CH5170 - Assignment 2

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Questions

1. Determine if they are convex, concave, or neither or both.
2. Determine stationary points where $\nabla(f) = 0$.
3. Determine the character of the stationary point:, viz., local minimum, local maximum, saddle point. Are the minima and maxima (if they exist) global?
4. Sketch the contour curves as accurately as possible. In particular, pay attention to the orientation and aspect ratios.
5. Starting from the initial point [3,3],find a directions in which f decreases most.
6. If you were to go along such a direction, how far can you go before f starts to increase.
7. Starting from the initial point [3,3], find a direction in which f increases most.
8. If you were to go along such a direction, how far can you go before f starts to decrease.
9. Starting from [3,3], can you identify directions such that f does not change (locally)
10. Summarize your results in the form of the following table:

Function	λ_1	λ_1	Geometric Interpretation	Stationary point	Nature of Stationary point	Nature of Contours

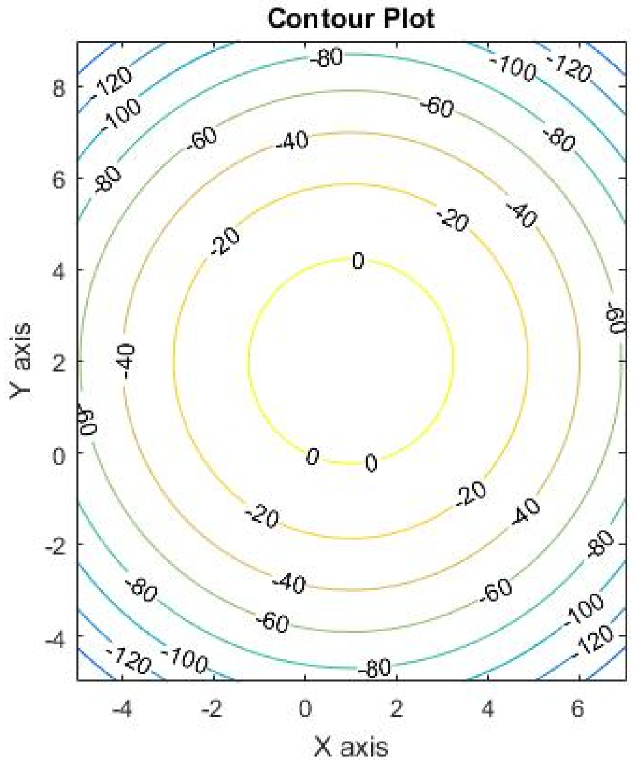
where λ_1 and λ_2 are the eigenvalues. Use keywords valley/hill/saddle to classify Geometric Nature.
Use keywords circle,ellipse, parabolas, straight lines

Answers

Problem 1

$$f(x) = \begin{pmatrix} 4 \\ 8 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} x$$

1. The function is concave because the Hessian matrix is Negative definite .
   The eigen values $\lambda_1, \lambda_2 = -4$.
2. $\nabla f = 0$ for $x^* = (1, 2)^T$.
3. Since the function is a quadratic function , Hessian is ND ,we say that x^* is Strict Global Maxima.
4. The contours are as follows:

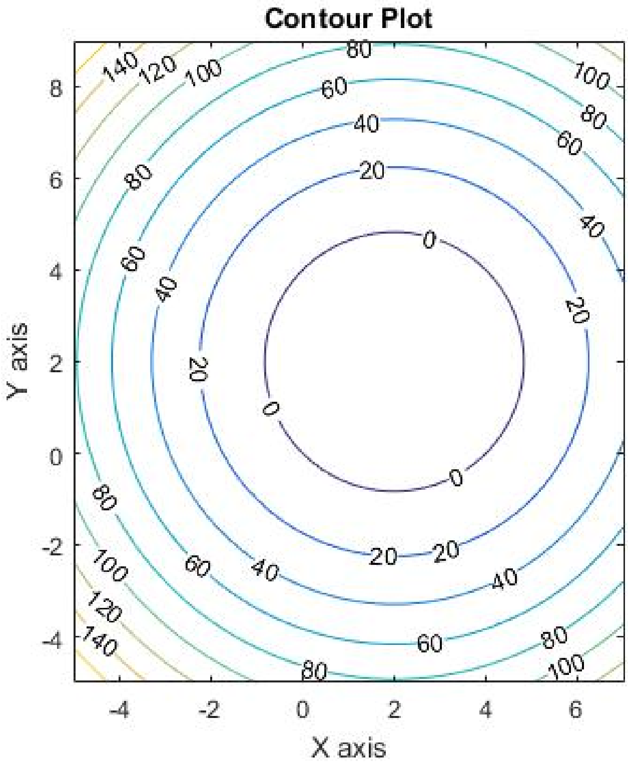
   

   The contours are circles since the eigen values are equal.
5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(8, 4)^T$.
6. The function is monotonically decreasing along positive α ,hence the function will continue to decrease for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} < 0$ always.)
7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3, 3)^T$ is $(-8, -4)^T$.
8. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.25 units$.
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction = (4,-8).

Problem 2

$$f(x) = \begin{pmatrix} -8 \\ -8 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} x$$

1. The function is concave because the Hessian matrix is Positive definite .
   The eigen values $\lambda_1, \lambda_2 = 4$.
2. $\nabla f = 0$ for $x^* = (2,2)^T$.
3. Since the function is a quadratic function , Hessian is PD ,we say that x^* is Strict Global Minima.
4. The contours are as follows:



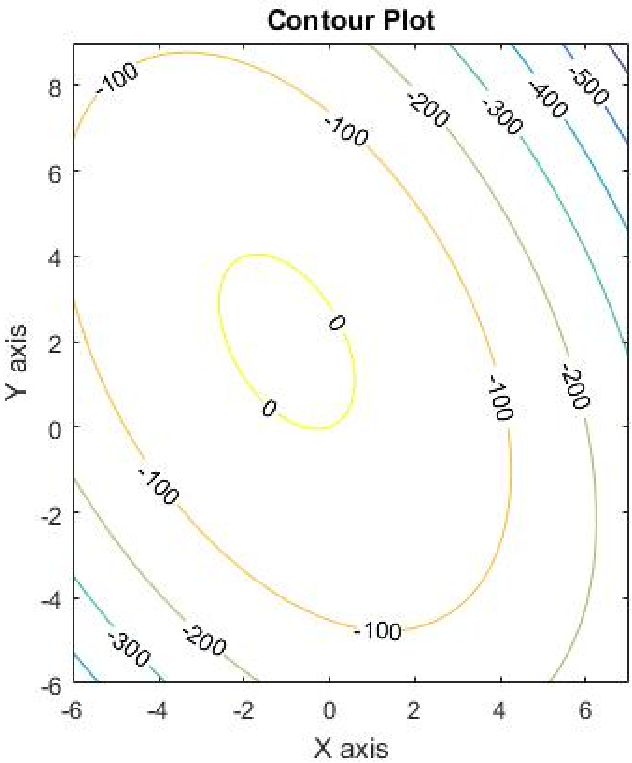
   The contours are circles since the eigen values are equal.

5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3,3)^T$ is $(-4,-4)^T$.
6. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.25 units$.
7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3,3)^T$ is $(4,4)^T$.
8. The function is monotonically increasing along positive $alpha$,hence the function will continue to increase for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} > 0$ always.)
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction = (-4,4).

Problem 3

$$f(x) = \begin{pmatrix} -3.07 \\ 8.54 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} -10 & -3.56 \\ -3.46 & -6 \end{pmatrix} x$$

1. The function is concave because the Hessian matrix is Negative definite .
   The eigen values $\lambda_1 = -11.9964, \lambda_2 = -4.0036$.
2. $\nabla f = 0$ for $x^* = (-0.9988, 1.9993)^T$.
3. Since the function is a quadratic function , Hessian is ND ,we say that x^* is Strict Global Maxima.
4. The contours are as follows:

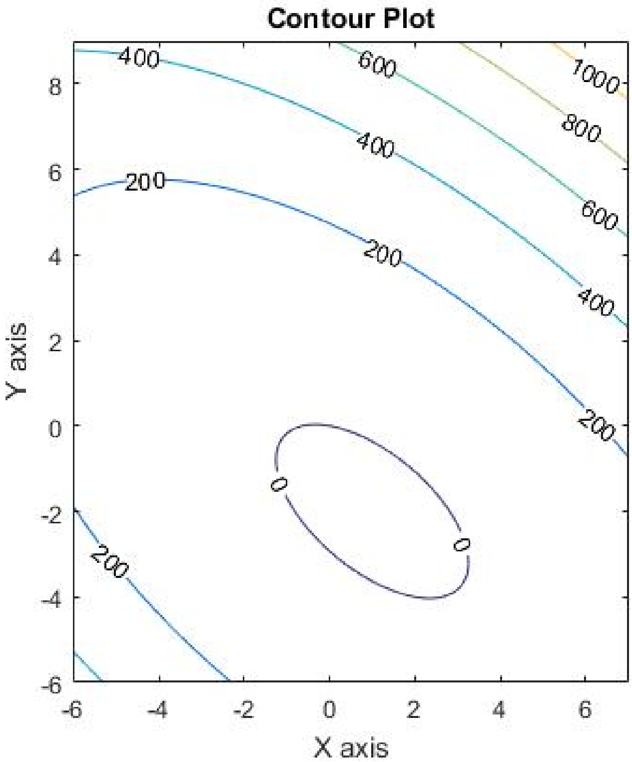
   

   The contours are ellipses since the eigen values are not equal.
5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3,3)^T$ is $(43.45, 19.84)^T$.
6. The function is monotonically decreasing along positive α ,hence the function will continue to decrease for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} < 0$ always.)
7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3,3)^T$ is $(-43.45, -19.84)^T$.
8. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.0839 units$.
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction = (19.84,-43.45).

Problem 4

$$f(x) = \begin{pmatrix} 2.86 \\ 16.17 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} 8.96 & 5.91 \\ 5.91 & 11.04 \end{pmatrix} x$$

1. The function is concave because the Hessian matrix is Positive definite .
   The eigen values $\lambda_1 = 3.9992, \lambda_2 = 16.0008$.
2. $\nabla f = 0$ for $x^* = (1, -2)^T$.
3. Since the function is a quadratic function , Hessian is PD ,we say that x^* is Strict Global Minima.
4. The contours are as follows:



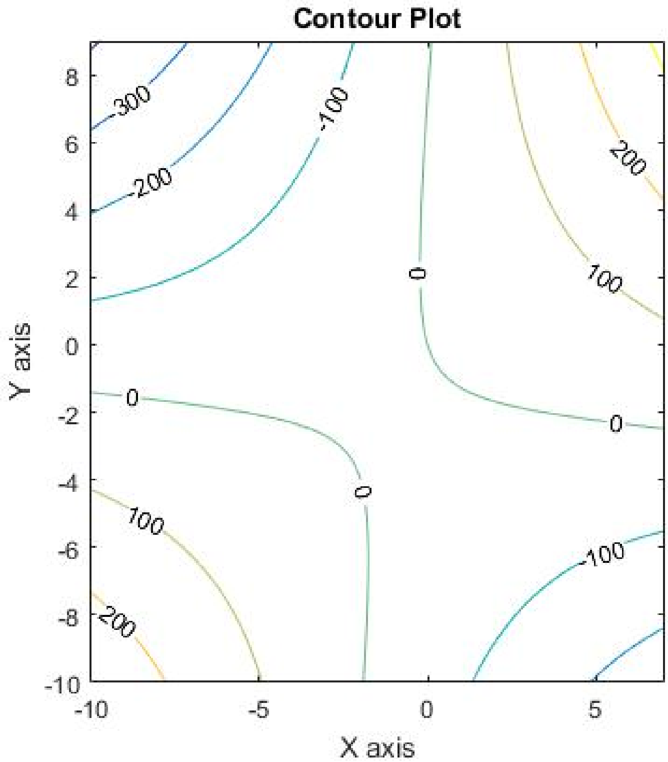
   The contours are ellipses since the eigen values are not equal.

5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(-47.47, -67.02)^T$.
6. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.0628 units$.
7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3, 3)^T$ is $(47.47, 67.02)^T$.
8. The function is monotonically increasing along positive α ,hence the function will continue to increase for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} > 0$ always.)
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction = (-67.02,47.47).

Problem 5

$$f(x) = \begin{pmatrix} 8.57 \\ 2.55 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} 0.69 & 3.94 \\ 3.94 & -0.69 \end{pmatrix} x$$

1. The function is neither concave nor convex because the Hessian matrix is Indefinite .
   The eigen values $\lambda_1 = -4, \lambda_2 = 4$.
2. $\nabla f = 0$ for $x^* = (-0.9975, -2.0004)^T$.
3. Since the function is a quadratic function , Hessian is Indefinite ,we say that x^* is a Saddle point.
4. The contours are as follows:

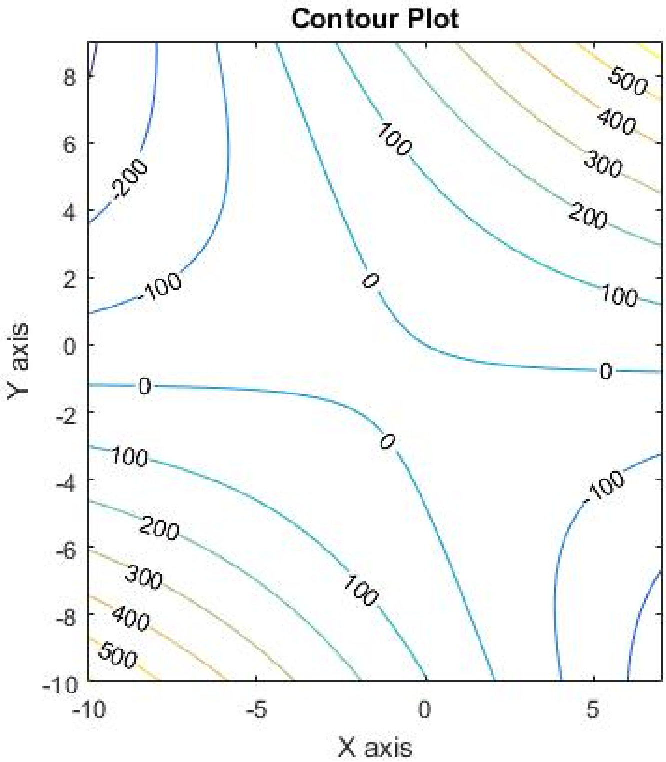
   

   Since the eigen values are not equal,not of same sign it is a hyperbola.
5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(-22.4600, -12.3000)^T$.
6. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.2709 units$.
7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3, 3)^T$ is $(22.4600, 12.3000)^T$.
8. The function is monotonically increasing along positive α ,hence the function will continue to increase for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} > 0$ at x^*.)
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f.
   Hence direction $= (-12.3, 22.46)^T$.

Problem 6

$$f(x) = \begin{pmatrix} 5.59 \\ 9.69 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} -0.05 & 5.64 \\ 5.64 & 4.05 \end{pmatrix} x$$

1. The function is neither concave nor convex because the Hessian matrix is Indefinite . The eigen values $\lambda_1 = -4.001, \lambda_2 = 8.001$.
2. $\nabla f = 0$ for $x^* = (-1, -1)^T$.
3. Since the function is a quadratic function , Hessian is Indefinite ,we say that x^* is a Saddle point.
4. The contours are as follows:



Since the eigen values are not equal,not of same sign it is a hyperbola.

5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(-22.3600, -38.7600)^T$.
6. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.1264 units$.
7. The direction of maximum increase is always ∇f. Hence the direction of maximum increase at point $(3, 3)^T$ is $(22.3600, 38.7600)^T$.
8. The function is monotonically increasing along positive α ,hence the function will continue to increase for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} > 0$ at x^*.)
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction $= (-38.7600, 22.3600)^T$.

Problem 7

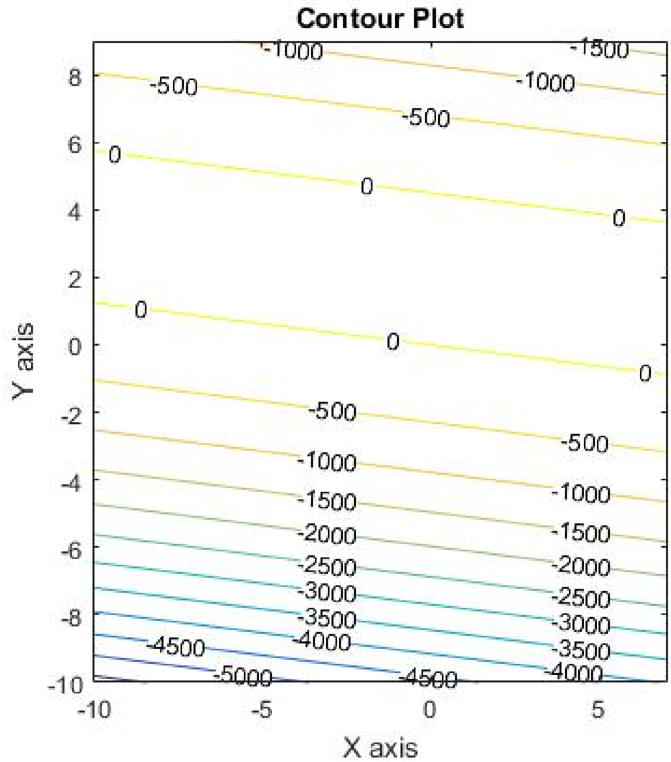
$$f(x) = \begin{pmatrix} 18 \\ 144 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} -1 & -8 \\ -8 & -64 \end{pmatrix} x$$

1. The function is non strict concave because the Hessian matrix is NSD .
   The eigen values $\lambda_1 = 0, \lambda_2 = -65$.

2. $\nabla f = 0$ for x^* along direction (-0.9923 ,0.1240).

3. Since the function is a quadratic function , Hessian is NSD ,we say that x^* is a non strict maxima.

4. The contours are as follows:



   Since one eigen value is zero,x^* exists we get straight lines.

5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3,3)^T$ is $(9.0000, 72.0000)^T$.

6. The function is monotonically decreasing along positive α ,hence the function will continue to decrease for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} < 0$ at x^*.)

7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3,3)^T$ is $(-9.0000, -72.0000)^T$.

8. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.0154 units$.

9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f.
   Hence direction $= (72.00, -9.00)^T$.

Problem 8

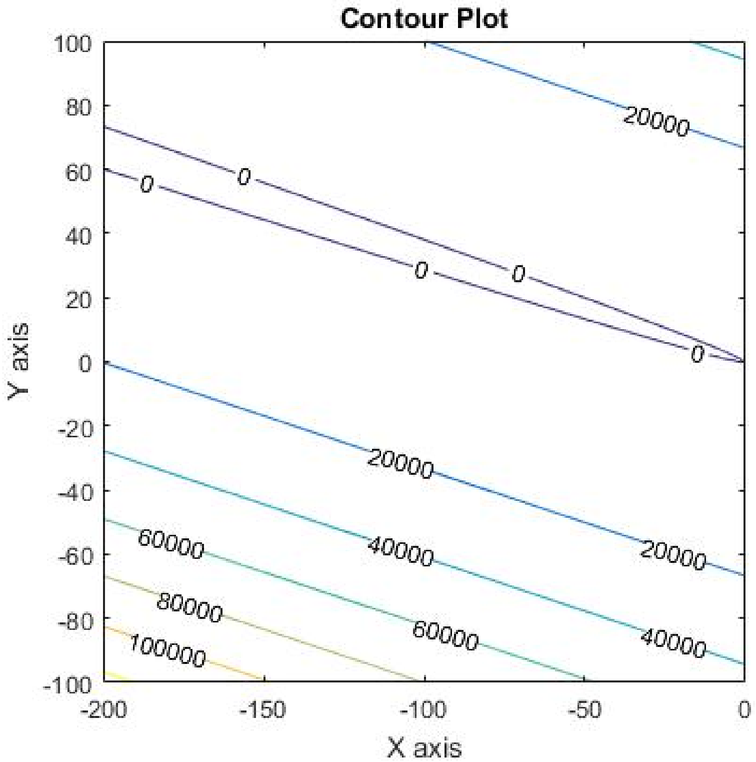
$$f(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} x$$

1. The function is non strict convex because the Hessian matrix is PSD .
   The eigen values $\lambda_1 = 0, \lambda_2 = 10$.

2. $\nabla f = 0$ for no x^*.

3. x^* doesn't exist.

4. The contours are as follows:

   Since one eigen value is zero,x^* doesn't exist ,the contour is a parabola.

5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(-13.0000, -36.0000)^T$.

6. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.1001 units$.

7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3, 3)^T$ is $(13.0000, 36.0000)^T$.

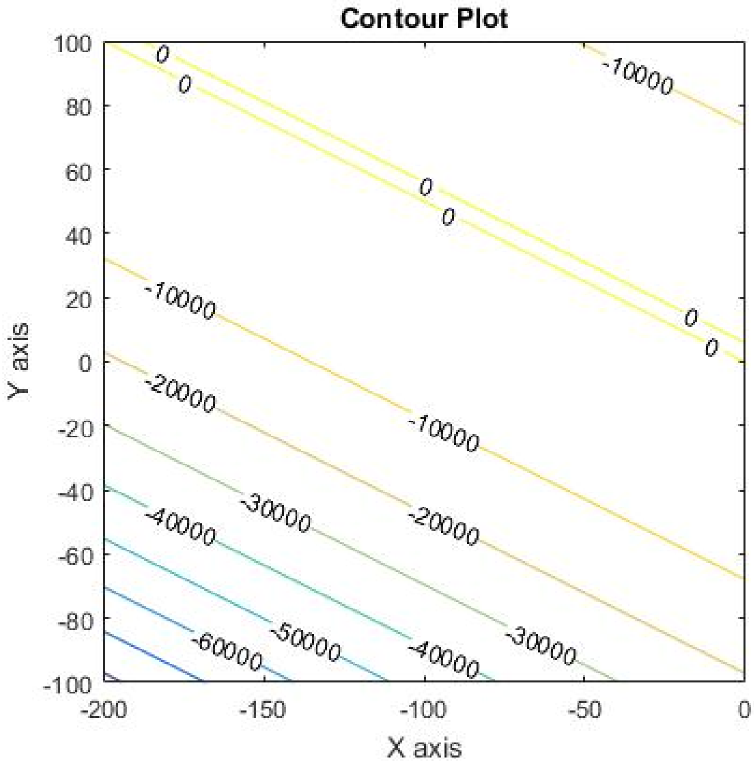
8. The function is monotonically increasing along positive α ,hence the function will continue to increase for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} > 0$.)

9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction $= (-36.00, 13.00)^T$.

Problem 9

$$f(x) = \begin{pmatrix} 6 \\ 12 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} x$$

1. The function is non strict concave because the Hessian matrix is NSD .
   The eigen values $\lambda_1 = -5, \lambda_2 = 0$.
2. $\nabla f = 0$ for x^* along direction (-0.8944 ,0.4472).
3. Since the function is a quadratic function , Hessian is NSD ,we say that x^* is a non strict maxima.
4. The contours are as follows:

   Since one eigen value is zero,x^* exists we get straight lines.
5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(3.0000, 6.0000)^T$.
6. The function is monotonically decreasing along positive α ,hence the function will continue to decrease for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} < 0$ at x^*.)
7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3, 3)^T$ is $(-3.0000, -6.0000)^T$.
8. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.2000 units$.
9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction $= (6.00, -3.00)^T$.

Problem 10

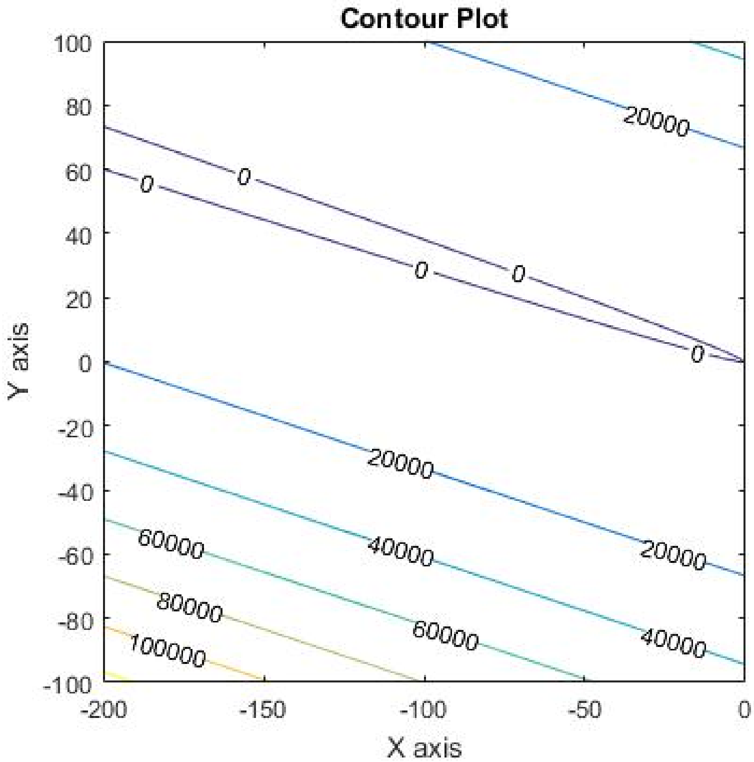
$$f(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T x + \frac{1}{2} x^T \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} x$$

1. The function is non strict convex because the Hessian matrix is PSD .
   The eigen values $\lambda_1 = 0, \lambda_2 = 26$.

2. $\nabla f = 0$ for no x^*.

3. x^* doesn't exist.

4. The contours are as follows:



   Since one eigen value is zero,x^* doesn't exist ,the contour is a parabola.

5. The direction of maximum decrease is always $-\nabla f$. Hence the direction of maximum decrease at point $(3, 3)^T$ is $(-19.0000, -90.0000)^T$.

6. Maximum distance $\alpha = -\frac{\nabla f^T p}{p^T H p}$.Hence $\alpha = 0.0385 units$.

7. The direction of maximum increase is always ∇f.
   Hence the direction of maximum increase at point $(3, 3)^T$ is $(19.0000, 90.0000)^T$.

8. The function is monotonically increasing along positive α ,hence the function will continue to increase for all positive displacements along the line. (ie $\frac{\delta^2 f}{\delta \alpha^2} > 0$.)

9. The direction at which there is no local change to $f(x)$ is the direction perpendicular to ∇f. Hence direction $= (-90.00, 19.00)^T$.

Consolidated data

S.No	λ_1	λ_1	Geometric Interpretation	Stationary point	Nature of Stationary point	Contours
1	-4	-4	Circular Hill	$(1,2)^T$	Strict Global Maxima	Circle
2	4	4	Circular Valley	$(2,2)^T$	Strict Global Minima	Circle
3	-11.99	-4.00	Elliptic Hill	$(-0.99, 1.99)^T$	Strict Global Maxima	Ellipse
4	3.99	16.00	Elliptic Valley	$(3.99, 16.00)^T$	Strict Global Minima	Ellipse
5	-4	4	Saddle	$(-0.99, -2.00)^T$	Saddle	Hyperbola
6	-4.00	8.00	Saddle	$(-1, -1)^T$	Saddle	Hyperbola
7	0	-65	Hill	$(-0.99, 0.12)^T$	Minima	St Lines
8	0	10	Valley	NULL	NULL	Parabola
9	-5	0	Hill	$(-0.89, 0.44)^T$	Maxima	St Lines
10	0	26	Valley	NULL	NULL	Parabola