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ON THE FLOW PROLONGATION OF VECTOR FIELDS

Abstract

Let F be a fiber product preserving bundle functor of the base order r on the category $\mathcal{FM}_m$ and η a projectable vector field on $Y \to M$ over a vector field ξ on M, $m = \dim M$. We construct a natural map transforming $F\eta$ and $j^r\xi$ into the flow prolongation of η and deduce its basic properties. Our main tool is a similar construction in the case of products of two manifolds and of product vector fields.

Аннотация

I. Kolář **О продолжении потока векторных полей**

Пусть F — естественный функтор базового порядка r на категории расслоений $\mathcal{FM}_m$, который сохраняет произведение, и η — векторное поле на $Y \to M$, проектирующееся в векторное поле ξ на M, $m = \dim M$. Мы определяем естественное отображение, преобразующее $F\eta$ и $j^r\xi$ в продолжение поля η, построенное с помощью потока, и изучаем его свойства.

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The prolongation of vector fields by means of flows is of general character. If F is a bundle functor on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms, then every vector field $\xi : M \to TM$ on an m-dimensional manifold M induces the flow prolongation $\mathcal{F}\xi : FM \to TFM$ by

$$\mathcal{F}\xi = \frac{\partial}{\partial t}\Big|_0 F(\mathrm{F}l_t^\xi)\,.$$

If F is a bundle functor on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms, then every projectable vector field η on such fibered manifold Y induces the flow prolongation $\mathcal{F}\eta : FY \to TFY$ in the same way.

A specific procedure can be used in the case F is a product preserving bundle functor on the category $\mathcal{M}f$ of all manifolds, so that F is a Weil functor determined by the Weil algebra $A = F\mathbb{R}$, see [8] for a survey. Then there exists a natural exchange diffeomorphism $\varkappa_M : FTM \to TFM$ such that $\mathcal{F}\xi = \varkappa_M \circ F\xi$, where $F\xi : FM \to FTM$ is the functorial prolongation of $\xi : M \to TM$. Further, let $\mathcal{FM}_m$ be the category of fibered manifolds with m-dimensional bases and fiber preserving maps with local diffeomorphisms as base maps. In [7], a fiber product preserving bundle functor F on $\mathcal{FM}_m$ was characterized in terms of Weil algebras. This enabled us to construct, [6], for every fibered manifold $p : Y \to M$, $\dim M = m$, a map

$$\mu_Y^F : J^rTM \times_{FTM} FTY \to TFY$$

with the property

$$\mathcal{F}\eta = \mu_Y^F \circ (j^r\xi \times_M F\eta) \tag{1}$$

for every projectable vector field η on Y over ξ on M. In [3] we outlined how this map can be used for prolongating projectable tangent valued forms from Y to FY.

In the present paper we study μ_Y^F systematically. In Section 1 we summarize all necessary facts concerning the Weil algebra approach. Section 2 is based on the fact that the construction of product fibered manifolds defines an injection $\mathcal{M}f_m \times \mathcal{M}f \hookrightarrow \mathcal{FM}_m$, so that F restricts to a bundle functor G on $\mathcal{M}f_m \times \mathcal{M}f$. There is a natural inclusion $FY \hookrightarrow G(M,Y)$. We construct a map

$$\mu_{M,N}^G : J^rTM \times_M G(M,TN) \to TG(M,N)$$

with the property that μ_Y^F is the restriction and corestriction of $\mu_{M,Y}^G$. The basic properties of $\mu_{M,N}^G$ and μ_Y^F are studied in Sections 2 and 3. In particular, we deduce that μ_Y^F is determined by the requirement (1). In Section 4 we consider another fiber product preserving bundle functor E on $\mathcal{FM}_m$ and we describe μ^{FE} of the iteration $F \circ E$ in terms of μ^F and μ^E. In the last section we show how μ_Y^F can be used for constructing the prolongation of a projectable tangent valued k-form on Y to FY. We remark that the relations of the latter construction to the theory of connections are discussed in [3].

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [8].

1. The Weil algebra approach

Consider a Weil algebra $A = \mathbb{R} \times N_A$, where N_A is the nilpotent part. The smallest integer r satisfying $N_A^{r+1} = 0$ is called the order of A, the dimension k of the vector space N_A/N_A^2 is called the width of A. Every choice of $a_1, \dots, a_k \in N_A$ such that

$$a_1 + N_A^2, \dots, a_k + N_A^2 \tag{2}$$

is a basis of N_A/N_A^2 determines an algebra epimorphism $\pi : \mathbb{D}_k^r \to A$, where $\mathbb{D}_k^r = J_0^r(\mathbb{R}^k, \mathbb{R})$.

In the algebraic approach by A. Weil, the bundle $T^A M$ of all A-velocities on M is defined as the set of all algebra homomorphisms from the algebra $C^\infty(M, \mathbb{R})$ of all smooth functions on M into A. However, we use the following jet-like approach. Two maps $\gamma, \delta : \mathbb{R}^k \to M$ are said to determine the same A-velocity $j^A\gamma = j^A\delta$, if, for every smooth function $\varphi : M \to \mathbb{R}$,

$$\pi\big(j_0^r(\varphi \circ \gamma)\big) = \pi\big(j_0^r(\varphi \circ \delta)\big) .$$

This definition is independent of the choice of (2). For every smooth map $f : M \to N$, we define $T^A f : T^A M \to T^A N$ by $T^A f(j^A\gamma) = j^A(f \circ \gamma)$. Clearly, T^A is a product preserving bundle functor on $\mathcal{M}f$.

Our starting point was the fundamental result that every product preserving bundle functor F on $\mathcal{M}f$ is a Weil functor T^A, where $A = F\mathbb{R}$ with the multiplication induced from the multiplication of reals, see [8] for a survey. In particular, the functor T_k^r of (k,r)-velocities corresponds to $\mathbb{D}_k^r$. Moreover, the natural transformations $T^A \to T^{\overline{A}}$ of two such functors are in bijection with the algebra homomorphisms $A \to \overline{A}$. We write $\mu_M : T^A M \to T^{\overline{A}} M$ for the natural transformation determined by μ :

$A \to \overline{A}$. In particular, we denote by $\varkappa_M^A : T^ATM \to TT^AM$ the exchange diffeomorphism mentioned in the introduction, which corresponds to the exchange map $A \otimes \mathbb{D}_1^1 \to \mathbb{D}_1^1 \otimes A$.

In [7], W. Mikulski and the author characterized a fiber product preserving bundle functor F on $\mathcal{FM}_m$ in a similar way. Having in mind the inclusion $\mathcal{M}f_m \times \mathcal{M}f \hookrightarrow \mathcal{FM}_m$, we start with a bundle functor G on $\mathcal{M}f_m \times \mathcal{M}f$. We write, analogously to the jet bundles,

$$G_x(M,N) \quad or \quad G(M,N)_y \quad or \quad G_x(M,N)_y\,, \qquad x \in M,\ y \in N\,,$$

for the submanifold of $G(M,N)$ with the first projection x or the second projection y or both, respectively. We say that G is of order r in the first factor, if for every local diffeomorphisms $f_1, f_2 : M \to \overline{M}$ and every map $g : N \to \overline{N}$, $j_x^r f_1 = j_x^r f_2$ implies $G_x(f_1,g) = G_x(f_2,g) : G_x(M,N) \to G_{\overline{x}}(\overline{M},\overline{N})$, where $x \in M$, $\overline{x} = f_1(x) = f_2(x) \in \overline{M}$. We say G preserves products in the second factor, if

$$G(M, N \times Q) = G(M,N) \times_M G(M,Q)\,,$$

where Q is another manifold. In [7] it is deduced that the bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ of order r in the first factor and preserving products in the second factor are in bijection with the pairs (A,H), where A is a Weil algebra and $H : G_m^r \to Aut\,A$ is a group homomorphism of the r-jet group G_m^r in dimension m into the group of all algebra automorphisms of A. Every $z \in G_m^r$ induces a diffeomorphism $H(z)_N : T^AN \to T^AN$. This defines an action H_N of G_m^r on T^AN. Since every $H(z)$ is a natural transformation, $T^Ag : T^AN \to T^A\overline{N}$ is G_m^r-equivariant for every map $g : N \to \overline{N}$. According to [7], $G(M,N)$ is an associated bundle to the r-th order frame bundle P^rM of M

$$G(M,N) = P^rM[T^AN, H_N] \quad \text{and} \quad G(f,g) = P^r\underline{f}[T^Ag] \tag{3}$$

is the induced morphism of associated bundles, $\underline{f} : M \to \overline{M}$ being a local diffeomorphism. If $\overline{G} = (\overline{A},\overline{H})$, $\overline{H} : G_m^r \to Aut\,\overline{A}$ is another such functor, then the natural transformations $G \to \overline{G}$ are in bijection with the equivariant algebra homomorphisms $A \to \overline{A}$.

For example, for the r-th order jet functor J^r on $\mathcal{M}f_m \times \mathcal{M}f$ we have $A = \mathbb{D}_m^r$ and H is the canonical action $C : G_m^r \times \mathbb{D}_m^r \to \mathbb{D}_m^r$, $C(z)(a) = a \circ z$ defined by the jet composition.

Consider a bundle functor F on $\mathcal{FM}_m$. The definition of the order of F is based on the concept of (q,s,r)-jet, $s \ge q \le r$, [8]. For a fibered manifold morphism $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$, where $\overline{Y} \to \overline{M}$ is another fibered manifold, $j_y^{q,s,r} f$ is identified with the triple

$$\left(j_y^q f, j_y^s(f|Y_x), j_x^r \underline{f}\right), \qquad y \in Y,\ x = p(y).$$

We say that F is of order (q, s, r), if for every Y, $\overline{Y}$ and every two $\mathcal{FM}_m$-morphisms $f, g : Y \to \overline{Y}$,

$$j_y^{q,s,r} f = j_y^{q,s,r} g \quad \textit{implies} \quad Ff|F_yY = Fg|F_yY\,, \qquad y \in Y\,.$$

The integer r is called the base order of F. In [7] it is deduced that the fiber product preserving bundle functors of the base order r on $\mathcal{FM}_m$ are in bijection with the triples (A, H, t), where A and H are as above and $t : \mathbb{D}_m^r \to A$ is an equivariant algebra homomorphism. We have

$$FY = \left\{\{u, Z\} \in P^r M[T^A Y], t_M(u) = T^A p(Z)\right\} \tag{4}$$

where $t_M : T_m^r M \to T^A M$ and the inclusion $P^r M \subset T_m^r M$ is taken into account. Further, $Ff : FY \to F\overline{Y}$ is the restriction and corestriction of $P^r \underline{f}[T^A f]$. Moreover, if $\overline{F} = (\overline{A}, \overline{H}, \overline{t})$ is another such functor, then the natural transformations $F \to \overline{F}$ are in bijection with the equivariant algebra homomorphisms $\mu : A \to \overline{A}$ satisfying $\overline{t} = \mu \circ t$.

Clearly, if G is the restriction of $F = (A, H, t)$ to $\mathcal{M}f_m \times \mathcal{M}f$, then $G = (A, H)$.

2. The product case

Having a vector field ξ on M and a vector field η on N, we write (ξ, η) for the product vector field on $M \times N$. Consider a principal bundle $P(M, G)$ and a left G-space S. If η is a right invariant vector field on P and ζ is a left invariant vector field on S, then the vector field (η, ζ) on $P \times S$ is projectable to the associated bundle $P[S]$. The underlying vector field on $P[S]$ will be denoted by $\{\eta, \zeta\}$.

Consider the functor $G = (A, H)$ on $\mathcal{M}f_m \times \mathcal{M}f$. Hence $\mathcal{P}^r\xi$ is a right invariant vector field on $P^r M$ and $\mathcal{T}^A\eta$ is a left invariant vector field on $T^A N$. Then (3) implies that the flow prolongation $\mathcal{G}(\xi, \eta)$ of (ξ, η) satisfies

$$\mathcal{G}(\xi, \eta) = \{\mathcal{P}^r\xi, \mathcal{T}^A\eta\}\,. \tag{5}$$

We are going to construct a map

$$\mu_{M,N}^G : J^r TM \times_M G(M, TN) \to TG(M, N)\,.$$

By (3) and (4), $J^rTM \subset P^rM[T^r_mTM]$ and $G(M,TN) = P^rM[T^ATN]$. Consider $\{u,W\} \in J^rTM$, $\{u,Z\} \in G(M,TN)$, $u \in P^rM$, $W \in T^r_mTM$, $Z \in T^ATN$. Write $\overline{\varkappa}_M : T^r_mTM \to TT^r_mM$ for the exchange map of T^r_m. Then $\overline{\varkappa}_M(W) = \frac{\partial}{\partial t}\big|_0 \gamma(t)$, $\gamma : \mathbb{R} \to T^r_mM$ and $\varkappa^A_N(Z) = \frac{\partial}{\partial t}\big|_0 \zeta(t)$, $\zeta : \mathbb{R} \to T^AN$. By (4), $\gamma(0) = u$. Since $P^rM \subset T^r_mM$ is an open subset, the values of $\gamma(t)$ locally lie in P^rM. Then we define

$$\mu^G_{M,N}\Big(\{u,W\},\{u,Z\}\Big) = \frac{\partial}{\partial t}\Big|_0 \{\gamma(t),\zeta(t)\}\,. \tag{6}$$

This is independent of the choice of the frame u. Indeed, if we replace u by $u \circ g$, $g \in G^r_m$, we have $TC(g)_M\big(\overline{\varkappa}_M(W)\big) = \frac{\partial}{\partial t}\big|_0 \big(\gamma(t) \circ g^{-1}\big)$, $TH(g)_N\big(\varkappa^A_N(Z)\big) = \frac{\partial}{\partial t}\big|_0 H(g)_N\big(\zeta(t)\big)$ and $\{\gamma(t),\zeta(t)\} = \{\gamma(t) \circ g^{-1}, H(g)_N\big(\zeta(t)\big)\}$.

For a vector field $\eta : N \to TN$, we have $(id_M,\eta) : M \times N \to M \times TN$. Hence $G(id_M,\eta) : G(M,N) \to G(M,TN)$ and we can construct the map

$$j^r\xi \times_M G(\underset{M}{id},\eta) : G(M,N) \to J^rTM \times_M G(M,TN)\,.$$

Proposition 1 *For every pair of vector fields ξ on M and η on N,*

$$\mu^G_{M,N} \circ \big(j^r\xi \times_M G(\underset{M}{id},\eta)\big) = \mathcal{G}(\xi,\eta)\,. \tag{7}$$

Proof. By Section 1, we have $j^r\xi = \{\mathcal{P}^r\xi, \mathcal{T}^r_m\xi\}$ and $G(id_M,\eta) = id_{P^rM}[T^A\eta]$. Then (6) implies

$$\mu^G_{M,N} \circ \big(j^r\xi \times_M G(\underset{M}{id},\eta)\big) = \{\mathcal{P}^r\xi, \mathcal{T}^A\eta\} = \mathcal{G}(\xi,\eta)\,.$$

□

The following assertion clarifies that $\mu^G_{M,N}$ is fully determined by (7).

Proposition 2 *If $\psi : J^rTM \times_M G(M,TN) \to TG(M,N)$ is a fibered manifold morphism over the identity of $G(M,N)$ satisfying*

$$\psi(j^r\xi \times_M G\big(\underset{M}{id},\eta)\big) = \mathcal{G}(\xi,\eta)$$

for every pair ξ and η of vector fields, then $\psi = \mu^G_{M,N}$.

Proof. We start from the fact that $T^r_kM = J^r_0(\mathbb{R}^k,M)$ and $\mathcal{T}^r_k\xi = \overline{\varkappa}_m \circ T^r_k\xi$ imply that for every $X \in TT^r_kM$ over $v \in T^r_kM$ there exists a vector field ξ on M satisfying $\mathcal{T}_k^r\xi(v) = X$. For an arbitrary Weil algebra A, the epimorphism $\pi : \mathbb{D}_k^r \to A$ from Section 1 induces a surjective map $\pi_N : T_k^r N \to T^A N$. This implies that for every $Z \in TT^A N$ over $z \in T^A N$ there exists a vector field η on N such that $\mathcal{T}^A\eta(z) = Z$. Hence for every u, W, Z from the left hand side of (6) there exists ξ and η such that $j_x^r\xi = \{u, W\}$ and $\mathcal{T}^A\eta(z) = Z$. Then $\psi\big(\{u, W\}, \{u, Z\}\big) = \mathcal{G}(\xi, \eta)\big(\{u, z\}\big)$, so that $\psi = \mu_{M,N}^G$. □

The following result is important for Section 5. We define

$$\widetilde{\mu}_{M,N}^G : J^rTM \times_M G(M, TN) \to J^rTM \times_{TM} TG(M, N),$$
$$\widetilde{\mu}_{M,N}^G(X_1, X_2) = \big(X_1, \mu_{M,N}^G(X_1, X_2)\big).$$

Proposition 3 *$\widetilde{\mu}_{M,N}^G$ is a diffeomorphism.*

Proof. Consider $\frac{\partial}{\partial t}\big|_0 \varphi(t) \in TG(M, N)$ and $j_x^r\xi \in J^rTM$ satisfying $\xi(x) = \frac{\partial}{\partial t}\big|_0 \underline{\varphi}(t)$, where $\underline{\varphi}$ is the underlying curve on M. Fix $u \in P_x^r M$ and construct $\gamma(t)$ from $j_x^r\xi$ and u in the same way as in (6). Since $G(M, N) = P^rM[T^AN]$, we can write $\varphi(t) = \{\gamma(t), \zeta(t)\}$, $\zeta : \mathbb{R} \to T^AN$. Then $\left\{u, \frac{\partial}{\partial t}\big|_0 \zeta(t)\right\} \in G(M, TN)$. By construction, $\widetilde{\mu}_{M,N}^G\left(j_x^r\xi, \left\{u, \frac{\partial}{\partial t}\big|_0 \zeta(t)\right\}\right) = \left(j_x^r\xi, \frac{\partial}{\partial t}\big|_0 \varphi(t)\right)$. □

We are going to deduce the local expression of $\mu_{M,N}^G$. Assume $M = \mathbb{R}^m$, $N = \mathbb{R}^n$. Write $h : \mathfrak{g}_m^r \times A \to A$ for the tangent map of $H : G_m^r \times A \to A$ at $(e, 0)$, i.e.

$$h\left(\frac{\partial}{\partial t}\Big|_0 g(t)\right)\left(\frac{\partial}{\partial t}\Big|_0 a(t)\right) = \frac{\partial}{\partial t}\Big|_0 H\left(g(t)\right)\left(a(t)\right), \qquad g : \mathbb{R} \to G_m^r, \ a : \mathbb{R} \to A \tag{8}$$

The Lie algebra $\mathfrak{g}_m^r$ of G_m^r is identified with the set of all r-jets at 0 of the vector fields on $\mathbb{R}^m$ with zero value at 0. Hence $J^rT\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m \times \mathfrak{g}_m^r$. On the other hand, $G(\mathbb{R}^m, \mathbb{R}^m) = \mathbb{R}^m \times A^n$, $G(\mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^n) = \mathbb{R}^m \times A^n \times A^n$, $TG(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times \mathbb{R}^m \times A^n \times A^n$. Hence

$$\mu_{\mathbb{R}^m,\mathbb{R}^n}^G : \mathbb{R}^m \times \mathbb{R}^m \times \mathfrak{g}_m^r \times A^n \times A^n \to \mathbb{R}^m \times \mathbb{R}^m \times A^n \times A^n . \tag{9}$$

The curve γ from (6) can be considered in the form (γ_1, γ_2), $\gamma_1 : \mathbb{R} \to \mathbb{R}^m$, $\gamma_2 : \mathbb{R} \to G_m^r$, $\gamma_1(0) = 0$, $\gamma_2(0) = e$, and $\zeta : \mathbb{R} \to A^n$. We have

$$\left\{\big(\gamma_1(t), \gamma_2(t)\big), \zeta(t)\right\} = \left\{\big(\gamma_1(t), \widetilde{e}(t)\big), H\big(\gamma_2(t)\big)\big(\zeta(t)\big)\right\} , \tag{10}$$

where $\widehat{e}$ is the constant curve whose only value is the unit of G^r_m. Write $z \in \mathbb{R}^m \times \mathbb{R}^m$, $u \in \mathfrak{g}^r_m$, a^p, b^p, $a, b \in A$, $p = 1, \dots, n$ for the variables on the left hand side of (9). Then (10) implies

$$\mu^G_{\mathbb{R}^m,\mathbb{R}^n}(z, u, a^p, b^p) = \big(z, a^p, b^p + h(u)(a^p)\big)\,. \tag{11}$$

To rewrite (11) in a form more suitable from the general point of view, [6], we take into account the fact $\mathbb{R}^n = V$ is a vector space only. Then $T^A V = V \otimes A$ and we have

$$\mu^G_{\mathbb{R}^m,V} : T\mathbb{R}^m \times \mathfrak{g}^r_m \times V \otimes A \times V \otimes A \to T\mathbb{R}^m \times V \otimes A \times V \otimes A\,. \tag{12}$$

If we consider a decomposable tensor $v \otimes a$ in the third entry and arbitrary w in the fourth entry on the left hand side, (11) can be rewritten as

$$\mu^G_{\mathbb{R}^m,V}(z, u, v \otimes a, w) = \big(z, v \otimes a, w + v \otimes h(u)(a)\big)\,. \tag{13}$$

For nondecomposable tensors, (13) is extended by bilinearity.

By (3), for every fibered manifold $q : Q \to N$, $G(id_M, q) : G(M, Q) \to G(M, N)$ is a fibered manifold. If Q is a vector bundle, the vector addition is a map $a : Q \times_N Q \to Q$. Using the standard technique of prolongating the diagrams expressing some properties of the structures in question, we deduce that each fiber of $G(M, Q) \to Q(M, N)$ is an Abelian group with respect to $G(id_M, a)$. Moreover, the multiplication of vectors by reals can be interpreted as a map $m : \mathbb{R} \times Q \to Q$. Then

$$G(\underset{M}{id}, m) : G(M, \mathbb{R}) \times_M G(M, Q) \to G(M, Q)\,. \tag{14}$$

But $G(M, \mathbb{R}) = P^r M[A]$. Hence (14) is an action of the algebra bundle $P^r M[A]$ on $G(M, Q)$ that is analogous to the action of the algebra A on the tangent bundle of the A-velocities bundle of a manifold, [8]. Clearly, $M \times \mathbb{R}$ is an invariant subbundle of $P^r M[A]$. The restriction of (14) to $M \times \mathbb{R}$ defines the multiplication by reals on $G(M, Q)$ and one sees easily that $G(M, Q) \to G(M, N)$ is a vector bundle. In particular, $G(M, TN) \to G(M, N)$ is a vector bundle. Then (13) implies

Proposition 4 *$\mu^G_{M,N}$ is bilinear in $J^r TM$ and $G(M, TN)$.*

3. The fibered case

Consider a bundle functor $F = (A, H, t)$ on $\mathcal{FM}_m$. We have an induced map $\widetilde{t}_Y : J^r Y \to FY$ defined in the following way. Every section

$s : M \to Y$ can be interpreted as a base preserving morphism $\widetilde{s}$ of the fibered manifold $id_M : M \to M$ with one-point fibers into Y. Then we set

$$\widetilde{t}_Y(j^r_x s) = F\widetilde{s}(x)\,.$$

In particular, we have $\widetilde{t}_{TM} : J^rTM \to FTM$. On the other hand, $FTp : FTY \to FTM$, so that we can construct the fiber product $J^rTM \times_{FTM} FTY$. Since $FTY \subset G(M,TY)$, we have

$$J^rTM \times_{FTM} FTY \subset J^rTM \times_M G(M,TY)\,.$$

Moreover, $FY \subset G(M,Y)$ implies $TFY \subset TG(M,Y)$.

Consider $\mu^G_{M,Y} : J^rTM \times_M G(M,TY) \to TG(M,Y)$. We denote by

$$\mu^F_Y : J^rTM \times_{FTM} FTY \to TFY$$

the map determined the following assertion.

Proposition 5 *The values of the restriction of $\mu^G_{M,Y}$ to $J^rTM \times_{FTM} FTY$ lie in TFY.*

Proof. In the notation from Section 2, the condition $\big(\{u,W\},\{u,Z\}\big) \in J^rTM \times_{FTM} FTY$ implies $t_M\big(\gamma(t)\big) = T^A p\big(\zeta(t)\big)$. Hence $\{\gamma(t),\zeta(t)\}$ lies in FY. □

The basic properties of μ^F_Y follow directly from the results of Section 2. First of all, if we interpret a projectable vector field η on Y over ξ on M as a morphism $(Y \to M) \to (TY \to M)$, we obtain

Corollary 1 *We have*

$$\mu^F_Y \circ (j^r\xi \times_M F\eta) = \mathcal{F}\eta\,. \tag{15}$$

Moreover, if a morphism $\psi : J^rTM \times_{FTM} FTY \to TFY$ over the identity of FY has this property, then $\psi = \mu^F_Y$.

In the case $F = J^r$, we have $J^rTM \times_{J^rTM} J^rTY = J^rTY$, so that $\mu^{J^r}_Y : J^rTY \to TJ^rY$. Corollary 1 implies that $\mu^{J^r}_Y$ coincides with a map introduced in another way by L. Mangiarotti and M. Modugno, [9].

Analogously to Section 2, we define

$$\widetilde{\mu}^F_Y : J^rTM \times_{FTM} FTY \to J^rTM \times_{TM} TFY\,,$$
$$\widetilde{\mu}^F_Y(X_1,X_2) = \big(X_1, \mu^F_Y(X_1,X_2)\big)\,.$$

Then Proposition 3 yields

Corollary 2 *$\widetilde{\mu}_Y^F$ is a diffeomorphism.*

To find the local expression of μ_Y^F, we consider the product fibered manifold $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and write $\mathbb{R}^n = V$. Then $TY = \mathbb{R}^m \times \mathbb{R}^m \times V \times V$ and $F(TY \to M) = \mathbb{R}^m \times (\mathbb{R} \times N_A)^m \times V \otimes A \times V \otimes A$. But $\widetilde{t}_{T\mathbb{R}^m}$ maps N_A^m into $\mathfrak{g}_m^r$. Hence

$$J^r T\mathbb{R}^m \times_{FT\mathbb{R}^m} F\big(T(\mathbb{R}^m \times V)\big) = T\mathbb{R}^m \times \mathfrak{g}_m^r \times V \otimes A \times V \otimes A\,.$$

Then Proposition 5 yields formally the same expression as (13)

$$\mu^F_{\mathbb{R}^m \times V}(z, u, v \otimes a, w) = \big(z, v \otimes a, w + v \otimes h(u)(a)\big)\,. \tag{16}$$

Analogously to Section 2, $F(TY \to M)$ is a vector bundle over FY. Proposition 4 implies directly

Corollary 3 *μ_Y^F is bilinear in J^rTM and FTY.*

By the geometrical character of our construction, μ_Y^F is functorial in both Y and F.

4. The iteration

If E is another fiber product preserving bundle functor on $\mathcal{FM}_m$ of the base order s, then the iteration FE preserves fiber products as well and is of the base order $r + s$. We are going to describe μ^{FE} in terms of μ^F and μ^E. A complete characterization of FE can be found in [4]. But we are not obliged to quote that result here, for our proof will be based on Corollary 1 only.

For a vector field ξ on M, $j^s\xi$ can be interpreted as a morphism from $M \xrightarrow{id} M$ into $J^sTM \to M$. Clearly, $F(j^s\xi)(x)$ depends on $j_x^{r+s}\xi$ only. Hence we can define a map $I^s_{F,M} : J^{r+s}TM \to J^rTM \times_{FTM} FJ^sTM$ by

$$I^s_{F,M}(j_x^{r+s}\xi) = \big(j_x^r\xi, F(j^s\xi)(x)\big)\,.$$

We have $\mu_Y^E : J^sTM \times_{ETM} ETY \to TEY$, so that

$$F\mu_Y^E : FJ^sTM \times_{FETM} FETY \to FTEY\,.$$

Moreover,

$$\mu^F_{EY} : J^rTM \times_{FTM} F\big(T(EY)\big) \to T(FEY)\,.$$

Proposition 6 *We have*

$$\mu_Y^{FE} = \mu_{EY}^F \circ (\underset{J^rTM}{id} \times_{FTM} F\mu_Y^E) \circ (I_{F,M}^s \times_{FETM} \underset{FETY}{id}).$$

Proof. For every η over ξ, we first find

$$(I_{F,M}^s \times_{FETM} \underset{FETY}{id})(j^{r+s}\xi \times_M FE\eta) = j^r\xi \times_M Fj^s\xi \times_M FE\eta.$$

Applying $id_{J^rTM} \times_{FTM} F\mu_Y^E$ to the right hand side, we obtain

$$j^r\xi \times_M F(\mathcal{E}\eta).$$

Finally,

$$\mu_{EY}^F(j^r\xi \times_M F(\mathcal{E}\eta)) = \mathcal{F}(\mathcal{E}\eta) = (\mathcal{F}\mathcal{E})(\eta).$$

Hence our claim follows from Corollary 1. □

5. Tangent valued forms

In [3] we studied in detail the flow prolongation of projectable semibasic tangent valued k-forms on a fibered manifold $Y \to M$. Then we outlined how μ_Y^F can be used for prolongating arbitrary projectable tangent valued k-forms. Now we are able to complete our first draft by a complete proof. We restrict ourselves to the case $k = 1$, for the case $k > 1$ is only technically more complicated.

For a projectable 1-form φ on Y over $\underline{\varphi}$ on M, $\mathcal{F}\varphi$ is introduced by the commutative diagram

$$\begin{tikzcd}
J^rTM \times_{FTM} FTY \arrow[r, "J^r\underline{\varphi}\times_{F\underline{\varphi}}F\varphi"] \arrow[d, shift left, "\tilde{\mu}_Y^F"] & J^rTM \times_{FTM} FTY \arrow[d, "\mu_Y^F"] \\
J^rTM \times_{TM} TFY \arrow[u, shift left] \arrow[r, "\mathcal{F}\varphi"] & TFY
\end{tikzcd}$$

Proposition 7 *$\mathcal{F}\varphi$ is a bilinear morphism.*

Proof. It suffices to deduce that $J^r\underline{\varphi} \times_{F\underline{\varphi}} F\varphi$ is bilinear. By Corollary 3, μ_Y^F is bilinear, so that $\tilde{\mu}_Y^F$ is bilinear too.

In general, consider a vector bundle $Q \to Y$, another fibered manifold $Z \to M$, a vector bundle $S \to Z$ and a linear morphism $\psi : Q \to S$ over a base preserving morphism $\chi : Y \to Z$. The fact that the linearity of ψ can be expressed by the commutativity of certain simple diagrams implies that $F\psi : FQ \to FS$ is a linear morphism from $FQ \to FY$ into $FS \to FZ$ over $F\chi : FY \to FZ$. In particular, $\underline{\varphi} : TM \to TM$ is a linear morphism, so that $J^r\underline{\varphi} : J^rTM \to J^rTM$ and $F\underline{\varphi} : FTM \to FTM$ are too. Moreover, since $\varphi : TY \to TY$ is a linear morphism, $F\varphi : FTY \to FTY$ is too. □

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