Computer Organization & Architecture

Chapter 9 – Unsigned Multiplication

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Content of this lecture

- 9.3 Multiplication of Unsigned Numbers
  - Manual Multiplication
  - Array Multiplier
  - Sequential Multiplier
  - Summary

Manual Multiplication (1)

- Unsigned & Positive Signed Numbers
  - Example: M= 1101, Q= 1011, calculate P= M×Q

```
    1 1 0 1        Multiplicand
  × 1 0 1 1        Multiplier
  ---------
    1 1 0 1
  1 1 0 1          Partial Products
0 0 0 0
1 1 0 1
---------------
1 0 0 0 1 1 1 1    Product
```

Manual Multiplication (2)

- Manual Multiplication Algorithm
  - Multiplication of the multiplicand by one bit of the multiplier
    - If the multiplier bit is 1, the multiplicand is entered in the appropriate position to be added to added to the partial product. If the multiplier bit is 0, then 0s are entered.
  - Note
    - The multiplication product of two n-bit binary integers results in a product of up to 2n bits in length.

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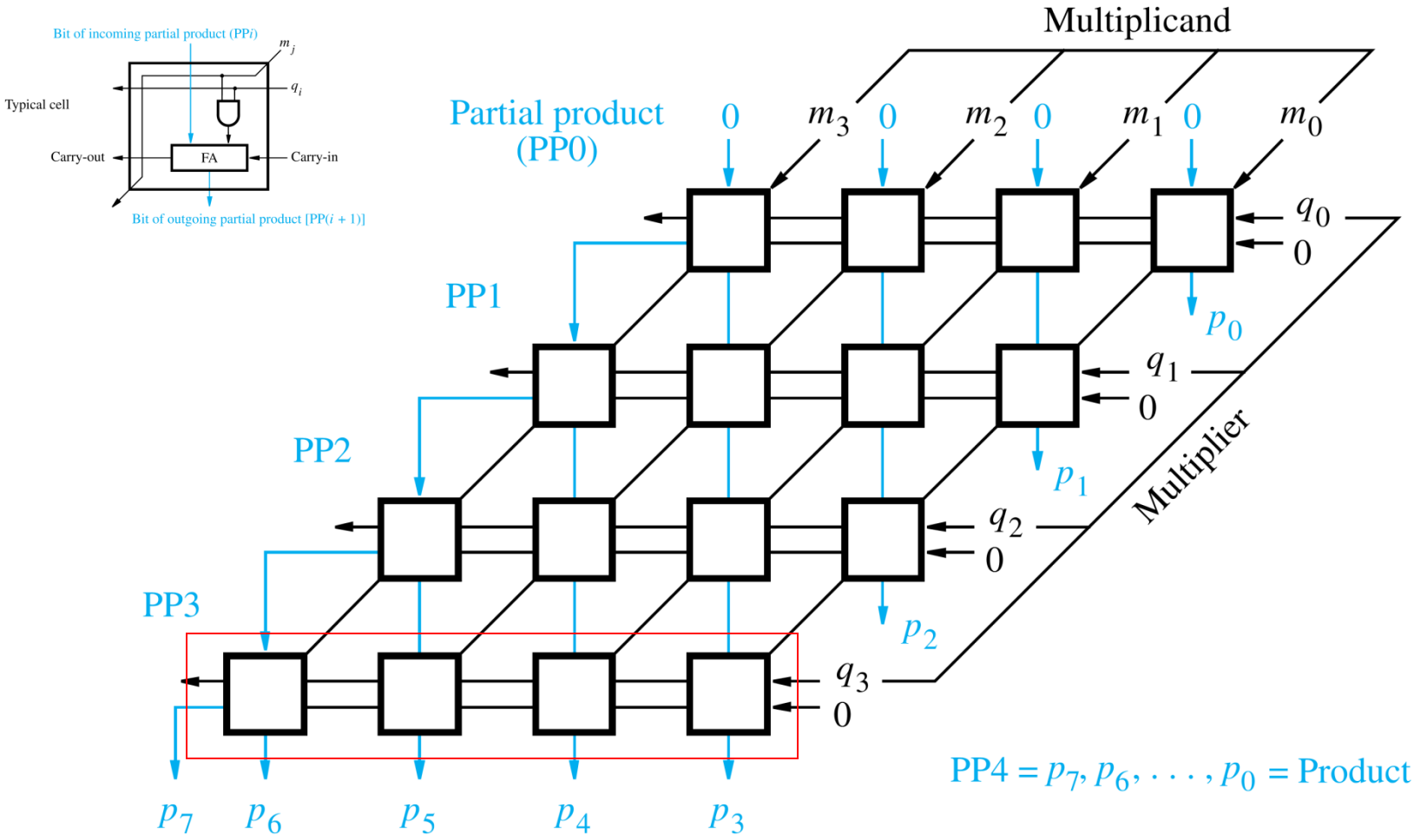
Array Multiplier (1)

- Assume that
  - $M = m_3m_2m_1m_0 \quad Q = q_3q_2q_1q_0$
  - $P = M \times Q = p_7p_6p_5p_4p_3p_2p_1p_0$

$$
\begin{array}{cccccccc}
 & & & m_3 & m_2 & m_1 & m_0 & \\
 & & \times & q_3 & q_2 & q_1 & q_0 & \\
\hline
 & & & m_3q_0 & m_2q_0 & m_1q_0 & m_0q_0 & \\
 & & m_3q_1 & m_2q_1 & m_1q_1 & m_0q_1 & & \\
 & m_3q_2 & m_2q_2 & m_1q_2 & m_0q_2 & & & \\
m_3q_3 & m_2q_3 & m_1q_3 & m_0q_3 & & & & \\
\hline
p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & p_0
\end{array}
$$

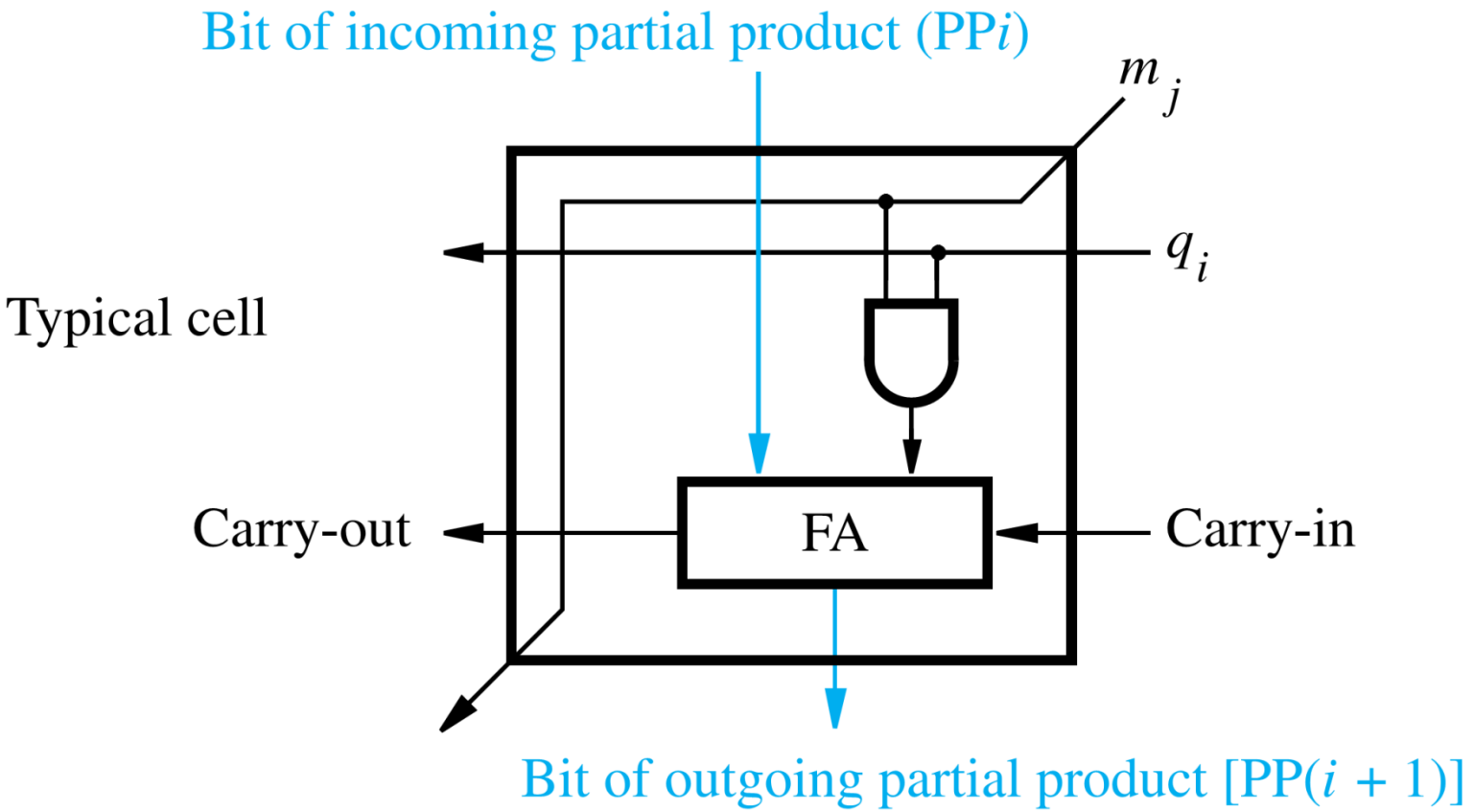
Array Multiplier (2)

- Array Implementation



Array Multiplier (3)

- Array Implementation (ctd.)



Array Multiplier (4)

- Question: When calculating P = M × Q (M and Q are all n bit unsigned binary numbers) , how many FAs and AND gates do we need (using n × n array multiplier)?

A:

FAs : n(n-1)

AND gates : n^2

n= 32, 992 FAs, 1024 AND gates

n= 64, 4032 FAs, 4096 AND gates

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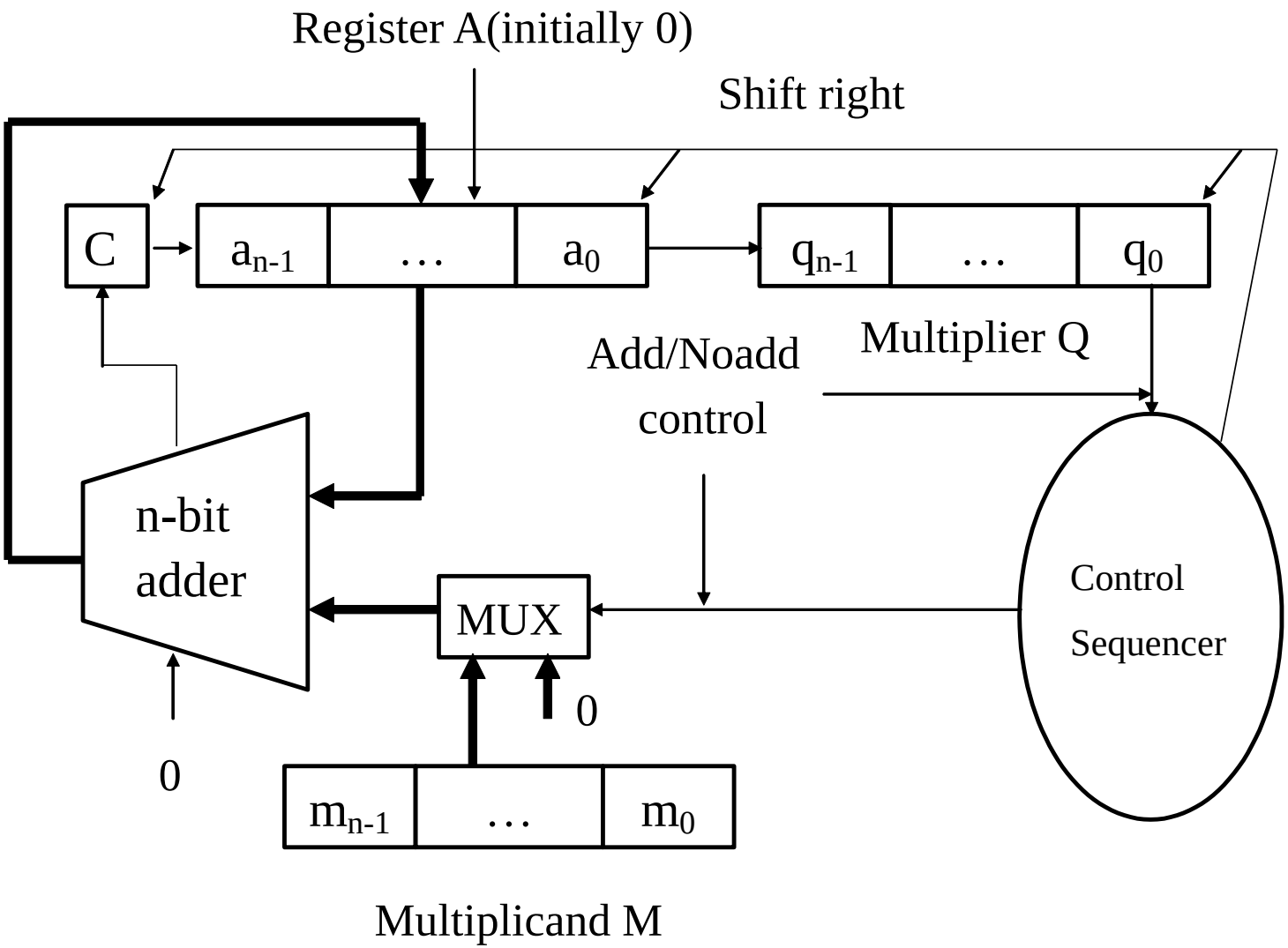
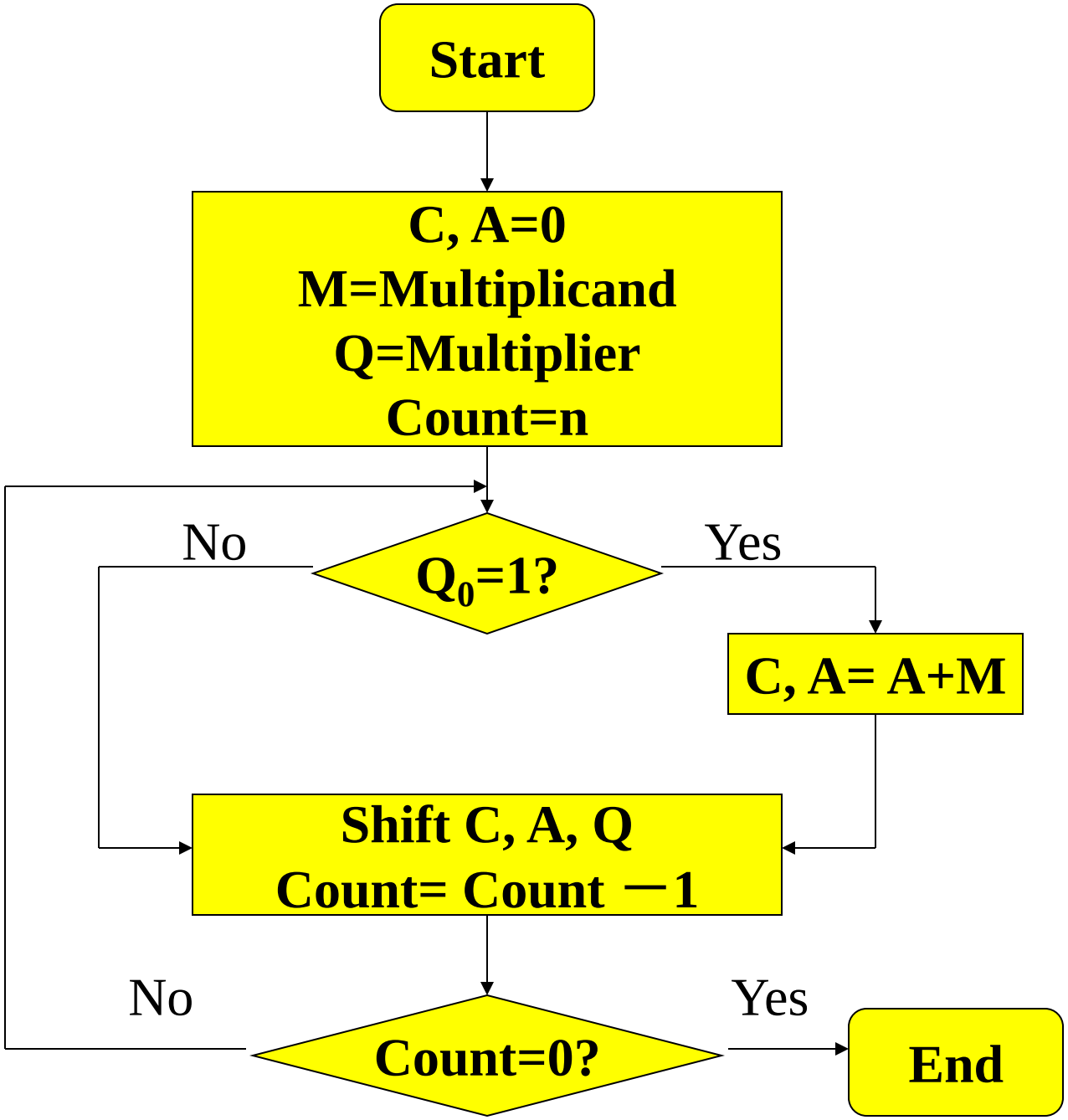
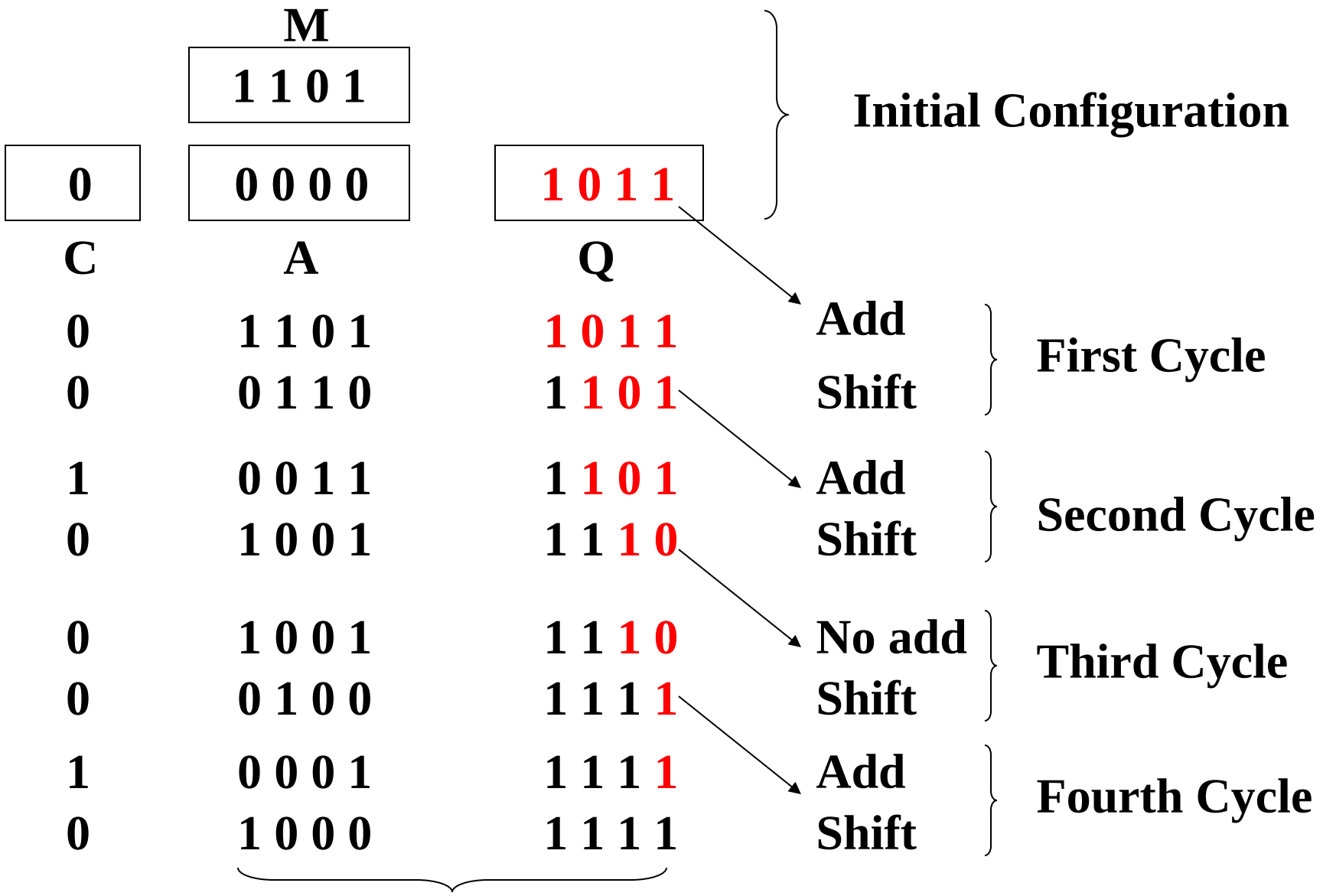
Sequential Multiplier (1)

- Register Configuration




Start
C, A=0
M=Multiplicand
Q=Multiplier
Count=n
No
Q_0=1?
Yes
C, A= A+M
Shift C, A, Q
Count= Count －1
No
Count=0?
Yes
End

Example:M= 1101, Q= 1011, calculate P= M×Q



Summary

- 知识点Sequential Multiplier
- 掌握程度
  - 给定两个整数，能够用机器算法计算出结果，写出整个计算步骤。

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Chapter 9 – Signed Multiplication

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- 9.4 Multiplication of Signed Numbers
  - Signed-Operand Multiplication
  - Booth Algorithm
  - Multiplication Summary
  - Solved Problem Example 9.2 (P374)
  - Summary

How do we handle signed numbers ?

- Solution 1: Convert to unsigned, multiply, then convert back to signed.
  - Works but is slow and a bit cumbersome.
- Solution 2: Directly handle the multiplication with the signed numbers.
  - Booth algorithm recodes the bits to allow for such a computation.
  - Works with 2’s complement numbers directly.

Signed-Operand Multiplication (1)

- If the multiplicand or the multiplier is negative, how should we do multiplication straightly?
- Case1: a negative multiplicand and a positive multiplier.
  - Example: M = 1001, Q = 0011, calculate P = M×Q
    - P = 00011011
    - If M, Q, and P are unsigned non-negative numbers, then: M = 9, Q = 3, P=27
    - If M, Q, and P are signed 2's complement numbers, then: M = -7, Q = 3, P = 27

Signed-Operand Multiplication (2)

- Case 1: a negative multiplicand and a positive multiplier. (ctd.)
  - Example: M = 1001, Q =0011, calculate P = M×Q

$$
\begin{array}{r}
1\ 0\ 0\ 1 \\
\times\ 0\ 0\ 1\ 1 \\
\hline
1\ 1\ 1\ 1\ 1\ 0\ 0\ 1 \\
1\ 1\ 1\ 1\ 0\ 0\ 1 \\
0\ 0\ 0\ 0\ 0\ 0 \\
0\ 0\ 0\ 0\ 0 \\
\hline
1\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \quad = -21
\end{array}
$$

Signed-Operand Multiplication (3)

- Case 2: a negative multiplicand and a negative multiplier
  - Example: M = 1101, Q = 1011, calculate P = M×Q
    - Straightforward multiplication P = 10001111
    - If M, Q, and P are unsigned non-negative numbers, then: M = 13, Q = 11, P = 143
    - If M, Q, and P are signed 2's complement numbers, then: M = - 3, Q = -5, P = - 113

Signed-Operand Multiplication (4)

- Case 2: a negative multiplier and a negative multiplicand (ctd.)
  - Example: M = 1101, Q = 1011, calculate P = M×Q

```
   0 0 1 1
 × 0 1 0 1
-----------
0 0 0 0 0 0 1 1
0 0 0 0 0 0 0 0
0 0 0 0 1 1
0 0 0 0 0
-----------------
0 0 0 0 1 1 1 1   =15
```

M = 0011, 2's complement of M

N = 0101, 2's complement of N

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  - Multiplication Summary
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  - Summary

Booth Algorithm (1)

- Booth Algorithm
  - Reducing number of partial products.
  - Fewer partial products generated for groups of consecutive 0′s and 1′s.
    - Group of consecutive 0′s in multiplier - no new partial product - only shift partial product right one bit position for every 0.
    - Group of m consecutive 1's in multiplier - less than m partial products generated.
      - Notice the following equality (Booth did)
        - $2^J + 2^{J-1} + 2^{J-2} + \ldots + 2^K = 2^{J+1} - 2^K$
      - Example: 0011110 = 0100000 – 0000010 (decimal notation: 30 = 32 – 2)

Booth Algorithm (2)

- Recoding of a multiplier

Multiplier		Version of Multiplicand Selected by bit i
Bit i	Bit i-1	
0	0	0×M
0	1	+1×M
1	0	- 1×M
1	1	0×M

Booth Algorithm (3)

- Recoding of a multiplier (ctd.)
  - Why it works?
    - $b \times (a_{31}a_{30}\ldots a_0)$

      $= a_0 \times b \times 2^0 + a_1 \times b \times 2^1 + \ldots + a_{31} \times b \times 2^{31}$

      $= (0 - a_0) \times b \times 2^0 + (a_0 - a_1) \times b \times 2^1 + \ldots$

      $+(a_{30} - a_{31}) \times b \times 2^{31} + a_{31} \times b \times 2^{32}$

Booth Algorithm(4)

- Examples of Recoding of a Multiplier
  - Example 1
    - 0 0 1 0 1 1 0 0 1 1 1 0 1 0 1 1 0 0
    - 0+1 -1+1 0-1 0+1 0 0-1+1-1+1 0 -1 0 0
  - Example 2

$$
\begin{array}{r}
0\ 1\ 1\ 0\ 1\ (+13) \\
\times\ 1\ 1\ 0\ 1\ 0\ (-6) \\
\hline
\end{array}
$$

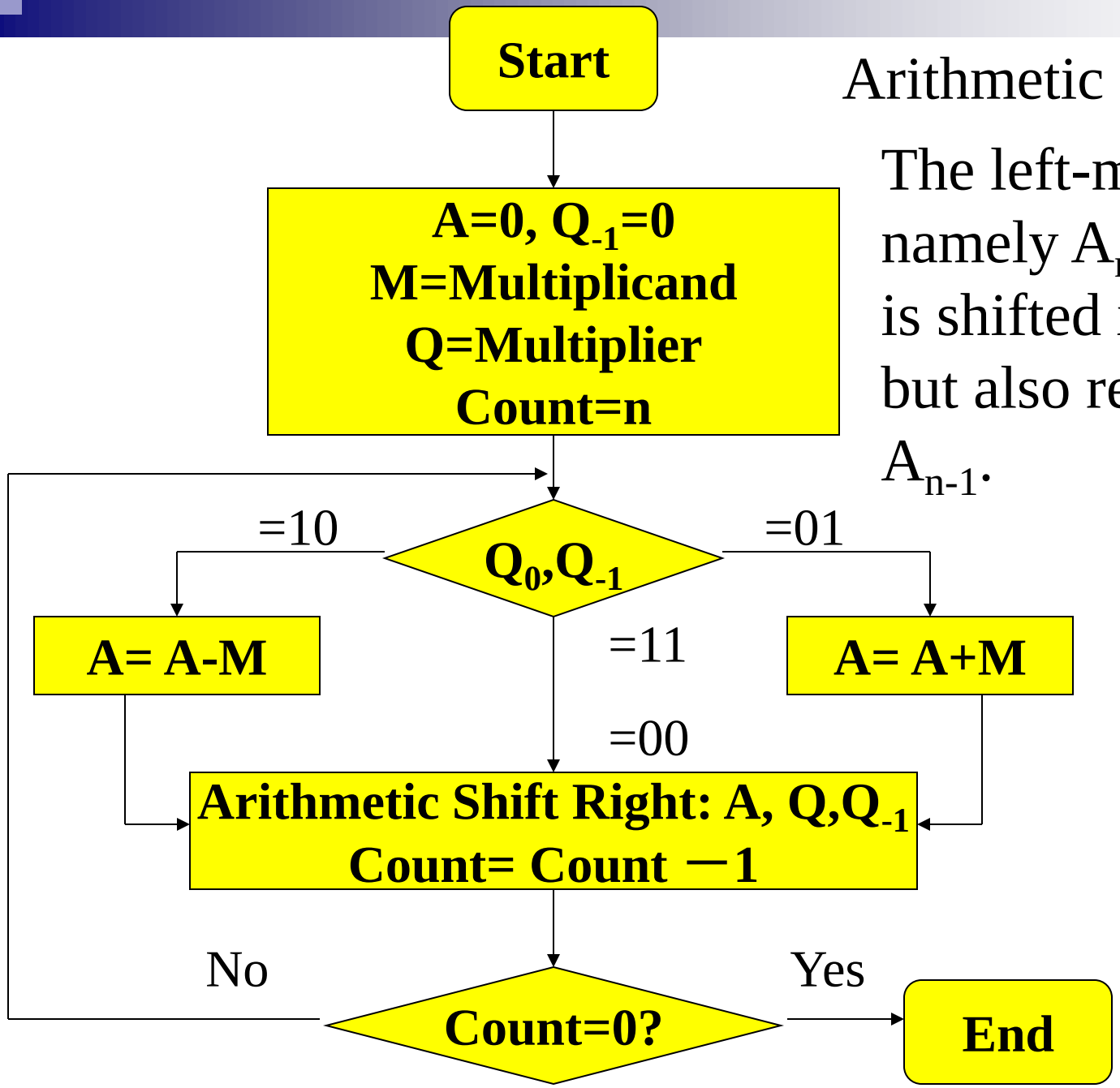
Booth Algorithm (5)

- Examples of Recoding of a multiplier(ctd.)
  - Example 2 (ctd.)

$$
\begin{array}{cccccccccc}
 & & & & & 0 & 1 & 1 & 0 & 1 \\
 & & & & & 0 & -1 & +1 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & & \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

Booth Algorithm (6)

- Hardware Organization
  - As before, the multiplicand and multiplier are placed in the Q and M registers, respectively.
  - There is a 1-bit register placed logically to the right of the least significant bit (Q_0) of the Q register and designated Q_{-1}.
  - The results of the multiplication will appear in the A and Q registers.
  - Control logic scans the bits of the multiplier one at a time. Now, as each bit is examined, the bit to its right is also examined.



Arithmetic Shift Right:

The left-most of A, namely A_{n-1}, not only is shifted into A_{n-2}, but also remains in A_{n-1}.

Booth Algorithm (7)

- Example

A	Q	Q_{-1}	M		
0000	0011	0	0111	Initial Values	
1001	0011	0	0111	A A - M	First Cycle
1100	1001	1	0111	Shift	
1110	0100	1	0111	Shift	Second Cycle
0101	0100	1	0111	A A + M	Third Cycle
0010	1010	0	0111	Shift	
0001	0101	0	0111	Shift	Fourth Cycle

Booth Algorithm (8)

- Performance/Efficiency
  - Good for sequences of 3 or more 1s.
    - Replaces 3 (or more) adds with 1 add and 1 subtract.

Good	0	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1
multiplier	0	0	0	+1	0	0	0	0	–1	0	0	0	+1	0	0	–1

  - Doesn't matter for sequences of 2 1s.
    - Replaces 2 adds with 1 add and 1 subtract (add = subtract).

Ordinary	1	1	0	0	0	1	0	1	1	0	1	1	1	1	0	0
multiplier	0	–1	0	0	+1	–1	+1	0	– 1	+1	0	0	0	–1	0	0

Booth Algorithm (9)

- Performance/Efficiency (ctd.)
  - Actually bad for singleton 1s.
    - Replaces 1 add with 1 add and 1 subtract.

Worst-case	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
multiplier	+1	–1	+1	–1	+1	–1	+1	–1	+1	–1	+1	–1	+1	–1	+1	–1

Booth Algorithm (10)

- Advantages
  - Handle both positive and negative multipliers uniformly.
  - Achieve some efficiency in the number of additions required when the multiplier has a few large blocks of 1s.
  - On average, the speed of doing multiplication with the Booth algorithm is the same as with the normal algorithm.

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Multiplication Summary (1)

- Multiplication involves 2 basic operations: generation of partial products + their accumulation
- 2 ways to speed up
  - Reducing number of partial products and/or
  - Accelerating accumulation
- 3 types of high-speed multipliers
  - Sequential multiplier - generates partial products sequentially and adds each newly generated product to previously accumulated partial product

Multiplication Summary (2)

- 3 types of high-speed multipliers (ctd.)
  - Parallel multiplier - generates partial products in parallel, accumulates using a fast multi-operand adder
  - Array multiplier - array of identical cells generating new partial products; accumulating them simultaneously
    - No separate circuits for generation and accumulation
    - Reduced execution time but increased hardware complexity

Solved Problems

- Example 9.2

Problem: Assuming 6-bit 2's-complement number representation, multiply the multiplicand A = 110101 by the multiplier B = 011011 using both the normal Booth algorithm and the bit-pair recoding Booth algorithm, following the pattern used in Figure 9.15.

Solution: The multiplications are performed as follows:

(*a*) Normal Booth algorithm

						1	1	0	1	0	1
					×	+1	0	−1	+1	0	−1
0	0	0	0	0	0	0	0	1	0	1	1
										0	
1	1	1	1	1	1	0	1	0	1		
0	0	0	0	0	1	0	1	1			
							0				
1	1	1	0	1	0	1					
1	1	1	0	1	1	0	1	0	1	1	1

Summary

- 知识点 Booth Algorithm
  - Recoding of multiplier
  - Using recoded multiplier to multiply
- 掌握程度
  - 使用布斯算法熟练转换乘数
  - 使用手工算法将转换后的乘数与被乘数相乘

Exercise (1)

- 1. In the hardware circuit of implementing Booth algorithm, the multiplier is placed in the Q register. And a 1-bit register Q-1 is placed logically to the right of the least significant bit (Q0) of the Q register. The results of the multiplication will appear in the A and Q registers. At the end of each cycle of computing the product, A, Q, Q-1 are _____ one bit position.
  - A. shifted left
  - B. shifted right
  - C. arithmetically shifted left
  - D. arithmetically shifted right

Exercise (2)

- 2. About Booth algorithm, which of the following is not true?
  - A. It can handle both positive and negative multipliers uniformly.
  - B. It achieves some efficiency in the number of additions required when the multiplier has a few large blocks of 1s.
  - C. Its speed of doing multiplication is always faster than the normal algorithm.
  - D. The multiplier 111100110 will generate three partial products.

Exercise (3)

- 3. What are the main advantages of the Booth algorithm?