Sturm-Liouville Theory

MSO-203B

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Overview

Sturm-Liouville Theory:-

- Introduction and Definitions.

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- Examples.

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- Examples.
- Properties of Sturm-Liouville Problem.

Introduction

Definition

Recall that a second order ordinary differential operator is of the form

$$L(y) = p_0 y'' + q_0 y' + r_0 y$$

for some p_0, q_0 and r_0 are real and continuously differentiable function in $I := [a, b]$.

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Remark

Note that this is true if $p_0' = q_0$ and $q = r_0$

Sturm-Liouville Problem

Sturm-Liouville Equation

We are interested in studying the BVP of the form

$$\begin{aligned} (py')' + qy + \lambda ry &= 0 \\ B_1(y) = 0,\ B_2(y) &= 0 \end{aligned} \tag{1}$$

where p, q and r are real, continuous function such that p and r are strictly positive in I and λ is a real parameter.

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Eigenvalues and Eigenvectors

The values of λ for which the BVP (1) has a nontrivial solution are called the Eigenvalues of the operator L and the solutions are called eigenfunction.

Boundary Conditions

Seperated Boundary Condition

If we have,

$$\begin{aligned} B_1(y) &:= \alpha_1 y(a) + \alpha_2 y'(a) = 0;\ \alpha_1^2 + \alpha_2^2 \neq 0 \\ B_2(y) &:= \beta_1 y(b) + \beta_2 y'(b) = 0;\ \beta_1^2 + \beta_2^2 \neq 0 \end{aligned} \tag{2}$$

The BVP (1) with (2) is called a regular Sturm-Liouville Problem.

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$$B_1(y) := \alpha_1 y(a) + \alpha_2 y'(a) = 0;\ \alpha_1^2 + \alpha_2^2 \neq 0 \qquad (2)$$
$$B_2(y) := \beta_1 y(b) + \beta_2 y'(b) = 0;\ \beta_1^2 + \beta_2^2 \neq 0$$

The BVP (1) with (2) is called a regular Sturm-Liouville Problem.

Periodic Boundary Condition

If $p(a) = p(b)$ and

$$B_1(y) := y(a) = y(b) \qquad (3)$$
$$B_2(y) := y'(a) = y'(b)$$

Then the BVP (1) with (3) is called a periodic Sturm-Liouville Problem.

Examples

Sturm-Liouville Problem

Consider the problem

$$y'' + \lambda y = 0 \text{ in } I = [0, \pi] \tag{4}$$

Examples of Boundary Conditions

1. $y(0) = 0$ and $y'(\pi) = 0$.
   Then Equation (4) along with the above boundary condition is a Regular Sturm-Liouville Problem(RSLP).

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Examples of Boundary Conditions

1. $y(0) = 0$ and $y'(\pi) = 0$.
   Then Equation (4) along with the above boundary condition is a Regular Sturm-Liouville Problem(RSLP).
2. $y(0) = y(\pi)$ and $y'(0) = y'(\pi)$.
   Then Equation (4) along with the above boundary condition is called a Periodic Sturm-Liouville Problem (PSLP).

Example 1

RSL Problem

Consider the problem:

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Solution for $\lambda < 0$

Let $\lambda < 0$ then $\lambda = -\mu^2$.
The general solution is given as

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \tag{5}$$

y satisfies the boundary condition iff $A = B = 0$. So $y(x) = 0$ and hence there are no negative eigenvalues.

RSL Problem

Solution for $\lambda = 0$

$y(x) = 0$ is the only solution satisfying the equation
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Solution for $\lambda > 0$

Let $\lambda > 0$ then $\lambda = \mu^2$ and the general solution is given by
$y(x) = A\cos(\mu x) + B\sin \mu x.$
Now y satisfies the boundary condition iff $A = 0$ and $B\cos(\mu\pi) = 0.$
But, $B\cos(\mu\pi) = 0$ iff $B = 0$ or $\cos(\mu\pi) = 0.$
When $A = B = 0$ then $y(x) = 0$ is the only solution hence this condition doesn't give us eigenvalue.

RSL Problem

Solution for $\lambda > 0$

If $y \neq 0$ and $B \neq 0$ then $\cos(\mu\pi) = 0$ holds which implies $\mu = \frac{2n-1}{2}$ for $n \in \mathbb{Z}$. Thus the eigenvalues are given by $\phi_n(x) = B\sin(\frac{2n-1}{2}x)$ for $n \in \mathbb{Z}$.

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Remark

- All the eigenvalues are real.

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Remark

- All the eigenvalues are real.
- Eigenfunctions corresponding to each eigenvalues form a vector space of dim 1.

Example 2

PSL Problem

Consider a problem $y'' + \lambda y = 0,\ y(0) - y(\pi) = 0;\ \ y'(0) - y'(\pi) = 0$

Solution for $\lambda < 0$

Let $\lambda < 0$ then $\lambda = -\mu^2$.
The general solution is given as

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \tag{6}$$

y satisfies the boundary condition iff $A = B = 0$. So $y(x) = 0$ and hence there are no negative eigenvalues.

Solution for $\lambda = 0$

If $\lambda = 0$ then the general solution is $y(x) = A + Bx$ and since y satisfies the boundary conditions iff $B = 0$. Thus A is arbitrary and hence 0 is an eigenvalue with eigenfunction being a non-zero constant.

Solution for $\lambda > 0$

If $\lambda > 0$ then $\lambda = \mu^2$ and the general solution is given by $y(x) = A\cos(\mu x) + B\sin(\mu x)$.
Since y satisfies the boundary condition if

$$\begin{aligned} A\sin(\mu\pi) + B(1 - \cos(\mu\pi)) &= 0 \\ A(1 - \cos(\mu\pi)) - B\sin(\mu\pi) &= 0 \end{aligned} \tag{7}$$

PSL Problem

Solution for $\lambda > 0$

Equation (7) has a nontrivial solution iff $\cos(\mu\pi) = 1$ which implies $\mu = \pm 2n$ with $n \in \mathbb{Z}$. Hence $\lambda = 4n^2$ for $n \in \mathbb{Z}$.
Thus positive eigenvalues are given by $\lambda_n = 4n^2$ for $n \in \mathbb{Z}$ and the eigenfunctions corresponding to λ_n are given by $\phi_n(x) = \cos(2nx)$ and $\chi_n(x) = \sin(2nx)$.

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Remark

- All eigenvalues are non-negative.
- Corresponding to the positive eigenvalues we have two linearly independent eigenfunctions.

Properties of Regular SLBVP

Theorem 1

The eigenvalues of a regular SLBVP

$$\begin{aligned}(py')' + qy + \lambda ry &= 0 \\ \alpha_1 y(a) + \alpha_2 y'(a) &= 0;\ \alpha_1^2 + \alpha_2^2 \neq 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0;\ \beta_1^2 + \beta_2^2 \neq 0\end{aligned} \tag{8}$$

are real provided p and r are strictly positive in I.

Proof of Theorem 1

Let $\lambda \in \mathbb{C}$ be a complex eigenvalue and u be the corresponding eigenfunction. Hence (λ, u) satisfies

$$(pu')' + qu = \lambda ru \tag{9}$$

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0;\ \alpha_1^2 + \alpha_2^2 \neq 0 \tag{10}$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0;\ \beta_1^2 + \beta_2^2 \neq 0 \tag{11}$$

Taking the complex conjugate we have,

$$(p\bar{u}')' + q\bar{u} = \bar{\lambda} r\bar{u} \tag{12}$$

$$\alpha_1 \bar{u}(a) + \alpha_2 \bar{u}'(a) = 0;\ \alpha_1^2 + \alpha_2^2 \neq 0 \tag{13}$$

$$\beta_1 \bar{u}(b) + \beta_2 \bar{u}'(b) = 0;\ \beta_1^2 + \beta_2^2 \neq 0 \tag{14}$$

Proof of Theorem 1

Multiplying $\bar{u}$ and u with (9) and (12) and subtracting we obtain by integrating on I

$$p(\bar{u}u' - u\bar{u}')|_a^b = (\bar{\lambda} - \lambda) \int_a^b r u \bar{u}$$

which implies $(\bar{\lambda} - \lambda) \int_a^b r|u|^2 = 0$ since $p(\bar{u}u' - u\bar{u}')|_a^b = 0$.

Properties of Regular SLBVP

Proof of Theorem 1

Multiplying $\bar{u}$ and u with (9) and (12) and subtracting we obtain by integrating on I

$$p(\bar{u}u' - u\bar{u}')|_a^b = (\bar{\lambda} - \lambda)\int_a^b ru\bar{u}$$

which implies $(\bar{\lambda} - \lambda)\int_a^b r|u|^2 = 0$ since $p(\bar{u}u' - u\bar{u}')|_a^b = 0$.

Conclusion

Since r is strictly positive and u is a non-trivial solution we have $\bar{\lambda} = \lambda$.

Properties of Regular SLBVP

Theorem 2

Eigenfunctions corresponding to distinct eigenvalues are orthogonal w.r.t the weight function $r(x)$ on I i.e, if u and v are eigenfunctions corresponding to distinct eigenvalues λ and μ respectively then

$$\int_a^b r(x)uv = 0$$

Proof of Theorem 2

Following the proof of the previous theorem for (λ, u) and (μ, v) we have,

$$\int_a^b [p(uv' - vu')]' + \int_a^b (\lambda - \mu) ruv = 0$$

Incorporating the boundary condition we get, $p(uv' - vu')|_a^b = 0$ and hence

$$\int_a^b ruv = 0$$

since $\lambda \neq \mu$.

Properties of Regular SLBVP

Theorem 3

The eigenvalues of a regular SLBVP are simple i.e, An eigenfunction corresponding to an eigenvalue is unique upto a constant multiple.

Conclusion

- The dimension of the space of eigenfunctions corresponding to an eigenvalue is 1.
- This is not true if we have Periodic Boundary condition.

Proof of Theorem 3

Let u and v be two eigenvalues corresponding to λ.
Following the proof of the previous theorems we can get

$$p(uv' - vu')|_a^b = c$$

where c is a constant.
The above expression implies that $p(x)\mathbb{W}(u, v)(x) = c$ for all $x \in I$.
Again from the boundary conditions we get, $\mathbb{W}(u, v)(a) = 0$.
This implies that $\mathbb{W}(u, v)(x) = 0$ for all $x \in I$.
Hence, u and v are linearly independent.

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