1 STRESS COMPONENTS

In this section, the closed form of the stress equations (see section ??) are discussed. The normal stress equation is given by Eq. (??). On taking fourier transform from $z-t$ domain to $k_z-\omega$ domain using Eq. (??),

$$\hat{\tau}_{rr}(r,\theta,k_z,\omega) = (\lambda+2\mu)\,\frac{\partial \hat{W}_e}{\partial r} + \frac{\lambda}{r}\left(\hat{W}_e + \frac{\partial \hat{\Theta}_e}{\partial \theta}\right) - j\lambda k_z \hat{U}_e. \tag{1}$$

On substituting the displacement components Eq. (??), Eq. (??) and Eq. (??) and simplifying, the closed form of normal stress is given by

$$\begin{aligned}
\hat{\tau}_{rr}(r,\theta,k_z,\omega) = \Bigg\{ & - j\lambda k_z[J_0(r\beta_1) + J_1(r\beta_1)\cos(\theta)] + \\
& \chi_1\Bigg[\frac{2\mu[J_1(r\beta_1) - J_0(r\beta_1)\cos(\theta)]}{r} + \frac{4\mu J_1(r\beta_1)\cos(\theta)}{r^2\beta_1} - (\lambda+2\mu)[J_0(r\beta_1) + \\
& J_1(r\beta_1)\cos(\theta)]\beta_1\Bigg]\Bigg\}\hat{P}_1 + \Bigg\{\frac{1}{r^2\beta_1}\sin(\theta)\Bigg[- 2r\mu\beta_1\chi_1 J_0(r\beta_1) \\
& + J_1(r\beta_1)\{4\mu\chi_1 - r^2\beta_1[j\lambda k_z + (\lambda+2\mu)\beta_1\chi_1]\}\Bigg]\Bigg\}\hat{P}_2 + \Bigg\{ - j\lambda k_z[Y_0(r\beta_1) \\
& + Y_1(r\beta_1)\cos(\theta)] + \frac{\chi_1}{r^2\beta_1}\Bigg[- r\beta_1 Y_0(r\beta_1)[2\mu\cos(\theta) + r\beta_1(\lambda+2\mu)] \\
& + Y_1(r\beta_1)\{4\mu\cos(\theta) + r\beta_1[2\mu - r\beta_1(\lambda+2\mu)\cos(\theta)]\}\Bigg]\Bigg\}\hat{Q}_1 \\
& + \Bigg\{\frac{1}{r^2\beta_1}\sin(\theta)\Bigg[- 2r\mu\beta_1\chi_1 Y_0(r\beta_1) + Y_1(r\beta_1)\{4\mu\chi_1 - r^2\beta_1[j\lambda k_z + \\
& (\lambda+2\mu)\beta_1\chi_1]\}\Bigg]\Bigg\}\hat{Q}_2 + \Bigg\{ - j\lambda k_z[J_0(r\beta_2) + J_1(r\beta_2)\cos(\theta)] \\
& + \chi_2\Bigg[\frac{2\mu[J_1(r\beta_2) - 2J_0(r\beta_2)\cos(\theta)]}{r} + \frac{8\mu J_1(r\beta_2)\cos(\theta)}{r^2\beta_2} - (\lambda+2\mu) \\
& [J_0(r\beta_2) + J_1(r\beta_2)\cos(\theta)]\beta_2\Bigg]\Bigg\}\hat{R}_1 + \Bigg\{ - j\lambda k_z[Y_0(r\beta_2) + Y_1(r\beta_2)\cos(\theta)] \\
& + \frac{\chi_2}{r^2\beta_2}\Bigg[- r\beta_2 Y_0(r\beta_2)[4\mu\cos(\theta) + r\beta_2(\lambda+2\mu)] + Y_1(r\beta_2)\{8\mu\cos(\theta) \\
& + r\beta_2[2\mu - r\beta_2(\lambda+2\mu)\cos(\theta)]\}\Bigg]\Bigg\}\hat{R}_2 + \Bigg\{\frac{2\mu\cos(\theta)}{r^2}\Bigg[- 2J_1(r\beta_2) \\
& + r\beta_2 J_0(r\beta_2)\Bigg]\Bigg\}\hat{S}_1 + \Bigg\{\frac{2\mu\cos(\theta)}{r^2}\Bigg[- 2Y_1(r\beta_2) + r\beta_2 Y_0(r\beta_2)\Bigg]\Bigg\}\hat{S}_2 \\
& + \Bigg\{\frac{2\mu\sin(\theta)}{r^2}\Bigg[- 2J_1(r\beta_2) + r\beta_2 J_0(r\beta_2)\Bigg]\Bigg\}\hat{T}_1 \\
& + \Bigg\{\frac{2\mu\sin(\theta)}{r^2}\Bigg[- 2Y_1(r\beta_2) + r\beta_2 Y_0(r\beta_2)\Bigg]\Bigg\}\hat{T}_2.
\end{aligned} \tag{2}$$

The shear stresses are given by Eq. (**??**), Eq. (**??**) and Eq. (**??**). These shear stresses are transformed to wavenumber-frequency domain and can be represented as

$$\hat{\tau}_{rz}(r,\theta,k_z,\omega) = \mu\left(-jk_z\hat{W}_e + \frac{\partial \hat{U}_e}{\partial r}\right), \tag{3}$$

$$\hat{\tau}_{r\theta}(r,\theta,k_z,\omega) = \mu\left(\frac{1}{r}\frac{\partial \hat{W}_e}{\partial \theta} + \frac{\partial \hat{\Theta}_e}{\partial r} - \frac{\hat{\Theta}_e}{r}\right), \tag{4}$$

and

$$\hat{\tau}_{z\theta}(r,\theta,k_z,\omega) = \mu\left(-jk_z\hat{\Theta}_e + \frac{1}{r}\frac{\partial \hat{U}_e}{\partial \theta}\right) \tag{5}$$

The displacement components are substituted in above equation and simplified as

$$\begin{aligned}
\hat{\tau}_{rz}(r,\theta,k_z,\omega) = \Bigg\{ &\frac{\mu}{r\beta_1}\{r\beta_1 J_0(r\beta_1)\cos(\theta) - J_1(r\beta_1)[\cos(\theta) + r\beta_1]\} \\
&(\beta_1 - jk_z\chi_1)\Bigg\}\hat{P}_1 + \Bigg\{\frac{\mu\sin(\theta)}{r\beta_1}[-J_1(r\beta_1) + r\beta_1 J_0(r\beta_1)](\beta_1 - jk_z\chi_1)\Bigg\}\hat{P}_2 \\
&+ \Bigg\{\frac{\mu}{r\beta_1}\{r\beta_1 Y_0(r\beta_1)\cos(\theta) - Y_1(r\beta_1)[\cos(\theta) + r\beta_1]\}(\beta_1 - jk_z\chi_1)\Bigg\}\hat{Q}_1 \\
&+ \Bigg\{\frac{\mu\sin(\theta)}{r\beta_1}[-Y_1(r\beta_1) + r\beta_1 Y_0(r\beta_1)](\beta_1 - jk_z\chi_1)\Bigg\}\hat{Q}_2 \\
&+ \Bigg\{\mu J_0(r\beta_2)\cos(\theta)(\beta_2 - jk_z\chi_2) + \frac{\mu J_1(r\beta_2)}{r\beta_2}\{-\beta_2[\cos(\theta) + r\beta_2] \\
&+ jk_z[2\cos(\theta) + r\beta_2]\chi_2\}\Bigg\}\hat{R}_1 + \Bigg\{\mu Y_0(r\beta_2)\cos(\theta)(\beta_2 - jk_z\chi_2) \\
&+ \frac{\mu Y_1(r\beta_2)}{r\beta_2}\{-\beta_2[\cos(\theta) + r\beta_2] + jk_z[2\cos(\theta) + r\beta_2]\chi_2\}\Bigg\}\hat{R}_2 \\
&+ \Bigg\{\frac{-j\mu k_z}{r}J_1(r\beta_2)\cos(\theta)\Bigg\}\hat{S}_1 + \Bigg\{\frac{-j\mu k_z}{r}Y_1(r\beta_2)\cos(\theta)\Bigg\}\hat{S}_2 \\
&+ \Bigg\{\frac{-j\mu k_z}{r}J_1(r\beta_2)\sin(\theta)\Bigg\}\hat{T}_1 + \Bigg\{\frac{-j\mu k_z}{r}Y_1(r\beta_2)\sin(\theta)\Bigg\}\hat{T}_2,
\end{aligned} \tag{6}$$

$$\hat{\tau}_{r\theta}(r,\theta,k_z,\omega) = \Bigg\{\frac{2}{r}\mu J_2(r\beta_1)\sin(\theta)\chi_1\Bigg\}\hat{P}_1 + \Bigg\{\frac{-2}{r}\mu J_2(r\beta_1)\cos(\theta)\chi_1\Bigg\}\hat{P}_2$$

$$+\left\{\frac{2}{r}\mu Y_2(r\beta_1)\sin(\theta)\chi_1\right\}\hat{Q}_1+\left\{\frac{-2}{r}\mu Y_2(r\beta_1)\cos(\theta)\chi_1\right\}\hat{Q}_2$$
$$+\left\{\frac{\mu\sin(\theta)\chi_2}{r^2\beta_2}[-4r\beta_2J_0(r\beta_2)+J_1(r\beta_2)(8-r^2\beta_2^2)]\right\}\hat{R}_1$$
$$+\left\{\frac{\mu\sin(\theta)\chi_2}{r^2\beta_2}[-4r\beta_2Y_0(r\beta_2)+Y_1(r\beta_2)(8-r^2\beta_2^2)]\right\}\hat{R}_2$$
$$+\left\{\frac{\mu\sin(\theta)}{r^2}[2r\beta_2J_0(r\beta_2)+J_1(r\beta_2)(-4+r^2\beta_2^2)]\right\}\hat{S}_1$$
$$+\left\{\frac{\mu\sin(\theta)}{r^2}[2r\beta_2Y_0(r\beta_2)+Y_1(r\beta_2)(-4+r^2\beta_2^2)]\right\}\hat{S}_2$$
$$+\left\{\frac{1}{r^2}\Big[-\mu r\beta_2J_0(r\beta_2)(2\cos(\theta)+r\beta_2)+\mu J_1(r\beta_2)\{4\cos(\theta)$$
$$+r\beta_2[2-r\beta_2\cos(\theta)]\}\Big]\right\}\hat{T}_1+\left\{\frac{1}{r^2}\Big[-\mu r\beta_2Y_0(r\beta_2)(2\cos(\theta)+r\beta_2)$$
$$+\mu Y_1(r\beta_2)\{4\cos(\theta)+r\beta_2[2-r\beta_2\cos(\theta)]\}\Big]\right\}\hat{T}_2 \quad (7)$$

and

$$\hat{\tau}_{z\theta}(r,\theta,k_z,\omega)=\left\{\frac{-\mu}{r\beta_1}J_1(r\beta_1)\sin(\theta)(\beta_1-jk_z\chi_1)\right\}\hat{P}_1+\left\{\frac{\mu}{r\beta_1}J_1(r\beta_1)\right.$$
$$\left.\cos(\theta)(\beta_1-jk_z\chi_1)\right\}\hat{P}_2+\left\{\frac{-\mu}{r\beta_1}Y_1(r\beta_1)\sin(\theta)(\beta_1-jk_z\chi_1)\right\}\hat{Q}_1$$
$$+\left\{\frac{\mu}{r\beta_1}Y_1(r\beta_1)\cos(\theta)(\beta_1-jk_z\chi_1)\right\}\hat{Q}_2+\left\{\frac{\mu\sin(\theta)}{r}[-J_1(r\beta_2)\right.$$
$$\left.+jrk_z\chi_2J_2(r\beta_2)]\right\}\hat{R}_1+\left\{\frac{\mu\sin(\theta)}{r}[-Y_1(r\beta_2)+jrk_z\chi_2Y_2(r\beta_2)]\right\}\hat{R}_2$$
$$+\left\{\frac{\mu\sin(\theta)}{r}jk_z[-J_1(r\beta_2)+r\beta_2J_0(r\beta_2)]\right\}\hat{S}_1$$
$$+\left\{\frac{\mu\sin(\theta)}{r}jk_z[-Y_1(r\beta_2)+r\beta_2Y_0(r\beta_2)]\right\}\hat{S}_2$$
$$+\left\{\frac{j\mu k_z}{r}\{-r\beta_2J_0(r\beta_2)\cos(\theta)+J_1(r\beta_2)[\cos(\theta)+r\beta_2)]\}\right\}\hat{T}_1$$
$$+\left\{\frac{j\mu k_z}{r}\{-r\beta_2Y_0(r\beta_2)\cos(\theta)+Y_1(r\beta_2)[\cos(\theta)+r\beta_2)]\}\right\}\hat{T}_2. \quad (8)$$

From section **??** and section **??**, there are ten independent unknown variables which have to be solved to find the pressure inside the elastic tube. The closed form expressions of boundary conditions required for solving the unknown variables are discussed in 2.

2 BOUNDARY CONDITIONS

In this section, closed form expressions of the boundary conditions that are required to compute the unknown variables are given.

2.1 Radial stress at the inner surface of the tube

The radial component of the normal stress τ_{rr} is equal to the negative of the acoustic pressure p_f at the inner surface of the tube. This can be expressed as

$$\hat{\tau}_{rr}(a, k_\theta, k_z, \omega) = -\hat{p}_f(a, k_\theta, k_z, \omega) \tag{9}$$

Substituting τ_{rr} from Eq. (2) and $\hat{p}_f$ from Eq. (**??**), the above equation can be represented as,

$$M_{1,1}\hat{P}_1 + M_{1,2}\hat{P}_2 + M_{1,3}\hat{Q}_1 + M_{1,4}\hat{Q}_2 + M_{1,5}\hat{R}_1 + M_{1,6}\hat{R}_2 + M_{1,7}\hat{S}_1 + M_{1,8}\hat{S}_2 + M_{1,9}\hat{T}_1 + M_{1,10}\hat{T}_2 + M_{1,11}\hat{P}_{f1} + M_{1,12}\hat{P}_{f2} = 0, \tag{10}$$

where

$$M_{1,1} = \sum_{N=0}^{1}\left\{ - j\lambda k_z[J_0(a\beta_1) + J_1(a\beta_1)\cos(\theta)] + \chi_1\left[\frac{2\mu[J_1(a\beta_1) - J_0(a\beta_1)\cos(\theta)]}{a} + \frac{4\mu J_1(a\beta_1)\cos(\theta)}{a^2\beta_1} - (\lambda + 2\mu)[J_0(a\beta_1) + J_1(a\beta_1)\cos(\theta)]\beta_1\right]\right\}\mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{11}$$

$$M_{1,2} = \sum_{N=0}^{1}\left\{\frac{1}{a^2\beta_1}\sin(\theta)\left[- 2a\mu\beta_1\chi_1 J_0(a\beta_1) + J_1(a\beta_1)\{4\mu\chi_1 - a^2\beta_1[j\lambda k_z + (\lambda + 2\mu)\beta_1\chi_1]\}\right]\right\}\mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{12}$$

$$M_{1,3} = \sum_{N=0}^{1}\left\{ - j\lambda k_z[Y_0(a\beta_1) + Y_1(a\beta_1)\cos(\theta)] + \chi_1\left[\frac{2\mu[Y_1(a\beta_1) - Y_0(a\beta_1)\cos(\theta)]}{a} + \frac{4\mu Y_1(a\beta_1)\cos(\theta)}{a^2\beta_1} - (\lambda + 2\mu)[Y_0(a\beta_1) + Y_1(a\beta_1)\cos(\theta)]\beta_1\right]\right\}\mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{13}$$

$$M_{1,4} = \sum_{N=0}^{1}\left\{\frac{1}{a^2\beta_1}\sin(\theta)\left[- 2a\mu\beta_1\chi_1 Y_0(a\beta_1) + Y_1(a\beta_1)\{4\mu\chi_1 - a^2\beta_1[j\lambda k_z + (\lambda + 2\mu)\beta_1\chi_1]\}\right]\right\}\mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{14}$$

$$M_{1,5} = \sum_{N=0}^{1}\left\{ - j\lambda k_z[J_0(a\beta_2) + J_1(a\beta_2)\cos(\theta)] + \chi_2\left[\frac{2\mu[J_1(a\beta_2) - 2J_0(a\beta_2)\cos(\theta)]}{a} + \frac{8\mu J_1(a\beta_2)\cos(\theta)}{a^2\beta_2} - (\lambda + 2\mu)[J_0(a\beta_2) + J_1(a\beta_2)\cos(\theta)]\beta_2\right]\right\}\mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{15}$$

$$M_{1,6} = \sum_{N=0}^{1}\left\{ - j\lambda k_z[Y_0(a\beta_2) + Y_1(a\beta_2)\cos(\theta)] + \frac{\chi_2}{a^2\beta_2}\left[- a\beta_2 Y_0(a\beta_2)[4\mu\cos(\theta) + a\beta_2(\lambda + 2\mu)] + Y_1(a\beta_2)\{8\mu\cos(\theta) + a\beta_2[2\mu - a\beta_2(\lambda + 2\mu)\cos(\theta)]\}\right]\right\}\mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{16}$$

$$M_{1,7} = \sum_{N=0}^{1} \left\{ \frac{2\mu\cos(\theta)}{a^2} \left[-2J_1(a\beta_2) + a\beta_2 J_0(a\beta_2) \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{17}$$

$$M_{1,8} = \sum_{N=0}^{1} \left\{ \frac{2\mu\cos(\theta)}{a^2} \left[-2Y_1(a\beta_2) + a\beta_2 Y_0(a\beta_2) \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{18}$$

$$M_{1,9} = \sum_{N=0}^{1} \left\{ \frac{2\mu\sin(\theta)}{a^2} \left[-2J_1(a\beta_2) + a\beta_2 J_0(a\beta_2) \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{19}$$

$$M_{1,10} = \sum_{N=0}^{1} \left\{ \frac{2\mu\sin(\theta)}{a^2} \left[-2Y_1(a\beta_2) + a\beta_2 Y_0(a\beta_2) \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{20}$$

$$M_{1,11} = \sum_{N=0}^{1} \left[J_0(\alpha a) + J_1(\alpha a)\cos(\theta) \right] \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{21}$$

$$M_{1,12} = \sum_{N=0}^{1} \left[J_1(\alpha a)\sin(\theta) \right] \mathrm{e}^{-j2\pi k_\theta \theta/N}. \tag{22}$$

2.2 Radial stress at the outer surface of the tube

The radial component of the normal stress τ_{rr} is equal to the negative of the external turbulent pressure p_0 at the outer surface of the tube. This can be expressed as

$$\hat{\tau}_{rr}(b, k_\theta, k_z, \omega) = -\hat{p}_0(k_z, \omega). \tag{23}$$

Substituting τ_{rr} from Eq. 2 and $\hat{p}_0$ from Eq. (**??**), the above equation can be represented as,

$$M_{2,1}\hat{P}_1 + M_{2,2}\hat{P}_2 + M_{2,3}\hat{Q}_1 + M_{2,4}\hat{Q}_2 + M_{2,5}\hat{R}_1 + M_{2,6}\hat{R}_2 + M_{2,7}\hat{S}_1 + M_{2,8}\hat{S}_2 + M_{2,9}\hat{T}_1 + M_{2,10}\hat{T}_2 + M_{2,11}\hat{P}_{f1} + M_{2,12}\hat{P}_{f2} = l_2, \tag{24}$$

where

$$M_{2,1} = \sum_{N=0}^{1} \left\{ -j\lambda k_z [J_0(b\beta_1) + J_1(b\beta_1)\cos(\theta)] + \chi_1 \left[\frac{2\mu[J_1(b\beta_1) - J_0(b\beta_1)\cos(\theta)]}{a} + \frac{4\mu J_1(b\beta_1)\cos(\theta)}{b^2\beta_1} - (\lambda + 2\mu)[J_0(b\beta_1) + J_1(b\beta_1)\cos(\theta)]\beta_1 \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{25}$$

$$M_{2,2} = \sum_{N=0}^{1} \left\{ \frac{1}{b^2\beta_1} \sin(\theta) \left[-2b\mu\beta_1\chi_1 J_0(a\beta_1) + J_1(b\beta_1)\{4\mu\chi_1 - b^2\beta_1[j\lambda k_z + (\lambda + 2\mu)\beta_1\chi_1]\} \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{26}$$

$$M_{2,3} = \sum_{N=0}^{1} \left\{ -j\lambda k_z [Y_0(b\beta_1) + Y_1(b\beta_1)\cos(\theta)] + \chi_1 \left[\frac{2\mu[Y_1(b\beta_1) - Y_0(b\beta_1)\cos(\theta)]}{b} + \frac{4\mu Y_1(b\beta_1)\cos(\theta)}{b^2\beta_1} - (\lambda + 2\mu)[Y_0(b\beta_1) + Y_1(b\beta_1)\cos(\theta)]\beta_1 \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{27}$$

$$M_{2,4} = \sum_{N=0}^{1} \left\{ \frac{1}{b^2\beta_1} \sin(\theta) \left[-2b\mu\beta_1\chi_1 Y_0(b\beta_1) + Y_1(b\beta_1)\{4\mu\chi_1 - b^2\beta_1[j\lambda k_z + (\lambda + 2\mu)\beta_1\chi_1]\} \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{28}$$

$$M_{2,5} = \sum_{N=0}^{1} \left\{ -j\lambda k_z [J_0(b\beta_2) + J_1(b\beta_2)\cos(\theta)] + \chi_2 \left[\frac{2\mu[J_1(b\beta_2) - 2J_0(b\beta_2)\cos(\theta)]}{b} + \frac{8\mu J_1(b\beta_2)\cos(\theta)}{b^2\beta_2} - (\lambda + 2\mu)[J_0(b\beta_2) + J_1(b\beta_2)\cos(\theta)]\beta_2 \right] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{29}$$

$$M_{2,6} = \sum_{N=0}^{1} \left\{ - j\lambda k_z [Y_0(b\beta_2) + Y_1(b\beta_2)\cos(\theta)] + \frac{\chi_2}{b^2\beta_2}\Big[- b\beta_2 Y_0(b\beta_2)[4\mu\cos(\theta) + b\beta_2(\lambda+2\mu)] + Y_1(b\beta_2)\{8\mu\cos(\theta) + b\beta_2[2\mu - b\beta_2(\lambda+2\mu)\cos(\theta)]\}\Big]\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{30}$$

$$M_{2,7} = \sum_{N=0}^{1} \left\{ \frac{2\mu\cos(\theta)}{b^2}\Big[- 2J_1(b\beta_2) + b\beta_2 J_0(b\beta_2)\Big]\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{31}$$

$$M_{2,8} = \sum_{N=0}^{1} \left\{ \frac{2\mu\cos(\theta)}{b^2}\Big[- 2Y_1(b\beta_2) + b\beta_2 Y_0(b\beta_2)\Big]\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{32}$$

$$M_{2,9} = \sum_{N=0}^{1} \left\{ \frac{2\mu\sin(\theta)}{b^2}\Big[- 2J_1(b\beta_2) + b\beta_2 J_0(b\beta_2)\Big]\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{33}$$

$$M_{2,10} = \sum_{N=0}^{1} \left\{ \frac{2\mu\sin(\theta)}{b^2}\Big[- 2Y_1(b\beta_2) + b\beta_2 Y_0(b\beta_2)\Big]\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{34}$$

$$M_{2,11} = 0, \tag{35}$$

$$M_{2,12} = 0, \tag{36}$$

$$\tag{37}$$

$$l_2 = - \sum_{N=0}^{1} \left\{ C\rho^2 \nu_*^3 b^2 \Big[(k_z b)^2 + \frac{1}{12}\Big]\Big[\frac{(\omega b - u_c k_z b)^2}{h^2\nu_*^2} + (k_z b)^2 + b^{-2}\Big]^{-2.5}\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}. \tag{38}$$

2.3 The shear stress τ_{rz} at the inner surface of the tube

The shear free boundary condition at $r - z$ plane at the inner surface of the tube can be expressed as

$$\hat{\tau}_{rz}(a, k_\theta, k_z, \omega) = 0. \tag{39}$$

Substituting $\hat{\tau}_{rz}$ from Eq. (6), the above equation can be represented as

$$M_{3,1}\hat{P}_1 + M_{3,2}\hat{P}_2 + M_{3,3}\hat{Q}_1 + M_{3,4}\hat{Q}_2 + M_{3,5}\hat{R}_1 + M_{3,6}\hat{R}_2 + M_{3,7}\hat{S}_1 + M_{3,8}\hat{S}_2 + M_{3,9}\hat{T}_1 + M_{3,10}\hat{T}_2 + M_{3,11}\hat{P}_{f1} + M_{3,12}\hat{P}_{f2} = 0, \tag{40}$$

where

$$M_{3,1} = \sum_{N=0}^{1} \left\{ \frac{\mu}{a\beta_1}\{a\beta_1 J_0(a\beta_1)\cos(\theta) - J_1(a\beta_1)[\cos(\theta) + a\beta_1]\}(\beta_1 - jk_z\chi_1)\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{41}$$

$$M_{3,2} = \sum_{N=0}^{1} \left\{ \frac{\mu\sin(\theta)}{a\beta_1}[-J_1(a\beta_1) + a\beta_1 J_0(a\beta_1)](\beta_1 - jk_z\chi_1)\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{42}$$

$$M_{3,3} = \sum_{N=0}^{1} \left\{ \frac{\mu}{a\beta_1}\{a\beta_1 Y_0(a\beta_1)\cos(\theta) - Y_1(a\beta_1)[\cos(\theta) + a\beta_1]\}(\beta_1 - jk_z\chi_1)\right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{43}$$

$$M_{3,4} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a\beta_1} [-Y_1(a\beta_1) + a\beta_1 Y_0(a\beta_1)](\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{44}$$

$$M_{3,5} = \sum_{N=0}^{1} \left\{ \mu J_0(a\beta_2)\cos(\theta)(\beta_2 - jk_z\chi_2) + \frac{\mu J_1(a\beta_2)}{a\beta_2} \{-\beta_2[\cos(\theta) + a\beta_2] + jk_z[2\cos(\theta) + a\beta_2]\chi_2\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{45}$$

$$M_{3,6} = \sum_{N=0}^{1} \left\{ \mu Y_0(a\beta_2)\cos(\theta)(\beta_2 - jk_z\chi_2) + \frac{\mu Y_1(a\beta_2)}{a\beta_2} \{-\beta_2[\cos(\theta) + a\beta_2] + jk_z[2\cos(\theta) + a\beta_2]\chi_2\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{46}$$

$$M_{3,7} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{a} J_1(a\beta_2)\cos(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{47}$$

$$M_{3,8} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{a} Y_1(a\beta_2)\cos(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{48}$$

$$M_{3,9} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{a} J_1(a\beta_2)\sin(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{49}$$

$$M_{3,10} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{a} Y_1(a\beta_2)\sin(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{50}$$

$$M_{3,11} = 0, \tag{51}$$

$$M_{3,12} = 0. \tag{52}$$

2.4 The shear stress τ_{rz} at the outer surface of the tube

The shear free boundary condition at $r - z$ plane at the outer surface of the tube can be expressed as

$$\hat{\tau}_{rz}(b, k_\theta, k_z, \omega) = 0. \tag{53}$$

Substituting $\hat{\tau}_{rz}$, the above equation can be represented as

$$M_{4,1}\hat{P}_1 + M_{4,2}\hat{P}_2 + M_{4,3}\hat{Q}_1 + M_{4,4}\hat{Q}_2 + M_{4,5}\hat{R}_1 + M_{4,6}\hat{R}_2 + M_{4,7}\hat{S}_1 + M_{4,8}\hat{S}_2 + M_{4,9}\hat{T}_1 + M_{4,10}\hat{T}_2 + M_{4,11}\hat{P}_{f1} + M_{4,12}\hat{P}_{f2} = 0, \tag{54}$$

where

$$M_{4,1} = \sum_{N=0}^{1} \left\{ \frac{\mu}{b\beta_1} \{b\beta_1 J_0(b\beta_1)\cos(\theta) - J_1(b\beta_1)[\cos(\theta) + b\beta_1]\}(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{55}$$

$$M_{4,2} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b\beta_1} [-J_1(b\beta_1) + b\beta_1 J_0(b\beta_1)](\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{56}$$

$$M_{4,3} = \sum_{N=0}^{1} \left\{ \frac{\mu}{b\beta_1} \{b\beta_1 Y_0(b\beta_1)\cos(\theta) - Y_1(b\beta_1)[\cos(\theta) + b\beta_1]\}(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{57}$$

$$M_{4,4} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b\beta_1} [-Y_1(b\beta_1) + b\beta_1 Y_0(b\beta_1)](\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{58}$$

$$M_{4,5} = \sum_{N=0}^{1} \left\{ \mu J_0(b\beta_2)\cos(\theta)(\beta_2 - jk_z\chi_2) + \frac{\mu J_1(b\beta_2)}{b\beta_2} \{-\beta_2[\cos(\theta) + b\beta_2] + jk_z[2\cos(\theta) + b\beta_2]\chi_2\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{59}$$

$$M_{4,6} = \sum_{N=0}^{1} \left\{ \mu Y_0(b\beta_2)\cos(\theta)(\beta_2 - jk_z\chi_2) + \frac{\mu Y_1(b\beta_2)}{b\beta_2} \{-\beta_2[\cos(\theta) + b\beta_2] + jk_z[2\cos(\theta) + b\beta_2]\chi_2\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{60}$$

$$M_{4,7} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{b} J_1(b\beta_2)\cos(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{61}$$

$$M_{4,8} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{b} Y_1(b\beta_2)\cos(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{62}$$

$$M_{4,9} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{b} J_1(b\beta_2)\sin(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{63}$$

$$M_{4,10} = \sum_{N=0}^{1} \left\{ \frac{-j\mu k_z}{b} Y_1(b\beta_2)\sin(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{64}$$

$$M_{4,11} = 0, \tag{65}$$

$$M_{4,12} = 0. \tag{66}$$

2.5 The shear stress $\tau_{r\theta}$ at the inner surface of the tube

$$\tau_{r\theta}(a, k_z, k_\theta, \omega) = 0. \tag{67}$$

Substituting $\hat{\tau}_{r\theta}$ from Eq. (7), the above equation can be represented as

$$M_{5,1}\hat{P}_1 + M_{5,2}\hat{P}_2 + M_{5,3}\hat{Q}_1 + M_{5,4}\hat{Q}_2 + M_{5,5}\hat{R}_1 + M_{5,6}\hat{R}_2 + M_{5,7}\hat{S}_1 + M_{5,8}\hat{S}_2 + M_{5,9}\hat{T}_1 + M_{5,10}\hat{T}_2 + M_{5,11}\hat{P}_{f1} + M_{5,12}\hat{P}_{f2} = 0, \tag{68}$$

where

$$M_{5,1} = \sum_{N=0}^{1} \left\{ \frac{2}{a} \mu J_2(a\beta_1)\sin(\theta)\chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{69}$$

$$M_{5,2} = \sum_{N=0}^{1} \left\{ \frac{-2}{a} \mu J_2(a\beta_1)\cos(\theta)\chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{70}$$

$$M_{5,3} = \sum_{N=0}^{1} \left\{ \frac{2}{a} \mu Y_2(a\beta_1)\sin(\theta)\chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{71}$$

$$M_{5,4} = \sum_{N=0}^{1} \left\{ \frac{-2}{a} \mu Y_2(a\beta_1) \cos(\theta) \chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{72}$$

$$M_{5,5} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta) \chi_2}{a^2 \beta_2} [-4a\beta_2 J_0(a\beta_2) + J_1(a\beta_2)(8 - a^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{73}$$

$$M_{5,6} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta) \chi_2}{a^2 \beta_2} [-4a\beta_2 Y_0(a\beta_2) + Y_1(a\beta_2)(8 - a^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{74}$$

$$M_{5,7} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a^2} [2a\beta_2 J_0(a\beta_2) + J_1(a\beta_2)(-4 + a^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{75}$$

$$M_{5,8} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a^2} [2a\beta_2 Y_0(a\beta_2) + Y_1(a\beta_2)(-4 + a^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{76}$$

$$M_{5,9} = \sum_{N=0}^{1} \left\{ \frac{1}{a^2} \Big[-\mu a\beta_2 J_0(a\beta_2)(2\cos(\theta) + a\beta_2) + \mu J_1(a\beta_2)\{4\cos(\theta) + a\beta_2[2 - a\beta_2 \cos(\theta)]\} \Big] \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{77}$$

$$M_{5,10} = \sum_{N=0}^{1} \left\{ \frac{1}{a^2} \Big[-\mu a\beta_2 Y_0(a\beta_2)(2\cos(\theta) + a\beta_2) + \mu Y_1(a\beta_2)\{4\cos(\theta) + a\beta_2[2 - a\beta_2 \cos(\theta)]\} \Big] \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{78}$$

$$M_{5,11} = 0, \tag{79}$$

$$M_{5,12} = 0. \tag{80}$$

2.6 The shear stress $\tau_{r\theta}$ at the outer surface of the tube

The shear free boundary condition at $r - \theta$ plane at the outer surface of the tube can be expressed as

$$\tau_{r\theta}(b, k_z, k_\theta, \omega) = 0. \tag{81}$$

Substituting $\hat{\tau}_{r\theta}$, the above equation can be represented as

$$M_{6,1}\hat{P}_1 + M_{6,2}\hat{P}_2 + M_{6,3}\hat{Q}_1 + M_{6,4}\hat{Q}_2 + M_{6,5}\hat{R}_1 + M_{6,6}\hat{R}_2 + M_{6,7}\hat{S}_1 + M_{6,8}\hat{S}_2 + M_{6,9}\hat{T}_1 + M_{6,10}\hat{T}_2 + M_{6,11}\hat{P}_{f1} + M_{6,12}\hat{P}_{f2} = 0, \tag{82}$$

where,

$$M_{6,1} = \sum_{N=0}^{1} \left\{ \frac{2}{b} \mu J_2(b\beta_1) \sin(\theta) \chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{83}$$

$$M_{6,2} = \sum_{N=0}^{1} \left\{ \frac{-2}{b} \mu J_2(b\beta_1) \cos(\theta) \chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{84}$$

$$M_{6,3} = \sum_{N=0}^{1} \left\{ \frac{2}{b} \mu Y_2(b\beta_1) \sin(\theta) \chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta / N}, \tag{85}$$

$$M_{6,4} = \sum_{N=0}^{1} \left\{ \frac{-2}{b} \mu Y_2(b\beta_1) \cos(\theta) \chi_1 \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{86}$$

$$M_{6,5} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta) \chi_2}{b^2 \beta_2} [-4b\beta_2 J_0(b\beta_2) + J_1(b\beta_2)(8 - b^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{87}$$

$$M_{6,6} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta) \chi_2}{b^2 \beta_2} [-4b\beta_2 Y_0(b\beta_2) + Y_1(b\beta_2)(8 - b^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{88}$$

$$M_{6,7} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b^2} [2b\beta_2 J_0(b\beta_2) + J_1(b\beta_2)(-4 + b^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{89}$$

$$M_{6,8} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b^2} [2b\beta_2 Y_0(b\beta_2) + Y_1(b\beta_2)(-4 + b^2\beta_2^2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{90}$$

$$M_{6,9} = \sum_{N=0}^{1} \left\{ \frac{1}{b^2} \Big[-\mu b\beta_2 J_0(b\beta_2)(2\cos(\theta) + b\beta_2) + \mu J_1(b\beta_2)\{4\cos(\theta) + b\beta_2[2 - b\beta_2 \cos(\theta)]\} \Big] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{91}$$

$$M_{6,10} = \sum_{N=0}^{1} \left\{ \frac{1}{b^2} \Big[-\mu b\beta_2 Y_0(b\beta_2)(2\cos(\theta) + b\beta_2) + \mu Y_1(b\beta_2)\{4\cos(\theta) + b\beta_2[2 - b\beta_2 \cos(\theta)]\} \Big] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{92}$$

$$M_{6,11} = 0, \tag{93}$$

$$M_{6,12} = 0. \tag{94}$$

2.7 The shear stress $\tau_{z\theta}$ at the inner surface of the tube

The shear free boundary condition at $z - \theta$ plane at the inner surface of the tube can be expressed as

$$\tau_{z\theta}(a, k_z, k_\theta, \omega) = 0. \tag{95}$$

Sustituting $\hat{\tau}_{z\theta}$ from Eq. (8), the above equation can be represented as

$$M_{7,1}\hat{P}_1 + M_{7,2}\hat{P}_2 + M_{7,3}\hat{Q}_1 + M_{7,4}\hat{Q}_2 + M_{7,5}\hat{R}_1 + M_{7,6}\hat{R}_2 + M_{7,7}\hat{S}_1 + M_{7,8}\hat{S}_2 + M_{7,9}\hat{T}_1 + M_{7,10}\hat{T}_2 + M_{7,11}\hat{P}_{f1} + M_{7,12}\hat{P}_{f2} = 0, \tag{96}$$

where

$$M_{7,1} = \sum_{N=0}^{1} \left\{ \frac{-\mu}{a\beta_1} J_1(a\beta_1) \sin(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{97}$$

$$M_{7,2} = \sum_{N=0}^{1} \left\{ \frac{\mu}{a\beta_1} J_1(a\beta_1) \cos(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{98}$$

$$M_{7,3} = \sum_{N=0}^{1} \left\{ \frac{-\mu}{a\beta_1} Y_1(a\beta_1) \sin(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{99}$$

$$M_{7,4} = \sum_{N=0}^{1} \left\{ \frac{\mu}{a\beta_1} Y_1(a\beta_1)\cos(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{100}$$

$$M_{7,5} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a} [-J_1(a\beta_2) + jak_z\chi_2 J_2(a\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{101}$$

$$M_{7,6} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a} [-Y_1(a\beta_2) + jak_z\chi_2 Y_2(a\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{102}$$

$$M_{7,7} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a} jk_z[-J_1(a\beta_2) + a\beta_2 J_0(a\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{103}$$

$$M_{7,8} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{a} jk_z[-Y_1(a\beta_2) + a\beta_2 Y_0(a\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{104}$$

$$M_{7,9} = \sum_{N=0}^{1} \left\{ \frac{j\mu k_z}{a} \{-a\beta_2 J_0(a\beta_2)\cos(\theta) + J_1(a\beta_2)[\cos(\theta) + a\beta_2)]\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{105}$$

$$M_{7,10} = \sum_{N=0}^{1} \left\{ \frac{j\mu k_z}{a} \{-a\beta_2 Y_0(a\beta_2)\cos(\theta) + Y_1(a\beta_2)[\cos(\theta) + a\beta_2)]\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{106}$$

$$M_{7,11} = 0, \tag{107}$$

$$M_{7,12} = 0. \tag{108}$$

2.8 The shear stress $\tau_{z\theta}$ at the outer surface of the tube

The shear free boundary condition at $z - \theta$ plane at the outer surface of the tube can be expressed as

$$\tau_{z\theta}(b, k_z, k_\theta, \omega) = 0. \tag{109}$$

Substituting $\hat{\tau}_{z\theta}$, the above equation can be represented as

$$M_{8,1}\hat{P}_1 + M_{8,2}\hat{P}_2 + M_{8,3}\hat{Q}_1 + M_{8,4}\hat{Q}_2 + M_{8,5}\hat{R}_1 + M_{8,6}\hat{R}_2 + M_{8,7}\hat{S}_1 + M_{8,8}\hat{S}_2 + M_{8,9}\hat{T}_1 + M_{8,10}\hat{T}_2 + M_{8,11}\hat{P}_{f1} + M_{8,12}\hat{P}_{f2} = 0, \tag{110}$$

where

$$M_{8,1} = \sum_{N=0}^{1} \left\{ \frac{-\mu}{b\beta_1} J_1(b\beta_1)\sin(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{111}$$

$$M_{8,2} = \sum_{N=0}^{1} \left\{ \frac{\mu}{b\beta_1} J_1(b\beta_1)\cos(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{112}$$

$$M_{8,3} = \sum_{N=0}^{1} \left\{ \frac{-\mu}{b\beta_1} Y_1(b\beta_1)\sin(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{113}$$

$$M_{8,4} = \sum_{N=0}^{1} \left\{ \frac{\mu}{b\beta_1} Y_1(b\beta_1)\cos(\theta)(\beta_1 - jk_z\chi_1) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{114}$$

$$M_{8,5} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b} [-J_1(b\beta_2) + jbk_z\chi_2 J_2(b\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{115}$$

$$M_{8,6} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b} [-Y_1(b\beta_2) + jbk_z\chi_2 Y_2(b\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{116}$$

$$M_{8,7} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b} jk_z[-J_1(b\beta_2) + b\beta_2 J_0(b\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{117}$$

$$M_{8,8} = \sum_{N=0}^{1} \left\{ \frac{\mu \sin(\theta)}{b} jk_z[-Y_1(b\beta_2) + b\beta_2 Y_0(b\beta_2)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{118}$$

$$M_{8,9} = \sum_{N=0}^{1} \left\{ \frac{j\mu k_z}{b} \{-b\beta_2 J_0(b\beta_2)\cos(\theta) + J_1(b\beta_2)[\cos(\theta) + b\beta_2)]\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{119}$$

$$M_{8,10} = \sum_{N=0}^{1} \left\{ \frac{j\mu k_z}{b} \{-b\beta_2 Y_0(b\beta_2)\cos(\theta) + Y_1(b\beta_2)[\cos(\theta) + b\beta_2)]\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{120}$$

$$M_{8,11} = 0, \tag{121}$$

$$M_{8,12} = 0. \tag{122}$$

2.9 Radial displacement at the inner surface of the tube

The radial displacement of the elastic tube is equal to that of fluid particle at the interior surface of the tube. This can be expressed as

$$\hat{W}_e(a, k_\theta, k_z, \omega) = \hat{U}_{fr}(a, k_\theta, k_z, \omega). \tag{123}$$

Substituting $\hat{W}_e$ from Eq. (**??**) and $\hat{U}_{fr}$ from Eq. (**??**), the above equation can be represented as

$$M_{9,1}\hat{P}_1 + M_{9,2}\hat{P}_2 + M_{9,3}\hat{Q}_1 + M_{9,4}\hat{Q}_2 - M_{9,5}\hat{R}_1 - M_{9,6}\hat{R}_2 + M_{9,7}\hat{S}_1 + M_{9,8}\hat{S}_2 + M_{9,9}\hat{T}_1 + M_{9,10}\hat{T}_2 - M_{9,11}\hat{P}_{f1} + M_{9,12}\hat{P}_{f2} = 0, \tag{124}$$

where

$$M_{9,1} = \sum_{N=0}^{1} \left\{ \frac{-\chi_1}{2} \{2J_1(a\beta_1) + [J_2(a\beta_1) - J_0(a\beta_1)]\cos(\theta)\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{125}$$

$$M_{9,2} = \sum_{N=0}^{1} \left\{ \frac{\chi_1}{2} [J_0(a\beta_1) - J_2(a\beta 1)]\sin(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{126}$$

$$M_{9,3} = \sum_{N=0}^{1} \left\{ \frac{-\chi_1}{2} \{2Y_1(a\beta_1) + [Y_2(a\beta_1) - Y_0(a\beta_1)]\cos(\theta)\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{127}$$

$$M_{9,4} = \sum_{N=0}^{1} \left\{ \frac{\chi_1}{2} [Y_0(a\beta_1) - Y_2(a\beta 1)]\sin(\theta) \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{128}$$

$$M_{9,5} = \sum_{N=0}^{1} \left\{ -\chi_2[J_1(a\beta_2) + J_2(a\beta_2)\cos(\theta)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{129}$$

$$M_{9,6} = \sum_{N=0}^{1} \left\{ -\chi_2[Y_1(a\beta_2) + Y_2(a\beta_2)\cos(\theta)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{130}$$

$$M_{9,7} = \sum_{N=0}^{1} \left\{ \frac{1}{a}[J_1(a\beta_2)\cos(\theta)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{131}$$

$$M_{9,8} = \sum_{N=0}^{1} \left\{ \frac{1}{a}[Y_1(a\beta_2)\cos(\theta)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{132}$$

$$M_{9,9} = \sum_{N=0}^{1} \left\{ \frac{1}{a}[J_1(a\beta_2)\sin(\theta)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{133}$$

$$M_{9,10} = \sum_{N=0}^{1} \left\{ \frac{1}{a}[Y_1(a\beta_2)\sin(\theta)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{134}$$

$$M_{9,11} = \sum_{N=0}^{1} \left\{ \frac{1}{a\rho\omega^2} \{a\alpha J_0(a\alpha)\cos(\theta) - J_1(a\alpha)[a\alpha + \cos(\theta)]\} \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}, \tag{135}$$

$$M_{9,12} = \sum_{N=0}^{1} \left\{ \frac{\sin(\theta)}{a\rho\omega^2} [a\alpha J_0(a\alpha) - J_1(a\alpha)] \right\} \mathrm{e}^{-j2\pi k_\theta \theta/N}. \tag{136}$$

2.10 Azimuthal displacement at the inner surface of the tube

The azimuthal displacement of the the elastic tube is equal to that of the fluid particle at the interior surface of the tube. This can be expressed as

$$\hat{\Theta}_e(a, k_\theta, k_z, \omega) = \hat{U}_{f\theta}(a, k_\theta, k_z, \omega). \tag{137}$$

Substituting $\hat{\Theta}_e$ from Eq. (**??**) and $\hat{U}_{f\theta}$ from Eq. (**??**), the above equation can be represented as

$$M_{10,1}\hat{P}_1 + M_{10,2}\hat{P}_2 + M_{10,3}\hat{Q}_1 + M_{10,4}\hat{Q}_2 + M_{10,5}\hat{R}_1 + M_{10,6}\hat{R}_2 + M_{10,7}\hat{S}_1 + M_{10,8}\hat{S}_2 + M_{10,9}\hat{T}_1 + M_{10,10}\hat{T}_2 + M_{10,11}\hat{P}_{f1} + M_{10,12}\hat{P}_{f2} = 0, \tag{138}$$

where

$$M_{10,1} = \sum_{N=0}^{1} \left\{ \frac{-\chi_1}{a\beta_1}[J_1(a\beta_1)\sin(\theta)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{139}$$

$$M_{10,2} = \sum_{N=0}^{1} \left\{ \frac{\chi_1}{a\beta_1}[J_1(a\beta_1)\cos(\theta)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{140}$$

$$M_{10,3} = \sum_{N=0}^{1} \left\{ \frac{-\chi_1}{a\beta_1}[Y_1(a\beta_1)\sin(\theta)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{141}$$

$$M_{10,4} = \sum_{N=0}^{1} \left\{ \frac{\chi_1}{a\beta_1}[Y_1(a\beta_1)\cos(\theta)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{142}$$

$$M_{10,5} = \sum_{N=0}^{1} \left\{ -\chi_2 J_2(a\beta_2)\sin(\theta) \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{143}$$

$$M_{10,6} = \sum_{N=0}^{1} \left\{ -\chi_2 Y_2(a\beta_2)\sin(\theta) \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{144}$$

$$M_{10,7} = \sum_{N=0}^{1} \left\{ \frac{\sin(\theta)}{a}[J_1(a\beta_2) - a\beta_2 J_0(a\beta_2)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{145}$$

$$M_{10,8} = \sum_{N=0}^{1} \left\{ \frac{\sin(\theta)}{a}[Y_1(a\beta_2) - a\beta_2 Y_0(a\beta_2)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{146}$$

$$M_{10,9} = \sum_{N=0}^{1} \left\{ \beta_2 J_0(a\beta_2)\cos(\theta) - \frac{1}{a}\{J_1(a\beta_2)[a\beta_2 + \cos(\theta)]\} \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{147}$$

$$M_{10,10} = \sum_{N=0}^{1} \left\{ \beta_2 Y_0(a\beta_2)\cos(\theta) - \frac{1}{a}\{Y_1(a\beta_2)[a\beta_2 + \cos(\theta)]\} \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{148}$$

$$M_{10,11} = \sum_{N=0}^{1} \left\{ \frac{-1}{a\rho\omega^2}[J_1(a\alpha)\sin(\theta)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}, \tag{149}$$

$$M_{10,12} = \sum_{N=0}^{1} \left\{ \frac{1}{a\rho\omega^2}[J_1(a\alpha)\cos(\theta)] \right\} \mathbf{e}^{-j2\pi k_\theta \theta/N}. \tag{150}$$

3 MATRIX EQUATION

There are ten independent unknown variables that is required to be solved for estimating on-axis flow noise but the boundary conditions involves all twelve unknown variables. This section discusses the conversion of the non linear equation into linear equation and also shows the matrix form of the equations.

$$\begin{bmatrix} a_{11} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & a_{4,9} & a_{4,10} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & a_{5,9} & a_{5,10} \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & a_{6,9} & a_{6,10} \\ a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} & a_{7,8} & a_{7,9} & a_{7,10} \\ a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & a_{8,8} & a_{8,9} & a_{8,10} \\ a_{9,1} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & a_{9,8} & a_{9,9} & a_{9,10} \\ a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} \end{bmatrix} \begin{Bmatrix} \hat{P}_2 \\ \hat{Q}_1 \\ \hat{Q}_2 \\ \hat{R}_1 \\ \hat{R}_2 \\ \hat{S}_2 \\ \hat{T}_1 \\ \hat{T}_2 \\ \hat{P}_{f1} \\ \hat{P}_{f2} \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ b_10 \end{Bmatrix}$$

where

$$a_{11} = (M_{1,2} \times M_{2,1} - M_{2,2} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,2} \times M_{3,1} - M_{3,2} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{151}$$

$$a_{1,2} = (M_{1,3} \times M_{2,1} - M_{2,3} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,3} \times M_{3,1} - M_{3,3} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{152}$$

$$a_{1,3} = (M_{1,4} \times M_{2,1} - M_{2,4} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,4} \times M_{3,1} - M_{3,4} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{153}$$

$$a_{1,4} = (M_{1,5} \times M_{2,1} - M_{2,5} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,5} \times M_{3,1} - M_{3,5} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{154}$$

$$a_{1,5} = (M_{1,6} \times M_{2,1} - M_{2,6} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,6} \times M_{3,1} - M_{3,6} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{155}$$

$$a_{1,6} = (M_{1,8} \times M_{2,1} - M_{2,8} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,8} \times M_{3,1} - M_{3,8} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{156}$$

$$a_{1,7} = (M_{1,9} \times M_{2,1} - M_{2,9} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,9} \times M_{3,1} - M_{3,9} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{157}$$

$$a_{1,8} = (M_{1,10} \times M_{2,1} - M_{2,10} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,10} \times M_{3,1} - M_{3,10} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{158}$$

$$a_{1,9} = (M_{1,11} \times M_{2,1} - M_{2,11} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,11} \times M_{3,1} - M_{3,11} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{159}$$

$$a_{1,10} = (M_{1,12} \times M_{2,1} - M_{2,12} \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (M_{2,12} \times M_{3,1} - M_{3,12} \times M_{2,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{160}$$

$$a_{2,1} = (M_{2,2} \times M_{3,1} - M_{3,2} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,2} \times M_{4,1} - M_{4,2} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{161}$$

$$a_{2,2} = (M_{2,3} \times M_{3,1} - M_{3,3} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,3} \times M_{4,1} - M_{4,3} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{162}$$

$$a_{2,3} = (M_{2,4} \times M_{3,1} - M_{3,4} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,4} \times M_{4,1} - M_{4,4} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{163}$$

$$a_{2,4} = (M_{2,5} \times M_{3,1} - M_{3,5} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,5} \times M_{4,1} - M_{4,5} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{164}$$

$$a_{2,5} = (M_{2,6} \times M_{3,1} - M_{3,6} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,6} \times M_{4,1} - M_{4,6} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{165}$$

$$a_{2,6} = (M_{2,8} \times M_{3,1} - M_{3,8} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,8} \times M_{4,1} - M_{4,8} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{166}$$

$$a_{2,7} = (M_{2,9} \times M_{3,1} - M_{3,9} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,9} \times M_{4,1} - M_{4,9} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{167}$$

$$a_{2,8} = (M_{2,10} \times M_{3,1} - M_{3,10} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,10} \times M_{4,1} - M_{4,10} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{168}$$

$$a_{2,9} = (M_{2,11} \times M_{3,1} - M_{3,11} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,11} \times M_{4,1} - M_{4,11} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{169}$$

$$a_{2,10} = (M_{2,12} \times M_{3,1} - M_{3,12} \times M_{2,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}) - (M_{3,12} \times M_{4,1} - M_{4,12} \times M_{3,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}), \tag{170}$$

$$a_{3,1} = (M_{3,2} \times M_{4,1} - M_{4,2} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,2} \times M_{5,1} - M_{5,2} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{171}$$

$$a_{3,2} = (M_{3,3} \times M_{4,1} - M_{4,3} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,3} \times M_{5,1} - M_{5,3} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{172}$$

$$a_{3,3} = (M_{3,4} \times M_{4,1} - M_{4,4} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,4} \times M_{5,1} - M_{5,4} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{173}$$

$$a_{3,4} = (M_{3,5} \times M_{4,1} - M_{4,5} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,5} \times M_{5,1} - M_{5,5} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{174}$$

$$a_{3,5} = (M_{3,6} \times M_{4,1} - M_{4,6} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,6} \times M_{5,1} - M_{5,5} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{175}$$

$$a_{3,6} = (M_{3,8} \times M_{4,1} - M_{4,8} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,8} \times M_{5,1} - M_{5,7} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{176}$$

$$a_{3,7} = (M_{3,9} \times M_{4,1} - M_{4,9} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,9} \times M_{5,1} - M_{5,9} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{177}$$

$$a_{3,8} = (M_{3,10} \times M_{4,1} - M_{4,10} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,10} \times M_{5,1} - M_{5,10} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{178}$$

$$a_{3,9} = (M_{3,11} \times M_{4,1} - M_{4,11} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,11} \times M_{5,1} - M_{5,11} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{179}$$

$$a_{3,10} = (M_{3,12} \times M_{4,1} - M_{4,12} \times M_{3,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}) - (M_{4,12} \times M_{5,1} - M_{5,12} \times M_{4,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{180}$$

$$a_{4,1} = (M_{4,2} \times M_{5,1} - M_{5,2} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,2} \times M_{6,1} - M_{6,2} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{181}$$

$$a_{4,2} = (M_{4,3} \times M_{5,1} - M_{5,3} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,3} \times M_{6,1} - M_{6,3} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{182}$$

$$a_{4,3} = (M_{4,4} \times M_{5,1} - M_{5,4} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,4} \times M_{6,1} - M_{6,4} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{183}$$

$$a_{4,4} = (M_{4,5} \times M_{5,1} - M_{5,5} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,5} \times M_{6,1} - M_{6,5} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{184}$$

$$a_{4,5} = (M_{4,6} \times M_{5,1} - M_{5,6} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,6} \times M_{6,1} - M_{6,6} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{185}$$

$$a_{4,6} = (M_{4,8} \times M_{5,1} - M_{5,8} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,8} \times M_{6,1} - M_{6,8} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}),, \tag{186}$$

$$a_{4,7} = (M_{4,9} \times M_{5,1} - M_{5,9} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,9} \times M_{6,1} - M_{6,9} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{187}$$

$$a_{4,8} = (M_{4,10} \times M_{5,1} - M_{5,10} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,10} \times M_{6,1} - M_{6,10} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{188}$$

$$a_{4,9} = (M_{4,11} \times M_{5,1} - M_{5,11} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,11} \times M_{6,1} - M_{6,11} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{189}$$

$$a_{4,10} = (M_{4,12} \times M_{5,1} - M_{5,12} \times M_{4,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}) - (M_{5,12} \times M_{6,1} - M_{6,12} \times M_{5,1}) \times (M_{4,7} \times M_{5,1} - M_{5,7} \times M_{4,1}), \tag{190}$$

$$a_{5,1} = (M_{5,2} \times M_{6,1} - M_{6,2} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,2} \times M_{7,1} - M_{7,2} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{191}$$

$$a_{5,2} = (M_{5,3} \times M_{6,1} - M_{6,3} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,2} \times M_{7,1} - M_{7,3} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{192}$$

$$a_{5,3} = (M_{5,4} \times M_{6,1} - M_{6,4} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,4} \times M_{7,1} - M_{7,4} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{193}$$

$$a_{5,4} = (M_{5,5} \times M_{6,1} - M_{6,5} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,5} \times M_{7,1} - M_{7,5} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{194}$$

$$a_{5,5} = (M_{5,6} \times M_{6,1} - M_{6,6} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,6} \times M_{7,1} - M_{7,6} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{195}$$

$$a_{5,6} = (M_{5,8} \times M_{6,1} - M_{6,8} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,8} \times M_{7,1} - M_{7,8} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{196}$$

$$a_{5,7} = (M_{5,9} \times M_{6,1} - M_{6,9} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,9} \times M_{7,1} - M_{7,9} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{197}$$

$$a_{5,8} = (M_{5,10} \times M_{6,1} - M_{6,10} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,10} \times M_{7,1} - M_{7,10} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{198}$$

$$a_{5,9} = (M_{5,11} \times M_{6,1} - M_{6,11} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,11} \times M_{7,1} - M_{7,11} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{199}$$

$$a_{5,10} = (M_{5,12} \times M_{6,1} - M_{6,12} \times M_{5,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}) - (M_{6,12} \times M_{7,1} - M_{7,12} \times M_{6,1}) \times (M_{5,7} \times M_{6,1} - M_{6,7} \times M_{5,1}), \tag{200}$$

$$a_{6,1} = (M_{6,2} \times M_{7,1} - M_{7,2} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,2} \times M_{8,1} - M_{8,2} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{201}$$

$$a_{6,2} = (M_{6,3} \times M_{7,1} - M_{7,3} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,3} \times M_{8,1} - M_{8,3} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{202}$$

$$a_{6,3} = (M_{6,4} \times M_{7,1} - M_{7,4} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,4} \times M_{8,1} - M_{8,4} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{203}$$

$$a_{6,4} = (M_{6,5} \times M_{7,1} - M_{7,5} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,5} \times M_{8,1} - M_{8,5} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{204}$$

$$a_{6,5} = (M_{6,6} \times M_{7,1} - M_{7,6} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,6} \times M_{8,1} - M_{8,6} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{205}$$

$$a_{6,6} = (M_{6,8} \times M_{7,1} - M_{7,8} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,8} \times M_{8,1} - M_{8,8} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{206}$$

$$a_{6,7} = (M_{6,9} \times M_{7,1} - M_{7,9} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,9} \times M_{8,1} - M_{8,9} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{207}$$

$$a_{6,8} = (M_{6,10} \times M_{7,1} - M_{7,10} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,10} \times M_{8,1} - M_{8,10} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{208}$$

$$a_{6,9} = (M_{6,11} \times M_{7,1} - M_{7,11} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,11} \times M_{8,1} - M_{8,11} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{209}$$

$$a_{6,10} = (M_{6,12} \times M_{7,1} - M_{7,12} \times M_{6,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}) - (M_{7,12} \times M_{8,1} - M_{8,12} \times M_{7,1}) \times (M_{6,7} \times M_{7,1} - M_{7,7} \times M_{6,1}), \tag{210}$$

$$a_{7,1} = (M_{7,2} \times M_{8,1} - M_{8,2} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,2} \times M_{9,1} - M_{9,2} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{211}$$

$$a_{7,2} = (M_{7,3} \times M_{8,1} - M_{8,3} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,3} \times M_{9,1} - M_{9,3} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{212}$$

$$a_{7,3} = (M_{7,4} \times M_{8,1} - M_{8,4} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,4} \times M_{9,1} - M_{9,4} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{213}$$

$$a_{7,4} = (M_{7,5} \times M_{8,1} - M_{8,5} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,5} \times M_{9,1} - M_{9,5} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{214}$$

$$a_{7,5} = (M_{7,6} \times M_{8,1} - M_{8,6} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,6} \times M_{9,1} - M_{9,6} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{215}$$

$$a_{7,6} = (M_{7,8} \times M_{8,1} - M_{8,8} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,8} \times M_{9,1} - M_{9,8} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{216}$$

$$a_{7,7} = (M_{7,9} \times M_{8,1} - M_{8,9} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,9} \times M_{9,1} - M_{9,9} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{217}$$

$$a_{7,8} = (M_{7,10} \times M_{8,1} - M_{8,10} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,10} \times M_{9,1} - M_{9,10} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{218}$$

$$a_{7,9} = (M_{7,11} \times M_{8,1} - M_{8,11} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,11} \times M_{9,1} - M_{9,11} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{219}$$

$$a_{7,10} = (M_{7,12} \times M_{8,1} - M_{8,12} \times M_{7,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}) - (M_{8,12} \times M_{9,1} - M_{9,12} \times M_{8,1}) \times (M_{7,7} \times M_{8,1} - M_{8,7} \times M_{7,1}), \tag{220}$$

$$a_{8,1} = (M_{8,2} \times M_{9,1} - M_{9,2} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,2} \times M_{10,1} - M_{10,2} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{221}$$

$$a_{8,2} = (M_{8,3} \times M_{9,1} - M_{9,3} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,3} \times M_{10,1} - M_{10,3} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{222}$$

$$a_{8,3} = (M_{8,4} \times M_{9,1} - M_{9,4} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,4} \times M_{10,1} - M_{10,4} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{223}$$

$$a_{8,4} = (M_{8,5} \times M_{9,1} - M_{9,5} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,5} \times M_{10,1} - M_{10,5} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{224}$$

$$a_{8,5} = (M_{8,6} \times M_{9,1} - M_{9,6} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,6} \times M_{10,1} - M_{10,6} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{225}$$

$$a_{8,6} = (M_{8,8} \times M_{9,1} - M_{9,8} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,8} \times M_{10,1} - M_{10,8} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{226}$$

$$a_{8,7} = (M_{8,9} \times M_{9,1} - M_{9,9} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,9} \times M_{10,1} - M_{10,9} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{227}$$

$$a_{8,8} = (M_{8,10} \times M_{9,1} - M_{9,10} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,10} \times M_{10,1} - M_{10,10} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{228}$$

$$a_{8,9} = (M_{8,11} \times M_{9,1} - M_{9,11} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,11} \times M_{10,1} - M_{10,11} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{229}$$

$$a_{8,10} = (M_{8,12} \times M_{9,1} - M_{9,12} \times M_{8,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}) - (M_{9,12} \times M_{10,1} - M_{10,12} \times M_{9,1}) \times (M_{8,7} \times M_{9,1} - M_{9,7} \times M_{8,1}), \tag{230}$$

$$a_{9,1} = (M_{9,2} \times M_{10,1} - M_{10,2} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,2} \times M_{1,1} - M_{1,2} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{231}$$

$$a_{9,2} = (M_{9,3} \times M_{10,1} - M_{10,3} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,3} \times M_{1,1} - M_{1,3} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{232}$$

$$a_{9,3} = (M_{9,4} \times M_{10,1} - M_{10,4} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,4} \times M_{1,1} - M_{1,4} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{233}$$

$$a_{9,4} = (M_{9,5} \times M_{10,1} - M_{10,5} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,5} \times M_{1,1} - M_{1,5} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{234}$$

$$a_{9,5} = (M_{9,6} \times M_{10,1} - M_{10,6} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,6} \times M_{1,1} - M_{1,6} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{235}$$

$$a_{9,6} = (M_{9,8} \times M_{10,1} - M_{10,8} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,8} \times M_{1,1} - M_{1,8} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{236}$$

$$a_{9,7} = (M_{9,9} \times M_{10,1} - M_{10,9} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,9} \times M_{1,1} - M_{1,9} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{237}$$

$$a_{9,8} = (M_{9,10} \times M_{10,1} - M_{10,10} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,10} \times M_{1,1} - M_{1,10} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{238}$$

$$a_{9,9} = (M_{9,11} \times M_{10,1} - M_{10,11} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,11} \times M_{1,1} - M_{1,11} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{239}$$

$$a_{9,10} = (M_{9,12} \times M_{10,1} - M_{10,12} \times M_{9,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) - (M_{10,12} \times M_{1,1} - M_{1,12} \times M_{10,1}) \times (M_{9,7} \times M_{10,1} - M_{10,7} \times M_{9,1}), \tag{240}$$

$$a_{10,1} = (M_{10,2} \times M_{1,1} - M_{1,2} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,2} \times M_{2,1} - M_{2,2} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{241}$$

$$a_{10,2} = (M_{10,3} \times M_{1,1} - M_{1,3} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,3} \times M_{2,1} - M_{2,3} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{242}$$

$$a_{10,3} = (M_{10,4} \times M_{1,1} - M_{1,4} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,4} \times M_{2,1} - M_{2,4} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}),, \tag{243}$$

$$a_{10,4} = (M_{10,5} \times M_{1,1} - M_{1,5} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,5} \times M_{2,1} - M_{2,5} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{244}$$

$$a_{10,5} = (M_{10,6} \times M_{1,1} - M_{1,6} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,6} \times M_{2,1} - M_{2,6} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{245}$$

$$a_{10,6} = (M_{10,8} \times M_{1,1} - M_{1,8} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,8} \times M_{2,1} - M_{2,8} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{246}$$

$$a_{10,7} = (M_{10,9} \times M_{1,1} - M_{1,9} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,9} \times M_{2,1} - M_{2,9} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{247}$$

$$a_{10,8} = (M_{10,10} \times M_{1,1} - M_{1,10} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,10} \times M_{2,1} - M_{2,10} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{248}$$

$$a_{10,9} = (M_{10,11} \times M_{1,1} - M_{1,11} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,11} \times M_{2,1} - M_{2,11} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{249}$$

$$a_{10,10} = (M_{10,12} \times M_{1,1} - M_{1,12} \times M_{10,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}) - (M_{1,12} \times M_{2,1} - M_{2,12} \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}), \tag{250}$$

$$b_1 = (-l_2 \times M_{1,1}) \times (M_{2,7} \times M_{3,1} - M_{3,7} \times M_{2,1}) - (l_2 \times M_{3,1}) \times (M_{1,7} \times M_{2,1} - M_{2,7} \times M_{1,1}), \tag{251}$$

$$b_2 = (l_2 \times M_{3,1}) \times (M_{3,7} \times M_{4,1} - M_{4,7} \times M_{3,1}), \tag{252}$$

and

$$b_10 = (l_2 \times M_{1,1}) \times (M_{10,7} \times M_{1,1} - M_{1,7} \times M_{10,1}) \tag{253}$$

The above relations from $a_{1,1}$ to b_10 are the conversions done to make the system of non linear equations in twelve variables to system of linear equation in ten variable to solve for the ten independent unknown variables.