MT240:Complex Variables and Transforms

Homework 4

Due Date: 18, June ,2021
Max Marks : 100

Spring 2021

Tips to avoid plagiarism:

- Do not copy the solutions of your classmates.
- You are encouraged to discuss the problems with your classmates in whatever way you like but, make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1 (20)

Find the path and sketch it .

a) $z(t) = 1 + i + e^{-\pi i t} \quad 0 \leq t \leq 2$
b) $z(t) = 5e^{-it} \quad 0 \leq t \leq \frac{\pi}{2}$
c) $z(t) = t + it^3 \quad -2 \leq t \leq 2$

Problem 2 (20)

Let C be the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1,1) and (2,3).

Find the value of

$$\oint 12z^2 - 4iz\, dz$$

Problem 3 (20)

Integrate f(z) around unit circle clockwise ,Indicate whether cauchy's theorem applies or not .

a) $f(z) = \mathrm{Im(z)}$
b) $f(z) = \frac{1}{z^4-1}$
c) $f(z) = \frac{-1}{2z-1}$
d) $f(z) = z^3\cot(\mathrm{z})$

Problem 4 (20)

Integrate it as indicted .

1) $\oint \frac{\sin z}{4z^2-8iz}$ C, consist of boundaries of squares with vertices $\pm 3, \pm 3i$ counter clockwise , ± 1 , $\pm 1i$ clockwise .
2) $\oint \frac{expz^2 dz}{z^2(z-1-i)}$ C, consist of $|z| = 2$ counter clockwise and $|z| = 1$ clockwise.

Problem 5 (20)

Integrate it by first method ,if it does not apply ,state why it does not apply and then use 2nd method .

$$\oint sec^2 z\, dz\ any\ path\ from\ \frac{\pi}{4}\ to\ \frac{\pi i}{4}$$

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CVT - Assignment #4

Q:-1

ⓐ $z(t) = 1 + i + e^{-\pi i t}$ $\quad 0 \le t \le 2$

Solve

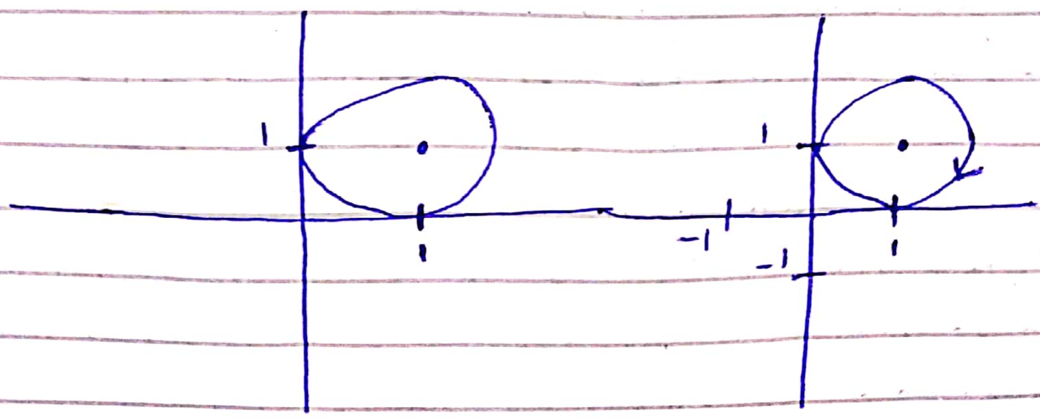
$= 1 + i + \cos\pi t - i\sin\pi t$

$x = 1 + \cos\pi t$, $\cos\pi t = x - 1$
$y = 1 - \sin\pi t$, $\sin\pi t = 1 - y$

Now $(\cos\pi t)^2 + (\sin\pi t)^2 = 1 = (x-1)^2 + (1-y)^2$

Here $r = 1$ & center $= (1,1)$

while the path would be clockwise for $0 \le t \le +2\pi$



(b) $z(t) = 5e^{-it}$ $\qquad 0 \le t \le \frac{\pi}{2}$

solve

$= 5[\cos t - i \sin t]$

$x = 5\cos t \qquad \cos t = x/5$

$y = -5\sin t \qquad \sin t = -y/5$

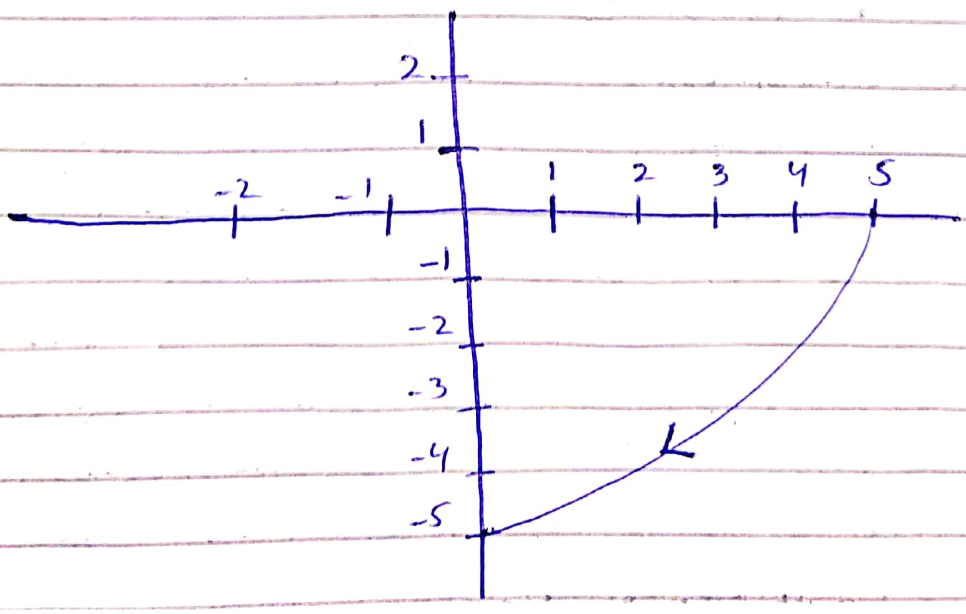
And $\cos^2 t + \sin^2 t = 1 = \left(\frac{x}{5}\right)^2 + \left(\frac{-y}{5}\right)^2$

$x^2 + y^2 = 5^2$

$r = 5$ & center $= (0,0)$

The path would be clockwise from $(0 \le t \le \frac{\pi}{2})$ means

$(0 \le t \le -\frac{\pi}{2})$



(c) $z(t) = t + it^3 \quad -2 \le t \le 2$

Solve

Here $z(t) = x + iy$

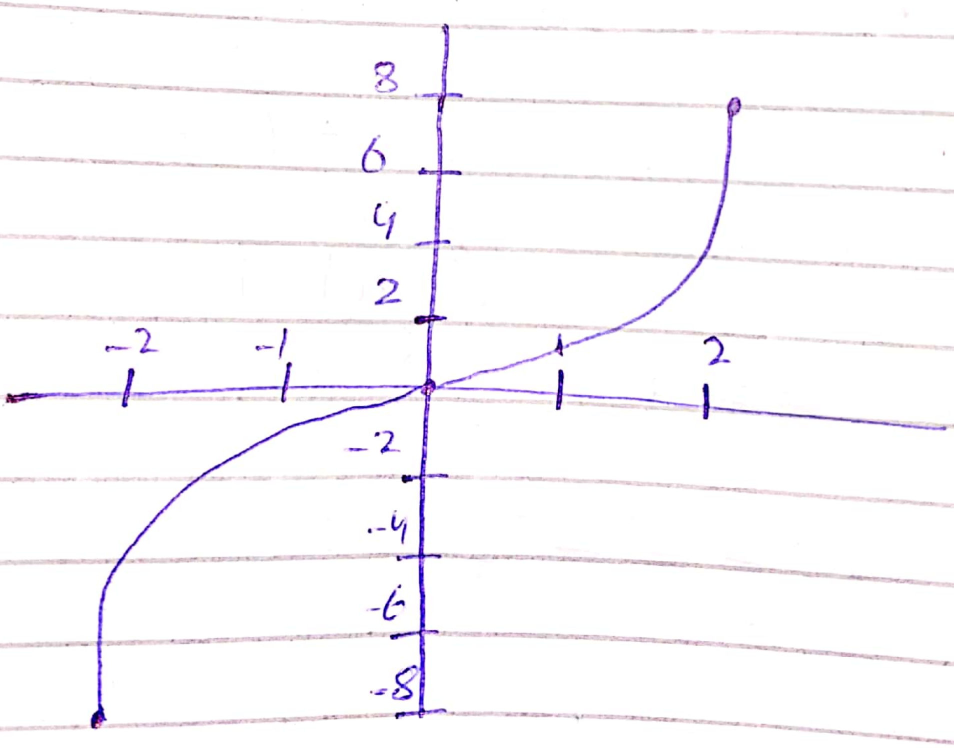
$x = t$

$y = t^3 \quad ; \quad y = x^3$

for $t = -2$, $\quad t = 2$, $\quad t = 0$

$x = -2$, $\quad x = 2$, $\quad x = 0$

$y = -8$, $\quad y = 8$, $\quad y = 0$



Problem 2

$y = x^3 - 3x^2 + 4x - 1$ joining $(1,1)$ and $(2,3)$

Find $\oint_c (12z^2 - 4iz)\,dz$

Solve

using $y = mx + c$

$(-1+y) = \left[\frac{3-1}{2-1}\right](x - 1)$

$(-1+y) = 2(x-1)$

$y = 2x-1$, $y^2 = 4x^2+1-4x$

$dy = 2dx$,

$z = x + iy$

$dz = dx + i\,dy = \cancel{dx + 2i\,d}$

$= dx + 2i\,dx$

$dz = (1+2i)\,dx$

$\int_1^2 (12z^2 - 4iz)\,dz = \int_1^2 \left((12(x+iy)^2 - 4i(x+iy)\right)(1+2i)\,dx$

$\int_1^2 (1+2i)\left[12(x^2 - y^2 + 2xyi) - 4ix + 4y\right]$

$(1+2i)\int_1^2 12(x^2 - 4x^2 - 1 + 4x + 4x^2 - 2x) - 4ix + 8x - 4)\,dx$

$$(1+2i)\int_1^2 -36x^2+48ix^2+x(32-28i)-16$$

$$(1+2i)\left[\left[-12x^3\right]_1^2+\left[16ix^3\right]_1^2+\left[x^2(16-14i)\right]_1^2-\left[16x\right]_1^2\right.$$

$$(1+2i)\left(-84-16+48-42i+112i\right)$$

$$(1+2i)\left(-52+70i\right)$$

$$=\boxed{-192-34i}$$

Problem 3

(a) $f(z) = Im(z)$

Solve

$$z = x + iy = \cos t + i \sin t$$

$$Im(z) = \sin t$$
$$dz = (-\sin t + i \cos t)\, dt$$

$$\int_C f(z)\, dz = \int_{2\pi}^{0} (\sin t)(-\sin t + i\cos t)\, dt$$

$$= -\int_{2\pi}^{0} \sin^2 t + i \int_{2\pi}^{0} \sin t \cos t\, dt$$

$$= -\int_{2\pi}^{0} \frac{1-\cos 2t}{2} + i \int_{2\pi}^{0} \sin t \cos t\, dt$$

$$= -\frac{1}{2}\int_{2\pi}^{0} 1 + \frac{1}{2}\int_{2\pi}^{0} \cos 2t + i\int_{2\pi}^{0} \sin t \cos t\, dt$$

$$= -\frac{1}{2}\left[-2\pi\right] + \frac{1}{2}\left[\frac{\sin 2t}{2}\right]_{2\pi}^{0} + i\left[\frac{\sin^2 t}{2}\right]_{2\pi}^{0}$$

$$= \pi + 0 + 0$$

$$= \boxed{\pi}$$

As $Im(z)$ is non analytic and (π) is greater than (one)(unit circle's radius) so Cauchy's theorum cannot be applied.

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(b) $f(z) = \dfrac{1}{z^4 - 1}$

Solve

$z^4 - 1 = 0$ for non analyticity

$z^4 = 1$

$z = 1$

and we have a unit circle while this point lies on the circle thats why Cauchy's integral theorum can~~not~~ apply.

and

$$\int f(z) = \int \frac{1}{z^4 - 1} = 0$$

(c) $f(z) = \frac{-1}{2z-1}$

Solve

$2z-1 = 0, \quad z = 1/2$

as it lies outside the contor thats why CIT can apply on it means

$$\int f(z)\,dz = \int \frac{-1}{2z-1}\,dz = \boxed{0}$$

(d) $f(z) = z^3 \cot(z) =$

Solve

$$z^3 \frac{\cos(z)}{\sin(z)}$$

Hence sin(z) is in denominator and ~~anat~~ non analytical points lies ~~insid~~ outside the contore Then ER CIT can be applied.

$$\int z^3 \cot(z) = 0$$

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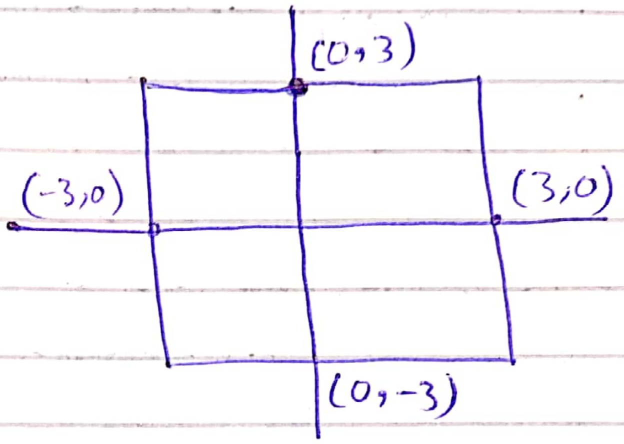
Problem 4

(1) $\oint \frac{\sin z}{4z^2 - 8iz} dz$



Solve

$4z^2 - 8iz = 0$

$4z(z-2i) = 0$

$z = 0 \quad , \quad z = 2i$

These both points lie inside the square.

$$\oint_C \frac{\sin z}{4z^2 - 8iz} dz = \frac{1}{4} \int_{C_1} \frac{\sin z \, dz}{z(z-2i)} + \frac{1}{4} \int_{C_2} \frac{\sin z \, dz}{z-2i}$$

Here $c_1 = 0, \; c_2 = 2i$

$$\oint_{C_1} \frac{\sin z / z-2i}{z} dz$$

Let $f(z) = \frac{\sin z}{z-2i}$

$f(0) = 0$

$$\oint_{C_1} \frac{\sin z / (z-2i)}{z} dz = 2\pi i f(0) = 0 \quad (1)$$

Similarly,

$$\oint_{C_2} \frac{\sin z / z}{z - 2i} \, dz$$

let $f(z) = \frac{\sin z}{z}$

$$f(2i) = \frac{\sin 2i}{2i} = \frac{i \sinh 2}{2i}$$

$$f(2i) = \frac{\sinh 2}{2} = \frac{1}{2}\left[\frac{e^{2} - e^{-2}}{2}\right]$$

$$f(2i) = \frac{e^{2} - e^{-2}}{4} \quad \text{—— } (2)$$

$$\oint_{C_2} \frac{\sin z / z}{z - 2i} \, dz = 2\pi i f(2i)$$

$$= 2\pi i \left[\frac{e^{2} - e^{-2}}{4}\right]$$

$$= \frac{\pi i (e^{2} - e^{-2})}{2}$$

Now $\oint_{C} \frac{\sin z}{4z^2 - 8iz} \, dz = \frac{1}{4} 2\pi i f(0) + \frac{1}{4} \pi i \left[\frac{e^{2} - e^{-2}}{2}\right]$

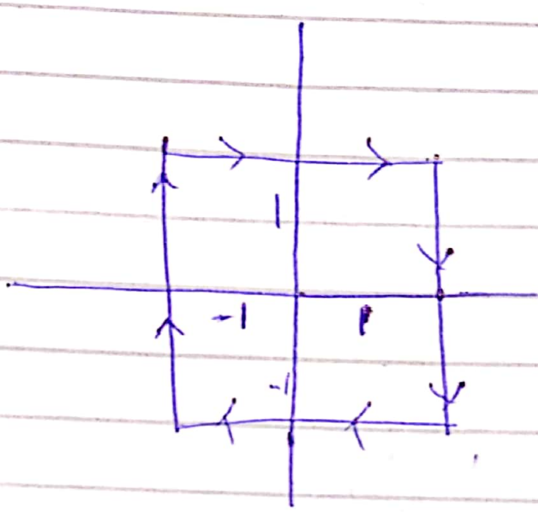
$$= \frac{\pi i}{16}\left[e^{2} - e^{-2}\right] \text{ A}$$

$$= \frac{\pi i}{8}\left[e^{2} - e^{-2}\right] \text{ A}$$

② for ± 1, $\pm 1i$ clockwise

Solve

$$-\oint_C \frac{\sin z}{4z^2-8iz}\, dz$$



$$-\int_C \frac{\sin z}{4z^2-8iz}\, dz$$

$$-\frac{1}{4}\oint_{C_1} \frac{\sin z / z-2i}{z}\, dz - \frac{1}{4}\oint_{C_2} \frac{\sin z / z}{z-2i}\, dz$$

$c_1 = 0 = z_1$, $c_2 = 2i$

Let $f(z) = \dfrac{\sin z}{z-2i}$

$$f(0) = 0$$

$$\oint_{C_1} \frac{\sin z / z-2i}{z}\, dz = 2\pi i (f(0))$$

$$= \boxed{0} \quad \text{—— ③}$$

Now let $f(z) = \frac{\sin z}{z}$

$$f(2i) = \frac{\sin 2i}{2i}$$

$$= \frac{i \sinh 2h}{2i}$$

$$f(2i) = \cancel{e^{2}} = \frac{e^{2} - e^{-2}}{4}$$

$$\oint_{c2} \frac{\sin z / z}{z - 2i} \, dz = 2\pi i \, f(2i)$$

$$= 2\pi i \left[\frac{e^{2} - e^{-2}}{4} \right]$$

$$= \pi i \left[\frac{e^{2} - e^{-2}}{2} \right]$$

$$= \int_{c} \frac{\sin z}{4z^{2} - 8iz} \, dz = 0 - \frac{1}{4} \left[\pi i \left[\frac{e^{2} - e^{-2}}{2} \right] \right]$$

$$= -\pi i \left[\frac{e^{2} - e^{-2}}{8} \right] \text{ A}$$

Problem 5

$\oint \sec^2 z \, dz$ any path from $\frac{\pi}{4}$ to $+\frac{\pi i}{4}$

Solve

$$\int \sec^2 z \, dz = \tan z$$

$$\int \sec^2 z \, dz = F\left[\frac{\pi i}{4}\right] - F\left[\frac{\pi}{4}\right]$$

$$= \tan\left[\frac{\pi i}{4}\right] - \cancel{F}\left[\tan\left(\frac{\pi}{4}\right)\right]$$

$$= \tan\frac{\pi i}{4} - 1$$

$$= \frac{\sin\frac{\pi i}{4}}{\cos\frac{\pi i}{4}}$$

$$\sin\frac{\pi i}{4} = \frac{e^{-\pi/4} - e^{\pi/4}}{2i}, \quad \cos\frac{\pi i}{4} = \frac{e^{-\pi/4} + e^{\pi/4}}{2}$$

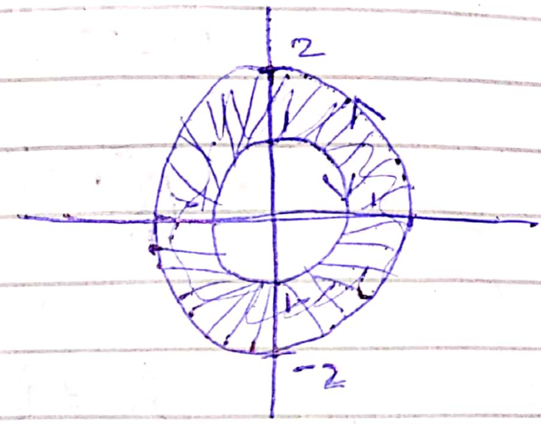
$$\tan\frac{\pi i}{4} = \frac{e^{-\pi/4} - e^{\pi/4}}{i\left(e^{-\pi/4} + e^{\pi/4}\right)}$$

$$= \frac{-i\left[\frac{-e^{+\pi/4} - e^{-(\pi/4)}}{2}\right] \times 2}{2\left[\frac{e^{\pi/4} + e^{-\pi/4}}{2}\right]}$$

$$\oint \sec^2 z \, dz = (-i)\left(\frac{-\sinh \pi/4}{\cosh\frac{\pi}{4}}\right) = \boxed{i\tanh\frac{\pi}{4} - 1}$$

Q 4

(2) $\oint \frac{\exp z^2 \, dz}{z^2(z-1-i)}$ $|z| = 2$ counter
$|z| = 1$ clockwise



$$z^2(z-1-i) = 0$$

$$z^2 = 0 \,, \quad z-1-i = 0$$
$$z = 0 \,, \quad z = 1+i$$
$$|z| = \sqrt{2} < 2$$

Here one point $(z=0)$ lies ~~in~~ outside the contore Hence cauch'ys integral theorum can apply on it.

$$\oint_{C_1=0} \frac{\exp z^2 \, dz}{z^2(z-1-i)} = 0$$

While other point $(z = 1+i)$ lies inside the contore, it shoes CIT cannot be applied ont.

$$\int_{C_2=1+i} \frac{\exp z^2 \, dz}{z^2(z-1-i)} \Rightarrow \int_{1+i} \frac{f(z)}{z-1-i} \quad \text{— (2)}$$

Here $f(z) = \frac{\exp z^2}{z^2}$ — (3)

$$f(1+i) = \frac{\exp(1+i)^2}{(1+i)^2}$$

$$= \frac{\exp(1+2i-1)}{(1+2i-1)}$$

$$= \frac{\exp(2i)}{2i}$$

$$= \frac{\cos 2 + i \sin 2}{2i}$$

$$f(1+i) = \cancel{\cos}\ \frac{\sin 2}{2} - i\frac{\cos 2}{2} \quad \text{—— (4)}$$

from (2) $$\int_{1+i} \frac{f(z)}{z-1-i} = \frac{1}{2\pi i} \oint \frac{f(z)}{z-1-i}\, dz$$

$$= 2\pi i\left(f(1+i)\right) \quad \text{—— (5)}$$

Substituting (4) in (5)

$$= 2\pi i\left[\frac{\sin 2}{2} - i\frac{\cos 2}{2}\right]$$

$$= \pi i \sin 2 - i^2 \pi \cos 2$$

$$= \pi \cos 2 + \pi i \sin 2$$

$$= \pi(\cos 2 + i \sin 2) \text{ Ans}$$