Definition of Stress

Consider a small area δA on the surface of a body (Fig. 1.1). The force acting on this area is δF
This force can be resolved into **two perpendicular components**

- The component of force acting normal to the area called **normal** force and is denoted by δF_n
- The component of force acting along the plane of area is called **tangential** force and is denoted by δF_t

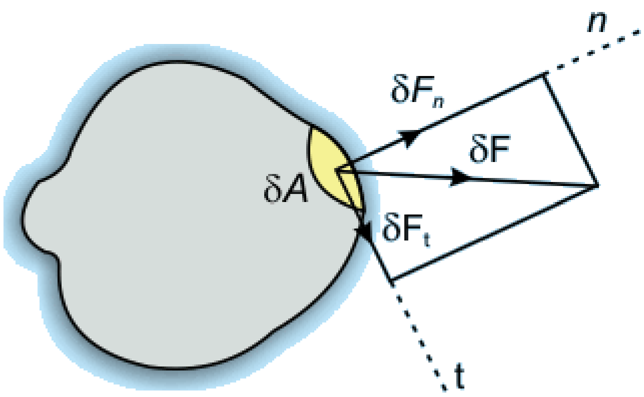


Fig 1.1 Normal and Tangential Forces on a surface

When they are expressed as force per unit area they are called as **normal stress** and **tangential stress** respectively**.** The tangential stress is also

called shear stress

The normal stress

$$\sigma = \lim_{\delta A \to 0} \left(\frac{\delta F_n}{\delta A} \right) \tag{1.1}$$

And shear stress

$$\tau = \lim_{\delta A \to 0} \left(\frac{\delta F_t}{\delta A} \right) \tag{1.2}$$

Definition of Fluid

- A fluid is a substance that **deforms continuously** in the face of tangential or shear stress, **irrespective of the magnitude of shear stress** .This continuous deformation under the application of shear stress constitutes a flow.

- In this connection fluid can also be defined as the **state of matter that cannot sustain any shear stress.**

Example : Consider Fig 1.2

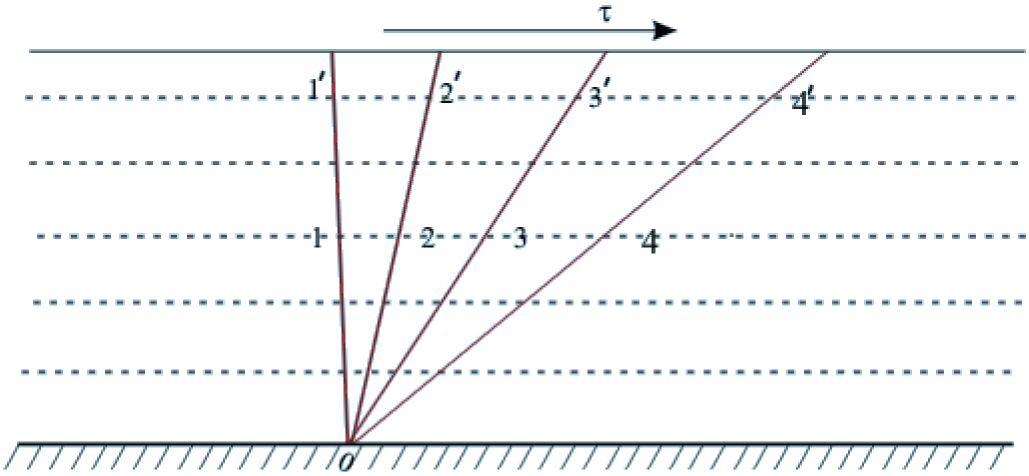


Fig 1.2 Shear stress on a fluid body

If a shear stress τ is applied at any location in a fluid, the element 011' which is initially at rest, will move to 022', then to 033'. Further, it moves to 044' and continues to move in a similar fashion.

In other words, the **tangential stress in a fluid body depends on velocity of deformation and vanishes as this velocity approaches zero. A good example is Newton's parallel plate experiment where dependence of shear force on the velocity of deformation was established. Distinction Between Solid and Fluid:**

Solid	Fluid
▪ More Compact Structure	▪ Less Compact Structure
▪ Attractive Forces between the molecules are larger therefore more closely packed	▪ Attractive Forces between the molecules are smaller therefore more loosely packed
▪ Solids can resist tangential stresses in static condition	▪ Fluids cannot resist tangential stresses in static condition.
▪ Whenever a solid is subjected to shear stress a. It undergoes a definite deformation α or breaks b. α is proportional to shear stress upto some limiting condition	▪ Whenever a fluid is subjected to shear stress a. No fixed deformation b. Continious deformation takes place until the shear stress is applied
▪ Solid may regain partly or fully its original shape when the tangential	▪ A fluid can never regain its original shape, once it has been distorded by

stress is removed	the shear stress

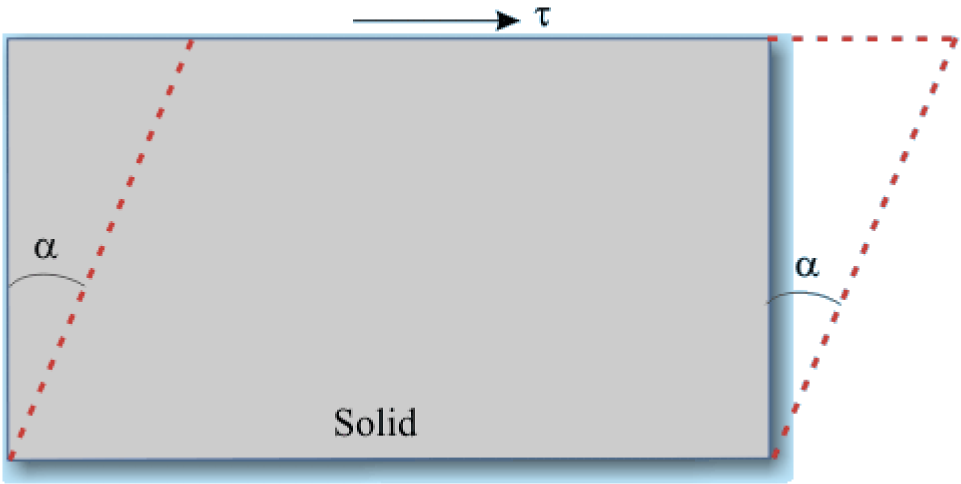


Fig 1.3 Deformation of a Solid Body

Concept of Continuum

- The concept of continuum is a kind of idealization of the continuous description of matter where the properties of the matter are considered as continuous functions of space variables. Although any matter is composed of several molecules, the concept of continuum assumes a continuous distribution of mass within the matter or system with no empty space, instead of the actual conglomeration of separate molecules.

- Describing a fluid flow quantitatively makes it necessary to assume that flow variables (pressure , velocity etc.) and fluid properties vary continuously from one point to another. Mathematical description of flow on this basis have proved to be reliable and treatment of fluid medium as a continuum has firmly become established. For example density at a point is normally defined as

$$\rho = \lim_{\Delta\forall \to 0}\left(\frac{m}{\Delta\forall}\right)$$

Here $\Delta\forall$ is the volume of the fluid element and m is the mass

- If $\Delta\forall$ is very large ρ is affected by the inhomogeneities in the fluid medium. Considering another extreme if $\Delta\forall$ is very small, random movement of atoms (or molecules) would change their number at different times. In the continuum approximation point density is defined at the smallest magnitude of $\Delta\forall$, before statistical fluctuations become significant. This is called continuum limit and is denoted by $\Delta\forall_c$.

$$\rho = \lim_{\Delta\forall \to \Delta\forall_C} \left(\frac{m}{\Delta\forall} \right)$$

▪

Concept of Continuum - contd from previous slide

- One of the factors considered important in determining the validity of continuum model is molecular density. It is the distance between the molecules which is characterised by mean free path (λ). It is calculated by finding statistical average distance the molecules travel between two successive collisions. If the mean free path is very small as compared with some characteristic length in the flow domain (i.e., the molecular density is very high) then the gas can be treated as a continuous medium. If the mean free path is large in comparison to some characteristic length, the gas cannot be considered continuous and it should be analysed by the molecular theory.

- A dimensionless parameter known as Knudsen number, $K_n = \lambda / L$, where λ is the mean free path *and L* is the characteristic length. It describes the degree of departure from continuum.

  Usually when $K_n > 0.01$, the concept of continuum does not hold good.

  Beyond this critical range of Knudsen number, the flows are known as

  slip flow ($0.01 < K_n < 0.1$),

  transition flow ($0.1 < K_n < 10$) and

  free-molecule flow ($K_n > 10$).

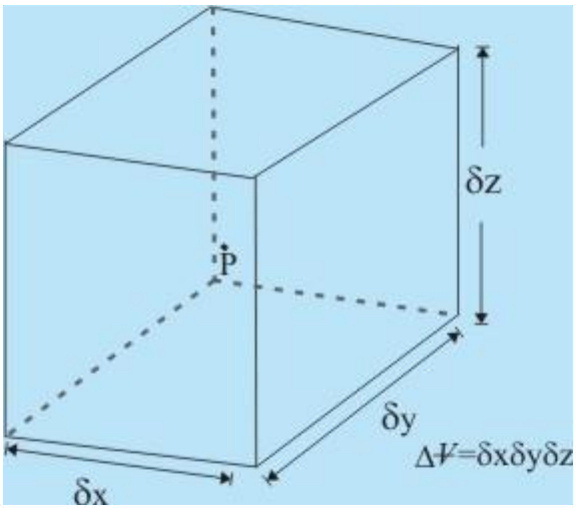
  However, for the flow regimes considered in this course **, K_n is always less than 0.01 and it is usual to say that the fluid is a continuum.**

  Other factor which checks the validity of continuum is the elapsed time between collisions. The time should be small enough so that the random statistical description of molecular activity holds good.

  In continuum approach, fluid properties such as density, viscosity, thermal conductivity, temperature, etc. can be expressed as continuous functions of space and time.

Fluid Properties :

Characteristics of a continuous fluid which are independent of the motion of the fluid are called basic properties of the fluid. Some of the basic properties are as discussed below.

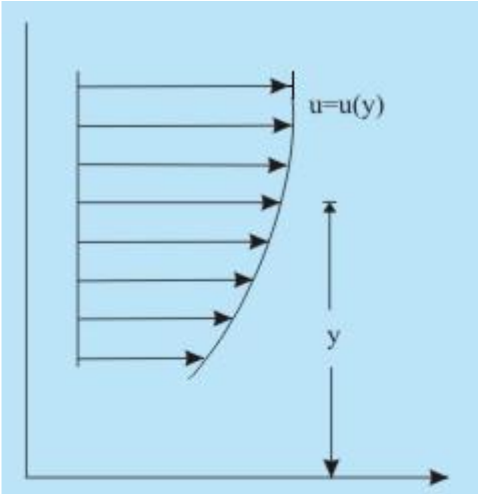
Property	Symbol	Definition	Unit
Density	ρ	The density *p* of a fluid is its mass per unit volume . If a fluid element enclosing a point *P* has a volume $\Delta\forall$ and mass Δm (Fig. 1.4), then density (ρ)at point *P* is written as $\rho = \lim_{\Delta\forall \to \Delta\forall_C} \left(\frac{m}{\Delta\forall}\right)$ However, in a medium where continuum model is valid one can write - $\rho = \lim_{\Delta\forall \to 0} \left(\frac{m}{\Delta\forall}\right) = \left[\frac{dm}{d\forall}\right]_{\forall}$ (1.3)  Fig 1.4 A fluid element enclosing point P	kg/m^3
Specific Weight	γ	The **specific weight** is the weight of fluid per unit volume. The specific weight is given by $\gamma = \rho g$ (1.4) Where *g* is the gravitational acceleration. Just as weight must be clearly distinguished from mass, so must the specific weight be distinguished from density.	N/m^3
Specific	v	The **specific volume** of a fluid is the volume occupied by unit mass of fluid.	m^3

Volume		Thus $v = \frac{1}{\rho}$ (1.5)	/kg
Specific Gravity	s	For liquids, it is the ratio of density of a liquid at actual conditions to the density of pure water at 101 kN/m^2 , and at 4°C. The specific gravity of a gas is the ratio of its density to that of either hydrogen or air at some specified temperature or pressure. **However, there is no general standard; so the conditions must be stated while referring to the specific gravity of a gas.**	-

Viscosity (μ) :

- Viscosity is a fluid property whose effect is understood when the fluid is in motion.
- In a flow of fluid, when the fluid elements move with different velocities, each element will feel some resistance due to fluid friction within the elements.
- Therefore, shear stresses can be identified between the fluid elements with different velocities.
- The relationship between the shear stress and the velocity field was given by Sir Isaac Newton.

Consider a flow (Fig. 1.5) in which all fluid particles are moving in the same direction in such a way that the fluid layers move parallel with different velocities.



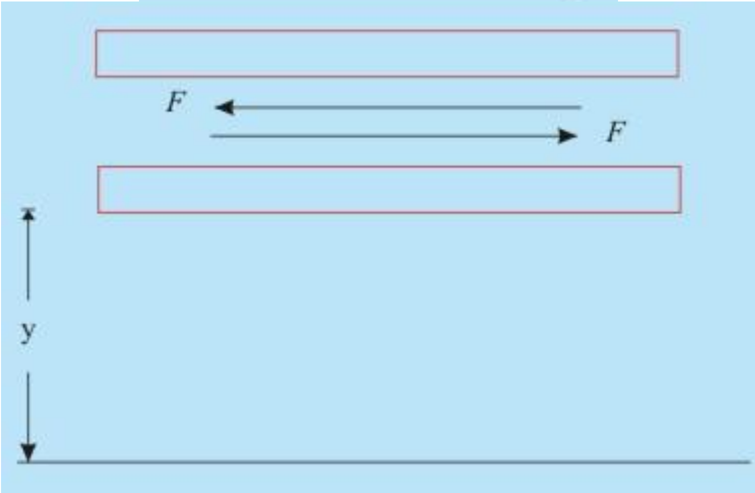


Fig 1.5 Parallel flow of a fluid

Fig 1.6 Two adjacent layers of a moving fluid.

- The upper layer, which is moving faster, tries to draw the lower slowly moving layer along with it by means of a force *F* along the direction of flow on this layer. Similarly, the lower layer tries to retard the upper one, according to Newton's third law, with an equal and opposite force *F* on it (Figure 1.6).

- Such a fluid flow where x-direction velocities, for example, change with y-coordinate is called **shear flow** of the fluid.

- Thus, the dragging effect of one layer on the other is experienced by a tangential force *F* on the respective layers. If *F* acts over an area of contact *A*, then the shear stress τ is defined as

$$\tau = F/A$$

Viscosity (μ) :

- Newton postulated that τ is proportional to the quantity $\Delta u/\Delta y$ where Δy is the distance of separation of the two layers and Δu is the difference in their velocities.

- In the limiting case of , $\Delta u / \Delta y$ equals du/dy, the velocity gradient at a point in a direction perpendicular to the direction of the motion of the layer.

- According to Newton τ and du/dy bears the relation

$$\tau = \mu \frac{du}{dy} \tag{1.7}$$

where, the constant of proportionality μ is known as the **coefficient of viscosity** or simply viscosity which is a property of the fluid and depends on its state. Sign of τ depends upon the sign of du/dy. For the profile shown in Fig. 1.5, du/dy is positive everywhere and hence, τ is positive. Both the velocity and stress are considered positive in the positive direction of the coordinate parallel to them.
Equation

$$\tau = \mu \frac{du}{dy}$$

defining the viscosity of a fluid, is known as Newton's law of viscosity. Common fluids, viz. water, air, mercury obey Newton's law of viscosity and are known as *Newtonian fluids.*

Other classes of fluids, viz. paints, different polymer solution, blood do not obey the typical linear relationship, of τ and du/dy and are known as **non-Newtonian fluids**. In non-newtonian fluids viscosity itself may be a function of deformation rate as you will study in the next lecture.

Causes of Viscosity

- The causes of viscosity in a fluid are possibly attributed to two factors:

(i) intermolecular force of cohesion
(ii) molecular momentum exchange

- Due to strong cohesive forces between the molecules, any layer in a moving fluid tries to drag the adjacent layer to move with an equal speed and thus produces the effect of viscosity as discussed earlier. Since cohesion decreases with temperature, the liquid viscosity does likewise.

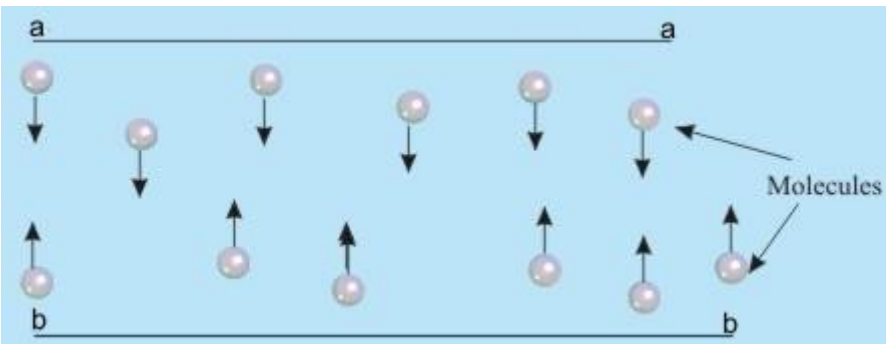


Fig 1.7 Movement of fluid molecules between two adjacent

moving layers

Click to play the Demonstration

- Molecules from layer aa in course of continous thermal agitation migrate into layer bb
- Momentum from the migrant molecules from layer aa is stored by molecules of layer bb by way of collision
- Thus layer bb as a whole is speeded up
- Molecules from the lower layer bb arrive at aa and tend to retard the layer aa
- Every such migration of molecules causes forces of acceleration or deceleration to drag the layers so as to oppose the differences in velocity between the layers and produce the effect of viscosity.

Causes of Viscosity - contd from previous slide...

- As the random molecular motion increases with a rise in temperature, the viscosity also increases accordingly. Except for very special cases (e.g., at very high pressure) the viscosity of both liquids and gases ceases to be a function of pressure.
- For Newtonian fluids, the coefficient of viscosity depends strongly on temperature but varies very little with pressure.
- For liquids, molecular motion is less significant than the forces of cohesion, thus **viscosity of liquids decrease with increase in temperature.**
- For gases,molecular motion is more significant than the cohesive forces, thus **viscosity of gases increase with increase in temperature.**

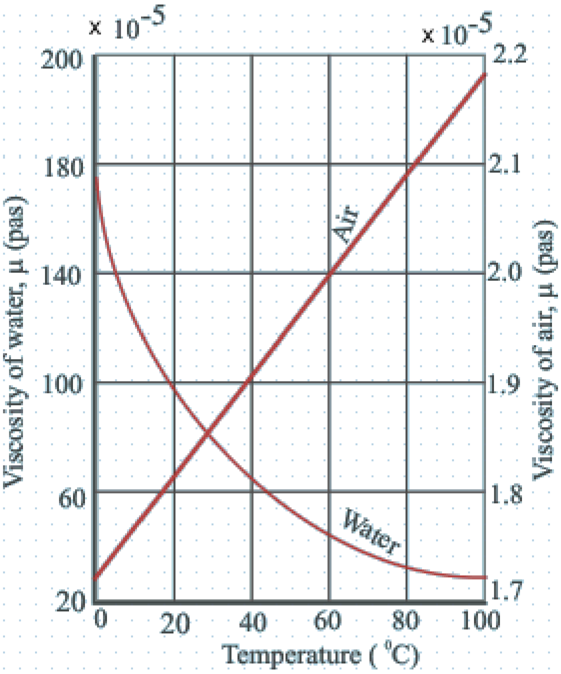


Fig 1.8: Change of Viscosity of Water and Air under 1 atm

No-slip Condition of Viscous Fluids

- It has been established through experimental observations that the relative velocity between the solid surface and the adjacent fluid particles is zero whenever a viscous fluid flows over a solid surface. This is known as no-slip condition.
- This behavior of no-slip at the solid surface is not same as the wetting of surfaces by the fluids. For example, mercury flowing in a stationary glass tube will not wet the surface, but will have zero velocity at the wall of the tube.
- The wetting property results from surface tension, whereas the no-slip condition is a consequence of fluid viscosity.

Ideal Fluid

- Consider a hypothetical fluid having a zero viscosity ($\mu = 0$). Such a fluid is called an *ideal fluid* and the resulting motion is called as **ideal** or **inviscid flow**. **In an ideal flow, there is no existence of shear force because of vanishing viscosity.**

$$\tau = \mu \frac{du}{dy} = 0 \qquad \text{since } \mu = 0$$

- All the **fluids in reality have viscosity** ($\mu > 0$) and hence they are termed as real fluid and their motion is known as viscous flow.
- Under certain situations of very high velocity flow of viscous fluids, an accurate analysis of flow field away from a solid surface can be made from the ideal flow theory.

Non-Newtonian Fluids

- There are certain fluids where the linear relationship between the shear stress and the deformation rate (velocity gradient in parallel flow) as expressed by the $\tau = \mu \frac{du}{dy}$ is not valid. For these fluids the viscosity varies with rate of deformation.
- Due to the deviation from Newton's law of viscosity they are commonly termed as **non-Newtonian fluids**. Figure 2.1 shows the class of fluid for which this relationship is nonlinear.

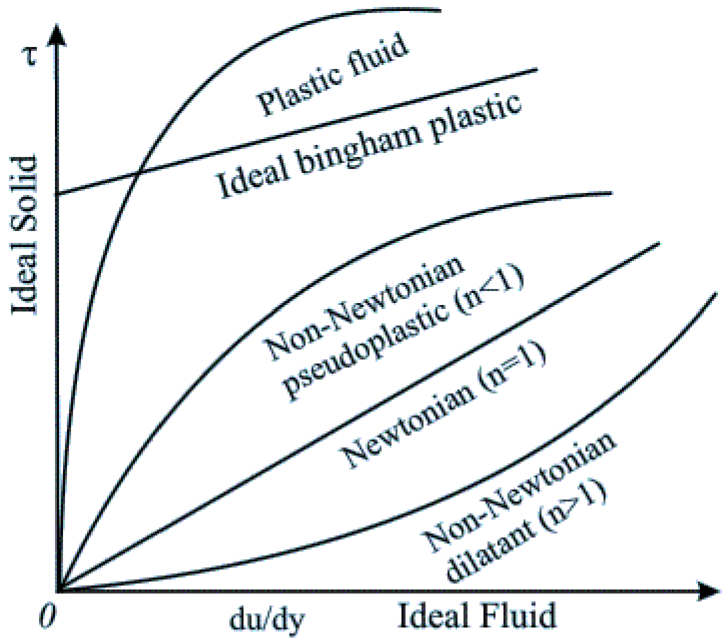


Figure 2.1 Shear stress and deformation rate relationship of different fluids

- The abscissa in Fig. 2.1 represents the behaviour of ideal fluids since for the ideal fluids the resistance to shearing deformation rate is always zero, and hence they exhibit zero shear stress under any condition of flow.

- The ordinate represents the ideal solid for there is no deformation rate under any loading condition.

- The Newtonian fluids behave according to the law that shear stress is linearly proportional to velocity gradient or rate of shear strain $\tau = \mu\, du/dy$. Thus for these fluids, the plot of shear stress against velocity gradient is a straight line through the origin. The slope of the line determines the viscosity.

- The non-Newtonian fluids are further classified as pseudo-plastic, dilatant and Bingham plastic.

Compressibility

- Compressibility of any substance is the measure of its change in volume under the action of external forces.

- The normal compressive stress on any fluid element at rest is known as hydrostatic pressure p and arises as a result of innumerable molecular collisions in the entire fluid.

- The degree of compressibility of a substance is characterized by the bulk modulus of elasticity E defined as

$$E = \lim_{\Delta\forall \to 0}\left(\frac{-\Delta p}{\Delta\forall\ /\forall}\right) \qquad (2.3)$$

- Where $\Delta\forall$ and Δp are the changes in the volume and pressure respectively, and $\forall$is the initial volume. The negative sign (-sign) is included to make E positive, since increase in pressure would decrease the volume i.e for $\Delta p>0$, $\Delta\forall<0$) in volume.
- For a given mass of a substance, the change in its volume and density satisfies the relation

$\Delta m = 0, \quad \Delta(\rho\forall) = 0$

$$\frac{\Delta\forall}{\forall} = -\frac{\Delta\rho}{\rho} \qquad (2.4)$$

using $E = \lim_{\Delta\forall \to 0}\left(\frac{-\Delta p}{\Delta\forall\ /\forall}\right)$ & $\frac{\Delta\forall}{\forall} = -\frac{\Delta\rho}{\rho}$

we get

$$E = \lim_{\Delta\rho \to 0}\left(\frac{\Delta p}{\Delta\rho / \rho}\right) = \rho\frac{dp}{d\rho} \qquad (2.5)$$

- Values of *E* for liquids are very high as compared with those of gases (except at very high pressures). Therefore, liquids are usually termed as incompressible fluids though, in fact, no substance is theoretically incompressible with a value of *E* as ∞.
- For example, the bulk modulus of elasticity for water and air at atmospheric pressure are approximately 2 x 10^6 kN/m ^{2} and 101 kN/m ^{2} respectively. It indicates that air is about 20,000 times more compressible than water. Hence water can be treated as incompressible.
- For gases another characteristic parameter, known as compressibility *K*, is usually defined , it is the reciprocal of *E*

$$K = \frac{1}{E} = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right) = -\frac{1}{\forall}\left(\frac{d\forall}{dp}\right) \qquad (2.6)$$

  - K is often expressed in terms of specific volume $\forall$.
- For any gaseous substance, a change in pressure is generally associated with a change in volume and a change in temperature simultaneously. A **functional relationship between the pressure, volume and temperature at any equilibrium state is known as thermodynamic equation of state for the gas.**

For an ideal gas, the thermodynamic equation of state is given by

$$p = \rho RT \qquad (2.7)$$

-
- where T is the temperature in absolute thermodynamic or gas temperature scale (which are, in fact, identical), and R is known as the characteristic gas constant, the value of which depends upon a particular gas. However, this equation is also valid for the real gases which are thermodynamically far from their liquid phase. For air, the value of R is 287 J/kg K.

- K and E generally depend on the nature of process

Distinction between an Incompressible and a Compressible Flow

- In order to know, if it is necessary to take into account the compressibility of gases in fluid flow problems, we need to consider whether the change in pressure brought about by the fluid motion causes large change in volume or density.

  Using Bernoulli's equation

  $p + (1/2)\rho V^2$= constant *(V* being the velocity of flow), change in pressure, Δp, in a flow field, is of the order of $(1/2)\rho V^2$ (dynamic head).
  Invoking this relationship into

$$E = \lim_{\delta\rho \to 0} \left(\frac{\Delta p}{\Delta \rho / \rho} \right) = \rho \frac{dp}{d\rho}$$

- we get ,

$$\frac{\Delta \rho}{\rho} \approx \frac{1}{2} \frac{\rho V^2}{E} \qquad (2.12)$$

-

  So if Δρ/ρ is very small, the flow of gases can be treated as incompressible with a good degree of approximation.

- According to Laplace's equation, the velocity of sound is given by

$$a = \sqrt{\frac{E}{\rho}}$$

- Hence

$$\frac{\Delta \rho}{\rho} \approx \frac{1}{2} \frac{V^2}{a^2} \approx \frac{1}{2} Ma^2$$

-

  where, Ma is the ratio of the velocity of flow to the acoustic velocity in the flowing medium at the condition and is known as **Mach number**. So we can conclude that the compressibility of gas in a flow can be neglected if $\Delta\rho/\rho$ is considerably smaller than unity, i.e. $(1/2)Ma^2 << 1$.

- In other words, if the flow velocity is small as compared to the local acoustic velocity, compressibility of gases can be neglected. **Considering a maximum relative change in density of 5 per cent as the criterion of an incompressible flow, the upper limit of Mach number becomes approximately 0.33**. In the case of air at standard pressure and temperature, the acoustic velocity is about 335.28 m/s. Hence a Mach number of 0.33 corresponds to a velocity of about 110 m/s. Therefore flow of air up to a velocity of 110 m/s under standard condition can be considered as incompressible flow.

Surface Tension of Liquids

- The phenomenon of surface tension arises due to the two kinds of intermolecular forces

  (i) **Cohesion :** The force of attraction between the molecules of a liquid by virtue of which they are bound to each other to remain as one assemblage of particles is known as the force of cohesion. This property enables the liquid to resist tensile stress.

  (ii) **Adhesion :** The force of attraction between unlike molecules, i.e. between the molecules of different liquids or between the molecules of a liquid and those of a solid body when they are in contact with each other, is known as the force of adhesion. This force enables two different liquids to adhere to each other or a liquid to adhere to a solid body or surface.

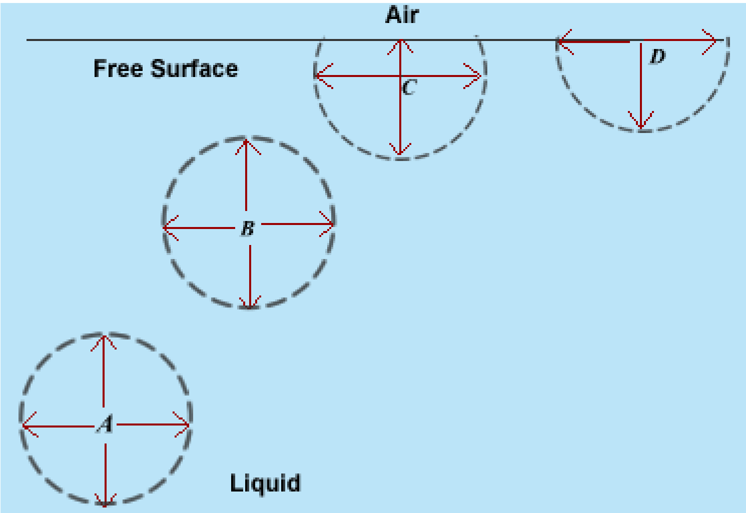


Figure 2.3 The intermolecular cohesive force field in a bulk of liquid with a free surface

A and B experience equal force of cohesion in all directions, C experiences a net force interior of the liquid The net force is maximum for D since it is at surface

- Work is done on each molecule arriving at surface against the action of an inward force. Thus mechanical work is performed in creating a free surface or in increasing the area of the surface. Therefore, a surface requires mechanical energy for its formation and the existence of a free surface implies the presence of stored mechanical energy known as free surface energy. Any system tries to attain the condition of stable equilibrium with its potential energy as minimum. Thus a quantity of liquid will adjust its shape until its surface area and consequently its free surface energy is a minimum.

- The magnitude of surface tension is defined as the tensile force acting across imaginary short and straight elemental line divided by the length of the line.

- The dimensional formula is *F/L* or MT^{-2} . It is usually expressed in N/m in SI units.

- Surface tension is a binary property of the liquid and gas or two liquids which are in contact with each other and defines the interface. It decreases slightly with increasing temperature. The surface tension of water in contact with air at 20°C is about 0.073 N/m.

- It is due to surface tension that a curved liquid interface in equilibrium results in a greater pressure at the concave side of the surface than that at its convex side.

Capillarity

- The interplay of the forces of cohesion and adhesion explains the phenomenon of capillarity. When a liquid is in contact with a solid, if the forces of adhesion between the molecules of the liquid and the solid are greater than the forces of cohesion among the liquid molecules themselves, the liquid molecules crowd towards the solid surface. The area of contact between the liquid and solid increases and the liquid thus wets the solid surface.

- The reverse phenomenon takes place when the force of cohesion is greater than the force of adhesion. These adhesion and cohesion properties result in the phenomenon of capillarity by which a liquid either rises or falls in a tube dipped into the liquid depending upon whether the force of adhesion is more than that of cohesion or not (Fig.2.4).

- The angle θ as shown in Fig. 2.4, is the area wetting contact angle made by the interface with the solid surface.

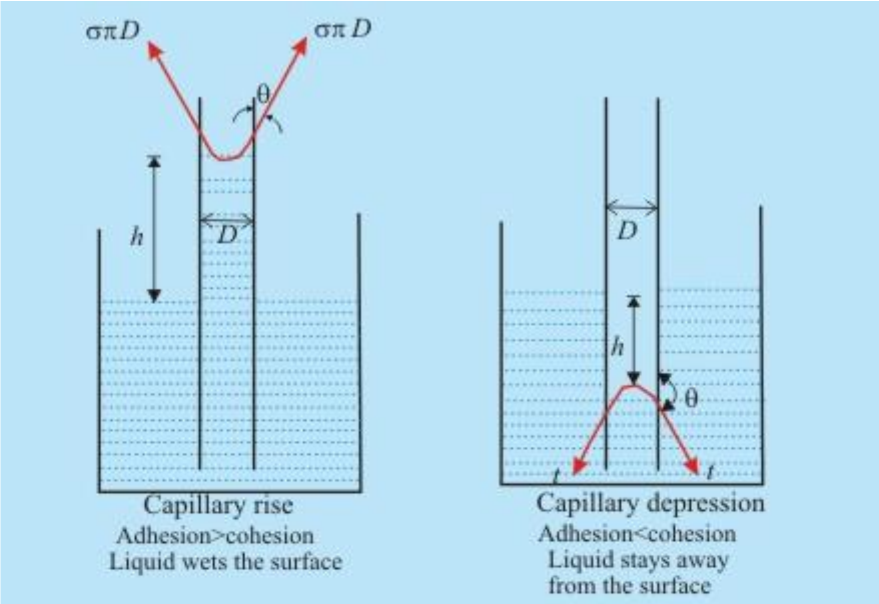
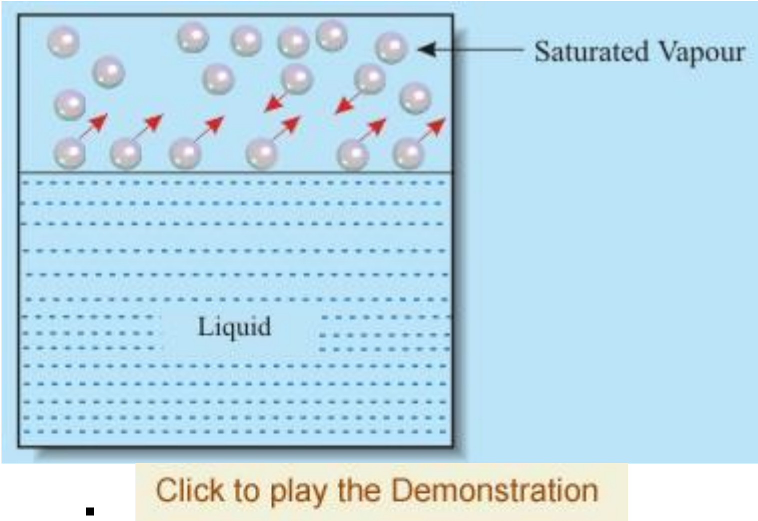


Fig 2.4 Phenomenon of Capillarity

- For pure water in contact with air in a clean glass tube, the capillary rise takes place with $\theta = 0$. Mercury causes capillary depression with an angle of contact of about 130^0 in a clean glass in contact with air. Since *h* varies inversely with D as found from Eq. ($h = \frac{4\sigma\cos\theta}{\rho g D}$), an appreciable capillary rise or depression is observed in tubes of small diameter only.
- **Vapour pressure**
- All liquids have a tendency to evaporate when exposed to a gaseous atmosphere. The rate of evaporation depends upon the molecular energy of the liquid which in turn depends upon the type of liquid and its temperature. The vapour molecules exert a partial pressure in the space above the liquid, known as vapour pressure. If the space above the liquid is confined (Fig. 2.5) and the liquid is maintained at constant temperature, after sufficient time, the confined space above the liquid will contain vapour molecules to the extent that some of them will be forced to enter the liquid. Eventually an equilibrium condition will evolve when the rate at which the number of vapour molecules striking back the liquid surface and condensing is just equal to the rate at which they leave from the surface. The space above the liquid then becomes saturated with vapour. The vapour pressure of a given liquid is a function of temperature only and is equal to the saturation pressure for boiling corresponding to that temperature. Hence, the vapour pressure increases with the increase in temperature. Therefore the phenomenon of boiling of a liquid is closely related to the vapour pressure. In fact, when the vapour pressure of a liquid becomes equal to the total pressure impressed on its surface, the liquid starts boiling. This concludes that boiling can be achieved either by raising the temperature of the liquid, so that its vapour pressure is elevated to the ambient pressure, or by lowering the pressure of the ambience (surrounding gas) to the liquid's vapour pressure at the existing temperature.



- **Figure 2.5 To and fro movement of liquid molecules from an interface in a confined space as a closed surrounding**
- Exercise Problems - Chapter 1
- 1. A thin film of liquid flows down an inclined channel. The velocity distribution in the flow is given by

$$u = \frac{1}{2\mu}\left(h^2 - y^2\right)\rho g \sin\alpha$$

- where, h = depth of flow, α = angle of inclination of the channel to the horizontal, u = velocity at a depth h below the free surface, ρ = density of liquid, μ = dynamic viscosity of the fluid. Calculate the shear stress: (a) at the bottom of the channel (b) at mid-depth and (c) at the free surface. The coordinate y is measured from the free surface along its normal

[(a) $\rho g h \,\mathrm{Sin}\,\alpha$, (b) $\frac{\rho g h}{2}\,\mathrm{Sin}\,\alpha$, (c) 0]

- 2. Two discs of 250 mm diameter are placed 1.5 mm apart and the gap is filled with an oil. A power of 500 W is required to rotate the upper disc at 500 rpm while keeping the lower one stationary. Determine the viscosity of the oil.

[0. 71 kg/ms]

- 3. Eight kilometers below the surface of the ocean the pressure is 100 MPa. Determine the specific weight of sea water at this depth if the specific weight at the surface is 10 kN/m^3 and the average bulk modulus of elasticity of water is 2.30 GPa. Neglect the variation of g.

[10. 44 kN/m^3]

- 4. The space between two large flat and parallel walls 20 mm apart is filled with a liquid of absolute viscosity 0.8 Pas. Within this space a thin flat plate 200 mm × 200mm is towed at a velocity of 200 mm/s at a distance of 5 mm from one wall. The plate and its movement are parallel to the walls. Assuming a linear velocity distribution between the plate and the walls, determine the force exerted by the liquid on the plate.

[1. 71 N]

- 5. What is the approximate capillary rise of water in contact with air (surface tension 0.073 N/m) in a clean glass tube of 5mm in diameter?

[5.95]

-

Recap

In this chapter you have learnt the following

- A fluid is a substance that deforms continuously when subjected to even an infinitesimal shear stress. Solids can resist tangential stress at static conditions undergoing a definite deformation while a fluid can do it only at dynamic conditions undergoing a continuous deformation as long as the shear stress is applied.

- The concept of continuum assumes a continuous distribution of mass within the matter or system with no empty space. In the continuum approach, properties of a system can be expressed as continuous functions of space and time. A dimensionless parameter known as **Knudsen number** $K_n = \lambda / L$ where λ is the mean free path and L is the characteristic length, aptly describes the degree of departure from continuum. The concept of continuum usually holds good when $K_n < 0.01$.

- Viscosity is a property of a fluid by virtue of which it offers resistance to flow. The shear stress at a point in a moving fluid is directly proportional to the rate of shear strain. For a one dimensional flow, $\tau = \mu \frac{du}{dy}$. The constant of proportionality μ is known as coefficient of viscosity or simply the viscosity. The relationship is known as the Newton's law of viscosity and the fluids which obey this law are known as **Newtonian fluids.**

- The relationship between the shear stress and the rate of shear strain is known as the constitutive equation. The fluids whose constitutive equations are not linear through origin (do not obey the Newton's law of viscosity) are known as **non-Newtonian fluids**. For a Newtonian fluid, viscosity is a function of temperature only. With an increase in temperature, the viscosity of a liquid decreases, while that of a gas increases. For non-Newtonian fluid, the viscosity depends not only on temperature but also on the deformation rate of the fluid. Kinematic viscosity *v* is defined as μ / ρ.

- Compressibility of a substance is the measure of its change in volume or density under the action of external forces. It is usually characterized by the

bulk modulus of elasticity

$$E = \lim_{\Delta\forall \to 0} \frac{-\Delta p}{\Delta\forall / \forall}$$

- A flow is said to be incompressible when the change in its density due to the change in pressure brought about by the fluid motion is negligibly small. When the flow velocity is equal to or less than 0.33 times of the local acoustic speed, the relative change in density of the fluid, due to flow, becomes equal to or less than 5 per cent respectively, and hence the flow is considered to be incompressible

- The force of attraction between the molecules of a fluid is known as cohesion, while that between the molecules of a fluid and of a solid is known as adhesion. The interplay of these two intermolecular forces explains the phenomena of surface tension and capillary rise or depression. A free surface of the liquid is always under stretched condition implying the existence of a tensile force on the surface. The magnitude of this force per unit length of an imaginary line drawn along the liquid surface is known as the surface tension coefficient or simply the **surface tension**.

Forces on Fluid Elements

Fluid Elements - Definition:

Fluid element can be defined as an infinitesimal region of the fluid continuum in isolation from its surroundings.

Two types of forces exist on fluid elements

- **Body Force**: distributed over the entire mass or volume of the element. It is usually expressed per unit mass of the element or medium upon which the forces act. Example: Gravitational Force, Electromagnetic force fields etc.
- **Surface Force**: Forces exerted on the fluid element by its surroundings through direct contact at the surface.
  Surface force has two components:
  - Normal Force: along the normal to the area
  - Shear Force: along the plane of the area.

The ratios of these forces and the elemental area in the limit of the area tending to zero are called the normal and shear stresses respectively.

The shear force is zero for any fluid element at rest and hence the only surface force on a fluid element is the normal component.

Normal Stress in a Stationary Fluid

Consider a stationary fluid element of tetrahedral shape with three of its faces coinciding with the coordinate planes x, y and z.

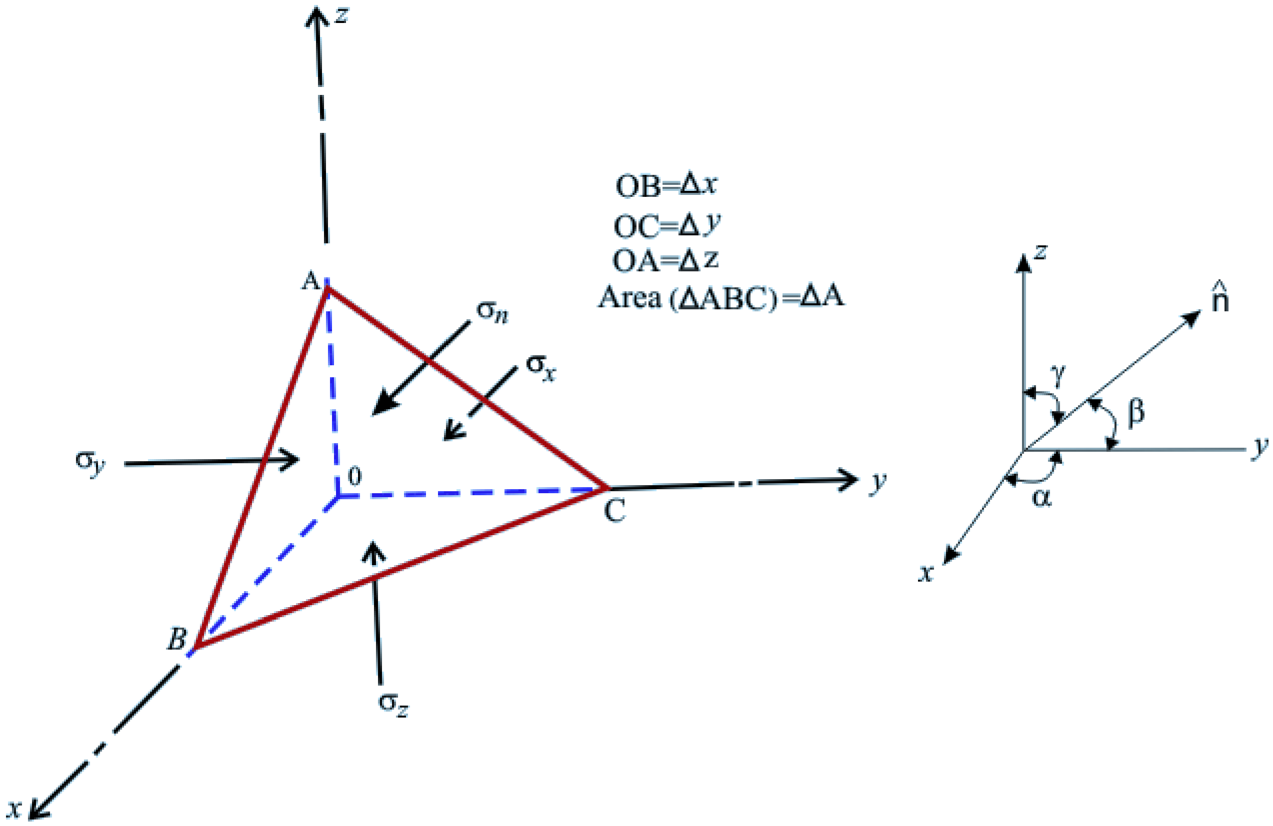


Fig 3.1 State of Stress in a Fluid Element at Rest

Since a fluid element at rest can develop neither shear stress nor tensile stress, the normal stresses acting on different faces are compressive in nature.

Suppose, ΣF_x, ΣF_y and ΣF_z are the net forces acting on the fluid element in positive x,y and z directions respectively. The direction cosines of the normal to the inclined plane of an area ΔA are cos α, cos β and cos γ .Considering gravity as the only source of external body force, acting in the -ve z direction, the equations of static equlibrium for the tetrahedronal fluid element can be written as

$$\sum F_x = \sigma_x \left(\frac{\Delta y \Delta z}{2}\right) - \sigma_n \Delta A \cos\alpha = 0 \qquad (3.1)$$

$$\sum F_y = \sigma_y \left(\frac{\Delta x \Delta z}{2}\right) - \sigma_n \Delta A \cos\beta = 0 \qquad (3.2)$$

$$\sum F_z = \sigma_z \left(\frac{\Delta y \Delta x}{2}\right) - \sigma_n \Delta A \cos\gamma - \frac{\rho g}{6}(\Delta x \Delta y \Delta z) = 0 \qquad (3.3)$$

where $\left(\frac{\Delta x\ \Delta y\ \Delta z}{6}\right)$ = Volume of tetrahedral fluid element

Pascal's Law of Hydrostatics

Pascal's Law

The normal stresses at any point in a fluid element at rest are directed towards the point from all directions and they are of the equal magnitude.

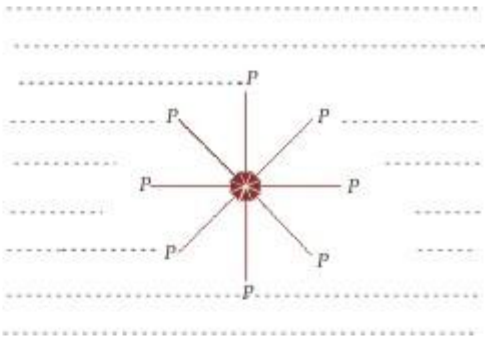


Fig 3.2 State of normal stress at a point in a fluid body at rest

Derivation:
The inclined plane area is related to the fluid elements (refer to Fig 3.1) as follows

$$\Delta A \cos\alpha = \left(\frac{\Delta y \Delta z}{2}\right) \tag{3.4}$$

$$\Delta A \cos\beta = \left(\frac{\Delta x \Delta z}{2}\right) \tag{3.5}$$

$$\Delta A \cos\gamma = \left(\frac{\Delta x \Delta y}{2}\right) \tag{3.6}$$

Substituting above values in equation 3.1- 3.3 we get

$$\sigma_x = \sigma_y = \sigma_z = \sigma_n \tag{3.7}$$

Conclusion:

The state of normal stress at any point in a fluid element at rest is same and directed towards the point from all directions. These stresses are denoted by a scalar quantity p defined as the hydrostatic or thermodynamic pressure.
Using "+" sign for the tensile stress the above equation can be written in terms of pressure as

$$\sigma_x = \sigma_y = \sigma_z = -\mathrm{p} \tag{3.8}$$

Fundamental Equation of Fluid Statics

The fundamental equation of fluid statics describes the spatial variation of hydrostatic pressure p in the continuous mass of a fluid.

Derivation:

Consider a fluid element at rest of given mass with volume *V* and bounded by the surface *S*.

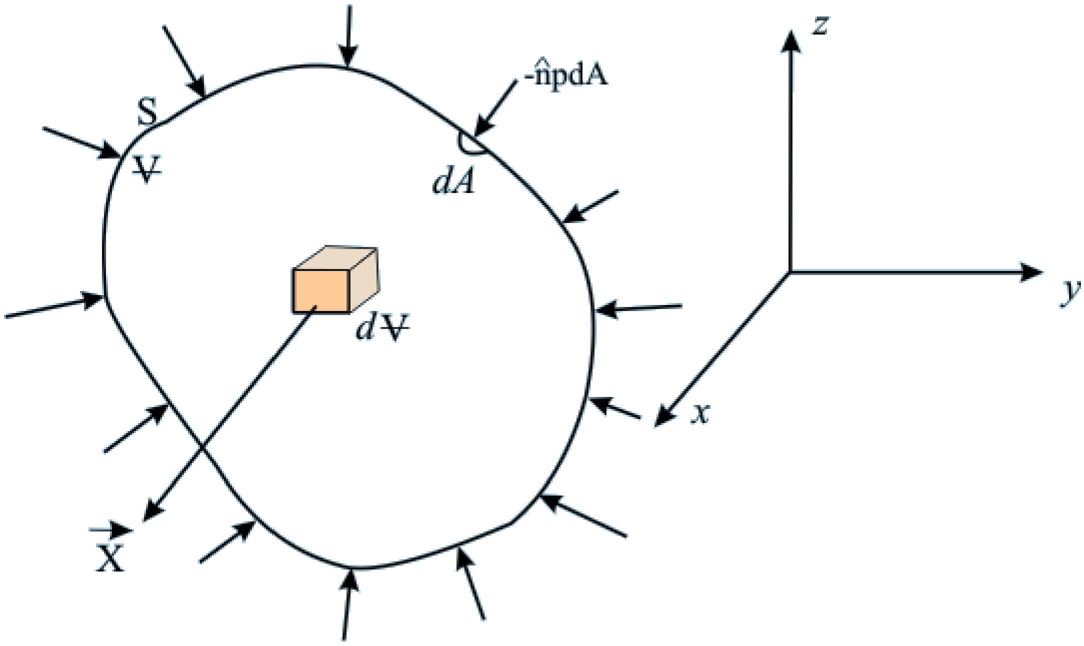


Fig 3.3 External Forces on a Fluid Element at Rest

The fluid element stays at equilibrium under the action of the following two forces

- **The Resultant Body Force**

$\vec{F}_B = \iiint_{\forall} \vec{X}\,\rho\, d\forall$	(3.9)	$d\forall$: element of volume $\rho\, d\forall$: mass of the element $\vec{X}$: body Force per unit mass acting on the elementary volume

- **The Resultant Surface Force**

$\vec{F}_s = -\iint_{s} \hat{n} p\, dA$	(3.10)	dA : area of an element of surface $\hat{n}$: the unit vector normal to the elemental surface,taken positive when directed outwards

Using Gauss divergence theorem, Eq (3.10) can be written as

$$\vec{F}_s = -\iint_s \hat{n} \mathrm{p} dA = -\iint_s \mathrm{p}(\hat{n}\, dA)$$

$$\vec{F}_s = -\iiint_{\forall} (\nabla \mathrm{p})\, d\forall \qquad (3.11)$$

[Click here to see the derivation]

For the fluid element to be in equilibrium , we have

$$\vec{F}_B + \vec{F}_s = \iiint_{\forall} \left(\vec{X}\rho - \nabla \mathrm{p}\right) d\forall = 0 \qquad (3.12)$$

The equation is valid for any volume of the fluid element, no matter how small, thus we get

$$\vec{X}\rho - \nabla p = 0$$

$$\nabla p = \vec{X}\rho \qquad (3.13)$$

This is the fundamental equation of fluid statics.

Fundamental Fluid Static Equations in Scalar Form

Considering gravity as the only external body force acting on the fluid element, Eq. (3.13) can be expressed in its scalar components with respect to a cartesian coordinate system (see Fig. 3.3) as

$\frac{\partial p}{\partial x} = 0$ (in x direction) (3.13a)	X_z: the external body force per unit mass in the positive direction of z (vertically upward), equals to the negative value of g (the acceleration due to gravity).
$\frac{\partial p}{\partial y} = 0$ (in y direction) (3.13b)	
$\frac{\partial p}{\partial z} = X_z \rho = -\rho g$ (in z direction) (3.13c)	

From Eqs (3.13a)-(3.13c), it can be concluded that the pressure p is a function of z only.

Thus, Eq. (3.13c) can be re-written as,

$$\boxed{\frac{\partial p}{\partial z} = \rho g \qquad (3.14)}$$

Constant and Variable Density Solution

Constant Density Solution

The explicit functional relationship of hydrostatic pressure *p* with *z* can be obtained by integrating the Eq. (3.14).

For an incompressible fluid, the density ρ is constant throughout. Hence the Eq. (3.14) can be integrated and expressed

as

$$p = -\rho g z + C \qquad (3.15)$$

where C is the integration constant.

If we consider an expanse of fluid with a free surface, where the pressure is defined as $p = p_0$,which is equal to atmospheric pressure.

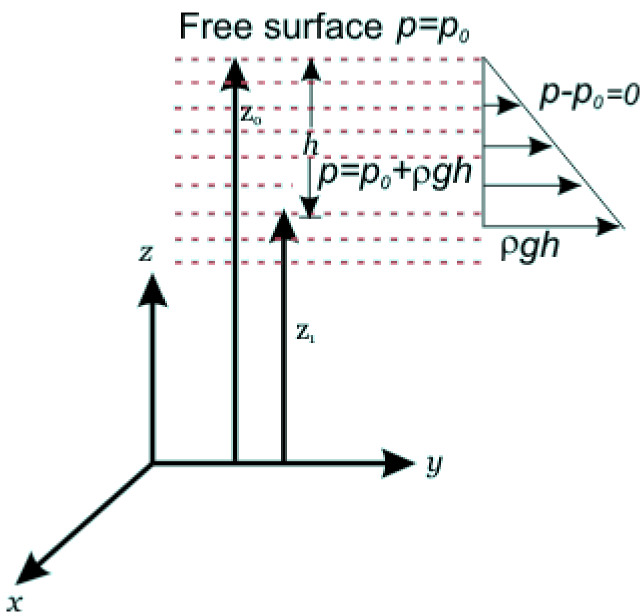


Fig 3.4 Pressure Variation in an Incompressible Fluid at rest with a Free Surface

Eq. (3.15) can be written as,

$$p - p_0 = \rho g (z_0 - z_1) = \rho g h \qquad (3.16a)$$

Therefore, Eq. (3.16a) gives the expression of hydrostatic pressure p at a point whose vertical depression from the free surface is h.

Similarly,

$$p_1 - p_2 = \rho g (z_1 - z_2) = \rho g h \qquad (3.16b)$$

Thus, the difference in pressure between two points in an incompressible fluid at rest can be expressed in terms of the vertical distance between the points. This result is known as **Torricelli's principle**, which is the basis for differential pressure measuring` devices. The pressure p_0 at free surface is the local atmospheric pressure.

Therefore, it can be stated from Eq. (3.16a), that the pressure at any point in an expanse of a fluid at rest, with a free surface exceeds that of the local atmosphere by an amount ρgh, where h is the vertical depth of the point from the free surface.

Variable Density Solution: As a more generalised case, for compressible fluids at rest, the pressure variation at rest depends on how the fluid density changes with height *z* and pressure p. For example this can be done for special cases of "isothermal and non-isothermal fluids"

Units and scales of Pressure Measurement

Pascal (N/m^2) is the unit of pressure .

Pressure is usually expressed with reference to either absolute zero pressure (a complete vacuum)or local atmospheric pressure.

- The absolute pressure: It is the difference between the value of the pressure and the absolute zero pressure.

  $$p_{abs} = p - 0 = p$$

- Gauge pressure: It is the diference between the value of the pressure and the local atmospheric pressure(p_{atm})

  $$p_{gauge} = p - p_{atm}$$

- Vacuum Pressure: If $p<p_{atm}$ then the gauge pressure (p_{gauge}) becomes negative and is called the vacuum pressure.But one should always remember that hydrostatic pressure is always compressive in nature

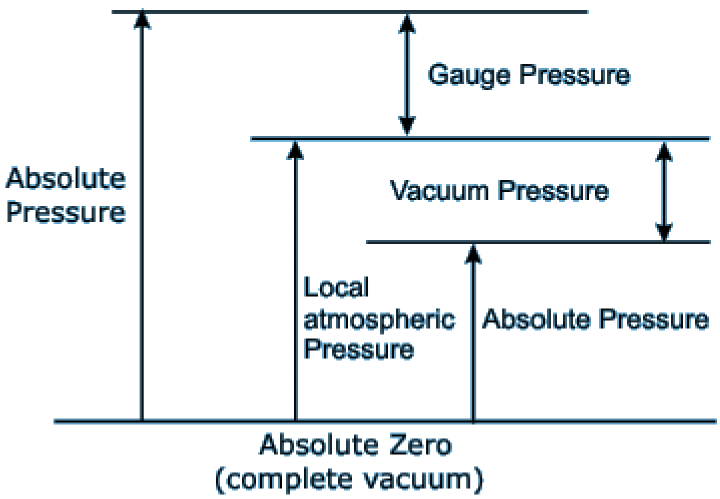


Fig 4.1 The Scale of Pressure

At sea-level, the international standard atmosphere has been chosen as P_{atm} = 101.32 kN/m^2

Piezometer Tube

The direct proportional relation between gauge pressure and the height h for a fluid of constant density enables the pressure to be simply visualized in terms of the vertical height, $h = p/\rho g$.

The height h is termed as pressure head corresponding to pressure p. For a liquid without a free surface in a closed pipe, the pressure head $p/\rho g$ at a point corresponds to the vertical height above the point to which a free surface would rise, if a small tube of sufficient length and open to atmosphere is connected to the pipe

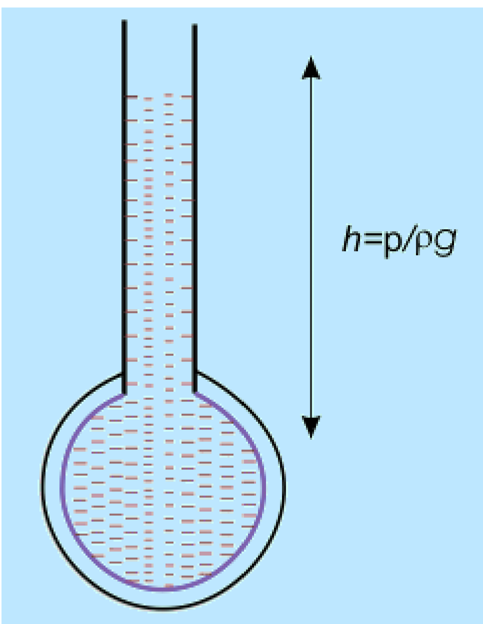
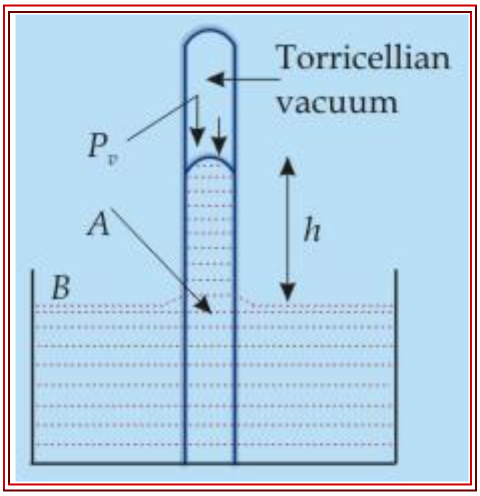


Fig 4.2 A piezometer Tube

Such a tube is called a piezometer tube, and the height h is the measure of the gauge pressure of the fluid in the pipe. If such a piezometer tube of sufficient length were closed at the top and the space above the liquid surface were a perfect vacuum, the height of the column would then correspond to the absolute pressure of the liquid at the base. This principle is used in the well known mercury barometer to determine the local atmospheric pressure.

The Barometer

Barometer is used to determine the local atmospheric pressure. Mercury is employed in the barometer because its density is sufficiently high for a relative short column to be obtained. and also because it has very small vapour pressure at normal temperature. High density scales down the pressure head(h) to repesent same magnitude of pressure in a tube of smaller height.



Click to play the Demonstration

Fig 4.3 A Simple Barometer

Even if the air is completely absent, a perfect vacuum at the top of the tube is never possible. The space would be occupied by the mercury vapour and the pressure would equal to the vapour pressure of mercury at its existing temperature. This almost vacuum condition above the mercury in the barometer is known as Torricellian vacuum.

The pressure at A equal to that at B (Fig. 4.3) which is the atmospheric pressure p_{atm} since A and B lie on the same horizontal plane. Therefore, we can write

$$p_B = p_{atm} = p_v + \rho g h \qquad (4.1)$$

The vapour pressure of mercury p_v, can normally be neglected in comparison to p_{atm}.
At 20^0C,P_v is only 0.16 p_{atm}, where p_{atm} =1.0132 X10^5 Pa at sea level. Then we get from Eq. (4.1)

$$h = p_{atm} / \rho g = \frac{1.0132 \times 10^5 \ N/m^2}{(13560 \, kg/m^3)(9.81 N/Kg)} = 0.752 m \ of \ Hg$$

For accuracy, small corrections are necessary to allow for the variation of ρ with temperature, the thermal expansion of the scale (usually made of brass). and surface tension effects. If water was used instead of mercury, the corresponding height of the column would be about 10.4 m provided that a perfect vacuum could be achieved above the water. However, the vapour pressure of water at ordinary temperature is appreciable and so the actual height at, say, 15°C would be about 180 mm less than this value. Moreover. with a tube smaller in diameter than about 15 mm, surface tension effects become significant.

Manometers for measuring Gauge and Vacuum Pressure

Manometers are devices in which columns of a suitable liquid are used to measure the difference in pressure between two points or between a certain point and the atmosphere.

Manometer is needed for measuring large gauge pressures. It is basically the modified form of the piezometric tube. A common type manometer is like a transparent "U-tube" as shown in Fig. 4.4.

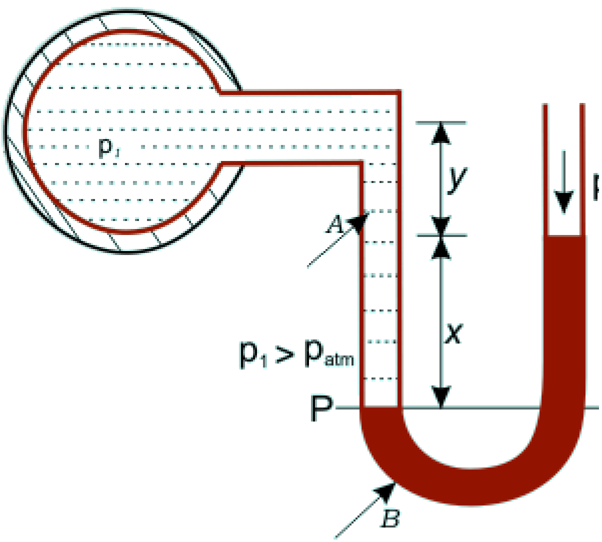


Fig 4.4 A simple manometer to measure gauge pressure

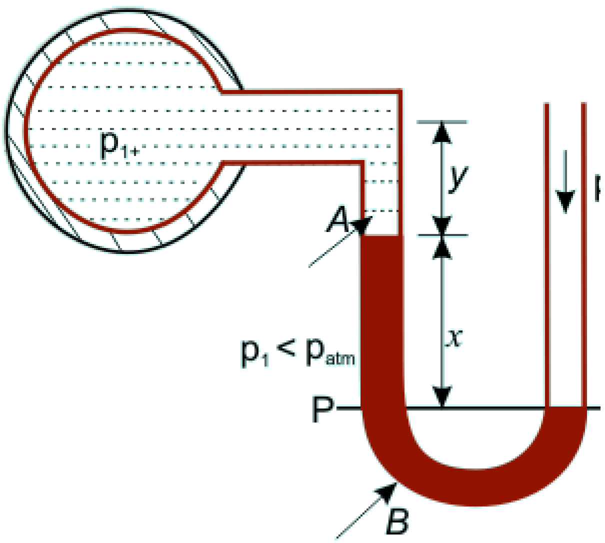


Fig 4.5 A simple manometer to measure vacuum pressure

One of the ends is connected to a pipe or a container having a fluid (A) whose pressure is to be measured while the other end is open to atmosphere. The lower part of the U-tube contains a liquid immiscible with the fluid A and is of greater density than that of A. This fluid is called the manometric fluid.

The pressures at two points *P* and *Q* (Fig. 4.4) in a horizontal plane within the continuous expanse of same fluid (the liquid B in this case) must be equal. Then equating the pressures at *P* and *Q* in terms of the heights of the fluids above those points, with the aid of the fundamental equation of hydrostatics (Eq 3.16), we have

$$p_1 + \rho_A g(y + x) = p_{atm} + \rho_B g x$$

Hence,

$$p_1 - p_{atm} = (\rho_B - \rho_A)gx - \rho_A gy$$

where p_1 is the absolute pressure of the fluid A in the pipe or container at its centre line, and p_{atm} is the local atmospheric pressure. When the pressure of the fluid in the container is lower than the atmospheric pressure, the liquid levels in the manometer would be adjusted as shown in Fig. 4.5. Hence it becomes,

$$p_1 + \rho_A gy + \rho_B gx = p_{atm}$$

$$p_{atm} - p_1 = (\rho_A y + \rho_B x)g \qquad (4.2)$$

Manometers to measure Pressure Difference

A manometer is also frequently used to measure the pressure difference, in course of flow, across a restriction in a horizontal pipe.

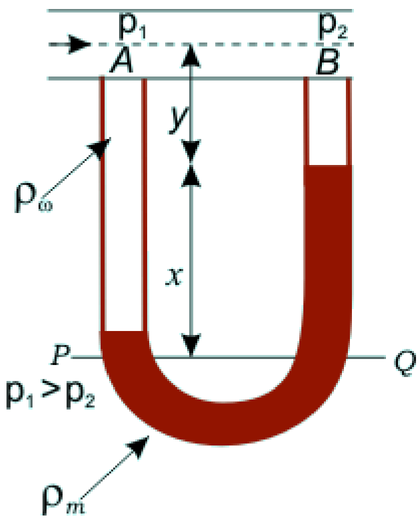


Fig 4.6 Manometer measuring pressure difference

The axis of each connecting tube at A and B should be perpendicular to the direction of flow and also for the edges of the connections to be smooth. Applying the principle of hydrostatics at P and Q we have,

$$p_1 + (y + x)\rho_w g = p_2 + y\rho_w g + \rho_m gx$$

$$p_1 - p_2 = (\rho_m - \rho_w)gx \qquad (4.3)$$

where, ρ_m is the density of manometric fluid and ρ_w is the density of the working fluid flowing through the pipe.

We can express the difference of pressure in terms of the difference of heads (height of the working fluid at equilibrium).

$$h_1 - h_2 = \frac{p_1 - p_2}{\rho_w g} = \left(\frac{\rho_m}{\rho_w} - 1\right)x \qquad (4.4)$$

Inclined Tube Manometer

- For accurate measurement of small pressure differences by an ordinary u-tube manometer, it is essential that the ratio ρ_m/ρ_w should be close to unity. This is not possible if the working fluid is a gas; also having a manometric liquid of density very close to that of the working liquid and giving at the same time a well defined meniscus at the interface is not always possible. For this purpose, an inclined tube manometer is used.

- If the transparent tube of a manometer, instead of being vertical, is set at an angle θ to the horizontal (Fig. 4.7), then a pressure difference corresponding to a vertical difference of levels x gives a movement of the meniscus s = x/sinθ along the slope.

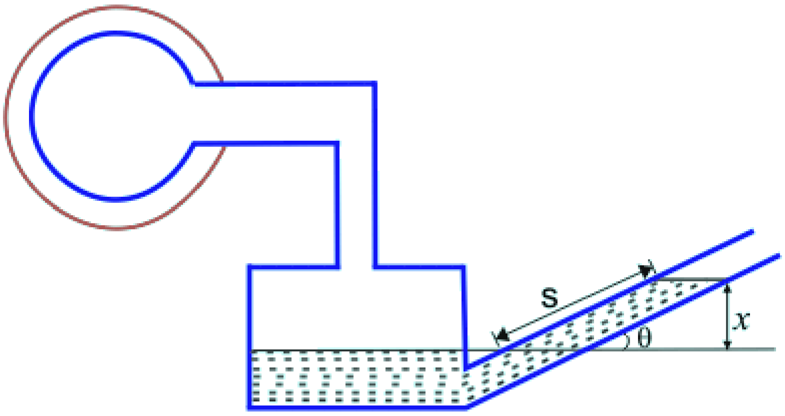
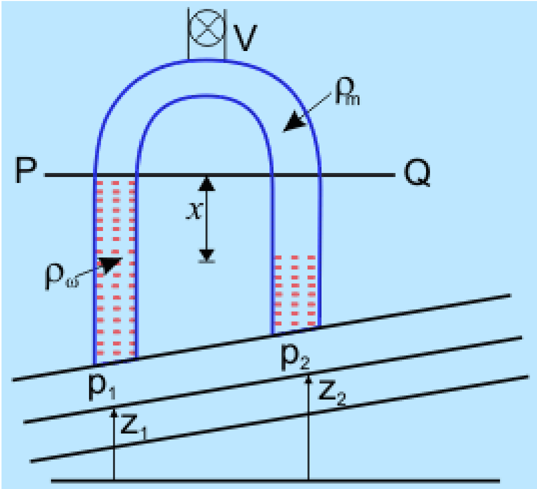


Fig 4.7 An Inclined Tube Manometer

- If θ is small, a considerable mangnification of the movement of the meniscus may be achieved.

- Angles less than 5^0 are not usually satisfactory, because it becomes difficult to determine the exact position of the meniscus.

- One limb is usually made very much greater in cross-section than the other. When a pressure difference is applied across the manometer, the movement of the liquid surface in the wider limb is practically negligible compared to that occurring in the narrower limb. If the level of the surface in the wider limb is assumed constant, the displacement of the meniscus in the narrower limb needs only to be measured, and therefore only this limb is required to be transparent.
- **Inverted Tube Manometer**
- For the measurement of small pressure differences in liquids, an inverted U-tube manometer is used.



- Fig 4.8 An Inverted Tube Manometer

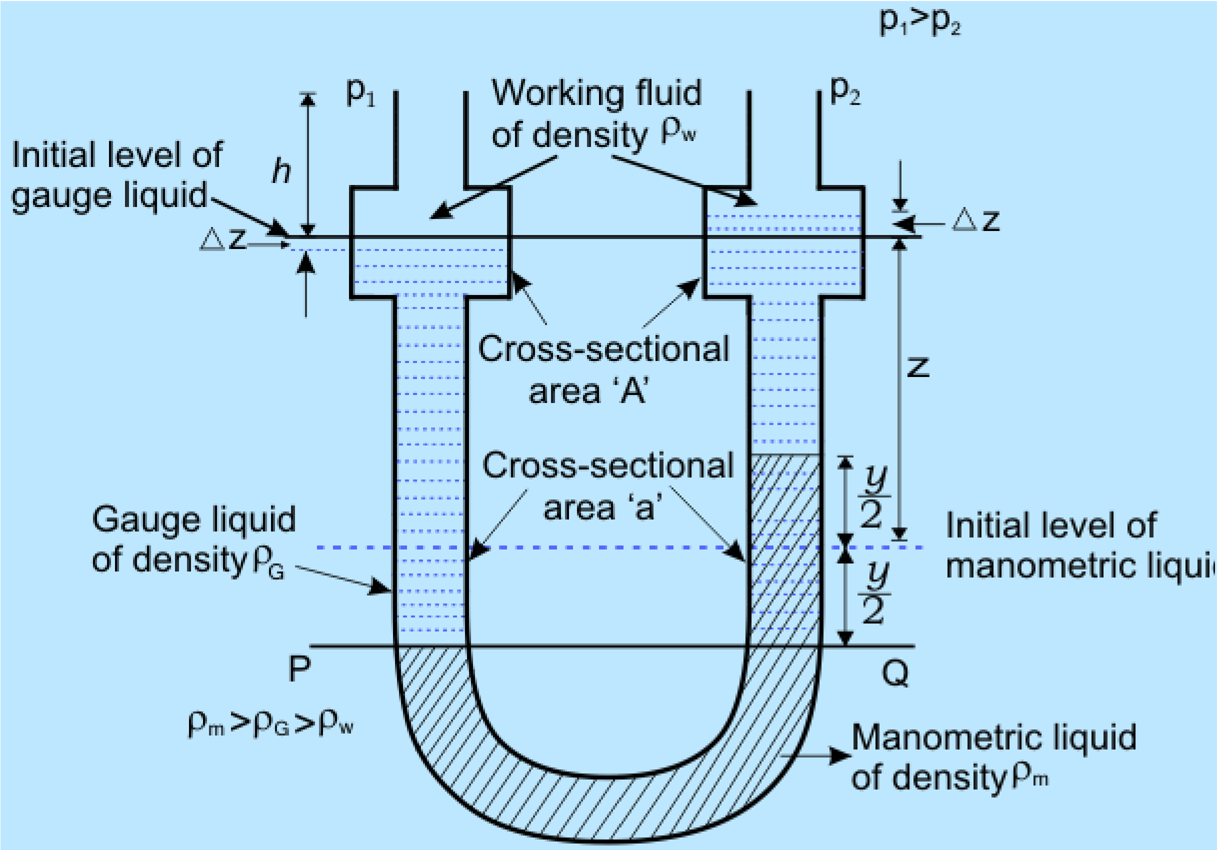
- Here $\rho_m < \rho_w$ and the line PQ is taken at the level of the higher meniscus to equate the pressures at P and Q from the principle of hydrostatics. It may be written that

$$p_1^* - p_2^* = (\rho_w - \rho_m)gx$$

- where p^* represents the **piezometric pressure**, $p + \rho gz$ (z being the vertical height of the point concerned from any reference datum). In case of a horizontal pipe ($z_1 = z_2$) the difference in piezometric pressure becomes equal to the difference in the static pressure. If $(\rho_w - \rho_m)$ is sufficiently small, a large value of x may be obtained for a small value of $p_1^* - p_2^*$. Air is used as the manometric fluid. Therefore, ρ_m is negligible compared with ρ_w and hence,

$$p_1^* - p_2^* \approx \rho_w gx \qquad (4.5)$$

- Air may be pumped through a valve V at the top of the manometer until the liquid menisci are at a suitable level.
- **Micromanometer**
- When an additional gauge liquid is used in a U-tube manometer, a large difference in meniscus levels may be obtained for a very small pressure difference.



- Fig 4.9 A Micromanometer

- The equation of hydrostatic equilibrium at PQ can be written as

$$p_1 + \rho_w g(h+\Delta z) + \rho_G g\left(z - \Delta z + \frac{y}{2}\right) = p_2 + \rho_w g(h-\Delta z) + \rho_G g\left(z + \Delta z - \frac{y}{2}\right) + \rho_m g y$$

- where ρ_w, ρ_G and ρ_m are the densities of working fluid, gauge liquid and manometric liquid respectively.
  From continuity of gauge liquid,

$$A\Delta z = a\frac{y}{2} \tag{4.6}$$

-

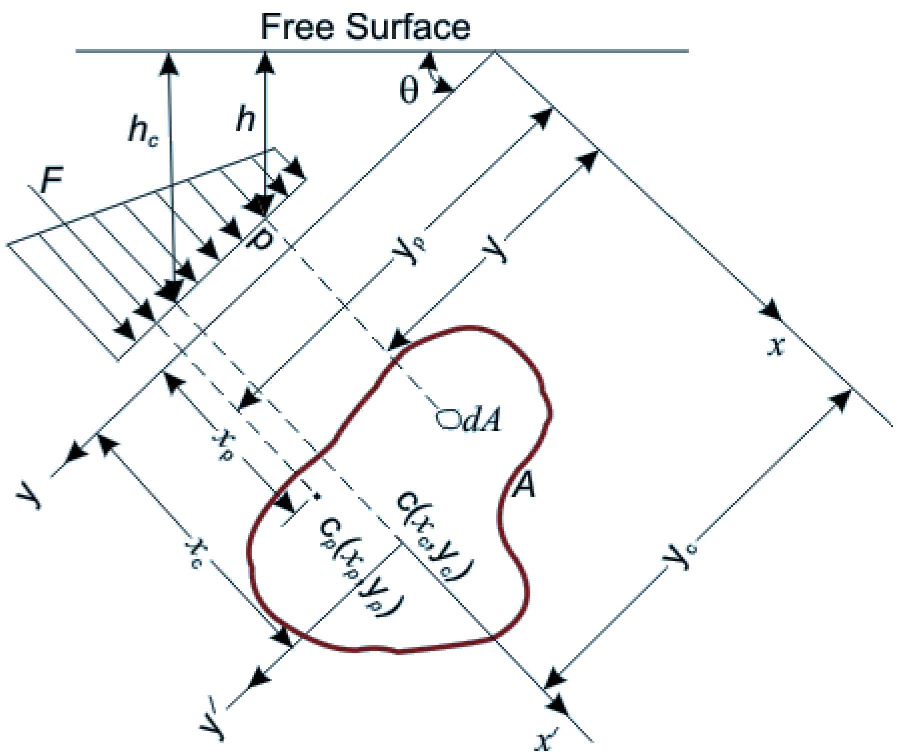
$$p_1 - p_2 = gy\left[\rho_m - \rho_G\left(1 - \frac{a}{A}\right) - \rho_w\frac{a}{A}\right] \tag{4.7}$$

- If a is very small compared to A

$$p_1 - p_2 \approx (\rho_m - \rho_G)gy \tag{4.8}$$

- With a suitable choice for the manometric and gauge liquids so that their densities are close $(\rho_m \approx \rho_G)$ a reasonable value of y may be achieved for a small pressure difference.
- **Hydrostatic Thrusts on Submerged Plane Surface**

- Due to the existence of hydrostatic pressure in a fluid mass, a normal force is exerted on any part of a solid surface which is in contact with a fluid. The individual forces distributed over an area give rise to a resultant force.
- **Plane Surfaces**
- Consider a plane surface of arbitrary shape wholly submerged in a liquid so that the plane of the surface makes an angle θ with the free surface of the liquid. We will assume the case where the surface shown in the figure below is subjected to hydrostatic pressure on one side and atmospheric pressure on the other side.



- Fig 5.1 Hydrostatic Thrust on Submerged Inclined Plane Surface
- Let p denotes the gauge pressure on an elemental area dA. The resultant force F on the area A is therefore

$$F = \iint_A p\, dA \tag{5.1}$$

- According to Eq (3.16a) Eq (5.1) reduces to

$$F = \rho g \iint h dA = \rho g \sin\theta \iint y dA \tag{5.2}$$

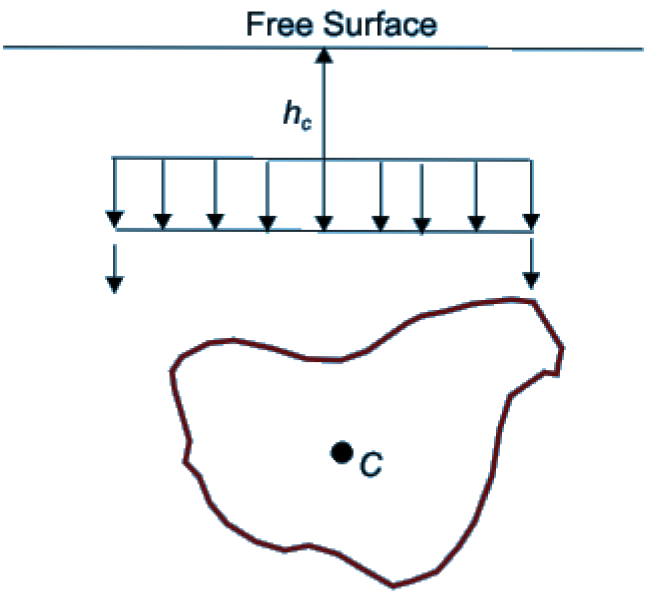
- Where **h is the vertical depth of the elemental area dA from the free surface** and the distance y is measured from the x-axis, the line of intersection between the extension of the inclined plane and the free surface (Fig. 5.1). The ordinate of the centre of area of the plane surface A is defined as

$$y_c = \frac{1}{A}\iint y dA \tag{5.3}$$

- Hence from Eqs (5.2) and (5.3), we get

$$F = \rho g y_c \sin\theta A = \rho g h_c A \tag{5.4}$$

- where $h_c (= y_c \sin\theta)$ is the vertical depth (from free surface) of centre c of area .
- Equation (5.4) implies that the hydrostatic thrust on an inclined plane is equal to the pressure at its centroid times the total area of the surface, i.e., the force that would have been experienced by the surface if placed horizontally at a depth h_c from the free surface (Fig. 5.2).



- Fig 5.2 Hydrostatic Thrust on Submerged Horizontal Plane Surface

- The point of action of the resultant force on the plane surface is called the centre of pressure c_p . Let x_p and y_p be the distances of the centre of pressure from the y and x axes respectively. Equating the moment of the resultant force about the x axis to the summation of the moments of the component forces, we have

$$y_p F = \int y dF = \rho g \sin\theta \iint y^2 dA \tag{5.5}$$

- Solving for y_p from Eq. (5.5) and replacing F from Eq. (5.2), we can write

$$y_p = \frac{\iint_A y^2 dA}{\iint_A y dA} \tag{5.6}$$

- In the same manner, the x coordinate of the centre of pressure can be obtained by taking moment about the y-axis. Therefore,

$$x_p F = \int x dF = \rho g \sin\theta \iint xy dA$$

- From which,

$$x_p = \frac{\iint\limits_A xy dA}{\iint\limits_A y\, dA} \qquad (5.7)$$

- The two double integrals in the numerators of Eqs (5.6) and (5.7) are the moment of inertia about the x-axis $I_{xx}$and the product of inertia I_{xy} about x and y axis of the plane area respectively. By applying the theorem of parallel axis

$$I_{xx} = \iint y^2\, dA = I_{x'x'} + Ay_c^2 \qquad (5.8)$$

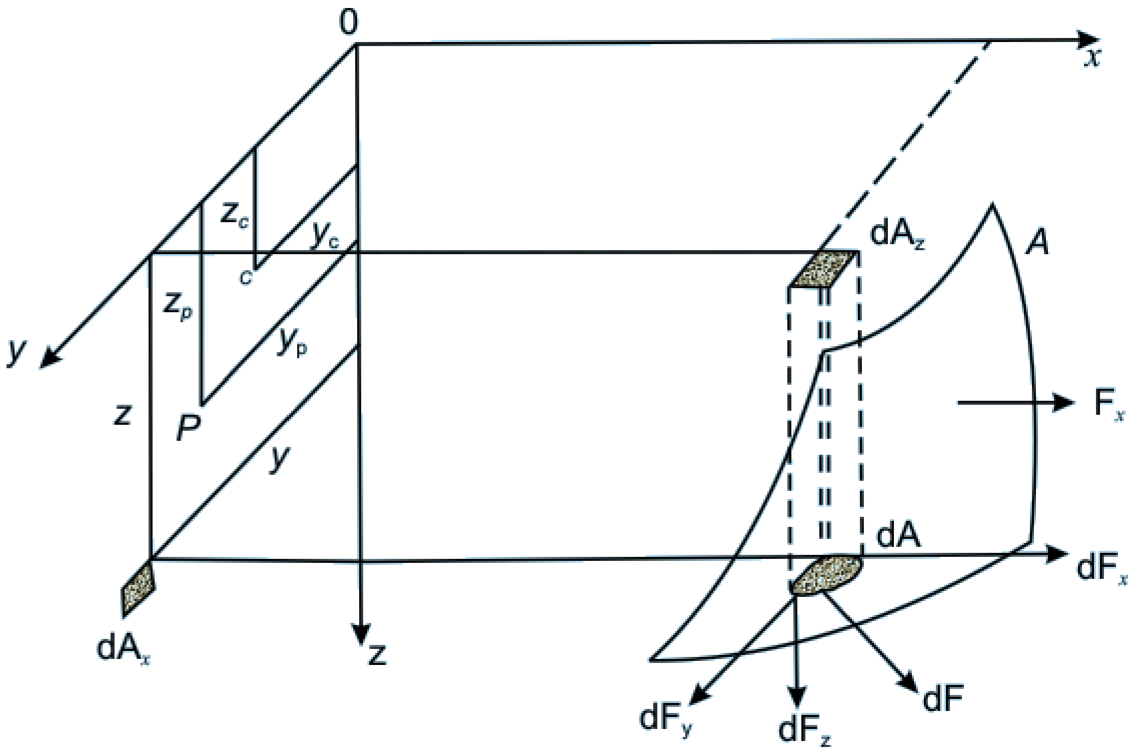
$$I_{xy} = \iint xy dA = I_{x'y'} + Ax_c y_c \qquad (5.9)$$

- where, $I_{x'x'}$ and $I_{x'y'}$ are the moment of inertia and the product of inertia of the surface about the centroidal axes $(x'-y'), x_c$, and y_c are the coordinates of the center c of the area with respect to x-y axes.
- With the help of Eqs (5.8), (5.9) and (5.3), Eqs (5.6) and (5.7) can be written as

$$y_p = \frac{I_{x'x'}}{Ay_c} + y_c \qquad (5.10a)$$

$$x_p = \frac{I_{x'y'}}{Ay_c} + x_c \qquad (5.10b)$$

- The first term on the right hand side of the Eq. (5.10a) is always positive. Hence, the centre of pressure is always at a higher depth from the free surface than that at which the centre of area lies. This is obvious because of the typical variation of hydrostatic pressure with the depth from the free surface. When the plane area is symmetrical about the y' axis, $I_{x'y'} = 0$, and $x_p = x_c$.
- **Hydrostatic Thrusts on Submerged Curved Surfaces**
- On a curved surface, the direction of the normal changes from point to point, and hence the pressure forces on individual elemental surfaces differ in their directions. Therefore, a scalar summation of them cannot be made. Instead, the resultant thrusts in certain directions are to be determined and these forces may then be combined vectorially. An arbitrary submerged curved surface is shown in Fig. 5.3. A rectangular Cartesian coordinate system is introduced whose xy plane coincides with the free surface of the liquid and z-axis is directed downward below the x - y plane.



- Fig 5.3 Hydrostatic thrust on a Submerged Curved Surface

- Consider an elemental area dA at a depth z from the surface of the liquid. The hydrostatic force on the elemental area dA is

$$d\,F = \rho\, g\, z\, dA \tag{5.11}$$

- and the force acts in a direction normal to the area dA. The components of the force dF in x, y and z directions are

$$dF_x = ldF = l\rho gzdA \quad (5.12a)$$

$$dF_y = mdF = m\rho gzdA \quad (5.12b)$$

$$dF_z = ndF = n\rho gzdA \quad (5.13c)$$

- Where l, m and n are the direction cosines of the normal to dA. The components of the surface element dA projected on yz, xz and xy planes are, respectively

$$dA_x = l\,dA \quad (5.13a)$$

$$dA_y = m\,dA \quad (5.13b)$$

$$dA_z = n\,dA \quad (5.13c)$$

- Substituting Eqs (5.13a-5.13c) into (5.12) we can write

$$dF_x = \rho g\, z\, dA_x \tag{5.14a}$$

$$dF_y = \rho g\, z\, dA_y \tag{5.14b}$$

$$dF_z = \rho g\, z\, dA_z \tag{5.14c}$$

- Therefore, the components of the total hydrostatic force along the coordinate axes are

$$F_x = \iint \rho g z dA_x = \rho g z_c A_x \tag{5.15a}$$

$$F_y = \iint \rho g z dA_y = \rho g z_c A_y \tag{5.15b}$$

$$F_z = \iint \rho g z dA_z \tag{5.15c}$$

- where z_c is the z coordinate of the centroid of area A_x and A_y (the projected areas of curved surface on yz and xz plane respectively). If z_p and y_p are taken to be the coordinates of the point of action of F_x on the projected area A_x on yz plane, , we can write

$$z_p = \frac{1}{A_x z_c} \iint z^2 dA_x = \frac{I_{yy}}{A_x z_c} \tag{5.16a}$$

$$y_p = \frac{1}{A_x z_c} \iint yz dA_x = \frac{I_{yz}}{A_x z_c} \tag{5.16b}$$

- where I_{yy} is the moment of inertia of area A_x about y-axis and I_{yz} is the product of inertia of A_x with respect to axes y and z. In the similar fashion, z_p' and x_p' the coordinates of the point of action of the force F_y on area A_y, can be written as

$$z'_p = \frac{1}{A_y z_c} \iint z^2 dA_y = \frac{I_{xx}}{A_y z_c} \tag{5.17a}$$

$$x'_p = \frac{1}{A_y z_c} \iint xz dA_y = \frac{I_{xz}}{A_y z_c} \tag{5.17b}$$

- where I_{xx} is the moment of inertia of area A_y about x axis and I_{xz} is the product of inertia of A_y about the axes x and z.
- We can conclude from Eqs (5.15), (5.16) and (5.17) that for a curved surface, the component of hydrostatic force in a horizontal direction is equal to the hydrostatic force on the projected plane surface perpendicular to that direction and acts through the centre of pressure of the projected area. From Eq. (5.15c), the vertical component of the hydrostatic force on the curved surface can be written as

$$F_z = \rho g \iint z\, dA_z = \rho g \forall \tag{5.18}$$

▪

- where ∀is the volume of the body of liquid within the region extending vertically above the submerged surface to the free surfgace of the liquid. Therefore, the vertical component of hydrostatic force on a submerged curved surface is equal to the weight of the liquid volume vertically above the solid surface of the liquid and acts through the center of gravity of the liquid in that volume.

Buoyancy

- When a body is either wholly or partially immersed in a fluid, a lift is generated due to the net vertical component of hydrostatic pressure forces experienced by the body.
- This lift is called the buoyant force and the phenomenon is called buoyancy
- Consider a solid body of arbitrary shape completely submerged in a homogeneous liquid as shown in Fig. 5.4. **Hydrostatic pressure forces act on the entire surface of the body.**

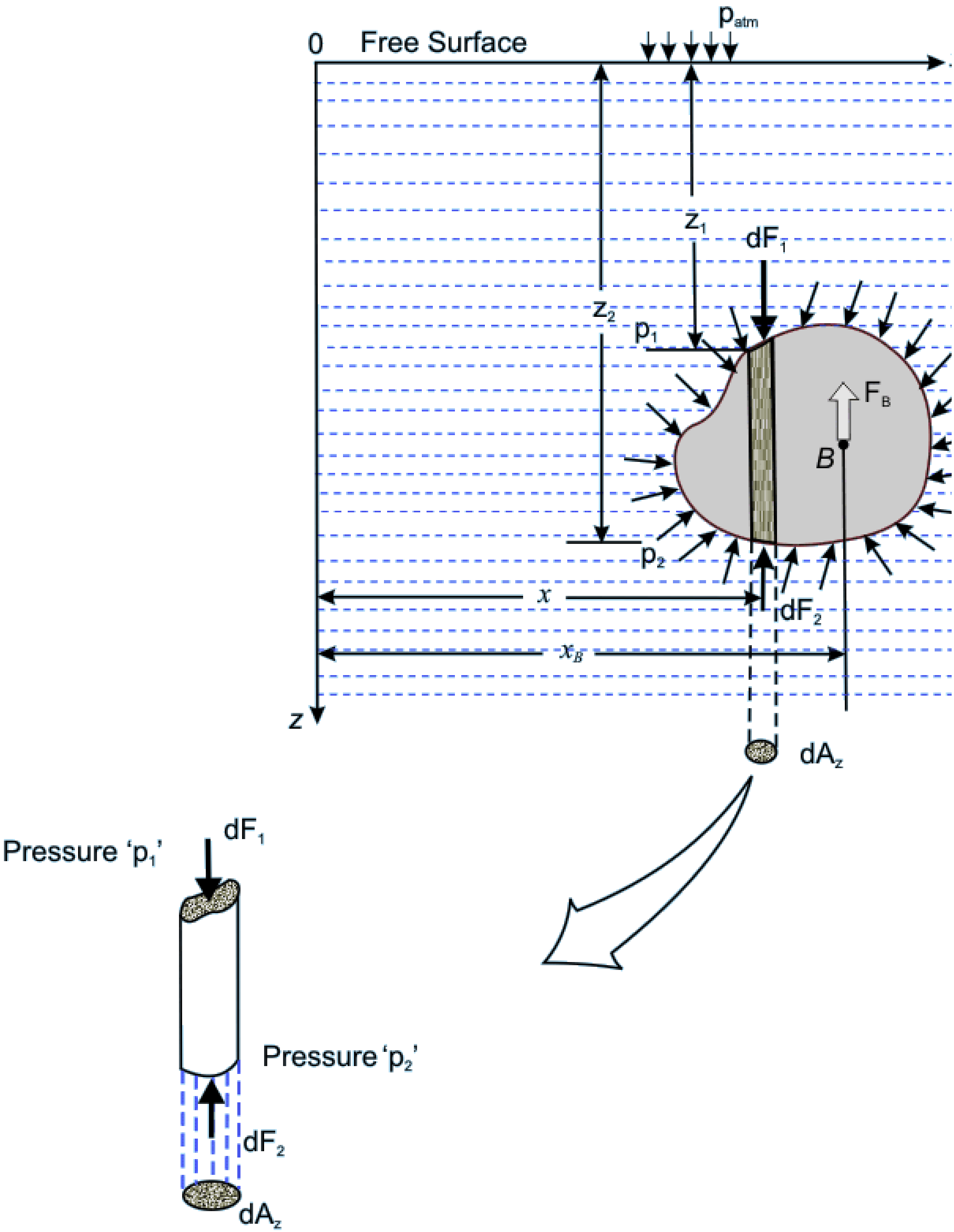


Fig 5.4 Buoyant Force on a Submerged Body

To calculate the vertical component of the resultant hydrostatic force, the body is considered to be divided into a number of elementary vertical prisms. The vertical forces acting on the two ends of such a prism of cross-section dA_z (Fig. 5.4) are respectively

$$dF_1 = (p_{atm} + p_1)dA_z = (p_{atm} + \rho g z_1)dA_z \tag{5.19a}$$

$$dF_2 = (p_{atm} + p_2)dA_z = (p_{atm} + \rho g z_2)dA_z \tag{5.19b}$$

Therefore, the buoyant force (the net vertically upward force) acting on the elemental prism of volume $d\forall$ is -

$$dF_B = dF_2 - dF_1 = \rho g(z_2 - z_1)dA_z = \rho g d\forall \tag{5.19c}$$

Hence the buoyant force F_B on the entire submerged body is obtained as

$$F_B = \iiint_{\forall} \rho g d\forall = \rho g \forall \tag{5.20}$$

Where $\forall$ **is the total volume of the submerged body.** The line of action of the force F_B can be found by taking moment of the force with respect to z-axis. Thus

$$x_B F_B = \int x dF_B \tag{5.21}$$

Substituting for dF_B and F_B from Eqs (5.19c) and (5.20) respectively into Eq. (5.21), the **x coordinate of the center of the buoyancy is** obtained as

$$x_B = \frac{1}{\forall}\iiint_{\forall} x d\forall \tag{5.22}$$

which is **the centroid of the displaced volume**. It is found from Eq. (5.20) that the buoyant force F_B equals to the weight of liquid displaced by the submerged body of volume $\forall$. This phenomenon was discovered by Archimedes and is known as the Archimedes principle.

ARCHIMEDES PRINCIPLE

The buoyant force on a submerged body

- The Archimedes principle states that the buoyant force on a submerged body is equal to the weight of liquid displaced by the body, and acts vertically upward through the centroid of the displaced volume.
- Thus the net weight of the submerged body, (the net vertical downward force experienced by it) is reduced from its actual weight by an amount that equals the buoyant force.

The buoyant force on a partially immersed body

- According to Archimedes principle, the buoyant force of a partially immersed body is equal to the weight of the displaced liquid.
- Therefore the buoyant force depends upon the density of the fluid and the submerged volume of the body.
- For a floating body in static equilibrium and in the absence of any other external force, the buoyant force must balance the weight of the body.

Stability of Unconstrained Submerged Bodies in Fluid

- The equilibrium of a body submerged in a liquid requires that the weight of the body acting through its cetre of gravity should be colinear with an equal hydrostatic lift acting through the centre of buoyancy.
- In general, if the body is not homogeneous in its distribution of mass over the entire volume, the location of **centre of gravity G does not coincide with the centre of volume, i.e., the centre of buoyancy B.**
- Depending upon the relative locations of G and B, a floating or submerged body attains three different states of equilibrium-

Let us suppose that a body is given a small angular displacement and then released. Then it will be said to be in

- Stable Equilibrium: If the body **returns to its original position** by retaining the originally vertical axis as vertical.
- Unstable Equilibrium: If the body **does not return to its original position but moves further** from it.
- Neutral Equilibrium: If the body **neither returns to its original position nor increases its displacement further**, it will simply adopt its new position.

Stable Equilibrium

Consider a submerged body in equilibrium whose centre of gravity is located below the centre of buoyancy (Fig. 5.5a). If the body is tilted slightly in any direction, the buoyant force and the weight always produce a restoring couple trying to return the body to its original position (Fig. 5.5b, 5.5c).

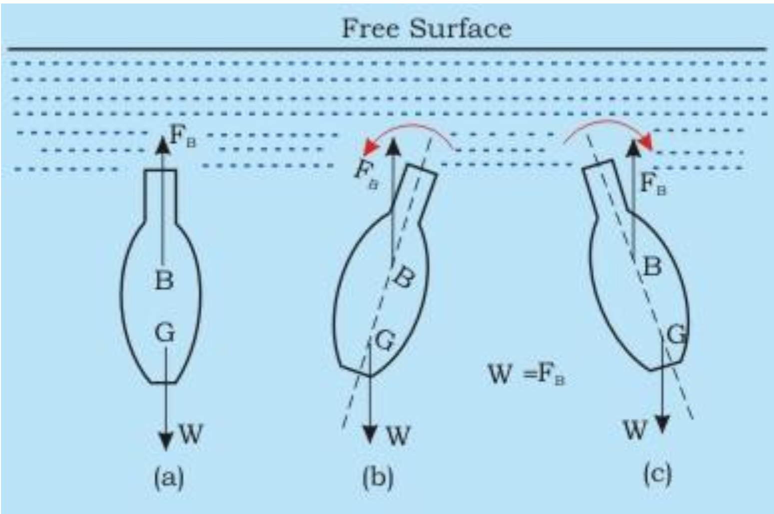
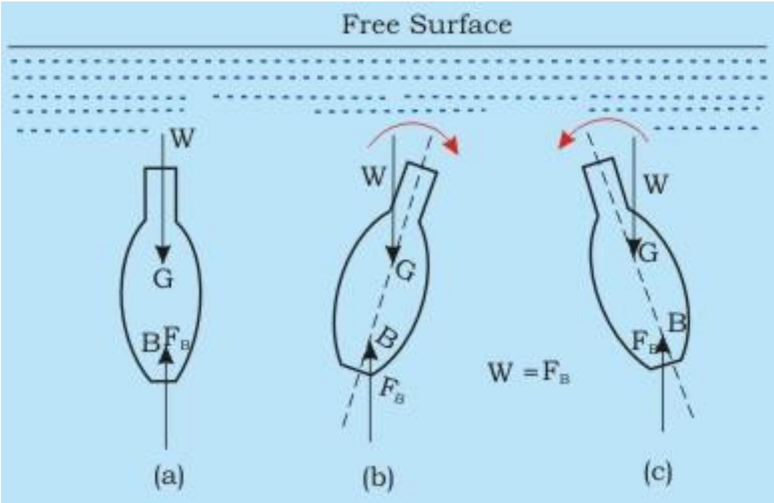


Fig 5.5 A Submerged body in Stable Equilibrium

Unstable Equilibrium

On the other hand, if point G is above point B (Fig. 5.6a), any disturbance from the equilibrium position will create a destroying couple which will turn the body away from its original position (5.6b, 5.6c).



Click to play the Demonstration

Fig 5.6   A Submerged body in Unstable Equilibrium

Neutral Equilibrium

When the centre of gravity G and centre of buoyancy B coincides, the body will always assume the same position in which it is placed (Fig 5.7) and hence it is in neutral equilibrium.

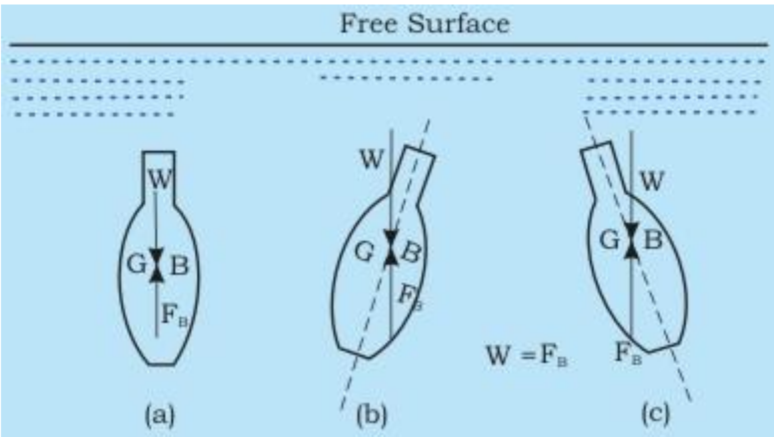


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Fig 5.7   A Submerged body in Neutral Equilibrium

Therefore, it can be concluded that **a submerged body will be in stable, unstable or neutral equilibrium if its centre of gravity is below, above or coincident with the center of buoyancy respectively (Fig. 5.8).**

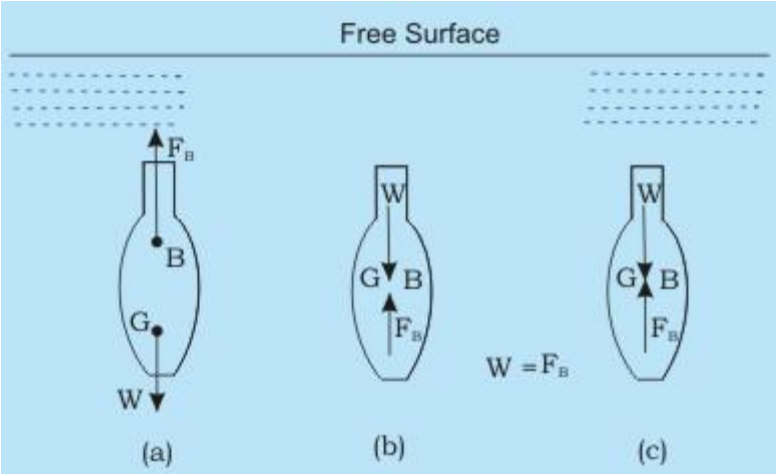


Fig 5.8 States of Equilibrium of a Submerged Body

(a) STABLE EQUILIBRIUM (B) UNSTABLE EQUILIBRIUM (C) NEUTRAL EQUILIBRIUM

Stability of Floating Bodies in Fluid

- When the body undergoes an angular displacement about a horizontal axis, the shape of the immersed volume changes and so the centre of buoyancy moves relative to the body.
- As a result of above observation stable equlibrium can be achieved, under certain condition, even when G is above B.
  Figure 5.9a illustrates a floating body -a boat, for example, in its equilibrium position.

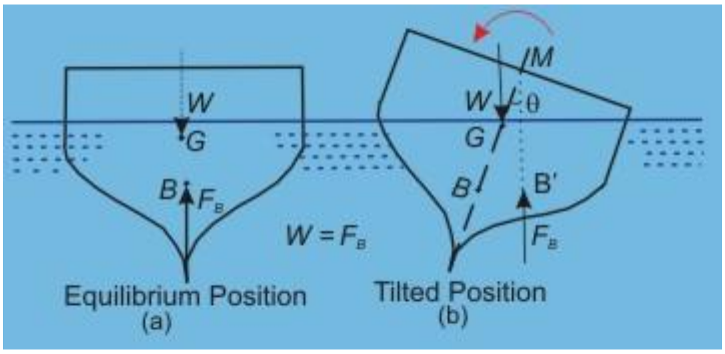


Fig 5.9 A Floating body in Stable equilibrium

Important points to note here are

a. The force of buoyancy F_B is equal to the weight of the body W

b. Centre of gravity G is above the centre of buoyancy in the same vertical line.

c. Figure 5.9b shows the situation after the body has undergone a small angular displacement θ with respect to the vertical axis.

d. The centre of gravity G remains unchanged relative to the body (This is not always true for ships where some of the cargo may shift during an angular displacement).

e. During the movement, the volume immersed on the right hand side increases while that on the left hand side decreases. Therefore the centre of buoyancy moves towards the right to its new position B'.

Let the new **line of action of the buoyant force** (which **is always vertical**) through B' intersects the axis BG (the old vertical line containing the centre of gravity G and the old centre of buoyancy B) at M. For small values of θ the **point M is** practically constant in position and is **known as metacentre**. For the body shown in Fig. 5.9, M is above G, and the couple acting on the body in its displaced position is a restoring couple which tends to turn the body to its original position. If M were below G, the couple would be an overturning couple and the original equilibrium would have been unstable. When M coincides with G, the body will assume its new position without any further movement and thus will be in neutral equilibrium. T**herefore, for a floating body, the stability is determined not simply by the relative position of B and G, rather by the relative position of M and G**. The distance of metacentre above G along the line BG is known as metacentric height GM which can be written as

$$GM = BM - BG$$

Hence the **condition of stable equilibrium for a floating body** can be expressed **in terms of metacentric height** as follows:

GM > 0 (M is above G)	**Stable equilibrium**
GM = 0 (M coinciding with G)	**Neutral equilibrium**
GM < 0 (M is below G)	**Unstable equilibrium**

The angular displacement of a boat or ship about its longitudinal axis is known as 'rolling' while that about its transverse axis is known as "pitching".

Floating Bodies Containing Liquid

If a floating body carrying liquid with a free surface undergoes an angular displacement, the liquid will also move to keep its free surface horizontal. Thus not only does the centre of buoyancy B move, but also the centre of gravity G of the floating body and its contents move in the same direction as the movement of B. Hence the stability of the body is reduced. For this reason, liquid which has to be carried in a ship is put into a number of separate compartments so as to minimize its movement within the ship.

Period of Oscillation

The restoring couple caused by the buoyant force and gravity force acting on a floating body displaced from its equilibrium placed from its equilibrium position is $W.GM\sin\theta$ (Fig. 5.9). Since the torque equals to mass moment of inertia (i.e., second moment of mass) multiplied by angular acceleration, it can be written

$$W(GM)\sin\theta = -I_M \frac{d^2\theta}{dr^2} \qquad (5.23)$$

Where I_M represents the mass moment of inertia of the body about its axis of rotation. The minus sign in the RHS of Eq. (5.23) arises since the torque is a retarding one and decreases the angular acceleration. If θ is small, sin θ=θ and hence Eq. (5.23) can be written as

$$\frac{d^2\theta}{dt^2} + \frac{W.GM}{I_M}\theta = 0 \qquad (5.24)$$

Equation (5.24) represents a simple harmonic motion. The time period (i.e., the time of a complete oscillation from one side to the other and back again) equals to $2\pi(I_M/W.GM)^{1/2}$. The oscillation of the body results in a flow of the liquid around it and this flow has been disregarded here. In practice, of course, viscosity in the liquid introduces a damping action which quickly suppresses the oscillation unless further disturbances such as waves cause new angular displacements.

Exercise Problems - Chapter 2

1. For the system shown in Fig 5.10,determine the air pressure p_A which will make the

[3.33 kPa]

pressure at N one fourth of that at M.

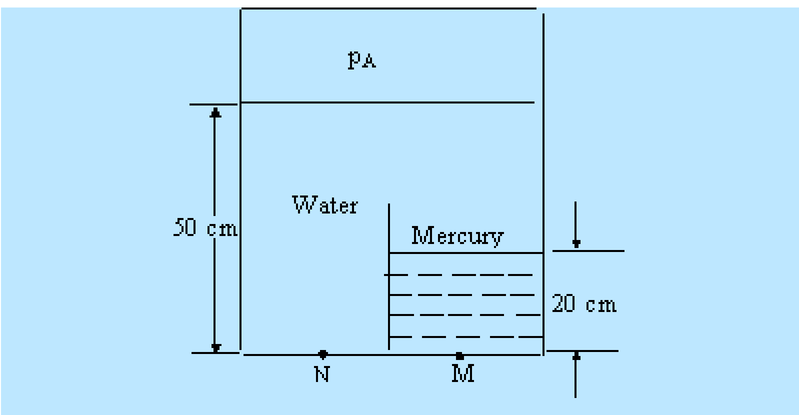


Fig 5.10

2. Consider the pipe and manometer system as shown in Fig 5.11. The pipe contains water. Find the value of manometer reading h, and the difference in pressure between A and B if

there is no flow. If there is a flow from A towards B and the manometer reading is h = 60 mm, then determine the static
pressure difference $p_A - p_B$

[0, 2.94 kPa; 3.53 kPa]

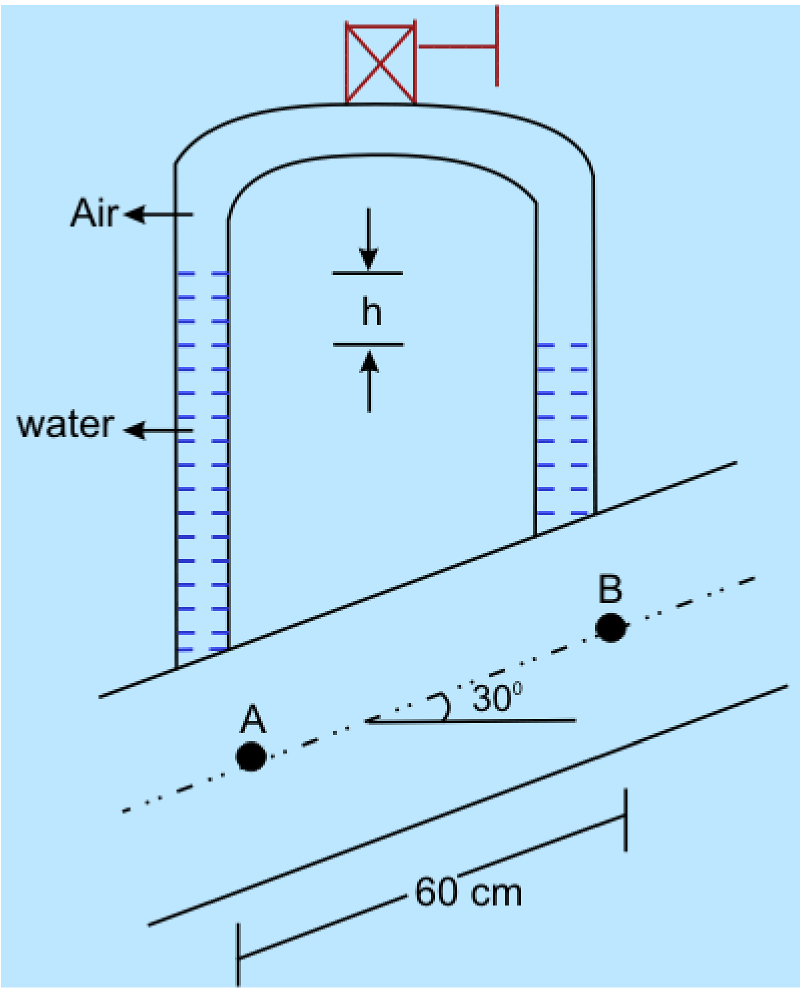


Fig 5.11

3. Determine the air pressure above the water surface in the tank if a force of 8 kN is required to hold the hinged door in
position as shown in Fig 5.12.

[10.76 kPa]

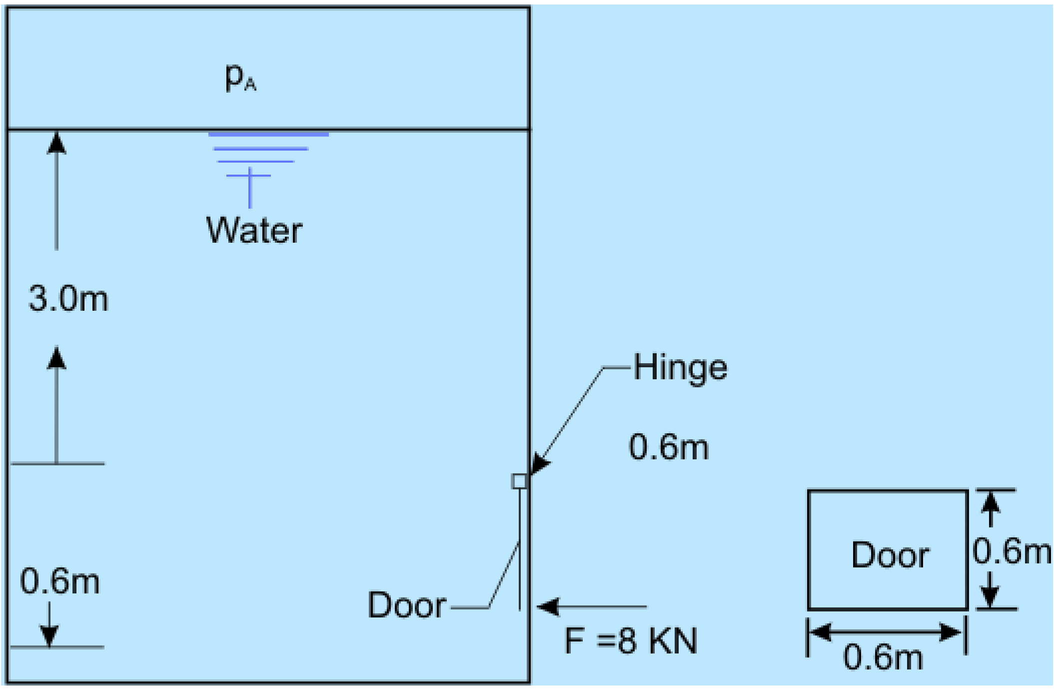


Fig 5.12

4. The profile of the inner face of a dam takes the form of a parabola with the equation 18y = x^2 , where y is the height above the base and x is the horizontal distance of the face from the vertical reference line. The water level is 27m above the base. Determine the thrust on the dam (per meter with) due to the water pressure, its inclination to the vertical and the point

[5.28 MN/m, 42^o 33', 30.29 m from face]

where the line of action of this force intersects the free water surface

5. A solid uniform cylinder of length 150 mm and diameter 75 mm is to float upright in

[0.641 kg and 0.663 kg]

water. Determine the limits within which its mass should lie.

6. A long prism, the cross-section of which is an equilateral traingle of side **a**, floats in water with one side horizontal and submerged to a depth **h**. Find

(a) **h/a** as a function of the specific gravity, **S** of the prism.
(b) The metacentric height in terms of side **a**, for small angle of rotation if specific gravity, **S=0.8**.

7. A metal sphere of volume $\forall_m = 0 \cdot 1 \ m^3$, specific gravity $s_m = 2$ and fully immersed in water is attached by a flexible wire to a buoy of volume $\forall_B = 1 \ m^3$ and specific gravity $s_B = 0 \cdot 1$. Calculate the tension T in the wire and volume of the buoy that is submerged. Refer to Fig 5.13.

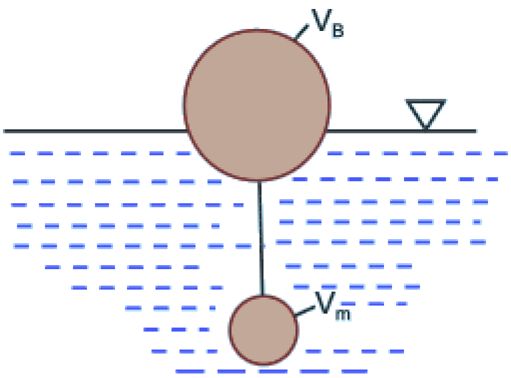


Fig 5.13

Recap

In this course you have learnt the following

- Forces acting on a fluid element in isolation are of two types;
  - Body force : Body forces act over the entire volume of the fluid element and are caused by external agencies
  - Surface force. Surface forces, resulting from the action of surrounding mass on the fluid element, appear on its surfaces.

- Normal stresses at any point in a fluid at rest, being directed towards the point from all directions, are of equal magnitude. The scalar magnitude of the stress is known as hydrostatic or thermodynamic pressure.

- The fundamental equations of fluid statics are written as $\partial p/\partial x = 0$, $\partial p/\partial y = 0$, $\partial p/\partial z = -\rho g$ with respect to a cartesian frame of reference with x - y plane as horizontal and axis z being directed vertically upwards. For an incompressible fluid, pressure P at a depth h below the free surface can be written as p = Po + ρ gh, where Po is the local atmospheric pressure.

- At sea-level, the international standard atmospheric pressure has been chosen as Patm = 101.32 kN/m2. The pressure expressed as the difference between its value and the local atmospheric pressure is known as gauge pressure.

- Piezometer tube measures the gauge pressure of a flowing liquid in terms of the height of liquid column. Manometers are devices in which columns of a suitable liquid are used to measure the difference in pressure between two points or between a certain point and the atmosphere. A simple U-tube manometer is modified as inclined tube manometer, inverted tube manometer and micro manometer to measure a small difference in pressure through a relatively large deflection of liquid columns.

- The hydrostatic force on anyone side of a submerged plane surface is equal to the product of the area and the pressure at the centre of area. The force acts in a direction perpendicular to the surface and its point of action, known as pressure centre, is always at a higher depth than that at which the centre of area lies. The distance of centre of pressure from the centre of area along the axis of symmetry is given by $y_p - y_c = I_{x'x'} / A y_c$

- For a curved surface, the component of hydrostatic force in any horizontal direction is equal to the hydrostatic force on the projected plane surface on a vertical plane perpendicular to that direction and acts through the centre of pressure for the projected plane area. The vertical component of hydrostatic force on a submerged curved surface is equal to the weight of the liquid volume vertically above the submerged surface to the level of the free surface of liquid and acts through the centre of gravity of the liquid in that volume.

- When a solid body is either wholly or partially immersed in a fluid, the hydrostatic lift due to net vertical component of the hydrostatic pressure forces experienced by the body is called the buoyant force. The buoyant force on a submerged or floating body is equal to the weight of liquid displaced by the body and acts vertically upward through the centroid of displaced volume known as centre of buoyancy.

- The equilibrium of floating or submerged bodies requires that the weight of the body acting through its centre of gravity has to be colinear with an equal buoyant force acting through the centre of buoyancy. A submerged body will be in stable, unstable or neutral equilibrium if its centre of gravity is below, above or coincident with the centre of buoyancy respectively. Metacentre of a floating body is defined as the point of intersection of the centre line of cross-section containing the centre of gravity and centre of buoyancy with the vertical line through new centre of buoyancy due to any small angular displacement of the body. For stable equilibrium of floating bodies, metacentre M has to be above the centre of gravity G. M coinciding with G or lying below G refers to the situation of neutral and unstable equilibrium respectively. The distance of metacentre from centre of gravity along the centre line of cross-section is known as metacentric height and is given by.

$$MG = \left(I_{yy} / \forall\right) - BG$$

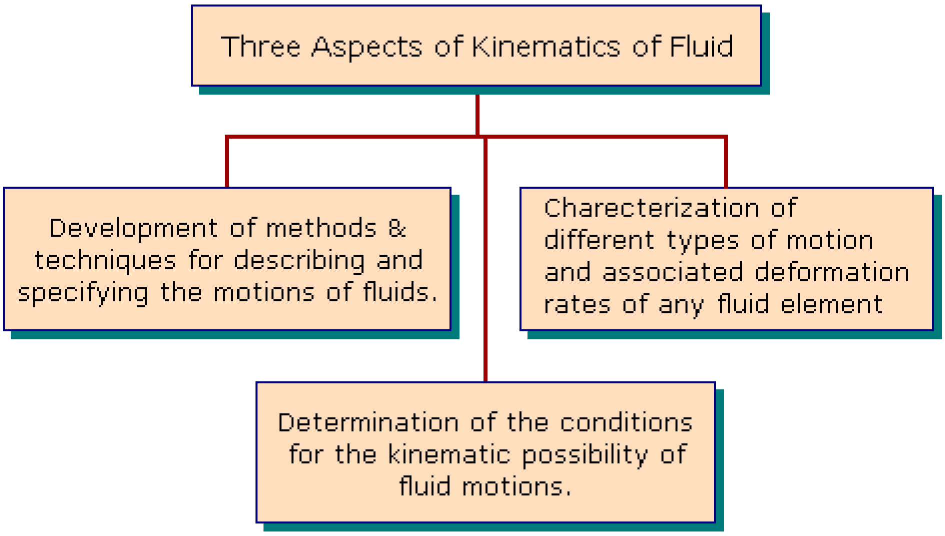
KINEMATICS OF FLUID

Introduction

Kinematics is the geometry of Motion.

Kinematics of fluid describes the fluid motion and its consequences without consideration of the nature of forces causing the motion.

The subject has **three main aspects:**



Scalar and Vector Fields

Scalar: Scalar is a quantity which can be expressed by a single number representing its **magnitude.**
Example: mass, density and temperature.

Scalar Field

If at every point in a region, a scalar function has a defined value, the region is called a **scalar field**.
Example: **Temperature distribution in a rod.**

Vector: Vector is a quantity which is specified by **both magnitude and direction**.
Example: Force, Velocity and Displacement.

Vector Field

If at every point in a region, a vector function has a defined value, the region is called a **vector field**.
Example: velocity field of a flowing fluid **.**

Flow Field

The region in which the flow parameters i.e. velocity, pressure etc. are defined at each and every point at any instant of time is called a **flow field**.

Thus a flow field would be specified by the velocities at different points in the region at different times.

Description of Fluid Motion

A. Lagrangian Method

- Using Lagrangian method, the fluid motion is described by tracing the **kinematic behaviour of each particle** constituting the flow.
- Identities of the particles are made by specifying their initial position (spatial location) at a given time. The position of a particle at any other instant of time then becomes a function of its identity and time.

Analytical expression of the last statement :

$\vec{S} = S(\vec{S}_o, t)$ — $\vec{S}$ is the position vector of a particle (with respect to a fixed point of reference) at a time t. $\vec{S}_0$ is its initial position at a given time $t = t_0$ (6.1)

Equation (6.1) can be written into scalar components with respect to a rectangular cartesian frame of coordinates as:

$x = x(x_0, y_0, z_0, t)$ (6.1a)

$y = y(x_0, y_0, z_0, t)$ (6.1b)

$z = z(x_0, y_0, z_0, t)$ (6.1c)

(where, x_0, y_0, z_0 are the initial coordinates and x, y, z are the coordinates at a time t of the particle.)

Hence $\vec{S}$ in can be expressed as

$\vec{S} = \vec{i}x + \vec{j}y + \vec{k}z$	$\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors along x, y and z axes respectively.

velocity and acceleration

The **velocity** $\vec{V}$ and **acceleration** $\vec{a}$ of the fluid particle can be obtained from the material derivatives of the position of the particle with respect to time. Therefore,

$$\vec{V} = \left[\frac{d\vec{s}}{dt}\right]_{s_0} \quad (6.2a)$$

In terms of scalar components,

$u = \left[\frac{dx}{dt}\right]_{x_o, y_o, z_o}$	(6.2b)
$v = \left[\frac{dy}{dt}\right]_{x_o, y_o, z_o}$	(6.2c)
$w = \left[\frac{dz}{dt}\right]_{x_o, y_o, z_o}$	(6.2d)

where **u, v, w are the components of velocity in x, y, z directions respectively.**

Similarly, for the **acceleration**,

$$\vec{a} = \left[\frac{d^2\vec{S}}{dt^2}\right]_{S_o} \tag{6.3a}$$

and hence,

$$a_x = \left[\frac{d^2x}{dt^2}\right]_{x_o,y_o,z_o} \tag{6.3b}$$

$$a_y = \left[\frac{d^2y}{dt^2}\right]_{x_o,y_o,z_o} \tag{6.3c}$$

$$a_z = \left[\frac{d^2z}{dt^2}\right]_{x_o,y_o,z_o} \tag{6.3d}$$

where **a_x, a_y, a_z are accelerations in x, y, z directions respectively**.

Advantages of Lagrangian Method:

1. Since motion and trajectory of each fluid particle is known, its history can be traced.
2. Since particles are identified at the start and traced throughout their motion, conservation of mass is inherent.

Disadvantages of Lagrangian Method:

1. The solution of the equations presents appreciable mathematical difficulties except certain special cases and therefore, the method is rarely suitable for practical applications.
2. B. Eulerian Method
3. The method was developed by **Leonhard Euler**.
4. This method is of greater advantage since it avoids the determination of the movement of each individual fluid particle in all details.
5. It **seeks the velocity $\vec{V}$ and its variation with time t at each and every location ($\vec{S}$) in the flow field.**
6. In Eulerian view, all hydrodynamic parameters are functions of location and time.
7. **Mathematical representation** of the flow field in **Eulerian method**:

$$\vec{V} = V(\vec{S}, t) \tag{6.4}$$

8. where

$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$ and $\vec{S} = \vec{i}x + \vec{j}y + \vec{k}z$

9.
10. Therefore,

$$\boxed{u = u\,(x, y, z, t)}$$

$v = v\ (x, y, z, t)$
$w = w\ (x, y, z, t)$

Relation between Eulerian and Lagrangian Method

The **Eulerian** description can be written as :

$$\frac{d\vec{S}}{dt} = V\left(\vec{S}, t\right) \qquad (6.5)$$

or

$$\frac{dx}{dt} = u(x, y, z, t)$$

$$\frac{dy}{dt} = \mathrm{v}(x, y, z, t)$$

$$\frac{dz}{dt} = \mathrm{w}(x, y, z, t)$$

The integration of Eq. (6.5) yields the constants of integration which are to be found from the initial coordinates of the fluid particles.

Hence, the solution of Eq. (6.5) gives the equations of Lagrange as,

$$\vec{S} = S(\vec{S}_o, t)$$

or

$$x = x\left(x_0, y_0, z_0, t\right)$$

$$y = y\left(x_0, y_0, z_0, t\right)$$

$$z = z\left(x_0, y_0, z_0, t\right)$$

Above relation are same as **Lagrangian** formulation.

In principle, **the Lagrangian method of description can always be derived from the Eulerian method.**

Problem

In a one-dimensional flow field, the velocity at a point may be given in the Eulerian system by u=x+2t. Determine the displacement of a fluid particle whose initial position is x_0 at initial time to in the Lagrangian system.

Solution

Given u=x+2t

$$\therefore \frac{dx}{dt} = x + 2t$$

gives on integration

$$xe^{-t} = c - 2e^{-t}(t+1)$$

$$\Rightarrow x = ce^{t} - 2(t+1) \qquad \text{(A)}$$

where, c is the constant of integration. C can be found by initial conditions-

For $t = t_0$, $x = x_0$

Therefore,

$$x_0 = ce^{t_0} - 2(t_0 + 1)$$

$$\Rightarrow c = \frac{x_0 + 2(t_0 + 1)}{e^{t_0}} \qquad \text{(B)}$$

Putting value of c from (B) into (A)

$$x = [x_0 + 2(t_0 + 1)]e^{(t - t_0)} - 2(t+1)$$

This equation represents the Lagrangian version of the fluid particle having the identity $x = x_0$ at $t = t_0$

Variation of Flow Parameters in Time and Space

Hydrodynamic parameters like pressure and density along with flow velocity may vary from one point to another and also from one instant to another at a fixed point.

According to **type of variations**, categorizing the flow:

Steady and Unsteady Flow

- **Steady Flow**
  A steady flow is defined as a flow in which the various hydrodynamic parameters and fluid properties at any point do not change with time.

In **Eulerian approach**, a steady flow is described as,

$$\vec{V} = V(\vec{S})$$

and

$$\vec{a} = a(\vec{S})$$

Implications:

1. Velocity and acceleration are functions of space coordinates only.

2. In a steady flow, the hydrodynamic parameters may vary with location, but the spatial distribution of these parameters remain invariant with time.

In the **Lagrangian approach**,

1. Time is inherent in describing the trajectory of any particle.

2. In steady flow, the velocities of all particles passing through any fixed point at different times will be same.

3. Describing velocity as a function of time for a given particle will show the velocities at different points through which the particle has passed providing the information of velocity as a function of spatial location as described by **Eulerian method**. Therefore, the Eulerian and Lagrangian approaches of describing fluid motion become identical under this situation.

- **Unsteady Flow**
  An unsteady Flow is defined as a flow in which the hydrodynamic parameters and fluid properties changes with time.

Uniform and Non-uniform Flows

- **Uniform Flow**

The flow is defined as uniform flow when in the flow field the **velocity and other hydrodynamic parameters do not change** f**rom point to point at any instant of time.**

For a uniform flow, the velocity is a function of time only, which can be expressed in Eulerian description as

$$\vec{V} = V(t)$$

Implication:

1. For a uniform flow, there will be no spatial distribution of hydrodynamic and other parameters.

2. **Any hydrodynamic parameter will have a unique value in the entire field**, irrespective of whether it

   changes with time − **unsteady uniform flow** OR

   does not change with time − **steady uniform flow.**

3. Thus ,steadiness of flow and uniformity of flow does not necessarily go together.

- **Non-Uniform Flow**

When the **velocity and other hydrodynamic parameters changes from one point to another** the flow is defined as **non-uniform**.

Important points:

1. For a non-uniform flow, the changes with position may be found either in the direction of flow or in directions perpendicular to it.

2.Non-uniformity in a direction perpendicular to the flow is always encountered near solid boundaries past which the fluid flows.

Reason: All fluids possess **viscosity** which reduces the relative velocity (of the fluid w.r.t. to the wall) to zero at a solid boundary. This is known as **no-slip condition.**

Four possible combinations

Type	Example
1. Steady Uniform flow	Flow at constant rate through a duct of uniform cross-section (The region close to the walls of the duct is disregarded)
2. Steady non-uniform flow	Flow at constant rate through a duct of non-uniform cross-section (tapering pipe)
3. Unsteady Uniform flow	Flow at varying rates through a long straight pipe of uniform cross-section. (Again the region close to the walls is ignored.)
4. Unsteady non-uniform flow	Flow at varying rates through a duct of non-uniform cross-section.

Material Derivative and Acceleration

- **Let the position of a particle at any instant t in a flow field be given by the space coordinates (x, y, z) with respect to a rectangular cartesian frame of reference.**
- **The velocity components u, v, w of the particle along x, y and z directions respectively can then be written in Eulerian form as**

$$u = u\,(x, y, z, t)$$
$$v = v\,(x, y, z, t)$$
$$w = w\,(x, y, z, t)$$

- After an infinitesimal time interval t , let the particle move to a new position given by the coordinates $(x + \Delta x,\ y + \Delta y\ ,\ z + \Delta z)$.
- Its velocity components at this new position be $u + \Delta u$, $v + \Delta v$ and $w + \Delta w$.
- Expression of velocity components in the **Taylor's series** form:

$$u+\Delta u = u(x,y,z,t)+\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\frac{\partial u}{\partial z}\Delta z+\frac{\partial u}{\partial t}\Delta t+ higher\ order\ terms\, in\ \Delta x,\ \Delta y,\ \Delta z\ and$$

$$v+\Delta v = v(x,y,z,t)+\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\frac{\partial v}{\partial z}\Delta z+\frac{\partial v}{\partial t}\Delta t+ higher\ order\ terms\, in\ \Delta x,\ \Delta y,\ \Delta z\ and\ \Delta t$$

$$w+\Delta w = w(x,y,z,t)+\frac{\partial w}{\partial x}\Delta x+\frac{\partial w}{\partial y}\Delta y+\frac{\partial w}{\partial z}\Delta z+\frac{\partial w}{\partial t}\Delta t+ higher\ order\ terms\, in\ \Delta x,\ \Delta y,\ \Delta z\ and$$

The increment in space coordinates can be written as -

$$\Delta x = u\Delta t,\quad \Delta y = v\Delta t \quad and \quad \Delta z = w\Delta t$$

Substituting the values of $\Delta x,\ \Delta y,\ \Delta z$ in above equations, we have

$$\frac{\Delta u}{\Delta t} = u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}$$ etc

- In the limit $\Delta t \to 0$, the **equation** becomes

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \qquad (7.1a)$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \qquad (7.1b)$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \qquad (7.1c)$$

Material Derivation and Acceleration...contd. from previous slide

- The above equations tell that the operator for **total differential** with respect to time, D/Dt in a **convective field** is related to the **partial differential** as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z} \qquad (7.2)$$

Explanation of equation 7.2 :

- The total differential D/Dt is known as the material or **substantial derivative** with respect to time.
- The **first term** $\partial/\partial t$ in the right hand side of is known as **temporal or local derivative** which expresses the rate of change with time, at a fixed position.
- The **last three terms** in the right hand side of are together known as **convective derivative** which represents the time rate of change due to change in position in the field.

Explanation of equation 7.1 (a, b, c):

- The terms in the left hand sides of Eqs (7.1a) to (7.1c) are defined as x, y and z components of **substantial or materia**l acceleration.
- The first terms in the right hand sides of Eqs (7.1a) to (7.1c) represent the respective **local or temporal** accelerations, while the other terms are **convective** accelerations.

Thus we can write,

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (7.2a)$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (7.2b)$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (7.2c)$$

(Material or substantial acceleration) = (temporal or local acceleration) + (convective acceleration)

Important points:

1. In a steady flow, the **temporal** acceleration is zero, since the velocity at any point is invariant with time.

2. In a uniform flow, on the other hand, the **convective** acceleration is zero, since the velocity components are not the functions of space coordinates.

3. In a steady and uniform flow, both the temporal and convective acceleration vanish and hence there exists no material acceleration.

Existence of the components of acceleration for different types of flow is shown in the table below.

Type of Flow	Material Acceleration	
	Temporal	**Convective**
1. Steady Uniform flow	0	0
2. Steady non-uniform flow	0	exists
3. Unsteady Uniform flow	exists	0
4. Unsteady non-uniform flow	exists	exists

Problem

Given the velocity field

$$\vec{V} = (4 + xy + 2t)\hat{i} + 6x^3\hat{j} + (3xt^2 + z)\hat{k}$$

Find the acceleration of fluid particle -

a. as a function of x,y,z and t
b. at (1,1,1) and time t=1

Solution

a) From Equation 6.7a to 6.7c we have

$$a_x = \frac{DU}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{DV}{Dt} = \frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}$$

$$a_z = \frac{Dz}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

From the given velocity field,

$$u = 4 + xy + 2t;\ v = 6x^3;\ z = 3xt^2 + z$$

Therefore,

$$a_x = 2 + (4 + xy + 2t)(y) + 6x^3(x) + (3xt^2 + z)(0) \qquad \text{(i)}$$

$$a_y = 0 + (4 + xy + 2t)(18x^2) + 6x^3(0) + (3xt^2 + z)(0) \qquad \text{(ii)}$$

$$a_z = 6xt + (4 + xy + 2t)(3t^2) + 6x^3(0) + (3xt^2 + z)(1) \qquad \text{(iii)}$$

Combining (i), (ii) and (iii) total acceleration is

$$\begin{aligned}\vec{a} &= a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \\ &= (2 + 4y + xy^2 + 2ty + 6x^4)\hat{i} + (72x^2 + 18x^3y + 36tx^2)\hat{j} \\ &\quad + (6xt + 12t^2 + 3xyt^2 + 6t^3 + 2 + 3xt^2)\hat{k}\end{aligned}$$

b) At 1,1,1 and t=1 acceleration vector is -

$$\vec{a} = (2+4+1+2+6)\,\hat{i} + (72+18+36)\,\hat{j} + (6+12+3+6+2+3)\,\hat{k} = 15\hat{i} + 126\hat{j} + 32\hat{k}$$

- In vector form, Components of **Acceleration** in **Cylindrical Polar Coordinate System (r, θ , z)**

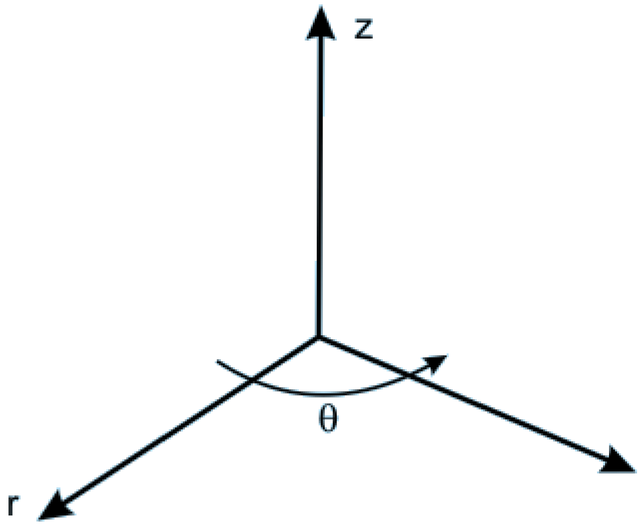


Fig 7.1 Velocity Components in a cylindrical Polar Coordinate System

- In a cylindrical polar coordinate system (Fig. 7.1), the components of acceleration in **r, θ and z** directions can be written as

$$a_r = \frac{DV_r}{Dt} - \frac{V_\theta^2}{r} = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_\theta = \frac{DV_\theta}{Dt} + \frac{V_r V_\theta}{r} = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$

$$a_z = \frac{DV_z}{Dt} = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}$$

Explanation of the additional terms appearing in the above equation:

1. The term $-\frac{V_\theta^2}{r}$ appears due to an inward radial acceleration arising from a change in the direction of V_θ (velocity component in the **azimuthal** direction) with θ as shown in Fig. 7.1. This is known as **centripetal** acceleration.
2. The term $V_r V_\theta / r$ represents a component of acceleration in **azimuthal direction** caused by **a change in the direction V_r of with θ**

Streamlines

Definition: Streamlines are the Geometrical representation of the of the flow velocity.

Description:

- In the **Eulerian** method, the velocity vector is defined as a function of time and space coordinates.
- If for a fixed instant of time, a **space curve** is drawn so that it is **tangent** everywhere to the **velocity** vector, then this curve is called a **Streamline**.

Therefore, the Eulerian method gives a series of instantaneous streamlines of the state of motion (Fig. 7.2a).

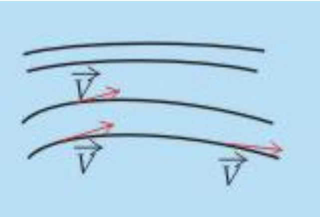


Fig 7.2a Streamlines

Alternative Definition:

A streamline at any instant can be defined as an imaginary curve or line in the flow field so that the tangent to the curve at any point represents the direction of the **instantaneous velocity** at that point.

Comments:

- In an **unsteady flow** where the velocity vector changes with time, the pattern of streamlines also **changes from instant to instant.**
- In a **steady flow**, the orientation or the pattern of streamlines will be **fixed**.

From the above definition of streamline, it can be written as

$$\vec{V} \times d\vec{S} = 0 \qquad (7.3)$$

Description of the terms:

1. $d\vec{S}$ is the length of an infinitesimal line segment along a streamline at a point .

2. $\vec{V}$ is the instantaneous velocity vector.

The above expression therefore represents the **differential equation of a streamline**. In a cartesian coordinate-system, representing

$$\vec{S} = \vec{i}dx + \vec{j}dy + \vec{k}dz \qquad \vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

the above equation (Equation 7.3) may be simplified as

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad (7.4)$$

Stream tube:

A bundle of neighboring streamlines may be imagined to form a passage through which the fluid flows. This passage is known as a **stream-tube**.

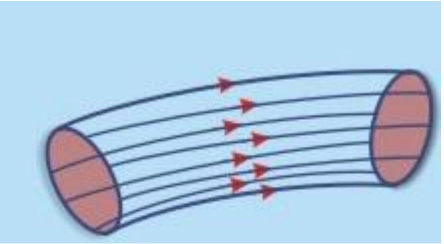


Fig 7.2b Stream Tube

Properties of Stream tube:

1. The stream-tube is bounded on all sides by streamlines.

2. Fluid velocity does not exist across a streamline, no fluid may enter or leave a stream-tube except through its ends.

3. The entire flow in a flow field may be imagined to be composed of flows through stream-tubes arranged in some arbitrary positions.

Path Lines

Definition: A path line is the trajectory of a fluid particle of **fixed identity** as defined by Eq. (6.1).

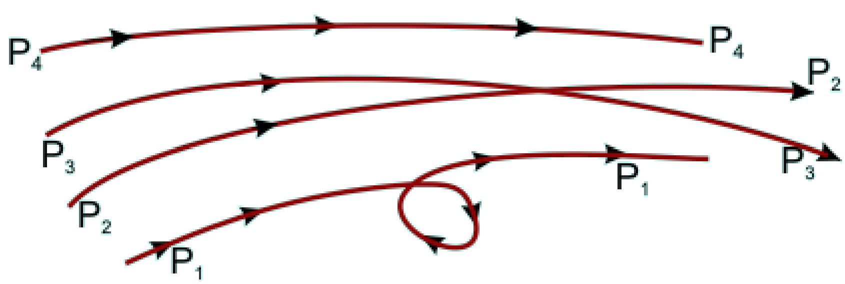


Fig 7.3 Path lines

A family of path lines represents the **trajectories of different particles**, say, P_1, P_2, P_3, etc. (Fig. 7.3).

Differences between Path Line and Stream Line

Path Line	Stream Line
▪ This refers to a path followed by a fluid particle over a period of time.	▪ This is an imaginary curve in a flow field for a fixed instant of time, tangent to which gives the instantaneous velocity at that

	point .
▪ Two path lines can intersect each other as or a single path line can form a loop as different particles or even same particle can arrive at the same point at different instants of time.	▪ Two stream lines can never intersect each other, as the instantaneous velocity vector at any given point is unique.

Note: In a steady flow **path lines** are **identical** to **streamlines** as the **Eulerian and Lagrangian** versions become the **same**.

Problem1:

A velocity field is given by

$$\vec{U} = (1 + At + Bt^2)\hat{i} + x\hat{j}$$

a) Find the equation of the streamline at t $=t_0$ passing through the point (x_0,y_0).

b) Obtain the path line of a fluid element which comes to (x_0, y_0) at $t=t_0$.

c) Show that, if A=0 and B=0 (i.e. steady flow), the streamline and path line coincide.

Solution:

a) **Streamline**: Here $U_x=(1+At+Bt^2)$ and $U_y=x$.

Since the slope of the streamline (dy/dx) is the same as the slope (U_y/U_x) of the velocity vector.

Therefore $$\frac{dy}{dx} = \frac{x}{(1 + At_0 + Bt_0^2)} \quad at \quad t = t_0$$

Integrating this with the condition $x=x_0$, $y=y_0$ gives the Streamline

$$(y - y_0)(1 + At_0 + Bt_0^2) = \frac{(x^2 - x_0^2)}{2}$$

b) **Path line:** Consider a fluid element passing through (x_0, y_0) at $t=t_0$. Its co-ordinates (x,y) at other values of t (which define the pathline) can

be expressed as

$$x = x(x_0, y_0, t)$$
$$y = y(x_0, y_0, t)$$

Since,

$$\frac{dx}{dt} = U_x = (1 + At + Bt^2)$$

And,

$$\frac{dy}{dt} = U_y = x$$

Integrating the first equation gives,

$$x - x_0 = (t - t_0) + A(t^2 - t_0^2)/2 + B\frac{(t^3 - t_0^3)}{3}$$

Now, $$\frac{dy}{dt} = x = x_0 + (t - t_0) + A(t^2 - t_0^2)/2 + B\frac{(t^3 - t_0^3)}{3}$$

$$y - y_0 = x_0(t - t_0) + \frac{(t^2 - t_0^2)}{2} - t_0(t - t_0) + A\left[\frac{(t^3 - t_0^3)}{6} - \frac{t_0^2(t - t_0)}{2}\right] + B\left[\frac{(t^4 - t_0^4)}{12} - \frac{t_0^3(t - t_0)}{3}\right]$$

These equations of x, y are **parametric equation of path line**.

The time t can be eliminated between them to give an equation for y in terms of x.

c) When A=B=0, then the equation of streamline becomes

$$y - y_0 = \frac{x^2 - x_0^2}{2}$$

and the parametric equations of the path line becomes;

$$x - x_0 = t - t_0$$

$$y - y_0 = \left[x_0 + \frac{t + t_0}{2} - t_0\right](t - t_0)$$

$$= \left[x_0 + \frac{t - t_0}{2}\right](x - x_0)$$

$$= \left[x_0 + \frac{x - x_0}{2}\right](x - x_0)$$

$$= \frac{x + x_0}{2}(x - x_0)$$

$$= \frac{x + x_0}{2}(x - x_0)$$

Therefore,

$$y - y_0 = \frac{x^2 - x_0^2}{2}$$

which is equivalent to **streamline.**

Problem2:

A two-dimensional flow field is defined as

$$\vec{V} = \hat{i}y - \hat{j}x$$

Define the equation of **Streamline** passing through the point (1,0)

Solution:

The **equation of Streamline** is

$$\vec{V} \times d\vec{S} = 0$$

or,

$$udy - vdx = 0$$

Hence,

$$dy/dx = v/u = -\frac{x}{y}$$

or,

$$ydy = -xdx$$

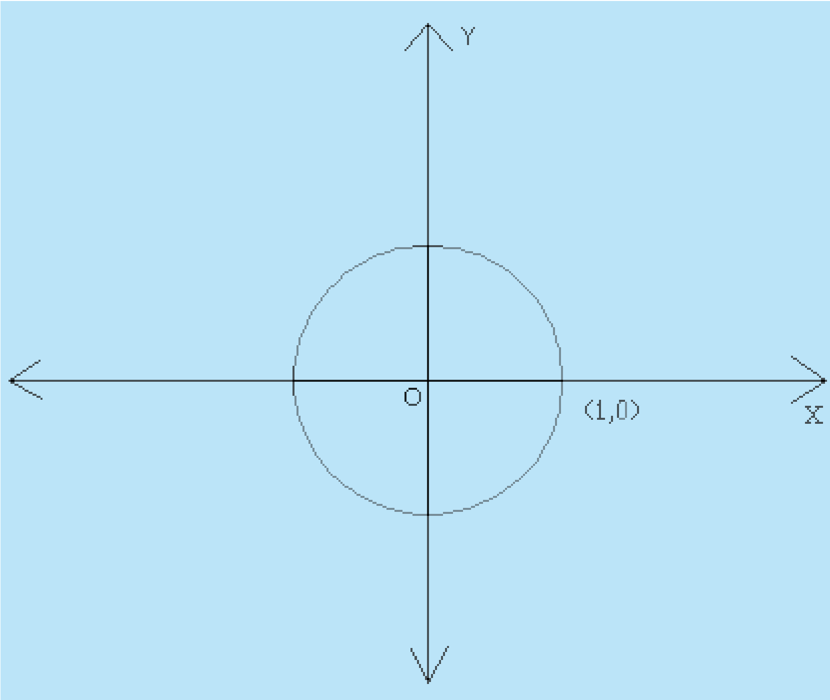
Integration of equation above gives

$$x^2 + y^2 = k$$

where k is constant

For stream line passing through (1,0) , $k = 1$

Hence, the required equation is: $x^2 + y^2 = 1$



Streak Lines

Definition: **A streak line** is the locus of the temporary locations of all particles that have passed though a fixed point in the flow field at any instant of time.

Features of a Streak Line:

- While a path line refers to the identity of a fluid particle, a streak line is specified by a fixed point in the flow field.
- It is of particular interest in experimental flow visualization.
- **Example:** If dye is injected into a liquid at a fixed point in the flow field, then at a later time t, the dye will indicate the end points of the path lines of particles which have passed through the injection point.
- The equation of a streak line at time t can be derived by the **Lagrangian method**.

If a fluid particle $(\vec{S}_o)$ passes through a fixed point $(\vec{S}_1)$ in course of time t, then the Lagrangian method of description gives the equation

$$S(\vec{S}_o, t) = \vec{S}_1 \qquad (7.5)$$

Solving for $\vec{S}_0$,

$$\vec{S}_o = F(\vec{S}_1, t) \qquad (7.6)$$

If the positions $(\vec{S}_o)$ of the particles which have passed through the fixed point $(\vec{S}_1)$ are determined, then **a streak line** can be drawn through these points.

Equation: The equation of the streak line at a time t is given by

$$\vec{S} = f(\vec{S}_o, t) \tag{7.7}$$

Substituting Eq. (7.5) into Eq. (7.6) we get the **final form of equation of the streak line**,

$$\vec{S} = f\left[F(\vec{S}_1, t), t\right] \tag{7.8}$$

Difference between Streak Line and Path Line

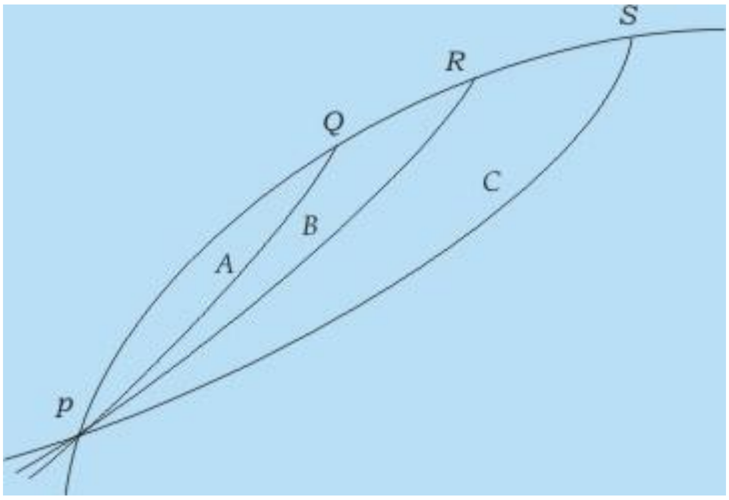


Fig 7.4 Description of a Streak line

Above diagram can be described by the following points:

Describing a **Path Line:**

a) Assume P be a fixed point in space through which particles of different identities pass at different times.

b) In an **unsteady flow**, the velocity vector at P will change with time and hence the particles arriving at P at **different times** will traverse

different paths like PAQ, PBR and PCS which represent the path lines of the particle.

Describing a **Streak Line:**

a) Let at any instant these particles arrive at points Q, R and S.

b) Q, R and S represent the end points of the trajectories of these three particles at the instant.

c) The curve joining the points S, R, Q and the fixed point P will define the **streak line** at that instant.

d) The fixed point P will also lie on the line, since at any instant, there will be always a particle of some identity at that point.

Above points show the differences.

Similarities:

a) For a **steady flow**, the velocity vector at any point is invariant with time

b) The path lines of the particles with different identities passing through P at different times will not differ

c) The path line would coincide with one another in a single curve which will indicate the streak line too.

Conclusion: Therefore, in a **steady flow**, the path lines, streak lines and streamlines are identical.

One, Two and Three Dimensional Flows

- Fluid flow is **three-dimensional** in nature.
- This means that the flow parameters like velocity, pressure and so on vary in all the three coordinate directions.

Sometimes simplification is made in the analysis of different fluid flow problems by:

- Selecting the appropriate coordinate directions so that appreciable variation of the hydro dynamic parameters take place in only two directions or even in only one.

One-dimensional flow

- All the flow parameters may be expressed as functions of time and one space coordinate only.
- The single space coordinate is usually the distance measured along the centre-line (not necessarily straight) in which the fluid is flowing.
- **Example:** the flow in a pipe is considered one-dimensional when variations of pressure and velocity occur along the length of the pipe, but any variation over the cross-section is assumed negligible.
- In reality, flow is never one-dimensional because **viscosity** causes the velocity to decrease to zero at the solid boundaries.
- If however, the **non uniformity of the actual flow is not too great**, valuable results may often be obtained from a "**one dimensional analysis**".
- The **average values** of the flow parameters at any given section (perpendicular to the flow) are assumed to be applied to the entire flow at that section.

Two-dimensional flow

- All the flow parameters are functions of time and two space coordinates (say x and y).
- No variation in z direction.
- The same streamline patterns are found in all planes perpendicular to z direction at any instant.

Three dimensional flow

- The hydrodynamic parameters are functions of three space coordinates and time.

Translation of a Fluid Element

The movement of a fluid element in space has three distinct features simultaneously.

- Translation
- Rate of deformation
- Rotation.

Figure 7.4 shows the picture of a pure translation in absence of rotation and deformation of a fluid element in a two-dimensional flow described by a rectangular cartesian coordinate system.

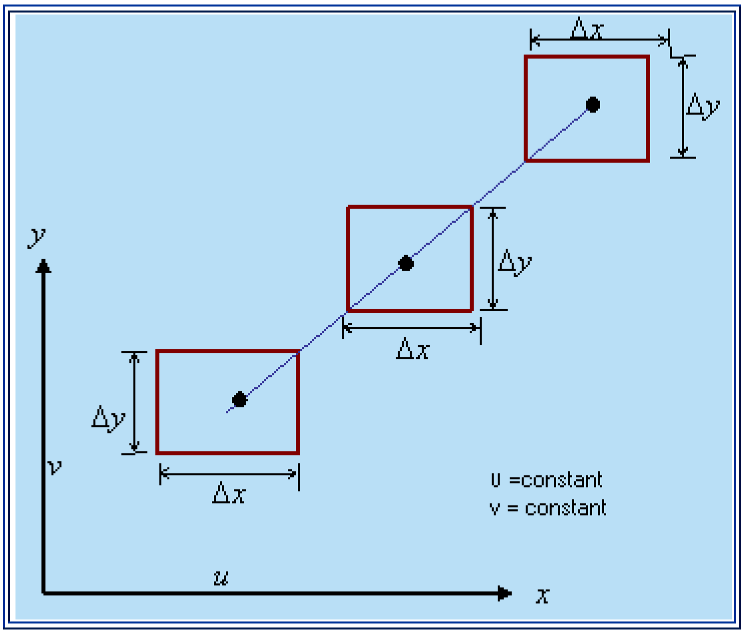
In absence of deformation and rotation,

a) There will be no change in the length of the sides of the fluid element.

b) There will be no change in the included angles made by the sides of the fluid element.

c) The sides are displaced in parallel direction.

This is possible when the flow velocities u (the x component velocity) and v (the y component velocity) are neither a function of x nor of y, i.e., the flow field is totally **uniform.**



Click to play the Demonstration

Fig 8.1 Fluid Element in pure translation

If a component of flow velocity becomes the function of only **one space coordinate** along which that velocity component is defined.

For example,

- if $u = u(x)$ and $v = v(y)$, the fluid element ABCD suffers a change in its linear dimensions along with translation
- there is no change in the included angle by the sides as shown in Fig. 7.5.

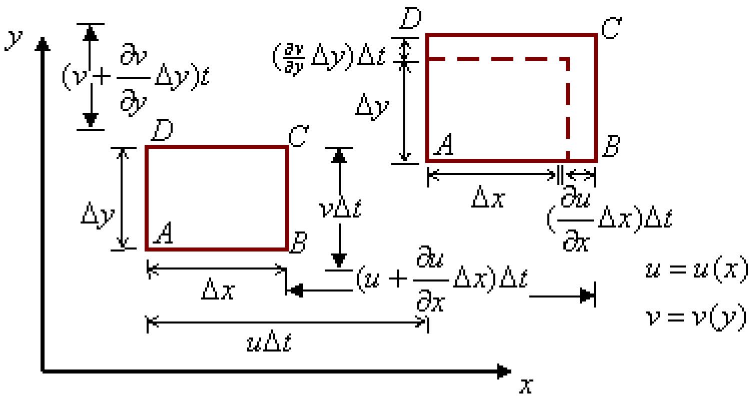


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Fig 8.2 Fluid Element in Translation with Continuous Linear Deformation

The relative displacement of point B with respect to point A per unit time in x direction is

$$\frac{\partial u}{\partial x}\Delta x$$

Similarly, the relative displacement of D with respect to A per unit time in y direction is

$$\frac{\partial v}{\partial y}\Delta y$$

Translation with Linear Deformations

Observations from the figure:

Since u is not a function of y and *v* is not a function of x

- All points on the linear element AD move with same velocity in the x direction.
- All points on the linear element AB move with the same velocity in y direction.
- Hence the sides move parallel from their initial position without changing the included angle.

This situation is referred to as **translation with linear deformation**.

Strain rate:

The changes in lengths along the coordinate axes per unit time per unit original lengths are defined as the **components of linear deformation or strain rate in the respective directions**.

Therefore, **linear strain rate** component in the x direction

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}$$

and, linear strain rate component in y direction

$$\dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}$$

Rate of Deformation in the Fluid Element

Let us consider both the velocity component u and v are functions of x and y, i.e.,

$$u = u(x,y)$$

$$v = v(x,y)$$

Figure 8.3 represent the above condition

Observations from the figure:

- Point B has a relative displacement in y direction with respect to the point A.
- Point D has a relative displacement in x direction with respect to point A.
- The included angle between AB and AD changes.
- The fluid element suffers a continuous angular deformation along with the linear deformations in course of its motion.

Rate of Angular deformation:

The rate of angular deformation is defined as the **rate of change of angle** between the linear segments AB and AD which were initially perpendicular to each other.

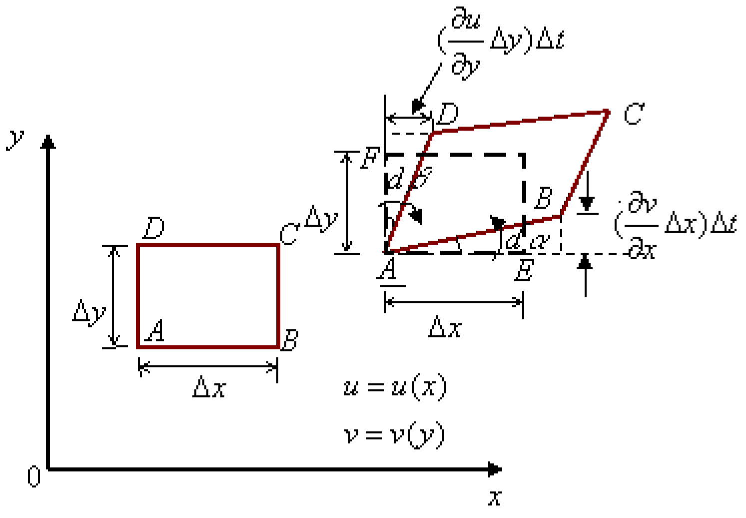

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Fig 8.3 Fluid element in translation with simultaneous linear and angular deformation rates

From the above figure rate of angular deformation,

$$\dot{\gamma}_{xy} = \left(\frac{d\alpha}{dt} + \frac{d\beta}{dt} \right) \tag{8.1}$$

From the geometry

$$d\alpha = \frac{\partial v}{\partial x} dt \tag{8.2a}$$

$$d\alpha = \lim_{\Delta t \to 0} \left(\frac{\frac{\partial v}{\partial x} \Delta x \, \Delta t}{\Delta x \left(1 + \frac{\partial u}{\partial x} \Delta t \right)} \right) = \frac{\partial v}{\partial x} dt$$

$$d\beta = \lim_{\Delta t \to 0} \left(\frac{\frac{\partial u}{\partial y} \Delta y \, \Delta t}{\Delta y \left(1 + \frac{\partial v}{\partial y} \Delta t \right)} \right) = \frac{\partial u}{\partial y} dt$$

$$d\beta = \frac{\partial u}{\partial y} dt \tag{8.2b}$$

Hence,

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{8.3}$$

Finally

$$\dot{\gamma}_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{8.4}$$

Rotation

Figure 8.3 represent the situation of rotation

Observations from the figure:

- The transverse displacement of B with respect to A and the lateral displacement of D with respect to A (Fig. 8.3) can be considered as the rotations of the linear segments AB and AD about A.
- This brings the concept of rotation in a flow field.

Definition of rotation at a point:

The rotation at a point is defined as the **arithmetic mean** of the **angular velocities** of two perpendicular linear segments meeting at that point.

Example: The angular velocities of AB and AD about A are

$\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ respectively.

Considering the **anticlockwise direction as positive**, the rotation at A can be written as,

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) \tag{8.5a}$$

or

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{8.5b}$$

The suffix z in ω represents the rotation about z-axis.

When $u = u\,(x, y)$ and $v = v\,(x, y)$ the **rotation and angular deformation of a fluid element exist simultaneously**.

Special case : Situation of pure Rotation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\,, \quad \dot{\gamma}_{xy} = 0 \quad \text{and} \quad \omega_z = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Observation:

- The linear segments AB and AD move with the same angular velocity (both in magnitude and direction).
- The included angle between them remains the same and no angular deformation takes place. This situation is known as **pure rotation.**

Vorticity

Definition: The vorticity Ω in its simplest form is defined as a vector which is equal to **two times the rotation vector**

$$\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{V} \tag{8.6}$$

For an **irrotational** flow, vorticity components are zero.

Vortex line:

If tangent to an imaginary line at a point lying on it is in the direction of the Vorticity vector at that point , the line is a **vortex line**.

The **general equation** of the **vortex line** can be written as,

$$\vec{\Omega}\times d\vec{s} = 0 \qquad (8.6b)$$

In a rectangular cartesian cartesian coordinate system, it becomes

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \qquad (8.6c)$$

where,

$$\Omega_x = 2\omega_x$$

$$\Omega_y = 2\omega_y$$

$$\Omega_z = 2\omega_z$$

Vorticity components as vectors:

The vorticity is actually an **anti symmetric tensor** and its three distinct elements transform like the components of a vector in cartesian coordinates.

This is the reason for which the vorticity components can be treated as vectors.

Existence of Flow

- A fluid must obey the law of conservation of mass in course of its flow as it is a material body.
- For a Velocity field to exist in a fluid continuum, the velocity components must obey the **mass conservation principle**.
- Velocity components which follow the mass conservation principle are said to constitute a possible fluid flow
- Velocity components violating this principle, are said to describe an impossible flow.
- The existence of a physically possible flow field is verified from the principle of conservation of mass.

The detailed discussion on this is deferred to the next chapter along with the discussion on principles of **conservation of momentum and energy**.

Exercise Problems - Chapter 3

1. The velocity field for a steady flow in a rectangular cartesian system is given by $\vec{V} = 18t\,\vec{i} + (3y+8)\vec{j} + 6z\vec{k}$ What is the path line of the particle which is at (5, 3, 4) at t = 1s ?

[$9(x+4) - \{In\ (3y+8) + Inz + 4.77\}^2 = 0$]

2. Verify whether the following flow fields are rotational. If so, determine the components of rotation about the coordinate axes

(i) $u = xyz, v = xz, w = \frac{1}{2}yz^2 - xy$, (ii) $u = xy, v = \frac{1}{2}\left(x^2 - y^2\right)$

(iii) $V_r = A/r, V_\theta = Br, V_z = 0$

[(i) $\omega_x = \frac{1}{2}\left(z^2 - 2x\right), \ \omega_y = \frac{y}{2}(x+1), \omega_z = \frac{z}{2}(1-x)$; (ii) irrotational; (iii) $\omega_z = B$]

3. For a steady two-dimensional incompressible flow through a nozzle, the velocity field is given by $\vec{V} = u_0\left(1 + 3x/L\right)\vec{i}$ where x is the distance along the axis of the nozzle from its inlet plane and L is the length of the nozzle. Find

(i) an expression of the acceleration of a particle flowing through the nozzle and

(ii) the time required for a fluid particle to travel from the inlet to the exit of the nozzle

[(i) $a_x = \frac{3u_0^2}{L}\left(1 + \frac{3x}{L}\right)$, (ii) $t = \frac{L}{3u_0} In\, 4$]

4. The velocity field for a steady two-dimensional flow in a cartesian coordinate system is given by $\vec{V} = by\,\vec{i} - ax\,\vec{j}$, where a and b are constants. Find the equation of stream line passing through the point $\left(1/\sqrt{2a}, 1/\sqrt{2b}\right)$. Find also the condition for irrotationality of the flow.

[$ax^2 + by^2 = 1, a + b = 0$]

5. Show that the velocity field given by $\vec{V} = (a - by - cz)\vec{i} + (d + bx - ez)\vec{j} + (f + cx + ey)\vec{k}$ of a fluid represents a rigid body motion.

6. For a flow field u is given by $e^{-x}\cosh y + 1$. (a) Find v, if it vanishes on $y = 0$
(b) Also find the stream function that will give these velocity components

[(a) $e^{-x}\sinh y$ (b) $\psi = e^{-x}\sinh y + y + const$]

7. A two-dimensional flow field is given by $u = 2y, v = x$. Find the velocity and acceleration in flow field at point A ($x = 3.5, y = 1.2$)

Velocity = $2..4\hat{i} + 3.5\hat{j}$

Acceleration= $7\hat{i} + 2.4\hat{j}$

Recap

In this course you have learnt the following

- Kinematics of fluid deals with the geometry of fluid motion. It characterizes the different types of motion and associated deformation rates of fluid element.

- The fluid motion is described by two methods, namely, Lagrangian method and Eulerian method. In the Lagrangian view, the velocity and other hydrodynamic parameters are specified for particles or elements of given identities, while, in the Eulerian view, these parameters are expressed as functions of location and time. The Lagrangian version of a flow field can be obtained from the integration of the set of equations describing the flow in the Eulerian version.

- A flow is defined to be steady when the hydrodynamic parameters and fluid properties at any point do not change with time. Flow in which any of these parameters changes with time is termed as unsteady. A flow may appear steady or unsteady depending upon the choice of coordinate axes. A flow is said to be uniform when no hydrodynamic parameter changes from point to point at any instant of time, or else the flow is non-uniform.

- The total derivative of velocity with respect to time is known as material or substantial acceleration, while the partial derivative of velocity with respect to time for a fixed location is known as temporal acceleration.
  Material acceleration = temporal acceleration + convective acceleration.

- A streamline at any instant of time is an imaginary curve or line in the flow field so that the tangent to the curve at any point represents the direction of the instantaneous velocity at that point. A path line is the trajectory of a fluid particle of a given identity. A streak line at any instant of time is the locus of temporary locations of all particles that have passed through a fixed point in the flow. In a steady flow, the streamlines, path lines and streak lines are identical.

- Flow parameters, in general, become functions of time and space coordinates. A one dimensional flow is that in which the flow parameters are functions of time and one space coordinate only.

- A fluid motion consists of translation, rotation and continuous deformation. In an uniform flow, the fluid elements are simply translated without any deformation or rotation. The deformation and rotation of fluid elements are caused by the variations in velocity components with the space coordinates. The linear deformation or strain rate is defined as the rate of change of length of a linear fluid element per unit original length. The rate of angular deformation at a point is defined as the rate of change of angle between two linear elements at that point which were initially perpendicular to each other.

The rotation at a point is defined as the arithmetic mean of the angular velocities of two perpendicular linear segments meeting at that point. The rotation of a fluid element in absence of any deformation is known as pure or rigid body rotation. When the components of rotation at all points in a flow become zero, the flow is said to be irrotational.

- The vorticity is actually an antisymmetric tensor but it is defined as a vector that equals to two times the rotation vector. Vorticity is zero for an irrotational flow.
- The existence of a physically possible flow field is verified from the principle of conservation of mass.

System

Definition

- ***System*:** A quantity of matter in space which is analyzed during a problem.
- **Surroundings:** Everything external to the system.
- **System Boundary:** A separation present between system and surrounding.

Classification of the system boundary:-

- Real solid boundary
- Imaginary boundary

The system boundary may be further classified as:-

- Fixed boundary or Control Mass System
- Moving boundary or Control Volume System

The choice of boundary depends on the problem being analyzed.

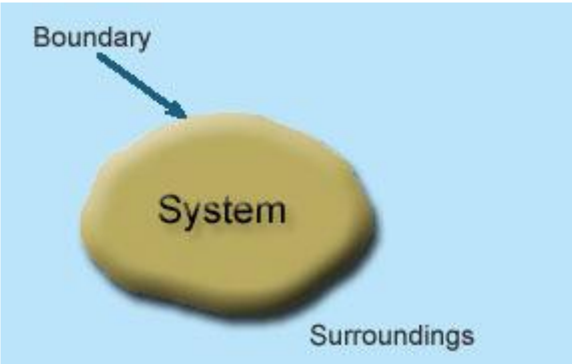
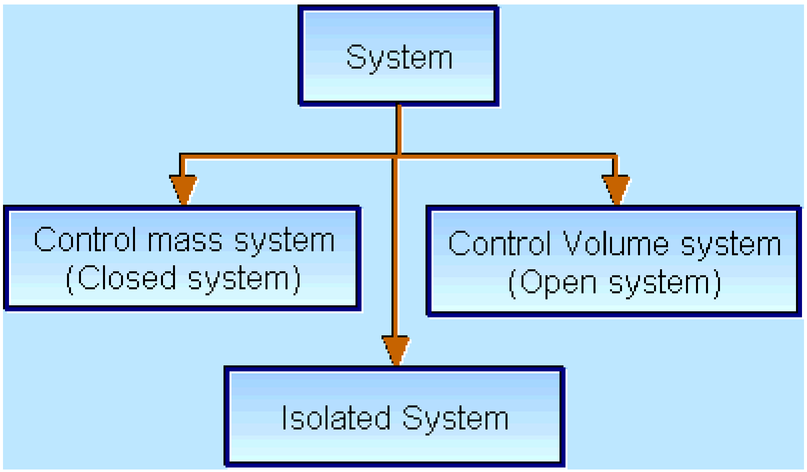


Fig 9.1 System and Surroundings

Classification of Systems



Types of System

Control Mass System (Closed System)

1. Its a system of **fixed mass** with **fixed identity**.
2. This type of system is usually referred to as **"closed system"**.

3. There is no mass transfer across the system boundary.
4. Energy transfer may take place into or out of the system.

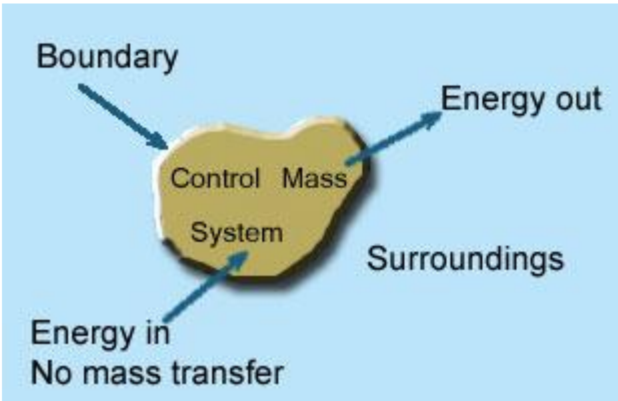


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Fig 9.2 A Control Mass System or Closed System

Control Volume System (Open System)

1. Its a system of **fixed volume**.
2. This type of system is usually referred to as **"open system"** or a **"control volume"**
3. Mass transfer can take place across a control volume.
4. Energy transfer may also occur into or out of the system.
5. A control volume can be seen as a fixed region across which mass and energy transfers are studied.
6. Control Surface- Its the boundary of a control volume across which the transfer of both mass and energy takes place.

7. The mass of a control volume (open system) may or may not be fixed.
8. When the net influx of mass across the control surface equals zero then the mass of the system is fixed and vice-versa.
9. The identity of mass in a control volume always changes unlike the case for a control mass system (closed system).
10. Most of the engineering devices, in general, represent an open system or control volume.

Example:-

- Heat exchanger - Fluid enters and leaves the system continuously with the transfer of heat across the system boundary.
- Pump - A continuous flow of fluid takes place through the system with a transfer of mechanical energy from the surroundings to the system.

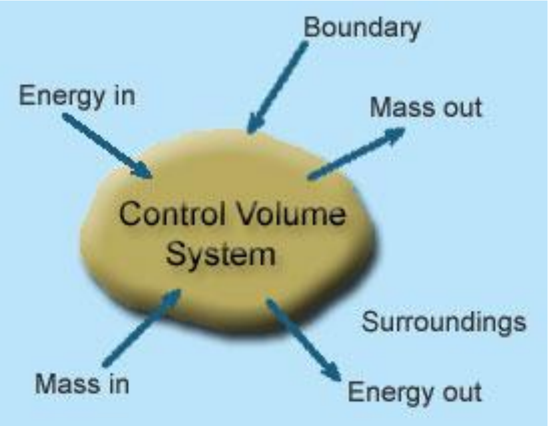

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Fig 9.3 A Control Volume System or Open System

Isolated System

1. Its a system of **fixed mass** with *same* **identity and fixed energy**.
2. No interaction of mass or energy takes place between the system and the surroundings.
3. In more informal words an isolated system is like a closed shop amidst a busy market.

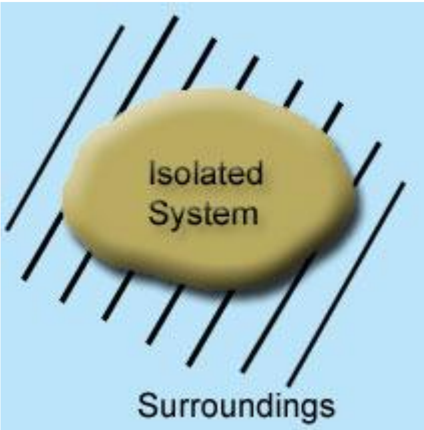

Fig 9.4 An Isolated System

Conservation of Mass - The Continuity Equation

Law of conservation of mass

The law states that *mass can neither be created nor be destroyed*. Conservation of mass is inherent to a control mass system (closed system).

- The mathematical expression for the above law is stated as:

$$\Delta m/\Delta t = 0, \quad \text{where } m = \text{mass of the system}$$

- For a control volume (Fig.9.5), the principle of conservation of mass is stated as

Rate at which mass enters = Rate at which mass leaves the region + Rate of accumulation of mass in the region

OR

Rate of accumulation of mass in the control volume
+ Net rate of mass efflux from the control volume = 0 (9.1)

Continuity equation

The above statement expressed analytically in terms of velocity and density field of a flow is known as the **equation of continuity**.

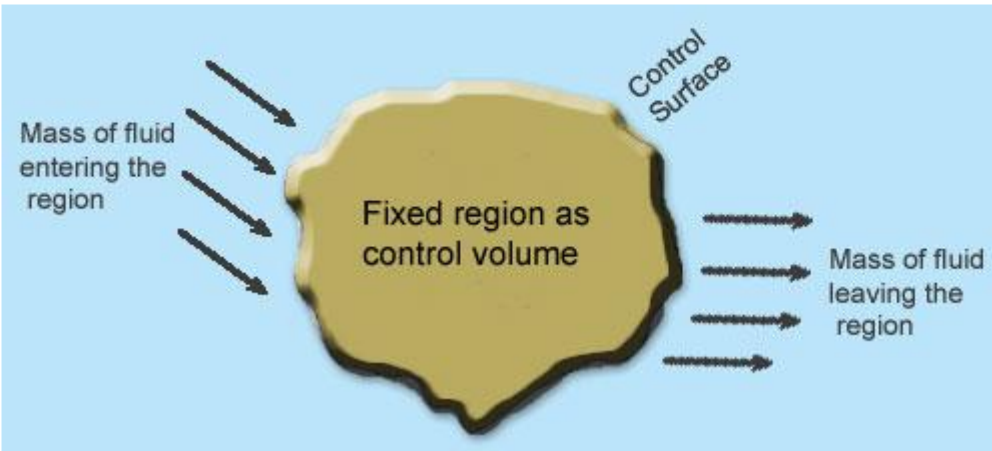

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Fig 9.5 A Control Volume in a Flow Field

Continuity Equation - Differential Form

Derivation

1. The point at which the continuity equation has to be derived, is enclosed by an elementary control volume.
2. The influx, efflux and the rate of accumulation of mass is calculated across each surface within the control volume.

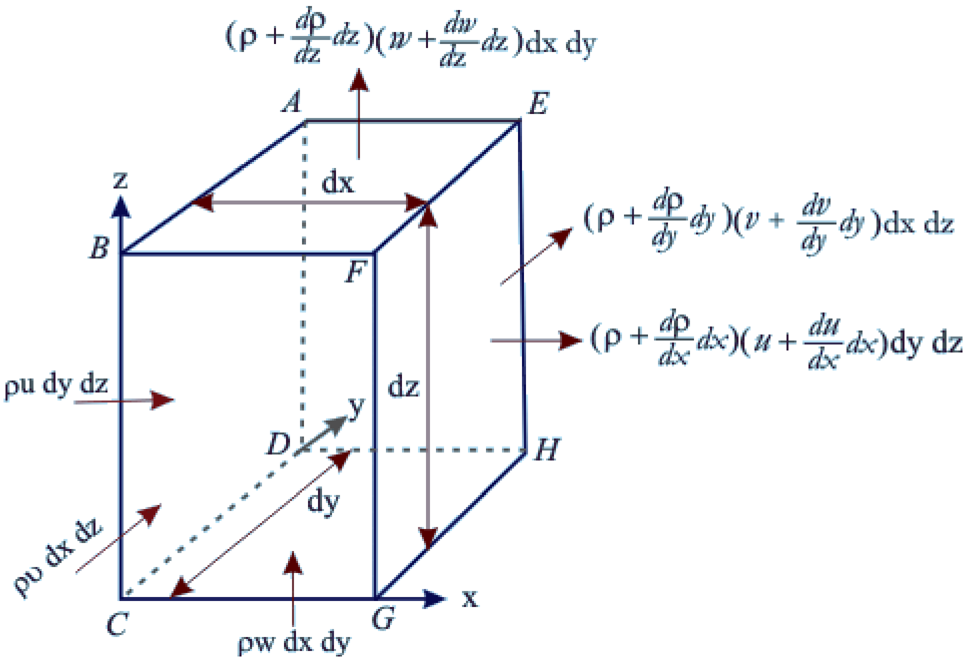


Fig 9.6 A Control Volume Appropriate to a Rectangular Cartesian Coordinate System

Consider a rectangular parallelopiped in the above figure as the control volume in a rectangular cartesian frame of coordinate axes.

- Net efflux of mass along x -axis must be the excess outflow over inflow across faces normal to x -axis.
- Let the fluid enter across one of such faces ABCD with a velocity u and a density ρ.The velocity and density with which the fluid will leave the face EFGH will be $u+\frac{\partial u}{\partial x}dx$ and $\rho+\frac{\partial \rho}{\partial x}dx$ respectively (neglecting the higher order terms in δx).
- Therefore, the rate of mass entering the control volume through face ABCD = $\rho u\ dy\ dz$.
- The rate of mass leaving the control volume through face EFGH will be

$$=\left(\rho+\frac{\partial \rho}{\partial x}dx\right)\left(u+\frac{\partial u}{\partial x}dx\right)dydz$$

$$= \left(\rho u + \frac{\partial}{\partial x}(\rho u) dx \right) dydz$$ (neglecting the higher order terms in dx)

- Similarly influx and efflux take place in all y and z directions also.
- Rate of accumulation for a point in a flow field

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \rho (d\forall) = \frac{\partial \rho}{\partial t} d\forall$$

- Using, Rate of influx = Rate of Accumulation + Rate of Efflux

$$\rho udydz + \rho vdxdz + \rho wdxdy = \frac{\partial \rho}{\partial t} d\forall + (\rho + \frac{\partial \rho}{\partial x} dx)(u + \frac{\partial u}{\partial x} dx) dydz$$
$$+ (\rho + \frac{\partial \rho}{\partial y} dy)(v + \frac{\partial v}{\partial y} dy) dxdz + (\rho + \frac{\partial \rho}{\partial z} dz)(w + \frac{\partial w}{\partial z} dz) dxdy$$

- Transferring everything to right side

$$0 = \left[\left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right) + \left(\rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} \right) + \left(\rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z} \right) \right] dxdydz + \left(\frac{\partial \rho}{\partial t} \right) d\forall$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] d\forall = 0 \qquad (9.2)$$

This is the Equation of Continuity for a compressible fluid in a rectangular cartesian coordinate system.

Continuity Equation - Vector Form

- The continuity equation can be written in a vector form as

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) . \left[\rho u \hat{i} + \rho v \hat{j} + \rho w \hat{k} \right] = 0$$

or,

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{V}) = 0 \qquad (9.3)$$

where $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ is the velocity of the point

- In case of a **steady flow**,

$$\frac{\partial \rho}{\partial t} = 0$$

- Hence Eq. (9.3) becomes

$$\nabla.(\rho\vec{V}) = 0 \qquad (9.4)$$

- In a rectangular cartesian coordinate system

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (9.5)$$

- Equation (9.4) or (9.5) represents the **continuity equation for a steady flow**.
- In case of an incompressible flow,
  ρ = constant

- Hence,

$$\frac{\partial \rho}{\partial t} = 0$$

- Moreover

$$\nabla.(\rho\vec{V}) = \rho\nabla.(\vec{V})$$

- Therefore, the **continuity equation for an incompressible flow** becomes

$$\nabla.(\vec{V}) = 0 \qquad (9.6)$$

$$\text{or,} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (9.7)$$

- In cylindrical polar coordinates eq.9.7 reduces to

$$\frac{1}{R}\frac{\partial}{\partial R}(R^2 V_R) + \frac{1}{\sin\varphi}\frac{\partial V_\theta}{\partial \theta} + \frac{1}{\sin\varphi}\frac{\partial(V_\varphi \sin\varphi)}{\partial \varphi} = 0$$

- Eq. (9.7) can be written in terms of the strain rate components as

$$\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = 0 \tag{9.8}$$

Continuity Equation - A Closed System Approach

We know that the conservation of mass is inherent to the definition of a closed system as Dm/Dt = 0 (where m is the mass of the closed system).

However, the general form of continuity can be derived from the basic equation of mass conservation of a system.

Derivation :-

Let us consider an elemental closed system of volume V and density ρ.

$$\frac{Dm}{Dt} = 0 \Rightarrow \frac{D}{Dt}(\rho\Delta\forall) = 0$$

$$\Rightarrow \Delta\forall\frac{D\rho}{Dt} + \rho\frac{D(\Delta\forall)}{Dt} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \frac{\rho}{\Delta\forall}\frac{D(\Delta\forall)}{Dt} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \frac{\rho}{\Delta\forall}\frac{D\Delta\forall}{Dt} = 0$$

Now $\frac{1}{\Delta\forall}\frac{D\Delta\forall}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ (dilation per unit volume)

$$\Rightarrow \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \left(u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x}\right) + \left(v\frac{\partial\rho}{\partial y} + \rho\frac{\partial v}{\partial y}\right) + \left(w\frac{\partial\rho}{\partial z} + \rho\frac{\partial w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

In vector notation we can write this as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V}) = 0$$

The above equations are same as that formulated from Control Volume approach

Stream Function

Let us consider a two-dimensional incompressible flow parallel to the x - y plane in a rectangular cartesian coordinate system. The flow field in this case is defined by

$$u = u(x, y, t)$$
$$v = v(x, y, t)$$
$$w = 0$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (10.1) \qquad\qquad (10.1)$$

If a function $\psi(x, y, t)$ is defined in the manner

$$u = \frac{\partial \psi}{\partial y} \qquad (10.2a)$$

$$v = -\frac{\partial \psi}{\partial x} \qquad (10.2b)$$

so that it automatically satisfies the equation of continuity (Eq. (10.1)), then the function is known as stream function.
Note that for a **steady flow, ψ is a function of two variables x and y only.**

Constancy of ψ on a Streamline

Since ψ is a point function, it has a value at every point in the flow field. Thus a change in the stream function ψ can be written as

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

The equation of a streamline is given by

$$\frac{u}{dx} = \frac{v}{dy} \qquad \text{or} \qquad udy - vdx = 0 (\text{since tangent dy/dx equals the velocity v/u})$$

It follows that dψ = 0 on a streamline.This implies the value of ψ is constant along a streamline. Therefore, the equation of a streamline can be expressed in terms of stream function as

$$\psi(x, y) = \text{constant} \qquad (10.3)$$

Once the function ψ is known, streamline can be drawn by joining the same values of ψ in the flow field.

Stream function for an irrotational flow

In case of a two-dimensional irrotational flow

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \qquad \Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)=0$$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x^2}-\frac{\partial^2 \psi}{\partial y^2}=0 \qquad \Rightarrow \frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0$$

$$\Rightarrow \psi_{xx}+\psi_{yy}=0$$

$$\Rightarrow \nabla^2 \psi = 0$$

Conclusion drawn:For an irrotational flow, stream function satisfies the Laplace's equation

Physical Significance of Stream Funtion ψ

Figure 10.1 illustrates a two dimensional flow.

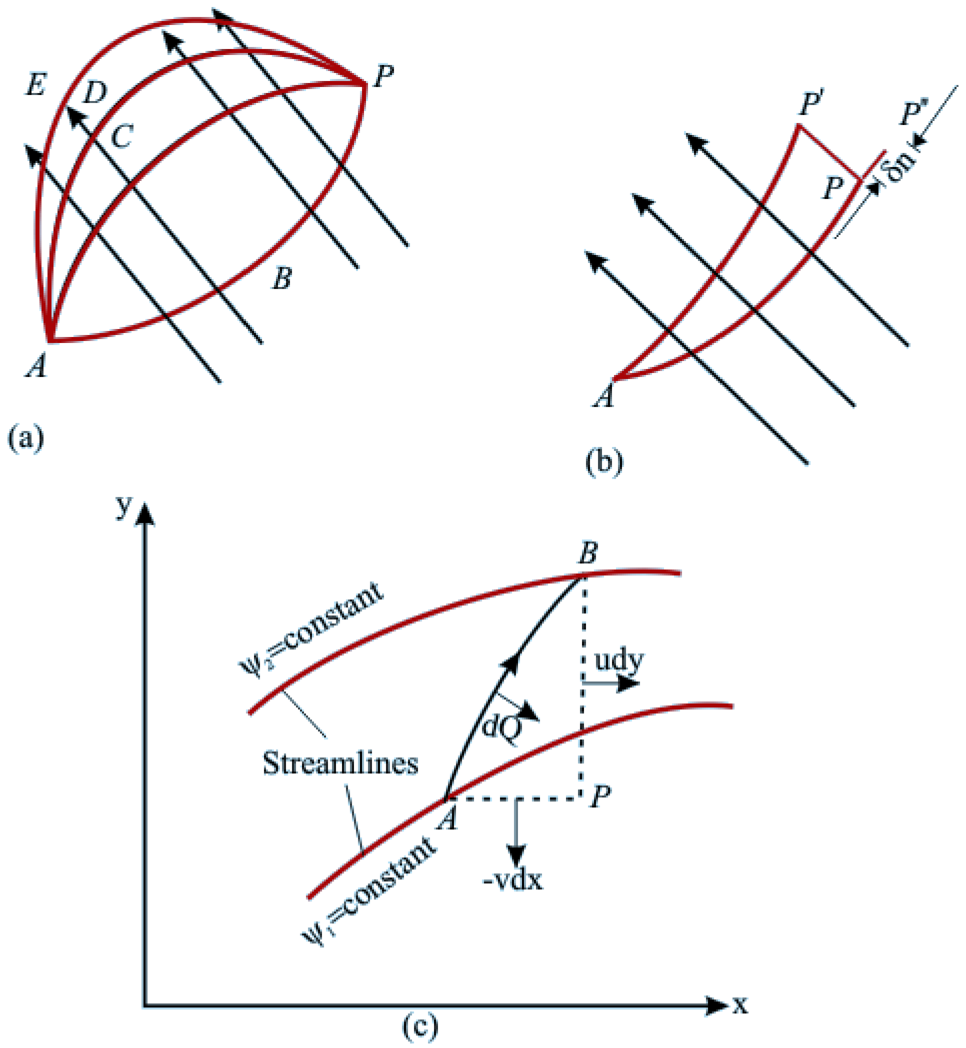


Fig 10.1 Physical Interpretation of Stream Function

Let A be a fixed point, whereas P be any point in the plane of the flow. The points A and P are joined by the arbitrary lines ABP and ACP. For an incompressible steady flow, the volume flow rate across ABP into the space ABPCA (considering a unit width in a direction perpendicular to the plane of the flow) must be equal to that across ACP. A number of different paths connecting A and P (ADP, AEP,...) may be imagined but the volume flow rate across all the paths would be the same. This implies that the **rate of flow across any curve between A and P depends only on the end points A and P.**

Since A is fixed, the rate of flow across ABP, ACP, ADP, AEP (any path connecting A and P) is a function only of the position P. This function is known as the stream function ψ.

The value of ψ at P represents the volume flow rate across any line joining P to A.
The value of ψ at A is made arbitrarily zero. If a point P' is considered (Fig. 10.1b),PP' being along a streamline, then the rate of flow across the curve joining A to P' must be the same as across AP, since, by the definition of a streamline, there is no flow across PP'

The value of ψ thus remains same at P' and P. Since P' was taken as any point on the streamline through P, it follows that ψ is constant along a streamline. Thus the flow may be represented by a series of streamlines at equal increments of ψ.

In fig (10.1c) moving from A to B net flow going past the curve AB is

$$\int dQ = \int_A^B (udy - vdx)$$

$$= \int_A^B (\psi_y dy + \psi_x dx) \qquad \left[\text{since } u = \psi_y \text{ and } v = -\psi_x\right]$$

$$\int dQ = \int_A^B d\psi$$

$$\therefore Q = \int_A^B d\psi = \psi_2 - \psi_1$$

The stream function, in a polar coordinate system is defined as

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad V_\theta = -\frac{\partial \psi}{\partial r}$$

The expressions for V_r and V_θ in terms of the stream function automatically satisfy the equation of continuity given by

$$\frac{\partial}{\partial r}(V_r r) + \frac{\partial}{\partial \theta}(V_\theta) = 0$$

Stream Function in Three Dimensional and Compressible Flow

Stream Function in Three Dimensional Flow

In case of a three dimensional flow, it is not possible to draw a streamline with a single stream function.

An axially symmetric three dimensional flow is similar to the two-dimensional case in a sense that the flow field is the same in every plane containing the axis of symmetry.

The equation of continuity in the cylindrical polar coordinate system for an incompressible flow is given by the following equation

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

For an axially symmetric flow (the axis r = 0 being the axis of symmetry), the term $\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$ =0 ,and simplified equation is satisfied by functions defined as

$$rV_r = -\frac{\partial \psi}{\partial z}, \qquad rV_z = \frac{\partial \psi}{\partial r} \qquad (10.4)$$

The function ψ , defined by the Eq.(10.4) in case of a three dimensional flow with an axial symmetry, is called the **stokes stream function.**

Stream Function in Compressible Flow

For compressible flow, stream function is related to mass flow rate instead of volume flow rate because of the extra density term in the continuity equation (unlike incompressible flow)

The continuity equation for a steady two-dimensional compressible flow is given by

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

Hence a stream function ψ is defined which will satisfy the above equation of continuity as

$$\rho u = \rho_o \frac{\partial \psi}{\partial y}$$

$$\rho v = -\rho_o \frac{\partial \psi}{\partial x} \qquad (10.5)$$

[where ρ_0 is a reference density**]**

ρ_0 is used to retain the unit of ψ same as that in the case of an incompressible flow. Physically, the difference in stream function between any two streamlines multiplied by the reference density ρ_0 will give the mass flow rate through the passage of unit width formed by the streamlines.

Continuity Equation: Integral Form

Let us consider a control volume $\forall$ bounded by the control surface S. The efflux of mass across the control surface S is given by

$$\iint_S \rho \vec{V}.d\vec{A}$$

where $\vec{V}$ is the velocity vector at an elemental area(which is treated as a vector by considering its positive direction along the normal drawn outward from the surface).

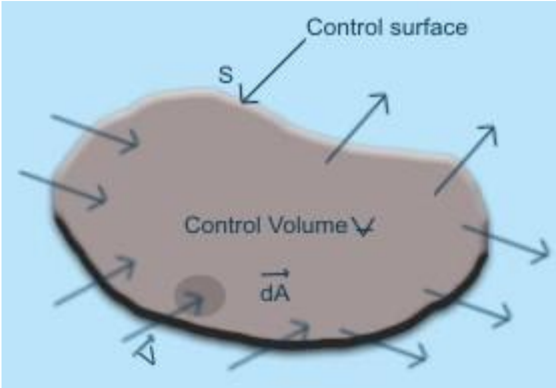


Fig 10.2 A Control Volume for the Derivation of Continuity Equation (integral form)

The rate of mass accumulation within the control volume becomes

$$\frac{\partial}{\partial t}\iiint_{V} \rho . d\forall$$

where d∀ is an elemental volume, ρ is the density and ∀is the total volume bounded by the control surface S. Hence, the continuity equation becomes (according to the statement given by Eq. (9.1))

$$\frac{\partial}{\partial t}\iiint_{V} \rho . d\forall + \iint_{S} \rho \vec{V} . d\vec{A} = 0 \qquad (10.6) \tag{10.6}$$

The second term of the Eq. (10.6) can be converted into a volume integral by the use of the Gauss divergence theorem as

$$\iint_{S} \rho \vec{V} . d\vec{A} = \iiint_{V} \nabla . (\rho \vec{V}) d\forall$$

Since the volume ∀does not change with time, the sequence of differentiation and integration in the first term of Eq.(10.6) can be interchanged.
Therefore Eq. (10.6) can be written as

$$\iiint_{V} \left[\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{V}) \right] d\forall = 0 \tag{10.7}$$

Equation (10.7) is valid for any arbitrary control volume irrespective of its shape and size. So we can write

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{V}) = 0 \tag{10.8}$$

Conservation of Momentum: Momentum Theorem

In Newtonian mechanics, the conservation of momentum is defined by Newton’s second law of motion.

Newton's Second Law of Motion

- The rate of change of momentum of a body is proportional to the impressed action and takes place in the direction of the impressed action.
- If a force acts on the body ,linear momentum is implied.
- If a torque (moment) acts on the body,angular momentum is implied.

Reynolds Transport Theorem

A study of fluid flow by the Eulerian approach requires a mathematical modeling for a control volume either in differential or in integral form. Therefore the physical statements of the principle of conservation of mass, momentum and energy with reference to a control volume become necessary.

This is done by invoking a theorem known as the Reynolds transport theorem which relates the control volume concept with that of a control mass system in terms of a general property of the system.

Statement of Reynolds Transport Theorem

The theorem states that "the time rate of increase of property N within a control mass system is equal to the time rate of increase of property N within the control volume plus the net rate of efflux of the property N across the control surface".

Equation of Reynolds Transport Theorem

After deriving Reynolds Transport Theorem according to the above statement we get

$$\left(\frac{dN}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho \, d\forall + \iint_{CS} \eta\rho\vec{V}.\,d\vec{A} \qquad (10.9)$$

In this equation

N - flow property which is transported

η - intensive value of the flow property

Application of the Reynolds Transport Theorem to Conservation of Mass and Momentum

Conservation of mass The constancy of mass is inherent in the definition of a control mass system and therefore we can write

$$\left(\frac{dm}{dt}\right)_{CMS} = 0 \qquad (10.13a)$$

To develop the analytical statement for the conservation of mass of a control volume, the Eq. (10.11) is used with N = m (mass) and η = 1 along with the Eq. (10.13a).

This gives

$$\frac{\partial}{\partial t}\iiint_{CV} \rho d\forall + \iint_{CS} \rho(\vec{V}_r . d\vec{A}) = 0 \qquad (10.13b)$$

The Eq. (10.13b) is identical to Eq. (10.6) which is the integral form of the continuity equation derived in earlier section. At steady state, the first term on the left hand side of Eq. (10.13b) is zero. Hence, it becomes

$$\iint_{CS} \rho\left(\vec{V}_r . d\vec{A}\right) = 0 \qquad (10.13c)$$

Conservation of Momentum or Momentum Theorem The principle of conservation of momentum as applied to a control volume is usually referred to as the momentum theorem.

Linear momentum The first step in deriving the analytical statement of linear momentum theorem is to write the Eq. (10.11) for the property N as the linear - momentum $m\vec{V}$ and accordingly η as the velocity $(\vec{V})$. Then it becomes

$$\frac{d}{dt}(m\vec{V})_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}\rho d\forall + \iint_{CS} \vec{V}\rho(\vec{V}_r . d\vec{A}) \qquad (10.14)$$

The velocity $(\vec{V})$ defining the linear momentum in Eq. (10.14) is described in an inertial frame of reference. Therefore we can substitute the left hand side of Eq. (10.14) by the external forces $\sum \vec{F}$ on the control mass system or on the coinciding control volume by the direct application of Newton’s law of motion. This gives

$$\sum \vec{F} = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}\rho d\forall + \iint_{CS} \vec{V}\rho(\vec{V}_r . d\vec{A}) \qquad (10.15)$$

This Equation is the analytical statement of linear momentum theorem.

In the analysis of finite control volumes pertaining to practical problems, it is convenient to describe all fluid velocities in a frame of coordinates attached to the control volume. Therefore, an equivalent form of Eq.(10.14) can be obtained, under the situation, by substituting N as and accordingly η as $\vec{V}_r$, we get

$$\frac{d}{dt}(m\vec{V}_r)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}_r\rho d\forall + \iint_{CS} \vec{V}_r\rho(\vec{V}_r . d\vec{A}) \qquad (10.16)$$

With the help of the Eq. (10.12) the left hand side of Eq. can be written as

$$\frac{d}{dt}(m\vec{V}_r)_{CMS} = m\left(\frac{d\vec{V}_r}{dt}\right)_{CMS}$$

$$= m\frac{d}{dt}(\vec{V} - \vec{V}_C)_{CMS}$$

$$= m\left(\frac{d\vec{V}}{dt}\right)_{CMS} - m\vec{a}_c$$

where $a_c\left(= d\vec{V}_c / dt\right)$ is the rectilinear acceleration of the control volume (observed in a fixed coordinate system) which may or may not be a function of time. From Newton's law of motion

$$m\left(\frac{d\vec{V}}{dt}\right)_{CMS} = \sum \vec{F}$$

Therefore, $$m\left(\frac{d\vec{V}_r}{dt}\right)_{CMS} = \sum \vec{F} - ma_c \qquad (10.17)$$

The Eq. (10.16) can be written in consideration of Eq. (10.17) as

$$\sum \vec{F} - ma_c = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}_r \rho d\forall + \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18a)$$

At steady state, it becomes

$$\sum \vec{F} - ma_c = \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18b)$$

In case of an inertial control volume (which is either fixed or moving with a constant rectilinear velocity), $\vec{a}_c = 0$ and hence Eqs (10.18a) and (10.18b) becomes respectively

$$\sum \vec{F} = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}_r \rho d\forall + \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18c)$$

and $$\sum \vec{F} = \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18d)$$

The Eqs (10.18c) and (10.18d) are the useful forms of the linear momentum theorem as applied to an inertial control volume at unsteady and steady state respectively, while the Eqs (10.18a) and (10.18b) are the same for a non-inertial control volume having an arbitrary rectilinear acceleration.

In general, the external forces $\sum \vec{F}$ in Eqs (10.14, 10.18a to 10.18c) have two components - the body force and the surface force. Therefore we can write

$$\sum \vec{F} = \iiint_{CV} \vec{F}_B d\forall + \vec{F}_S \qquad (10.18e)$$

where $\vec{F}_B$ is the body force per unit volume and $\vec{F}_s$ is the area weighted surface force.

Angular Momentum

The angular momentum or moment of momentum theorem is also derived from Eq.(10.10) in consideration of the property N as the angular momentum and accordingly η as the angular momentum per unit mass. Thus,

$$\frac{d}{dt}\left(A_{CMS}\right) = \frac{\partial}{\partial t}\iiint_{CV} \rho(\vec{r}\times\vec{V}_r)d\forall + \iint_{CS} (\vec{r}\times\vec{V}_r)\rho(\vec{V}_r.d\vec{A}) \qquad (10.19)$$

where $A_{\text{Control mass system}}$ is the **angular momentum of the control mass system**. . It has to be noted that the origin for the angular momentum is the origin of the position vector $\vec{r}$

The term on the left hand side of Eq.(10.19) is the time rate of change of angular momentum of a control mass system, while the first and second terms on the right hand side of the equation are the time rate of increase of angular momentum within a control volume and rate of net efflux of angular momentum across the control surface.

The velocity $(\vec{V})$ defining the angular momentum in Eq.(10.19) is described in an inertial frame of reference.Therefore, the term $d/dt\left(A_{CMS}\right)$ can be substituted by the net moment ΣM applied to the system or to the coinciding control volume. Hence one can write Eq. (10.19) as

$$\sum M = \frac{\partial}{\partial t}\iiint_{CV} \rho(\vec{r}\times\vec{V})d\forall + \iint_{CS} (\vec{r}\times\vec{V})\rho(\vec{V}_r.d\vec{A}) \qquad (10.20a)$$

At steady state

$$\frac{\partial}{\partial t}\iiint_{CV}\rho(\vec{r}\ X\ \vec{V})d\forall = 0$$

$$\sum M = \iint_{CS}(\vec{r}\times\vec{V})\rho(\vec{V}_r.d\vec{A}) \tag{10.20b}$$

Analysis Of Finite Control Volumes - the application of momentum theorem

We'll see the application of momentum theorem in some practical cases of inertial and non-inertial control volumes.

Inertial Control Volumes
Applications of momentum theorem for an inertial control volume are described with reference to three distinct types of practical problems, namely

- Forces acting due to internal flows through expanding or reducing pipe bends.
- Forces on stationary and moving vanes due to impingement of fluid jets.
- Jet propulsion of ship and aircraft moving with uniform velocity.

Non-inertial Control Volume
A good example of non-inertial control volume is a rocket engine which works on the principle of jet propulsion.

We shalll discuss each example seperately in the following slides.

Forces due to Flow Through Expanding or Reducing Pipe Bends

Let us consider, a fluid flow through an expander shown in Fig. 11.1a below. The expander is held in a vertical plane. The inlet and outlet velocities are given by V_1 and V_2 as shown in the figure. The inlet and outlet pressures are also prescribed as p_1 and p_2. The velocity and pressure at inlet and at outlet sections are assumed to be uniform. The problem is usually posed for the estimation of the force required at the expander support to hold it in position.

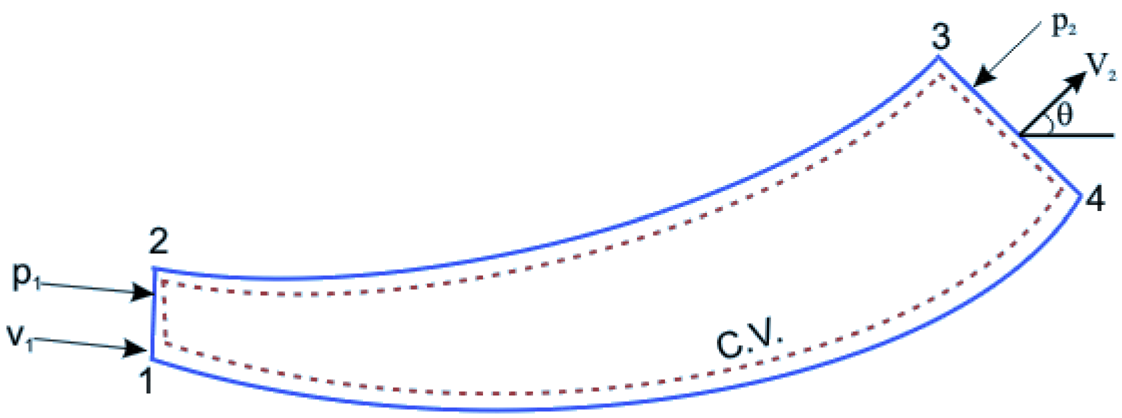


Fig 11.1a   Flow of a fluid through an expander

For the solution of this type of problem, a control volume is chosen to coincide with the interior of the expander as shown in Fig. 11.1a. The control volume being constituted by areas 1-2, 2-3, 3-4, and 4-1 is shown separately in Fig.11.1b.

The external forces on the fluid over areas 2-3 and 1-4 arise due to net efflux of linear momentum through the interior surface of the expander. Let these forces be F_x and F_y. Since the control volume 1234 is stationary and at a steady state, we apply Eq.(10.18d) and have for x and y components

$$\dot{m}V_2\cos\theta - \dot{m}V_1 = p_1A_1 - p_2A_2\cos\theta + F_x \tag{11.1a}$$

and $$\dot{m}V_2\sin\theta - 0 = -p_2A_2\sin\theta + F_y - Mg \tag{11.1b}$$

or, $$F_x = \dot{m}(V_2\cos\theta - V_1) + p_2A_2\cos\theta - p_1A_1 \tag{11.2a}$$

and $$F_y = \dot{m}V_2\sin\theta + p_2A_2\sin\theta + Mg \tag{11.2b}$$

where $\dot{m}$ = mass flow rate through the expander. Analytically it can be expressed as

$$\dot{m} = \rho A_1V_1 = \rho A_2V_2$$

where A_1 and A_2 are the cross-sectional areas at inlet and outlet of the expander and the flow is considered to be incompressible.
M represents the mass of fluid contained in the expander at any instant and can be expressed as

$M = \rho\forall$ where $\forall$ is the internal volume of the expander.

Thus, the forces F_x and F_y acting on the control volume (Fig. 11.1b) are exerted by the expander. According to Newton's third law, the expander will experience the forces R_x ($= -F_x$) and R_y ($= -F_y$) in the x and y directions respectively as shown in the free body diagram of the expander. in fig 11.1c.

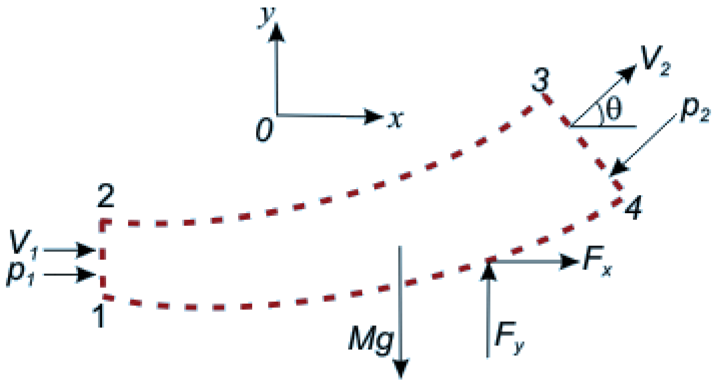


Fig 11.1b   Control Volume Comprising the fluid

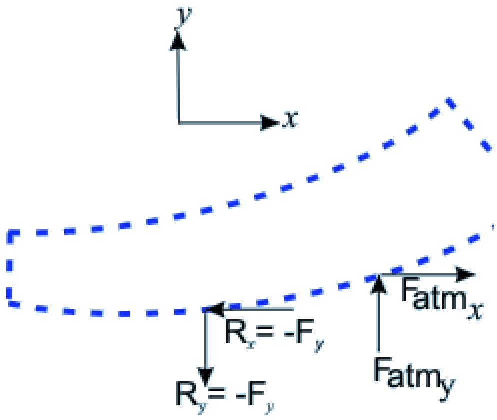


Fig 11.1c   Free Body Diagram of the

contained in the expander at any instant Expander

The expander will also experience the atmospheric pressure force on its outer surface. This is shown separately in Fig. 11.2.

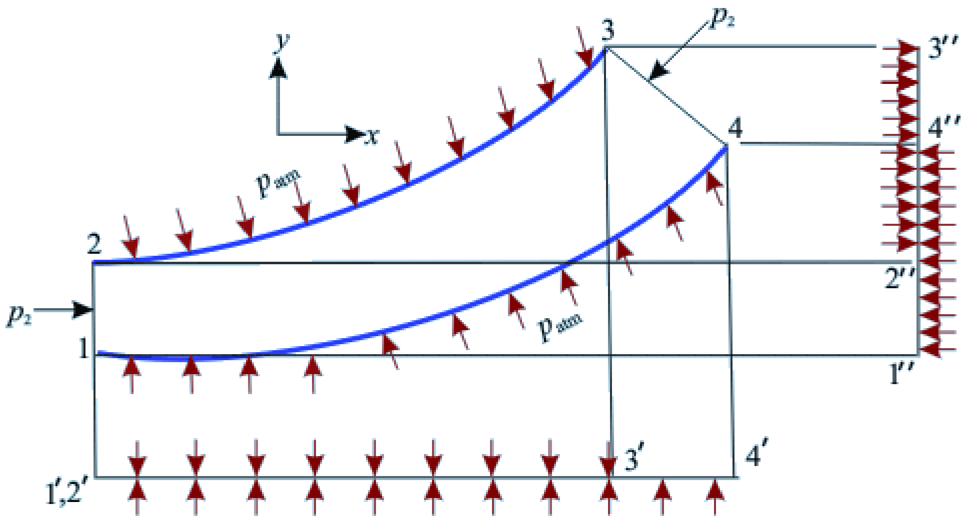


Fig 11.2 Effect of atmospheric pressure on the expander

From Fig.11.2 the net x and y components of the atmospheric pressure force on the expander can be written as

$$F_{atm-x} = p_{atm} A_2 \cos\theta - p_{atm} \cdot A_1$$

$$F_{atm-y} = p_{atm} \cdot A_2 \sin\dot{\theta}$$

The net force on the expander is therefore,

$$E_x = R_x + F_{atm-x} = -F_x + F_{atm-x} \quad (11.3a)$$

$$E_y = R_y + F_{atm-y} = -F_y + F_{atm-y} \quad (11.3b)$$

or,

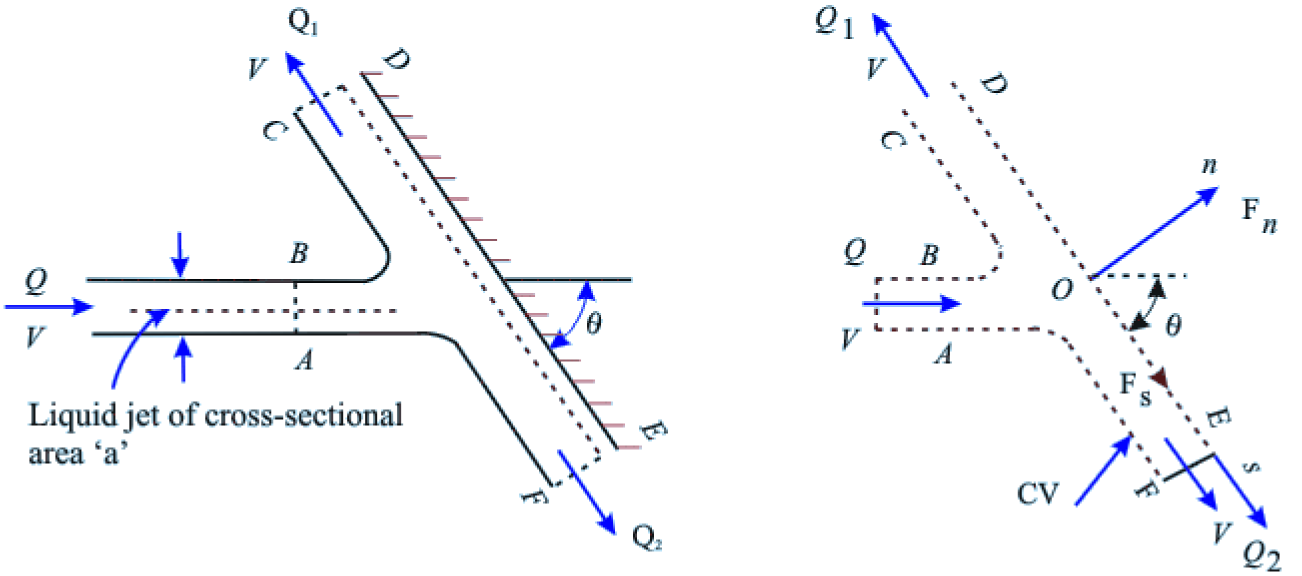
$$E_x = -\dot{m}(V_2 \cos\theta - V_1) - (p_2 - p_{atm})A_2 \cos\theta + (p_1 - p_{atm})A_1 \quad (11.4a)$$

$$E_y = -\dot{m}V_2 \sin\theta - (p_2 - p_{atm})A_2 \sin\theta - Mg \quad (11.4b)$$

Note: At this stage that if F_x and F_y are calculated from the Eqs (11.2a) and (11.2b) with p_1 and p_2 as the gauge pressures instead of the absolute ones the net forces on the expander E_x and E_y will respectively be equal to $-F_x$ and $-F_y$.

Dynamic Forces on Plane Surfaces due to the Impingement of Liquid Jets

Force on a stationary surface Consider a stationary flat plate and a liquid jet of cross sectional area "a" striking with a velocity V at an angle θ to the plate as shown in Fig. 11.3a.



(a) Jet striking on a stationary plate (b) The appropriate control volume

Fig 11.3 Impingement of liquid Jets on a Stationary Flat Plate

To calculate the force required to keep the plate stationary, a control volume ABCDEFA (Fig. 11.3a) is chosen so that the control surface DE coincides with the surface of the plate. The control volume is shown separately as a free body in Fig. 11.3b. Let the volume flow rate of the incoming jet be Q and be divided into Q_1 and Q_2 gliding along the surface (Fig. 11.3a) with the same velocity V since the pressure throughout is same as the atmospheric pressure, the plate is considered to be frictionless and the influence of a gravity is neglected (i.e. the elevation between sections CD and EF is negligible).

Coordinate axes are chosen as *0s* and *0n* along and perpendicular to the plate respectively. Neglecting the viscous forces. (the force along the plate to be zero),the momentum conservation of the control volume ABCDEFA in terms of *s* and *n* components can be written from Eq.(10.18d) as

$$F_s = 0 = \rho Q_2 V + \rho Q_1(-V) - \rho Q V \cos\theta \tag{11.5a}$$

and

$$F_n = 0 - \rho Q (V \sin\theta) \tag{11.5b}$$

where F_s and F_n are the forces acting on the control volume along *0s* and *0n* respectively, From continuity,

$$Q = Q_1 + Q_2 \tag{11.6}$$

With the help of Eqs (11.5a) and (11.6), we can write

$$Q_1 = \frac{Q}{2}(1 - \cos\theta) \tag{11.7a}$$

$$Q_2 = \frac{Q}{2}(1 + \cos\theta) \tag{11.7b}$$

The net force acting on the control volume due to the change in momentum of the jet by the plate is F_n along the direction "*On*" and is given by the Eq. (11.7b) as

$$F_n = -\rho Q V \sin\theta \tag{11.7c}$$

Hence, according to Newton's third law, the force acting on the plate is

$$F_p = -F_n = \rho Q V \sin\theta \tag{11.8}$$

If the cross-sectional area of the jet is "a", then the volume flow rate Q striking the plate can be written as Q = aV. Equation (11.8) then becomes

$$F_p = \rho a V^2 \sin\theta \tag{11.9}$$

Stationary vane problem

Consider a jet that is deflected by a stationary vane, such as is given in Fig. 11.4. If the jet speed and diameter are 25 m/s and 25 cm, respectively and jet is deflected 60^0, what force is exerted by the jet on the vane?

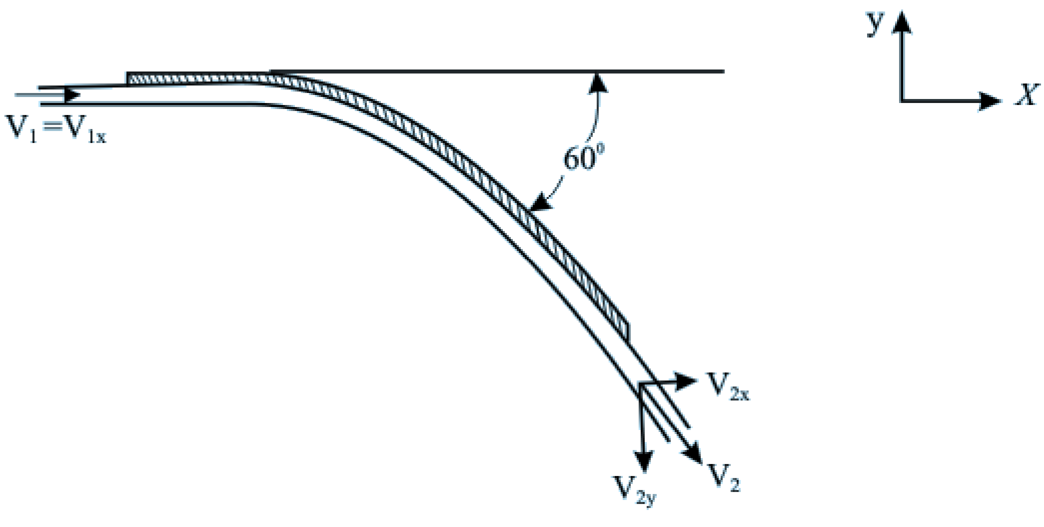


Fig 11.4

First solve for F_x , the x-component of force of the vane on the jet -

$$F_x = \rho Q (V_{2x} - V_{1x})$$

Here, the final velocity in the x-direction is given as

$$V_{2x} = 25 \cos 60^0$$

Hence,

$$V_{2x} = 25 \times 0 \cdot 5 \quad = 12 \cdot 5 \; m/s$$

also,

$$V_{1x} = 25 \text{ m/s}$$

and

$$Q = V_1 A_1 = 25 m/s \times \frac{\pi \times (0 \cdot 25)^2}{4} m^2 = 1 \cdot 227 m^3/s$$

Therefore,

$$\begin{aligned} F_x &= 1000 \; kg/m^3 \times 1 \cdot 227 \; m^3/s \times (12 \cdot 5 - 25) \; m/s \\ &= -15 \cdot 3398 \; KN \end{aligned}$$

similarly determined, the y-component of force on the jet is

$$\begin{aligned} F_y &= 1000 \; kg/m^3 \times 1 \cdot 227 \; m^3/s \times (-21 \cdot 65 - 0) \; m/s \\ &= -26 \cdot 5646 \; KN \end{aligned}$$

Then the force on the vane will be the reactions to the forces of the vane on the jet, or

$$F_x = +15 \cdot 3398 \; KN$$

$$F_y = +26 \cdot 5646 \; KN$$

Force on a moving surface

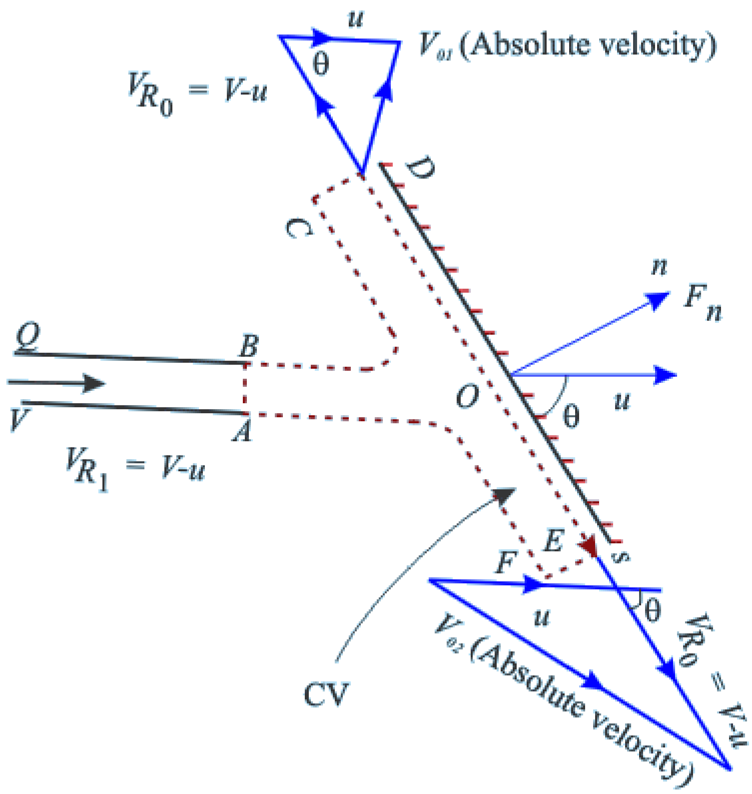


Fig 11.5 Impingement of liquid jet on a moving flat plate

If the plate in the above problem moves with a uniform velocity u in the direction of jet velocity V (Fig. 11.5). The volume of the liquid striking the plate per unit time will be

$$Q = a(V - u) \qquad (11.10)$$

Physically, when the plate recedes away from the jet it receives a less quantity of liquid per unit time than the actual mass flow rate of liquid delivered, say by any nozzle. When u = V, Q = 0 and when u > V, Q becomes negative. This implies physically that when the plate moves away from the jet with a velocity being equal to or greater than that of the jet, the jet can never strike the plate.

The control volume ABCDEFA in the case has to move with the velocity u of the plate. Therefore we have to apply Eq. (10.18d) to calculate the forces acting on the control volume. Hence the velocities relative to the control volume will come into picture. The velocity of jet relative to the control volume at its inlet becomes $V_{R1} = V - u$

Since the pressure remains same throughout, the magnitudes of the relative velocities of liquid at outlets become equal to that at inlet, provided the friction between the plate and the liquid is neglected. Moreover, for a smooth shockless flow, the liquid has to glide along the plate and hence the direction of V_{R0}, the relative velocity of the liquid at the outlets, will be along the plate. The absolute velocities of the liquid at the outlets can be found out by adding vectorially the plate velocity u and the relative velocity of the jet V - u with respect to the plate. This is shown by the velocity triangles at the outlets (Fig. 11.5). Coordinate axes fixed to the control volume ABCDEFA are chosen as ”0s” and ”0n” along and perpendicular to the plate respectively.

The force acting on the control volume along the direction "*0s*" will be zero for a frictionless flow. The net force acting on the control volume will be along "*0n*" only. To calculate this force F_n, the momentum theorem with respect to the control volume ABCDEFA can be written as

$$F_n = 0 - \rho Q[(V - u)\sin\theta]$$

Substituting Q from Eq (11.10),

$$F_n = -\rho a(V - u)^2 \sin\theta$$

Hence the force acting on the plate becomes

$$F_p = -F_n = \rho a(V - u)^2 \sin\theta \qquad (11.11)$$

If the plate moves with a velocity u in a direction opposite to that of V (plate moving towards the jet), the volume of liquid striking the plate per unit time will be Q = a(V + u) and, finally, the force acting on the plate would be

$$F_p = -F_n = \rho a(V + u)^2 \sin\theta \qquad (11.12)$$

From the comparison of the Eq. (11.9) with Eqs (11.11) and (11.12), conclusion can be drawn that for a given value of jet velocity V, the force exerted on a moving plate by the jet is either greater or lower than that exerted on a stationary plate depending upon whether the plate moves towards the jet or-away from it respectively.
The power developed due to the motion of the plate can be written (in case of the plate moving in the same direction as that of the jet) as

$$P = F_p \,.\, U$$

$$P = F_p \sin\theta |u| = \rho a(V - u)^2 u \sin^2\theta \qquad (11.13)$$

Dynamic Forces on Curve Surfaces due to the Impingement of Liquid Jets

The principle of fluid machines is based on the utilization of useful work due to the force exerted by a fluid jet striking and moving over a series of curved vanes in the periphery of a wheel rotating about its axis. The force analysis on a moving curved vane is understood clearly from the study of the inlet and outlet velocity triangles as shown in Fig. 11.6.

The fluid jet with an absolute velocity V_1 strikes the blade at the inlet. The relative velocity of the jet V_{r1} at the inlet is obtained by subtracting vectorially the velocity u of the vane from V_1. The jet strikes the blade without shock if β_1 (Fig. 11.6) coincides with the inlet angle at the tip of the blade. **If friction is neglected and pressure remains constant, then the relative velocity at the outlet is equal to that at the inlet ($V_{r2} = V_{r1}$)**.

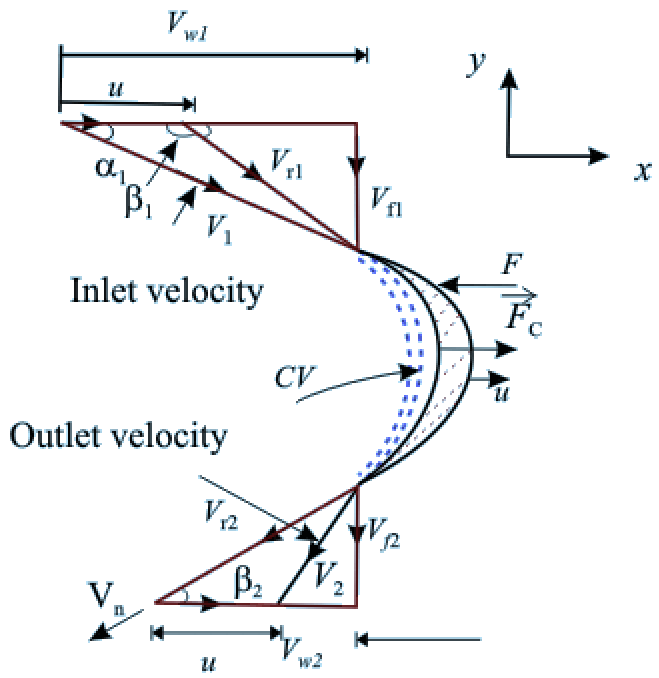


Fig 11.6 Flow of Fluid along a Moving Curved Plane

The control volume as shown in Fig. 11.6 is moving with a uniform velocity u of the vane.Therefore we have to use Eq.(10.18d) as the momentum theorem of the control volume at its steady state. Let F_c be the force applied on the control volume by the vane.Therefore we can write

$$F_c = \dot{m}V_{r2}\cos\beta_2 - \dot{m}V_{r1}\cos(180^0 - \beta_1)$$

$$= \dot{m}\, V_{w2} + \dot{m}V_{w1}$$

$$= \dot{m}(V_{w1} + V_{w2})$$

To keep the vane translating at uniform velocity, u in the direction as shown. the force F has to act opposite to F_c Therefore,

$$F = -F_c = -\dot{m}(V_{w1} + V_{w2}) \qquad (11.14)$$

From the outlet velocity triangle, it can be written

$$(V_{w2} + u)^2 = V_{r2}^2 - V_{f2}^2$$

or, $$V_{w2}^2 + u^2 + 2V_{w2}u = V_{r2}^2 - V_{f2}^2$$

or, $$V_2^2 - V_{f2}^2 + u^2 + 2V_{w2}u = V_{r2}^2 - V_{f2}^2$$

or, $$V_{w2}u = \frac{1}{2}[V_{r2}^2 - V_2^2 - u^2] \qquad (11.15a)$$

Similarly from the inlet velocity triangle. it is possible to write

$$V_{w1}u = \frac{1}{2}[-V_{r1}^2 + V_1^2 + u^2] \qquad (11.15b)$$

Addition of Eqs (11.15a) and (11.15b) gives

$$(V_{w1} + V_{w2})u = \frac{1}{2}(V_1^2 - V_2^2)$$

Power developed is given by

$$P = \dot{m}(V_{w1} + V_{w2})u = \frac{\dot{m}}{2}(V_1^2 - V_2^2) \qquad (11.16)$$

The efficiency of the vane in developing power is given by

$$\eta = \frac{\dot{m}(V_{w1} + V_{w2})u}{\frac{\dot{m}}{2}V_1^2} = 1 - \frac{V_2^2}{V_1^2} \qquad (11.17)$$

Propulsion of a Ship

Jet propulsion of ship is found to be less efficient than propulsion by screw propeller due to the large amount of frictional losses in the pipeline and the pump, and therefore, it is used rarely. Jet propulsion may be of some advantage in propelling a ship in a very shallow water to avoid damage of a propeller.

Consider a jet propelled ship, moving with a velocity V, scoops water at the bow and discharges astern as a jet having a velocity Vr relative to the ship.The control volume is taken fixed to the ship as shown in Fig. 11.7.

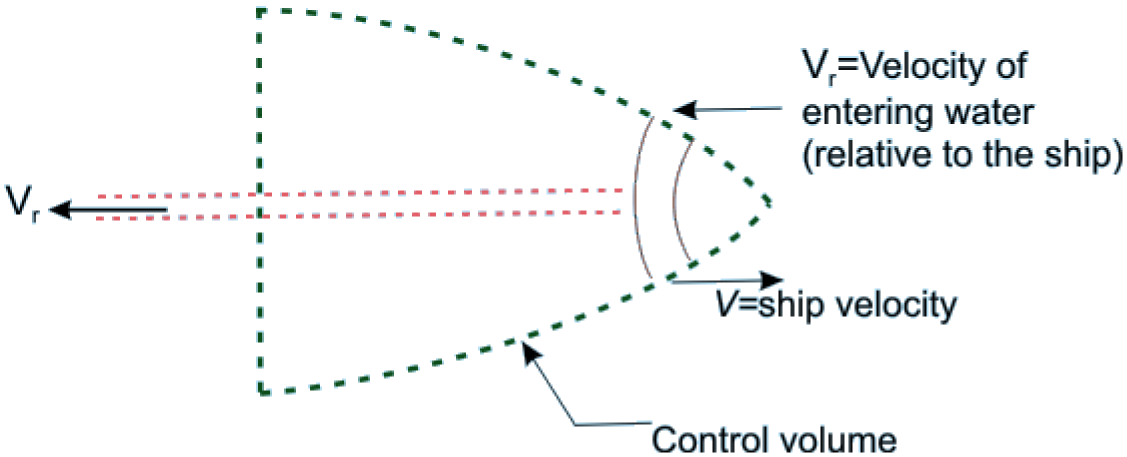


Fig 11.7 A control volume for a moving ship

Following the momentum theorem as applied to the control volume shown. We can write

$$F_c = \dot{m}[-V_r + (V)]$$

$$= \dot{m}(V - V_r)$$

Where F_c is the external force on the control volume in the direction of the ship's motion. The forward propulsive thrust F on the ship is given by

$$F = -F_c = \dot{m}(V_r - V) \qquad (11.18)$$

Propulsive power is given by

$$P = \dot{m}(V_r - V)V \qquad (11.19)$$

Jet Engine

A jet engine is a mechanism in which air is scooped from the front of the engine and is then compressed and used in burning of the fuel carried by the engine to produce a jet for propulsion. The usual types of jet engines are turbojet, ramjet and pulsejet.

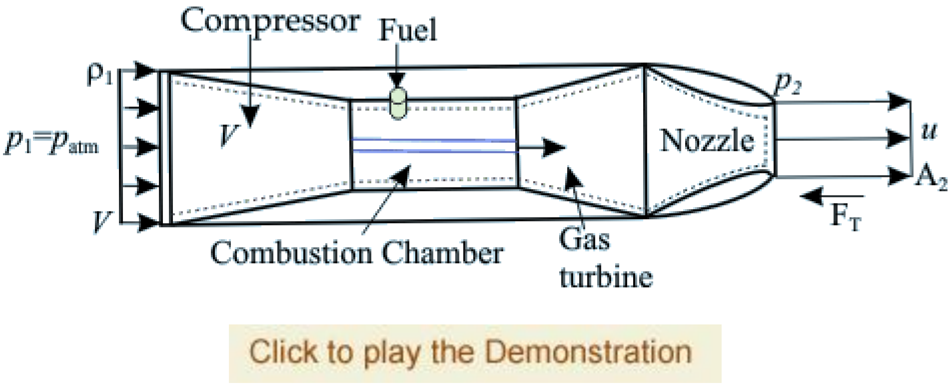


Fig 11.8 A Turbojet Engine

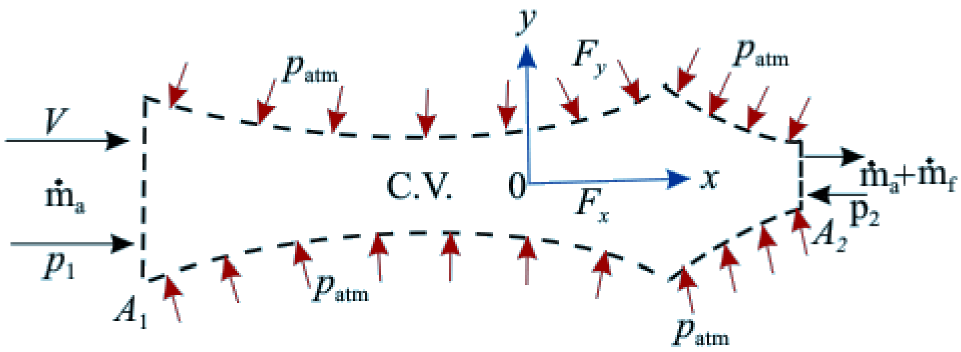


Fig 11.9 An Appropriate Control Volume Comprising the Stream of Fluid Flowing through the Engine

A turbojet engine consists essentially (Fig. 11.8) of -

- a compressor,
- a combustion chamber,
- a gas turbine and
- a nozzle.

A portion of the thermal energy of the product of combustion is used to run the gas turbine to drive the compressor. The remaining part of thermal energy is converted into kinetic energy of the jet by a nozzle. At high speed fiight, jet engines are advantageous since a propeller has to rotate at high speed to create a large thrust. This will result in excessive blade stress and a decrease in the efficiency for blade tip speeds near and above sonic level. In a jet propelled aircraft, the spent gases are ejected to the surroundings at high velocity usually equal to or greater than the velocity of sound in the fluid at that state.

In many cases, depending upon the range of fight speeds, the jet is discharged with a velocity equal to sonic velocity in the medium and the pressure at discharge does not fall immediately to the ambient pressure. In these cases, the discharge pressure p_2 at the nozzle exit becomes higher than the ambient pressure p_{atm}. Under the situation of uniform velocity of the aircraft, we have to use Eg. (10.18d) as the momentum theorem for the control volume as shown in Fig. 11.9 and can write

$$\left(\dot{m}_a + \dot{m}_f\right)u - \dot{m}_a V = F_x - (p_2 - p_{atm})A_2$$

$$\text{or,}\quad F_x = (p_2 - p_{atm})A_2 + \left(\dot{m}_a + \dot{m}_f\right)u - \dot{m}_a V$$

where, F_x is the force acting on the control volume along the direction of the coordinate axis "OX" fixed to the control volume, V is the velocity of the aircraft, u is the relative velocity of the exit jet with respect to the aircraft, $\dot{m}_a$ and $\dot{m}_f$ are the mass flow rate of air, and mass burning rate of fuel respectively. Usually $\dot{m}_f$ is very less compared to $\dot{m}_a (\dot{m}_f / \dot{m}_a$ usually varies from 0.01 to 0.02 in practice).

The propulsive thrust on the aircraft can be written as

$$F_T = -F_x = -[m_a(u - V) + (p_2 - p_{atm})A_2] \qquad (11.20)$$

$$(\text{since, } \dot{m}_f << \dot{m}_a)$$

The terms in the bracket are always positive. Hence, the negative sign in F_T represents that it acts in a direction opposite to ox, i.e. in the direction of the motion of the jet engine. The propulsive power is given by

$$P = [\dot{m}_a(u - V) + (p_2 - p_{atm})A_2]\,V \qquad (11.21)$$

Non-inertial Control Volume

Rocket engine

Rocket engine works on the principle of jet propulsion.

- The gases constituting the jet are produced by the combustion of a fuel and appropriate oxidant carried by the engine. Therefore, no air is required from outside and a rocket can operate satisfactorily in a vacuum.
- A large quantity of oxidant has to be carried by the rocket for its operation to be independent of the atmosphere.
- At the start of journey, the fuel and oxidant together form a large portion of the total load carried by the rocket.
- Work done in raising the fuel and oxidant to a great height before they are burnt is wasted.
- Therefore, to achieve the efficient use of the materials, the rocket is accelerated to a high velocity in a short distance at the start. This period of rocket acceleration is of practical interest.

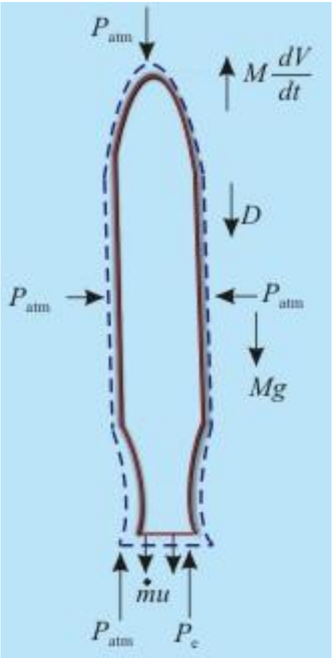


Fig 11.10 A Control Volume for a Rocket Engine

Let $\dot{m}$ be the rate at which spent gases are discharged from the rocket with a velocity u relative to the rocket (Fig. 11.10) Both $\dot{m}$ and u are assumed to be constant.

Let M and V be the instantaneous mass and velocity (in the upward direction) of the rocket. The control volume as shown in Fig. 11.10 is an accelerating one. Therefore we have to apply Eq. (10.18b) as the momentum theorem of the control volume. This gives

$$\sum F - M\frac{dV}{dt} = \dot{m}[(-u) - 0]$$

$$\sum F = M\frac{dV}{dt} - \dot{m}u \tag{11.22}$$

where ΣF is the sum of the external forces on the control volume in a direction vertically upward. If p_e and p_a be the nozzle exhaust plane gas pressure and ambient pressure respectively and D is the drag force to the motion of the rocket, then one can write

$$\sum F = (p_e - p_a)A_e - Mg - D \tag{11.23}$$

Where, A_e is outlet area of the propelling nozzle. Then Eq. (11.22) can be written as

$$M\frac{dV}{dt} = \dot{m}u + (p_e - p_a)A_e - Mg - D$$

In absence of gravity and drag, Eq (11.23) becomes

$$M\frac{dV}{dt} = \dot{m}u + (p_e - p_a)A_e$$

Application of Moment of Momentum Theorem

Let us take an example of a sprinkler like turbine as shown in Fig. 12.2. The turbine rotates in a horizontal plane with angular velocity ω. The radius of the turbine is r. Water enters the turbine from a vertical pipe that is coaxial with the axis of rotation and exits through the nozzles of cross sectional area 'a' with a velocity V_e relative to the nozzle.

A control volume with its surface around the turbine is also shown in the fig below.

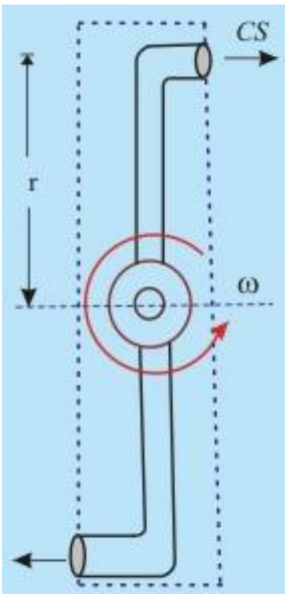


Fig 12.1 A Sprinkler like Turbine

Application of Moment of Momentum Theorem (Eq. 10.20b) gives

$$M_{zc} = \dot{m}\left(\vec{r} \times \vec{V}\right) \tag{12.1}$$

When M_{zc} is the moment applied to the control volume. The mass flow rate of water through the turbine is given by

$$\dot{m} = \rho 2 V_e a$$

The velocity $\vec{V}$ must be referenced to an inertial frame so that

$$\vec{r} \times \vec{V} = -r\vec{i}_r \times \left(V_e - \omega r\right)\vec{i}_\theta = -r\left(V_e - \omega r\right)\vec{i}_z$$

$$M_{zc} = -\dot{m}r\left(V_e - \omega r\right) \tag{12.2}$$

The moment M_z acting on the turbine can be written as

$$M_z = -M_{zc} = \dot{m}r\left(V_e - \omega r\right) \tag{12.3}$$

The power produced by the turbine is given by

$$P = M_z \omega \tag{12.4}$$

Euler's Equation: The Equation of Motion of an Ideal Fluid

This section is not a mandatory requirement. One can skip this section (if he/she does not like to spend time on Euler's equation) and go directly to Steady Flow Energy Equation.

Using the Newton's second law of motion the relationship between the velocity and pressure field for a flow of an inviscid fluid can be derived. The resulting equation, in its differential form, is known as Euler's Equation. The equation is first derived by the scientist Euler.

Derivation:
Let us consider an elementary parallelopiped of fluid element as a control mass system in a frame of rectangular cartesian coordinate axes as shown in Fig. 12.3. The external forces acting on a fluid element are the body forces and the surface forces.

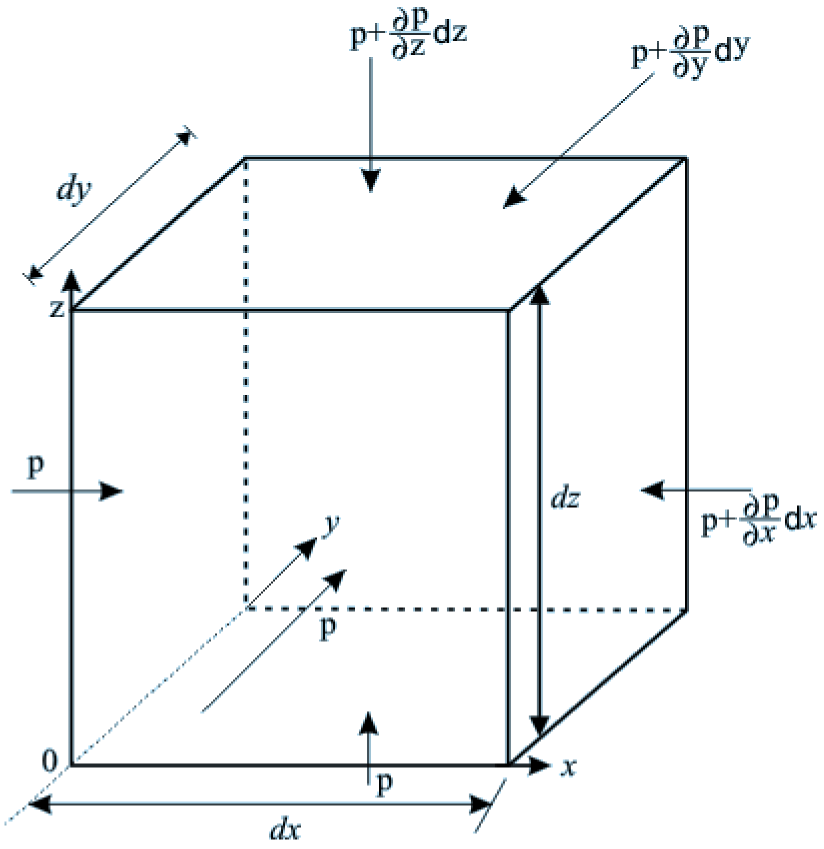


Fig 12.2 A Fluid Element appropriate to a Cartesian Coordinate System used for the derivation of Euler's Equation

Let X_x, X_y, X_z be the components of body forces acting per unit mass of the fluid element along the coordinate axes x, y and z respectively. The body forces arise due to external force fields like gravity, electromagnetic field, etc., and therefore, the detailed description of X_x, X_y and X_z are provided by the laws of physics describing the force fields. The surface forces for an inviscid fluid will be the pressure forces acting on different surfaces as shown in Fig. 12.3. Therefore, the net forces acting on the fluid element along x, y and z directions can be written as

$$F_x = \rho X_x dx\ dy\ dz + p.dy\ dz - (p + \frac{\partial p}{\partial x}dx)dy\ dz = (\rho X_x - \frac{\partial p}{\partial x})dx\ dy\ dz$$

$$F_y = \rho X_y dx\ dy\ dz + p.dx\ dz - (p + \frac{\partial p}{\partial y}dy)dx\ dz = (\rho X_y - \frac{\partial p}{\partial y})dx\ dy\ dz$$

$$F_z = \rho X_z dx\ dy\ dz + p.dy\ dx - (p + \frac{\partial p}{\partial z}dz)dx\ dz = (\rho X_z - \frac{\partial p}{\partial z})dx\ dy\ dz$$

Since each component of the force can be expressed as the rate of change of momentum in the respective directions, we have

$$\frac{D}{Dt}(\rho\ dx\ dy\ dz\ u) = \left(\rho X_x - \frac{\partial p}{\partial x}\right)dxdydz \qquad (12.5a)$$

$$\frac{D}{Dt}(\rho\ dx\ dy\ dz\ \mathrm{v}) = \left(\rho X_y - \frac{\partial p}{\partial y}\right) dxdydz \qquad (12.5b)$$

$$\frac{D}{Dt}(\rho\ dx\ dy\ dz\ \mathrm{w}) = \left(\rho X_z - \frac{\partial p}{\partial z}\right) dxdydz \qquad (12.5c)$$

s the mass of a control mass system does not change with time, $\rho dxdydz$ is constant with time and can be taken common. Therefore we can write Eqs (12.5a to 12.5c) as

$$\frac{Du}{Dt} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (12.6a)$$

$$\frac{Dv}{Dt} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (12.6b)$$

$$\frac{Dw}{Dt} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (12.6c)$$

Expanding the material accelerations in Eqs (12.6a) to (12.6c) in terms of their respective temporal and convective components, we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (12.7a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (12.7b)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (12.7c)$$

The Eqs (12.7a, 12.7b, 12.7c) are valid for both incompressible and compressible flow. By putting u = v = w = 0, as a special case, one can obtain the equation of hydrostatics . Equations (12.7a), (12.7b), (12.7c) can be put into a single vector form as

$$\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{X} \qquad (12.7d)$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}.\nabla)\vec{V} = \vec{X} - \frac{1}{\rho}\nabla p \qquad (12.7e)$$

where $\vec{V}$ the velocity vector and the body force vector per unit volume $\rho\vec{X}$ are defined as

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

$$\rho\vec{X} = \vec{i}\rho X_x + \vec{j}\rho X_y + \vec{k}\rho X_z$$

Equation (12.7d) or (12.7e) is the well known **Euler's equation** in vector form, while Eqs (12.7a) to (12.7c) describe the Euler's equations in a rectangular Cartesian coordinate system.

Euler's Equation along a Streamline

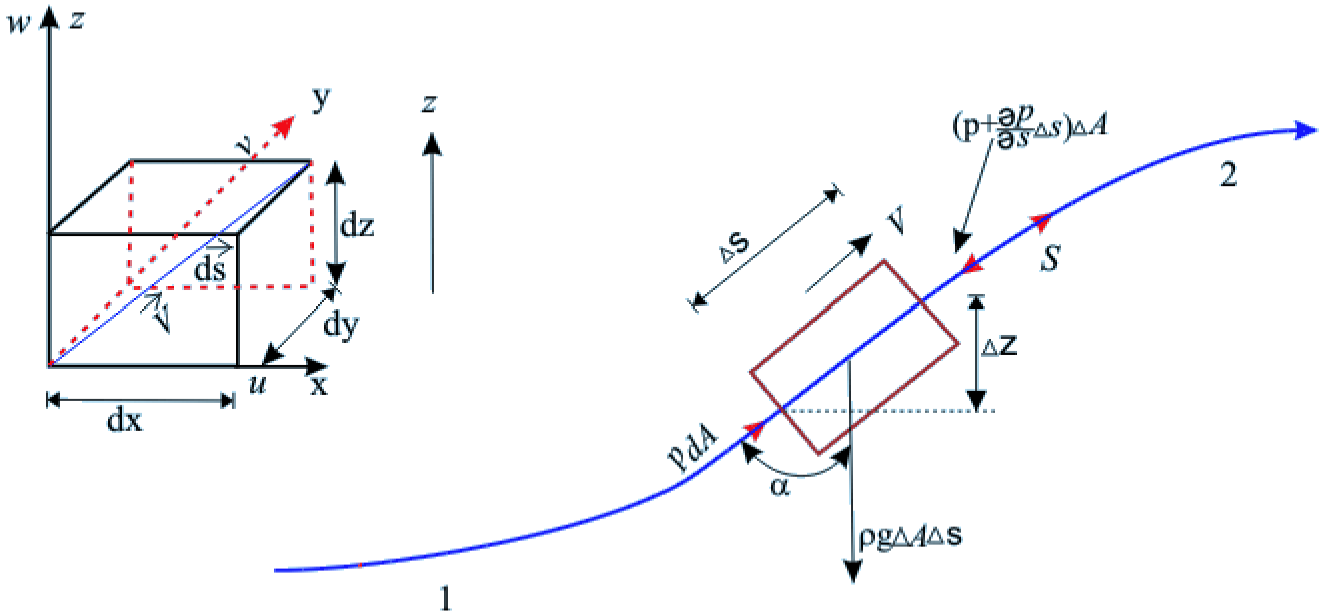


Fig 12.3 Force Balance on a Moving Element Along a Streamline

Derivation

Euler's equation along a streamline is derived by applying Newton's second law of motion to a fluid element moving along a streamline. Considering gravity as the only body force component acting vertically downward (Fig. 12.3), the net external force acting on the fluid element along the directions can be written as

$$F_s = -\frac{\partial p}{\partial s}\Delta s\Delta A - \rho\Delta s\Delta A g\cos\alpha \tag{12.8}$$

where ΔA is the cross-sectional area of the fluid element. By the application of Newton's second law of motion in s direction, we get

$$\rho\Delta s\Delta A\frac{DV}{Dt} = -\frac{\partial p}{\partial s}\Delta s\Delta A - \rho\Delta s\Delta A g\cos\alpha \tag{12.9}$$

From geometry we get

$$\cos\alpha = \lim_{\Delta s\to 0}\frac{\Delta z}{\Delta s} = \frac{dz}{ds}$$

Hence, the final form of Eq. (12.9) becomes

$$\rho\frac{DV}{Dt} = -\frac{\partial p}{\partial s} - \rho g\frac{dz}{ds}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{dz}{ds} \qquad (12.10)$$

Equation (12.10) is the Euler's equation along a streamline.

Let us consider $d\vec{s}$ along the streamline so that

$$d\vec{s} = \vec{i}dx + \vec{j}dy + \vec{k}dz$$

Again, we can write from Fig. 12.3

$$\frac{dx}{ds} = \frac{u}{V}, \quad \frac{dy}{ds} = \frac{v}{V} \text{ and } \frac{dz}{ds} = \frac{w}{V}$$

The equation of a streamline is given by

$$\vec{V}\times d\vec{S} = 0$$

or, $\begin{vmatrix} i & j & k \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$ which finally leads to

$$udy = vdx; \qquad udz = wdx; \qquad vdz = wdy$$

Multiplying Eqs (12.7a), (12.7b) and (12.7c) by dx, dy and dz respectively and then substituting the above mentioned equalities, we get

$$\rho\left(u\frac{\partial u}{\partial t}.\frac{ds}{V} + u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + u\frac{\partial u}{\partial z}dz\right) = -\frac{\partial p}{\partial x}dx + X_x dx$$

$$\rho\left(v\frac{\partial v}{\partial t}.\frac{ds}{V} + v\frac{\partial v}{\partial x}dx + v\frac{\partial v}{\partial y}dy + v\frac{\partial v}{\partial z}dz\right) = -\frac{\partial p}{\partial y}dy + X_y dy$$

$$\rho\left(w\frac{\partial w}{\partial t}.\frac{ds}{V} + w\frac{\partial w}{\partial x}dx + w\frac{\partial w}{\partial y}dy + w\frac{\partial w}{\partial z}dz\right) = -\frac{\partial p}{\partial z}dy + X_z dz$$

Adding these three equations, we can write

$$\rho\left(\frac{ds}{V}\cdot\frac{\partial}{\partial t}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)dx+\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)dy+\frac{\partial}{\partial z}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)dz\right)$$

$$= \rho\left(\frac{ds}{V}\cdot\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)dx+\frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)dy+\frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)dz\right)$$

$$= \rho\left[\frac{\partial V}{\partial t}+V\left(\frac{\partial V}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial V}{\partial y}\cdot\frac{dy}{ds}+\frac{\partial V}{\partial z}\cdot\frac{dz}{ds}\right)\right]=-\left(\frac{\partial p}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial p}{\partial y}\cdot\frac{dy}{ds}+\frac{\partial p}{\partial z}\cdot\frac{dz}{ds}\right)-\rho g\frac{dz}{ds}$$

Hence, $$\boxed{\rho\left[\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s}\right]=-\frac{\partial p}{\partial s}-\rho g\frac{dz}{ds}}$$

This is the more popular form of Euler's equation because the velocity vector in a flow field is always directed along the streamline.

Euler’s Equation in Different Conventional Coordinate System

Euler’s equation in different coordinate systems can be derived either by expanding the acceleration and pressure gradient terms of Eq. (12.7d), or by the application of Newton’s second law to a fluid element appropriate to the coordinate system.

Euler's Equation in Different Conventional Coordinate Systems

Coordinate System	Euler's Equation (Equation of motion for an inviscid flow)	
Rectangular Cartesian coordinate	x direction	$\frac{Du}{Dt}=X_x-\frac{1}{\rho}\frac{\partial p}{\partial x}$
	y direction	$\frac{Dv}{Dt}=X_y-\frac{1}{\rho}\frac{\partial p}{\partial y}$
	z direction	$\frac{Dw}{Dt}=X_z-\frac{1}{\rho}\frac{\partial p}{\partial z}$

Cylindrical Polar Coordinate	r direction	$\frac{DV_r}{Dt} - \frac{V_\theta^2}{r} = X_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$
	θ direction	$\frac{DV_\theta}{Dt} - \frac{V_r V_\theta}{r} = X_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$
	z direction	$\frac{DV_z}{Dt} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$
Spherical Polar Coordinate	R direction	$\frac{DV_R}{Dt} - \frac{V_\phi^2 + V_\theta^2}{R} = X_R - \frac{1}{\rho}\frac{\partial p}{\partial R}$
	θ direction	$\frac{DV_\theta}{Dt} + \frac{V_R V_\theta}{R} + \frac{V_\phi V_\theta \cot\phi}{R} = X_\theta - \frac{1}{R\sin\phi}\frac{\partial p}{\partial \theta}$
	φ direction	$\frac{DV_\phi}{Dt} + \frac{V_R V_\phi}{R} - \frac{V_\theta^2 \cot\phi}{R} = X_\phi - \frac{1}{\rho R}\frac{\partial p}{\partial \phi}$

A Control Volume Approach for the Derivation of Euler's Equation

Euler's equations of motion can also be derived by the use of the momentum theorem for a control volume.

Derivation
In a fixed x, y, z axes (the rectangular cartesian coordinate system), the parallelopiped which was taken earlier as a control mass system is now considered as a control volume (Fig. 12.4).

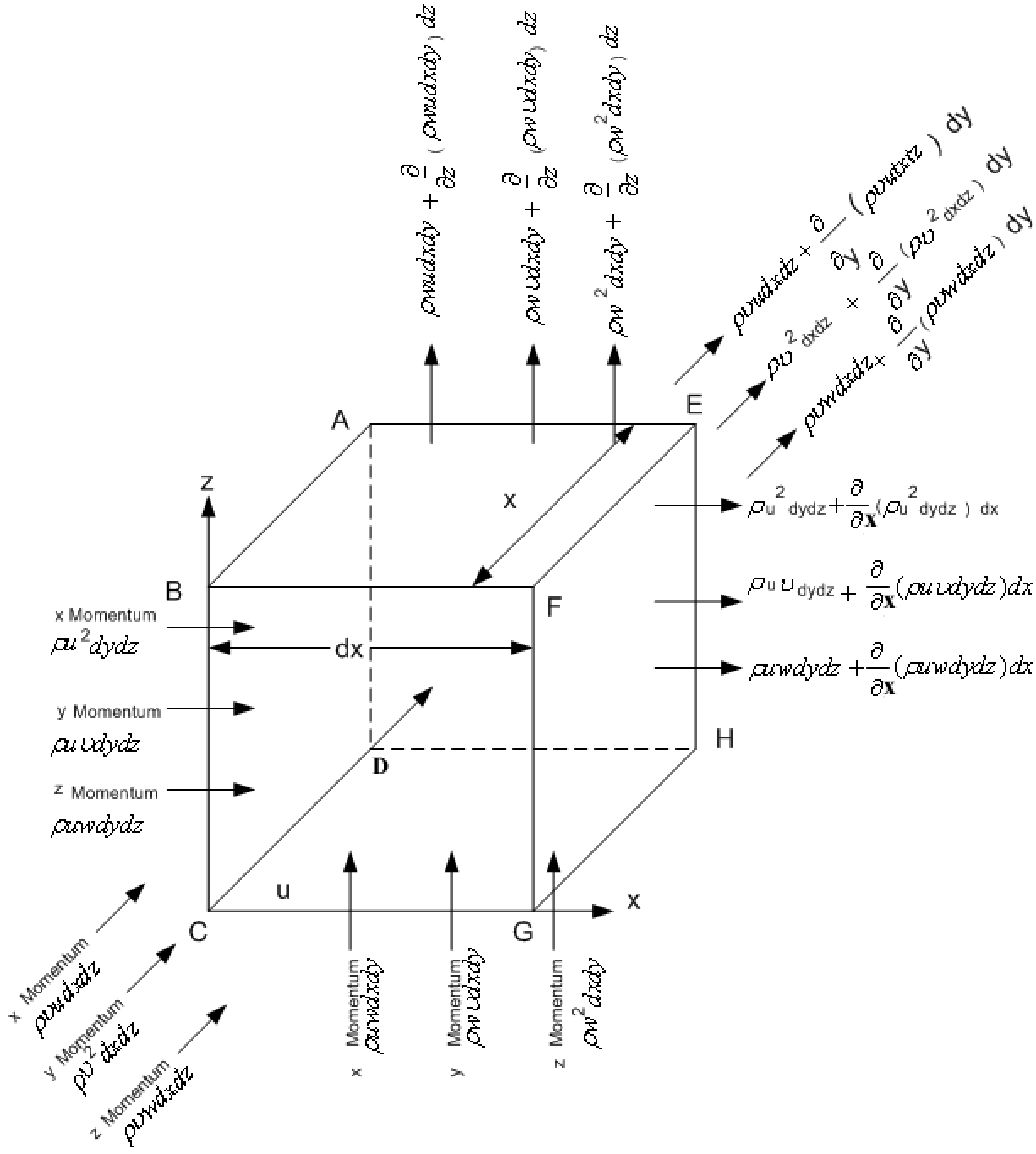


Fig 12.4   A Control Volume used for the derivation of Euler's Equation

We can define the velocity vector $\vec{V}$ and the body force per unit volume $\rho\vec{X}$ as

$$\vec{V} = \vec{i}\,u + \vec{j}v + \vec{k}w$$

$$\rho\vec{X} = \vec{i}\,\rho X_x + \vec{j}\,\rho X_y + \vec{k}\rho X_z$$

The rate of x momentum influx to the control volume through the face ABCD is equal to ρu^2 dy dz. The rate of x momentum efflux from the control volume through the face EFGH equals $\rho u^2 dy\, dz + \frac{\partial}{\partial x}\left(\rho u^2 dy\, dz\right) dx$

Therefore the rate of net efflux of x momentum from the control volume due to the faces perpendicular to the x direction (faces ABCD and EFGH) = $\frac{\partial}{\partial x}(\rho u^2)\mathrm{d}\forall$ where, $\mathrm{d}\forall$, the elemental volume = dx dy dz.

Similarly,

The rate of net efflux of x momentum due to the faces perpendicular to the y direction (face BCGF and ADHE) = $\frac{\partial}{\partial y}(\rho uv)\mathrm{d}\forall$

The rate of net efflux of x momentum due to the faces perpendicular to the z direction (faces DCGH and ABFE) = $\frac{\partial}{\partial z}(\rho uw)\mathrm{d}\forall$

Hence, the net rate of x momentum efflux from the control volume becomes

$$\left[\frac{\partial}{\partial z}(\rho u^2)+\frac{\partial}{\partial z}(\rho uv)+\frac{\partial}{\partial z}(\rho uw)\right]\mathrm{d}\forall$$

The time rate of increase in x momentum in the control volume can be written as

$\frac{\partial}{\partial t}(\rho u\mathrm{d}\forall) = \frac{\partial}{\partial t}(\rho u)\mathrm{d}\forall$ (Since, $\mathrm{d}\forall$, by the definition of control volume, is invariant with time)

Applying the principle of momentum conservation to a control volume (Eq. 4.28b), we get

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}\left(\rho u^2\right)+\frac{\partial}{\partial y}(\rho uv)+\frac{\partial}{\partial z}(\rho uw) = \rho X_x - \frac{\partial p}{\partial x} \qquad (12.11a)$$

The equations in other directions y and z can be obtained in a similar way by considering the y momentum and z momentum fluxes through the control volume as

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}\left(\rho v^2\right)+\frac{\partial}{\partial z}(\rho vw) = \rho X_y - \frac{\partial p}{\partial y} \qquad (12.11b)$$

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho uw)+\frac{\partial}{\partial y}(\rho vw)+\frac{\partial}{\partial z}\left(\rho w^2\right) = \rho X_z - \frac{\partial p}{\partial z} \qquad (12.11c)$$

The typical form of Euler's equations given by Eqs (12.11a), (12.11b) and (12.11c) are known as the conservative forms.

Conservation of Energy

The principle of conservation of energy for a control mass system is described by the first law of thermodynamics

Heat Q added to a control mass system- the work done W by the control mass system = change in its internal energy E

The internal energy depends only upon the initial and final states of the system. It can be written in the form of the equation as

$$Q - W = \Delta E = E_2 - E_1 \tag{13.1a}$$

Equation (13.1a) can be expressed on the time rate basis as

$$\frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t} = \frac{dE}{dt} \tag{13.1b}$$

Where δQ and δW are the amount of heat added and work done respectively during a time interval of δt. To develop the analytical statement for the conservation of energy of a control volume, the Eq. (10.10) is used with N = E (the internal energy) and η = e (the internal energy per unit mass) along with the Eq. (13.1b). This gives

$$\frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t} = \frac{d}{dt}\iiint_{CV} \rho e \, d\forall + \iint_{CS} \rho e \vec{V}.d\vec{A} \tag{13.2}$$

The Eq. (13.2) is known as the **general energy equation** for a control volume.

The first term on the right hand side of the equation is the time rate of increase in the internal energy within a control volume and the second term is the net rate of energy efflux from the control volume.

Different forms of energy associated with moving fluid elements comprising a control volume are -

1. Potential energy
The concept of potential energy in a fluid is essentially the same as that of a solid mass. The potential energy of a fluid element arises from its existence in a conservative body force field. This field may be a magnetic, electrical, etc. In the absence of any of such external force field, the earth's gravitational effect is the only cause of potential energy. If a fluid mass m is stored in a reservoir and its C.G. is at a vertical distance z from an arbitrary horizontal datum plane, then the potential energy is mgz and the potential energy per unit mass is gz. The arbitrary datum does not play a vital role since the difference in potential energy, instead of its absolute value, is encountered in different practical purposes.

2. Kinetic Energy
If a quantity of a fluid of mass m flows with a velocity V, being the same throughout its mass, then the total kinetic energy is $mV^2/2$ and the kinetic energy

per unit mass is $V^2/2$. For a stream of real fluid, the velocities at different points will not be the same. If V is the local component of velocity along the direction of flow for a fluid flowing through an open channel or closed conduit of cross-sectional area A, the total kinetic energy at any section is evaluated by summing up the kinetic energy flowing through differential areas as

$$K.E. = \int_A \frac{\rho V^3}{2} dA$$

The average velocity at a cross-section in a flowing stream is defined on the basis of volumetric flow rate as,

$$V_{av} = \frac{\int_A V dA}{A}$$

The kinetic energy per unit mass of the fluid is usually expressed as $\alpha(V_{av}^2/2)$ where α is known as the **kinetic energy correction factor**.
Therefore, we can write

$$\alpha \frac{V_{av}^2}{2} \rho V_{av} A = \int_A \frac{V^3 \rho}{2} dA$$

Hence,

$$\alpha = \frac{A^2 \int V^3 \rho \, dA}{\rho \left[\int V dA \right]^3} \qquad (13.3a)$$

For an incompressible flow,

$$\alpha = \frac{A^2 \int_A V^3 dA}{\left[\int_A V dA \right]^3} \qquad (13.3b)$$

3. Intermolecular Energy
The intermolecular energy of a substance comprises the potential energy and kinetic energy of the molecules. The potential energy arises from intermolecular forces. For an ideal gas, the potential energy is zero and the intermolecular energy is, therefore, due to only the kinetic energy of molecules. The kinetic energy of the molecules of a substance depends on its temperature.

4. Flow Work

Flow work is the work done by a fluid to move against pressure.
For a flowing stream, a layer of fluid at any cross-section has to push the adjacent neighboring layer at its downstream in the direction of flow to make its way through and thus does work on it. The amount of work done can be calculated by considering a small amount of fluid mass $A_1 \rho_1$ dx to cross the surface AB from left to right (Fig. 13.1). The work done by this mass of fluid then becomes equal to $p_1 A_1$ dx and thus the flow work per unit mass can be expressed as

$p_1 A_1 dx / A_1 \rho_1 dx = p_1 / \rho_1$ (where p_1 is the pressure at section AB (Fig 13.1)

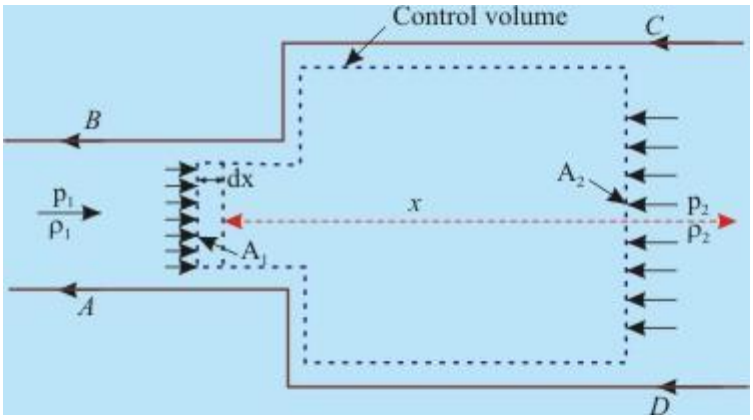


Fig 13.1 Work done by a fluid to flow against pressure

Therefore the flow work done per unit mass by a fluid element entering the control volume ABCDA (Fig. 13.1) is p_1 / ρ_1 Similarly, the flow work done per unit mass by a fluid element leaving the control volume across the surface CD is p_2/ρ_1

Important- In introducing an amount of fluid inside the control volume, the work done against the frictional force at the wall can be shown to be small as compared to the work done against the pressure force, and hence it is not included in the flow work.

Although 'flow work' is not an intrinsic form of energy, it is sometimes referred to as 'pressure energy' from a view point that by virtue of this energy a mass of fluid having a pressure p at any location is capable of doing work on its neighboring fluid mass to push its way through.
Steady Flow Energy Equation

The energy equation for a control volume is given by Eq. (13.2). At steady state, the first term on the right hand side of the equation becomes zero and it becomes

$$\frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t} = \iint_{CS} \rho e \vec{V} . d\vec{A} \qquad (13.4)$$

In consideration of all the energy components including the flow work (or pressure energy) associated with a moving fluid element, one can substitute 'e' in Eq. (13.4) as

$$e = u + \frac{p}{\rho} + \frac{V^2}{2} + gz$$

and finally we get

$$\frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t} = \iint_{cs} \left(u + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) \rho \vec{V}.d\vec{A} \qquad (13.5)$$

The Eq. (13.5) is known as steady flow energy equation.

Bernoulli's Equation

Energy Equation of an ideal Flow along a Streamline

Euler’s equation (the equation of motion of an inviscid fluid) along a stream line for a steady flow with gravity as the only body force can be written as

$$V\frac{dV}{ds} = -\frac{1}{\rho}\frac{dp}{ds} - g\frac{dz}{ds} \qquad (13.6)$$

Application of a force through a distance ds along the streamline would physically imply work interaction. Therefore an equation for conservation of energy along a streamline can be obtained by integrating the Eq. (13.6) with respect to ds as

$$\int V\frac{dV}{ds}ds = -\int \frac{1}{\rho}\frac{dp}{ds}ds - \int g\frac{dz}{ds}ds$$

$$\text{or,} \quad \frac{V^2}{2} + \int \frac{dp}{\rho} + gz = C \qquad (13.7)$$

Where C is a constant along a streamline. In case of an incompressible flow, Eq. (13.7) can be written as

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = C \qquad (13.8)$$

The Eqs (13.7) and (13.8) are based on the assumption that no work or heat interaction between a fluid element and the surrounding takes place. The first term of the Eq. (13.8) represents the flow work per unit mass, the second term represents the kinetic energy per unit mass and the third term represents the potential energy per unit mass. Therefore the sum of three terms in the left hand side of Eq. (13.8) can be considered as the total mechanical energy per unit mass which remains constant along a streamline for a steady inviscid and incompressible flow of fluid. Hence the Eq. (13.8) is also known as **Mechanical energy equation**.

This equation was developed first by Daniel Bernoulli in 1738 and is therefore referred to as Bernoulli's equation. Each term in the Eq. (13.8) has the dimension of energy per unit mass. The equation can also be expressed in terms of energy per unit weight as

$$\frac{P}{\rho g}+\frac{V^2}{2g}+z=C_1(\text{constant}) \tag{13.9}$$

In a fluid flow, the energy per unit weight is termed as head. Accordingly, equation 13.9 can be interpreted as

Pressure head + Velocity head + Potential head =Total head (total energy per unit weight).

Bernoulli's Equation with Head Loss

The derivation of mechanical energy equation for a real fluid depends much on the information about the frictional work done by a moving fluid element and is excluded from the scope of the book. However, in many practical situations, problems related to real fluids can be analysed with the help of a modified form of Bernoulli's equation as

$$\frac{P_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{P_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_f \tag{13.10}$$

where, h_f represents the frictional work done (the work done against the fluid friction) per unit weight of a fluid element while moving from a station 1 to 2 along a streamline in the direction of flow. The term h_f is usually referred to as head loss between 1 and 2, since it amounts to the loss in total mechanical energy per unit weight between points 1 and 2 on a streamline due to the effect of fluid friction or viscosity. It physically signifies that the difference in the total mechanical energy between stations 1 and 2 is dissipated into intermolecular or thermal energy and is expressed as loss of head h_f in Eq. (13.10). The term head loss, is conventionally symbolized as h_L instead of h_f in dealing with practical problems. For an inviscid flow $h_L = 0$, and the total mechanical energy is constant along a streamline.

Exercise Problems - Chapter 4

1. Which of the following velocity fields are kinematically possible for an incompressible flow ?

(i) $u = x^2 + y^2$, $v = y^2 + z^2$, $w = -2 (x + y) z$

(ii) $\vec{v} = x^2 y\,\vec{i} + (x+y+z)\,\vec{j} + (z^2+x^2)\,\vec{k}$

(iii) $u = -\frac{kx}{y}, \quad v = k \log(xy)$

(iv) $u = \dfrac{k(x^2 - y^2)}{(x^2 + y^2)^2}$, $v = \dfrac{2kxy}{(x^2 + y^2)^2}$

(k is a constant) [(i) yes, (ii) No, (iii) yes, (iv) No]

2. The x component of velocity in a two-dimensional incompressible flow is prescribed as $u = By^3 - Ax^4$, where A and B are constants. Find out the y component of velocity. Assume that for all values of x, v = 0 at y = 0. Check whether the flow is irrotational.

[$v = 4\,Ax^3y$, No]

3. Consider a vertical nozzle of inlet and outlet diameters of 0.6 m and 0.3 m respectively as shown in Fig 13.2. The pressure at section 1 is 20 kPa (gauge), and the volume flow rate is 0.6 m^3/s. Find

(i) the velocities at section 1 and section 2,

(ii) total force acting on the walls of the nozzle.
[Neglect frictional resistance]

[(i) V_1 = 2.12 m/s, V_2 = 8.45 m/s
(ii) 0.517 kN (vertically upwards)]

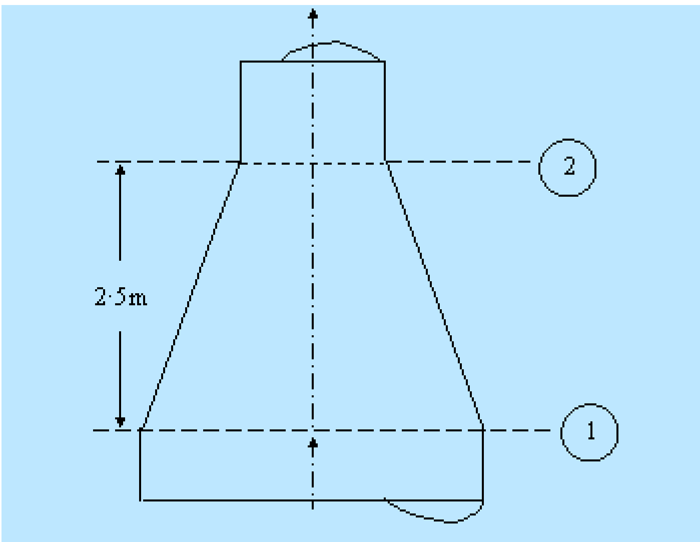


Fig 13.2

4. Water flows through a 5 m high conical vertical pipe whose diameter changes from 0.5m at the top end to 1.5 m at the bottom end. Measurements indicate that when velocity at the smaller section is 18 m/s, the frictional head loss is 1m of water for flow in either direction. For a pressure of 1.8 m of water gauge at the smaller section, determine the pressure (in meter of water gauge) at the larger section when the flow is (i) in the downward direction, (ii) in the upward direction.

[(i) 24. 11m, (ii) 26.11 m]

5. A force of 1 kN is required to hold the plate in position for a flow of oil of specific gravity of 0. 8 as shown in Fig.13.3. Find the velocity V of the flow of oil

[42 m/s]

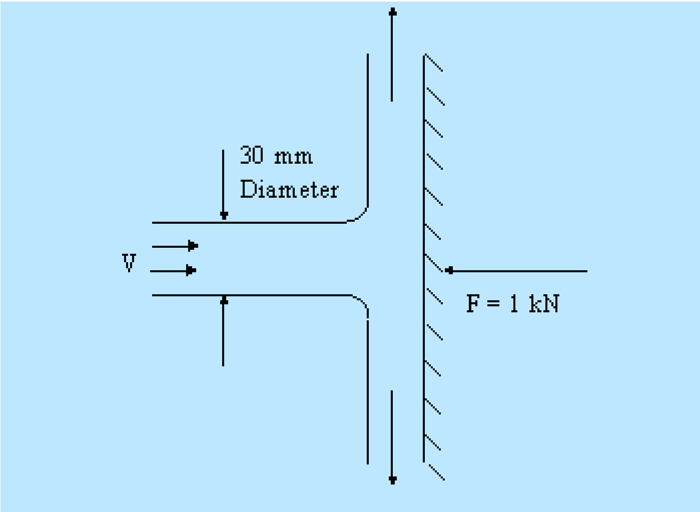


Fig 13.3

6. Water flows as two free jets from the tee attached to the pipe shown in Figure 13.4 below. The exit speed is 15 m/s. If viscous effects and gravity are negligible, determine the x and y components of the force that the pipe exerts on the tee.

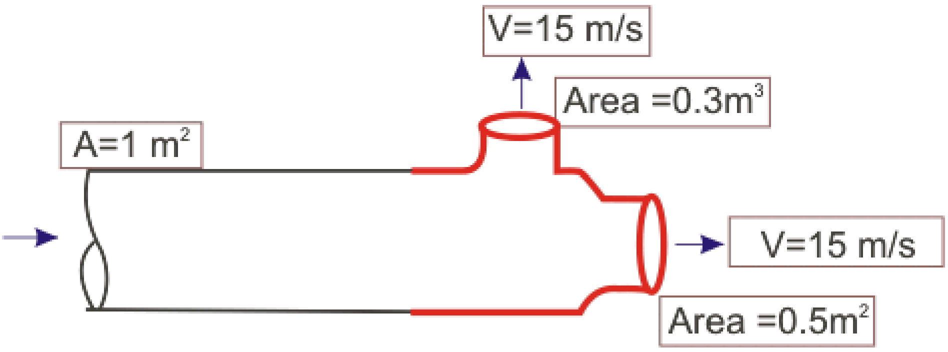


Fig 13.4

7. A horizontal jet of water with velocity V and cross sectional area A impinges on a stationary vane, which deflects the jet through an angle θ (see Fig 13.5). Derive expressions for the horizontal and vertical force components X and Y acting on the vane. Neglect effects of gravity and friction.

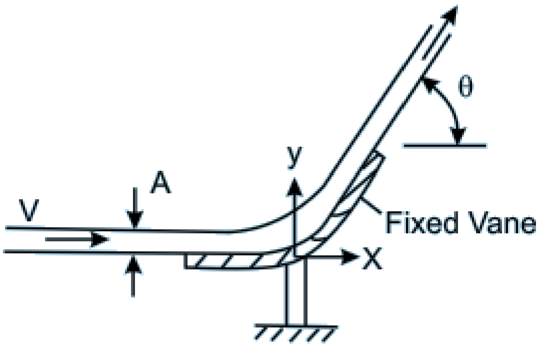


Fig 13.5

Recap

In this course you have learnt the following

- A control mass or closed system is characterized by a fixed quantity of mass of a given identity, while in an open system or control volume mass may change continuously due to the flow of mass across the system boundary.

- Continuity equation is the equation of conservation of mass in a fluid flow. The general form of the continuity equation for an unsteady compressible flow is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V})$$ where, $\vec{V}$ is the velocity vector

- The concept of stream function is a consequence of continuity. In a two dimensional incompressible flow, the difference in stream functions between two points gives the volume flow rate (per unit width in a direction perpendicular to the plane of flow) across any line joining the points. The value of stream function is constant along a streamline.

- The equation of motion (conservation of momentum) of an inviscid flow is known as Euler's equation. The general form of Euler's equation is given by $\rho D\vec{V} / Dt = \rho \vec{X} - \nabla p$, where $\vec{X}$ is the body force vector per unit mass and $\vec{V}$ is the velocity vector. Euler's equation along a streamline, with gravity as the only body force, can be written as

$$\rho \frac{DV}{Dt} = -\frac{\partial p}{\partial s} - \rho g \frac{dz}{ds}$$ (where s represents the coordinate along the streamline.)

- Reynolds transport theorem states the relation between equations applied to a system and those applied to a control volume. The statement of the law of conservation of momentum as applied to a control volume is known as momentum theorem. This theorem states that the resultant force (or torque) acting on a control volume is equal to the time rate of increase in linear momentum (or angular momentum) within the control volume plus the rate of net efflux of linear momentum (or angular momentum) from the control volume.

- A fluid element in motion possesses intermolecular energy, kinetic energy and potential energy. The work required by a fluid element to move against pressure is known as flow work. It is loosely termed as pressure energy. The shaft work is the work interaction between the control volume and the surrounding that takes place by the action of
  shear force such as the torque exerted on a rotating shaft. The equation for conservation of energy of a steady, in viscid and incompressible flow in a conservative body force field is known as Bernoulli's equation.
  Bernoulli's equation in the case of gravity as the only body force field is given by

  $$\frac{p}{\rho g}+\frac{V^2}{2g}+z=C$$ (The value of C remains constant along a streamline.)

- The loss of mechanical energy due to friction in a real fluid is considered in Bernoulli's equation as

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_f$$

where, h_f = frictional work done or loss of mechanical energy due to friction per unit weight of a fluid element while movingfrom station 1 to 2 along a streamline. The term h_f is usually referred to as head loss.

Bernoulli's Equation In Irrotational Flow

In the previous lecture (lecture 13) we have obtained Bernoulli's equation

$$\boxed{\frac{p}{\rho}+\frac{V^2}{2}+gz=C}$$

- This equation was obtained by integrating the Euler's equation (the equation of motion) with respect to a displacement '**ds**' along a streamline. Thus, the value of C in the above equation is constant only along a streamline and should essentially vary from streamline to streamline.
- The equation can be used to define relation between flow variables at point B on the streamline and at point A, along the same streamline. So, in order to apply this equation, one should have knowledge of velocity field beforehand. This is one of the limitations of application of Bernoulli's equation.

Irrotationality of flow field

Under some special condition, the constant C becomes invariant from streamline to streamline and the Bernoulli's equation is applicable with same value of C to the entire flow field. The typical condition is the irrotationality of flow field.

Click here to play the demonstration

Proof:

Let us consider a steady two dimensional flow of an ideal fluid in a rectangular Cartesian coordinate system. The velocity field is given by

$$\vec{V} = \vec{i}u + \vec{j}v$$

hence the condition of irrotationality is

$$\nabla \times \vec{V} = \left\{ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad (14.1)$$

The steady state Euler's equation can be written as

$$\rho \left\{ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right\} = -\frac{\partial p}{\partial x} \qquad (14.2a)$$

$$\rho \left\{ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right\} = -\frac{\partial p}{\partial y} \; -\rho g \qquad (14.2b)$$

We consider the y-axis to be vertical and directed positive upward. From the condition of irrotationality given by the Eq. (14.1), we substitute $\frac{\partial v}{\partial x}$ in place of $\frac{\partial u}{\partial y}$ in the Eq. 14.2a and $\frac{\partial u}{\partial y}$ in place of $\frac{\partial v}{\partial x}$ in the Eq. 14.2b. This results in

$$\left\{ u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} \right\} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (14.3a)$$

$$\left\{ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} \right\} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \; -g \qquad (14.3b)$$

Now multiplying Eq.(14.3a) by 'dx' and Eq.(14.3b) by 'dy' and then adding these two equations we have

$$u\left\{ \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \right\} + v\left\{ \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \right\} = -\frac{1}{\rho}\left\{ \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \right\} - g\,dy \qquad (14.4)$$

The Eq. (14.4) can be physically interpreted as the equation of conservation of energy for an arbitrary displacement

$d\vec{r} = \vec{i}dx + \vec{j}dy$. Since, u, v and p are functions of x and y, we can write

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{14.5a}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \tag{14.5b}$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \tag{14.5c}$$

With the help of Eqs (14.5a), (14.5b), and (14.5c), the Eq. (14.4) can be written as

$$udu + vdv = -\frac{1}{\rho}dp - gdy$$

$$d\left\{\frac{u^2}{2}\right\} + d\left\{\frac{v^2}{2}\right\} = -\frac{1}{\rho}dp - gdy$$

$$d\left\{\frac{u^2+v^2}{2}\right\} = -\frac{1}{\rho}dp - gdy$$

$$d\left\{\frac{V^2}{2}\right\} = -\frac{1}{\rho}dp - gdy \tag{14.6}$$

The integration of Eq. 14.6 results in

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gy = C \tag{14.7a}$$

For an **incompressible flow**,

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gy = C} \tag{14.7b}$$

The constant C in Eqs (14.7a) and (14.7b) has the same value in the entire flow field, since no restriction was made in the choice of dr which was considered as an arbitrary displacement in evaluating the work.

Note: In deriving Eq. (13.8) the displacement **ds** was considered along a streamline. Therefore, the total mechanical energy remains constant everywhere in an inviscid and irrotational flow, while it is constant only along a streamline for an inviscid but rotational flow.

The equation of motion for the flow of an inviscid fluid can be written in a vector form as

$$\boxed{\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{X}}$$

where $\vec{X}$ is the body force vector per unit mass

Plane Circular Vortex Flows

- Plane circular vortex flows are defined as flows where streamlines are concentric circles. Therefore, with respect to a polar coordinate system with the centre of the circles as the origin or pole, the velocity field can be described as

$$V_\theta \neq 0 \qquad\qquad V_r = 0$$

where V_θ and V_r are the tangential and radial component of velocity respectively.

- The equation of continuity for a two dimensional incompressible flow in a polar coordinate system is

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

which for a plane circular vortex flow gives $\frac{\partial V_\theta}{\partial \theta} = 0$ i.e. V_θ is not a function of θ. Hence, V_θ is a function of r only.

- We can write for the variation of total mechanical energy with radius as

$$\frac{\partial H}{\partial r} = \frac{V_\theta}{g}\left(\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right) \qquad (14.8)$$

Click to see the Derivation

Free Vortex Flows

- Free vortex flows are the plane circular vortex flows where the total mechanical energy remains constant in the entire flow field. There is neither any addition nor any destruction of energy in the flow field.
- Therefore, the total mechanical energy does not vary from streamline to streamline. Hence from Eq. (14.8), we have,

$$\frac{\partial H}{\partial r} = \frac{V_\theta}{g}\left(\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right) = 0$$

or, $$\frac{1}{r}\left[\frac{d}{dr}(V_\theta r)\right] = 0 \qquad (14.9)$$

- Integration of Eq 14.9 gives

$$V_\theta = \frac{C}{r} \tag{14.10}$$

- The Eq. (14.10) describes the velocity field in a free vortex flow, where C is a constant in the entire flow field. The vorticity in a polar coordinate system is defined by -

$$\Omega = \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}$$

- In case of vortex flows, it can be written as

$$\Omega = \frac{dV_\theta}{dr} + \frac{V_\theta}{r}$$

- For a free vortex flow, described by Eq. (14.10),Ω becomes zero. Therefore we conclude that a free vortex flow is irrotational, and hence, it is also referred to as **irrotational vortex**.
- It has been shown before that the total mechanical energy remains same throughout in an irrotational flow field. Therefore, irrotationality is a direct consequence of the constancy of total mechanical energy in the entire flow field and vice versa.
- The interesting feature in a free vortex flow is that as $r \to 0,\ V_\theta \to \infty$[Eq. (14.10)]. It mathematically signifies a point of singularity at r = 0 which, in practice, is impossible. In fact, the definition of a free vortex flow cannot be extended as r = 0 is approached.
- In a real fluid, friction becomes dominant as **r→0** and so a fluid in this central region tends to rotate as a solid body. Therefore, the singularity at r = 0 does not render the theory of irrotational vortex useless, since, in practical problems, our concern is with conditions away from the central core.

Pressure Distribution in a Free Vortex Flow

- Pressure distribution in a vortex flow is usually found out by integrating the equation of motion in the r direction. The equation of motion in the radial direction for a vortex flow can be written as

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{V_\theta^2}{r} - g\cos\theta \tag{14.11}$$

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{V_\theta^2}{r} - g\frac{dz}{dr} \tag{14.12}$$

- Integrating Eq. (14.12) with respect to dr, and considering the flow to be incompressible we have,

$$\frac{p}{\rho} = \int \frac{V_\theta^2}{r} dr - gz + A \tag{14.13}$$

- For a free vortex flow,

$$V_\theta = \frac{C}{r}$$

- Hence Eq. 14.13 becomes

$$\frac{p}{\rho} = -\frac{C^2}{2r^2} - gz + A \tag{14.14}$$

- If the pressure at some radius **$r = r_a$,** is known to be the atmospheric pressure p_{atm} then equation (14.14) can be written as

$$\frac{p - p_{atm}}{\rho} = \frac{C^2}{2}\left(\frac{1}{r_a^2} - \frac{1}{r^2}\right) - g(z - z_a)$$
$$= \frac{\left(V_\theta^2\right)_{r=r_a}}{2} - \frac{V_\theta^2}{2} - g(z - z_a) \tag{14.15}$$

where z and z_a are the vertical elevations (measured from any arbitrary datum) at r and r_a.

- Equation (14.15) can also be derived by a straight forward application of Bernoulli's equation between any two points at $r = r_a$ and $r = r$.

- **In a free vortex flow total mechanical energy remains constant.** There is neither any energy interaction between an outside source and the flow, nor is there any dissipation of mechanical energy within the flow. The fluid rotates by virtue of some rotation previously imparted to it or because of some internal action.

- Some examples are a whirlpool in a river, the rotatory flow that often arises in a shallow vessel when liquid flows out through a hole in the bottom (as is often seen when water flows out from a bathtub or a wash basin), and flow in a centrifugal pump case just outside the impeller.

Cylindrical Free Vortex

- A cylindrical free vortex motion is conceived in a cylindrical coordinate system with axis z directing vertically upwards (Fig. 14.1), where at each horizontal cross-section, there exists a planar free vortex motion with tangential velocity given by Eq. (14.10).

- The total energy at any point remains constant and can be written as

$$\frac{p}{\rho}+\frac{C^2}{2r^2}+gz=H\,(Cons.) \qquad (14.16)$$

- The pressure distribution along the radius can be found from Eq. (14.16) by considering z as constant; again, for any constant pressure p, values of z, determining a surface of equal pressure, can also be found from Eq. (14.16).

- If p is measured in gauge pressure, then the value of z, where p = 0 determines the free surface (Fig. 14.1), if one exists.

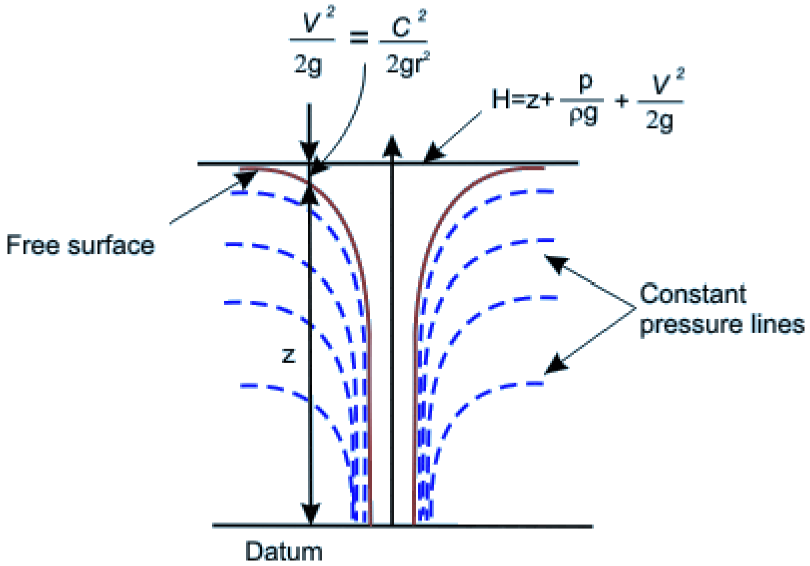


Fig 14.1 Cylindrical Free Vortex

Forced Vortex Flows

- Flows where streamlines are concentric circles and the tangential velocity is directly proportional to the radius of curvature are known as **plane circular forced vortex flows**.

- The flow field is described in a polar coordinate system as,

$$V_\theta = \omega r \qquad (14.17a)$$

and $V_r = 0$ (14.17b)

- All fluid particles rotate with the same angular velocity ω like a solid body. Hence a forced vortex flow is termed as a **solid body rotation**.

- The vorticity Ω for the flow field can be calculated as

$$\Omega = \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}$$

$$= \omega - \mathbf{0} + \omega = \mathbf{2}\omega$$

- Therefore, a forced vortex motion is not irrotational; rather it is a rotational flow with a constant vorticity 2ω. Equation (14.8) is used to determine the distribution of mechanical energy across the radius as

$$\frac{dH}{dr} = \frac{V_\theta}{g}\left(\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right) = \frac{2\omega^2 r}{g}$$

- Integrating the equation between the two radii on the same horizontal plane, we have,

$$H_2 - H_1 = \frac{\omega^2}{g}(r_2^2 - r_1^2) \qquad (14.18)$$

- Thus, we see from Eq. (14.18) that the total head (total energy per unit weight) increases with an increase in radius. The total mechanical energy at any point is the sum of kinetic energy, flow work or pressure energy, and the potential energy.

- Therefore the difference in total head between any two points in the same horizontal plane can be written as,

$$H_2 - H_1 = \left[\frac{p_2}{\rho g} - \frac{p_1}{\rho g}\right] + \left[\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right]$$
$$= \frac{p_2}{\rho g} - \frac{p_1}{\rho g} + \frac{\omega^2}{2g}(r_2^2 - r_1^2)$$

- Substituting this expression of H_2-H_1 in Eq. (14.18), we get

$$\frac{p_2 - p_1}{\rho} = \frac{\omega^2}{2}(r_2^2 - r_1^2)$$

- The same equation can also be obtained by integrating the equation of motion in a radial direction as

$$\int_1^2 \frac{1}{\rho}\frac{dp}{dr}dr = \int_1^2 \frac{V_\theta^2}{r}dr = \omega^2 \int_1^1 r\,dr$$
$$\frac{p_2 - p_1}{\rho} = \frac{\omega^2}{2}(r_2^2 - r_1^2)$$

- To maintain a forced vortex flow, mechanical energy has to be spent from outside and thus an external torque is always necessary to be applied continuously.
- Forced vortex can be generated by rotating a vessel containing a fluid so that the angular velocity is the same at all points.

Losses Due to Geometric Changes

- In case of flow of a real fluid, the major source for the loss of its total mechanical energy is the viscosity of fluid which causes friction between layers of fluid and between the solid surface and adjacent fluid layer.

Loss of Energy

- It is the role of friction, as an agent, to convert a part of the mechanical energy into intermolecular energy. This part of the mechanical energy converted into the intermolecular energy is termed as the **loss of energy**.
- When the path of the fluid is suddenly changed in course of its flow through a closed duct due to any abrupt change in the geometry of the duct then apart from the losses due to friction between solid surface and fluid layer past it, the loss of mechanical energy is also incurred. In long ducts, these losses are very small compared to the frictional loss, and hence they are often termed as **minor losses**.
- But minor losses may, however, outweight the friction loss in short pipes or ducts. The source of these losses is usually confined to a very short length of the duct, but the turbulence produced may persist for a considerable distance downstream.

Example of some minor Loss

- Losses Due to Sudden Enlargement
- Exit Loss
- Losses Due to Sudden Contraction
- Entry Loss

We'll discuss them individually in the next consequent slides

Losses Due to Sudden Enlargement

- If the cross-section of a pipe with fluid flowing through it, is abruptly enlarged (Fig. 14.2a) at certain place, fluid emerging from the smaller pipe is unable to follow the abrupt deviation of the boundary.
- The streamline takes a typical diverging pattern (shown in Fig. 14.2a). This creates pockets of turbulent eddies in the corners resulting in the dissipation of mechanical energy into intermolecular energy.

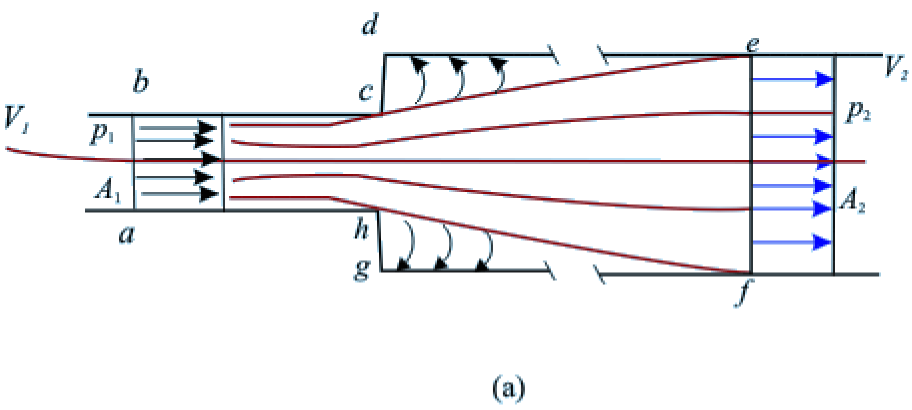
Basic mechanism of this type of loss

- The fluid flows against an adverse pressure gradient. The upstream pressure p_1 at section a-b is lower than the downstream pressure p_2 at section e-f since the upstream velocity V_1 is higher than the downstream velocity V_2 as a consequence of continuity.

- The fluid particles near the wall due to their low kinetic energy cannot overcome the adverse pressure hill in the direction of flow and hence follow up the reverse path under the favourable pressure gradient (from p_2 to p_1).

- This creates a zone of recirculating flow with turbulent eddies near the wall of the larger tube at the abrupt change of cross-section, as shown in Fig. 14.2a, resulting in a loss of total mechanical energy.

- For high values of Reynolds number, usually found in practice, the velocity in the smaller pipe may be assumed sensibly uniform over the crosssection. Due to the vigorous mixing caused by the turbulence, the velocity becomes again uniform at a far downstream section e-f from the enlargement (approximately 8 times the larger diameter).



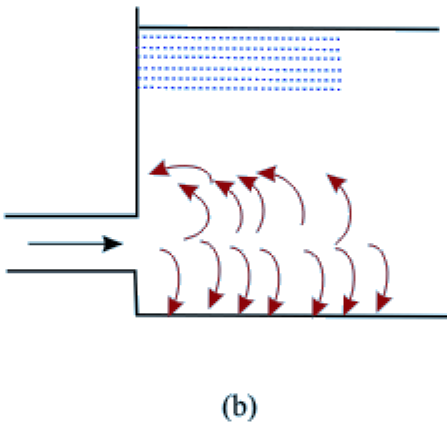


Fig 14.2 (a) Flow through abrupt but finite enlargement
(b) Flow at Infinite enlargement (Exit Loss)

- A control volume abcdefgh is considered (Fig. 14.2a) for which the momentum theorem can be written as

$$p_1A_1 + p'(A_2 - A_1) - p_2A_2 = \rho Q(V_2 - V_1) \tag{14.20}$$

where A_1, A_2 are the cross-sectional areas of the smaller and larger parts of the pipe respectively, Q is the volumetric fllow rate and p' is the mean pressure of the eddying fluid over the annular face, gd. It is known from experimental evidence, the $p' = p_1$.

- Hence the Eq. (14.20) becomes

$$(p_2 - p_1)A_2 = \rho Q(V_1 - V_2) \tag{14.21}$$

- From the equation of continuity

$$Q = V_2 A_2 \tag{14.22}$$

- With the help of Eq. (14.22), Eq. (14.21) becomes

$$p_2 - p_1 = \rho V_2 (V_1 - V_2) \tag{14.23}$$

- Applying Bernoulli's equation between sections ab and ef in consideration of the flow to be incompressible and the axis of the pipe to be horizontal, we can write

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gh_L$$

$$\frac{p_2 - p_1}{\rho} = \frac{V_1^2 - V_2^2}{2} - gh_L \tag{14.24}$$

where h_L is the loss of head. Substituting $(p_2 - p_1)$ from Eq. (14.23) into Eq. (14.24), we obtain

$$h_L = \frac{(V_1 - V_2)^2}{2g} = \quad \frac{V_1^2}{2g}\left[(1\text{-}(\frac{A_1}{A_2})\right]^2 \tag{14.25}$$

- In view of the assumptions made, Eq.(14.25) is subjected to some inaccuracies, but experiments show that for coaxial pipes they are within only a few per cent of the actual values.

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Exit Loss

- If, in Eq.(14.25), $A_2 \rightarrow \infty$, then the head loss at an abrupt enlargement tends to $V_1^2/2g$. The physical resemblance of this situation is the submerged outlet of a pipe discharging into a large reservoir as shown in Fig.14.2b.
- Since the fluid velocities are arrested in the large reservoir, the entire kinetic energy at the outlet of the pipe is dissipated into intermolecular energy of the reservoir through the creation of turbulent eddies.

- In such circumstances, the loss is usually termed as the **exit loss** for the pipe and **equals to the velocity head at the discharge end of the pipe.**

Losses Due to Sudden Contraction

- An abrupt contraction is geometrically the reverse of an abrupt enlargement (Fig. 14.3). Here also the streamlines cannot follow the abrupt change of geometry and hence gradually converge from an upstream section of the larger tube.
- However, immediately downstream of the junction of area contraction, the cross-sectional area of the stream tube becomes the minimum and less than that of the smaller pipe. This section of the stream tube is known as **vena contracta**, after which the stream widens again to fill the pipe.
- The velocity of flow in the converging part of the stream tube from Sec. 1-1 to Sec. c-c (vena contracta) increases due to continuity and the pressure decreases in the direction of flow accordingly in compliance with the Bernoulli's theorem.
- In an accelerating flow,under a favourable pressure gradient, losses due to separation cannot take place. But in the decelerating part of the flow from Sec. c-c to Sec. 2-2, where the stream tube expands to fill the pipe, losses take place in the similar fashion as occur in case of a sudden geometrical enlargement. Hence eddies are formed between the vena contracta c-c and the downstream Sec. 2-2.
- The flow pattern after the vena contracta is similar to that after an abrupt enlargement, and the loss of head is thus confined between Sec. c-c to Sec. 2-2. Therefore, **we can say that the losses due to contraction is not for the contraction itself, but due to the expansion followed by the contraction.**

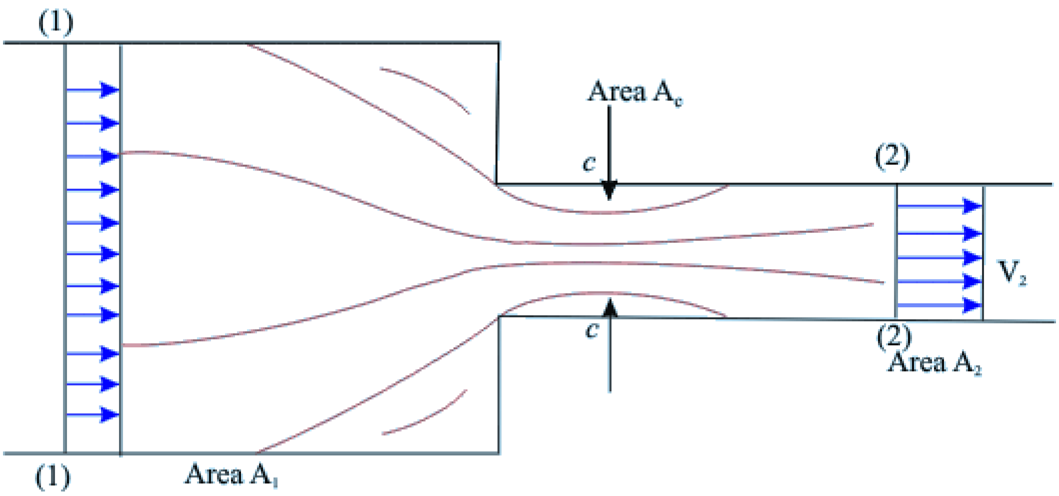

Fig 14.3 Flow through a sudden contraction

- Following Eq. (14.25), the loss of head in this case can be written as

$$h_L = \frac{V_2^2}{2g}\left[(\frac{A_2}{A_c}) - 1\right]^2 = \frac{V_2^2}{2g}\left[(\frac{1}{C_c}) - 1\right]^2 \qquad (14.26)$$

where A_c represents the **cross-sectional area of the vena contracta**, and C_c is the **coefficient of contraction** defined by

$$C_c = \frac{A_c}{A_2} \qquad (14.27)$$

- Equation (14.26) is usually expressed as

$$h_L = K\frac{V_2^2}{2g} \qquad (14.28)$$

where,

$$K = \left[(\frac{1}{C_c}) - 1\right]^2 \qquad (14.29)$$

- Although the area A_1 is not explicitly involved in the Eq. (14.26), **the value of C_c depends on the ratio A_2/A_1**. For coaxial circular pipes and at fairly high Reynolds numbers. Table 14.1 gives representative values of the coefficient K.

Table 14.1

A_2/A_1	0	0.04	0.16	0.36	0.64	1.0
K	0.5	0.45	0.38	0.28	0.14	0

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Entry Loss

- As $A_1 \to \infty$, the value of K in the Eq. (14.29) tends to 0.5 as shown in Table 14.1. This limiting situation corresponds to the flow from a large reservoir into a sharp edged pipe, provided the end of the pipe does not protrude into the reservoir (Fig. 14.4a).

- The loss of head at the entrance to the pipe is therefore given by $0.5\frac{V_2^2}{2g}$ and is known as **entry loss**.

- A protruding pipe (Fig. 14.4b) causes a greater loss of head, while on the other hand, if the inlet of the pipe is well rounded (Fig. 14.4c), the fluid can follow the boundary without separating from it, and the entry loss is much reduced and even may be zero depending upon the rounded geometry of the pipe at its inlet.

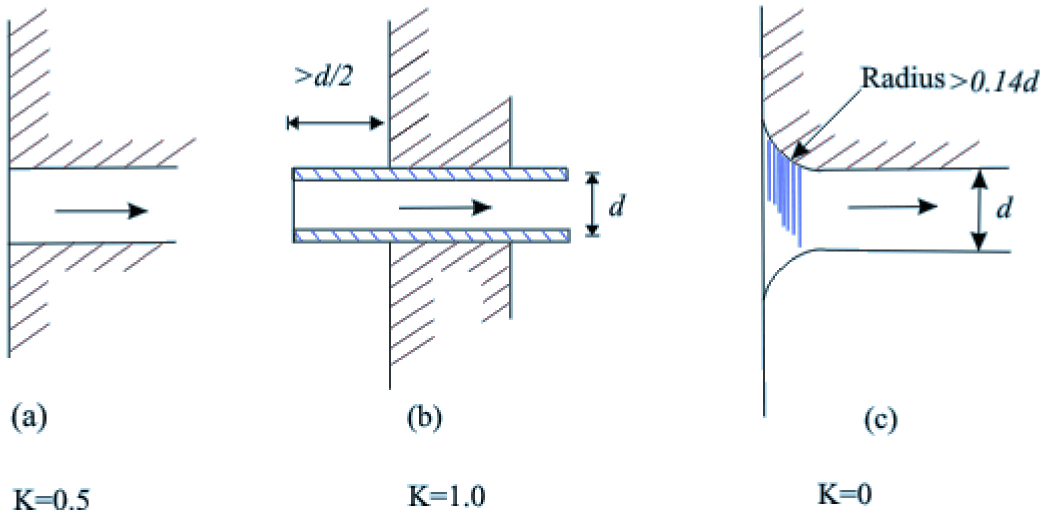

Fig 14.4 Flow from a reservoir to a sharp edges pipe

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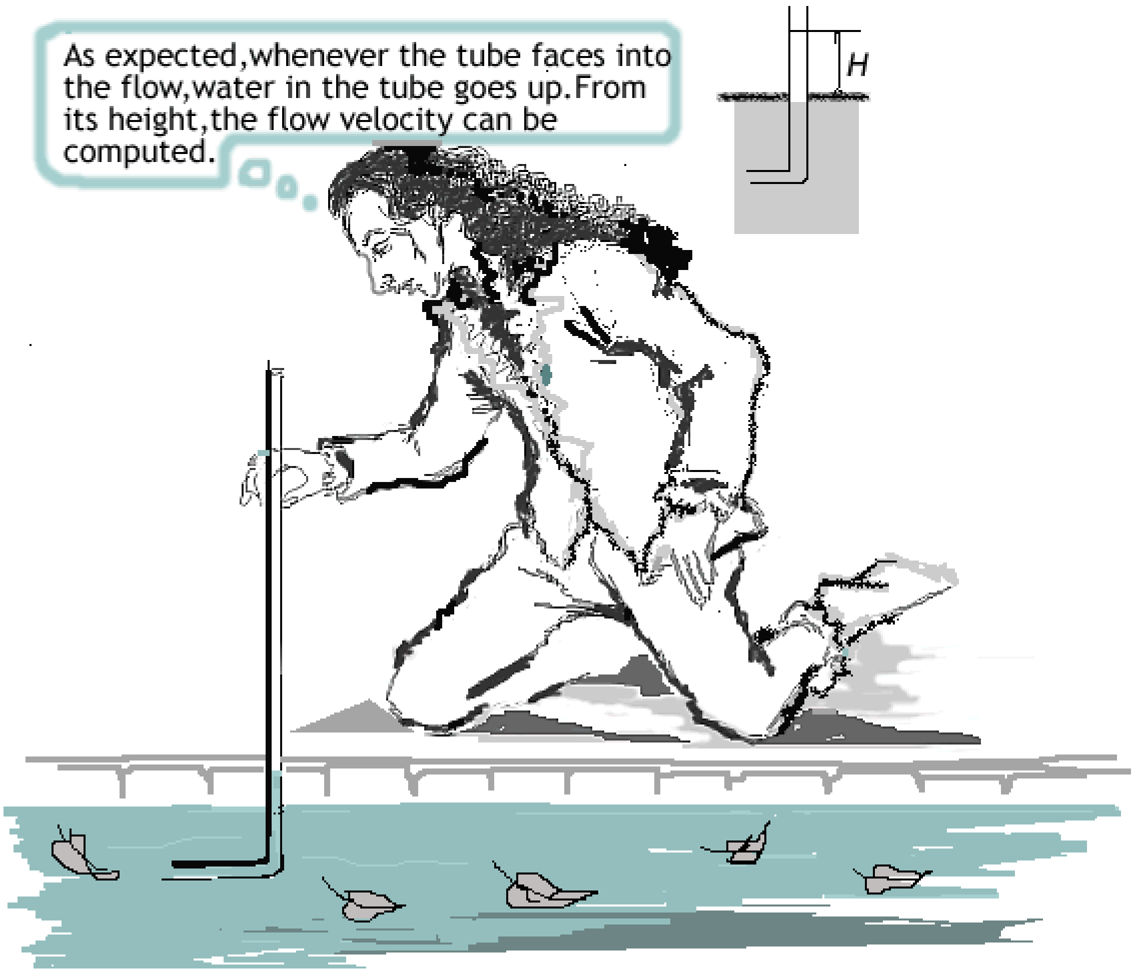
Measurement Of Flow Rate Through Pipe

Flow rate through a pipe is usually measured by providing a coaxial area contraction within the pipe and by recording the pressure drop across the contraction. Therefore the determination of the flow rate from the measurement of pressure drop depends on the straight forward application of Bernoulli's equation.

Three different flow meters operate on this principle.

- Venturimeter
- Orificemeter
- Flow nozzle.



Venturimeter

Construction: A venturimeter is essentially a short pipe (Fig. 15.1) consisting of two conical parts with a short portion of uniform cross-section in between. This short portion has the minimum area and is known as the throat. The two conical portions have the same base diameter, but one is having a shorter length with a larger cone angle while the other is having a larger length with a smaller cone angle.

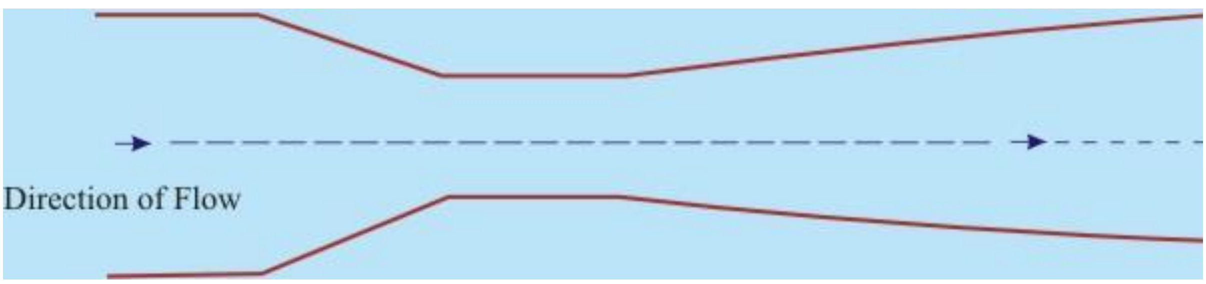


Fig 15.1 A Venturimeter

Working:

- The venturimeter is always used in a way that the upstream part of the flow takes place through the short conical portion while the downstream part of the flow through the long one.

- This ensures a rapid converging passage and a gradual diverging passage in the direction of flow to avoid the loss of energy due to separation. In course of a flow through the converging part, the velocity increases in the direction of flow according to the principle of continuity, while the pressure decreases according to Bernoulli's theorem.

- The velocity reaches its maximum value and pressure reaches its minimum value at the throat. Subsequently, a decrease in the velocity and an increase in the pressure takes place in course of flow through the divergent part. This typical variation of fluid velocity and pressure by allowing it to flow through such a constricted convergent-divergent passage was first demonstrated by an Italian scientist Giovanni Battista Venturi in 1797.

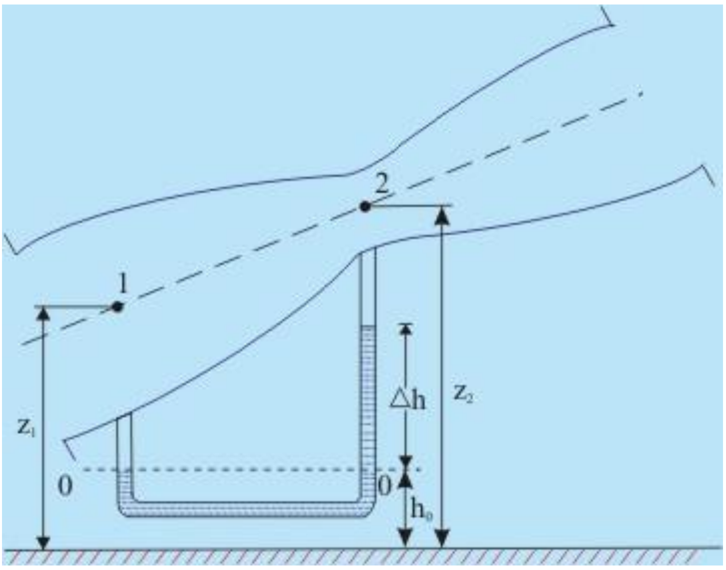


Fig 15.2 Measurement of Flow by a Venturimeter

- Figure 15.2 shows that a venturimeter is inserted in an inclined pipe line in a vertical plane to measure the flow rate through the pipe. Let us consider a steady, ideal and one dimensional (along the axis of the venturi meter) flow of fluid. Under this situation, the velocity and pressure at any section will be uniform.

- Let the velocity and pressure at the inlet (Sec. 1) are V_1 and p_1 respectively, while those at the throat (Sec. 2) are V_2 and p_2. Now, applying Bernoulli's equation between Secs 1 and 2, we get

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2 \qquad (15.1)$$

$$\frac{V_2^2 - V_1^2}{2g} = \frac{p_1 - p_2}{2g} + z_1 - z_2 \qquad (15.2)$$

where ρ is the density of fluid flowing through the venturimeter.

- From continuity,

$$V_1 A_1 = V_2 A_2 \qquad (15.3)$$

where A_1 and A_2 are the cross-sectional areas of the venturi meter at its throat and inlet respectively.

- With the help of Eq. (15.3), Eq. (15.2) can be written as

$$\frac{V_2^2}{2g}\left(1 - \frac{A_2^2}{A_1^2}\right) = \left(\frac{p_1}{2g} + z_1\right) - \left(\frac{p_2}{2g} + z_2\right)$$

$$V_2 = \frac{1}{\sqrt{1 - \frac{A_2^2}{A_1^2}}}\sqrt{2g(h_1^* - h_2^*)} \qquad (15.4)$$

where h_1^* and h_2^* are the piezometric pressure heads at sec. 1 and sec. 2 respectively, and are defined as

$$h_1^* = \frac{p_1}{\rho g} + z_1 \qquad (15.5a)$$

$$h_2^* = \frac{p_2}{\rho g} + z_2 \qquad (15.5b)$$

- Hence, the volume flow rate through the pipe is given by

$$Q = A_2 V_2 = \frac{A_2}{\sqrt{1 - \frac{A_2^2}{A_1^2}}}\sqrt{2g(h_1^* - h_2^*)} \qquad (15.6)$$

(Venturimeter continued....),

...Venturimeter... contd from previous slide

- If the pressure difference between Sections 1 and 2 is measured by a manometer as shown in Fig. 15.2, we can write

$$p_1 + \rho g(z_1 - h_o) = p_2 + \rho g(z_2 - h_o - \Delta h) + \Delta h \rho_m g$$

$$\text{or,} \quad (p_1 + \rho g z_1) - (p_2 + \rho g z_2) = (\rho_m - \rho) g \Delta h$$

$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = \left(\frac{\rho_m}{\rho} - 1\right)\Delta h$$

$$\text{or,} \quad h_1^* - h_2^* = \left(\frac{\rho_m}{\rho} - 1\right)\Delta h \qquad (15.7)$$

where ρ_m is the density of the manometric liquid.

- Equation (15.7) shows that a manometer always registers a direct reading of the difference in piezometric pressures. Now, substitution of $h_1^* - h_2^*$ from Eq. (15.7) in Eq. (15.6) gives

$$Q = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g(\rho_m / \rho - 1)\Delta h} \qquad (15.8)$$

- If the pipe along with the venturimeter is horizontal, then $z_1 = z_2$; and hence $h_1^* - h_2^*$ becomes $h_1 - h_2$, where h_1 and h_2 are the static pressure heads $\left(h_1 = \frac{p_1}{\rho g}, h_2 = \frac{p_2}{\rho g}\right)$
- The manometric equation Eq. (15.7) then becomes

$$h_1 - h_2 = \left[\frac{\rho_m}{\rho} - 1\right]\Delta h$$

- Therefore, it is interesting to note that the final expression of flow rate, given by Eq. (15.8), in terms of manometer deflection Δh, remains the same irrespective of whether the pipe-line along with the venturimeter connection is horizontal or not.
- Measured values of Δh, the difference in piezometric pressures between Secs I and 2, for a real fluid will always be greater than that assumed in case of an ideal fluid because of frictional losses in addition to the change in momentum.
- Therefore, Eq. (15.8) always overestimates the actual flow rate. In order to take this into account, a multiplying factor C_d, called the coefficient of discharge, is incorporated in the Eq. (15.8) as

$$Q_{actual} = C_d \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g(\rho_m / \rho - 1)\Delta h}$$

- The **coefficient of discharge** C_d is always less than unity and is defined as

$$C_d = \frac{\text{Actual rate of discharge}}{\text{Theoretical rate of discharge}}$$

where, the theoretical discharge rate is predicted by the Eq. (15.8) with the measured value of Δh, and the actual rate of discharge is the discharge rate measured in practice. **Value of C_d for a venturimeter usually lies between 0.95 to 0.98.**

Orificemeter

Construction: An orificemeter provides a simpler and cheaper arrangement for the measurement of fow through a pipe. An orificemeter is essentially a thin circular plate with a sharp edged concentric circular hole in it.

Working:

- The orifice plate, being fixed at a section of the pipe, (Fig. 15.3) creates an obstruction to the flow by providing an opening in the form of an orifice to the flow passage.

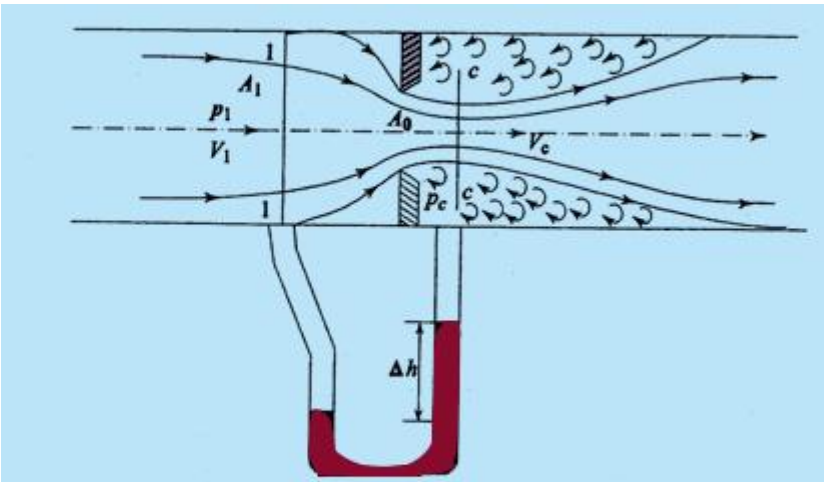


Fig 15.3 Flow through an Orificemeter

- The area A_0 of the orifice is much smaller than the cross-sectional area of the pipe. The flow from an upstream section, where it is uniform, adjusts itself in such a way that it contracts until a section downstream the orifice plate is reached, where the vena contracta is formed, and then expands to fill the passage of the pipe.

- One of the pressure tapings is usually provided at a distance of one diameter upstream the orifice plate where the flow is almost uniform (Sec. 1-1) and the other at a distance of half a diameter downstream the orifice plate.

- Considering the fluid to be ideal and the downstream pressure taping to be at the vena contracta (Sec. c-c), we can write, by applying Bernoulli's theorem between Sec. 1-1 and Sec. c-c,

$$\frac{p_1^*}{\rho g} + \frac{V_1^2}{2g} = \frac{p_c^*}{\rho g} + \frac{V_c^2}{2g} \qquad (15.10)$$

where p_1^* and p_c^* are the piezometric pressures at Sec.1-1 and c-c respectively.

- From the equation of continuity,

$$V_1 A_1 = V_c A_c \qquad (15.11)$$

where A_c is the area of the vena contracta.

- With the help of Eq. (15.11), Eq. (15.10) can be written as,

$$V_c = \sqrt{\frac{2(p_1^* - p_c^*)}{\rho\left(1 - A_c^2 / A_1^2\right)}} \qquad (15.12)$$

Correction in Velocity

- Recalling the fact that the measured value of the piezometric pressure drop for a real fluid is always more due to friction than that assumed in case of an inviscid flow, a coefficient of velocity C_v (always less than 1) has to be introduced to determine the actual velocity V_c when the pressure drop $p_1^* - p_c^*$ in Eq. (15.12) is substituted by its measured value in terms of the manometer deflection **'Δh'**

Hence,

$$V_c = C_v \sqrt{\frac{2\rho g(\rho_m / \rho - 1)\Delta h}{1 - A_c^2 / A_1^2}} \qquad (15.13)$$

where 'Δh' is the difference in liquid levels in the manometer and ρ_m is the density of the manometric liquid.

Volumetric flow rate

$$Q = A_c V_c \qquad (15.14)$$

- If a **coefficient of contraction** C_c is defined as, $C_c = A_c/A_0$, where A_0 is the area of the orifice, then Eq.(15.14) can be written, with the help of Eq. (15.13),

$$Q = C_c A_0 C_v \sqrt{\frac{2g(\rho_m/\rho - 1)\Delta h}{1 - C_c^2 A_0^2 / A_1^2}}$$

$$= C_c A_0 C_v \sqrt{\frac{2g}{1 - C_c^2 A_0^2 / A_1^2}} \sqrt{(\rho_m/\rho - 1)\Delta h}$$

$$= C\sqrt{(\rho_m/\rho - 1)\Delta h} \tag{15.15}$$

$$\text{with,} \qquad C = C_d A_0 \sqrt{\frac{2g}{1 - C_c^2 A_0^2 / A_1^2}}, \; where \; (C_d = C_v C_c)$$

The value of C depends upon the ratio of orifice to duct area, and the Reynolds number of flow.

- The main job in measuring the flow rate with the help of an orificemeter, is to find out accurately the value of C at the operating condition.

- The downstream manometer connection should strictly be made to the section where the vena contracta occurs, but this is not feasible as the vena contracta is somewhat variable in position and is difficult to realize.

- In practice, various positions are used for the manometer connections and C is thereby affected. **Determination of accurate values of C of an orificemeter at different operating conditions is known as calibration of the orifice meter.**

Flow Nozzle

- The flow nozzle as shown in Fig.15.4 is essentially a venturi meter with the divergent part omitted. Therefore the basic equations for calculation of flow rate are the same as those for a venturimeter.

- The dissipation of energy downstream of the throat due to flow separation is greater than that for a venturimeter. But this disadvantage is often offset by the lower cost of the nozzle.

- The downstream connection of the manometer may not necessarily be at the throat of the nozzle or at a point sufficiently far from the nozzle.

- The deviations are taken care of in the values of C_d, **The coefficient C_d depends on the shape of the nozzle, the ratio of pipe to nozzle diameter and the Reynolds number of flow.**

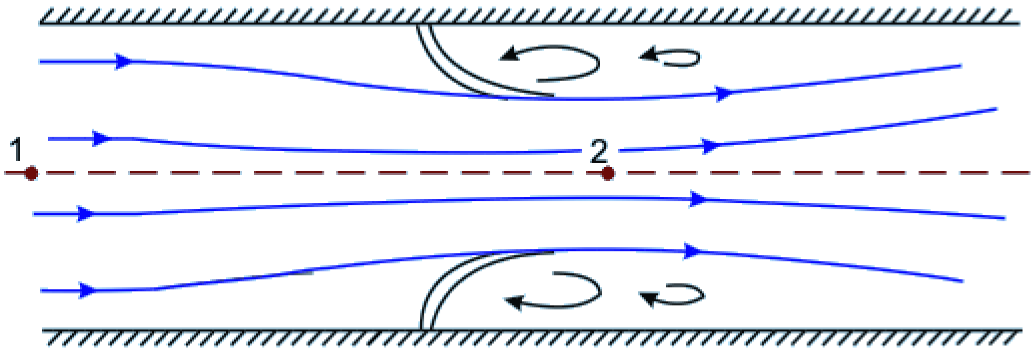


Fig 15.4 A Flow nozzle

- A comparative picture of the typical values of C_d, accuracy, and the cost of three flow meters (venturimeter, orificemeter and flow nozzle) is given below:

Type of Flowmeter	Accuracy	Cost	Loss of Total Head	Typical Values of C_d
Venturimeter	High	High	Low	0.95 to 0.98
Orificemeter	Low	Low	High	0.60 to 0.65
Flow Nozzle	Intermediate between a venturimeter and an orificemeter			0.70 to 0.80

Concept of Static Pressure

- **The thermodynamic or hydrostatic pressure caused by molecular collisions is known as static pressure in a fluid flow and is usually referred to as the pressure *p*.**

- When the fluid is at rest, this pressure p is the same in all directions and is categorically known as the **hydrostatic pressure**.

- For the flow of a real and Stoksian fluid (the fluid which obeys Stoke's law) the static or thermodynamic pressure becomes equal to the arithmetic average of the normal stresses at a point. The static pressure is a parameter to describe the state of a flowing fluid.

- Let us consider the flow of a fluid through a closed passage as shown in Fig. 16.1a.

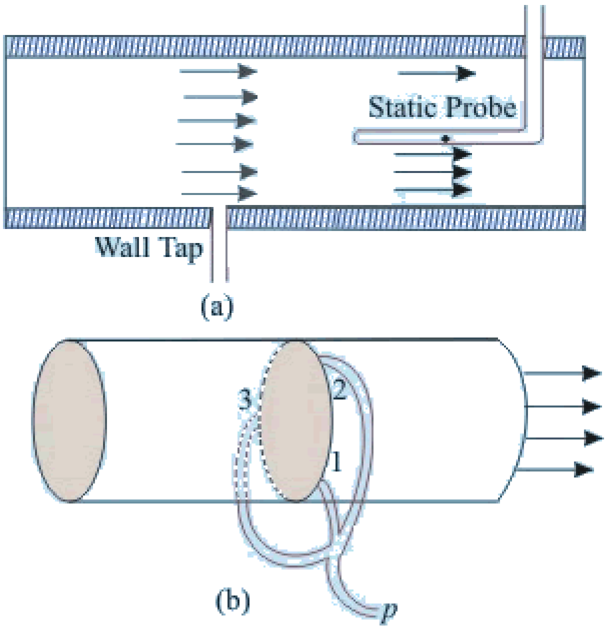


Fig 16.1 Measurement of Static pressure
(a) Single Wall tap (b) Multiple Wall Tap

- If a hole is made at the wall and is connected to any pressure measuring device, it will then sense the static pressure at the wall. This type of hole at the wall is known as a **wall tap.**

- The fact that a wall tap actually senses the static pressure can be appreciated by noticing that there is no component of velocity along the axis of the hole.

- In most circumstances, for example, in case of parallel flows, the static pressure at a cross-section remains the same. The wall tap under this situation registers the static pressure at that cross-section.

- In practice, instead of a single wall tap, a number of taps along the periphery of the wall are made and are mutually connected by flexible tubes (Fig. 16.1b) in order to register the static pressure more accurately.

Hydrostatic, Hydrodynamic, Static and Total Pressure

- Let us consider a fluid flowing through a pipe of varying cross sectional area. Considering two points A and B as shown in Figure 16.1(c), such that A and B are at a height Z_A and Z_B respectively from the datum.

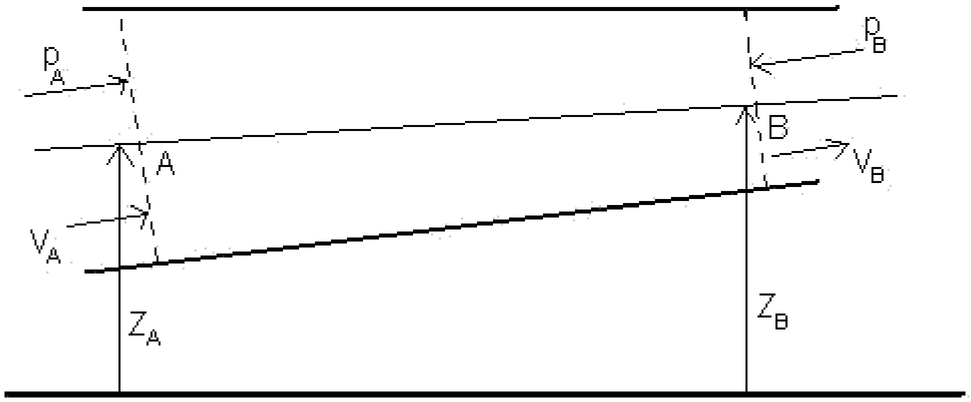


Figure 16.1 (c)

- If we consider the fluid to be stationary, then, $\left(\frac{\partial p}{\partial z}\right)_{hs} = -\rho g$ where the subscript 'hs' represents the hydrostatic case.

So, $p_{Ahs} - p_{Bhs} = \rho g(Z_B - Z_A)$ (16.1)

where p_{Ahs} is the hydrostatic pressure at A and p_{Bhs} is the hydrostatic pressure at B.

- Thus, from above we can conclude that the Hydrostatic pressure at a point in a fluid is the pressure acting at the point when the fluid is at rest or pressure at the point due to weight of the fluid above it.

- Now if we consider the fluid to be moving, the pressure at a point can be written as a sum of two components, Hydrodynamic and Hydrostatic.

$p_A = p_{Ahs} + p_{Ahd}$ (16.2)

where p_{Ahs} is the hydrostatic pressure at A and p_{Ahd} is the hydrodynamic pressure at A.

- Using equation (16.2) in Bernoulli's equation between points A and B.

$$\frac{p_{Ahd} - p_{Bhd}}{\rho g} + \left[\frac{p_{Ahs} - p_{Bhs}}{\rho g} + (Z_A - Z_B)\right] = \frac{V_B^2 - V_A^2}{2g} \qquad (16.3)$$

From equation (16.1), the terms within the square bracket cancel each other.

Hence,

$$\frac{p_{Ahd} - p_{Bhd}}{\rho g} = \frac{V_B^2 - V_A^2}{2g} \qquad (16.4)$$

$$p_{Ahd} + \frac{\rho V_A^2}{2} = p_{Bhd} + \frac{\rho V_B^2}{2} = C = p_0 \qquad (16.5)$$

- Equations (16.4) and (16.5) convey the following. **The pressure at a location has both hydrostatic and hydrodynamic components. The difference in kinetic energy arises due to hydrodynamic components only.**
- In a frictionless flow, the sum of flow work due to hydrodynamic pressure and the kinetic energy is conserved. Such conservation shall apply to the entire flow field if the flow is irrotational.
- **The hydrodynamic component is often called static pressure and the velocity term, dynamic pressure. The sum of two, p_0 is known as total pressure. This is conserved in isentropic, irrotational flow.**

Stagnation Pressure

- The stagnation pressure at a point in a fluid flow is the pressure which could result if the fluid were brought to rest isentropically.
- The word isentropically implies the sense that the entire kinetic energy of a fluid particle is utilized to increase its pressure only. This is possible only in a reversible adiabatic process known as **isentropic process**.

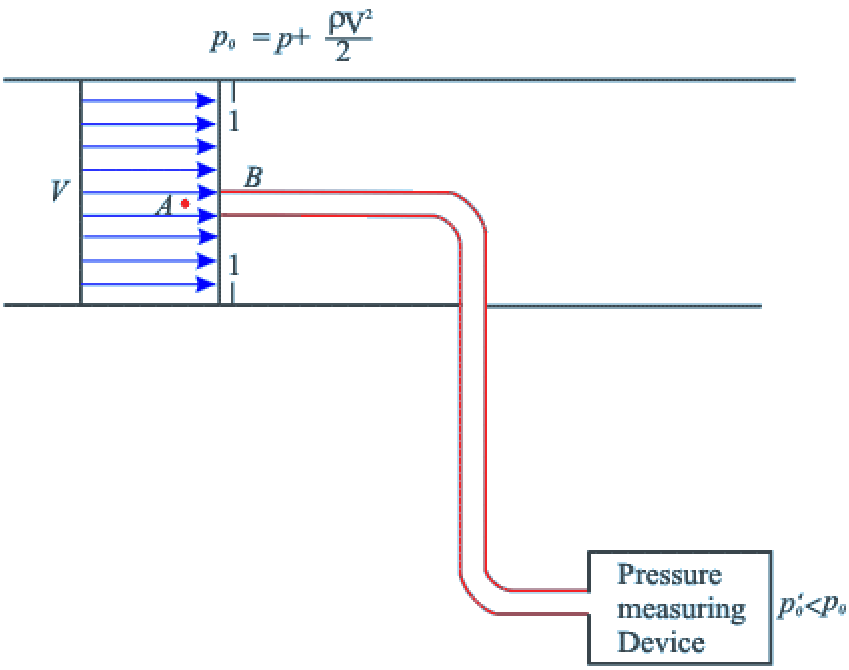


Fig 16.2 Measurement of Stagnation Pressure

- Let us consider the flow of fluid through a closed passage (Fig. 16.2). At Sec. l-l let the velocity and static pressure of the fluid be uniform. Consider a point A on that section just in front of which a right angled tube with one end facing the flow and the other end closed is placed.

- When equilibrium is attained, the fluid in the tube will be at rest, and the pressure at any point in the tube including the point B will be more than that at A where the flow velocity exists.

- By the application of Bernoulli's equation between the points B and A, in consideration of the flow to be inviscid and incompressible, we have,

$$p_0 = p + \frac{\rho V^2}{2} \qquad (16.6)$$

where p and V are the pressure and velocity respectively at the point A at Sec. I-I, and p_0 is the pressure at B which, according to the definition, refers to the stagnation pressure at point A.

- It is found from Eq. (16.6) that the stagnation pressure p_0 consists of two terms, the **static pressure**, **p** and the term $\boldsymbol{\rho V^2/2}$ which is known as **dynamic pressure**. Therefore Eq. (16.6) can be written for a better understanding as

$$V = \sqrt{2\left(\frac{p_0 - p}{\rho}\right)} \qquad (16.7)$$

- Therefore, it appears from Eq.(16.7), that from a measurement of both static and stagnation pressure in a flowing fluid, the velocity of flow can be determined.

- But it is difficult to measure the stagnation pressure in practice for a real fluid due to friction. The pressure p_0' in the stagnation tube indicated by any pressure measuring device (Fig. 16.2) will always be less than p_0, since a part of the kinetic energy will be converted into intermolecular energy due to fluid friction). This is taken care of by an empirical factor C in determining the velocity from Eq. (16.7) as

$$V = C\sqrt{2\left(\frac{p_0 - p}{\rho}\right)} \qquad (16.8)$$

Pitot Tube for Flow Measurement

Construction: The principle of flow measurement by Pitot tube was adopted first by a French Scientist Henri Pitot in 1732 for measuring velocities in the river. A right angled glass tube, large enough for capillary effects to be negligible, is used for the purpose. One end of the tube faces the flow while the other end is open to the atmosphere as shown in Fig. 16.3a.

Working:

- The liquid flows up the tube and when equilibrium is attained, the liquid reaches a height above the free surface of the water stream.

- Since the static pressure, under this situation, is equal to the hydrostatic pressure due to its depth below the free surface, the difference in level between the liquid in the glass tube and the free surface becomes the measure of dynamic pressure. Therefore, we can write, neglecting friction,

$$\boxed{p_0 - p = \frac{\rho V^2}{2} = h\rho g}$$

where p_0, p and V are the stagnation pressure, static pressure and velocity respectively at point A (Fig. 16.3a).

$$\boxed{\mathbf{V} = \sqrt{2gh}}$$

- Such a tube is known as a **Pitot tube** and provides one of the most accurate means of measuring the fluid velocity.

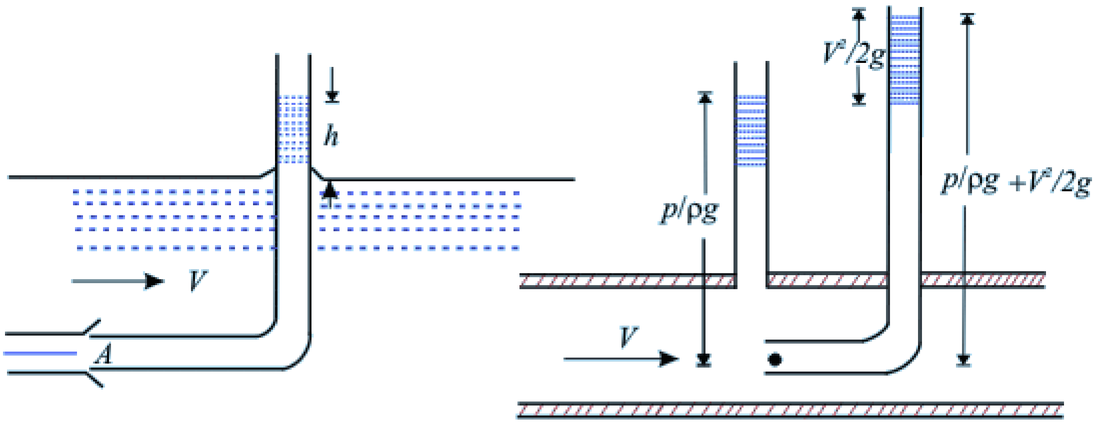


Fig 16.3 Simple Pitot Tube (a) tube for measuring the Stagnation Pressure
(b) Static and Stagnation tubes together

- For an open stream of liquid with a free surface, this single tube is suffcient to determine the velocity. But for a fluid flowing through a closed duct, the Pitot tube measures only the stagnation pressure and so the static pressure must be measured separately.

- Measurement of static pressure in this case is made at the boundary of the wall (Fig. 16.3b). The axis of the tube measuring the static pressure must be perpendicular to the boundary and free from burrs, so that the boundary is smooth and hence the streamlines adjacent to it are not curved. This is done to sense the static pressure only without any part of the dynamic pressure.

- A Pitot tube is also inserted as shown (Fig. 16.3b) to sense the stagnation pressure. The ends of the Pitot tube, measuring the stagnation pressure, and the piezometric tube, measuring the static pressure, may be connected to a suitable differential manometer for the determination of flow velocity and hence the flow rate.

Pitot Static Tube

- The tubes recording static pressure and the stagnation pressure (Fig. 16.3b) are usually combined into one instrument known as **Pitot static tube** (Fig. 16.4).

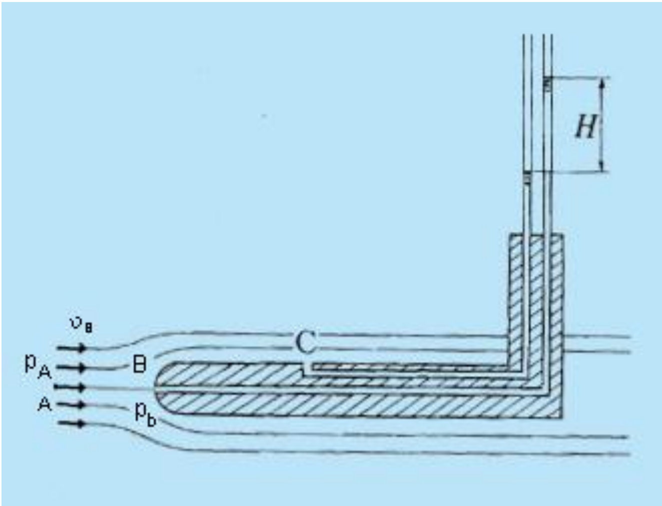


Fig 16.4 Pitot Static Tube

- The tube for sensing the static pressure is known as static tube which surrounds the pitot tube that measures the stagnation pressure.
- Two or more holes are drilled radially through the outer wall of the static tube into annular space. The position of these static holes is important. Downstream of the nose N, the flow is accelerated somewhat with consequent reduction in static pressure. But in front of the supporting stem, there is a reduction in velocity and increase in pressure.
- The static holes should therefore be at the position where the two opposing effects are counterbalanced and the reading corresponds to the undisturbed static pressure. Finally the flow velocity is given by

$$V = C\sqrt{2\left(\frac{\Delta p}{\rho}\right)} \qquad (16.9)$$

where Δp is the difference between stagnation and static pressures.

- **The factor C takes care of the non-idealities, due to friction, in converting the dynamic head into pressure head and depends, to a large extent, on the geometry of the pitot tube. The value of C is usually determined from calibration test of the pitot tube.**

Flow Through Orifices And Mouthpieces

- **An orifice is a small aperture through which the fluid passes.** The thickness of an orifice in the direction of flow is very small in comparison to its other dimensions.

- If a tank containing a liquid has a hole made on the side or base through which liquid flows, then such a hole may be termed as an orifice.The rate of flow of the liquid through such an orifice at a given time will depend partly on the shape, size and form of the orifice.

- An orifice usually has a sharp edge so that there is minimum contact with the fluid and consequently minimum frictional resistance at the sides of the orifice. If a sharp edge is not provided, the flow depends on the thickness of the orifice and the roughness of its boundary surface too.

Flow from an Orifice at the Side of a Tank under a Constant Head

- Let us consider a tank containing a liquid and with an orifice at its side wall as shown in Fig. 16.5. The orifice has a sharp edge with the bevelled side facing downstream. Let the height of the free surface of liquid above the centre line of the orifice be kept fixed by some adjustable arrangements of inflow to the tank.

- The liquid issues from the orifice as a free jet under the influence of gravity only. The streamlines approaching the orifice converges towards it. Since an instantaneous change of direction is not possible, the streamlines continue to converge beyond the orifice until they become parallel at the Sec. c-c (Fig. 16.5).

- For an ideal fluid, streamlines will strictly be parallel at an infinite distance, but however fluid friction in practice produce parallel flow at only a short distance from the orifice. The area of the jet at the Sec. c-c is lower than the area of the orifice. The Sec. c-c is known as the **vena contracta**.

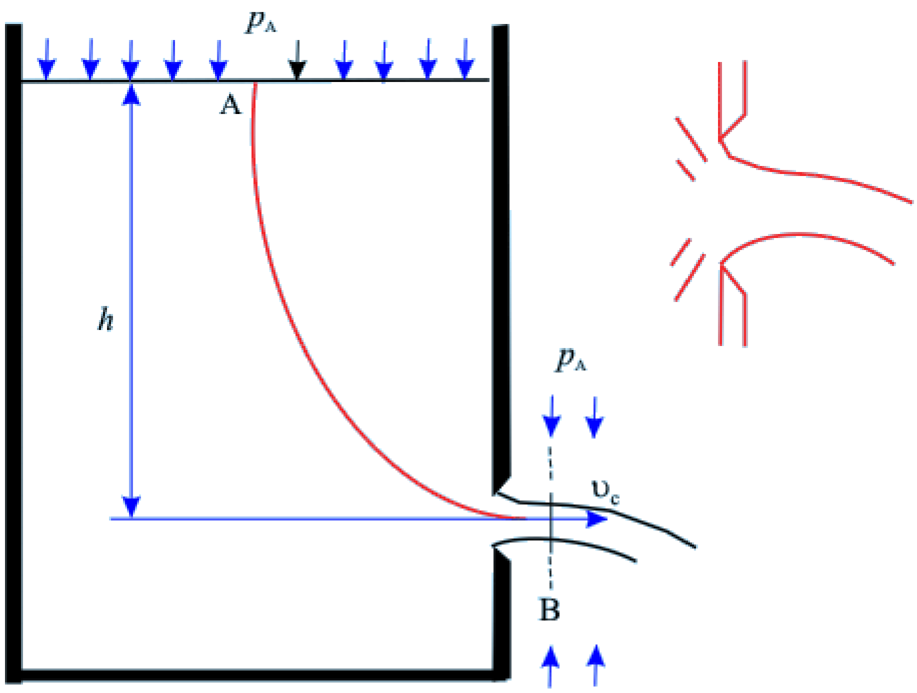


Fig 16.5 Flow from a Sharp edged Orifice

- **The contraction of the jet can be attributed to the action of a lateral force on the jet due to a change in the direction of flow velocity when the fluid approaches the orifice.** Since the streamlines become parallel at vena contracta, the pressure at this section is assumed to be uniform.

- If the pressure difference due to surface tension is neglected, the pressure in the jet at vena contracta becomes equal to that of the ambience surrounding the jet.

- Considering the flow to be steady and frictional effects to be negligible, we can write by the application of Bernoulli's equation between two points 1 and 2 on a particular stream-line with point 2 being at vena contracta (Fig 16.5).

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_{atm}}{\rho g}+\frac{V_2^2}{2g}+0 \qquad (16.10)$$

- The horizontal plane through the centre of the orifice has been taken as datum level for determining the potential head.

- If the area of the tank is large enough as compared to that of the orifice, the velocity at point 1 becomes negligibly small and pressure p1 equals to the hydrostatic pressure p_1 equals to the hydrostatic pressure at that point as $p_1=p_{atm}+\rho g(h-z_1)$.

- Therefore, Eq. (16.10) becomes

$$\frac{p_{atm}}{\rho g}+(h-z_1)+0+z_1=\frac{p_{atm}}{\rho g}+\frac{V_2^2}{2g} \qquad (16.11)$$

or, $V_2 = \sqrt{2gh}$ (16.12)

- If the orifice is small in comparison to h, the velocity of the jet is constant across the vena contracta. The Eq. (16.12) states that the velocity with which a jet of liquid escapes from a small orifice is proportional to the square root of the head above the orifice, and is known as **Torricelli's formula**.
- The velocity V_2 in Eq. (16.12) represents the ideal velocity since the frictional effects were neglected in the derivation. Therefore, a multiplying factor C_v known as **coefficient of velocity** is introduced to determine the actual velocity as

$$V_{2actual} = C_v \sqrt{2gh}$$

- Since the role of friction is to reduce the velocity, C_v is always less than unity. The **rate of discharge** through the orifice can then be written as,

$$Q = a_c C_v \sqrt{2gh} \qquad (16.13)$$

where a_c is the cross-sectional area of the jet at vena contracta.

- Defining a **coefficient of contraction** C_c as the **ratio of the area of vena contracta to the area of orifiice**, Eq. (16.8) can be written as

$$Q = a_0 C_c C_v \sqrt{2gh} \qquad (16.14)$$

where, a_0 is the cross-sectional area of the orifice. The product of C_c and C_v is written as C_d and is termed as **coefficient of discharge**. Therefore,

$$Q = a_0 C_d \sqrt{2gh}$$

$$C_d = \frac{Q}{a_0 \sqrt{2gh}} = \frac{Actual\ discharge}{Ideal\ discharge}$$

Exercise Problems - Chapter 5

1. An open cylindrical tank 2m high and 1 m in diameter, is filled with water to a depth of 1m. If the cylinder rotates about its vertical axis. Determine

(a) the maximum angular velocity of the cylinder without spilling any water

(b) the depth at the center and gauge pressure at the bottom 0·2 m from the center at the condition of maximum angular velocity

(6.26 rad/s, zero, 783.75 Pa.)

2. The velocity distribution for laminar flow between two parallel plates is given by

$$V = V_o\left(1 - \frac{y^2}{h^2}\right)$$

where V_o is the center plane velocity, h is the half spacing between the plates and y is the normal distance from the center plane. Determine the

(a) average velocity
(b) momentum correction factor

3. Flow of air at 50°C is measured by a pilot-static tube. The differential reading in a water manometer is 24 mm. Determine the velocity of air if the coefficient of tube is 0·95. Assume the density of air to be constant at 1·2 kg/m^3

(6 m/s)

4. Consider a short cylindrical duct whose cross-section enlarges abruptly from a diameter D_1 to a diameter D_2 . Find the ratio D_1 /D_2 so that the pressure drop for a given flow rate of a fluid through the duct is independent of the direction of flow. Neglect the losses due to skin friction. (Take co efficient of contraction C_c = 0.6)

(0.577)

5. A venturimeter is placed at 30° to the horizontal (sloping upwards in the direction of flow) to a pipe line carrying an oil of specific gravity 0·8. A differential manometer with mercury as the manometric fluid is attached to the inlet and throat of the venturimeter. The manometer shows a deflection (the difference in height between the menisci of mercury at the two limbs) of 100mm. The pipe diameter is 200 mm, while the diameter of venturi throat is 100 mm.

(a) Find the volume flow rate of oil if the coefficient of discharge of the venturimeter is 0·96.
(b) What will be the reading of differential manometer if the venturimeter is turned horizontal? The length of venturimeter between the inlet and the throat is 320 mm.

(0.044 m^3 /s, 110 mm)

6. Fig. 16.6 shows the weight W supported by the pressure difference created in the venturi and applied across the piston. At the inlet the air velocity is 10 m/s and the area is 100 cm^2. The throat area is 4 cm^2. Given that the air density is equal to 1.2 kg/m^3, calculate the weight W that will be in equilibrium. Assume ideal flow in the duct, no friction between the piston and cylinder, and the piston has no mass.

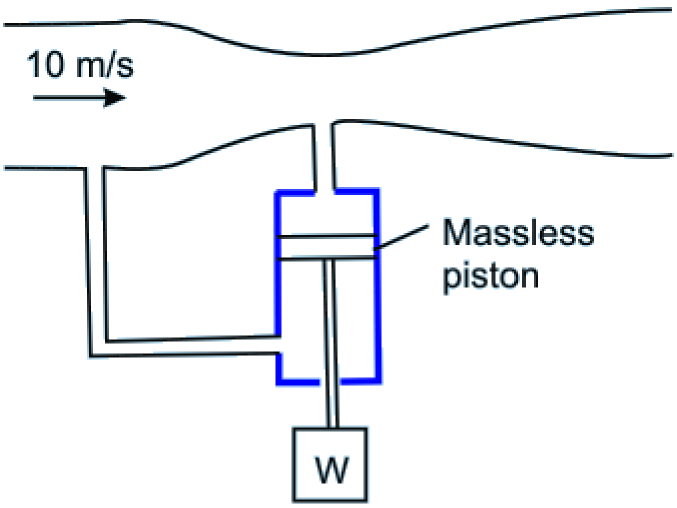


Figure 16.6

(18.72 N)

7. A fire-fighting hose with an exit nozzle diameter = 50 mm discharges water at the rate of 3000 litres per minute. The height of the nozzle exit from the pump is 4 m and water level in the sump is 1 m below the pump (see Fig. 16.7). Calculate the head and power developed by the pump. Assume there are no losses in the pipeline and pump.

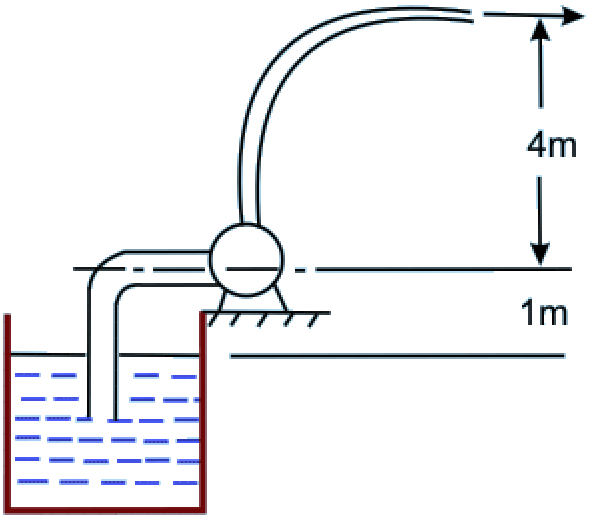
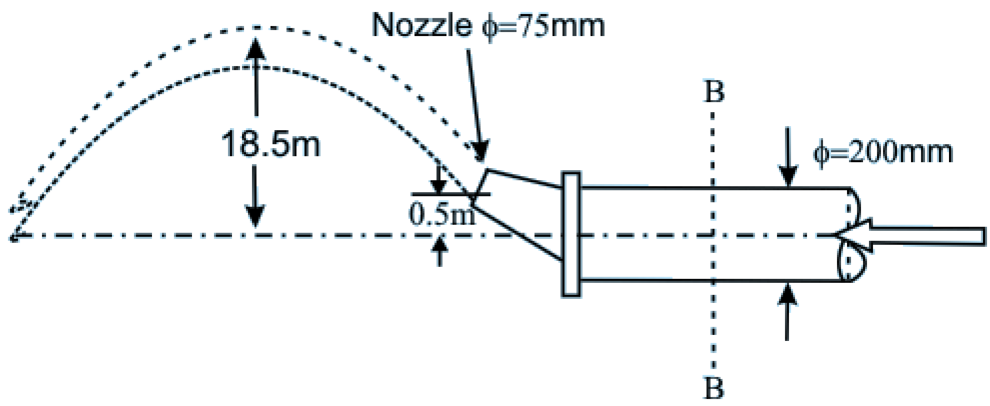


Figure 16.7

(38.17 m, 18.72 kw)

8. A free jet of water is produced using a 75 mm diameter nozzle attached to a 200 mm diameter pipe, as shown in Figure 16.8. If the average velocity of water at plane B is 3.8 m/s, calculate the velocity of water at point A in the free jet. Neglect friction losses in the nozzle and pipe.



(19.4 m/s)

Figure 16.8

Recap

In this course you have learnt the following

- The total mechanical energy of a fluid element in an inviscid and irrotational flow remains the same everywhere in the flow field, while it does so only along a streamline in an inviscid but rotational flow.
- Flows having only tangential velocities with streamlines as concentric circles are known as plane circular vortex flows. A free vortex flow is an irrotational vortex flow where the total mechanical energy of the fluid elements remains same in the entire flow field and the tangential velocity is inversely proportional to the radius of curvature. A forced vortex flow is a rotational vortex flow where the tangential velocity is directly proportional to the radius of curvature. Pressure in vortex flows increases with an increase in the radius of curvature. Spiral vortex flows are obtained as a result of superimposition of a plane circular vortex flow with a purely radial flow.
- Apart from losses due to friction, the loss of mechanical energy is incurred, in course of flow through a closed duct, when the path of the fluid stream is suddenly changed due to any abrupt change in the geometry of the duct. In long ducts, these losses are very small as compared to the friction loss and hence they are termed as minor losses. These include (i) losses due to an abrupt enlargement of the cross-section of a duct, (ii) losses due to an abrupt contraction of the cross-section of a duct, (iii) losses due to the exist from a small pipe or duct to a large reservoir, and (iv) losses due to the entrance from a large reservoir to a small pipe or duct.
- Venturimeter, Orificemeter and Flow nozzle are the typical flow meters which measure the rate of flow of a fluid through a pipe by providing a coaxial area contraction within the pipe and thus creating a pressure drop across the contraction. The flow rate is measured by determining the velocity of flow at the constricted section in terms of the pressure drop by the application of

Bernoulli's equation.

- A venturimeter is a short pipe consisting of two conical parts with a sort uniform cross-section, in between, known as throat.

- An orificementer is a thin circular plate with a sharp edged concentric circular hole in it.

- A flow nozzle is a short conical tube providing only a convergent passage to the flow. In a comparison between the three flow meters, a venturimeter is the most accurate but the most expensive, while the orificemeter is the least expensive but the least accurate. Flow nozzle falls in between these two.

- The static pressure in a fluid is the thermodynamic pressure defining the state of fluid and becomes equal to the arithmetic average of the normal stresses at a point in case of a real and Stoksian fluid. The stagnation pressure at a point in a fluid flow is the pressure which could result if the fluid were brought to rest isentropically. The difference between the stagnation and static, pressure is the pressure equivalence of the velocity head ($1/2\rho V^2$) and is known as dynamic pressure.

- An instrument which contains tubes to record the stagnation and static pressures in a flow to finally determine the flow velocity and flow rate is known as a Pitot static tube.

- An orifice is a small aperture through which the fluid passes. The liquid from a tank is usually discharged through a small orifice at its side. A drowned or submerged orifice is one which does not discharge into open atmosphere, but discharge into liquid of the same kind. The discharge through an orifice is increased by fitting a short length of pipe to the outside known as external mouthpiece. The discharge rate is increased due to a decrease in the pressure at vena contracta within the mouthpiece resulting in an increase in the effective head causing the flow.

Principles of Physical Similarity - An Introduction

Laboratory tests are usually carried out under altered conditions of the operating variables from the actual ones in practice. These variables, in case of experiments relating to fluid flow, are pressure, velocity, geometry of the working systems and the physical properties of the working fluid.

The pertinent questions arising out of this situation are:

1. How to apply the test results from laboratory experiments to the actual problems?

2. Is it possible, to reduce the large number of experiments to a lesser one in achieving the same objective?

Answer of the above two questions lies in the principle of physical similarity. This principle is useful for the following cases:

1. To apply the results taken from tests under one set of conditions to another set of conditions

and

2. To predict the influences of a large number of independent operating variables on the performance of a system from an experiment with a limited number of operating variables.

Concept and Types of Physical Similarity

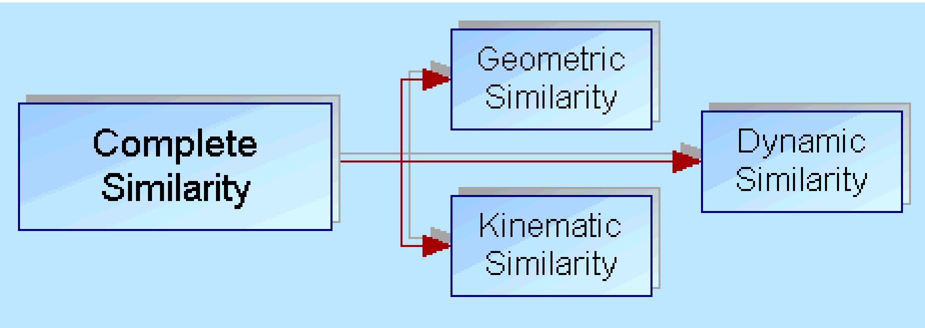
The primary and fundamental requirement for the **physical similarity** between two problems is that the **physics of the problems must be the same**.

For an example, two flows: one governed by viscous and pressure forces while the other by gravity force cannot be made physically similar. Therefore, the laws of similarity have to be sought between problems described by the same physics.

Definition of physical similarity as a general proposition.

Two systems, described by the same physics, operating under different sets of conditions are said to be physically similar in respect of certain specified physical quantities; when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

In the field of mechanics, there are three types of similarities which constitute the complete similarity between problems of same kind.



Geometric Similarity : If the specified physical quantities are geometrical dimensions, the similarity is called Geometric Similarity,

Kinematic Similarity : If the quantities are related to motions, the similarity is called Kinematic Similarity

Dynamic Similarity : If the quantities refer to forces, then the similarity is termed as Dynamic Similarity.

Geometric Similarity

- Geometric Similarity implies the similarity of shape such that, the **ratio of any length in one system to the corresponding length in other system is the same everywhere**.

- This ratio is usually known as **scale factor**.

Therefore, geometrically similar objects are similar in their shapes, i.e., proportionate in their physical dimensions, but differ in size.

In investigations of physical similarity,

- the full size or **actual scale systems** are known as **prototypes**

- the **laboratory scale systems** are referred to as **models**

- use of the same fluid with both the prototype and the model is not necessary

- model need not be necessarily smaller than the prototype. The flow of fluid through an injection nozzle or a carburettor , for example, would be more easily studied by using a model much larger than the prototype.

- the model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity, and properties of the fluid.

If l_1 and l_2 are the two characteristic physical dimensions of any object, then the requirement of geometrical similarity is

$$\frac{l_{1m}}{l_{1p}} = \frac{l_{2m}}{l_{2p}} = l_r$$

(**model ratio**)

(The second suffices *m* and *p* refer to model and prototype respectively) where l_r is the scale factor or sometimes known as the model ratio. Figure 5.1 shows three pairs of geometrically similar objects, namely, a right circular cylinder, a parallelopiped, and a triangular prism.

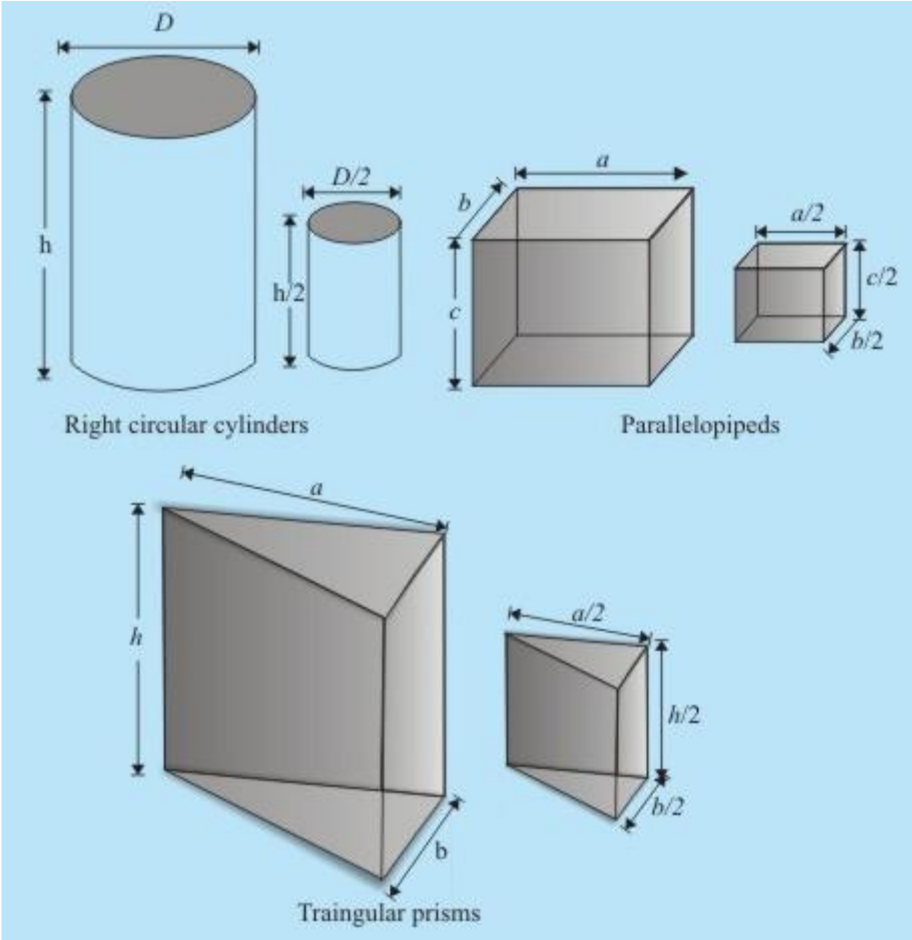


Fig 17.1 Geometrically Similar Objects

In all the above cases model ratio is 1/2

Geometric similarity is perhaps the most obvious requirement in a model system designed to correspond to a given prototype system.

A perfect geometric similarity is not always easy to attain. **Problems in achieving perfect geometric similarity** are:

- For a small model, the surface roughness might not be reduced according to the scale factor (unless the model surfaces can be made very much smoother than those of the prototype). If for any reason the scale factor is not the same throughout, a distorted model results.

- Sometimes it may so happen that to have a perfect geometric similarity within the available laboratory space, physics of the problem changes. For example, in case of large prototypes, such as rivers, the size of the model is limited by the available floor space of the laboratory; but if a very low scale factor is used in reducing both the horizontal and vertical lengths, this may result in a stream so shallow that surface tension has a considerable effect and, moreover, the flow may be laminar instead of turbulent. In this situation, a distorted model may be unavoidable (a lower scale factor "for horizontal lengths while a relatively higher scale factor for vertical lengths. The extent to which perfect geometric similarity should be sought therefore depends on the problem being investigated, and the accuracy required from the solution.

Kinematic Similarity

Kinematic similarity refers to **similarity of motion.**

Since motions are described by distance and time, it implies **similarity of lengths (i.e., geometrical similarity**) and, in addition**, similarity of time intervals**.

If the corresponding lengths in the two systems are in a fixed ratio, the velocities of corresponding particles must be in a fixed ratio of magnitude of corresponding time intervals.

If the ratio of corresponding lengths, known as the **scale factor, is l_r** and **the ratio of corresponding time intervals is t_r**, then the magnitudes of corresponding **velocities are in the ratio l_r/t_r** and the magnitudes of corresponding **accelerations are in the ratio l_r/t^2_r.**

A well-known **example** of kinematic similarity is found in a planetarium. Here the galaxies of stars and planets in space are reproduced in accordance with a certain length scale and in simulating the motions of the planets, a fixed ratio of time intervals (and hence velocities and accelerations) is used.

When fluid motions are kinematically similar, the **patterns formed by streamlines are geometrically similar** at corresponding times.

Since the impermeable boundaries also represent streamlines, **kinematically similar flows are possible only past geometrically similar boundaries.**

Therefore, **geometric similarity is a necessary condition for the kinematic similarity** to be achieved, but not the sufficient one.

For example, geometrically similar boundaries may ensure geometrically similar streamlines in the near vicinity of the boundary but not at a distance from the boundary.

Dynamic Similarity

Dynamic similarity is the **similarity of forces** .

In dynamically similar systems, the **magnitudes of forces** at correspondingly similar points in each system are **in a fixed ratio.**

In a system involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows:

Viscous Force (due to viscosity)	$\vec{F}_v$
Pressure Force (due to different in pressure)	$\vec{F}_p$
Gravity Force (due to gravitational attraction)	$\vec{F}_g$
Capillary Force (due to surface tension)	$\vec{F}_c$
Compressibility Force (due to elasticity)	$\vec{F}_e$

According to Newton 's law, the resultant F_R of all these forces, will cause the acceleration of a fluid element. Hence

$$\vec{F_R} = \vec{F_v} + \vec{F_p} + \vec{F_g} + \vec{F_c} + \vec{F_e} \quad (17.1)$$

Moreover, the **inertia force $\vec{F_i}$ is defined as equal and opposite to the resultant accelerating force $\vec{F_R}$**

$$\vec{F_i} = -\vec{F_R}$$

Therefore Eq. 17.1 can be expressed as

$$\vec{F_v} + \vec{F_p} + \vec{F_g} + \vec{F_c} + \vec{F_e} + \vec{F_i} = 0$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both the prototype and the model. **The inertia force $\vec{F_i}$ is usually taken as the common one to describe the ratios** as (or putting in other form we equate the the non dimensionalised forces in the two systems)

$$\frac{|\vec{F_v}|}{|\vec{F_i}|}, \frac{|\vec{F_p}|}{|\vec{F_i}|}, \frac{|\vec{F_g}|}{|\vec{F_i}|}, \frac{|\vec{F_c}|}{|\vec{F_i}|}, \frac{|\vec{F_e}|}{|\vec{F_i}|}$$

Magnitudes of Different Forces

A fluid motion, under all such forces is characterised by

1. Hydrodynamic parameters like pressure, velocity and acceleration due to gravity,
2. Rheological and other physical properties of the fluid involved, and
3. Geometrical dimensions of the system.

It is important to express the magnitudes of different forces in terms of these parameters, to know the extent of their influences on the different forces acting on a flluid element in the course of its flow.

Inertia Force $\vec{F_i}$

- The inertia force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.
- The mass of a fluid element is proportional to ρl^3 where, ρ is the density of fluid and l is the characteristic geometrical dimension of the system.
- The acceleration of a fluid element in any direction is the rate at which its velocity in that direction changes with time and is therefore proportional in magnitude to some characteristic velocity V divided by some specified interval of time t. The time interval t is proportional to the characteristic length l divided by the characteristic velocity V, so that the acceleration becomes proportional to V^2/l.

The magnitude of inertia force is thus proportional to

$$\boxed{\rho l^3 V^2 / l = \rho l^2 V^2}$$

This can be written as,

$$\left|\vec{F}_i\right| \propto \rho l^2 V^2 \qquad (18.1a)$$

Viscous Force $\vec{F}_v$

The viscous force arises from shear stress in a flow of fluid.

Therefore, we can write

Magnitude of viscous force $\vec{F}_v$ = shear stress X surface area over which the shear stress acts

Again, shear stress = μ (viscosity) X rate of shear strain

where, rate of shear strain $\propto$ velocity gradient $\propto \frac{V}{l}$ and surface area $\propto l^2$

Hence

$$\left|\vec{F}_v\right| \propto \mu \frac{V}{l} l^2$$

$$\propto \mu V l \qquad (18.1b)$$

Pressure Force $\vec{F}_p$

The pressure force arises due to the difference of pressure in a flow field.

Hence it can be written as

$$\left|\vec{F}_p\right| \propto \Delta p . l^2 \qquad (18.1c)$$

(where, Δp is some characteristic pressure difference in the flow.)

Gravity Force $\vec{F}_g$

The gravity force on a fluid element is its weight. Hence,

$$\left|\vec{F}_g\right| \propto \rho l^3 g \qquad (18.1d)$$

(where g is the acceleration due to gravity or weight per unit mass)

Capillary or Surface Tension Force $\vec{F}_c$

The capillary force arises due to the existence of an interface between two fluids.

- The surface tension force acts tangential to a surface .
- It is equal to the coefficient of surface tension σ multiplied by the length of a linear element on the surface perpendicular to which the force acts.

Therefore,

$$\left|\vec{F}_c\right| \propto \sigma l \qquad (18.1e)$$

Compressibility or Elastic Force $\vec{F}_e$

Elastic force arises due to the compressibility of the fluid in course of its flow.

- For a given compression (a decrease in volume), the increase in pressure is proportional to the bulk modulus of elasticity E
- This gives rise to a force known as the elastic force.

Hence, for a given compression $\Delta \mathbf{p} \propto \mathbf{E}$

$$\left|\vec{F}_e\right| \propto E l^2 \qquad (18.1f)$$

The flow of a fluid in practice does not involve all the forces simultaneously.

Therefore, the pertinent dimensionless parameters for dynamic similarity are derived from the ratios of significant forces causing the flow.

Dynamic Similarity of Flows governed by Viscous, Pressure and Inertia Forces

The criterion of dynamic similarity for the **flows controlled by viscous, pressure and inertia forces** are derived from the ratios of the representative magnitudes of these forces with the help of Eq. (18.1a) to (18.1c) as follows:

$$\frac{\text{Viscous force}}{\text{Inertia Force}} = \frac{\left|\vec{F}_v\right|}{\left|\vec{F}_i\right|} \propto \frac{\mu V l}{\rho V^2 l^2} = \frac{\mu}{\rho V l} \qquad (18.2a)$$

$$\frac{\text{Pressure force}}{\text{Inertia Force}} = \frac{|\vec{F}_p|}{|\vec{F}_i|} \propto \frac{\Delta p l^2}{\rho V^2 l^2} = \frac{\Delta p}{\rho V^2} \tag{18.2b}$$

The term $\rho l V/\mu$ **is known as Reynolds number, Re** after the name of the scientist who first developed it and is thus proportional to the magnitude ratio of inertia force to viscous force .(Reynolds number plays a vital role in the analysis of fluid flow)

The term $\Delta p/\rho V^2$ **is known as Euler number, Eu** after the name of the scientist who first derived it. The dimensionless terms Re and Eu represent the critieria of dynamic similarity for the flows which are affected only by viscous, pressure and inertia forces. Such instances, for example, are

1. the full flow of fluid in a completely closed conduit,
2. flow of air past a low-speed aircraft and
3. the flow of water past a submarine deeply submerged to produce no waves on the surface.

Hence, for a complete dynamic similarity to exist between the prototype and the model for this class of flows, the Reynolds number, Re and Euler number, Eu have to be same for the two (prototype and model). Thus

$$\frac{\rho_p\, l_p V_p}{\mu_p} = \frac{\rho_m\, l_m V_m}{\mu_m} \tag{18.2c}$$

$$\frac{\Delta P_p}{\rho_p V_p^2} = \frac{\Delta P_m}{\rho_m V_m^2} \tag{18.2d}$$

where, the suffix p and suffix m refer to the parameters for prototype and model respectively.

In practice, the pressure drop is the dependent variable, and hence it is compared for the two systems with the help of Eq. (18.2d), while the equality of Reynolds number (Eq. (18.2c)) along with the equalities of other parameters in relation to kinematic and geometric similarities are maintained.

- The characteristic geometrical dimension l **and** the reference velocity **V** in the expression of the Reynolds number **may be any geometrical dimension and any velocity which are significant in determining the pattern of flow.**
- For internal flows through a closed duct, the hydraulic diameter of the duct D_h and the average flow velocity at a section are invariably used for l and V respectively.
- The hydraulic diameter D_h is defined as $D_h = 4A/P$ where A and P are the cross-sectional area and wetted perimeter respectively.

Dynamic Similarity of Flows with Gravity, Pressure and Inertia Forces

A flow of the type in which **significant forces** are **gravity force, pressure force and inertia force,** is found **when a free surface is present.**

Examples can be

1. the flow of a liquid in an open channel.
2. the wave motion caused by the passage of a ship through water.
3. the flows over weirs and spillways.

The condition for dynamic similarity of such flows requires

- the equality of the Euler number Eu (the magnitude ratio of pressure to inertia force),

and

- the equality of the magnitude ratio of gravity to inertia force at corresponding points in the systems being compared.

Thus ,

$$\frac{\text{Gravity force}}{\text{Inertia Force}} = \frac{\left|\vec{F}_g\right|}{\left|\vec{F}_i\right|} \propto \frac{\rho l^3 g}{\rho V^2 l^2} = \frac{lg}{V^2} \tag{18.2e}$$

- In practice, it is often convenient to use the square root of this ratio so to deal with the first power of the velocity.
- From a physical point of view, equality of $(lg)^{1/2}/V$ implies equality of lg/V^2 as regard to the concept of dynamic similarity.

The **reciprocal of the term** $(lg)^{1/2}/V$ **is known as Froude number (** after William Froude who first suggested the use of this number in the study of naval architecture.)

Hence **Froude number,** $Fr = V/(lg)^{1/2}$ **.**

Therefore, **the primary requirement for dynamic similarity between the prototype and the model involving flow of fluid with gravity as the significant force, is the equality of Froude number, Fr,** i.e.,

$$\frac{(l_p g_p)^{1/2}}{V_p} = \frac{(l_m g_m)^{1/2}}{V_m} \tag{18.2f}$$

Dynamic Similarity of Flows with Surface Tension as the Dominant Force

Surface tension forces are important in certain classes of practical problems such as ,

1. flows in which capillary waves appear
2. flows of small jets and thin sheets of liquid injected by a nozzle in air
3. flow of a thin sheet of liquid over a solid surface.

Here the significant parameter for dynamic similarity is the magnitude ratio of the surface tension force to the inertia force.

This can be written as

$$\frac{|\vec{F}_e|}{|\vec{F}_i|} \propto \frac{\sigma l}{\rho V^2 l^2} = \frac{\sigma}{\rho V^2 l} \qquad (18.2g)$$

The term $\sigma / \rho V^2 l$ **is usually known as Weber number, Wb** (after the German naval architect Moritz Weber who first suggested the use of this term as a relevant parameter.)

Thus for dynamically similar flows $(Wb)_m = (Wb)_p$

i.e.,

$$\frac{\sigma_m}{\rho_m V_m^2 L_m} = \frac{\sigma_p}{\rho_p V_p^2 L_p^2}$$

Dynamic Similarity of Flows with Elastic Force

When the compressibility of fluid in the course of its flow becomes significant, the elastic force along with the pressure and inertia forces has to be considered.

Therefore, the magnitude ratio of inertia to elastic force becomes a relevant parameter for dynamic similarity under this situation.

Thus we can write,

$$\boxed{\frac{\text{Inertia force}}{\text{Elastic Force}} = \frac{|\vec{F}_i|}{|\vec{F}_e|} \propto \frac{\rho V^2 l^2}{E l^2} = \frac{\rho V^2}{E}} \qquad (18.2h)$$

The parameter $\rho V^2 / E$ **is known as Cauchy number** ,(after the French mathematician A.L. Cauchy)

If we consider the **flow to be isentropic** , then it can be written

$$\frac{|\vec{F}_i|}{|\vec{F}_e|} \propto \frac{\rho V^2}{E_s} \qquad (18.2i)$$

(where E_s is the isentropic bulk modulus of elasticity)

Thus for dynamically similar flows $(cauchy)_m=(cauchy)_p$

ie., $$\frac{\rho_m V_m^2}{(E_s)_m} = \frac{\rho_p V_p^2}{(E_s)_p}$$

- The velocity with which a sound wave propagates through a fluid medium equals to $\sqrt{E_s/\rho}$.
- Hence, the term $\rho V^2/E_s$ can be written as V^2/a^2 where **a is the acoustic velocity** in the fluid medium.

The ratio V/a is known as Mach number, Ma (after an Austrian physicist Earnst Mach)

It has been shown in Chapter 1 that the effects of compressibility become important when the Mach number exceeds 0.33.

The situation arises in the flow of air past high-speed aircraft, missiles, propellers and rotory compressors. In these cases equality of Mach number is a condition for dynamic similarity. Therefore,

$$(Ma)_p=(Ma)_m$$

i.e.

$$\boxed{V_p/a_p = V_m/a_m} \qquad (18.2j)$$

Ratios of Forces for Different Situations of Flow

Pertinent Dimensionless term as the croterion of dynamic similarity in different situations of fluid flow	Representative magnitude ration of the forces	Name	Recommended symbol
$\rho l V/\mu$	$\frac{\text{Inertia force}}{\text{Viscous force}}$	Reynolds number	Re
$\Delta p/\rho V^2$	$\frac{\text{Pressure force}}{\text{Inertia force}}$	Euler number	Eu
	$\frac{\text{Inertia force}}{\text{Gravity force}}$	Froude number	Fr

$V/(lg)^{1/2}$			
$\sigma/\rho V^2 l$	$\dfrac{\text{Surface Tesion force}}{\text{Inertia force}}$	Weber number	Wb
	$\dfrac{\text{Inertia force}}{\text{Elastic force}}$	Mach number	Ma

The Application of Dynamic Similarity - The Dimensional Analysis

The concept:

A physical problem may be characterised by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables.

This gives a clue to the reduction in the number of parameters requiring separate consideration in an experimental investigation.

For an **example,** if the Reynolds number Re = $\rho V D_h/\mu$ is considered as the independent variable, in case of a flow of fluid through a closed duct of hydraulic diameter D_h, then a change in Re may be caused through a change in flow velocity V only. Thus a range of Re can be covered simply by the variation in V without varying other independent dimensional variables ρ,D_h and μ.

In fact, the variation in the Reynolds number physically implies the variation in any of the dimensional parameters defining it, though the change in Re, may be obtained through the variation in anyone parameter, say the velocity V.

A number of such **dimensionless parameters** in relation to dynamic similarity are shown in Table 5.1. Sometimes it becomes diffcult to derive these parameters straight forward from an estimation of the representative order of magnitudes of the forces involved. An alternative **method of determining these dimensionless parameters by a mathematical technique is known as dimensional analysis** .

The Technique:

The requirement of dimensional homogeneity imposes conditions on the quantities involved in a physical problem, and these restrictions, placed in the form of an algebraic function by the requirement of dimensional homogeneity, play the central role in dimensional analysis.

There are two existing approaches;

- one due to Buckingham known as **Buckingham's pi theorem**
- other due to Rayleigh known as **Rayleigh's Indicial method**

In our next slides we'll see few examples of the dimensions of physical quantities.

Dimensions of Physical Quantities

All physical quantities are expressed by magnitudes and units.

*For **example*** , the velocity and acceleration of a fluid particle are 8m/s and 10m/s^2 respectively. Here the dimensions of velocity and acceleration are ms^{-1} and ms^{-2} respectively.

In SI (System International) units, the primary physical quantities which are assigned base dimensions are the mass, length, time, temperature, current and luminous intensity. Of these, the first four are used in fluid mechanics and they are symbolized as M (mass), L (length), T (time), and θ (temperature).

- Any physical quantity can be expressed in terms of these primary quantities by using the basic mathematical definition of the quantity.
- The resulting expression is known as the dimension of the quantity.

Let us take some **examples:**

1. **Dimension of Stress**

Shear stress τ is defined as force/area. Again, force = mass × acceleration

Dimensions of acceleration = Dimensions of velocity/Dimension of time.

$$= \frac{\text{Dimension of Distance}}{(\text{Dimension of Time})^2}$$

$$= \frac{L}{T^2}$$

Dimension of area = (Length)^2 =L^2

Hence, dimension of shear stress

$$\tau = \left(ML/T^2\right)/L^2 = ML^{-1}T^{-2} \qquad (19.1)$$

2. Dimension of Viscosity

Consider Newton's law for the definition of viscosity as

$$\tau = \mu du/dy$$

or,

$$\mu = \frac{\tau}{(du/dy)}$$

The dimension of velocity gradient du/dy can be written as

dimension of du/dy= dimension of u/dimension of y = (L / T)/L = T^{-1}

The dimension of shear stress τ is given in Eq. (19.1).

Hence dimension of

$$\mu = \frac{\text{Dimension of } \tau}{\text{Dimension of du/dy}} = \frac{ML^{-1}T^{-2}}{T^{-1}}$$

$$= ML^{-1}T^{-1}$$

Dimensions of Various Physical Quantities in Tabular Format

Physical Quantity	Dimension
Mass	M
Length	L
Time	T
Temperature	θ
Velocity	LT^{-1}
Angular velocity	T^{-1}
Acceleration	LT^{-2}
Angular Acceleration	T^{-2}
Force, Thrust, Weight	MLT^{-2}
Stress, Pressure	$ML^{-1}T^{-2}$
Momentum	MLT^{-1}
Angular Momentum	$ML^{2}T^{-1}$
Moment, Torque	$ML^{2}T^{-2}$
Work, Energy	$ML^{2}T^{-2}$
Power	$ML^{2}T^{-3}$
Stream Function	$L^{2}T^{-1}$
Vorticity, Shear Rate	T^{-1}
Velocity Potential	$L^{2}T^{-1}$
Density	ML^{-3}
Coefficient of Dynamic Viscosity	$ML^{-1}T^{-1}$
Coefficient of Kinematic Viscosity	$L^{2}T^{-1}$

Surface Tension	MT^{-2}
Bulk Modulus of Elasticity	$ML^{-1}T^{-2}$

Buckingham's Pi Theorem

Assume, a physical phenomenon is described by **m number of independent variables like x_1 , x_2 , x_3 , ..., x_m**

The phenomenon may be expressed analytically by an implicit functional relationship of the controlling variables as

$$f(x_1, x_2, x_3, \ldots\ldots, x_m) = 0 \qquad (19.2)$$

Now if **n be the number of fundamental dimensions like mass, length, time, temperature etc** ., involved in these m variables, then according to Buckingham's p theorem -

The phenomenon can be described in terms of (m - n) independent dimensionless groups like π_1 ,π_2 , ..., π_{m-n} , where p terms, represent the dimensionless parameters and consist of different combinations of a number of dimensional variables out of the m independent variables defining the problem.

Therefore. the analytical version of the phenomenon given by Eq. (19.2) can be reduced to

$$\boxed{F(\pi_1, \pi_2, \ldots\ldots\ldots, \pi_{m-n}) = 0} \qquad (19.3)$$

according to Buckingham's pi theorem

- This physically implies that the **phenomenon** which is basically described by m independent dimensional variables, **is ultimately controlled by (m-n) independent dimensionless parameters known as π terms.**

Alternative Mathematical Description of (π) Pi Theorem

A physical problem described by m number of variables involving n number of fundamental dimensions (n < m) leads to a system of n linear algebraic equations with m variables of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots a_{1m}x_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2m}x_m = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots a_{nm}x_m = b_n \qquad (19.4)$$

or in a matrix form,

$$\boxed{Ax = b} \qquad (19.5)$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$

Determination of π terms

- A group of n (n = number of fundamental dimensions) variables out of m (m = total number of independent variables defining the problem) variables is first chosen to form a basis so that all n dimensions are represented . These n variables are referred to as repeating variables.

- Then the p terms are formed by the product of these repeating variables raised to arbitrary unknown integer exponents and anyone of the excluded (m -n) variables.

*For **example*** , if x_1 x_2 ...x_n are taken as the repeating variables. Then

$$\boxed{\begin{gathered} \pi_2 = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n}\, x_{n+2} \\ \pi_1 = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n}\, x_{n+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \pi_{m-n} = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n}\, x_m \end{gathered}}$$

- The sets of integer exponents **a_1, a_2 . . . a_n are different for each p term.**
- Since p terms are dimensionless, it requires that when all the variables in any p term are expressed in terms of their fundamental dimensions, the exponent of all the fundamental dimensions must be zero.

- This leads to a system of n linear equations in a, a_2 . . . a_n which gives a unique solution for the exponents. This gives the values of a_1 a_2 . . . a_n for each p term and hence the p terms are uniquely defined.

In selecting the repeating variables, the following points have to be considered:

1. The repeating variables must include among them all the n fundamental dimensions, not necessarily in each one but collectively.
2. The dependent variable or the output parameter of the physical phenomenon should not be included in the repeating variables.

No physical phenomena is represented when -

- **m < n** because there is no solution **and**
- **m = n** because there is a unique solution of the variables involved and hence all the parameters have fixed values.

. **Therefore all feasible phenomena are defined with m > n .**

- **When m = n + 1,** then, according to the Pi theorem, the number of pi term is one and the phenomenon can be expressed as

$$f(\pi_1) = 0$$

where, the non-dimensional term π_1 is some specific combination of n + 1 variables involved in the problem.

When m > n+ 1 ,

1. the number of π terms are more than one.
2. A number of choices regarding the repeating variables arise in this case.

Again, it is true that if one of the repeating variables is changed, it results in a different set of π terms. Therefore the interesting question is **which set of repeating variables is to be chosen** , to arrive at the correct set of π terms to describe the problem. The **answer to this question lies in the fact that different sets of π terms resulting from the use of different sets of repeating variables are not independent. Thus, anyone of such interdependent sets is meaningful in describing the same physical phenomenon.**

From any set of such π terms, one can obtain the other meaningful sets from some combination of the π terms of the existing set without altering their total numbers (m-n) as fixed by the Pi theorem.

Example

Consider pressure drop in a tube of length ℓ , hydraulic diameter d, surface roughness $\in$, with fluid of density ρ and viscosity μ moving with average velocity v

This can be expressed as

$$f\,(\Delta P, U, d, \ell, \in, \rho, \mu) = 0$$

Now m=7 since the phenomenon involves 7 independent parameters.

We select ρ, U, d as repeating variables (so that all 3 dimensions are represented)

Now 4 π (7 - 3) parameters are determined as

$$\Pi_1 = \rho^{a1} U^{b1} d^{c1} \Delta p$$
$$\Pi_2 = \rho^{a2} U^{b2} d^{c2} \mu$$
$$\Pi_3 = \rho^{a3} U^{b3} d^{c3} \in$$
$$\Pi_4 = \rho^{a4} U^{b4} d^{c4} \ell$$

Now basic units

$$\rho \rightarrow ML^{-3}$$
$$U \rightarrow LT^{-1}$$
$$d \rightarrow L$$
$$\Delta p \rightarrow ML^{-1}T^{-2}$$
$$\mu \rightarrow ML^{-1}T^{-1}$$
$$\in \rightarrow L$$
$$\ell \rightarrow L$$

All Π parameters → $M^0 L^0 T^0$

The above four equations yield

$$a_1 = -1;\ b_1 = -2;\ c_1 = 0$$
$$a_2 = -1;\ b_2 = -1;\ c_2 = -1$$
$$a_3 = 0;\ b_3 = 0;\ c_3 = -1$$
$$a_4 = 0;\ b_4 = 0;\ c_4 = -1$$

Thus writing

$$\Pi_1 = f(\pi_2, \Pi_3, \Pi_4)$$

implies

$$\frac{\Delta p}{\rho U^2} = f\left(\frac{\mu}{\rho U d}, \frac{\in}{d}, \frac{L}{d}\right)$$

$$\frac{\rho U d}{\mu} = Re\ (\text{Reynolds no})$$

Therefore,
$$\frac{\Delta p}{\rho U^2} = f\left(Re, \frac{\epsilon}{d}, \frac{L}{d}\right)$$

ϵ/d is called relative roughness

Similarly, other sets of Π parameters can be chosen to describe the phenomena. Thus though it does not give the actual relationship, but it puts the data in a compact form

Rayleigh's Indicial Method

This alternative method is also b**ased on the fundamental principle of dimensional homogeneity** of physical variables involved in a problem.

Procedure-

1. The dependent variable is identified and expressed as a product of all the independent variables raised to an unknown integer exponent.
2. Equating the indices of n fundamental dimensions of the variables involved, n independent equations are obtained .
3. These n equations are solved to obtain the dimensionless groups.

Example

Let us illustrate this method by solving the pipe flow problem

. **Step 1 -** ----- Here, the dependent variable Δp/l can be written as

$$\frac{\Delta p}{l} = AV^a D_h^b \rho^c \mu^d$$ (where, A is a dimensionless constant.)

Step 2 -----Inserting the dimensions of each variable in the above equation, we obtain,

$$ML^{-2}T^{-2} = A(LT^{-1})^a (L)^b (ML^{-3})^c (ML^{-1}T^{-1})^d$$

Equating the indices of M, L, and T on both sides, we get **,**

c + d = 1
a + b - 3c - d = -2
-a - d = -2

Step 3 -----There are three equations and four unknowns. Solving these equations in terms of the unknown d, we have

a = 2- d
b = -d - 1
c = 1- d

Hence , we can be written

$$\frac{\Delta p}{l} = AV^{2-d} D_h^{-d-1} \rho^{1-d} \mu^d$$

$$\frac{\Delta p}{l} = \frac{AV^2 \rho}{D_h} \left(\frac{\mu}{VD_h \rho} \right)^d$$

or, $$\frac{\Delta p D_h}{l \rho V^2} = A \left(\frac{\mu}{VD_h \rho} \right)^d$$

Therefore we see that there are two independent dimensionless terms of the problem, namely,

- **Both Buckingham's method and Rayleigh's method of dimensional analysis determine only the relevant independent dimensionless parameters of a problem, but not the exact relationship between them.**

For example, the numerical values of A and d can never be known from dimensional analysis. They are found out from experiments.

If the system of equations is solved for the unknown c, it results,

$$\frac{\Delta p}{l} \frac{D_h^2}{V \mu} = A \left(\frac{VD_h \rho}{\mu} \right)^c$$

Therefore different interdependent sets of dimensionless terms are obtained with the change of unknown indices in terms of which the set of indicial equations are solved. This is similar to the situations arising with different possible choices of repeating variables in Buckingham's Pi theorem.

Exercise Problems - Chapter 6

1. A 1/6 model automobile is tested in a wind tunnel with same air properties as the prototype. The prototype automobile runs on the roads at a velocity of 60 km/hr. For

dynamically similar conditions, the drag measured on the model is 500 N. Determine the drag of the prototype and the power required to overcome this drag.

(500N, 8.33 KN)

2. A model is built of a flow phenomenon which is governed by the action of gravity and surface tension force. Show that the length scale ratio which will ensure complete similarity between model and the prototype is $L_r = \sqrt{\sigma_r / \rho_r}$

3. The speed of propagation U of a capillary wave in deep water is known to be a function of density ρ , wave length λ , and surface tension α . Using Dimensional Analysis, find out a relationship of U with ρ , λ , and α . (b) For a given surface tension and wavelength, how does the propagation speed changes if the density is halved ?

(increased by a factor of $\sqrt{2}$)

4. A hydraulic jump occurring in a stilling basin is to be studied in a 1:36 scale model. The prototype jump has an initial velocity of 10 m/s, an entrance Froude number of 6.0 and a power loss of 2 kW per meter width of basin. Determine (a) the corresponding model velocity, (b) model Froude number and (c) power loss per meter width of the model

(a 1·67 m/s, (b) 6· 0, (c) 0·26 W)

5. A model of a reservoir having a free water surface within it is drained in 3 minutes by opening a sluice gate. The geometrical scale of the model is 1/100. How long would it take to empty the prototype?

(30 minutes)

6. Assuming that nothing is known about the particle motion under gravity beyond $x = G(v_0, g, t)$ where x , v_0 , g, and t are respectively the displacement, initial velocity, gravitational acceleration, and time. Perform a dimensional analysis to explain the situation.

7. The tensile force inside a pendulum is known to depend on the mass, length, period, and angular amplitude of the pendulum. Perform a dimensional analysis.

8. The pressure drop in pipe flows of liquids is found to depend on the time required to pass a volume of a given liquid through, on this volume, and on the density as well as the viscosity of the liquid. Perform a dimensional analysis in a step by step manner, with the pressure drop and the density displayed as leading quantities.

9. Using the long steps of dimensional analysis, reduce the relationship n = G (*g*, *A*, ρ, *M*) for the frequency n of the wing beat of a flying insect, where g stands for the gravitational acceleration; A, the wing area; ρ, the air density; and M, the mass of the insect. Choose *n* and *A* as the leading quantities.

10. The shape of a drop of liquid pulsates as it falls. The period of oscillation is observed to depend on the surface tension, the mean radius of the drop, and the liquid density. Perform a dimensional analysis to express the period of oscillation..

11. Liquid flows across an orifice loses useful power which is dependent on the liquid density and viscosity, the volume flow rate, and the orifice diameter. Perform a dimensional analysis (with the objective of analysing power loss).

12. During the flow through a pipe, it is observed that there exists a critical average flow velocity $\overline{V}_{cr}$, beyond which the flow becomes turbulent. It is also known that $\overline{V}_{cr}$ is influenced by the diameter of the pipe, the density and the viscosity of the fluid. Perform a dimensional analysis to explain the situation.

In this course you have learnt the following

- Physical similarities are always sought between the problems of same physics. The complete physical similarity requires geometric similarity, kinematic similarity and dynamic similarity to exist simultaneously.

- In geometric similarity, the ratios of the corresponding geometrical dimensions between, the systems remain the same. In kinematic similarity, the ratios of corresponding motions and in dynamic similarity, the ratios of corresponding forces between the systems remain the same.

- For prediction of the performance characteristics of actual systems in practice from the results of model scale experiments in laboratories, complete physical similarity has to be achieved between the prototype and the model.

- Dimensional homogeneity of physical quantities implies that the number of dimensionless independent variables are smaller as compared to the number of their dimensional counterparts to describe a physical phenomenon. The dimensionless variables represent the criteria of similarity. Buckingham's π theorem states that if a physical problem is described by m dimensional variables which can be expressed by n fundamental dimensions, then the number of independent dimensionless variables defining the problem will be m-n. These dimensionless variables are known as π terms. The independent π terms of a physical problem are determined either by Buckingham's π theorem or by Rayleigh's indicial method.

Analysis of Inviscid, Incompressible, Irrotation Flows

Incompressible flow is a constant density flow.

Let us visualize a fluid element of defined mass, moving along a streamline in an incompressible flow.

Due to constant density , we can write

$$\nabla.\vec{V} = 0 \tag{20.1}$$

Irrotational Flow

- **if the fluid element does not rotate as it moves along the streamline, or to be precise, if its motion is translational (and deformation with no rotation) only, the flow is termed as irrotational.**

The **rate of rotation of the fluid element** can be measured as the **average rate of rotation of two perpendicular line segments.**

The average rate of rotation ω_z about z-axis is expressed in terms of the gradients of velocity components as

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Similarly, the other two components of rotation are

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] \quad \text{and} \quad \omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

ω_x , ω_y and ω_z are components of $\vec{\omega}$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V})$$

In a two-dimensional flow, ω_z is the only non-trivial component of the rate of rotation called in-plane component of vorticity and computed as $\frac{1}{2}(\nabla \times \vec{V})\hat{k}$

Thus **for irrotational flow, vorticity is zero i.e.** $\vec{\varpi} = 0$

Potential Flow Theory

Let us imagine a pathline of a fluid particle shown in Fig. 20.1.

Rate of spin of the particle is ω_z . The flow in which this spin is zero throughout is known as irrotational flow .

For irrotational flows, $\nabla \times \vec{V} = 0$

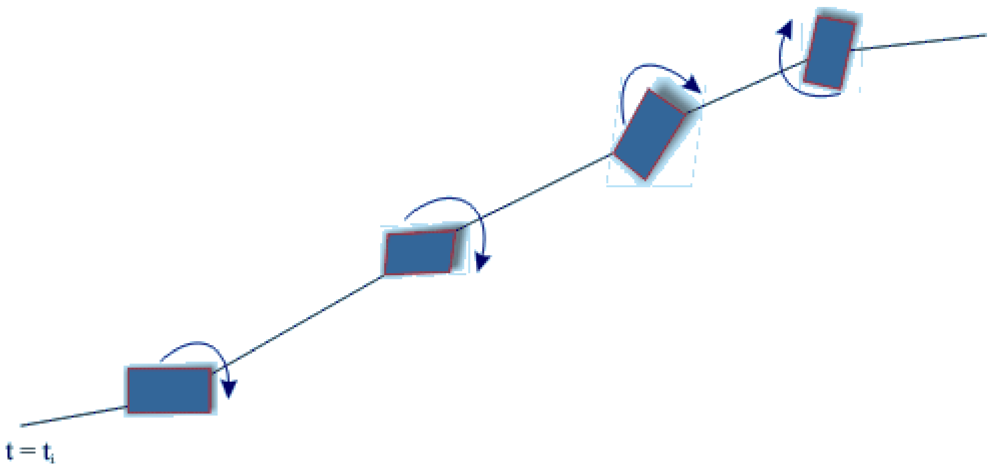


Fig 20.1 Pathline of a Fluid Particle

Velocity Potential and Stream Function

Since for irrotational flows $\nabla \times \vec{V} = 0$.

the velocity for an irrotational flow, can be expressed as the gradient of a scalar function called the velocity potential, denoted by Φ

$$\vec{V} = \nabla \phi \qquad (20.2)$$

Combination of Eqs (20.1) and (20.2) yields

$$\nabla^2 \phi = 0 \qquad \text{(Laplace's equation)} \qquad (20.3)$$

For irrotational flows

$\vec{\varpi} = 0$

For two-dimensional case (as shown in Fig 20.1)

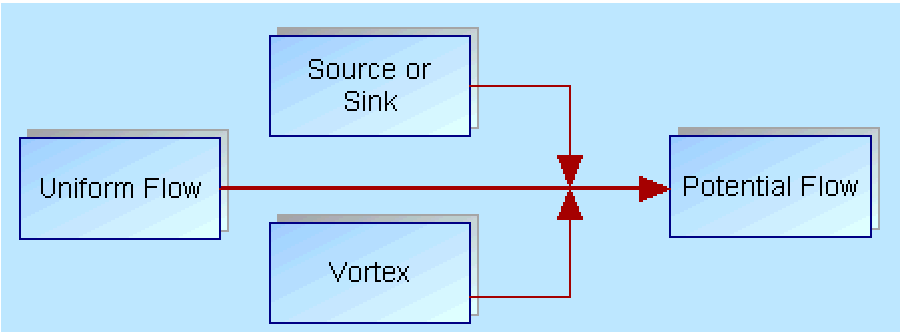
$$\varpi = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0$$
$$\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$-\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0 \qquad \left[u = \frac{\partial \psi}{\partial y}; v = -\frac{\partial \psi}{\partial x}\right]$$
$$-\left(+\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}\right) = 0$$
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$
$$\nabla^2 \psi = 0$$

which is again Laplace's equation.

- From Eq. (20.3) we see that **an inviscid, incompressible, irrotational flow is governed by Laplace's equation.**
- Laplace's equation is linear, hence any number of particular solutions of Eq.(20.3) added together will yield another solution .
- A complicated flow pattern for an inviscid, incompressible, irrotational flow can be synthesized by adding together a number of elementary flows (provided they are also inviscid, incompressible and irrotational)----- **The Superposition Principle**

The analysis of Laplace's Eq. (20.3) and finding out the potential functions are known as Potential Flow Theory and the inviscid, incompressible, irrotational flow is often called as Potential Flow .

There are some elementary flows which constitute several complex potential-flow problems.



Uniform Flow

- Velocity does not change with y-coordinate
- There exists only one component of velocity which is in the x direction.
- Magnitude of the velocity is U_0 .

Since $\vec{V} = \nabla \phi$

$$\hat{i}U_0 + \hat{j}0 = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y}$$

or,

$$\frac{\partial \phi}{\partial x} = U_0, \quad \frac{\partial \phi}{\partial y} = 0$$

Thus,

$$\phi = U_0 x + C_1 \tag{20.4}$$

Using stream function ψ for uniform flow

$$\frac{\partial \psi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial y} = U_0$$

so

$$\psi = U_0 y + K_1 \tag{20.5}$$

The constants of integration C_1 and K_1 are arbitrary.

The values of ψ and Φ for different streamlines and velocity potential lines may change but flow pattern is unaltered

. The **constants of integration may be omitted, without any loss of generality** and it is possible to write

$$\psi = U_0 y, \qquad \phi = U_0 x \tag{20.6}$$

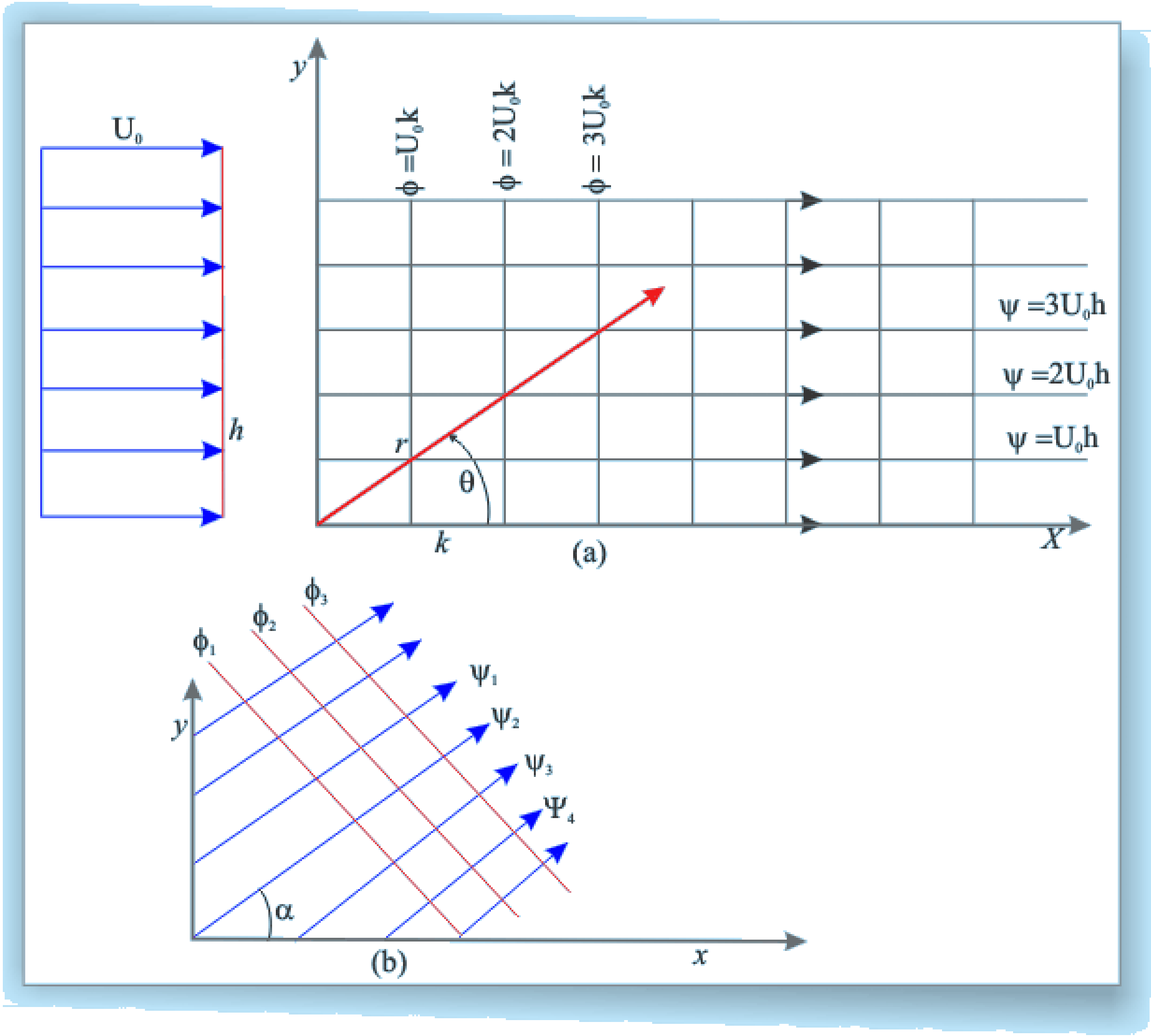


Fig 20..2 (a) Flownet for a Uniform Stream (b) Flownet for uniform Stream with an Anglea with x-axis

These are plotted in Fig. 20.2(a) and consist of a rectangular mesh of straight streamlines and orthogonal straight potential-lines (remember **streamlines and potential lines are always orthogonal**). It is conventional to put arrows on the streamlines showing the direction of flow.

In terms of polar (r - θ) coordinate, Eq. (20.6) becomes

$$\boxed{\psi = U_0 r \sin\theta, \quad \phi = U_0 r\cos\theta} \tag{20.7}$$

Flow at an angle

If we consider a uniform stream at an angle α to the x-axis as shown in Fig. 20.2b. we require that

$$u = U_0 \cos\alpha = \frac{d\psi}{dy} = \frac{d\phi}{dx}$$

and

$$v = U_0 \sin\alpha = -\frac{d\psi}{dx} = \frac{d\phi}{dy} \qquad (20.8)$$

Integrating. we obtain for a uniform velocity U_0 at an angle α, the stream function and velocity potential respectively as

$$\boxed{\psi = U_0(y\cos\alpha - x\sin\alpha), \quad \phi = U_0(x\cos\alpha + y\sin\alpha)} \qquad (20.9)$$

Source or Sink

Source flow -

- A flow with straight streamlines emerging from a point.
- Velocity along each streamline varies inversely with distance from the point (shown in Fig. 20.3).
- Only the radial component of velocity is non-trivial. (v_θ=0, v_z=0).

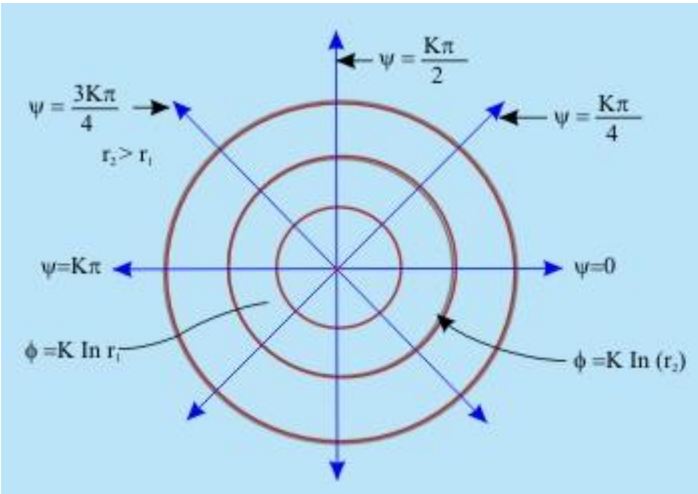


Fig 20.3 Flownet for a source flow

In a steady source flow the amount of fluid crossing any given cylindrical surface of radius r and unit length is constant ($\dot{m}$)

that is ($\dot{m}$)$_{in}$= ($\dot{m}$)$_{out}$

$$\dot{m} = 2\pi r v_r \rho$$

$$v_r = \frac{\dot{m}}{2\pi\rho}.\frac{1}{r} = \frac{\Lambda}{2\pi}.\frac{1}{r} = \frac{K}{r} \qquad (20.10a)$$

(which shows that velocity is inversely proportional to the distance)

where, K is the source strength $K = \frac{\dot{m}}{2\pi\rho} = \frac{\Lambda}{2\pi}$ and Λ is the volume flow rate

The **definition of stream function in cylindrical polar coordinate** states that

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{\partial\psi}{\partial r} \qquad (20.11)$$

For the source flow,

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{K}{r} \qquad (20.12)$$

$$-\frac{\partial\psi}{\partial r} = 0 \qquad (20.13)$$

Combining Eqs (20 .12) and (20.13) , we get

$$\psi = K\theta + C_1 \qquad (20.14)$$

Thus

$$if \quad \psi = k\tan^{-1}\left(\frac{y}{x}\right)$$

$$u = \frac{\partial\psi}{\partial y} = k\left(\frac{x}{x^2+y^2}\right) \qquad \text{and} \quad v = -\frac{\partial\psi}{\partial x} = k\left(\frac{y}{x^2+y^2}\right)$$

Because the flow is irrotational, we can write

$$\hat{r}v_r + \hat{\theta}v_\theta = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

or

$$v_r = \frac{\partial\phi}{\partial r} \qquad \text{and} \qquad v_\theta = 0 = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

or

$$v_r = \frac{\partial\phi}{\partial r} = \frac{K}{r} \quad \text{or} \quad \phi = K\ln r + C_2 \qquad (20.15)$$

The integration constants C_1 and C_2 in Eqs (20.14) and (20.15) have no effect on the basic structure of velocity and pressure in the flow.

The equations for streamlines and velocity potential lines for source flow become

ϕ=Klnr $\qquad \psi = K\theta$ (20.16)

K = source strength and is proportional to Λ Λ = the rate of volume flow from the source per unit depth perpendicular to the page

Sink flow

- When Λ **is negative** , we get sink flow,
- here the flow is in the opposite direction of the source flow.

In Fig. 20.3, the point 0 is the origin of the radial streamlines. We visualize that point O is a point source or sink that induces radial flow in the neighbourhood .

The point source or sink is a point of singularity in the flow field (because v_r becomes infinite).

The stream function and velocity potential function are

ϕ=-Klnr $\qquad \psi = -K\theta$ (20.17)

Concept of Circulation in a Free Vortex Flow

Free Vortex Flow

- Fluid particles move in circles about a point.
- The only non-trivial velocity component is tangential.
- This tangential speed varies with radius r so that same circulation is maintained.
- Thus,all the **streamlines are concentric circles** about a given point where the velocity along each streamline is inversely proportional to the distance from the centre. This **flow is necessarily irrotational.**

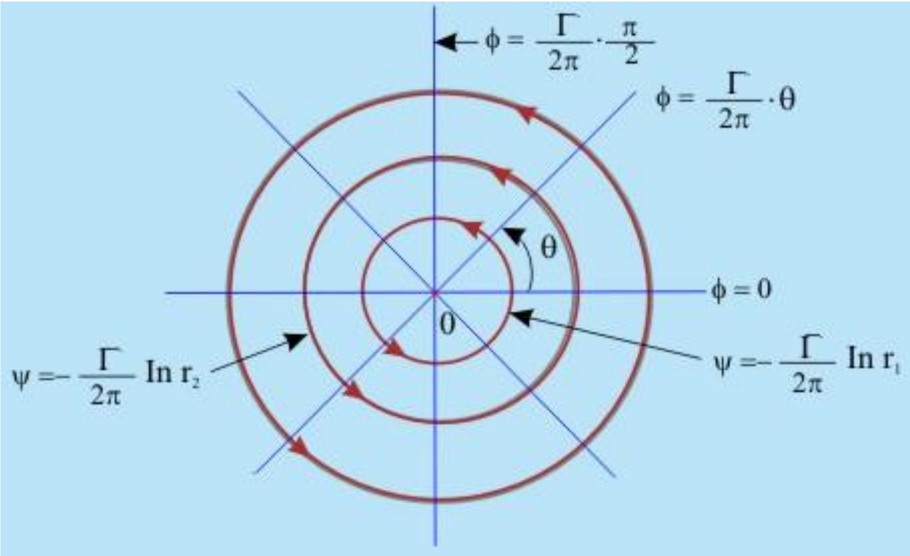


Fig 21.1 Flownet for a vortex (free vortex)

Velocity components

In a purely circulatory (free vortex flow) motion, the tangential velocity can be written as

$$v_\theta = \frac{\text{Circulation constant}}{\text{r}}$$

or,

$$v_\theta = \frac{\Gamma/2\pi}{r} \qquad (21.1)$$

where Γ is circulation

For purely circulatory motion we can also write

$$v_r = 0 \qquad (21.2)$$

Stream Function

Using the definition of stream function, we can write

$$v_\theta = -\frac{\partial \psi}{\partial r} \quad \text{and} \quad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Combining Eqs (21.1) and (21.2) with the above said relations for stream function, it is possible to write

$$\psi = -\frac{\Gamma}{2\pi}\ln r + C_1 \qquad (21.3)$$

Velocity Potential Function

Because of irrotationality, it should satisfy

$$\hat{r}v_r + \hat{\theta}v_\theta = \hat{r}\frac{\partial \phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Eqs (21.1) and (21.2) and the above solution of Laplace's equation yields

$$\phi = \frac{\Gamma}{2\pi}\theta + C_2 \qquad (21.4)$$

Since, the integration constants C_1 and C_2 have no effect on the structure of velocities or pressures in the flow. We can ignore the integration constants without any loss of generality.

It is clear that the **streamlines for vortex flow are circles while the potential lines are radial** . **These are given by**

$$\boxed{\psi = -\frac{\Gamma}{2\pi}\ln r \qquad \text{and} \qquad \phi = \frac{\Gamma}{2\pi}\theta} \qquad (21.5)$$

- In Fig. 21.1, point 0 can be imagined as a point vortex that induces the circulatory flow around it.
- The point vortex is a singularity in the flow field (v_θ becomes infinite).
- Point 0 is simply a point formed by the intersection of the plane of a paper and a line perpendicular to the plane.
- This line is called vortex filament of strength $\Gamma$where $\Gamma$is the circulation around the vortex filament .

Circulation is defined as

$$\Gamma = \int \vec{V}.d\vec{s} \qquad (21.6)$$

This circulation constant denotes the algebraic strength of the vortex filament contained within the closed curve. From Eq. (21.6) we can write

$$\Gamma = \int \vec{V}.d\vec{s} = \int (udx + vdy + wdz)$$

For a two-dimensional flow

$$\Gamma = \int (udx + vdy)$$

or,

$$\Gamma = \int V \cos\alpha ds \quad \text{(according to Fig. 21.2)} \tag{21.7}$$

Consider a fluid element as shown in Fig. 21.2. **Circulation is positive in the anticlockwise direction** (not a mandatory but general convention).

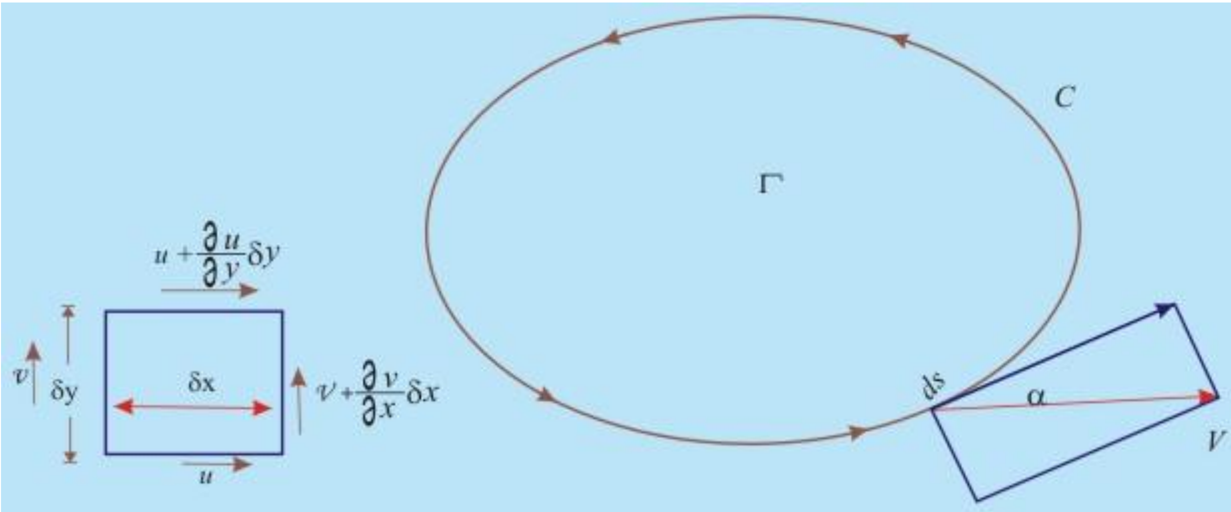


Fig 21.2 Circulation in a flow field

$$\delta\Gamma = u\delta x + \left(v + \frac{dv}{dx}\delta x\right)\delta y - \left(u + \frac{du}{dy}\delta y\right)\delta x - v\delta y$$

$$= \delta x \delta y\left(\frac{dv}{dx} - \frac{du}{dy}\right) = \delta x\delta y(2\omega_z) \qquad \text{where} \quad \delta A = \delta x \delta y$$

After simplification

$$\frac{\delta\Gamma}{\delta A} = 2\omega_z = \Omega_z \tag{21.8}$$

Physically, **circulation per unit area is the vorticity of the flow** .

Now, for a free vortex flow, the tangential velocity is given by Eq. (21.1) as

$$v_\theta = \frac{\Gamma/2\pi}{r} = \frac{C}{r}$$

For a circular path (refer Fig.21.2)

$$v_\theta = \frac{\Gamma/2\pi}{r} = \frac{C}{r}$$

Thus,

$$\Gamma = \int_0^{2\pi} vr\,d\theta$$

$$= 2\pi C$$

Therefore

$$\boldsymbol{\Gamma = 2\pi C} \qquad (21.9)$$

It may be noted that although free vortex is basically an irrotational motion, the circulation for a given path containing a singular point (including the origin) is constant (2πC) and independent of the radius of a circular streamline.

- However, **circulation calculated in a free vortex flow along any closed contour excluding the singular point (the origin), should be zero.**

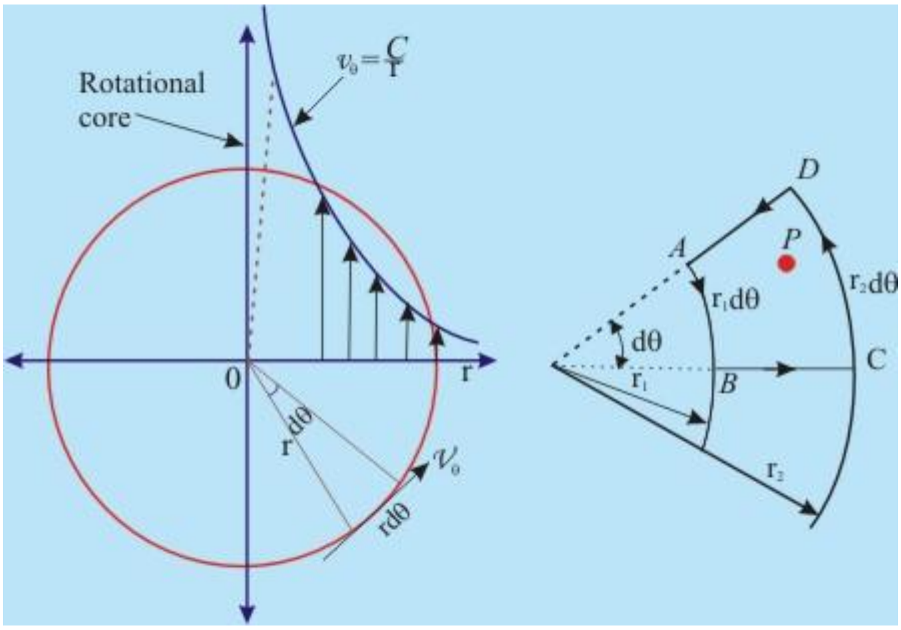


Fig 21.3 (a) Free Vortex Flow

Considering Fig 21.3 (a) and taking a closed contour ABCD in order to obtain circulation about the point, P around ABCD it may be shown that

$$\Gamma_{ABCD} = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA} = -\frac{C}{r_1} r_1 d\theta + 0 + \frac{C}{r_2} r_2 d\theta + 0 = 0$$

Forced Vortex Flow

- If there exists a solid body rotation at constant ω (induced by some external mechanism), the flow should be called a forced vortex motion (Fig. 21.3 (b).

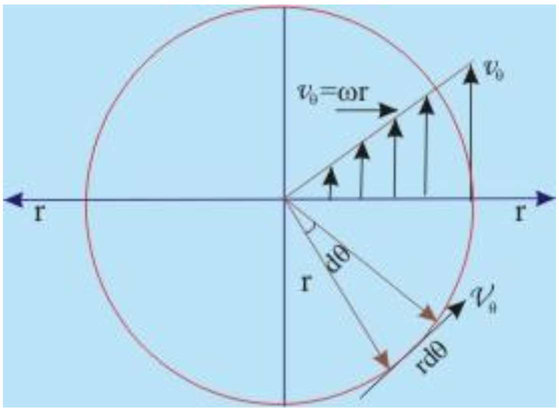


Fig 21.3 (b)   Forced Vortex Flow

we can write

$$v_\theta = \omega r \quad \text{and}$$

$$\Gamma = \int v_\theta ds = \int_0^{2\pi} \omega r . r d\theta = 2\pi r^2 \omega \qquad (21.10)$$

Equation (21.10) predicts that

1. The circulation is zero at the origin
2. It increases with increasing radius.
3. The variation is parabolic.

It may be mentioned that the **free vortex (irrotational) flow at the origin is impossible** because of mathematical singularity. However, physically there should exist a rotational (forced vortex) core which is shown by the dotted line (in Fig. 21.3a).

Below are given two statements which are related to Kelvin's circulation theorem (stated in 1869) and Cauchy's theorem on irrotational motion (stated in 1815) respectively

1. The circulation around any closed contour is invariant with time in an inviscid fluid.--- **Kelvin's Theorem**
2. A body of inviscid fluid in irrotational motion continues to move irrotationally.------------ **Cauchy's Theorem**

Combination of Fundamental Flows

1) Doublet

We can now form different flow patterns by superimposing the velocity potential and stream functions of the elementary flows stated above.

In order to develop a doublet, imagine a source and a sink of equal strength K at equal distance s from the origin along x-axis as shown in Fig. 21.4.

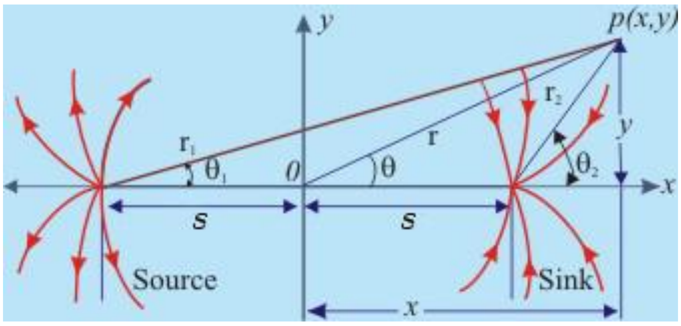


Fig 21.4 Superposition of a Source and a Sink

From any point p(x, y) in the field, r_1 and r_2 are drawn to the source and the sink. The polar coordinates of this point (r, θ) have been shown.

The potential functions of the two flows may be superimposed to describe the potential for the combined flow at P as

$$\phi = k\ln r_1 - k\ln r_2 \tag{21.11}$$

Similarly,

$$\psi = k(\theta_1 - \theta_2) = -k\alpha \tag{21.12}$$

where $\alpha = (\theta_2 - \theta_1)$

Expanding θ_1 and θ_2 in terms of coordinates of p and s

$$\tan\theta_1 = \frac{y}{x+s} \qquad \tan\theta_2 = \frac{y}{x-s} \tag{21.13}$$

$$r_1 = \sqrt{\left(r^2 + s^2 + 2rs\cos\theta\right)} \quad \text{and} \quad r_2 = \sqrt{\left(r^2 + s^2 - 2rs\cos\theta\right)} \tag{21.14}$$

Using

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}$$

we find

$$\tan\alpha = \left[\frac{yx + ys - yx + ys}{x^2 - s^2}\right] \Bigg/ \left(1 + \frac{y^2}{x^2 - s^2}\right)$$

or, $$\tan\alpha = \frac{2ys}{x^2 + y^2 - s^2}$$

Hence the stream function and the velocity potential function are formed by combining Eqs (21.12) and (21.13), as well as Eqs(21.11) and (21.14) respectively

$$\psi = -k\alpha$$

Hence $$\psi = -k\tan^{-1}\left(\frac{2ys}{x^2+y^2-s^2}\right)$$ ------- **Stream Function** (21.15)

$$\phi = k\ln r_1 - k\ln r_2 = k\ln\frac{r_1}{r_2}$$

$$\phi = \frac{k}{2}\ln\left(\frac{r^2+s^2+2rs\cos\theta}{r^2+s^2-2rs\cos\theta}\right)$$ ----- **Potential Function** (21.16)

Doublet is a special case when a source as well as a sink are brought together in such a way that

- $s \rightarrow 0$ and at the same time the
- strength $\frac{\Lambda}{\pi}\left(k/2\right)$ is increased to an infinite value.

These are assumed to be accomplished in a manner which makes the product of s and $\frac{\Lambda}{\pi}\left(k/2\right)$ (in limiting case) a finite value χ

This gives us

$$\boxed{\psi = -\frac{\chi\sin\theta}{r} \quad \text{and} \quad \phi = \frac{\chi\cos\theta}{r}}$$

Streamlines, Velocity Potential for a Doublet

We have seen in the last lecture that the streamlines associated with the doublet are

$$-\frac{\chi\sin\theta}{r} = C_1$$

If we replace sinθ by y/r, and the minus sign be absorbed in C_1 , we get

$$-\frac{\chi\sin\theta r}{r^2} = C_1$$

$$\Rightarrow \frac{\chi y}{r^2} = C_1 \qquad (21.17a)$$

Putting $r^2 = x^2 + y^2$ we get

$$x^2 + y^2 - \frac{\chi}{C_1} y = 0 \qquad (21.17b)$$

Equation (21.17b) represents a family of circles with

- **radius :** $\frac{\chi}{2C_1}$
- **centre :** $\left(0, \frac{\chi}{2C_1}\right)$

- For x = 0, there are two values of y, one of them=0.
- The centres of the circles fall on the y-axis.
- On the circle, where y = 0, x has to be zero for all the values of the constant.
- family of circles formed(due to different values of C_1) is tangent to x-axis at the origin.

These streamlines are illustrated in Fig. 21.5.

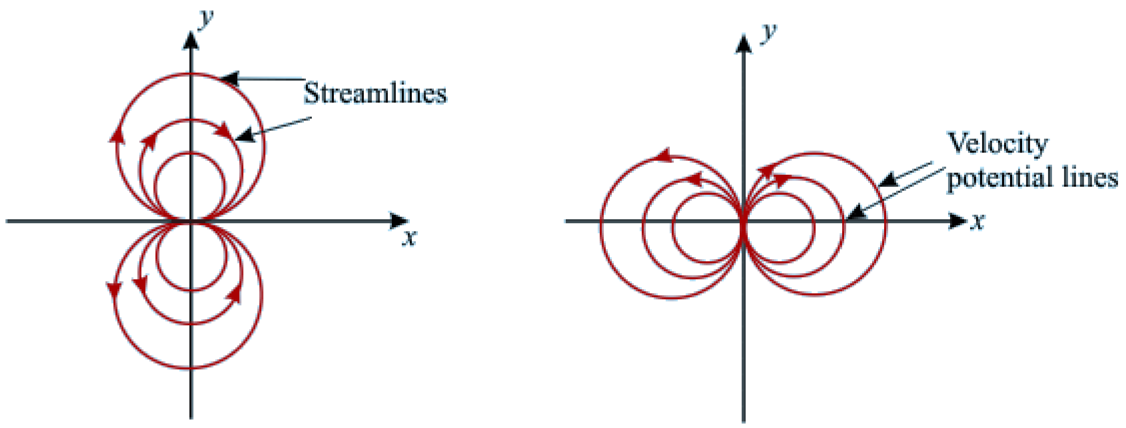


Fig 21.5 Streamlines and Velocity Potential Lines for a Doublet

Due to the initial positions of the source and the sink **in the development of the doublet** , it is certain that

- **the flow will emerge in the negative x direction from the origin**

and

- **it will converge via the positive x direction of the origin.**

Velocity potential lines

$$\frac{\chi \cos\theta}{r} = K_1$$

In cartresian coordinate the equation becomes

$$x^2 + y^2 - \frac{\chi}{K_1}x = 0 \tag{21.18}$$

Once again we shall obtain **a family of circles**

- **radius:** $\frac{\chi}{2K_1}$
- **centre:** $\left(\frac{\chi}{2K_1}, 0\right)$
- The centres will fall on x-axis.
- For y = 0 there are two values of x, one of which is zero.
- When x = 0, y has to be zero for all values of the constant.
- These circles are tangent to y-axis at the origin.

In addition to the determination of the stream function and velocity potential, it is observed that for a doublet

$$v_r = \frac{d\phi}{dr} = -\frac{\chi \cos\theta}{r^2}$$

As the centre of the doublet is approached; the radial velocity tends to be infinite.

It shows that the doublet flow has a singularity.

Since the circulation about a singular point of a source or a sink is zero for any strength, it is obvious that the **circulation about the singular point in a doublet flow must be zero i.e. doublet flow Γ=0**

$$\Gamma = \int \vec{V}.d\vec{s} = 0 \tag{21.19}$$

Applying Stokes Theorem between the line integral and the area-integral

$$\Gamma = \iint \left(\nabla \times \vec{V}\right) d\vec{A} = 0 \tag{21.20}$$

From Eq. 21.20 the obvious conclusion is $\nabla \times \vec{V} = 0$ i.e., **doublet flow is an irrotational flow.**

Flow About a Cylinder without Circulation

- Inviscid-incompressible flow about a cylinder in uniform flow is equivalent to the **superposition of a uniform flow and a doublet.**

- The doublet has its axis of development parallel to the direction of the uniform flow (x-axis in this case).
- The potential and stream function for this flow will be the sum of those for uniform flow and doublet.

Potential Function

$$\phi = U_0 x + \frac{\chi \cos\theta}{r}$$

Stream function

$$\psi = U_0 y - \frac{\chi \sin\theta}{r}$$

Streamlines

In two dimensional flow, a streamline may be interpreted as

- the edge of a surface, on which the velocity vector is always tangential.

and

- there is no flow in the direction normal to the surface (characteristic of a solid impervious boundary).

Hence, a **streamline** may also be considered as the **contour of an impervious two-dimensional body** .

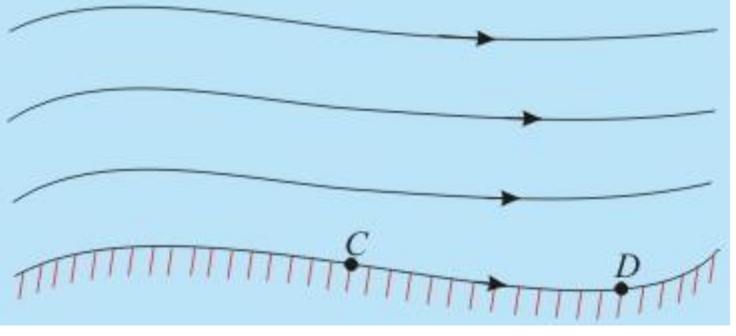


Fig 22.1 Surface Streamline

Figure 22.1 shows a set of streamlines.

1. The **streamline C-D** may be considered as the **edge of a two-dimensional body .**
2. other streamlines form the flow about the boundary.

In order to form a flow about the body of interest, a streamline has to be determined which encloses an area whose shape is of practical importance in fluid flow. This streamline describes the boundary of a two-dimensional solid body. The remaining streamlines outside this solid region, constitute the flow about this body.

If we look for the streamline whose value is zero, we will obtain

$$U_0 y - \frac{\chi \sin\theta}{r} = 0 \tag{22.1}$$

replacing y by rsinθ, we have

$$\sin\theta\left(U_0 r - \frac{\chi}{r}\right) = 0 \tag{22.2}$$

Solution of Eq. 22.2

1. If θ = 0 or θ = π, the equation is satisfied. This indicates that the x-axis is a part of the streamline Ψ = 0.
2. When the quantity in the parentheses is zero, the equation is **identically satisfied** . Hence it follows that

$$r = \left(\frac{\chi}{U_0}\right)^{1/2} \tag{22.3}$$

Interpretation of the solution

There is a circle of radius $r = \left(\frac{\chi}{U_0}\right)^{1/2}$ which is an intrinsic part of the streamline Ψ = 0.

This is shown in Fig.22.2

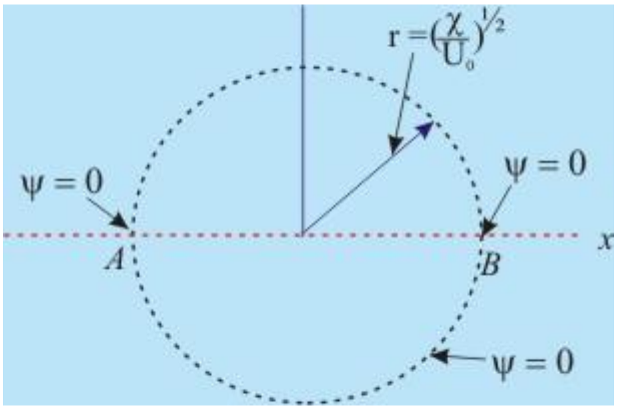


Fig 22.2 Streamline ψ = 0 in a Superimposed Flow of Doublet and Uniform Stream

Stagnation Points

Let us look at the points of intersection of the circle and x- axis , i.e. the points A and B in the above figure. The polar coordinate of these points are

$$r=\left(\frac{\chi}{U_0}\right)^{1/2},\theta=\pi$$ for point A

$$r=\left(\frac{\chi}{U_0}\right)^{1/2},\theta=0$$ for point B

The velocity at these points are found out by taking partial derivatives of the velocity potential in two orthogonal directions and then substituting the proper values of the coordinates.

Since, $$\phi=U_0 r\cos\theta+\frac{\chi\cos\theta}{r}$$

$$v_r=\frac{\partial\phi}{\partial r}=U_0\cos\theta-\frac{\chi\cos\theta}{r^2} \tag{22.4a}$$

$$v_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$$=\frac{1}{r}\left[-U_0 r\sin\theta-\frac{\chi\sin\theta}{r}\right] \tag{22.4b}$$

$$=-U_0\sin\theta-\frac{\chi\sin\theta}{r^2}$$

At point A $$\left[\theta=\pi,r=\left(\frac{\chi}{U_0}\right)^{1/2}\right]$$

$$v_r=0,v_\theta=0$$

At point B $$\left[\theta=0,r=\left(\frac{\chi}{U_0}\right)^{1/2}\right]$$

$$v_r = 0, v_\theta = 0$$

The points A and B are the stagnation points through which the flow divides and subsequently reunites forming a zone of circular bluff body.

The circular region, enclosed by part of the streamline $\psi = 0$ could be imagined as a solid cylinder in an inviscid flow. At a large distance from the cylinder the flow is moving uniformly in a cross-flow configuration.

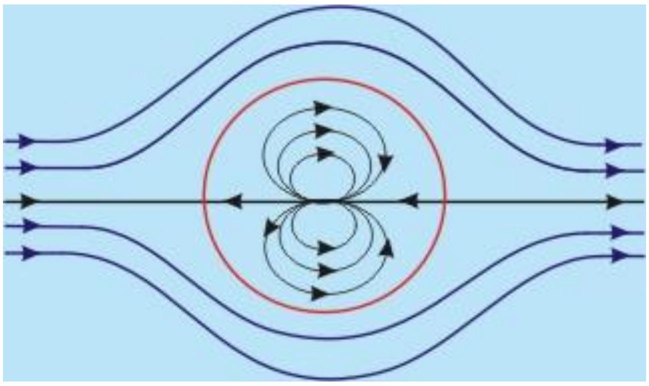


Fig. 22.3 Inviscid Flow past a Cylinder

Figure 22.3 shows the streamlines of the flow.

The **streamlines outside the circle** describe the **flow pattern of the inviscid irrotational flow across a cylinder.**

The **streamlines inside the circle may be disregarded** since this region is considered as a solid obstacle.

Lift and Drag for Flow Past a Cylinder without Circulation

Pressure in the Cylinder Surface

Pressure becomes uniform at large distances from the cylinder (where the influence of doublet is small).

Let us imagine the pressure p_0 is known as well as uniform velocity U_0 .
We can apply Bernoulli's equation between infinity and the points on the boundary of the cylinder.

Neglecting the variation of potential energy between the aforesaid point at infinity and any point on the surface of the cylinder, we can write

$$\frac{p_0}{\rho g} + \frac{U_0^2}{2g} = \frac{p_b}{\rho g} + \frac{U_b^2}{2g} \tag{22.5}$$

where the subscript b represents the surface on the cylinder.

Since fluid cannot penetrate the solid boundary, the velocity **U_b should be only in the transverse direction** , or in other words, only v_θ component of velocity is present on the streamline $\psi = 0$.

Thus at $$r = \left(\frac{\chi}{U_0}\right)^{1/2}$$

$$U_b = v_\theta \Big|_{at\ r=\left(\frac{\chi}{U_0}\right)^{1/2}} = \frac{1}{2}\frac{\partial \phi}{\partial r}\Big|_{at\ r=\left(\frac{\chi}{U_0}\right)^{1/2}} = -2U_0 \sin\theta \qquad (22.6)$$

From eqs (22.5) and (22.6) we obtain

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g} - \frac{(2U_0 \sin\theta)^2}{2g}\right] \qquad (22.7)$$

Lift and Drag

Lift :force acting on the cylinder (per unit length) in the direction normal to uniform flow.

Drag: force acting on the cylinder (per unit length) in the direction parallel to uniform flow.

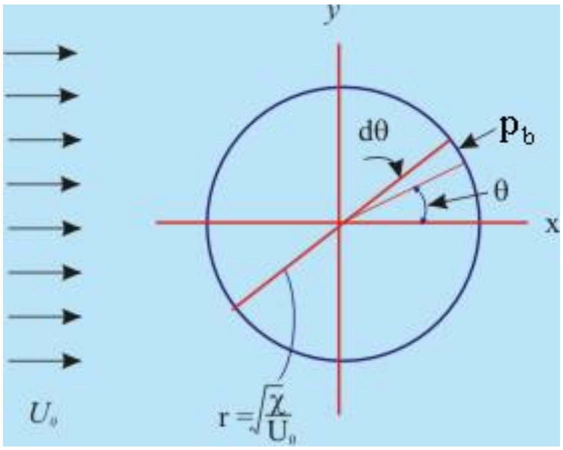


Fig 22.4 Calculation of Drag in a Cylinder

The drag is calculated by integrating the force components arising out of pressure, in the x direction on the boundary. Referring to Fig.22.4, the drag force can be written as

$$D=-\int_0^{2\pi} p_b \cos\theta r d\theta \qquad rd\theta = ds \rightarrow$$ infinitesimal length on the circumference

Since, $$r=\left(\frac{\chi}{U_0}\right)^{1/2}$$

$$D=-\int_0^{2\pi} p_b \cos\theta \left(\frac{\chi}{U_0}\right)^{1/2} d\theta$$

or,

$$D=-\int_0^{2\pi} \rho g \left(\frac{\chi}{U_0}\right)^{1/2}\left[\frac{U_0^2}{2g}+\frac{p_0}{\rho g}-\frac{(2U_0 \sin\theta)^2}{2g}\right]\cos\theta d\theta$$

$$D=-\int_0^{2\pi}\left[p_0+\frac{\rho U_0^2}{2}\left(1-4U_0\sin^2\theta\right)\right]\left(\frac{\chi}{U_0}\right)^{1/2}\cos\theta d\theta \tag{22.8}$$

Similarly, the lift force may be calculated as

$$L=-\int_0^{2\pi} p_b \sin\theta \left(\frac{\chi}{U_0}\right)^{1/2} d\theta \tag{22.9}$$

The Eqs (22.8) and (22.9) produce **D=0 and L=0 after** the integration is carried out.

However, **in reality, the cylinder will always experience some drag force. This contradiction between the inviscid flow result and the experiment is usually known as D 'Almbert paradox.**

Bernoulli's equation can be used to calculate the pressure distribution on the cylinder surface

$$\frac{p_b(\theta)}{\rho g}+\frac{U_b^2(\theta)}{2g}=\frac{p_0}{\rho g}+\frac{U_0^2}{2g}$$

$$\frac{p_b(\theta)-p_0}{\rho}=\frac{U_0^2}{2}\left[1-4\sin^2\theta\right]$$

The pressure coefficient , c_p is therefore

$$C_p=\frac{p_b(\theta)-p_0}{\frac{1}{2}\rho U_0^2}=\left[1-4\sin^2\theta\right] \tag{22.10}$$

The pressure distribution on a cylinder is shown in Figure below

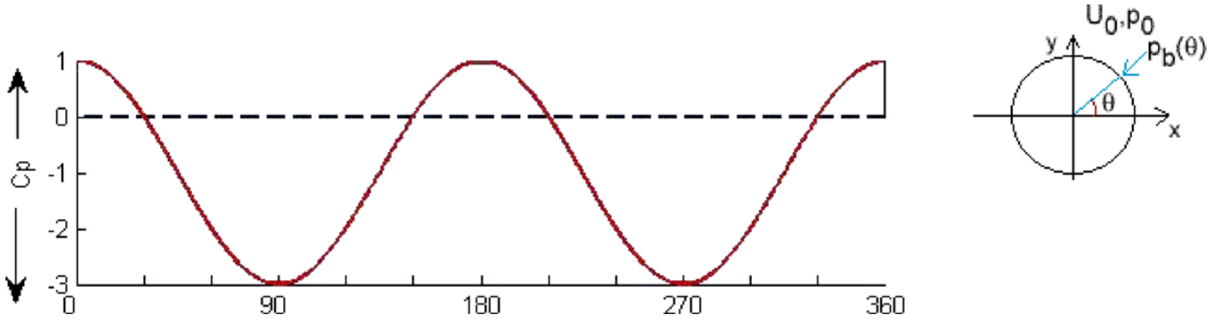


Fig 22.5 Variation of coefficient of pressure with angle

Analysis of Potential Flows through Complex Variables

The properties exhibited by the velocity potential and stream function of two dimensional irrotational flow of an inviscid fluid are identical to those exhibited by the real and imaginary part of an analytic function of a complex variable. It is natural to combine ϕ and ψ into an analytic function of a complex variable $z = x + Iy$ in the region of z plane occupied by the flow. Here, $I = \sqrt{-1}$ is called **imaginary unit**.

An analytic function, $$f(z) = f(x + Iy) = u(x, y) + I\,v(x, y) \tag{22.11}$$

and $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{22.12}$$

These are known as **Cauchy-Riemann** condition. Also, u and v are real single valued continuous functions. We get from the above

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\,\partial y}$$

Therefore, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Consider $$F(z) = \phi + I\psi \tag{22.13}$$

where ϕ is velocity potential function and ψ is stream function. This leads to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which means

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Finally we get

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

This completes the definition

$$\begin{aligned} \nabla \phi &= \hat{i}\,\frac{\partial \phi}{\partial x} + \hat{j}\,\frac{\partial \phi}{\partial y} = \hat{i}u + \hat{j}v \\ \nabla \psi &= \hat{i}\,\frac{\partial \psi}{\partial x} + \hat{j}\,\frac{\partial \psi}{\partial y} = -\hat{i}v + \hat{j}u \end{aligned} \tag{22.14}$$

Also, $$\nabla \phi . \nabla \psi = 0 \tag{22.15}$$

Therefore ϕ and ψ are perpendicular to each other.

Let us consider another function or complex potential

$$F(z) = z^2 = (x + Iy)^2$$

Which gives, $$F(z) = (x^2 - y^2) + 2I\,xy \tag{22.16}$$

Therefore, we get

$$\phi = x^2 - y^2 \quad \text{and} \quad \psi = 2xy$$

$$F = \phi + I\,\psi \quad \text{and} \quad z = x + I\,y$$

$$\frac{\partial \phi}{\partial x} + I\,\frac{\partial \psi}{\partial x} = \frac{dF}{dz}.\frac{dz}{dx}\,;$$ which means

$$\frac{dz}{dx} = 1 \quad \text{and} \quad \frac{dF}{dz}$$ is the complex velocity

Therefore,

$$\frac{dF}{dz} = \frac{\partial \phi}{\partial x} + I\,\frac{\partial \psi}{\partial x} = u - Iv$$

$$\left|\frac{dF}{dz}\right| = \sqrt{u^2 + v^2} \qquad (22.17)$$

..contd...Analysis of Potential Flows through Complex Variables

Now consider another situation, where the complex potential is given by

$$F(z) = Az = A(x + I\,y) \qquad (22.18)$$

and $\phi = Ax$ and $\psi = Ay$

$$\frac{dF}{dz} = A = u - I\,v$$ entailing $u = A$ and $v = 0$

$$u = \frac{\partial \phi}{\partial x} = A = U_0$$ and $$v = \frac{\partial \phi}{\partial y} = 0$$

Therefore $Az = F(z)$ signifies uniform flow. The flow was earlier represented via Figure 20.2(a)

We may choose yet another complex potential, given by

$$F(z) = A\,e^{-I\alpha}\,z \qquad (22.19)$$

or,

$$F(z) = A(\cos\alpha - I\sin\alpha)\;(x + I\,y)$$

or,

$$F(z) = A\,(x\cos\alpha + y\sin\alpha) + I\,A(y\cos\alpha - x\sin\alpha)$$

$$\frac{dF}{dz} = Ae^{-I\alpha} = A\,(\cos\alpha - I\sin\alpha) = u - I\,v \qquad (22.20)$$

This signifies

$$u = A\cos\alpha = U_0\cos\alpha$$
$$v = A\sin\alpha = U_0\sin\alpha$$

$$\phi = A\,(\,x\cos\alpha + y\sin\alpha\,) \qquad (22.21)$$

$$\psi = A\,(\,y\cos\alpha - x\sin\alpha\,) \qquad (22.22)$$

The flow is basically elementary uniform flow at an angle as represented by Figure 20.2 (b).

Consider another complex potential given by

$$F(z) = A\ln(z), \quad \text{where} \quad z = re^{I\theta} \tag{22.23}$$

$$F(z) = A\ln\left(re^{I\theta}\right)$$
$$F(z) = A\ln r + A\ln e^{I\theta}$$
$$F(z) = A\ln r + I A\theta$$

We obtain $\phi = A\ln r$ and $\psi = A\theta$

If A is positive, ψ is in the outward direction and it is a source flow (Figure 20.3). If A is negative, ψ is in the inward direction and it is sink flow.

The radial and tangential component of velocities are given by

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{K}{r}$$
$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

Let

$$2\pi r v_r \rho = \dot{m}$$, where $\dot{m}$ is the mass flux

$$v_r = \frac{\dot{m}}{2\pi r\rho} = \frac{K}{r}$$

The quantity K is,

$$K = \frac{\dot{m}}{2\pi\rho} = \frac{\Lambda}{2\pi}$$ and Λ is the volume flow rate

..contd...Analysis of Potential Flows through Complex Variables

Let us combine a source and sink now. Refer to Figure 21.4. The complex potential is given by

$$F(z) = K\ln(z+s) - K\ln(z-s) \tag{22.24}$$

This follows

$$\phi = K \ln r_1 - K \ln r_2$$
$$= K \ln \left(\frac{r_1}{r_2}\right)$$
$$\psi = K\,(\theta_1 - \theta_2) = -K\,(\theta_2 - \theta_1)$$

with $\tan\theta_1 = \dfrac{y}{x+s}$ and $\tan\theta_2 = \dfrac{y}{x-s}$

We know,

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

or, $$(\theta_2 - \theta_1) = \tan^{-1}\left(\frac{2\,y\,s}{x^2 + y^2 - s^2}\right)$$

Therefore, $$\psi = -K \tan^{-1}\left(\frac{2\,y\,s}{x^2 + y^2 - s^2}\right) \tag{22.25}$$

We also find $r_2^2 = r^2 \sin^2\theta + (r\cos\theta - s)^2$ and $r_1^2 = r^2 + s^2 + 2rs\cos\theta$

That results in

$$\phi = \frac{K}{2} \ln\left(\frac{1 + \frac{2s}{r}\cos\theta + \frac{s^2}{r^2}}{1 - \frac{2s}{r}\cos\theta + \frac{s^2}{r^2}}\right) \tag{22.26}$$

Under the limiting condition of $s \rightarrow 0$ (the flow becomes a doublet)

$$\phi = \frac{K}{2}\left[\ln\left(1 + \frac{2s}{r}\cos\theta\right) - \ln\left(1 - \frac{2s}{r}\cos\theta\right)\right]$$

We also know,

$$\ln(1 \pm \varepsilon) = \pm\varepsilon + \frac{\varepsilon^2}{2} \pm \ldots\ldots\ldots\ldots\ldots\ldots [\varepsilon \text{ is } << 1]$$
$$\phi = \frac{K}{2}\cdot\frac{2s}{r}\cos\theta\cdot 2 = \frac{2\,sK}{r}\cos\theta = \frac{\chi\cos\theta}{r}$$

The strength of the doublets given by $\chi = 2Ks = \frac{\Lambda}{\pi} s$

We get family of velocity potential from $\phi =$ constant lines

$$\phi = \frac{\chi \cos\theta}{r} = \frac{\chi x}{x^2 + y^2}$$

or, $$x^2 + y^2 - \frac{\chi x}{\phi} = 0$$

or, $$\left(x - \frac{\chi}{2\phi}\right)^2 + y^2 = \left(\frac{\chi}{2\phi}\right)^2 \qquad (22.27)$$

This is the equation for $\phi =$ constant lines (also see Figure 21.5)

Now from equation (22.25), for the limiting case of $s \to 0$, one can write

$$\psi = -K \frac{2ys}{x^2 + y^2 - s^2}$$ [as $\tan^{-1}\alpha = \alpha$ under the limiting case]

or, $$\psi = -\frac{\Lambda}{2\pi} \frac{2ys}{x^2 + y^2 - s^2}$$

or, $$\psi = -\chi \frac{y}{x^2 + y^2} = -\frac{\chi \sin\theta}{r}$$

The streamlines associated with the doublet are

$\psi =$ constant lines, which can be expressed as

$$\psi = -\frac{\chi \sin\theta}{r} = -\frac{\chi y}{r^2} = -\frac{\chi y}{x^2 + y^2}$$

or, $$\psi = -\frac{\chi y}{x^2 + y^2} = C_1$$

or, $$x^2 + y^2 + \frac{\chi y}{C_1} = 0$$

or, $$x^2 + \left(y + \frac{\chi}{2\psi}\right)^2 = \left(\frac{\chi}{2\psi}\right)^2 \qquad (22.28)$$

This is the equation for $\psi =$ constant lines (see Figure 21.5)

Therefore for the doublet

$$\psi = -\frac{\chi \sin\theta}{r} \quad , \quad \phi = \frac{\chi \cos\theta}{r}$$

and, $$F(z) = \phi + I\psi$$

It can also be written as

$$F(z) = \frac{\Lambda}{2\pi}\ln(z+s) - \frac{\Lambda}{2\pi}\ln(z-s) \tag{22.30}$$

$$F(z) = \frac{\Lambda}{2\pi}\ln\left(\frac{z+s}{z-s}\right)$$

$$F(z) = \frac{\Lambda}{2\pi}\ln\left(\frac{1+\frac{s}{z}}{1-\frac{s}{z}}\right)$$

$$F(z) = \frac{\Lambda}{2\pi}\ln\left\{\left(1+\frac{s}{z}\right)\left[1+\frac{s}{z}+\left(\frac{s}{z}\right)^2+\left(\frac{s}{z}\right)^3+\cdots\right]\right\}$$

$$F(z) = \frac{\Lambda}{2\pi}\ln\left\{\left[1+\frac{2s}{z}+2\left(\frac{s}{z}\right)^2+2\left(\frac{s}{z}\right)^3+\cdots\right]\right\}$$

If $s \rightarrow 0$ the $F(z)$ represents a doublet.

Under the limiting conditions,

$$F(z) = \frac{\Lambda}{2\pi}\ln\left(1+\frac{2s}{z}\right)$$

$$F(z) = \frac{\Lambda}{2\pi}\ln\left[\frac{2s}{z}-\frac{1}{2}\left(\frac{2s}{z}\right)^2+\frac{1}{3}\left(\frac{2s}{z}\right)^3+\cdots\right]$$

$$F(z) = \frac{\Lambda}{2\pi}\cdot\frac{2s}{z}$$

$$F(z) = \frac{\chi}{z} \tag{22.31}$$

Thus, by using the elementary complex potential for source, sink, doublet, uniform flow, vortex flow etc more complicated fields can be constructed via the method of superposition.

Especially, external flow past objects of various shapes can be simulated. However, for the first course, we shall follow simpler approach and construct various complex flows without using the route of complex potentials. We shall take up such exercises in the following lectures.

Flow Past a Source

When a uniform flow is added to that due to a source -

- fluid emitted from the source is swept away in the downstream direction
- stream function and velocity potential for this flow will be the sum of those for uniform flow and source

Stream function; $\psi = U_0 r \sin\theta + \kappa\theta$

Velocity Potential; $\phi = U_0 r \cos\theta + \kappa \ ln\ r$

So $$\upsilon_r = \frac{\partial\phi}{\partial r} = U_0 \cos\theta + \frac{\kappa}{r}$$

and $$\upsilon_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\left(-U_0 r \sin\theta\right) = -U_0 \sin\theta$$

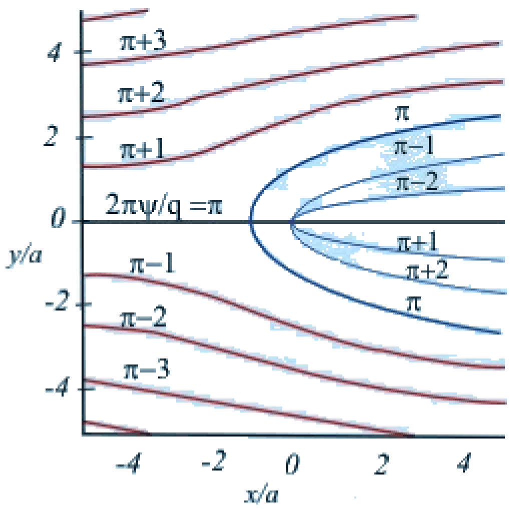


Fig 23.1 The streamlines of the flow past a line source for equal increments of 2πψ/q
The Plane coordinates are x/a, y/a where a=k/u

Explanation of Figure

- At the point x = -a, y = 0 fluid velocity is zero.
- This is called stagnation point of the flow
- Here the source flow is turned around by the oncoming uniform flow
- The parabolic streamline passing through stagnation point $\psi = \pm\pi\kappa$ seperates uniform flow from the source flow.

- The streamline becomes parallel to x axis as $x \to \infty \quad where\ y = \pm \pi a$

Flow Past Vortex

when uniform flow is superimposed with a vortex flow -

- Flow will be asymmetrical about x - axis
- Again stream function and velocity potential will be the sum of those for uniform flow and vortex flow

Stream Function: $$\psi = U_0 r \sin\theta - \frac{\Gamma}{2\pi} In\ r$$

Velocity Potential: $$\phi = U_0 r \cos\theta + \frac{\Gamma}{2\pi}\theta$$

so that; $$\upsilon_r = \frac{\partial\phi}{\partial r} = U_0 \cos\theta$$

$$\upsilon_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_0 \sin\theta + \frac{\Gamma}{2\pi r}$$

Flow About a Rotating Cylinder

Magnus Effect

Flow about a rotating cylinder is equivalent to the combination of flow past a cylinder and a vortex.
As such in addition to superimposed uniform flow and a doublet, a vortex is thrown at the doublet centre which will simulate a rotating cylinder in uniform stream.

The pressure distribution will result in a force, a component of which will culminate in lift force

The phenomenon of generation of lift by a rotating object placed in a stream is known as Magnus effect.

Velocity Potential and Stream Function

The velocity potential and stream functions for the combination of doublet, vortex and uniform flow are

$$\phi = U_0 x + \frac{\chi\cos\theta}{r} - \frac{\Gamma}{2\pi}\theta \qquad (23.1)$$

(clockwise rotation)

$$\psi = U_0 y - \frac{\chi \sin\theta}{r} + \frac{\Gamma}{2\pi}\ln r$$ (clockwise rotation) (23.2)

By making use of either the stream function or velocity potential function, the velocity components are (putting x= rcosθ and y= rsinθ)

$$v_r = \frac{\partial \phi}{\partial r} = (U_0 - \frac{\chi}{r^2})\cos\theta \qquad (23.3)$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -(U_0 + \frac{\chi}{r^2})\sin\theta - \frac{\Gamma}{2\pi r} \qquad (23.4)$$

Stagnation Points

At the stagnation points the velocity components must vanish. From Eq. (23.3), we get

$$\cos\theta\left(U_0 - \frac{\chi}{r^2}\right) = 0 \qquad (23.5)$$

Solution :

1. From Eq. (23.5) it is evident that a zero radial velocity component may occur at

- $\theta = \pm\frac{\pi}{2}$ and
- along the circle, $r = \left(\frac{\chi}{U_0}\right)^{1/2}$.

Eq. (23.4) depicts that a zero transverse velocity requires

$$\sin\theta = \frac{-\Gamma/2\pi r}{U_0 + \left(\frac{\chi}{r^2}\right)} \quad \text{or} \quad \theta = \sin^{-1}\left[\frac{-\Gamma/2\pi r}{U_0 + \frac{\chi}{r^2}}\right] \qquad (23.6)$$

At the stagnation point, both radial and transverse velocity components must be zero .

Thus the location of stagnation point occurs at

$$r = \left(\frac{\chi}{U_0}\right)^{1/2}$$

$$\theta = \sin^{-1}\left\{\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right\}$$

There will be two stagnation points since there are two angles for a given sine except for $\sin^{-1}(\pm 1)$

Determination of Stream Line

The streamline passing through these points may be determined by evaluating ψ at these points.

Substitution of the stagnation coordinate (r, θ) into the stream function (Eq. 23.2) yields

$$\psi = \left[U_0\left(\frac{\chi}{U_0}\right)^{1/2} - \frac{\chi}{\left(\frac{\chi}{U_0}\right)^{1/2}}\right]\sin\ \sin^{-1}\left[\frac{\Gamma}{4\pi(\chi U_0)^{1/2}}\right] + \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

$$\psi = \left[(U_0\chi)^{1/2} - (U_0\chi)^{1/2}\right]\left[\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right] + \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

or,

$$\psi_{stag} = \frac{\Gamma}{2\pi} ln\left(\frac{\chi}{U_0}\right)^{1/2} \tag{23.7}$$

Equating the general expression for stream function to the above constant, we get

$$U_0 r\ \sin\theta - \frac{\chi \sin\theta}{r} + \frac{\Gamma}{2\pi} ln\ r = \frac{\Gamma}{2\pi} ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

By rearranging we can write

$$\sin\theta\left[U_0 r - \frac{\chi}{r}\right] + \frac{\Gamma}{2\pi}\left[\ln r - \ln\left(\frac{\chi}{U_0}\right)^{1/2}\right] = 0 \tag{23.8}$$

All points along the circle $\Gamma = \left(\frac{\chi}{U_0}\right)^{1/2}$ **satisfy Eq. (23.8)** , since for this value of r, each quantity within parentheses in the equation is zero.

Considering the interior of the circle (on which $\psi = 0$) to be a solid cylinder, the outer streamline pattern is shown in Fig 23.2.

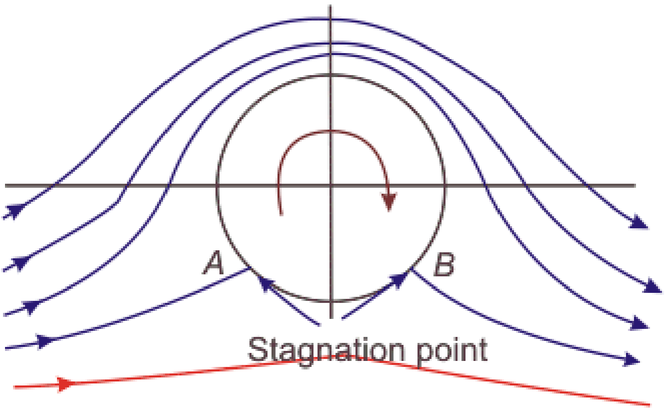


Fig 23.2 Flow Past a Cylinder with Circulation

At the stagnation point

$$\theta = \sin^{-1}\left[\frac{-\left(\Gamma/2\pi\right)}{2\left(\chi U_0\right)^{1/2}}\right]$$

$$\theta = \sin^{-1}\left[\frac{-\Gamma/2\pi}{2U_0 r}\right]$$

The limiting case arises for $\frac{\left(\Gamma/2\pi\right)}{U_0 r} = 2$, where $\theta = \sin^{-1}(-1) = -90^0$ and two stagnation points meet at the bottom as shown in Fig. 23.3.

In the case of a circulatory flow past the cylinder, the streamlines are symmetric with respect to the y-axis. The presures at the points on the cylinder surface are symmetrical with respect to the y-axis. There is no symmetry with respect to the x-axis. Therefore a resultant force acts on the cylinder in the direction of the y-axis, and the resultant force in the direction of the x-axis is equal to zero as in the flow without circulation; that is, the D'Alembert paradox takes place here as well. Thus, in the presence of circulation, different flow patterns can take place and, therefore, it is necessary for the uniqueness of the solution, to specify the magnitude of circulation.

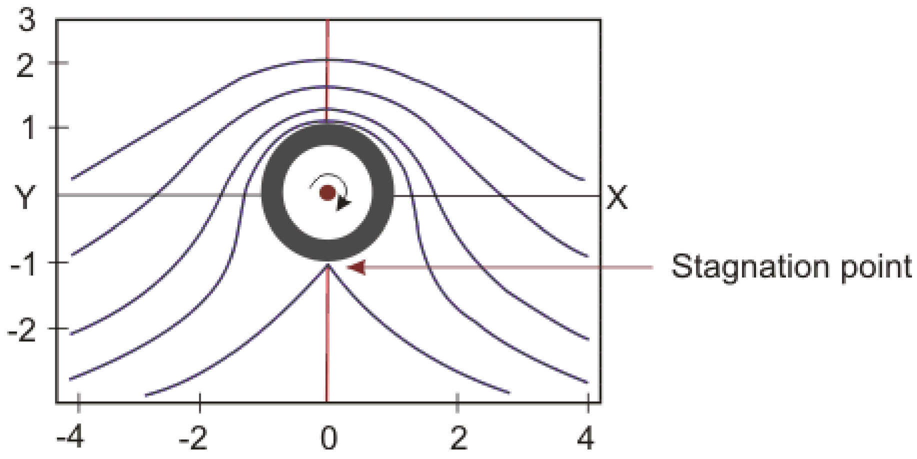


Fig 23.3 Flow Past a Circular Cylinder with Circulation Value $\frac{(\Gamma/2\pi)}{U_0 r} = 2$

However, in all these cases the **effects of the vortex and doublet become negligibly small as one moves a large distance from the cylinder.**

The flow is assumed to be uniform at infinity.

We have already seen that the change in strength Γ of the vortex changes the flow pattern, particularly the position of the stagnation points but the radius of the cylinder remains unchanged.

Lift and Drag for Flow About a Rotating Cylinder

The pressure at large distances from the cylinder is uniform and given by p_0.

Deploying Bernoulli's equation between the points at infinity and on the boundary of the cylinder,

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho_g} - \frac{U_b^2}{2g}\right] \tag{23.9}$$

Hence,

$$U_b = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -2U_0 \sin\theta - \frac{r}{2\pi}\left[\frac{U_0}{\chi}\right]^{1/2} \tag{23.10}$$

From Eqs (23.9) and (23.10) we can write

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g}\right] - \left[\frac{-2U_0 \sin\theta - \frac{\tau}{2\pi}\left(\frac{U_0}{\chi}\right)^{1/2}}{2g}\right]^2 \qquad (23.11)$$

The lift may calculated as

$$L = -\int_0^{2\pi} p_b \sin\theta \left[\frac{\chi}{U_0}\right]^{1/2} d\theta$$

$$L = -\int_0^{2\pi} \left\{ \frac{\rho U_0^2}{2} + p_0 - \frac{\rho\left[-2U_0 \sin\theta - \frac{\Gamma}{2\pi}\left(\frac{U_0}{\chi}\right)^{1/2}\right]^2}{2} \right\} \left[\frac{x}{U_0}\right]^{1/2} (\sin\theta) d\theta$$

or,

$$L = -\int_0^{2\pi} \left[\frac{\rho U_0^2}{2}\left(\frac{\chi}{U_0}\right)^{1/2} \sin\theta + p_0\left(\frac{\chi}{U_0}\right)^{1/2} \sin\theta - \frac{\rho}{2}\left\{4U_0^2 \sin^2\theta + \frac{4U_0 r \sin\theta}{2\pi}\left(\frac{U_0}{\chi}\right)^{1/2} + \frac{\Gamma^2}{4\pi^2}\right.\right.$$

$$L = -\int_0^{2\pi} \left[\frac{\rho U_0^2}{2}\left(\frac{\chi}{U_0}\right)^{1/2} \sin\theta + p_0\left(\frac{\chi}{U_0}\right)^{1/2} \sin\theta - 2\rho U_0^2 \sin^3\theta\left(\frac{\chi}{U_0}\right)^{1/2} - \frac{\rho U_0 \Gamma}{\pi}\sin^2\theta - \frac{\rho\Gamma^2}{8\pi^2}\right.$$

$$L = \rho U_0 \Gamma \qquad (23.12)$$

The drag force , which includes the multiplication by cosθ (and integration over 2π) is zero.

- Thus the inviscid flow also demonstrates lift.
- lift becomes a simple formula involving only the density of the medium, free stream velocity and circulation.
- in two dimensional incompressible steady flow about a boundary of any shape, the lift is always a product of these three quantities.----- **Kutta- Joukowski theorem**

Aerofoil Theory

Aerofoils are streamline shaped wings which are used in airplanes and turbo machinery. These shapes are such that the drag force is a very small fraction of the lift. The following nomenclatures are used for defining an aerofoil

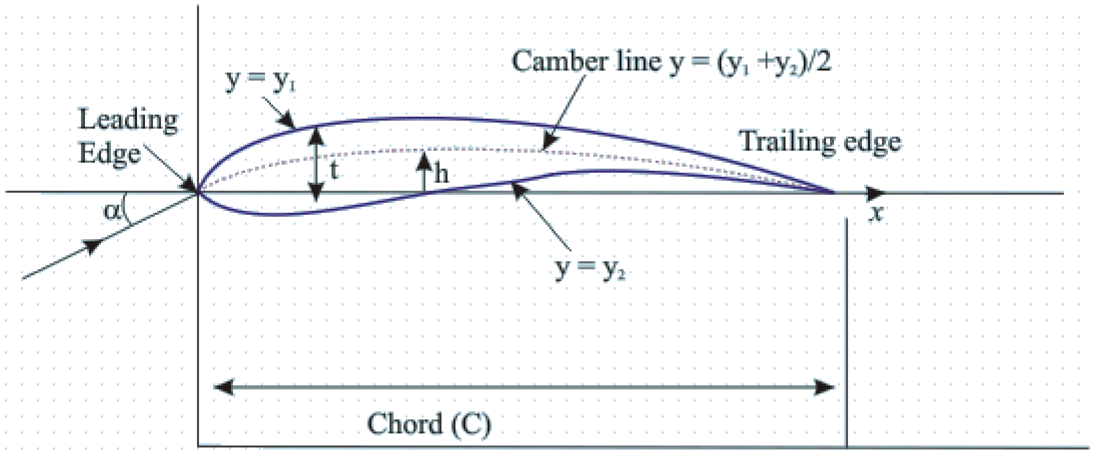


Fig 23.4 Aerofoil Section

- The **chord** (C) is the distance between the leading edge and trailing edge.
- The length of an aerofoil, normal to the cross-section (i.e., normal to the plane of a paper) is called the **span** of a aerofoil.
- The **camber line** represents the mean profile of the aerofoil. Some important geometrical parameters for an aerofoil are the ratio of maximum thickness to chord (t/C) and the ratio of maximum camber to chord (h/C). When these ratios are small, an aerofoil can be considered to be thin. For the analysis of flow, a thin aerofoil is represented by its camber.

The theory of thick cambered aerofoils uses a complex-variable mapping which transforms the inviscid flow across a rotating cylinder into the flow about an aerofoil shape with circulation.

Flow Around a Thin Aerofoil

- Thin aerofoil theory is based upon the superposition of uniform flow at infinity and a continuous distribution of clockwise free vortex on the camber line having circulation density $\gamma(s)$ per unit length .
- The circulation density $\gamma(s)$ should be such that the resultant flow is tangent to the camber line at every point.
- Since the slope of the camber line is assumed to be small, $\gamma(s)ds = \gamma(\eta)d\eta$. The total circulation around the profile is given by

$$\Gamma = \int_0^C \gamma(\eta)\, d\eta \qquad (23.13)$$

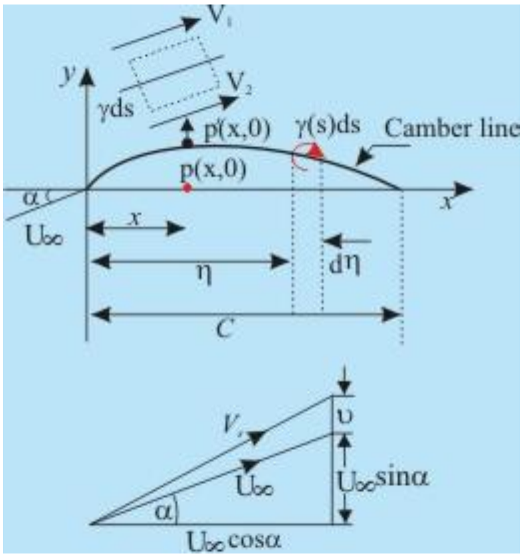


Fig 23.5 Flow Around Thin Aerofoil

A vortical motion of strength $\gamma d\eta$ at x= η develops a velocity at the **point p** which may be expressed as

$$d\upsilon = \frac{\gamma(\eta)d\eta}{2\pi(\eta - x)}\ acting\ upwards$$

The total induced velocity in the upward direction at **point p** due to the entire vortex distribution along the camber line is

$$\upsilon(x) = \frac{1}{2\pi}\int_0^C \frac{\gamma(\eta)d\eta}{(\eta - x)} \qquad (23.14)$$

For a small camber (having small α), this expression is identically valid for the induced velocity at **point p'** due to the vortex sheet of variable strength $\gamma(s)$ on the camber line. The resultant velocity due to U_∞ and v(x) must be tangential to the camber line so that the slope of a camber line may be expressed as

$$\frac{dy}{dx} = \frac{U_\infty \sin\alpha + \upsilon}{U_\infty \cos\alpha} = \tan\alpha + \frac{\upsilon}{U_\infty \cos\alpha}$$

$$\frac{dy}{dx} = \alpha + \frac{\nu}{U_\infty}\left[\sin ce\ \ \alpha\ is\ verysmall\right] \qquad (23.15)$$

From Eqs (23.14) and (23.15) we can write

$$\frac{dy}{dx} = \alpha + \frac{1}{2\pi U_\infty}\int_0^c \frac{\gamma(\eta)d\eta}{\eta - x}$$

Consider an element ds on the camber line. Consider a small rectangle (drawn with dotted line) around ds. The upper and lower sides of the rectangle are very close to each other and these are parallel to the camber line. The other two sides are normal to the camber line. The circulation along the rectangle is measured in clockwise direction as

$V_1 ds - V_2 ds = \gamma ds$ [normal component of velocity at the camber line should be zero]

or $V_1 - V_2 = \gamma$

If the mean velocity in the tangential direction at the camber line is given by $Vs = (V_1 + V_2)/2$, it can be rewritten as

$V_1 = Vs + \frac{\gamma}{2}$ and $V_2 = V_s - \frac{\gamma}{2}$

if v is very small $[v << U_\infty], V_s$ becomes equal to U_∞. The difference in velocity across the camber line brought about by the vortex sheet of variable strength $\gamma(s)$ causes pressure difference and generates lift force.

Generation of Vortices Around a Wing

- **The lift around an aerofoil is generated following Kutta-Joukowski theorem** . Lift is a product of ρ , U_∞ and the circulation Γ.

  $Lift = \rho U_\infty \Gamma$

- When the motion of a wing starts from rest, vortices are formed at the trailing edge.
- At the start, there is a velocity discontinuity at the trailing edge. This is eventual because near the trailing edge, the velocity at the bottom surface is higher than that at the top surface. This discrepancy in velocity culminates in the formation of vortices at the trailing edge.
- Figure 23.6(a) depicts the formation of starting vortex by impulsively moving aerofoil. However, the starting vortices induce a counter circulation as shown in Figure 23.6(b). The circulation around a path (ABCD) enclosing the wing and just shed (starting) vortex must be zero. Here we refer to **Kelvin's theorem** once again.

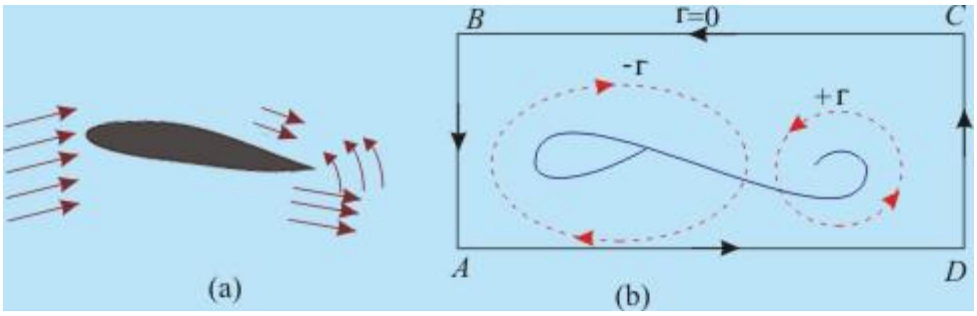


Fig 23.6 Vortices Generated when an Aerofoil Just Begins to Move

- Initially, the flow starts with the zero circulation around the closed path. Thereafter, due to the change in angle of attack or flow velocity, if a fresh starting vortex is shed, the circulation around the wing will adjust itself so that a net zero vorticity is set around the closed path.
- Real wings have finite span or finite aspect ratio (AR) λ , defined as

$$\lambda = \frac{b^2}{A_s} \tag{23.16}$$

where b is the span length, A_s is the plan form area as seen from the top..

- For a wing of finite span, the end conditions affect both the lift and the drag. In the leading edge region, pressure at the bottom surface of a wing is higher than that at the top surface. The longitudinal vortices are generated at the edges of finite wing owing to pressure differences between the bottom surface directly facing the flow and the top surface.

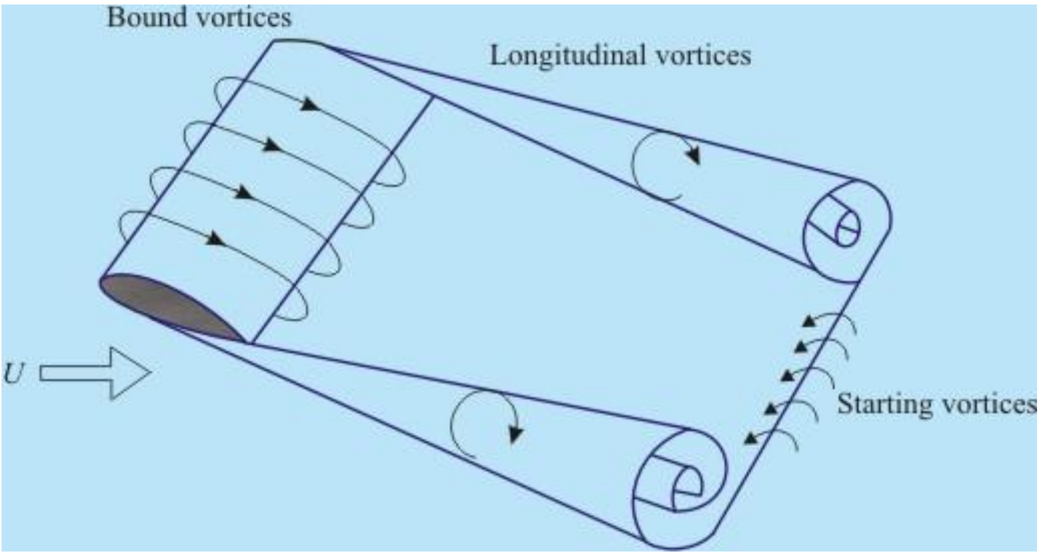


Fig 23.7 Vortices Around a Finite Wing

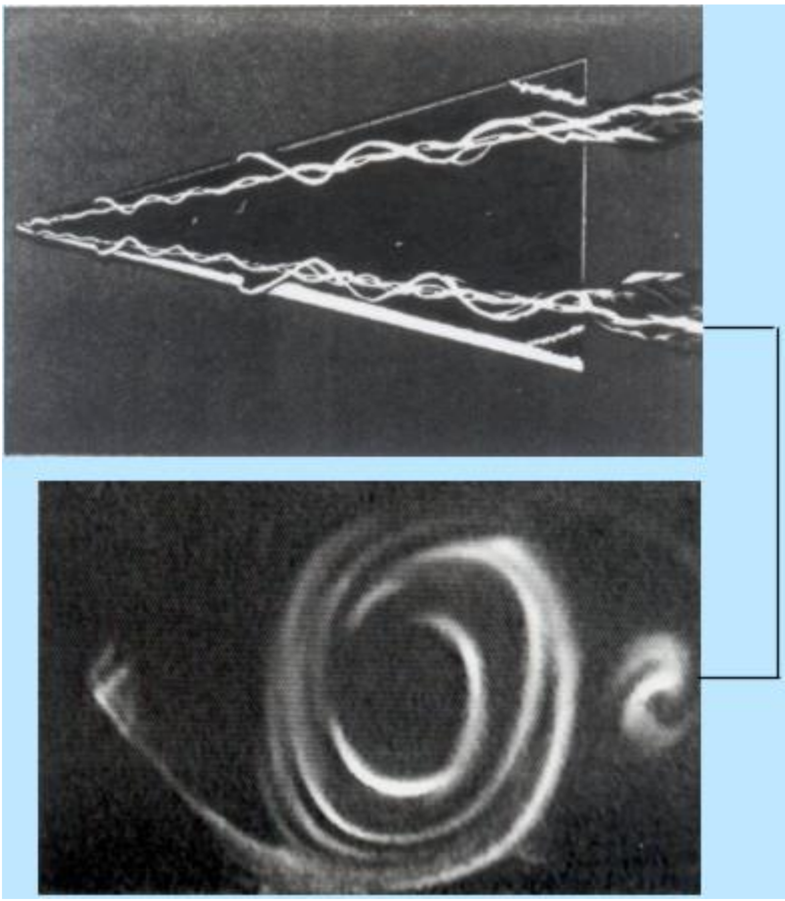


Fig 23.8   Generation of Longitudinal Vortices

Exercise Problems - Chapter 7

1. Determine whether the function

$$A(r + a^2 / r)\cos\theta - (B / 2\pi)\theta$$

is a possible velocity potential for inviscid incompressible flow.

2. Find the stream function and the velocity potential for flow around a closed contour formed by superposition of a source, sink and a rectilinear flow. The source and sink are of equal strength and they are located at equal distances on either side of the origin along the x-axis.

3. Verify the stream function and the velocity potential for flow around a circular cylinder with Circulation satisfying their respective Laplace equations.

4. A kite is having cross-sectional area of .5 m ×.5 m. The weight of kite is 4.9 N and it flies in the air at an angle of 10^0 with the horizontal plane. The string of the kite makes an angle of 45^0 with the horizontal plane. The tension applied by the person flying the kite is N. If the

velocity of the wind at the layer where the kite is flying is 15 m/s, find out the lift and drag coefficients of the kite.

Take ρ of air as 1.2 kg/m , $F_D = C_D \; \rho \; A\dfrac{U_0^2}{2}$ and $F_L = C_L \; \rho \; A\dfrac{U_0^2}{2}$

$\left[C_D = 0.419,\; C_L = 0.564\right]$

5. In a tornado (free vortex motion), at a radius of 20 cm the velocity (tangential component is the only non-trivial component) and pressures are 22.5 m/s and 117.72 kpa absolute. Find out the pressure at a radius of 4.5 m. The fluid is air and assumed to have density equal to 1.2 kg/m^3 .

[196.705 kpa]

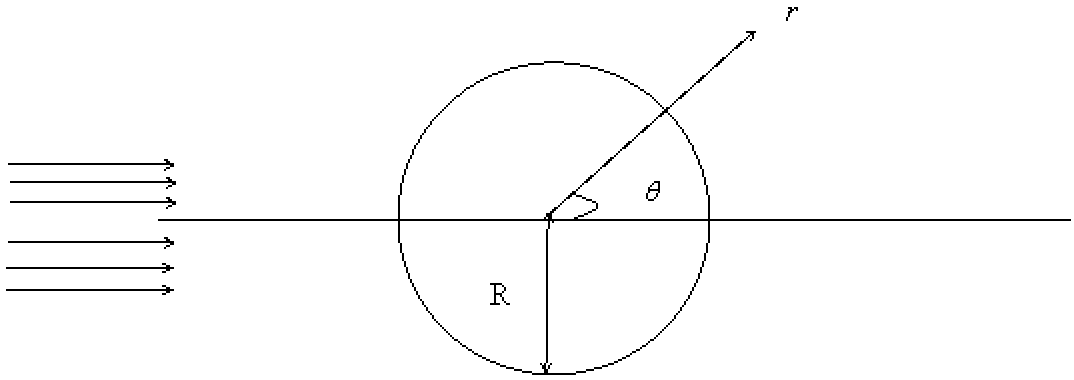
6. Flow past a rotating cylinder can be simulated by superposition of a doublet, a uniform flow and a vortex. The peripheral velocity of the rotating cylinder alone is given by v_θ at $r = R$ (R is the radius of the cylinder). Use the expression for the combined velocity potential for the superimposed uniform flow, doublet and vortex flow (clockwise rotation) and show that the resultant velocity at any point on the cylinder is given by $-2U_0 \; \sin\theta - v_\theta \; (at \; r = R)$. The angle θ is the angular position of the point of interest. A cylinder rotates at 600 rpm around its own axis which is perpendicular to the uniform air stream density 1.24 kg/m^3 having a velocity of 40 m/s. The cylinder is 2 m in diameter. Find out (a) circulation, Γ (b) lift per unit length and the (c) position of the stagnation points.

[394.78 m^2 / s, 19581.29 N/m, -51.75^0 and 231.75^0]

7. In the early attempts for the development of the aeroplane, rotating cylinders were used as the airfoils. Consider such a cylinder having a diameter of 1m and a length of 10 m. If this cylinder is rotated at 120 rpm while the plane moves at a speed of 120 km/hour through the air at 700 m standard atmosphere ($\rho = 0.59$ kg/m^3), estimate the minimum lift that could be developed disregarding end effects.

[3880 N]

8. If a cylinder (placed with its axis normal to the free stream) is to be used as a pitot-static tube for the measurement of the free-stream velocity, find the locations of piezo-metric holes on the cylinder



Assume ideal fluid flow. The velocity components in the flow field are

$$v_r = U_0 \left(1 - \frac{R^2}{r^2}\right) \cos\theta$$

$$v_\theta = -U_0 \left(1 + \frac{R^2}{r^2}\right) \sin\theta$$

Note: A Pitot-static tube measures the velocity head, i.e., the difference between the stagnation pressure and static pressure in the flow.

$[180^0, 210^0]$

Recap

In this course you have learnt the following

- Irrotationality leads to the condition $\nabla \times \vec{V} = 0$ which demands $\vec{V} = \nabla \phi$, where φ is known as a potential function. For a potential flow $\nabla^2 \phi = 0$.
- The stream function also obeys the Laplace's equation $\nabla^2 \psi = 0$ for the potential flows. Laplace's equation is linear, hence any number of particular solutions of Laplace' s equation added together will yield another solution. So a complicated flow for an in viscid, incompressible, irrotational condition can be synthesized by adding together a number of elementary flows which are also inviscid, incompressible and irrotational. This is called the method of superposition.
- Some inviscid flow configurations of practical importance are solved by using the method of superposition. The circulation in a flow field is defined as $\Gamma = \int \vec{V} . d\vec{s}$. Subsequently , the velocity may be defined as circulation per unit area . The circulation for a closed path in an irrotational flow field is zero.

However , the circulation for a given closed path is an irrotational flow containinga finite number of singular points is a non -zero constant.

- The lift around an immersed body is generated when the flow field processes circulation. The lift around a body of any shape is given by $L = \rho U_0 \Gamma$, where ρ is the density and U_0 is the velocity in the streamwise direction.

General Viscosity Law

Newton's viscosity law is

$$\tau = \mu \frac{\partial V}{\partial n} \qquad (24.1)$$

where,

τ = Shear Stress,
n is the coordinate direction normal to the solid-fluid interface,
μ is the coefficient of viscosity, and
V is velocity.

The above law is valid for parallel flows.

Considering Stokes' viscosity law: **shear stress is proportional to rate of shear strain** so that

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \qquad (24.2a)$$

$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \qquad (24.2b)$$

$$\tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \qquad (24.2c)$$

τ **has two subscripts---**

first subscript : denotes the direction of the normal to the plane on which the stress acts, while the
second subscript : denotes direction of the force which causes the stress.

The expressions of Stokes' law of viscosity for normal stresses are

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} + \mu'\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.3a)$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial \upsilon}{\partial y} + \mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.3b)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} + \mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.3c)$$

where μ' is a proportionality factor and it is related to the second coefficient of viscosity μ_1 by the relationship $\mu_1 = \mu' + \frac{2}{3}\mu$.

We have already seen that the thermodyamic pressure is $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$
Now if we add the three equations 24.3(a),(b) and (c) , we obtain,

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + 2\mu\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] + 3\mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right]$$

or

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + (2\mu + 3\mu')\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.4)$$

- For incompressible fluids, $\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} = 0$
  So, $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ is satisfied eventually. **This is known as Thermodynamic pressure.**
- For compressible fluids, **Stokes' hypothesis** is $\mu' + \frac{2}{3}\mu = 0$.
- Invoking this to Eq. (24.4), will finally result in $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ (same as for incompressible fluid).
- Interesting historical aspects of the Stoke's assumption $3\mu' + 2\mu = 0$ can be found in Truesdell $(1952)^{\dagger}$.
- --

  † Truesdell , C.A. "Stoke's Principle of Viscosity", Journal of Rational Mechanics and Analysis, Vol.1, pp.228-231,1952.

- --
- Generally, **fluids obeying the ideal gas equation follow this hypothesis and they are called Stokesian fluids** .

- The second coefficient of viscosity, μ_1 has been verified to be negligibly small. Substituting μ for μ' in 24.3a, 24.3b, 24.3c we obtain

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.5a)$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial \upsilon}{\partial y} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.5b)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (24.5c)$$

In deriving the above stress-strain rate relationship, it was assumed that a fluid has the following properties

- Fluid is homogeneous and isotropic, i.e. the relation between components of stress and those of rate of strain is the same in all directions.
- Stress is a linear function of strain rate.
- The stress-strain relationship will hold good irrespective of the orientation of the reference coordinate system.

The stress components must reduce to the hydrostatic pressure "**p**" (**typically** thermodynamic pressure = hydrostatic pressure) when all the gradients of velocities are zero.

Navier-Strokes Equation

- Generalized equations of motion of a real flow named after the inventors CLMH Navier and GG Stokes are derived from the **Newton's second law**
- **Newton's second law** states that the **product of mass and acceleration is equal to sum of the external forces acting on a body.**
- External forces are of two kinds-
  - one acts throughout the mass of the body ----- **body force** (gravitational force, electromagnetic force)
  - another acts on the boundary---------------------- **surface force** (pressure and frictional force).

Objective - We shall consider a differential fluid element in the flow field (Fig. 24.1). Evaluate the surface forces acting on the boundary of the rectangular parallelepiped shown below.

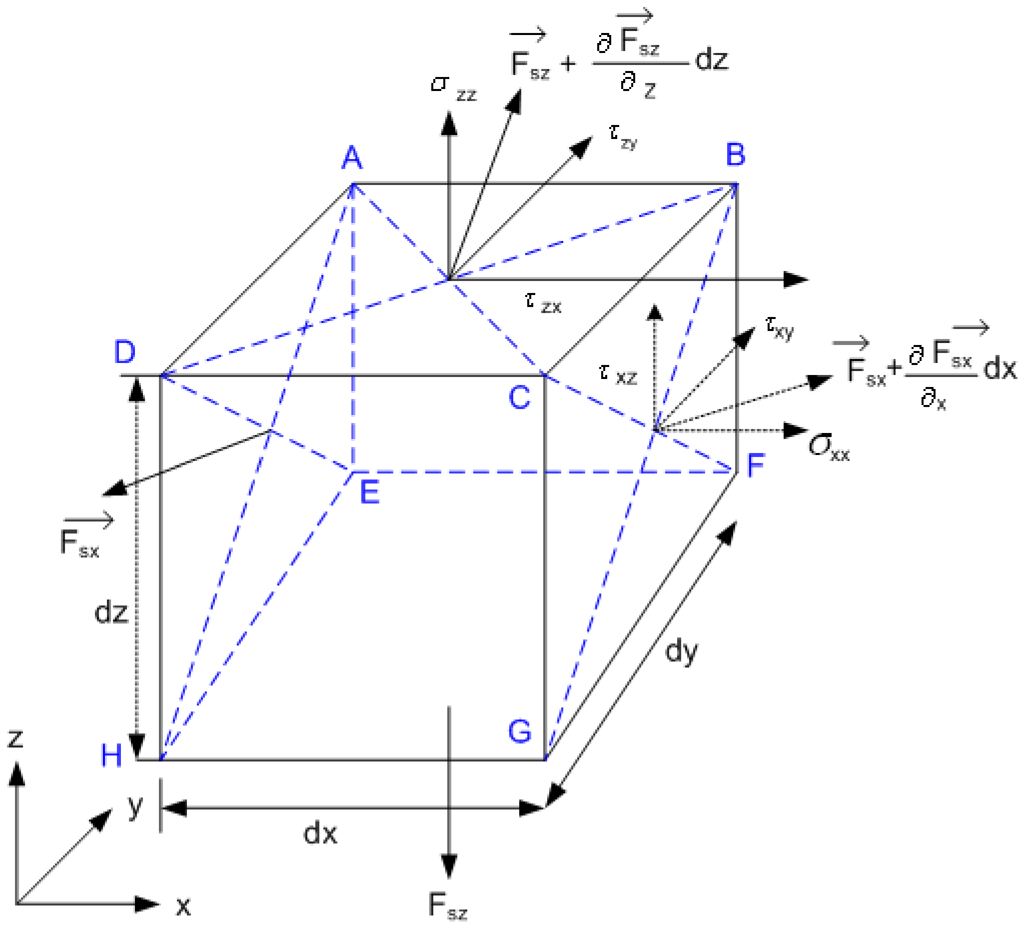


Fig. 24.1 Definition of the components of stress and their locations in a differential fluid element

- Let the body force per unit mass be

$$\vec{f}_b = \hat{i}\, f_x + \hat{j}\, f_y + \hat{k}\, f_z \tag{24.6}$$

and surface force per unit volume be

$$\vec{F} = \hat{i}\, F_x + \hat{j}\, F_y + \hat{k}\, F_z \tag{24.7}$$

- Consider surface force on the surface AEHD, per unit area,

$$\vec{F}_{sx} = \hat{i}\, \sigma_{xx} + \hat{j}\, \tau_{xy} + \hat{k}\, \tau_{xz}$$

[Here second subscript x denotes that the surface force is evaluated for the surface whose outward normal is the x axis]

- Surface force on the surface *BFGC* per unit area is

$$\vec{F}_{sx} + \frac{\partial \vec{F}_{sx}}{\partial x} dx$$

- Net force on the body due to imbalance of surface forces on the above two surfaces is

$$\frac{\partial \vec{F}_{sx}}{\partial x} dx\, dy\, dz \qquad (24.8)$$

(since area of faces AEHD and BFGC is dydz)

- Total force on the body due to net surface forces on all six surfaces is

$$\left(\frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \right) dx\, dy\, dz \qquad (24.9)$$

- And hence, the resultant surface force *dF*, per unit volume, is

$$d\vec{F} = \frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \qquad (24.10)$$

(since Volume= d*x* d*y* dz)

- The quantities $\vec{F}_{sx}$, $\vec{F}_{sy}$ and $\vec{F}_{sz}$ are vectors which can be resolved into normal stresses denoted by σ and shearing stresses denoted by τ as

$$\begin{aligned} \vec{F}_{sx} &= \hat{i}\, \sigma_{xx} + \hat{j}\, \tau_{xy} + \hat{k}\, \tau_{xz} \\ \vec{F}_{sy} &= \hat{i}\, \tau_{yx} + \hat{j}\, \sigma_{yy} + \hat{k}\, \tau_{yz} \\ \vec{F}_{sz} &= \hat{i}\, \tau_{zx} + \hat{j}\, \tau_{zy} + \hat{k}\, \sigma_{zz} \end{aligned} \qquad (24.11)$$

The **stress system has nine scalar quantities. These nine quantities form a stress tensor.**

Nine Scalar Quantities of Stress System - Stress Tensor

The set of nine components of stress tensor can be described as

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \qquad (24.12)$$

- The stress tensor is ***symmetric***,
- This means that two shearing stresses with subscripts which differ only in their sequence are equal. For example $\tau_{xz} = \tau_{zx}$

- Considering the equation of motion for instantaneous rotation of the fluid element (Fig. 24.1) about y axis, we can write

$$\begin{aligned}\dot{\omega}_y \; dI_y &= \left(\tau_{xz}\; dy\; dz\right)dx - \left(\tau_{zx}\; dx\; dy\right)dz \\ &= \left(\tau_{xz} - \tau_{zx}\right) d\forall\end{aligned}$$

where $d\forall$ =dxdydz is the volume of the element, $\dot{\omega}_y$ is the angular acceleration

dI_y is the moment of inertia of the element about y-axis

- Since dI_y is proportional to fifth power of the linear dimensions and $d\forall$ is proportional to the third power of the linear dimensions, the left hand side of the above equation and the second term on the right hand side vanishes faster than the first term on the right hand side on contracting the element to a point.
- Hence, the result is

$$\tau_{xz} = \tau_{zx}$$

From the similar considerations about other two remaining axes, we can write

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

which has already been observed in Eqs (24.2a), (24.2b) and (24.2c) earlier.

- Invoking these conditions into Eq. (24.12), the stress tensor becomes

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \qquad (24.13)$$

- Combining Eqs (24.10), (24.11) and (24.13), the resultant surface force per unit volume becomes

$$\begin{aligned} d\overline{F} = &\hat{i}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) \\ &+ \hat{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) \\ &+ \hat{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \end{aligned} \qquad (8.14)$$

- As per the velocity field,

$$\frac{D\vec{V}}{Dt} = \hat{i}\frac{Du}{Dt} + \hat{j}\frac{Dv}{Dt} + \hat{k}\frac{Dw}{Dt} \tag{24.15}$$

By Newton's law of motion applied to the differential element, we can write

$$\rho(dx\,dy\,dz)\frac{D\vec{V}}{Dt} = \left(d\vec{F}\right)(dx\,dy\,dz) + \rho\vec{f}_b\,(dx\,dy\,dz)$$

or, $$\rho\frac{D\vec{V}}{Dt} = d\vec{F} + \rho\,\vec{f}_b$$

Substituting Eqs (24.15), (24.14) and (24.6) into the above expression, we obtain

$$\rho\frac{Du}{Dt} = \rho f_x + \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\right) \tag{24.16a}$$

$$\rho\frac{Dv}{Dt} = \rho f_y + \left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z}\right) \tag{24.16b}$$

$$\rho\frac{Dw}{Dt} = \rho f_z + \left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right) \tag{24.16c}$$

Since $$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla\cdot\vec{V}\right)$$

$$\frac{\partial\sigma_{xx}}{\partial x} = -\,{\partial p}/{\partial x} + {\partial}/{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$

Similarly others follow.

- So we can express $\frac{Du}{Dt}$, $\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$ in terms of field derivatives,

$$\boxed{\rho\frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad (24.17a)}$$

$$\rho\frac{Dv}{Dt}=\rho f_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \tag{24.17b}$$

$$\rho\frac{Dw}{Dt}=\rho f_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \tag{24.17c}$$

- These differential equations are known as Navier-Stokes equations.
- At this juncture, discuss the equation of continuity as well, which has a **general (conservative) form**

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{24.18}$$

- In case of incompressible flow ρ = constant. Therefore, equation of continuity for incompressible flow becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{24.19}$$

- Invoking Eq. (24.19) into Eqs (24.17a), (24.17b) and (24.17c), we get

$$\begin{aligned}\rho\frac{Du}{Dt}&=\rho\ f_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}\right)\right]+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\left(\frac{\partial}{\partial y}\frac{\partial v}{\partial x}\right)+\mu\left(\frac{\partial}{\partial z}\frac{\partial w}{\partial x}\right)\\&=\rho\ f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\cancelto{0}{\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}}\right)\end{aligned}$$

Similarly others follow

Thus,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{24.20a}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho f_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{24.20b}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{24.20c}$$

Vector Notation & derivation in Cylindrical Coordinates - Navier-Stokes equation

- Using, vector notation to write Navier-Stokes and continuity equations for incompressible flow we have

$$\rho \frac{D\vec{V}}{Dt} = \rho\, \vec{f}_b - \nabla p + \mu \nabla^2 \vec{V} \qquad (24.21)$$

and

$$\nabla \cdot \vec{V} = 0 \qquad (24.22)$$

- we have **four unknown quantities, u, v, w and p** ,
- we also have **four equations, - equations of motion in three directions and the continuity equation.**
- In principle, these equations are solvable but to date generalized solution is not available due to the complex nature of the set of these equations.
- The highest order terms, which come from the viscous forces, are linear and of second order
- The first order convective terms are non-linear and hence, the set is termed as **quasi-linear.**
- Navier-Stokes equations in cylindrical coordinate (Fig. 24.2) are useful in solving many problems. If , and denote the velocity components along the radial, cross-radial and axial directions respectively, then for the case of incompressible flow, Eqs (24.21) and (24.22) lead to the following system of equations:

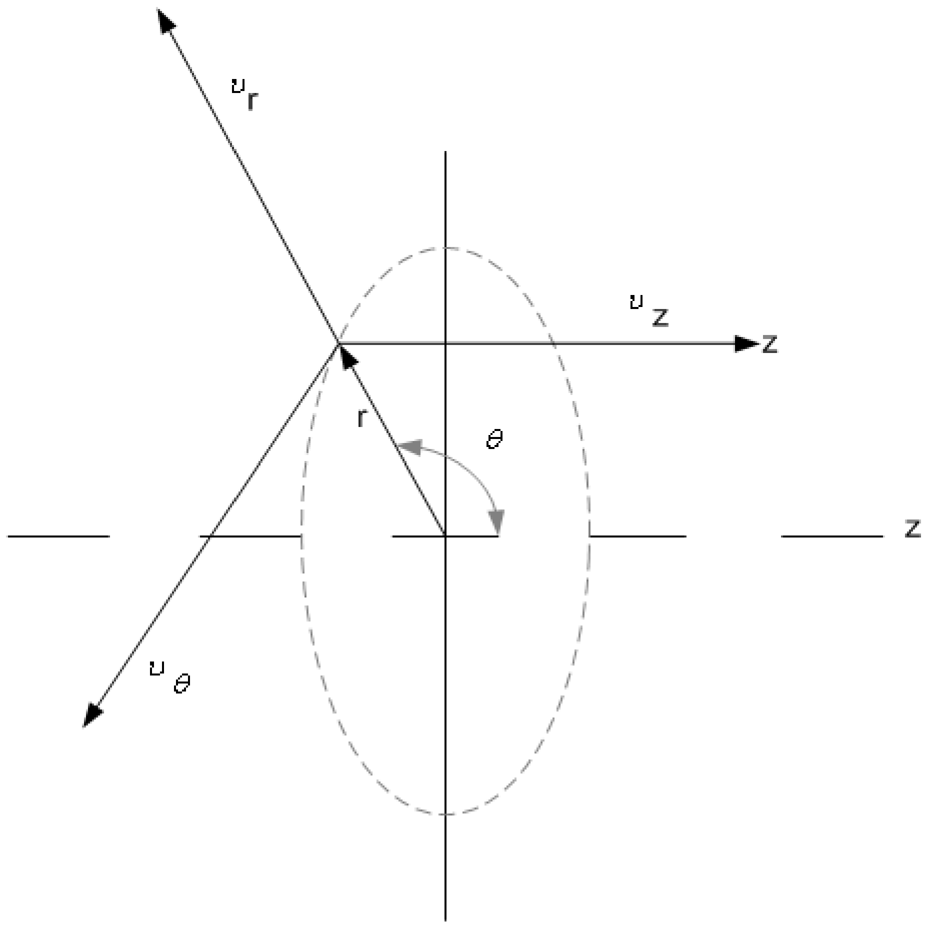


FIG 24.2 Cylindrical polar coordinate and the velocity components

$$\rho\left(\frac{\partial \upsilon_r}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_\theta}{r}\cdot\frac{\partial \upsilon_r}{\partial \theta}-\frac{\upsilon_\theta^2}{r}+\upsilon_z\frac{\partial \upsilon_r}{\partial z}\right)=\rho f_r-\frac{\partial p}{\partial r}$$
$$+\mu\left(\frac{\partial^2 \upsilon_r}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \upsilon_r}{\partial r}-\frac{\upsilon_r}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 \upsilon_r}{\partial \theta^2}-\frac{2}{r^2}\cdot\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\partial^2 \upsilon_r}{\partial z^2}\right) \qquad (24.23a)$$

$$\rho\left(\frac{\partial \upsilon_\theta}{\partial t}+\upsilon_r\frac{\partial \upsilon_\theta}{\partial r}+\frac{\upsilon_\theta}{r}\cdot\frac{\partial \upsilon_\theta}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_\theta}{\partial z}+\frac{\upsilon_r\upsilon_\theta}{r}\right)=\rho f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left(\frac{\partial^2 \upsilon_\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \upsilon_\theta}{\partial r}-\frac{\upsilon_\theta}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 \upsilon_\theta}{\partial \theta^2}+\frac{2}{r^2}\cdot\frac{\partial \upsilon_r}{\partial \theta}+\frac{\partial^2 \upsilon_\theta}{\partial z^2}\right) \qquad (24.23b)$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}$$
$$+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right) \qquad (24.23c)$$

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \qquad (24.24)$$

A general way of deriving the Navier-Stokes equations from the basic laws of physics.

- **Consider a general flow field as represented in Fig. 25.1.**
- **Imagine a closed control volume,** $\forall_0$ **within the flow field. The control volume is fixed in space and the fluid is moving through it. The control volume occupies reasonably large finite region of the flow field.**
- **A control surface , A_0 is defined as the surface which bounds the volume** $\forall_0$**.**
- **According to Reynolds transport theorem, "The rate of change of momentum for a system equals the sum of the rate of change of momentum inside the control volume and the rate of efflux of momentum across the control surface".**

- **The rate of change of momentum for a system (in our case, the control volume boundary and the system boundary are same) is equal to the net external force acting on it.**

Now, we shall transform these statements into equation by accounting for each term,

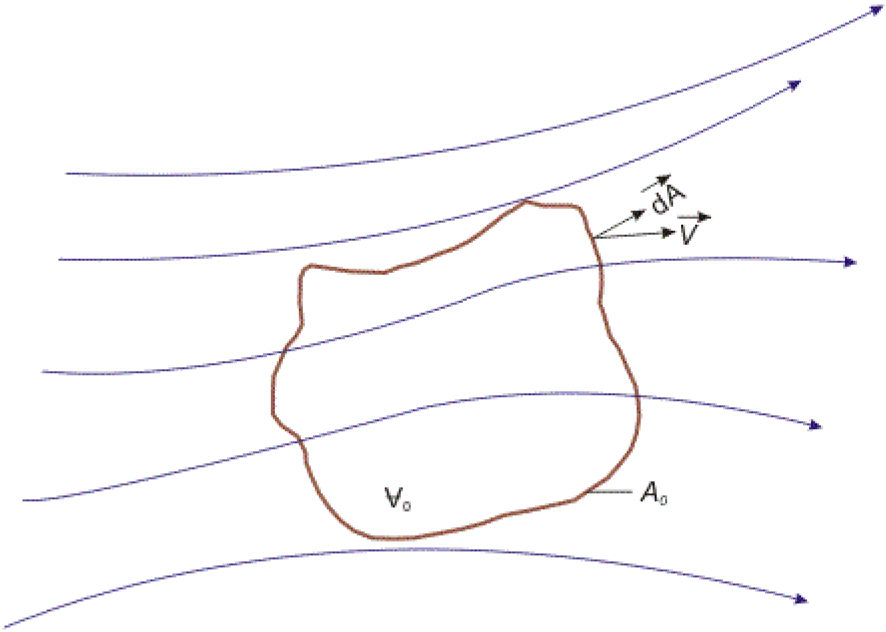


FIG 25.1 Finite control volume fixed in space with the fluid moving through it

- **Rate of change of momentum inside the control volume**

$$= \frac{\partial}{\partial t}\int\!\!\int\!\!\int_{\forall_0} \rho \vec{V} d\forall$$

$$= \int\!\!\int\!\!\int_{\forall_0} \frac{\partial}{\partial t}(\rho \vec{V}) d\forall \qquad (25.1)$$

(since t is independent of space variable)

- **Rate of efflux of momentum through control surface**

$$\int\limits_{A_0}\int \rho\vec{V}\left(\vec{V}.d\vec{A}\right) = \int\limits_{A_0}\int \rho\vec{V}\vec{V}.\vec{n}dA$$

$$= \int\int\limits_{\forall_0}\int\left\{\vec{V}\left(\nabla.\rho\vec{V}\right) + \rho\vec{V}.\nabla\vec{V}\right\}d\forall \qquad (25.2)$$

- **Surface force acting on the control volume**

$$= \int\limits_{A_0}\int d\vec{A}\,\sigma$$

(σ is symmetric stress tensor)

$$= \int\int\limits_{\forall_0}\int\left(\nabla.\sigma\right)d\forall \qquad (25.3)$$

- **Body force acting on the control volume**

$$\int\int\limits_{\forall_0}\int \rho\vec{f}_b d\forall \qquad (25.4)$$

$\vec{f}_b$ in Eq. (25.4) is the body force per unit mass.

- **Finally, we get,**

$$\int\int\limits_{\forall_0}\int\left(\frac{\partial}{\partial t}\left(\rho\vec{V}\right) + \left\{\vec{V}\left(\nabla.\rho\vec{V}\right) + \rho\vec{V}.\nabla\vec{V}\right\}\right)d\forall$$
$$= \int\int\limits_{\forall_0}\int\left(\nabla.\sigma + \rho\vec{f}_b\right)d\forall$$

or

or, $$\rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\frac{\partial\rho}{\partial t} + \rho\vec{V}.\nabla\vec{V} + \vec{V}\left(\nabla.\rho\vec{V}\right) = \nabla.\sigma + \rho\vec{f}_b$$

or $$\rho\left(\frac{\partial\vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}\right) + \vec{V}\left(\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{V}\right) = \nabla.\sigma + \rho\vec{f}_b \qquad (25.5)$$

We know that $\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{V} = 0$ is the general form of **mass conservation equation** (popularly known as the **continuity equation**), valid for both **compressible** and **incompressible** flows.

- Invoking this relationship in Eq. (25.5), we obtain

$$\rho\left(\frac{\partial \vec{V}}{\partial t}+\vec{V}.\nabla \vec{V}\right)=\nabla.\sigma+\rho \vec{f}_b$$

or $$\rho \frac{D\vec{V}}{Dt}=\nabla.\sigma+\rho \vec{f}_b \qquad (25.6)$$

- Equation (25.6) is referred to as **Cauchy's equation of motion** . In this equation, σ is the stress tensor,
- After having substituted σ we get

$$\nabla.\sigma=-\nabla p+(\mu'+\mu)\nabla(\nabla.\vec{V})+\mu\nabla^2\vec{V} \qquad (25.8)$$

From **Stokes's hypothesis** we get, $$\mu'+\frac{2}{3}\mu=0 \qquad (25.9)$$

Invoking above two relationships into Eq.(25.6) we get

$$\rho \frac{D\vec{V}}{Dt}=-\nabla p+\mu\nabla^2\vec{V}+\frac{1}{3}\mu\nabla(\nabla.\vec{V})+\rho \vec{f}_b \qquad (25.10)$$

This is the most general form of Navier-Stokes equation.

Exact Solutions Of Navier-Stokes Equations

Consider a class of flow termed as parallel flow in which **only one velocity term is nontrivial** and all the fluid particles move in one direction only.

- We **choose** x **to be the direction along which all fluid particles travel** , i.e. $u\neq 0, v=w=0$. Invoking this in continuity equation, we get

$$\frac{\partial u}{\partial x}+\cancelto{0}{\frac{\partial v}{\partial y}}+\cancelto{0}{\frac{\partial w}{\partial z}}=0$$

$\frac{\partial u}{\partial x}=0$ which means $u=u(y,z,t)$

- Now. Navier-Stokes equations for incompressible flow become

$$\frac{\partial u}{\partial t}+u\cancel{\frac{\partial u}{\partial x}}^{0}+v\cancel{\frac{\partial u}{\partial y}}^{0}+w\cancel{\frac{\partial u}{\partial z}}^{0}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\cancel{\frac{\partial^2 u}{\partial x^2}}^{0}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\cancel{\frac{\partial v}{\partial t}}^{0}+u\cancel{\frac{\partial v}{\partial x}}^{0}+v\cancel{\frac{\partial v}{\partial y}}^{0}+w\cancel{\frac{\partial v}{\partial z}}^{0}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\cancel{\frac{\partial^2 v}{\partial x^2}}^{0}+\cancel{\frac{\partial^2 v}{\partial y^2}}^{0}+\cancel{\frac{\partial^2 v}{\partial z^2}}^{0}\right]$$

$$\cancel{\frac{\partial w}{\partial t}}^{0}+u\cancel{\frac{\partial w}{\partial x}}^{0}+v\cancel{\frac{\partial w}{\partial y}}^{0}+w\cancel{\frac{\partial w}{\partial z}}^{0}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\cancel{\frac{\partial^2 w}{\partial x^2}}^{0}+\cancel{\frac{\partial^2 w}{\partial y^2}}^{0}+\cancel{\frac{\partial^2 w}{\partial z^2}}^{0}\right]$$

So, we obtain

$$\frac{\partial p}{\partial t}=\frac{\partial p}{\partial z}=0$$ which means $p = p(x)\ alone$

and $$\frac{\partial u}{\partial t}=-\frac{1}{\rho}.\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] \qquad (25.11)$$

Parallel Flow in a Straight Channel

Consider **steady flow between two infinitely broad parallel plates** as shown in Fig. 25.2.

Flow is independent of any variation in z direction, hence, z dependence is gotten rid of and Eq. (25.11) becomes

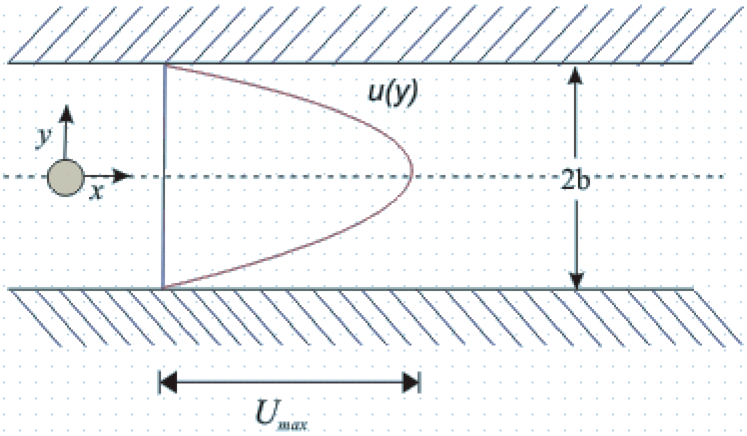


FIG 25.2 Parallel flow in a straight channel

$$\frac{dp}{dx}=\mu\frac{\partial^2 u}{\partial y^2} \qquad (25.12)$$

The boundary conditions are at y = b, u = 0; and y = -b, u = O.

- From Eq. (25.12), we can write

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

or $$u = \frac{1}{2\mu}\cdot\frac{dp}{dx}y^2 + C_1 y + C_2$$

- Applying the boundary conditions, the constants are evaluated as

$$C_1 = 0 \quad \text{and} \quad C_2 = -\frac{1}{\mu}\cdot\frac{dp}{dx}\cdot\frac{b^2}{2}$$

So, the solution is

$$u = -\frac{1}{2\mu}\cdot\frac{dp}{dx}\left(b^2 - y^2\right) \qquad (25.13)$$

which implies that the velocity profile is parabolic.

Average Velocity and Maximum Velocity

- To establish the relationship between the maximum velocity and average velocity in the channel, we analyze as follows

  At y = 0, $u = U_{max}$; this yields

$$U_{max} = -\frac{b^2}{2\mu}\cdot\frac{dp}{dx} \qquad (25.14a)$$

On the other hand, the average velocity,

$$U_{av} = \frac{Q}{2b} = \frac{flow\ rate}{flow\ area} = \frac{1}{2b}\int_{-b}^{b} u\,dy$$

or $$U_{av} = \frac{2}{2b}\int_{0}^{b} -\frac{1}{2\mu}\cdot\frac{dp}{dx}\left(b^2 - y^2\right)dy$$

$$= \frac{1}{2\mu}\frac{dp}{dx}\cdot\frac{1}{b}\left\{\left[b^2 y\right]_0^b - \left(\frac{y^3}{3}\right)_0^b\right\}$$

Finally, $$U_{av} = -\frac{1}{2\mu}\frac{dp}{dx}.\frac{2}{3}b^2 \qquad (25.14b)$$

So, $\frac{U_{av}}{U_{max}} = \frac{2}{3}$ or $U_{max} = \frac{3}{2}U_{av}$ (25.14c)

- The shearing stress at the wall for the parallel flow in a channel can be determined from the velocity gradient as

$$\tau_{yx}\big|_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = b\frac{dp}{dx} = -2\mu\frac{U_{max}}{b}$$

Since the upper plate is a "minus y surface", a negative stress acts in the positive x direction, i.e. to the right.

- The local friction coefficient, C_f is defined by

$$C_f = \frac{\left|\left(\tau_{yx}\right)_b\right|}{\frac{1}{2}\rho U_{av}^2} = \frac{3\mu U_{av}/b}{\frac{1}{2}\rho U_{av}^2}$$

$$C_f = \frac{12}{\frac{\rho U_{av}(2b)}{\mu}} = \frac{12}{Re} \qquad (25.14d)$$

where $Re = U_{av}(2b)/\nu$ is the Reynolds number of flow based on average velocity and the channel height (2b).

- Experiments show that Eq. (25.14d) is valid in the laminar regime of the channel flow.
- The maximum Reynolds number value corresponding to fully developed laminar flow, for which a stable motion will persist, is 2300.
- In a reasonably careful experiment, laminar flow can be observed up to even Re = 10,000.
- But the value below which the flow will always remain laminar, i.e. **the critical value of Re is 2300.**

Couette Flow

Couette flow is the flow between **two parallel plates** (Fig. 26.1). Here, one plate is at **rest** and the other is **moving** with a velocity *U* . Let us assume the plates are infinitely large in *z* direction, so the ***z* dependence is not there**.

The **governing equation** is

$$\frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$$

flow is independent of any variation in z-direction.
The **boundary conditions** are ---(i)**At y = 0, u = 0** (ii)**At y = h, u = U.**

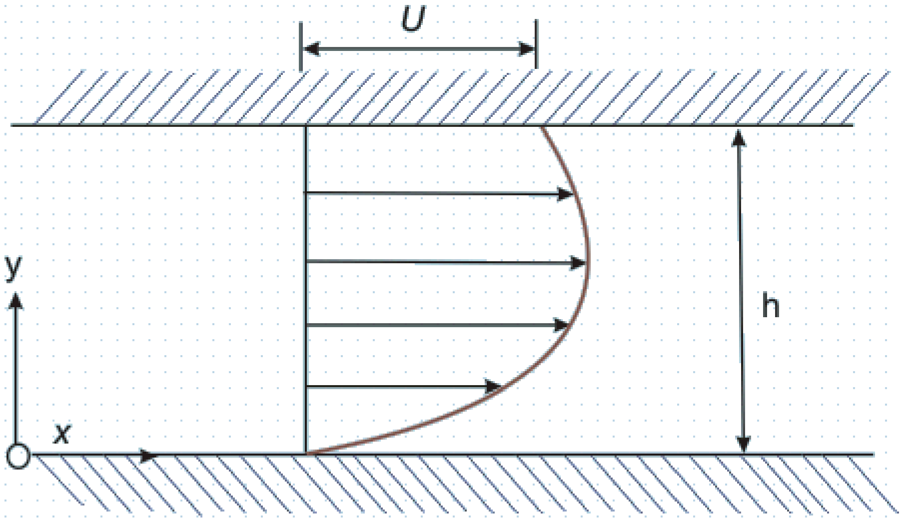


FIG 26.1 Couette flow between two parallel flat plates

- We get,

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y + C_2$$

Invoking the condition (**at y = 0, u = 0**), C_2 becomes equal to zero.

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y$$

Invoking the other condition (**at y = h, u = U**),

$$C_1 = \frac{U}{h} - \frac{1}{2\mu} \cdot \frac{dp}{dx} h$$

So, $$u = \frac{y}{h} U - \frac{h^2}{2\mu} \cdot \frac{dp}{dx} \cdot \frac{y}{h} \left(1 - \frac{y}{h}\right) \qquad (26.1)$$

Equation (26.1) can also be expressed in the form

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U} \cdot \frac{dp}{dx} \cdot \frac{y}{h}\left(1 - \frac{y}{h}\right)$$

or, $$\frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (26.2a)$$

Where

$$P = -\frac{h^2}{2\mu U}\left(\frac{dp}{dx}\right),$$

Equation (26.2a) describes the velocity distribution in non-dimensional form across the channel with **P** as a parameter known as the **non-dimensional pressure gradient .**

- When **P = 0**, the velocity distribution across the channel is reduced to

$$\frac{u}{U} = \frac{y}{h}$$

This particular case is known as **simple Couette flow.**

- When **P > 0** , i.e. for a ***negative*** or ***favourable*** **pressure gradient** $(-dp/dx)$in the direction of motion, the velocity is positive over the whole gap between the channel walls. For negative value of P (**P < 0**), there is a ***positive*** or ***adverse*** **pressure gradient** in the direction of motion and the velocity over a portion of channel width can become negative and **back flow may occur** near the wall which is at rest. **Figure 26.2a** shows the effect of dragging action of the upper plate exerted on the fluid particles in the channel for different values of pressure gradient.

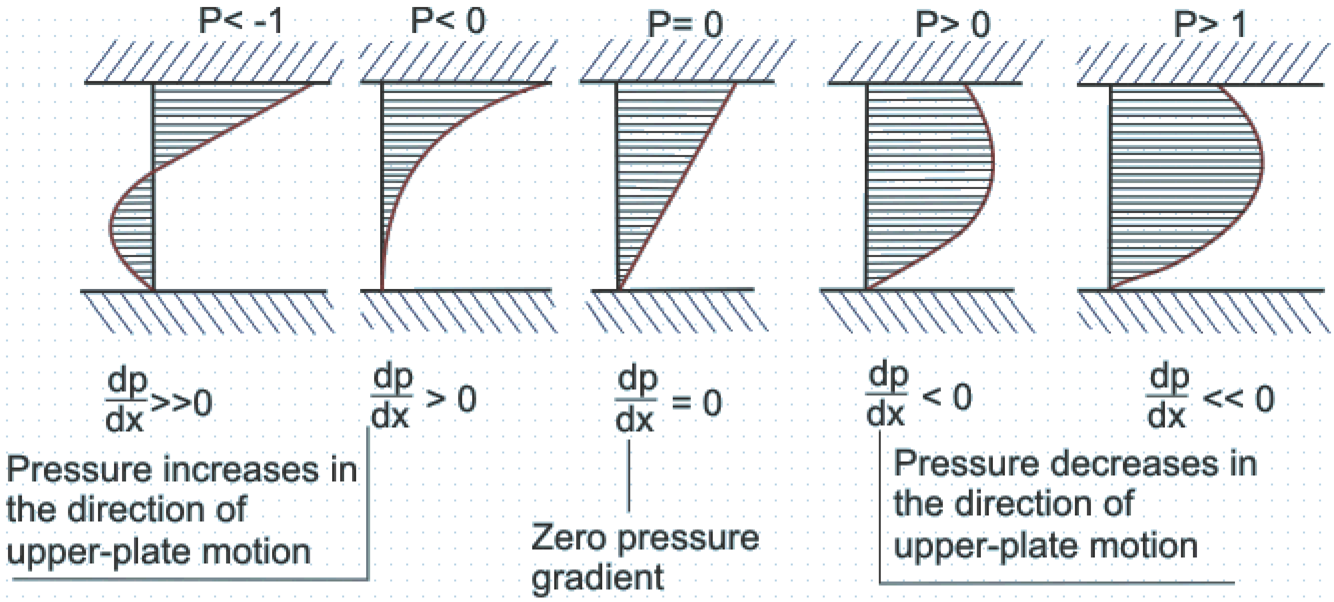


FIG 26.2a - Velocity profile for the Couette flow for various values of pressure gradient

Maximum and minimum velocities

The quantitative description of non-dimensional velocity distribution across the channel, depicted by Eq. (26.2a), is shown

in **Fig. 26.2b**.

- The location of maximum or minimum velocity in the channel is found out by setting du/dy **=0**. From Eq. (26.2a), we can write

$$\frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left(1 - 2\frac{y}{h}\right)$$

Setting $\frac{du}{dy} = 0$ gives

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P} \qquad 26.2b$$

- It is interesting to note that **maximum velocity for P = 1 occurs at y/h = 1 and equals to *U* .** For **P > 1**, the maximum velocity occurs at a location $\frac{y}{h} < 1$ **.**
- This means that with P > 1, the fluid particles attain a velocity higher than that of the moving plate at a location somewhere below the moving plate.
- For **P = -1**, the minimum velocity occurs, at $\frac{y}{h} = 0$. For **P < -1**, the minimum velocity occurs at a location $\frac{y}{h} > 1$.

- This means that there occurs a back flow near the fixed plate. The values of maximum and minimum velocities can be determined by substituting the value of *y* from Eq. (26.2b) into Eq. (26.2a) as

$$u_{max} = \frac{U(1+P)^2}{4P} \qquad for\ P \geq 1$$

$$u_{min} = \frac{U(1+P)^2}{4P} \qquad for\ P \leq 1 \qquad (26.2c)$$

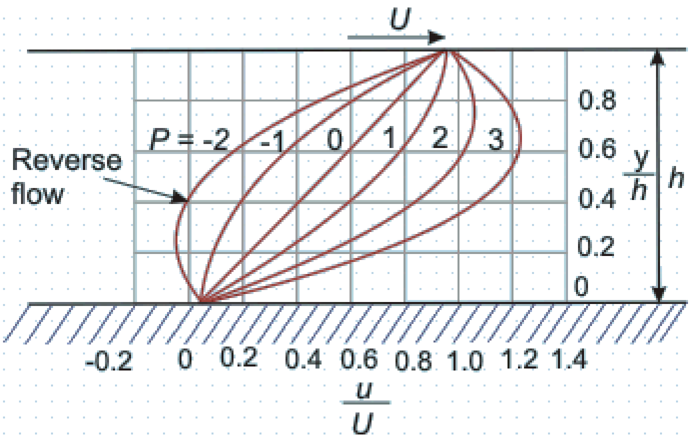


FIG 26.2b - Velocity distribution of the Couette flow

Hagen Poiseuille Flow

- Consider **fully developed** laminar flow through a straight tube of circular cross-section as in Fig. 26.3. Rotational symmetry is considered to make the flow two-dimensional axisymmetric.
- Let us take z-axis as the axis of the tube along which all the fluid particles travel, i.e.

$$v_z \neq 0,\ v_r = 0, v_\theta = 0$$

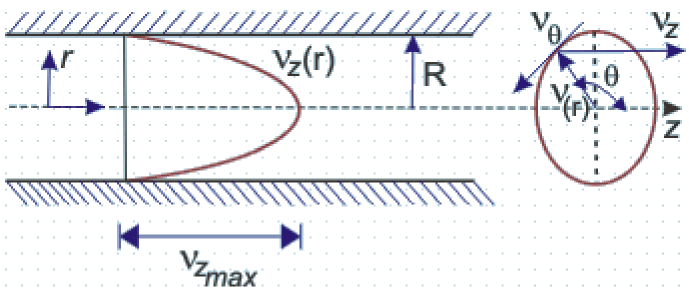


Fig 26.3 - Hagen-Poiseuille flow through a pipe

- Now, from continuity equation, we obtain

$$\cancel{\frac{\partial v_r}{\partial r}}^{0} + \cancel{\frac{v_r}{r}}^{0} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{[For rotational symmetry, } \frac{1}{r}\cdot\frac{\partial v_\theta}{\partial \theta} = 0 \text{ .]}$$

which means $v_z = v_z(r,t)$

- Invoking $\left[v_r = 0, v_\theta = 0, \frac{\partial v_z}{\partial z} = 0,\ and \frac{\partial}{\partial \theta}(anyquantity) = 0\right]$ in the

Navier-Stokes equations, we obtain

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r}\right) \quad (26.3)$$

(in the z-direction)

- For **steady flow**, the governing equation becomes

$$\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} = \frac{1}{\mu}\frac{dp}{dz} \quad (26.4)$$

The ***boundary conditions*** are- **(i)** At r = 0, $v_z$is finite and **(ii)** r = R, $v_z = 0$ yields

- Equation (26.4) can be written as

$$r\frac{d^2 \upsilon_z}{dr^2} + \frac{d\upsilon_z}{dr} = \frac{1}{\mu} \cdot \frac{dp}{dz} r$$

or, $\frac{d}{dr}\left(r\frac{d\upsilon_z}{dr}\right) = \frac{1}{\mu} \cdot \frac{dp}{dz} r$

or, $r\frac{d\upsilon_z}{dr} = \frac{1}{2\mu} \cdot \frac{dp}{dz} r^2 + A$

or, $\frac{d\upsilon_z}{dr} = \frac{1}{2\mu} \cdot \frac{dp}{dz} r + \frac{A}{r}$

or, $\upsilon_z = \frac{1}{4\mu} \cdot \frac{dp}{dz} r^2 + A\, In\, r + B$

- At r =0, $v_z$is finite which means A should be equal to zero and at r = R, v_z= 0 yields

$B = -\frac{1}{4\mu} \cdot \frac{dp}{dz} \cdot R^2$

$$v_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right) \quad (26.5)$$

- **This shows that the axial velocity profile in a fully developed laminar pipe flow is having parabolic variation along r.**
- At r = 0, as such, $v_z = v_{z_{\max}}$

$$v_z = v_{z_{\max}} = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right) \qquad (26.6a)$$

- The average velocity in the channel,

$$v_{z_{av}} = \frac{Q}{\pi R^2} = \frac{\int_0^R 2\pi r v_z(r)dr}{\pi R^2}$$

or, $$v_{z_{av}} = \frac{2\pi \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right]}{\pi R^2}$$

$$v_{z_{av}} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2} v_{z\,\max} \qquad (26.6b)$$

or $$v_{z_{\max}} = 2v_{z_{av}} \qquad (26.6c)$$

- Now, the discharge (Q) through a pipe is given by

$$Q = \pi R^2 v_{z_{av}} \qquad (26.7)$$

or, $$Q = \pi R^2 \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right)$$ [From Eq. 26.6b]

or $$Q = -\frac{\pi D^4}{128\,\mu}\left(\frac{dp}{dz}\right) \qquad (26.8)$$

Applications-

- Equation (26.8) is commonly used in the measurement of viscosity with the help of capillary tube viscometers . Such a viscometer consists of a constant head tank to supply liquid to a capillary tube (Fig. 26.4).

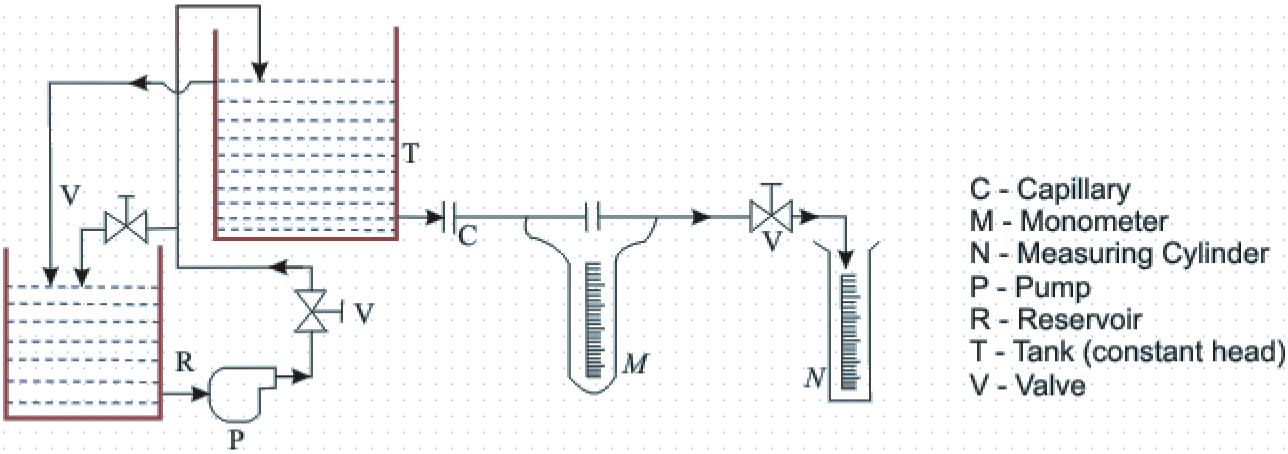


FIG 26.4 Schematic diagram of the experimental facility for determination of viscosity

- Pressure drop readings across a specified length in the developed region of the flow are taken with the help of a **manometer.** The developed flow region is ensured by providing the necessary and sufficient entry length.
- From Eq. (26.8), the expression for viscosity can be written as

$$\mu = -\frac{\pi D^4}{128Q}\cdot\frac{dp}{dl} = \frac{\pi D^4}{128Q}\frac{(p_1 - p_2)}{l}$$

- The **volumetric flow rates (Q)** are measured by collecting the liquid in a measuring cylinder. The **diameter (D)** of the capillary tube is known beforehand. Now the viscosity of a flowing fluid can easily be evaluated.
- Shear stress profile across the cross-section can also be determined from this information. Shear stress at any point of the pipe flow is given by

$$\tau\Big|_r = \mu\frac{dv_z}{dr}$$

From Eq. (26.5) $$\frac{dv_z}{dr} = \frac{R^2}{4\mu}\left(\frac{dp}{dz}\right)\frac{2r}{R^2}$$

or $$\frac{dv_z}{dr} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right)\cdot r \qquad (26.9a)$$

which means $$\tau\big|_r = \frac{1}{2}\left(\frac{dp}{dz}\right)\cdot r \qquad (26.9b)$$

This also indicates that τ varies linearly with the radial distance from the axis.

- **At the wall, τ assumes the maximum value.**

$$r = R, \qquad \tau = \tau_{max} = \frac{1}{2}\left(\frac{dp}{dz}\right)R$$

Again, over a **pipe length of *l***, the total shear force is

$$F_s = \tau_{max}\, 2\pi R\, l$$

or

$$F_s = \frac{1}{2}\left(\frac{p_2 - p_1}{l}\right) . 2\pi R^2\, l$$

or

$F_s = -\pi R^2 \times$ [Pressure drop between the specified length]

as it should be. **Negative sign indicates that the force is acting in opposite to the flow direction.**

- However, from Eq. (26.6b), we can write

$$(v_z)_{av} = -\frac{1}{8\mu}\left(\frac{dp}{dz}\right)R^2 \qquad (26.9c)$$

or

$$-\left(\frac{dp}{dz}\right) = \frac{8\mu (v_z)_{av}}{R^2} \qquad (26.10)$$

Losses and Friction Factors

- Over a finite length l , the head loss $h_f = \frac{pressure drop}{\rho g}$ (26.11)
  Combining Eqs (26.10) and (26.11), we get

$$h_f = \frac{32\mu (v_{z_{av}})}{D^2} \cdot l \cdot \frac{1}{\rho g}$$

or

$$h_f = \frac{32\mu (v_{z_{av}})^2 l}{D^2 \cdot \rho g} \cdot \frac{1}{(v_{z_{av}})} \qquad (26.12)$$

- On the other hand, the head loss in a pipe flow is given by **Darcy-Weisbach formula** as

$$h_f = \frac{f l (v_{z_{av}})^2}{2gD} \qquad (26.13)$$

where "f" is Darcy friction factor . Equations (26.12) and (26.13) yield

$$\frac{32\mu(v_{z_{av}})^2 l}{D^2 \rho g} \cdot \frac{1}{(v_{z_{av}})} = \frac{f\,l(v_{z_{av}})^2}{2gD}$$

which finally gives $f = \frac{64}{\mathrm{Re}}$, where $\mathrm{Re} = \frac{\rho(v_{z_{av}})D}{\mu}$ is the Reynolds number.

- So, for a fully developed laminar flow, the **Darcy (or Moody) friction factor** is given by

$$f = \frac{64}{\mathrm{Re}} \qquad (26.14a)$$

Alternatively, the **skin friction coefficient** for Hagen-Poiseuille flow can be expressed by

$$C_f = \frac{\left|\tau_{at\,r=R}\right|}{\frac{1}{2}\rho(v_{z_{av}})^2}$$

With the help of Eqs (26.9b) and (26.9c), it can be written

$$C_f = \frac{16}{\mathrm{Re}} \qquad (26.14b)$$

The skin friction coefficient C_f is called as Fanning's friction factor . From comparison of Eqs (26.14a) and (26.14b), it appears

$$f = 4C_f$$

- For fully developed turbulent flow, the analysis is much more complicated, and we generally depend on experimental results. Friction factor for a wide range of Reynolds number (10^4 to 10^8) can be obtained from a look-up chart . Friction factor, for high Reynolds number flows, is also a function of tube surface condition. However, in circular tube, flow is laminar for Re ≤ 2300 and turbulent regime starts with Re ≥ 4000.
- The surface condition of the tube is another responsible parameter in determination of friction factor.
- Friction factor in the turbulent regime is determined for different degree of surface-roughness $\left(\frac{\epsilon}{D_h}\right)$ of the pipe, where ε is the dimensional roughness and D_h is usually the hydraulic diameter of the pipe .
- Friction factors for different Reynolds number and surface-roughness have been determined experimentally by various investigators and the comprehensive results are

expressed through a graphical presentation which is known as **Moody Chart** after **L.F. Moody** who compiled it.

- The hydraulic diameter which is used as the characteristic length in determination of friction factor, instead of ordinary geometrical diameter, is defined as

$$D_h = \frac{4A_w}{P_w} \qquad (26.15)$$

where A_w is the flow area and P_w is the wetted perimeter .

- **Kinetic energy correction factor ,** α The kinetic energy associated with the fluid flowing with its profile through elemental area $dA = \left[\frac{1}{2}\rho v_z dA\right] v_z^2$ and the total kinetic energy passing through per unit time $= \frac{\rho}{2}\int v_z^3 dA$.
- This can be related to the kinetic energy due to average velocity($v_{z_{av}}$), through a correction factor, α as $\left[\frac{\rho}{2}\left(v_{z_{av}}\right)^3 A\right]\alpha = \frac{\rho}{2}\int v_z^3 dA$

or $$\alpha = \frac{1}{A}\int\left[\frac{v_z}{\left(v_{z_{av}}\right)}\right]^3 dA$$

- Here, for Hagen-Poiseuille flow,

$$\alpha = \frac{1}{A}\int_0^R \left[\frac{\left(v_{z_{\max}}\right)\left(1-\left(\frac{r}{R}\right)^2\right)}{\left(v_{z_{\max}}\right)\frac{1}{2}}\right]^3 2\pi r\, dr = 2 \qquad (26.16)$$

Flow between Two Concentric Rotating Cylinders

- Another example which leads to an exact solution of Navier-Stokes equation is the flow between **two concentric rotating cylinders.**
- Consider flow in the annulus of two cylinders (Fig. 26.5), where r_1 and r_2 are the radii of inner and outer cylinders, respectively, and the cylinders move with different rotational speeds ω_1 and ω_2 respectively

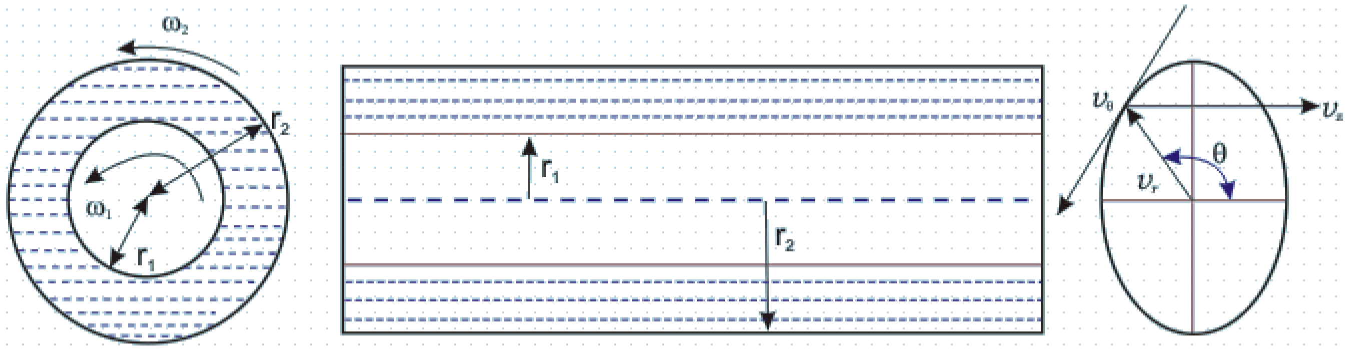


FIG 26.5 - Flow between two concentric rotating cylinders

- From the physics of the problem we know, , $v_z = 0, v_r = 0$.
- From the continuity Eq. and these two conditions, we obtain

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

which means v_θ is not a function of θ. Assume z dimension to be large enough so that end effects can be neglected and $\frac{\partial}{\partial z}$ (any property) = 0.

- This implies $v_\theta = v_\theta(r)$. With these simplifications and assuming that " θ **symmetry**" holds good, Navier-Stokes equation reduces to

$$\rho \frac{v_\theta^2}{r} = \frac{dp}{dr} \qquad (26.17)$$

and
$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r} \cdot \frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0 \qquad (26.18)$$

- Equation (26.17) signifies that the centrifugal force is supplied by the radial pressure, exerted by the wall of the enclosure on the fluid. In other words, it describes the **radial pressure distribution.**
  From Eq. (26.18), we get

$$\frac{d}{dr}\left[\frac{1}{r} \cdot \frac{d}{dr}(r v_\theta)\right] = 0$$

$$\frac{d}{dr}(r v_\theta) Ar \quad \text{or} \quad v_\theta = \frac{Ar}{2} + \frac{B}{r} \qquad (26.19)$$

- For the azimuthal component of velocity, v_θ, the boundary conditions are: at $r = r_1, v_\theta = r_1\omega_1$ at $r = r_2, v_\theta = r_2\omega_2$.
- Application of these boundary conditions on Eq. (26.19) will produce

$$A = 2\left[\omega_1 - \frac{r_2^2}{\left(r_2^2 - r_1^2\right)}(\omega_1 - \omega_2)\right]$$

and

$$B = \frac{r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right)}(\omega_1 - \omega_2)$$

- Finally, the velocity distribution is given by

$$v_\theta = \frac{1}{\left(r_2^2 - r_1^2\right)}\left[\left(\omega_2 r_2^2 - \omega_1 r_1^2\right)r + \frac{r_1^2 r_2^2(\omega_1 - \omega_2)}{r}\right] \qquad (26.20)$$

Calculation of Stress and Torque Transmitted

Now, $\tau_{r\theta} = \mu\dot{\gamma}_{r\theta}$ is the general stress-strain relation.

or $$\tau_{r\theta} = \mu\left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\cdot\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right)$$

- In our case,

$$\tau_{r\theta} = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

or $$\tau_{r\theta} = \mu r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) \qquad (26.21)$$

- Equations (26.20) and (26.21) yields

$$\tau_{r\theta} = \frac{2\mu}{r_2^2 - r_1^2}(\omega_2 - \omega_1)r_1^2\, r_2^2\,\frac{1}{r^2} \qquad (26.22)$$

- Now,

$$\tau_{r\theta}\Big|_{r=r_1} = \frac{2\mu r_2^2(\omega_2 - \omega_1)}{r_2^2 - r_1^2}$$

and,

$$\tau_{r\theta}\Big|_{r=r_2} = \frac{2\mu r_1^2(\omega_2 - \omega_1)}{r_2^2 - r_1^2}$$

- For the case, when the inner cylinder is at rest and the outer cylinder rotates, the torque transmitted by the outer cylinder to the fluid is

$$T_2 = \frac{2\mu r_1^2 \omega_2}{r_2^2 - r_1^2} \cdot 2\pi r_2 l r_2$$

or

$$T_2 = 4\pi\mu l \frac{r_2^2 r_1^2}{r_2^2 - r_1^2}\omega_2 \qquad (26.23)$$

where *l* is the length of the cylinder.

- The moment T_1, with which the fluid acts on the inner cylinder has the same magnitude. If the angular velocity of the external cylinder and the moment acting on the inner cylinder are measured, the coefficient of viscosity can be evaluated by making use of the Eq. (26.23).

Low Reynolds Number Flow Around a Sphere

- Stokes obtained the solution for the pressure and velocity field for the slow motion of a viscous fluid past a sphere. In his analysis, Stokes neglected the inertia terms of Navier-Stokes equations.
- Avoiding details, integrating the pressure distribution and the shearing stress over the surface of a sphere of radius *R* , Stokes found that the drag *D* of the sphere, which is placed in a parallel stream of uniform velocity U_∞ , is given by

$$D = 6\pi\mu R U_\infty \qquad (27.1)$$

This is the well-known **Stokes' equation** for the drag of a sphere.

- It can be shown that one third of the total drag is due to pressure distribution and the remaining two third arises from frictional forces. If the drag coefficient is defined according to the relation

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 A} \qquad (27.2)$$

where $\left(A = \frac{\pi}{4}d^2\right)$ is the frontal area of the sphere, then

$$C_D = \frac{6\pi\mu R U_\infty}{\frac{1}{2}\rho U_\infty^2 \frac{\pi}{4}d^2}$$

or $$C_D = \frac{24}{Re}, \quad Re = \frac{U_\infty d}{\nu} \qquad (27.3)$$

- A comparison between Stokes' drag coefficient in Eq. (27.3) and experiments is shown in Fig. 27.1. The approximate solution due to Stokes' is valid for **Re < 1.**

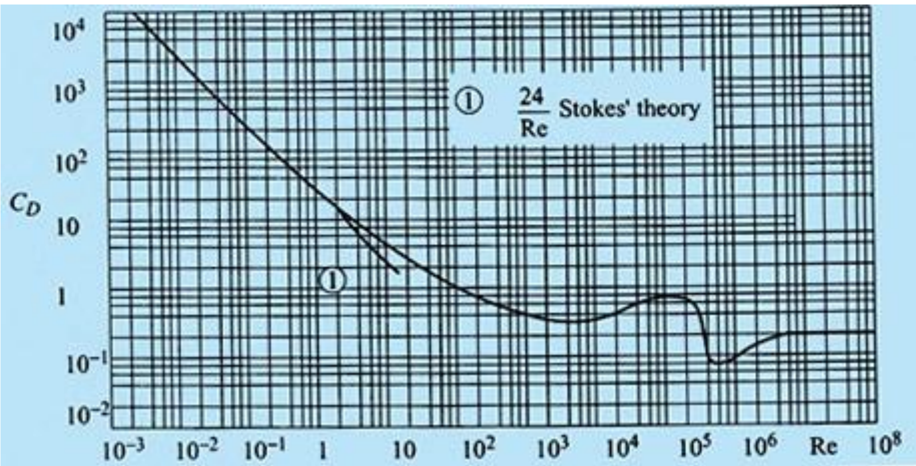


FIG 27.1 - Comparison between Stokes' drag coefficient and experimental drag coefficient

- An important application of Stokes' law is the **determination of viscosity** of a **viscous fluid** by measuring the **terminal velocity** of a falling sphere. In this device, a sphere is dropped in a transparent cylinder containing the fluid under test. If the specific weight of the sphere is close to that of the liquid, the sphere will approach a small constant speed after being released in the fluid. Now we can apply Stokes' law for steady creeping flow around a sphere where the drag force on the sphere is given by Eq. (27.1).
- With the sphere, falling at a constant speed, the acceleration is zero. This signifies that the falling body has attained terminal velocity and we can say that the sum of the buoyant force and drag force is equal to weight of the body.

$$\frac{4}{3}\pi R^3 \rho_s\, g = \frac{4}{3}\pi R^3 \rho_l\, g + 6\pi\mu V_T R \qquad (27.4)$$

where ρ_s is the density of the sphere, ρl is density of the liquid and V_T is the terminal velocity.

- Solving for μ, we get

$$\mu = \frac{2}{9}\frac{gR^2}{V_T}(\rho_s - \rho_l) \tag{27.5}$$

The **terminal velocity** V_T can be measured by observing the time for the sphere to cross a known distance between two points after its acceleration has ceased.

Theory of Hydrodynamic Lubrication

- Thin film of oil, confined between the interspace of moving parts, may acquire high pressures up to 100 MPa which is capable of supporting load and reducing friction. The salient features of this type of motion can be understood from a study of slipper bearing (Fig. 27.2). The slipper moves with a constant velocity *U* past the bearing plate. This slipper face and the bearing plate are not parallel but slightly inclined at an angle of α. A typical bearing has a gap width of 0.025 mm or less, and the convergence between the walls may be of the order of **1/5000**. It is assumed that the sliding surfaces are very large in transverse direction so that the problem can be considered two-dimensional.

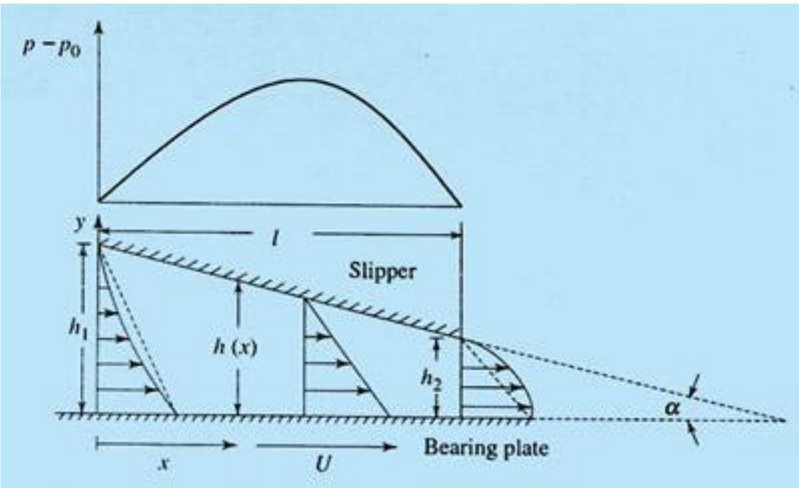


Fig 27.2 - Flow in a slipper bearing

- For the analysis, we may assume that the slipper is at rest and the plate is forced to move with a constant velocity *U* .
- The height *h(x)* of the wedge between the block and the guide is assumed to be very small as compared with the length l of the block.
- The essential difference between this motion and that discussed in Lecture 26 (Couette flow) is that here the **two walls are inclined at an angle** to each other.
- Due to the gradual reduction of narrowing passage, the convective acceleration $u\frac{\partial u}{\partial x}$ is distinctly not zero.
- For all practical purposes, inertia terms can be neglected as compared to viscous term. This can be justified in following way

$$\frac{\text{Inertia forece}}{\text{Viscous force}} = \frac{\rho u(\partial u/\partial x)}{\mu\left(\partial^2 u/\partial y^2\right)} = \frac{\rho U^2/l}{\mu U/h^2} = \frac{\rho Ul}{\mu}\left(\frac{h}{l}\right)^2$$

The inertia force can be neglected with respect to viscous force if the modified Reynolds number,

$$R^* = \frac{Ul}{\nu}\left(\frac{h}{l}\right)^2 << 1$$

- The equation for motion in *y* direction can be omitted since the *v* component of velocity is very small with respect to *u* . Besides, in the x-momentum equation, $\partial^2 u/\partial x^2$ can be neglected as compared with $\partial^2 u/\partial y^2$ because the former is smaller than the latter by a factor of the order of $(h/l)^2$. With these simplifications the equations of motion reduce to

$$\mu\frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx} \qquad (27.6)$$

The equation of continuity can be written as :

$$Q = \int_0^{h(x)} u\,dy \qquad (27.7)$$

The **boundary conditions** are:
at y = 0, u = U at x = 0, p = p_0
at *y = h, u = 0 and at x = l, p =* p_0 (27.8)

- Integrating Eq. (27.6) with respect to *y* , we obtain

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

- Application of the **kinematic boundary conditions** (at y=0, u =U and y = h, U=0), yields

$$u = U\left(1 - \frac{y}{h}\right) - \frac{h^2}{2\mu}\cdot\frac{dp}{dx}\left(1 - \frac{y}{h}\right)\frac{y}{h} \qquad (27.9)$$

Note that $\frac{dp}{dx}$ is constant as far as integration along *y* is concerned, but *p* and $\frac{dp}{dx}$ vary along *x* -axis.

- At the point of maximum pressure, $\frac{dp}{dx}$=0 hence

$$u = U\left(1 - \frac{y}{h}\right) \tag{27.10}$$

- Equation (27.10) depicts that the velocity profile along *y* is linear at the location of maximum pressure. The gap at this location may be denoted as h^*.
- Substituting Eq. (27.9) into Eq. (27.8) and integrating, we get

$$Q = \frac{Uh}{2} - \frac{p'h^3}{12\mu}$$

or
$$p' = 12\mu\left(\frac{U}{2h^2} - \frac{Q}{h^3}\right) \tag{27.11}$$

where $p' = dp/dx$

- **Integrating Eq. (27.11) with respect to *x* , we obtain**

$$\int \frac{dp}{dx} dx = 6\mu U \int \frac{dx}{(h_1 - \alpha x)^2} - 12\mu Q \int \frac{dx}{(h_1 - \alpha x)^3} + C_3 \tag{27.12a}$$

or
$$p = \frac{6\mu U}{\alpha(h_1 - \alpha x)} - \frac{6\mu Q}{\alpha(h_1 - \alpha x)^2} + C_3 \tag{27.12b}$$

where $\alpha = (h_1 - h_2)/l$ and C_3 is a constant

- Since the **pressure must be the same ($p = p_0$), at the ends of the bearing,** namely, $p = p_0$ **at** x = 0 and $p = p_0$ **at** x=l, the unknowns in the above equations can be determined by applying the pressure boundary conditions. We obtain

$$Q = \frac{Uh_1 h_2}{h_1 + h_2} \quad and \quad C_3 = p_0 - \frac{6\mu U}{\alpha(h_1 + h_2)}$$

- With these values inserted, the equation for pressure distribution (27.12) becomes

$$p = p_0 + \frac{6\mu Ux(h - h_2)}{h^2(h_1 + h_2)}$$

or
$$p - p_0 = \frac{6\mu Ux(h - h_2)}{h^2(h_1 + h_2)} \tag{27.13}$$

- It may be seen from Eq. (27.13) that, **if the gap is uniform,** i.e. $h = h_1 = h_2$**,** the **gauge pressure will be zero.** Furthermore, it can be said that very **high pressure** can be developed by keeping the **film thickness very small.**
- Figure 27.2 shows the distribution of pressure throughout the bearing.

...Theory of Hydrodynamic Lubrication... cont from previous slide

- **The total load bearing capacity per unit width is**

$$P = \int_0^l (p - p_0)dx = \frac{6\mu U}{h_1 + h_2}\int_0^l \frac{x(h - h_2)}{h^2}dx$$

After substituting $h=h_1 - \alpha x$ with $\alpha=(h_1-h_2)/ l$ in the above equation and performing the integration,

$$P = \frac{6\mu U l^2}{(h_1 - h_2)^2}\left[In\frac{h_1}{h_2} - 2\left\{\frac{h_1 - h_2}{h_1 + h_2}\right\}\right] \qquad (27.14)$$

- **The shear stress at the bearing plate is**

$$\tau_0 = -\mu\frac{\partial u}{\partial y}\Big|_{y=0} = \left(p'\frac{h}{2} + \mu\frac{U}{h}\right) \qquad (27.15)$$

Substituting the value of p' from Eq. (27.14) and then invoking the value of Q in Eq. (27.15), the final expression for shear stress becomes

$$\tau_0 = 4\mu\frac{U}{h} - \frac{6\mu U h_1 h_2}{h^2(h_1 + h_2)}$$

- **The drag force required to move the lower surface at speed *U* is expressed by**

$$D = \int_0^1 \tau_0 dx = \frac{\mu U l}{h_1 - h_2}\left[4 ln\frac{h_1}{h_2} - 6\frac{h_1 - h_2}{h_1 + h_2}\right] \qquad (27.16)$$

- Michell thrust bearing, named after A.G.M. Michell, works on the principles based on the theory of **hydrodynamic lubrication** . The journal bearing (Fig. 27.3) develops its force by the same action, except that the surfaces are curved.

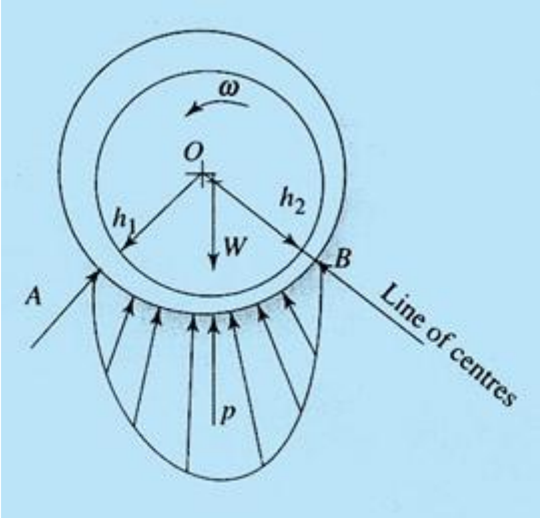


FIG 27.3 Hydrodynamic action of a journal bearing

Exercise Problems - Chapter 8

1. Water flows between two very large horizontal parallel flat plates 30 mm apart. If the average velocity of water is 0.2 m/s, what is the shear stress
(a) at the lower plate and
(b) at the middle plane?
Take $\mu = 1.1 \times 10^{-3} Ns/m^2$

(a) 0.022 N/m^2 (b) 0

2. A viscous fluid flows steadily, parallel to the axis in the annular space between two coaxial cylinders having radii r_1 and r_2 respectively. Show that the volumetric flow rate is given by

$$Q = \frac{\pi r_1^4}{8\mu}\left(-\frac{dp}{dz}\right)\left[\left(n^4 - 1\right) - \frac{\left(n^2 - 1\right)^2}{\ln n}\right]$$

where $n = r_2/r_1$

3. An endless belt passes upward through a chemical both with speed V and drags a layer of liquid of thickness H, density ρ, and viscosity μ . Gravity pulls the liquid down, however, the upward movement of the belt keeps it from running off. Assuming laminar and fully developed flow, solve the velocity distribution across the layer of liquid, if the atmosphere exerts no-shear on its surface. Also determine the rate at which the liquid is being dragged up by the belt.

4. Derive a formula for the terminal velocity of a sphere with Re (based on the diameter) much less than 1. Apply the result to the settlement speed of blood cells in plasma, using a radius of 3×10^{-3} mm for the cell, a difference of densities between cell and the plasma of 0.07×10^{-6} kg/mm^3, a plasma viscosity of 1.27×10^{-3} Ns/m^2

This problem should be considered as a take-home assignment

5. Refer to Figure 27.4. This problem illustrates the secret of the strength of cello-tape joints.

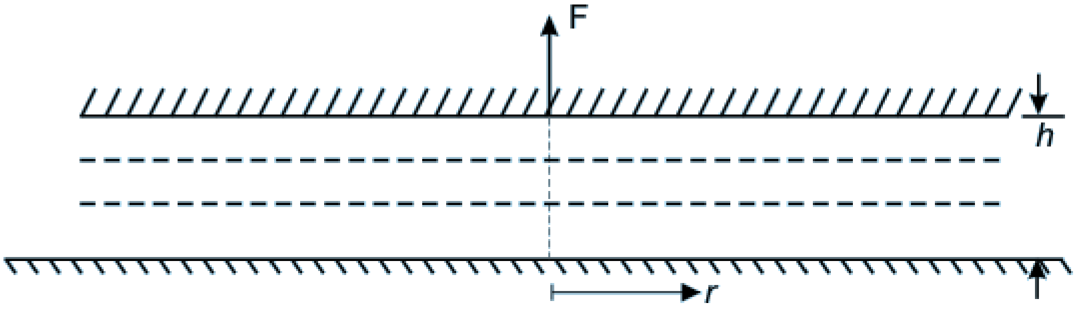


Figure 27.4

A disc of radius r is at a uniform distance h from a large flat plate, and the gap h is filled with a viscous liquid. The disc moves upwards at *v =dh/dt*, in response to a central force F. Due to the symmetry of the problem (in the θ direction), the governing equations are υ

Continuity: $\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial \upsilon}{\partial z}=0\ ,\quad u = radial\ velocity$

momentum:
$$\begin{cases} u\frac{\partial u}{\partial r}+\upsilon\frac{\partial u}{\partial u}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right) \\ \\ u\frac{\partial \upsilon}{dr}+\nu\frac{\partial \upsilon}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 \upsilon}{\partial r^2}+\frac{\partial^2 \upsilon}{\partial z^2}\right) \end{cases}$$

a) Show that if $h << R$ and $\frac{h}{\nu}\frac{dh}{dt} << 1,$ inertia terms can be neglected in the momentum equations and we get the following :

Continuity Equation remains unchanged

$$\frac{\partial p}{\partial r}=\mu\frac{\partial^2 u}{\partial z^2}\quad ;\quad \frac{\partial p}{\partial z}=0$$

b) Solve these differential equations to get

$$F=\frac{3}{2}\cdot\pi\mu\frac{R^4}{h^3}\frac{dh}{dt}$$

c) Estimate F if R=1 cm, $\frac{dh}{dt}=1mm/s,$ *h* =0.1 mm $\mu = 4.2\times10^{-6}\,N-cm^{-2}-s$

6.Find the load that the following step bearing (Figure 27.5) can carry:

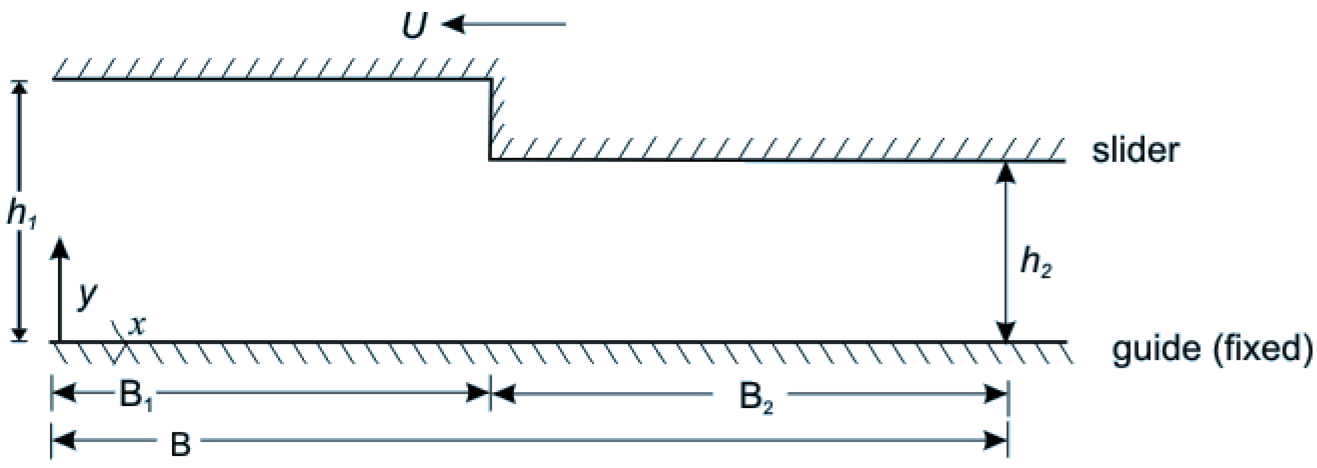


Figure 27.5

The gap is uniform (=h_1) over length B_1 and also uniform $(= h_2 \neq h_1)$ over length B_2 .

Recap

In this course you have learnt the following

- The boundary layer is the thin layer of fluid adjacent to the solid surface. Phenomenologically, the effect of viscosity is very prominent within this layer.
- The main-stream velocity undergoes a change from zero at the solid surface to the full magnitude through the boundary layer.Effectively, the boundary layer theory is a complement to the inviscid flow theory.
- The governing equation for the boundary layer can be obtained through correct reduction of the **Navier-Stokes equations** within the thin layer referred above. There is no variation in pressure in y direction within the boundary layer.
- The pressure is impressed on the boundary layer by the outer inviscid flow which can be calculated using **Bernoulli's equation**.
- The boundary layer equation is a second order non-linear partial differential equation. The exact solution of this equation is known as **similarity solution**. For the flow over a flat plate, the similarity solution is often referred to as **Blasius solution.** Complete analytical treatment of this solution is beyond the scope of this text. However, the momentum integral equation can be derived from the boundary layer equation which is amenable to analytical treatment.
- The solutions of the momentum integral equation are called approximate solutions of the boundary layer equation.
- The boundary layer equations are valid up to the point of separation. At the point of separation, the flow gets detached from the solid surface due to excessive

adverse pressure gradient.

- Beyond the point of separation, the flow reversal produces eddies. During flow past bluff-bodies, the desired pressure recovery does not take place in a separated flow and the situation gives rise to **pressure drag or form drag.**

Introduction

- The **boundary layer** of a flowing fluid is **the thin layer close to the wall**
- In a flow field, **viscous stresses are very prominent within this layer**.
- Although the layer is thin, it is very important to know the details of flow within it.
- The **main-flow velocity** within this layer **tends to zero** while approaching the wall **(no-slip condition)**.
- Also the gradient of this velocity component in a direction normal to the surface is large as compared to the gradient in the streamwise direction.

Boundary Layer Equations

- In 1904, **Ludwig Prandtl,** the well known German scientist, introduced the concept of boundary layer and **derived the equations for boundary layer flow** by correct reduction of Navier-Stokes equations.

- He hypothesized that **for fluids having relatively small viscosity**, **the effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies over which the fluid flows.**

- Thus, close to the body is the boundary layer where **shear stresses exert an increasingly larger effect** on the fluid **as one moves from free stream towards the solid boundary.**

- However, **outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer (since the velocity gradient** $\partial u / \partial y$ **is negligible)**,---------

    1. the fluid particles experience **no vorticity** and therefore**,**
    2. the flow is similar to a **potential flow.**

- Hence, the ***surface*** **at the boundary layer interface** is a rather fictitious one, that **divides rotational and irrotational flow.** Fig 28.1 shows Prandtl's model regarding boundary layer flow.

- Hence with the exception of the immediate vicinity of the surface, the flow is frictionless (inviscid) and the velocity is *U* (the potential velocity).

- In the region, very near to the surface (in the thin layer), there is friction in the flow which signifies that the fluid is retarded until it adheres to the surface (**no-slip condition).**

- The transition of the mainstream velocity from zero at the surface (with respect to the surface) to full magnitude takes place across the boundary layer.

About the boundary layer

- Boundary layer **thickness is** δ which is a **function of** the coordinate direction ***x*** .
- The thickness is considered to be **very small compared to the characteristic length *L*** of the domain.
- In the normal direction, **within this thin layer**, the gradient $\partial u / \partial y$ **is very large compared to** the gradient in the flow direction $\partial u / \partial x$ **.**

Now we take up the Navier-Stokes equations for : steady, two dimensional, laminar, incompressible flows.

Considering the Navier-Stokes equations together with the equation of continuity, the following dimensional form is obtained.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{28.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{28.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{28.3}$$

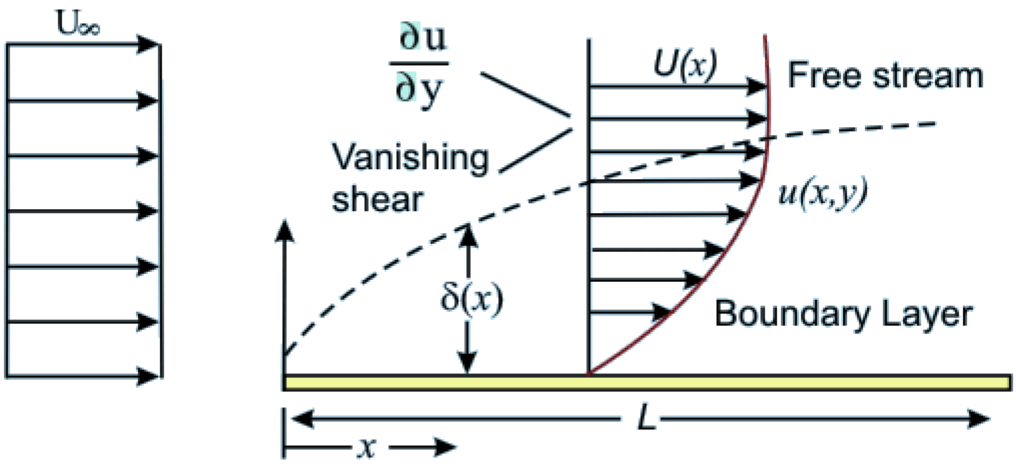


Fig 28.1 Boundary layer and Free Stream for Flow Over a flat plate

- u - velocity component along x direction.
- v - velocity component along y direction
- p - static pressure
- ρ - density.
- μ - dynamic viscosity of the fluid

- The equations are now non-dimensionalised.
- The l**ength and the velocity scales** are chosen as ***L*** **and** U_∞ **respectively.**

- The non-dimensional variables are:

$$u^* = \frac{u}{U_\infty}, v^* = \frac{v}{U_\infty}, p^* = \frac{p}{\rho U_\infty^2}$$

$$x^* = \frac{x}{L}, y^* = \frac{y}{L}$$

where U_∞ is the dimensional free stream velocity and the pressure is non-dimensionalised by twice the dynamic pressure $p_d = (1/2)\rho U_\infty^2$.

Using these non-dimensional variables, the Eqs (28.1) to (28.3) become

$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right]$ click for details	(28.4)
$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right]$	(28.5)
$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$	(28.6)

where the Reynolds number,

$$\mathrm{Re} = \frac{\rho U_\infty L}{\mu}$$

Order of Magnitude Analysis

- Let us examine what happens to the *u* velocity as we go across the boundary layer. At the **wall** the *u* velocity is **zero** [with respect to the wall and absolute zero for a stationary wall (which is normally implied if not stated otherwise)].
The value of *u* on the **inviscid side**, that is on the free stream side beyond the

boundary layer is ***U.***
For the case of external flow over a flat plate, this ***U*** is equal to U_∞.

- Based on the above, we can identify the following scales for the boundary layer variables:

Variable	***Dimensional scale***	***Non-dimensional scale***
u	U_∞	1
x	L	1
y	δ	$\varepsilon = \delta / L$

- The **symbol** ε **describes a value much smaller than 1.**
- Now we analyse equations 28.4 - 28.6, and look at the order of magnitude of each individual term

Eq 28.6 - the continuity equation

One **general rule** of incompressible fluid mechanics is that **we are not allowed to drop any term from the continuity equation**.

- From the scales of boundary layer variables, the derivative $\partial u^* / \partial x^*$ is of the order 1.
- The second term in the continuity equation $\partial v^* / \partial y^*$ should also be of the order 1.The reason being v^* has to be of the order ϵ because y^* becomes $\varepsilon (= \delta / L)$ at its maximum.

Eq 28.4 - x direction momentum equation

- Inertia terms are of the order 1.
- $\partial^2 u / \partial x^2$ is of the order 1
- $\partial^2 u / \partial y^2$ is of the order $(1/\varepsilon^2)$.

However after multiplication with 1/Re, the sum of the two second order derivatives should produce at least one term which is of the same order of magnitude as the inertia terms. This is possible only if the Reynolds number (Re) is of the order of $(1/\varepsilon^2)$.

- It follows from that $-\partial p^* / \partial x^*$ will not exceed the order of 1 so as to be in balance with the remaining term.

- Finally, Eqs (28.4), (28.5) and (28.6) can be rewritten as

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right] \tag{28.4}$$

$$(1)\frac{(1)}{(1)} \qquad (\varepsilon)\frac{(1)}{(\varepsilon)}=(1) \qquad (\varepsilon^2)\left[\frac{(1)}{(1)}+\frac{1}{(\varepsilon^2)}\right]$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right] \tag{28.5}$$

$$(1)\frac{(\varepsilon)}{(1)} \qquad (\varepsilon)\frac{(\varepsilon)}{(\varepsilon)}=(?) \qquad (\varepsilon^2)\left[\frac{(\varepsilon)}{(1)}+\frac{\varepsilon}{(\varepsilon^2)}\right]$$

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{28.6}$$

$$\frac{(1)}{(1)} \qquad \frac{(\varepsilon)}{(\varepsilon)}$$

As a consequence of the order of magnitude analysis, $\partial^2 u^*/\partial x^{*2}$ can be dropped from the *x* direction momentum equation, because on multiplication with $1/Re$ it assumes the smallest order of magnitude.

Eq 28.5 - *y* direction momentum equation.

- All the terms of this equation are of a smaller magnitude than those of Eq. (28.4).
- This equation can only be balanced if $\partial p^*/\partial y^*$ is of the same order of magnitude as other terms.
- Thus they momentum equation reduces to

$$\frac{\partial p^*}{\partial y^*}=O(\varepsilon) \tag{28.7}$$

- This means that the **pressure across the boundary layer does not change**. The **pressure is impressed on the boundary layer**, and its value is determined by hydrodynamic considerations.
- This also implies that the **pressure *p* is only a function of *x***. The pressure forces on a body are solely **determined by the inviscid flow outside the boundary layer.**
- The application of Eq. (28.4) at the outer edge of boundary layer gives

$$u^*\frac{\partial u^*}{\partial x^*}=-\frac{\partial p^*}{\partial x^*} \tag{28.8a}$$

In dimensional form, this can be written as

$$U\frac{dU}{dx}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (28.8b)$$

On integrating Eq (28.8b) the well known Bernoulli's equation is obtained

$$p+\frac{1}{2}\rho U^2 = \textit{a constant} \qquad (28.9)$$

- Finally, it can be said that by the order of magnitude analysis, the Navier-Stokes equations are simplified into equations given below.

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial y^{*2}} \qquad (28.10)$$

-

$$\frac{\partial p^*}{\partial y^*}=0 \qquad (28.11)$$

-

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \qquad (28.12)$$

-
- These are known as Prandtl's boundary-layer equations.

The available boundary conditions are:

Solid surface $at\ y^*=0, u^*=0=v^*$

or $at\ y=0, u=0=v$ (28.13)

Outer edge of boundary-layer

$$at\ y^*=(\varepsilon)=\frac{\delta}{L}, u^*=1$$

or $at\ y=\delta, u=U(x)$ (28.14)

- The unknown pressure *p* in the x-momentum equation can be determined from Bernoulli's Eq. (28.9), if the inviscid velocity distribution *U(x)* is also known.

We solve the Prandtl boundary layer equations for $u^*(x,y)$ and $v^*(x,y)$ with *U* obtained from the outer inviscid flow analysis. The equations are solved by commencing at the leading edge of the body and moving downstream to the desired location

- it allows the no-slip boundary condition to be satisfied which constitutes a significant improvement over the potential flow analysis while solving real fluid flow problems.
- The **Prandtl boundary layer equations** are thus a simplification of the Navier-Stokes equations.

Boundary Layer Coordinates

- The boundary layer equations derived are in Cartesian coordinates.
- The Velocity components u and v represent x and y direction velocities respectively.
- For objects with small curvature, these equations can be used with -
    - x coordinate : streamwise direction
    - y coordinate : normal component
- They are called **Boundary Layer Coordinates.**

Application of Boundary Layer Theory

- The Boundary-Layer Theory is not valid beyond the point of separation.
- At the point of separation, boundary layer thickness becomes quite large for the thin layer approximation to be valid.
- It is important to note that boundary layer theory can be used to locate the point of seperation itself.
- In applying the boundary layer theory although U is the free-stream velocity at the outer edge of the boundary layer, it is interpreted as the fluid velocity at the wall calculated from inviscid flow considerations (known as **Potential Wall Velocity**)
- Mathematically, application of the boundary - layer theory converts the character of governing Navier-Stroke equations from elliptic to parabolic
- This allows the marching in flow direction, as the solution at any location is independent of the conditions farther downstream.

Blasius Flow Over A Flat Plate

- The classical problem considered by H. Blasius was
    1. Two-dimensional, steady, incompressible flow over a flat plate at zero angle of incidence with respect to the uniform stream of velocity U_∞ .
    2. The fluid extends to infinity in all directions from the plate.

The physical problem is already illustrated in Fig. 28.1

- Blasius wanted to determine
  (a) the velocity field solely within the boundary layer,
  (b) the boundary layer thickness (δ),
  (c) the shear stress distribution on the plate, and
  (d) the drag force on the plate.
- The Prandtl boundary layer equations in the case under consideration are

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{28.15}$$

$$\nu=\mu/\rho$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

The boundary conditions are

$$at\ y=0,\ u=v=0 \tag{28.16}$$

$$at\quad y=\infty,\quad u=U_\infty$$

- Note that the substitution of the term $-\frac{1}{\rho}\frac{dp}{dx}$ in the original boundary layer momentum equation in terms of the free stream velocity produces $U_\infty\frac{dU_\infty}{dx}$ which is equal to zero.
- Hence the governing Eq. (28.15) does not contain any pressure-gradient term.
- However, the characteristic parameters of this problem are U_∞, v, x, y that is, $u=u\left(U_\infty, v, x, y\right)$
- This relation has five variables U_∞, v, x, y.
- It involves two dimensions, length and time.
- Thus it can be reduced to a dimensionless relation in terms of (5-2) =3 quantities (**Buckingham Pi Theorem**)
- Thus a similarity variables can be used to find the solution
- Such flow fields are called self-similar flow field .

Law of Similarity for Boundary Layer Flows

- It states that the *u* component of velocity with two velocity profiles of *u(x,y)* at different *x* locations differ only by scale factors in *u* and *y* .
- Therefore, the velocity profiles *u(x,y)* at all values of *x* can be made

congruent if they are plotted in coordinates which have been made dimensionless with reference to the scale factors.

- The local free stream velocity *U(x)* at section *x* is an obvious scale factor for *u*, because the dimensionless *u(x)* varies between zero and unity with *y* at all sections.
- The scale factor for *y* , denoted by *g(x)* , is proportional to the local boundary layer thickness so that *y* itself varies between zero and unity.
- Velocity at two arbitrary *x* locations, namely x_1 and x_2 should satisfy the equation

$$\frac{u\left[x_1,\{y/g(x_1)\}\right]}{U(x_1)} = \frac{u\left[x_2,\{y/g(x_2)\}\right]}{U(x_2)} \qquad (28.17)$$

- Now, for Blasius flow, it is possible to identify g(x) with the boundary layers thickness δ we know

$$\varepsilon = \frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$$

Thus in terms of x we get

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{\frac{U_\infty x}{\nu}}}$$

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$$

i.e.,

$$\frac{u}{U_\infty} = F\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right) = F(\eta) \qquad (28.18)$$

where $\eta \sim \frac{y}{\delta}$ *and* $\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$

or more precisely,

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \qquad (28.19)$$

$$y = \eta\sqrt{\frac{\nu x}{U_\infty}}$$

$$dy = \sqrt{\frac{\nu x}{U_\infty}}d\eta$$

The stream function can now be obtained in terms of the velocity components as

$$\psi = \int u dy = \int U_\infty F(\eta)\sqrt{\frac{\nu x}{U_\infty}}d\eta = \sqrt{U_\infty \nu x}\int F(\eta)d\eta$$

or

$$\psi = \sqrt{U_\infty \nu x} f(\eta) + D \tag{28.20}$$

where D is a constant. Also $\int F(\eta)d\eta = f(\eta)$ and the constant of integration is zero if the stream function at the solid surface is set equal to zero.

Now, the velocity components and their derivatives are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}.\frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \tag{28.21a}$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{U_\infty \nu}\left[\frac{1}{2}.\frac{1}{\sqrt{x}}f(\eta) + \sqrt{x}f'(\eta)\left\{-\frac{1}{2}\frac{y}{\sqrt{\nu x / U_\infty}}\frac{1}{x}\right\}\right]$$

or

$$v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta f'(\eta) - f(\eta)\right] \tag{28.21b}$$

$$\frac{\partial u}{\partial x} = U_\infty f''(\eta)\frac{\partial \eta}{\partial x} = U_\infty f''(\eta)\left[-\frac{1}{2}.\frac{y}{\sqrt{\nu x / U_\infty}}\frac{1}{x}\right]$$

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2}.\frac{\eta}{x}.f''(\eta) \tag{28.21c}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta)\frac{\partial \eta}{\partial y} = U_\infty f''(\eta)\left[\frac{1}{\sqrt{\nu x / U_\infty}}\right]$$

$$\frac{\partial u}{\partial y} = U_\infty\sqrt{\frac{U_\infty}{\nu x}}f''(\eta) \tag{28.21d}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(\eta) \left\{ \frac{1}{\sqrt{\nu x / U_\infty}} \right\}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f'''(\eta) \qquad (28.21e)$$

- Substituting (28.2) into (28.15), we have

$$-\frac{U_\infty^2}{2} \frac{\eta}{x} . f'(\eta) f''(\eta) + \frac{U_\infty^2}{2x} \left[\eta f'(\eta) - f(\eta) \right] f''(\eta) = \frac{U_\infty^2}{x} f'''(\eta)$$

$$-\frac{1}{2} \frac{U_\infty^2}{x} f(\eta) f''(\eta) = \frac{U_\infty^2}{x} f'''(\eta)$$

or,

$$2 f'''(\eta) + f(\eta) f''(\eta) = 0 \qquad (28.22)$$

where

$$f(\eta) = \int F(\eta) d\eta + C$$

$$= \int \frac{u}{U_\infty} d\eta + C$$

and

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}$$

This is known as Blasius Equation .

Contd. from Previous Slide

- The boundary conditions as in Eg. (28.16), in combination with Eg. (28.21a) and (28.21b) become at $y = 0, u = 0$, therefore $\eta = 0 : f(\eta) = 0, f'(\eta) = 0$

at $y = \infty, u = U_\infty$ therefore $\eta = \infty : F(\eta) = f'(\eta) = 1$ (28.23)

Equation (28.22) is a **third order nonlinear differential equation .**

- Blasius obtained the solution of this equation in the form of series expansion through analytical techniques
- We shall not discuss this technique. However, we shall discuss a numerical technique to solve the aforesaid equation which can be understood rather easily.
- Note that the **equation for f does not contain x** .
- **Boundary conditions at $x = 0$ and $y = \infty$ merge into the condition $\eta \to \infty, u/U_\infty = f' = 1$. This is the key feature of similarity solution.**
- We can rewrite Eq. (28.22) as three first order differential equations in the following way

$$f' = G \tag{28.24a}$$

$$G' = H \tag{28.24b}$$

$$H' = -\frac{1}{2} fH \tag{28.24c}$$

- Let us next consider the boundary conditions.

1. The condition $f(0) = 0$ remains valid.
2. The condition $f'(0) = 0$ means that $G(0) = 0$ **.**
3. The condition $f'(\infty) = 1$ gives us $G(\infty) = 1$.

Note that the equations for *f* and *G* have initial values. However, the value for H(0) is not known. Hence, we do not have a usual initial-value problem.

Shooting Technique

We handle this problem as an initial-value problem by choosing values of $H(0)$ and solving by numerical methods $f(\eta), G(\eta)$, and $H(\eta)$.

In general, the condition $G(\infty) = 1$ will not be satisfied for the function G arising from the numerical solution.

We then choose other initial values of H so that eventually we find an $H(0)$ which results in $G(\infty) = 1$.

This method is called the shooting technique **.**

- In Eq. (28.24), the primes refer to differentiation wrt. the similarity variable η. The integration steps following Runge-Kutta method are given below.

$$f_{n+1} = f_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{28.25a}$$

$$G_{n+1} = G_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \tag{28.25b}$$

$$H_{n+1} = H_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{28.25c}$$

- One moves from η_n to $\eta_{n+1} = \eta_n + h$. A fourth order accuracy is preserved if *h* is constant along the integration path, that is, $\eta_{n+1} - \eta_n = h$ for all values of *n* . The values of *k*, *l* and *m* are as follows.
- For generality let the system of governing equations be

$$f' = F_1(f, G, H, \eta), G' = F_2(f, G, H, \eta) \;\&\; H' = F_3(f, G, H, \eta)$$
$$k_1 = hF_1(f_n, G_n, H_n, \eta_n)$$
$$l_1 = hF_2(f_n, G_n, H_n, \eta_n)$$
$$m_1 = hF_3(f_n, G_n, H_n, \eta_n)$$
$$k_2 = hF_1\left\{\left((f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1), (\eta_n + \frac{h}{2})\right)\right\}$$
$$l_2 = hF_2\left\{\left((f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1), (\eta_n + \frac{h}{2})\right)\right\}$$
$$m_2 = hF_3\left\{\left((f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1), (\eta_n + \frac{h}{2})\right)\right\}$$

In a similar way K_3, l_3, m_3 and k_4, l_4, m_4 mare calculated following standard formulae for the Runge-Kutta integration. For example, K_3 is given by
$k_3 = hF_1\left\{\left((f_n + \frac{1}{2}k_2), (G_n + \frac{1}{2}l_2), (H_n + \frac{1}{2}m_2), (\eta_n + \frac{h}{2})\right)\right\}$ The functions F_1, $F_2$and F_3 are G, H , - *f H / 2* respectively. Then at a distance $\Delta\eta$ from the wall, we have

$$f(\Delta\eta) = f(0) + G(0)\Delta\eta \tag{28.26a}$$

$$G(\Delta\eta) = G(0) + H(0)\Delta\eta \tag{28.26b}$$

$$H(\Delta\eta) = H(0) + H'(0)\Delta\eta \tag{28.26c}$$

$$H'(\Delta\eta) = -\frac{1}{2}f(\Delta\eta)H(\Delta\eta) \tag{28.26d}$$

- As it has been mentioned earlier $f''(0) = H(0) = \lambda$ is unknown. It must be determined such that the condition $f'(\infty) = G(\infty) = 1$ is satisfied.

The condition at infinity is usually approximated at a finite value of η (around $\eta = 10$). The process of obtaining λ accurately involves iteration and may be calculated using the procedure described below.

- For this purpose, consider Fig. 28.2(a) where the solutions of G versus η for two different values of $H(0)$ are plotted.
  The values of $G(\infty)$ are estimated from the G curves and are plotted in Fig. 28.2(b).
- The value of $H(0)$ now can be calculated by finding the value $\tilde{H}(0)$ at which the line 1-2 crosses the line $G(\infty) = 1$ By using similar triangles, it can be said that $\dfrac{\tilde{H}(0) - H(0)_1}{1 - G(\infty)_1} = \dfrac{H(0)_2 - H(0)_1}{G(\infty)_2 - G(\infty)_1}$. By solving this, we get $\tilde{H}(0)$.
- Next we repeat the same calculation as above by using $\tilde{H}(0)$ and the better of the two initial values of $H(0)$. Thus we get another improved value $\tilde{\tilde{H}}(0)$. This process may continue, that is, we use $\tilde{\tilde{H}}(0)$ and $\tilde{H}(0)$ as a pair of values to find more improved values for $H(0)$, and so forth. The better guess for H (0) can also be obtained by using the Newton Raphson Method. It should be always kept in mind that for each value of $H(0)$, the curve $G(\eta)$ versus η is to be examined to get the proper value of $G(\infty)$.
- The functions $f(\eta), f'(\eta) = G$ and $f''(\eta) = H$ are plotted in Fig. 28.3.The velocity components, *u* and *v* inside the boundary layer can be computed from Eqs (28.21a) and (28.21b) respectively.
- A sample computer program in FORTRAN follows in order to explain the solution procedure in greater detail. The program uses Runge Kutta integration together with the Newton Raphson method

Download the program

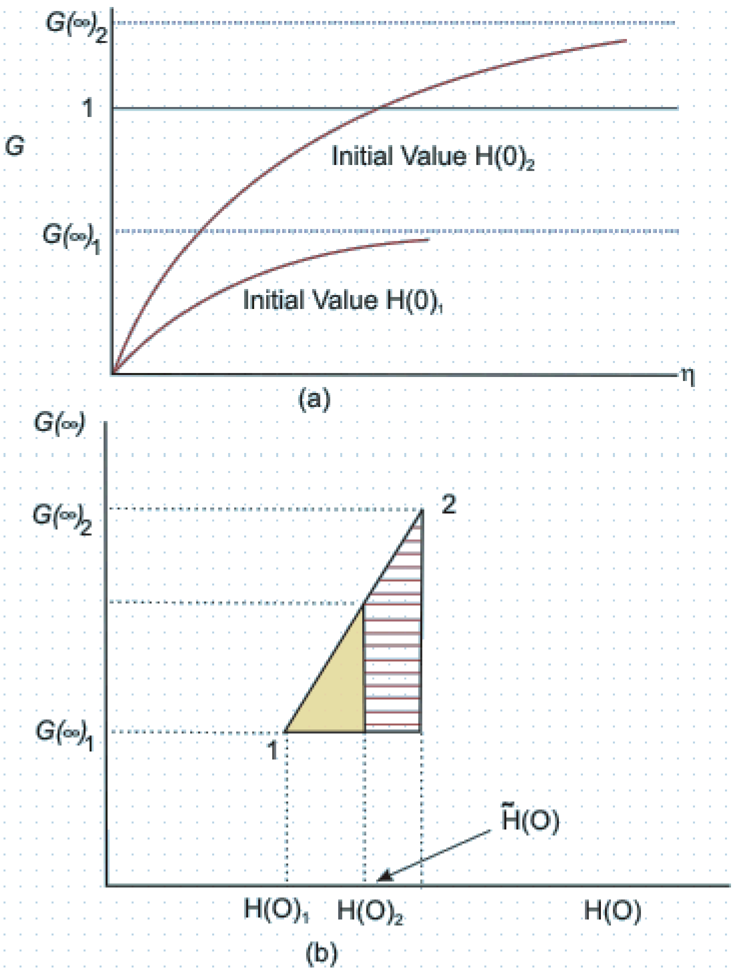


Fig 28.2 Correcting the initial guess for H(O)

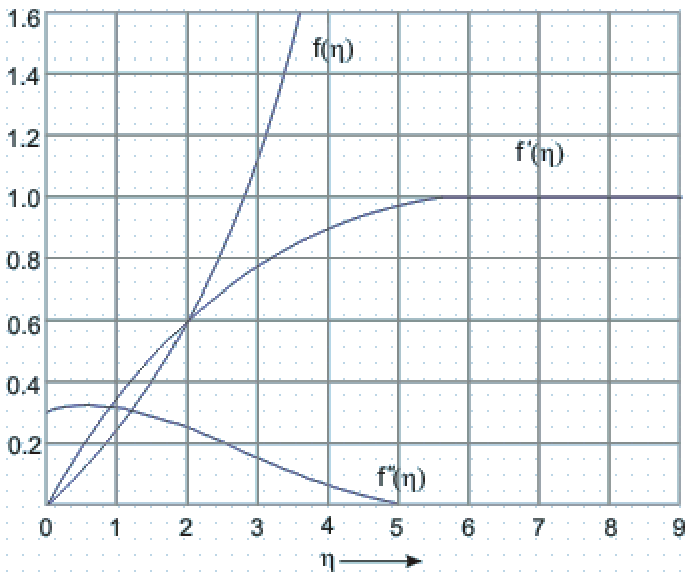


Fig 28.3 f, G and H distribution in the boundary layer

- Measurements to test the accuracy of theoretical results were carried out by many scientists. In his experiments, J. Nikuradse, found excellent agreement with the theoretical results with respect to velocity distribution (u/U_∞) within the boundary layer of a stream of air on a flat plate.

- In the next slide we'll see some values of the velocity profile shape $f'(\eta) = u/U_\infty = G$ and $f''(\eta) = H$ in tabular format.

Values of the velocity profile shape $f'(\eta) = u/U_\infty = G \text{ and } f''(\eta) = H$

Table 28.1 The Blasius Velocity Profile $G = u/U_\infty \; f \text{ and } H$

η	f	G	H
0	0	0	0.33206
0.2	0.00664	0.006641	0.33199
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
8.8	7.07923	1.00000	0.00000

Wall Shear Stress

- With the profile known, wall shear can be evaluated as

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

Now, $$\frac{\partial u}{\partial y} = U_\infty f'(\eta)\frac{\partial \eta}{\partial y}$$

$$\tau_\omega = \mu U_\infty f''(\eta)\frac{\partial \eta}{\partial y}\bigg|_{\eta=0}$$

or

$$= \mu U_\infty H \frac{\partial \eta}{\partial y}\bigg|_{\eta=0}$$

or

$[f''(0) = 0.3326]$ from Table 28.1

$$\tau_\omega = \frac{0.332\rho U_\infty^2}{\sqrt{\mathrm{Re}_x}} \qquad (29.1a)$$

(Wall Shear Stress)

and the local skin friction coefficient is $C_{fx} = \dfrac{\tau_\omega}{1/2\rho U_\infty^2}$

- Substituting from (29.1a) we get

$$C_{fx} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \qquad (29.1b)$$

(Skin Friction Coefficient)

-
- In 1951, Liepmann and Dhawan , measured the shearing stress on a flat plate directly. Their results showed a striking confirmation of Eq. (29.1).
- Total frictional force per unit width for the plate of length L is

$$F = \int_0^L \tau_\omega dx$$

$$F = \int_0^L \frac{0.332\rho U_\infty^2}{\sqrt{U_\infty / \nu}} \frac{dx}{\sqrt{x}}$$

or

$$F = \left[\frac{0.332\rho U_\infty^2}{\sqrt{U_\infty / \nu}} \times \frac{x^{1/2}}{1/2}\right]_0^L$$

or

$$F = 0.664 \times \rho U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}} \qquad (29.2)$$

and the average skin friction coefficient is

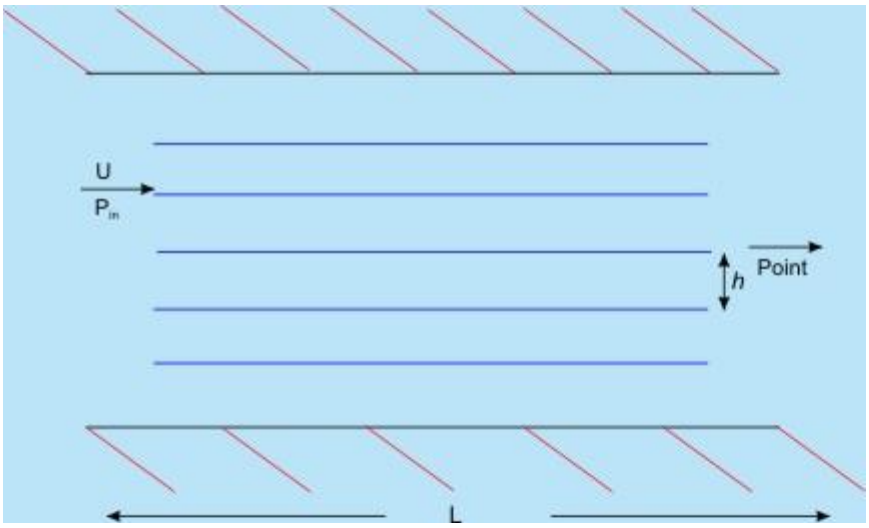
$$\overline{C_f} = \frac{F}{1/2\left(\rho U_\infty^2 L\right)} = \frac{1.328}{\sqrt{\mathrm{Re}_L}} \qquad (29.3)$$

where, $\mathrm{Re}_L = U_\infty L/\nu$.

For a flat plate of length L in the streamwise direction and width w perpendicular to the flow, the Drag D would be

$$D = F(2wL) = 0.664(2wL)\rho U_\infty^2\left(\frac{\nu L}{U_\infty}\right)^{1/2} = 1.328wL\left(\frac{\rho\mu U_\infty^3}{L}\right)^{1/2} \qquad (29.4)$$

Example



The above engineering system shows a series of thin parallel plates aligned with the intake flows. The plate spacing is h and the plate length is L.

Assume that the flow is incompressible. Derive expression for the pressure drop pin-pout between inflow and outflow streams for the high velocity flow where a boundary layer develops on each surface independent of adjacent plates.

Solution.

The drag force on each plate is

$$D = 1.328\ wL\left(\frac{\rho\mu U^3{}_\infty}{L}\right)^{1/2}$$

Now choosing one channel as control volume

$$\left(P_{in} - P_{out}\right)wh = D$$

or, $$\Delta P\ wh = 1.328\ w\left(\rho\mu U_\infty^3 L\right)^{1/2}$$

or, $$\Delta p = 1.328\left(\frac{\rho\mu U_\infty^3 L}{h^2}\right)^{1/2}$$ independent of the width of the plate

Boundary Layer Thickness

- Since $u/U_\infty \rightarrow 0.99, as \quad y \rightarrow \infty$, it is customary to select the boundary layer thickness δ as that point where u/U_∞ **approaches 0.99.**
- From Table 28.1, u/U_∞ **reaches 0.99 at η= 5.0** and we can write

$$\delta \Big/ \sqrt{\left(\frac{\nu x}{U_\infty}\right)} \approx 5.0$$

$$or \quad \delta \approx 5.0\sqrt{\left(\frac{\nu x}{U_\infty}\right)} = \frac{5.0x}{\sqrt{\mathrm{Re}_x}} \qquad (29.5)$$

-
- However, the aforesaid definition of boundary layer thickness is somewhat arbitrary, a physically more meaningful measure of boundary layer estimation is expressed through **displacement** thickness .

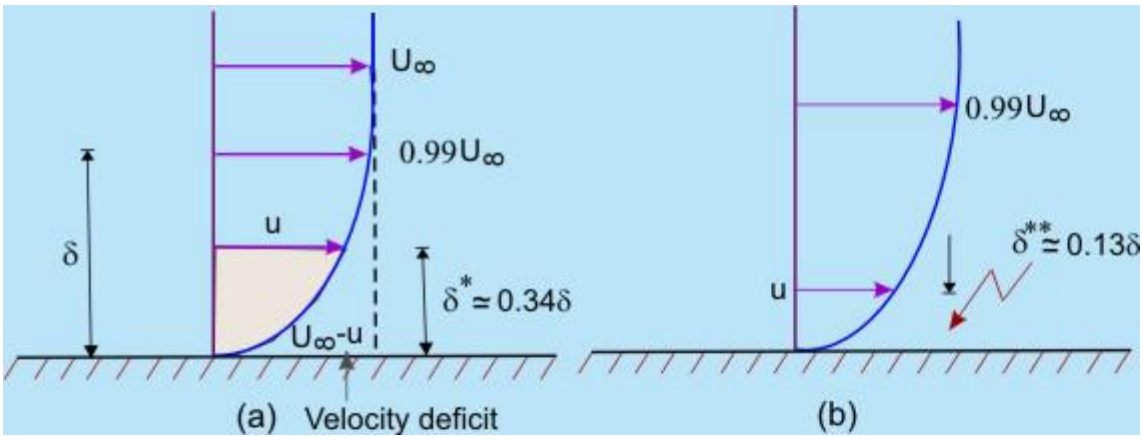


Fig. 29.1 (Displacement thickness) (b) Momentum thickness

- **Displacement thickness** (δ^*)**:** It is defined as the distance by which the external potential flow is displaced outwards due to the decrease in velocity in the boundary layer.

$$U_\infty \delta^* = \int_0^\infty (U_\infty - u)dy$$

Therefore,
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right)dy \qquad (29.6)$$

$$dy = \delta d\eta = \sqrt{\frac{\nu x}{U_\infty}} d\eta$$

-
- Substituting the values of u/U_∞ and η from Eqs (28.21a) and (28.19) into Eq.(29.6), we obtain

$$\delta^* = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty (1 - f')d\eta = \sqrt{\frac{\nu x}{U_\infty}} \lim_{\eta \to \infty} [\eta - f(\eta)]$$

or,
$$\delta^* = 1.7208 \sqrt{\frac{\nu x}{U_\infty}} = \frac{1.7208x}{\sqrt{\mathrm{Re}_x}} \qquad (29.7)$$

Following the analogy of the displacement thickness, a momentum thickness may be defined.

Momentum thickness (δ^{**}): It is defined as the loss of momentum in the boundary layer as compared with that of potential flow. Thus

$$\rho U_\infty^2 \delta^{**} = \int_0^\infty \rho u (U_\infty - u)dy$$

$$\delta^{**} = \int_0^\infty \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy \tag{29.8}$$

With the substitution of (u/U_∞) and η from Eg. (28.21a) and (28.19), we can evaluate numerically the value of δ^{**} for a flat plate as

$$\delta^{**} = \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty f'(1-f')d\eta$$

$$or \quad \delta^{**} = 0.664\sqrt{\frac{\nu x}{U_\infty}} = \frac{0.664x}{\sqrt{\mathrm{Re}_x}} \tag{29.9}$$

The relationships between $\delta, \delta^{*} and \delta^{**}$ have been shown in Fig. 29.1.

Momentum-Integral Equations For The Boundary Layer

- To employ boundary layer concepts in real engineering designs, we need approximate methods that would quickly lead to an answer even if the accuracy is somewhat less.
- **Karman and Pohlhausen** devised a simplified method by **satisfying only the boundary conditions of the boundary layer flow** rather than satisfying Prandtl's differential equations for each and every particle within the boundary layer. We shall discuss this method herein.
- Consider the case of steady, two-dimensional and incompressible flow, i.e. we shall refer to Eqs (28.10) to (28.14). Upon integrating the dimensional form of Eq. (28.10) with respect to *y = 0* (wall) to *y* = δ (where δ signifies the interface of the free stream and the boundary layer), we obtain

$$\int_0^\delta\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)dy = \int_0^\delta\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}\right)dy$$

or, $$\int_0^\delta u\frac{\partial u}{\partial x}dy+\int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta -\frac{1}{\rho}\frac{\partial p}{\partial x}dy+\int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy \tag{29.10}$$

- The second term of the left hand side can be expanded as

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = [vu]_0^\delta - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

or, $$\int_0^\delta v\frac{\partial u}{\partial y}dy = U_\infty v_\delta + \int_0^\delta u\frac{\partial u}{\partial x}dy\left(\sin ce\ \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}\right)$$ by continuity equation or, $$\int_0^\delta v\frac{\partial u}{\partial y}dy = -U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy \qquad (29.11)$$

- Substituting Eq. (29.11) in Eq. (29.10) we obtain

$$\int_0^\delta 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy = -\int_0^\delta \frac{1}{\rho}\frac{\partial p}{\partial x}dy - v\left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad (29.12)$$

- Substituting the relation between $\frac{\partial p}{\partial x}$ and the free stream velocity U_∞ for the inviscid zone in Eq. (29.12) we get

$$\int_0^\delta 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^\delta \frac{\partial u}{\partial x}dy - \int_0^\delta U_\infty\frac{dU_\infty}{dx}dy = -\left(\frac{\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}}{\rho}\right)$$

$$\int_0^\delta\left(2u\frac{\partial u}{\partial x} - U_\infty\frac{\partial u}{\partial x} - U_\infty\frac{dU_\infty}{dx}\right)dy = -\frac{\tau_w}{\rho}$$

which is reduced to

$$\int_0^\delta \frac{\partial}{\partial x}\left[u(U_\infty - u)\right]dy + \frac{dU_\infty}{dx}\int_0^\delta (U_\infty - u)dy = \frac{\tau_w}{\rho}$$

- Since the integrals vanish outside the boundary layer, we are allowed to increase the integration limit to infinity (i.e $\delta = \infty$.)

$$\int_0^\delta \frac{\partial}{\partial x}\left[u(U_\infty - u)\right]dy + \frac{dU_\infty}{dx}\int_0^\delta (U_\infty - u)dy = \frac{\tau_w}{\rho}$$

or, $$\frac{d}{dx}\int_0^\delta \left[u(U_\infty - u)\right]dy + \frac{dU_\infty}{dx}\int_0^\delta (U_\infty - u)dy = \frac{\tau_w}{\rho} \qquad (29.13)$$

- Substituting Eq. (29.6) and (29.7) in Eq. (29.13) we obtain

$$\frac{d}{dx}\left[U_\infty^2\delta^{**}\right]+\delta^*U_\infty\frac{dU_\infty}{dx}=\frac{\tau_w}{\rho} \qquad (29.14)$$

where $\delta^* = \int_0^\delta \left(1-\frac{u}{U_\infty}\right)dy$ is the displacement thickness

$\delta^{**} = \int_0^\delta \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy$ is momentum thickness

Equation (29.14) is known as momentum integral equation for two dimensional incompressible laminar boundary layer. The same remains valid for turbulent boundary layers as well.

Needless to say, **the wall shear stress** (τ_w) **will be different for laminar and turbulent flows.**

- The term $U_\infty\frac{dU_\infty}{dx}$ signifies space-wise acceleration of the free stream. Existence of this term means that free stream pressure gradient is present in the flow direction.
- For example, we get finite value of $U_\infty\frac{dU_\infty}{dx}$ outside the boundary layer in the entrance region of a pipe or a channel. For external flows, the existence of $U_\infty\frac{dU_\infty}{dx}$ depends on the shape of the body.
- During the flow over a flat plate, $U_\infty\frac{dU_\infty}{dx}=0$ and the momentum integral equation is reduced to

$$\frac{d}{dx}\left[U_\infty^2\delta^{**}\right]=\frac{\tau_w}{\rho} \qquad (29.15)$$

Seperation of Boundary Layer

- It has been observed that the **flow is reversed at the vicinity of the wall** under certain conditions.
- The phenomenon is termed as **separation of boundary layer**.
- Separation takes place **due to excessive momentum loss near the wall in a boundary layer trying to move downstream against increasing pressure, i.e.,** $\frac{dp}{dx}>0$ **, which is called *adverse pressure gradient*.**
- Figure 29.2 shows the flow past a circular cylinder, in an infinite medium.
    1. Up to $\theta = 90^\circ$, the flow area is like a constricted passage and the flow behaviour is like that of a nozzle.
    2. Beyond $\theta = 90^\circ$ the flow area is diverged, therefore, the flow behaviour is much similar to a diffuser

This dictates the inviscid pressure distribution on the cylinder which is shown by a firm line in Fig. 29.2.

Here

p_∞ **: pressure in the free stream**

U_∞ **: velocity in the free stream and**

p ***:* is the local pressure on the cylinder.**

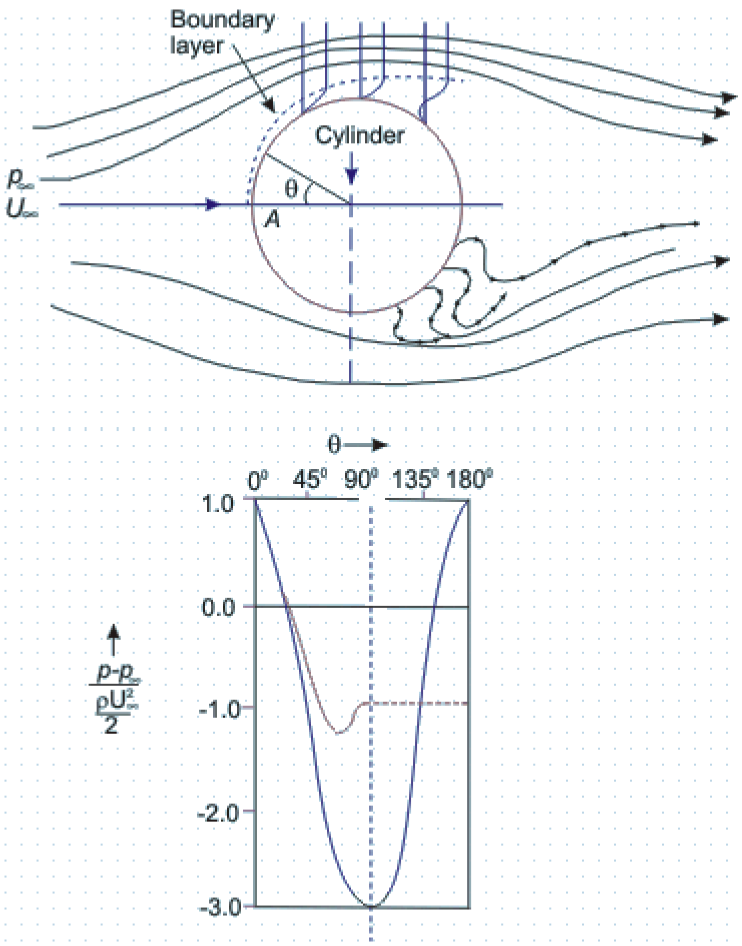


Fig. 29.2 Flow separation and formation of wake behind a circular cylinder

- Consider the forces in the flow field.
  In the **inviscid region,**
  1. **Until** $\theta = 90^\circ$ the pressure force and the force due to streamwise acceleration i.e. inertia forces are acting in the same direction (**pressure gradient** being **negative/favourable**)
  2. **Beyond** $\theta = 90^\circ$, the **pressure gradient is positive or adverse**. Due to the adverse pressure gradient the pressure force and the force due to acceleration will be opposing each other in the in viscid zone of this part.

-

  So long as no viscous effect is considered, the situation does not cause any sensation. In the **viscid region** (near the solid boundary),

  1. **Up to** $\theta = 90^\circ$ **, th**e viscous force opposes the combined pressure force and the force due to acceleration. Fluid particles overcome this viscous resistance **due to continuous conversion of pressure force into kinetic energy.**
  2. Beyond $\theta = 90^\circ$, within the viscous zone, the flow structure becomes different. It is seen that the force due to acceleration is opposed by both the viscous force and pressure force.
- Depending upon the magnitude of adverse pressure gradient, **somewhere around** $\theta = 90^\circ$ **, the fluid particles, in the boundary layer are separated from the wal**l and driven in the upstream direction. However, the far field external stream pushes back these separated layers together with it and develops **a broad pulsating wake behind the cylinder**.
- **The mathematical explanation of flow-separation :** The point of separation may be defined as the limit between forward and reverse flow in the layer very close to the wall, i.e., at the point of separation

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \tag{29.16}$$

  This means that the shear stress at the wall, $\tau_w = 0$. But at this point, the adverse pressure continues to exist and at the downstream of this point the flow acts in a reverse direction resulting in a back flow.

- We can also explain flow separation using the argument about the second derivative of velocity *u* at the wall. From the dimensional form of the momentum at the wall, where $u = v = 0$, we can write

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{1}{\mu}\frac{dp}{dx} \tag{29.17}$$

- Consider the situation due to a **favourable pressure gradient** where $\frac{dp}{dx} < 0$ we have,

  1. $\left(\partial^2 u / \partial y^2\right)_{wall} < 0$. (From Eq. (29.17))
  2. **As we proceed towards the free stream, the velocity** ***u*** **approaches** U_∞ **asymptotically,** so $\partial u / \partial y$ decreases at a continuously lesser rate in *y* direction.

3. This means that $\partial^2 u / \partial y^2$ remains less than zero near the edge of the boundary layer.
4. The curvature of a velocity profile $\partial^2 u / \partial y^2$ is always negative as shown in (Fig. 29.3a)

- Consider the case of **adverse pressure gradient**, $\partial p / \partial x > 0$

1. At the boundary, the curvature of the profile must be positive (since $\partial p / \partial x > 0$).
2. Near the interface of boundary layer and free stream the previous argument regarding $\partial u / \partial y$ and $\partial^2 u / \partial y^2$ still holds good and the curvature is negative.
3. Thus we observe that for an adverse pressure gradient, there must exist a point for which $\partial^2 u / \partial y^2 = 0$. This point is known as *point of inflection* of the velocity profile in the boundary layer as shown in Fig. 29.3b
4. However, point of separation means $\partial u / \partial y = 0$ at the wall.
5. $\partial^2 u / \partial y^2 > 0$ at the wall since separation can only occur due to adverse pressure gradient. But we have already seen that at the edge of the boundary layer, $\partial^2 u / \partial y^2 < 0$. It is therefore, clear that **if there is a point of separation, there must exist a point of inflection in the velocity profile.**

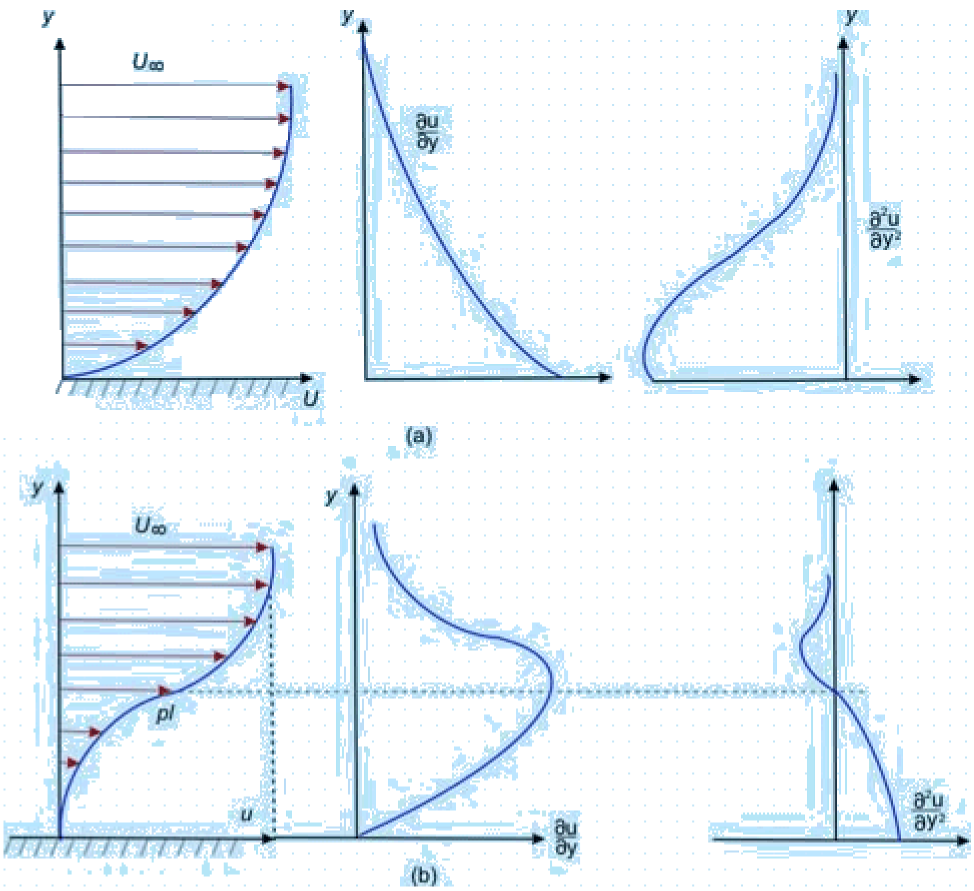


Fig. 29.3 Velocity distribution within a boundary layer

(a) Favourable pressure gradient, $\frac{dp}{dx} < 0$

(b) adverse pressure gradient, $\frac{dp}{dx} > 0$

1. Let us reconsider the flow past a circular cylinder and continue our **discussion on the wake behind a cylinder.** The pressure distribution which was shown by the firm line in Fig. 21.5 is obtained from the potential flow theory. However. somewhere near $\theta = 90^\circ$ **(in experiments it has been observed to be at $\theta = 81^\circ$) . the boundary layer detaches itself from the wall.**
2. Meanwhile, **pressure in the wake remains close to separation-point-pressure** since the eddies (formed as a consequence of the retarded layers being carried together with the upper layer through the action of shear) cannot convert rotational kinetic energy into pressure head. The actual pressure distribution is shown by the dotted line in Fig. 29.3.
3. Since the **wake zone pressure is less than that of the forward stagnation point** (pressure at point A in Fig. 29.3), the cylinder experiences a drag force which is basically attributed to the pressure difference.

The drag force, brought about by the pressure difference is known as *form drag* whereas the shear stress at the wall gives rise to *skin friction drag*. Generally, these two drag forces together are responsible for resultant drag on a body

Karman-Pohlhausen Approximate Method For Solution Of Momentum Integral Equation Over A Flat Plate

- The basic equation for this method is obtained by integrating the *x* direction momentum equation (boundary layer momentum equation) with respect to *y* from the wall (at y = 0) to a distance $\delta(x)$ which is assumed to be outside the boundary layer. Using this notation, we can rewrite the Karman momentum integral equation as

$$U_\infty^2 \frac{d\delta^{**}}{dx} + \left(2\delta^{**} + \delta^*\right)U_\infty \frac{dU_\infty}{dx} = \frac{\tau_w}{\rho} \qquad (30.1)$$

- The effect of pressure gradient is described by the second term on the left hand side. For pressure gradient surfaces in external flow or for the developing sections in internal flow, this term contributes to the pressure gradient.
- We assume **a velocity profile which is a polynomial of** $\eta = y/\delta$. η being a form of **similarity variable** , implies that **with the growth of boundary layer as distance *x* varies from the leading edge, the velocity profile** (u/U_∞) **remains geometrically similar.**
- We choose a velocity profile in the form

$$\frac{u}{U_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 \tag{30.2}$$

- In order to determine the constants $a_0, a_1, a_2 \ \ and \ \ a_3$ we shall prescribe the following boundary conditions

$$at \ \ y = 0, u = 0 \quad or \quad at \ \ \eta = 0, \frac{u}{U_\infty} = 0 \tag{30.3a}$$

$$at \ \ y = 0, \frac{\partial^2 u}{\partial y^2} = 0 \quad or \quad at \ \ \eta = 0, \frac{\partial^2}{\partial \eta^2}(u / U_\infty) = 0 \tag{30.3b}$$

- at

$$at \ \ y = \delta, u = U_\infty \quad or \quad at \ \ \eta = 1, \frac{u}{U_\infty} = 1 \tag{30.3c}$$

- at

$$at \ \ y = \delta, \frac{\partial u}{\partial y} = 0 \quad or \quad at \ \ \eta = 1, \frac{\partial (u / U_\infty)}{\partial \eta} = 0 \tag{30.3d}$$

-
- These requirements will yield $a_0 = 0, a_2 = 0, a_1 + a_3 = 1 \ \ and \ \ a_1 + 3a_3 = 0$ respectively

Finally, we obtain the following values for the coefficients in Eq. (30.2),

$a_0 = 0, a_1 = 3/2, a_2 = 0 \ \ and \ \ +a_3 = -1/2$ and the velocity profile becomes

$$\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \tag{30.4}$$

- For flow over a flat plate, $\frac{dp}{dx} = 0, \quad hence \ U_\infty \frac{dU_\infty}{dx} = 0$ and the governing Eq. (30.1) reduces to

$$\frac{d\delta^{**}}{dx} = \frac{\tau_w}{\rho U_\infty^2} \tag{30.5}$$

- Again from Eq. (29.8), the momentum thickness is

$$\delta^{**} = \int_0^{\infty} \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy \qquad or\ \ \delta^{**} = \int_0^{\delta} \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy$$

$$or\ \ \delta^{**} = \delta\int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1-\frac{3}{2}\eta+\frac{1}{2}\eta^3\right)d\eta$$

$$or\ \ \delta^{**} = \frac{39}{280}\delta$$

- The wall shear stress is given by

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg|_{y=0}$$

$$or\ \ \tau_w = \mu\left[\frac{\partial}{\delta\partial\eta}\left\{U_\infty\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\right\}\right]_{\eta=0}$$

$$or\ \ \tau_w = \frac{3\mu U_\infty}{2\delta}$$

- Substituting the values of δ^{**} and τ_w in Eq. (30.5) we get,

$$\frac{39}{280}\frac{d\delta}{dx} = \frac{3\mu U_\infty}{2\delta\rho U_\infty^2}$$

$$or\ \ \int \delta d\delta = \int \frac{140}{13}\frac{\mu}{\rho U_\infty}dx + C_1$$

$$or\ \ \frac{\delta^2}{2} = \frac{140}{13}\frac{\nu x}{U_\infty} + C_1 \qquad (30.6)$$

where C_1 is any arbitrary unknown constant.

- The condition at the leading edge ($at\ \ x=0, \delta=0$) yields $C_1 = 0$
  Finally we obtain,

$$\delta^2 = \frac{280}{13}\frac{\nu x}{U_\infty} \qquad (30.7)$$

$$or\ \delta = 4.64\sqrt{\frac{\nu x}{U_\infty}}$$

$$or\ \delta = \frac{4.64x}{\sqrt{\mathrm{Re}_x}} \qquad (30.8)$$

- This is the value of boundary layer thickness on a flat plate. Although, the method is an approximate one, the result is found to be reasonably accurate. The value is slightly lower than the exact solution of laminar flow over a flat plate . As such, **the accuracy depends on the order of the velocity profile.** We could have have used a fourth order polynomial instead --

$$\frac{u}{U_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 \qquad (30.9)$$

- In addition to the boundary conditions in Eq. (30.3), we shall require another boundary condition at

$$y = \delta, \frac{\partial^2 u}{\partial y^2} = 0 \;\; or\ at\ \ \eta = 1, \frac{\partial^2 (u/U_\infty)}{\partial \eta^2} = 0$$

- This yields the constants as $a_0 = 0, a_1 = 2, a_3 = -2\ \ and\ \ a_4 = 1$. Finally the velocity profile will be

$$\frac{u}{U_\infty} = 2\eta - 2\eta^3 + \eta^4$$

Subsequently, for a fourth order profile the growth of boundary layer is given by

$$\delta = \frac{5.83x}{\sqrt{\mathrm{Re}_x}} \qquad (30.10)$$

Integral Method For Non-Zero Pressure Gradient Flows

- A wide variety of "integral methods" in this category have been discussed by Rosenhead . The Thwaites method is found to be a very elegant method, which is an extension of the method due to Holstein and Bohlen . We shall discuss the **Holstein-Bohlen method** in this section**.**
- This is an **approximate method for solving boundary layer equations for two-dimensional generalized flow**. The integrated Eq. (29.14) for laminar flow with pressure gradient can be written as

$$\frac{d}{dx}\left[U^2 \delta^{**}\right] + \delta^* U \frac{dU}{dx} = \frac{\tau_\omega}{\rho}$$

or

$$U^2 \frac{d\delta^{**}}{dx} + \left(2\delta^{**} + \delta^*\right)U\frac{dU}{dx} = \frac{\tau_\omega}{\rho} \tag{30.11}$$

- The **velocity profile** at the boundary layer is considered to be a **fourth-order polynomial** in terms of the dimensionless distance $\eta = y/\delta$, and is expressed as

$$u/U = a\eta + b\eta^2 + c\eta^3 + d\eta^4$$

The **boundary conditions are**

$$\eta = 0 : u = 0, v = 0 \qquad \frac{v}{\delta^2}\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx}$$

$$\eta = 1 : u = U \qquad \frac{\partial u}{\partial \eta} = 0,\ \frac{\partial^2 u}{\partial \eta^2} = 0$$

- A dimensionless quantity, known as **shape factor** is introduced as

$$\lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} \tag{30.12}$$

- The following relations are obtained

$$a = 2 + \frac{\lambda}{6},\ b = -\frac{\lambda}{2},\ c = -2 + \frac{\lambda}{2},\ d = 1 - \frac{\lambda}{6}$$

- Now, the **velocity profile can be expressed as**

$$u/U = F(\eta) + \lambda G(\eta), \tag{30.13}$$

where

$$F(\eta) = 2\eta + 2\eta^3 + \eta^4,\ G(\eta) = \frac{1}{6}\eta(1-\eta)^3$$

- The shear stress $\tau_\omega = \mu(\partial u/\partial y)_{y=0}$ is given by

$$\frac{\tau_\omega \delta}{\mu U} = 2 + \frac{\lambda}{6} \tag{30.14}$$

- We use the following dimensionless parameters,

$$L = \frac{\tau_{\omega}\delta^{**}}{\mu U} = \frac{\delta^{**}}{\delta}\left(2 + \frac{\lambda}{6}\right) \qquad (30.15)$$

$$K = \frac{\left(\delta^{**}\right)^2}{\nu}\frac{dU}{dx} = \left(\frac{\delta^{**}}{\delta}\right)^2 \lambda \qquad (30.16)$$

$$H = \delta^{*} \big/ \delta^{**} \qquad (30.17)$$

- The integrated momentum Eq. (30.10) reduces to

$$U\frac{d\delta^{**}}{dx} + \delta^{**}(2 + H)\frac{dU}{dx} = \frac{\nu L}{\delta^{**}}$$

$$U\frac{d}{dx}\left[\frac{\left(\delta^{**}\right)^2}{\nu}\right] = 2\left[L - K(H + 2)\right] \qquad (30.18)$$

- The parameter ***L* is related to the skin friction**
- The parameter ***K* is linked to the pressure gradient.**
- If we take ***K* as** the **independent variable** . ***L* and *H*** can be shown to be the **functions of *K*** since

$$\frac{\delta^{*}}{\delta} = \int_0^1 \left[1 - F(\eta) - \lambda G(\eta)\right]d\eta = \frac{3}{10} - \frac{\lambda}{120} \qquad (30.19)$$

$$\frac{\delta^{*}}{\delta} = \int_0^1 \left(F(\eta) + \lambda G(\eta)\right)\left(1 - F(\eta) - \lambda G(\eta)\right)d\eta$$

$$= \frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} \qquad (30.20)$$

$$K = \frac{[\delta^{**}]^2}{\delta^2}\lambda = \lambda\left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right)^2 \qquad (30.21)$$

Therefore,

$$L = \left(2 + \frac{\lambda}{6}\right)\frac{\delta^{**}}{\delta} = \left(2 + \frac{\lambda}{6}\right)\left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right) = f_1(k)$$

$$H = \frac{\delta^{*}}{\delta^{**}} = \frac{(3/10) - (\lambda/120)}{(37/315) - (\lambda/945) - \left(\lambda^2/9072\right)} = f_2(k)$$

- The right-hand side of Eq. (30.18) is thus a function of *K* alone. Walz pointed out that this function can be approximated with a good degree of accuracy by a linear function of *K* so that

$$2[L - K(H-2)] = a - bK \quad [Walz's\ approximation]$$ **[Walz's approximation]**

- Equation (30.18) can now be written as

$$\frac{d}{dx}\left(\frac{U[\delta^{**}]^2}{\nu}\right) = a - (b-1)\frac{U[\delta^{**}]^2}{\nu}\frac{1}{U}\frac{dU}{dx}$$

Solution of this differential equation for the dependent variable $U[\delta^{**}]^2/\nu$ subject to the boundary condition *U = 0* when *x = 0* , gives

$$\frac{U[\delta^{**}]^2}{\nu} = \frac{a}{U^{b-1}}\int_0^x U^{b-1}dx$$

- **With *a* = 0.47 and *b* = 6. the approximation is particularly close between the stagnation point and the point of maximum velocity.**
- **Finally the value of the dependent variable is**

$$\left[\delta^{**}\right]^2 = \frac{0.47\nu}{U^6}\int_0^x U^5\,dx \qquad (30.22)$$

- By taking the limit of Eq. (30.22), according to L'Hopital's rule, it can be shown that

$$[\delta^{**}]^2\,|_{x=0} = 0.47\nu/6U'(0)$$

This corresponds to **K = 0.0783.**

- Note that $\left|\delta^{**}\right|$ is not equal to zero at the stagnation point. If $\left([\delta^{**}]^2/\nu\right)$ is determined from Eq. (30.22), *K(x)* can be obtained from Eq. (30.16).
- Table 30.1 gives the necessary parameters for obtaining results, such as velocity profile and shear stress τ_ω The approximate method can be applied successfully to a wide range of problems.

Table 30.1 Auxiliary functions after Holstein and Bohlen

λ	K	$f_1(K)$	$f_2(K)$
12	0.0948	2.250	0.356
10	0.0919	2.260	0.351
8	0.0831	2.289	0.340
7.6	0.0807	2.297	0.337
7.2	0.0781	2.305	0.333

7.0	0.0767	2.309	0.331
6.6	0.0737	2.318	0.328
6.2	0.0706	2.328	0.324
5.0	0.0599	2.361	0.310
3.0	0.0385	2.427	0.283
1.0	0.0135	2.508	0.252
0	0	2.554	0.235
-1	-0.0140	2.604	0.217
-3	-0.0429	2.716	0.179
-5	-0.0720	2.847	0.140
-7	-0.0999	2.999	0.100
-9	-0.1254	3.176	0.059
-11	-0.1474	3.383	0.019
-12	-0.1567	3.500	0

λ	K	$f_1(K)$	$f_2(K)$
0	0	0	0
0.2	0.00664	0.006641	0.006641
0.4	0.02656	0.13277	0.13277
0.8	0.10611	0.26471	0.26471
1.2	0.23795	0.39378	0.39378
1.6	0.42032	0.51676	0.51676
2.0	0.65003	0.62977	0.62977
2.4	0.92230	0.72899	0.72899
2.8	1.23099	0.81152	0.81152
3.2	1.56911	0.87609	0.87609
3.6	1.92954	0.92333	0.92333
4.0	2.30576	0.95552	0.95552
4.4	2.69238	0.97587	0.97587
4.8	3.08534	0.98779	0.98779
5.0	3.28329	0.99155	0.99155
8.8	7.07923	1.00000	1.00000

- As mentioned earlier, K and λ are related to the pressure gradient and the shape factor.
- Introduction of K and λ in the integral analysis enables extension of Karman-Pohlhausen method for solving flows over curved geometry. However, the **analysis is not valid for the geometries, where** $\lambda < -12$ **and** $\lambda > +12$

Point of Seperation

For point of seperation, $\tau_\omega = 0$

$$\Rightarrow \frac{\mu U}{\delta}\left(2 + \frac{\lambda}{6}\right)$$

or, $$2 + \frac{\lambda}{6} = 0$$

or, $$\lambda = -12$$

Entry Flow In A Duct -

- Growth of boundary layer has a remarkable influence on flow through a constant area duct or pipe.
  Consider a flow entering a pipe with uniform velocity.
  1. The boundary layer starts growing on the wall at the entrance of the pipe.
  2. Gradually it becomes thicker in the downstream.
  3. The flow becomes fully developed when the boundary layers from the wall meet at the axis of the pipe.
- The velocity profile is **nearly rectangular** at the entrance and it gradually changes to a parabolic profile at the fully developed region.
- Before the boundary layers from the periphery meet at the axis, there prevails a core region which is uninfluenced by viscosity.
- Since the volume-flow must be same for every section and the boundary-layer thickness increases in the flow direction, the inviscid core accelerates, and there is a corresponding fall in pressure.
- **Entrance length** : It can be shown that for laminar incompressible flows, the velocity profile approaches the parabolic profile through a distance L_e from the entry of the pipe. This is known as entrance length and is given by

$$\frac{Le}{D} \approx 0.05\mathrm{Re}, \qquad where \ \ \mathrm{Re} = \frac{U_{av}D}{\nu}$$

For a Reynolds number of 2000, this distance, the entrance length is about 100 pipe-diameters. For turbulent flows, the entrance region is shorter, since the turbulent boundary layer grows faster.

- **At the entrance region**,

1. The velocity gradient is steeper at the wall, causing a higher value of shear stress as compared to a developed flow.

2. Momentum flux across any section is higher than that typically at the inlet due to the change in shape of the velocity profile.
3. Arising out of these, an additional pressure drop is brought about at the entrance region as compared to the pressure drop in the fully developed region.

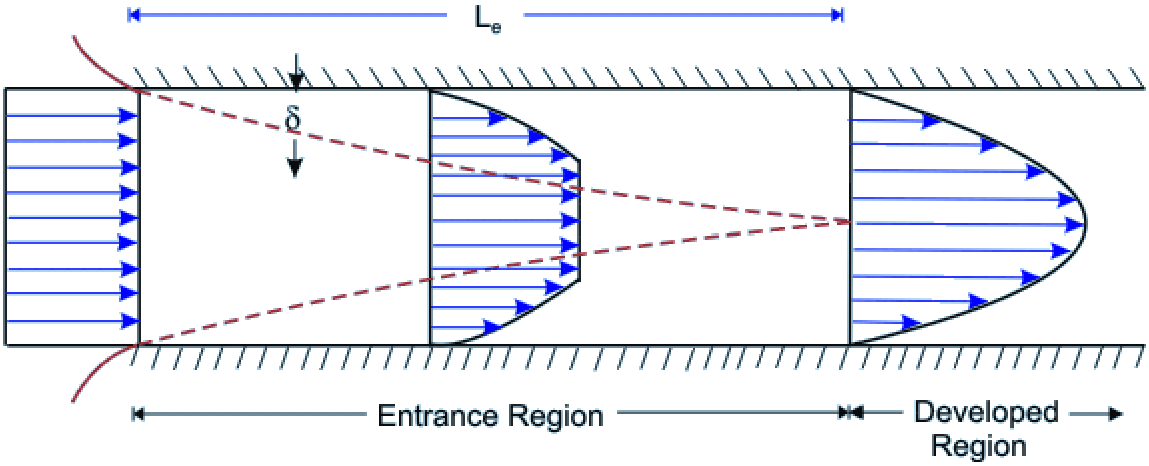


Fig. 31.1 Development of boundary layer in the entrance region of a duct

Control Of Boundary Layer Separation -

- The total drag on a body is attributed to form drag and skin friction drag. In some flow configurations, the contribution of form drag becomes significant.
- In order **to reduce the form drag, the boundary layer separation** should be **prevented or delayed** so that **better pressure recovery takes place** and the form drag is reduced considerably. There are some popular methods for this purpose which are stated as follows.
    i. By giving the profile of the body a streamlined shape(as shown in *Fig. 31.2*).
        1. This has an elongated shape in the rear part to reduce the magnitude of the pressure gradient.
        2. The optimum contour for a streamlined body is the one for which the wake zone is very narrow and the form drag is minimum.

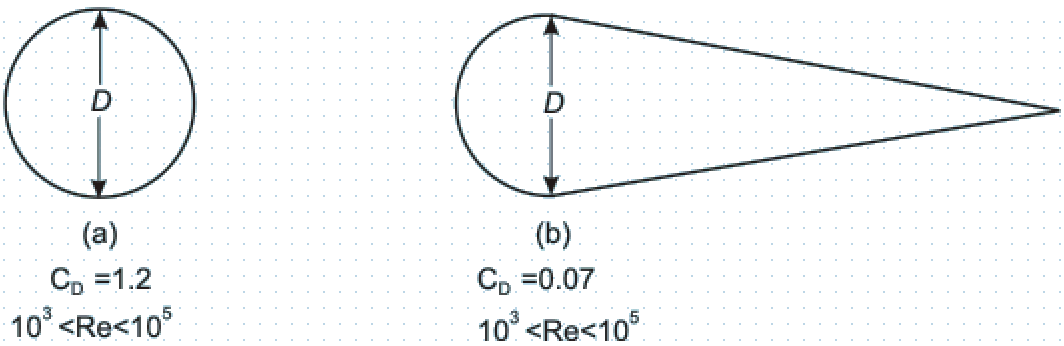


Fig. 31.2 Reduction of drag coefficient (C_D) by giving the profile a streamlined shape

ii. **The injection of fluid through porous wall can also control the boundary layer separation. This is generally accomplished by** blowing high energy fluid particles tangentially from the location where separation would have taken place otherwise. This is shown in Fig. 31.3.

1. The **injection of fluid promotes turbulence**
2. This **increases skin friction.** But the **form drag is reduced** considerably due to suppression of flow separation
3. The reduction in form drag is quite significant and **increase in skin friction drag can be ignored.**

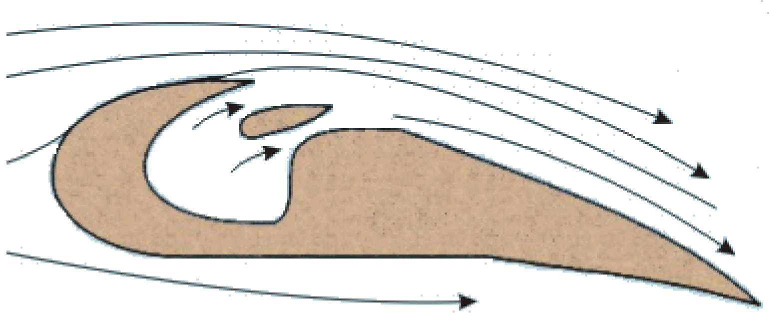


Fig. 31.3 Boundary layer control by blowing

Mechanisms of Boundary Layer Transition

- One of the interesting problems in fluid mechanics is the physical mechanism of transition from laminar to turbulent flow. The problem evolves about the generation of both steady and unsteady vorticity near a body, its subsequent molecular diffusion, its kinematic and dynamic convection and redistribution downstream, and the resulting feedback on the velocity and pressure fields near the body. We can perhaps realise the complexity of the transition problem by examining the behaviour of a real flow past a cylinder.

  Figure 31.4 (a) shows the flow past a cylinder for a very low **Reynolds number** (~ 1). The **flow smoothly divides and reunites around the cylinder.**

- At a **Reynolds number of about 4**, the **flow (boundary layer) separates in the downstream** and the wake is formed by **two symmetric eddies** . The eddies remain steady and **symmetrical** but grow in size **up to a Reynolds number of about 40** as shown in Fig. 31.4(b).

- At a **Reynolds number above 40** , **oscillation in the wake** induces **asymmetry** and finally the wake starts **shedding vortices** into the stream. This situation is termed as **onset of periodicity** as shown in Fig. 31.4(c) and the wake keeps on undulating **up to a Reynolds number of 90** .

- At a **Reynolds number above 90** , the **eddies are shed alternately from a top and bottom** of the cylinder and the regular pattern of **alternately shed clockwise and counterclockwise vortices form *Von Karman vortex street*** as in Fig. 31.4(d).

- Periodicity is eventually induced in the flow field with the vortex-shedding phenomenon.

- The periodicity is characterised by the **frequency of vortex shedding** f

- **In non-dimensional form, the vortex shedding frequency is expressed as** fD/U_{ref} known as the **Strouhal number** named after V. Strouhal, a German physicist who experimented with wires singing in the wind. The Strouhal number shows a slight but continuous variation with Reynolds number around a value of 0.21. The boundary layer on the cylinder surface remains laminar and separation takes placeat about 81^0 from the forward stagnation point.

- **At about Re = 500** , multiple frequencies start showing up and the **wake tends to become Chaotic.**

- As the Reynolds number becomes higher, the boundary layer around the cylinder tends to become turbulent. The wake, of course, shows fully turbulent characters (Fig31.4 (e)).

- For larger Reynolds numbers, the boundary layer becomes turbulent. A turbulent boundary layer offers greater resistance to seperation than a laminar boundary layer. As a consequence the seperation point moves downstream and the seperation angle is delayed to 110^0 from the forward stagnation point (Fig 31.4 (f)).

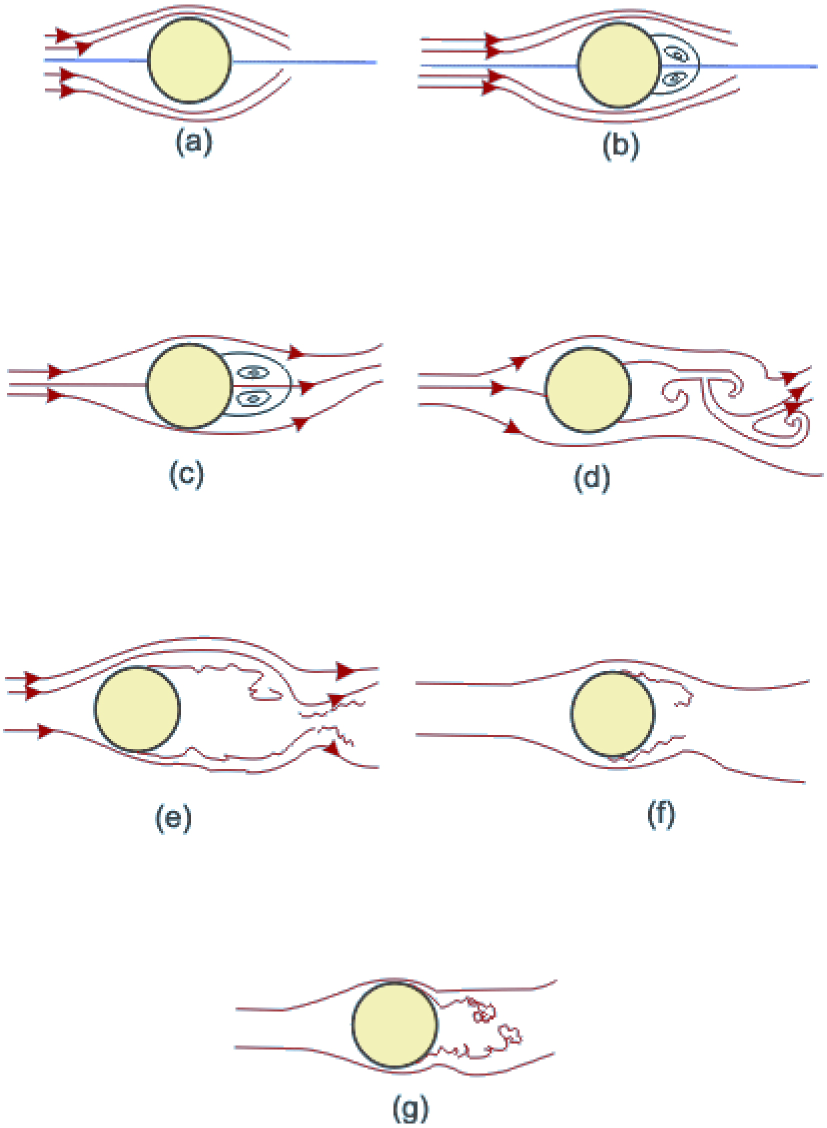


Fig. 31.4 Influence of Reynolds number on wake-zone aerodynamics

- Experimental flow visualizations past a circular cylinder are shown in *Figure 31.5 (a) and (b)*

Fig 31.5 (a) Flow Past a Cylinder at Re=2000 [Photograph courtesy Werle and Gallon (ONERA)]

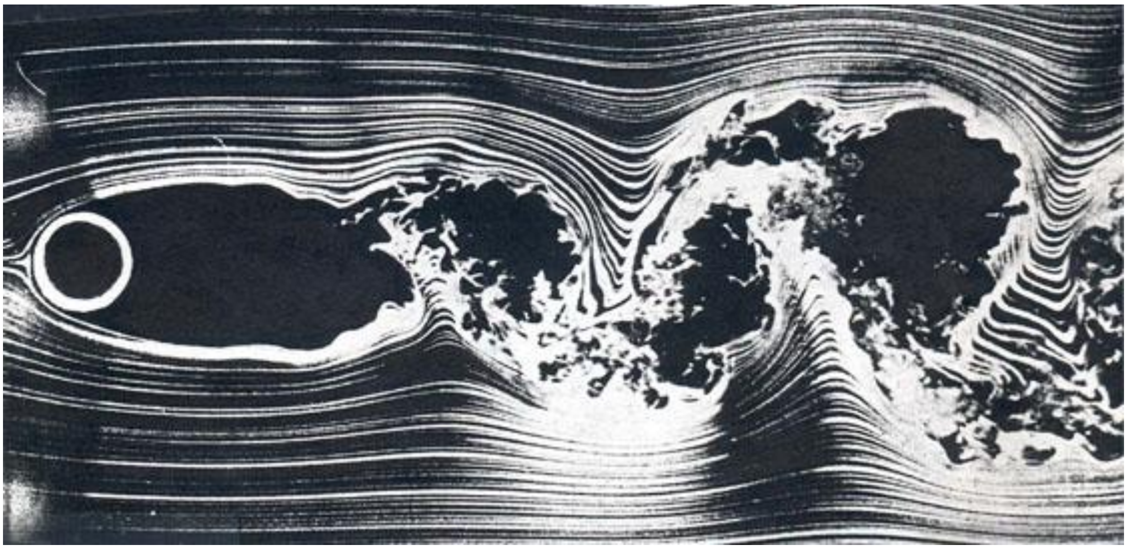


Fig 31.5 (b) Flow Past a Cylinder at Re=10000 [Photograph courtesy Thomas Corke and Hasan Najib (Illinois Institute of Technology, Chicago)]

- A very interesting sequence of events begins to develop when the Reynolds number is increased beyond 40, at which point the wake behind the cylinder becomes unstable. Photographs show that the wake develops a slow oscillation in which the velocity is periodic in time and downstream distance. The amplitude of the oscillation increases downstream. The oscillating wake rolls up into two staggered rows of vortices with opposite sense of rotation.

- Karman investigated the phenomenon and concluded that a nonstaggered row of vortices is unstable, and a staggered row is stable only if the ratio of lateral distance between the vortices to their longitudinal distance is 0.28. Because of the similarity of the wake with footprints in a street, the staggered row of vortices behind a blue body is called a **Karman Vortex Street** . The vortices move downstream at a speed smaller than the upstream velocity *U*.

- In the range 40 < Re < 80, the vortex street does not interact with the pair of attached vortices. As Re is increased beyond 80 the vortex street forms closer to the cylinder, and the attached eddies themselves begin to oscillate. Finally the attached eddies periodically break off alternately from the two sides of the cylinder.

- While an eddy on one side is shed, that on the other side forms, resulting in an unsteady flow near the cylinder. As vortices of opposite circulations are shed off alternately from the two sides, the circulation around the cylinder changes sign, resulting in an oscillating "lift" or lateral force. If the frequency of vortex shedding is close to the natural frequency of some mode of vibration of the cylinder body, then an appreciable lateral vibration culminates.

- An understanding of the transitional flow processes will help in practical problems either by improving procedures for predicting positions or for determining methods of advancing or retarding the transition position.

- The **critical value at which the transition occurs in pipe flow is** $Re_{cr} = 2300$. The actual value depends upon the disturbance in flow. Some experiments have shown the critical Reynolds number to reach as high as 10,000. The precise upper bound is not known, but the lower bound appears to be $Re_{cr} = 2300$. **Below this value, the flow remains laminar even when subjected to strong disturbances.**

- In the case of flow through a channel, $2300 \le Re_{cr} \le 2600$, the flow alternates randomly between laminar and partially turbulent. **Near the centerline, the flow is more laminar than turbulent, whereas near the wall, the flow is more turbulent than laminar.** For flow over a flat plate, turbulent regime is observed between Reynolds numbers $U_\infty x/\nu$ of 3.5×10^5 and 10^6.

Several Events Of Transition -

Transitional flow consists of several events as shown in *Fig. 31.8*. Let us consider the events one after another.

1. Region of instability of small wavy disturbances-

Consider a laminar flow over a flat plate aligned with the flow direction (*Fig. 31.8*).

- In the presence of an adverse pressure gradient, at a high Reynolds number (water velocity approximately 9-cm/sec), **two-dimensional waves appear.**
- **These waves are called *Tollmien-Schlichting* wave(** In 1929, Tollmien and Schlichting predicted that the waves would form and grow in the boundary layer).
- These waves can be made visible by a method known as tellurium method.

2. Three-dimensional waves and vortex formation-

- Disturbances in the free stream or **oscillations in the upstream boundary layer can generate wave growth,** which has a variation **in the span wise direction.**
- This leads an initially two-dimensional wave to **a three-dimensional form.**
- In many such transitional flows, periodicity is observed in the span wise direction.
- This is accompanied by the appearance of vortices whose axes lie in the direction of flow.

3. Peak-Valley development with streamwise vortices-

- As the three-dimensional wave propagates downstream, the **boundary layer flow develops into a complex stream wise vortex system.**
- Within this vortex system, **at some spanwise location, the velocities fluctuate violently** .
- These locations are **called peaks and the neighbouring locations of the peaks are valleys (***Fig. 31.9***).**

4. Vorticity concentration and shear layer development-

At the spanwise locations corresponding to the peak, the instantaneous streamwise velocity profiles demonstrate the following

- Often, an inflexion is observed on the velocity profile.
- The inflectional profile appears and disappears once after each cycle of the basic wave.

5. Breakdown-

The instantaneous velocity profiles produce high shear in the outer region of the boundary layer.

- The velocity fluctuations develop from the shear layer at a higher frequency than that of the basic wave.
- These velocity fluctuations have a strong ability to amplify any slight three-dimensionality, which is already present in the flow field.
- As a result, **a staggered vortex pattern evolves with the streamwise wavelength twice the wavelength of Tollmien-Schlichting wavelength .**
- The span wise wavelength of these structures is about one-half of the stream wise value.
- The high frequency fluctuations are referred as **hairpin eddies.**

This is known as breakdown.

6. Turbulent-spot development-

- The hairpin-eddies travel at a speed grater than that of the basic (primary) waves.
- As they travel downstream, eddies spread in the spanwise direction and towards the wall.
- The vortices begin a cascading breakdown into smaller vortices.
- In such a fluctuating state, intense local changes occur at random locations in the shear layer near the wall in the form of turbulent spots.

- Each spot grows almost linearly with the downstream distance.

The creation of spots is considered as the main event of transition .

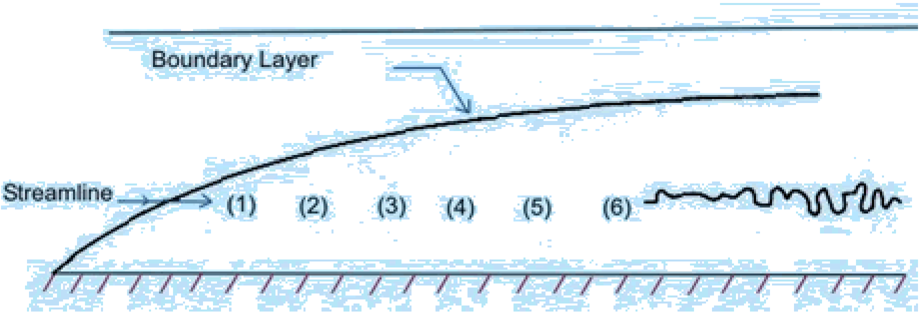


Fig. 31.8 Sequence of event involved in transition

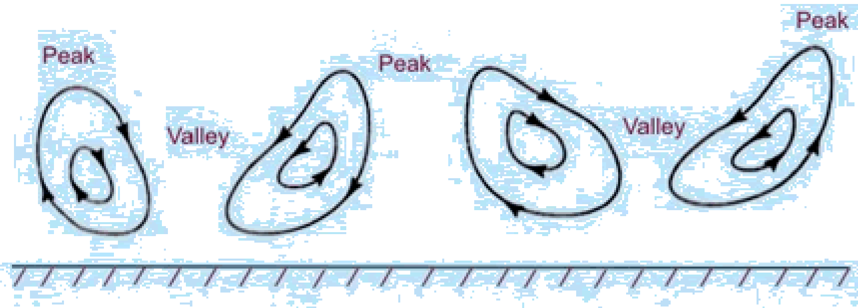


Fig. 31.9 Cross-stream view of the streamwise vortex system

Exercise Problems - Chapter 9

1.Two students are asked to solve the Blasius flow over a flat plate to determine the variation of boundary layer thickness as a function of the Reynolds number. One student solves the problem by similarity method and arrives at the result $\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$. The other student chooses to solve the problem by using the momentum-integer equation and Karman-Pohlhausen method and funds that $\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$.
Which of the two results is expected to be closer to the experimental results and why?

2. A scientist claims that a highly viscous flow around a body can generate the same flow patterns as the flow of an inviscid and incompressible fluid around that body. According to our understanding, the Reynolds number for the first flow is very small, while the Reynolds number for the second flow can be taken to be ∞ (infinity). Do you think it is possible to get the same flow patterns for the two extreme values of Reynolds number? Please use mathematical analysis to prove or disprove the scientist's claim.

3. In boundary layer theory, a boundary layer can be characterized by any of the following quantities (i) Boundary layer thickness (ii) Displacement thickness (iii) Momentum thickness.

How do these quantities differ in their physical as well as mathematical definitions? For the flow over a flat plate, which of these is expected to have the highest value at a given location on the wall, and which the lowest?

4. What do you mean by the "point of separation" of a boundary layer? How will the velocity gradient $\frac{\partial u}{\partial y}$ and the second gradient $\frac{\partial^2 u}{\partial y^2}$ .Vary within the boundary layer at the point of separation? Please show the variation graphically. Here u is the velocity along the wall and y is the co-ordinate perpendicular to the wall.

5. Reduce the Prandtl's boundary layer equations to a simpler form than that given by equations (28.10) - (28.12) for -

(a) Flow over a flat plate.
(b) The case $\tau_{yx} = C_1$ (a constant)
(c) The case where velocity (v) is directly proportional to kinematic viscosity (ν)
(d) Also solve the Prandtl's boundary layer equations for v = ν assuming pressure gradient $\frac{\partial p}{\partial x}$ =0.

6. Water of kinematic viscosity (ν) equal to 9.29×10^{-7} m^2 /s is flowing steadily over a smooth flat plate at zero angle of incidence, with a velocity of 1.524 m/s. The length of the plate is 0.3048 m. Calculate-

(a) The thickness of the boundary layer at 0.1524 m from the leading edge.
(b) Boundary layer rate of growth at 0.1524 m from the leading edge.
(c) Total drag coefficient on the plate.

7. Use the Prandtl's boundary layer equations and show that the velocity profile for a laminar flow past a flat plate has an infinite radius of curvature on the surface of the plate.

8. Air is flowing over a smooth flat plate at a velocity of 4.39 m/s. The density of air is 1.031 Kg/m^3 and the kinematic viscosity is 1.34×10^{-5} m^2 /s. The length of the plate is 12.2 m in the direction of the flow. Find-

(a) The boundary layer thickness at 15.24 cm from the leading edge.
(b) The drag coefficient (C_{Df}).

9. Show that the shape factor (H) has the value $\approx$ 2.6 for the boundary layer flow over a flat plate. Also calculate the position where the flow is critical for flow velocity of 3.048 m/s and kinematic viscosity 9.29×10^{-7} m^2 /s.
Given that at the critical location Reynold's Number (based on distance from the leading edge surface) is related to shape factor (H) by-

$$\log(R_{critical}) = H.$$

10. Determine the distance downstream from the bow of a ship moving at 3.9 m/s relative to still water at which the boundary layer will become turbulent. Also find the boundary layer

thickness and total friction drag coefficient for this portion of the surface of the ship. Given the kinematic viscosity = 1.124×10^{-6} m^2 /s.

Recap

In this course you have learnt the following

- The boundary layer is the thin layer of fluid adjacent to the solid surface. Phenomenologically, the effect of viscosity is very prominent within this layer.
- The main-stream velocity undergoes a change from zero at the solid surface to the full magnitude through the boundary layer.Effectively, the boundary layer theory is a complement to the inviscid flow theory.
- The governing equation for the boundary layer can be obtained through correct reduction of the *Navier-Stokes equations* within the thin layer referred above. There is no variation in pressure in y direction within the boundary layer.
- The pressure is impressed on the boundary layer by the outer inviscid flow which can be calculated using **Bernoulli's equation.**
- The boundary layer equation is a second order non-linear partial differential equation. The exact solution of this equation is known as **similarity solution**. For the flow over a flat plate, the similarity solution is often referred to as **Blasius solution.** Complete analytical treatment of this solution is beyond the scope of this text. However, the momentum integral equation can be derived from the boundary layer equation which is amenable to analytical treatment.
- The solutions of the momentum integral equation are called approx imate solutions of the boundary layer equation.
- The boundary layer equations are valid up to the point of separation. At the point of separation, the flow gets detached from the solid surface due to excessive adverse pressure gradient.
- Beyond the point of separation, the flow reversal produces eddies. During flow past bluff-bodies, the desired pressure recovery does not take place in a separated flow and the situation gives rise to **pressure drag** or **form drag.**

Introduction

- The turbulent motion is an **irregular** motion.
- Turbulent fluid motion can be considered as an irregular condition of flow in which various quantities (such as velocity components and pressure) show a **random variation with time and space** in such a way that the statistical average of those quantities can be quantitatively expressed.

- It is postulated that the fluctuations inherently come from **disturbances** (such as roughness of a solid surface) and they may be either dampened out due to viscous damping or may grow by drawing energy from the free stream.

- At a **Reynolds number less than the critical**, the kinetic energy of flow is not enough to sustain the random fluctuations against the viscous damping and in such cases **laminar flow** continues to exist.

- At somewhat **higher Reynolds number** than the critical Reynolds number, the kinetic energy of flow supports the growth of fluctuations and **transition to turbulence** takes place.

Characteristics Of Turbulent Flow

- The most important characteristic of turbulent motion is the fact that **velocity and pressure** at a point **fluctuate with time** in a random manner.

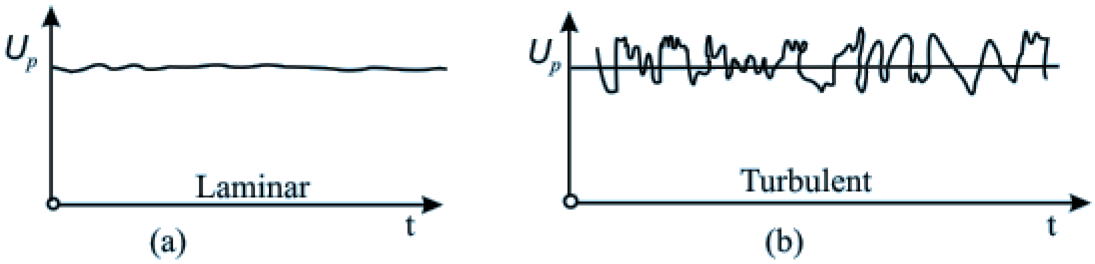


Fig. 32.1 Variation of horizontal components of velocity for laminar and turbulent flows at a point P

- The mixing in turbulent flow is more due to these fluctuations. As a result we can see more uniform velocity distributions in turbulent pipe flows as compared to the laminar flows .

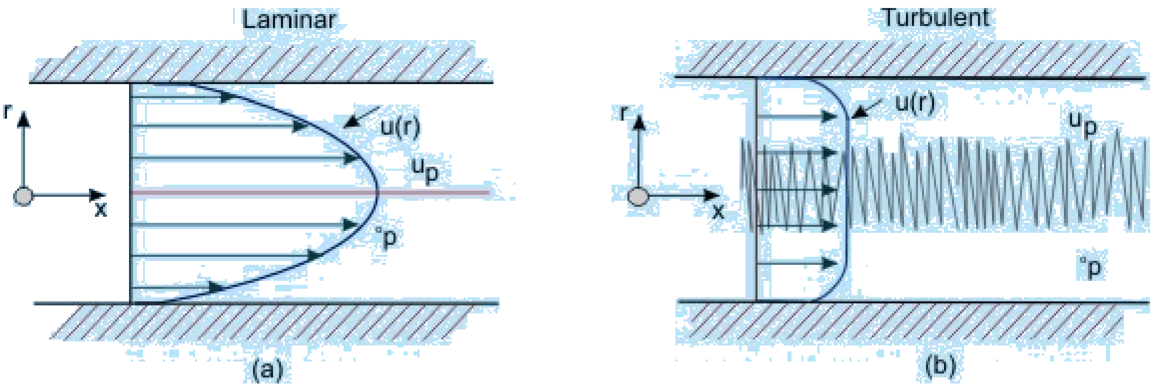


Fig. 32.2 Comparison of velocity profiles in a pipe for (a) laminar and (b) turbulent flows

- **Turbulence can be generated by -**
  1. frictional forces at the confining solid walls
  2. the flow of layers of fluids with different velocities over one another

The turbulence generated in these two ways are considered to be different.

Turbulence generated and continuously affected by fixed walls is designated as **wall turbulence** , and turbulence generated by two adjacent layers of fluid in absence of walls is termed as **free turbulence** . One of the effects of viscosity on turbulence is to make the flow more homogeneous and less dependent on direction.

- Turbulence can be categorised as below -
    - **Homogeneous Turbulence**: Turbulence has the same structure quantitatively in all parts of the flow field.
    - **Isotropic Turbulence**: The statistical features have no directional preference and perfect disorder persists.
    - **Anisotropic Turbulence**: The statistical features have directional preference and the mean velocity has a gradient.

- **Homogeneous Turbulence** : The term homogeneous turbulence implies that the velocity fluctuations in the system are random but the average turbulent characteristics are independent of the position in the fluid, i.e., invariant to axis translation.

Consider the root mean square velocity fluctuations

$$u' = \sqrt{\overline{u^2}}\ , \quad v' = \sqrt{\overline{v^2}}\ , \quad w' = \sqrt{\overline{w^2}}$$

In homogeneous turbulence, the rms values of u', v' and w' can all be different, but each value must be constant over the entire turbulent field. Note that even if the rms fluctuation of any component, say u' s are constant over the entire field the instantaneous values of u necessarily differ from point to point at any instant.

- **Isotropic Turbulence**: The velocity fluctuations are independent of the axis of reference, i.e. invariant to axis rotation and reflection. <u>Isotropic turbulence is by its definition always homogeneous</u> . In such a situation, the gradient of the mean velocity does not exist, the mean velocity is either zero or constant throughout.

In isotropic turbulence fluctuations are independent of the direction of reference and

$$\sqrt{\overline{u^2}} = \sqrt{\overline{v^2}} = \sqrt{\overline{w^2}} \quad \text{or} \quad u' = v' = w'$$

It is re-emphasised that even if the rms fluctuations at any point are same, their instantaneous values necessarily differ from each other at any instant.

- **Turbulent flow is diffusive and dissipative** . In general, turbulence brings about better mixing of a fluid and produces an additional diffusive effect. Such a diffusion is termed as "Eddy-diffusion ".(Note that this is different from molecular diffusion)
  At a large Reynolds number there exists a continuous transport of energy from the

free stream to the large eddies. Then, from the large eddies smaller eddies are continuously formed. Near the wall smallest eddies destroy themselves in dissipating energy, i.e., converting kinetic energy of the eddies into intermolecular energy.

Laminar-Turbulent Transition

- For a turbulent flow over a flat plate,

Boundary layer starts as laminar flow at leading edge $\xrightarrow{\text{subsequently}}$ Flow turns into transition flow $\xrightarrow[\text{thereafter}]{\text{shortly}}$ turns into turbulent flow

- The turbulent boundary layer continues to grow in thickness, with a small region below it called a **viscous sublayer**. In this sub layer, the flow is well behaved,just as the laminar boundary layer (Fig. 32.3)

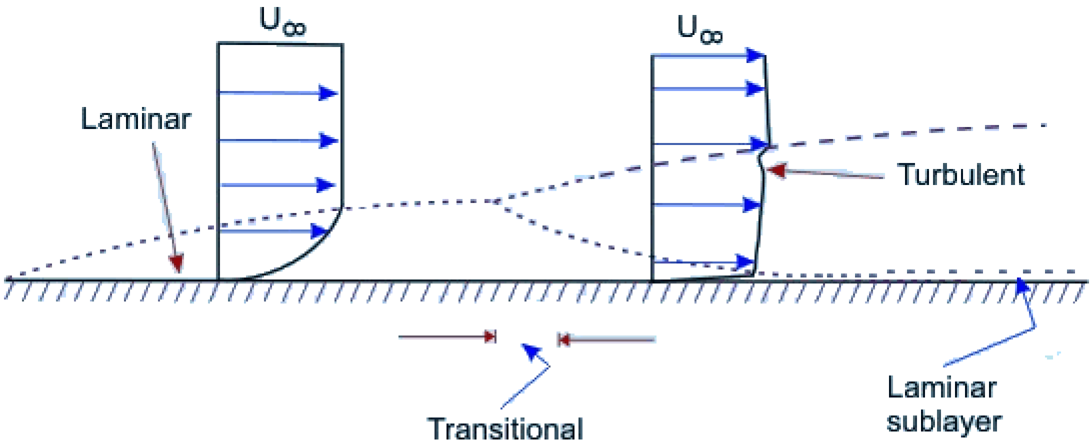


Fig. 32.3 Laminar - turbulent transition

Illustration

- Observe that at a certain axial location, the laminar boundary layer tends to become unstable. Physically this means that the disturbances in the flow grow in amplitude at this location.

Free stream turbulence, wall roughness and acoustic signals may be among the sources of such disturbances. **Transition to turbulent flow is thus initiated with the instability in laminar flow**

- The possibility of instability in boundary layer was felt by **Prandtl** as early as 1912.The theoretical analysis of **Tollmien and Schlichting** showed that unstable waves could exist if the **Reynolds number was 575**.

The Reynolds number was defined as

$$\mathrm{Re} = U_\infty \delta^* / \nu$$

where U_∞ is the free stream velocity , δ^* is the displacement thickness and ν is the kinematic viscosity .

- **Taylor** developed an alternate theory, which assumed that the transition is caused by a momentary separation at the boundary layer associated with the free stream turbulence.
  In a pipe flow the initiation of turbulence is usually observed at **Reynolds numbers ($U_\infty D/\nu$)in the range of 2000 to 2700**.

The development starts with a laminar profile, undergoes a transition, changes over to turbulent profile and then stays turbulent thereafter (Fig. 32.4). The length of development is of the order of 25 to 40 diameters of the pipe.

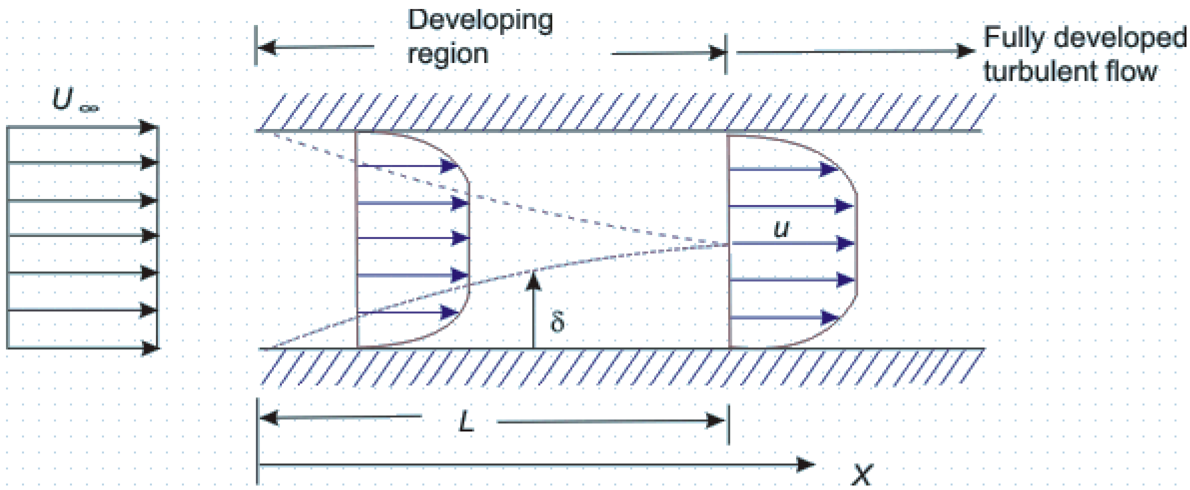


Fig. 32.4 Development of turbulent flow in a circular duct

Correlation Functions

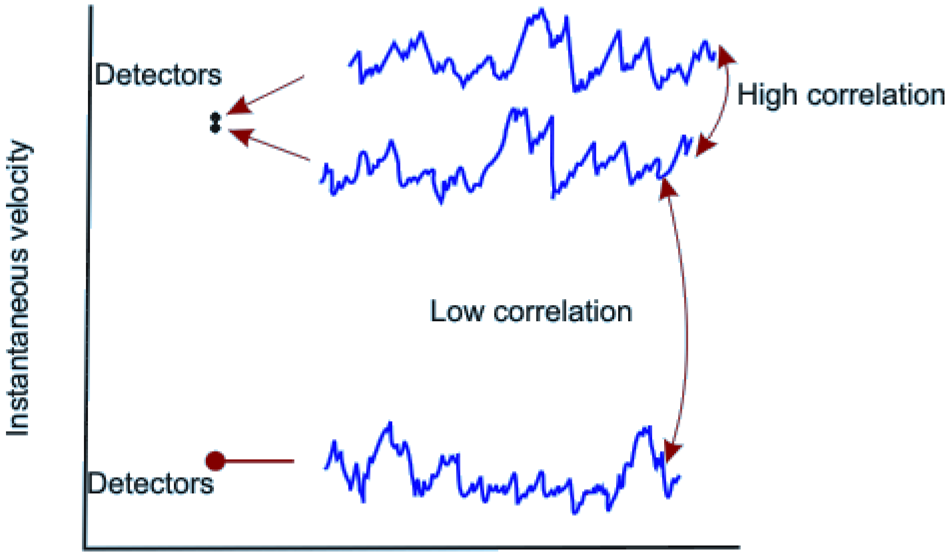


Fig 32.5 Velocity Correlation

- A statistical correlation can be applied to fluctuating velocity terms in turbulence. Turbulent motion is by definition eddying motion. Not withstanding the circulation strength of the individual eddies, a high degree of correlation exists between the velocities at two points in space, if the distance between the points is smaller than the diameter of the eddy. Conversely, if the points are so far apart that the space, in between, corresponds to many eddy diameters (Figure 32.5), little correlation can be expected.

- Consider a statistical property of a random variable (velocity) at two points separated by a distance r. An Eulerian correlation tensor (nine terms) at the two points can be defined by

$$Q = \overline{u(x)u(x+r)}$$

In other words, the dependence between the two velocities at two points is measured by the correlations, i.e. the time averages of the products of the quantities measured at two points. The correlation of the u'components of the turbulent velocity of these two points is defined as

$$\overline{u'(x)u'(x+r)}$$

It is conventional to work with the non-dimensional form of the correlation, such as

$$R(r) = \frac{\overline{u'(x)u'(x+r)}}{\left(\overline{u'^2}(x)\overline{u'^2}(x+r)\right)^{1/2}}$$

A value of **R(r) of unity signifies a perfect correlation** of the two quantities involved and their motion is in phase**.** Negative value of the correlation function implies that the time averages of the velocities in the two correlated points have different signs. Figure 32.6 shows typical variations of the correlation R with increasing separation r .

The positive correlation indicates that the fluid can be modelled as travelling in lumps. Since swirling motion is an essential feature of turbulent motion, these lumps are viewed as eddies of various sizes. The correlation R(r) is a measure of the strength of the eddies of size larger than r. Essentially the velocities at two points are correlated if they are located on the same eddy

- To describe the evolution of a fluctuating function *u'(t)*, we need to know the manner in which the value of *u'* at different times are related. For this purpose the correlation function

$$R(\tau) = \frac{\overline{u'(t)u'(t+\tau)}}{u'^2}$$

between the values of *u'* at different times is chosen and is called **autocorrelation function.**

- The correlation studies reveal that the turbulent motion is composed of eddies which are convected by the mean motion . The eddies have a wide range variation in their size. The size of the large eddies is comparable with the dimensions of the neighbouring objects or the dimensions of the flow passage.

The size of the smallest eddies can be of the order of 1 mm or less. However, the smallest eddies are much larger than the molecular mean free paths and the turbulent motion does obey the principles of continuum mechanics.

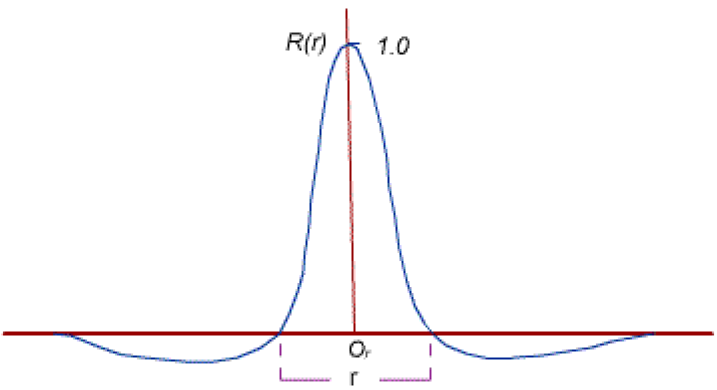


Fig 32.6 Variation of R with the distance of separation, r

Reynolds decomposition of turbulent flow :

- **The Experiment:** In 1883, O. Reynolds conducted experiments with pipe flow by feeding into the stream a thin thread of liquid dye. For low Reynolds numbers, the dye traced a straight line and did not disperse. With increasing velocity, the dye thread got mixed in all directions and the flowing fluid appeared to be uniformly colored in the downstream flow.

The Inference: It was conjectured that on the main motion in the direction of the pipe axis, there existed a superimposed motion all along the main motion at right angles to it. The superimposed motion causes exchange of momentum in transverse direction and the velocity distribution over the cross-section is more uniform than in laminar flow. This description of turbulent flow which consists of superimposed streaming and fluctuating (eddying) motion is well known as **Reynolds decomposition of turbulent flow.**

- Here, we shall discuss different descriptions of mean motion. Generally, for Eulerian velocity *u* , the following two methods of averaging could be obtained.

  (i) Time average for a stationary turbulence:

$$\overline{u}^{t}(x_0) = \lim_{t_1 \to \infty} \frac{1}{2t_1} \int_{-t_1}^{t_1} u(x_0, t)dt$$

(ii) Space average for a homogeneous turbulence:

$$\overline{u}^{s}(t_0) = \lim_{x \to \infty} \frac{1}{2x} \int_{-x}^{x} u(x, t_0)dx$$

For a stationary and homogeneous turbulence, it is assumed that the two averages lead to the same result: $\overline{u}^t = \overline{u}^s$ and the assumption is known as the **ergodic hypothesis**.

- In our analysis, average of any quantity will be evaluated as a *time average* . Take a finite time interval t_1. This interval must be larger than the time scale of turbulence. Needless to say that it must be small compared with the period t_2 of any slow variation (such as periodicity of the mean flow) in the flow field that we do not consider to be chaotic or turbulent **.**

Thus, for a parallel flow, it can be written that the axial velocity component is

$$u(y,t) = \overline{u}(y) + u'(\Gamma, t) \qquad (32.1)$$

As such, the time mean component $\overline{u}(y)$ determines whether the turbulent motion is steady or not. The symbol Γ signifies any of the space variables.

- While the motion described by Fig.32.6(a) is for a turbulent flow with steady mean velocity the Fig.32.6(b) shows an example of turbulent flow with unsteady mean velocity. The time period of the high frequency fluctuating component is t_1 whereas the time period for the unsteady mean motion is t_2 and for obvious reason $t_2 >> t_1$. Even if the bulk motion is parallel, the fluctuation *u* ' being random varies in all directions.
- The continuity equation, gives us

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Invoking Eq.(32.1) in the above expression, we get

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (32.2)$$

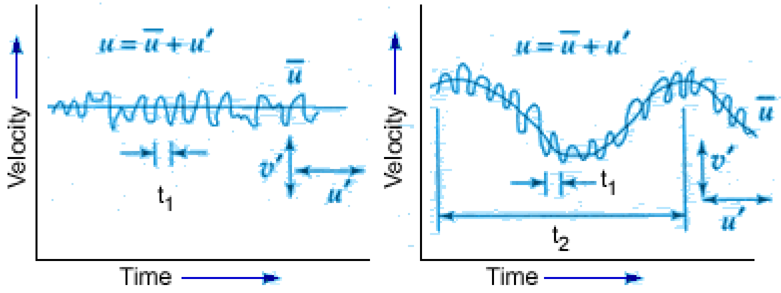


Fig 32.6 Steady and unsteady mean motions in a turbulent flow

Since $\frac{\partial u'}{\partial x} \neq 0$, Eq.(32.2) depicts that *y* and *z* components of velocity exist even for the parallel flow if the flow is turbulent. We have-

$$\begin{aligned} u(y,t) &= u(y) + u'(\Gamma,t) \\ v &= 0 + v'(\Gamma,t) \\ w &= 0 + w'(\Gamma,t) \end{aligned} \qquad (32.3)$$

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- However, the fluctuating components do not bring about the bulk displacement of a fluid element. The instantaneous displacement is $u'dt$**,** and that is not responsible for the bulk motion**.** We can conclude from the above

$$\int_{-T}^{T} u'dt = 0 \quad (t_1 < T \le t_2)$$

Due to the interaction of fluctuating components, macroscopic momentum transport takes place. Therefore, interaction effect between two fluctuating components over a long period is non-zero and this can be expressed as

$$\int_{-T}^{T} u'v'dt \neq 0$$

Taking time average of these two integrals and write

$$\bar{u}' = \frac{1}{2T}\int_{-T}^{T} u'\,dt = 0 \qquad (32.4a)$$

and

$$\overline{u'v'} = \frac{1}{2T}\int_{-T}^{T} u'v'\,dt \neq 0 \qquad (32.4b)$$

- Now, we can make a general statement with any two fluctuating parameters, say, with *f* ' and *g*' as

$$\overline{f}' = \overline{g'} = 0 \qquad (32.5a)$$

$$\overline{f'g'} \neq 0 \qquad (32.5b)$$

The time averages of the spatial gradients of the fluctuating components also follow the same laws, and they can be written as

$$\left.\begin{aligned} &\overline{\frac{\partial f'}{\partial s}} = \overline{\frac{\partial^2 f'}{\partial s^2}} = 0 \\ &\overline{\frac{\partial (f'g')}{\partial s}} \neq 0 \end{aligned}\right\} \qquad (32.6)$$

- The **intensity of turbulence** or **degree of turbulence** in a flow is described by the relative magnitude of the root mean square value of the fluctuating components with respect to the time averaged main velocity. The mathematical expression is given by

$$\mathrm{Tu} = \frac{1}{U_\infty}\sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)} \qquad (32.7a)$$

The degree of turbulence in a wind tunnel can be brought down by introducing screens of fine mesh at the bell mouth entry. In general, at a certain distance from the screens, the turbulence in a wind tunnel becomes isotropic, i.e. the mean oscillation in the three components are equal,

$$\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$$

In this case, it is sufficient to consider the oscillation u' in the direction of flow and to put

$$\mathrm{Tu} = \frac{1}{U_\infty}\sqrt{\overline{u'^2}} \qquad (32.7b)$$

This simpler definition of turbulence intensity is often used in practice even in cases when turbulence is not isotropic.

Following Reynolds decomposition, it is suggested to separate the motion into a mean motion and a fluctuating or eddying motion. Denoting the time average of the u component of velocity by $\bar{u}$ and fluctuating component as u', we can write down the following,

$$u = \bar{u} + u', v = \bar{v} + v', w = \bar{w} + w', p = \bar{p} + p'$$

By definition, the time averages of all quantities describing fluctuations are equal to zero.

$$\bar{u'} = 0, \bar{v'} = 0, \bar{w'} = 0, \bar{p}' = 0 \qquad (32.8)$$

The fluctuations u', v' , and w' influence the mean motion $\bar{u}$, $\bar{v}$ and $\bar{w}$ in such a way that the mean motion exhibits an apparent increase in the resistance to deformation. In other words, the effect of fluctuations is an apparent increase in viscosity or macroscopic momentum diffusivity .

- **Rules of mean time - averages**

If *f* and *g* are two dependent variables and if *s* denotes anyone of the independent variables x, y

$$\overline{\bar{f}} = \bar{f}; \overline{f+g} = \bar{f} + \bar{g}; \overline{\bar{f}.g} = \bar{f}.\bar{g};$$
$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}; \int \overline{f ds} = \int \bar{f} ds$$

Intermittency

- Consider a turbulent flow confined to a limited region. To be specific we shall consider the example of a wake (Figure 33.1a), but our discussion also applies to a jet (Figure 33.1b), a shear layer (Figure 33.1c), or the outer part of a boundary layer on a wall.
- The fluid outside the turbulent region is either in irrotational motion (as in the case of a wake or a boundary layer), or nearly static (as in the case of a jet). Observations show that the instantaneous interface between the turbulent and nonturbulent fluid is very sharp.
- The thickness of the interface must equal the size of the smallest scales in the flow, namely the **Kolmogorov microscale**.

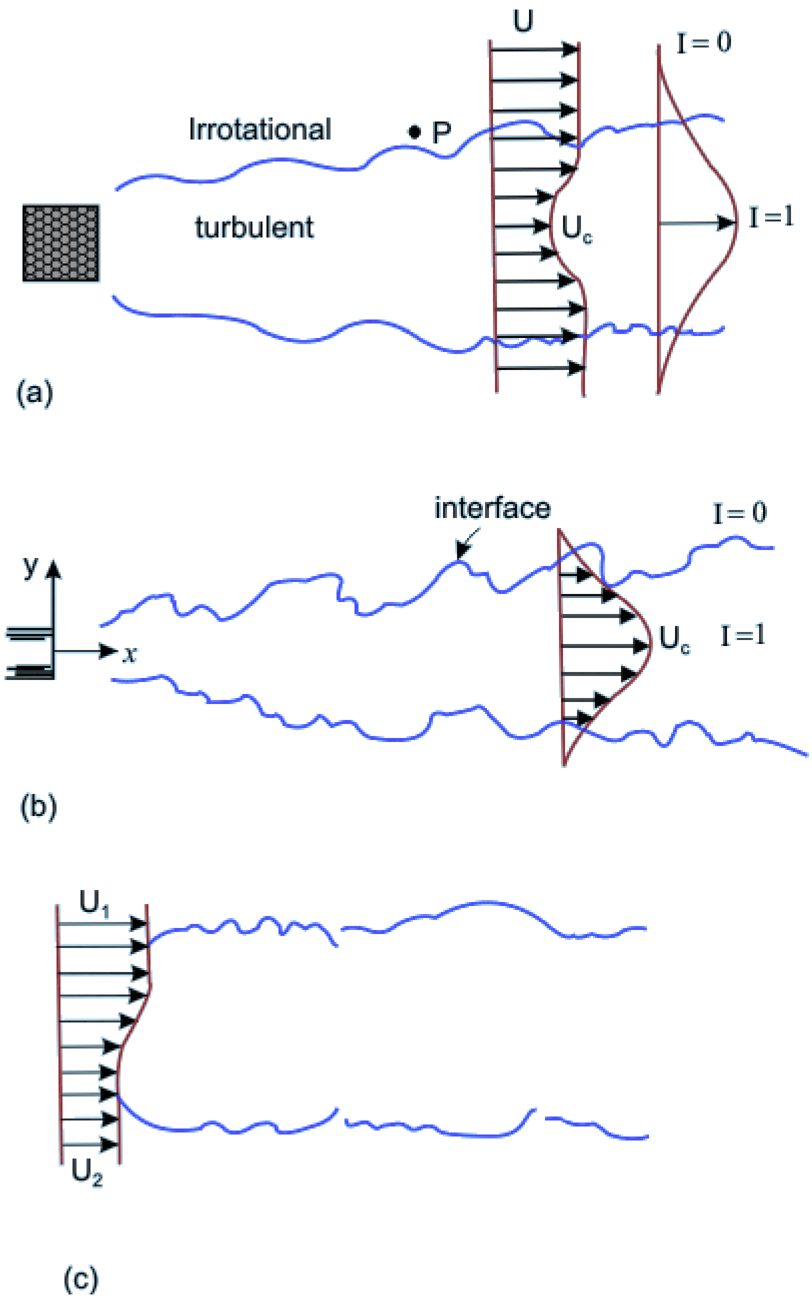

Figure 33.1 Three types of free turbulent flows; (a) wake (b) jet and (c) shear layer **[after P.K. Kundu and I.M. Cohen, Fluid Mechanics, Academic Press, 2002]**

- Measurement at a point in the outer part of the turbulent region (say at point P in Figure 33.1a) shows periods of high-frequency fluctuations as the point P moves into the turbulent flow and low-frequency periods as the point moves out of the turbulent region. Intermittency I is defined as the fraction of time the flow at a point is turbulent.

- The variation of I across a wake is sketched in Figure 33.1a, showing that I =1 near the center where the flow is always turbulent, and I = 0 at the outer edge of the flow domain.

Derivation of Governing Equations for Turbulent Flow

- For **incompressible flows**, the Navier-Stokes equations can be rearranged in the form

$$\rho\left[\frac{\partial u}{\partial t}+\frac{\partial(u^2)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\nabla^2 u \tag{33.1a}$$

$$\rho\left[\frac{\partial v}{\partial t}+\frac{\partial(uv)}{\partial x}+\frac{\partial(v^2)}{\partial y}+\frac{\partial(vw)}{\partial z}\right]=-\frac{\partial p}{\partial y}+\mu\nabla^2 v \tag{33.1b}$$

$$\rho\left[\frac{\partial w}{\partial t}+\frac{\partial(uw)}{\partial x}+\frac{\partial(vw)}{\partial y}+\frac{\partial(w^2)}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\nabla^2 w \tag{33.1c}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{33.2}$$

- Express the velocity components and pressure in terms of time-mean values and corresponding fluctuations. In **continuity equation**, this substitution and subsequent time averaging will lead to

$$\overline{\frac{\partial(\bar{u}+u')}{\partial x}}+\overline{\frac{\partial(\bar{v}+v')}{\partial y}}+\overline{\frac{\partial(w+w')}{\partial z}}=0$$

or,

$$\left(\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}\right)+\left(\frac{\partial\overline{u'}}{\partial x}+\frac{\partial\overline{v'}}{\partial y}+\frac{\partial\overline{w'}}{\partial z}\right)=0$$

Since,

$$\frac{\partial\overline{u'}}{\partial x}=\frac{\partial\overline{v'}}{\partial y}=\frac{\partial\overline{w'}}{\partial z}=0$$

We can write

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0 \tag{33.3a}$$

From Eqs (33.3a) and (33.2), we obtain

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \tag{33.3b}$$

- It is evident that **the time-averaged velocity components** and **the fluctuating velocity components**, each satisfy the continuity equation for incompressible flow.

- Imagine a two-dimensional flow in which the turbulent components are independent of the *z* -direction. Eventually, Eq.(33.3b) tends to

$$\frac{\partial u'}{\partial x} = -\frac{\partial v'}{\partial y} \qquad (33.4)$$

On the basis of condition (33.4), it is postulated that if at an instant there is an increase in u' in the *x* -direction, it will be followed by an increase in v' in the negative *y* -direction. In other words, $\overline{u'v'}$ **is non-zero and negative**. (see Figure 33.2)

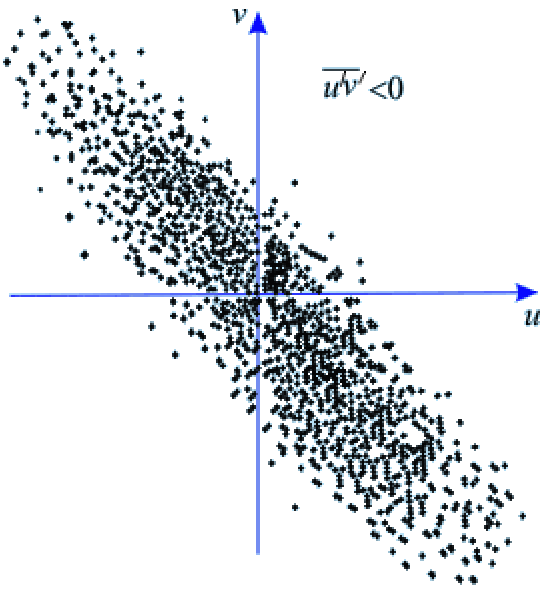

Fig 33.2 Each dot represents uv pair at an instant

- Invoking the concepts of eqn. (32.8) into the equations of motion eqn (33.1 a, b, c), we obtain expressions in terms of mean and fluctuating components. Now, forming time averages and considering the rules of averaging we discern the following. The terms which are linear, such as $\frac{\partial u'}{\partial t}$ and $\frac{\partial^2 u'}{\partial x^2}$ vanish when they are averaged [from (32.6)]. The same is true for the mixed terms like $\overline{u} \cdot u'$, or $\overline{u} \cdot v'$, but the quadratic terms in the fluctuating components remain in the equations. After averaging, they form $\overline{u'^2}$, $\overline{u'v'}$ etc.

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- If we perform the aforesaid exercise on the x-momentum equation, we obtain

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\cancelto{0}{\frac{\partial \overline{u'}}{\partial t}}+\frac{\partial\left(\bar{u}^2+\overline{u'^2}\right)}{\partial x}+\frac{\partial\left(\overline{u\cdot v}+\overline{u'v'}\right)}{\partial y}+\frac{\partial\left(\bar{u}\cdot\bar{w}+\overline{u'w'}\right)}{\partial z}\right\}$$

using rules of time averages,

$$\frac{\partial \overline{u}'}{\partial t}=0,\frac{\partial \overline{p}'}{\partial x}=0,\frac{\partial^2 \overline{u}'}{\partial x^2}=\frac{\partial^2 \overline{u}'}{\partial y^2}=\frac{\partial^2 \overline{u}'}{\partial z^2}=0$$

We obtain

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\frac{\partial\left(\bar{u}^2\right)}{\partial x}+\frac{\partial\left(\bar{u}\cdot\bar{v}\right)}{\partial y}+\frac{\partial\left(\bar{u}\cdot\bar{w}\right)}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}\right]$$

- Introducing simplifications arising out of continuity Eq. (33.3a), we shall obtain.

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}\right]$$

- Performing a similar treatment on y and z momentum equations, finally we obtain the momentum equations in the form.

In x direction,

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}\right] \qquad (33.5a)$$

In y direction,

$$\rho\left\{\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}-\rho\left[\frac{\partial}{\partial x}\overline{u'v'}+\frac{\partial}{\partial y}\overline{v'^2}+\frac{\partial}{\partial z}\overline{v'w'}\right] \qquad (33.5b)$$

In z direction,

$$\rho\left\{\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}-\rho\left[\frac{\partial}{\partial x}\overline{u'w'}+\frac{\partial}{\partial y}\overline{v'w'}+\frac{\partial}{\partial z}\overline{w'^2}\right] \qquad (33.5c)$$

- Comments on the governing equation :
    1. The left hand side of Eqs (33.5a)-(33.5c) are essentially similar to the

steady-state Navier-Stokes equations if the velocity components *u*, *v* and *w* are replaced by $\bar{u}$, $\bar{v}$ and $\bar{w}$.

2. The same argument holds good for the first two terms on the right hand side of Eqs (33.5a)-(33.5c).
3. However, the equations contain some additional terms which depend on turbulent fluctuations of the stream. **These additional terms can be interpreted as components of a stress tensor.**

- Now, the resultant surface force per unit area due to these terms may be considered as

In x direction,

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\left[\frac{\partial}{\partial x}\sigma'_{xx}+\frac{\partial}{\partial y}\tau'_{yx}+\frac{\partial}{\partial z}\tau'_{zx}\right] \quad (33.6a)$$

In y direction,

$$\rho\left\{\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}+\left[\frac{\partial}{\partial x}\tau'_{xy}+\frac{\partial}{\partial y}\sigma'_{yy}+\frac{\partial}{\partial z}\tau'_{zy}\right] \quad (33.6b)$$

In z direction,

$$\rho\left\{\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}+\left[\frac{\partial}{\partial x}\tau'_{xz}+\frac{\partial}{\partial y}\tau'_{yz}+\frac{\partial}{\partial z}\sigma'_{zz}\right] \quad (33.6c)$$

- Comparing Eqs (33.5) and (33.6), we can write

$$\begin{bmatrix}\sigma'_{xx} & \tau'_{xy} & \tau'_{xz}\\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz}\\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz}\end{bmatrix}=-\rho\begin{bmatrix}\overline{u'^2} & \overline{u'v'} & \overline{u'w'}\\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'}\\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2}\end{bmatrix} \quad (33.7)$$

- It can be said that the mean velocity components of turbulent flow satisfy the same Navier-Stokes equations of laminar flow. However, for the turbulent flow, the laminar stresses must be increased by additional stresses which are given by the stress tensor (33.7). These additional stresses are known as **apparent stresses of turbulent flow** or **Reynolds stresses** . Since turbulence is considered as eddying motion and the aforesaid additional stresses are added to the viscous stresses due to mean motion in order to explain the complete stress field, it is often said that the apparent stresses are caused by eddy viscosity . The total stresses are now

$$\left.\begin{aligned}\sigma_{xx} &= -\bar{p} + 2\mu\frac{\partial\bar{u}}{\partial x} - \overline{\rho u'^2}\\ \tau_{xy} &= \mu\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right) - \overline{\rho u'v'}\end{aligned}\right] \tag{33.8}$$

and so on. The apparent stresses are much larger than the viscous components, and the viscous stresses can even be dropped in many actual calculations .

Turbulent Boundary Layer Equations

- For a two-dimensional flow (w = 0)over a flat plate, the thickness of turbulent boundary layer is assumed to be much smaller than the axial length and the **order of magnitude analysis** may be applied. As a consequence, the following inferences are drawn:

$$(a) \quad \frac{\partial\bar{p}}{\partial y} = 0,$$

$$(b) \quad \frac{\partial\bar{p}}{\partial x} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}x}$$

$$(c) \quad \frac{\partial^2\bar{u}}{\partial x^2} << \frac{\partial^2\bar{u}}{\partial y^2},$$

$$(d) \quad \frac{\partial}{\partial x}\left(-\overline{\rho u'^2}\right) << \frac{\partial}{\partial y}\left(-\overline{\rho u'v'}\right),$$

- The turbulent boundary layer equation together with the equation of continuity becomes

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0 \tag{33.9}$$

-

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{1}{\rho}\cdot\frac{\partial\bar{p}}{\partial x} + \frac{\partial}{\partial y}\left\{\nu\frac{\partial\bar{u}}{\partial y} - \overline{u'v'}\right\} \tag{33.10}$$

- A comparison of Eq. (33.10) with laminar boundary layer Eq. (23.10) depicts that: *u*, *v* and *p* are replaced by the time average values $\bar{u}, \bar{v}$ and $\bar{p}$,and laminar viscous force per unit volume $\frac{\partial(\tau_l)}{\partial y}$ is replaced by $\frac{\partial}{\partial y}(\tau_l+\tau_t)$ where $\tau_l = \mu\frac{\partial \overline{u}}{\partial y}$ is the laminar shear stress and $\tau_t = -\rho u'v'$ is the turbulent shear stress.

Boundary Conditions

- All the components of apparent stresses vanish at the solid walls and only stresses which act near the wall are the viscous stresses of laminar flow. The boundary conditions, to be satisfied by the mean velocity components, are similar to laminar flow.
- A very thin layer next to the wall behaves like a near wall region of the laminar flow. This layer is known as **laminar sublayer** and its velocities are such that the viscous forces dominate over the inertia forces. No turbulence exists in it (see Fig. 33.3).
- For a developed turbulent flow over a flat plate, in the near wall region, inertial effects are insignificant, and we can write from Eq.33.10,

$$\nu\frac{\partial^2\overline{u}}{\partial y^2} - \frac{\partial\left(\overline{u'v'}\right)}{\partial y} = 0$$

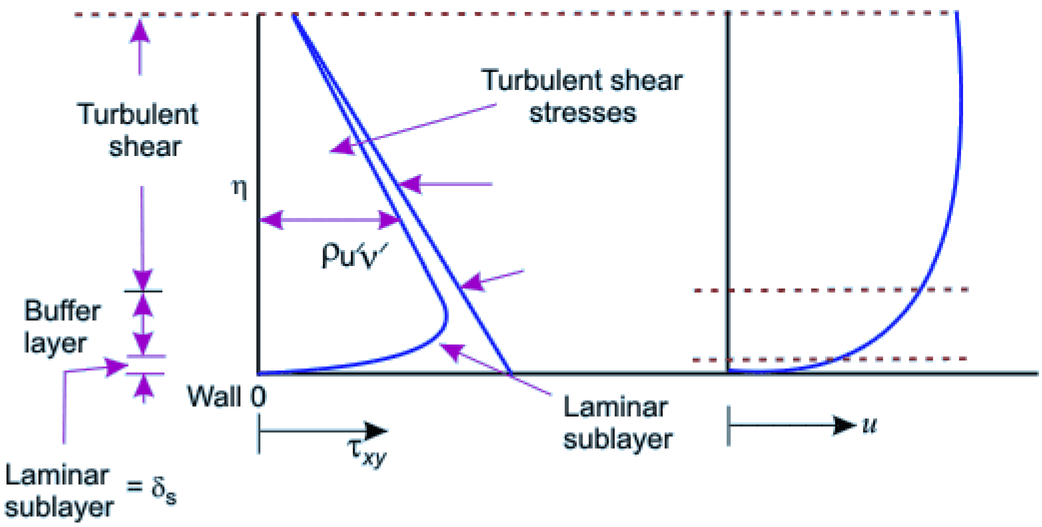

Fig 33.3 Different zones of a turbulent flow past a wall

which can be integrated as , $\frac{\nu\partial\overline{u}}{\partial y} - \overline{u'v'}$ =constant

- We know that the fluctuating components, do not exist near the wall, the shear stress on the wall is purely viscous and it follows

$$\nu\frac{\partial\overline{u}}{\partial y}\bigg|_{y=0} = \frac{\tau_w}{\rho}$$

However, the wall shear stress in the vicinity ofthe laminar sublayer is estimated as

$$\tau_w = \mu\left[\frac{U_s - 0}{\delta_s - 0}\right] = \mu\frac{U_s}{\delta_s} \qquad (33.11a)$$

where U_s is the fluid velocity at the edge of the sublayer. The flow in the sublayer is specified by a velocity scale (characteristic of this region).

- We define the **friction velocity**,

$$u_\tau = \left[\frac{\tau_w}{\rho}\right]^{1/2} \qquad (33.11b)$$

as our velocity scale. Once u_τ is specified, the structure of the sub layer is specified. It has been confirmed experimentally that the turbulent intensity distributions are scaled with u_τ. For example, maximum value of the $\overline{u'^2}$ is always about $8u_\tau^2$. The relationship between u_τ and the U_s can be determined from Eqs (33.11a) and (33.11b) as

$$u_\tau^2 = \nu\frac{U_s}{\delta_s}$$

Let us assume $U_s = \bar{C}U_\infty$. Now we can write

$$u_\tau^2 = \bar{C}\nu\frac{U_\infty}{\delta_s} \qquad (33.12a)$$

where $\bar{C}$ is a proportionality constant

or

$$\frac{\delta_s u_\tau}{\nu} = \bar{C}\left[\frac{U_\infty}{u_\tau}\right] \qquad (33.12b)$$

Hence, a non-dimensional coordinate may be defined as, $\eta = \frac{yu_\tau}{\nu}$ which will help us estimating different zones in a turbulent flow. **The thickness** of **laminar sublayer or viscous sublayer is considered to be** $\eta \approx 5$.

Turbulent effect starts in the zone of $\eta > 5$ and in a zone of $5 < \eta < 70$, laminar and turbulent motions coexist. This domain is termed as **buffer zone**. Turbulent effects far outweight the laminar effect in the zone beyond $\eta = 70$ and this regime is termed as turbulent core .

- For flow over a flat plate, the turbulent shear stress ($-\rho\overline{u'v'}$) is constant throughout in the y direction and this becomes equal to τ_w at the wall. In the event of flow through a channel, the turbulent shear stress ($-\rho\overline{u'v'}$) varies with *y* and it is possible to write

$$\frac{\tau_t}{\tau_w} = \frac{\zeta}{h} \qquad (33.12c)$$

where the channel is assumed to have a height ***2h*** and ζ is the distance measured from the centreline of the channel $(= h - y)$. Figure 33.1 explains such variation of turbulent stress.

Shear Stress Models

- In analogy with the coefficient of viscosity for laminar flow, J. Boussinesq introduced a **mixing coefficient** μ_T for the Reynolds stress term, defined as

$$\tau_t = -\rho\overline{u'v'} = \mu_t\frac{\partial\bar{u}}{\partial y}$$

- Using μ_T the shearing stresses can be written as

$$\tau_l = \rho\nu\frac{\partial u}{\partial y}, \tau_t = \mu_t\frac{\partial\bar{u}}{\partial y} = \rho\nu_t\frac{\partial\bar{u}}{\partial y}$$

such that the equation

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{1}{\rho}.\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{\nu\frac{\partial\bar{u}}{\partial y} - \overline{u'v'}\right\}$$

may be written as

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = \frac{1}{\rho}.\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{(\nu + \nu_t)\frac{\partial\bar{u}}{\partial y}\right\} \qquad (33.13)$$

The term ν_t is known as **eddy viscosity** and the model is known as **eddy viscosity model** .

- Unfortunately the value of ν_t is not known. The term ν is a property of the fluid whereas ν_t is attributed to random fluctuations and is not a property of the fluid. However, it is necessary to find out empirical relations between ν_t, and the mean velocity. The following section discusses relation between the aforesaid apparent or eddy viscosity and the mean velocity components

Prandtl's Mixing Length Hypothesis

- Consider a fully developed turbulent boundary layer . The stream wise mean velocity varies only from streamline to streamline. The main flow direction is assumed parallel to the x-axis (Fig. 33.4).

- The time average components of velocity are given by $\bar{u} = \bar{u}(y), \bar{v} = 0, \bar{w} = 0$. The fluctuating component of transverse velocity v'transports mass and momentum across a plane at y_1 from the wall. The shear stress due to the fluctuation is given by

$$\tau_t = -\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y} \qquad (33.14)$$

- Fluid, which comes to the layer y_1 from a layer $(y_1 - l)$ has a positive value of v'. If the lump of fluid retains its original momentum then its velocity at its current location y_1 is smaller than the velocity prevailing there. The difference in velocities is then

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y1} \qquad (33.15)$$

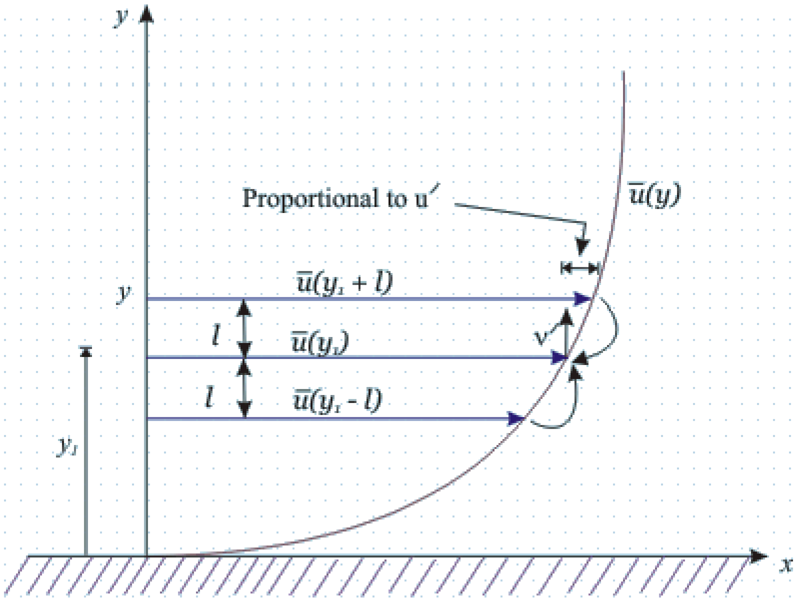

Fig. 33.4 One-dimensional parallel flow and Prandtl's mixing length hypothesis

The above expression is obtained by expanding the function in a Taylor series and neglecting all higher order terms and higher order derivatives. l is a small length scale known as Prandtl's mixing length . Prandtl proposed that the transverse displacement of any fluid particle is, on an average, 'l' .

continued..

- Consider another lump of fluid with a negative value of v'. This is arriving at y_1 from $(y_1 + l)$. If this lump retains its original momentum, its mean velocity at the current lamina $y_1$will be somewhat more than the original mean velocity of y_1. This difference is given by

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y1} \qquad (33.16)$$

- The velocity differences caused by the transverse motion can be regarded as the turbulent velocity components at y_1.
- We calculate the time average of the absolute value of this fluctuation as

$$\left|\overline{u'}\right| = \frac{1}{2}\left(\left|\Delta u_1\right| + \left|\Delta u_2\right|\right) = l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|_{y1} \qquad (33.17)$$

- Suppose these two lumps of fluid meet at a layer y_1The lumps will collide with a velocity $2u'$ and diverge. This proposes the possible existence of transverse velocity component in both directions with respect to the layer at y_1. Now, suppose that the two lumps move away in a reverse order from the layer $y_1$with a velocity $2u'$. The empty space will be filled from the surrounding fluid creating transverse velocity components which will again collide at y_1. Keeping in mind this argument and the physical explanation accompanying Eqs (33.4), we may state that

$$\left|\overline{v'}\right| \sim \left|\overline{u'}\right|$$

or, $$\left|\overline{v'}\right| = (const)\left|\overline{u'}\right| = (const)\, l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|$$

along with the condition that the moment at which $u'$is positive, v' is more likely to be negative and conversely when $u'$is negative. Possibly, we can write at this stage

$$\overline{u'v'} = -C_1\left|\overline{u'}\right|\left|\overline{v'}\right|$$

$$\overline{u'v'} = -C_2 l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \qquad (33.18)$$

where C_1 and C_2 are different proportionality constants. However, the constant C_2 can now be included in still unknown mixing length and Eg. (33.18) may be rewritten as

$$\overline{u'v'} = -l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

- For the expression of turbulent shearing stress $\tau_t$we may write

$$\mu_t = \rho l^2\left|\frac{\partial \bar{u}}{\partial y}\right| \qquad (33.19)$$

- After comparing this expression with the eddy viscosity Eg. (33.14), we may arrive at a more precise definition,

$$\tau_t = \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \left(\frac{\partial \bar{u}}{\partial y}\right) = \mu_t \frac{\partial \bar{u}}{\partial y} \qquad (33.20a)$$

where the apparent viscosity may be expressed as

$$\mu_t = \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \qquad (33.20b)$$

and the apparent kinematic viscosity is given by

$$\nu_t = l^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \qquad (33.20c)$$

- The decision of expressing one of the velocity gradients of Eq. (33.19) in terms of its modulus as $\left|\frac{\partial \bar{u}}{\partial y}\right|$ was made in order to assign a sign to τ_t according to the sign of $\frac{\partial \bar{u}}{\partial y}$.
- Note that the apparent viscosity and consequently,the mixing length are not properties of fluid. They are dependent on turbulent fluctuation.
- But how to determine the value of "l" the mixing length? Several correlations, using experimental results for τ_t have been proposed to determine l.

  However, so far the most widely used value of mixing length in the regime of isotropic turbulence is given by

$$l = \chi y \qquad (33.21)$$

where y is the distance from the wall and χ is known as **von Karman constant** (≈ 0.4).

Universal Velocity Distribution Law And Friction Factor In Duct Flows For Very Large Reynolds Numbers

- For flows in a rectangular channel at very large Reynolds numbers the laminar sublayer can practically be ignored. The channel may be assumed to have a width *2h* and the *x* axis will be placed along the bottom wall of the channel.

- Consider a turbulent stream along a smooth flat wall in such a duct and denote the distance from the bottom wall by *y*, while *u(y)* will signify the velocity. In the neighbourhood of the wall, we shall apply

$$l = \chi y$$

- According to Prandtl's assumption, the turbulent shearing stress will be

$$\tau_t = \rho \chi^2 y^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{34.1}$$

At this point, Prandtl introduced an additional assumption which like a plane Couette flow takes a constant shearing stress throughout, i.e

$$\tau_t = \tau_w \tag{34.2}$$

where $\tau_w$denotes the shearing stress at the wall.

- Invoking once more the friction velocity $u_\tau = \left[\frac{\tau_w}{\rho} \right]^{1/2}$, we obtain

$$u_\tau^2 = \chi^2 y^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{34.3}$$

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_\tau}{\chi y} \tag{34.4}$$

On integrating we find

$$\bar{u} = \frac{u_\tau}{\chi} \ln y + C \tag{34.5}$$

- Despite the fact that Eq. (34.5) is derived on the basis of the friction velocity in the neighbourhood of the wall because of the assumption that $\tau_w = \tau_t$= constant, we shall use it for the entire region. At $y = h$ (at the horizontal mid plane of the channel), we have $\bar{u} = U_{max}$. The constant of integration is eliminated by considering

$$U_{\max} = \frac{u_\tau}{\chi} \ln h + C$$

$$C = U_{\max} - \frac{u_\tau}{\chi} \ln h$$

Substituting C in Eq. (34.5), we get

$$\frac{U_{max} - \bar{u}}{u_\tau} = \frac{1}{\chi}\ln\left(\frac{h}{y}\right) \qquad (34.6)$$

Equation (34.6) is known as **universal velocity defect law** of Prandtl and its distribution has been shown in Fig. 34.1

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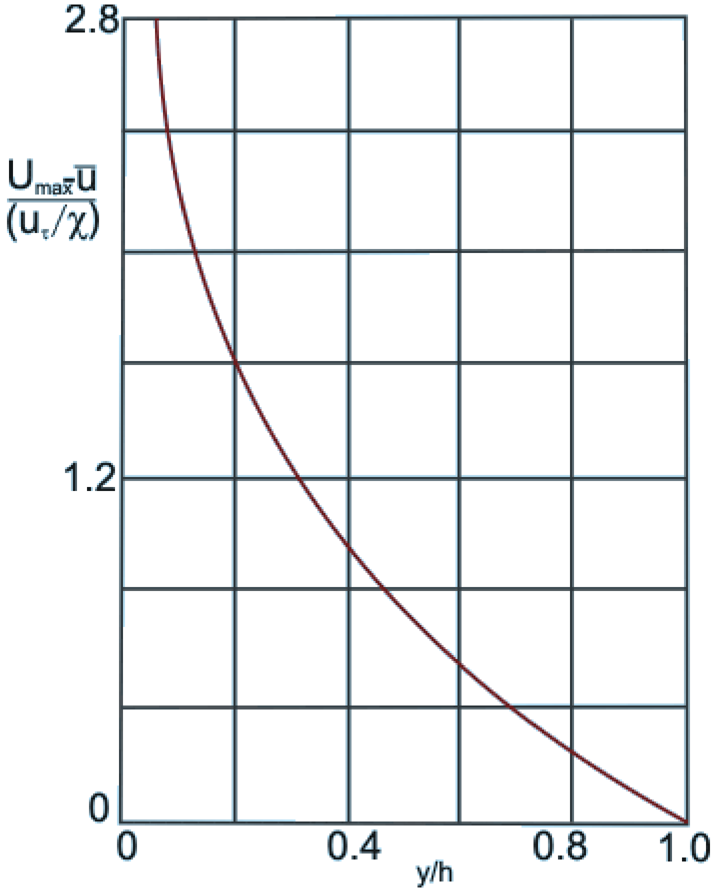

Fig 34.1 Distibution of universal velocity defect law of Prandtl in a turbulent channel flow

Here, we have seen that the friction velocity u_τ is a reference parameter for velocity.Equation (34.5) can be rewritten as

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi}\ln y + C'$$

where $c' = \frac{c}{u_\tau}$

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- The no-slip condition at the wall cannot be satisfied with a finite constant of integration. This is expected that the appropriate condition for the present problem should be that y_0 at a very small distance $y = y_0$ from the wall. Hence, Eq. (34.5) becomes

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi}\left(\ln y - \ln y_0\right) \tag{34.7}$$

- The distance y_0 is of the order of magnitude of the thickness of the viscous layer. Now we can write Eq. (34.7) as

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi}\left(\ln y\frac{u_\tau}{\nu} - \ln \beta\right)$$

$$\frac{\bar{u}}{u_\tau} = A_1 \ln \eta + D_1 \tag{34.8}$$

where $A_1 = (1/\ \chi)$, the unknown β is included in D_1.

Equation (34.8) is generally known as the **universal velocity profile** because of the fact that it is applicable from moderate to a very large Reynolds number.

However, the constants A_1 and D_1 have to be found out from experiments. The aforesaid profile is not only valid for channel (rectangular) flows, it retains the same functional relationship for circular pipes as well . It may be mentioned that even without the assumption of having a constant shear stress throughout, the universal velocity profile can be derived.

- Experiments, performed by J. Nikuradse, showed that Eq. (34.8) is in good agreement with experimental results. Based on Nikuradse's and Reichardt's experimental data, the empirical constants of Eq. (34.8) can be determined for a **smooth pipe** as

$$\frac{\bar{u}}{u_\tau} = 2.5 \ln \eta + 5.5 \tag{34.9}$$

This velocity distribution has been shown through curve (b) in Fig. 34.2

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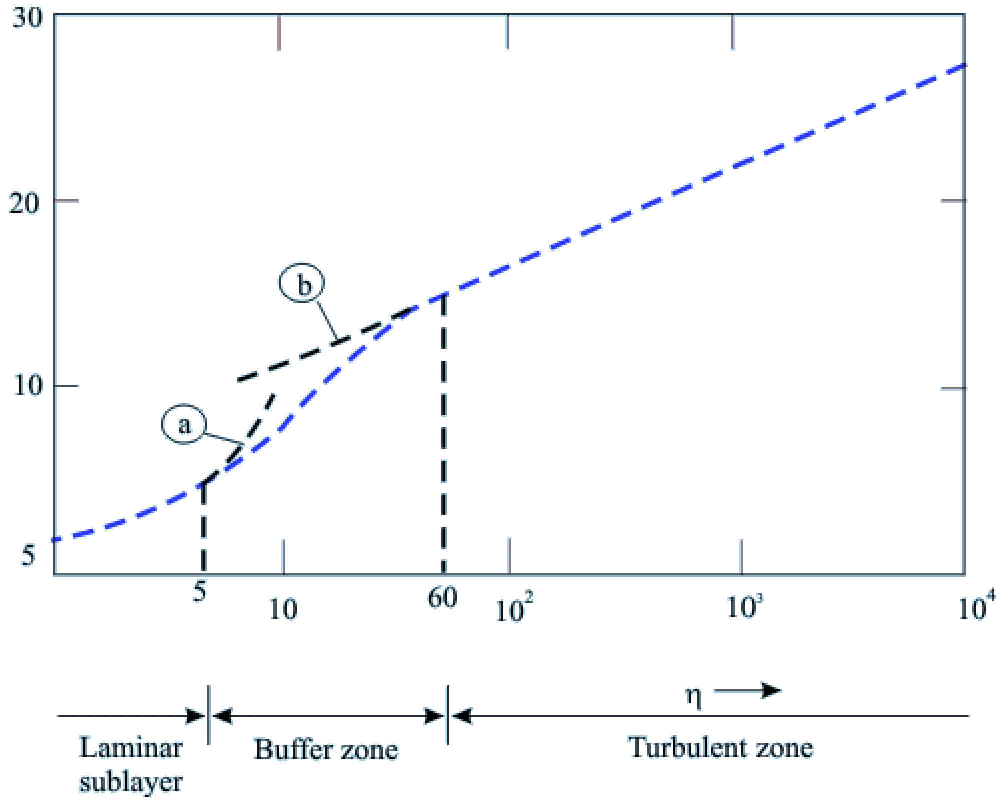


Fig 34.2   The universal velocity distribution law for smooth pipes

- However, the corresponding friction factor concerning Eq. (34.9) is

$$\frac{1}{\sqrt{f}} = 2.0\log_{10}\left(\mathrm{Re}\sqrt{f}\right) - 0.8 \tag{34.10}$$

the universal velocity profile does not match very close to the wall where the viscous shear predominates the flow.

- Von Karman suggested a modification for the **laminar sublayer** and the **buffer zone** which are

$$\frac{\bar{u}}{u_\tau} = \eta = \frac{u_\tau y}{\nu} \text{for } \eta < 5.0 \tag{34.11}$$

$$\frac{\bar{u}}{u_\tau} = 11.5\log_{10}\frac{u_\tau y}{\nu} - 3.0 \text{for } 5 < \eta < 60 \tag{34.12}$$

Equation (34.11) has been shown through curve(a) in Fig. 34.2.

- It may be worthwhile to mention here that a surface is said to be hydraulically smooth so long

$$0 \le \frac{\varepsilon_p u_\tau}{\nu} \le 5 \qquad (34.13)$$

where ε_p is the average height of the protrusions inside the pipe.

Physically, the above expression means that for smooth pipes protrusions will not be extended outside the laminar sublayer. If protrusions exceed the thickness of laminar sublayer, it is conjectured (also justified though experimental verification) that some additional frictional resistance will contribute to pipe friction due to the form drag experienced by the protrusions in the boundary layer.

- In rough pipes experiments indicate that the velocity profile may be expressed as:

$$\frac{\bar{u}}{u_\tau} = 2.5 \ln \frac{y}{\varepsilon_p} + 8.5 \qquad (34.14)$$

At the centre-line, the maximum velocity is expressed as

$$\frac{U_{max}}{u_\tau} = 2.5 \ln \frac{R}{\varepsilon_p} + 8.5 \qquad (34.15)$$

Note that ν no longer appears with R and ε_p. This means that for completely rough zone of turbulent flow, the profile is independent of Reynolds number and a strong function of pipe roughness .

- However, for pipe roughness of varying degrees, the recommendation due to **Colebrook and White works** well. Their formula is

$$\frac{1}{\sqrt{f}} = 1.74 - 2.0 \log_{10}\left[\frac{\varepsilon_p}{R} + \frac{18.7}{\mathrm{Re}\sqrt{f}}\right] \qquad (34.16)$$

where R is the pipe radius

For $\varepsilon_p \to 0$, this equation produces the result of the smooth pipes (Eq.(34.10)). For $\mathrm{Re} \to \infty$, it gives the expression for friction factor for a completely rough pipe at a very high Reynolds number which is given by

$$f = \frac{1}{\left(2\log\frac{R}{\varepsilon_p} + 1.74\right)^2} \qquad (34.17)$$

Turbulent flow through pipes has been investigated by many researchers because of its enormous practical importance.

Fully Developed Turbulent Flow In A Pipe For Moderate Reynolds Numbers

- The entry length of a turbulent flow is much shorter than that of a laminar flow, J. Nikuradse determined that a fully developed profile for turbulent flow can be observed after an entry length of 25 to 40 diameters. We shall focus to fully developed turbulent flow in this section.
- Considering a fully developed turbulent pipe flow (Fig. 34.3) we can write

$$2\pi R\tau_w = -\left(\frac{dp}{dx}\right)\pi R^2 \qquad (34.18)$$

or

$$\left(-\frac{dp}{dx} = \frac{2\tau_w}{R}\right) \qquad (34.19)$$

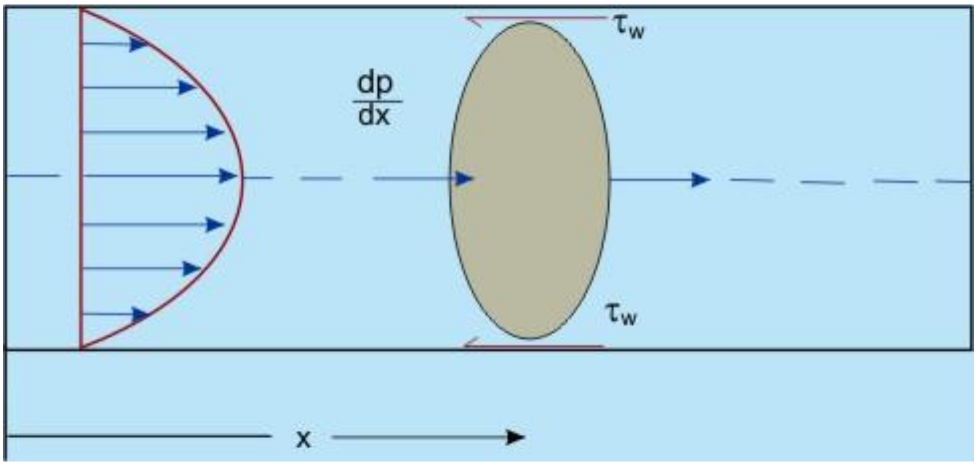


Fig. 34.3 Fully developed turbulent pipe flow

It can be said that in a fully developed flow, the pressure gradient balances the wall shear stress only and has a constant value at any x. However, the friction factor (Darcy friction factor) is defined in a fully developed flow as

$$-\left(\frac{dp}{dx}\right) = \frac{\rho f U_{av}^2}{2D} \qquad (34.20)$$

Comparing Eq.(34.19) with Eq.(34.20), we can write

$$\tau_w = \frac{f}{8}\rho U_{av}^2 \qquad (34.21)$$

H. Blasius conducted a critical survey of available experimental results and established the empirical correlation for the above equation as

$$f = 0.3164\,\mathrm{Re}^{-0.25} \text{ where } \mathrm{Re} = \rho U_{av} D/\mu \qquad (34.22)$$

- It is found that the Blasius's formula is valid in the range of Reynolds number of Re $\leq 10^5$. At the time when Blasius compiled the experimental data, results for higher Reynolds numbers were not available. However, later on, J. Nikuradse carried out experiments with the laws of friction in a very wide range of Reynolds numbers, $4 \times 10^3 \leq \mathrm{Re} \leq 3.2 \times 10^6$. The velocity profile in this range follows:

$$\frac{u}{\bar{u}} = \left[\frac{y}{R}\right]^{1/n} \qquad (34.23)$$

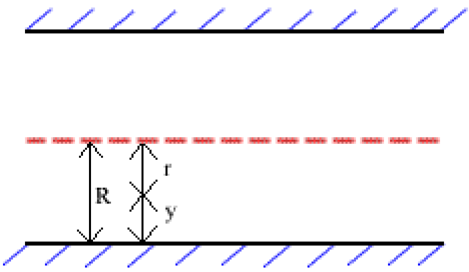
where $\bar{u}$ is the time mean velocity at the pipe centre and y is the distance from the wall . The exponent *n* varies slightly with Reynolds number. In the range of Re ~ 10^5, *n* is 7.

Fully Developed Turbulent Flow In A Pipe For Moderate Reynolds Numbers

- The ratio of $\bar{u}$ and U_{av} for the aforesaid profile is found out by considering the volume flow rate Q as

$$Q = \pi R^2 U_{av} = \int_0^R 2\pi r u\, dr$$

$$r = R - y$$



From equation (34.23)

$$\pi R^2 U_{av} = 2\pi \bar{u} \int_0^R (R-y)(y/R)^{1/n}(-dy)$$

or

$$\pi R^2 U_{av} = 2\pi \bar{u}\left[\frac{n}{n+1}\left(R^{\frac{n-1}{n}} y^{\frac{n+1}{n}}\right) - \frac{n}{2n+1}\left(y^{\frac{2n+1}{n}} R^{-\frac{1}{n}}\right)\right]_0^R$$

or

$$\pi R^2 U_{av} = 2\pi \bar{u}\left[R^2 \frac{n}{n+1} - \frac{n}{2n+1} R^2\right]$$

or

$$\pi R^2 U_{av} = 2\pi R^2 \bar{u}\left[\frac{n^2}{(n+1)(2n+1)}\right]$$

or

$$\frac{U_{av}}{\bar{u}} = \frac{2n^2}{(n+1)(2n+1)} \qquad (34.24a)$$

- Now, for different values of *n* (for different Reynolds numbers) we shall obtain different values of $U_{av}/\bar{u}$ from Eq.(34.24a). On substitution of Blasius resistance formula (34.22) in Eq.(34.21), the following expression for the shear stress at the wall can be obtained.

$$\tau_w = \frac{0.3164}{8} \mathrm{Re}^{-0.25} \rho U_{av}^2$$

putting $\mathrm{Re} = \rho U_{av} 2R/\mu$

and where $\nu = \mu/\rho$

$$\tau_w = 0.03955 \rho U_{av}^2 \left(\frac{\nu}{2RU_{av}}\right)^{1/4}$$

or

$$\tau_w = 0.03325 \rho U_{av}^{7/4} \left(\frac{\nu}{R}\right)^{1/4}$$

or

$$\tau_w = 0.03325\rho\left(\frac{U_{av}}{\bar{u}}\right)^{7/4}(\bar{u})^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

- For n=7, $U_{av}/\bar{u}$ becomes equal to 0.8. substituting $U_{av}/\bar{u} = 0.8$ in the above equation, we get

$$\tau_w = 0.03325\rho(0.8)^{1/4}(\bar{u})^{7/4}(\nu/R)^{1/4}$$

Finally it produces

$$\tau_w = 0.0225\rho(\bar{u})^{7/4}(\nu/R)^{1/4} \qquad (34.24b)$$

or

$$u_\tau^2\rho = 0.0225\rho(\bar{u})^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

where u_τ is friction velocity. However, u_τ^2 may be spitted into $u_\tau^{1/4}$ and $u_\tau^{7/4}$ and we obtain

$$\left(\frac{\bar{u}}{u_\tau}\right)^{7/4} = 44.44\left(\frac{u_\tau R}{\nu}\right)^{1/4}$$

or

$$\frac{\bar{u}}{u_\tau} = 8.74\left(\frac{u_\tau R}{\nu}\right)^{1/7} \qquad (34.25a)$$

- Now **we can assume that the above equation is not only valid at the pipe axis (y = R) but also at any distance** from the wall y and a **general form is proposed as**

$$\frac{\bar{u}}{u_\tau} = 8.74\left(\frac{y u_\tau}{\nu}\right)^{1/7} \qquad (34.25b)$$

- Concluding Remarks :

1. It can be said that (1/7)th power velocity distribution law (24.38b) can be derived from Blasius's resistance formula (34.22) .
2. Equation (34.24b) gives the shear stress relationship in pipe flow at a moderate Reynolds number, i.e $\mathrm{Re} \le 10^5$. Unlike very high Reynolds number flow, here laminar effect cannot be neglected and the laminar sub layer brings about remarkable

influence on the outer zones.

3. The friction factor for pipe flows, f , defined by Eq. (34.22) is valid for a specific range of Reynolds number and for a particular surface condition.

Skin Friction Coefficient For Boundary Layers On A Flat Plate

- Calculations of skin friction drag on lifting surface and on aerodynamic bodies are somewhat similar to the analyses of skin friction on a flat plate. Because of zero pressure gradient, the flat plate at zero incidence is easy to consider. In some of the applications cited above, the pressure gradient will differ from zero but the skin friction will not be dramatically different so long there is no separation.

- We begin with the momentum integral equation for flat plate boundary layer which is valid for both laminar and turbulent flow.

$$\frac{d}{dx}\left(U_\infty^2 \delta^{**}\right) = \frac{\tau_w}{\rho} \qquad (34.26a)$$

- Invoking the definition of C_{fx} $\left(C_{fx} = \frac{\tau_w}{1/2\rho U_\infty^2}\right)$, Eq.(34.26a) can be written as

$$C_{fx} = 2\frac{d\delta^{**}}{dx} \qquad (34.26b)$$

- Due to the similarity in the laws of wall, correlations of previous section may be applied to the flat plate by substituting δ for *R* and U_∞ for the time mean velocity at the pipe centre.The rationale for using the turbulent pipe flow results in the situation of a turbulent flow over a flat plate is to consider that the time mean velocity, at the centre of the pipe is analogous to the free stream velocity, both the velocities being defined at the edge of boundary layer thickness.

Finally, the velocity profile will be [following Eq. (34.24)]

$$\frac{u}{U_\infty} = \left[\frac{y}{\delta}\right]^{1/7} \quad \text{for } \mathrm{Re} \le 10^5 \qquad (34.27)$$

Evaluating momentum thickness with this profile, we shall obtain

$$\delta^{**} = \int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]dy = \frac{7}{72}\delta \qquad (34.28)$$

Consequently, the law of shear stress (in range of $\mathrm{Re} \leq 10^5$) for the flat plate is found out by making use of the pipe flow expression of Eq. (34.24b) as

$$\tau_w = 0.0225\rho\left(\bar{u}\right)^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

$$\frac{\tau_w}{\rho\left(\bar{u}\right)^2} = 0.0225\left[\frac{\nu}{R\bar{u}}\right]^{1/4}$$

Substituting U_∞ for $\bar{u}$ and δ for *R* in the above expression, we get

$$\frac{\tau_w}{\rho\left(\bar{u}\right)^2} = 0.0225\left[\frac{\nu}{\delta U_\infty^2}\right]^{1/4} \qquad (34.29)$$

Once again substituting Eqs (34.28) and (34.29) in Eq.(34.26), we obtain

$$\frac{7}{72}\cdot\frac{d\delta}{dx} = 0.0225\left[\frac{\nu}{\delta U_\infty^2}\right]^{1/4}$$

$$\delta^{1/4}\frac{d\delta}{dx} = 0.2314\left[\frac{\nu}{U_\infty^2}\right]^{1/4}$$

$$\delta^{5/4} = 0.2892x\left(\frac{\nu}{U_\infty^2}\right)^{1/4} + C$$

Continued...Skin Friction Coefficient For Boundary Layers On A Flat Plate

- For simplicity, if we assume that the turbulent boundary layer grows from the leading edge of the plate we shall be able to apply the boundary conditions x = 0, δ = 0 which will yield *C* = 0, and Eq. (34.30) will become From Eqs (34.26b), (34.28) and (34.31), it is possible to calculate the **average skin friction coefficient** on a flat plate as

$$\left(\frac{\delta}{x}\right)^{5/4} = 0.2892\left[\frac{\nu}{xU_\infty}\right]^{1/4}$$

or, $$\frac{\delta}{x} = 0.37\left[\frac{\nu}{xU_\infty}\right]^{1/5}$$

or, $$\frac{\delta}{x} = 0.37\left(\mathrm{Re}_x\right)^{-1/5} \qquad (34.31)$$

Where $\mathrm{Re}_x = \left(U_\infty x\right)/\nu$

From Eqs (34.26b), (34.28) and (34.31), it is possible to calculate the **average skin friction coefficient** on a flat plate as

$$\overline{C}_f = 0.072(\mathrm{Re}_L)^{-1/5} \tag{34.32}$$

It can be shown that Eq. (34.32) predicts the average skin friction coefficient correctly in the regime of Reynolds number below 2×10^6.

- This result is found to be in good agreement with the experimental results in the range of Reynolds number between $5 \times 10^5$and 10^7 which is given by

$$\overline{C}_f = 0.074(\mathrm{Re}_L)^{-1/5} \tag{34.33}$$

Equation (34.33) is a widely accepted correlation for the average value of turbulent skin friction coefficient on a flat plate.

- With the help of Nikuradse's experiments, Schlichting obtained the semi empirical equation for the average skin friction coefficient as

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} \tag{34.34}$$

Equation (34.34) was derived asssuming the flat plate to be completely turbulent over its entire length . In reality, a portion of it is laminar from the leading edge to some downstream position. For this purpose, it was suggested to use

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} - \frac{A}{\mathrm{Re}} \tag{34.35a}$$

where A has various values depending on the value of Reynolds number at which the transition takes place.

- If the trasition is assumed to take place around a Reynolds number of 5×10^5, the average skin friction correlation of Schlichling can be written as

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} - \frac{1700}{\mathrm{Re}} \tag{34.35b}$$

All that we have presented so far, are valid for a smooth plate.

- Schlichting used a logarithmic expression for turbulent flow over a rough surface and derived

$$\overline{C}_f = \left(1.89 + 1.62\log\frac{L}{\varepsilon_p}\right)^{2.5} \qquad (34.36)$$

Exercise Problems - Chapter 10

1.Estimate the power required to move a flat plate, 15 m. long and 4 m. wide, in oil $(\rho = 800 kg/m^3, \nu = 10^{-5} m^2/sec)$ at 4m/sec, under the following cases:

a) The boundary layer is assumed laminar over the entire surface of the plate. (Ans. 1665.5 N-m/sec)

b) Transition to turbulence occurs at $Re = 3\times10^5$ and plate is smooth.(Ans. 9486 N-m/sec)

c) The boundary layer is turbulent over the entire plate which is smooth.(Ans. 10023.94 N-m/sec)

d) The boundary layer is turbulent over the entire rough plate with $\frac{u_\infty \varepsilon_p}{\nu} = 10^3$.(Ans. 17200 N-m/sec)

2. Water $(\rho = 1000 kg/m^3, \nu = 2\times10^{-6} m^2/sec)$ is transported through a horizontal pipeline, 800 m. long, with a maximum velocity of 3m/sec. If the Reynolds number is , find the diameter of the pipe (with and without the use of Moody Diagram).

Also calculate the thickness of laminar sub-layer and the buffer layer, and find the power required to maintain the flow. Calculate your results for a fully rough pipe with $\frac{u_\tau \varepsilon_p}{\nu} = 100$.

(Ans. Diameter of the pipe 0.8 m., laminar sub-layer thickness 0.1 mm, buffer layer thickness 1.3 mm, power required 50250 W)

3. Find the frictional drag on the top and sides of a box-shaped moving van 2.4 m wide, 3.0 m high, and 10.5 m long traveling at 100km/h through air ($\nu = 1.4\times10^{-5}$). Assume that the vehicle has a rounded nose so that the flow does not seperate from the top and side. also assume that a turbulent boundary layer starts immediately at the leading edge.

Also, find the thickness of the boundary layer and the shear stress at the trailing edge.

(Ans. Drag = 105.9 N, B.L. = 0.136m, Shear stress = 0.904 Pa)

Recap

In this course you have learnt the following

- Turbulent motion is an irregular motion of fluid particles in a flow field.

However, for homogeneous and isotropic turbulence, the flow field can be described by time-mean motions and fluctuating components. This is called Reynolds decomposition of turbulent flow.

- In a three dimensional flow field, the velocity components and the pressure can be expressed in terms of the time-averages and the corresponding fluctuations. Substitution of these dependent variables in the Navier-Stokes equations for incompressible flow and subsequent time averaging yield the governing equations for the turbulent flow. The mean velocity components of turbulent flow satisfy the same Navier-Stokes equations for laminar flow. However, for the turbulent flow, the laminar stresses are increased by additional stresses arising out of the fluctuating velocity components. These additional stresses are known as apparent stresses of turbulent flow or Reynolds stresses.In analogy with the laminar shear stresses, the turbulent shear stresses can be expressed in terms of mean velocity gradients and a mixing coefficient known as eddy viscosity. The eddy viscosity (v_t) can be expressed as $v_t = l^2 \left|\frac{d\bar{u}}{dy}\right|$, where l is known as Prandtl's mixing length.

- For fully developed turbulent duct flows at high Reynolds numbers, the velocity profile is given by

$$\frac{\bar{u}}{u_\tau} = A_1 \ln \eta + D$$

where $\bar{u}$ is the time mean velocity at any $\eta \left(= y u_\tau / v\right)$ and u_τ is the friction velocity given by $\sqrt{\tau_w / \rho}$. The constants A_1 and D_1 are determined experimentally. For the smooth pipes, A_1 and D_1 are 2.5 and 5.5 respectively. Corresponding friction factor, f is given by

$$\frac{1}{\sqrt{f}} = 2\log_{10}\left(\text{Re}\sqrt{f}\right) - 0.8$$

Introduction

- A complete analytical solution for the equation of motion in the case of a laminar flow is available, even the advanced theories in the analysis of turbulent flow depend at some point on experimentally derived information. Flow through pipes is usually turbulent in practice.

- One of the most important items of information that an hydraulic engineer needs is the power required to force fluid at a certain steady rate through a pipe or pipe network system. This information is furnished in practice through some routine solution of pipe flow problems with the help of available empirical and theoretical information.

- This lecture deals with the typical approaches to the solution of pipe flow problems in practice.

Concept of Friction Factor in a pipe flow:

- The friction factor in the case of a pipe flow was already mentioned in lecture 26.
- We will elaborate further on friction factor or friction coefficient in this section.
- Skin friction coefficient for a fully developed flow through a closed duct is defined as

$$C_f = \frac{\tau_w}{(1/2)\rho V^2} \tag{35.1}$$

where, V is the average velocity of flow given by $V = Q/A$, Q and A are the volume flow rate through the duct and the cross-sectional area of the duct respectively.

From a force balance of a typical fluid element (Fig. 35.1) in course of its flow through a duct of constant cross-sectional area, we can write

$$\tau_w = \frac{\Delta p^* A}{SL} \tag{35.2}$$

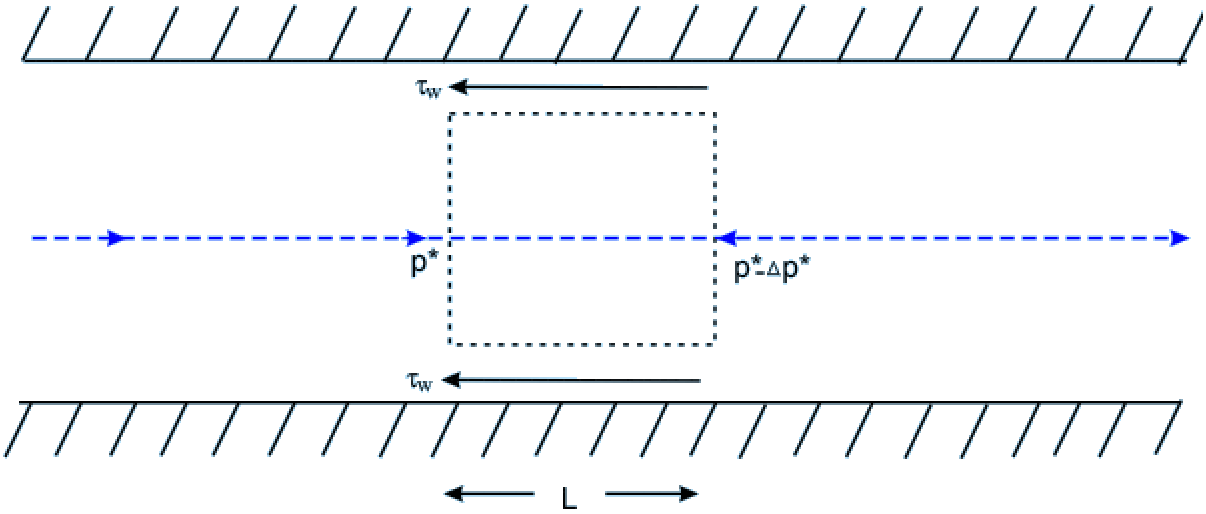


FIG 35.1 Force Balance of a fluid element in the course of flow through a duct

where, τ_w is the shear stress at the wall and Δp^* is the piezometric pressure drop over a length of **L** . **A** and **S** are respectively the cross-sectional area and wetted perimeter of the duct.
Substituting the expression (35.2) in Eq. (35.1), we have,

$$C_f = \frac{\Delta p^* A}{SL(1/2)\rho V^2} = \frac{1}{4}\frac{D_h}{L}\frac{\Delta p^*}{(1/2)\rho V^2} \tag{35.3}$$

where, $D_h = 4A/S$ and is known as the **hydraulic diameter** .

In case of a circular pipe, $D_h=D$, the diameter of the pipe. The coefficient C_f defined by Eqs (35.1) or (35.3) is known as **Fanning's friction factor** .

- To do away with the factor 1/4 in the Eq. (35.3), Darcy defined a friction factor **f (Darcy's friction factor)** as

$$f = \frac{D_h}{L} \frac{\Delta p^*}{(1/2)\rho V^2} \qquad (35.4)$$

- Comparison of Eqs (35.3) and (35.4) gives $f = 4C_f$. Equation (35.4) can be written for a pipe flow as

$$f = \frac{D_h}{L} \frac{\Delta p^*}{(1/2)\rho V^2} \qquad (35.5)$$

- Equation (35.5) is written in a different fashion for its use in the solution of pipe flow problems in practice as

$$\Delta p^* = f \cdot \frac{L}{D_h} \cdot \frac{\rho}{2} V^2 \qquad (35.6a)$$

or in terms of head loss (energy loss per unit weight)

$$h_f = \frac{\Delta p^*}{\rho g} = \frac{f L V^2}{2 g D_h} \qquad (35.6b)$$

where, h_f represents the loss of head due to friction over the length L of the pipe.

- Equation (35.6b) is frequently used in practice to determine h_f
- In order to evaluate h_f, we require to know the value of f. The value of f can be determined from Moody's Chart.

Variation of Friction Factor

- In case of a laminar fully developed flow through pipes, the friction factor, f is found from the exact solution of the Navier-Stokes equation as discussed in lecture 26. It is given by

$$f = \frac{64}{\mathrm{Re}} \qquad (35.7)$$

- In the case of a turbulent flow, friction factor depends on both the Reynolds number and the roughness of pipe surface.
- Sir Thomas E. Stanton (1865-1931) first started conducting experiments on a number of pipes of various diameters and materials and with various fluids. Afterwards, a German engineer Nikuradse carried out experiments on flows through pipes in a very

wide range of Reynolds number.

- A comprehensive documentation of the experimental and theoretical investigations on the laws of friction in pipe flows has been presented in the form of a diagram, as shown in Fig. 35.2, by **L.F. Moody** to show the variation of friction factor, f with the pertinent governing parameters, namely, the Reynolds number of flow and the relative roughness ε / D of the pipe. This diagram is known as **Moody's Chart** which is employed till today as the best means for predicting the values of f .

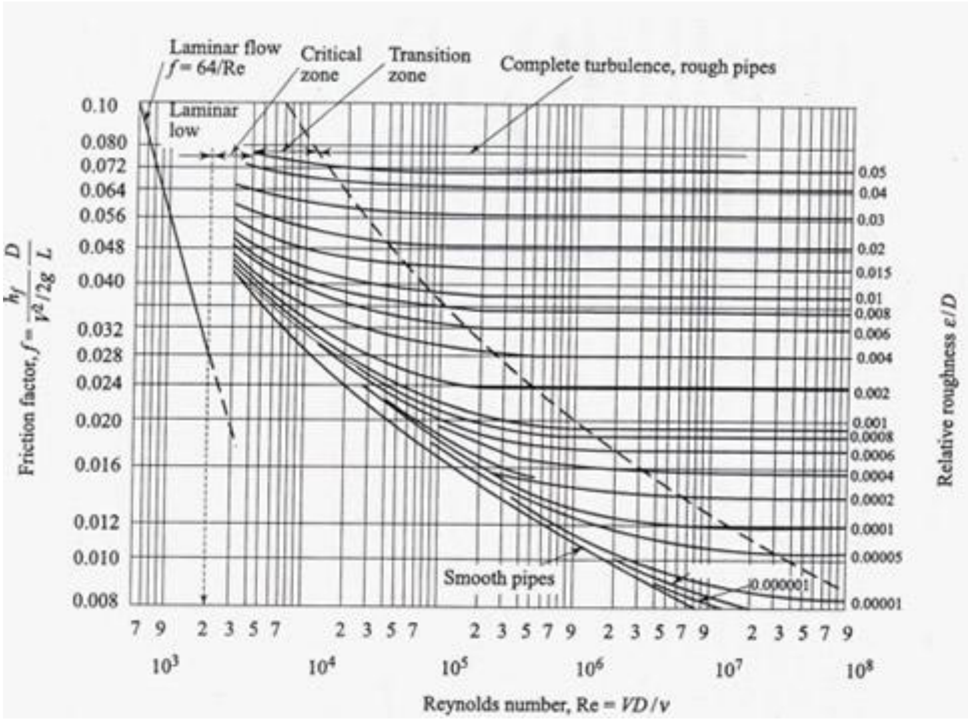


Fig. 35.2 Friction Factors for pipes (adapted from Trans. ASME, 66,672, 1944)

Figure 35.2 depicts that

- The friction factor *f* at a given Reynolds number, in the turbulent region, depends on the relative roughness, defined as the ratio of average roughness to the diameter of the pipe, rather than the absolute roughness.
- For moderate degree of roughness, a pipe acts as a smooth pipe up to a value of Re where the curve of f vs Re for the pipe coincides with that of a smooth pipe. This zone is known as the **smooth zone of flow** .
- The region where f vs Re curves (Fig. 35.2) become horizontal showing that **f** is independent of Re, is known as the **rough zone** and the intermediate region between the smooth and rough zone is known as the **transition zone**.
- The position and extent of all these zones depend on the relative roughness of the pipe. In the smooth zone of flow, the laminar sublayer becomes thick, and hence, it covers appreciably the irregular surface protrusions. Therefore all the curves for smooth flow coincide.
- With increasing Reynolds number, the thickness of sublayer decreases and hence the surface bumps protrude through it. The higher is the roughness of the pipe, the lower is the value of Re at which the curve of f vs Re branches off from smooth pipe curve (Fig. 35.2).

- In the rough zone of flow, the flow resistance is mainly due to the form drag of those protrusions. The pressure drop in this region is approximately proportional to the square of the average velocity of flow. Thus f becomes independent of Re in this region.

In practice, there are three distinct classes of problems relating to flow through a single pipe line as follows:

1. The flow rate and pipe diameter are given. One has to determine the loss of head over a given length of pipe and the corresponding power required to maintain the flow over that length.
2. The loss of head over a given length of a pipe of known diameter is given. One has to find out the flow rate and the transmission of power accordingly.
3. The flow rate through a pipe and the corresponding loss of head over a part of its length are given. One has to find out the diameter of the pipe.

In the first category of problems, the friction factor f is found out explicitly from the given values of flow rate and pipe diameter. Therefore, the loss of head h_f and the power required, P can be calculated by the straightforward application of Eq.(35.6b).

Now Let's do some Examples

Problem

Water at 15^{0}C flow through a 200m long galvanized steel pipe of diameter 250 mm and at 0.225 m^3/s. Note that ninematic viscosity of water at 15^{0}c=1.14×10^{-5} m^2/s and average surface roughness for galvanized steel=0.15 mm.

Determine -

(a) Loss of head due to friction.

(b) Pumping power required to maintain the above flow.

(a)

Average flow velocity $$\bar{V} = \frac{Q}{\pi/4d^2} = \frac{0.225}{\pi/4 \times 0.25^2} = 4.58\ m/s$$

Therefore, Reynolds number $$\mathrm{Re} = \frac{\bar{V}D}{\nu} = \frac{4.58 \times 0.25}{1.14 \times 10^{-5}} = 1 \times 10^{-5}$$

Relative roughness $$\varepsilon/D = \frac{0.15}{250} = 0.0006$$

From Fig. 35.2, f = 0.02

Hence, using Eq. (35.6b)

$$h_f = f\frac{L}{D_h}\left(\frac{V^2}{2g}\right)$$ [for circular pipes $D_h = D$]

$$h_f = 0.02 \times \frac{200}{0.25} \times \left(\frac{4.58^2}{2 \times 9.81}\right) = 17.11\,m$$

(b)

Power required to maintain a flow at the rate of *Q* under a loss of head of h_f is given by

$$p = \rho g h_f Q = 10^3 \times 9.81 \times 17.11 \times 0.225 = 37.77 \times 10^3 \mathrm{w} = 37.77 \mathrm{\ kw}$$

$$p = \rho g h_f Q = 10^3 \times 9.81 \times 17.11 \times 0.225W = 37.77kW$$

Problem

Oil flows through a cast iron pipe of 250 mm diameter such that the loss of head over a pipe length of 100 m is 4 m of the oil. Determine the flow rate of oil through the pipe.

Given: Kinematic viscosity of the oil = 10^{-5} m^2/s
Average surface roughness of iron = 0.25 mm

Since the velocity is unknown, Re is unknown.

Relative rougness $= \frac{0.25}{250} = 0.001$

A guess of the friction factor at this relative roughness is made from Fig. 35.2 as *f* = 0.02
Then Eq. (35.6b) gives a first trial

$$h_f = f\left(\frac{L}{Dn}\right)\left(\frac{V^2}{2g}\right)$$

$$\Rightarrow \quad 4 = 0.02 \times \frac{100}{0.25} \times \frac{V^2}{2 \times 9.81}$$

$$\Rightarrow \quad V = 3.132\ m/s$$

Hence,

$$\Rightarrow \quad \mathrm{Re} = \frac{VD}{\nu} = \frac{3.132 \times 0.25}{10.5} = 7.8 \times 10^4$$

At this Re, *f* =0.023 (Fig 35.2). The second step of iteration involves recalculation of *v* with f=0.0225 as

$$4 = 0.023 \times \frac{100}{0.25} \times \frac{V^2}{2 \times 9.81}$$

This gives $V = 2.92\,m/s$ and $\mathrm{Re} = \dfrac{2.92 \times 0.25}{10^{-5}} = 7.4 \times 10^4$

The value of *f* at this Re = 0.0235.

Therefore, the flow rate $Q = V \times \pi/4d^2 = 2.92 \times \pi/4 \times 0.25^2 = 0.143\ \mathrm{m}^3/s$

Problem

Water flows through a galvanized iron pipe at 0.09 m^3/s. Determine the size of the pipe needed to transmit water a distance 200 m with a head loss 10 m.

Given: Kinematic viscosity of the water = 1.14 $\times 10^{-5}\ m^2/s$
Average surface roughness for galvanized iron = 0.15 mm

$$V = \frac{Q}{\pi/4\, D^2} = \frac{0.09}{\pi D^2/4}$$

From (35.6b),

$$h_f = f\left(\frac{L}{D_h}\right)\left(\frac{V^2}{2g}\right)$$

$$\Rightarrow\ 10 = f\left(\frac{200}{D}\right) \times \left(\frac{0.09}{\pi D^2/4}\right)^2 \times \frac{1}{2 \times 9.81}$$

$$\Rightarrow\ D^5 = 0.013\, f \qquad (35.8)$$

and

$$\mathrm{Re} = \frac{0.09}{\pi D^2/4} \times \frac{D}{1.14 \times 10^{-5}} = \frac{10^5}{D} \qquad (35.9)$$

First, a guess in *f* is made as 0.024.

Then from Eq. (35.8) D = 0.2 m and from Eq. (35.9) $\mathrm{Re} = 5 \times 10^5$

The relative roughness $\in/D = \dfrac{0.15}{200} = 0.00075$

With the values of Re and $\in/D$, the updated value of *f* is found from Fig. 35.2 as 0.018. With this value of *f*,recalculation of *D* and Re from Eqs (35.8) and (35.9) gives

D =0.188 m
Re=5.323×10^5

Also,

$$\epsilon / D = \frac{0.15}{0.188 \times 10^3} = 0.0008$$

The new values of Re and ϵ / D predict $f \sim 0.018$

Hence D=0.188 m

Concept of Flow Potential and Flow Resistance

- Consider the flow of water from one reservoir to another as shown in Fig. 35.3. The two reservoirs A and B are maintained with constant levels of water. The difference between these two levels is ΔH as shown in the figure. Therefore water flows from reservoir *A* to reservoir *B* .

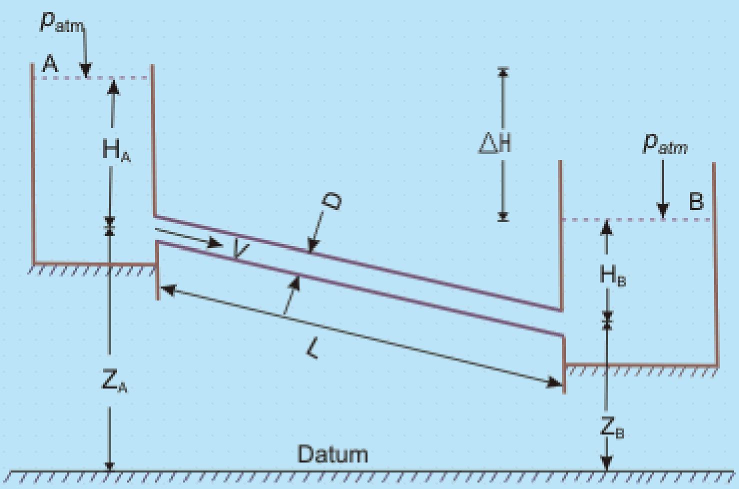


Fig 35.3 Flow of liquid from one reservoir to another

- Application of Bernoulli's equation between two points *A* and *B* at the free surfaces in the two reservoirs gives

$$\frac{p_{atm}}{\rho g} + H_A + Z_A = \frac{p_{atm}}{\rho g} + H_B + Z_B + h_f$$

$$\Delta H = (Z_A + H_A) - (Z_B + H_B) = h_f \qquad (35.10)$$

where h_f is the loss of head in the course of flow from *A* to *B* .

- Therefore, Eq. (35.10) states that under steady state, the head causing flow ΔH becomes equal to the total loss of head due to the flow.
- Considering the possible hydrodynamic losses, the total loss of head h_f can be written in terms of its different components as

$$h_f = \frac{0.5V^2}{2g} + f\frac{L}{D}\frac{V^2}{2g} + \frac{V^2}{2g} = \left(1.5 + f\frac{L}{D}\right)\frac{V^2}{2g} \qquad (35.11)$$

Loss of heat at entry to the pipe from reservoir A	Friction loss in pipe over its length L	Exit loss to the reservoir B

where, *V* is the average velocity of flow in the pipe.

(contd from previous...) Concept of Flow Potential and Flow Resistance

The velocity *V* in the above equation is usually substituted in terms of flow rate *Q* , since, under steady state, the flow rate remains constant throughout the pipe even if its diameter changes. Therefore, replacing V in Eq. (35.11) as $V = 4Q/\pi D^2$ we finally get

$$h_f = \left[8\left(1.5 + f\frac{L}{D}\right)\frac{1}{\pi^2 D^4 g}\right]Q^2$$

or, $$h_f = RQ^2 \qquad (35.12)$$

where $$R = \left[\frac{8}{\pi^2 D^4 g}\left(1.5 + f\frac{L}{D}\right)\right] \qquad (35.13)$$

The term *R* is defined as the **flow resistance** .

In a situation where ***f*** becomes independent of **Re**, the flow resistance expressed by Eg. (35.13) becomes simply a function of the pipe geometry. With the help of Eq. (35.10), Eq. (35.12) can be written as

$$\Delta H = RQ^2 \qquad (35.14)$$

- ΔH in Eq. (35.14) is the head causing the flow and is defined as the difference in flow potentials between A and B.

  This equation is comparable to the voltage-current relationship in a purely resistive electrical circuit. In a purely resistive electrical circuit, $\Delta V = RI$, where ΔV is the voltage or electrical potential difference across a resistor whose resistance is *R* and the electrical current flowing through it is *I*.
- The difference however is that while the voltage drop in an electrical circuit is linearly proportional to current, the difference in the flow potential in a fluid circuit is proportional to the square of the flow rate.
- Therefore, the fluid flow system as shown in Fig. 35.3 and described by Eq. (35.14) can be expressed by an equivalent electrical network system as shown in Fig. 35.4.

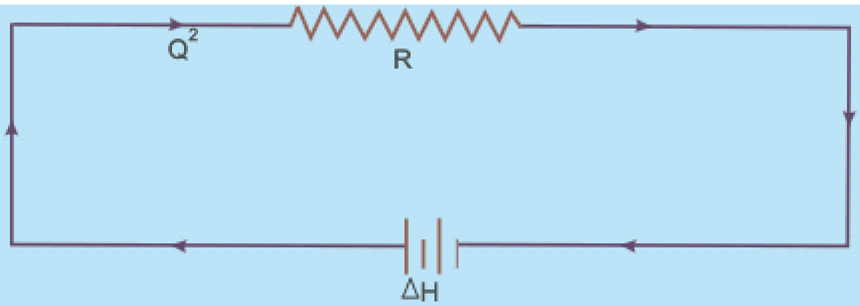


Fig 35.4 Equivalent electrical network system for a simple pipe flow problem shown in Fig.35.3

Flow Through Branched Pipes

In several practical situations, flow takes place under a given head through different pipes jointed together either in series or in parallel or in a combination of both of them.

Pipes in Series

- If a pipeline is joined to one or more pipelines in continuation, these are said to constitute pipes in series. A typical example of pipes in series is shown in Fig. 36.1. Here three pipes *A*, *B* and *C*are joined in series.

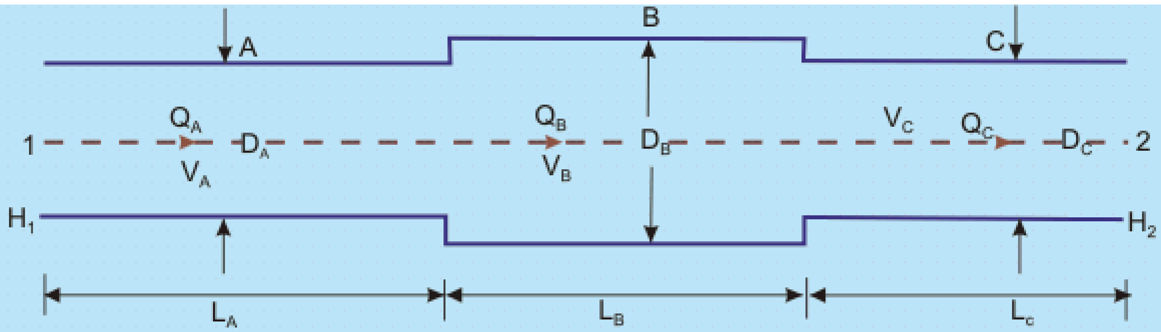


Fig 36.1 Pipes in series

In this case, rate of flow *Q* remains same in each pipe. Hence,

$$Q_A = Q_B = Q_C = Q$$

- If the total head available at Sec. 1 (at the inlet to pipe A) is H_1 which is greater than H_2, the total head at Sec. 2 (at the exit of pipe C), then the flow takes place from 1 to 2 through the system of pipelines in series.
- Application of Bernoulli's equation between Secs.1 and 2 gives

$$H_1 - H_2 = h_f$$

where, h_f is the loss of head due to the flow from 1 to 2. Recognizing the minor and major losses associated with the flow, h_f can be written as

$$h_f = f_A \frac{L_A}{D_A}\frac{V_A^2}{2g} + \frac{(V_A - V_B)^2}{2g} + f_B \frac{L_B}{D_B}\frac{V_B^2}{2g} + \left(\frac{1}{C_C} - 1\right)^2 \frac{V_C^2}{2g} + f_C \frac{LC}{D_C}\frac{V_C^2}{2g} \quad (36.1)$$

Friction loss in pipe A	Loss due to enlargement at entry to pipe B	Friction loss in pipe B	Loss due to abrupt contraction at entry to pipe C	Friction loss in pipe C

The subscripts *A, B* and *C* refer to the quantities in pipe A, B and C respectively. C_c is the coefficient of contraction.

- The flow rate Q satisfies the equation

$$Q = \frac{\pi D_A^2}{A} V_A = \frac{\pi D_B^2}{4} V_B = \frac{\pi D_C^2}{4} V_C \quad (36.2)$$

Velocities V_A, V_B and V_C in Eq. (36.1) are substituted from Eq. (36.2), and we get

$$h_f = \Bigg[\underbrace{\frac{8}{g\pi^2} f_A \frac{L_A}{D_A^5}}_{(R_1)} + \underbrace{\frac{8}{g\pi^2}\left(1 - \frac{D_A^2}{D_B^2}\right)^2 \frac{1}{D_A^4}}_{(R2)} + \underbrace{\frac{8}{g\pi^2} f_B \frac{L_B}{D_B^5}}_{(R3)} + \underbrace{\frac{8}{g\pi^2}\left(\frac{1}{C_C} - 1\right)^2 \frac{1}{D_C^4}}_{(R4)} + \underbrace{\frac{8}{g\pi^2} f_c \frac{L_C}{D_C^5}}_{(R5)}\Bigg] Q^2 \quad (36.3)$$

$$h_f = RQ_2$$

$$R = R_1 + R_2 + R_3 + R_4 + R_5 \quad (36.4)$$

Equation (36.4) states that the total flow resistance is equal to the sum of the different resistance components. Therefore, the above problem can be described by an equivalent electrical network system as shown in Fig. 36.2.

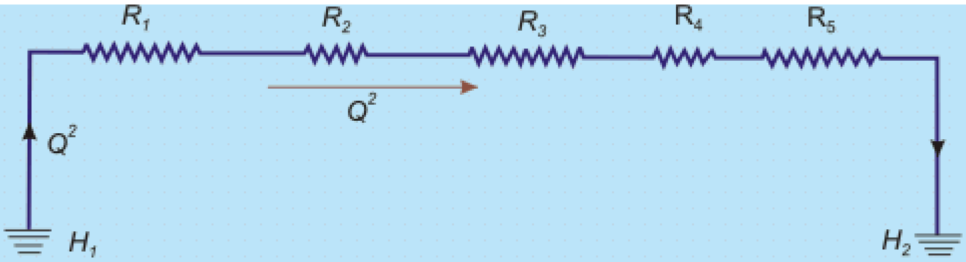


Fig 36.2 Equivalent electrical network system for through pipes in series

Pipes In Parallel

- When two or more pipes are connected, as shown in Fig. 36.3, so that the flow divides and subsequently comes together again, the pipes are said to be in parallel.
- In this case (Fig. 36.3), equation of continuity gives

$$Q = Q_A + Q_B \qquad (36.5)$$

where, *Q* is the total flow rate and Q_A and Q_B are the flow rates through pipes *A* and *B* respectively.

- Loss of head between the locations 1 and 2 can be expressed by applying Bernoulli's equation either through the path 1-A-2 or 1-B-2.
- Therefore, we can write

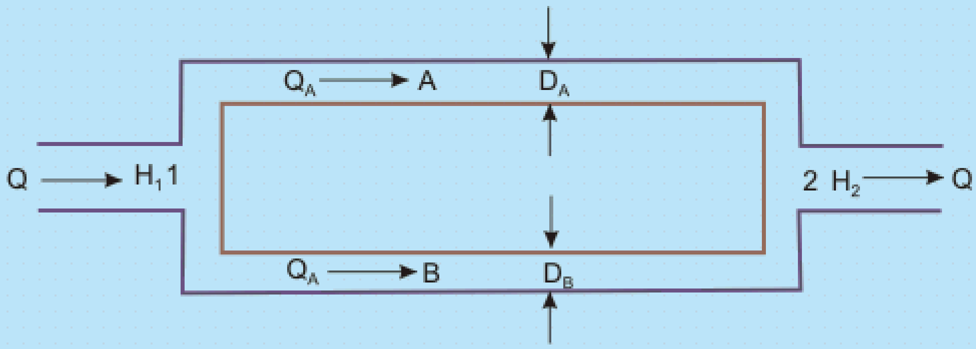


Fig 36.3 Pipes in Parallel

$$H_1 - H_2 = f_A \frac{L_A}{D_A}\frac{V_A^2}{2g} = \frac{8L_A}{\pi^2 D_A^5 g} f_A Q_A^2$$

and

$$H_1 - H_2 = f_B \frac{L_B}{D_B}\frac{V_B^2}{2g} = \frac{8L_B}{\pi^2 D_B^5 g} f_B Q_B^2$$

Equating the above two expressions, we get -

$$Q_A^2 = \frac{R_B}{R_A} Q_B^2 \tag{36.6}$$

where, $R_A = \dfrac{8L_A}{\pi^2 D_A^5 g} f_A$

$$R_B = \frac{8L_B}{\pi^2 D_B^5 g} f_B$$

Equations (36.5) and (36.6) give -

$$Q_A = \frac{K}{1+K} Q, \; Q_B = \frac{1}{1+K} Q \tag{36.7}$$

where, $K = \sqrt{R_B / R_A}$ (36.8)

- The flow system can be described by an equivalent electrical circuit as shown in Fig. 36.4.

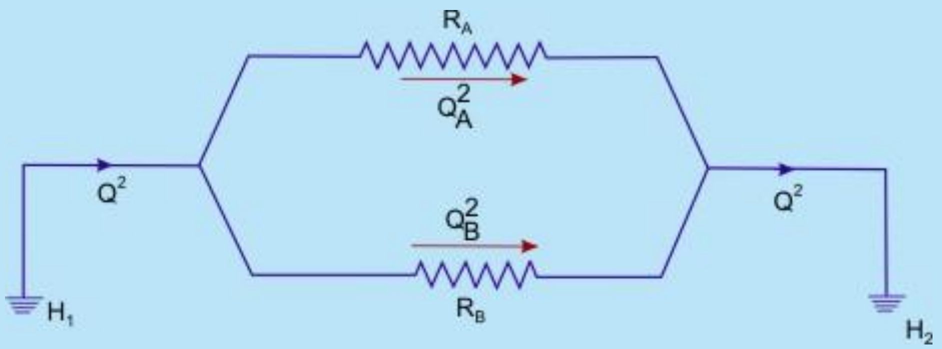


Fig 36.4 Equivalent electrical network system for flow through pipes in parallel

From the above discussion on flow through branched pipes (pipes in series or in parallel, or in combination of both), the following principles can be summarized:

1. The friction equation must be satisfied for each pipe.
2. There can be only one value of head at any point.
3. Algebraic sum of the flow rates at any junction must be zero. i.e., the total mass flow rate towards the junction must be equal to the total mass flow rate away from it.
4. Algebraic sum of the products of the flux (Q^2) and the flow resistance (the sense being determined by the direction of flow) must be zero in any closed hydraulic circuit.

The principles 3 and 4 can be written analytically as

$\sum Q = 0$ at a node (Junction) (36.9)

$$\sum R|Q|Q = 0 \quad \text{in a loop} \qquad (36.10)$$

While Eq. (36.9) implies the principle of continuity in a hydraulic circuit, Eq. (36.10) is referred to aspressure equation of the circuit.

Pipe Network: Solution by Hardy Cross Method

- The distribution of water supply in practice is often made through a pipe network comprising a combination of pipes in series and parallel. The flow distribution in a pipe network is determined from Eqs(36.9) and (36.10).
- The solution of Eqs (36.9) and (36.10) for the purpose is based on an iterative technique with an initial guess in *Q*
- The method was proposed by Hardy-Cross and is described below:
  - The flow rates in each pipe are assumed so that the continuity (Eq. 36.9) at each node is satisfied. Usually the flow rate is assumed more for smaller values of resistance *R*and vice versa.
  - If the assumed values of flow rates are not correct, the pressure equation Eq. (36.10) will not be satisfied. The flow rate is then altered based on the error in satisfying the Eq. (36.10).
- Let Q_0 be the correct flow in a path whereas the assumed flow be Q. The error d*Q* in flow is then defined as

$$Q = Q_0 + dQ \qquad (36.11)$$

Let $$h = R|Q|Q \qquad (36.12a)$$

and $$h' = R|Q_0|Q_0 \qquad (36.12b)$$

Then according to Eq. (36.10)

$$\sum h' = 0 \quad \text{in a loop} \qquad (36.13a)$$

and $$\sum h = e \quad \text{in a loop} \qquad (36.13b)$$

Where *'e'* is defined to be the error in pressure equation for a loop with the assumed values of flow rate in each path.
From Eqs (36.13a) and (36.13b) we have

$$\sum (h - h') = e$$

or, $$\sum dh = e \qquad (36.14)$$

Where dh (= h - h') is the error in pressure equation for a path. Again from Eq. (36.12a), we can write

$$\frac{dh}{dQ} = 2R|Q|$$

or, $$dh = 2R|Q|dQ \qquad (36.15)$$

Substituting the value of dh from Eq. (36.15) in Eq. (36.14) we have

$$\sum 2R\,|Q|\,dQ = e$$

Considering the error dQ to be the same for all hydraulic paths in a loop, we can write

$$dQ = \frac{e}{\sum 2R\,|Q|} \qquad (36.16)$$

The Eq. (36.16) can be written with the help of Eqs (36.12a) and (36.12b) as

$$dQ = \frac{\sum R\,|Q|\,Q}{\sum 2R\,|Q|} \qquad (36.17)$$

The error in flow rate d*Q* is determined from Eq. (36.17) and the flow rate in each path of a loop is then altered according to Eq. (36.11).

The Hardy-Cross method can also be applied to a hydraulic circuit containing a pump or a turbine. The pressure equation (Eq. (36.10)) is only modified in consideration of a head source (pump) or a head sink (turbine) as

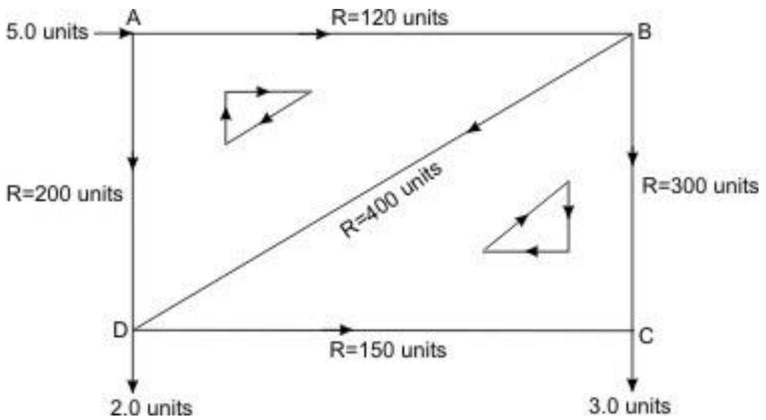
$$-\Delta H + \sum R\,|Q|\,Q = 0 \qquad (36.18)$$

where ΔH is the head delivered by a source in the circuit. Therefore, the value of ΔH to be substituted in Eq. (36.18) will be positive for a pump and negative for a turbine.

Problem

A pipe network with two loops is shown in Fig. 36.5. Determine the flow in each pipe for an inflow of 5 units at the junction A and outflows of 2.0 units and 3.0 units at junctions *D* and *C* respectively. The resistance *R*for different pipes are shown in the figure.

Flow direction is assumed positive clockwise for both the loops *ABD* and *BCD* . The iterative solutions based on Hardy-Cross method has been made. The five trials have been made and the results of each trial is shown in Fig. 36.5; for each trial, d *Q* is calculated from Eq. (36.17). After fifth trial, the error d *Q* is so small that it changes the flow only in the third place of decimal. Hence the calculation has not been continued beyond the fifth trial.



First Trial

Loop ABD		Loop BCD	
R|Q|Q	2R|Q|	R|Q|Q	2R|Q|
$120 \times 2^2 = 480$	$2 \times 120 \times 2 = 480$	$300 \times (1.2)^2 = 432$	2 ×□300 ×1.2 = 720
$400 \times (0.8)^2 = 256$	$2 \times 400 \times 0.8 = 640$	$-150 \times (1.8)^2 = -486$	2 ×□150 ×1.8 = 540
$-200 \times 3^2 = -1800$	$2 \times 200 \times 3 = 1200$	$-400 \times (0.8)^2 = -256$	2 ×□400 ×0.8 = 640
Σ R|Q|Q = -1064	2Σ R|Q| = 2320	Σ R|Q|Q = - 310	2Σ R|Q| = 1900

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-1064}{2320} = -0.46$$

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-300}{1900} = -0.16$$

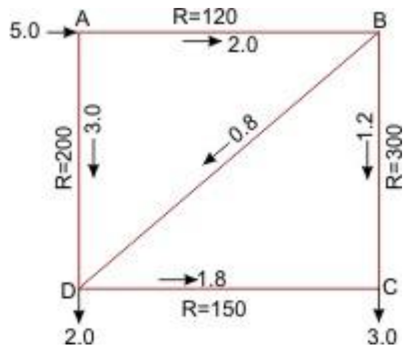


Fig 36.6 a Flow Distribution in a pipe network after first trial

Second Trial

Loop ABD		Loop BCD	
R|Q|Q	2R|Q|	R|Q|Q	2R|Q|
$120 \times (2.46)^2 = 726.19$	2 ×□120 ×2.46 = 590.40	$300 \times (1.36)^2 = 554.88$	2 ×□300 ×1.36 = 816
$400 \times (1.10)^2 = 484.00$	2 ×□400 ×1.10 = 880.00	$-150 \times (1.64)^2 = -403.44$	2 ×□150 ×1.64 = 492
$-1200 \times (2.54)^2 = -1290.32$	2 ×□200 × 2.54 = 1016.00	$-400 \times (1.10)^2 = -484.00$	2 ×□400 ×1.10= 880
Σ R|Q|Q = -50.13	2Σ R|Q| = 2486.40	Σ R|Q|Q = - 332.56	2Σ R|Q| = 2188

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-50.13}{2486.40} = -0.02$$

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-332.56}{2188} = -0.15$$

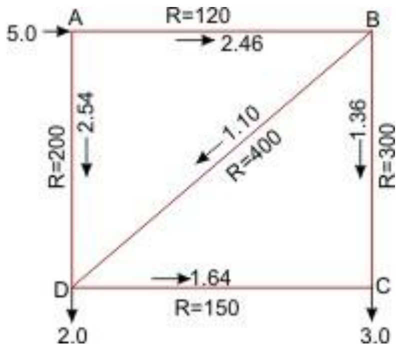


Fig 36.6 a Flow Distribution in a pipe network after second trial

Third Trial

Loop ABD		Loop BCD	
R|Q|Q	2R|Q|	R|Q|Q	2R|Q|
$120 \times (2.48)^2 = 738.05$	$2 \times 120 \times 2.48 = 595.20$	$300 \times (1.51)^2 = 684.03$	$2 \times 300 \times 1.51 = 906.00$
$400 \times (0.97)^2 = 376.36$	$2 \times 400 \times 0.97 = 776.00$	$-150 \times (1.49)^2 = -333.01$	$2 \times 150 \times 1.49 = 447$
$-200 \times (2.52)^2 = -1270.08$	$2 \times 200 \times 2.52 = 1008.00$	$-400 \times (0.97)^2 = -376.36$	$2 \times 400 \times 0.97 = 776.00$
Σ R|Q|Q = -155.67	2Σ R|Q| = 2379.20	Σ R|Q|Q = - 25.34	2Σ R|Q| = 2129

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-155.67}{2379.67} = -0.06$$

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-25.34}{2129} = -0.01$$

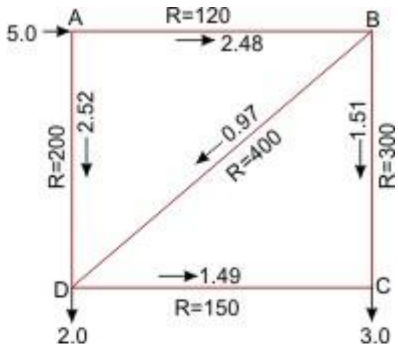


Fig 36.6 a Flow Distribution in a pipe network after third trial

Fourth Trial

Loop ABD		Loop BCD	
R|Q|Q	2R|Q|	R|Q|Q	2R|Q|
$120 \times (2.54)^2 = 774.20$	$2 \times 120 \times 2.54 = 609.60$	$300 \times (1.52)^2 = 693.12$	$2 \times 300 \times 1.52 = 912.00$
$400 \times (1.02)^2 = 416.16$	$2 \times 400 \times 1.02 = 816.00$	$-150 \times (1.48)^2 = -328.56$	$2 \times 150 \times 1.48 = 444.00$
$-200 \times (2.46)^2 = -1210.32$	$2 \times 200 \times 2.46 = 984.00$	$-400 \times (1.02)^2 = -416.16$	$2 \times 400 \times 1.02 = 816.00$
Σ R|Q|Q = -19.96	2Σ R|Q| = 2409.60	Σ R|Q|Q = - 51.6	2Σ R|Q| = 2172

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-19.96}{2409.60} = -0.008$$

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-51.6}{2172} = -0.02$$

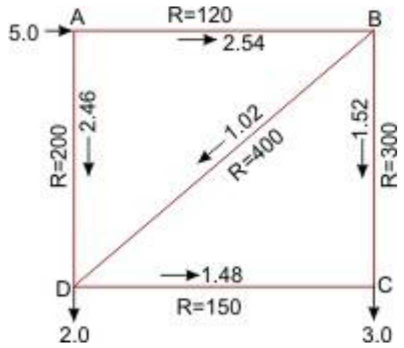


Fig 36.6 a Flow Distribution in a pipe network after fourth trial

Fifth Trial

Loop ABD		Loop BCD	
R|Q|Q	2R|Q|	R|Q|Q	2R|Q|
120 × (2.58)^2 = 779.08	2 ×□120 × 2.58 = 619.20	300 × (1.54)^2 = 711.48	2 ×□300 × 1.54 = 924.00
400 × (1.008)^2 = 406.42	2 ×□400 × 1.008 = 806.40	-150 × (1.46)^2 = -319.74	2 ×□150 ×□1.46 = 438.00
-200 × (2.452)^2 = -1202.46	2 ×□200 × 2.452 = 980.80	-400 × (1.08)^2 = -406.42	2 ×□400 × 1.008 = 806.00

Σ R|Q|Q = -16.96 2Σ R|Q| = 2406.40 Σ R|Q|Q = - 14.68 2Σ R|Q| = 2168.40

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-16.96}{2406.40} = -0.007$$

$$dQ = \frac{\Sigma R|Q|Q}{\Sigma 2R|Q|} = \frac{-14.68}{2168.40} = -0.007$$

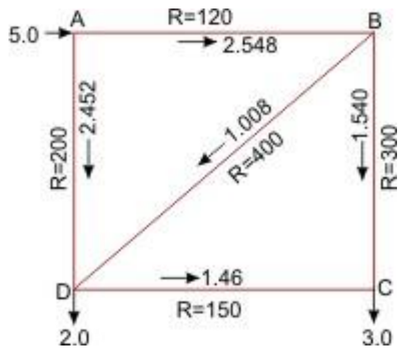


Fig 36.6 a Flow Distribution in a pipe network after fifth trial

Flow Through Pipes With Side Tappings

- In course of flow through a pipe, a fluid may be withdrawn from the side tappings along the length of the pipe as shown in Fig. 37.1

- If the side tappings are very closely spaced, the loss of head over a given length of pipe can be obtained as follows:

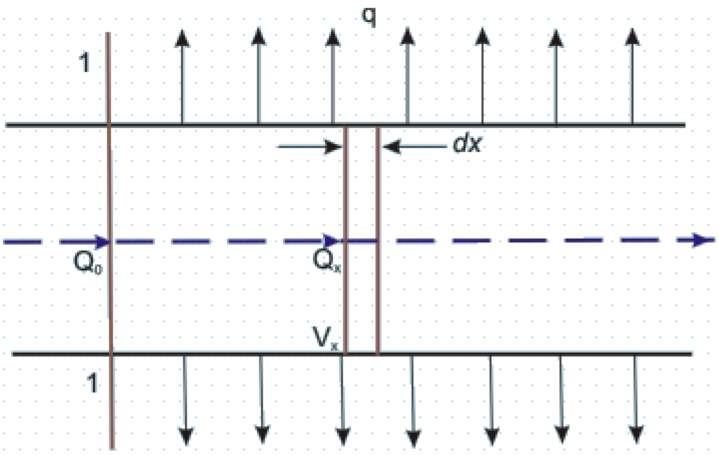


Fig. 37.1 Flow through pipes with side tappings

- The rate of flow through the pipe, under this situation, decreases in the direction of flow due to side tappings. Therefore, the average flow velocity at any section of the pipe is not constant.
- The frictional head loss dh_f over a small length dx of the pipe at any section can be written as

$$dh_f = f\frac{dx}{D}\frac{V_x^2}{2g} \tag{37.1}$$

where, V_x is the average flow velocity at that section.

- If the side tappings are very close together, Eq. (37.1) can be integrated to determine the loss of head due to friction over a given length L of the pipe, provided, V_x can be replaced in terms of the length of the pipe.
- Let us consider, for this purpose, a Section 1-1 at the upstream just after which the side tappings are provided. If the tappings are uniformly and closely spaced, so that the fluid is removed at a uniform rate q per unit length of the pipe, then the volume flow rate Qx at a distance x from the inlet Section 1-1 can be written as

$$Q_x = Q_0 - qx$$

where, Q_0 is the volume flow rate at Sec.1-1.

- Hence,

$$V_x = \frac{4Q_x}{\pi D^2} = \frac{4Q_0}{\pi D^2}\left(1 - \frac{q}{Q_0}x\right) \tag{37.2}$$

Substituting V_x from Eq. (37.2) into Eq. (37.1), we have,

$$dh_f = \frac{16Q_0^2 f}{2\pi^2 D^5 g}\left(1 - \frac{q}{Q_0}x\right)^2 dx \tag{37.3}$$

Therefore, the loss of head due to friction over a length L is given by

$$h_f = \int_0^L dh_f = \frac{8Q_0^2 f L}{\pi^2 D^5 g}\left(1 - \frac{q}{Q_0}L + \frac{1}{3}\frac{q^2}{Q_0^2}L^2\right) \tag{37.4a}$$

- Here, the friction factor f has been assumed to be constant over the length L of the pipe. If the entire flow at Sec.1-1 is drained off over the length L , then,

$$Q_0 - qL = 0 \quad or \quad \frac{q}{Q_0} = \frac{1}{L}$$

Equation (37.4a), under this situation, becomes

$$h_f = \frac{8}{3}\frac{Q_0^2 f L}{\pi^2 D^5 g} = \frac{1}{3} f \frac{L}{D} V_0^2 \frac{1}{2g} \tag{37.4b}$$

- where, V0 is the average velocity of flow at the inlet Section 1-1.

  Equation (37.4b) indicates that the loss of head due to friction over a length L of a pipe, where the entire flow is drained off uniformly from the side tappings, becomes one third of that in a pipe of same length and diameter, but without side tappings.

Losses In Pipe Bends

- Bends are provided in pipes to change the direction of flow through it. An additional loss of head, apart from that due to fluid friction, takes place in the course of flow through pipe bend.
- The fluid takes a curved path while flowing through a pipe bend as shown in Fig. 37.2.

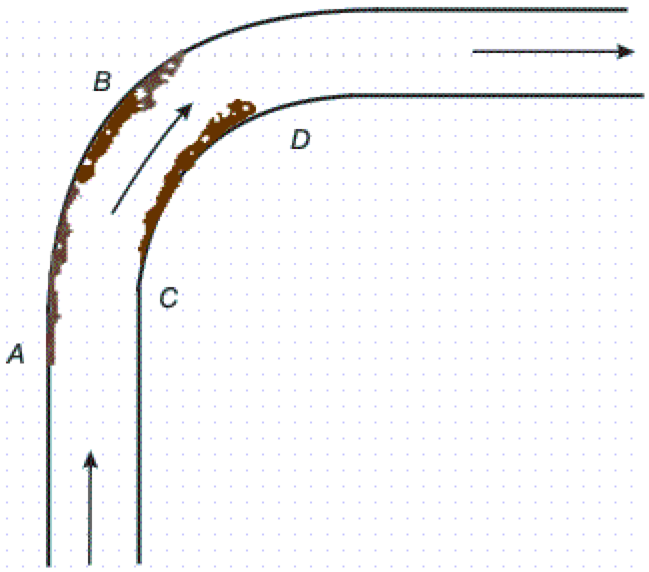


Fig. 37.2 Flow through pipe bend

Whenever a fluid flows in a curved path, there must be a force acting radially inwards on the fluid to provide the inward acceleration, known as centripetal acceleration .

This results in an increase in pressure near the outer wall of the bend, starting at some point A (Fig. 37.2) and rising to a maximum at some point B . There is also a reduction of pressure near the inner wall giving a minimum pressure at C and a subsequent rise from C to D . Therefore between A and Band between C and D the fluid experiences an adverse pressure gradient (the pressure increases in the direction of flow).

Fluid particles in this region, because of their close proximity to the wall, have low velocities and cannot overcome the adverse pressure gradient and this leads to a separation of flow from the boundary and consequent losses of energy in generating local eddies. Losses also take place due to a secondary flow in the radial plane of the pipe because of a change in pressure in the radial depth of the pipe.

This flow, in conjunction with the main flow, produces a typical spiral motion of the fluid which persists even for a downstream distance of fifty times the pipe diameter from the central plane of the bend. This spiral motion of the fluid increases the local flow velocity and the velocity gradient at the pipe wall, and therefore results in a greater frictional loss of head than that which occurs for the same rate of flow in a straight pipe of the same length and diameter.

The additional loss of head (apart from that due to usual friction) in flow through pipe bends is known as bend loss and is usually expressed as a fraction of the velocity head as $KV^2/2g$, where V is the average velocity of flow through the pipe. The value of K depends on the total length of the bend and the ratio of radius of curvature of the bend and pipe diameter R/D. The radius of curvature R is usually taken as the radius of curvature of the centre line of the bend. The factor K varies slightly with Reynolds number Re in the typical range of Re encountered in practice, but increases with surface roughness.

Losses In Pipe Fittings

- An additional loss of head takes place in the course of flow through pipe fittings like valves, couplings and so on. In-general, more restricted the passage is, greater is the loss of head.
- For turbulent flow, the losses are proportional to the square of the average flow velocity and are usually expressed by $KV^2/2g$, where V is the average velocity of flow. The value of Kdepends on the exact shape of the flow passages. Typical values of K are

Approximate Loss Coefficients, K for Commercial Pipe Fittings .

Type and position of fittings	Values of *K*
Globe valve,wide open	**10**
Gate valve, wide open	**0.2**
Gate valve, three-quarters open	**1.15**
Gate valve, half open	**5.6**
Gate valve, quarter open	**24**
Pump foot valve	**1.5**
90°elbow(threaded)	**0.9**
45°elbow(threaded)	**0.4**
Side outlet of T junction	**1.8**

- Since the eddies generated by fittings persist for some distance downstream, the total loss of head caused by two fittings close together is not necessarily the same as the sum of the losses which,each alone would cause.
  These losses are sometimes expressed in terms of an equivalent length of an unobstructed straight pipe in which an equal loss would occur for the same average flow velocity. That is

$$K\frac{V^2}{2g} = f\frac{L_e}{D}\frac{V^2}{2g} \quad or \quad \frac{L_e}{D} = \frac{K}{f} \qquad (37.5)$$

where, L_e represents the equivalent length which is usually expressed in terms of the pipe diameter as given by Eq. (37.5). Thus L_e/D depends upon the friction factor f , and therefore on the Reynolds number and roughness of the pipe.

.

Power Transmission By A Pipeline

- In certain occasions, hydraulic power is transmitted by conveying fluid through a pipeline. For example, water from a reservoir at a high altitude is often conveyed by a pipeline to an impulse hydraulic turbine in an hydroelectric power station. The hydrostatic head of water is thus transmitted by a pipeline. Let us analyse the efficiency of power transmission under this situation.

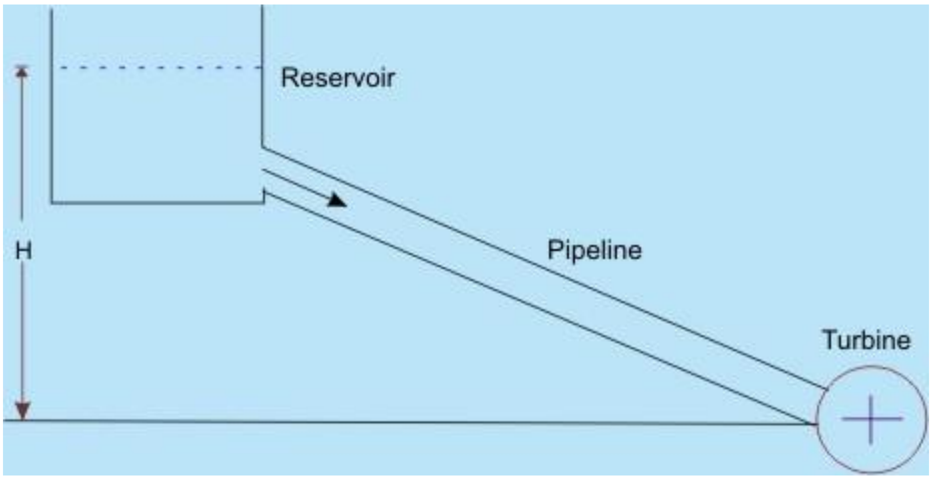


Fig. 37.3 Transmission of hydraulic power by a pipeline to a turbine

The potential head of water in the reservoir = H (the difference in the water level in the reservoir and the turbine center)

The head available at the pipe exit (or at the turbine entry) $= H_E = H - h_f$

Where h_f is the loss of head in the pipeline due to friction.

- Assuming that the friction coefficient and other loss coefficients are constant, we can write

  $$h_f = RQ^2$$

Where Q is the volume flow rate and R is the hydraulic resistance of the pipeline. Therefore, the power available P at the exit of the pipeline becomes

$$P = \rho g Q H_E = \rho g Q (H - RQ^2)$$

For P to be maximum, for a given head H, dP/dQ should be zero. This gives

$$H - 3RQ^2 = 0 \tag{37.6}$$

or, $$RQ^2 = h_f = \frac{H}{3}$$

$[d^2 P / dQ^2]$ is always negative which shows that P has only a maximum value (not a minimum) with Q.

- From Eq. (37.6), we can say that maximum power is obtained when one third of the head available at the source (reservoir) is lost due to friction in the flow.
- The efficiency of power transmission η_p is defined as

$$\eta_p = \frac{\rho g Q\left(H - RQ^2\right)}{\rho g Q H} = 1 - \frac{RQ^2}{H} \qquad (37.7)$$

1. The efficiency η_p equals to unity for the trivial case of Q = 0.
2. For flow to commence and hence η_p is a monotonically decreasing function of Q from a maximum value of unity to zero.
3. The zero value of η_p corresponds to the situation given by $RQ^2 = H \left(or\ ,\ Q = \sqrt{H/R}\right)$ when the head H available at the reservoir is totally lost to overcome friction in the flow through the pipe.

- The efficiency of transmission at the condition of maximum power delivered is obtained by substituting RQ2 from Eq. (37.6) in Eq. (37.7) as

$$\eta_{at\ P=P_{\max}} = 1 - \frac{H/3}{H} = \frac{2}{3}$$

Therefore the maximum power transmission efficiency through a pipeline is 67%.

Exercise Problems - Chapter 11

1. Calculate the force F required on the piston to discharge $500\ \mathrm{mm}^3/s$ of water through a syringe (see Fig. 37.4), taking into account the frictional loss in the syringe needle only. Assume fully developed laminar flow in the syringe needle. Take the dynamic viscosity of water $\mu = 10^{-3}\ \mathrm{Ns/m}^2$.

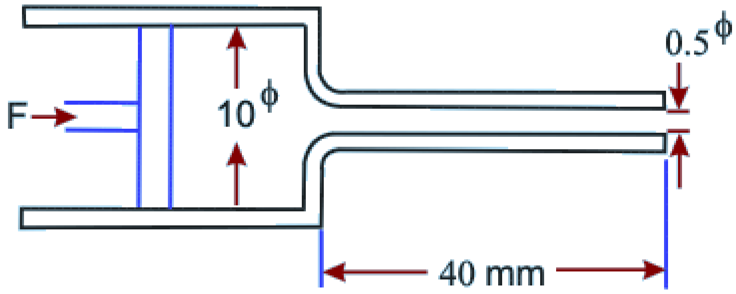


Figure 37.4

2. A hydrocarbon oil (viscosity 0.025 pa-s and density 900 kg/m3) is transported using a 0.6 m diameter, 10 km long pipe. The maximum allowable pressure drop across the pipe length is 1 MPa. Due to a maintenance schedule on this pipeline, it is required to use a 0.4 m diameter, 10 km long pipe to pump the oil at the same volumetric flow rate as in the earlier case. Estimate the pressure drop for the 0.4 m diameter pipe. Assume both pipes to be hydrodynamically smooth and in the range of operating conditions, the Fanning friction factor is given by:

$$f = 0.079\, Re^{-0.25}$$

3. Two reservoirs 1 and 2 are connected as shown in the Fig 37.5 through a turbine T. Given the friction factor relation

$$\frac{1}{\sqrt{f}} = 2\log_{10}\left(Re\sqrt{f}\right) - 0.8$$

for the connecting pipes, the turbine characteristics $H = 10^3 Q\, m$ of water [Q in m3/s] and an ideal draft tube at the discharge end, find (a) the volume flow rate between the two reservoirs and (b) the power developed by the turbine. Note:

$$f = \frac{(\Delta p/L)d}{\frac{1}{2}\rho U^2}$$

$$\mathrm{Re} = \rho U d/\mu$$

$$\rho = 1000 kg/m^3$$

$$\mu = 80\times 10^{-5}\, pas.$$

Use an initial guess for power developed by the turbine as 1 MW. Show only two iterations . Also H is head available at the turbine.

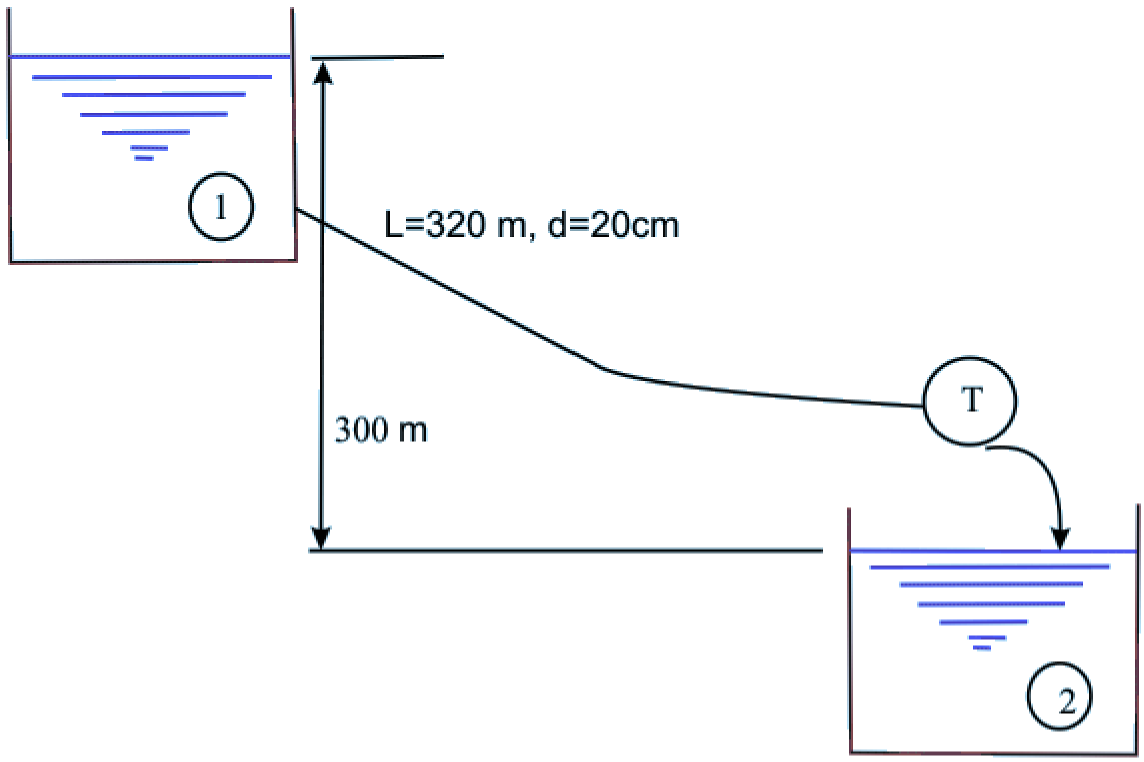


figure 37.5

Recap

In this course you have learnt the following

- The Fanning's friction coefficient for a flow through a closed duct, $C_f = \tau_w / (1/2)\rho v^2$ in terms of wall shear stress, and $C_f = (\frac{1}{4})(D_h/L)\Delta P^*/(1/2)\ \rho V^2$ in terms of piezometric pressure drop Darcy's friction factor is defined as $f = 4C_f$

- Loss of head in a pipe flow is expressed in terms of Darcy's friction factor as $h_f = f(L/D)(V^2/2g)$

- Friction factor in case of laminar fully developed flow is found by N-S equation and is given by f = 64/Re**.** Friction factor for turbulent flow depends both on Re and the roughness at pipe surface.

- Flows, in practice, takes place through several pipes together either in series or parallel or in combination of both of them. The relationship between the head causing the flow ΔH and flow rate Q can be expressed as $\Delta H = RQ^2$, where R is the flow resistance in the hydraulic path.

- The loss of head due to friction over a length L of a pipe. Where the entire flow is drained off uniformly from the side tappings, becomes 1/3 of that in a pipe of same length and diameter, but without side tappings.

- An additional head loss over that due to pipe friction takes place in a flow through pipe bends and pipe fittings like valves, couplings and so on.

- The hydraulic power can be transmitted by a pipeline. For a maximum power transmission, the head due to friction in the flow equals to one third of the head at source to be transmitted. The maximum power transmitted efficiency is 67%.

Introduction

- **Compressible flow** is often called as **variable density flow**. For the flow of all liquids and for the flow of gases under certain conditions, the density changes are so small that assumption of constant density remains valid.

- Let us consider a small element of fluid of volume $\forall$. The pressure exerted on the element by the neighbouring fluid is p . If the pressure is now increased by an amount dp , the volume of the element will correspondingly be reduced by the amount d $\forall$ .The **compressibility** of the fluid K is thus defined as

$$K = \frac{1}{\rho} \cdot \frac{d\rho}{dp} \qquad (38.1)$$

However, when a gas is compressed, its temperature increases. Therefore, the above mentioned definition of compressibility is not complete unless temperature condition is specified. When the temperature is maintained at a constant level, the **isothermal compressibility** is defined as

$$K_T = -\frac{1}{\forall}\left(\frac{d\forall}{dp}\right)_T \qquad (38.2)$$

- Compressibility is a property of fluids. Liquids have very low value of compressibility (for ex. compressibility of water is 5 $\times$ 10^{-10} m^2/N at 1 atm under isothermal condition), while gases have very high compressibility (for ex. compressibility of air is 10^{-5} m^2/N at 1 atm under isothermal condition).

- If the fluid element is considered to have unit mass and v is the specific volume (volume per unit mass) , the density is $\rho = \frac{1}{v}$. In terms of density; Eq. (38.1) becomes

$$\rho = \frac{1}{v} \Rightarrow d\rho = -\frac{1}{v^2} dv \qquad (38.3)$$

$$K = \frac{1}{\rho} \cdot \frac{d\rho}{dp}$$

We can say that from Eqn (38.1) for a change in pressure, dp, the change in density is

$$d\rho = \rho \cdot K \cdot dp \qquad (38.4)$$

- If we also consider the fluid motion, we shall appreciate that the flows are initiated and maintained by changes in pressure on the fluid. It is also known that high pressure gradient is responsible for high speed flow. However**, for a given pressure gradient dp , the change in density of a liquid will be much smaller than the change in density of a gas** (as seen in Eq. (38.4)).

So, for flow of gases, moderate to high pressure gradients lead to substantial changes in the density. Due to such pressure gradients, gases flow with high velocity. **Such flows, where ρ is a variable, are known as compressible flows.**

- Recapitulating Chapter 1, we can say that the proper criterion for a nearly incompressible flow is a small Mach number,

$\mathrm{Ma} = \dfrac{V}{a} << 1$	(38.5)

- where V is the flow velocity and a is the speed of sound in the fluid. For small Mach number, changes in fluid density are small everywhere in the flow field.

- In this chapter we shall treat compressible flows which have Mach numbers greater than 0.3 and exhibit appreciable density changes. The **Mach number is the most important parameter in compressible flow analysis**. Aerodynamicists make a distinction between different regions of Mach number.

  Categories of flow for external aerodynamics.

  - Ma < 0.3: **incompressible flow**; change in density is negligible.
  - 0.3< Ma < 0.8: **subsonic flow**; density changes are significant but shock waves do not appear.
  - 0.8< Ma < 1.2: **transonic flow**; shock waves appear and divide the subsonic and supersonic regions of the flow. Transonic flow is characterized by mixed regions of locally subsonic and supersonic flow
  - 1.2 < Ma < 3.0: **supersonic flow**; flow field everywhere is above acoustic speed. Shock waves appear and across the shock wave, the streamline changes direction discontinuously.
  - 3.0< Ma : **hypersonic flow**; where the temperature, pressure and density of the flow increase almost explosively across the shock wave.

- **For internal flow**, it is to be studied **whether the flow is subsonic (** Ma < 1) **or supersonic** (Ma > 1)**.** The effect of change in area on velocity changes in subsonic and supersonic regime is of considerable interest. By and large, in this chapter we shall mostly focus our attention to internal flows.

Perfect Gas

- A perfect gas is one in which **intermolecular forces are neglected**. The equation of state for a perfect gas can be derived from kinetic theory. It was synthesized from laboratory experiments by Robert Boyle, Jacques Charles, Joseph Gay-Lussac and

John Dalton. For a perfect gas, it can be written

$$p\forall = MRT \qquad (38.6)$$

- where p is pressure (N/m^2), $\forall$ is the volume of the system (m^3), M is the mass of the system (kg), R is the characteristic gas constant (J/kg K) and T is the temperature (K). This equation of state can be written as

$$pv = RT \qquad (38.7)$$

- where v is the specific volume (m^3/kg). Also,

$$p = \rho RT \qquad (38.8)$$

where ρ is the density (kg/m^3).

- In another approach, which is particularly useful in chemically reacting systems, the equation of state is written as

$$p\forall = N\Re T \qquad (38.9)$$

- where N is the number of moles in the system, and $\Re$ is the universal gas constant which is same for all gases

  .

- Recall that a mole of a substance is that amount which contains a mass equal to the molecular weight of the gas and which is identified with the particular system of units being used. For example, in case of oxygen (O_2), 1 kilogram-mole (or kg. mol) has a mass of 32 kg. Because the masses of different molecules are in the same ratio as their molecular weights; 1 mol of different gases always contains the same number of molecules, i.e. 1 kg-mol always contains 6.02×10^{26} molecules, independent of the species of the gas. Dividing Eq. (38.9) by the number of moles of the system yields

$$p\forall^1 = \Re T \qquad (38.10)$$

- $\forall^1$: Vol. per unit mole
- If Eq. (38.9) is divided by the mass of the system, we can write

$$pv = \eta \Re T \qquad (38.11)$$

- where v is the specific volume as before and η is the mole-mass ratio (kg- mol/kg).

Also, Eq. (38.9) can be divided by system volume, which results in

$p = C\Re T$	(38.12)

- where C is the concentration (kg - mol/m^3)

- The equation of state can also be expressed in terms of particles. If N_A is the number of molecules in a mole (**Avogadro** constant, which for a kilogram- mole is 6.02 $\times 10^{26}$ particles), from Eq. (38.12) we obtain

$p = (N_A C)\left(\frac{\Re}{N_A}\right)T$	(38.13)

In the above equation, $N_A C$ is the number density, i.e. number of particles per unit volume and $\frac{\Re}{N_A}$ is the gas constant per particle, which is nothing but Boltzmann constant. Finally, Eq. (38.13) can be written as

$p = n\kappa T$	(38.14)

where n**: number density**
κ **: Boltzmann constant.**

- It is interesting to note that there exist a variety of gas constants whose use depends on the equation in consideration.

1.Universal gas constant- When the equation deals with moles, it is in use. It is same for all the gases.

$\Re$ = 8314 J/(Kg-mol-K)

2.Characteristic gas constant- When the equation deals with mass, the characteristic gas constant (R) is used. It is a gas constant per unit mass and it is different for different gases. As such $R = \Re / M$, where M is the molecular weight. For air at standard conditions,

R = 287 J/(kg-K)

3.Boltzmann constant- When the equation deals with molecules, Boltzmann constant is used. It is a gas constant per unit molecule .

κ = 1.38 X 10^{-23} J / K

Application of the perfect gas theory

a. It has been experimentally determined that at low pressures (1 atm or less) and at high temperature (273 K and above), the value of $\frac{p\nu}{RT}$ (the well known compressibility z, of a gas) for most pure gases differs from unity by a quanity less than one percent (the well known compressibility z, of a gas).
b. Also, that at very low temperatures and high pressures the molecules are densely packed. Under such circumstances, the gas is defined as real gas and the perfect gas equation of state is replaced by the famous **Van-der-Waals equation** which is

$$\left(p+\frac{a}{\nu^2}\right)(\nu-b)=R\,T \qquad (38.15)$$

where a and b are constants and depend on the type of the gas.

In conclusion, it can be said that for a wide range of applications related to compressible flows, the temperatures and pressures are such that the equation of state for the perfect gas can be applied with high degree of confidence.

Internal Energy and Enthalpy

- Microscopic view of a gas is a collection of particles in random motion. Energy of a particle consists of **translational energy, rotational energy, vibrational energy** and **specific electronic energy**. All these energies summed over all the particles of the gas, form the specific internal energy, e , of the gas.

- Imagine a gas in thermodynamic equilibrium,i.e., gradients in velocity, pressure, temperature and chemical concentrations do not exist.

Then the enthalpy, h , is defined as $h = e + p\upsilon$, where υ is the specific volume.

$$e = e(T, \forall)$$
$$h = h(T, p) \qquad (38.16)$$

If the gas is not chemically reacting and the intermolecular forces are neglected, the system can be called as a **thermally perfect gas**, where internal energy and enthalpy are functions of temperature only. One can write

$$e = e(T)$$
$$h = h(T)$$
$$de = c_v dT$$
$$dh = c_p dT \qquad (38.17)$$

For a calorically perfect gas,

$e = c_v T$ $h = c_p T$	(38.18)

Please note that in most of the compressible flow applications, the pressure and temperatures are such that the gas can be considered as calorically perfect.

- For calorically perfect gases, we assume constant specific heats and write

$c_p - c_v = R$	(38.19)

- The specific heats at constant pressure and constant volume are defined as

$c_p = \left(\frac{\partial h}{\partial T}\right)_p \qquad c_v = \left(\frac{\partial e}{\partial T}\right)_v$	(38.20)

Equation (38.19), can be rewritten as

$1 - \frac{c_v}{c_p} = \frac{R}{c_p}$	(38.21)

Also $\frac{c_p}{c_v} = \gamma$. So we can rewrite Eq. (38.21) as

$1 - \frac{1}{\gamma} = \frac{R}{c_p}$	(38.22)
$c_p = \frac{\gamma R}{\gamma - 1}$	

In a similar way, from Eq. (38.19) we can write

$c_v = \frac{R}{\gamma - 1}$	(38.23)

First Law of Thermodynamics

- Let us imagine a control-mass system with a fixed mass of gas. If δq amount of heat is added to the system across the system-boundary and if δw is the work done on the system by the surroundings, then there will be an eventual change in internal energy of the system which is denoted by de and we can write

$$de = \delta q + \delta w \qquad (38.24)$$

This is **first law of thermodynamics**. Here, de is an exact differential and its value depends only on initial and final states of the system**.** However, δq and δw are dependent on the path of the process. A process signifies the way by which heat can be added and the work is done on/by the system. (Note that heat added to system is taken as positive and work done on the system is taken as positive)

- In this chapter we are interested in isentropic process which is a combination of adiabatic (no heat is added to or taken away from the system) and reversible process (occurs through successive stages, each stage consists of an infinitesimal small gradient and is an equilibrium state). **In an isentropic process, entropy of a system remains constant** (as seen in the following lecture).

Entropy and Second Law of Thermodynamics

- Equation (38.24) does not tell us about the direction (i.e., a hot body with respect to its surrounding will gain temperature or cool down) of the process. To determine the proper direction of a process, we define a new state variable, entropy, which is

$$ds = \frac{\delta q_{rev}}{T} \qquad (38.25)$$

where *s* is the entropy of the system, δq_{rev} is the heat added reversibly to the system and T is the temperature of the system. It may be mentioned that Eqn. (38.25) is valid if both external and internal irreversibilities are maintained during the process of heat addition

- **Entropy** is a **state variable** and it can be associated with any type of process, reversible or irreversible. An effective value of δq_{rev} can always be assigned to relate initial and end points of an irreversible process, where the actual amount of heat added is δq . One can write

$$ds = \frac{\delta q}{T} + ds_{irrev} \qquad (38.26)$$

It states that the change in entropy during a process is equal to actual heat added divided by the temperature plus a contribution from the irreversible dissipative phenomena. It may be mentioned that ds_{irrev} implies internal irreversibilities if T is the temperature at the system boundary. If T is the temperature of the surrounding ds_{irrev} implies both external and internal irreversibilities. The irreversible phenomena always increases the entropy, hence

$$ds_{irrev} \geq 0 \qquad (38.27)$$

- Significance of greater than sign is understandable. The equal sign represents a reversible process. On combining Eqs (38.26) and (38.27) we get.

$$ds \geq \frac{\delta q}{T} \qquad (38.28)$$

If the process is adiabatic, $\delta q = 0$, Eq. (38.28) yields

$ds \geq 0$	(38.29)

- Equations (38.28) and (38.29) are the expressions for the second law of thermodynamics. The second law tells us in what direction the process will take place. The direction of a process is such that the change in entropy of the system plus surrounding is always positive or zero (for a reversible adiabatic process). In conclusion, it can be said that the **second law governs the direction of a natural process**.

- For a reversible process, it can be said that $\delta W = -pdv$ where dv is change in volume and from the first law of thermodynamics it can be written as

$\delta q - pdv = de$	(38.30)

- If the process is reversible, we use the definition of entropy in the form $\delta q_{rev} = Tds$ the Eq. (38.30) then becomes,

$Tds - pdv = de$ $Tds = de + pdv$	(38.31)

- Another form can be obtained in terms of enthalpy. For example, by definition

$h = e + pv$	

Differentiating, we obtain

$dh = de + pdv + vdp$	(38.32)

Combining Eqs (38.31) and (38.32), we have

$Tds = dh + vdp$	(38.33)

- Equations (38.31) and (38.33) are termed as **first Tds equation** and **second Tds equation**, respectively.

- For a thermally perfect gas, we have $dh = c_p dT$ (from Eq. 38.20) , substitute this in Eq. (38.33) to obtain

$ds = c_p \dfrac{dT}{T} - v\dfrac{dp}{T}$	(38.34)

Further substitution of $pv = RT$ into Eq. (38.34) yields

$ds = c_p \dfrac{dT}{T} - R\dfrac{dp}{p}$	(38.35)

Integrating Eq. (38.35) between states 1 and 2,

$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R\, ln \frac{p_2}{p_1}$	(38.36)

If c_p is a variable, we shall require gas tables; but for constant c_p, we obtain the analytic expression

$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$	(38.37)

In a similar way, starting with Eq. (38.31) and making use of the relation $de = c_v dT$ the change in entropy can also be written as

$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$	(38.38)

Isentropic Relation

- An isentropic process is a reversible-adiabatic process. For an adiabatic process δq =0, and for a reversible process, ds_{irrev} =0. From Eq. (38.26), for an isentropic process, $ds = 0$. However, in Eq. (38.37), substitution of isentropic condition yields

	$c_p \ln \frac{T_2}{T_1} = R \ln \frac{p_2}{p_1}$

or	$\ln \frac{p_2}{p_1} = \frac{c_p}{R} \ln \frac{T_2}{T_1}$

or	$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{c_p/R}$	(38.39)

Using $c_p = \frac{\gamma R}{\gamma - 1}$, we have

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \qquad (38.40)$$

Considering Eq. (38.38), in a similar way, yields

$$c_v \ln \frac{T_2}{T_1} + R \ln \frac{\forall_2}{\forall_1} = 0$$

$$\ln \frac{\forall_2}{\forall_1} = -\frac{c_v}{R} \ln \frac{T_2}{T_1}$$

or $$\frac{\forall_2}{\forall_1} = \left(\frac{T_2}{T_1}\right)^{-c_v/R} \qquad (38.41)$$

Using $c_v = \dfrac{R}{\gamma-1}$, we get

$$\frac{\forall_2}{\forall_1} = \left(\frac{T_2}{T_1}\right)^{-\frac{1}{\gamma-1}} \qquad (38.42)$$

- Using $\rho_2/\rho_1 = \forall_1/\forall_2$ we can write

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}} \qquad (38.43)$$

- Combining Eq. (38.40) with Eq. (38.43), we find,

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \qquad (38.44)$$

This is a key relation to be remembered throughout the chapter.

Speed of Sound

- The so-called sound speed is the **rate of propogation of a pressure pulse** of infinitesimal strength through a still fluid. It is a **thermodynamic property** of a fluid.

- A pressure pulse in an incompressible flow behaves like that in a rigid body. A displaced particle displaces all the particles in the medium. In a compressible fluid, on the other hand, displaced mass compresses and increases the density of neighbouring mass which in turn increases density of the adjoining mass and so on. Thus, a disturbance in the form of an elastic wave or a pressure wave travels through the medium. If the amplitude and theerfore the strength of the elastic wave is infinitesimal, it is termed as acoustic wave or sound wave.

- Figure 39.1(a) shows an infinitesimal pressure pulse propagating at a speed " a " towards still fluid (*V* = 0) at the left. The fluid properties ahead of the wave are p,T and ρ , while the properties behind the wave are p+dp, T+dT and $\rho + d\rho$. The fluid velocity dV is directed toward the left following wave but much slower.

- In order to make the analysis steady, we superimpose a velocity " a " directed towards right, on the entire system (Fig. 39.1(b)). The wave is now stationary and the fluid appears to have velocity " a " on the left and (a - dV) on the right. The flow in Fig. 39.1 (b) is now steady and one dimensional across the wave. Consider an area A on the wave front. A mass balance gives

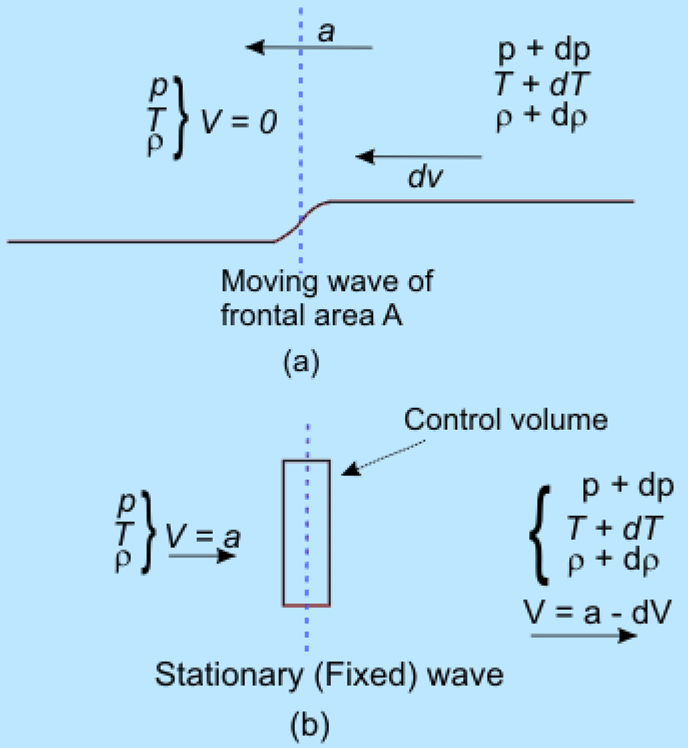


Fig 39.1: Propagation of a sound wave
(a) Wave Propagating into still Fluid (b) Stationary Wave

$$\rho A a = (\rho + d\rho) A (a - dV)$$

$$dV = a\left[\frac{d\rho}{\rho + d\rho}\right] \qquad (39.1)$$

This shows that

(a) $dv > 0$ if dρ is positive.

(b) A compression wave leaves behind a fluid moving in the direction of the wave (Fig. 39.1(a)).

(c) Equation (39.1) also signifies that the fluid velocity on the right is much smaller than the wave speed " a ". Within the framework of infinitesimal strength of the wave (sound wave), this " a " itself is very small.

- Applying the momentum balance on the same control volume in Fig. 39.1 (b). It says that the net force in the x direction on the control volume equals the rate of outflow of x momentum minus the rate of inflow of x momentum. In symbolic form, this yields

$$pA - (p + dp)A = (A\rho a)(a - dV) - (A\rho a)(a)$$

In the above expression, $A\rho a$ is the mass flow rate. The first term on the right hand side represents the rate of outflow of x-momentum and the second term represents the rate of inflow of x momentum.

- Simplifying the momentum equation, we get

$$dp = \rho a dV \qquad (39.2)$$

If the wave strength is very small, the pressure change is small.

Combining Eqs (39.1) and (39.2), we get

$$a^2 = \frac{dp}{d\rho}\left(1 + \frac{d\rho}{\rho}\right) \qquad (39.3a)$$

The larger the strength $d\rho/\rho$ of the wave ,the faster the wave speed; i.e., powerful explosion waves move much faster than sound waves.In the limit of infinitesimally small strength, $d\rho \to 0$ we can write

$$a^2 = \frac{dp}{d\rho} \qquad (39.3b)$$

Note that

(a) In the limit of infinitesimally strength of sound wave, there are no velocity gradients on either side of the wave. Therefore, the frictional effects (irreversible) are confined to the interior of the wave.

(b) Moreover, the entire process of sound wave propagation is adiabatic because there is no temperature gradient except inside the wave itself.

(c) So, for sound waves, we can see that the process is reversible adiabatic or**isentropic**.

So the correct expression for the sound speed is

$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$	(39.4)

For a perfect gas, by using of $\frac{p}{\rho^\gamma} = const$, and $p = \rho RT$, we deduce the speed of sound as

$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$	(39.5)

For air at sea-level and at a temperature of 15^0C, a=340 m/s

Pressure Field Due to a Moving Source

- Consider a point source emanating infinitesimal pressure disturbances in a still fluid, in which the speed of sound is "a". If the point disturbance, is stationary then the wave fronts are concentric spheres. As shown in Fig. 39.2(a), wave fronts are present at intervals of Δt .

- Now suppose that source moves to the left at speed U < a**.** Figure 39.2(b) shows four locations of the source, 1 to 4, at equal intervals of time Δt , with point 4 being the current location of the source.

- At point 1, the source emanated a wave which has spherically expanded to a radius $3a\Delta t$ in an interval of time $3\Delta t$. During this time the source has moved to the location 4 at a distance of $3u\Delta t$ from point 1. The figure also shows the locations of the wave fronts emitted while the source was at points 2 and 3, respectively.

- When the source speed is supersonic U > a (Fig. 39.2(c)), the point source is ahead of the disturbance and an observer in the downstream location is unaware of the approaching source. The disturbance emitted at different

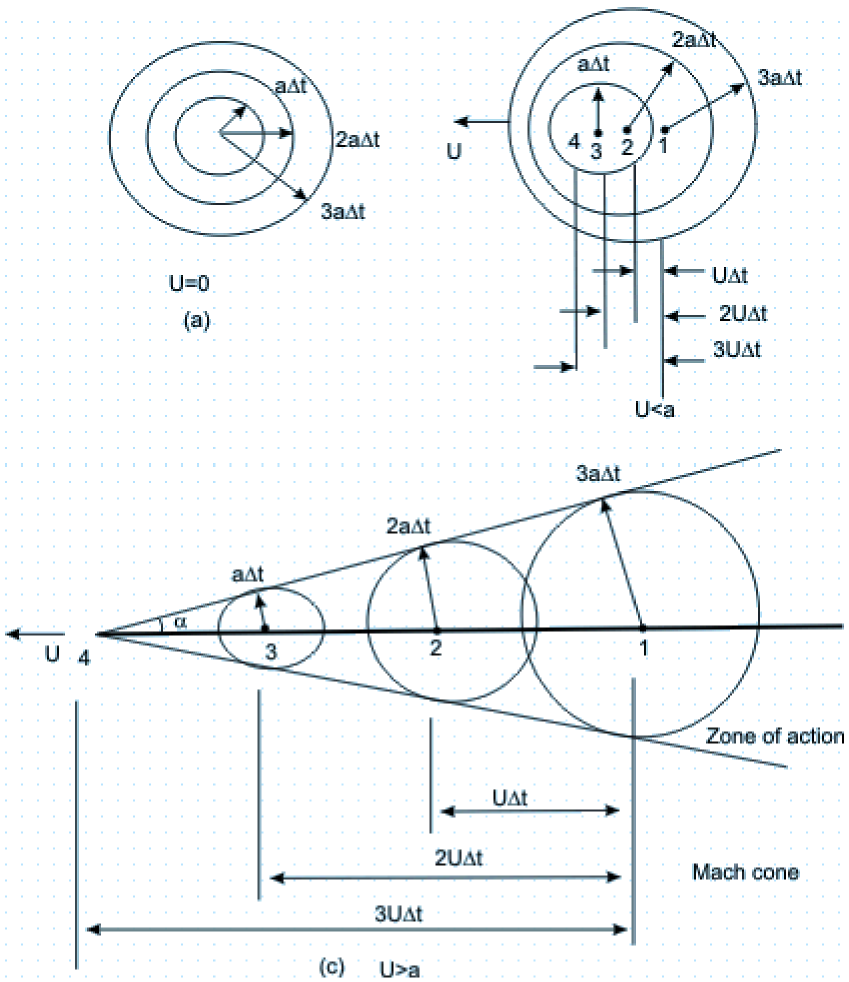


Fig 39.2 Wave fronts emitted from a point source in a still fluid when the source speed is (a) U = 0 (still Source) (b) U < a (Subsonic) (c) U > a (Supersonic)

[see animated view here]

points of time are enveloped by an imaginary conical surface known as **"Mach Cone"**. **The half angle of the cone** α **, is known as Mach angle** and given by

$$\sin\alpha = \frac{a\Delta t}{U\Delta t} = \frac{1}{Ma}$$

$$\alpha = \sin^{-1}(1/Ma)$$

- Since the disturbances are confined to the cone, **the area within the cone is known as zone of action**and the **area outside the cone is zone of silence** .

  An observer does not feel the effects of the moving source till the Mach Cone covers his position.

Basic Equations for One-Dimensional Flow

- Here we will study a class of compressible flows that can be treated as one dimensional flow. Such a simplification is meaningful for flow through ducts where the centreline of the ducts does not have a large curvature and the cross-section of the ducts does not vary abruptly.

- In one dimension, the flow can be studied by ignoring the variation of velocity and other properties across the normal direction of the flow. However, these distributions are taken care of by assigning an average value over the cross-section (Fig. 39.3).

- The area of the duct is taken as A(x) and the flow properties are taken as p(x), ρ(x), V(x) etc. The forms of the basic equations in a one-dimensional compressible flow are;

  - **Continuity Equation**
  - **Energy Equation**
  - **Bernoulli and Euler Equations**
  - **Momentum Principle for a Control Volume**

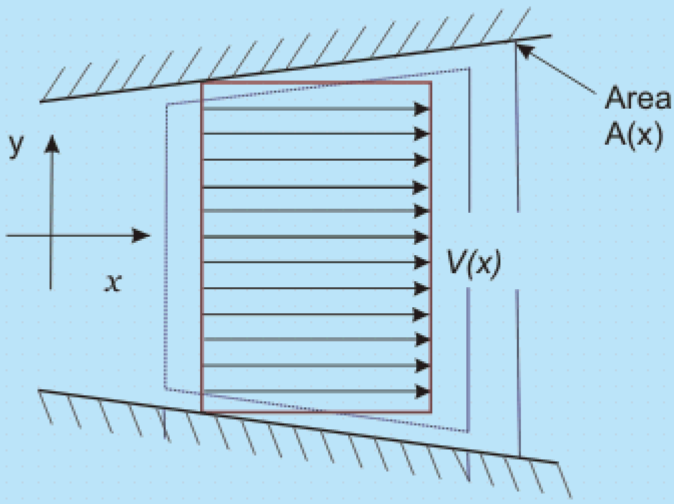


Fig 39.3 One-Dimensional Approximation

Continuity Equation

For steady one-dimensional flow, the equation of continuity is

$$\rho(x).\mathrm{V}(x).\mathrm{A}(x) = const$$

Differentiating(after taking log), we get

$$\frac{d\rho}{\rho} + \frac{d\mathrm{V}}{\mathrm{V}} + \frac{d\mathrm{A}}{\mathrm{A}} = 0 \qquad (39.6)$$

Energy Equation

Consider a control volume within the duct shown by dotted lines in Fig. 39.3. The first law of thermodynamics for a control volume fixed in space is

$$\frac{d}{dt}\iiint \rho(e+\frac{V^2}{2})d\forall + \iint (e+\frac{V^2}{2})\rho V dA = \iint V(\tau . dA) - \iint q . dA \qquad (39.7)$$

where $\frac{V^2}{2}$ is the kinetic energy per unit mass.

Let us discuss the various terms from above equation:

- The first term on the left hand side signifies the rate of change of energy (internal + kinetic) within the control volume
- The second term depicts the flux of energy out of control surface.
- The first term on the right hand side represents the work done on the control surface
- The second term on the right means the heat transferred through the control surface.

It is to be noted that dA is directed along the outward normal.

- Assuming steady state, the first term on the left hand side of Eq. (39.7) is zero. Writing $\rho_2 V_2 A_2 = \rho_1 V_1 A_1 = \dot{m}$ (where the subscripts are for Sections 1 and 2), the second term on the left of Eq. (39.7) yields

$$\iint (e+\frac{V^2}{2})\rho V dA = m\left[(e_2+\frac{V_2^{\,2}}{2})-(e_1+\frac{V_1^{\,2}}{2})\right]$$

The work done on the control surfaces is

$$\iint V(\tau . dA) = V_1 p_1 A_1 - V_2 p_2 A_2$$

The rate of heat transfer to the control volume is

$$-\iint q . dA = Q\dot{m}$$

where Q is the heat added per unit mass (in J/kg).

- Invoking all the aforesaid relations in Eq. (39.7) and dividing by $\dot{m}$, we get

$$e_2+\frac{V_2^{\,2}}{2}-e_1-\frac{V_1^{\,2}}{2} = -\frac{1}{\dot{m}}\left[p_2 V_2 A_2 - p_1 V_1 A_1\right]+Q \qquad (39.8)$$

We know that the density ρ is given by $\dot{m}$ /VA , hence the first term on the right may be expressed in terms of $\forall$ (specific volume=1/ρ). Equation (39.8) can be rewritten as

$$e_2+\frac{V_2^{\,2}}{2}-e_1-\frac{V_1^{\,2}}{2} = p_1\forall_1 - p_2\forall_2 + Q \qquad (39.9)$$

- NOTE:- $p_1\forall_1$ is the work done (per unit mass) by the surrounding in pushing fluid into the control volume. Following a similar argument, is the $p_2\forall_2$ work done by the fluid inside the control volume on the surroundings in pushing fluid out of the control volume.

- Since $h = e + p\forall$ Eq. (39.9) gets reduced to

$$h_2 + \frac{V_2^2}{2} = h_1 + \frac{V_1^2}{2} + Q \qquad (39.10)$$

This is **energy equation**, which is valid even in the presence of friction or non-equilibrium conditions between secs 1 and 2.

- It is evident that **the sum of enthalpy and kinetic energy remains constant in an adiabatic flow**. Enthalpy performs a similar role that internal energy performs in a nonflowing system. The difference between the two types of systems is the flow work p $\forall$ required to push the fluid through a section.

Bernoulli and Euler Equations

- For inviscid flows, the steady form of the momentum equation is the Euler equation,

$$\frac{dp}{\rho} + VdV = 0 \qquad (39.11)$$

Integrating along a streamline, we get the Bernoulli's equation for a compressible flow as

$$\int \frac{dp}{\rho} + \frac{V^2}{2} = const \qquad (39.12)$$

- For adiabatic frictionless flows the Bernoulli's equation is identical to the energy equation. Recall, that this is an isentropic flow, so that the **Tds equation** is given by

$$Tds = dh - vdp$$

For isentropic flow, ds=0

Therefore,

$$dh = \frac{dp}{\rho}$$

Hence, the Euler equation (39.11) can also be written as

$$VdV + dh = 0$$

This is identical to the adiabatic form of the energy Eq. (39.10).

Momentum Principle for a Control Volume

For a finite control volume between Sections 1 and 2 (Fig. 39.3), the momentum principle is

$$p_1 A_1 + p_2 A_2 + F = \dot{m} V_2 - \dot{m} V_1$$
$$= \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1 \qquad (39.13)$$

where F is the x-component of resultant force exerted on the fluid by the walls. Note that the momentum principle, Eq. (39.13), is applicable even when **there are frictional dissipative processes** within the control volume.

Stagnation and Sonic Properties

- The stagnation properties at a point are defined as those which are to be obtained if the local flow were imagined to cease to zero velocity isentropically. As we will see in the later part of the text, stagnation values are useful reference conditions in a compressible flow.

Let us denote stagnation properties by subscript zero. Suppose the properties of a flow (such as T, p , ρ etc.) are known at a point, the stangation enthalpy is, thus, defined as

$$h_0 = h + \frac{1}{2} V^2$$

where h is flow enthalpy and V is flow velocity.

- For a perfect gas , this yields,

$$c_p T_0 = c_p T + \frac{1}{2} V^2 \qquad (40.1)$$

which defines the **Stagnation Temperatur** (T_0)

Now, $\frac{T_0}{T}$ can be expressed as

$$\frac{T_0}{T} = 1 + \frac{V^2}{2 c_p T} = 1 + \frac{\gamma - 1}{2} \cdot \frac{V^2}{\gamma RT}$$

Since,

$$c_p - \frac{c_p}{\gamma} = R$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} Ma^2 \qquad \left(Ma = \frac{V}{a} = \frac{V}{\sqrt{\gamma RT}} \right) \qquad (40.2)$$

If we know the local temperature (**T**) and Mach number (**Ma**) , we can find out the stagnation temperature **T_0** .

- Consequently, **isentropic(adiabatic) relations** can be used to obtain **stagnation pressure** and **stagnation density** as

$$\frac{p_0}{p} = \left(\frac{T_0}{T} \right)^{\frac{\gamma}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2} Ma^2 \right]^{\frac{\gamma}{\gamma - 1}} \qquad (40.3)$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T} \right)^{\frac{1}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2} Ma^2 \right]^{\frac{1}{\gamma - 1}} \qquad (40.4)$$

Values of T_0/T , p_0/p and ρ_0/ρ as a function of Mach number can be generated using the above relationships and the tabulated results are known as **Isentropic Table** . Interested readers are suggested to refer the following books

1. J.Spruk, Fluid Mechanics, Springer, Heidelberg , NewYork, 1997
2. K.Muradidhar and G.Biswas, Advanced Engineering Fluid Mechanics, Second Edition, Narosa, 2005

Contd.

Note that in general the stagnation properties can vary throughout the flow field.

Let us consider some special cases :-

Case 1: Adiabatic Flow:

$h + \frac{V^2}{2}$ (from eqn 39.10) is constant throughout the flow. It follows that the h_0 , T_0 , a_0 are constant throughout an adiabatic flow, even in the presence of friction.

Hence, all stagnation properties are constant along an isentropic flow. If such a flow starts from a large reservoir where the fluid is practically at rest, then the properties in the reservoir are equal to the stagnation properties everywhere in the flow Fig (40.1)

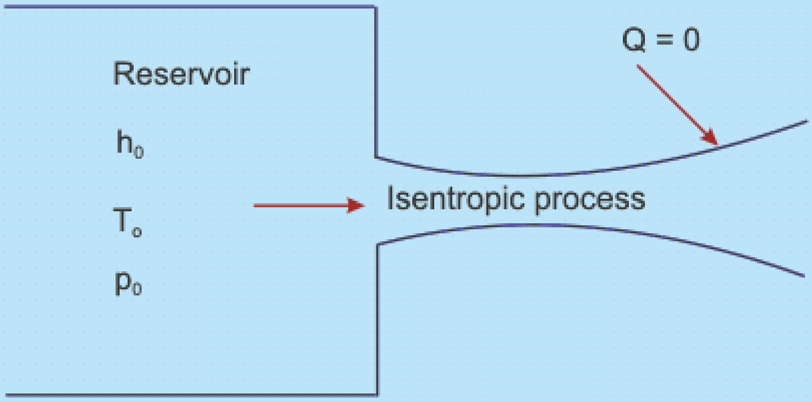


Fig 40.1: An isentropic process starting from a reservoir

Case 2: Sonic Flow (Ma=1)

The sonic or critical properties are denoted by asterisks: p*, ρ*, a*, and T* . These properties are attained if the local fluid is imagined to expand or compress isentropically until it reaches Ma = 1.

Important-

The total enthalpy, hence T_0 , is conserved as long as the process is adiabatic, irrespective of frictional effects.

From Eq. (40.1), we note that

$$V^2 = 2c_p(T_0 - T)$$

$$V = \left[\frac{2\gamma R}{\gamma - 1}(T_0 - T)\right]^{\frac{1}{2}} \qquad (40.5a)$$

This gives the relationship between the fluid velocity V, and local temperature (T), in an adiabatic flow.

Putting T=0 we obtain maximum attainable velocity as,

$$V_{max} = \left[\frac{2\gamma RT_0}{\gamma - 1}\right]^{\frac{1}{2}} \qquad (40.5b)$$

Considering the condition, when Mach number, Ma=1, for a compressible flow we can write from Eq. (40.2), (40.3) and (40.4),

$$\frac{T_0}{T^*} = \frac{1+\gamma}{2} \quad (40.6a)$$

$$\frac{p_0}{p^*} = \left(\frac{1+\gamma}{2}\right)^{\frac{\gamma}{\gamma-1}} \quad (40.6b)$$

$$\frac{\rho_0}{\rho^*} = \left(\frac{1+\gamma}{2}\right)^{\frac{1}{\gamma-1}} \quad (40.6c)$$

- For diatomic gases, like air $\gamma = 1.4$, the numerical values are
  $\frac{T^*}{T_0} = 0.8333$, $\frac{p^*}{p_0} = 0.5282$, $\frac{\rho^*}{\rho} = 0.6339$
- The fluid velocity and acoustic speed are equal at sonic condition and is

$$V^* = a^* = \sqrt{\gamma R T^*} \quad (40.7a)$$

or $$V^* = \left[\frac{2\gamma}{\gamma+1} R T_0\right]^{\frac{1}{2}} \quad (40.7b)$$

We shall employ both stagnation conditions and critical conditions as reference conditions in a variety of one dimensional compressible flows.

Effect of Area Variation on Flow Properties in Isentropic Flow

In considering the effect of area variation on flow properties in isentropic flow, we shall determine the effect on the velocity V and the pressure p .

From Eq . (39.11), we can write

$$\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) = 0$$

$$dp = -\rho V dV$$

Dividing by ρV^2 , we obtain

$$\frac{dp}{\rho V^2} = -\frac{dV}{V} \quad (40.8)$$

A convenient differential form of the continuity equation can be obtained from Eq. (39.6) as

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho}$$

Substituting from Eq. (40.8),

$$\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho}$$

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{dp/d\rho}\right] \tag{40.9}$$

Invoking the relation (39.3b) for isentropic process in Eq. (40.9), we get

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{a^2}\right] = \frac{dp}{\rho V^2}\left[1 - Ma^2\right] \tag{40.10}$$

- From Eq.(40.10), we see that for Ma<1 an area change causes a pressure change of the same sign, i.e. positive dA means positive dp for Ma<1 . For Ma>1 , an area change causes a pressure change of opposite sign.

- Again, substituting from Eq. (40.8) into Eq. (40.10), we obtain

$$\frac{dA}{A} = -\frac{dV}{V}\left[1 - Ma^2\right] \tag{40.11}$$

From Eq. (40.11) we see that Ma<1 an area change causes a velocity change of opposite sign, i.e. positive dA means negative dV for Ma<1 . For Ma>1 an area change causes a velocity change of same sign.

These results can be summarized in fig 40.2. Equations (40.10) and (40.11) lead to the following important conclusions about compressible flows:

1. At subsonic speeds**(**Ma<1**)** a decrease in area increases the speed of flow. A subsonic nozzle should have a convergent profile and a subsonic diffuser should possess a divergent profile. The flow behaviour in the regime of Ma<1 is therefore qualitatively the same as in incompressible flows.
2. In supersonic flows **(**Ma>1**)** the effect of area changes are different. According to Eq. (40.11), a supersonic nozzle must be built with an increasing area in the flow direction. A supersonic diffuser must be a converging channel. Divergent nozzles are used to produce supersonic flow in missiles and launch vehicles.

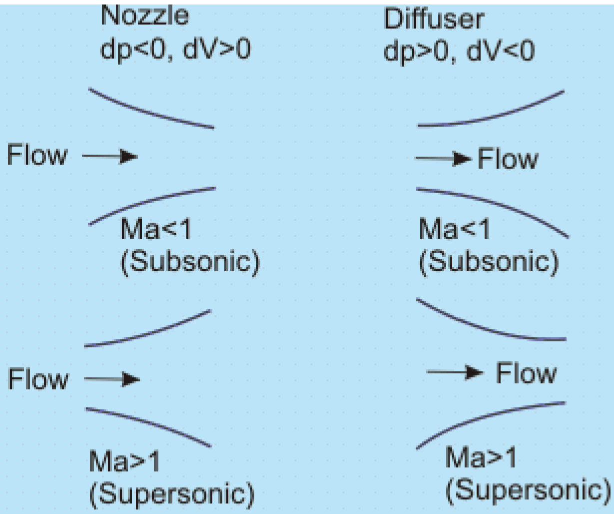


Fig 40.2 Shapes of nozzles and diffusers in subsonic and supersonic regimes

Convergent - Divergent Nozzle

Suppose a nozzle is used to obtain a supersonic stream starting from low speeds at the inlet (Fig. 40.3). Then the Mach number should increase from Ma=0 near the inlet to Ma>1 at the exit. It is clear that the nozzle must converge in the subsonic portion and diverge in the supersonic portion. Such a nozzle is called a **convergent-divergent nozzle**. A convergent-divergent nozzle is also called a **de laval nozzle**, after Carl G.P. de Laval who first used such a configuration in his steam turbines in late nineteenth century.

From **Fig. 40.3** it is clear that the Mach number must be unity at the throat, where the area is neither increasing nor decreasing. This is consistent with Eq. (40.11) which shows that dV can be nonzero at the throat only if Ma =1. It also follows that the **sonic velocity can be achieved only at the throat of a nozzle or a diffuser**.

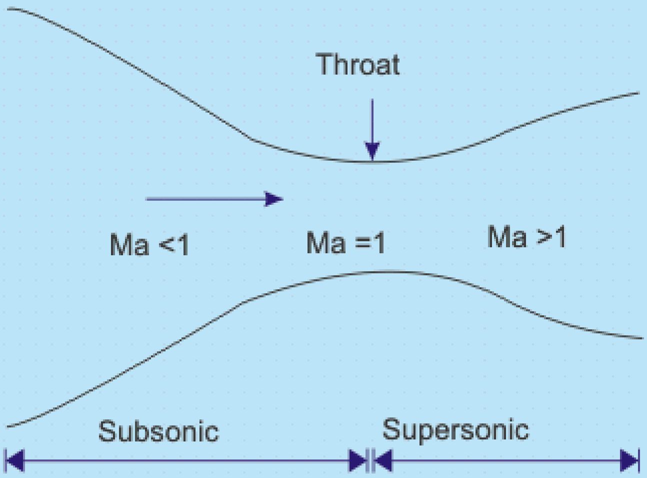


Fig 40.3 A Convergent-Divergent Nozzle

The condition, however, does not restrict that Ma must necessarily be unity at the throat. According to Eq. (40.11), a situation is possible where Ma ≠ 1 at the throat if dV = 0 there. For an example, the flow in a convergent-divergent duct may be subsonic everywhere with Ma increasing in the convergent portion and decreasing in the divergent portion with Ma ≠ 1 at the throat (see Fig. 40.4).

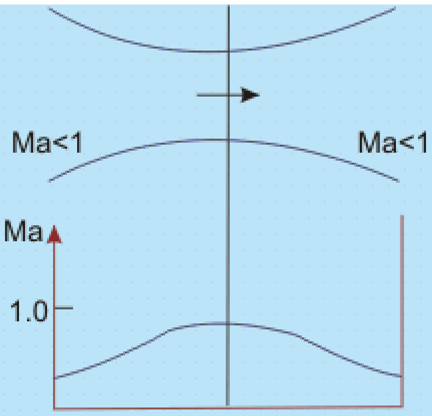


Fig 40.4 Convergent-Divergent duct with Ma ≠ 1 **at throat**

The first part of the duct is acting as a nozzle, whereas the second part is acting as a diffuser. Alternatively, we may have a convergent divergent duct in which the flow is supersonic everywhere with Ma decreasing in the convergent part and increasing in the divergent part and again Ma ≠ 1 at the throat (see Fig. 40.5)

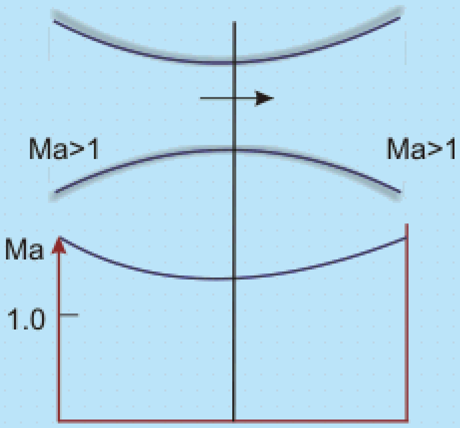


Fig 40.5 Convergent-Divergent duct with Ma ≠ 1 **at throat**

Isentropic Flow in a Converging Nozzle

Consider the mass flow rate of an ideal gas through a converging nozzle. If the flow is isentropic, we can write

$$\dot{m} = \rho A V$$

where V is flow velocity, A is area, ρ is the density of the field.

This can equivalently be written as-

$$\frac{\dot{m}}{A} = \frac{p}{RT} \cdot a \cdot Ma$$

$$\frac{\dot{m}}{A} = \frac{p}{RT} \cdot \sqrt{\gamma RT} \cdot Ma$$

$$\frac{\dot{m}}{A} = \frac{p}{\sqrt{T}} \cdot \sqrt{\frac{\gamma}{R}} \cdot Ma$$

$$\frac{\dot{m}}{A} = \frac{p}{p_0} \cdot p_0 \cdot \sqrt{\frac{T_0}{T}} \cdot \sqrt{\frac{1}{T_0}} \cdot \sqrt{\frac{\gamma}{R}} \cdot Ma$$

$$\frac{\dot{m}}{A} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \left(\frac{T_0}{T}\right)^{\frac{1}{2}} \frac{p_0}{\sqrt{T_0}} \cdot \sqrt{\frac{1}{T_0}} \cdot Ma$$

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}} \cdot \left(\frac{T_0}{T}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}} \cdot \frac{1}{\left[1 + \frac{\gamma-1}{2} Ma^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (40.12)$$

In the expression (40.12), p_0, T_0, γ and R are constant

- The discharge per unit area $\frac{\dot{m}}{A}$ is a function of Ma only. There exists a particular value of Ma for which it is maximum. Differentiating with respect to Ma and equating it to zero, we get

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0}{\sqrt{T_0}} \cdot \frac{1}{\left[1 + \frac{\gamma+1}{2} Ma^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$+ \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}} \left[-\frac{\gamma+1}{2(\gamma-1)} \left\{1 + \frac{\gamma-1}{2} Ma^2\right\} \left(\frac{\gamma+1}{2(\gamma-1)} - 1\right) \left\{\frac{\gamma-1}{2} 2Ma\right\}\right] = 0$$

$$\Rightarrow 1 - \frac{Ma^2 (\gamma+1)}{2\left\{1 + \frac{\gamma-1}{2} Ma^2\right\}} = 0$$

$$\Rightarrow \; Ma^2 (\gamma+1) = 2 + (\gamma-1) Ma^2$$

$$\Rightarrow \; Ma = 1$$

Hence, discharge is maximum when Ma = 1.

- We know that $V = aMa = \sqrt{\gamma RT}\,Ma.$. By logarithmic differentiation, we get

$$\frac{dV}{V} = \frac{dMa}{Ma} + \frac{1}{2}\frac{dT}{T} \tag{40.13}$$

We also know that

$$\frac{T}{T_0} = \left[1 + \frac{\gamma - 1}{2} Ma^2\right]^{-1}$$

By logarithmic differentiation, we get

$$\frac{dT}{T} = -\frac{(\gamma - 1)Ma^2}{1 + \frac{\gamma - 1}{2}Ma^2} \cdot \frac{dMa}{Ma} \tag{40.14}$$

From Eqs(40.13) and (40.14) , we get

$$\frac{dV}{V} = \frac{dMa}{Ma}\left[1 - \frac{\frac{\gamma - 1}{2}Ma^2}{1 + \frac{\gamma - 1}{2}Ma^2}\right]$$

$$\Rightarrow \frac{dV}{V} = \frac{dMa}{Ma}\left[\frac{1}{1 + \frac{\gamma - 1}{2}Ma^2}\right] \tag{40.15}$$

From Eqs (40.11) and (40.15), we get

$$\frac{dA}{A}\frac{1}{(Ma^2 - 1)} = \frac{1}{1 + \frac{\gamma - 1}{2}Ma^2}\frac{dMa}{Ma}$$

$$\frac{dA}{A} = \frac{(Ma^2 - 1)}{1 + \frac{\gamma - 1}{2}Ma^2}\frac{dMa}{Ma} \tag{40.16}$$

By substituting Ma=1 in Eq. (40.16), we get dA = 0 or A = constant.

- Ma=1 can occur only at the throat and nowhere else, and this happens only when the discharge is maximum. **When Ma = 1 , the discharge is maximum and the nozzle is said to be choked.**

  The properties at the throat are termed as critical properties which are already expressed through Eq. (40.6a), (40.6b) and (40.6c). By substituting Ma = 1 in Eq. (40.12), we get

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0}{\sqrt{T_0}} \cdot \frac{1}{\left[\frac{(\gamma+1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (40.17)$$

(as we have earlier designated critical or sonic conditions by a superscript asterisk). Dividing Eq. (40.17) by Eq. (40.12) we obtain

$$\frac{A}{A^*} = \frac{1}{Ma}\left[\left\{\frac{2}{\gamma+1}\right\}\left\{1+\frac{(\gamma-1)}{2}Ma^2\right\}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (40.18)$$

From Eq. (40.18) we see that a choice of Ma gives a unique value of A/A* . The following figure shows variation of A / A * with Ma (Fig 40.6). Note that the curve is double valued; that is, for a given value of A/A* (other than unity), there are two possible values of Mach number. **This signifies the fact that the supersonic nozzle is diverging.**

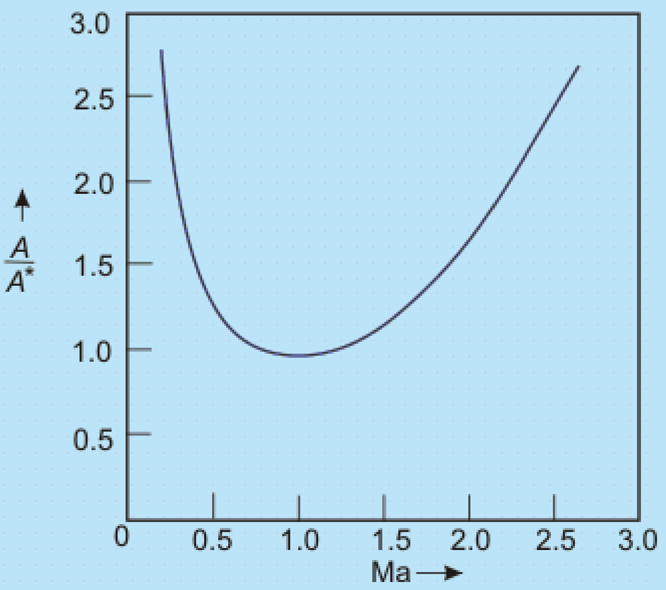


Fig 40.6: Variation of A/A* with Ma in isentropic flow for γ = 1.4

The area ratio, as a function of Mach number, is also included in the Isentropic Table (see Spruk [1], Muralidhar and Biswas [2].

Pressure Distribution and Choking in a Converging Nozzle

- Consider a convergent nozzle as shown in Fig. 40.7(a). Figure 40.7(b) shows the pressure ratio p/p_0 along the length of the nozzle.

- The inlet conditions of the gas are at the stagnation state (p_0, T_0) which are constants. The pressure at the exit plane of the nozzle is denoted by P_E and the back pressure is P_B which can be varied by the adjustment of the valve. At the condition $P_0 = P_E = P_B$ there shall be no flow through the nozzle.

- The pressure is P_0 throughout, as shown by condition (i) in Fig. 40.7(b). **As P_B is gradually reduced, the flow rate shall increase.** The pressure will decrease in the direction of flow as shown by condition (ii) in Fig. 40.7(b). The exit plane pressure P_E shall remain equal to P_B so long as the maximum discharge condition is not reached. Condition (iii) in Fig. 40.7(b) illustrates the pressure distribution in the maximum discharge situation.

- When $\frac{\dot{m}}{A}$ attains its maximum value, given by substituting Ma = 1 in Eq. (40.12), P_E is equal to p* . Since the nozzle does not have a diverging section, further reduction in back pressure P_B will not accelerate the flow to supersonic condition. As a result, the exit pressure P_E shall continue to remain at p * even though $P_B$is lowered further.
- The convergent-nozzle discharge against the variation of back pressure is shown in Fig. 40.8. We are aware, that the maximum value of (m/A) at Ma = 1 is stated as the choked flow. With a given nozzle, the flow rate cannot be increased further. Thus neither the nozzle exit pressure, nor the mass flow rate are affected by lowering P_B below p *.

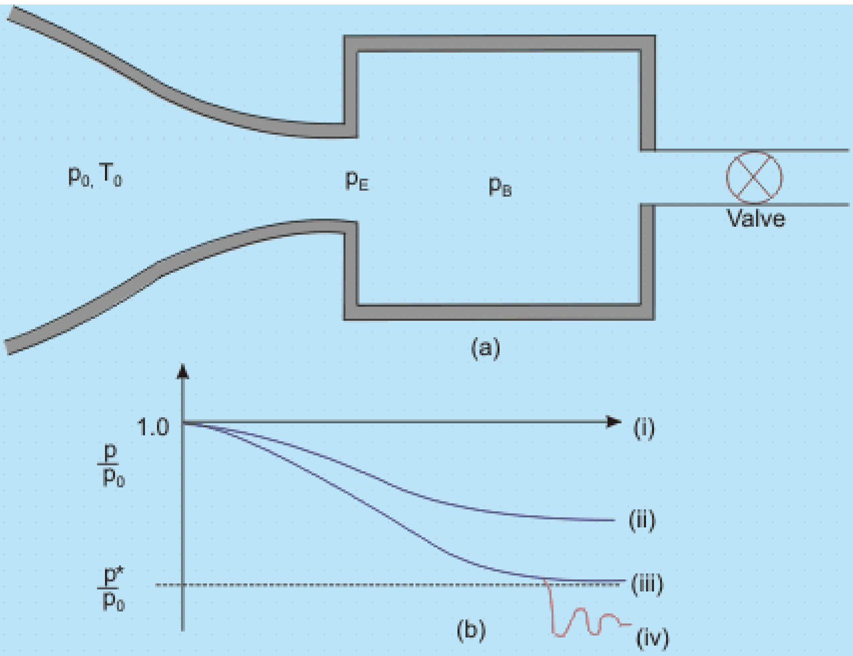


Fig 40.7 (a) Compressible flow through a converging nozzle
(b) Pressure distribution along a converging nozzle for different values of back pressure

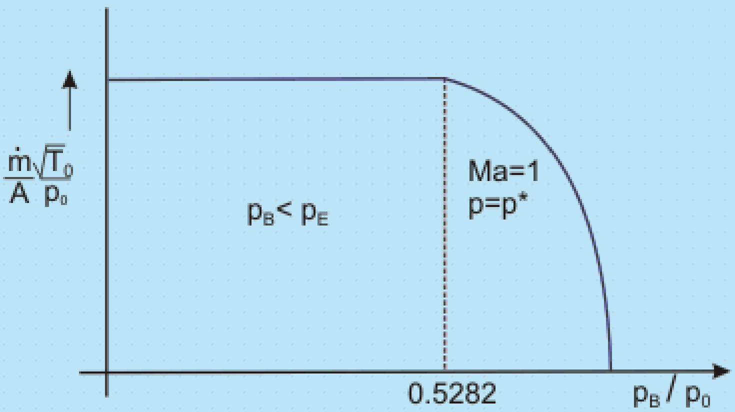


Fig. 40.8 Mass flow rate and the variation of back pressure in a converging nozzle

However for P_B less than p *, the flow leaving the nozzle has to expand to match the lower back pressure as shown by condition (iv) in Fig. 40.7(b). This expansion process is three-dimensional and the pressure distribution cannot be predicted by one-dimensional theory. Experiments reveal that a series of shocks form in the exit stream, resulting in an increase in entropy.

Isentropic Flow in a Converging-Diverging Nozzle

- Consider the flow in a convergent-divergent nozzle (Fig. 40.9). The upstream stagnation conditions are assumed constant; the pressure in the exit plane of the nozzle is denoted by P_E ; the nozzle discharges to the back pressure, P_B .

- With the valve initially closed, there is no flow through the nozzle; the pressure is constant at P_0. Opening the valve slightly produces the pressure distribution shown by curve (i). Completely subsonic flow is discerned.

- Then P_B is lowered in such a way that sonic condition is reached at the throat (ii). The flow rate becomes maximum for a given nozzle and the stagnation conditions.

- On further reduction of the back pressure, the flow upstream of the throat does not respond. However, if the back pressure is reduced further (cases (iii) and (iv)), the flow initially becomes supersonic in the diverging section, but then adjusts to the back pressure by means of a normal shock standing inside the nozzle. In such cases, the position of the shock moves downstream as P_B is decreased, and for curve (iv) the normal shock stands right at the exit plane.

- The flow in the entire divergent portion up to the exit plane is now supersonic. When the back pressure is reduced even further (v), there is no normal shock anywhere within the nozzle, and the jet pressure adjusts to P_B by means of oblique shock waves outside the exit plane. A converging diverging nozzle is generally intended to produce supersonic flow near the exit plane.

- If the back pressure is set at (vi), the flow will be isentropic throughout the nozzle, and supersonic at nozzle exit. Nozzles operating at P_B (corresponding to curve (vi) in Fig. 40.8) are said to be at design conditions. Rocket-propelled vehicles use converging-diverging nozzles to accelerate the exhaust gases to the maximum possible velocity to produce high thrust.

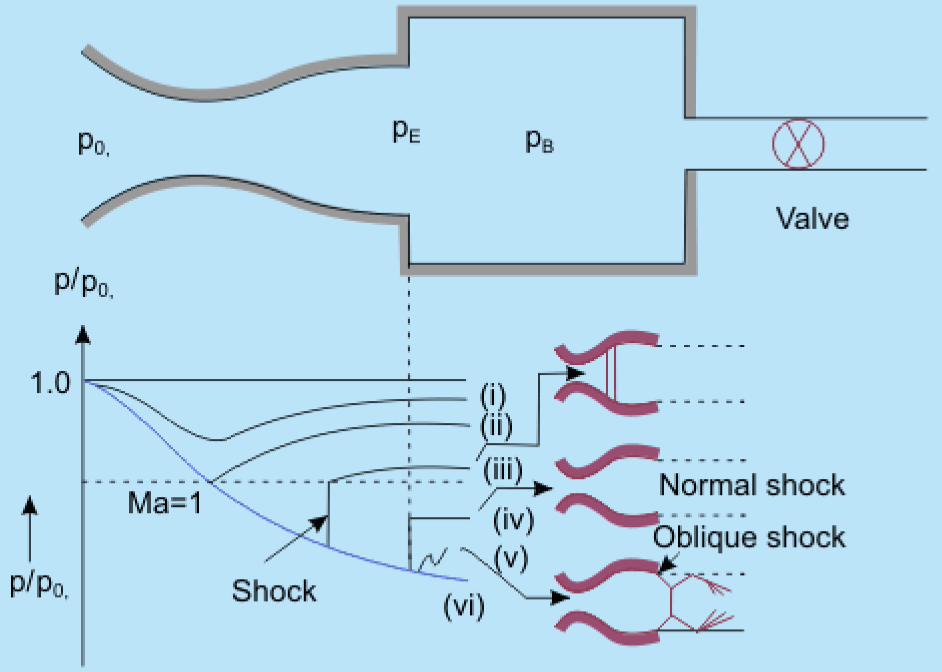


Fig 40.9: Pressure Distribution along a Converging-Diverging Nozzle for different values of back pressure P_B

Normal Shocks

- Shock waves are highly localized irreversibilities in the flow .
- Within the distance of a mean free path, the flow passes from a supersonic to a subsonic state, the velocity decreases suddenly and the pressure rises sharply. A shock is said to have occurred if there is an abrupt reduction of velocity in the downstream in course of a supersonic flow in a passage or around a body.
- **Normal shocks** are substantially **perpendicular** to the flow and **oblique shocks** are inclined at **any angle**.
- Shock formation is possible for confined flows as well as for external flows.
- Normal shock and oblique shock may mutually interact to make another shock pattern.

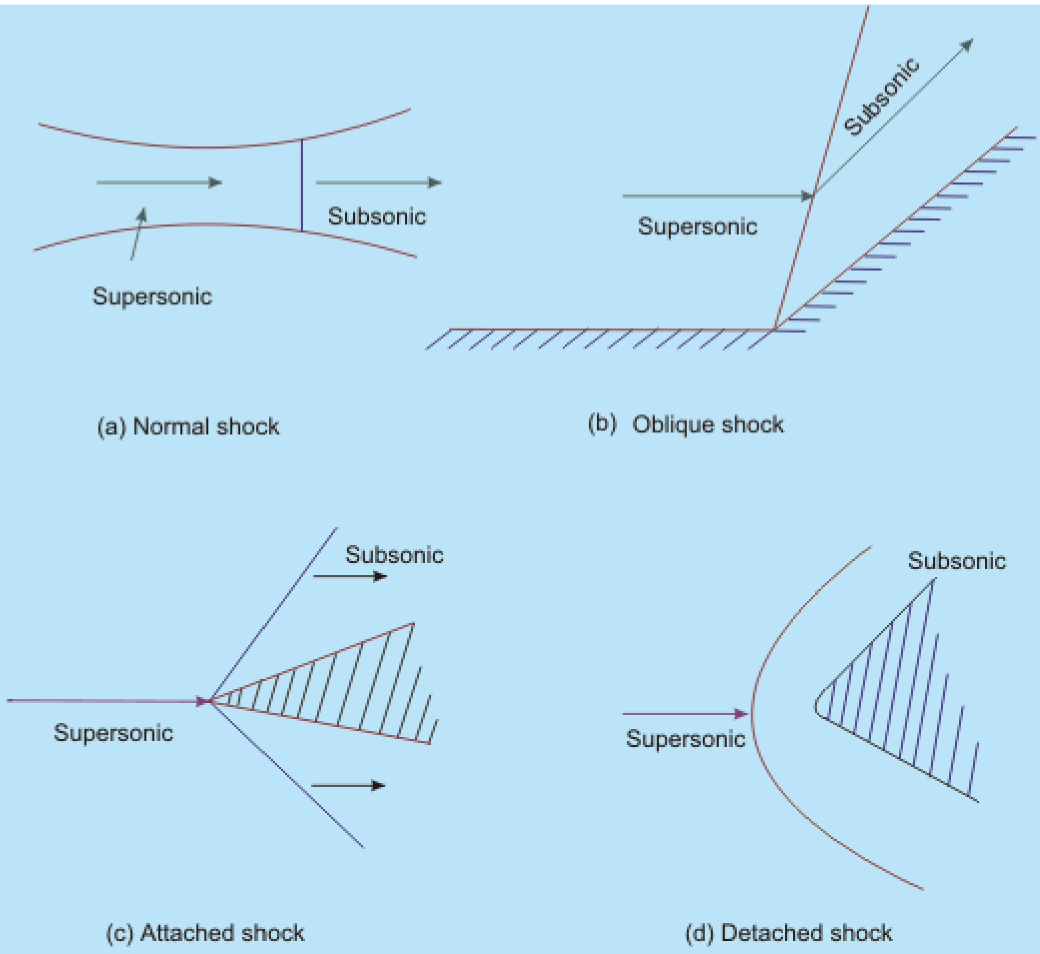


Fig 41.1 Different type of Shocks

Figure below shows a control surface that includes a normal shock.

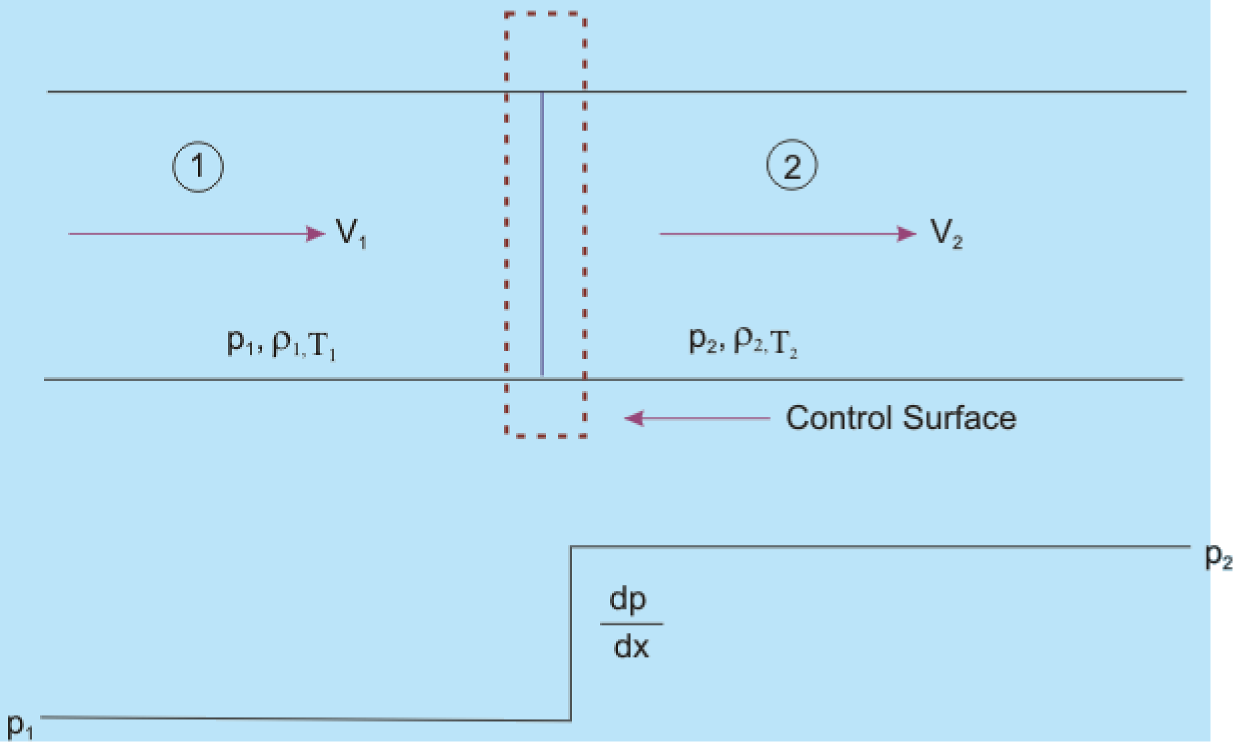


Fig 41.2 One Dimensional Normal Shock

- The fluid is assumed to be in thermodynamic equilibrium upstream and downstream of the shock, the properties of which are designated by the subscripts 1 and 2, respectively. (Fig 41.2).

Continuity equation can be written as

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2 = G \qquad (41.1)$$

where G is the mass velocity kg/ m^2 s, and $\dot{m}$ is mass flow rate

From momentum equation, we can write

$$p_1 - p_2 = \frac{\dot{m}}{A}(V_2 - V_1) = \rho_2 V_2^2 - \rho_1 V_1^2 \qquad (41.2a)$$

$$\Rightarrow p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \qquad (41.2b)$$

$$\Rightarrow \quad F_1 = F_2$$

where p + ρV^2 is termed as **Impulse Function** .

The energy equation is written as

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{01} = h_{02} = h_0 \qquad (41.3)$$

where h_0 is **stagnation enthalpy**.

From the second law of thermodynamics, we know

$$s_2 - s_1 \geq 0$$

To calculate the entropy change, we have

$$T ds = dh - \forall dp$$

For an ideal gas

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

For an ideal gas the equation of state can be written as

$$p = \rho R T \qquad (41.4)$$

For constant specific heat, the above equation can be integrated to give

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \qquad (41.5)$$

Equations (41.1), (41.2a), (41.3), (41.4) and (41.5) are the governing equations for the flow of an ideal gas through normal shock.

If all the properties at state 1 (upstream of the shock) are known, then we have six unknowns $T_2, p_2, \rho_2, V_2, h_2, s_2$ in these five equations.

We know relationship between h and T [Eq. (38.17)] for an ideal gas, $dh = c_p dT$. For an ideal gas with constant specific heats,

$$\Delta h = h_2 - h_1 = c_p (T_2 - T_1) \qquad (41.6)$$

Thus, we have the situation of six equations and six unknowns.

- If all the conditions at state "1"(immediately upstream of the shock) are known, how many possible states 2 (immediate downstream of the shock) are there? **The mathematical answer indicates that there is a unique state 2 for a given state 1.**

Fanno Line Flows

- If we consider a problem of frictional adiabatic flow through a duct, the governing Eqs (41.1), (41.3), (38.8), (41.5) and (41.6) are valid between any two points "1" and "2".

- Equation (41.2a) requires to be modified in order to take into account the frictional force, R_x, of the duct wall on the flow and we obtain

$$R_x + p_1 A - p_2 A = \dot{m} V_2 - \dot{m} V_1$$

So, for a frictional flow, we have the situation of six equations and seven unknowns.

- If all the conditions of "1" are known, the no. of possible states for "2" is 2. With an infinite number of possible states "2" for a given state "1", what do we observe if all possible states "2" are plotted on a **T - s**diagram, The locus of all possible states "2" reachable from state "1" is a continuous curve passing through state "1". The question is how to determine this curve? The simplest way is to assume different values of $\mathbf{T_2}$. For an assumed value of $\mathbf{T_2}$, the corresponding values of all other properties at " 2 " and $R_x$can be determined.

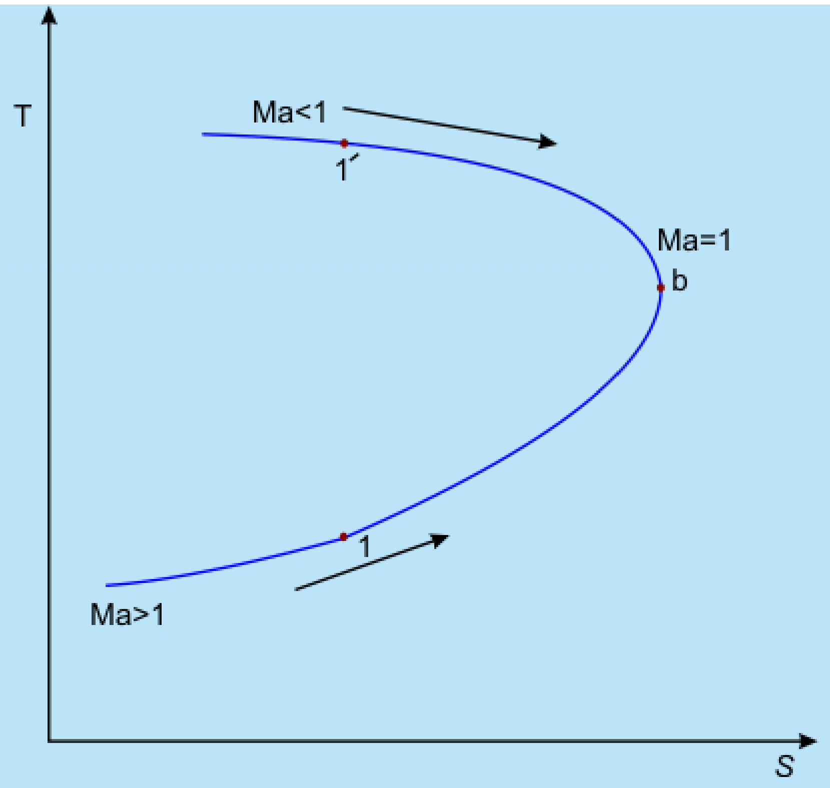


Fig 41.3 Fanno line representation of constant area adiabatic flow

- **The locus of all possible downstream states is called Fanno line** and is shown in Fig. 41.3. **Point " b " corresponds to maximum entropy where the flow is sonic.** This point splits the Fanno line into**subsonic (upper)** and **supersonic (lower)** portions.

- If the inlet flow is supersonic and corresponds to point 1 in Fig. 41.3, then friction causes the downstream flow to move closer to point "b" with a consequent decrease of Mach number towards unity.

- Note that each point on the curve between point 1 and "b" corresponds to a certain duct length L. As L is made larger, the conditions at the exit move closer to point "b". Finally, for a certain value of L, the flow becomes sonic. Any further increase in L is not possible without a drastic revision of the inlet conditions.

- Consider the alternative case where the inlet flow is subsonic, say, given the point 1' in Fig. 41.3. As L increases, the exit conditions move closer to point "b". If L is increased to a sufficiently large value, then point "b" is reached and the flow at the exit becomes sonic. The flow is again choked and any further increase in L is not possible without an adjustment of the inlet conditions.

Rayleigh Line Flows

- Consider the effects of heat transfer on a frictionless compressible flow through a duct, the governing Eq. (41.1), (41.2a), (41.5), (38.8) and (41.6) are valid between any two points "1" and "2".

- Equation (41.3) requires to be modified in order to account for the heat transferred to the flowing fluid per unit mass, dQ , and we obtain

$$dQ = h_{02} - h_{01} \qquad (41.8)$$

-
- So, for frictionless flow of an ideal gas in a constant area duct with heat transfer, we have a situation of six equations and seven unknowns. If all conditions at state "1" are known, there exists infinite number of possible states "2". With an infinite number of possible states "2" for a given state "1", we find if all possible states "2" are plotted on a T- s diagram, The locus of all possible states "2" reachable from state "1" is a continuous curve passing through state "1".

- Again, the question arises as to how to determine this curve? The simplest way is to assume different values of T_2 . For an assumed value of T_2, the corresponding values of all other properties at "2" and dQ can be determined. The results of these calculations are shown in the figure below. The curve in Fig. 41.4 is called the **Rayleigh line**.

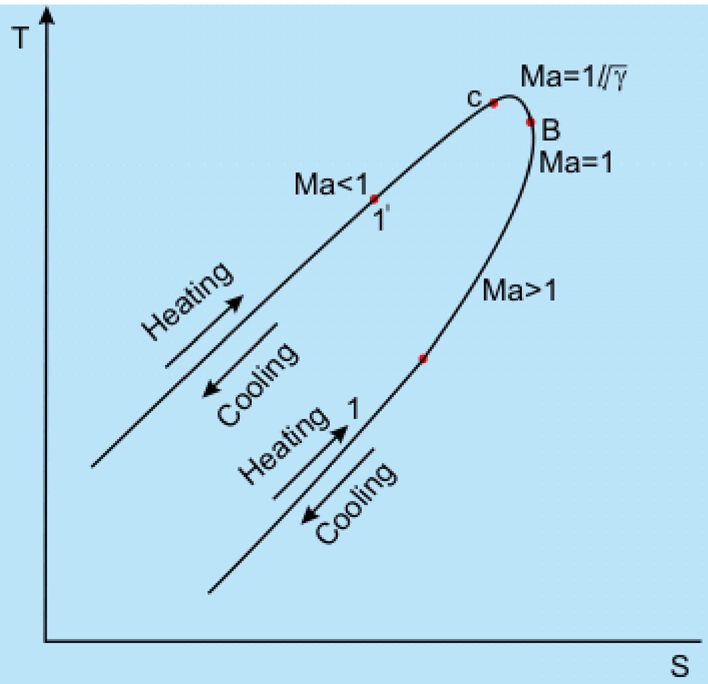


Fig 41.4 Rayleigh line representation of frictionless flow in a constant area duct with heat transfer

- At the point of maximum temperature (point "c" in Fig. 41.4) , the value of Mach number for an ideal gas is $1/\sqrt{\gamma}$. **At the point of maximum entropy(point "b") , the Mach number is unity.**

- On the upper branch of the curve, the flow is always subsonic and it increases monotonically as we proceed to the right along the curve. At every point on the lower branch of the curve, the flow is supersonic, and it decreases monotonically as we move to the right along the curve.

- Irrespective of the initial Mach number, with heat addition, the flow state proceeds to the right and with heat rejection, the flow state proceeds to the left along the Rayleigh line. For example , Consider a flow which is at an initial state given by 1 on the Rayleigh line in fig. 41.4. If heat is added to the flow, the conditions in the downstream region 2 will move close to point "b". **The velocity reduces due to increase in pressure and density, and Ma approaches unity.** If dQ is increased to a sufficiently high value, then point "b" will be reached and flow in region 2 will be sonic. The flow is again choked, and any further increase in dQ is not possible without an adjustment of the initial condition. The flow cannot become subsonic by any further increase in dQ .

The Physical Picture of the Flow through a Normal Shock

- It is possible to obtain physical picture of the flow through a normal shock by employing some of the ideas of Fanno line and Rayleigh line Flows. Flow through a normal shock must satisfy Eqs (41.1), (41.2a), (41.3), (41.5), (38.8) and (41.6).

- Since all the condition of state "1" are known, there is no difficulty in locating state "1" on T-s diagram. In order to draw a **Fanno line** curve through state "1", we require a locus of mathematical states that satisfy Eqs (41.1), (41.3), (41.5), (38.8) and (41.6). The Fanno line curve does not satisfy Eq. (41.2a).

- While **Rayleigh line curve** through state "1" gives a locus of mathematical states that satisfy Eqs (41.1), (41.2a), (41.5), (38.8) and (41.6). The Rayleigh line does not satisfy Eq. (41.3). Both the curves on a same T-s diagram are shown in Fig. 41.5.

- We know normal shock should satisfy all the six equations stated above. At the same time, for a given state" 1", the end state "2" of the normal shock must lie on both the Fanno line and Rayleigh line passing through state "1." Hence, the intersection of the two lines at state "2" represents the conditions downstream from the shock.

- In Fig. 41.5, the flow through the shock is indicated as transition from state "1" to state "2". This is also consistent with directional principle indicated by the second law of thermodynamics, i.e. $s_2 > s_1$.

- From Fig. 41.5, it is also evident that the flow through a normal shock signifies a change of speed from supersonic to subsonic. Normal shock is possible only in a flow which is initially supersonic.

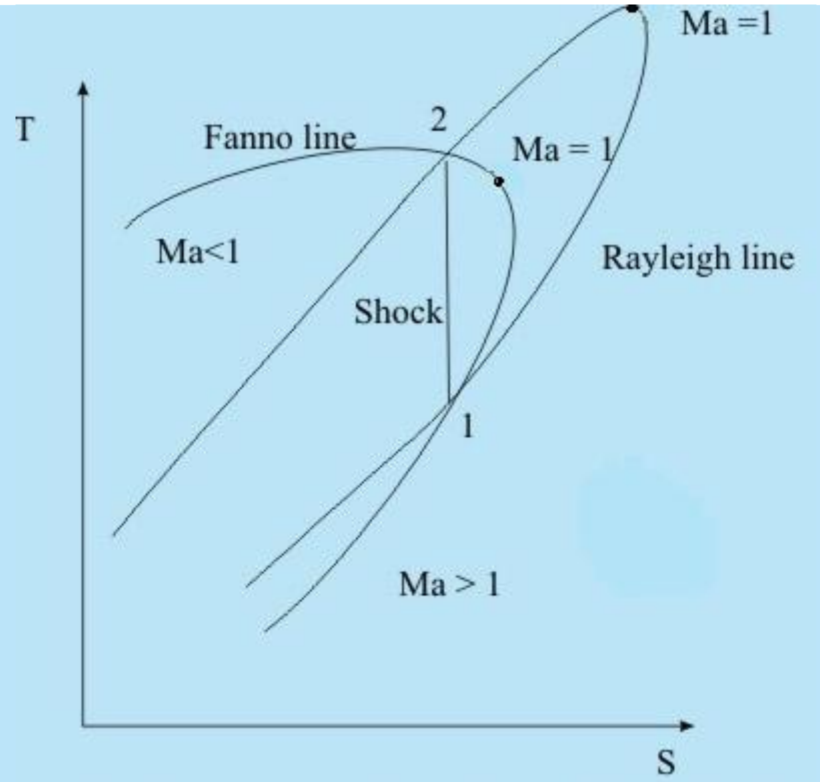


Fig 41.5 Intersection of Fanno line and Rayleigh line and the solution for normal shock condition

Calculation of Flow Properties Across a Normal Shock

- The easiest way to analyze a normal shock is to consider a control surface around the wave as shown in Fig. 41.2. **The continuity equation** (41.1), **the momentum equation** (41.2) and the **energy equation**(41.3) have already been discussed earlier. The energy equation can be simplified for an ideal gas as

$$T_{01} = T_{02} \tag{40.9}$$

- By making use of the equation for the speed of sound eq. (39.5) and the equation of state for ideal gas eq. (38.8), the continuity equation can be rewritten to include the influence of Mach number as:

$$\frac{p_1}{RT_1} Ma_1 \sqrt{\gamma RT_1} = \frac{p_2}{RT_2} Ma_2 \sqrt{\gamma RT_2} \tag{40.10}$$

Introducing the Mach number in momentum equation, we have

$$\rho_2 V_2^2 - \rho_1 V_1^2 = p_1 - p_2$$

$$p_1 + \frac{p_1}{RT_1} V_1^2 = p_2 + \frac{p_2}{RT_2} V_2^2$$

Therefore ,

$$p_1\left(1 + \gamma Ma_1^2\right) = p_2\left(1 + \gamma Ma_2^2\right) \tag{40.11}$$

Rearranging this equation for the static pressure ratio across the shock wave, we get

$$\frac{p_2}{p_1} = \frac{\left(1 + \gamma\ Ma_1^2\right)}{\left(1 + \gamma\ Ma_2^2\right)} \tag{40.12}$$

- As already seen, the Mach number of a normal shock wave is always greater than unity in the upstream and less than unity in the downstream, the static pressure always increases across the shock wave.

- The energy equation can be written in terms of the temperature and Mach number using the stagnation temperature relationship (40.9) as

$$\frac{T_2}{T_1} = \frac{\{1 + (\gamma - 1)/2\} Ma_1^2}{\{1 + (\gamma - 1)/2\} Ma_2^2} \tag{40.13}$$

Substituting Eqs (40.12) and (40.13) into Eq. (40.10) yields the following relationship for the Mach numbers upstream and downstream of a normal shock wave:

$$\frac{Ma_1}{1+\gamma\, Ma_1^2}\left[1+\frac{\gamma-1}{2}Ma_1^2\right]^{\frac{1}{2}} = \frac{Ma_2}{1+\gamma\, Ma_2^2}\left[1+\frac{\gamma-1}{2}Ma_2^2\right]^{\frac{1}{2}} \qquad (40.14)$$

Then, solving this equation for Ma_2 as a function of Ma_1 we obtain two solutions. One solution is trivial $Ma_2 = Ma_1$, which signifies no shock across the control volume. The other solution is

$$Ma_2^2 = \frac{(\gamma-1)Ma_1^2+2}{2\gamma Ma_1^2-(\gamma-1)} \qquad (40.15)$$

$Ma_1 = 1$ in Eq. (40.15) results in $Ma_2 = 1$

Equations (40.12) and (40.13) also show that there would be no pressure or temperature increase across the shock. In fact, the shock wave corresponding to $Ma_1 = 1$ is the sound wave across which, by definition, pressure and temperature changes are infinitesimal. Therefore, it can be said that the sound wave represents a degenerated normal shock wave. The pressure, temperature and Mach number (Ma_2) behind a normal shock as a function of the Mach number Ma_1, in front of the shock for the perfect gas can be represented in a tabular form (known as Normal Shock Table). The interested readers may refer to Spurk[1] and Muralidhar and Biswas[2].

Oblique Shock

- The discontinuities in supersonic flows do not always exist as normal to the flow direction. There are oblique shocks which are inclined with respect to the flow direction. Refer to the shock structure on an obstacle, as depicted qualitatively in Fig.41.6.
- The segment of the shock immediately in front of the body behaves like a normal shock.

- Oblique shock can be observed in following cases-
    1. **Oblique shock formed as a consequence of the bending of the shock in the free-stream direction** (shown in Fig.41.6)
    2. In a supersonic flow through a duct, viscous effects cause the shock to be oblique near the walls, the shock being normal only in the core region.
    3. **The shock is also oblique when a supersonic flow is made to change direction near a sharp corner**

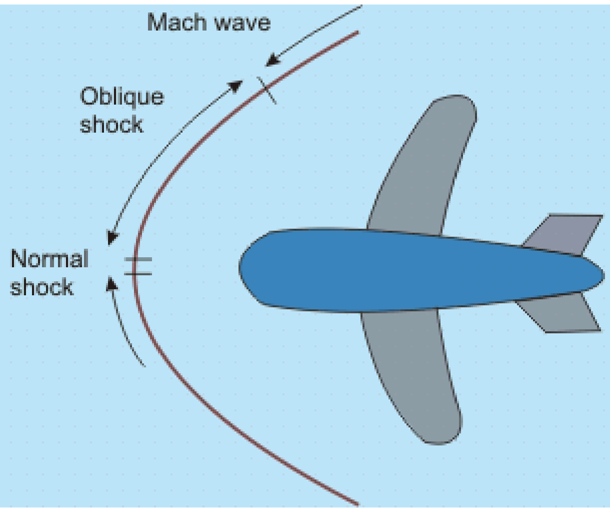


Fig 41.6 Normal and oblique Shock in front of an Obstacle

- The relationships derived earlier for the normal shock are valid for the velocity components normal to the oblique shock. The oblique shock continues to bend in the downstream direction until the Mach number of the velocity component normal to the wave is unity. At that instant, **the oblique shock degenerates into a so called Mach wave across which changes in flow properties are infinitesimal.**

- Let us now consider a two-dimensional oblique shock as shown in Fig.41.7 below

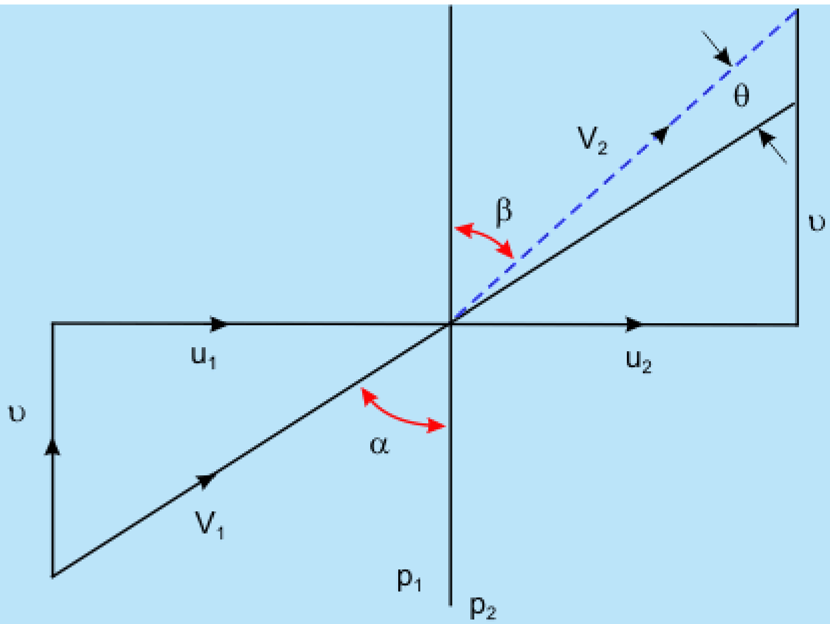


Fig 41.7 Two dimensional Oblique Shock

For analyzing flow through such a shock, it may be considered as a normal shock on which a velocity υ (parallel to the shock) is superimposed. The change across shock front is determined in the same way as for the normal shock. The equations for mass, momentum and energy conservation , respectively, are

$$\rho_1 u_1 = \rho_2 u_2 \tag{41.16}$$

$$\rho_1 u_1 (u_1 - u_2) = p_2 - p_1 \tag{41.17}$$

$$h_{01} = h_{02}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$\frac{\gamma}{\gamma - 1} \cdot \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma - 1} \cdot \frac{p_2}{\rho_2} + \frac{u_2^2}{2} \tag{41.18}$$

These equations are analogous to corresponding equations for normal shock. In addition to these, we have

$$\frac{u_1}{a_1} = Ma_1 \sin\alpha \quad \text{and} \quad \frac{u_2}{a_2} = Ma_2 \sin\beta$$

Modifying normal shock relations by writing $Ma_1 \sin\alpha$ and $Ma_2 \sin\beta$ in place of Ma_1 and Ma_2, we obtain

$$\frac{p_2}{p_1} = \frac{2\gamma Ma_1^2 \sin\alpha - \gamma + 1}{\gamma + 1} \tag{41.19}$$

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{\tan\alpha}{\tan\beta} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) + Ma_1^2 \sin^2\alpha} \tag{41.20}$$

$$Ma_2^2 \sin^2\beta = \frac{2 + (\gamma - 1) Ma_1^2 \sin^2\alpha}{1 + \tan^2\alpha (\tan\beta / \tan\alpha)} \tag{41.21}$$

Note that although $Ma_2 \sin\beta$ <1, Ma_2 might be greater than 1. So **the flow behind an oblique shock may be supersonic although the normal component of velocity is subsonic.**

In order to obtain the angle of deflection of flow passing through an oblique shock, we use the relation

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

$$= \frac{\tan\alpha - (\tan\beta / \tan\alpha) \tan\alpha}{1 + \tan^2\alpha (\tan\beta / \tan\alpha)}$$

Having substituted $(\tan\beta / \tan\alpha)$ from Eq. (41.20), we get the relation **(see steps here)**

$$\tan\theta = \frac{Ma_1^2 \sin 2\alpha - 2\cot\alpha}{Ma_1^2 (\gamma + \cos 2\alpha) + 2} \tag{41.22}$$

Sometimes, a design is done in such a way that an oblique shock is allowed instead of a normal shock. The losses for the case of oblique shock are much less than those of normal shock.This is the reason for making the nose angle of the fuselage of a supersonic aircraft small.

Supplementary Questions

I. Air enters a diffuser at 27^{0}C and 1 N / M^2. The approach velocity is 300m/sec, assume the flow to be isentropic. If the velocity of the air leaving the diffuser is 60m/sec, calculate the entrance and exit Mach numbers, the static pressure at exit and the percent change in cross-sectional area between entrance and exit.

Ans. Ma(entrance) = 0.864, Ma(exit) = 0.1616, p(exit) = 1.6 N/m ^{2} , increase in area=258%

II. Carbon dioxide discharges to the atmosphere through a 10mm diameter hole in the wall of a tank in which the pressure is 8 bars(gauge) and temperature is 20^0C. What is the velocity of the jet ? Take γ =1.3 and R=188 J/KgK and atmospheric presuure=1 bar.

Ans. 249.54 m/sec

III. Air flows isentropically through a duct. At a given point the area is $0.5\ m^2$ and the Mach number is 0.4. At another point in the duct, the area is 0.36 m^2. What is the Mach number at the second point? What would the area be at a point where Ma=1?

Ans. Ma=0.64 or 1.4516, A= 0.3144 m^2

IV. The stagnation temperature and stagnation pressure of air in a reservoir supplying a convergent-divergent nozzle are are 450K and 4 N/m^2 respectively. The nozzle throat area is 1 cm^2 and the nozzle exit area is 3 cm^2. A shock is noted at a position in the diverging portion where the area is 2 cm^2.

(i) What are the exit pressure, temperature and velocity?(The Mach no. just before the shock is 2.1972)

Ans. p(exit)= 2.3386 N/m^2, T(exit)=441 K, V(exit)=137.2 m/sec

(ii) What value of the back pressure would cause the shock to stand in the exit plane of the nozzle so that the flow through the nozzle is completely supersonic?

Ans. p= 1.505 N/m^2

(iii) What value of the back pressure would result in completely isentropic flow interior and exterior to the nozzle?

Ans. p= 0.1892 N/m^2(supersonic flow at exit); $3.8928\ N/m^2 \le p \le 4\ N/m^2$ (subsonic flow at exit)

Recap

In this course you have learnt the following

- Fluid density varies mainly due to a large Ma flow. This leads to a situation where continuity & momentum equation can be coupled to the energy equation and the equation of state to solve four unknowns- P, T, V, ρ.
- The stagnation enthalpy and hence T_0 are conserved in isentropic flows. The effect of area variation in on flow properties in an isentropic flow is of great significance. This reveals the phenomenon of choking at the sonic velocity in the throat of a nozzle.
- At choke condition, the ratio of throat pressure to stagnation pressure is constant and is equal to 0.528 for γ=1.4.

- At supersonic velocities, the normal shock wave appears across which the gas discontinuity reverts to the subsonic conditions.

- Fanno and Rayleigh line flows both entail choking of the exit flow. The conditions before and after a normal shock are defined by the points of intersection of Fanno and Rayleigh lines on a T-s Diagram.

- If a supersonic flow is made to change its direction, the oblique shock is evolved. The oblique shock continues to bend in the downstream direction until the Mach Number of the velocity component normal to the wave is unity.