ASSIGNMENT VI MSO 202 A

ROUCHÉ'S THEOREM AND MÖBIUS TRANSFORMATIONS

Exercises 0.2-0.4 relies on the following theorem:

Theorem 0.1 (Rouché's Theorem)**.** *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside the circle C.*

Exercise 0.2 : Let a, b, c be positive real numbers such that $a+c < b$. Consider the polynomial $p(z) = az^7 + bz^3 + c$. Then $p(z)$ has exactly 3 zeros in the open unit disc $\mathbb{D}$.

Solution. Apply Rouché's Theorem to $f(z) = bz^3$ and $g(z) = az^7 + c$:

$$|f(z)| = b > a + c \geq |az^7 + c| = |g(z)| \text{ on } |z| = 1.$$

Exercise 0.3 : Show that the functional equation $\lambda = z + e^{-z}$ $(\lambda > 1)$ has exactly one (real) solution in the right half plane.

Solution. Apply Rouché's Theorem to $f(z) = z - \lambda$ and $g(z) = e^{-z}$: Indeed, for $z = x + iy$, $x > 0$ on $|z| = 2\lambda$,

$$|g(z)| = e^{-x} \leq 1 < |z - \lambda| = |f(z)|.$$

Exercise 0.4 : Find the number of zeros of $3e^z - z$ in the closed unit disc centered at the origin.

Solution. The number of zeros are 0. To see that, apply Rouché's Theorem to $f(z) = 3e^z$ and $g(z) = z$: Indeed, for $z = x + iy$ on $|z| = 1$,

$$|g(z)| = |z| = 1 < 3e^{-1} \leq 3e^x = |f(z)|.$$

Further, note that for no z with $|z| = 1$, $3e^z = z$ (this can be seen by taking modulus on both sides).

Exercise 0.5 : If γ is a line and $f(z) = \frac{1}{z}$ then $f(\gamma)$ is a circle or line.

Solution. Suppose γ represents line $a\text{Re}(z) + b\text{Im}(z) = c$ for real numbers a, b, c. Then $f(\gamma)$ is obtained by replacing z by $\frac{1}{w}$:

$$a\text{Re}(1/w) + b\text{Im}(1/w) = c, \text{ that is }, a\text{Re}(w) - b\text{Im}(w) = c|w|^2.$$

If $c = 0$ then $f(\gamma)$ is a line. Otherwise, let $\alpha = (a + ib)/(2c)$, and note that $2\text{Re}(\alpha w) = |w|^2$. Thus

$$|w - \alpha|^2 = |w|^2 - 2\text{Re}(\alpha w) + |\alpha|^2 = |\alpha|^2.$$

Hence $f(\gamma)$ represents the circle centered at α and of radius $|\alpha|$.

Exercise 0.6 : Show that for any $z_1, z_2 \in \mathbb{D}$, there exists a bijective holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_1) = z_2$.

Solution. For $|a| < 1$, consider the Möbius transformation $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$. We have seen in the lecture notes that $\psi_a : \mathbb{D} \to \mathbb{D}$ is bijective, holomorphic with inverse equal to ψ_a itself. Let $f = \psi_{z_2} \circ \psi_{z_1}$ and note that $f(z_1) = \psi_{z_2} \circ \psi_{z_1}(z_1) = \psi_{z_2}(0) = z_2$.

Exercise 0.7 : Consider the Möbius transformation $f(z) = \frac{az+b}{cz+d}$ for real numbers a, b, c, d. Show that if $ad - bc > 0$ then f maps the open upper half plane $\mathbb{U} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ onto itself.

Solution. Let $z = x + iy \in \mathbb{C}$ be such that $y > 0$. Note that

$$f(z) = \frac{(ax + b + iay)(cx + d - ciy)}{|cz + d|^2}.$$

Thus the imaginary part of $f(z)$ is equal to $\frac{y(ad-bc)}{|cz+d|^2}$, which is clearly positive.

Remark 0.8 : It turns out that any bijective, holomorphic function which maps $\mathbb{U}$ onto itself is given by $f(z) = \frac{az+b}{cz+d}$ for real numbers a, b, c, d such that $ad - bc > 0$.