Revision

Approaches to reduce interpolation errors

- Selection of interpolation points
- Piecewise fitting of polynomials

Splines – Linear (C^0 continuous)
Quadratic (C^1
Cubic (C^2 continuous)

Cubic splines

$$S(x) = \begin{cases} q_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3 & x_0 \le x \le x_1 \\ \vdots \\ q_n(x) = a_n + b_n x + c_n x^2 + d_n x^3 & x_{n-1} \le x \le x_n \end{cases}$$

4n unknowns

Conditions

1. $q_i(x_{i-1}) = y_{i-1}$ — n
2. $q_i(x_i) = y_i$ — n
3. $q_i'(x_i) = q_{i+1}'(x_i)$ — $n-1$
4. $q_i''(x_i) = q_{i+1}''(x_i)$ — $n-1$

Remaining 2 conditions from BCs

1. Natural spline $q_1''(x_0) = q_n''(x_n) = 0$
2. Not-a-knot spline
3. Clamped
4. Periodic

Solve system of $4n$ equations

Alternate approach – make second derivatives unknown

→ Second derivative is a linear polynomial between any two intervals

Set of $n-1$ equations in $(n+1)$ unknowns

BC 1

$$\begin{bmatrix} & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & & & \\ & & h_{n-1} & 2(h_{n-1}+h_n) & h_n \\ & & & & \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix} = 6 \begin{bmatrix} g_2 - g_1 \\ g_3 - g_2 \\ \vdots \\ g_n - g_{n-1} \end{bmatrix}$$

BC_2

$$h_i = x_i - x_{i-1} \qquad \sigma_i = q_i''(x_i)$$

$$g_i = \frac{y_i - y_{i-1}}{h_i} \qquad = q_{i+1}''(x_i)$$

Natural splines

$(n-1)$ eq.

$$\sigma_0 = \sigma_n = 0$$

$$\begin{bmatrix} 2(h_1+h_2) & h_2 & & \\ h_2 & 2(h_2+h_3) & h_3 & \\ & h_3 & 2(h_3+h_4) & \\ & & & \ddots \\ & & & & 2(h_{n-1}+h_n) \end{bmatrix}$$

Example

i	x_i	y_i	$h_i = x_i - x_{i-1}$	g_i	σ_i	A_i	B_i	C_i	D_i
0	3	2.5			0				
1	4.5	1.0	1.5	$= \frac{1-2.5}{1.5} = -1$	1.6971	—			
2	7	2.5	2.5	0.6	-1.5331	—			
3	9	0.5	2.0	-1.0	0	—			

$x_{i-1} \le x \le x_i$

Natural spline $\quad \sigma_0 = \sigma_3 = 0 \qquad \sigma_1 = ? \quad \sigma_2 = ?$

$$\begin{bmatrix} 2(h_1+h_2) & h_2 \\ h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = 6 \begin{bmatrix} g_2 - g_1 \\ g_3 - g_2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2.5 \\ 2.5 & 9.0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = 6 \begin{bmatrix} 1.6 \\ -1.6 \end{bmatrix}$$

$\sigma_1 = 1.6971$

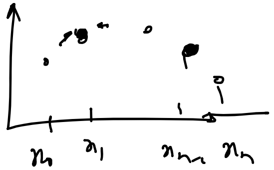
$\sigma_2 = -1.5331$

$$q_i(x) = A_i (x - x_{i-1})^3 - B_i (x - x_i)^3 + C_i (x - x_{i-1}) - D_i (x - x_i)$$

$A_i = \dfrac{\sigma_i}{6 h_i} \qquad C_i = \dfrac{y_i}{h_i} - \dfrac{\sigma_i h_i}{6}$

$B_i = \dfrac{\sigma_{i-1}}{6 h_i} \qquad D_i = \dfrac{y_{i-1}}{h_i} - \dfrac{\sigma_{i-1} h_i}{6}$

Not-a-knot spline



$$q_1'''(x_1) = q_2'''(x_1)$$

$$q_{n-1}'''(x_{n-1}) = q_n'''(x_n)$$

$$q_i''(x) = \frac{\sigma_i}{h_i}(x - x_{i-1}) - \frac{\sigma_{i-1}}{h_i}(x - x_i)$$

Differentiate

$$q_i'''(x) = \frac{\sigma_i}{h_i} - \frac{\sigma_{i-1}}{h_i}$$

If apply the conditions

$i=1$

$$\frac{\sigma_1}{h_1} - \frac{\sigma_0}{h_1} = \frac{\sigma_2}{h_2} - \frac{\sigma_1}{h_1}$$

$$\Rightarrow \sigma_0 h_2 - \sigma_1 (h_1 + h_2) + \sigma_2 h_1 = 0 \quad -(1)$$

$$\sigma_{n-1} h_n - \sigma_{n-1}(h_{n-1} + h_n) + \sigma_n h_{n-1} = 0 \quad -(2)$$

$$\begin{bmatrix} h_2 & -(h_1+h_2) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & \\ & & & & & \\ & & & h_n & -(h_{n-1}+h_n) & h_{n-1} \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Remark 1 – Inverse interpolation

(x_i, y_i) $\quad i = 0, 1, \dots n$

$P_n(x)$

Given x^* $\quad y^* = P_n(x^*)$

Suppose y^* $\quad$ Find x^*

First approach $\quad (y_i, x_i) \quad P_n(y)$

$$x^* = P_n(y^*)$$

Second approach $\quad \Rightarrow P_n(x^*) = y^*$

Finding a root

x	2	3	4	5
y	8	27	64	125

$y^* = 100$

Remark 2 Multi dimensional Interpolation

$y \backslash x$	2	3	4	5		
1	-	-	-	.	$P_{y=1}(x_*)$	y'_1
1.5	-	.	-		$P_{y=1.5}(x_*)$	y'_2
2.0	.	.	.		$P_{y=2}(x_*)$	y'_3
2.5	-	.	.		$P_{y=2.5}(\,)$	y'_4

$$z = f(x, y)$$

$x = 2.5,$
$y = 1.7$

Given x^*, y^*
Find $z^* = ?$

Successive one dimensional interpolation

Direct fit

$$P_n(x,y) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$
$$+ b_1 y + b_2 y^2 + b_3 y^3 + \dots + b_n y^n$$
$$+ c_1 xy + c_2 x^2 y + \dots$$

Numerical Integration & Differentiation

INTEGRATION

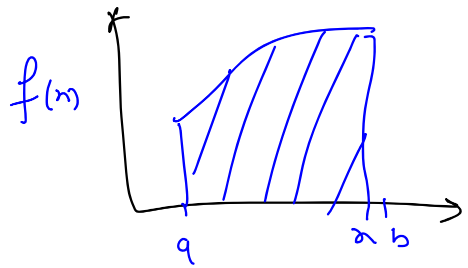
$$I = \int_a^b f(x)\, dx$$

Definite Integrals

$$= F(b) - F(a)$$

where $F(x)$ is an anti derivative of $f(x)$

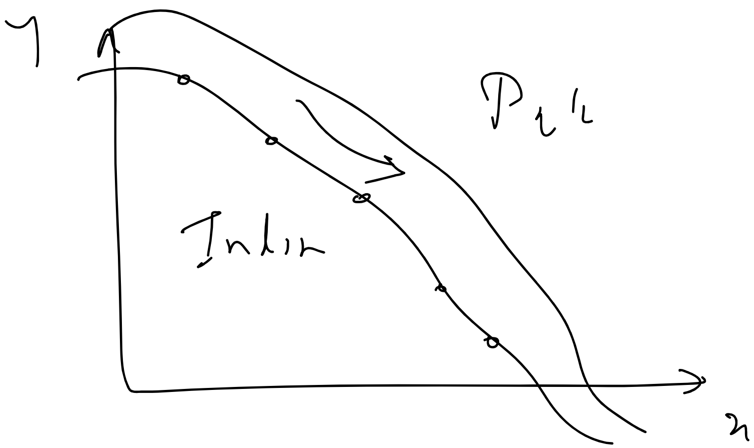
$$f(x) = \frac{dF(x)}{dx}$$



Numerical Integration (Quadrature)

Need → 1. Sometimes functions can be evaluated only at discrete points

2. The function is so "complex" that analytical expression does not exists



Approach

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b \left(P_n(x) + R_n(x)\right) dx$$

$$= \int_a^b P_n(x)\,dx + \int_a^b R_n(x)\,dx$$

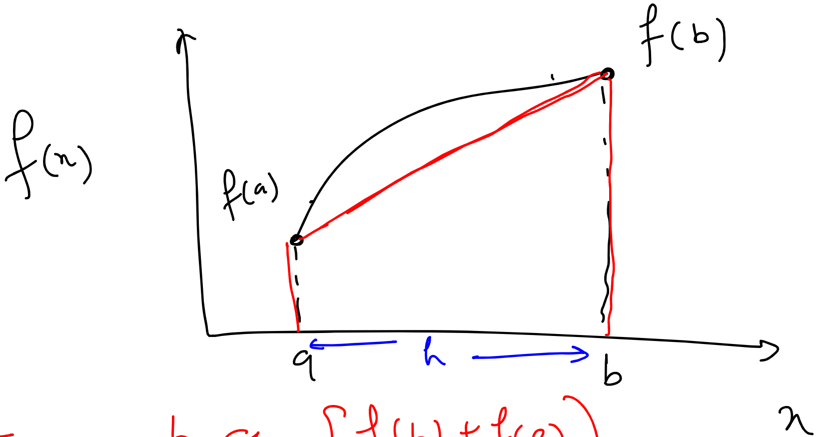
$$= I + \epsilon$$

Newton - Cotes formulas for integration

Applicable when function values are available at equal intervals.

Linear polynomial

Only One Interval



$$I = \frac{b-a}{2} \cdot \left[f(b) + f(a) \right]$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$I = \int_a^b P_1(x)\, dx$$

$$= \int_a^b \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)\, dx$$

$x_0 = a$

$x_1 = b$

$x_1 - x_0 = b - a = h$

$f(x_0) = f(a)$

$f(x_1) = f(b)$

$$= \int_a^b -\frac{(x-b)}{h} f(a) + \frac{(x-a)}{h} f(b)\, dx$$

$$\boxed{I = \frac{h}{2} \left[f(a) + f(b) \right]}$$

$$R_1(x) = \frac{f''(\xi)}{2!}(x-x_0)(x-x_1)$$

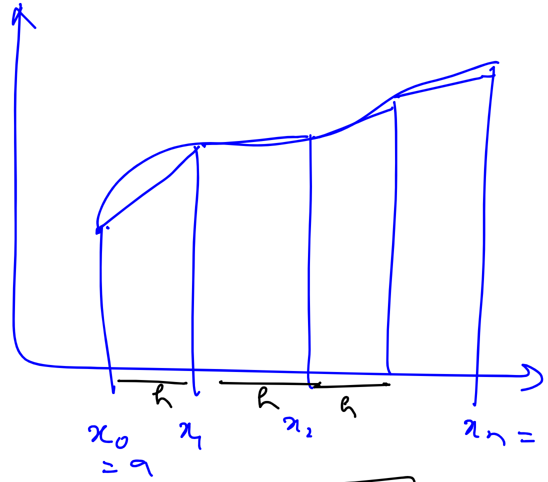
$$E = \int_a^b R_1(x)\,dx$$

$$= \int_a^b \frac{f''(\xi)}{2!}(x-a)(x-b)$$

$$\boxed{E = -\frac{1}{12} f''(\xi)\, h^3} \qquad O(h^3)$$

$$x_0 \le \xi \le x_1$$

<u>Multiple application</u>



$$I = \frac{h}{2}\left[f(x_1) + f(x_0)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$+ \frac{h}{2}\left[f(x_2) + f(x_3)\right] + \dots$$

$$+ \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$E = \sum_{i=1}^{n} -\frac{1}{12} f''(\xi_i)\, h^3$$

$$= -\frac{1}{12} h^3\, n\, f''(\bar{\xi})$$

$$a \le \bar{\xi} \le b$$

$$= -\frac{1}{12} (b-a)\, h^2 f''(\bar{\xi})$$

$$(b-a) = nh$$

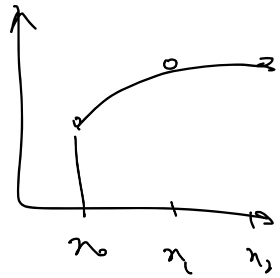
$$= O(h^2)$$

$$h/2 \longrightarrow E_h/4 = E_{h/2}$$

Quadratic Polynomial

$$P_3(x) = L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2)$$

Simpson's $1/3^{rd}$ rule



$$I = \frac{h}{3}\left[f(x_0) + 4 f(x_1) + f(x_2) \right]$$

$$E = -\frac{1}{90} h^5 f^{IV}(\xi)$$

Multiple app

$$I = \frac{h}{3}\left[f(x_0) + 4 \sum_{i=1,3}^{n-1} f(x_i) + 2 \sum_{i=2,4}^{n-2} f(x_i) + f(x_n) \right]$$

$$E \approx O(h^4)$$

$$(h/2) \longrightarrow E_h/16 = E_{h/2}$$