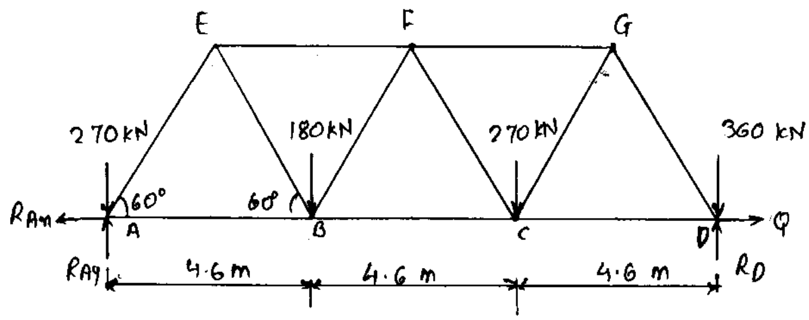
Solutions to H/W Problems on Energy Method

Solution to problem 2.7 :



To find:
Horizontal displacement of poin D.

* Assume: a fictitious horizontal force Q at D as shown in the figure

* Equilibrium:

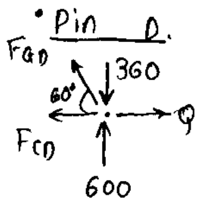
- Reaction Rd:

$\Sigma M_A = 0 \Rightarrow R_d \times 3 \times 4.6 - 360 \times 3 \times 4.6 - 270 \times 2 \times 4.6 - 180 \times 4.6 = 0.$

$\Rightarrow R_d = 600$ kN.

- Forces in the bars:

- Pin D.



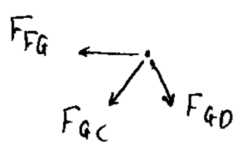
$\Sigma F_y = 0 \Rightarrow F_{GD} \sin 60 - 360 + 600 = 0$ —— (1)

$\Rightarrow F_{GD} = -277.13$ kN.

$\Sigma F_x = 0 \Rightarrow -F_{CD} + Q - F_{GD} \cos 60 = 0.$ —— (2)

$\therefore F_{CD} = Q + 138.56.$

- Pin G:



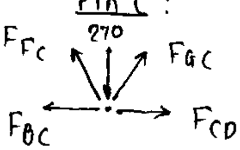
$\Sigma F_y = 0 \Rightarrow -F_{GC} \sin 60 - F_{GD} \sin 60 = 0$ —— (3)

$\Sigma F_x = 0 \Rightarrow -F_{FG} - F_{GC} \cos 60 + F_{GD} \cos 60 = 0$ —— (4)

(3) and (4) $\Rightarrow F_{GC} = 277.13$ kN

$F_{FG} = -277.13$ kN.

- Pin C :



$\Sigma F_y = 0 \Rightarrow F_{FC} \sin 60 - 270 + F_{GC} \sin 60 = 0$ —— (5)

$\Sigma F_x = 0 \Rightarrow -F_{BC} - F_{FC} \cos 60 + F_{GC} \cos 60 + F_{CD} = 0$ —— (6)

(5) and (6) $\Rightarrow F_{FC} = 34.64$ kN

$F_{BC} = (259.8 + Q)$ kN.

(problem 2.7 contd.)

- Pin F :

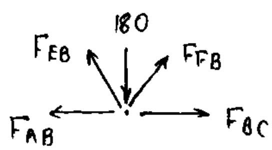
$$\Sigma F_y = 0 \Rightarrow -F_{FB}\sin 60 - F_{FC}\sin 60 = 0 \quad \text{———} (7)$$

$$\Sigma F_x = 0 \Rightarrow -F_{FE} - F_{FB}\cos 60 + F_{FC}\cos 60 + F_{FG} = 0 \quad — (8)$$

$$(7) \text{ and } (8) \Rightarrow \quad F_{FB} = -34.64 \text{ kN}$$

$$F_{FE} = -242.49 \text{ kN}.$$

- Pin B :



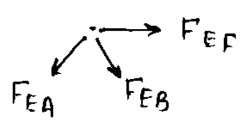
$$\Sigma F_y = 0 \Rightarrow F_{EB}\sin 60 - 180 + F_{FB}\sin 60 = 0 \quad \text{——} (9)$$

$$\Sigma F_x = 0 \Rightarrow -F_{AB} - F_{EB}\cos 60 + F_{FB}\cos 60 + F_{BC} = 0 \quad — (10)$$

$$(9) \text{ and } (10) \Rightarrow \quad F_{EB} = 242.29 \text{ kN}$$

$$F_{AB} = (121.24 + Q) \text{ kN}.$$

- Pin E :



$$\Sigma F_y = 0 \Rightarrow -F_{EA}\sin 60 - F_{EB}\sin 60 = 0 \quad \text{——} (11)$$

$$\Rightarrow F_{EA} = -242.49 \text{ kN}.$$

* Strain Energy :

$$U = \int_0^L \frac{1}{2AE}\left[F_{AE}^2 + F_{EB}^2 + F_{AB}^2 + F_{EF}^2 + F_{FB}^2 + F_{FC}^2 + F_{BC}^2 + F_{CG}^2 + F_{FG}^2 + F_{GD}^2 + F_{CD}^2\right] ds$$

$$= \int_0^L \frac{10^6}{2AE}\Big[(-242.49)^2 + (242.49)^2 + (121.24+Q)^2 + (-242.49)^2 + (-34.64)^2 + (34.64)^2 + (259.8+Q)^2 + (277.13)^2 + (-277.13)^2 + (-277.13)^2 + (138.56+Q)^2\Big] ds.$$

* Horizontal Deflection at point D

- Castigliano's theorem:

$$\delta_{DH} = \frac{\partial U}{\partial Q}\Big|_{Q=0} = \int_0^L \frac{10^3}{2AE}\left[2(121.24+Q) + 2(259.8+Q) + 2(138.56+Q)\right] ds \Big|_{Q=0}$$

$$= \frac{10^3 \times L}{AE}\left[121.24 + 259.8 + 138.56\right]$$

(problem 2.7 contd.)

$L = 4.6\ m \qquad A = 3250\ mm^2 = 3250 \times 10^{-6} m^2$

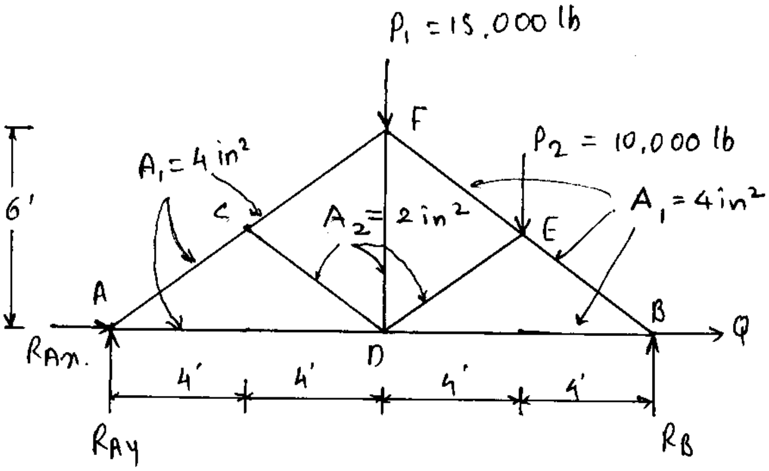
$E = 210\ GPa = 210 \times 10^9\ N/m^2$ (steel).

$$\therefore \delta_{DH} = \frac{4.6 \times 519.6 \times 10^3}{3250 \times 10^{-6} \times 210 \times 10^9}$$

$$= 3.5 \times 10^{-3}\ m$$

$$= 3.5\ mm.$$

Solution to problem 2.8 :



To find : Horizontal displacement of point B

* Assume :

A fictitious horizontal force Q at point B as shown in the figure.

* Equilibrium :

- Reactions :

$$\Sigma M_A = 0 \Rightarrow -P_1 \times 8 - P_2 \times 12 + R_B \times 16 = 0.$$

$$\therefore R_B = \frac{15{,}000 \times 8 + 10{,}000 \times 12}{16} = 15{,}000 \text{ lb.}$$

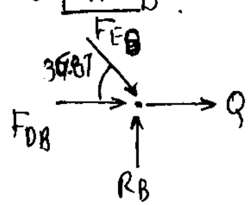
$$\Sigma F_y = 0 \Rightarrow R_{Ay} - P_1 - P_2 + R_B = 0$$

$$\Rightarrow R_{Ay} = 15{,}000 + 10000 - 15000 = 10{,}000 \text{ lb}$$

$$\Sigma F_x = 0 \Rightarrow R_{Ax} + Q = 0 \Rightarrow R_{Ax} = -Q.$$

- Determination of forces :

- Pin B :



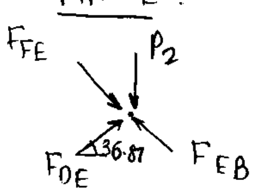
$$\Sigma F_y = 0 \Rightarrow -F_{EB} \sin 36.87 + R_B = 0 \quad \text{— 1)}$$

$$\Sigma F_x = 0 \Rightarrow F_{DB} + F_{EB} \cos 36.87 + Q = 0. \quad \text{— 2)}$$

(1) & (2) $\Rightarrow F_{EB} = 25000$ lb.,

$F_{DB} = -20\,000 - Q$

- Pin E :

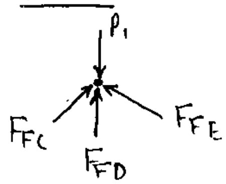


$$\Sigma F_y = 0 \Rightarrow -P_2 - F_{FE} \sin 36.87 + F_{EB} \sin 36.87 + F_{DE} \cos 36.87 = 0 \quad \text{— 3)}$$

$$\Sigma F_x = 0 \Rightarrow F_{FE} \cos 36.87 - F_{EB} \cos 36.87 + F_{DE} \cos 36.87 = 0 \quad \text{— (4)}$$

(3) & (4) $\Rightarrow F_{FE} = 16{,}670$ lb, $F_{DE} = 8330$ lb

- Pin F :

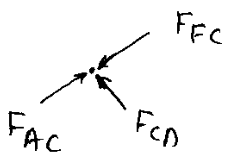


$$\Sigma F_x = 0 \Rightarrow -F_{FE}\cos 36.87 + F_{FC}\cos 36.87 = 0 \quad -5)$$

$$\Sigma F_y = 0 \Rightarrow -P_1 + F_{FE}\sin 36.87 + F_{FC}\sin 36.87 + F_{FD} = 0 \quad -6)$$

$$(5)\ \&(6) \Rightarrow F_{FC} = 16{,}670 \text{ lb},$$
$$F_{FD} = -5000 \text{ lb}.$$

- Pin C :

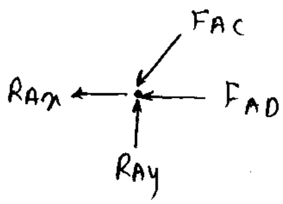


$$\Sigma F_x = 0 \Rightarrow -F_{FC}\cos 36.87 + F_{AC}\cos 36.87 - F_{CD}\cos 36.87 = 0 \quad — 7)$$

$$\Sigma F_y = 0 \Rightarrow -F_{FC}\sin 36.87 + F_{AC}\sin 36.87 + F_{CD}\sin 36.87 = 0. \quad —(8)$$

$$(7)\ \&\ (8) \Rightarrow F_{AC} = 16{,}670 \text{ lb},$$
$$F_{CD} = 0 \text{ lb}.$$

- Pin A :



$$\Sigma F_x = 0 \Rightarrow -F_{AD} - F_{AC}\overset{\cos}{\cancel{\sin}}\, 36.87 - R_{Ax} = 0 \quad —(9)$$

$$\Sigma F_y = 0 \Rightarrow -F_{AC}\overset{\sin}{\cancel{\cos}}\, 36.87 + R_{Ay} = 0. \quad —(10)$$

$$(9)\ \&(10) \Rightarrow F_{AD} = -(13{,}330 + Q),$$
$$F_{AC} = 16{,}670 \text{ lb}.$$

* Strain energy :

$$U = U_{AD} + U_{DB} + U_{AC} + U_{CD} + U_{CF} + U_{FE} + U_{FD} + U_{DE} + U_{EB}.$$

$$= \frac{1}{2}\Bigg[\frac{[-(13330+Q)]^2 96}{A_1E} + \frac{[-(20000+Q)]^2 96}{A_1E} + \frac{16{,}670^2 \times 60}{A_1E} + 0 + \frac{16{,}670^2 \times 60}{A_1E}$$
$$+ \frac{16{,}670^2 \times 60}{A_1E} + \frac{(-5000)^2 \times 72}{A_2E} + \frac{8330^2 \times 60}{A_2E} + \frac{25000^2 \times 60}{A_1E}\Bigg].$$

* Castigliano's Theorem. (Length in inches)

$$\delta_{BH} = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} \quad (\text{in } +\text{ve } x \text{ direction}).$$

$$\therefore \delta_{BH} = \frac{1}{2A_1E}\Big[96 \times 2\,(13330 + Q) + 96 \times 2\,(20000 + Q)\Big]\Big|_{Q=0}.$$

(problem 2.8 contd.)

$$\therefore \delta_{BH} = \frac{192}{2A_1E}\left[13330 + 20000\right]$$

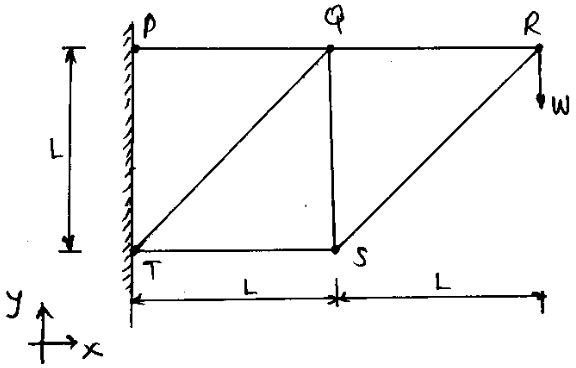
$$= \frac{192 \times 33330}{2 \times 4 \times 10 \times 10^6}$$

$\left(\begin{array}{c} E = 10 \times 10^6 \text{ psi} \\ \text{for} \\ \text{Aluminium} \end{array}\right)$

$$= .80 \times 10^{-3} \text{ in.}$$

$$= .08 \text{ in}$$

Solution to problem 2.10:



Given:

A and E are same for each bar.

To find:

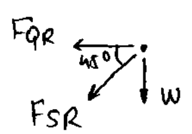
i) Forces in all the rods

ii) vertical deflection of R.

* Equilibrium:

• Determination of forces:

• **Pin R:**



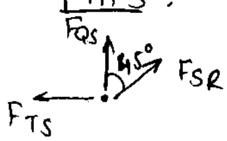
$$\Sigma F_x = 0 \Rightarrow -F_{QR} - \frac{F_{SR}}{\sqrt{2}} = 0 \quad \text{—— 1)}$$

$$\Sigma F_y = 0 \Rightarrow -W - \frac{F_{SR}}{\sqrt{2}} = 0 \quad \text{—— 2)}$$

1) and 2) $\Rightarrow$ $F_{SR} = -\sqrt{2}\,W$ (C)

$F_{QR} = W$ (T).

• **Pin S:**



$$\Sigma F_x = 0 \Rightarrow -F_{TS} + \frac{F_{SR}}{\sqrt{2}} = 0$$

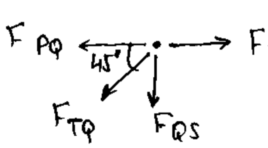
$$\Rightarrow -F_{TS} - W = 0 \quad \text{—— 3)}$$

$$\Sigma F_y = 0 \Rightarrow F_{QS} + \frac{F_{SR}}{\sqrt{2}} = 0$$

$$\Rightarrow F_{QS} - W = 0 \quad \text{—— 4)}$$

3) and 4) $\Rightarrow$ $F_{QS} = W$ (T) ; $F_{TS} = -W$ (C).

• **Pin Q:**



$$\Sigma F_x = 0 \Rightarrow -F_{PQ} - \frac{F_{QT}}{\sqrt{2}} + F_{QR} = 0$$

$$\Rightarrow -F_{PQ} - \frac{F_{QT}}{\sqrt{2}} + W = 0 \quad \text{—— 5)}$$

$$\Sigma F_y = 0 \Rightarrow -\frac{F_{QT}}{\sqrt{2}} - F_{QS} = 0 \quad \text{—— 6)}$$

5) and 6) $\Rightarrow$ $F_{QT} = -\sqrt{2}\,W$ (C) ; $F_{PQ} = 2W$ (T).

(problem 2.10 contd.)

* Strain energy:

$$U = U_{PQ} + U_{QR} + U_{TS} + U_{QT} + U_{QS} + U_{SR}.$$

$$= \frac{1}{2}\left[\frac{(2W)^2 L}{AE} + \frac{W^2 L}{AE} + \frac{(-W)^2 L}{AE} + \frac{(-\sqrt{2}W)^2\sqrt{2}L}{AE} + \frac{W^2 L}{AE} + \frac{(-\sqrt{2}W)^2\sqrt{2}L}{AE} \right]$$

$$= \frac{(7+4\sqrt{2})W^2 L}{2AE}.$$

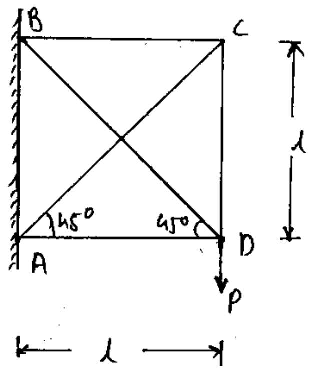
* Castigliano's Theorem:

$$\delta_{RV} = \frac{\partial U}{\partial W} \quad (\text{downward})$$

$$= \frac{(7+4\sqrt{2})\,2WL}{2AE}$$

$$= \frac{(7+4\sqrt{2})WL}{AE}.$$

Solution to problem 2.42:

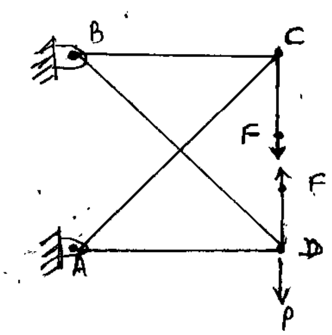


Given:

All struts are of same material and same crosssection.

To find:

load in member CD

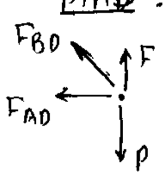


* Statically indeterminate problem.

- Cut the link CD
- Let F be the internal force in link CD
- Now, the internal forces in other links can be determined in terms of P and F

* Determination of forces:

- Pin D:



$$\Sigma F_y = 0 \Rightarrow \frac{F_{BD}}{\sqrt{2}} + F - P = 0 \quad \text{—— 1)}$$

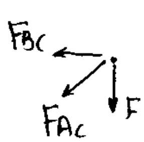
$$\Sigma F_x = 0 \Rightarrow -\frac{F_{BD}}{\sqrt{2}} - F_{AD} = 0 \quad \text{—— 2)}$$

1) and 2) $\Rightarrow$

$$F_{BD} = \sqrt{2}\,(P-F)$$

$$F_{AD} = -(P-F).$$

- Pin C:



$$\Sigma F_y = 0 \Rightarrow -\frac{F_{AC}}{\sqrt{2}} - F = 0. \quad \text{—— (3)}$$

$$\Rightarrow F_{AC} = -\sqrt{2}\,F$$

$$\Sigma F_x = 0 \Rightarrow -\frac{F_{AC}}{\sqrt{2}} - F_{BC} = 0. \quad \text{—— (4)}$$

$$\Rightarrow F_{BC} = -F$$

(Problem 2.42 contd)

* Strain Energy

$$U = U_{AD} + U_{AC} + U_{BD} + U_{BC} + U_{CD}$$

$$= \frac{1}{2AE}\left\{ [-(P-F)]^2 \ell^2 + (-\sqrt{2}F)^2 \sqrt{2}\ell + [\sqrt{2}(P-F)]^2 \sqrt{2}\ell + (-F)^2 \ell^2 + F^2 \ell^2 \right\}$$

$$= \frac{\ell^2}{2AE}\left[(1+2\sqrt{2})(P-F)^2 + (2+2\sqrt{2})F^2 \right]$$

* Castigliano's Theorem

Since F is an internal force,

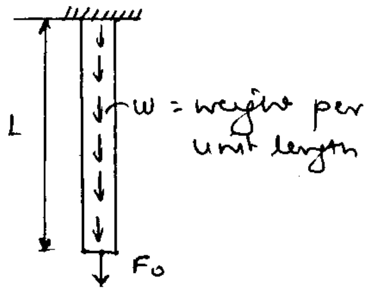
$$\frac{\partial U}{\partial F} = 0$$

$$\therefore \frac{\ell^2}{AE}\left[-2(1+2\sqrt{2})(P-F) + (2+\sqrt{2})2F \right] = 0$$

$$\therefore \frac{2\ell^2}{AE}\left[-(1+2\sqrt{2})P + (3+4\sqrt{2})F \right] = 0$$

$$\therefore F = \frac{1+2\sqrt{2}}{3+4\sqrt{2}} P$$

Solution to problem 2.45:



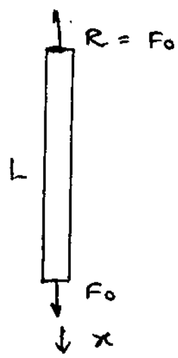
To find:

Energy stored in a bar

i) without inclusion of ω

ii) with inclusion of ω.

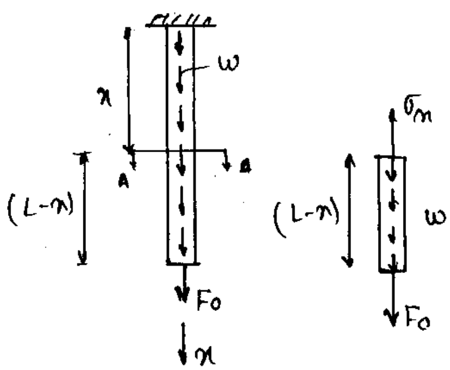
(i) Without w:



Strain Energy $= \int_L \frac{F_0^2}{2AE}\, dx$

$= \frac{F_0^2 L}{2AE}$.

(ii) with w:



Consider the lower part of bar after cutting thr' at a distance ... from fixed end by plane AA.

Equilibrium gives:

$$-\sigma_x A + \omega(L-x) + F_0 = 0.$$

$$\therefore \sigma_x = \frac{F_0 + \omega(L-x)}{A}.$$

Strain Energy:

$$U = \int_0^L \frac{(\sigma_x)^2}{2E} . A\, dx$$

$$= \int_0^L \frac{[F_0 + \omega(L-x)]^2}{2AE}\, dx.$$

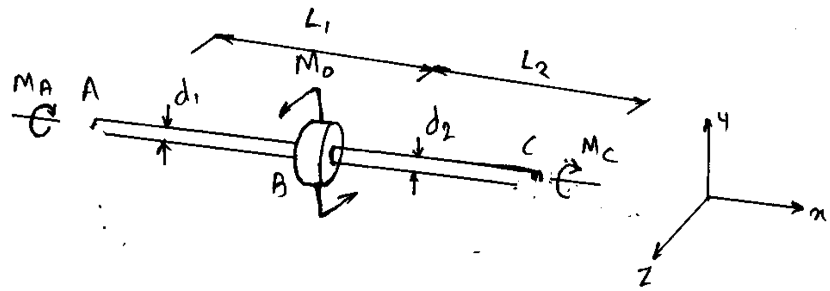
(problem 2.45 contd.)

$$\therefore \; U = \frac{1}{2AE}\left\{ F_0^2 x + 2F_0 w\left(Lx - \frac{x^2}{2}\right) + w^2\left(L^2 x - 2L\frac{x^2}{2} + \frac{x^3}{3}\right)\right\}\Big|_0^L$$

$$= \frac{1}{2AE}\left[F_0^2 L + 2F_0 w\left(L^2 - \frac{L^2}{2}\right) + w^2\left(L^3 - L^3 + L^3/3\right)\right]$$

$$= \frac{1}{2AE}\left[F_0^2 L + F_0 w L^2 + w^2 L^3/3\right].$$

Solution to problem 6.10:



Note: Force reactions at the supports not shown

To find: Twisting couples exerted on ends of shaft A and C.

* We will assume an external couple applied at $C = M_c = M$.

Then. $\Sigma M_x = 0 \Rightarrow M_A = M_0 - M_c$

* Strain energy:

$$U = \int_0^{L_1} \frac{M_A^2}{2GI_{z_1}} dz + \int_0^{L_2} \frac{(M_c)^2}{2GI_{z_2}} dz.$$

$$= \frac{M_A^2}{2GI_{z_1}} L_1 + \frac{M_c^2}{2GI_{z_2}} L_2.$$

$$= \frac{(M_0 - M_c)^2}{2GI_{z_1}} L_1 + \frac{M_c^2}{2GI_{z_2}} L_2.$$

* Compatibility:

As end C is fixed

$$\frac{\partial U}{\partial M_c} = 0 \Rightarrow \frac{L_1}{2GI_{z_1}}\left[2(M_0 - M_c)(-1)\right] + \frac{L_2}{2GI_{z_2}} \cdot 2M_c = 0.$$

$$\therefore M_c = \frac{M_0 \, L_1/GI_{z_1}}{\dfrac{L_2}{GI_{z_2}} + \dfrac{L_1}{GI_{z_1}}}$$

$$= \frac{M_0}{1 + \dfrac{L_2}{L_1}\dfrac{I_{z_1}}{I_{z_2}}}$$

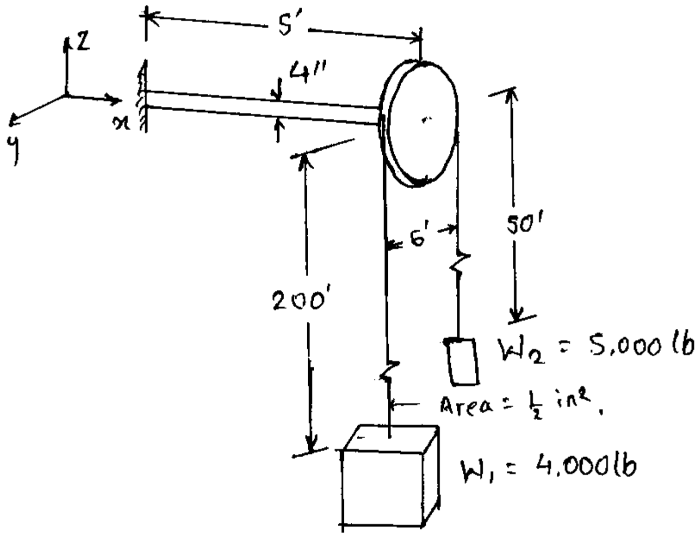
$$\therefore \quad M_c = \frac{M_0}{1 + \frac{L_2}{L_1}\left(\frac{d_1}{d_2}\right)^4}$$

$$M_A = M_0 - \frac{M_0}{1 + \frac{L_2}{L_1}\left(\frac{d_1}{d_2}\right)^4}.$$

$$= M_0\left[1 - \frac{1}{1 + \frac{L_2}{L_1}\left(\frac{d_1}{d_2}\right)^4}\right].$$

$$= \frac{M_0}{1 + \frac{L_1}{L_2}\left(\frac{d_2}{d_1}\right)^4}.$$

Solution to problem 6.23



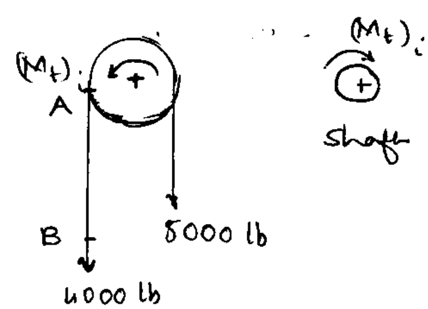
Given:

displacement of cage after people stepped in = 0.2"

To find:

Weight of people.

- Before passengers stepped in:



* Torque on shaft :

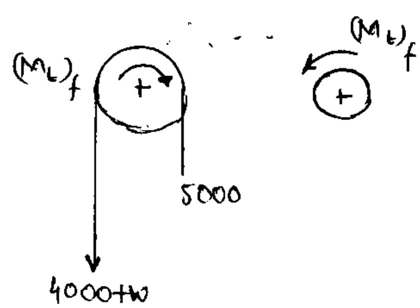
$$(M_t)_i = (5000 - 4000) \times 3$$

$$= 3000 \text{ lb.ft} \quad \underline{(CW)}$$

* Tension in cable

$$T_i = 4000 \text{ lb}$$

- After passengers stepped in:



* Torque on shaft:

$$(M_t)_f = (4000 + W - 5000) \times 3$$

$$= 3W - 3000 \text{ lb.ft} \quad \underline{CCW}$$

* Tension in cable

$$T_f = 4000 + W \text{ lb}$$

- Increase in torque (ΔM_t) = $3W$ lb.ft. $\underline{(CCW)}$
- Increase in Tension (ΔT) = W lb

(problem 6.23 contd.)

- Increase in strain energy :

$$\Delta U = \int_0^{5\times12} \frac{\Delta m_t^2\, dx}{2GI_p} + \int_0^{200\times12} \frac{\Delta T_1^2}{2AE}\, dz$$

$$= \int_0^{60} \frac{(3W\times12)^2}{2GI_p} dx + \int_0^{2400} \frac{W^2}{2AE}\, dz$$

$$= \frac{60\times(36)^2 W^2}{2GI_p} + \frac{2400 W^2}{2AE}.$$

- deflection due to additional load W :

* Castigliano's Theorem:⇒

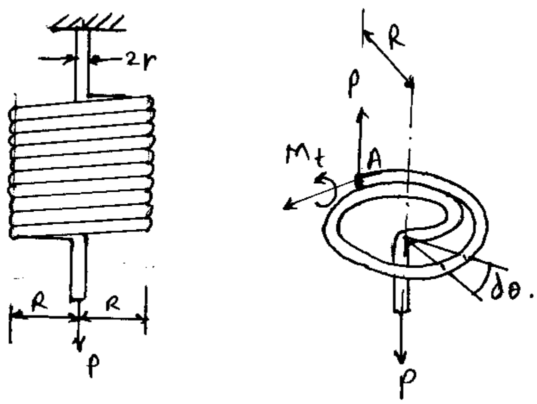
$$\Delta\delta = \frac{\partial \Delta U}{\partial W} = \frac{60\times36^2 W}{GI_p} + \frac{2400 W}{AE}.$$

$$\therefore 0.2 = \left(\frac{60\times36^2}{12\times10^6\times\frac{\pi}{32}\cdot4^4} + \frac{2400}{\frac{1}{2}\times30\times10^6}\right)W.$$

$$\therefore W = 478.66 \text{ lb.}$$

- Steel
  $E = 30\times10^6$ psi
  $G = 12\times10^6$ psi
- Cable
  $A = \frac{1}{2}$ in^2
- Shaft
  $d = 4''$

Solution to problem 6.30



* Take a section at any point A along the spring. The F.B.D. of the part is shown in adjescent figure. Note that twisting moment M_t is ~~indepent on~~ independent of the position of point A on spring.

Equilibrium gives

$$M_t = PR.$$

* Strain energy : (due to twisting moment M_t. only)

$$U = \int_0^L \frac{M_t^2}{2GI_z}\, dz$$

$$= \int_0^{2\pi n} \frac{(PR)^2}{2GI_z} R\, d\theta. \quad \text{where, } n = \text{no. of turns.}$$

$$= \frac{P^2R^3}{2GI_z} 2\pi n$$

[The strain energy due to shear force P is neglected because of the assumption: deflection of spring is due primarily to twisting.]

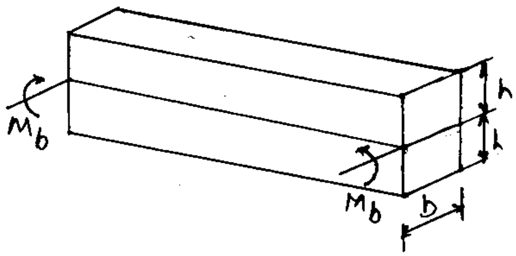
*. deflection in the direction of P: Castigliano's Theorem ⇒

$$\delta = \frac{\partial U}{\partial P} = \frac{PR^3}{GI_z} 2\pi n.$$

$$= \frac{PR^3\, 2\pi n}{G\, \frac{\pi r^4}{2}} = \frac{4PR^3 n}{Gr^4}.$$

Solution to problem 7.29:

(a) (I) When soldered togather the two bars behave as one solid bar of height 2h.



Strain energy:

$$* \quad U = \frac{M_b^2 L}{2E(I_{zz})_c}$$

$$* \quad L\frac{d\phi}{ds} = \frac{\partial U}{\partial M_b}$$

$$\therefore \cancel{L}\frac{d\phi}{ds} = \frac{\cancel{2} M_b \cancel{L}}{\cancel{2}E(I_{zz})_c}$$

[Pure bending:

$$U = \int_0^L \tfrac{1}{2} M_b\, d\phi$$

$$= \int_0^L \tfrac{1}{2} M_b \frac{d\phi}{ds}\, dx$$

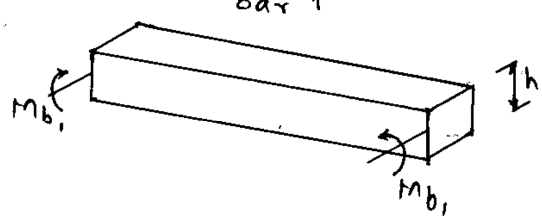
$$= \tfrac{1}{2} M_b \frac{d\phi}{ds} L$$
]

$$* \quad \frac{M_b}{d\phi/ds} = E(I_{zz})_c = \frac{2}{3}Ebh^3.$$

$\therefore$ stiffness: $(K_b)_c = \frac{2}{3}Ebh^3$. — (I)

(II) When two bars are considered separately:

bar 1



* Suppose that total bending moment M_b is ~~devided~~ shared ~~for~~ by ~~each~~ two bars as

$$M_{b_1} + M_{b_2} = M_b. \quad \text{——} \; 1)$$

* for bar 1: $\quad U_1 = \dfrac{M_{b_1}^2 L}{2E(I_{zz})_s}$

Similarly for bar 2: $\quad U_2 = \dfrac{M_{b_2}^2 L}{2E(I_{zz})_s}$.

$$* \quad L\left(\frac{d\phi}{ds}\right)_1 = \frac{\partial U_1}{\partial M_{b_1}} = \frac{2M_{b_1}L}{2E(I_{zz})_s}$$

$$\text{or} \quad \left(\frac{d\phi}{ds}\right)_1 = \frac{M_{b_1}}{E(I_{zz})_s} \quad \text{——} \; (a)$$

(problem 7.29 contd.)

Similarly $\left(\frac{d\phi}{ds}\right)_2 = \frac{M_{b_2}}{E(I_{zz})_s}$ —— (b)

* The curvatures of both beams will be almost same.

$\therefore$ Compatibility:

$$\left(\frac{\partial\phi}{\partial s}\right)_1 = \left(\frac{\partial\phi}{\partial s}\right)_2$$

(a) & (b) $\Rightarrow M_{b_1} = M_{b_2}$. — 2)

* From equations 1) and 2)

$$M_{b_1} = M_{b_2} = \frac{M_b}{2}.$$

* Stiffness in separate beams case:

$$(k_b)_s = \frac{M_b}{d\phi/ds} = \frac{2(M_b)_1}{(d\phi/ds)_1} = 2E(I_{zz})_s \quad \text{from (a)}$$

$$= 2E\frac{1}{12}bh^3$$

$\therefore (k_b)_s = \frac{1}{6}Ebh^3$ —— (II)

(III). ratio of stiffness for the two cases:

$$\text{(I) \& (II)} \Rightarrow \frac{(k_b)_c}{(k_b)_s} = \frac{\frac{2}{3}Ebh^3}{\frac{1}{6}Ebh^3} = 4.$$

b) * The ratio of maximum bending stress is equal to the ratio of maximum bending strains.

Max. bending strain is given by the expression

$$\text{max. strain} = \frac{d\phi}{ds}\left(\frac{1}{2}\cdot\text{beam height}\right).$$

(problem 7.29 contd.)

* For same bending moment M_b the maximum strains will be.

$$(\epsilon_{max})_c = \frac{d\phi}{ds} \times \left(\frac{1}{2} \times 2h\right) = \frac{M_b}{(k_b)_c} \cdot h$$

$$(\epsilon_{max})_s = \frac{d\phi}{ds} \times \left(\frac{1}{2} \times h\right) = \frac{M_b}{(k_b)_s} \cdot \frac{h}{2}.$$

* ratio of maximum stress :

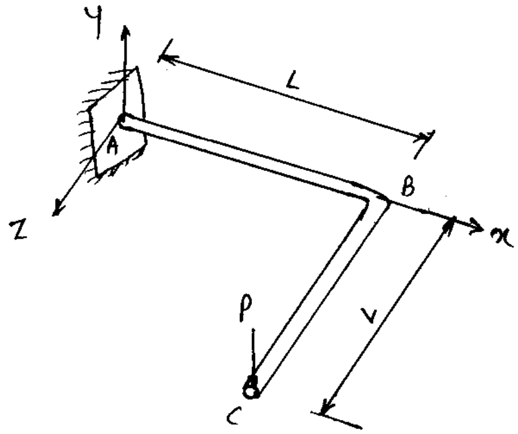
$$\frac{(\sigma_{max})_c}{(\sigma_{max})_s} = \frac{\frac{M_b}{(k_b)_c} \cdot h}{\frac{M_b}{(k_b)_s} \cdot \frac{h}{2}} = 2 \cdot \left(\frac{k_{b_s}}{k_{b_c}}\right)$$

$$= 2 \cdot \frac{1}{4}$$

$$= \frac{1}{2}.$$

(Note: Part (b) cannot be done by the energy method)

Solution to problem 8.6:



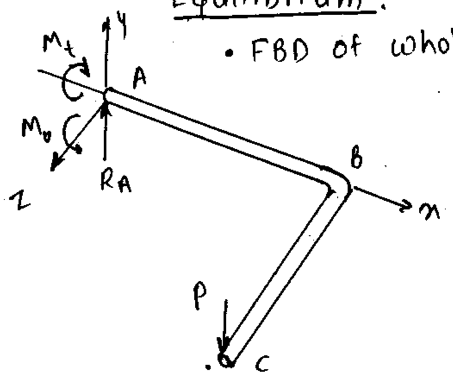
Given:
- Material properties: E, ν
- Radius of bar $= r$.

To find:

deflection at C.

* Equilibrium:

- FBD of whole rod:

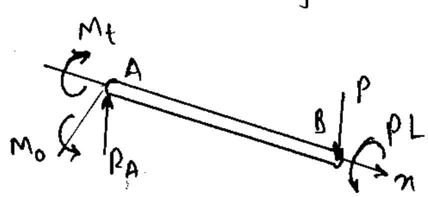


$$\Sigma F_y = 0 \Rightarrow R_A = P.$$

$$\Sigma M_x = 0 \Rightarrow M_t = PL$$

$$\Sigma M_z = 0 \Rightarrow M_0 = PL.$$

* Bending and Twisting moment expressions:



- Part AB

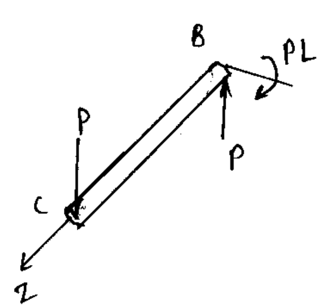
Bending Moment:

$$M_b = -M_0 + R_A x$$

$$= P(x - L) \qquad 0 \leq x \leq L.$$

Twisting moment:

$$M_t = PL \qquad 0 \leq x \leq L.$$



- Part BC:

Bending moment only:

$$M_b = -PL + Pz$$

$$= P(z - L) \qquad 0 \leq z \leq L.$$

(problem 8.6 contd.)

* Strain energy:

$$U = \int_0^L \frac{M_b^2}{2EI}\,dx + \underbrace{\int_0^L \frac{M_t^2}{2GI_p}\,dx}_{\text{part AB}} + \underbrace{\int_0^L \frac{M_b^2}{2EI}\,dz}_{\text{part BC}}$$

$$= \frac{1}{2EI}\int_0^L P^2(x^2 - 2Lx + L^2)\,dx + \frac{1}{2GI_p}\int_0^L P^2L^2\,dx$$

$$+ \frac{1}{2EI}\int_0^L P^2(z^2 - 2Lz + L^2)\,dz$$

$$= \frac{1}{2EI}P^2\left(\frac{L^3}{3} - 2L\frac{L^2}{2} + L^2 \cdot L\right) + \frac{1}{2GI_p}P^2L^2 \cdot L$$

$$+ \frac{1}{2EI}P^2\left(\frac{L^3}{3} - 2L\frac{L^2}{2} + L^2 L\right)$$

$$= \frac{P^2L^3}{3EI} + \frac{P^2L^3}{2GI_p}.$$

- Substitute $I_p = 2I$, $G = \dfrac{E}{2(1+\nu)}$.

$$U = \frac{P^2L^3}{3EI} + \frac{P^2L^3 \cdot \cancel{2}(1+\nu)}{\cancel{2}E\,(2I)} = \frac{P^2L^3}{6EI}(5+3\nu)$$

- Substitute $I = \frac{1}{4}\pi r^4$

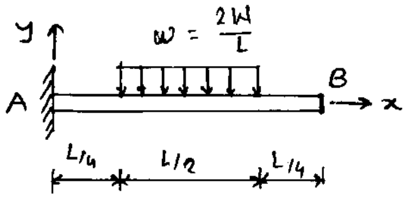
$$\therefore U = \frac{4P^2L^3}{6\pi E r^4}(5+3\nu)$$

* Deflection under P:

$$v|_P = \frac{\partial U}{\partial P} = \frac{4(2P)L^3}{6\pi E r^4}(5+3\nu)$$

$$= \frac{4}{3}\frac{PL^3}{\pi E r^4}(5+3\nu).$$

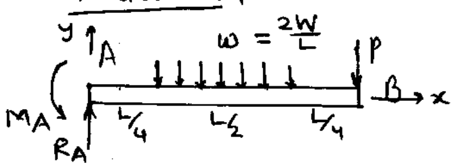
Solution to problem 8.12



To find: deflection at right end.

* Apply fictitious load P at the right end.

* **Reactions**:



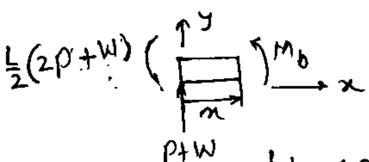
$$\Sigma F_y = 0 \Rightarrow R_A = P + W$$

$$\Sigma M_A = 0 \Rightarrow M_A - W \cdot \frac{L}{2} - P \cdot L = 0.$$

$$\therefore M_A = \frac{L}{2}(2P + W).$$

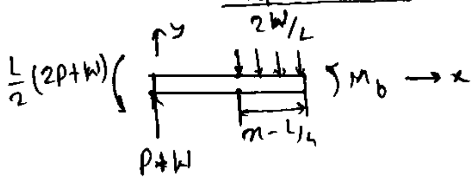
* **Bending Moment**:

• $0 < x < L/4$



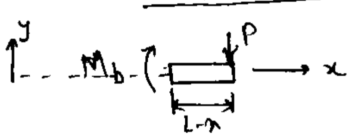
$$M_b = (P+W)x - \frac{L}{2}(2P+W)$$

• $L/4 < x < 3L/4$



$$M_b = (P+W)x - \frac{L}{2}(2P+W) - \frac{W}{L}(x - L/4)^2$$

• $3L/4 < x < L$



$$M_b = -P(L-x).$$

* **Strain Energy**:

$$U = \int_0^{L/4} \frac{\left[(P+W)x - \frac{L}{2}(2P+W)\right]^2}{2EI} dx + \int_{L/4}^{3L/4} \frac{\left[(P+W)x - \frac{L}{2}(2P+W) - \frac{W}{L}(x - L/4)^2\right]^2}{2EI} dx$$

$$+ \int_{3L/4}^{L} \frac{\left[-P(L-x)\right]^2}{2EI} dx ,$$

(problem 8.12 contd.)

Deflection at right end:

$$\delta = \frac{\partial U}{\partial P}\bigg|_{P=0}.$$

$$= \int_0^{L/4} \frac{2\left[(P+W)x - \frac{L}{2}(2P+W)\right](x-L)}{2EI}\,dx\,\bigg|_{P=0}$$

$$+ \int_{L/4}^{3L/4} \frac{2\left[(P+W)x - \frac{L}{2}(2P+W) - \frac{W}{L}(x-L/4)^2\right](x-L)}{2EI}\,dx\,\bigg|_{P=0}$$

$$+ \int_{3L/4}^{L} \frac{2\left[-P(L-x)\right](L-x)}{2EI}\,dx\,\bigg|_{P=0}.$$

$$= \int_0^{L/4} \frac{(Wx - \frac{L}{2}W)(x-L)}{EI}\,dx + \int_{L/4}^{3L/4} \frac{\left[Wx - \frac{L}{2}W - \frac{W}{L}(x-L/4)^2\right](x-L)}{EI}\,dx + 0$$

$$= \frac{1}{EI}\left[\frac{Wx^3}{3} - \frac{L}{2}W\frac{x^2}{2} - WL\frac{x^2}{2} + W\frac{L^2}{2}x\right]_0^{L/4}$$

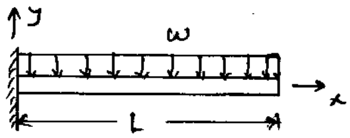
$$+ \frac{1}{EI}\left[W\frac{x^3}{3} - \frac{3WL}{2}\frac{x^2}{2} - \frac{W}{L}\left(\frac{x^4}{4} - \frac{3L}{2}\frac{x^3}{3} + \frac{9L^2}{16}\frac{x^2}{2}\right) + \frac{L^2}{2}Wx + \frac{W}{L}\left(\frac{L^3}{16}x\right)\right]_{L/4}^{3L/4}$$

$$= \frac{WL^3}{EI}\left[\frac{27}{192} - \frac{25}{64} + \frac{3}{8} - \frac{112}{52} + \frac{78}{384}\right]$$

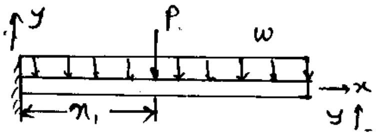
$$= \frac{7WL^3}{64EI}.$$

Solution to problem 8.15 :

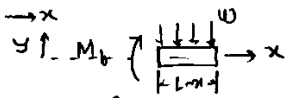
$\boxed{\text{Case 1}}$:



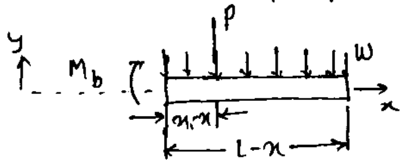
* consider a fictitious force P at a distance x_1 from the fixed end.



* Bending Moments :



$$M_b = -\frac{w(L-x)^2}{2} \qquad (x_1 < x < L)$$



$$M_b = -w\frac{(L-x)^2}{2} - P(x_1 - x). \qquad (0 < x < x_1)$$

* Strain energy :

$$U = \int_0^{x_1}\left[-\frac{w(L-x)^2}{2} - P(x_1-x)\right]^2 \times \frac{1}{2EI}\,dx + \int_{x_1}^{L} -\frac{w(L-x)^2}{2}\cdot\frac{1}{2EI}\,dx$$

* deflection at point x_1 :

$$v = -\frac{\partial U}{\partial P}\bigg|_{P=0} = \int_0^{x_1}\frac{1}{EI}\left(\frac{w(L-x)^2}{2} + P(x_1-x)\right)(x_1-x)\,dx\bigg|_{P=0} + 0.$$

$$= \int_0^{x_1}\frac{1}{2EI}\,w(L-x)^2(x_1-x)\,dx.$$

$$= \frac{w}{2EI}\int_0^{x_1}(L^2x_1 - 2Lx_1x + x_1x^2 - L^2x + 2Lx^2 - x^3)\,dx.$$

$$= \frac{w}{2EI}\left(L^2x_1^2 - 2L\frac{x_1^3}{2} + \frac{x_1^4}{3} - L^2\frac{x_1^2}{2} + 2L\frac{x_1^3}{3} - \frac{x_1^4}{4}\right)$$

$$= \frac{w}{2EI}\left(\frac{L^2x_1^2}{2} - \frac{Lx_1^3}{3} + \frac{x_1^4}{12}\right).$$

Replacing x_1 by x to get expression for deflection we get

$$v = -\frac{w}{2EI}\left(\frac{1}{2}L^2x^2 - \frac{1}{3}Lx^3 + \frac{1}{12}x^4\right).$$

$$* \; v_{max}\Big|_{x=L} = -\frac{w}{2EI}\left[\frac{L^4}{2} - \frac{L^4}{3} + \frac{L^4}{12}\right] = -\frac{wL^4}{8EI}.$$

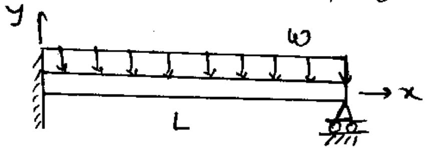
$$|v_{max}| = \frac{wL^4}{8EI}.$$

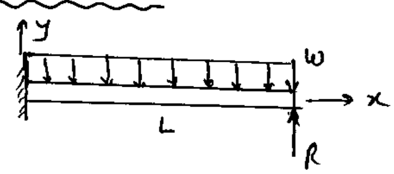
$$* \; M_{b\,max}\Big|_{x=0} = -\frac{wL^2}{2}$$

$$\therefore \; |M_{b\,max}| = \frac{wL^2}{2}.$$

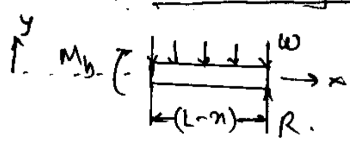
$\boxed{\text{Case 2}}$:

- First we will find the value of R. (Reaction at support end).





* Bending Moments :



$$M_b = -w\frac{(L-x)^2}{2} + R(L-x).$$

* Strain energy :

$$U = \int_0^L \frac{1}{2EI}\left[-w\frac{(L-x)^2}{2} + R(L-x)\right]^2 dx.$$

* Castigliano's Theorem :

$$\delta = \frac{\partial U}{\partial R} = \int_0^L \frac{1}{EI}\left[-w\frac{(L-x)^2}{2} + R(L-x)\right](L-x)\,dx.$$

$$= \int_0^L \frac{1}{EI}\left(-\frac{w(L-x)^3}{2} + R(L-x)^2\right] dx.$$

$$= \frac{1}{EI}\left(-w\left(L^3.L - 3L^2.\frac{L^2}{2} + 3L.\frac{L^3}{3} - \frac{L^4}{4}\right) + R\left(L^2.L - 2L.\frac{L^2}{2} + \frac{L^3}{3}\right)\right).$$

$$= \frac{1}{EI}\left[-w\left(\frac{1}{8}L^4\right) + R.\frac{L^3}{3}\right].$$

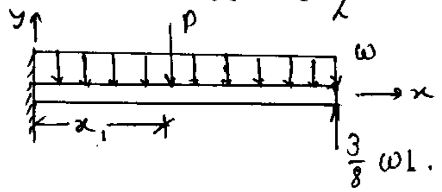
(problem 8.15 (contd.)

* compatibility:

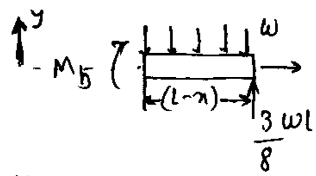
$$\delta = 0 \Rightarrow \frac{1}{EI}\left[-\omega\left(\frac{L^4}{8}\right) + R\left(\frac{L^3}{3}\right)\right] = 0.$$

$$\Rightarrow R = \frac{3}{8}\omega L.$$

- Now consider a fictitious load P acting at x_1 from fixed end.

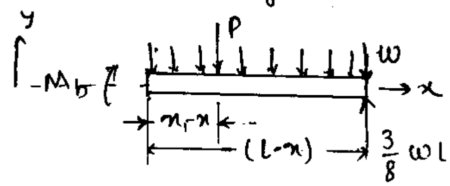


* Bending Moments:



$x_1 < x < L$

$$M_b = -\omega\frac{(L-x)^2}{2} + \frac{3}{8}\omega L\,(L-x).$$



$0 < x < x_1$

$$M_b = -\frac{\omega(L-x)^2}{2} + \frac{3}{8}\omega L\,(L-x) - P\,(x_1 - x).$$

* strain energy:

$$U = \int_0^{x_1} \frac{1}{2EI}\left[-\omega\frac{(L-x)^2}{2} + \frac{3}{8}\omega L(L-x) - P(x_1-x)\right]^2 dx$$

$$+ \int_{x_1}^{L} \frac{1}{2EI}\left[-\omega\frac{(L-x)^2}{2} + \frac{3}{8}\omega L\,(L-x)\right]^2 dx.$$

* Deflection at x_1 :

$$v = -\frac{\partial U}{\partial P}\Big|_{P=0} = \int_0^{x_1} \frac{2}{2EI}\left[-\frac{\omega(L-x)^2}{2} + \frac{3}{8}\omega L\,(L-x)\right]\left[-(x_1-x)\right] dx.$$

$$= \int_0^{x_1} \frac{1}{EI}\left[\frac{\omega}{2}(x_1-x)(L^2 - 2Lx + x^2) - \frac{3\omega L}{8}(x_1-x)(L-x)\right] dx$$

$$= \frac{1}{EI}\left[\frac{\omega}{2}\left(L^2x_1^2 - \frac{L^2x_1^2}{2} + \frac{x_1^4}{3} - \frac{x_1^4}{4} - \frac{2Lx_1^3}{2} + \frac{2Lx_1^3}{3}\right)\right.$$

$$\left. - \frac{3\omega L}{8}\left(Lx_1^2 - (x_1+L)\frac{x_1^2}{2} + x_1^3/3\right)\right].$$

(problem 8.15 contd.)

$$v = -\frac{1}{EI} w \left[\frac{x_1^4}{24} - \frac{5}{48} x_1^3 L + \frac{1}{16} L^2 x_1^2 \right].$$

Replacing x_1 by x we get the expression for v as

$$v = -\frac{w}{EI} \left(\frac{x^4}{24} - \frac{5}{48} x^3 L + \frac{1}{16} L^2 x^2 \right).$$

* Maximum deflection:

maximum deflection will be at

$$\frac{\partial v}{\partial x} = 0 \Rightarrow -\frac{w}{EI} \left(\frac{4x^3}{24} - \frac{15 x^2}{48} L + \frac{2}{16} L^2 x \right) = 0.$$

$$\Rightarrow x = 0.578 L, \ 1.3 L.$$

$x = 0.578 L$ is the only possible solution.

$$\therefore v_{max} = -\frac{w}{EI} \left[\frac{(0.578 L)^4}{24} - \frac{5}{48} (0.578 L)^3 L + \frac{1}{16} L^2 (0.578 L)^2 \right].$$

$$= -0.26 \frac{WL^4}{48EI}$$

$$|v_{max}| = 0.26 \frac{WL^4}{48EI}.$$

* Max. Bending moment:

Bending moment equation for $P=0$ is

$$M_b = -\frac{w(L-x)^2}{2} + \frac{3}{8} wL(L-x).$$

For extremum value of M_b: It will occur at pt. where

$$\frac{\partial M_b}{\partial x} = 0 \Rightarrow w(L-x) - \frac{3}{8} wL = 0. \Rightarrow x = \frac{5}{8} L.$$

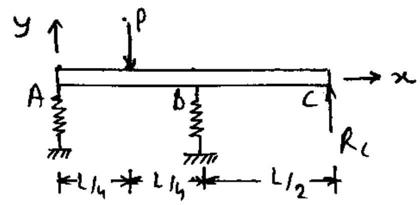
$$\therefore M_{b\,max}\Big|_{x=\frac{5}{8}L} = 0.07 wL^2.$$

But M_b at $x=0$ $= \frac{wL^2}{8}$

$$\therefore M_{b\,max} = \frac{wL^2}{8} \text{ at } x=0.$$

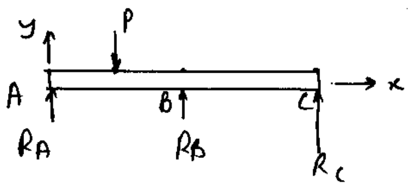
Solution to problem 8.20:

* Replace the spring c by a force.



Assume: all springs are in compression.

F.B.D.



* Equilibrium:

$\Sigma F_y = 0 \Rightarrow R_A + R_B + R_c = P$ — 1)

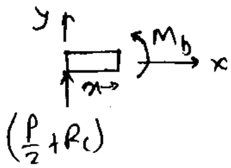
$\Sigma M_A = 0 \Rightarrow \frac{L}{2} R_B = L/4\, P - R_c L$ — 2)

② $\Rightarrow R_B = \frac{P}{2} - 2R_c$ —— 3)

① & ③ $\Rightarrow R_A = \frac{P}{2} + R_c$.

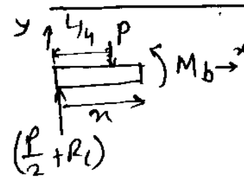
* Bending moments:

$0 < \frac{L}{4} < x$



$M_b = \left(\frac{P}{2} + R_c\right) x$.

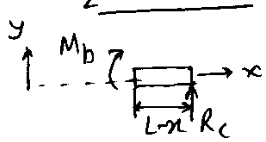
$\frac{L}{4} < x < \frac{L}{2}$



$M_b = \left(\frac{P}{2} + R_c\right)x - P\left(x - \frac{L}{4}\right)$

$= P\frac{L}{4} + \left(R_c - \frac{P}{2}\right)x$.

$\frac{L}{2} < x < L$



$M_b = R_c(L - x)$

* Strain energy (of the beam):

$$U = \frac{1}{2EI}\left[\int_0^{L/4}\left(\frac{P}{2} + R\right)^2 x^2\,dx + \int_{L/4}^{L/2}\left[\frac{PL}{4} + (R_c - P/2)x\right]^2 dx + \int_{L/2}^{L} R_c^2 (L-x)^2\,dx.\right.$$

$$= \frac{1}{2EI}\left[\left(\frac{P}{2} + R_c\right)^2 \frac{(L/4)^3}{3} + \frac{P^2L^2}{16}\cdot\frac{L}{4} + \frac{2PL}{4}\left(R_c - \frac{P}{2}\right)\frac{x^2}{2}\Big|_{L/4}^{L/2} + (R_c - P_2)^2\frac{x^3}{3}\Big|_{L/4}^{L/2} + R_c^2\frac{(L-x)^3}{3}\Big|_{L/2}^{L}\right.$$

(problem 8.20 contd.)

$$\therefore U = \frac{L^3}{768\,EI}\left[32R_c^2 + 6PR_c + P^2\right]$$

* Total strain energy (Including that of springs):

$$U_T = \frac{L^3}{768\,EI}\left(32R_c^2 + 6PR_c + P^2\right) + \frac{R_A^2}{2k} + \frac{R_B^2}{2K}.$$

$$= \frac{L^3}{768\,EI}\left[32R_c^2 + 6PR_c + P^2\right] + \frac{1}{2k}\left[\frac{P^2}{4} + PR_c + R_c^2 + \frac{P^2}{4} - 2PR_c + 4R_c^2\right]$$

$$= \frac{L^3}{768\,EI}\left[32R_c^2 + 6PR_c + P^2\right] + \frac{1}{4k}\left[10R_c^2 - 2PR_c + P^2\right].$$

* Castigliano's Theorem:

$$\frac{\partial U_T}{\partial R_c} = \frac{L^3}{768\,EI}\left[64R_c + 6P\right] + \frac{1}{4k}\left[20R_c - 2P\right].$$

* Compatibility:

$$\frac{\partial U}{\partial R_c} = -\frac{R_c}{k}.$$

$$\therefore \frac{L^3}{384\,EI}\left(32R_c + 3P\right) + \frac{1}{2k}\left(10R_c - P\right) = -\frac{R_c}{k}.$$

$$(32R_c + 3P) + \frac{384}{2}\frac{EI}{kL^3}\left(10R_c - P\right) = -384\frac{EI}{kL^3}R_c$$

Let $\alpha = \frac{EI}{kL^3}$

$$\therefore 32R_c + 3P + 192\alpha\left(10R_c - P\right) = -384\alpha R_c.$$

$$\therefore R_c = \frac{(-3 + 192\alpha)}{(32 + 2304\alpha)}P.$$

(problem 8.20 contd.)

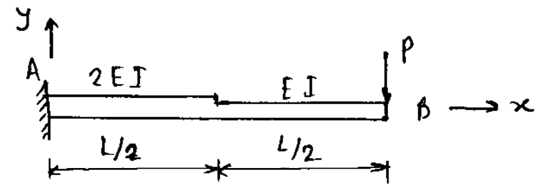
$$R_B = \frac{P}{2} - 2R_C = \frac{P}{2} + \frac{6 - 384\alpha}{32 + 2304\alpha} P.$$

$$= \frac{22 + 768\alpha}{32 + 2304\alpha} P.$$

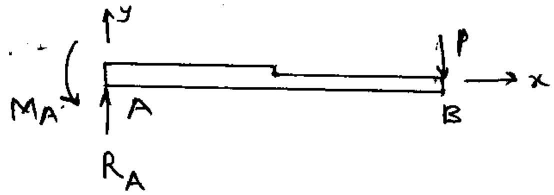
$$R_A = \frac{P}{2} + R_C = \frac{P}{2} + \left(-\frac{(3 - 192\alpha)}{32 + 2304\alpha} P\right)$$

$$= \frac{13 + 1344\alpha}{32 + 2304\alpha} P.$$

Solution to problem 8·22 :



* Reactions :

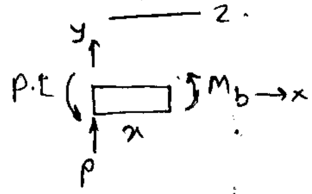


$M_A = P \cdot L$

$R_A = P.$

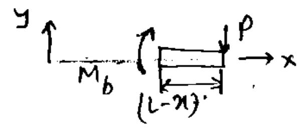
* Bending moments :

• $0 < x < \frac{L}{2}$.



$M_b = P(x - L)$

• $\frac{L}{2} < x < L$



$M_b = -P(L - x)$.
$= P(x - L)$

* Strain energy :

$$U = \int_0^{L/2} \frac{M_b^2}{2(2EI)} dx + \int_{L/2}^{L} \frac{M_b^2}{2EI} dx.$$

$$= \int_0^{L/2} \frac{1}{4EI} [P^2 (x - L)^2] dx + \int_{L/2}^{L} \frac{1}{2EI} [P^2 (-)^2]$$

$$= \frac{P^2}{4EI} \left[\frac{x^3}{3} - Lx^2 + L^2 x \right]_0^{L/2} + \frac{P^2}{2EI} \left[\frac{x^3}{3} - Lx^2 + L^2 x \right]_{L/2}^{L}$$

$$= \frac{P^2}{4EI} \left[\frac{L^3}{3 \cdot 2^3} - L \frac{L^2}{2^2} + L^2 \frac{L}{2} \right] + \frac{P^2}{2EI} \left[\frac{L^3}{3} - \frac{L^3}{3 \cdot 2^3} - \cancel{L^3} + \frac{L^3}{2^2} + L^3 - \frac{L^3}{2} \right]$$

$$= \frac{9 P^2 L^3}{96 EI}$$

* Castigliano's Theorem : $\delta = \frac{\partial U}{\partial P} = \frac{9 P L^3}{48 EI} = \frac{3 PL^3}{16 EI}$.