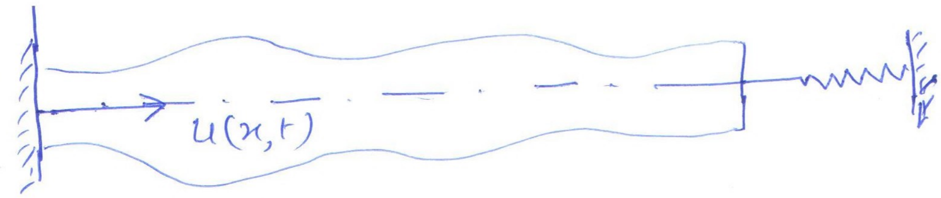
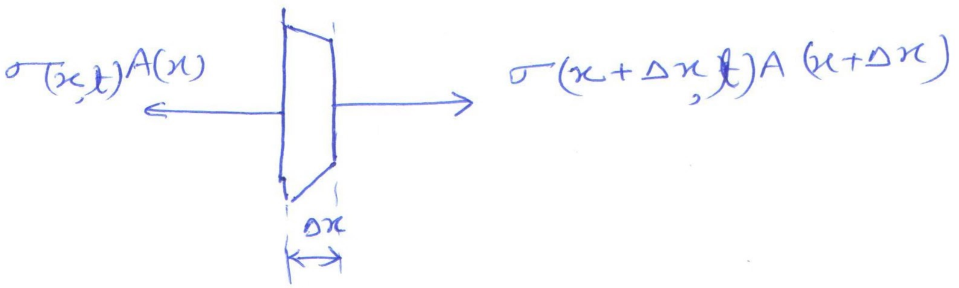
Q 5.1



After discretization of domain, and consider a element of length Δx at distance x



Force balance

$$\sigma(x+\Delta x,t)A(x+\Delta x) - \sigma(x,t)A(x) = \rho A(x)\,\Delta x\,\frac{\partial^2 u}{\partial t^2} \quad \text{— (1)}$$

$$\sigma(x+\Delta x,t)A(x+\Delta x) = \left[\sigma(x,t) + \frac{\partial \sigma(x,t)}{\partial x}\Delta x + \Delta(0)\right]\left[A(x) + \frac{\partial A(x)}{\partial x}\Delta x + \Delta(0)\right]$$

$$= \sigma(x,t)A(x) + \frac{\partial}{\partial x}\left[\sigma(x,t)A(x)\right]\Delta x + \Delta(0) \quad \text{— (2)}$$

$\therefore$ form eq (1) & eq (2)

$$\frac{\partial}{\partial x}\left[\sigma(x,t)A(x)\right]\Delta x + \Delta(0) = \rho A(x)\,\Delta x\,\frac{\partial^2 u}{\partial t^2}$$

as $\Delta x \to 0$

as $\Delta x \to 0$

$$\frac{\partial}{\partial x}\left[\sigma(x,t)A(x)\right] = \rho A(x) \frac{\partial^2 u}{\partial t^2} \quad — (3)$$

strain - displacement relationship & Hook's Law

$$\sigma(x,t) = E\,\epsilon_x(x,t) = E\,u_{,x}(x,t) \quad — (4)$$

from eq^n (3) & (4)

$$\boxed{\frac{\partial}{\partial x}\left[E\,u_{,x}(x,t)\,A(x)\right] = \rho A(x) \frac{\partial^2 u(x,t)}{\partial t^2}}$$

B.C.

at $x = 0$

displacement

$$u(0,t) = 0$$

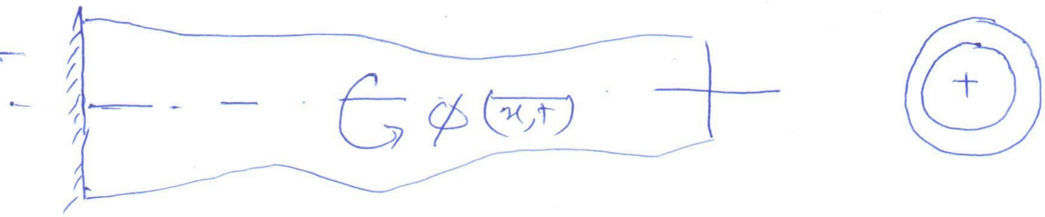
at $x = L$

longitudinal force

$$\sigma(L,t)\,A(L) = -K\,u(L,t)$$

$$A(L)\,E\,u_{,x}(L,t) = -K\,u(L,t)$$

5.2



ρ, A, G

at distance x, consider a element of length Δx

ϕ = angle of twist

ψ = angular deformation of a longitudinal line

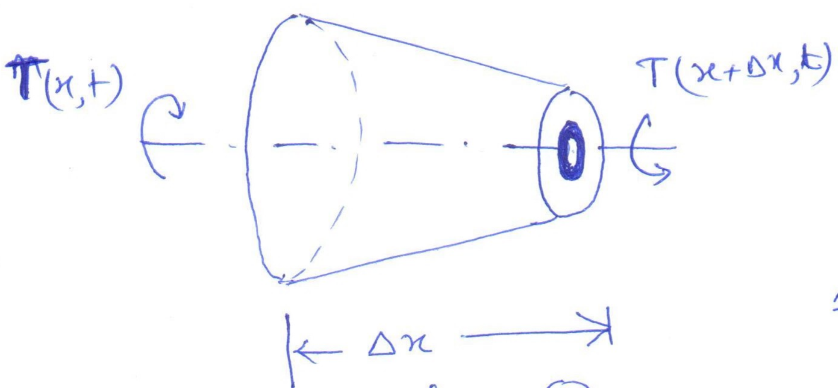


fig : 1

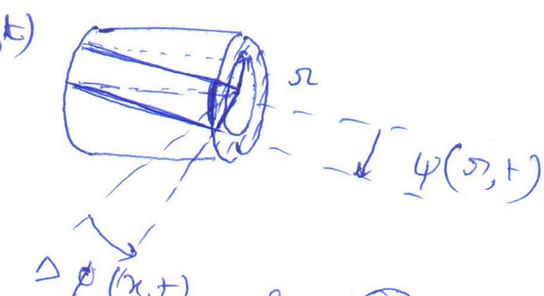


fig : 2

Kinematic relation from figure 2

$$r \, \Delta\phi(x,t) = \Delta x \, \psi(r,t) \qquad \text{—— (1)}$$

Shear stress relation

$$\tau_{x\phi}(r,t) = G \, \psi(r,t) \qquad \text{—— (2)}$$

from eq^n (1) & (2) as $\Delta x \to 0$

$$\tau_{x\phi}(r,t) = G \, r \, \phi_{,x} \qquad \text{—— (3)}$$

As we Know

$$\left[\frac{\tau}{r} = \frac{T}{I_p} = G\frac{\Delta\phi}{\Delta x}\right] \quad - \quad (4)$$

moment
of momentum balance for element as shown in fig. (1)

$$\left[\int_{A(x+\cancel{\phi}\Delta \cancel{x})} \rho r^2 \Delta x \, dA\right] \phi_{,tt} = G I_p (x+\Delta x)\, \phi_{,x}(x+\Delta x, t)$$

$$- G I_p(x)\, \phi_{,x}(x,t)$$

$$\left[\int_A r^2 dA = I_p .\right]$$ we Know that

and as $\Delta x \to 0$ [similar to previous question]

$$\rho I_p \phi_{,tt} = \frac{\partial}{\partial x}\left(G I_p \frac{\partial \phi}{\partial x}\right)$$

B.C.

at $x = 0$

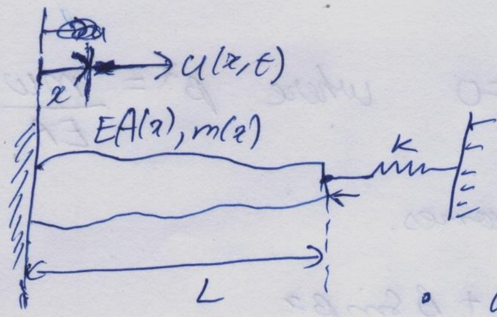
$\phi(L,t) = 0$

at $x = L$

$$\tau(L,t) = G r \frac{\partial}{\partial x}\phi = 0$$

Q 5.3

(1)



- Equation of motion of beam

$$\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u(x,t)}{\partial x}\right) = m(x)\frac{\partial^2 u(x,t)}{\partial t^2}$$

for $0 \le x \le L$

- with bcs. $u(0,t) = 0$

& $EA(x)\frac{du(x,t)}{dx}\Big|_{x=L} = +ku(x,t)\Big|_{x=L}$

- Here $u(x,t)$ is longitudinal displacement of rod at position x & time t.

- Assumptions. $EA(x) = $ constant ; $m(x) = $ constant

- $$EA\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2}$$

- using separation of variables method [above eq^n is homogenous with homogenous bcs]

So let $u(x,t) = g(x).h(t)$

$$\Rightarrow EA\frac{d^2u}{dx^2} = m\frac{d^2u}{dt^2}$$

$$\Rightarrow EA.h(t)\frac{d^2g(x)}{dx^2} = mg(x)\frac{d^2h(t)}{dt^2}$$

$$\Rightarrow \frac{EA}{m}\frac{d^2g(x)}{g(x)dx^2} = \frac{d^2h(t)}{h(t)dt^2}$$

- LHS is f^n of x & RHS is f^n of t

So these both should be equal to constant.

$$\frac{EA}{m}\frac{d^2g}{g\,dx^2} = \frac{d^2h}{h\,dt^2} = -\omega^2$$

$$\Rightarrow \frac{d^2g}{g\,dx^2} + \frac{m\omega^2}{EA} = 0$$

$$\& \quad \frac{d^2h}{h\,dt^2} + \omega^2 = 0$$

So, $\frac{d^2 g(x)}{dx^2} + \beta^2 g(x) = 0$ where $\beta^2 = \frac{m\omega^2}{EA}$

- Sol^n of above eq^n becomes.

$$g(x) = A\cos\beta x + B\sin\beta x$$

- B.Cs, $u(0,t) = 0 \Rightarrow g(0) = 0$

$$EA\frac{du(L,t)}{dx}\Big|_{x=L} = +ku(L,t) \Rightarrow EA\frac{dg(x)}{dx}\Big|_{x=L} = +kg(x)\Big|_{x=L}$$

- $g(0) = 0 \Rightarrow A = 0$
- $EA\frac{dg(x)}{dx}\Big|_{x=L} = +k\,g(x)\Big|_{x=L} \Rightarrow B\beta\cos\beta L = +kB\sin\beta L$

$$\Rightarrow \left(\frac{+\beta}{k} = \tan\beta L\right) \rightarrow (1)$$

$\Rightarrow$ Characteristic Equation

eq^n (1) gives eigen values (eigen frequencies)

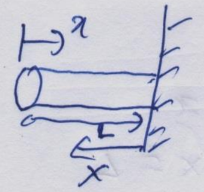
$$\left(g_r(x) = B_r\sin\beta_r x\right) \Rightarrow \text{modes shapes}$$

with $B_r = 1$ for $r = 1, 2, 3, \ldots$

normalised $\underset{\sim}{g_r}(x) = \sin\beta_r x \rightarrow$ Normalised mode shapes.

Q5) 3

2)



$X = L - x$

at $x=0$, $X = L$

$x = L$, $X = 0$

$$\frac{\partial}{\partial X}\left(GJ(X)\frac{\partial\theta(X,t)}{\partial X}\right) = I(X)\frac{\partial^2\theta(X,t)}{\partial t^2} \quad ; 0 \le X \le L$$

with bcs at $X=0$, $\theta(X,t)\big|_{X=0} = 0$

at $X=L$, $GJ(X)\frac{\partial\theta(X,t)}{\partial X}\Big|_{X=L} = 0$

$\theta(X,t)$ is twist at position X & time t

Assumptions $GJ(X) = \text{constant}$, $I(X) = \text{constant}$

$$GJ \frac{\partial^2\theta(X,t)}{\partial X^2} = I\frac{\partial^2\theta(X,t)}{\partial t^2}$$

Using separation of variables.

$\theta(X,t) = \phi(X)h(t)$

$$\Rightarrow GJ\frac{d^2(\phi(X)h(t))}{dX^2} = m\frac{d^2(\phi(X)h(t))}{dt^2}$$

$$\Rightarrow \frac{GJ}{I}\frac{d^2\phi(X)}{\phi(X)\,dX^2} = \frac{d^2h(t)}{h(t)\,dt}$$

LHS is f^n of X & RHS is f^n of t

$$\Rightarrow \frac{GJ}{I}\frac{d^2\phi(X)}{\phi(X)\,dX^2} = \frac{d^2h}{h\,dt} = -\omega^2$$

$$\Rightarrow \frac{d^2\phi}{\phi\, dx^2} + \frac{I\omega^2}{GJ} = 0$$

$$\& \quad \frac{d^2h}{dt^2} + \omega^2 = 0$$

$$\frac{d^2\phi}{\phi\, dx^2} + \beta^2 \phi(x) = 0 \quad , \beta^2 = \frac{I\omega^2}{GJ}$$

Sol^n of above eq^n $\phi(x) = A\cos\beta x + B\sin\beta x$

BCs $\phi'(x)\big|_{x=0} = 0 \Rightarrow A = 0$

$\phi'(x)\big|_{x=L} \Rightarrow B\beta\cos\beta x\big|_{x=L} = 0$

$\Rightarrow (\cos\beta L = 0) \Rightarrow$ [characteris eq^n]

$\Rightarrow \beta L = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \beta_n = (2n+1)\dfrac{\pi}{2L}$

$\omega_n = \sqrt{\dfrac{GJ}{I}}\, \beta_n$ (eigen frequencies)

$\phi_n(x) = B_n \sin\left(\beta_n \sqrt{\dfrac{GJ}{I}} x\right)$

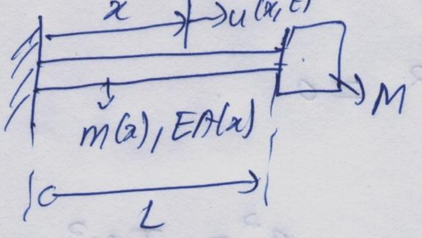
$\Rightarrow \phi_n(L-x) = B_n \sin\left(\beta_n \sqrt{\dfrac{GJ}{I}}(L-x)\right)$

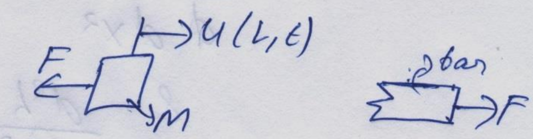
$\Rightarrow \alpha_n(x) = B_n \sin\left(\beta_n \sqrt{\dfrac{GJ}{I}}(L-x)\right) \Rightarrow$ modes shapes or eigenfn.

with $B_n = 1$ for $n = 1, 2, 3 \ldots$

normalised $\underset{\sim}{\alpha}_n(x) = \sin\left(\beta_n \sqrt{\dfrac{GJ}{I}}(L-x)\right) \Rightarrow$ Normalised eigen fn.

Q 5.4)





FBD of mass M at $x=L$

EOM. $\dfrac{\partial}{\partial x}\left[EA(x)\dfrac{\partial u(x,t)}{\partial x}\right] = m(x)\dfrac{\partial^2 u(x,t)}{\partial t^2}$, $0 \le x \le L$ $\rightarrow$ (1)

Assumption $EA(x) = $ Const., $m(x) = $ const.

BCs

- $u(0,t) = 0$ (fixed end)
- $F(L,t) = EA\dfrac{\partial u(x,t)}{\partial x}\Big|_{x=L}$
- $F(L,t) = -\dfrac{M\partial^2 u(x,t)}{\partial t^2}\Big|_{x=L}$

$\Rightarrow -EA\dfrac{\partial u(x,t)}{\partial x}\Big|_{x=L} = \dfrac{M\partial^2 u(x,t)}{\partial t^2}\Big|_{x=L}$, at $(x=L)$

eq^n (1) & BCs forms the Boundary value problem.

eq^n(1) & BCs are homogenous

So eq^n(1) can be solved by using separation of variable methods.

$u(x,t) = g(x)h(t)$

$\Rightarrow \dfrac{EA}{m}\dfrac{d^2 g(x)}{g(x)dx^2} = \dfrac{d^2 h(t)}{h(t)dt^2} = -\omega^2$

$\Rightarrow \dfrac{EA}{m}\dfrac{d^2 g(x)}{g(x)\,dx^2} + \omega^2 = 0$; Let $\beta^2 = \dfrac{\omega^2 m}{EA}$.

$u(x,t) = C\,U(x)\cos(\omega t - \phi)$, where $U(x) = A\cos\beta x + B\sin\beta x$

- $\because U(0) = 0 \Rightarrow A = 0$

- $EA\dfrac{dU(x)}{dx} = \omega^2 M U(x)$, $x = L$

$\Rightarrow EA\,\beta\cos\beta x = \omega^2 M\sin\beta x$

$\Rightarrow \beta\cos\beta x = \beta^2\sin\beta x$

$\Rightarrow \beta(\beta\sin\beta x - \cos\beta x) = 0$

$\Rightarrow \beta = 0$ or $\boxed{\tan\beta x = \dfrac{1}{\beta}}$ $\Rightarrow$ Characteristic eq^n

$U_n(x) = B_n\sin\beta_n x$, where $n = 1, 2, 3, \ldots$

Characteristic eq^n given eigen ~~function~~ values (eigen values)

$U_n(x) \Leftrightarrow$ eigen functions

for $B_n = 1$ for $n = 1, 2, 3, \ldots$

normalised $U_n(x) = \underset{\sim}{U_n}(x) = \sin\beta_n x$

Q 5.8

Let us consider a hanging cable or rope of linear mass density μ.

In limit of small displacements from equilibrium, the tension in the cable acts mainly in the vertical direction. $T(y) = \mu g y$.

net transversal force on a infinitesimal mass element of dm is due to dT_x.

$$dm\,\ddot{x} = dT_x = d(\mu g y \sin\theta) \rightarrow dx/dy = \tan\theta$$

$$\approx d(\mu g y \cancel{\tan\theta})$$

$$\frac{dm}{dy}\ddot{x} = \mu g \frac{d}{dy}\left(y\left(\frac{dx}{dy}\right)\right)$$

$$\frac{dm}{dy} = \mu \qquad \Rightarrow \quad \ddot{x} = g\frac{d}{dy}\left(y\frac{dx}{dy}\right)$$

$$x(y,t) = f(y)g(t)$$

$$\ddot{g}(t) = -\omega^2 g(t)$$

$$\frac{d}{dy}\left(y\left(\frac{d}{dy}f(y)\right)\right) = -\frac{\omega^2}{g}f(y) \qquad -(1)$$

$$u = \sqrt{\alpha y} \quad (\text{substitute})$$

with. $\alpha = 4\omega^2/g$.

eq 1 becomes.

$$\frac{d^2 f(u)}{du^2} + \frac{1}{u}\frac{d}{du}f(u) + f(u) = 0.$$

$\rightarrow$ Bessel differential equation.

The above eqn has 2 independent solutions

$J_0(u)$, the zeroth order Bessel fn. and $N_0(u)$, the zeroth order Neumann function.

- $N_0(u)$ goes to $-\infty$ to $u=0$, so it doesnot contribute

we apply boundary condition at $x(L,t)=0$

$$\Rightarrow \alpha_M = \tau_M^2/L \quad \text{where } \tau_M \text{ is the } M^{th} \text{ root of } J_0.$$

$$\omega_M = \frac{\tau_M}{2}\sqrt{g/L}.$$

M^{th} natural frequency.

general soln, combining the time dependent and position dependent parts is.

$$x(y,t) = Re\left\{ C J_0\left(\sqrt{\frac{y}{L}}\,\tau_M\right) e^{-i(\omega_M t-\phi)}\right\}.$$

Refer to webpages.ursinus.edu/lriley/courses/p212/lectures/no...

for mode shapes of hanging cable.

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Q 5.15

From section 5.9 of the book.

$$y(x,0) = y_0(x) = Ax(1 - x/L)$$

GDE:- $-EI\dfrac{\partial^4 y(x,t)}{\partial x^4} + f(x,t) = m\dfrac{\partial^2 y(x,t)}{\partial x^2}$ $\quad 0<x<L$

$f(x,t) = 0.$

$y(0,t) = 0 \quad EI\dfrac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=0} = 0.$

$y(L,t) = 0 \quad EI\dfrac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=L} = 0.$

$\omega_r = (r\pi)^2\sqrt{\dfrac{EI}{mL^4}}$ = natural frequencies.

$Y_r(x) = \sqrt{\dfrac{2}{mL}}\sin\left(\dfrac{r\pi x}{L}\right)$

By modal analysis (Refer to section 5.9 of the book)

The general response is

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\left[\frac{1}{\omega_r}\int_0^t Q_r(\tau)\sin\omega_r(t-\tau)\,d\tau + q_{r0}\cos\omega_r t + \frac{\dot{q}_{r0}}{\omega_r}\sin\omega_r t\right]$$

$\Rightarrow Q_r(t) = 0 \quad q_{r0} = \displaystyle\int_0^L m\,y_0(x)\,Y_r(x)\,dx.$

since $f(x) = 0$.

$\dot{q}_{r0} = 0$ since $v_0(x) = 0$.

$$q_{r0} = \int_0^L m\,Ax(1 - x/L)\sqrt{\frac{2}{mL}}\sin\left(\frac{r\pi x}{L}\right)dx.$$

$$q_{r0} = -\left[\frac{L^2 m A\sqrt{2}\sqrt{\frac{1}{mL}}\left(r\pi\sin(r\pi)+2\cos(r\pi)-2\right)}{r^3\pi^3}\right]$$

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\, q_{r0}\cos\omega_r t \,.$$

where $Y_r(x)$, q_{r0} and ω_r are described above.