INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

PDE: Fundamental Operators

MSO 203B

November 8, 2016

1. Prove that for $u \in C^2(\bar{\Omega})$ and $i = 1, 2...$:
   a) $\int_\Omega u_{x_i} dx = \int_{\partial\Omega} u\gamma_i ds.$
   b) $\int_\Omega u_{x_i} v dx = -\int_\Omega u v_{x_i} dx + \int_{\partial\Omega} uv\gamma_i ds.$
   c) $\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial\phi}{\partial\gamma} ds$

   using the Gauss Divergence Theorem and where γ is the unit outward normal.

2. If ϕ exists and is harmonic everywhere inside the closed curve $\mathbb{C}$ bounding a region $\mathbb{R}$ then prove that
$$\int_{\mathbb{C}} \frac{\partial\phi}{\partial\gamma} ds = 0$$
   where γ is the unit outward normal.

3. Prove that the equation
$$-\Delta u = f \text{ in } \Omega \quad u = g \text{ on } \partial\Omega$$
   admits a unique solution without using Maximum Principle.

4. (Stability of Solution) Let u_1 be the solution of
$$-\Delta u = f \text{ in } \Omega$$
$$u = h_1 \text{ on } \partial\Omega$$
   and u_2 satisfies
$$-\Delta u = f \text{ in } \Omega$$
$$u = h_2 \text{ on } \partial\Omega$$

then prove that

$$\max_{x\in\Omega}|u_2(x) - u_1(x)| \le \max_{x\in\partial\Omega}|h_2(x) - h_1(x)|$$

5. Solve the following:

$$-\Delta u = 0 \text{ in } \mathbb{R}^+ \times (0, b)$$

subject to the boundary condition

$$u(x,0) = 0;\ \ u(x,b) + \gamma u_y(x,b) = 0$$
$$u(0,y) = f(y) \text{ where } \gamma > 0$$

6. Solve the following problem:

$$u_{tt} = u_{xx};\ \ x \ge 0,\ \ t \ge 0$$
$$u(0,t) = 0;\ \ t \ge 0$$
$$u(x,0) = f(x);\ \ u_t(x,t) = g(x);\ \ x \ge 0$$

also assume that u is bounded as $x \to \infty$.

7. Find the solution to the heat equation

$$u_t = ku_{xx};\ \text{ on } (x,t) \in (0,L) \times (0,\infty)$$

subject to the boundary conditions

$$u(x,0) = f(x);\ \ u(0,t) = 0;\ \ u(L,t) = 0$$

8. Solve the following heat problem:

$$u_t = u_{xx},\ \text{ for } (x,t) \in [0,\pi] \times (0,\infty)$$
$$u(0,t) = 0,\ \ u_x(\pi,t) = 0$$
$$u(x,0) = 3\sin\left(\tfrac{5x}{2}\right)$$