MT240:Complex Variables and Transforms

# Homework 6

Due Date: 24, July ,2021
Max Marks : 100
*Spring 2021*

Tips to avoid plagiarism:

- Do not copy the solutions of your classmates.
- You are encouraged to discuss the problems with your classmates in whatever way you like but, make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

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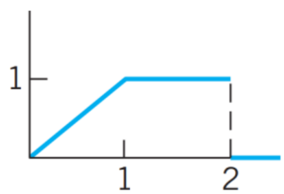
## Problem 1

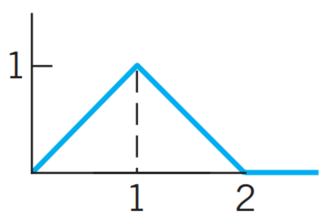
Find the Laplace transform of the following functions given below . Show detail of your work.

a) $z(t) = e^{2t} sinht$

b)




c)




## Problem 2

Laplace transforms of functions are given below. Find their inverse Laplace transforms f(t), where L and n are constants . Show detail of your work .

a) $\frac{s}{L^2s^2+n^2\pi^2}$

b) $\frac{s+10}{s^2-s-2}$

## Problem 3

Find the Laplace transform for part (a) and find inverse transform for part (b). Show details of your work.

a) $0.5e^{-4.5t}sin2\pi t$
b) $\frac{2s-1}{s^2-6s+18}$

## Problem 4

Solve IVPs by Laplace Transform . Show all details.

a) $y'' - \frac{1}{4}y = 0 \quad y(0) = 12, \quad y'(0) = 0$
b) $y'' - 4y' + 3y = 6t - 8 \quad y(0) = 0, \quad y'(0) = 0$

## Problem 5

Solve the shifted data IVPs by Laplace Transform.

$$y'' + 3y' - 4y = 6e^{2t-3}$$

Where $y'(1.5) = 5$ and $y(1.5) = 4$.

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CVT Assignment 6

Q.1

(a) $z(t) = e^{2t} \sinh t$

Solve

$$= e^{2t}\left[\frac{e^{t} - e^{-t}}{2}\right] \Rightarrow \frac{1}{2}(e^{3t} - e^{t})$$

$$z(t) = \frac{e^{3t}}{2} - \frac{e^{t}}{2} \quad \text{--- (1)}$$

laplace of 'z(t) $= \mathcal{L}\left[\frac{e^{3t}}{2} - \frac{e^{t}}{2}\right]$

$$= \frac{1}{2}\left[\mathcal{L}(e^{3t}) - \mathcal{L}(e^{t})\right]$$

$$z(s) = \frac{1}{2}\left[\frac{1}{s-3} - \frac{1}{s-1}\right]$$

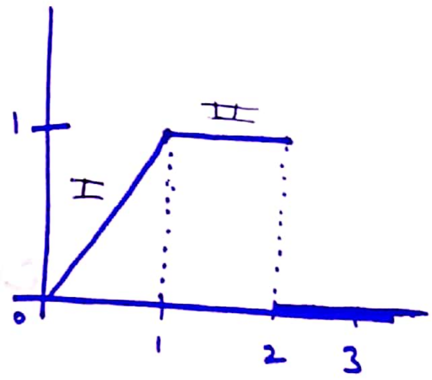
$$\boxed{z(s) = \frac{1}{2(s-3)} - \frac{1}{2(s-1)}}$$ Answere.

ⓑ

Solve $y = y_{I} + y_{II}$

$y_{I} = \left[\frac{1-0}{1-0}\right]x + 0$, $\boxed{y_{I} = t} = x$

$y_{II} = 1$




$y_{I} = t(u(t))$ , $y_{II} = u(t-1)$

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^1 t e^{-st} dt + \int_1^2 e^{-st} dt +$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{-s}\left[e^{-2s} - e^{-s}\right]$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s} + \frac{e^{-s}}{s}$$

$$= \frac{e^{-s}}{s}\left[-e^{-s} - \frac{1}{s}\right] = -\frac{e^{-s}}{s}\left(e^{-s} + \frac{1}{s}\right) \text{ Ans}$$

$$\boxed{F(s) = -\frac{e^{-2s}}{s} - \frac{e^{-s}}{s^2}}$$

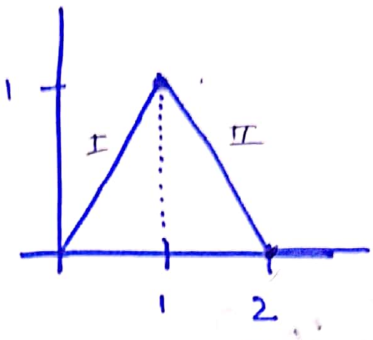
(c)

Solve

for (II), $y = \left\{\frac{0-1}{2-1}\right\}x + c$

$y = -x + c \quad \therefore (2,0),\ c = 2$

$y = -x + 2$




$$f(x) = \begin{cases} x & 0<x<1 \\ -x+2 & 1<x<2 \end{cases} \Rightarrow \begin{cases} t & 0<t<1 \\ -t+2 & 1<t<2 \end{cases} = f(t)$$

$$\mathcal{L}(f(t)) = \mathcal{L}(f(t)) = \int_{-\infty}^{\infty} f(t)$$

$$\mathcal{L}(f(t)) = \int_0^1 t e^{-st} dt - \int_1^2 t e^{-st} + \int_1^2 2e^{-st}$$

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^1 - \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2 + 2\left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \left[\frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}\right] + 2\left[\frac{e^{-2s}}{-s} + \frac{e^{-s}}{s}\right]$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s} + \frac{2e^{-s}}{s}$$

$$= -\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$

$$F(s) = \mathcal{L}(f(t)) = \frac{1}{s^2}\left(e^{-2s} - 2e^{-s}\right)$$ Answer

Q: 2

ⓐ $\dfrac{s}{L^2 s^2 + n^2\pi^2}$

Solve

$$f(s) = \frac{s}{L^2 s^2 + n^2\pi^2} = \frac{s}{L^2\left[s^2 + \frac{n^2\pi^2}{L^2}\right]}$$

$$f(s) = \frac{1}{L^2}\left[\frac{s}{(s)^2 + \left[\frac{n\pi}{L}\right]^2}\right]$$

$$f(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left[\frac{1}{L^2}\left[\frac{s}{(s)^2 + \left(\frac{n\pi}{L}\right)^2}\right]\right]$$

$$f(t) = \frac{1}{L^2}\left\{\mathcal{L}^{-1}\left(\frac{s}{(s)^2 + \left(\frac{n\pi}{L}\right)^2}\right)\right\}$$

$\therefore\ \mathcal{L}\sin\omega t = \dfrac{\omega}{s^2+\omega^2}$

$\therefore\ \mathcal{L}\cos\omega t = \dfrac{s}{s^2+\omega^2}$

$\therefore\ \mathcal{L}^{-1}\left(\dfrac{s}{s^2+\omega^2}\right) = \cos\omega t$

$$= \frac{1}{L^2}\left[\cos\left(\frac{n\pi}{L}\right)t\right]$$

$$\boxed{f(t) = \frac{\cos\left[\frac{n\pi}{L}\right]t}{L^2}}$$ Ans

$$f(t) = \frac{1}{L^2}\cos\left(\frac{n\pi}{L}t\right)u(t)$$ Ans

(b) $\dfrac{S+10}{S^2-S-2}$

Solve
$$f(s) = \frac{S}{S^2-S-2} + \frac{10}{S^2-S-2}$$

$$= \frac{S}{S^2-S+(0.5)^2-2.25} + \frac{10}{S^2-S+(0.5)^2-2.25}$$

$$= \frac{S}{(S-0.5)^2-(1.5)^2} + \frac{10}{(S-0.5)^2-(1.5)^2}$$

$$f(s) = \frac{S+10}{(S-2)(S+1)} = \frac{A}{S-2} + \frac{B}{S+1}$$ $\therefore A = 4$, $\therefore B = -3$

$$f(s) = \frac{4}{S-2} - \frac{3}{S+1}$$

Now Taking inverse laplace on both sides.

$$\mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left[\frac{4}{S-2} - \frac{3}{S+1}\right]$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{4}{S-2}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{S+1}\right]$$

$$f(t) = 4e^{2t}u(t) - 3e^{-t}u(t)$$

$$\boxed{f(t) = (4e^{2t} - 3e^{-t})\,u(t) \quad \text{A}}$$

Q:: 3

(a) $0.5\, e^{-4.5t} \sin 2\pi t$

Solve

Let $f(t) = \sin 2\pi t$

$$f(s) = \frac{2\pi}{s^2 + (2\pi)^2} \quad \text{—— (1)}$$

$$f(s-(-4.5)) = \frac{2\pi}{(s-(-4.5))^2 + (2\pi)^2}$$

$$f(s+4.5) = \frac{2\pi}{(s+4.5)^2 + (2\pi)^2}$$

$$\mathcal{L}(0.5\, e^{-4.5t} \sin 2\pi t) = 0.5 \left[\frac{2\pi}{(s+\frac{9}{2})^2 + 4\pi^2}\right]$$

$$F(s) = \frac{\pi}{s^2 + \frac{81}{4} + 9s + 4\pi^2} \quad \text{—— (2)}$$

$$F(s) = \cancel{\frac{9\pi}{9s^2+81+}} = \frac{9\pi}{9s^2 + s + \frac{9}{4} + \frac{4\pi^2}{9}}$$

$$F(s) = \cancel{\frac{9\pi}{(3s+}} = \frac{\pi}{s^2 + 9s + \frac{81}{4} + 4\pi^2} \quad \text{:: from (2)}$$

$$= \frac{4\pi}{4s^2 + 36s + 81 + \pi^2} = \frac{4\pi}{((2s)^2 + 9^2 + 2(2)(9)) + \pi^2}$$

$$\boxed{F(s) = \frac{4\pi}{(2s+9)^2 + \pi^2}}$$

Ans

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(b) $\dfrac{2S-1}{S^2-6S+18}$

Solve $= \dfrac{2S-1}{S^2-2(3)(1)S+9+9} = \dfrac{2S-1}{(S-3)^2+3^2}$

$$= \frac{2S}{(S-3)^2+3^2} - \frac{1}{(S-3)^2+3^2}$$

$$f(S) = 2\left[\frac{S}{(S-3)^2+3^2}\right] - \frac{1}{(S-3)^2+3^2}$$

Now taking the inverse laplace of $f(S)$.

$$\mathcal{L}^{-1}(f(S)) = \mathcal{L}^{-1}\left[2\left[\frac{S}{(S-3)^2+3^2}\right] - \frac{1}{(S-3)^2+3^2}\right]$$

$$= \mathcal{L}^{-1}\left[\left[2\left[\frac{S\cancel{+}3}{(S-3)^2+3^2} + \frac{3}{(S-3)^2+3^2}\right]\right] - \frac{1}{(S-3)^2+3^2}\right]$$

$$= \mathcal{L}^{-1}\left[2\left(\frac{(S-3)}{(S-3)^2+3^2} + \frac{3}{(S-3)^2+3^2}\right) - 0.333\left[\frac{3}{(S-3)^2+3^2}\right]\right]$$

$\because \mathcal{L}(e^{at}\cos wt) = \dfrac{S-a}{(S-a)^2+w^2}$ $\qquad \because \mathcal{L}(e^{at}\sin wt) = \dfrac{w}{(S-a)^2+w^2}$

$$f(t) = 2\left[e^{3t}\cos 3t + e^{3t}\sin 3t\right] - 0.333\, e^{3t}\sin 3t$$

$$f(t) = e^{3t}\left[2\cos 3t + 2\sin 3t - 0.333\sin 3t\right]$$

$$f(t) = e^{3t}\left[2\cos 3t + 1.666\sin 3t\right] \text{ A}$$

Q. 4

ⓐ $y'' - \frac{1}{4}y = 0$ $\qquad y(0) = 12 \quad , \; y'(0) = 0$

Solve

$$y'' - \frac{y}{4} = 0$$

Taking laplace transfrom on b/s

$$s^2 Y - s y(0) - y'(0) - \frac{Y}{4} = 0$$

$$s^2 Y - 12s - 0 - \frac{Y}{4} = 0$$

$$Y\left[s^2 - \frac{1}{4}\right] = 12s \quad , \quad Y = \frac{12s}{s^2 - \frac{1}{4}}$$

$$Y = 12\left[\frac{s}{s^2 - \left(\frac{1}{2}\right)^2}\right] = \frac{12s}{\left(s+\frac{1}{2}\right)\left(s-\frac{1}{2}\right)} = \frac{A}{s+\frac{1}{2}} + \frac{B}{s-\frac{1}{2}}$$

$$Y = 12s = A\left(s - \frac{1}{2}\right) + B\left(s + \frac{1}{2}\right) \quad \therefore \; A = 6 \, , \; B = 6$$

$$Y = \frac{6}{s+\frac{1}{2}} + \frac{6}{s-\frac{1}{2}} \quad \text{—— ①}$$

Now taking the inverse lapplace on b/s

$$y(t) = 6e^{-\frac{t}{2}} + 6e^{t/2}$$

$$y(t) = 6\left(e^{-t/2} + e^{t/2}\right) = 12\left[\frac{e^{t/2} + e^{-t/2}}{2}\right]$$

$$\boxed{y(t) = 12 \cosh \frac{t}{2}} \quad \text{or} \quad \boxed{12 \cos (0{\cdot}5ht)}$$

(b) $y'' - 4y' + 3y = 6t - 8$ $\quad y(0)=0, \; y'(0)=0$

Solve

$$y'' - 4y' + 3y = 6t - 8$$

Taking laplace transform on b/s

$$s^2Y - sy(0) - y'(0) - 4sY + 4y(0) + 3Y = \frac{6}{s^2} - \frac{8}{s}$$

substituting initial conditions

$$s^2Y - 0 - 0 - 4sY + 0 + 3Y = \frac{6 - 8s}{s^2}$$

$$Y[s^2 - 4s + 3] = \frac{6 - 8s}{s^2}$$

$$Y = \left[\frac{6-8s}{s^2}\right] \times \left[\frac{1}{s^2-4s+3}\right] = \left(\frac{6-8s}{s^2}\right)\left[\frac{1}{(s-1)(s-3)}\right] -$$

$$6 - 8s = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3} \quad \text{— (1)}$$

$A = 0, \; B = 2, \; C = 1, \; D = -1$

$$Y = \frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s-3} \quad \text{— (2)}$$

Taking inverse laplace on b/s

$$y(t) = 2t + e^{t} - e^{3t}$$

$$y(t) = (2t + e^{t} - e^{3t})\, u(t)$$

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Q5

$$y'' + 3y' - 4y = 6e^{2t-3}$$

$y'(1.5) = 5$ , $y(1.5) = 4$

Solve

$$y'' + 3y' - 4y = 6e^{2t-3} \quad \text{--- (1)}$$

Taking laplace on b/s

$\cdot\; s^2 Y - s y(0)$

Let $t_0 = 1.5$

$$t = \tilde{t} + 1.5 \quad \text{--- (2)}$$

(1) becomes

$$\tilde{y}'' + 3\tilde{y}' - 4\tilde{y} = 6e^{2(\tilde{t}+1.5)-3}$$

$$\tilde{y}'' + 3\tilde{y}' - 4\tilde{y} = 6e^{2\tilde{t}} \quad \text{--- (3)}$$

Taking laplace on b/s

$$s^2\tilde{Y} - s\tilde{Y}(0) - \tilde{y}'(0) + 3s\tilde{Y} - 3\tilde{y}(0) - 4\tilde{Y} = \frac{6}{s-2}$$

$$s^2\tilde{Y} - s[4] - 5 + 3s\tilde{Y} - 3(4) - 4(\cancel{5})\tilde{Y} = \frac{6}{s-2}$$

$$s^2\tilde{Y} - s4 - 5 + 3s\tilde{Y} - 20 - 4\tilde{Y} = \frac{6}{s-2}$$

$$\tilde{Y}[s^2 + 3s - 4] = \frac{6}{s-2} + 4s + 25$$

$$\tilde{Y} = \frac{6 + 4s^2 - 8s + 25s - 50}{s-2} \times \frac{1}{s^2 + 3s - 4}$$

$$\tilde{Y} = \frac{4s^2 + 17s - 44}{(s-2)(s-1)(s+4)}$$

$$\tilde{Y} = \frac{1}{s-2} + \frac{23}{5(t-1)} - \frac{8}{5(t+4)}$$

$$\tilde{Y} = \frac{1}{s-2} + \frac{23}{5} \times \frac{1}{(s-1)} - \frac{8}{5} \times \frac{1}{(s+4)} \quad \text{—— (4)}$$

Taking inverse laplace on b/s

$$\tilde{y} = e^{+2\tilde{t}} + \frac{23}{5} e^{\tilde{t}} - \frac{8}{5} e^{-4\tilde{t}}$$

$$\tilde{y} = e^{2\tilde{t}} + \frac{23}{5} e^{\tilde{t}} - \frac{8}{5} e^{-4\tilde{t}} \quad \text{—— (5)}$$

using eq (2)

$$\tilde{t} = t - 1.5$$

(5) becomes

$$y(\tilde{t}) = e^{2(t-1.5)} + \frac{23}{5} e^{(t-1.5)} - \frac{8}{5} e^{-4(t-1.5)}$$

$$y(\tilde{t}) = e^{2t-3} + \frac{23}{5} e^{t-1.5} - \frac{8}{5} e^{-4t+6}$$

$$\boxed{y(t-1.5) = e^{2t-3} + \frac{23}{5} e^{t-1.5} - \frac{8}{5} e^{-4t+6}} \quad A$$

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