Pr 1(a).

$$\underset{\sim}{A} = A_{ij}\, \underset{\sim}{e_i}\, \underset{\sim}{e_j}$$

$$\underset{\sim}{A}^{-1} = A^{-1}_{ij}\, \underset{\sim}{e_i}\, \underset{\sim}{e_j}$$

$$\underset{\sim}{A}^{-T} = A^{-1}_{ji}\, \underset{\sim}{e_i}\, \underset{\sim}{e_j}$$

(b)

$$\underset{\sim}{A}^{-T} : \underset{\sim}{A} = A^{-1}_{ji}\, \underset{\sim}{e_i}\, \underset{\sim}{e_j} : A_{mn}\, \underset{\sim}{e_m}\, \underset{\sim}{e_n}$$

$$= A^{-1}_{ji}\, \delta_{im}\, \delta_{jn}\, A_{mn}$$

$$= A^{-1}_{ji}\, A_{ij} = \delta_{nn} = 3.$$

Pr. 2 ~~(c)~~

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})\, \underset{\sim}{A} = (u_i v_j\, \underset{\sim}{e_i} \underset{\sim}{e_j}) \cdot A_{mn}\, \underset{\sim}{e_m}\, \underset{\sim}{e_n}$$

$$= u_i v_j\, \underset{\sim}{e_i}\, \delta_{jm}\, A_{mn}\, \underset{\sim}{e_n}$$

$$= u_i v_m A_{mn}\, \underset{\sim}{e_i}\, \underset{\sim}{e_n}$$

$$= u_i v_k A_{kj}\, \underset{\sim}{e_i}\, \underset{\sim}{e_j}$$

(i,j)th component of

$$\underset{\sim}{u} \otimes (\underset{\sim}{A}^T \underset{\sim}{v}) = \left\{ \underset{\sim}{u} \otimes (\underset{\sim}{A}^T \underset{\sim}{v}) \right\}_{ij}$$

$$= u_i (\underset{\sim}{A}^T \underset{\sim}{v})_j = u_i A_{mj} v_m$$

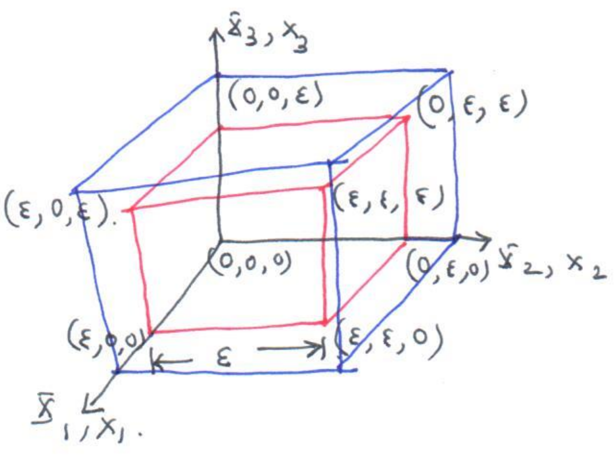
$$= u_i v_k A_{kj}$$

Pr. 3.

$$x_1 = \lambda_1 X_1$$

$$x_2 = \lambda_2 X_2$$

$$x_3 = \lambda_3 X_3.$$



Sphere centered at origin:

$$X_1^2 + X_2^2 + X_3^3 = \varepsilon^2.$$

After deformation,

$$X_1 \to x_1, \quad X_2 \to x_2, \quad X_3 \to x_3$$

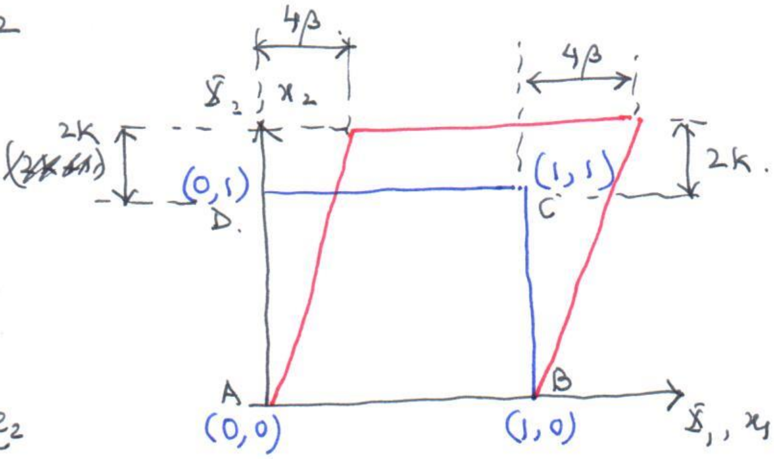
$$\frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} + \frac{x_3^2}{\lambda_3^2} = \varepsilon^2$$

$$\Rightarrow \frac{x_1^2}{\varepsilon^2\lambda_1^2} + \frac{x_2^2}{\varepsilon^2\lambda_2^2} + \frac{x_3^2}{\varepsilon^2\lambda_3^2} = 1. \Rightarrow \left(\frac{x_1}{\lambda_1}\right)^2 + \left(\frac{x_2}{\lambda_2}\right)^2 + \left(\frac{x_3}{\lambda_3}\right)^2 = \varepsilon^2$$

which is an ellipsoid with semi-axes $\varepsilon\lambda_1, \varepsilon\lambda_2, \varepsilon\lambda_3$.

3(b)

$$x_1 = \beta X_2^2 t^2 + X_1.$$
$$x_2 = k X_2 t + X_2$$
$$x_3 = X_3.$$



$$\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X}$$
$$= (\beta X_2^2 t^2 + X_1 - X_1)\,\underset{\sim}{e_1} + (k X_2 t + X_2 - X_2)\,\underset{\sim}{e_2}$$

$$= \beta X_2^2 t^2\,\underset{\sim}{e_1} + k X_2 t\,\underset{\sim}{e_2}$$

$$\nabla_0 \underset{\sim}{u} = \left(\frac{\partial}{\partial X_k}\underset{\sim}{e_k}\right)\left(\beta X_2^2 t^2\,\underset{\sim}{e_1} + k X_2 t\,\underset{\sim}{e_2}\right)$$

$$= \frac{\partial}{\partial X_1}(\beta X_2^2 t^2)\,\underset{\sim}{e_1}\underset{\sim}{e_1} + \frac{\partial}{\partial X_2}(\beta X_2^2 t^2)\,\underset{\sim}{e_2}\underset{\sim}{e_1}$$
$$+ \frac{\partial}{\partial X_1}(k X_2 t)\,\underset{\sim}{e_1}\underset{\sim}{e_2} + \frac{\partial}{\partial X_2}(k X_2 t)\,\underset{\sim}{e_2}\underset{\sim}{e_2}$$

$$= 2\beta X_2 t^2\,\underset{\sim}{e_2}\underset{\sim}{e_1} + kt\,\underset{\sim}{e_2}\underset{\sim}{e_2} \Rightarrow \text{at } t = 2,$$

$$\nabla_0 \underset{\sim}{u} = 8\beta X_2\,\underset{\sim}{e_2}\underset{\sim}{e_1} + 2k\,\underset{\sim}{e_2}\underset{\sim}{e_2}$$

Pr4 (a)

$$\varepsilon_{11} = 2Cx_1x_2$$

$$\varepsilon_{22} = -2\nu Cx_1x_2$$

$$\varepsilon_{12} = (1+\nu)C(a^2 - x_2^2).$$

$$\frac{\partial u_1}{\partial x_1} = 2Cx_1x_2 \Rightarrow u_1 = Cx_1^2x_2 + f(x_2)$$

$$\frac{\partial u_2}{\partial x_2} = -2\nu Cx_1x_2 \Rightarrow u_2 = -\nu Cx_1x_2^2 + g(x_1)$$

$$\therefore \left.\begin{aligned} \frac{\partial u_1}{\partial x_2} &= Cx_1^2 + \frac{df}{dx_2} \\ \frac{\partial u_2}{\partial x_1} &= -\nu Cx_2^2 + \frac{dg}{dx_1} \end{aligned}\right\} \Rightarrow \varepsilon_{12} = \frac{1}{2}\left\{Cx_1^2 - \nu Cx_2^2 + \frac{df}{dx_2} + \frac{dg}{dx_1}\right\}.$$

$$\therefore \frac{1}{2}\left\{Cx_1^2 - \nu Cx_2^2 + \frac{df}{dx_2} + \frac{dg}{dx_1}\right\} = (1+\nu)C(a^2 - x_2^2).$$

$$\Rightarrow Cx_1^2 - \nu Cx_2^2 + \frac{df}{dx_2} + \frac{dg}{dx_1} - 2(1+\nu)C(a^2 - x_2^2) = 0.$$

$$\Rightarrow \underbrace{\left(Cx_1^2 + \frac{dg}{dx_1}\right)}_{A} + \underbrace{\left(\frac{df}{dx_2} - \nu Cx_2^2 - 2(1+\nu)C(a^2 - x_2^2)\right)}_{-A.} = 0.$$

$$\therefore \frac{dg}{dx_1} + Cx_1^2 = 0$$

$$g = Ax_1 - \frac{Cx_1^3}{3} + P.$$

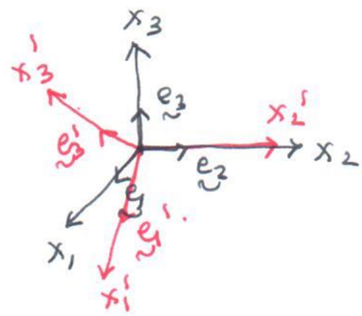
$$\frac{df}{dx_2} = \nu Cx_2^2 + 2(1+\nu)C(a^2 - x_2^2). - A$$

$$\therefore f = \nu C\frac{x_2^3}{3} + 2(1+\nu)C\left(a^2x_2 - \frac{x_2^3}{3}\right) - Ax_2 + Q$$

$$= \{2(1+\nu)Ca^2 - A\}x_2 - \frac{C}{3}x_2^3(2+\nu) + Q.$$

Pr 5:

$$\underset{\sim}{\sigma} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{pmatrix} \text{MPa}$$



$$\underset{\sim}{Q} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

as $Q_{ij} = \cos(\underset{\sim}{e}_i', \underset{\sim}{e}_j)$.

Here

$$\underset{\sim}{e}_1' = \frac{1}{\sqrt{2}}\underset{\sim}{e}_1 - \frac{1}{\sqrt{2}}\underset{\sim}{e}_3$$

$$\underset{\sim}{e}_2' = \underset{\sim}{e}_2$$

$$\underset{\sim}{e}_3' = \frac{1}{\sqrt{2}}\underset{\sim}{e}_1 + \frac{1}{\sqrt{2}}\underset{\sim}{e}_3$$

$Q_{11} = \frac{1}{\sqrt{2}}$ $\quad Q_{12} = 0$ $\quad Q_{13} = -1/\sqrt{2}.$

$Q_{21} = 0$ $\quad Q_{22} = 1$ $\quad Q_{23} = 0.$

$Q_{31} = \frac{1}{\sqrt{2}}$ $\quad Q_{32} = 0$ $\quad Q_{33} = 1/\sqrt{2}.$

(b)
$$\underset{\sim}{t} = \underset{\sim}{\sigma}\,\underset{\sim}{n} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{pmatrix}\begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix}.$$

$$= \sqrt{2}\,\underset{\sim}{e}_1 + \left(\frac{1}{\sqrt{2}} - \sqrt{2}\right)\underset{\sim}{e}_2 + \frac{1}{\sqrt{2}}\underset{\sim}{e}_3$$

$$t_n = \underset{\sim}{t}\cdot\underset{\sim}{n} = 1 + \frac{1}{2} = \frac{3}{2} \text{ MPa}.$$

$$|\underset{\sim}{t}|^2 = 2 + \frac{1}{2} + \frac{1}{2} = 3.$$

$$\sigma_s^2 = |\underset{\sim}{t}|^2 - t_n^2 = \frac{3}{4} \Rightarrow \sigma_s = \frac{\sqrt{3}}{2} \text{ MPa}.$$

Also, $\underset{\sim}{\sigma}' = \underset{\sim}{Q}\,\underset{\sim}{\sigma}\,\underset{\sim}{Q}^T = \begin{pmatrix} 1.5 & 2.12 & 0.5 \\ 2.12 & 3.0 & -0.707 \\ 0.5 & -0.707 & 1.5 \end{pmatrix}$ MPa.

5(c). $I_1 = \mathrm{tr}\,\underset{\sim}{\sigma} = 6$

$$I_2 = \frac{1}{2}\left[\left(\mathrm{tr}\,\underset{\sim}{\sigma}\right)^2 - \mathrm{tr}\left(\underset{\sim}{\sigma}^2\right)\right] = 6.$$

$$I_3 = \det \underset{\sim}{\sigma} = -3.$$

Note $\underset{\sim}{\sigma}^2 = \underset{\sim}{\sigma}\,\underset{\sim}{\sigma}$

$$= \begin{pmatrix} 5 & 5 & -2 \\ 5 & 14 & -8 \\ -2 & -8 & 5 \end{pmatrix}.$$

5(d). $\sigma_1, \sigma_2, \sigma_3 = 4.53, 1.83, -0.36$ MPa.

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = 2.4452 \text{ MPa}.$$

$$\underset{\sim}{P} = \begin{pmatrix} -0.2318 & 0.9168 & -0.3253 \\ 0.5474. & -0.1535 & -0.8227 \\ 0.8041 & 0.3688 & 0.4662 \end{pmatrix}.$$

Pr 6.

$$\int_V \sigma_{ij}\, dV \cancel{= \int_V \sigma_{ij}\,\partial} = \int_V \sigma_{ik}\,\delta_{kj}\, dV.$$

$$= \int_V \sigma_{ik}\frac{\partial x_j}{\partial x_k}\, dV = \int_V \left\{\frac{\partial}{\partial x_k}\left(\sigma_{ik} x_j\right) - \frac{\partial \sigma_{ik}}{\partial x_k} x_j\right\} dV$$

in absence of body forces

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{is the eqbm eqn}$$

$$\Rightarrow \int_V \sigma_{ij}\, dV = \int_V \frac{\partial}{\partial x_k}\left(\sigma_{ik} x_j\right) dV = \int_{\partial V} \sigma_{ik} n_k x_j\, dS$$

(by application of divergence theorem)

$$= \int_{\partial V} t_i x_j\, dS$$ (by Cauchy relation $t_i = \sigma_{ij} n_j$)

$$\Rightarrow \int_V \underset{\sim}{\sigma}\, dV = \int_{\partial V} \underset{\sim}{t} \otimes \underset{\sim}{x}\, dS$$