KRP — Assignment 2

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⋆ **This assignment, due on 21st April at 23:59, contributes to 10% of the final marks for this course. Please be advised that only Questions 1 − 8 are mandatory. Nevertheless, students can earn up to one bonus mark by completing Question 9. This bonus mark can potentially augment a student's overall marks but is subject to a maximum total of 100 for the course. By providing bonus marks, we aim to incentivize students to excel in their studies and reward those with a remarkable grasp of the course materials.**

Question 1. Bisimulation Invariance

In the lecture, we defined bisimulation for $\mathcal{ALC}$ and showed bisimulation invariance of $\mathcal{ALC}$ (Theorem 3.2).

- Define a notion of "$\mathcal{ALCN}$-bisimulation" that is appropriate for $\mathcal{ALCN}$ in the sense that bisimilar elements satisfy the same $\mathcal{ALCN}$-concepts.
- Use the definition to show that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

Question 2. Bisimulation over Filtration

Let C be an $\mathcal{ALC}$-concept, $\mathcal{T}$ an $\mathcal{ALC}$-TBox, $\mathcal{I}$ an interpretation and $\mathcal{J}$ its filtration w.r.t. $\mathsf{sub}(C) \cup \mathsf{sub}(\mathcal{T})$ (see Definition 3.14 for the definition of filtration). Show the truth or falsity of the following statement:

- the relation $\rho = \{(\mathsf{d}, [\mathsf{d}]) \mid \mathsf{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

Hint: If the above relation ρ were a bisimulation, why do we have to explicitly prove Lemma 3.15 in the lecture? Wouldn't Lemma 3.15 then be a consequence of Theorem 3.2?

Question 3. Bisimulation within the Same Interpretation (2 marks)

We define "*bisimulations on* $\mathcal{I}$" as bisimulations between an interpretation $\mathcal{I}$ and itself. Let $\mathsf{d}, \mathsf{e} \in \Delta^{\mathcal{I}}$ be two elements. We write $\mathsf{d} \approx_{\mathcal{I}} \mathsf{e}$ if they are bisimilar, i.e., if there is a bisimulation ρ on $\mathcal{I}$ such that $\mathsf{d} \, \rho \, \mathsf{e}$.

- Show that $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

Consider the interpretation $\mathcal{J}$ defined like the filtration, but with $\approx_{\mathcal{I}}$ instead of $\simeq$.

- Show that $\rho = \{(\mathsf{d}, [\mathsf{d}]_{\approx_{\mathcal{I}}}) \mid \mathsf{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.
- Show that, if $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, then so is $\mathcal{J}$.
- Why can't we use the previous result to show the finite model property for $\mathcal{ALC}$?

Question 4. Closure under Disjoint Union

Recall Theorem 3.8 from the lecture, which says that the disjoint union of a family of models of an $\mathcal{ALC}$-TBox $\mathcal{T}$ is again a model of $\mathcal{T}$. Note that the disjoint union is only defined for concept and role names.

- Extend the notion of disjoint union to individual names such that the following holds: for any family $(\mathcal{I}_\nu)_{\nu\in\Omega}$ of models of an $\mathcal{ALC}$-knowledge base $\mathcal{K}$, the disjoint union $\biguplus_{\nu\in\Omega}\mathcal{I}_\nu$ is also a model of $\mathcal{K}$.

Question 5. Closure under Disjoint Union
Let $\mathcal{K} = \{\mathcal{T}, \mathcal{A}\}$ be a consistent $\mathcal{ALC}$-KB. We write $C \sqsubseteq_\mathcal{K} D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$.

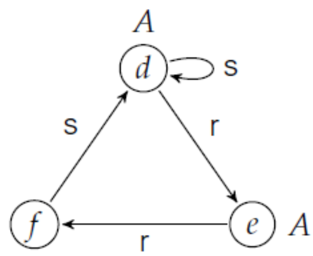
- Prove that for all $\mathcal{ALC}$-concepts C and D we have $C \sqsubseteq_\mathcal{K} D$ iff $C \sqsubseteq_\mathcal{T} D$.

Hint: Use the modified definition of disjoint union from the previous question.

Question 6. Finite Model Property (2 marks)
Let C be an $\mathcal{ALC}$-concept that is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$. Show truth or falsity of the following statement:

- for all $m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.
- Does it hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$"?

Question 7. Unravelling
Draw the unraveling of the following interpretation $\mathcal{I}$ at d up to depth 5, i.e., restricted to d-paths of length at most 5 (see Definition 3.21):



Question 8. Tree Model Property

- Show the truth or falsity of the following statement: if $\mathcal{K}$ is an $\mathcal{ALC}$-KB and C an $\mathcal{ALC}$-concept such that C is satisfiable w.r.t. $\mathcal{K}$, then C has a tree model w.r.t. $\mathcal{K}$.

Question 9 (with 1 bonus mark). Bisimulation Invariance
Interpretations of $\mathcal{ALC}$ can be represented as graphs, with edges labeled by roles and nodes labeled by sets of concept names. More precisely, in such a graph:

each node corresponds to an element in the domain of the interpretation and it is labeled with all the concept names to which this element belongs in the interpretation;

an edge with label r between two nodes says that the corresponding two elements of the interpretation are related by the role r.

- Show that: the description logic $\mathcal{S}$ (i.e., $\mathcal{ALC}$ with transitive roles) is more expressive than $\mathcal{ALC}$.