Chapter 2
Algorithm Analysis

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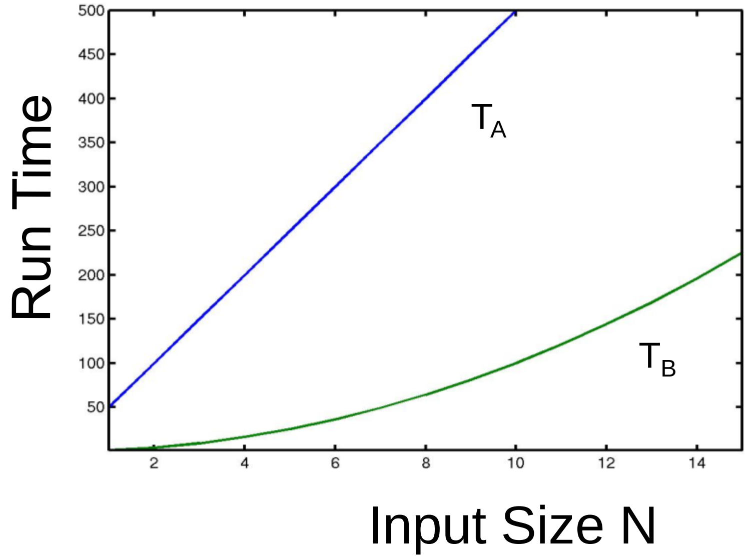
Chapter 2
Algorithm Analysis

- Definitions of Big-Oh and Other Notations
- Common Functions and Growth Rates
- Simple Model of Computation
- Worst Case vs. Average Case Analysis
- How to Perform Analyses
- Comparative Examples

Algorithm Analysis

- 1. Why do we analyze algorithms?

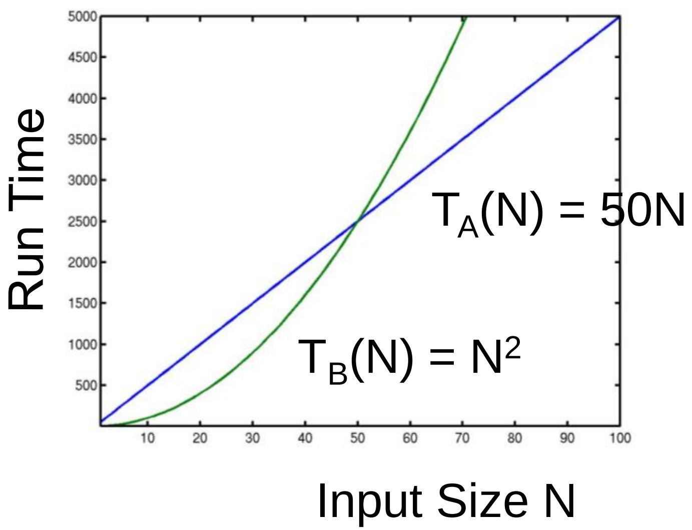
- Suppose you are given two algorithms A and B for solving a problem.
- The running times $T_A(N)$ and $T_B(N)$ of A and B as a function of input size N are given



Which is better?

Algorithm Analysis

- 1. Why do we analyze algorithms?
  - For large N, the running time of A and B is:

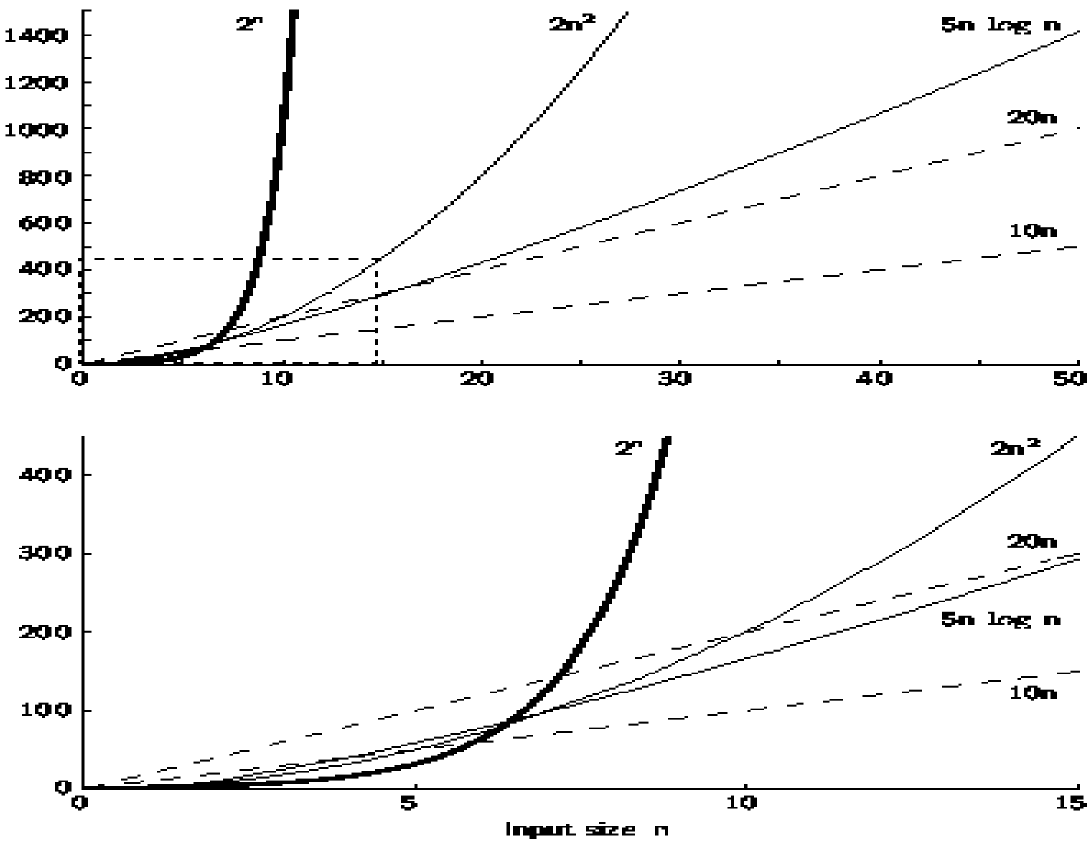


Now which algorithm would you choose?

Algorithm Analysis

- 1. Why do we analyze algorithms?
- 2. How do we measure the efficiency of an algorithm?
  - A. Measuring its running time on my computer and compare its time to that of another algorithm that has already been analyzed.
  - B. Count how many instructions it will execute for an arbitrary input data set.

Growth Rate Graph



$c_1 \log n < c_2 n < c_3 n \log n < c_4 n^2 < c_5 2^n < c_6 n!$

Exercises

P71-Exercises 2.1

1) $2/N$, 37, $\sqrt{N}$, N, $N \log \log N$, $N \lg N$, $N \log(N^2)$, $N \log^2 N$, $N^{1.5}$, N^2, $N^2 \log N$, N^3, $2^{N/2}$, 2^N.

2) $N \log N$ and $N \log (N^2)$ grow at the same rate

Algorithm Analysis

- Suppose there are n inputs.

- We'd like to find a time function **T(n)** that shows how the execution time depends on **n**.

$\boxed{T(n) = 3n + 4}$ $\boxed{T(n) = 2^n}$ $\boxed{T(n) = 2}$

Mathematical Definitions

- **Big-Oh**

  $T(N) = O(f(N))$ if there are positive constants c and n_0 such that $T(N) \leq cf(N)$ when $N \geq n_0$.

- **Big-Omega**

  $T(N) = \Omega(g(N))$ if there are positive constants c and $n0$ such that $T(N) \geq cg(N)$ when $N \geq n0$.

- **Big-Theta**

  $T(N) = \Theta(h(N))$ if and only if $T(N) = O(h(N))$ and $T(N) = \Omega(h(N))$.

- **Little-oh**

  $T(N) = o(p(N))$ if, for all positive constants c, there exists an n_0 such that $T(N) < cp(N)$ when $N > n_0$.

"Big-Oh"

$T(N) = O(f(N))$ if there are positive constants c and n_0 such that $T(N) \le cf(N)$ when $N \ge n_0$.

We say "T(N) has order f(N)." means "f(N) is an **upper bound** on T(N)"

We try to simplify T(N) into one or more **common functions**.

Ex. 1 $T(N) = 3N + 4$

T(N) is linear. Intuitively, f(N) should be N.

More formally,

$T(N) = 3N + 4 \le 3N + 4N, \ N \ge 1$

$T(N) \le 7N, N \ge 1$

So T(N) is of order N.

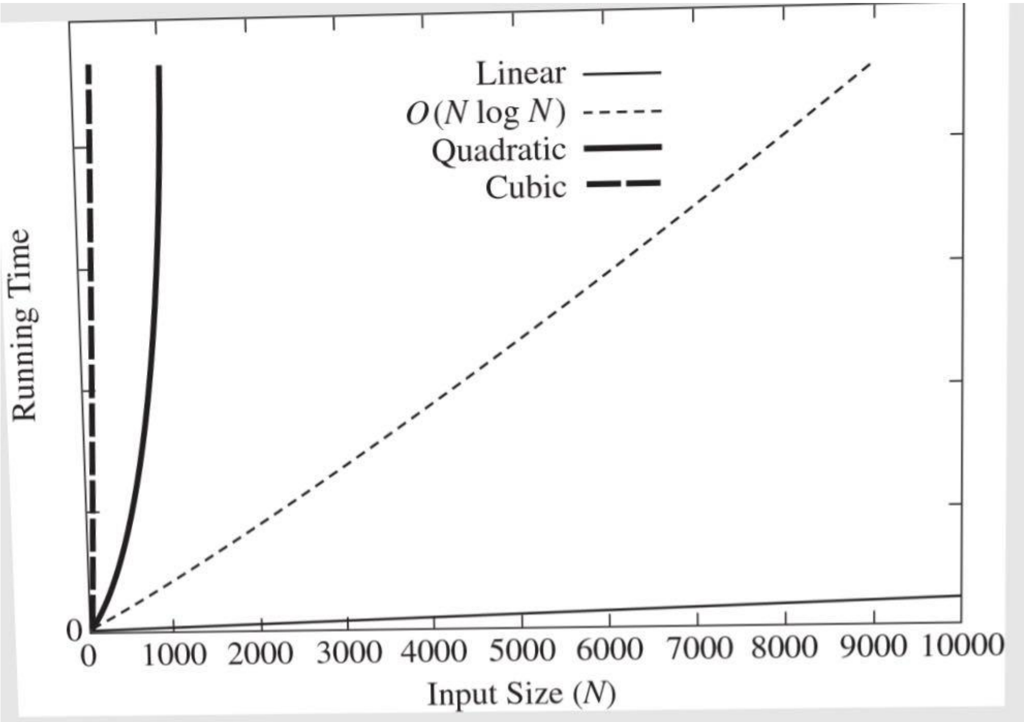
"Common Functions to Use"

$O(1)$	constant
$O(\log n)$	log base 2
$O(n)$	linear
$O(n \log n)$	
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(2^n)$ or $O(e^n)$	exponential
$O(n+m)$	
$O(nm)$	
$O(n^m)$	

Growth Rates

The growth rate is the rate at which the cost of the algorithm grows as the input size grows

The idea of the definitions is to establish a relative order among functions.



“simplifying rules”

Rule 1: If $T1(N) = O(f(N))$ and $T2(N) = O(g(N))$, then
(a) $T1(N) + T2(N) = O(f(N) + g(N))$
(b) $T1(N) * T2(N) = O(f(N) * g(N))$.

Suppose we get $T_1(N) = O(f(N))$ and $T_2(N)=O(g(N))$,

$f(N) = 4N^2 + 6$, $g(N) = 3N$, $T(N) = T_1(N)+T_2(N) =?$

$$
\begin{aligned}
T(N) = T_1(N)+T_2(N) &= O(f(N) + g(N)) \\
&= 4N^2 + 3N + 6 \\
&= O(N^2)
\end{aligned}
$$

intuitively $O(\max(f(N), g(N)))$

“simplifying rules”

Rule 2: If T(N) is a polynomial of degree k, then

$$T(N) = \Theta\ (N^k).$$

Rule 3: $\log^k N = O(N)$ for any constant k.

Rule 4: If $T(n) = O(g(n))$ and $g(n) = O(h(n))$, then $T(n) = O(h(n))$. – transitive*

Rule 5: If $T(n) = O(kg(n))$ for any constant $k > 0$, then $T(n) = O(g(n))$. – ignore the constant*.

“Common Functions to Use”

Suppose we get $T(N) = 4N^2 + 3N + 6$.

Is $T(N) = O(N^2)$? Or Is $T(N) = O(N^3)$?

Generally, we look for the smallest $f(N)$ that bounds $T(N)$.

We want a common function that is a least upper bound.

If $T(N) = c_k N^k + c_{k-1} N^{k-1} + \ldots + c_0$.

$T(N) = O(N^k)$.

N^k is the dominant term.

Model of Computation

- Simple Model of Computation
  - › instructions are executed sequentially
  - › has the standard repertoire of simple instructions
    - it takes exactly one time unit to do addition, multiplication, comparison, and assignment
  - › assume that the model has fixed-size integers and no fancy operations
  - › assume infinite memory

What to Analyze

- What to Analyze
  - Running time required
  - Memory or disk space required to run the program and store the data structure
- Main factors
  - the algorithm used
  - the input to the algorithm
  - Not include the programming language, compiler,…

Analyze the algorithms rather than the programs

How to Analyze

- How to analyze
  - Asymptotic algorithm analysis
    - Measures the efficiency of an algorithm, as the input size becomes large – growth rate.
    - best-case often of little interest
    - average-case often reflects typical behavior
    - worst-case represents a guarantee for performance on any possible input.

,

Need to know enough about the input data distribution to do average-case analysis

How to Analyze

Given an array containing **n** integers, suppose the sequential search algorithm is adopted

Q1: The cost of finding the largest value

Always c*n

Q2: The cost for finding a particular value K

May be different for different inputs

Best case:

if the first integer is K – examine 1 value

Worst case:

if only the last integer is K – examine n values

Average case:

If the sequential search is performed on different inputs for many times – examine n/2 values on average

How to Analyze

Step 1. Counting	T(N)
Step 2. Simplifying	O(f(N))

```
int sumit( int v[ ],  int num)  {    //Loop
   sum = 0;
   for (i = 0;  i  <  num;  i++)
      sum = sum + v[i];
   return sum
}
```

T(num) = c1 + c2 * num + c3

= c2 * num + (c1 + c3)

= O(num) throw away leading constants

We say T(n) of the algorithm is in O(n)

How to Analyze

```
// Nested Loops

int sum(int n) {
  int sum = 0;
  for (i=1; i<=n; i++)
    for (j=1; j<=n; j++)
      sum++;
}
```

$T(n) = c1 + c2 * n^2$

$= O(n^2)$

We say T(n) of the algorithm is in $O(n^2)$

How to Analyze

```
// compare the cost of the two loop codes
sum = 0;
for(k=1; k<=n; k*=2)  //do log(n) times
   for(j=1; j<=n; j++)   //do n times
     sum++;
```

In this double loop, the cost is

$$T(n) = \Theta\left(\sum_{i=1}^{\log n} n\right) = \Theta(n \log n)$$

How to Analyze

```
// compare the cost of the two loop codes
sum = 0;
for(k=1; k<=n; k*=2)    //do log(n) times
  for(j=1; j<=k; j++)     //do k times
    sum++;
```

In this double loop, the cost is

$$\Theta\left(\sum_{i=1}^{\log n} 2^i\right) = \Theta(2^{\log n}) = \Theta(n)$$

$$\sum_{0}^{\log n} 2^i = 2^{\log n + 1} - 1 = 2n - 1$$

How to Analyze

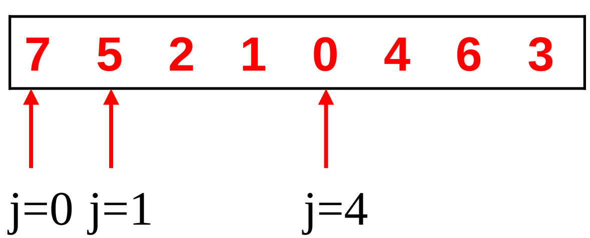
```
// assume array A contains n values,
//  random takes constant time c1 and
// sort takes cnlogn steps
for (i=0; i<n; i++) {
  for (j=0; j<n; j++)
    A[j] = random(n);
    sort(A, n);
}
```

Determine Θ in average case

$\Theta(n(c_1 n+cn\log n))= \Theta(n^2\log n)$

How to Analyze

```
// assume A[] contains a random permutation
// of the values from 0 to n-1
sum=0;
for(i=0; i<n; i++)
  for(j=0; A[j] !=i; j++)
    sum++;
```



Determine Θ in average case

$\Theta(n*n/2)=\Theta(n^2)$

How to Analyze

```
// A simple assignment to an integer variable
a = b;
```

$T(N) = \Theta(1)$

How to Analyze

- While loop – similar to for loop
- If-then-else statement – the greater of the costs for the then and else clauses.
- Switch statement – the most expensive branch
- Subroutine call – add the cost of subroutine
- Recursive subroutine – express the cost by a recurrence relation and then find the closed-form solution.

How to Analyze

```
// Here || is the string concatenation operator
string t (int n){
   if (n == 1) return '(1) ';
   else return '(' || n || t(n - 1) || ') '
}
```

$T(n) = T(n-1) + c$, for $n>1$, with $T(1) = c_1$

That is ,

$T(n) = c*(n-1) + c_1 = \Theta(n)$

How to Analyze

// Multiple Parameters

// A picture with P pixels; each pixel take one of C color

// values; Sort the colors w.r.t. the number of pixels with

// the color (Θ(C log C))

```
for (i=0; i<C; i++)  // Initialize count
   count[i] = 0;
for (i=0; i<P; i++)  // Look at all pixels
   count[value(i)]++; // Increment count
sort(count);          // Sort pixel counts
```

The cost is Θ (C) + Θ(P) + Θ(C log C) = Θ(P + C log C)

Can we drop Θ(P) or Θ(C log C)?

notes

Best, Worst, Average-Cases

vs.

Upper, Lower Bounds

- U/L bounds refer to the algorithm's growth rate.
- B/W/A cases refer to a certain type of inputs which cause the shortest/average/longest running time among all the inputs in study.
- U/L bounds can be used to describe the running time of an algorithm in its [best, worst, average] case

Notes

- For an algorithm with $T(n) = c_1n^2+c_2n$ in the average case ($c_1, c_2>0$) ,
  - › $c_1n^2+c_2n <= (c_1+c_2)n^2$ for all $n>1$, $T(n) <= cn^2$ for $c=c_1+c_2$ and $n_0=1$.
  - › $c_1n^2+c_2n >= c_1n^2$ for all $n>1$, $T(n) >= c_1n^2$ for $n_0=1$.
  - › $T(n)$ is in $\Theta(n^2)$ in the average case

- Reading the value from the first position in an array takes constant time regardless of the size of the array.
  - › $T(n) = c$ for the (best, worst, and average) cases.
  - › Traditionally, we say that the algorithm is in $O(1)$.

第1周SPOC线上学习内容

- 观看SPOC上的课程内容（包括视频及课件, 教材）复习巩固：

  1. Introducation、2. Algorithm Analysis

  学习：

  3. Lists, Stacks, and Queues