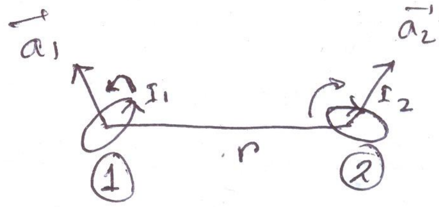
1. Tiny loops ≡ magnetic dipoles



a) $\vec B_1$ = field due to loop ①

$$= \frac{\mu_0 I_1}{4\pi r^3}\left(3(\vec a_1\cdot\hat r)\hat r - \vec a_1\right)$$

$$\begin{cases} \vec m_1 = I_1 \vec a_1 \\ m_2 = I_2 \vec a_2 \end{cases}$$

Flux through loop ②

$$\Phi = \vec B_1\cdot\vec a_2 = \frac{\mu_0 I_1}{4\pi r^3}\left[3(\vec a_1\cdot\hat r)(\vec a_2\cdot\hat r) - \vec a_1\cdot\vec a_2\right]$$

$$= M I_1 \quad (M = \text{mutual inductance})$$

$$\Rightarrow M = \frac{\mu_0}{4\pi r^3}\left[3(\vec a_1\cdot\hat r)(\vec a_2\cdot\hat r) - \vec a_1\cdot\vec a_2\right]$$

b) emf induced in loop ① to set up the current I_2 in loop ② ; $\mathcal{E}_1 = -M\frac{dI_2}{dt}$

Work done per unit time to run the current I_1 against the back emf $\mathcal{E}_1$

$$\frac{dW}{dt} = -\mathcal{E}_1 I_1 \quad [\text{(-)ve sign for work done against the emf}]$$

$$= M I_1 \frac{dI_2}{dt}$$

$$\Rightarrow W = \int M I_1 \frac{dI_2}{dt}\, dt = M I_1 I_2$$

$$\left[3(\vec a_1\cdot\hat r)(\vec a_2\cdot\hat r) - \vec a_1\cdot\vec a_2\right]$$

$$= \frac{\mu_0}{4\pi r^3} I_1 I_2 \left[3(\vec a_1\cdot\hat r)(\vec a_2\cdot\hat r) - \vec m_1\cdot\vec m_2\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[3(\vec m_1\cdot\hat r)(\vec m_2\cdot\hat r) - \vec m_1\cdot\vec m_2\right]$$

$$U = -\vec m_2\cdot\vec B_1 = -\frac{\mu_0}{4\pi r^3}\left[3(\vec m_1\cdot\hat r)(\vec m_2\cdot\hat r) - \vec m_1\cdot\vec m_2\right]$$

(*) Interaction energy of two dipoles $= -U$

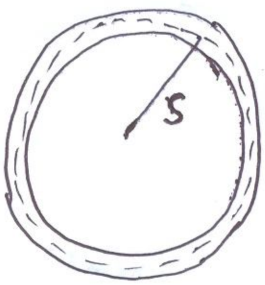
- Work done against the induced emf
- In calc. of interaction energy $|\vec m_1|$ & $|\vec m_2|$ are kept fixed & we consider only the energy required to bring them & rotate them to the desired orientation. Here we consider the energy required to sustain the current (i.e. $|\vec m|$).

2. Iron ring wound with N turns of current carrying wire
$\equiv$ a toroidal solenoid

$\Rightarrow$ Symmetry requires that inside the solenoid, the $\vec{H}$ field have the form

$$\begin{cases} \vec{H} = H(r)\,\hat{\phi} & \text{[inside the iron, we consider } \vec{H} \text{ in place of } \vec{B}\text{]} \\ \& \ \vec{H} = 0 \text{ outside} & \text{(for any cross-sectional shape)} \end{cases}$$

[See Example 5.10 in Griffiths book]



By considering an Amperian loop of radius s

$$\oint \vec{H}\cdot d\vec{l} = I_{encl.} = NI$$

N = # of total turns
I = current in each turn.

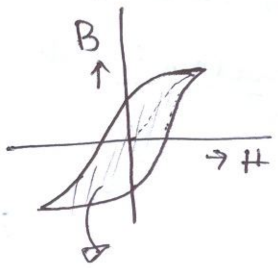
$$\Rightarrow \vec{H} = \frac{NI}{2\pi s}\,\hat{\phi}$$

Since radius $r \gg$ cross-sectional area A, we can consider a constant H field)
$\vec{H} = \frac{NI}{2\pi r}\hat{\phi}$ over the whole cross-section (i.e.

Now I is varied to change $\vec{H}$ & $\vec{B}$.
$\Rightarrow$ Power must be supplied $P = -I\mathcal{E}$ where $\mathcal{E} = -\frac{d\Phi}{dt} = -NA\frac{dB}{dt}$
= back emf.

$$\Rightarrow P = +\left(\frac{2\pi r H}{\cancel{N}}\right)\cancel{N}\cdot A\cdot\frac{dB}{dt} = (2\pi r A)\,H\frac{dB}{dt} = VH\frac{dB}{dt}$$

$V = 2\pi r A$ = volume of the iron ring.



$\Rightarrow$ Total energy supplied

$$W = \int P\,dt = V\oint H\frac{dB}{dt}\,dt = V\oint H\,dB.$$

$$\oint H\,dB = \frac{W}{V} = \text{energy loss per unit volume per hysteresis loop.}$$

= area enclosed by the hysteresis loop in B-H plane.

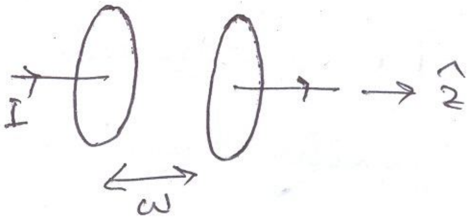
$\Rightarrow$ $\boxed{\text{energy loss per unit volume = area enclosed by the hysteresis loop.}}$

(*) if you consider a linear medium, $\vec{H} = \mu\vec{B}$,
then $\oint \vec{H}\,d\vec{B} = 0$ [work done = change in magnetic energy given by $U_m = \frac{1}{2\mu_0}\int B^2\,d\tau$]

~~for linear medium~~
But for ferromagnet (e.g. iron), $\oint H\,dB \neq 0$ $\Rightarrow$ Process of changing $\vec{B}$ is not a conservative process.

3.

a) $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$



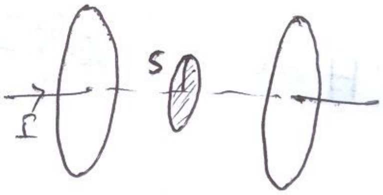
Let, at any time t, Q = charge on the plate

$\Rightarrow$ current $I = \frac{dQ}{dt}$

$\Rightarrow Q = It$ $\left[\begin{array}{l}\text{as at } t=0,\ I=0 \\ \& \ I = \text{constant}\end{array}\right]$ ⇐ given.

$$\vec{E}(t) = \frac{\sigma(t)}{\epsilon_0}\hat{z} = \frac{1}{\epsilon_0}\left[\frac{Q(t)}{\pi a^2}\right]\hat{z} = \frac{It}{\pi a^2 \epsilon_0}\hat{z}$$

b) Displacement current density



$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$
$$= \epsilon_0 \frac{\partial}{\partial t}\left(\frac{It}{\pi a^2 \epsilon_0}\right)\hat{z}$$
$$= \cancel{\epsilon_0}\frac{I}{\pi a^2 \cancel{\epsilon_0}}\hat{z} = \frac{I}{\pi a^2}\hat{z}$$

displacement current through the circle of radius s

$$I_d = J_d \cdot \pi s^2 = \frac{I s^2}{a^2}$$

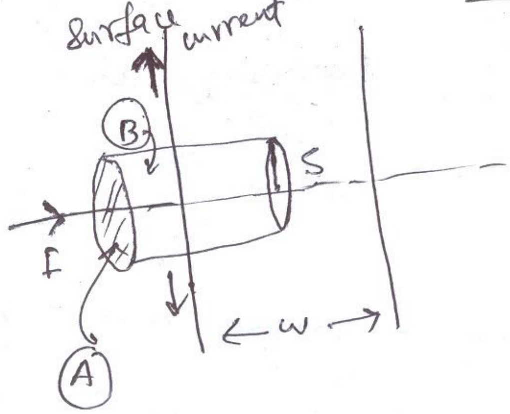
Ampere's law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 [\overbrace{\cancel{\int \vec{J}\cdot d\vec{a}}}^{=0} + \int \vec{J}_d \cdot d\vec{a}]$

$$\underbrace{\oint \vec{B}\cdot d\vec{l}}_{(B\cdot 2\pi s)} = \mu_0 I_{d\,\text{encl.}} = \mu_0 I s^2/a^2$$

$$\Rightarrow B = \frac{\mu_0}{2\pi}\frac{Is}{a^2} \quad (s<a)$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{2\pi}\frac{Is}{a^2}\hat{\phi}$$

3. c) Surface current flows radially outward over the left plate



Current through the flat surface (A)

$I_A = I =$ current through the wire.

Current through the curved surface (B)

$I_B = \left(I - I_{dencl}\right) = I - \dfrac{I s^2}{a^2}$

↑ displacement current passing out through the circle of radius s.

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl.} = \mu_0 (I_A - I_B) = \mu_0 \dfrac{I s^2}{a^2}$

$\qquad = \mu_0 \left(I - (I - I s^2/a^2)\right)$

$\Rightarrow B \cdot 2\pi s = \mu_0 \left(I - (I - I s^2/a^2)\right)$

$\Rightarrow B = \dfrac{\mu_0}{2\pi} \dfrac{I s}{a^2}$ ⟶ same as obtained in (b).

Another way to calculate I_B:

Let $Q'(t)$ be the charge on the plate outside the cylinder of radius "s".

$Q'(t) = \dfrac{Q(t)}{\pi a^2}\left(\pi a^2 - \pi s^2\right)$ [as surface charge is uniform over the plate]

$\Rightarrow$ surface current $I_B = \dfrac{dQ'}{dt} = \dfrac{dQ}{dt}\left(1 - s^2/a^2\right) = \underline{I\left(1 - \dfrac{s^2}{a^2}\right)}$

4. Faraday's law $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

as $E = 0 \Rightarrow \frac{\partial B}{\partial t} = 0 \Rightarrow B =$ constant inside the conductor.

For a superconductor $B = 0$ inside

$\Rightarrow \nabla \times \bar{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$

$B = 0,\ E = 0 \Rightarrow \vec{J} = 0 \Rightarrow$ No volume current.

$\Rightarrow$ There can be only surface current.

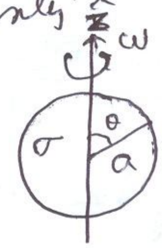
Since $B = 0$ inside a superconductor, when the sphere is ~~coll~~ cooled down below T_c, the induced surface current should be such that it produces $-B_0 \hat{k}$ (inside) to cancel the applied magnetic field $(B_0 \hat{k})$.

We know that a rotating shell with surface charge density σ produces uniform magnetic field inside the sphere. [Griffiths example 5.11]

Due to the rotating sphere

$\vec{B} = \frac{2}{3} \mu_0 \sigma \omega a \hat{z}$

$\left\{ \begin{array}{l} \sigma = \text{surface charge density} \\ \omega = \text{angular velocity} \\ a = \text{radius.} \end{array} \right.$



So, we need

$\frac{2}{3} \sigma \omega a \mu_0 = -B_0$

$\Rightarrow (\sigma \omega a) = -\frac{3}{2} \left(\frac{B_0}{\mu_0} \right)$

Surface current $\vec{K} = \sigma \vec{v} = (\sigma \omega a) \sin\theta\ \hat{\phi}$

$= -\frac{3}{2} \frac{B_0}{\mu_0} \sin\theta\ \hat{\phi}$