Function Approximation [Curve Fitting]

Data have Scatter [Regression]

Given (x_i, y_i) $\quad \hat{y}_i = \sum_{j=0}^{m} a_0 \phi_j(x_i)$

ϕ_j — basis functions

Principle of least squares $\quad \Phi^T \Phi a = \Phi^T Y$ — normal equation

- $\Phi^T \Phi$ has large condition number hence need for orthogonal basis functions

- Gram-Schmidt process
  - Legendre polynomials
- Three term relationship for any polynomial

Function $\quad f(x) \quad g(x) = \sum_{j=0}^{m} a_0 \phi_j(x)$

$x \in (a, b)$

$$\Phi^T \Phi a = \Phi^T f$$

Data are precise [Interpolation]

⇒ Polynomials

Given $(n+1)$ data points

$(x_i, y_i) \quad i = 0, 1, 2, \ldots n$

Objective – Fit an n^{th} order polynomial

Though the polynomial is unique, there are variety of formats in which the polynomial can be expressed

- Direct fit (Standard format)
- Lagrange polynomials
- Newtons Divided Difference polynomials

1. DIRECT FIT

Given $(x_i, y_i) \quad i = 0, 1, 2, \ldots n$

To find nth order polynomial

$$P_n(x_i) = a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_n x_i^n$$

Substitute the values

$$P_n(x_0) = y_0 = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n$$

$$P_n(x_1) = y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n$$

$$\vdots$$

$$P_n(x_n) = y_n = a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Vandermonde matrix

System of $(n+1)$ equations

For Vandermonde matrix determinant will be non-zero

Unique solution.

Example

$x^* = 0.22$

$y(x^*) = 0.5234$

x	0	0.1	0.2	0.3
y	0.5	0.501	0.516	0.581

$\hookrightarrow f(x) = 10x^4 + 0.5$

Linear polynomial

x	0.2	0.3
y	0.516	0.581

$$\begin{bmatrix} 1 & 0.2 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0.516 \\ 0.581 \end{bmatrix}$$

$a_0 = 0.386$

$a_1 = 0.65$

$P_1(x) = 0.386 + 0.65x$

$P_1(x^*) = P_1(0.22) = 0.529$ (1.07%)

Quadratic polynomial

$$\begin{bmatrix} 1 & 0.2 & 0.2^2 \\ 1 & 0.3 & 0.3^2 \\ 1 & 0.1 & 0.1^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$a_0 = 0.536$

$a_1 = -0.6$

$a_2 = 2.5$

$y^* = 0.525$

$= 0.306\%$

Cubic

$y_* = 0.5238$

$= 0.086\%$

Vandermonde matrix has a large condition number

(b) Lagrange Polynomials

Consider 2 points

$(x_0, y_0) \quad (x_1, y_1)$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

Lagrange polynomials

$P_1(x_0) = y_0$

$P_1(x_1) = y_1$

$P_1(x) = L_0(x)\, y_0 + L_1(x)\, y_1$

$L_0, L_1 \rightarrow$ Lagrange poly.

Quadratic

$(x_0, y_0) \quad (x_1, y_1) \quad (x_2, y_2)$

$$P_2(x) = \underbrace{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}_{L_0(x)} y_0 + \underbrace{\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}}_{L_1(x)} y_1 + \underbrace{\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}_{L_2(x)} y_2$$

$$P_2(x) = L_0(x)\, y_0 + L_1(x)\, y_1 + L_2(x)\, y_2$$

nth order

$$P_n(x) = \sum_{j=0}^{n} L_j(x)\, y_j$$

$$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

$\prod \rightarrow$ product

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Exampl $x^* = 0.22$

X	0	0.1	0.2	0.3
Y	0.5	0.501	0.516	0.581

Linear $x^* = 0.22$

$$\left[\begin{array}{l} x_0 = 0.2 \\ x_1 = 0.3 \end{array}\right.$$

$$P_1(x) = \frac{(x^* - x_1)}{(x_0 - x_1)} y_0 + \frac{(x^* - x_0)}{(x_1 - x_0)} y_1$$

$$= \left(\frac{0.22 - 0.3}{(0.2 - 0.3)}\right) \cdot y_0 + \frac{(0.22 - 0.2)}{(0.3 - 0.2)} y_1$$

$$= 0.8\, y_0 + 0.2\, y_1$$

$$= \underline{\underline{0.529}}$$

Quadratic

x_0	x_1	x_2
0.2	0.3	0.1

$$P_2(x) = L_0(x) y_0 + L_1(x) y_1 + L_2(x) y_2$$

$$= 0.96\, y_0 + 0.12\, y_1 - 0.08\, y_2$$

$$= 0.526$$

3. Newton Divided Difference Polynomials

The divided difference is defined as the ratio of difference between function values at two points by the difference between the two points.

Example

$$f(x_i)$$

$$(x_0, y_0) \quad (x_1, y_1) \quad (x_2, y_2)$$

First divided difference

$$f(x_1, x_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = f(x_0, x_1)$$

$$f(x_2, x_1) = \frac{y_2 - y_1}{x_2 - x_1}$$

Second divided difference

$$f(x_2, x_1, x_0) = \frac{f(x_2, x_1) - f(x_1, x_0)}{x_2 - x_0}$$

$$\vdots$$

nth divided difference

$$f(x_n, x_{n-1}, \ldots x_0) = \frac{f(x_n, x_{n-1}, \ldots x_1) - f(x_{n-1}, x_{n-2}, \ldots x_0)}{x_n - x_0}$$

$$P_n(x) = \overset{y_0}{f(x_0)} + f(x_1, x_0)(x - x_0)$$

$$+ f(x_2, x_1, x_0)(x - x_0)(x - x_1)$$

$$+ f(x_3, x_2, x_1, x_0)(x - x_0)(x - x_1)(x - x_2)$$

$$\vdots$$

$$+ f(x_n, x_{n-1}, \ldots x_0)(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})$$

$$P_n(x_0) = f(x_0) = y_0$$

$$P_n(x_1) = \overset{= y_0}{f(x_0)} + \frac{y_1 - y_0}{\cancel{x_1 - x_0}} \cdot (\cancel{x_1 - x_0})$$

$$= y_1$$

$$P_n(x_2) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)$$

$$+ \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{\cancel{x_2 - x_0}} \cdot (\cancel{x_2 - x_0})(x_2 - x_1)$$

$$= y_2$$

Example

x	0	0.1	0.2	0.3
y	0.5	0.501	0.516	0.581

$x^* = 0.22$

$y^* = 0.5234$

Divided difference

i	x_i	y_i	First	Second	Third
0	0.2	0.516	$\frac{0.581-0.516}{0.3-0.2} = 0.65$	$\frac{0.4-0.65}{0.1-0.2} = 2.5$	$\frac{1.3-2.5}{0-0.2} = 6.$
1	0.3	0.581	0.4	$\frac{0.1-0.4}{0-0.3}$ 1.3	
2	0.1	0.501	0.1		
3	0.0	0.5			

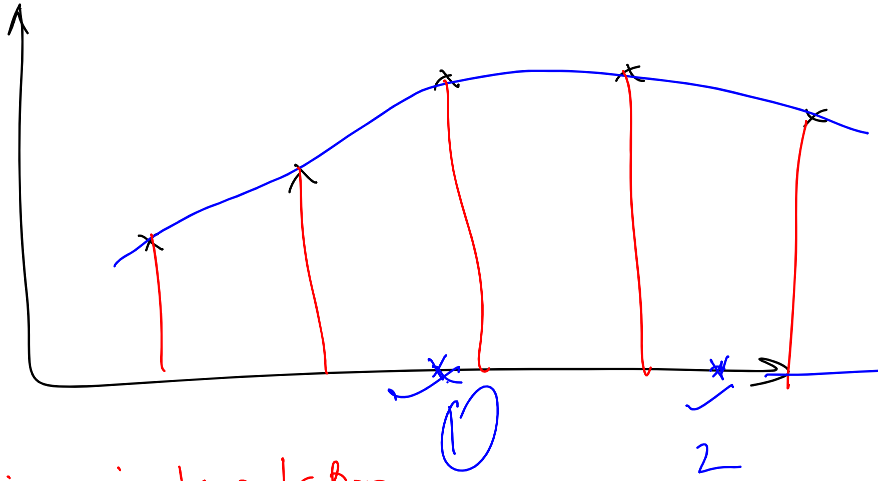
$P_1(x) = y_0 + 0.65(x-0.2)$

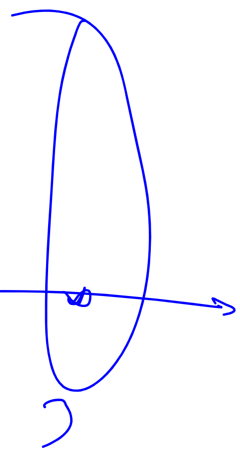
$P_1(0.22) = 0.529$

$P_2(x) = y_0 + 0.65(x-0.2)$

$+ 2.5(x-0.2)(x-0.3)$

$+ 6(\quad)(\quad)(\quad)$





Error in interpolation

3 > 2 > 1

Taylor series

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