A F Mujibur Rahman Foundation-Bangladesh Mathematical Society

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Questions and Solutions for Chattogram and Barishal Region

C+B-1: xoy-axes are rotated (origin fixed) through an angle $\theta = 45^\circ$ and is formed $x'oy'$-axes. Sketch the equation of the curve $x^2 - xy + y^2 = 2$ showing both xoy and $x'oy'$-axes.

Solution: $x^2 - xy + y^2 = 2$ ---------------------------- (1)

We know that, $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta - y'\cos\theta$

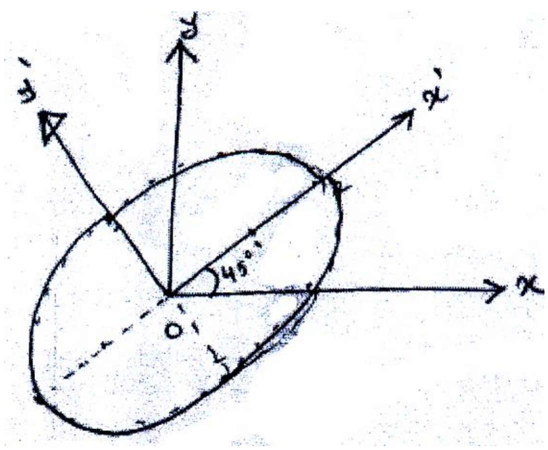
Here, $\theta = 45^0$

$\therefore\ x = \frac{1}{\sqrt{2}}(x' - y')$, $\qquad y = \frac{1}{\sqrt{2}}(x' + y')$

$(1) \Rightarrow\ x'^2 + 3y'^2 = 4$

or, $\frac{x'^2}{2^2} + \frac{y'^2}{\left(\frac{R}{\sqrt{3}}\right)^2} = 1$ ----------------------- (2)

Horizontal ellipse.



C+B-2: Suppose the vector point function $\vec{F} = (yz\sec^2 x - 2xy)\hat{i} + (z\tan x - x^2 + 2y)\hat{j} + (y\tan x - 3)\hat{k}$ is irrotational. Find a scalar point function $\phi(x, y, z)$ such that $\vec{F} = \vec{\nabla}\phi$.

Solution: Let $\varphi = \varphi(x,\ y,\ z)$

$$\therefore\ d\varphi=\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy+\frac{\partial\varphi}{\partial z}dz$$

$$=\left(\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j}+\frac{\partial\varphi}{\partial z}\hat{k}\right).\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)$$

$$=\underline{\nabla}\varphi\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)$$

$$=\underline{F}\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)\qquad\qquad \because\ =\underline{F}=\underline{\nabla}\varphi$$

$$=\left(yz\sec^2 x-2xy\right)dx+\left(z\tan x-x^2 2y\right)dy+\left(y\tan x-3\right)dz$$

$$=\left(yz\sec^2 xdx+z\tan dy+y\tan xdz\right)+2ydy-\left(2xydx+x^2dy\right)-3dz$$

$$d\varphi=d\left(yz\tan x\right)+2ydy-d\left(x^2y\right)-3dz$$

or, $\int d\varphi=\int d\left(yz\tan x\right)+\int 2ydy-\int d\left(x^2y\right)-\int 3dz$

$$\therefore\ \varphi=yz\tan x+y^2-x^2y-3z+c$$

C+B-3: Find all integer solutions of the system of equations

$$x+y+z=3$$

$$x^3+y^3+z^3=3$$

Solution: $(x,\ y,\ z)\to$ solution

$$(x+y+z)^3-\left(x^3+y^3+z^3\right)=3(x+y)(y+z)(z+x)$$

$$\Rightarrow\ 8=(3-z)(3-x)(3-y)$$

Since $6=(3-z)+(3-x)+(3-y)$ factorization of 8, the solutions are

(1, 1, 1), (–5, 4, 4), (4, –5, 4), (4, 4, –5).

C+B-4: Prove that the average of the numbers $n\sin n^0\ (n=2,\ 4,\ 6,\ \cdots,\ 180)$ is $\cot 1^0$.

Solution: Clain: $2\sin 2+4\sin 4+\cdots+178\sin)78=90\cot 1$

$$\Rightarrow 2\sin 2.\sin 1+2\left(2\sin 4.\sin 1\right)+\cdots+89\left(2\sin 178.\sin 1\right)=90\cos 1$$

Let $= 2\sin 2.\sin 1 + 2(2\sin 4.\sin 1) + \cdots + 89(2\sin 178.\sin 1)$

$= (\cos 1 - \cos 3) + 2(\cos 3 - \cos 5) + \cdots + 89(\cos 177 - \cos 179)$

$= \cos 1 + \cos 3 + \cdots + \cos 177 - 89\cos 179$

$= \cos 1 + (\cos 3 + \cos 177) + \cdots + (\cos 89 + \cos 1) - 89\cos 179$

$= \cos 1 + 49\cos 1 = 9\cos 1$ = RHS.

C+B-5: Find the volume of the solid made up by the revolution of the area bounded by $y = x^3, y = x$ and $x = 2$ about the line$x = 2$.

Solution: Volume of the solid revolving the curve $y = x^3$ about the line $x = 2$ from $y = 1$ to $y = 8$ is $\int_1^8 \pi\left(1 - y^{\frac{1}{3}}\right)^2 dy = \frac{31\pi}{10}$ and the volume of the solid revolving the line $y = x$ about $x = 2$ from $y = 1$ to $y = 2$ is $\frac{\pi}{3}$. So the required volume is $\frac{31\pi}{10} - \frac{\pi}{3} = \frac{81\pi}{30}$ $unit^3$

C+B-6: Find the value of $\prod_{i=0}^{n}(1+\frac{i^5}{n^5})^{\frac{i^4}{n^5}}$, as $\to \infty$.

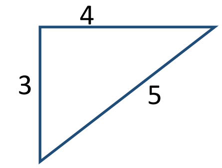
Solution: $Exp\left[\lim_{n\to\infty}\sum_{i=0}^{n}\frac{i^4}{n^5}\ln\left(1+\frac{i^5}{n^5}\right)\right] = Exp\left[\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{n}\frac{i^4}{n^4}\ln\left(1+\frac{i^5}{n^5}\right)\right]$

$$= Exp\left[\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{i^4}{n^4}\ln\left(1+\frac{i^5}{n^5}\right)\right] = Exp\left[\int_0^1 x^4 \ln(1+x^5)dx\right]$$

$$= Exp\left[\frac{1}{5}\ln(2) - \frac{1}{5}\ln(1)\right] = e^{\ln 2^5} = 64$$

C+B-7: Find the length of the curve along the diameter of the circle$x + 2y - 2z - 2 = 0$, $x^2 + y^2 + z^2 - 2x - 8y + 2y - 7 = 0$ on the sphere $x^2 + y^2 + z^2 - 2x - 8y + 2y - 7 = 0$ cut by the plane$x + 2y - 2z - 2 = 0$.

Solution: Centre of the sphere: $(1,4,-1)$, radius $= 5$, and the distance of the given plane from the centre of the sphere is 3. So the radius of the circle is $\sqrt{5^2-3^2}=4$.



Angle between 3 and 5 is $\tan^{-1} 4/3$. So the length is $10\tan^{-1}\frac{4}{3}$

C+B-8: Let $f: R \to R$ be a twice differentiable function. Suppose $f(0)=0$. Prove that there exist $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f''(c) = f(c)\left(1+\tan^2 c\right)$.

Solution: Let $g(x) = f(x)\cos x$.

Since $g\left(-\frac{\pi}{2}\right) = g(0) = g\left(\frac{\pi}{2}\right) = 0$, by Rolle's theorem there exists some $c_1 \in \left(-\frac{\pi}{2}, 0\right)$ and $c_2 \in \left(0, \frac{\pi}{2}\right)$ such that $g'(c_1) = g'(c_2) = 0$

Now consider the function $h(x) = \dfrac{g'(x)}{\cos^2 x} = \dfrac{f'(x)\cos x - f(x)\sin x}{\cos^2 x}$

We have, $h(c_1) = h(c_2) = 0$, so by Rolle's theorem there exist $c \in (c_1, c_2)$ for which

$$0 = h'(c) = \frac{g''(c)\cos^2 c + 2\cos c \sin c g'(c)}{\cos^4 c}$$

$$= \frac{\left(f''(c)\cos c - 2f'(c)\sin c - f(c)\cos c\right)\cos c + 2\sin c\left(f'(c)\cos c - f(c)\sin c\right)}{\cos^3 c}$$

$$= \frac{f''(c)\cos^2 c - f(c)(\cos^2 c + 2\sin^2 c}{\cos^3 c}$$

$$0 = \frac{1}{\cos c} f''(c) - f(c)\left(1+2\tan^2 c\right)$$

$$\Rightarrow f''(c) = f(c)(1+2\tan^2 c) \text{ (Proved)}$$

C+B-9: If $n \in N$ and $0 < a < 1$ then show that $(1-na) < (1-a)^n < \dfrac{1}{1+na}$

Solution: We know from Weiertrass's Inequality.

$$(1-a_1)(1-a_2)\cdots(1-a_n) > 1-s$$

$$\Rightarrow (1-a_1)(1-a_2)\cdots(1-a_n) \geq 1-(a_1+a_2+\cdots a_n) \qquad [\text{where} \quad s = a_1+a_2+\cdots+a_n]$$

Let $a_1 = a_2 = \cdots = a_n = a$

$$(1-a)(1-a)\cdots(1-a) > 1-(a+a+\cdots+a)$$

$$\Rightarrow (1-a)^n > (1-na)$$

$$\Rightarrow (1-na) < (1-a)^n \text{ -- (i)}$$

Again we also know

$$(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+s}$$

$$(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+(a_1+a_2+\cdots+a_n)}$$

Let $a_1 = a_2 = \cdots = a_n = a$

then $(1-a)(1-a)\cdots(1-a) < \dfrac{1}{1+(a+a+\cdots+a)}$

$$\Rightarrow (1-a)^n < \frac{1}{1+na} \text{ -- (ii)}$$

Equation (1) and Equation (ii)

$$(1-na) < (1-a)^n < \frac{1}{1+na}$$

C+B-10: A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass in released from a point 8 inche below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of motion.

Solution: Because we are using the engineering system of units, the measurements given in terms of inches must be converted into feet: 6 in. = $\frac{1}{2}$ ft; 8 in. = $\frac{2}{3}$ ft. In additon, we must convert the units of weight given in pounds into units of mass. From $m = W/g$ we have $m = \frac{2}{32} = \frac{1}{16}$

slug. Also, from Hooke's law, $2 = k\left(\frac{1}{2}\right)$ implies that the spring constant i k = 4 lb/ft. Hence (1) given

$$\frac{1}{16}\frac{d^2x}{dt^2} = -4x \quad \text{or} \quad \frac{d^2x}{dt^2} + 64x = 0.$$

The initial displacement and initial velocity are $x(0) = \frac{2}{3}, x'(0) = -\frac{4}{3}$, where the negative sign in the last condition is a consequence of the fact that the mass is given an initial velocity in the negative, or upward, direction.