COMS20011: Symbols, Patterns and Signals

Problem Sheet: Maximum likelihood

1. You were consulted by a Physics student who is trying to estimate the voltage (V) given current (I) and resistance (R) information. The student informs you that the physical model is
$$I = \frac{V}{R} + \epsilon$$
subject to a measurement error $\epsilon \sim \mathcal{N}(0, \sigma^2)$. Assuming *i.i.d* observations, the likelihood of $p(D|V)$ where V is the model's only parameter and D is the observed data is equal to:
$$p(D|V) = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(I_i - \frac{V}{R_i})^2}{\sigma^2}}$$
where I_i is the current value for observation i and R_i is the resistance value for observation i. Show that the Maximum Likelihood value of V is $V_{ML} = \sum \frac{I_i}{R_i} / \sum \frac{1}{R_i^2}$ (differentiate $\log p(D|V)$ wrt to the parameter, V, then find the maximum by solving for the setting of V for which the gradient is zero)

2. For a given probabilistic model,
$$p(D|\theta) = b\ e^{-(3-\theta)^2}$$
where b is a normalising constant and a known prior of
$$p(\theta) = c\, e^{-\theta(\theta-1)}$$
where c is a normalising constant, Find the maximum a posteriori value of θ (the posterior is $p(\theta|D) \propto P(\theta)P(D|\theta)$; find the maximum by computing the gradient of $\log p(\theta|D)$ and solving for where the gradient is zero). **Clearly show the steps you followed in finding the answer.**

3. Suppose that X is a discrete random variable with the following probability mass function, where $0 \leq \theta \leq 1$ is a parameter.

X	0	1	2	3
$P(X)$	$\frac{2\theta}{3}$	$\frac{\theta}{3}$	$\frac{2(1-\theta)}{3}$	$\frac{(1-\theta)}{3}$

The following 10 independent observations were taken from this distribution:

3	0	2	1	3	2	1	0	2	1

(a) What is the Maximum Likelihood estimate of θ

(b) Assume you have prior knowledge that $p(\theta) = b\,\theta\,(1-\theta)$, what would the Maximum a Posteriori (MAP) be?