Canonical Form

MSO-203B

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Canonical Form for 2nd Order linear PDE

Definition

Consider the 2nd Order linear PDE:

$$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{1}$$

We have seen the existence of a C^1 diffeomorphic change of variable such that $Lu = g$ is tranformed into

$$\bar{L}(w) = Aw_{\theta\theta} + 2Bw_{\theta\eta} + Cw_{\eta\eta} + Dw_{\theta} + Ew_{\eta} + Fw = G$$

where,

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where,

$$\begin{aligned} A(\theta, \eta) &= a{\theta_x}^2 + 2b\theta_x\theta_y + c{\theta_y}^2 \\ B(\theta, \eta) &= a\theta_x\eta_x + b(\theta_x\eta_y + \eta_x\theta_y) + c\eta_y\theta_y \\ C(\theta, \eta) &= a{\eta_x}^2 + 2b\eta_x\eta_y + c{\eta_y}^2 \end{aligned}$$

Definitions

- There exists a change of variable $(x, y) \to (\theta, \eta)$ such that if equation (1) is Hyperbolic then it can be reduced to $w_{\theta\eta} + l(w) = h$ or $w_{\theta\theta} - w_{\eta\eta} + l(w) = h$.

Canonical Form

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- There exists a change of variable $(x, y) \to (\theta, \eta)$ such that if equation (1) is Parabolic then it can be reduced to $w_{\eta\eta} + l(w) = h$.

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- There exists a change of variable $(x, y) \to (\theta, \eta)$ such that if equation (1) is Parabolic then it can be reduced to $w_{\eta\eta} + l(w) = h$.
- There exists a change of variable $(x, y) \to (\theta, \eta)$ such that if equation (1) is Elliptic if it can be reduced to $w_{\theta\theta} + w_{\eta\eta} + l(w) = h$.

where $l(w)$ contains the lower order terms and h is a smooth function.

Question

Observation

We already know that the equation remains invariant under a C^1 diffeomorphic change of variable, but it should be noted that the number of variables also remains unchanged.

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Question

How to explicitly find a change of variable so one can change equation (1) into the canonical form.

General Framework

Canonical Form- Hyperbolic Equation

Suppose equation (1) is Hyperbolic in Ω which means $b^2 - ac > 0$ at every point of Ω. We show the existence of a change of variable such that

$$A(\theta,\eta) = a{\theta_x}^2 + 2b\theta_x\theta_y + c{\theta_y}^2 = 0 \tag{2}$$
$$C(\theta,\eta) = a{\eta_x}^2 + 2b\eta_x\eta_y + c{\eta_y}^2 = 0 \tag{3}$$

Then the equation (1) reduced to $w_{\theta\eta} + l(w) = h$.

General Framework

Reduction Process

Assume $a, c \neq 0$ and note that expression (2) and (3) implies that θ and η are solutions of $a{\zeta_x}^2 + 2b\zeta_x\zeta_y + c{\zeta_y}^2 = 0$ which is same as

$$a[\zeta_x - \mu_1\zeta_y][\zeta_x - \mu_2\zeta_y] = 0 \tag{4}$$

where $\mu_1 = \frac{-b-\sqrt{b^2-ac}}{a}$ and $\mu_2 = \frac{-b+\sqrt{b^2-ac}}{a}$

General Framework

Reduction Process

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Observation

Note that μ_1 and μ_2 are real solutions of the equation $a\mu^2 + 2b\mu + c = 0$.

General Framework

To find the Change of variable

To find the nonsingular map $(x, y) \to (\theta, \eta)$ we choose θ to be the solution of $\zeta_x - \mu_1 \zeta_y = 0$ and η to be the solution of $\zeta_x - \mu_2 \zeta_y = 0$.

General Framework

To find the Change of variable

To find the nonsingular map $(x, y) \to (\theta, \eta)$ we choose θ to be the solution of $\zeta_x - \mu_1 \zeta_y = 0$ and η to be the solution of $\zeta_x - \mu_2 \zeta_y = 0$.

Solution

Solving this 1st order equations with the Method of Characteristics one finds that the solutions are constant along the Characteristics curves given by the ODE $\frac{dy}{dx} = -\mu_i$ for $i = 1, 2$

An example

Problem

Reduce the Tricomi equation

$$y u_{xx} + u_{yy} = 0 \tag{5}$$

into its canonical form.

An example

Problem

Reduce the Tricomi equation

$$yu_{xx} + u_{yy} = 0 \tag{5}$$

into its canonical form.

Solution

The equation is Hyperbolic in the region where $y < 0$. Hence one has

$$y\zeta_x^2 + \zeta_y^2 = 0$$

which reduces to the $\zeta_x - \frac{1}{(-y)^{\frac{1}{2}}}\zeta_y = 0$ and $\zeta_x + \frac{1}{(-y)^{\frac{1}{2}}}\zeta_y = 0$

An Example

Solution

The solutions of the equations are

$$\frac{2}{3}(-y)^{\frac{3}{2}} + x = C_1 \text{ and } -\frac{2}{3}(-y)^{\frac{3}{2}} + x = C_2$$

An Example

Solution

The solutions of the equations are

$$\frac{2}{3}(-y)^{\frac{3}{2}} + x = C_1 \text{ and } -\frac{2}{3}(-y)^{\frac{3}{2}} + x = C_2$$

Choosing the Variable

Set $\theta(x, y) = \frac{2}{3}(-y)^{\frac{3}{2}} + x$
and
$\eta(x, y) = -\frac{2}{3}(-y)^{\frac{3}{2}} + x$

Solution

The solutions of the equations are

$$\frac{2}{3}(-y)^{\frac{3}{2}} + x = C_1 \text{ and } -\frac{2}{3}(-y)^{\frac{3}{2}} + x = C_2$$

Choosing the Variable

Set $\theta(x, y) = \frac{2}{3}(-y)^{\frac{3}{2}} + x$
and
$\eta(x, y) = -\frac{2}{3}(-y)^{\frac{3}{2}} + x$

Changing the Variable

Define, $u(x, y) = w(\theta(x, y), \eta(x, y))$

The Canonical Form

Computing the COV

Using Change of Variable we have,

$$u_x = w_\theta + w_\eta$$
$$u_y = -(-y)^{\frac{1}{2}} w_\theta + (-y)^{\frac{1}{2}} w_\eta$$
$$u_{xx} = w_{\theta\theta} + 2w_{\theta\eta} + w_{\eta\eta}$$
$$u_{yy} = -yw_{\theta\theta} + 2yw_{\theta\eta} - yw_{\eta\eta} + \frac{1}{2}(-y)^{\frac{1}{2}}[w_\theta - w_\eta]$$

The Canonical Form

Computing the COV

Using Change of Variable we have,

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\begin{aligned}
u_x &= w_\theta + w_\eta \\
u_y &= -(-y)^{\frac{1}{2}} w_\theta + (-y)^{\frac{1}{2}} w_\eta \\
u_{xx} &= w_{\theta\theta} + 2w_{\theta\eta} + w_{\eta\eta} \\
u_{yy} &= -y w_{\theta\theta} + 2y w_{\theta\eta} - y w_{\eta\eta} + \frac{1}{2}(-y)^{\frac{1}{2}} [w_\theta - w_\eta]
\end{aligned}
$$

Reduced Form

Substituting in the equation one obtains,

$$
w_{\theta\eta} - \frac{1}{6(\theta - \eta)}(w_\theta - w_\eta) = 0
$$

Canonical Form-Parabolic Equation

General Framework

Suppose the equation

$$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{6}$$

is Parabolic in Ω which means $b^2 - ac = 0$ at every point of Ω. We will show the existence of a change of variable $(x, y) \to (\theta, \eta)$ such that

$$A(\theta, \eta) = a{\theta_x}^2 + 2b\theta_x\theta_y + c{\theta_y}^2 = 0$$
$$B(\theta, \eta) = a\theta_x\eta_x + b(\theta_x\eta_y + \eta_x\theta_y) + c\eta_y\theta_y = 0$$

Then the equation (6) reduced to $w_{\eta\eta} + l(w) = h$.

Canonical Form-Parabolic Equation

Remark

Using the invariance of the 2nd order equation under a C^1 diffeomorphism we have $B^2 - AC = 0$ which implies that ssuming $A = 0$ would imply $B = 0$.

Canonical Form-Parabolic Equation

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Using the invariance of the 2nd order equation under a C^1 diffeomorphism we have $B^2 - AC = 0$ which implies that ssuming $A = 0$ would imply $B = 0$.

Reducing the Problem

Since $A = 0$ we have, that θ satisfies $a{\zeta_x}^2 + 2b\zeta_x\zeta_y + c{\zeta_y}^2 = 0$ and since $b^2 - ac = 0$ we have,

$$a(\zeta_x^2 + 2\frac{b}{a}\zeta_x\zeta_y + \frac{b^2}{a^2}\zeta_y^2) = 0$$

which reduces to $(\zeta_x + \frac{b}{a}\zeta_y)^2 = 0$ since $a \neq 0$.

Canonical Form-Parabolic Equation

Finding θ

Hence we have θ is a solution of the equation $(\zeta_x + \frac{b}{a}\zeta_y)^2 = 0$, which is constant along the characteristics $\frac{dy}{dx} = \frac{b}{a}$.

Canonical Form-Parabolic Equation

Finding θ

Hence we have θ is a solution of the equation $(\zeta_x + \frac{b}{a}\zeta_y)^2 = 0$, which is constant along the characteristics $\frac{dy}{dx} = \frac{b}{a}$.

Finding η

To find η use the equation $\theta_x\eta_y - \eta_x\theta_y \neq 0$ and choose any such $\eta \in C^1$.

Canonical Form-Parabolic Equation

Finding θ

Hence we have θ is a solution of the equation $(\zeta_x + \frac{b}{a}\zeta_y)^2 = 0$, which is constant along the characteristics $\frac{dy}{dx} = \frac{b}{a}$.

Finding η

To find η use the equation $\theta_x\eta_y - \eta_x\theta_y \neq 0$ and choose any such $\eta \in C^1$.

Remark

There are infinitely many such η.

An Example

Problem

Reduce the following equation to its Canonical form:

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0 \ \text{ for } \ x > 0$$

An Example

Problem

Reduce the following equation to its Canonical form:

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0 \text{ for } x > 0$$

Solution

Note that the equation is parabolic since $b^2 - ac = 0$.

An example

Solution

From the 1st part we know that if the gievn equation is parabolic then there exists a change of variable $(x, y) \to (\theta, \eta)$ such that θ satisfies the equation $\zeta_x - \frac{y}{x}\zeta_y = 0$

An example

Solution

From the 1st part we know that if the gievn equation is parabolic then there exists a change of variable $(x, y) \to (\theta, \eta)$ such that θ satisfies the equation $\zeta_x - \frac{y}{x}\zeta_y = 0$

Finding θ

We have that θ is constant along the characteristics curve $xy = C_1$. Hence we choose $\theta(x, y) = xy$.

Solution

Finding η

We have that $\theta_x \eta_y - \eta_x \theta_y \neq 0$ which implies $y\eta_y - x\eta_x \neq 0$. Choose $\eta(x, y) = x$.

Solution

Finding η

We have that $\theta_x\eta_y - \eta_x\theta_y \neq 0$ which implies $y\eta_y - x\eta_x \neq 0$. Choose $\eta(x,y) = x$.

Applying COV

Define, $w(\theta,\eta) = u(x,y)$ and using COV we have,

$$
\begin{aligned}
u_x &= yw_\theta + w_\eta \\
u_y &= xw_\theta \\
u_{xx} &= y^2 w_{\theta\theta} + 2yw_{\theta\eta} + w_{\eta\eta} \\
u_{xy} &= xyw_{\theta\theta} + xw_{\theta\eta} + w_\theta \\
u_{yy} &= x^2 w_{\theta\theta}
\end{aligned}
$$

Reduced equation

Solution

Substituting the values in the original equation one has,

$$w_{\eta\eta} + \frac{1}{\eta} w_{\eta} = 0$$

which is the required Parabolic equation in the reduced form.

The End