MT240:Complex Variables and Transforms

Homework 3

Due Date: Thu, May 20
Max Marks : 100

Spring 2021

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- You are encouraged to discuss the problems with your classmates in whatever way you like but, make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1 (30 marks)

Find the derivatives of each of the following.

a) $\coth^{-1}(z \csc 2z)$
b) $\ln\{\cot^{-1} z^2\}$
c) $\tan^{-1}(z+3i)^{-1/2}$
d) $\ln\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right)$

Problem 2 (20 marks)

Let $w = t\sec(t - 3i)$ and $z = \sin^{-1}(2t - 1)$. Find dw/dz.

Problem 3 (20 marks)

Given $w = \cos\zeta, z = \tan(\zeta + \pi i)$. Find $d^2w/dz^2 \;\; at \; \zeta = 0$.

Problem 4 (30 marks)

Evaluate.

a) $\lim\limits_{z\to e^{\pi i/3}} \left(z - e^{\pi i/3}\right)\left(\frac{z}{z^3+1}\right)$

b) $\lim\limits_{z\to i} \frac{z^2-2iz-1}{z^4+2z^2+1}$

c) $\lim\limits_{z\to 0} \frac{z-\sin z}{z^3}$

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CVT_Assignment 3

Q:-1

(a) $\coth^{-1}(z\csc 2z) = y$

Solve

Differentiating wrt (z)

$$= \frac{1}{1-(z\csc 2z)^2} \times \frac{d}{dz}(z\csc 2z)$$

$$= \frac{1}{1-z^2\csc^2 2z}\left[\csc 2z - 2z\cot(2z)\csc(2z)\right]$$

$$= \frac{\csc 2z\left(1-2z\cot(2z)\right)}{1-z^2\csc^2(2z)}$$

$$= \frac{\csc(2z)\left(1-2z\cot(2z)\right)}{1-z^2\left(1+\cot^2(2z)\right)}$$

$$= \frac{\csc(2z)\left(1-2z\cot(2z)\right)}{1-z^2-z^2\cot^2(2z)}$$

(b) $\ln(\cot^{-1}(z^2))$

Solve

differentiating wrt z

$$= \frac{1}{\cot^{-1}(z^2)} \frac{d}{dz}(\cot^{-1}(z^2))$$

$$= \frac{1}{\cot^{-1} z^2} \times \frac{-1}{1+z^4} \times 2z$$

$$= \frac{-2z}{(\cot^{-1}(z^2))(1+z^4)}$$

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(c) $\tan^{-1}(z+3i)^{-\frac{1}{2}}$

Solve

Differentiating wrt z

$$y = \tan^{-1}(z+3i)^{-\frac{1}{2}}$$

$$\frac{dy}{dz} = \frac{1}{1+z+3i} \times (1)\left(-\frac{1}{2}\right)(z+3i)^{-\frac{3}{2}}$$

$$\frac{dy}{dz} = \frac{-(z+3i)^{-3/2}}{2(1+z+3i)}$$

$$\frac{dy}{dz} = \frac{-1}{2(1+z+3i)(z+3i)^{3/2}}$$

(d) $\ln\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right)$

Solve

Differentiating wrt z

$$= \frac{1}{\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right)} \times \frac{d}{dz}\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right)$$

$$= \frac{1}{\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 3i}\right)}\left(1 + \frac{1}{2}\left(z^2 - 3z + 2i\right)^{-\frac{1}{2}} \times (2z-3)\right)$$

$$= \frac{1 + \frac{2z-3}{2\sqrt{z^2 - 3z + 2i}}}{\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right)}$$

$$= \frac{2\sqrt{z^2 - 3z + 2i} + 2z - 3}{\left(2\sqrt{z^2 - 3z + 2i}\right)\left(z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right)}$$

$$= \frac{2\sqrt{z^2 - 3z + 2i} + 2z - 3}{2z\sqrt{z^2 - 3z + 2i} - 3\sqrt{z^2 - 3z + 2i} + 2\left(z^2 - 3z + 2i\right)}$$

$$= \frac{2z - 3 + 2\sqrt{z^2 - 3z + 2i}}{(2z-3)\left(\sqrt{z^2 - 3z + 2i}\right) + 2z^2 - 6z + 4i}$$

Q2

$$\omega = t\sec(t-3i) \quad , z = \sin^{-1}(2t-1)$$

Solve

$$\frac{dw}{dt} = \sec(t-3i) + t\sec(t-3i)\tan(t-3i)$$

$$\frac{dz}{dt} = \frac{1}{\sqrt{1-(2t-1)^2}} \times 2 \Rightarrow \frac{2}{\sqrt{\cancel{1}-4t^2-\cancel{1}+4t}} \Rightarrow \frac{\cancel{1}}{\sqrt{\cancel{t^2}+t}}$$

$$\frac{dz}{dt} = \frac{1}{\sqrt{t-t^2}}$$

Now $\frac{d\cancel{w}}{\cancel{d}t} = \frac{dw}{dz} = \frac{dw}{dt} \times \frac{dt}{dz}$

$$\frac{dw}{dz} = \frac{\sec(t-3i)\left[1+t\tan(t-3i)\right]}{\sqrt{t-t^2}}$$

OR

$$\frac{dw}{dz} = \frac{\sec(t-3i)\left(1+t\tan(t-3i)\right]}{\sqrt{t}-t}$$

Q:- 3

$$w = \cos \zeta \quad , \quad z = \tan(\zeta + \pi i)$$

find $\dfrac{d^2 w}{dz^2}$ at $\zeta = 0$

Solve

$$\cancel{\frac{d\omega}{dt}}\ \frac{dw}{d\zeta} = -\sin \zeta \quad , \quad \frac{d^2 w}{d\zeta^2} = -\cos \zeta \quad \text{--- (1)}$$

$$\frac{dz}{d\zeta} = \sec^2(\zeta + \pi i) \quad , \quad \frac{dz^2}{d\zeta^2} = \sec^4(\zeta + \pi i) \quad \text{--- (2)}$$

Now

$$\frac{d^2 w}{dz^2} = \frac{d^2 w}{d\zeta^2} \times \frac{d\zeta^2}{dz^2}$$

$$= \frac{-\cos \zeta}{\sec^4(\zeta + \pi i)}$$

$$\frac{d^2 w}{dz^2} = (-\cos \zeta)(\cos^4(\zeta + \pi i))$$

at $\zeta = 0$

$$\frac{d^2 w}{dz^2} = -\cos^4(\pi i) \quad \Delta$$

Q4

ⓐ $\lim_{z \to e^{\pi i/3}} \left(z - e^{\pi i/3}\right)\left(\frac{z}{z^3+1}\right)$

Solve

if we apply limit on the given function then it comes out to be zero so we have to simplify.

$$= \left(z - e^{\pi i/3}\right)\left[\frac{z}{z^3+1}\right]$$

$$= \frac{z^2 - ze^{\pi i/3}}{z^3+1}$$

Now differentiating (applying L'hopital's rule)

$$= \frac{2z - z - e^{\pi i/3}}{3z^2}$$

Now if we apply limits, the answere would also be zero so applying L'hopital's rule again.

$$= \frac{2}{6z} = \frac{1}{3z}$$

Now applying limits

$$\lim_{z \to e^{\pi i/3}} \left[\frac{1}{3z}\right] = \boxed{\frac{1}{3e^{\pi i/3}}}$$

$$\Rightarrow \boxed{\frac{e^{-\pi i/3}}{3}}$$

(b) $\lim_{z \to i} \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1} \approx \frac{0}{0}$

Solve

applying L'hopital's rule

$\lim_{z \to i} \frac{2z - 2i}{4z^3 + 4z} = \frac{0}{0} = \frac{0}{0}$

applying '' '' again

$\lim_{z \to i} \frac{2}{12z^2 + 4}$

Now, applying limits.

$\frac{2}{12(i^2) + 4} = \frac{2}{-12 + 4}$

$= \frac{2}{-8}$

$= \boxed{-\frac{1}{4}}$

(c) $\lim_{z \to 0} \frac{z - \sin z}{z^3}$

Solve if we put $z=0$, it beomes $\frac{0}{0}$

applying L'hopital's rule untill we get some result.

$$= \frac{z - \sin z}{z^3}$$

$$= \frac{1 - \cos z}{3z^2}$$

$$= \frac{\sin z}{6z}$$

$$= \frac{\cos z}{6}$$

Now applying limit

$$\lim_{z \to 0} \frac{\cos z}{6}$$

$$\Rightarrow \boxed{\frac{1}{6}}$$