathbf{y}}_T(l+1) + \Psi_l \hat{\mathbf{a}}_{T+1}$$

Summary

- Ultimate form of the h variates VSARIMA$\left(p, \{d_i\}_1^h, q\right) \times \left(P, \{D_i\}_1^h, Q\right)_s$, model: $\boldsymbol{\Phi}(B)\boldsymbol{\Phi}_s(B^s)\mathbf{D}(B)\mathbf{D}(B^s)\mathbf{y}_t = \boldsymbol{\Theta}(B)\boldsymbol{\Theta}_s(B^s)\mathbf{a}_t$.
- All the properties of univariate models have the twin in the multivariate world.

Univariate	Multivariate
ACF (MA cut-off)	CCM (VMA cut-off)
PACF (AR cut-off)	PAR (VAR cut-off)
EACF (ARMA north-west corner)	ECCM (ARMA north-west corner)
Stationarity $\phi(B) = 0$	Stationarity $\lvert\boldsymbol{\Phi}(B)\rvert = 0$
Invertibility $\theta(B) = 0$	Invertibility $\boldsymbol{\Theta}(B) = 0$
3 Explicit Forms (Difference Equation, Random Shock, Inverted)	
MLE based on $\mathrm{N}(0, \sigma_a^2)$	MLE based on $\mathrm{N}(0, \boldsymbol{\Sigma})$
Residuals diagnosed by ACF, PACF	Residuals diagnosed by CCM, PAR