1. Classify the PDE:-

(a) $u_t + u_x + uu_x = u_{xxt}$

Highest order term is 3 and the corresponding coefficient is 1.

Hence the eqn is a $\boxed{\text{semilinear}}$ PDE.

(b) $u_x^2 + u_y^2 = 1$

Highest order term is 1 and the highest order is nonlinearly depend upon u Hence $\boxed{\text{nonlinear}}$.

(c) $u_t + u^2 u_x = 0$.

~~Coeffi Highest~~ Coefficient of the highest order term contains a fxn of u. Hence Quasilinear PDE.

(d) $u_t + x^2 u_x = \sin(x)$

Purely linear Eqn

2 (a) Solve $y u_x - x u_y = 0$; $u(0,y) = 2y^2$ for $y>0$.

let $\Gamma(r)$ be the data curve given by $\Gamma(r) = \{(0, r, r^2) : r \in \mathbb{R}\}$

$\therefore$ the Char Eqn are

$\frac{dx}{ds}(r,s) = y$, $x(r,0) = 0$ — (i)

$\frac{dy}{ds}(r,s) = -x$, $y(r,0) = r$ — (ii)

$\frac{dz}{ds}(r,s) = 0$, $z(r,0) = r^2$. — (iii)

From (i) & (ii) ;

$x''(r,s) = -x(r,s)$ s.t $x(r,0) = 0$ & $x'(r,0) = r$.

$\therefore x(r,s) = r \sin s$.

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Purely linear Eqn

2(a) Solve $yu_x - xu_y = 0$; $u(0,y) = 2y^2$ for $y > 0$.

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∴ the Char Eqn are

$$\frac{dx}{ds}(r,s) = y \qquad x(r,0) = 0 \quad \text{—(i)}$$

$$\frac{dy}{ds}(r,s) = -x \qquad y(r,0) = r \quad \text{—(ii)}$$

$$\frac{dz}{ds}(r,s) = 0 \qquad z(r,0) = 2r^2. \quad \text{—(iii)}$$

From (i) & (ii);

$x''(r,s) = -x(r,s)$ s.t $x(r,0) = 0$ & $x'(r,0) = r$.

∴ $x(r,s) = r\sin s$.

and also, $y(r,s) = r\cos s$.

Hence, $x^2+y^2=r^2$. —(iv)

Again from (iii) ; $z(r,s) = 2r^2$. —(v)

Hence from (iv) & (v) one has

$z(r,s) = 2(x^2+y^2)$.

$\therefore u(x,y) = z(r,s) = 2(x^2+y^2)$.

(b)

$u_t + (x+t)u_x = t$; $u(x,0) = e^x$.

Let the data curve be given by $\Gamma(r) = \{(r,0,e^r); r\in\mathbb{R}\}$.

The char eqn is given by

$\frac{dx}{ds}(r,s) = x+t$ | $\frac{dt}{ds}(r,s) = 1$ | $\frac{dz}{ds}(r,s) = t$

$x(r,0) = r$ —(i) | $t(r,0) = 0$ —(ii) | $z(r,0) = e^r$ —(iii)

From (ii) ; $t(r,s) = s$. —(iv)

Using (iv) in (i) we have

$\frac{dx}{ds} = x+s$; $x(r,0) = r$.

$\Rightarrow x(r,s) = e^s\left[\int se^{-s} + g_1(r)\right] = -e^s.e^{-s}(s+1) + g_1(r)e^s$.

Putting $x(r,0) = r$ we have,

$r = -1 + g_1(r)e^0 \Rightarrow g_1(r) = r+1$.

$\therefore x(r,s) = -(s+1) + (r+1)e^s$.

Again from (iii),

$$z(r,s) = \frac{s^2}{2} + e^{r}.$$

$$\therefore u(x,t) = \frac{t^2}{2} + \exp\left((x+s+1)e^{-s} - 1\right).$$

(c) Solve :- $u_t + x u_x + u = 3x$; $u(x,0) = \tan^{-1}(x)$.

The data curve is given by $\Gamma(r) = \{(r, 0, \tan^{-1} r) ; r \in \mathbb{R}\}$.

The Char Eqn are

$$\frac{dx}{ds}(r,s) = x \qquad \frac{dt}{ds}(r,s) = 1 \qquad \frac{dz}{ds}(x,s) = 3x - z$$

$$x(r,0) = r \quad \text{— (i)} \qquad t(r,0) = 0 \quad \text{— (ii)} \qquad z(r,0) = \tan^{-1} r \quad \text{— (iii)}$$

From (i) ; $x(r,s) = r e^{s}$

From (ii) ; $t(r,s) = s$

From (iii) ; $z'(r,s) = 3re^{s} - z$

$$\Rightarrow \frac{dz}{ds} + z = 3re^{s}.$$

$$\Rightarrow z e^{s} = \frac{3}{2} r e^{2s} + g_1(r)$$

$$\Rightarrow z = \frac{3}{2} r e^{s} + g_1(r) e^{-s}.$$

$\because z(r,0) = \tan^{-1} r$.

$$\therefore g_1(r) = \tan^{-1}(r) - \frac{3}{2} r.$$

$$\therefore u(x,t) = \frac{3x}{2} + \left[\tan^{-1}(xe^{-t}) - \frac{3}{2} x e^{-t}\right].$$

(a) $x(y^2-z^2)u_x - y(u^2+x^2)u_y = (x^2+y^2)u$.

The char eqn are

$$\left.\begin{aligned} \frac{dx(s)}{ds} &= x(y^2-z^2) \\ \frac{dy(s)}{ds} &= -y(x^2+z^2) \end{aligned}\right\} - (1)$$

$$\frac{dz}{ds}(s) = (x^2+y^2)z.$$

(1) can also be written as

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{(x^2+y^2)z}$$

Using property of Ratio's we have,

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) - y^2(z^2+x^2) + z^2(x^2+y^2)} = \frac{dz}{(x^2+y^2)z}.$$

i.e,
$$\frac{x\,dx + y\,dy + z\,dz}{0} = \frac{dz}{(x^2+y^2)z}.$$

or, $x\,dx + y\,dy + z\,dz = 0$.

or, $d(x^2+y^2+z^2) = 0 \Rightarrow x^2+y^2+z^2 = c$. ($c$ - constant).

Again,
$$\frac{\frac{dx}{x} - \frac{dy}{y}}{y^2-z^2+z^2+x^2} = \frac{dz}{(x^2+y^2)z} \Rightarrow \frac{dx}{x} - \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow z = K\frac{x}{y}. \quad (K\text{ - constant})$$

$\therefore$ General Soln is $u(x,y) = \frac{x}{y} f(x^2+y^2+x^2)$.

(3) $u_x + u_y = 1$; $u(x,x) = 1$ has no solution

The initial curve in $\mathbb{R}^3$ is parametrized as $\Gamma = \{(r,r,1) : r \in \mathbb{R}\}$ and so the data curve is $\tilde{\Gamma} = \{(r,r) : r \in \mathbb{R}$

Γ is non-char ??

$(a,b)\cdot(-r_2'(r), r_1'(r)) = (1,1)\cdot(-1,1) = 0$.

Hence not a non-characteristic $\Rightarrow$ There is no guarantee of unique so

But again the G.S is given by $u(x,y) = y + f(x-y)$.

and $u(x,x) = 1 \Rightarrow f(0) = 1 - x$ which is not possible

~~Since the~~ Hence the problem has no sol^n.

(4) $u_x + u_y = 1$; $u(x,x) = x$.

The initial curve in $\mathbb{R}^3$ is parametrized as $\Gamma = \{(r,r,r) : r \in \mathbb{R}\}$ and so the data curve is $\tilde{\Gamma} = \{(r,r) : r \in \mathbb{R}^2\}$

Transversability condition

$(1,1)\,(-1,1) = 0$

not an non-char. $\Rightarrow$ there is no guarantee of unique sol^n.

$\therefore \Gamma$ is not an non-char.

Again the G.S is $u(x,y) = y + f(x-y)$.

Now, $u(x,x) = x \Rightarrow x = x + f(0) \Rightarrow f(0) = 1$.

$\therefore u(x,y) = y + f(x-y)$ is a solution for any ~~f~~ $f \in C^1(\mathbb{R})$ s.t. $f(0) = 1$.

$\therefore$ There are infinitely many sol^n

(5) $u_x - u_y = 1$; $u(x,0) = x^2$. (6)

The Initial Curve is parametrized by $\Gamma = \{(r, 0, r^2) : r \in \mathbb{R}\}$.

~~Use~~ Transversibility Condition

$(1,-1)\cdot(0,1) = -1 \neq 0$.

$\therefore$ The Cauchy problem has a unique soln in a nbd of Γ.

$\therefore$ The Char Eqns are

$\frac{dx(r,s)}{ds} = 1$, $x(r,0) = r$ | $\frac{dy}{ds}(r,s) = -1$, $y(r,0) = 0$ | $\frac{dz}{ds}(r,s) = 1$, $z(r,0) = r^2$.

$\therefore$ $x(r,s) = s + r$

& $y(r,s) = -s$.

& $z(r,s) = s + r^2$.

$\therefore u(x,y) = -y + (x+y)^2$.