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## Proof of variance of stationary time series

Asked 3 years, 11 months ago   Active 6 months ago   Viewed 1k times



4



Suppose that  $\{X_t\}$  is a weakly stationary time series with mean  $\mu = 0$  and a covariance function  $\gamma(h)$ ,  $h \geq 0$ ,  $E[X_t] = \mu = 0$  and  $\gamma(h) = \text{Cov}(X_t, X_{t+h}) = E[X_t X_{t+h}]$

Show that:

$$\text{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{u=1}^{n-1} \left(1 - \frac{u}{n}\right) \gamma(u).$$

So far, I've gotten this:

<https://stats.stackexchange.com/questions/240010/proof-of-variance-of-stationary-time-series>

$$\begin{aligned} \text{Var}(\bar{X}) &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j) \\ &= \frac{1}{n^2} \sum_{i-j=-n}^n (n - |i-j|) \gamma(i-j) \\ &= \frac{1}{n} \sum_{m=-n}^n \left(1 - \frac{|m|}{n}\right) \gamma(m) \end{aligned}$$

How am I supposed to come up with the  $\frac{\gamma(0)}{n} + \frac{2}{n}$ ?

time-series

self-study

variance

stationarity

edited May 11 '18 at 13:36



gung - Reinststate Monica

122k 41 321 611

asked Oct 13 '16 at 13:38



FBeller

125 4



gung - Reinststate Monica

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- 3 Hint: under stationarity, only the distance of two elements of the process  
– [Christoph Hanck](#) Oct 13 '16 at 15:13

related: [stats.stackexchange.com/questions/154070/...](https://stats.stackexchange.com/questions/154070/...) your question +

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This has caused... dozens of const... 1

## 2 Answers

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1



You are almost there! Now you just need to recognise that auto-correlation only depends on the lag, so you have  $\gamma(m) = \gamma(|m|)$ , which means that the entire summand depends on  $m$  only through  $|m|$  (i.e., it is symmetric around  $m = 0$ ). This allows you to split the sum into the middle element ( $m = 0$ ) and two lots of the symmetric part ( $|m| = 1, \dots, n-1$ ), which gives you:

$$\begin{aligned}\text{Var}(\bar{X}) &= \frac{1}{n} \sum_{m=-n}^n \left(1 - \frac{|m|}{n}\right) \gamma(m) \\ &= \frac{1}{n} \sum_{m=-n}^n \left(1 - \frac{|m|}{n}\right) \gamma(|m|) \\ &= \frac{1}{n} \left[ \gamma(0) + 2 \sum_{|m|=1}^n \left(1 - \frac{|m|}{n}\right) \gamma(|m|) \right] \\ &= \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{m=1}^n \left(1 - \frac{m}{n}\right) \gamma(m) \\ &= \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right) \gamma(m).\end{aligned}$$

(The last step follows from the fact that  $1 - \frac{m}{n} = 0$  for  $m = n$ .) This method of splitting symmetric sums around their mid-point is a common trick used in these kinds of cases to simplify the sum by taking it only over positive arguments. It is a worthwhile trick to learn in general.

edited May 7 '19 at 22:45

answered Jul 18 '18 at 2:28



**Ben**

59.7k

2

89

242



first, fixing the definition of the problem, the index is  $m$  instead of  $u$ , to make simpler I will use only the index  $i$  and  $j$ .

0

We want to prove that

$$\text{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) \gamma(i).$$

The begin is correct,

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j)$$

We can notice that  $\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$  and, from our assumptions about the problem, that  $\text{Cov}(X_i, X_i + h) = \text{Cov}(X_i, X_i - h) = \gamma(h)$  for any  $i$  and  $h$ .

We can visualize the sum of covariances in  $i$  and  $j$  as follows

$$\begin{vmatrix} \text{Cov}(1,1) & \text{Cov}(1,2) & \cdots & \text{Cov}(1,n-1) & \text{Cov}(1,n) \\ \text{Cov}(2,1) & \text{Cov}(2,2) & \cdots & \text{Cov}(2,n-1) & \text{Cov}(2,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \text{Cov}(n-1,1) & \text{Cov}(n-1,2) & \cdots & \text{Cov}(n-1,n-1) & \text{Cov}(n-1,n) \\ \text{Cov}(n,1) & \text{Cov}(n,2) & \cdots & \text{Cov}(n,n-1) & \text{Cov}(n,n) \end{vmatrix}$$

What is equal to

$$\begin{vmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{vmatrix}$$

To sum all the elements we can first sum the main diagonal, and as it is symmetric sum twice the other diagonals

$$\sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j) = n\gamma(0) + 2 \sum_{i=1}^{n-1} (n-i)\gamma(i)$$

Back to the main equation

$$\text{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{\gamma(0)}{n} + \frac{2}{n^2} \sum_{i=1}^{n-1} (n-i)\gamma(i) = \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) \gamma(i).$$

edited Mar 31 at 7:46

answered Oct 14 '16 at 17:07

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