FEM approximations: the basic idea

In FEM, the field variable $\boldsymbol{u} = u_i \boldsymbol{e}_i$ is approximated as

$$u_i^h(\boldsymbol{x}) = \sum_{I=1}^{N} N^I(\boldsymbol{x}) u_i^I,$$

where u_i^I are the values of the variable u_i at N discrete points $I \in [1, N]$ in the doaminn of interest.

Moreover, the *shape functions* $N_I(\boldsymbol{x})$ satisfy the Kronecker delta property

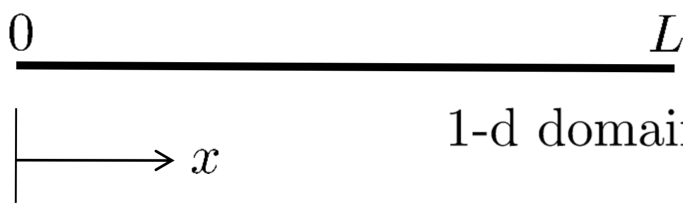
$$N^I(\boldsymbol{x}^J) = \delta_{IJ}.$$

This property is needed to ensure that

$$u_i^J = \sum_{I=1}^{N} N^I(\boldsymbol{x}^J) u_i^J.$$

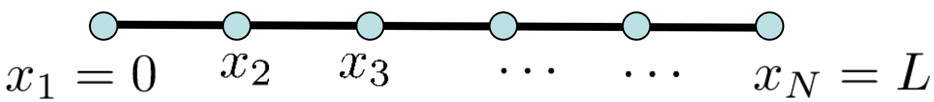
The other property needed of the shape functions is the *partition of unity*. Consider the case when u_i is a cosntant c_0. This will lead to the fact that

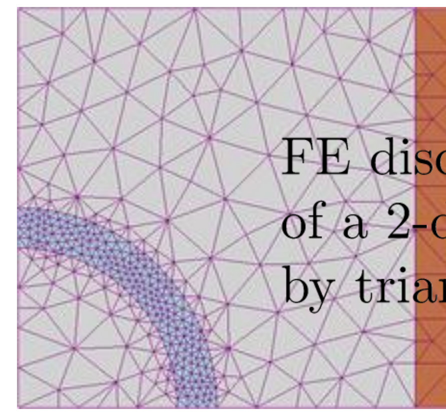
$$\sum_{I=1}^{N} N^I(\boldsymbol{x}) = 1.$$



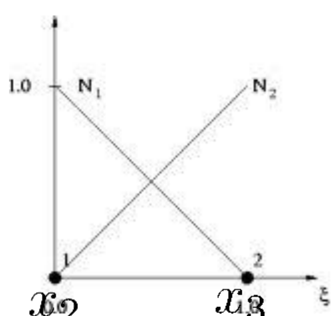
1-d domain

FE discretisation

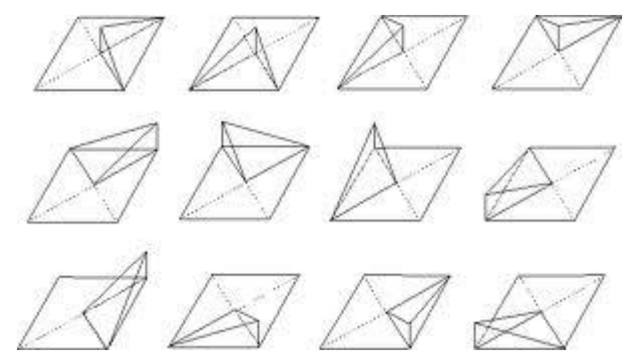




FE discretisation of a 2-d domain by triangles

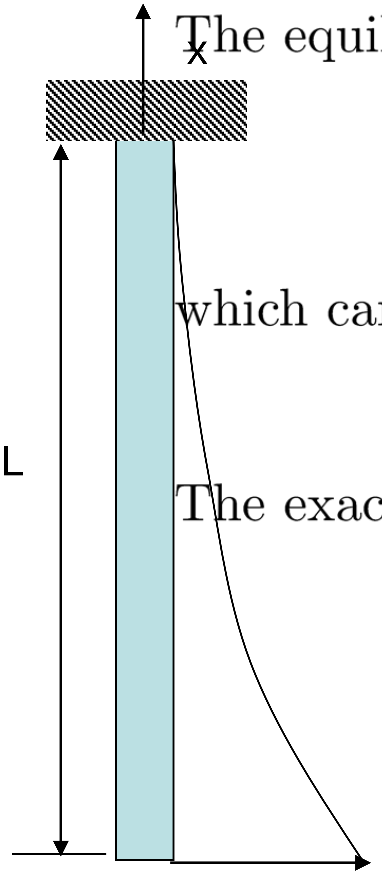


Typical shape functions



Typical shape functions

Simple cases: bars and beams



$$u(L) = -\frac{\rho g L^2}{2}$$

The equilibrium equation for the bar hanging under its own weight is

$$\frac{d\sigma_x}{dx} - \rho g = 0$$

which can be written as

$$E\frac{d^2u}{dx^2} - \rho g = 0.$$

The exact solution to this problem is easily obtained as

$$u(x) = \frac{\rho g x^2}{2E} + \frac{\rho g L x}{E}$$

Let us derive the weak form for the problem

$$AE\frac{d^2u}{dx^2} - A\rho g = 0.$$

Using the direct variational technique (we can easily verify that the operator $\mathcal{L} = AEd^2/dx^2$ is self-adjoint) we get

$$\Pi = (1/2)(\mathcal{L}u, u)_H - (f, u)_H,$$

which gives

$$\begin{aligned}
\Pi &= \frac{1}{2}\int_0^{-L} AE\frac{d^2u}{dx^2}udx + A\int_0^{-L} \rho gudx \\
&= \frac{AE}{2}\left[\frac{du}{dx}u\bigg|_0^{-L} - \int_0^{-L}\left(\frac{du}{dx}\right)^2 dx\right] + A\int_0^{-L} \rho gudx \\
&= \int_0^{-L}\left(\frac{du}{dx}\right)^2 dx + A\int_0^{-L} \rho gudx
\end{aligned}$$

The last step follows from the fact that $u(0) = 0$ and $du/dx(x = -L) = 0$.

The same weak form is obtained from the potential energy

$$\Pi = \frac{1}{2}\int_V \sigma_{ij}\epsilon_{ij}dV - \int_V u_i b_i dV - \int_{\partial V} t_i u_i dS - \sum_I F_i^I u_i^I$$

In the present case, the only non-zero stress is σ_{xx}, $u_1 = u$ the only non-zero displacement and $b_x = -\rho g$, so that

$$\Pi = \frac{1}{2}\int_0^{-L} AE\left(\frac{du}{dx}\right)^2 dx + A\int_0^{-L} \rho g u dx,$$

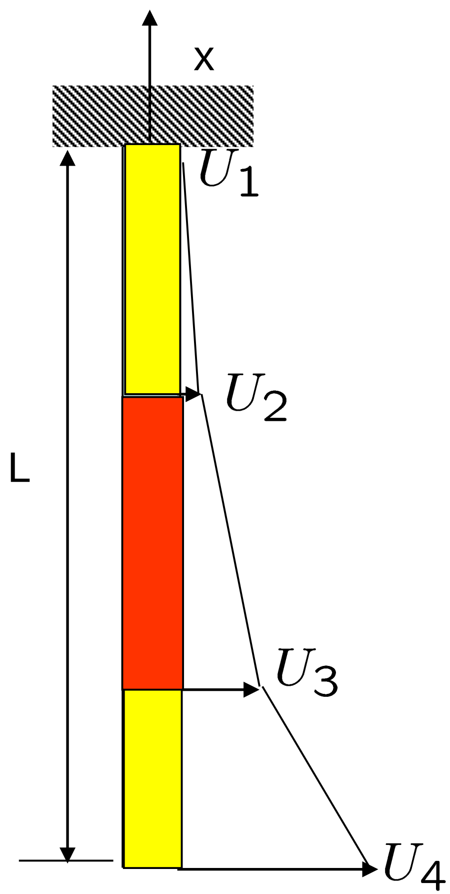
which is the same as what we got from the direct variational route.

We now need that

$$\delta\Pi = AE\int_0^{-L} \frac{du}{dx}\frac{d\delta u}{dx}dx + A\int_0^{-L} \rho g \delta u dx = 0.$$

If the domain is divided into $N+1$ nodes and N elements

$$\delta \Pi = \sum_{e=1}^{N} \delta \Pi^e = 0.$$



Each element is bounded by two nodes 1 and 2 having global coordinates x_1^e and x_2^e.. Assuming linear variation in u over each element,

$$u^h = N_1(x)u_1 + N_2(x)u_2 = \boldsymbol{N}\boldsymbol{U}.$$

From the properties of the shape functions, it is easy to see that

$$N_1 = \frac{x_2^e - x}{L^e}, N_2 = -\frac{x_1^e - x}{L^e}.$$

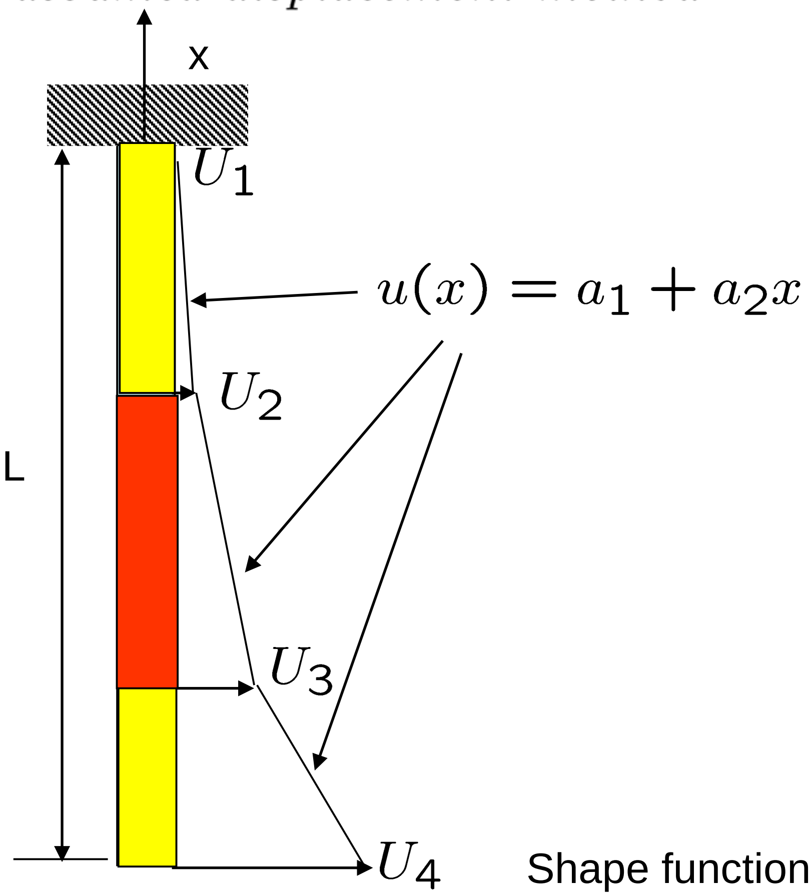
Also,

$$\delta u^h = N_1 \delta u_1 + N_2 \delta u_2 = \boldsymbol{N}\delta\boldsymbol{U},$$

and

$$\frac{du^h}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 = \boldsymbol{B}\boldsymbol{U}.$$

The *shape function matrix* $\boldsymbol{N}$ and the *strain displacement matrix* $\boldsymbol{B}$ are central to the FE approximation. They can also be derived through an alternative route called *the assumed displacement method.*



$$u(x) = \langle\ 1 \quad x\ \rangle \left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}$$

$$\boldsymbol{U} = \boldsymbol{A}\boldsymbol{a}$$

$$u(x) = \underbrace{\langle\ 1 \quad x\ \rangle \boldsymbol{A}^{-1}}_{\boldsymbol{N}} \boldsymbol{U}$$

Shape function matrix

$$\frac{du}{dx} = \langle\ 0 \quad 1\ \rangle \boldsymbol{A}^{-1}\boldsymbol{U} = \boldsymbol{B}\boldsymbol{U}$$

Strain displacement matrix

Now, after getting $\boldsymbol{N}$ and $\boldsymbol{B}$,

$$
\begin{aligned}
\delta\Pi^e &= AE\int_{x_1^e}^{x_2^e}\frac{du}{dx}\frac{d\delta u}{dx}dx + A\int_{x_1^e}^{x_2^e}\rho g\delta u dx. \\
&= \delta\boldsymbol{U}^T\int_{x_1^e}^{x_2^e}\boldsymbol{B}^T AE\boldsymbol{B}dx\boldsymbol{U} + \delta\boldsymbol{U}^T\int_{x_1^e}^{x_2^e}\rho g\boldsymbol{N}^T A dx \\
&= \delta\boldsymbol{U}^T\boldsymbol{K}^e\boldsymbol{U} - \delta\boldsymbol{U}^T\boldsymbol{F}^e
\end{aligned}
$$

Here, $\boldsymbol{K}^e$ and $\boldsymbol{F}^e$ are the element stiffness matrix and the element force vector given as

$$
\boldsymbol{K}^e = AE\begin{pmatrix} \int_{x_1^e}^{x_2^e}\frac{dN_1}{dx}\frac{dN_1}{dx} & \int_{x_1^e}^{x_2^e}\frac{dN_1}{dx}\frac{dN_2}{dx} \\ \int_{x_1^e}^{x_2^e}\frac{dN_2}{dx}\frac{dN_1}{dx} & \int_{x_1^e}^{x_2^e}\frac{dN_2}{dx}\frac{dN_2}{dx} \end{pmatrix}, \boldsymbol{F}^e = -A\rho g\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}.
$$

As $\delta\boldsymbol{U}$ is arbitrary, we have $\delta\Pi^e = 0$ and so

$$
\boldsymbol{K}^e\boldsymbol{U} = \boldsymbol{F}^e,
$$

which may be assembled to give the global stiffness and force.

In the case of the bar,

$$\frac{dN_1}{dx} = \frac{1}{L^e}, \frac{dN_2}{dx} = -\frac{1}{L^e},$$

so that, performing the integrations, we get

$$\boldsymbol{K}^e = \begin{pmatrix} \frac{AE}{L^e} & -\frac{AE}{L^e} \\ -\frac{AE}{L^e} & \frac{AE}{L^e} \end{pmatrix},$$

which is exactly the same as what we got from the fundamental definition of the stiffness matrix at the beginning of the course.

`Suggestion: write an UEL for a bar element with gravity load`

An Euler Bernoulli beam subjected to a distributed load $q(x)$ has the governing differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x).$$

The variational statement of the problem for an element is

$$\Pi^e = \frac{1}{2}\int_{x_1^e}^{x_2^e} w\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)dx - \int_{x_1^e}^{x_2^e} qwdx.$$

Through the usual procedure we ave adopted so many times before, we can write this as

$$\begin{aligned}\Pi^e = \int_{x_1^e}^{x_2^e} \frac{EI}{2}\left[\left(\frac{d^2w}{dx^2}\right)^2 - wq\right]dx \quad &- \quad w(x_1^e)Q_1^e - w(x_2^e)Q_3^e \\ &- \quad \left(-\frac{dw}{dx}\right)\bigg|_{x_1^e} Q_2^e - \left(-\frac{dw}{dx}\right)\bigg|_{x_2^e} Q_4^e\end{aligned}$$

where, Q_1^e, Q_3^e are shear forces and Q_2^e, Q_4^e are bending moments acting at the ends of the element.

Note that the FE approximation has to be twice differentiable as the variational principle contains d^2w/dx^2. Consider

$$w^h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

So that

$$\left\{ \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array} \right\} = \begin{pmatrix} 1 & x_1^e & x_1^{e2} & x_1^{e3} \\ 0 & -1 & -2x_1^e & -3x_1^{e2} \\ 1 & x_2^e & x_2^{e2} & x_2^{e3} \\ 0 & -1 & -2x_2^e & -3x_2^{e2} \end{pmatrix} \left\{ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \right\}$$

where $w(x_1^e) = w_1, w(x_2^e) = w_2$ and $(-dw/dx)_{x_1^e} = \theta_1, (-dw/dx)_{x_1^e} = \theta_2$.

Inverting the above equation, w^h can be written as

$$w^h = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2.$$

The shape functions are determined as

$$
\begin{aligned}
N_1 &= 1-3\left(\frac{x-x_1^e}{L^e}\right)^2+2\left(\frac{x-x_1^e}{L^e}\right)^3 \\
N_2 &= -(x-x_1^e)\left(1-\frac{x-x_1^e}{L^e}\right)^2 \\
N_3 &= 3\left(\frac{x-x_1^e}{L^e}\right)^2-2\left(\frac{x-x_1^e}{L^e}\right)^3 \\
N_4 &= -(x-x_1^e)\left[\left(\frac{x-x_1^e}{L^e}\right)^3-\frac{x-x_1^e}{L^e}\right]
\end{aligned}
$$

The interpolation of w is done by using the values of w at the nodes as well as the values of its derivatives, Such interpolation is called *cubic spline* interpolation.

Similar to the bar element we write:

$$\boldsymbol{U} = \left\{ \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array} \right\}, \text{ and } \boldsymbol{N}^{\mathrm{T}} = \left\{ \begin{array}{c} N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \right\}$$

so that

$$w^h = \boldsymbol{N}\boldsymbol{U}$$

and

$$\frac{d^2 w^h}{dx^2} = \boldsymbol{B}\boldsymbol{U}.$$

Also,

$$\begin{aligned} \delta\Pi^e \quad &= \quad \int_{x_1^e}^{x_2^e} \left[EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2} - q\delta w \right] dx - \delta w_1 Q_1^e - \\ & \delta w_2 Q_2^e - \left(-\frac{d\delta w}{dx} \right) \bigg|_{x_1^e} Q_2^e - \left(-\frac{d\delta w}{dx} \right) \bigg|_{x_2^e} Q_4^e. \end{aligned}$$

Again we get

$$\boldsymbol{K}^e = \int_{x_1^e}^{x_2^e} EI\boldsymbol{B}^T\boldsymbol{B}dx, \text{ and } \boldsymbol{F}^e = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^T q dx + \langle Q_1^e \; Q_2^e \; Q_3^e \; Q_4^e\rangle^T.$$

Explicitly,

$$\boldsymbol{K}^e = \frac{2E^eI^e}{(L^e)^3}\left(\begin{array}{cccc} 6 & -3L^e & -6 & -3L^e \\ -3L^e & 2(L^e)^2 & 3L^e & (L^e)^2 \\ -6 & 3L^e & 6 & 3L^e \\ -3L^e & (L^e)^2 & 3L^e & 2(L^e)^2 \end{array}\right),$$

and

$$\boldsymbol{F}^e = \frac{qL^e}{12}\left\{\begin{array}{c} 6 \\ -L^e \\ 6 \\ L^e \end{array}\right\} + \left\{\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{array}\right\}$$

Suggest: Write an UEL for a 2-d beam+truss element oriented at an arbitrary angle