Introduction to Complex Analysis
MSO 202 A

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Course Structure

- This course will be conducted in Flipped Classroom Mode.
- Every Friday evening, 3 to 7 videos of total duration 60 minutes will be released.
- The venue and timings of Flipped classrooms: W/Th 09:00-9:50 L7
- The timing of tutorial is M 09:00-9:50.
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  - R. Churchill and J. Brown, Complex variables and applications. Fourth edition. McGraw-Hill Book Co., New York, 1984. - an elementary text suitable for a one semester; emphasis on applications.
  - E. Stein and R. Shakarchi, Complex Analysis, Princeton University Press, 2006.- Modern treatment of the subject, but recommended for second reading.
  - Lecture notes and assignments by P. Shunmugaraj, (strongly recommended for students), http://home.iitk.ac.in/ psraj/
- Please feel free to contact me through chavan@iitk.ac.in

Syllabus

- Complex Numbers, Complex Differentiation and C-R Equations,
- Analytic Functions, Power Series and Derivative of Power Series,
- Complex Exponential, Complex Logarithm and Trigonometric Functions,
- Complex Integration, Cauchy's Theorem and Cauchy's Integral Formulas,
- Taylor series, Laurent series and Cauchy residue theorem,
- Mobius Transformation.

Complex Numbers

- real line: $\mathbb{R}$, real plane: $\mathbb{R}^2$
- A complex number : $z = x + iy$, where $x, y \in \mathbb{R}$ and i is an imaginary number that satisfies $i^2 + 1 = 0$.
- complex plane: $\mathbb{C}$
- $\text{Re}: \mathbb{C} \to \mathbb{R}$ by $\text{Re}\,(z) =$ real part of $z = x$
- $\text{Re}\,(z + w) = \text{Re}\,z + \text{Re}\,w$, $\text{Re}\,(a\ w) = a\,\text{Re}\,w$ if $a \in \mathbb{R}$

Remark Same observation holds for $\text{Im}: \mathbb{C} \to \mathbb{R}$ defined by $\text{Im}\,(z) =$ imaginary part of $z = y$.

Definition

A map f from $\mathbb{C}$ is $\underline{\mathbb{R}\text{-linear}}$ if $f(z+w) = f(z) + f(w)$ and $f(a\, z) = a\, f(z)$ for all $z, w \in \mathbb{C}$ and $a \in \mathbb{R}$.

Example

- Re and Im are $\mathbb{R}$-linear maps.
- $id(z) = z$ and $c(z) = \mathrm{Re}\,(z) - i\,\mathrm{Im}\,(z)$ are $\mathbb{R}$-linear maps.
- $H : \mathbb{C} \to \mathbb{R}^2$ defined by $H(z) = (\mathrm{Re}(z), \mathrm{Im}(z))$ is $\mathbb{R}$-linear.

Remark H is an $\mathbb{R}$-linear bijection from the real vector space $\mathbb{C}$ onto $\mathbb{R}^2$.

$\mathbb{C}$ (over $\mathbb{R}$) and $\mathbb{R}^2$ are same as vector spaces. But complex multiplication makes $\mathbb{C}$ different from $\mathbb{R}^2$:

$$z\ w := (\operatorname{Re} z\ \operatorname{Re} w - \operatorname{Im} z\ \operatorname{Im} w) + i(\operatorname{Re} z\ \operatorname{Im} w + \operatorname{Im} w\ \operatorname{Re} w).$$

In particular, any non-zero complex number z has a inverse:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2},$$

where

- $\bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z)$
- $|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$

Polar Decomposition

Any non-zero complex number z can be written in the polar form:

$$z = r\, e^{i \arg z},$$

where $r > 0$ and $\arg z \in \mathbb{R}$. Note that

- r is unique. Indeed, $r = |z|$.
- $\arg z$ is any real number satisfying

$$\frac{z}{|z|} = \cos(\arg z) + i \sin(\arg z).$$

- $\arg z$ is unique up to a multiple of 2π.

For $\theta \in [0, 2\pi)$, define rotation $r_\theta : \mathbb{C} \to \mathbb{C}$ by angle θ as

$$r_\theta(z) = e^{i\theta} z.$$

For $t \in (0, \infty)$, define dilation $d_t : \mathbb{C} \to \mathbb{C}$ of magnitude t as

$$d_t(z) = t\, z.$$

Example

For a non-zero w, define $m_w : \mathbb{C} \to \mathbb{C}$ by $m_w(z) = w\, z$. Then

$$m_w = d_{|w|} \circ r_{\arg w}.$$

Convergence in $\mathbb{C}$

Definition

Let $\{z_n\}$ be a sequence of complex numbers. Then

- $\{z_n\}$ is a <u>Cauchy sequence</u> if $|z_m - z_n| \to 0$ as $m, n \to \infty$.
- $\{z_n\}$ is a <u>convergent sequence</u> if $|z_n - z| \to 0$ for some $z \in \mathbb{C}$.

Theorem ($\mathbb{C}$ is complete)

Every Cauchy sequence in $\mathbb{C}$ is convergent.

Proof.

- $|z_m - z_n| \to 0$ iff $|\mathrm{Re}\,(z_m - z_n)| \to 0$ and $|\mathrm{Im}\,(z_m - z_n)| \to 0$.
- But Re and Im are $\mathbb{R}$-linear. Hence $\{z_n\}$ is Cauchy iff $\{\mathrm{Re}\,(z_n)\}$ and $\{\mathrm{Im}\,(z_n)\}$ are Cauchy sequences.
- However, any Cauchy sequence in $\mathbb{R}$ is convergent.

$\square$

Continuity

Definition

A function f defined on $\mathbb{C}$ is continuous at a if

$$z_n \to a \Longrightarrow f(z_n) \to f(a).$$

f is continuous if it is continuous at every point.

Example

- $H(z) = (\mathrm{Re}\,(z), \mathrm{Im}\,(z))$.
- $m_w(z) = w\,z$.
- $p(z) = a_0 + a_1 z + \cdots + a_n z^n$.
- $f(z) = |z|$.

Complex Differentiability

For $a \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$.

Definition

A function $f : \mathbb{D}_r(a) \to \mathbb{C}$ is <u>complex differentiable at a</u> if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \text{ for some } f'(a) \in \mathbb{C}.$$

f is <u>holomorphic</u> if it is complex differentiable at every point.

Remark

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - D_a(h)}{h} = 0,$$

where $D_a : \mathbb{C} \to \mathbb{C}$ is given by $D_a(h) = f'(a)\, h$.

Theorem

Every holomorphic function is continuous.

Example

$f(z) = z^n$ is holomorphic. Indeed, $f'(a) = na^{n-1}$:

$$\frac{(a+h)^n - a^n}{h} = (a+h)^{n-1} + (a+h)^{n-2}a + \cdots + a^{n-1} \to na^{n-1}.$$

More generally, $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ is holomorphic.

Example

$f(z) = \bar{z}$ is not complex differentiable at 0. Indeed, $\frac{\bar{h}}{h} \to +1$ along real axis and $\frac{\bar{h}}{h} \to -1$ along imaginary axis.

Example

For $b, d \in \mathbb{C}$, define $f(z) = \frac{z+b}{z+d}$. Then f is complex differentiable at any $a \in \mathbb{C} \setminus \{-d\}$.

Cauchy-Riemann Equations

Write $f : \mathbb{C} \to \mathbb{C}$ as $f = u + i\,v$ for real valued functions u and v. Assume that the partial derivatives of u and v exists. Consider

$$J_{u,v}(a) = \begin{bmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a). \end{bmatrix} \text{ (\underline{Jacobian matrix}).}$$

Recall that $H(z) = (\mathrm{Re}\,(z), \mathrm{Im}\,(z))$. Treating $(\mathrm{Re}\,(z), \mathrm{Im}\,(z))$ as a column vector, define $\mathbb{R}$-linear map $F_a : \mathbb{C} \to \mathbb{C}$ by

$$F_a(z) = H^{-1} \circ J_{u,v}(a) \circ H(z)$$

$$= (u_x(a)\mathrm{Re}\,(z) + u_y(a)\mathrm{Im}\,(z)) + i(v_x(a)\mathrm{Re}\,(z) + v_y(a)\mathrm{Im}\,(z)).$$

Question When $F_a(\alpha\, z) = \alpha F_a(z)$ for every $\alpha \in \mathbb{C}$ (or, when F_a is $\mathbb{C}$-linear) ?

Suppose that $F_a(i\,z) = iF_a(z)$. Letting $z = 1$, we obtain

$$F_a(i) = u_y(a) + iv_y(a), \;\; iF_a(1) = -v_x(a) + iu_x(a).$$

Thus we obtain $u_x = v_y$ and $u_y = -v_x$ (C-R Equations).

Interpretation Let $\nabla\, u = (u_x, u_y)$ and $\nabla\, v = (v_x, v_y)$. If f satisfies C-R equations then $\nabla u \cdot \nabla v = 0$. The level curves $u = c_1$ and $v = c_2$ are orthogonal, where they intersect.

- $f(z) = z$ then $u = x = c_1$ and $v = y = c_2$ (pair of lines).
- If $f(z) = z^2$ then $u = x^2 - y^2 = c_1$ and $v = 2xy = c_2$ (pair of hyperbolas).

Suppose $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at a. Note that

$$F_a(h) = (u_x(a) + iv_x(a))h_1 + (u_y(a) + iv_y(a))h_2.$$

However,

$$\lim_{h_1 \to 0} \frac{f(a+h_1) - f(a) - (u_x(a) + iv_x(a))h_1}{h_1} = 0,$$

$$\lim_{h_2 \to 0} \frac{f(a+ih_2) - f(a) - (u_y(a) + iv_y(a))h_2}{ih_2} = 0.$$

By uniqueness of limit, $u_x(a) + iv_x(a) = f'(a) = \frac{u_y(a)+iv_y(a)}{i}$, and

$$F_a(h) = f'(a)h,$$

and hence F_a is $\mathbb{C}$-linear. We have thus proved that the C-R equations are equivalent to $\mathbb{C}$-linearity of F_a!

Theorem (Cauchy-Riemann Equations)

If $f = u + i\,v$ and u, v have continuous partial derivatives then f is complex differentiable if and only if f satisfies C-R equations.

Corollary

If $f = u + i\,v$ is complex differentiable at a, then

$$|f'(a)|^2 = \det J_{u,v}(a).$$

In particular, $f : \mathbb{C} \to \mathbb{C}$ is constant if $f' = 0$.

Proof.

We already noted that $f'(a) = u_x(a) + iv_x(a)$, and hence $|f'(a)|^2 = u_x(a)^2 + v_x(a)^2$. However, by the C-R equations,

$$J_{u,v}(a) = \begin{bmatrix} u_x(a) & -v_x(a) \\ v_x(a) & u_x(a). \end{bmatrix},$$

so that $\det J_{u,v}(a) = u_x(a)^2 + v_x(a)^2 = |f'(a)|^2$. $\square$

Range of a Holomorphic Function

- Suppose $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with range contained in the real axis. Then $f = u + i\,v$ with $v = 0$. By C-R equations,
$$u_x = 0, \; u_y = 0.$$
Hence u is constant, and hence so is f.
- Suppose $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with range contained in a line. Note that for some $\theta \in \mathbb{R}$ and $c > 0$, the range of $g(z) = e^{i\theta} f(z) + c$ is contained in the real axis. By last case, g, and hence f is constant.

We will see later that the range of any non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ intersects every disc in the complex plane!

Power Series

Definition

A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n, \text{ where } a_n \in \mathbb{C}.$$

$\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n||z|^n < \infty$.

Definition (Domain of Convergence)

$D := \{w \in \mathbb{C} : \sum_{n=0}^{\infty} |a_n||w|^n < \infty\}$.

Note that

- $w_0 \in D \implies e^{i\theta} w_0 \in D$ for any $\theta \in \mathbb{R}$.
- $w_0 \in D \implies w \in D$ for any $w \in \mathbb{C}$ with $|w| \leqslant |w_0|$.

Conclude that D is either $\mathbb{C}$, $\mathbb{D}_R(0)$ or $\overline{\mathbb{D}}_R(0)$ for some $R \geqslant 0$.

Radius of Convergence

Definition

The radius of convergence (for short, RoC) of $\sum_{n=0}^{\infty} a_n z^n$ is defined as

$$R := \sup\{|z| : \sum_{n=0}^{\infty} |a_n||z|^n < \infty\}.$$

Theorem (Hadamard's Formula)

The RoC of $\sum_{n=0}^{\infty} a_n z^n$ is given by

$$R = \frac{1}{\limsup |a_n|^{1/n}},$$

where we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

Examples

- $\sum_{n=0}^{k} a_n z^n$, $a_n = 0$ for $n > k$, $R = \infty$.
- $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $a_n = \frac{1}{n!}$, $R = \infty$.
- $\sum_{n=0}^{\infty} z^n$, $a_n = 1$, $R = 1$.
- $\sum_{n=0}^{\infty} n! z^n$, $a_n = n!$, $R = 0$.

The coefficients of a power series may not be given by a single formula.

Example

Consider the power series $\sum_{n=0}^{\infty} z^{n^2}$. Then

$$a_k = 1 \text{ if } k = n^2, \text{ and } 0 \text{ otherwise.}$$

Clearly, $\limsup |a_n|^{1/n} = 1$, and hence $R = 1$.

Theorem

If the RoC of $\sum_{n=0}^{\infty} a_n z^n$ is R then the RoC of the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also R.

Proof.

Since $\lim_{n\to\infty} n^{1/n} = 1$, $R = \frac{1}{\limsup |n a_n|^{1/n}} = \frac{1}{\limsup |a_n|^{1/n}}$. $\square$

Example

Consider the power series $\sum_{n=0}^{\infty} a_n z^n$, where a_n is number of divisors of n^{1111}. Note that

$$1 \leq a_n \leq n^{1111}.$$

Note that $1 \leq \limsup |a_n|^{1/n} \leq \limsup (n^{1111})^{1/n} = 1$, and hence the RoC of $\sum_{n=0}^{\infty} a_n z^n$ equals 1.

Power series as Holomorphic function

Theorem

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with RoC equal to $R > 0$. Define $f : \mathbb{D}_R \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then f is holomorphic with $f'(z) = g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

- For z_0, find $h \in \mathbb{C}$, $r > 0$ with $\max\{|z_0|, |z_0 + h|\} < r < R$.
- $S_k(z) = \sum_{n=0}^{k} a_n z^n$, $E_k(z) = \sum_{n=k+1}^{\infty} a_n z^n$.
- $\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = A + (S_k'(z_0) - g(z_0)) + B$, where

$$A := \left(\frac{S_k(z_0+h) - S_k(z_0)}{h} - S_k'(z_0)\right), B := \left(\frac{E_k(z_0+h) - E_k(z_0)}{h}\right).$$

- $$|B| \leqslant \sum_{n=k+1}^{\infty} |a_n| \left| \frac{(z_0+h)^n - z_0^n}{h} \right| \leqslant \sum_{n=k+1}^{\infty} |a_n| n r^{n-1}.$$

Corollary

A power series is infinitely complex differentiable in the disc of convergence.

Let U be a subset of $\mathbb{C}$. We say that U is <u>open</u> if for every $z_0 \in U$, there exists $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq U$.

Definition

Let $U \subseteq$ be open. A function $f : U \to \mathbb{C}$ is said to be <u>analytic at z_0</u> if there exists a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ for all } z \in \mathbb{D}_r(z_0)$$

for some $r > 0$. A function f is <u>analytic</u> if it is analytic at $z_0 \in U$.

Example (Analyticity of Polynomials and Linear Equations)

Any polynomial $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ is analytic in $\mathbb{C}$. To see this, fix $z_0 \in \mathbb{C}$. We show that there exist unique scalars $a_0, \cdots, a_n$ such that

$$p(z) = a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n \text{ for every } z \in \mathbb{C}.$$

Comparing coefficients of $1, z, \cdots, z^{n-1}$ on both sides, we get

$$\begin{bmatrix} 1 & -z_0 & z_0^2 & \cdots & \\ 0 & 1 & -2z_0 & \cdots & \\ 0 & 0 & 1 & -3z_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \\ & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}.$$

Alternatively, the solution is given by $a_k = \frac{p^{(k)}(z_0)}{k!}$ $(k = 0, \cdots, n)$.

Exponential Function

Appeared $e^{i \arg z}$ in the polar decomposition of z.

Definition

The exponential function e^z is the power series given by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \ (z \in \mathbb{C}).$$

Since the radius of convergence of e^z is ∞, exponential is holomorphic everywhere in $\mathbb{C}$. Further,

$$(e^z)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z.$$

Thus e^z is a solution of the differential equation $f' = f$. Moreover, e^z is the only solution of the IVP: $f' = f,\ f(0) = 1$.

Certainly, e^z is not surjective as for no $z \in \mathbb{C}$, $e^z = 0$. If $w \neq 0$ then by polar decomposition, $w = |w|e^{i \arg w}$ $(0 \leqslant \arg w < 2\pi)$. Also, since $|w| = e^{\log |w|}$, we obtain

$$w = e^{\log |w| + i \arg w}.$$

Thus the range of e^z is the punctured complex plane $\mathbb{C} \setminus \{0\}$. Further, since $\arg z$ is unique up to a multiple of 2π, e^z is one-one in $\{z \in \mathbb{C} : 0 \leqslant \arg z < 2\pi\}$, but not in $\mathbb{C}$.

Theorem (Polynomials Vs Exponential)

If p is a polynomial then $\lim_{|z| \to \infty} |p(z)| = \infty$. *However,*

$$\lim_{|z| \to \infty} |e^z| \neq \infty.$$

Parametrized curves

- A parametrized curve is a function $z : [a, b] \to \mathbb{C}$. We also say that γ is a curve with parametrization z.
- A parametrized curve z is smooth if $z'(t)$ exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$.
- A parametrized curve z is piecewise smooth if z is continuous on $[a, b]$ and z is smooth on every $[a_k, a_{k+1}]$ for some points $a_0 = a < a_1 < \cdots < a_n = b$.
- A parametrized curve z is closed if $z(a) = z(b)$.

Example

- $z(t) = z_0 + re^{it}$ $(0 \leqslant t \leqslant 2\pi)$ (+ve orientation).
  $z(t) = z_0 + re^{-it}$ $(0 \leqslant t \leqslant 2\pi)$ ($-$ve orientation).
- Rectangle with vertices $R, R + iz_0, -R + iz_0, -R$ with +ve orientation is a parametrized curve, which is piecewise smooth but not smooth.

Integration along curves

Definition

Given a smooth curve γ parametrized by $z : [a, b] \to \mathbb{C}$, and f a continuous function on γ, define the integral of f along γ by

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Remark. If there is another parametrization $\tilde{z}(s) = z(t(s))$ for some continuously differentiable bijection $t : [a, b] \to [c, d]$ then, $\int_a^b f(z(t))z'(t)dt = \int_c^d f(\tilde{z}(s))\tilde{z}'(s)ds$.

Definition

In case γ is piecewise smooth, the integral of f along γ is given by

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt.$$

Examples

Example

Let γ be the circle $|z| = 1$, $f(z) = z^n$ for an integer n. Note that

$$\int_\gamma f(z)dz = \int_0^{2\pi} f(e^{it})(e^{it})'dt = \int_0^{2\pi} e^{int} ie^{it} dt.$$

- $n \neq -1$: $\int_\gamma f(z)dz = \int_0^{2\pi} \frac{d}{dt}\frac{e^{i(n+1)t}}{n+1} dt = \frac{e^{i(n+1)t}}{n+1}\Big|_0^{2\pi} = 0.$
- $n = -1$: $\int_\gamma f(z)dz = \int_0^{2\pi} idt = 2\pi i.$

Theorem (Cauchy's Theorem for Polynomials)

Let γ be the circle $|z - z_0| = R$ and let p be a polynomial. Then

$$\int_\gamma p(z)dz = 0.$$

Properties of Integrals over curves

Let $\gamma \subseteq U$ with parametrization z and $f : U \to \mathbb{C}$ be continuous.

- $\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz$.
- If γ^- (with parametrization $z^-(t) = z(b + a - t)$) is γ with reverse orientation, then

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz.$$

- If $\mathrm{length}(\gamma) := \int_\gamma |z'(t)|dt$ then

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma).$$

Theorem (Integral independent of curve)

Let $f : U \to \mathbb{C}$ be a continuous function such that $f = F'$ for a holomorphic function $F : U \to \mathbb{C}$. Let γ be a piecewise smooth parametrized curve in U such that $\gamma(a) = w_1$ and $\gamma(b) = w_2$. Then

$$\int_\gamma f(z)dz = F(w_2) - F(w_1).$$

In particular, if γ is closed then $\int_\gamma f(z)dz = 0$.

Proof.

We prove the result for smooth curves only. Note that

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt$$

$$= \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b)) - F(z(a)) = F(w_2) - F(w_1).$$

If γ is closed then $w_1 = w_2$, and hence $\int_\gamma f(z)dz = 0$. $\square$

Corollary

Let U be an open convex subset of $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be a holomorphic function. If $f' = 0$ then f is a constant function.

Proof.

Let $w_0 \in U$. We must check that $f(w) = f(w_0)$ for any $w \in U$. Let γ be a straight line connecting w_0 and w. By the last theorem,

$$0 = \int_\gamma f'(z)dz = f(w) - f(w_0),$$

and hence f is a constant function. $\square$

Example

There is no holomorphic function $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that

$$F'(z) = \frac{1}{z} \text{ for every } z \in \mathbb{C} \setminus \{0\}.$$

Can not define logarithm as a holomorphic function on $\mathbb{C} \setminus \{0\}$!

Logarithm as a Holomorphic Function

Define the logarithm function by

$$\log(z) = \log(r) + i\theta \text{ if } z = r\exp(i\theta),\ \theta \in (0, 2\pi).$$

Then log is holomorphic in the region $r > 0$ and $0 < \theta < 2\pi$.

Problem (Cauchy-Riemann Equations in Polar Co-ordinates)

The C-R equations are equivalent to $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

Hint. Treat u, v as functions in r and θ, and apply Chain Rule.

Some Properties of Logarithm.

- $e^{\log z} = e^{\log(|z|)+i\arg z} = |z|e^{i\arg z} = z$.
- $\log z$ can be defined in the region $r > 0$ and $0 \leqslant \theta < 2\pi$. But it is not continuous on the positive real axis.

Goursat's Theorem (Without Proof)

Theorem

If U is an open set and T is a triangle with interior contained in U then $\int_T f(z)dz = 0$ whenever f is holomorphic in U.

Corollary

If U is an open set and R is a rectangle with interior contained in U then $\int_R f(z)dz = 0$ whenever f is holomorphic in U.

Proof.

$E_1, \cdots, E_4$: sides of R, D: diagonal of R with $+$ve orientation, D^-: diagonal with $-$ve orientation. Since $\int_{D^-} f(z)dz = -\int_D f(z)dz$,
$\int_R f(z)dz = \int_{E_1 \cup E_2} f(z)dz + \int_{E_3 \cup E_4} f(z)dz$
$= \left(\int_{E_1 \cup E_2} f(z)dz + \int_D f(z)dz \right) + \left(\int_{E_3 \cup E_4} f(z)dz + \int_{D^-} f(z)dz \right) =$
$\int_{T_1} f(z)dz + \int_{T_2} f(z)dz = 0.$ $\square$

An Application I: $e^{-\pi x^2}$ is its own "Fourier transform"

Consider the function $f(z) = e^{-\pi z^2}$. For a fixed $x_0 \in \mathbb{R}$, let γ denote the rectagular curve with parametrization $z(t)$ given by

$$z(t) = t \text{ for } -R \leqslant t \leqslant R, \quad z(t) = R + it \text{ for } 0 \leqslant t \leqslant x_0,$$

$$z(t) = -t + ix_0 \text{ for } -R \leqslant t \leqslant R, \quad z(t) = -R - it \text{ for } -x_0 \leqslant t \leqslant 0.$$

Let $\gamma_1, \cdots, \gamma_4$ denote sides of γ. Note that $\int_\gamma e^{-\pi z^2} dz = \sum_{j=1}^4 \int_{\gamma_j} e^{-\pi z^2} dz$. Further, as $R \to \infty$, we obtain

- $\int_{\gamma_1} f(z)dz = \int_{-R}^R e^{-\pi t^2} dt \to 1$.
- $|\int_{\gamma_2} f(z)dz| \leqslant \int_0^{x_0} e^{-\pi(R^2 - t^2)} dt = e^{-\pi R^2} \int_0^{x_0} e^{\pi t^2} dt \to 0$.
- $\int_{\gamma_3} f(z)dz = -\int_{-R}^R e^{-\pi(t^2 - x_0^2 + 2itx_0)} dt \to -e^{\pi x_0^2} \int_{-\infty}^\infty e^{-\pi t^2} e^{-2itx_0} dt$.
- $|\int_{\gamma_4} f(z)dz| \leq \int_{-x_0}^0 e^{-\pi(R^2 - t^2)} dt = e^{-\pi R^2} \int_{-x_0}^0 e^{\pi t^2} dt \to 0$.

As a consequence of Goursat's Theorem, we see that $\int_\gamma e^{-\pi z^2} dz = 0$, and hence $\int_{-\infty}^\infty e^{-\pi t^2} e^{-2itx_0} dt = e^{-\pi x_0^2}$.

Application II: Existence of a Primitive in disc

Theorem

Let $\mathbb{D}$ denote the unit disc centered at 0 and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Then there exists a holomorphic function $F : \mathbb{D} \to \mathbb{C}$ such that $F' = f$.

Proof.

For $z \in \mathbb{D}$, define $F(z) = \int_{\gamma_1} f(w)dw + \int_{\gamma_2} f(w)dw$, where

$$\gamma_1(t) = t\,\mathrm{Re}\,(z)\ (0 \leqslant t \leqslant 1),\ \gamma_2(t) = \mathrm{Re}\,(z) + it\,\mathrm{Im}\,(z)\ (0 \leqslant t \leqslant 1).$$

Claim: $F'(z) = f(z)$. Indeed, for $h \in \mathbb{C}$ such that $z + h \in \mathbb{D}$, by Goursat's Theorem, $F(z+h) - F(z) = \int_{\gamma_3} f(w)dw$, where

$$\gamma_3(t) = (1-t)z + t(z+h)\ (0 \leqslant t \leqslant 1).$$

However, since f is (uniformly) continuous on γ_3, $\frac{1}{h}\int_{\gamma_3} f(w)dw = \frac{1}{h}\int_0^1 f(\gamma_3(t))\gamma_3'(t)dt = \int_0^1 f(\gamma_3(t))dt \to f(z)$. □

Cauchy's Theorem for a disc

Theorem

If f is a holomorphic function in a disc, then

$$\int_\gamma f(z)dz = 0$$

for any piecewise smooth, closed curve γ in that disc.

Corollary

If f is a holomorphic function in an open set containing some circle C, then

$$\int_C f(z)dz = 0.$$

Proof.

Let D be a disc containing the disc with boundary C. Now apply Cauchy's Theorem. $\square$

An Example

Consider $f(z) = \frac{1-e^{iz}}{z^2}$. Then f is holomorphic on $\mathbb{C} \setminus \{0\}$. Consider the indented semicircle γ (with $0 < r < R$) given by

$$z_1(t) = t \ (-R \leqslant t \leqslant -r), \ z_2(t) = re^{-it} \ (-\pi \leqslant t \leqslant 0),$$

$$z_3(t) = t \ (r \leqslant t \leqslant R), \ z_4(t) = Re^{it} \ (0 \leqslant t \leqslant \pi).$$

Since $z_1(-R) = -R = z_4(\pi)$, γ is closed. By Cauchy's Theorem,

$$\int_{-R}^{-r} \frac{1-e^{it}}{t^2} dt + \int_{-\pi}^{0} \frac{1-e^{iz_2(t)}}{z_2(t)^2}(-ire^{-it}) dt$$

$$+ \int_{r}^{R} \frac{1-e^{it}}{t^2} dt + \int_{0}^{\pi} \frac{1-e^{iz_4(t)}}{z_4(t)^2}(iRe^{it}) dt = 0.$$

Since $|f(x+iy)| \leq \frac{1+e^{-y}}{|z|^2} \leq \frac{2}{|z|^2}$, the 4th integral $\to 0$ as $R \to \infty$.

Thus we obtain

$$\int_{-\infty}^{-r} \frac{1-e^{it}}{t^2}dt + \int_{-\pi}^{0} \frac{1-e^{iz_2(t)}}{z_2(t)^2}(-ire^{-it})dt + \int_{r}^{\infty} \frac{1-e^{it}}{t^2}dt = 0.$$

Next, note that $\frac{1-e^{iz_2(t)}}{z_2(t)^2} = E(z_2(t)) - \frac{iz_2(t)}{z^2}$, where $E(z) = \frac{1+iz-e^{iz}}{z^2}$ is a bounded function near 0. It follows that

$$\int_{-\pi}^{0} \frac{1-e^{iz_2(t)}}{z_2(t)^2}(-ire^{-it})dt \to -\int_{-\pi}^{0} dt = -\pi \text{ as } r \to 0.$$

This yields the following:

$$\int_{-\infty}^{0} \frac{1-e^{it}}{t^2}dt + \int_{0}^{\infty} \frac{1-e^{it}}{t^2}dt = \pi.$$

Taking real parts, we obtain

$$\int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2}dx = \pi.$$

Cauchy Integral Formula I

The values of f at boundary determine its values in the interior!

Theorem

Let U be an set containing the disc $\mathbb{D}_R(z_0)$ centred at z_0 and suppose f is holomorphic in U. If C denotes the circle $\{z \in \mathbb{C} : |z - z_0| = R\}$ of positive orientation, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw \text{ for any } z \in \mathbb{D}_R(z_0).$$

Example

- $\int_{|w-i|=1} \frac{-w^2}{w^2+1} dw = \int_{|w-i|=1} \frac{-w^2/(w+i)}{w-i} dw = \pi.$
- $\displaystyle\int_{|w-\pi/2|=\pi} \frac{\sin(w)}{w(w-\pi/2)} dw = \frac{2}{\pi}\Big(\int_{|w-\pi/2|=\pi} \frac{\sin(w)}{w-\pi/2} dw - \int_{|w-\pi/2|=\pi} \frac{\sin(w)}{w} dw\Big) = 4i.$

An Application: Fundamental Theorem of Algebra

Corollary

Any non-constant polynomial p has a zero in $\mathbb{C}$.

Anton R. Schep, Amer. Math. Monthly, 2009 January.

If possible, suppose that p has no zeros, that is, $p(z) \neq 0$ for every $z \in \mathbb{C}$. Let $f(z) = \frac{1}{p(z)}$ and $z_0 = 0$ in CIF:

- $\frac{1}{p(0)} = \frac{1}{2\pi i} \int_{|w|=R} \frac{1/p(w)}{w}$,
- $\left| \frac{1}{2\pi} \int_{|w|=R} \frac{dw}{wp(w)} \right| \leq \max_{|w|=R} \left| \frac{1}{p(w)} \right| = \frac{1}{\min_{|w|=R} |p(w)|}$.
- $\min_{|w|=R} |p(w)| \leq |p(0)|$.
- $|p(z)| \geq |z|^n (1 - |a_{n-1}|/|z| - \cdots - |a_0|/|z^n|)$.
- $\lim_{R\to\infty} \min_{|w|=R} |p(w)| = \infty$.

This is not possible! $\square$

Proof of CIF I

Want to prove: If $f : U \to \mathbb{C}$ is holomorphic and $\overline{\mathbb{D}}_R(z_0) \subseteq U$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw \text{ for any } z \in \mathbb{D}_R(z_0).$$

For $0 < r, \delta < R$, consider the "keyhole" contour $\gamma_{r,\delta}$ with

- a big 'almost' circle $|w - z_0| = R$ of positive orientation,
- a small 'almost' circle $|w - z| = r$ of negative orientation,
- a corridor of width δ with two sides of opposite orientation.

$\frac{f(w)}{w-z}$ is holomorphic in the "interior" of $\gamma_{r,\delta}$. By Cauchy's Theorem,

$$\int_{\gamma_{r,\delta}} \frac{f(w)}{w - z} dw = 0.$$

$\gamma_{r,\delta}$ has three parts: big circle C, small circle C_r, and corridor.

- As $\delta \to 0$, integrals over sides of corridor get cancel.
- Note that

$$\int_{C_r} \frac{f(w) - f(z)}{w - z} dw + \int_{C_r} \frac{f(z)}{w - z} dw = \int_{C_r} \frac{f(w)}{w - z} dw.$$

As $r \to 0$, 1^{st} integral tends to 0 (since integrand is bounded near z), while 2^{nd} integral is equal to $-f(z)(2\pi i)$.

- As a result, we obtain

$$0 = \int_{\gamma_{r,\delta}} \frac{f(w)}{w - z} dw = \int_C \frac{f(w)}{w - z} dw - f(z)(2\pi i).$$

Maximum Modulus Principle for Polynomials

Problem

Let p be a polynomial. Show that if p is non-constant then $\max_{|z|\leq 1} |p(z)| = \max_{|z|=1} |p(z)|$.

Hint. If possible, there is $z_0 \in \mathbb{D}$ be such that $|p(z)| \leq |p(z_0)|$ for every $|z| \leq 1$. Write $p(z) = b_0 + b_1(z - z_0) + \cdots + b_n(z - z_0)^n$. If $0 < r < 1 - |z_0|$ then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |p(z_0 + re^{i\theta})|^2 d\theta = |b_0|^2 + |b_1|^2 r^2 + \cdots + |b_n|^2 r^{2n}.$$

However, $|b_0|^2 = |p(z_0)|^2$. Try to get a contradiction!

Growth Rate of Derivative

- $\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_C \frac{f(w)}{h}\left(\frac{1}{w-z-h} - \frac{1}{w-z}\right)dw$
  $= \frac{1}{2\pi i}\int_C f(w)\left(\frac{1}{(w-z-h)(w-z)}\right)dw.$
- Taking limit as $h \to 0$, we obtain

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw.$$

Corollary (Cauchy Estimates)

Under the hypothesis of CIF I,

$$|f'(z_0)| \leq \frac{\max\limits_{|z-z_0|=R} |f(z)|}{R}.$$

Entire Functions

Definition

f is <u>entire</u> if f is complex differentiable at every point in $\mathbb{C}$.

Theorem (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. If there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is a constant function.

Proof.

By Cauchy estimates, for any $R > 0$,

$$|f'(z_0)| \leqslant \frac{\max\limits_{|z-z_0|=R} |f(z)|}{R} \leqslant \frac{M}{R} \to 0 \text{ as } R \to \infty.$$

Thus $f'(z_0) = 0$. But z_0 was arbitrary, and hence $f' = 0$. $\square$

An Application: Range of Entire Functions

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. We contend that the range of f intersects every disc in the complex plane.

- On the contrary, assume that some disc $\mathbb{D}_R(z_0)$ does not intersect the range of f, that is,

$$|f(z) - z_0| \geqslant R \text{ for all } z \in \mathbb{C}.$$

- Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \frac{1}{f(z) - z_0}$.
- Note that g is entire such that $|g(z)| \leqslant \frac{1}{R}$ for all $z \in \mathbb{C}$.
- By Liouville's Theorem, g must be a constant function, and hence so is f. This is not possible.

Cauchy Integral Formula II

Corollary

Let U be an open set containing the disc $\mathbb{D}_R(z_0)$ and suppose f is holomorphic in U. If C denotes the circle $\{z \in \mathbb{C} : |z - z_0| = R\}$ of positive orientation, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw \text{ for any } z \in \mathbb{D}_R(z_0).$$

We have already seen a proof in case $n = 1$. Let try case $n = 2$.

- $\frac{f'(z+h)-f'(z)}{h} = \frac{1}{2\pi i} \int_C \frac{f(w)}{h} \left(\frac{1}{(w-z-h)^2} - \frac{1}{(w-z)^2} \right) dw$
  $= \frac{1}{2\pi i} \int_C f(w) \left(\frac{h+2(w-z)}{(w-z-h)^2(w-z)^2} \right) dw.$
- Taking limit as $h \to 0$, we obtain

$$f''(z) = \frac{2}{2\pi i} \int_C \frac{f(w)}{(w-z)^3} dw.$$

Holomorphic function is Analytic

Theorem

Suppose $\overline{\mathbb{D}}_R(z_0) \subseteq U$ and $f : U \to \mathbb{C}$ is holomorphic. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } z \in \mathbb{D}_R(z_0),$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$ for all integers $n \geqslant 0$.

Proof.

Let $z \in \mathbb{D}_R(z_0)$ and write

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}}.$$

Since $|w - z_0| = R$ and $z \in \mathbb{D}_R(z_0)$, there is $0 < r < 1$ such that

$$|z - z_0|/|w - z_0| < r.$$

Proof Continued.

Thus the series $\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$ converges uniformly for any w on $|w-z_0| = R$. We combine this with CIF I

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \text{ for any } z \in \mathbb{D}_R(z_0)$$

to conclude that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_C \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw$$

$$\overset{\text{uni cgn}}{=} \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{1}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

where we used CIF II. $\square$

Remark Once complex differentiable function is infinitely complex differentiable!

Taylor Series

We refer to the power series $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ as the Taylor series of f around z_0.

Example

Let us compute the Taylor series of $\log z$ in the disc $|z - i| = \frac{1}{2}$. Note that $a_0 = \log i$, $a_1 = \frac{1}{z}|_{z=i} = -i$, and more generally

$$a_n = \frac{f^{(n)}(i)}{n!} = (-1)^{n+1}\frac{1}{i^n}\frac{1}{n!}(n-1)! = \frac{-i^n}{n}.$$

Hence the Taylor series of $\log z$ is given by

$$\log i + \sum_{n=1}^{\infty} \frac{-i^n}{n}(z - i)^n \ (z \in \mathbb{D}_{\frac{1}{2}}(i)).$$

Theorem

An entire function f is given by $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$.

Corollary (Identity Theorem for entire functions)

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose $\{z_k\}$ of distinct complex numbers converges to $z_0 \in \mathbb{C}$. If $f(z_k) = 0$ for all $k \geqslant 1$ then $f(z) = 0$ for all $z \in \mathbb{C}$.

Proof.

Write $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ $(z \in \mathbb{C})$. *If* $f \neq 0$, *there is a smallest integer* n_0 *such that* $f^{(n_0)}(z_0) \neq 0$. *Thus* $f(z) =$
$$\sum_{n=n_0}^{\infty} a_n(z - z_0)^n = a_{n_0}(z - z_0)^{n_0}\left(1 + \sum_{n=1}^{\infty} \frac{a_{n_0+n}}{a_{n_0}}(z - z_0)^n)\right).$$
Since the "bracketed term" is non-zero at z_0, *one can find* $z_k \neq z_0$ *such that RHS is non-zero at* z_k. *But LHS is* 0 *at* z_k. *Not possible!* $\square$

Remark. 'Identity Theorem' does not hold for real differentiable functions.

Trigonometric Functions

Define $\sin z$ and $\cos z$ functions as follows:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

Note that $\sin z$ and $\cos z$ are entire functions (since RoC is ∞). We know the fundamental identity relating $\sin x$ and $\cos x$:

$$\sin^2 x + \cos^2 x = 1 \text{ for } x \in \mathbb{R}.$$

In particular, the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \sin^2 z + \cos^2 z - 1$ is entire and satisfies $f(x) = 0$ for $x \in \mathbb{R}$. Hence by the previous result,

$$\sin^2 z + \cos^2 z = 1 \text{ for } z \in \mathbb{C}.$$

A Problem

Note that there is an entire function f such that $f(z+1) = f(z)$ for all $z \in \mathbb{C}$, but f is not constant:

$$f(z) = e^{2\pi i z}.$$

Similarly, there exists a non-constant entire function f such that $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. However, if an entire function f satisfies both the above conditions, then it must be a constant!

Problem

Does there exist an entire function such that

$$f(z+1) = f(z), \;\; f(z+i) = f(z) \text{ for all } z \in \mathbb{C}\ ?$$

Hint. Show that f is bounded and apply Liouville's Theorem.

Zeros of a Holomorphic Function

Theorem (Identity Theorem)

Let U be an open connected subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ is a holomorphic function. Suppose $\{z_k\}$ of distinct numbers converges to $z_0 \in U$. If $f(z_k) = 0$ for all $k \geqslant 1$ then $f(z) = 0$ for all $z \in U$.

Definition

A complex number $a \in \mathbb{C}$ is a zero for a holomorphic function $f : U \to \mathbb{C}$ if $a \in U$ and $f(a) = 0$.

- The identity theorem says that the zeros of f has "isolated". This means that any closed disc contained in U contains at most finitely many zeros of f.
- However f can have infinitely many zeros: $\sin(z)$.
- The zeros of f is always countable.

Theorem

Suppose that f is a non-zero holomorphic function on a connected set U and $a \in U$ such that $f(a) = 0$. Then there exist $R > 0$, a holomorphic function $g : \mathbb{D}_R(a) \to \mathbb{C}$ with $g(z) \neq 0$ for all $z \in \mathbb{D}_R(a)$ and a unique integer $n > 0$ such that

$$f(z) = (z-a)^n g(z) \text{ for all } z \in \mathbb{D}_R(a) \subseteq U.$$

Proof.

Write $f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$, let $n \geqslant 1$ be a smallest integer such that $a_n \neq 0$ (which exists by the Identity Theorem). Then $f(z) = (z-a)^n g(z)$, where $g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n}$. Note that $g(a) = a_n \neq 0$, and hence by continuity of g, there exists $R > 0$ such that $g(z) \neq 0$ for all $z \in \mathbb{D}_R(a)$. □

We say that f has zero at a of order (or multiplicity) n. For example, z^n has zero at 0 of order n.

Zeros of $\sin(\pi z)$

Example

- $\sin(\pi z)$ has zeros at all integers; all are of order 1. Indeed, $\sin(\pi k) = 0$ and $\frac{d}{dz}\sin(\pi z)|_{z=k} = \pi\cos(\pi k) \neq 0$.
- If possible, suppose $\sin(\pi z_0) = 0$ for some $z_0 = x_0 + iy_0 \in \mathbb{C}$.
- By Euler's Formula, $\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$. Hence $e^{i\pi z_0} = e^{-i\pi z_0}$, that is, $e^{2i\pi z_0} = 1$. Taking modulus on both sides, we obtain $e^{-2\pi y_0} = 1$. Since e^x is one to one, $y_0 = 0$.
- Thus $e^{2i\pi x_0} = 1$, that is, $\cos(2\pi x_0) + i\sin(2\pi x_0) = 1$, and hence x_0 is an integer.

Problem

Show that all zeros of $\cos(\frac{\pi}{2}z)$ *are at odd integers.*

Singularities of a meromorphic function

By a deleted neighborhood of a, we mean the punctured disc

$$\mathbb{D}_R(a) \setminus \{a\} = \{z \in \mathbb{C} : 0 < |z - a| < R\}.$$

Definition

An isolated singularity of a function f is a complex number z_0 such that f is defined in a deleted neighborhood of z_0.

For instance, 0 is an isolated singularity of

- $f(z) = \frac{1}{z}$.
- $f(z) = \frac{\sin z}{z}$
- $f(z) = e^{\frac{1}{z}}$.

The singularities in these examples are different in a way.

Indeed, a holomorphic function can have three kinds of isolated singularities: pole, removable singularity, essential singularity

Poles

Definition

Let f be a function defined in a deleted neighborhood of a. We say that f has a pole at a if the function $\frac{1}{f}$, defined to be 0 at a, is holomorphic on $\mathbb{D}_R(a)$.

Example

- $\frac{1}{z-a}$ has a pole at a.
- 0 is not a pole of $\frac{\sin z}{z}$ (since $\frac{\sin z}{z} \to 1$ as $z \to 0$).
- The poles of a rational function (in a reduced form $\frac{p(z)}{q(z)}$) are precisely the zeros of $q(z)$. For instance, $\frac{z+1}{z+2}$ has only pole at $z = -2$ while the poles of $\frac{(z+1)\cdots(z+5)}{(z+2)\cdots(z+6)}$ are at $z = -1, -6$.

Theorem

Suppose that f has a pole at $a \in U$. Then there exist $R > 0$, a holomorphic function $h : \mathbb{D}_R(a) \to \mathbb{C}$ with $h(z) \neq 0$ for all $z \in \mathbb{D}_R(z_0)$ and a unique integer $n > 0$ such that

$$f(z) = (z-a)^{-n} h(z) \text{ for all } z \in \mathbb{D}_R(a) \setminus \{a\} \subseteq U.$$

Proof.

Note that $\frac{1}{f}$, with 0 at a, is a holomorphic function. Hence, by a result on Page 55, there exist $R > 0$, a holomorphic function $g : \mathbb{D}_R(a) \to \mathbb{C}$ with $g(z) \neq 0$ for all $z \in \mathbb{D}_R(a)$ and a unique integer $n > 0$ such that $\frac{1}{f(z)} = (z-a)^n g(z)$ for all $z \in \mathbb{D}_R(a)$. Now let $h(z) = \frac{1}{g(z)}$. □

We say that f has pole at a of <u>order</u> (or <u>multiplicity</u>) n. For example, $\frac{1}{z^n}$ has pole at 0 of order n.

Example

Let us find poles of $f(z) = \frac{1}{1+z^4}$.

- For this, let us first solve $1 + z^4 = 0$. Taking modulus on both sides of $z^4 = -1$, we obtain $|z| = 1$. Thus $z = e^{i\theta}$, and hence $e^{4i\theta} = e^{i\pi}$. This forces $4\theta = \pi + 2\pi k$ for integer k. Thus $e^{i\theta} = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$.
- Note that $\frac{1}{f(z)} = (z - e^{i\frac{\pi}{4}})^{-1}h(z)$, where $h(z) = (z - e^{i\frac{3\pi}{4}})(z - e^{i\frac{5\pi}{4}})(z - e^{i\frac{7\pi}{4}})$ is non-zero for every $z \in \mathbb{D}_R(e^{i\frac{\pi}{4}})$ for some $R > 0$.Thus $z = e^{i\frac{\pi}{4}}$ is a pole.
- Similar argument shows that $e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$ are poles of f.

Principal Part and Residue Part

Suppose that f has a pole of order n at a. By theorem on Page 60, there exist $R > 0$, a holomorphic function $h : \mathbb{D}_R(a) \to \mathbb{C}$ with $h(z) \neq 0$ for all $z \in \mathbb{D}_R(a)$ and a unique integer $n > 0$ such that

$$f(z) = (z-a)^{-n} h(z) \text{ for all } z \in \mathbb{D}_R(a) \setminus \{a\} \subseteq U.$$

Since h is holomorphic, $h(z) = b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots$,

$$f(z) = \frac{b_0}{(z-a)^n} + \frac{b_1}{(z-a)^{n-1}} + \frac{b_2}{(z-a)^{n-2}} + \cdots ,$$

which can be rewritten as

$$f(z) = \Big(\frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_{-1}}{z-a}\Big) + \Big(a_0 + a_1(z-a) + \cdots\Big),$$

$=$ Principal part $P(z)$ of f at $a + H(z)$.

Definition

The residue $\operatorname{res}_a f$ of f at a is defined as the coefficient a_{-1} of $\frac{1}{z-a}$.

The residue $\operatorname{res}_a f$ is special among all terms in the principal part $P(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_{-1}}{z-a}$ in the following sense:

- $\frac{a_{-k}}{(z-a)^k}$ has a primitive in a deleted neighborhood of a iff $k \neq 1$.
- If C_+ is the circle $|z-a| = R$ then $\frac{1}{2\pi i}\int_{C_+} P(z)dz = a_{-1}$.
- If f has a simple pole (pole of order 1) at a then $(z-a)f(z) = a_{-1} + a_0(z-a) + \cdots \to a_{-1} = \operatorname{res}_a f$ as $z \to a$:

$$\operatorname{res}_a f = \lim_{z\to a}(z-a)f(z).$$

- Suppose f has a pole of order 2. Then $(z-a)^2 f(z) = a_{-2} + a_{-1}(z-a) + a_0(z-a)^2 + \cdots$, and hence

$$\frac{d}{dz}(z-a)^2 f(z) = a_{-1} + 2a_0(z-a) + \cdots$$

  Thus we obtain $\operatorname{res}_a f = \lim_{z\to a} \frac{d}{dz}(z-a)^2 f(z)$.

Residue at poles of finite order

Theorem

If f has a pole of order n at a, then

$$res_a f = \lim_{z \to a} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-a)^n f(z).$$

Proof.

We already know

$$f(z) = \left(\frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_{-1}}{z-a}\right) + \left(a_0 + a_1(z-a) + \cdots\right),$$

$$(z-a)^n f(z) = \left(a_{-n} + \frac{a_{-n+1}}{z-a} + \cdots + (z-a)^{n-1} a_{-1}\right)$$

$$+ (z-a)^n \left(a_0 + a_1(z-a) + \cdots\right),$$

Now differentiate $(n-1)$ times and take limit as $z \to a$. $\square$

Example

Consider the function $f(z) = \frac{1}{1+z^2}$. Then f has simple poles at $z = \pm i$. Recall that

$$\operatorname{res}_a f = \lim_{z \to a} (z - a) f(z).$$

Thus we obtain

$$\operatorname{res}_i f = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i}.$$

$$\operatorname{res}_{-i} f = \lim_{z \to -i} (z + i) f(z) = \lim_{z \to -i} \frac{1}{z - i} = \frac{1}{-2i} = 2i.$$

The Residue Formula

Theorem

Suppose that $f : U \to \mathbb{C}$ is holomorphic except a pole at $a \in U$. Let $C \subseteq U$ be one of the following closed contour enclosing a in U and with "interior" contained in U: A circle, triangle, semicircle union segment etc. Then

$$\int_C f(z)dz = 2\pi i \, res_a \, f.$$

Example

Let $f(z) = \frac{1}{1+z^2}$. Let γ_R be union of $[-R, R]$ and semicircle C_R:

$$z_1(t) = t \ (-R \leqslant t \leqslant R), \ z_2(t) = Re^{it} \ (0 \leqslant t \leqslant \pi).$$

i is the only pole in the "interior" of γ_R if $R > 1$. Also, $\text{res}_i \, f = \frac{1}{2i}$.

By Residue Theorem, $\int_{-R}^{R} \frac{1}{1+x^2} dx + \int_{C_R} f(z)dz = \pi$. Let $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx + \lim_{R\to\infty} \int_{C_R} f(z)dz = \pi.$$

We claim that $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$. To see that,

$$\Big| \int_{C_R} f(z)dz \Big| \leqslant \int_0^{\pi} \Big| \frac{1}{1+R^2 e^{2it}} \Big| Rdt \leqslant \int_0^{\pi} \Big| \frac{1}{R^2-1} \Big| Rdt$$

$= \pi \frac{R}{R^2-1} \to 0$ as $R \to \infty$. This yields the formula:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

Proof of Residue Formula

Consider the keyhole contour $\gamma_{r,\delta}$ that avoids the pole a:
$\gamma_{r,\delta}$ consists of 'almost' C,

- a circle C_r: $|w-a|=r$ of negative orientation, and
- a corridor of width δ with two sides of opposite orientation.

Letting $\delta \to 0$, we obtain by Cauchy's Theorem that

$$\int_C f(z)dz + \int_{C_r} f(z)dz = 0.$$

However, we know that

$$f(z) = \Big(\frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_{-1}}{z-a}\Big) + \Big(a_0 + a_1(z-a) + \cdots\Big).$$

Now apply Cauchy's Integral Formula and Cauchy's Theorem to see that $\int_{C_r} f(z)dz = a_{-1}(-2\pi i)$ (as C_r has negative orientation).

Residue Formula: General Version

Theorem

Suppose that $f : U \to \mathbb{C}$ is holomorphic except pole at $a_1, \cdots, a_k$ in U. Let $C \subseteq U$ be one of the following closed contour enclosing $a_1, \cdots, a_k$ in U and with "interior" contained in U: A circle, triangle, semicircle union segment etc. Then

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{k} res_{a_i} f.$$

Example

Consider the function $\cosh(z) = \frac{e^z + e^{-z}}{2}$. Then $\cosh(\pi z)$ is an entire function with zeros at points z for which $e^{\pi z} = -e^{-\pi z}$, that is, $e^{2\pi z} = -1$. Solving this for z, we obtain $i/2$ and $3i/2$ as the only zeros of $\cosh(\pi z)$. Note that $\cosh(\pi z)$ is periodic of period $2i$.

Example Continued ...

- For $s \in \mathbb{R}$, consider now the function $f(z) = \frac{e^{-2\pi izs}}{\cosh(\pi z)}$.
- Check that f has simple poles at $a_1 = i/2$ and $a_2 = 3i/2$.
- Further, $\mathrm{res}_{a_1} f = \frac{e^{\pi s}}{\pi i}$ and $\mathrm{res}_{a_2} f = -\frac{e^{3\pi s}}{\pi i}$ (Verify).

Let γ denote the rectagular curve with parametrization

$$\gamma_1(t) = t \text{ for } -R \leqslant t \leqslant R, \quad \gamma_2(t) = R + it \text{ for } 0 \leqslant t \leqslant 2,$$

$$\gamma_3(t) = -t+2i \text{ for } -R \leqslant t \leqslant R, \quad \gamma_4(t) = -R-it \text{ for } -2 \leqslant t \leqslant 0.$$

By Residue Theorem

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{e^{\pi s}}{\pi i} - \frac{e^{3\pi s}}{\pi i}\right) = 2(e^{\pi s} - e^{3\pi s}).$$

Further, as $R \to \infty$, we obtain

- $\int_{\gamma_1} f(z)dz \to \int_{-\infty}^{\infty} \frac{e^{-2\pi its}}{\cosh(\pi t)} dt$.
- $|\int_{\gamma_2} f(z)dz| \leqslant \int_0^2 \frac{2e^{4\pi|s|}}{e^{\pi R} - e^{-\pi R}} dt \to 0$. Similarly, $\int_{\gamma_4} f(z)dz \to 0$.
- $\int_{\gamma_3} f(z)dz = -\int_{-R}^{R} \frac{e^{-2\pi izs}}{\cosh(\pi z)} dt \to -e^{4\pi s} \int_{-\infty}^{\infty} \frac{e^{-2\pi its}}{\cosh(\pi t)} dt$.

Example Continued ...

We club all terms together to obtain

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi its}}{\cosh(\pi t)} dt - e^{4\pi s} \int_{-\infty}^{\infty} \frac{e^{-2\pi its}}{\cosh(\pi t)} dt = \int_{\gamma} f(z) dz = 2(e^{\pi s} - e^{3\pi s}),$$

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi its}}{\cosh(\pi t)} dt = \frac{2}{1 - e^{4\pi s}} (e^{\pi s} - e^{3\pi s}).$$

However, $(e^{\pi s} - e^{3\pi s})(e^{\pi s} + e^{-\pi s}) = 1 - e^{4\pi s}$, and hence

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi its}}{\cosh(\pi t)} dt = \frac{2}{e^{\pi s} + e^{-\pi s}} = \cosh(\pi s).$$

Thus the "Fourier transform" of reciprocal of cosine hyperbolic function is reciprocal of cosine hyperbolic function itself.

Removable Singularity

Definition

Let U be an open subset of $\mathbb{C}$ and let $a \in U$. We say that a is a <u>removable singularity</u> of a holomorphic function $f : U \setminus \{a\} \to \mathbb{C}$ if there exists $\alpha \in \mathbb{C}$ such that $g : U \to \mathbb{C}$ below is holomorphic:

$$g(z) = f(z)\ (z \neq a),\ g(a) = \alpha$$

Example

Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1-\cos z}{z^2}$. Then 0 is a removable singularity of f. Indeed, define $g : U \to \mathbb{C}$ by

$$g(z) = f(z)\ (z \neq 0),\ g(0) = \frac{1}{2}.$$

Then g is complex differentiable at 0: $\frac{g(h)-g(0)}{h} = \frac{\frac{1-\cos h}{h^2} - \frac{1}{2}}{h} \to 0$. Hence g is holomorphic on $\mathbb{C}$.

Theorem

Let U be an open subset of $\mathbb{C}$ containing a. Let $f : U \setminus \{a\} \to \mathbb{C}$ be a holomorphic function. If $\alpha := \lim_{z \to a} f(z)$ exists and for some holomorphic function $F : \mathbb{D}_R(a) \to \mathbb{C}$,

$$f(z) - \alpha = (z - a)F(z) \ (z \in \mathbb{D}_R(a)),$$

then f has removable singularity at a.

Proof.

Define $g : U \to \mathbb{C}$ by

$$g(z) = f(z) \ (z \neq a), \ g(a) = \alpha.$$

We must check that g is complex differentiable at a. However,

$$\frac{g(h) - g(a)}{h - a} = \frac{f(h) - \alpha}{h - a} = F(h) \to F(a).$$

It follows that g is holomorphic on U. □

Example

Let $a = \pi/2$ and $f(z) = \frac{1-\sin z}{\cos z}$. Then

$$\cos z = \sum_{n=0}^{\infty} \left(\frac{d^n}{dz^n} \cos z|_{z=\pi/2} \right) (z-\pi/2)^n = (z-\pi/2) H(z), \ H(\pi/2) \neq 0,$$

$$1 - \sin z = \sum_{n=0}^{\infty} \left(\frac{d^n}{dz^n} (1 - \sin z)|_{z=\pi/2} \right) (z - \pi/2)^n = (z - \pi/2)^2 G(z)$$

It follows that $\alpha := \lim_{z\to\pi/2} f(z) = 0$. Also, for some $R > 0$,

$$f(z) - \alpha = (z - \pi/2) \frac{G(z)}{H(z)} \ (z \in \mathbb{D}_R(a)),$$

and hence $z = \pi/2$ is a removable singularity of f.

Laurent Series and Essential Singularity

Theorem

For $0 < r < R < \infty$, let $\mathbb{A}_{r,R}(z_0) : \{z \in \mathbb{C} : r < |z - z_0| < R\}$, suppose $f : \mathbb{A}_{r,R}(z_0) \to \mathbb{C}$ is holomorphic. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \text{ for all } z \in \mathbb{A}_{r,R}(z_0),$$

where $a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz$ for integers n and $r < \rho < R$.

We refer to the series appearing above as the Laurent series of f around z_0.

Outline of the Proof.

One needs Cauchy Integral Formula for the union of $|z - z_0| = r_1$ and $|z - z_0| = R_1$ (can be obtained from Cauchy's Theorem by choosing appropriate keyhole contour), where $r < r_1 < R_1 < R$. Thus for $z \in \mathbb{A}_{r,R}(z_0)$,

Outline of the Proof Continued.

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=R_1} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{|w-z_0|=r_1} \frac{f(w)}{w-z}dw.$$

One may argue as in the proof of Cauchy Integral Theorem to see that first integral gives the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ while second one leads to $\sum_{n=-\infty}^{1} a_n(z-z_0)^n$. $\square$

Definition

Let U be an open set and $z_0 \in U$ be an isolated singularity of the holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$. We say that z_0 is an essential singularity of f if infinitely many coefficients among $a_{-1}, a_{-2}, \cdots$, in the Laurent series of f are non-zero.

- The Laurent series of $f(z) = e^{1/z}$ around 0 is $1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$. Hence 0 is an essential singularity.
- Similarly, 0 is an essential singularity of $z^2 \sin(1/z)$.

Let us examine the Laurent series of f around z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \text{ for all } z \in \mathbb{A}_{r,R}(z_0),$$

- z_0 is a removable singularity if and only if $a_{-n} = 0$ for $n = 1, 2, \cdots$,
- z_0 is a pole of order k if and only if $a_{-n} = 0$ for $n = k + 1, k + 2, \cdots$, and $a_{-k} \neq 0$.
- z_0 is an essential singularity if and only if $a_{-n} \neq 0$ for infinitely many values of $n \geq 1$.

In particular, an isolated singularity is essential if it is neither a removable singularity nor a pole.

Counting Zeros and Poles

In an effort to understand "logarithm" of a holomorphic function $f : U \to \mathbb{C} \setminus \{0\}$, we must understand the change in the argument

$$\int_\gamma \frac{f'(z)}{f(z)} dz$$

of f as z traverses the curve γ. The argument principle says that for the unit circle γ, this is completely determined by the zeros and poles of f inside γ. We prove this in a rather special case, under the additional assumption that f has finitely many zeroes and poles.

Theorem (Argument Principle)

Suppose f is holomorphic except at poles in an open set containing a circle C and its interior. If f has no poles and zeros on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \text{ (number of zeros of } f \text{ inside } C)$$

$$- \text{ (number of poles of } f \text{ inside } C).$$

Here the number of zeros and poles of f are counted with their multiplicities.

Proof.

Let $z_1, \cdots, z_k$ (of multiplicities $n_1, \cdots, n_k$) and $p_1, \cdots, p_l$ (of multiplicities $m_1, \cdots, m_k$) denote the zeros and poles of f inside C respectively. If f has a zero at z_1 of order n_1 then

$$f(z) = (z - z_1)^{n_1} g(z)$$

in the interior of C for a non-vanishing function g near z_1.

Proof Continued.

Note that $\frac{f'(z)}{f(z)} = \frac{n_1(z-z_1)^{n_1-1}g(z)+(z-z_1)^{n_1}g'(z)}{(z-z_1)^{n_1}g(z)} = \frac{n_1}{z-z_1} + \frac{g'(z)}{g(z)}$.
Integrating both sides, we obtain

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = n_1 + \frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}dz.$$

Now g has a zero at z_2 of multiplicity n_2. By same argument to $g(z) = (z-z_2)^{n_2}h(z)$, we obtain

$$\frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}dz = n_2 + \frac{1}{2\pi i}\int_C \frac{h'(z)}{h(z)}dz.$$

Continuing this we obtain

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = n_1 + \cdots + n_k + \frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}dz \qquad (\star),$$

where $F(z)$ has no zeros.

Proof Continued.

Note that F has poles at $p_1, \cdots, p_l$ (of multiplicities $m_1, \cdots, m_k$) respectively. Write $F(z) = (z - p_1)^{-m_1} G(z)$ and note that

$$\frac{F'(z)}{F(z)} = \frac{-m_1(z - z_1)^{-m_1 - 1} G(z) + (z - p_1)^{-m_1} G'(z)}{(z - p_1)^{-m_1} G(z)}$$

$$= \frac{-m_1}{z - p_1} + \frac{G'(z)}{G(z)}.$$

It follows that $\int_C F'/F = -m_1$. Continuing this we obtain

$$\frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz = -m_1 - \cdots - m_l$$

(we need Cauchy's Theorem here). Now substitute this in $(\star)$. $\square$

Applications

Corollary

Suppose f is holomorphic in an open set containing a circle C and its interior. If f has no zeros on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \text{ (}\textit{number of zeros of f inside C}\text{)}.$$

Here the number of zeros of f are counted with their multiplicities.

Theorem (Rouché's Theorem)

Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f + g$ have the same number of zeros inside the circle C.

Outline of Proof of Rouché's Theorem.

Let $F_t(z) := f + tg$ for $t \in [0,1]$. By the corollary above,

$$\text{Number of zeros of } F_t(z) = \int_C \frac{f_t'(z)}{f_t(z)} dz$$

is an integer-valued, continuous function of t, and hence by Intermediate Value Theorem,

$$\text{Number of zeros of } F_0(z) = \text{ Number of zeros of } F_1(z).$$

But $F_0(z) = f$ and $F_1(z) = f(z) + g(z)$. $\square$

Example

Consider the polynomial $p(z) = 2z^{10} + 4z^2 + 1$. Then $p(z)$ has exactly 2 zeros in the open unit disc $\mathbb{D}$. Indeed, apply Rouché's Theorem to $f(z) = 4z^2$ and $g(z) = 2z^{10} + 1$:

$$|f(z)| = 4 > |2z^{10} + 1| = |g(z)| \text{ on } |z| = 1.$$

Example

Let p be non-constant polynomial. If $|p(z)| = 1$ whenever $|z| = 1$ then the following hold true:

- $p(z) = 0$ for z in the open unit disc. Indeed, by Maximum Modulus Principle, $|p(z)| \leq 1$. Hence, if $p(z) \neq 0$ then $\frac{1}{|p(z)|} \geq 1$ with maximum inside the disc, which is not possible.
- $p(z) = w_0$ has a root for every $|w_0| < 1$, that is, the range of p contains the unit disc. To see this, apply Rouché's Theorem to $f(z) = p(z)$ and $g(z) = -w_0$ to conclude that

  $$f(z) + g(z) = p(z) - w_0$$

  has a zero inside the disc.

Problem

Show that the functional equation $\lambda = z + e^{-z}$ $(\lambda > 1)$ *has exactly one (real) solution in the right half plane.*

Möbius Transformations

A Möbius transformation is a function of the form

$$f(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{C} \text{ such that } ad-bc \neq 0.$$

Note that f is holomorphic with derivative

$$f'(z) = \frac{ad-bc}{(cz+d)^2}.$$

This also shows that $f'(z) \neq 0$, and hence f is non-constant.

Example

- If $c = 0$ and $d = 1$ then $f(z) = az + b$ is a linear polynomial.
- If $a = 0$ and $b = 1$ then $f(z) = \frac{1}{cz+d}$ is a rational function.

The Möbius transformation $f(z) = \frac{az+b}{cz+d}$ is bijective with inverse

$$g(z) = \frac{-dz+b}{cz-a}.$$

Indeed, $f \circ g(z) = z = g \circ f(z)$ wherever f and g are defined.

Example

Let $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{a'z+b'}{c'z+d'}$ be Möbius transformations. Then $f \circ g$ is a also a Möbius transformation given by

$$f \circ g(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Lemma

If γ is a circle or a line and $f(z) = \frac{1}{z}$ then $f(\gamma)$ is a circle or line.

Proof.

Suppose γ is the circle $|z - a| = r$ (We leave the case of line as an exercise). Then $f(\gamma)$ is obtained by replacing z by $w = \frac{1}{z}$:
$|1/w - a| = r,$ that is, $1/|w|^2 - 2\text{Re}(a/\bar{w}) = r^2 - |a|^2$.

- If $r = |a|$ (that is, γ passes through 0), then $\text{Re}(aw) = 1/2$, which gives the line $\text{Re}(w)\text{Re}(a) - \text{Im}(w)\text{Im}(a) = \frac{1}{2}$.
- If $r \neq |a|$ then $1/(r^2 - |a|^2) - 2\frac{|w|^2}{r^2 - |a|^2}\text{Re}(a/\bar{w}) = |w|^2$. Thus

$$1/(r^2 - |a|^2) = |w|^2 + 2\text{Re}(w(a/(r^2 - |a|^2))$$

$$= |w|^2 + 2\text{Re}(w(a/(r^2 - |a|^2)) + |a|^2/(r^2 - |a|^2)^2 - |a|^2/(r^2 - |a|^2)^2$$

$$= |w - a/(r^2 - |a|^2)|^2 - |a|^2/(r^2 - |a|^2)^2.$$

Thus $f(\gamma)$ is the circle $|w - a/(r^2 - |a|^2)| = r/|r^2 - |a|^2|$.

$\square$

Theorem

Any Möbius transformation f maps circles and lines onto circles and lines.

Proof.

We consider two cases:

- $c = 0$: In this case f is linear and sends line to a line and circle to a circle.
- $c \neq 0$: Then $f(z) = f_1 \circ f_2 \circ f_3(z)$, where

$$f_1(z) = \frac{a}{c} - \left(\frac{ad - bc}{c}\right)z, \; f_2(z) = \frac{1}{z}, \text{ and } f_3(z) = cz + d.$$

Since f_1, f_2, f_3 map circles and lines onto circles and lines (by Lemma and Case $c = 0$), so does f. $\square$

Schwarz's Lemma (without Proof)

Theorem

If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map such that $f(0) = 0$ then $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$.

Problem

What are <u>all</u> the bijective holomorphic maps from $\mathbb{D}$ onto $\mathbb{D}$?

- $f(z) = az$ for $|a| = 1$.
- $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$ (Hint. By Cauchy Integral Formula, $|\psi_a(z)| \leq \max_{|w|=1} |\psi_a(w)|$, which is 1).

Corollary

If $f(0) = 0$ and $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic bijective map then f is a rotation: $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Proof.

By Schwarz's Lemma, $|f(z)| \leq |z|$. However, same argument applies to f^{-1}: $|f^{-1}(z)| \leq |z|$. Replacing z by $f(z)$, we obtain

Proof Continued.

$|z| \le |f(z)|$ implying $|f(z)| = |z|$. But then $f(z)/z$ attains max value 1 in $\mathbb{D}$. Hence $f(z)/z$ must be a constant function of modulus 1, that is, $f(z) = e^{i\theta} z$. $\square$

Theorem

If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic bijective map then f is a Möbius transformation:

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z} \text{ for some } a \in \mathbb{D} \text{ and } \theta \in \mathbb{R}.$$

Proof.

Note that $f(a) = 0$ for some $a \in \mathbb{D}$. Consider $f \circ \psi_a$ for $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$, and note that $f \circ \psi_a(0) = 0$. Also, $f \circ \psi_a$ is a holomorphic function on $\mathbb{D}$. Further, since $|\psi_a(z)| < 1$ whenever $|z| < 1$, $f \circ \psi_a$ maps $\mathbb{D} \to \mathbb{D}$. By last corollary, $f \circ \psi_a(z) = e^{i\theta} z$, that is, $f(z) == e^{i\theta} \psi_a^{-1}(z)$. However, by a routine calculation, $\psi_a^{-1}(z) = \psi_a(z)$. $\square$