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THE SEVEN CLASSES OF HARMONIC DIFFEOMORPHISMS AND THEIR GEOMETRY

Abstract

One of the present authors classified harmonic diffeomorphisms between Riemannian manifolds [1]. This paper is devoted to the study of the local and global geometry of certain classes of harmonic diffeomorphisms.

Аннотация

С.Е. Степанов, И.Г. Шандра **Семь классов гармонических диффеоморфизмов и их геометрия**

В [1] была получена классификация гармонических диффеоморфизмов между римановыми многообразиями (см. [1]). В данной работе изучается локальная и глобальная геометрия некоторых гармонических диффеоморфизмов.

1. Harmonic diffeomorphisms

1.1. Let (M, g) be a connected smooth pseudo-Riemannian manifold, ∇ its Levi-Civita connection, and R the corresponding curvature tensor defined by $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for smooth vector fields X and Y on (M, g). Let *Ric* and *Scal*, respectively, denote the Ricci tensor and the scalar curvature.

Next, let $f : (M, g) \to (M', g')$ be a smooth map between two pseudo-Riemannian manifolds and let $f^{-1}TM'$ be the pull-back bundle. The Levi-Civita connections on TM and TM' induce a connection $\overline{\nabla}$ in the bundle of 1-forms on M with values in $f^{-1}TM'$. Then $\wp_f = \overline{\nabla} df$ is a

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symmetric bilinear form on TM, called the *second fundamental form* of f [2]. The trace of $\wp_f$ with respect to g is called the *tension field* of $\wp_f$, and is denoted by $\tau(\wp_f)$. The map f is said to be *harmonic* if and only if it satisfies the *Euler-Lagrange equations* (see [3, 4, 5] for more details and references), that is

$$\tau(\wp_f) := trace_g \overline{\nabla} df = 0, \tag{1.1}$$

where $trace_g$ denotes the trace operator with respect to g.

Let's find out a form of the Euler-Lagrange equations in terms of local coordinate systems. Take coordinate neighbourhoods U with local coordinates $x^1, \ldots, x^m$ of (M, g) and U' with local coordinates $y^1, \ldots, y^{m'}$ of (M', g') such that $f(U) \subseteq U'$. The indices $i, j, k, l, \ldots$ run over the range $\{1, 2, \ldots, m\}$ and the indices $\alpha, \beta, \gamma, \delta, \ldots$ run over the range $\{1, 2, \ldots, m'\}$, where $m = dim\, M$ and $m' = dim\, M'$, respectively.

Suppose that $f : (M, g) \to (M', g')$ is represented by equations $y^\alpha = y^\alpha(x^1, \ldots, x^m)$ with respect to U and U'. We set $f_i^\alpha = (\partial_i y^\alpha)(x^1, \ldots, x^m)$, where $\partial_i = \frac{\partial}{\partial x^i}$. Then the differential df of the mapping f is represented by the matrix (f_i^α) with respect to the local coordinates $x^1, \ldots, x^m$ of (M, g) and $y^1, \ldots, y^{m'}$ of (M', g').

We denote by g_{ij} the local components of the pseudo-Riemannian metric g, by $g'_{\alpha\beta}$ those of the pseudo-Riemannian metric g'. The Christoffel symbols formed with g_{ij} and $g'_{\alpha\beta}$ we denote by Γ_{ij}^k and $\Gamma'^{\alpha}_{\beta\gamma}$, respectively.

In U we set

$$f_{ij}^\alpha = \overline{\nabla}_j f_i^\alpha, \tag{1.2}$$

where

$$\overline{\nabla}_j f_i^\alpha = \partial_i f_i^\alpha + \Gamma'^{\alpha}_{\beta\gamma} f_i^\beta f_j^\gamma - \Gamma_{ij}^l f_l^\alpha \tag{1.3}$$

is a local expression of the tensor field $\overline{\nabla} df$ and $f_{ij}^\alpha = f_{ji}^\alpha$. Differentiating (1.2) covariantly, we find the following

$$\overline{\nabla} f_{ij}^\alpha = \partial_k f_{ij}^\alpha + \Gamma'^{\alpha}_{\beta\gamma} f_{ij}^\beta f_k^\gamma - \Gamma_{jk}^l f_{li}^\alpha - \Gamma_{ik}^l f_{lj}^\alpha, \tag{1.4}$$

where $\overline{\nabla} f_{ij}^\alpha = \overline{\nabla}_k \overline{\nabla}_j f_i^\alpha$. Taking into account (1.2), (1.3), and (1.4), we obtain the formula of Ricci type [6]

$$\overline{\nabla}_k \overline{\nabla}_j f_i^\alpha - \overline{\nabla}_j \overline{\nabla}_k f_i^\alpha = R'^{\alpha}_{\beta\gamma\delta} f_i^\beta f_k^\delta f_j^\gamma - R_{ijk}^l f_l^\alpha, \tag{1.5}$$

where $R'^{\alpha}_{\beta\gamma\delta}$ and R_{ijk}^l are components of the curvature tensors of the Levi-Civita connections ∇ and $\overline{\nabla}$, respectively.

Now, using the (1.3), we find the coordinate form of the Euler-Lagrange equations

$$g^{ij}(\partial_i\partial_j f^\alpha - \Gamma^k_{ij}\partial_k f^\alpha + \Gamma'^\alpha_{\beta\gamma}\partial_i f^\beta \partial_j f^\gamma) = \Delta f^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}\partial_i f^\beta \partial_j f^\gamma = 0, \quad (1.6)$$

where Δ is the Laplace operator of the manifold (M, g). .

Remark 1.1. In the origin of normal coordinate system ($\Gamma^k_{ij} = 0$ and $\Gamma'^\alpha_{\beta\gamma} = 0$), each of the components of the harmonic mapping turns to be a harmonic function of the coordinates. Using the Cauchy-Riemann equations, we immediately obtain that every holomorphic mapping of Kählerian manifolds is harmonic [7]. We shall generalize this result (cf. the Theorem 1.3 and Corollary 1.4) in the third part of this section.

1.2. Let $f : (M, g) \to (M', g')$ be a diffeomorphism of equidimensional pseudo-Riemannian manifolds. If $dim\, M = dim\, M' = m$, we can assume that f is represented by the following equations [8]

$$y^i = x^i \quad (1.7)$$

with respect to the local coordinate systems $x^1, \ldots, x^m$ of (M, g) and $y^1, \ldots, y^m$ of (M', g'). In this case the differential df is represented by the matrix (δ^i_j), where δ^i_j is the Kronecker delta. If $f : (M, g) \to (M', g')$ is a harmonic diffeomorphism, the Euler-Lagrange equations (1.6) are equivalent to the following equations

$$\sum_{i,j} g^{ij}(\Gamma'^k_{ij} - \Gamma^k_{ij}) = 0, \quad (1.8)$$

where $T^k_{ij} = \Gamma'^k_{ij} - \Gamma^k_{ij}$ are local components of the deformation tensor $T = \nabla' - \nabla$ with respect to a common local coordinate system $x^1, \ldots, x^m$ [9].

We observe that the deformation tensor $T = \nabla' - \nabla$ is a section of the vector bundle $f^{-1}TM' \otimes S^2M$ for an arbitrary diffeomorphism $f : (M, g) \to (M', g')$. But if $f : (M, g) \to (M', g')$ is harmonic, then $T = \nabla' - \nabla$ is a section of the vector bundle $f^{-1}TM' \otimes S^2_0M$, where S^2_0M is the space of traceless symmetric 2-tensor fields on (M, g).

On the other hand, from definition of the deformation tensor and the Christoffel symbols we have

$$\nabla'_k g_{ij} = -g_{lj}T^l_{ik} - g_{il}T^l_{jk}; \quad \nabla_k g'_{ij} = g'_{lj}T^l_{ik} + g'_{il}T^l_{jk}; \quad (1.9)$$

and consequently

$$\begin{aligned} T^k_{ij} &= \tfrac{1}{2}g^{kl}(-\nabla'_i g_{jl} - \nabla'_j g_{il} + \nabla'_l g_{ij}); \\ T^k_{ij} &= \tfrac{1}{2}g'^{kl}(\nabla_i g'_{jl} + \nabla_j g'_{il} - \nabla_l g'_{ij}). \end{aligned} \quad (1.10)$$

Then we conclude that (1.8) is equivalent to

$$2g^{ik}\nabla'_i g_{kj} = g^{ik}\nabla'_j g_{ik} \tag{1.11}$$

or

$$2g^{ik}\nabla_i g'_{kj} = \nabla_i(g^{ik}g'_{ik}). \tag{1.12}$$

Now we can give the following theorem which provides an easy way to verify whether a diffeomorphism $f : (M, g) \to (M', g')$ is harmonic.

Theorem 1.1 *The diffeomorphism $f : (M, g) \to (M', g')$ of a pseudo-Riemannian manifold (M, g) with the Levi-Civita connection ∇ onto a pseudo-Riemannian manifold (M', g') with the Levi-Civita connection ∇' is harmonic if and only if one of the following equalities is true:*
(i) $\sum_{i,j} g^{ij}T^k_{ij} = 0$; (ii) $2g^{ik}\nabla'_i g_{kj} = g^{ik}\nabla'_j g_{ik}$; (iii) $2g^{ik}\nabla_i g'_{kj} = \nabla_j(g^{ik}g'_{ik})$;
where T^k_{ij}, g_{ij}, g'_{ij} are local components of the deformation tensor $T = \nabla' - \nabla$ and the pseudo-Riemannian metrics g and g' with respect to a common local coordinate system $x^1, \ldots, x^m$.

1.3. Here we shall give three examples of harmonic diffeomorphisms.

Example 1.1. We assume that $f = f'' \circ f'$ is a composition of a conformal diffeomorphism $f' : (M, g) \to (M', g')$ and a projective diffeomorphism $f'' : (M', g') \to (M'', g'')$.

It is well known that a diffeomorphism $f' : (M, g) \to (M', g')$ between pseudo-Riemannian manifolds is called conformal if $g' = e^{2\sigma}g$ for an arbitrary $\sigma \in C^\infty M$. In this case the deformation tensor $T' = \nabla' - \nabla$ has the local expression [9]

$$T'^k_{ij} = \Gamma'^k_{ij} - \Gamma^k_{ij} = \sigma_i\delta^k_j + \sigma_j\delta^k_i - \sigma^k g_{ij}, \tag{1.13}$$

where $\sigma_i = \partial_i\sigma$ and $\sigma^k = g^{ki}\sigma_i$, with respect to a common local coordinate system $x^1, \ldots, x^m$.

On the other hand, if an arbitrary geodesic in (M', g') is mapped into a geodesic in (M'', g''), then $f'' : (M', g') \to (M'', g'')$ is said to be a projective mapping [9]. In this case, with respect to a common local coordinate system $x^1, \ldots, x^m$ the deformation tensor $T'' = \nabla'' - \nabla'$ has the local expression

$$T''^k_{ij} = \Gamma''^k_{ij} - \Gamma'^k_{ij} = \psi_i\delta^k_j + \psi_j\delta^k_i, \tag{1.14}$$

where $\psi_j = \frac{1}{2(m+1)}\partial_j \ln\left|\frac{\det g''}{\det g'}\right| = \frac{1}{2(m+1)}\partial_j\left|\frac{\det g''}{\det g}\right| - \frac{m}{m+1}\sigma_j$.

Denote by $T = T' + T'' = \nabla'' - \nabla$ the deformation tensor of the composition $f = f'' \circ f' : (M, g) \to (M', g')$, then from (1.13) and (1.14) we obtain

$$\Gamma''^k_{ij} - \Gamma^k_{ij} = (\sigma_i + \psi_i)\delta^k_j + (\sigma_j + \psi_j)\delta^k_i - \sigma^k g_{ij}. \tag{1.15}$$

Since $f = f'' \circ f'$ is harmonic, Theorem 1.1 gives $(n-2)\sigma_j = 2\psi_j$. From this we obtain immediately the following (cf. [1]).

Theorem 1.2 *For any conformal diffeomorphism $f' : (M, g) \to (M', g')$ and a projective diffeomorphism $f'' : (M', g') \to (M'', g'')$ between m-dimensional ($m \geq 3$) pseudo-Riemannian manifolds the composition $f'' \circ f'$ is a harmonic diffeomorphism if and only if $g' = e^{2\sigma} g$ for $\sigma = (m^2 + m - 2)^{-1} \ln \left| \frac{\det g''}{\det g} \right| + \mathrm{const}$.*

Remark 1.2. If $m = 2$, from (1.13) it follows that any conformal diffeomorphism is harmonic. But if $m \geq 3$, any harmonic diffeomorphism $f : (M, g) \to (M', g')$ which is at the same time conformal, is a homothetic diffeomorphism [6].

Example 1.2. In this example we shall consider a holomorphic diffeomorphism $f : (M, g, J) \to (M', g', J')$ of a Kählerian manifold (M, g, J) onto a Kählerian manifold (M', g', J').

Let us recall necessary definitions. A manifold M endowed with a field J of endomorphisms of the tangent bundle TM such that $J^2 = Id_{TM}$, is called an *almost complex manifold.* Let g be a *Hermitian metric* on M, that is, a pseudo-Riemannian metric such that $g(J, J) = g$. An almost complex manifold (M, g, J) is a Kählerian manifold [11, 12] if $\nabla J = 0$. For Kählerian manifolds a map $f : (M, g, J) \to (M', g', J')$ is said to be *holomorphic* ([3] - [5]) if $f_* \circ J = J' \circ f_*$. In [7] Lichnerowicz proved that a holomorphic map $f : (M, g, J) \to (M', g', J')$ of a compact Kählerian manifold (M, g, J) onto a Kählerian manifold (M', g', J') is harmonic.

Example 1.3. Now let $f : (M, g, J) \to (M', g', J')$ be a holomorphic diffeomorphism of an *almost semi-Kählerian manifold* (M, g, J) onto a nearly Kählerian manifold (M', g', J').

Let us recall that an almost complex manifold (M, g, J) is *an almost semi-Kählerian manifold* (see [11, 12]) if

$$trace_g \nabla J = 0. \tag{1.16}$$

An almost complex manifold (M', g', J') is a *nearly Kählerian manifold* (see [11, 12]) if $\nabla' J'$ is a section of the tensor bundle $TM' \otimes \Lambda^2 M'$, where

$\Lambda^2 M'$ is the space of skew-symmetric 2-tensor fields on (M', g'). This condition is equivalent to

$$(\nabla_X J')X = 0 \tag{1.17}$$

for an arbitrary $X \in TM'$.

Theorem 1.3 *Let (M, g, J) be an almost Hermitian manifold. Then any holomorphic diffeomorphism $f : (M, g, J) \to (M', g', J')$, where (M', g', J') is a nearly Kählerian manifold, is harmonic if and only if the manifold (M, g, J) is an almost semi-Kählerian manifold.*

Proof. The diffeomorphism of an almost Hermitian manifold (M, g, J) onto an almost Hermitian manifold (M', g', J') is holomorphic if and only if with respect to a common local coordinate system $x^1, \ldots, x^m$

$$J_j^i = J_j'^i. \tag{1.18}$$

Differentiating (1.18) covariantly, we find

$$\nabla_k J_j^i = \nabla_k' J_j'^i + J_l'^i T_{jk}^l - J_j'^l T_{lk}^i. \tag{1.19}$$

We multiply (1.19) by g^{kj} and use (1.17), then we obtain

$$g^{kj} \nabla_k J_j^i = J_j'^i (g^{jk} T_{jk}^i). \tag{1.20}$$

From (1.20) we conclude that under assumptions of Theorem 1.3 the following statements

(i) (M, g, J) is an almost semi-Kählerian manifold;

(ii) $f : (M, g, J) \to (M', g', J')$ is a harmonic map

are equivalent. □

From (1.16) and (1.17) we obtain that a nearly Kählerian manifold is almost semi-Kählerian. As a consequence we have the following corollary.

Corollary 1.4 *Let (M, g, J) and (M', g', J') be nearly Kählerian manifolds. Then any diffeomorphism $f : (M, g, J) \to (M', g', J')$ is holomorphic if and only if $f : (M, g, J) \to (M', g', J')$ is a harmonic map.*

1.4. For the first time the Bochner technique [13] was used in studies of harmonic maps in the well-known paper by Eells and Sampson [4], where the first Weitzenbock formula was obtained. Now we give the following analogue of this formula for harmonic diffeomorphisms.

Theorem 1.5 *Let $f : (M,g) \to (M',g')$ be a harmonic diffeomorphism between Riemannian manifolds of some dimensional. Then*

$$\Delta \ln \tfrac{\det g'}{\det g} = 2(Skal - trace_g Ric') + 3\|sym\,T\|^2 - \|T\|^2. \tag{1.21}$$

Proof. From (1.5) one easily can obtain

$$\nabla'_i T^k_{jl} - \nabla'_l T^k_{ji} = \nabla_i T^k_{jl} - \nabla_l T^k_{ji} + T^n_{jl} T^k_{ni} - T^n_{ji} T^k_{nl} = R'^k_{jil} - R^k_{jil}. \tag{1.22}$$

By Theorem 1.1 and (1.22) we have

$$\begin{aligned} g^{ij}\nabla_i T^k_{jk} = g^{ij}(R^k_{ikj} - R'^k_{ikj}) + g^{ij} T^l_{ki} T^k_{lj} = g^{ij} R_{ij} - g^{ij} R'_{ij} + T_{ijk} T^{jik} = \\ = Scal - trace_g Ric' + \tfrac{3}{2}\|sym\,T\|^2 - \tfrac{1}{2}\|T\|^2, \end{aligned} \tag{1.23}$$

where the tensor field $sym\,T$ has the following local components

$$(sym\,T)_{ijk} = \frac{1}{3}(T_{ijk} + T_{jki} + T_{kij}).$$

By definitions of the deformation tensor T and the Christoffel symbols Γ'^k_{ij} and Γ^k_{ij} we have

$$T^k_{kj} = \Gamma'^k_{kj} - \Gamma^k_{kj} = \partial_j \ln \sqrt{\det g'} - \partial_j \ln \sqrt{\det g} = \partial_j \ln \sqrt{\tfrac{\det g'}{\det g}} \tag{1.24}$$

in a common local coordinate system $x^1, \ldots, x^m$. Finally from (1.23) and (1.24) we get (1.21). □

Remark 1.3. Other Weitzenbock formulas for harmonic maps can be found in [15], [16].

2. The seven classes of harmonic diffeomorphisms

2.1. In this section we shall find a classification of the harmonic diffeomorphisms with the use of representation theory.

Let x be an arbitrary point of (M,g), and $E^* = T^*_x M$. E^* is a pseudo-Euclidean vector space over $\mathbb{R}$ with inner product q induced from the metric g_x, E^* being the dual of $E = T_x M$. Then $T(E) = E^* \otimes S^2_0 E$ is a pseudo-Euclidean vector space with inner product defined by $q(T,T') = \sum_{i,j,k=1}^{m} T(e_i,e_j,e_k)T'(e_i,e_j,e_k)$, where $\{e_1,\ldots,e_m\}$ is an arbitrary orthonormal basis of $E = T_x M$ and $T, T' \in T(E)$.

Denote by $O(q)$ the group of linear transformations $\mathcal{A}$ under which the quadratic form q is invariant, $q(\mathcal{A},\mathcal{A}) = q$. It is customary to call $O(q)$ an orthogonal group. If $\mathcal{A}$ belongs to the orthogonal group $O(q)$ determined by q, then ${}^{tr}\mathcal{A}^{-1} = \mathcal{A}$ and, hence, the vector spaces E and E^* are isomorphic. The orthogonal group $O(q)$ also acts in $\mathcal{T}(E)$ according to the rule

$$(\mathcal{A}T)(a,b,c)T(\mathcal{A}^{-1}a,\mathcal{A}^{-1}b,\mathcal{A}^{-1}c)$$

for $a,b,c \in E$, $T \in \mathcal{T}(E)$, and $\mathcal{A} \in O(q)$, this turns $\mathcal{T}(E)$ into a representation space of the orthogonal group $O(q)$.

Next we set $T_{12}(c) = \sum\limits_{k=1}^{m}(e_k,e_k,c)$ for $c \in E$ and let $\mathcal{T}_1(E) = S_0^3E$; $\mathcal{T}_2(E) = \{T \in \mathcal{T}(E) \mid T(a,b,c) + T(b,c,a) + T(c,a,b) = 0,\ T_{12}(a) = 0\}$; $\mathcal{T}_3(E) = \{T \in \mathcal{T}(E) \mid T(a,b,c) = T_{12}(c)q(a,b) + T_{13}(b)q(a,c) - -\frac{2}{m}T_{23}(a)q(b,c)\}$ for $a,b,c \in E$.

We apply the Weyl theorem to find the number of irreducible subspaces of $\mathcal{T}(E) = E^* \otimes S_0^2E$ with respect to the action of $O(q)$ (cf. [4]). Then we have

Theorem 2.1 *For* $\dim E \geq 3$, $\mathcal{T}(E)$ *is the orthogonal direct sum of the subspaces* $\mathcal{T}_n(E)$, $n = 1,2,3$. *Moreover these spaces are invariant and irreducible under the action of* $O(q)$.

Let $f : (M,g) \to (M',g')$ be a diffeomorphism. In local considerations we may suppose that $M = M' = U$ is a domain in $\mathbb{R}^m$ and f is the identity map $Id_M : (M,g) \to (M',g')$. Therefore we will consider the deformation tensor T as a section of the vector bundle $TM \otimes S^2M$ (cf. also [9]). In particular, if f is harmonic, then the deformation tensor T is a section of the vector bundle $TM \otimes S_0^2M$. Moreover, we shall consider the deformation tensor T as a tensor field of type $(0,3)$.

The Theorem 2.1 implies that there are eight invariant subspaces, so we give

Definition. *Let* $G(E)$ *be an invariant subspace of* $\mathcal{T}(E)$. *We say that a harmonic diffeomorphism* $f : (M,g) \to (M',g')$ *is of type* G *if the deformation tensor* $T \in G(T_xM)$ *for all* $x \in M$.

Hence we may consider the seven classes of harmonic diffeomorphisms including the trivial class characterized by the condition $T = 0$. All these classes have many interesting properties.

2.2. Let $f : (M,g) \to (M',g')$ be a harmonic diffeomorphism of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ which is defined by the following identities:

$$g^{ij}T_{kij} = 0, \tag{2.1}$$

$$g^{ij}T_{ijk} = 0, \tag{2.2}$$

where $T_{ijk} = g_{il}T^l_{jk}$, T^l_{jk} are local components of the deformation tensor $T = \nabla' - \nabla$. Let $dV_{g'} = \sqrt{|det\, g'|}dx^1 \wedge \cdots \wedge dx^m$, $dV_g = \sqrt{|det\, g|}dx^1 \wedge \cdots \wedge dx^m$ be the volume elements of (M, g) and (M', g'), respectively. Then from (2.2) and (1.24) we get

$$dV_{g'} = e^C dV_g$$

for an arbitrary constant $C > 0$. Hence the harmonic diffeomorphism $f : (M, g) \to (M', g')$ is either *volume decreasing* [15] or *volume increasing.* Thus we have proved

Theorem 2.2 *Any harmonic diffeomorphism $f : (M, g) \to (M', g')$ of m-dimensional ($m \geq 3$) pseudo-Riemannian manifolds of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ is either volume decreasing or volume increasing.*

2.3. Let us consider the class $\mathcal{T}_1 \oplus \mathcal{T}_3$ which is defined by the following identities

$$T_{ijk} - T_{jik} = \frac{1}{m-1}(\theta_j g_{ik} - \theta_i g_{jk}), \tag{2.3}$$

where $\theta_j = T^k_{kj} = \partial_j \ln \sqrt{\left|\frac{det\, g'}{det\, g}\right|}$ with respect to a common local coordinate system $x^1, \ldots, x^m$. Using (1.9) one conclude fairly easy that the identities (2.3) are equivalent to

$$\nabla'_j G_{ik} - \nabla'_i G_{jk} = 0, \tag{2.4}$$

where

$$G_{ij} = e^{-\frac{1}{m-1}\theta} g_{ij} \tag{2.5}$$

for $\theta = \ln \sqrt{\left|\frac{det\, g'}{det\, g}\right|} + \text{const}$. Hence the tensor field $G = e^{-\frac{1}{m-1}\theta} g$ is a Codazzi tensor field on (M', g') [17]. Thus we have

Theorem 2.3 *A diffeomorphism $f : (M, g) \to (M', g')$ of m-dimensional ($m \geq 3$) pseudo-Riemannian manifolds is a harmonic diffeomorphism of class $\mathcal{T}_1 \oplus \mathcal{T}_3$ if and only if the tensor field $G = e^{-\frac{1}{m-1}\theta} g$ for $\theta = \ln \sqrt{\left|\frac{det\, g'}{det\, g}\right|} + \text{const}$ is a Codazzi tensor on (M', g').*

Now, let (M', g') be a pseudo-Riemannian manifold of nonzero constant sectional curvature C', then by (2.4) we have

$$G = Hess(F) + C'Fg', \tag{2.6}$$

where $Hess(F) = \nabla' dF$ for an arbitrary smooth function F on (M', g'). It is proved in [17] that any Codazzi tensor G has the form (2.6) on the pseudo-Riemannian manifold (M', g') of nonzero constant sectional curvature C'. Then from (2.5) and (2.6), we get

$$g = e^{\frac{1}{m-1}\theta}[Hess(F) + C'Fg']. \tag{2.7}$$

This implies

Theorem 2.4 *A diffeomorphism $f : (M, g) \to (M', g')$ of an m-dimensional $(m \geq 3)$ pseudo-Riemannian manifold (M, g) onto another pseudo-Riemannian manifold (M', g') of nonzero constant sectional curvature C' is a harmonic diffeomorphism of class $\mathcal{T}_1 \oplus \mathcal{T}_3$ if and only if the metric tensor g has the form $g = e^{\frac{1}{m-1}\theta}[Hess(F) + C'Fg']$ for $\theta =$* $\ln \sqrt{\left|\frac{\det g'}{\det g}\right|} + \text{const}$.

2.4. Any harmonic diffeomorphism $f : (M, g) \to (M', g')$ of class $\mathcal{T}_2 \oplus \mathcal{T}_3$ is characterized by the identities

$$T_{ijk} + T_{jik} + T_{kij} = \frac{2}{m+2}(\theta_i g_{jk} + \theta_j g_{ki} + \theta_k g_{ij}), \tag{2.8}$$

where $\theta_j = T^k_{kj} = \partial_j \ln \sqrt{\left|\frac{\det g'}{\det g}\right|}$ with respect to a common local coordinate system $x^1, \ldots, x^m$. Using (1.9) we conclude that (2.9) is equivalent to

$$\nabla'_i H_{jk} + \nabla'_j H_{ki} + \nabla'_k H_{ij} = 0, \tag{2.9}$$

where

$$H_{jk} = e^{-\frac{2}{m+2}\theta} g_{jk}. \tag{2.10}$$

We remark that in a pseudo-Riemannian manifold (M', g') the differential equations of the geodesics $\frac{dx^j}{dx} + \Gamma'^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$ admit linear quadratic first integrals of the form $H_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = \text{const}\,(H_{jk} = H_{kj})$ if and only if the equations (2.9) hold true. Hence follows

Theorem 2.5 *A diffeomorphism $f : (M, g) \to (M', g')$ between m-dimensional $(m \geq 3)$ pseudo-Riemannian manifolds is a harmonic diffeomorphism of class $\mathcal{T}_2 \oplus \mathcal{T}_3$ if and only if the differential equations of geodesics*

$$\frac{dx^i}{ds} + \Gamma'^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

in (M', g') *written with respect to a common local coordinate system* $x^1, \ldots, x^m$ *admit quadratic first integrals* $H_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = \text{const}$, *where* $H_{jk} = e^{-\frac{2}{m+2}\theta} g_{jk}$, *and* $\theta = \ln\sqrt{\left|\frac{\det g'}{\det g}\right|} + \text{const}$.

Now, let (M', g') be a locally flat pseudo-Riemannian manifold with the fundamental form $ds'^2 = g'_{ij}dx^i dx^j = \varepsilon_j (dx^j)^2$, where $\varepsilon_j = \pm 1$. Then the coefficients H_{jk} of quadratic first integrals of geodesics $H_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = const$ can be expressed in the following form (cf. [19])

$$H_{ij} = A_{ijkl}x^k x^l + B_{ijk}x^k + C_{ij}, \tag{2.11}$$

where $A_{ijkl} = A_{jikl}$, $B_{ijk} = B_{jik}$, $C_{ij} = C_{ji}$, and the $A's$, $B's$ and $C's$ are constants. By substituting (2.11) into (2.9) we find that these coefficients also have symmetries:

$$A_{ijkl} + A_{jkil} + A_{kijl} = 0; \quad B_{ijk} + B_{jki} + B_{kij} = 0.$$

From (2.10) and (2.11) we get

$$g_{ij} = e^{\frac{1}{m+2}\theta}(A_{ijkl}x^k x^1 + B_{ijk}x^k + C_{ij}) \tag{2.12}$$

We have proved the following result.

Theorem 2.6 *A diffeomorphism* $f : (M, g) \to (M', g')$ *of an* m*-dimensional* $(m \geq 3)$ *pseudo-Riemannian manifold* (M, g) *onto a locally flat pseudo-Riemannian manifold* (M', g') *is a harmonic diffeomorphism of class* $\mathcal{T}_1 \oplus \mathcal{T}_3$ *if and only if with respect to a common local coordinate system* $x^1, \ldots, x^m$ *such that* $ds'^2 = g'_{ij}dx^i dx^j = \varepsilon_j(dx^j)^2$ $(\varepsilon_j = \pm 1)$ *the metric tensor* g *has the following components*

$$g_{ij} = e^{\frac{1}{m+2}\theta}(A_{ijkl}x^k x^1 + B_{ijk}x^k + C_{ij}),$$

where $\theta = \ln\sqrt{\left|\frac{\det g'}{\det g}\right|} + \text{const}$, *the* $A's$, $B's$ *and* $C's$ *are constants such that* $A_{ijkl} = A_{jikl}$, $A_{jkil} + A_{jkil} + A_{kijl} = 0$, $B_{ijk} = B_{jik}$, $B_{ijk} + B_{jki} + B_{kij} = 0$, $C_{ij} = C_{ji}$.

We have established that the tensor field H with the local components (2.10) satisfies the equations (2.9). On a compact Riemannian manifold (M', g') of nonpositive sectional curvature such a tensor field is parallel [17]. This, by Shirokov's theorem [18], implies that either (M', g') is locally a Riemannian product, or $H = \mu g'$ for $\mu = const.$ In the second case $g' = \mu^{-1}e^{-\frac{2}{m+2}\theta}g$, then f is a conformal diffeomorphism. Thus, using Remark 2, we obtain

Theorem 2.7 *Let $f : (M,g) \to (M',g')$ be a diffeomorphism of an m-dimensional ($m \geq 3$) Riemannian manifold (M,g) onto a compact Riemannian manifold (M',g') of nonpositive sectional curvature. If f is a harmonic diffeomorphism of class $\mathfrak{T}_1 \oplus \mathfrak{T}_3$, then either (M',g') is locally a Riemannian product, or f is a homothetic diffeomorphism.*

Remark 2.1. P.A. Shirokov proved the following remarkable theorem [18]: a simply connected Riemannian manifold (M,g) is locally a Riemannian product of $(M_1,g_1),\ldots,(M_q,g_q)$ for $2 \leq q \leq m$ if there exists a symmetric 2-tensor field g' with q distinct eigenvalues $\lambda_1,\ldots,\lambda_q$ which is covariantly constant. Moreover, if (M,g) is not locally a Riemannian product, then for each covariantly constant symmetric tensor field g' there holds $g' = \mu g$ with constant μ.

2.5. The diffeomorphism $f : (M,g) \to (M',g')$ is a harmonic diffeomorphism of class $\mathfrak{T}_1$ if and only if the deformation tensor $T \in \mathfrak{T}_1(T_xM)$ for all $x \in M$. Then with respect to a common local coordinate system $x^1,\ldots,x^m$ we have

$$T_{ijk} = T_{jik}; \tag{2.13}$$

$$g^{ij}T_{kij} = 0 \tag{2.14}$$

It turns out that $f : (M,g) \to (M',g')$ is a harmonic diffeomorphism of class $\mathfrak{T}_1$ if and only if $\nabla'_k g_{ij} = \nabla'_i g_{kj}$ and $dV_{g'} = e^C dV_g$ for an arbitrary constant $C > 0$. Thus we get

Theorem 2.8 *A diffeomorphism $f : (M,g) \to (M',g')$ between m-dimensional ($m \geq 3$) pseudo-Riemannian manifolds is a harmonic diffeomorphism of class $\mathfrak{T}_1$ if and only if*

(i) *the tensor field is a Codazzi tensor on (M',g');*

(ii) *$dV_{g'} = e^C dV_g$ for an arbitrary constant $C > 0$, therefore the diffeomorphism $f : (M,g) \to (M',g')$ is either volume decreasing or volume increasing.*

From (1.21), (2.13), and the harmonicity of f, i.e. (2.14), we get

$$\Delta \ln \sqrt{\frac{\det g'}{\det g}} = Skal - trace_g Ric' + \|T\|^2.$$

Theorem 2.9 *Let (M,g) and (M',g') be Riemannian manifolds of a same dimension, and $f : (M,g) \to (M',g')$ be a harmonic diffeomorphism of class $\mathfrak{T}_1$. Assume in addition that (M,g) is compact. If (M,g)*

has non-negative scalar curvature, and (M', g') has non-positive Ricci curvature, then f is affine.

Proof. From the hypothesis we have $Skal - trace_g Ric' + \|T\|^2 \geq 0$. Therefore the Weitzenbock formula for harmonic diffeomorphisms of class $\mathcal{T}_1$ implies that

$$\Delta \ln \sqrt{\frac{\det g'}{\det g}} = Skal - trace_g Ric' + \|T\|^2 \geq 0.$$

Thus, if (M, g) is compact, then $T = 0$. Such a map $f : (M, g) \to (M', g')$ is said to be *affine* (see also [6]). In this case an arbitrary geodesic in (M, g) is mapped by f into a geodesic in (M', g') with the affine parameter preserved. □

2.6. Next, let $f : (M, g) \to (M', g')$ be a harmonic diffeomorphism of class $\mathcal{T}_2$, i.e. the deformation tensor $T \in \mathcal{T}_2(T_x M)$ for all $x \in M$. Then with respect to local coordinates we have

$$T_{ijk} + T_{jki} + T_{kij} = 0; \tag{2.15}$$

$$g^{jk} T_{jki} = 0. \tag{2.16}$$

From (1.11) we conclude that the identities (2.15) are equivalent to

$$\nabla'_i g_{kj} + \nabla'_k g_{ji} + \nabla'_j g_{ik} = 0. \tag{2.17}$$

Hence, for local coordinates $x^1, \ldots, x^m$ in (M', g') the differential equations of the geodesics $\frac{dx^i}{ds} + \Gamma'^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$ admit the quadratic first integrals $H_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = const$ with the coefficients $H_{jk} = g_{jk}$.

In addition we can say that the identities (2.15) are equivalent to $dV_{g'} = e^C dV_g$ for an arbitrary constant $C > 0$. This yields the following result.

Theorem 2.10 *A diffeomorphism $f : (M, g) \to (M', g')$ between m-dimensional ($m \geq 3$) pseudo-Riemannian manifolds is a harmonic diffeomorphism of class $\mathcal{T}_2$ if and only if*

(i) *the differential equations $\frac{dx^i}{ds} + \Gamma'^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$ of geodesics in (M', g') written with respect to a common local coordinate system $x^1, \ldots, x^m$ admit quadratic first integrals $H_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} =$ const, where $H_{jk} = g_{jk}$;*

(ii) *$dV_{g'} = e^C dV_g$ for an arbitrary constant $C > 0$, thus the diffeomorphism $f : (M, g) \to (M', g')$ is either volume decreasing or volume increasing.*

From (1.21), (2.15) and the harmonicity of f we get

$$\Delta \ln \frac{\det g'}{\det g} = 2(Skal - trace_g Ric') - \|T\|^2.$$

Theorem 2.11 *Let (M,g) and (M',g') be Riemannian manifolds of a same dimension, and $f:(M,g)\to(M',g')$ be a harmonic diffeomorphism of class $\mathfrak{T}_2$. Assume in addition that (M,g) is compact. If (M,g) has non-negative scalar curvature, and (M',g') has non-positive Ricci curvature, then f is affine.*

Proof. From the hypothesis we have $2(Skal-trace_g Ric')-\|T\|^2\le 0$. Therefore the Weitzenböck formula for harmonic diffeomorphisms of class $\mathfrak{T}_2$ implies that

$$\Delta \ln \tfrac{\det g'}{\det g} = 2(Skal - trace_g Ric') - \|T\|^2 \le 0.$$

Thus, if (M,g) is compact, then $T=0$. Such a map $f:(M,g)\to(M',g')$ is affine [6]. □

2.7. Let $f:(M,g)\to(M',g')$ be a harmonic diffeomorphism of class T_3. In this case the deformation tensor $T\in\mathfrak{T}_3(T_xM)$ for all $x\in M$, and hence we have

$$T_{ij}^k = \varphi_i\delta_j^k + \varphi_j\delta_i^k - \frac{2}{m}\varphi^k g_{ij}, \tag{2.18}$$

where $\varphi^k = g^{kj}\varphi_j$, and $\varphi_j = \frac{m}{(m+2)(m-1)}T_{kj}^k = \frac{m}{(m+2)(m-1)}\partial_j \ln\sqrt{\left|\frac{\det g'}{\det g}\right|}$. Using Theorem 1.2, we obtain the following

Theorem 2.12 *Let $f:(M,g)\to(M'',g'')$ be a harmonic diffeomorphism of class $\mathfrak{T}_3$ between m-dimensional $(m\ge 3)$ pseudo-Riemannian manifolds. Then f is a composition $f''\circ f'$ of a conformal diffeomorphism $f':(M,g)\to(M',g')$ and a projective diffeomorphism $f'':(M',g')\to(M'',g'')$ of pseudo-Riemannian manifolds.*

From (1.21) and (2.18) we get

$$\Delta \ln \tfrac{\det g'}{\det g} = 2(Skal - trace_g Ric') + \tfrac{m-2}{(m-1)(m+2)}\|\theta\|^2,$$

where $\theta = traceT = \frac{1}{2}grad \ln \frac{\det g'}{\det g}$.

Theorem 2.13 *Let (M,g) and (M',g') be Riemannian manifolds of a same dimension, and $f:(M,g)\to(M',g')$ be a harmonic diffeomorphism of class $\mathfrak{T}_3$. Assume in addition that (M,g) is compact. If (M,g) has non-negative scalar curvature, and (M',g') has non-positive Ricci curvature, then f is affine.*

Proof. From the hypothesis we have

$$2(Skal - trace_g Ric') + \frac{m-2}{(m-1)(m+2)}\|\theta\|^2 \ge 0.$$

Therefore the Weitzenbock formula for harmonic diffeomorphisms of class $\mathcal{T}_3$ implies that

$$\Delta \ln \frac{\det g'}{\det g} = 2(Skal - trace_g Ric') + \frac{m-2}{(m-1)(m+2)} ||\theta||^2 \geq 0.$$

Thus, if (M, g) is compact, then $\theta = 0$. A direct calculation shows that $T = 0$ in this case. Therefore a diffeomorphism $f : (M, g) \to (M', g')$ is affine [6]. □

2.8. Finally, let $f : (M, g) \to (M', g')$ be a harmonic diffeomorphism of the seventh class, i.e. the deformation tensor $T = 0$. Such a map $f : (M, g) \to (M', g')$ is affine [6]. Under this assumption, we have from (1.9) that $\nabla_k g'_{ij} = 0$. Using Shirokov's theorem [18], we obtain

Theorem 2.14 *Let $f : (M, g) \to (M', g')$ be a diffeomorphism of a simply connected Riemannian manifold (M, g) onto a Riemannian manifold (M', g'). If f is a harmonic diffeomorphism of the seventh class, then one of the following assertions is true:*

(i) *(M, g) is locally a Riemannian product;*

(ii) *f is a homothetic diffeomorphism.*

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