Name: Roll no:

ME623: Finite Element Methods in Engineering

End-Semester Examination, 29th April, 2018

Duration: 3 hours

Full Marks: 60

Important note:

- *Write the answer sought inside the box provided. Answers written elsewhere will not be counted.*
- *Nothing other than the specific answer must be written inside the boxes.*
- *Read instructions in each question carefully. The instructions about where and how you should show your calculations vary from question to question.*
- *You may use your laptop, coursenotes, books and programmes like Matlab.*
- *All figures are given in the last page of the question paper.*

"Tricks and treachery are the practise of fools, that don't have brains enough to be honest"

Benjamin Franklin.

Pr 1: Consider the structure shown in the figure for problem 1 in the last page of the question paper. The structure has 4 elements of three different types. Element 1 is a 4 noded iso-parametric plane stress element with unit thickness. 2 and 3 are truss elements and 4 is a beam element. The boundary conditions are as shown in the figure. For all elements modulus $E = 1000$ and $\nu = 0.25$ in consistent units. All dimensions in the figure are also given in consistent units. **For the following questions, use extra sheets to show your calculations. Only the final answers must be written in the boxes provided. Attach the extra sheets to your answer sheet. For all numerical answers, use upto 2 significant digits after decimal.**

(a) Derive the Jacobian J, its inverse J^{-1} and $J = \det J$ for element 1 at $(r, s) = (1/\sqrt{3}, -1/\sqrt{3})$ [5+1+1=7]

(c) Find the diagonal elements of the stiffness matrix for element 1 at a point $(r, s) = (1/\sqrt{3}, -1/\sqrt{3})$. [4]

224.5079	27.7778	-208.6586	24.8006	-71.7593	-45.9331	55.9099	-6.6453
27.7778	91.8661	51.0897	-49.6011	-65.1781	-55.5556	-13.6895	13.2906
-208.6586	51.0897	267.8092	-103.6681	12.6086	24.8006	-71.7593	27.7778
24.8006	-49.6011	-103.6681	207.3362	51.0897	-102.1795	27.7778	-55.5556
-71.7593	-65.1781	12.6086	51.0897	62.5291	27.7778	-3.3785	-13.6895
-45.9331	-55.5556	24.8006	-102.1795	27.7778	130.3561	-6.6453	27.3789
55.9099	-13.6895	-71.7593	27.7778	-3.3785	-6.6453	19.2278	-7.4430
-6.6453	13.2906	27.7778	-55.5556	-13.6895	27.3789	-7.4430	14.8861

$K_{11}^{(1)} = 430.56$
$K_{22}^{(1)} = 333.33$
$K_{12}^{(1)} = 166.67$
$K_{13}^{(1)} = -280.56$
$K_{57}^{(1)} = -280.56$
$K_{67}^{(1)} = 33.33.$

(e) Derive the stiffness matrix $K^{(2)}$ for element 2. Follow the local dof numbering shown. It is specified that $AE = 4000$ in consistent units. [3]

$$K^{(2)} = \begin{pmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 0 & 0 & 0 \\ -1000 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(f) Derive the stiffness matrix $K^{(2)}$ for element 3. Follow the local dof numbering shown. The value of AE is same as in element 2. [3]

$$K' = \begin{pmatrix} 2000 & 0 & 0 & -2000 & 0 & 0 \\ 0 & 1500 & 1500 & 0 & -1500 & 1500 \\ 0 & 1500 & 2000 & 0 & -1500 & 1000 \\ -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & -1500 & -1500 & 0 & 1500 & -1500 \\ 0 & 1500 & 1000 & 0 & -1500 & 2000 \end{pmatrix}$$

...ributed load $\underset{\sim}{t}$ is applied on $s = \pm 1$,

where $\underset{\sim}{t} = \begin{Bmatrix} 0 \\ \frac{p_0}{3} x \end{Bmatrix}$

On $s = \pm 1$, $x = N_3\big|_{s=1} x_3 + N_4\big|_{s=1} x_4$

$x = \frac{3}{2}(1+r)$.

$N_1\big|_{s=1} = N_2\big|_{s=1} = 0$

$N_4\big|_{s=1} = \frac{1}{2}(1-r)$

$$\Rightarrow \underset{\sim}{F_s} = \int_{-1}^{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1+r}{2} & 0 \\ 0 & \frac{1+r}{2} \\ \frac{1-r}{2} & 0 \\ 0 & \frac{1-r}{2} \end{pmatrix} \begin{Bmatrix} 0 \\ \frac{p_0}{2}(1+r) \end{Bmatrix} J_s\, dr.$$

$J_s = \frac{3}{2}$.

$$= \int_{-1}^{1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p_0}{4}(1+r)^2 \\ 0 \\ \frac{p_0}{4}(1-r^2) \end{Bmatrix} \frac{3}{2}\, dr = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p_0 \\ 0 \\ p_0/2. \end{Bmatrix}$$

the box below, scratch out the terms that you think are not relevant (depending on whether there is a contribution from the element in the superscipt). To the relevant terns add subscripts corresponding to the indices of the local stiffnesses that contribute to the given component of the global stiffness. The actual value of the component of the global stiffness has to be filled in the rightmost box. [4+2=6]

$$K_{22} = K^{(1)}_{\boxed{22}} + \cancel{K^{(2)}_{\square}} + \cancel{K^{(3)}_{\square}} + \cancel{K^{(4)}_{\square}} = \boxed{}$$ 333.33

$$K_{36} = K^{(1)}_{\boxed{56}} + \cancel{K^{(2)}_{\square}} + K^{(3)}_{\boxed{34}} + \cancel{K^{(4)}_{\square}} = \boxed{}$$ ~~[illegible]~~

$$K_{99} = \cancel{K^{(1)}_{\square}} + K^{(2)}_{\boxed{33}} + K^{(3)}_{\boxed{11}} + K^{(4)}_{\boxed{11}} = \boxed{}$$ 1000 + 353.5534 + 2000 = 3353.53

$$K_{10,10} = \cancel{K^{(1)}_{\square}} + K^{(2)}_{\boxed{44}} + K^{(3)}_{\boxed{22}} + K^{(4)}_{\boxed{22}} = \boxed{}$$ ~~2000~~ 0 + 353.5534 + 1500 = 1853.5534

$$K_{11,11} = \cancel{K^{(1)}_{\square}} + \cancel{K^{(2)}_{\square}} + \cancel{K^{(3)}_{\square}} + K^{(4)}_{\boxed{33}} = \boxed{}$$ 2000

(i) Element 1 has a distributed normal traction applied on the top edge as shown. Derive the global force vector $\boldsymbol{F}_s$ in terms of p_0. [3]

$$\boldsymbol{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p_0 \\ 0 \\ p_0/2 \end{Bmatrix}$$

Pr. 2: **The calculations for this problem must be shown in the space provided below the question. The final answer has to be written in the box** where X_i denotes the coordinates of a point in a fixed reference configuration of a body. The point deforms to $x_i = X_i + u_i$ under the action of forces. What is δE_{ij} if u_i is perturbed by δu_i? [4]

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).$$

$$\delta E_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial \bar{X}_j} + \frac{\partial \delta u_j}{\partial \bar{X}_i} + \frac{\partial \delta u_k}{\partial \bar{X}_i}\frac{\partial u_k}{\partial \bar{X}_j} + \frac{\partial u_k}{\partial \bar{X}_i}\frac{\partial \delta u_k}{\partial \bar{X}_j}\right).$$

$$\boxed{\delta E_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial \bar{X}_j} + \frac{\partial \delta u_j}{\partial \bar{X}_i} + \frac{\partial \delta u_k}{\partial \bar{X}_i}\frac{\partial u_k}{\partial \bar{X}_j} + \frac{\partial u_k}{\partial \bar{X}_i}\frac{\partial \delta u_k}{\partial \bar{X}_j}\right)}$$

(b) A self equilibriated stress field in a body with volume V and bounary ∂V (with normal n_i to ∂V) is one that satisfies the equilibrium equation

$$\sigma_{ij,j} + b_i = 0.$$

Consider an arbitrary function δu_i that satisfies $\delta u_i = 0$ on ∂V_u. A weak form of the above can be framed as:

$$\int_V \delta u_i \left(\sigma_{ij,j} + b_i\right) dV = 0.$$

Then. [3]

$\int_V \delta\epsilon_{ij}\sigma_{ij} dV = \int_V \delta u_i \, b_i \; dV + \int_{\partial V_t} \sigma_{ij} \, \delta u_i \; n_j dS.$

$$\int_V \delta u_i \left(\sigma_{ij,j} + b_i\right) dV = 0.$$

$$\Rightarrow \int_V \left(\delta u_i \, \sigma_{ij,j} + \delta u_i \, b_i\right) dV = \int_V \left[\left(\delta u_i \, \sigma_{ij}\right)_{,j} - \delta u_{i,j} \, \sigma_{ij} + \delta u_i b_i\right] dV$$

$$\Rightarrow \int_{\partial V_t} \sigma_{ij} n_j \, \delta u_i \, dS - \int_V \delta\varepsilon_{ij} \, \sigma_{ij} \, dV + \int_V \delta u_i \, b_i \, dV = 0$$

$$\Rightarrow \int_V \delta\varepsilon_{ij} \, \sigma_{ij} \, dV = \int_V \delta u_i \, b_i \, dV + \int_{\partial V_t} \sigma_{ij} \, n_j \, \delta u_i \, dS.$$

(c) A 2-d volume V is discretised by 4 noded quadrilateral elements. In each element, the field variable $\psi(x, y)$ is discretised as $\psi(x, y) = \sum_{I=1}^{4} N_I \Psi_I$, where Ψ_I are the nodal values of $\psi(x, y)$. Using the shape functions as the trial functions, apply Galerkin's method to derive the weak form. The field variable is governed by the equation

$$\nabla^2 \psi = 0,$$

and on the boundary ∂V, its value is specified as $\psi = \bar{\psi}$. The weak form is: [3]

$$\int_V N_I \, \nabla^2 \psi \, dV = \int_V N_I \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) dV = 0.$$

$$\int_V N_I \frac{\partial^2 \psi}{\partial x^2} dV = \int_V \left\{\frac{\partial}{\partial x}\left(N_I \frac{\partial \psi}{\partial x}\right) - \frac{\partial N_I}{\partial x}\frac{\partial \psi}{\partial x}\right\} dV$$

$$= \int_{\partial V} N_I \frac{\partial \psi}{\partial x} n_x \, dS - \int_V \frac{\partial N_I}{\partial x} \frac{\partial N_J}{\partial x} dV \, \Psi_J$$

$$\text{IIly} \int_V N_I \frac{\partial^2 \psi}{\partial x^2} dV = \int_{\partial V} N_I \frac{\partial \psi}{\partial y} n_y \, dS - \int_V \frac{\partial N_I}{\partial y} \frac{\partial N_J}{\partial y} dV \, \Psi_J.$$

$$\int_V \left(\frac{\partial N_I}{\partial x}\frac{\partial N_J}{\partial x} + \frac{\partial N_I}{\partial y}\frac{\partial N_J}{\partial y}\right) dV \Psi_J = 0.$$

Pr. 3: Consider the 1-dimensional 3 noded linear element shown in the figure below. Note that node 2 is situated at the quarter point. For this element, find the following. **All calculations must be shown in the space given below the question**

(a) The mapping between x and r in terms of L. [3]

$$N_1 = \frac{1}{2} r(r-1)$$

$$N_2 = 1 - r^2$$

$$N_3 = \frac{1}{2} r(1+r)$$

$$x = \frac{1}{2} r(r-1) \cdot 0 + (1-r^2)\frac{L}{4} + \frac{1}{2}(1+r)\, r \cdot L = \frac{L}{4}(1+r)^2.$$

$$x = \frac{L}{4}(1+r)^2$$

(b) The B matrix in terms of r and L. [5]

$$\frac{dN_1}{dr} = \frac{1}{2}(2r-1) \qquad \frac{dN_2}{dr} = 1-2r \qquad \frac{dN_3}{dr} = \frac{1}{2}(2r+1).$$

$$\frac{dx}{dr} = \frac{L}{2}(1+r).$$

$$\varepsilon_x = \frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx}$$

$$= \frac{2}{L(1+r)}\left(\frac{1}{2}(2r-1) \quad 1-2r \quad \frac{1}{2}(1+2r)\right)\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$B = \left(\frac{2\gamma - 1}{L(1+\gamma)} \quad \frac{2(1 - 2\gamma)}{L(1+\gamma)} \quad \frac{1 + 2\gamma}{L(1+\gamma)} \right).$$

(c) The $\boldsymbol{B}$ matrix in terms of x and L. [2]

$$x = \frac{L}{4}(1+\gamma)^2 \Rightarrow \gamma = 2\sqrt{\frac{x}{L}} - 1.$$

$$1 + \gamma = 2\sqrt{\frac{x}{L}}$$

$$1 - 2\gamma = -4\sqrt{\frac{x}{L}} + 3$$

$$1 + 2\gamma = 4\sqrt{\frac{x}{L}} - 1.$$

$$B = \left(-\left\{\frac{2}{L} + \frac{3}{2\sqrt{Lx}}\right\} \quad \left\{-\frac{4}{L} + \frac{2}{\sqrt{Lx}}\right\} \quad \left\{\frac{2}{L} - \frac{1}{2\sqrt{Lx}}\right\} \right).$$

(d) What is the variation of ϵ_{xx} very close to node 1?

$$\boxed{\varepsilon_{xx} \propto \frac{1}{\sqrt{x}}}$$

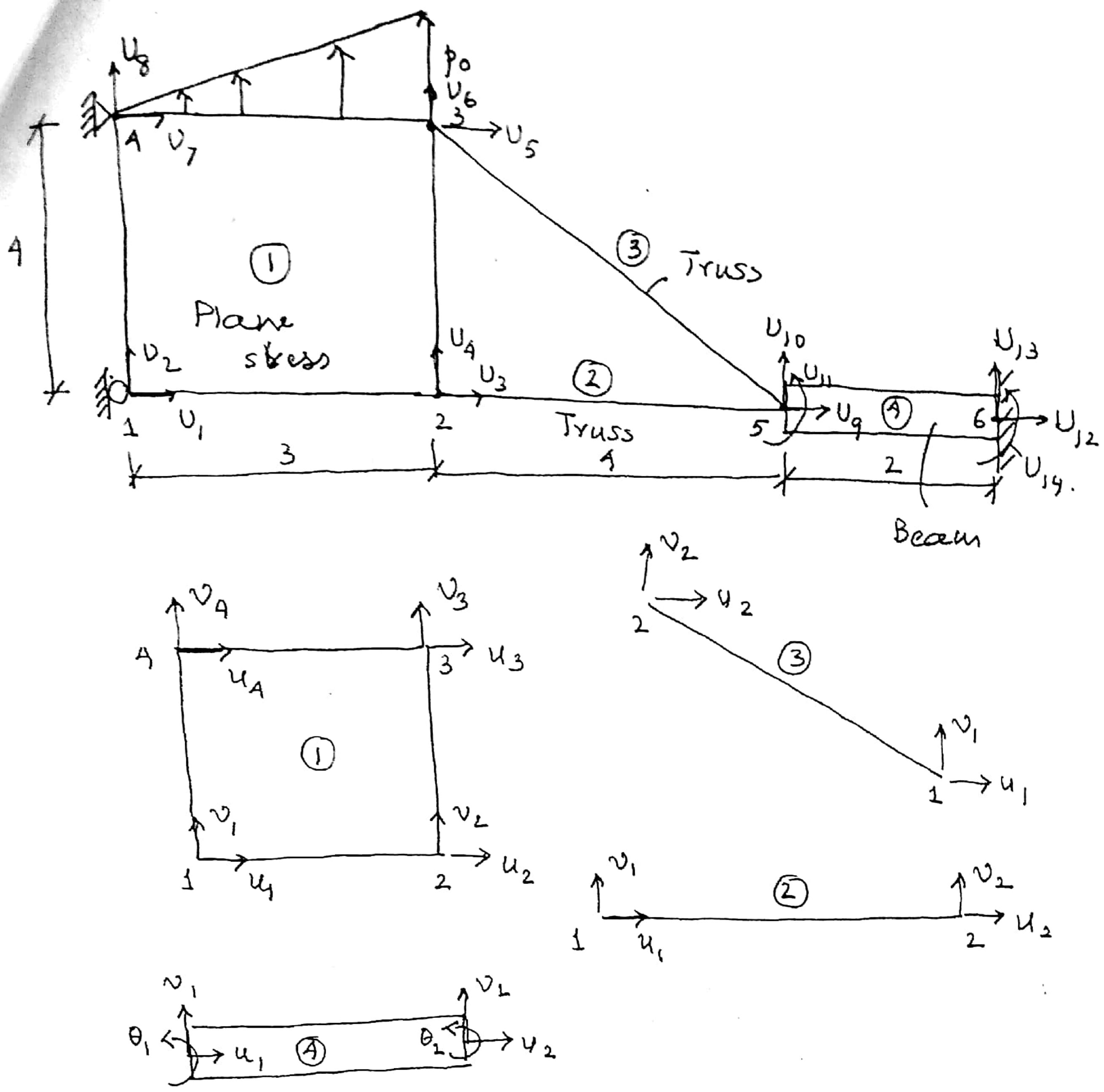

Figure for Problem 1

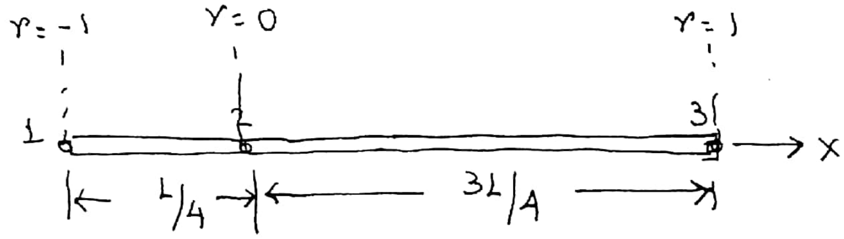

Figure for Problem 3