①

Problem 1:- Consider the SL problem

$$y'' + \lambda y = 0 \; ; \; 0 < x < 1$$

$$y(0) = 0$$

$$y(1) - y'(1) = 0 .$$

Find the eigenvalue & the eigenfxn of the problem.

Soln:- The characteristic Eqn is $m^2 + \lambda = 0$.

Case 1:- $\lambda < 0$ ~~[illegible]~~ let $\lambda = -\mu^2$; $\mu > 0$.

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$y(1) - y'(1) = 0 \Rightarrow c_1 e^{\mu} + c_2 e^{-\mu} - \mu c_1 e^{\mu} + \mu c_2 e^{-\mu} = 0$$

$$-c_2(1-\mu) e^{\mu} + c_2 (1+\mu) e^{-\mu} = 0$$

$$\Rightarrow \cancel{c_2(1+\mu)e^{\mu}} \Rightarrow c_1 \left[(\mu - 1) e^{\mu} + (1+\mu) e^{-\mu} \right] = 0$$

$$\Rightarrow c_1 = 0.$$

$\therefore$ There are no negative eigenvalue.

Case 2:- For $\lambda = 0$ the general soln is $y(x) = A + Bx$

$$y(0) = 0 \Rightarrow A + B \cdot 0 = 0 \Rightarrow A = 0.$$

$$y(1) - y'(1) = 0 \Rightarrow A + B \cdot 1 - B = 0$$

$\therefore B$ is arbitrary.

Hence, $\lambda = 0$ is an eigenvalue with eigenfxn.

$$y_0 = x$$

For $\lambda > 0$ set $\lambda = \mu^2$ (2)

and the General soln is

$$y(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

$y(0) = 0 \Rightarrow C_1 = 0$ & $y(x) = C_2 \sin(\mu x)$.

$y(1) - y'(1) = 0 \Rightarrow C_2 \sin \mu - C_2 \mu \cos \mu = 0$.

$\therefore$ μ must satisfy $\sin \mu = \mu \cos \mu$ for the eqn to have an eigenvalue.

$\therefore \tan \mu = \mu$.

Consider the fcn $g(\mu) = ~~\tan\mu - \mu~~ \sin \mu - \mu \cos \mu$.

$g(n\pi) = -(n\pi)(-1)^n$ and $g\left(n\pi + \frac{\pi}{2}\right) = (-1)^n$.

Thus, ~~between~~ $g(n\pi)\, g\left(n\pi + \frac{\pi}{2}\right) < 0$.

$\therefore$ Intermediate Value Theorem implies g has a zero in $n\pi$ and $\left(n\pi + \frac{\pi}{2}\right)$.

Again since $g'(\mu) = \mu \sin \mu$ doesn't change sign on $\left(n\pi, n\pi + \frac{\pi}{2}\right)$.

$\therefore$ g has a unique zero on $\left[n\pi, n\pi + \frac{\pi}{2}\right]$.

II^{ly} g has no zero in $\left(n\pi + \frac{\pi}{2}, (n+1)\pi\right)$

$\therefore$ The Eigenvalues are $\lambda_n = \mu_n^2$; $y_n = \sin(\mu_n x)$

where ν_n satisfies $\tan \nu_n = \nu_n$

$\nu_n \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$.

2.. Find the eigenvalues & eigenfxns of the SL problem

$$(x^2y')' + \lambda y = 0 \ , \ 1 < x < 2$$
$$y(1) = y(2) = 0 .$$

Soln:- Note that the ODE is the Cauchy-Euler problem
$$x^2y'' + 2xy' + \lambda y = 0$$

$\therefore$ Characteristic Eqn is $m^2 + 2m + \lambda = 0$

$\therefore$ Roots are $m_1 = \frac{-1-\sqrt{1-4\lambda}}{2}$ & $m_2 = \frac{-1+\sqrt{1-4\lambda}}{2}$.

Case 1:- If $\lambda < 1/4$ then both roots are real and distinct and the soln is given as
$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

The endpoints conditions are $y(1) = y(2) = 0$

$\therefore$ $c_1 + c_2 = 0$ & $c_1 2^{m_1} + c_2 2^{m_2} = 0$

$\Rightarrow$ $c_1 = c_2 = 0$ $(\because m_1 \neq m_2)$.

$\therefore$ When, $\lambda < 1/4$ ~~because~~ the eqn cannot have any eigenvalue.

Case 2:- If $\lambda = 1/4$ then $m_1 = m_2 = -1/2$. and the general soln is $y(x) = c_1 x^{-1/2} + B x^{-1/2} \ln x$.

Putting $y(1) = y(2) = 0$ we get,
$$c_1 = 0 \text{ and } c_2 2^{-1/2} \ln 2 = 0$$

Hence, $y \equiv 0$ and $\lambda = 1/4$ is not an eigenvalue.

Case 3 :- If $\lambda > 1/4$, set $1-4\lambda = -4\mu^2$; $\mu > 0$.

The Characterestic roots are $m = -\frac{1}{2} \pm i\mu$.

$\therefore$ General Soln is $y(x) = x^{-1/2}\left(C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)\right)$

$y(1) = C_1 = 0$

$$\& \ y(2) = \frac{C_1 \cos(\mu \ln 2) + C_2 \sin(\mu \ln 2)}{\sqrt{2}} = 0$$

Hence, $C_2 \sin(\mu \ln 2) = 0 \Rightarrow \sin(\mu \ln 2) = 0$ $[\because C_2 \neq 0]$

$\therefore \mu = \frac{n\pi}{\ln 2}$; $n \in \mathbb{N}$. ($\because \mu > 0$ assumed).

From $1-4\lambda = -4\mu^2$ we get

$$\lambda_n = 1/4 + \mu_n^2 = \frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2 \ ; n \in \mathbb{N}$$

and the eigenfunction are

$$y_n(x) = \frac{1}{\sqrt{x}} \sin \frac{n\pi \ln x}{\ln 2}.$$

Problem 3 :- Consider the problem which models the wave propagation in a nonhomogeneous string (the mass density of the string is not constant)

$$u_{tt} = (1+x)^2 u_{xx} \quad ; 0<x<1 \; ; t>0$$

$$u(0,t) = 0 \; ; u(1,t) = 0 \quad ; t>0$$

$$u(x,0) = f(x), \quad 0<x<1.$$

$$u_t(x,0) = g(x), \quad 0<x<1.$$

The mass density of the string at the pt x is $\frac{1}{(1+x)^2}$.

Using seperation of variable;

$$u(x,t) = X(x)T(t)$$

Hence, $X'' + \frac{\lambda}{(1+x)^2}X = 0$ — ① and $T'' + \lambda T = 0$ — ②

① is a regular S.L problem with weight $r(x) = \frac{1}{(1+x)^2}$.

Now we need to find the eigenfxn and the eigenvalue of ①

① can be written as,

$$(1+x)^2 X'' + \lambda X = 0$$

$\therefore$ Characteristic Eqn :- $m^2 - m + \lambda = 0$.

$\therefore$ the roots $m_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$.

Depending on the parameter λ consider the cases.

Case 1:- $\lambda < 1/4$ then m_1 & m_2 are real and distinct.

$\therefore$ G.S $\Rightarrow X(x) = C_1(1+x)^{m_1} + C_2(1+x)^{m_2}$

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Now, $X(0) = 0 \Rightarrow C_1 + C_2 = 0$

& $X(1) = C_1 2^{m_1} + C_2 2^{m_2} = 0$

Hence the only soln is $C_1 = C_2 = 0$

and such λ cannot be an eigenvalue

Case 2 :- If $\lambda = 1/4$ then $m_1 = m_2 = 1/2$.

the G.S is $C_1\sqrt{1+x} + C_2\sqrt{1+x}\ \ln(1+x) = \cancel{X}(x)$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(1) = C_1\sqrt{2} + C_2\sqrt{2}\ \ln 2 = 0 \Rightarrow C_2 = 0$

$\therefore \lambda = 1/4$ cannot be an eigenvalue

Case 3 :- If $\lambda > 1/4$ then m_1 and m_2 are complex conjugate

set, $\lambda = \mu^2 + 1/4$ with $\mu > 0$.

$\therefore$ roots $m_{1,2} = \frac{1}{2} \pm i\mu$.

$\therefore X(x) = A\sqrt{1+x}\cos(\mu\ln(1+x)) + B\sqrt{1+x}\sin(\mu\ln(1+x))$.

$X(0) = 0 \Rightarrow A = 0$ & $X(1) = 0 \Rightarrow B\sqrt{2}\sin(\mu\ln 2) = 0$.

$\therefore \sin(\mu\ln 2) = 0 \Leftrightarrow \mu = \frac{n\pi}{\ln 2}$ $(n \in \mathbb{N})$

$\therefore$ Eigenvalues and eigenfxns of ① are

$\lambda_n = 1/4 + \mu_n^2$; $X_n(x) = \sqrt{1+x}\sin(\mu_n\ln(1+x))$ where $\mu_n = \frac{n\pi}{\ln 2}$.

For each $n \in \mathbb{N}$ we get from eqn ②

$T_n^1(t) = \cos(\sqrt{\mu_n}\,t)$, $T_n^2(t) = \sin(\sqrt{\mu_n}\,t)$

$\therefore$ G.S $u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos(\sqrt{\mu_n}t) + B_n\sin(\sqrt{\mu_n}t)\right]\sqrt{1+x}\sin(\mu_n\ln(1+x))$.

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· Since the problem satisfy

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sqrt{1+x}\,\sin\left(\mu_n \ln(1+x)\right)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \sqrt{\mu_n}\, B_n \sqrt{1+x}\,\sin\left(\mu_n \ln(1+x)\right).$$

Problem 4 :- $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$ and has period 2π.

Find its Fourier Series and show $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Soln:- Fourier Representation of $f(x)$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 1.\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0.\,dx$$

$$= \frac{1}{\pi}.\pi = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 1.\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0.\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}$$

$$= \frac{1}{n\pi}\sin n\pi$$

$\therefore a_n = 0 \quad \forall n \in \mathbb{N}$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0.\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} = -\frac{1}{n\pi}\left(1-(-1)^n\right).$$

$\therefore b_{2n} = 0$ & $b_{2n+1} = -\frac{2}{n\pi}$.

$\therefore f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \dots\right]$

Picking $x = \pi/2$ we get

$$0 = \frac{1}{2} - \frac{2}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \dots\right]$$

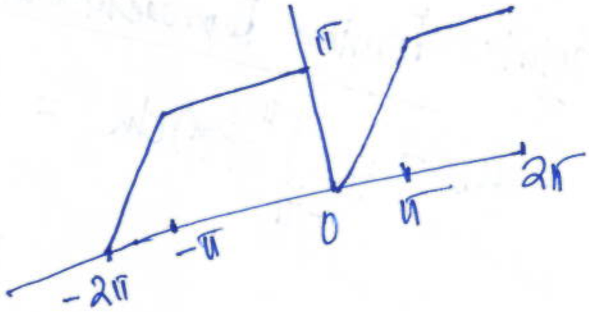
or, $0 = \frac{1}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$

Hence $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Problem 5:- Find the Fourier series of

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases}$$

and has period 2π.



and show

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Soln: $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,dx$

$= \frac{1}{\pi}\cdot\frac{\pi^2}{2} + \pi = \frac{3\pi}{2}$.

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\pi\cdot\cos nx\,dx$

$= \frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right] + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$

$= \frac{1}{\pi}\left[\frac{1}{n}(0-0) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}(0-0)$

$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$.

$$\therefore a_n = \begin{cases} -\frac{2}{n^2\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}\,dx\right] + \left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi + \frac{\sin n\pi - \sin 0}{n^2}\right) - \frac{1}{n}\left(1-(-1)^n\right)$$

$$= -\frac{(-1)^n}{n} - \frac{1}{n}\left(1-(-1)^n\right)$$

$$\therefore b_n = -\frac{1}{n}(-1)^n - \frac{1}{n} + \frac{(-1)^n}{n} = -\frac{1}{n}.$$

$$\therefore f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \cdots\right)$$
$$-\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

(ii) Put $x = 0$;

$$\pi/2 = 3\pi/4 - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Problem 6:- Find the fourier sine series of $f(x) = x$ on $[0,1]$

Soln:- $b_n = 2\int_0^1 x\sin(n\pi x)\,dx = -\frac{2x}{n\pi}\cos n\pi x\Big|_0^1$
$$+ \int_0^1 \frac{2}{n\pi}\cos(n\pi x)\,dx$$

$\therefore b_n = \dfrac{2(-1)^{n+1}}{n\pi}$.

$\therefore$ The Fourier Series of $f(x) = x$ is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x) = \frac{2}{\pi}\left\{\sin \pi x - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \dots\right\}$$

$\because f(x) = x$ is continuous the series converge to $f(x)$ for any $x \in (0,1)$.

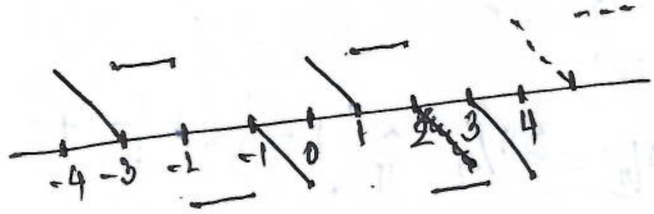
Problem 7:- Consider the fn

$$f(x) = \begin{cases} 1-x, & 0 < x \le 1. \\ 1, & 1 < x \le 2. \end{cases}$$

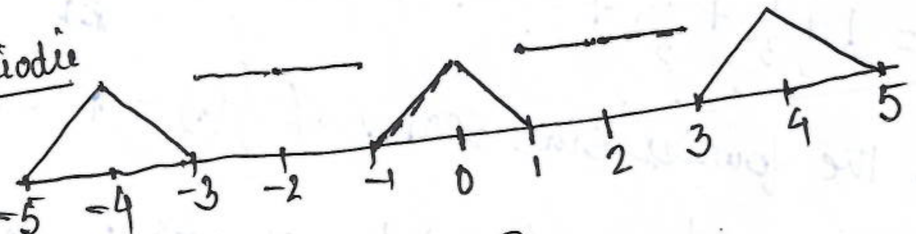
(i) Plot its odd & even periodic extensions over $(-4,4)$

(ii) Compute Fourier Cosine series.

Soln:- Odd Periodic



Even Periodic



Fourier Cosine Series:- $L = 2$

$$a_0 = \frac{1}{2}\int_0^1 (1-x)\,dx + \frac{1}{2}\int_1^2 1\,dx = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{4}.$$

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$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (L=2)$

$= \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$

$= \frac{2}{n\pi}\left[(1-x)\sin\frac{n\pi x}{2}\Big|_0^1 + \int_0^1 \sin\left(\frac{n\pi x}{2}\right)\right] + \frac{2}{n\pi}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right]$

$= -\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

$= \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

$\therefore a_n = \begin{cases} 0 & \text{if } n = 4k \\ \frac{4}{n^2\pi^2} - \frac{2}{n\pi} & \text{if } n = 4k+1 \\ \frac{8}{n^2\pi^2} & \text{if } n = 4k+2 \\ \frac{4}{n^2\pi^2} + \frac{2}{n\pi} & \text{if } n = 4k+3 \end{cases}$ —— (*)

$\therefore$ The Fourier Cosine series is

$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$ where a_n are as (*).

(8) $P_n(x)$ satisfies the eqn $((1-x^2)y')' + n(n+1)y = 0$ on $(-1,1)$

P_n forms an orthogonal system of polynomials on $(-1,1)$

$\therefore n(n+1)P_n(x) = [(x^2-1)P_n'(x)]'$ —— (i)

$m(m+1)P_m(x) = [(x^2-1)P_m'(x)]'$ —— (ii)

multiplying (i) with P_m and (ii) with P_n we have,

$$\{n(n+1) - m(m+1)\} \int_{-1}^{1} P_n(x) P_m(x)\, dx$$

$$= \left[(x^2-1)\left(P_n' P_m - P_m' P_n\right) \right]_{-1}^{1} = 0 .$$

$$\boxed{\|P_n(x)\|^2 = \frac{2}{2n+1} \quad \forall n.}$$

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{1}{2^n \lfloor n} \int_{-1}^{1} f(x) \frac{d^n}{dx^n}\{(x^2-1)^n\}\, dx$$

$$= \frac{1}{2^n \lfloor n} \left\{ \left[f(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\} \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} \{(x^2-1)^n\}\, dx \right\}$$

$$= - \frac{1}{2^n \lfloor n} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} \{(x^2-1)^n\}\, dx$$

Repeating this integration by parts $(n-1)$ more times gives:

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n \lfloor n} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\, dx$$

$$\Rightarrow \int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{(-1)^n}{2^n \lfloor n} \int_{-1}^{1} (x^2-1)^n P_n^{(n)}(x)\, dx$$

$$= \frac{(-1)^n}{2^n \lfloor n} \int_{-1}^{1} (x^2-1)^n \frac{1}{2^n \lfloor n} \frac{d^{2n}}{dx^{2n}} \{(x^2-1)^n\}\, dx$$

Hence, $\dfrac{d^{2n}}{dx^{2n}} (x^2-1)^n = 2n(2n-1) \cdots 3 \cdot 2 \cdot 1 = \lfloor 2n \quad \left(\because (x^2-1)^n = x^{2n} + \cdots + (-1)^n\right).$

we see, $$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{(-1)^n}{2^n \lfloor n} \int_{-1}^{1} (x^2-1)^n \frac{1}{2^n \lfloor n} \lfloor 2n\, dx = \frac{(-1)^n \lfloor 2n}{2^{2n} (\lfloor n)^2} \int_{-1}^{1} (x^2-1)^n\, dx$$

using the fact, $\displaystyle\int_{-1}^{1} (x^2-1)\, dx = \frac{2^{2n+2} \lfloor n \lfloor n+1 \, (-1)^n}{\lfloor 2n+2}.$

and hence, $\displaystyle\int_{-1}^{1} P_n^2\, dx = \frac{2}{2n+1}.$

Legendre - Fourier Series :-

Let $f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$ on $[-1,1]$

$\because (P_n)$ are orthogonal to each other.

$$\int_{-1}^{1} f(x) P_m(x)\,dx = \int_{-1}^{1} [C_0 P_m(x) + C_1 P_1(x) P_m(x) \ldots]\,dx$$

$$\Rightarrow \int_{-1}^{1} f(x) P_m(x)\,dx = \frac{2C_m}{2m+1} \quad \left[\because \int_{-1}^{1} P_m^2\,dx = \frac{2}{2m+1}\right]$$

$$\Rightarrow C_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\,dx .$$

Hence, f can be represented by $f(x) \simeq \sum_{n=0}^{\infty} C_n P_n(x)$ where C_n are given by $C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\,dx$.

$$\boxed{\text{L-F series of } f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & -1 < x < 0. \end{cases}}$$

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\,dx$$

$$= \frac{2n+1}{2} \int_{0}^{1} P_n(x)\,dx$$

$\therefore C_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$ | $C_1 = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{4}$ | $C_2 = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2-1)\,dx =$

Hence, $1 = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \ldots, \quad 0 < x < 1.$

and, $0 = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \ldots, \quad -1 < x < 0.$