MT240:Complex Variables and Transforms

Homework 2

Due Date: 5 May ,2021
Max Marks : 100

Spring 2020

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- You are encouraged to discuss the problems with your classmates in whatever way you like but, make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

Prove that the function $u = 2x(1 - y)$ is harmonic .Find a function v such that $f(z) = u + iv$ Is analytic . Express f(z) in terms of z.

Problem 2

Separate this into real and imaginary parts i.e find u(x,y), v(x,y) such that f(z)=u+iv .
$f(z) = z + 1/z$

Problem 3

Let $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$ while f(2i)=3+4i .

(a) Prove that $\lim_{z\to i} f(z)$ exists and determine its value.

(b) Is f(z) continuous at z = 2i? Explain.

(c) Is f(z) continuous at points z≠2i? Explain.

Problem 4

Show that the function $x^2 + iy^3$ is not analytic anywhere. Reconcile this with the fact that the Cauchy–Riemann equations are satisfied at x =0 ,y=0.

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CVT_Assignment 2

Problem 1

$u = 2x(1-y)$ is harmonic.

Solve

$$u = 2x - 2xy \quad \text{--- (1)}$$

$$\frac{du}{dx} = 2 - 2y \qquad \frac{du}{dy} = -2x$$

$$\frac{d^2u}{dx^2} = 0 \quad \text{--- (2)} \qquad \frac{d^2u}{dy^2} = 0 \quad \text{--- (3)}$$

A function is harmonic when $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

So using (2) and (3)

$$\boxed{0 + 0 = 0}, \quad \boxed{LHS = RHS}$$

$\boxed{\text{The function is harmonic.}}$

Find (v) such that $f(z) = u + iv$ is analytic

$z = u + iv \quad \therefore u = 2x - 2xy, \; v = ?$

using cauchy Riemann equation

$$\frac{du}{dx} = \frac{dv}{dy}, \quad \frac{du}{dy} = -\frac{dv}{dx}$$

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$\frac{du}{dx} = 2 - 2y$, $\frac{du}{dy} = -2x$

so

$2 - 2y = \frac{dv}{dy}$, $-2x = -\frac{dv}{dx}$

intigrating wrt y — integrating wrt x

$2y - y^2 = v$, $\frac{2x^2}{2} = v = x^2$

Hence $\boxed{v = x^2 - y^2 + 2y}$

prove

$\frac{du}{dx} = \frac{dv}{dy}$, $\frac{du}{dy} = -\frac{dv}{dx}$

$2 - 2y = 2 - 2y$, $-2x = -2x$

so

$z = u + iv$

$z = (2x(1-y)) + i(x^2 - y^2 + 2y)$

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Q:-4

$$f = x^2 + iy^3$$

Solve

here $u = x^2$, $v = y^3$

$\frac{du}{dx} = 2x$ — (1) , $\frac{dv}{dy} = 3y$ — (2)

for an analytic fuction

$$\frac{du}{dx} = \frac{dv}{dy} \quad \& \quad \frac{du}{dy} = -\frac{dv}{dx}$$

$\frac{du}{dy} = 0$ — (3) , $\frac{dv}{dx} = 0$ — (4)

from (1) and (2) the first condition is not satisfied while from (3) and (4) the 2nd condition is satisfied, Hence the given function $(f = x^2 + iy^3)$ is not analytic at $x=0$, $y=0$ the given function becomes $\boxed{f = 0 + 0i}$

$u = 0$, $v = 0$

$\frac{du}{dx} = 0 \Rightarrow \frac{dv}{dy} = 0$

$\frac{du}{dy} = 0$, $\frac{dv}{dx} = 0$

So

The function f at $x=0$, $y=0$ is analytic as the cauchy-Riemann equations $\left(\frac{du}{dx} = \frac{dv}{dy} \;\&\; \frac{du}{dy} = -\frac{dv}{dx}\right)$ are satisfied.

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Q:- 2 $f(z) = u + iv$; $f(z) = z + \frac{1}{z}$

find u, v

<u>Solve</u> let $z = x + iy$

$$f(z) = x + iy + \frac{1}{x+iy}$$

$$f(z) = \frac{(x+iy)^2 + 1}{x+iy}$$

$$f(z) = \frac{x^2 - y^2 + 2xyi + 1}{x+iy}$$

$$f(z) = \frac{(x^2 - y^2 + 2xyi + 1)}{x^2 - (iy)^2} \times (x - iy)$$

$$f(z) = \frac{(x^3 - xy^2 + 2x^2yi + x - x^2yi + y^3i - 2xy^2i^2 - iy)}{x^2 + y^2}$$

$$\times\ f(z) = (x^3 - xy^2 + 2x^2y + x) + i(y^3 - x^2\ \times$$

$$f(z) = \frac{(x^3 - xy^2 + x + 2xy^2) + i(y^3 + 2x^2y - y - x^2y)}{x^2 + y^2}$$

$$f(z) = \frac{(x^3 + xy^2 + x)}{x^2 + y^2} + i\frac{(y^3 + x^2y - y)}{x^2 + y^2}$$

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Hence $f(z) = u + iv$

By ~~comap~~ comparing.

$$u = \frac{x^3 + xy^2 + x}{x^2 + y^2} \quad , \quad v = \frac{y^3 + x^2 y - y}{x^2 + y^2}$$

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Q:- 3 $f(z) = \dfrac{z^2+4}{z-2i}$ if $z \neq 2i$

while $f(2i) = 3+4i$

(a) Prove:- $\lim_{z \to i} f(z)$ exist

$$\lim_{z \to i^-} f(z) = \frac{z^2+4}{z-2i}$$

applying limit

$$= \frac{(i^2)+4}{i-2i} \Rightarrow \frac{3}{-i}$$

$$= \frac{3 \times i}{-i \times i}, \quad \boxed{f(i) = 3i} \quad \text{--- (1)}$$

$$\lim_{z \to i^+} f(z) = \frac{z^2+4}{z-2i}$$

applying limit

$$= \frac{i^2+4}{i-2i} \quad \boxed{f(i^+) = 3i} \quad \text{--- (2)}$$

Hence:- $f(i) = f(i^-) = f(i^+)$

so the limit exits at $z = i$

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(b) $f(z)$ continuous at $z = 2i$.

solve

$$f(z) = \frac{z^2+4}{z-2i}$$

if we simply substitute $z = 2i$ in $f(z)$ then it would become a $\frac{0}{0}$ form so

$$f(z) = \frac{z^2+4}{z-2i}$$

$$f(z) = \frac{z^2-(2i)^2}{z-2i} \Rightarrow \frac{(z-2i)(z+2i)}{(z-2i)}$$

$$\boxed{f(z) = z+2i} \quad \text{--- } ①$$

so $\lim\limits_{z\to 2i} f(z) = \lim\limits_{z\to 2i} (z+2i)$

applying limit

$$= (2i+2i) = 4i \quad \text{--- } ②$$

Here

$f(2i) = 3+4i$ (given) $\neq \lim\limits_{z\to 2i} (f(z)$

so

the function is not continuous.

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(c) $f(z)$ continuous at points $z \neq 2i$

Solve

$$f(z) = \frac{z^2+4}{z-2i} = (z+2i)$$

For all the points other than $z = 2i$, the function will be ~~contineo~~ continuous.

And if $z = -2i$ then
$$\boxed{f(z) = 0}$$

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