ASSIGNMENT II MSO 202 A

POWER SERIES, ANALYTIC FUNCTIONS, AND INTEGRATION

Exercise 0.1 : Does there exist a holomorphic function $f = u + iv$ on the complex plane such that $u(x, y) = x^2$ and $v(x, y) = y^2$?

Exercise 0.2 : Find the radius of convergence (for short, RoC) of the following power series:

(1) $\sum_{n=1}^{\infty} \frac{z^n}{n}$.
(2) $\sum_{n=1}^{\infty} z^{n!}$.
(3) $\sum_{n=1}^{\infty} n^{(-1)^n} z^n$.
(4) $\sum_{n=2}^{\infty} (\log n)^2 z^n$.
(5) $\sum_{n=2}^{\infty} a_n z^n$, where a_n is the number of prime numbers less than or equal to n.

Exercise 0.3 : Show that $f(z) = \frac{1}{1-z}$ defines an analytic function on the unit disc centered at 0, that is, for every $|a| < 1$, $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ in some disc centered at a.

Exercise 0.4 : Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial and let γ denote the unit circle with parametrization $z(t) = e^{it},\ 0 \leqslant t \leqslant 2\pi$. Show that

$$\int_\gamma (p(z) + p(1/z))dz = (2\pi i)a_1.$$

Exercise 0.5 : Let γ be a circle of radius 2 centered at 0. Verify the following (*without* Cauchy Integral Formula):

(1) $\int_\gamma \frac{1}{z-1} dz = 2\pi i$.
(2) $\int_\gamma \frac{1}{z-3} dz = 0$.

Conclude that

$$\int_\gamma \frac{1}{(z-1)(z-3)} dz = -\pi i.$$

Exercise 0.6 : Let γ be the unit circle with following parametrizations:

$$\begin{aligned} z_1(t) &= e^{it} \ (0 \leq t \leq 2\pi), \\ z_2(t) &= e^{2it} \ (0 \leq t \leq 2\pi). \end{aligned}$$

Can you explain (with and without computations) why the integral of $\frac{1}{z}$ along the parametrizations z_1 and z_2 of the unit circle differ ?

Exercise 0.7 : Let $\mathbb{D}$ be the unit disc centered at 0 and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Prove that if $\mathrm{Re}(f'(z)) > 0$ for all $z \in \mathbb{D}$ then f is injective.