ME623: Finite Element Methods in Engineering Mechanics

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Coursenotes, assignments and announcements will all be made in Backpack (http://www.usebackpack.com).
Coursecode is 063085

The Google Calendar for the course is available at
https://calendar.google.com/calendar?cid=ZHIuc3VtaXQuYmFzdUBnbWFpbC5jb20

Evaluation Policy

The endsem and midsem exams will be the only exams that we will conduct. Each of these will carry 50% weight. These will be ‘pen and paper’ tests based on theoretical aspects of the FE method.

Several assignments meant to help your understanding of the techniques will be given from time to time. If you want to learn the method seriously, I suggest that you try these out.

Coding workshops will be held in the Linux Lab in the New Core Building.

Books

Elementary books covering a wide range of problems

- Concepts and applications of Finite element analysis: Cook, Malkus and Plesha, John Wiley and Sons, 2003.
- T.R. Chandrupatla and A.D. Belegundu, Introduction to Finite Elements in Engineering, Second Edition, Prentice-Hall, 1997.
- Daryl Logan, A First Course in Finite Element Method, Thomson, India Edition

Somewhat more advanced texts dealing less with procedures and more with ideas behind the technique

- K-J Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall of India, 1990
- O C Zienkiewicz and R L Taylor, The Finite Element Method: Its Basics and Fundamentals, Butterworth Heinemann, 2007
- J N Reddy, An Introduction To Finite Element Method, Tata McGraw Hill, 2005

Why do we need to simulate things?

Seeing is believing.

Sophisticated experiments can tell everything. Why do we need the FE method?

- Experimental results are subject to interpretation. Interpretations are as good as the competence of the experimenter.
- Experiments, especially sophisticated ones, can be expensive
- There are regimes of mechanical material behaviour that experiments cannot probe.
- Generality of behaviour is often not apparent from experiments.

Experiments and simulations are like two legs of a human being. You need both to walk and it does not matter which you use first!

FEA has become extremely important in engineering design

•The FEM has attained a level of generality that makes it suitable for a very wide range of engineering and scientific problems that require solution of governing partial differential equations with well defined boundary conditions. These include, solid and fluid mechanics, structural mechanics, biomechanics, quantum mechanics, electromagnetics, geomechanics …

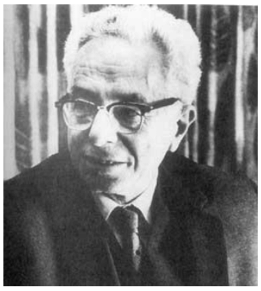
•The mathematical basis of the FEM is well understood and the discretisation equations for FEM arise out of well established principles.

•FEM simulations form important part of safety assessment procedures for Reactor Pressure Vessels in nuclear power plants, aircraft industry, crashworthiness of automobiles, earthquake resistance of buildings etc.

•Powerful commercial softwares that combine modelling algorithms with the power of FEM are now easily available. Extermely realistic problems may be solved given the enormous increase in computational power.

How did it evolve?

1943: Richard Courant, a mathematician described a piecewise polynomial solution for the torsion problem of a shaft of arbitrary cross section. Even holes.
The early ideas of FEA date back to a 1922 book
by Hurwitz and Courant.



His work was not noticed by engineers
and the procedure was impractical at the time due to the lack of digital computers.

1888-1972: b in Lublintz Germany

Student of Hilbert and Minkowski in Gottingen Germany

Ph.D in 1910 under Hilbert's supervision.

1934: moved to New York University, founded the Courant Institute

In the 1950s: work in the aircraft industry introduced FE to practicing engineers.
A classic paper described FE work that was prompted by a need to analyze
delta wings, which are too short for beam theory to be reliable.

1960: The name "finite element" was coined by structural engineer Ray Clough
of the University of California

Professor emeritus of Structural Engineering at UC Berkley

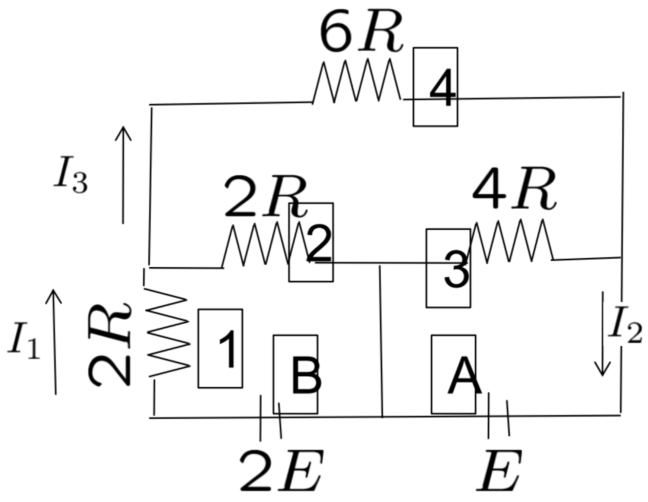
Ph.D from MIT

Well known earthquake engineer

By 1963 the mathematical validity of FE was recognized and the
method was expanded from its structural beginnings to include heat transfer,
groundwater flow, magnetic fields, and other areas.

Large general-purpose FE software began to appear in the 1970s.
By the late 1980s the software was available on microcomputers,
complete with color graphics and pre- and post-processors.
By the mid 1990s roughly 40,000 papers and books about
FE and its applications had been published.

Solving Engineering Problems



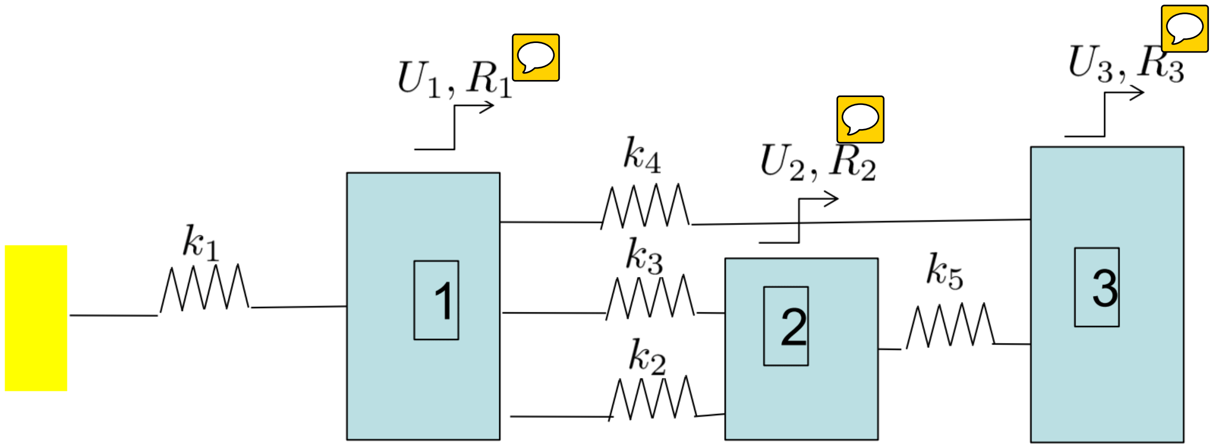
To solve engineeringing problems on the computer, one of the common goals is to cast the problem in a matrix form. Then we can use the numerical methods for solving linear system of equations. For example, for the simple circuit shown here,

$$
\begin{aligned}
2E &= 2RI_1 + 2R(I_1 - I_3) \\
E &= 4R(I_2 - I_1) \\
0 &= 6RI_3 + 4R(I_3 - I_2) + 2R(I_3 - I_1)
\end{aligned}
$$

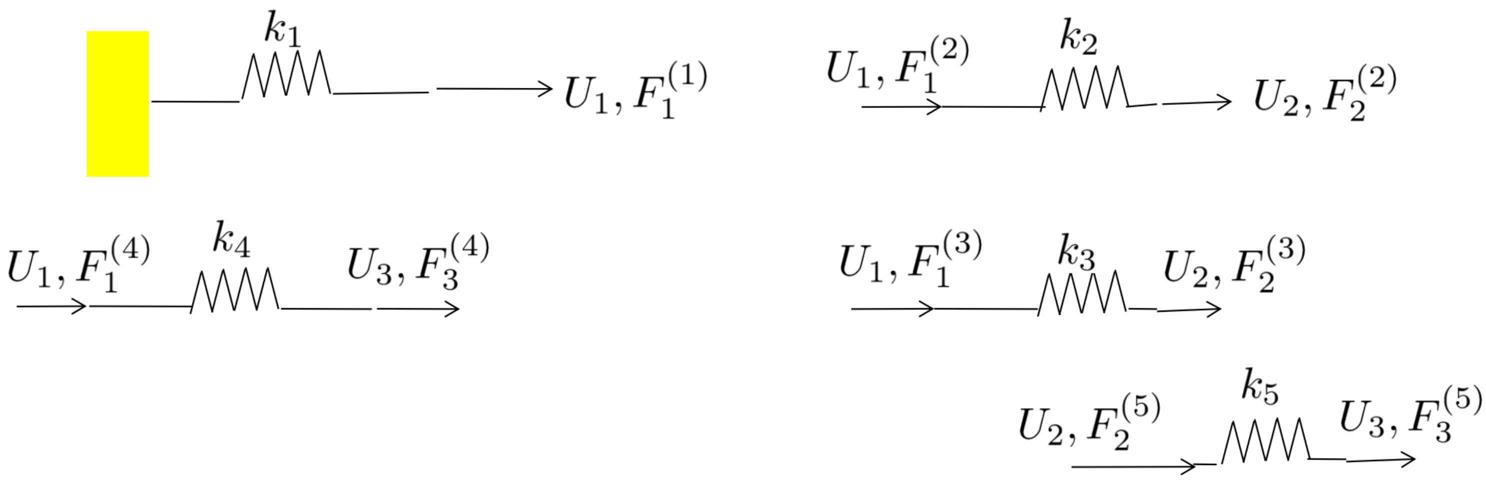
These equations can be cast in the form:

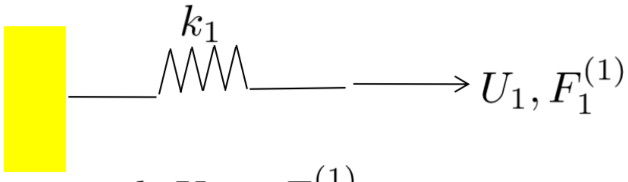
$$\begin{pmatrix} 4R & 0 & -2R \\ 0 & 4R & -4R \\ -2R & -4R & 12R \end{pmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} 2E \\ E \\ 0 \end{Bmatrix},$$

and hence can be easily solved for the unknowns I_1, I_2 and I_3. The procedure is similar for equivalent spring-mass problems.

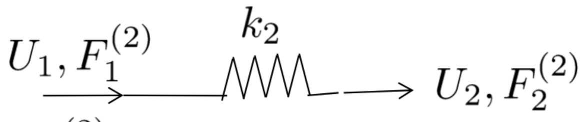


Define $F_i^{(j)}$ where $i = 1, 2, 3$ and $j = 1, \dots 5$

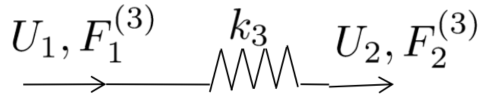




$$k_1 U_1 = F_1^{(1)}$$

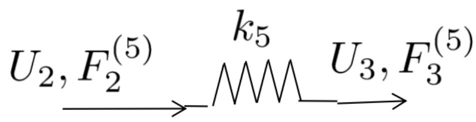


$$\begin{aligned} F_2^{(2)} &= k_2(U_2 - U_1) \\ F_1^{(2)} &= -k_2(U_2 - U_1) \end{aligned}$$



$$\begin{aligned} F_2^{(3)} &= k_3(U_2 - U_1) \\ F_1^{(3)} &= -k_3(U_2 - U_1) \end{aligned}$$

$$\begin{aligned} F_3^{(4)} &= k_4(U_3 - U_1) \\ F_1^{(4)} &= -k_4(U_3 - U_1) \end{aligned}$$



$$\begin{aligned} F_2^{(5)} &= k_5(U_3 - U_2) \\ F_2^{(5)} &= -k_5(U_3 - U_2) \end{aligned}$$

Finally, free body diagrams of each of the nodes $1, 2$ and 3 give

$$\begin{aligned}
F_1^{(1)} + F_1^{(2)} + F_1^{(3)} + F_1^{(4)} &= R_1 \\
F_2^{(2)} + F_2^{(3)} + F_2^{(5)} &= R_2 \\
F_3^{(4)} + F_3^{(5)} &= R_3.
\end{aligned}$$

Now we replace the forces on the left hand sides in terms of the stiffnesses and displacements. Also, U_1, U_2, U_3 are the unknowns in this problem. So, we get

$$\begin{pmatrix} (k_1+k_2+k_3+k_4) & -(k_2+k_3) & -k_4 \\ -(k_2+k_3) & (k_2+k_3+k_5) & -k_5 \\ -k_4 & -k_5 & (k_4+k_5) \end{pmatrix} \left\{ \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \right\} = \left\{ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \right\}.$$

The above set of equations govern the equilibium of the system of springs in this problem. They are expressed as (in analogy with a single linear spring)

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}.$$

The same problem can be solved using a energy based approach. It turns out that such an approach is more suited for numerical implementations. The energy principle (which we will deal with in detail later) states that the total potential energy of the system should be a minimum. The total potential energy of the system is

$$\Pi = \mathcal{U} - \mathcal{W},$$

where, $\mathcal{U}$ is the strain energy stored and $\mathcal{W}$ is the potential of the external forces.

The potential of the loads is

$$\mathcal{W} = U_1 R_1 + U_2 R_2 + U_3 R_3,$$

which can be expressed as

$$\mathcal{W} = \boldsymbol{U}^T \boldsymbol{F}.$$

Further, noting that the strain energy stored in a linear spring of stiffness k is $(1/2)ku^2$, the strain energy of the system is

$$\Pi = \frac{1}{2}\boldsymbol{U}^T \boldsymbol{K} \boldsymbol{U}. \text{ (verify this yourself.)}$$

Thus

$$\Pi = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} - \boldsymbol{U}^T\boldsymbol{F}.$$

Finding the extremum of Π with respect to U_i involves solving

$$\frac{\partial \Pi}{\partial U_i} = 0 \text{ for } i = 1, 2, 3.$$

The extremum is a minimum (can be checked formally) because the maximum is obviously infinity. Again, verify that, the above procedure applied to the Π derived above gives:

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}.$$

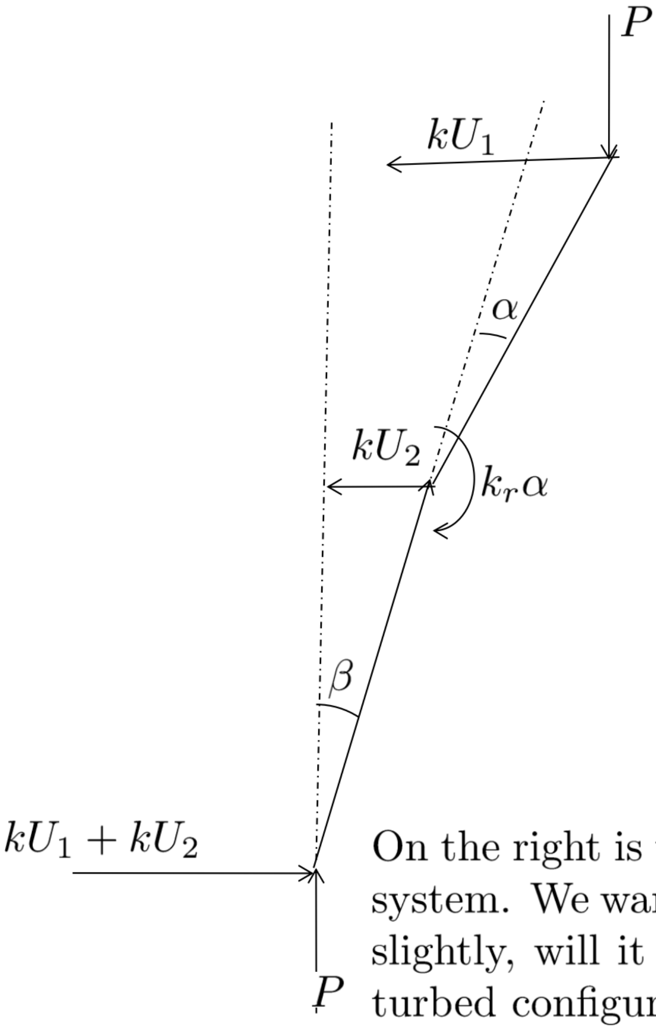
Minimising the potential energy of the system provides us with an alternate way to arrive at the equations of motion of a system.

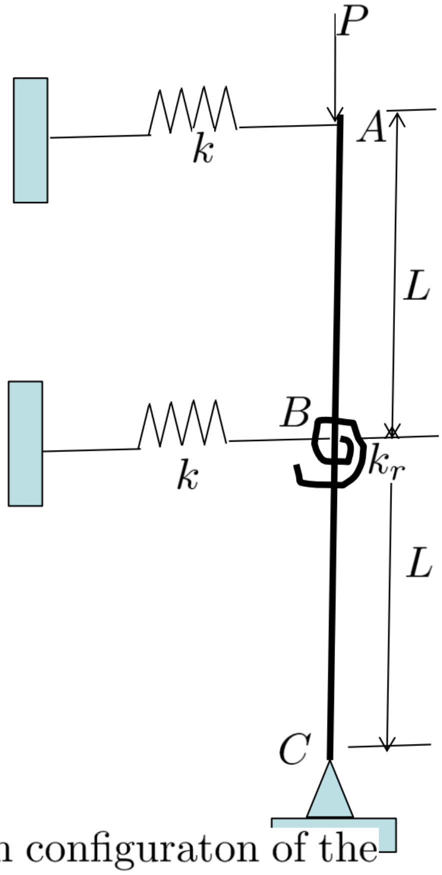
We can also investigate problems of stability where we ask if, given that a steady state solution to a problem is known, whether a small perturbation of the system will take it to another solution.

To investigate the stability of a system we will need to consider a generalised eigenvalue problem

$$\boldsymbol{K}\boldsymbol{U} = \lambda \boldsymbol{B}\boldsymbol{U}.$$

If a solution exists to this problem, λ_i and $\boldsymbol{U}_i$ that satisfy this are called *eigenvalues* and *eigenvectors* of $\boldsymbol{K}$. We will look at a simple problem to better understand this concept.





On the right is the equilibrium configuraton of the system. We want to know if we perturb the system slightly, will it be stable? On the left is the perturbed configuration and its free body diagram.

Strain energy stored in the system

$$\mathcal{U} = \frac{1}{2}\left(kU_1^2 + kU_2^2 + k_r\alpha^2\right).$$

Vertical movement of the external load is

$$\delta_v = L - L\cos(\alpha+\beta) + L - L\cos\beta = L\left[1 - \cos(\alpha+\beta) + 1 - \cos\beta\right].$$

Using a series approximation for the cosines assuming a small perturbation, we have

$$\delta_v = L\left[\frac{(\alpha+\beta)^2}{2} + \frac{\beta^2}{2}\right].$$

Further, for small perturbations

$$\alpha = \frac{U_1 - 2U_2}{L}, \beta = \frac{U_2}{L}, \alpha + \beta = \frac{U_1 - U_2}{L}.$$

Potential of the external loads is given as

$$\mathcal{W} = P\delta_v.$$

The potential energy thus can be written in terms of the displacements as

$$\begin{aligned}\Pi &= \frac{1}{2}kU_1^2 + \frac{1}{2}kU_2^2 + \frac{1}{2}k_r\left(\frac{U_1 - 2U_2}{L}\right)^2 \\ &- \frac{P}{2L}(U_1 - U_2)^2 - \frac{P}{2L}U_2^2.\end{aligned}$$

Now using

$$\frac{\partial \Pi}{\partial U_1} = \frac{\partial \Pi}{\partial U_2} = 0,$$

we get

$$\begin{pmatrix} kL + k_r/L & -2k_r/L \\ -2k_r/L & kL + 4k_r/L \end{pmatrix} \left\{ \begin{array}{c} U_1 \\ U_2 \end{array} \right\} = P \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \left\{ \begin{array}{c} U_1 \\ U_2 \end{array} . \right\}$$

Note that this is an eigenvalue problem and existence of non trivial solutions for the load P indicates that multiple solutions to this are possible.

Solving problems governed by differential equations: finite difference techniques

Mathematical model: an equation of motion

$$\frac{du}{dt} = f(t, u)$$

for $t > 0$ and $u = u_0$ at $t = 0$

Use

$$\left(\frac{du}{dt}\right)_i \simeq \frac{[u(t_{i+1}) - u(t_i)]}{t_{i+1} - t_i}$$

$\Rightarrow u_{i+1} = u_i + \Delta t f(u_i, t_i)$ Euler's explicit scheme or first order Runge Kutta scheme

Consider the motion of a pendulum:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$\theta(0) = \theta_0$ and $\dot{\theta}(0) = v_0$.
Solution to this system is

$$\theta(t) = \frac{v_0}{\lambda}\sin\lambda t + \theta_0\cos\lambda t$$

where $\lambda = \sqrt{g/l}$

Alternately,

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\lambda^2\theta \end{aligned}$$

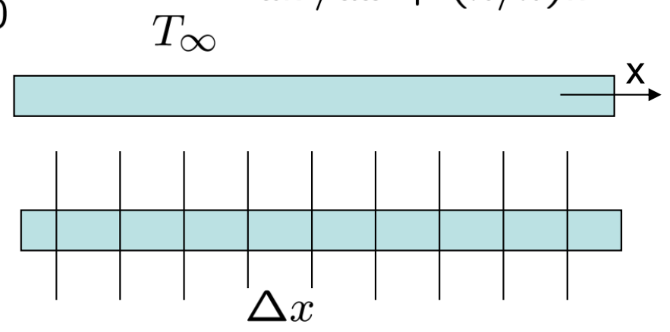
$\Rightarrow$

$$\begin{aligned} \theta_{i+1} &= \theta_i + \Delta t v_i \\ v_{i+1} &= v_i - \Delta t \lambda^2 \theta_i \end{aligned}$$

As another example, consider the problem of heat conduction in a bar with specified boundary conditions.

At $x = L$
$dT/dx + (h/k)T = 0$

At $x = 0$
$T = T_0$



The governing deq is

$$kA\frac{d^2T}{dx^2} + hP(T_\infty - T) = 0$$

Using $T_\infty = 0$ arbitrarily

$$\left(\frac{d^2T}{dx^2}\right)_{x=x_i} \simeq \left(\frac{T_{i-1} - 2T_i + T_{i+1}}{(\Delta x)^2}\right)$$

$\Rightarrow$

$$-T_{i-1} + [2 + (m\Delta x)^2]T_i - T_{i+1} = 0$$

where $m = \sqrt{hP/kA}$

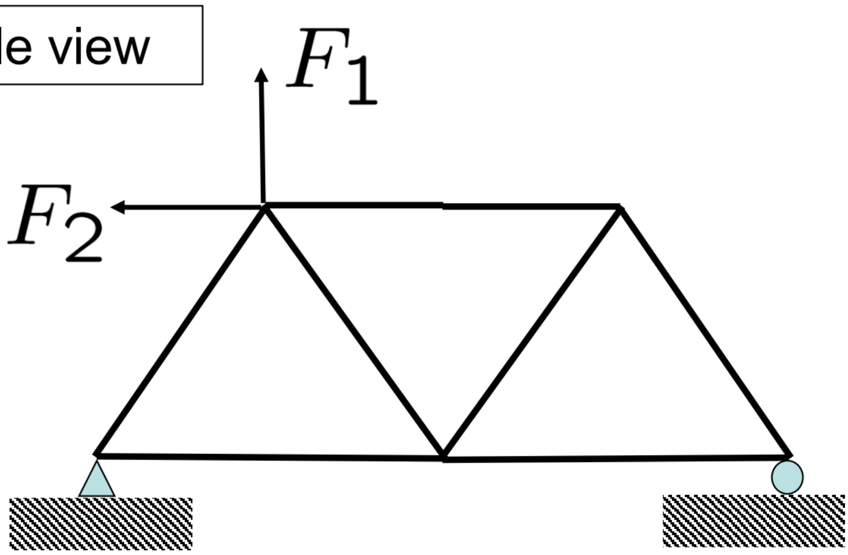
$$
\begin{aligned}
-T_0 + DT_1 - T_2 &= 0 \\
-T_1 + DT_2 - T_3 &= 0 \\
\cdots\cdots\cdots &\quad \cdots \\
-T_{N-1} + DT_N - T_{N+1} &= 0
\end{aligned}
$$

This is a *tridiagonal* system of equations, where, for a point i, the temperatures at $i-1$ and $i+1$ only appear in the ith equation. When put in the matrix form, only the diagonal and the sub-diagonals above and below it will be populated. Additionally,

$$
\begin{aligned}
\frac{T_{N+1} - T_N}{\Delta x} + \frac{h}{k} T_N &= 0 \\
\Rightarrow T_{N+1} &= \left(1 - \frac{h\Delta x}{k}\right) T_N
\end{aligned}
$$

The FEM procedure: a simple view

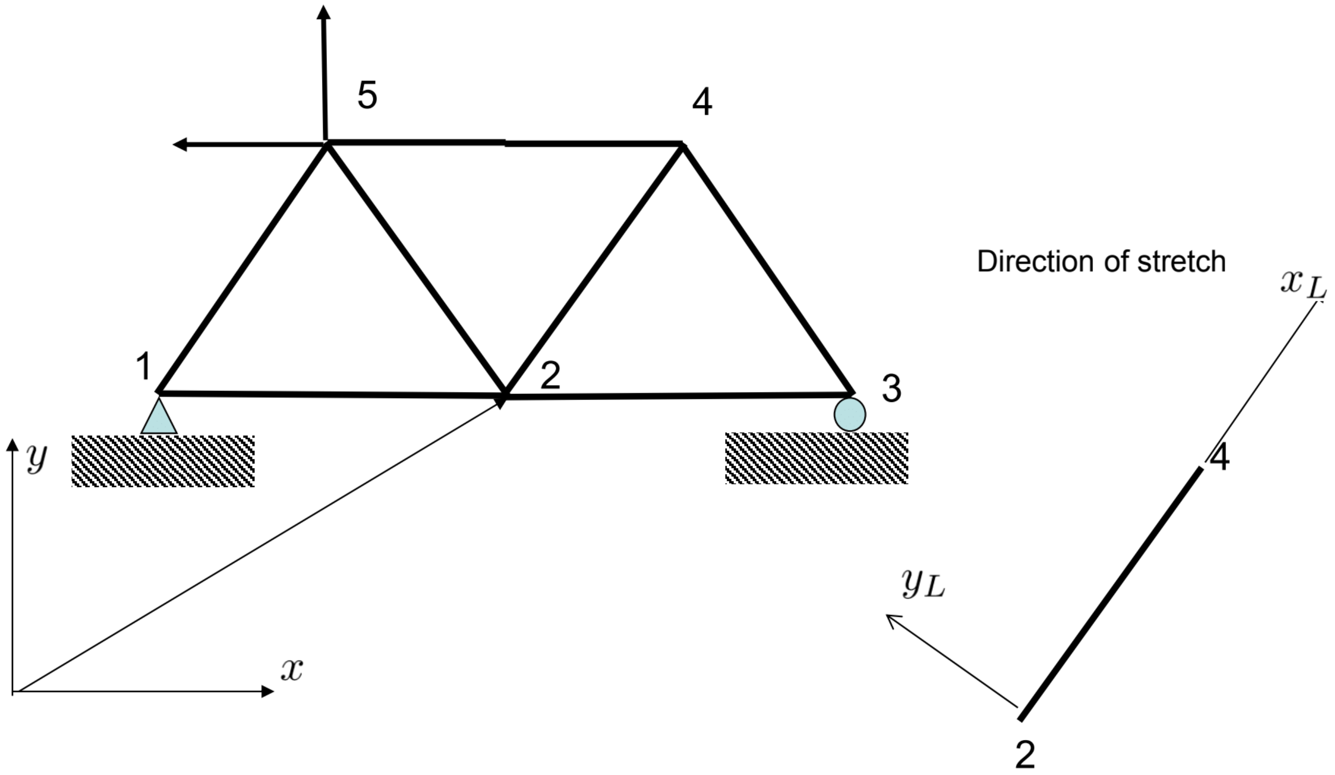
A 2-d truss with elements that can only withstand tension.



For any truss element, uniaxial elastic constitutive relations are valid.

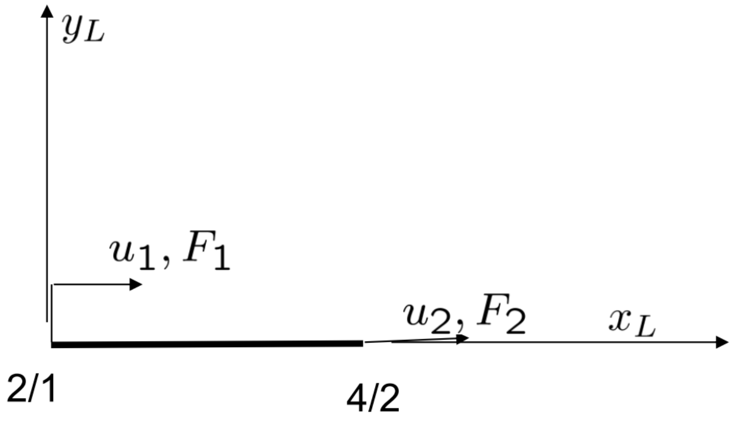
$$
\begin{aligned}
F &= \sigma A \\
&= EA\epsilon \\
&= EA\frac{l - l_0}{l_0} \\
&= k\delta
\end{aligned}
$$

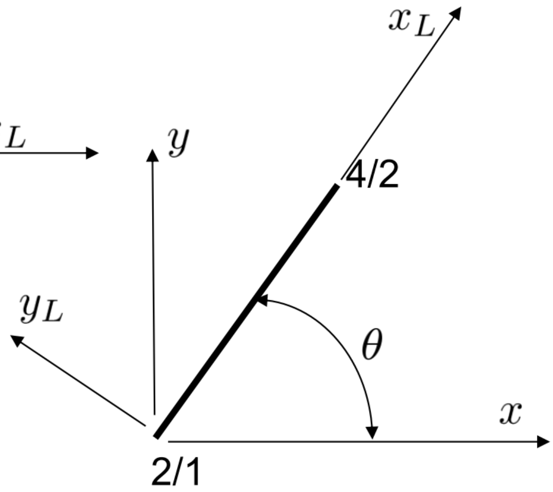
where, $k = \frac{AE}{l_0}$ and $\delta = l - l_0$.



Each node in a planar truss has 2 degrees of freedom (dof). Also, each node can be located in a global (x, y) coordinate system.

The equilibrium equation for every element in the truss can be written in a *local* coordinate system (x_L, y_L). In this system, the element has only one dof per node, it can only stretch along x_L. The orientation of a generic element can be defined by an angle θ wrt the global system.





$$\begin{aligned} F_2 &= k(u_2 - u_1) \\ F_1 &= k(-u_2 + u_1) \end{aligned}$$

$$\left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}$$

The force balance for a generic element can be expressed in the form of a 4×4 matrix by considering the possibility rigid body movement along y_L.

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} = \begin{pmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Local force vector (ndof X nnod,1)

Displacement vector (ndof X nnod, 1)

$$\boldsymbol{f} = \boldsymbol{K}_L \boldsymbol{u}$$

Local stiffness matrix in local coordinate system (ndof X nnod, ndof X nnod)

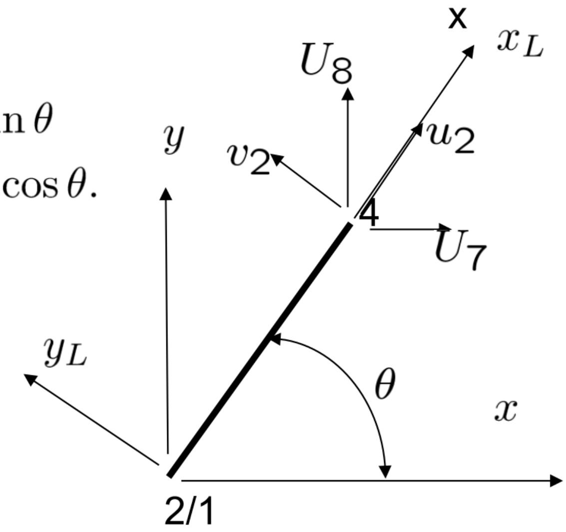
We will now express the local quantites in the global coordinates. The numbering for the global vectors is based on a continuous numbering of the dofs i.e. for node i the dof's are $(i-1)*ndof+1,\ldots,i*ndof$. Thus, the transformation for the displacement components at a node is given as:

$$
\begin{aligned}
u_2 &= U_7\cos\theta + U_8\sin\theta \\
v_2 &= -U_7\sin\theta + U_8\cos\theta.
\end{aligned}
$$

$$
\left\{\begin{array}{c} u_2 \\ v_2 \end{array}\right\} = \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right)\left\{\begin{array}{c} U_7 \\ U_8 \end{array}\right\}
$$

Also, in a similar manner, for the local node 1 with local degrees of freedom $1,2$ and global dof's $3,4$,

$$
\begin{aligned}
u_1 &= U_3\cos\theta + U_4\sin\theta \\
v_1 &= -U_3\sin\theta + U_4\cos\theta.
\end{aligned}
$$



The transformation from local to global quantities for both the nodes can be expressed in one equation as:

$$\left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \right\} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \left\{ \begin{array}{c} U_3 \\ U_4 \\ U_7 \\ U_8 \end{array} \right\}$$

Note that the transformation matrix is orthogonal.

$$\boldsymbol{u} = \boldsymbol{T}\boldsymbol{U}$$

Similarly the relation between the global and local components of the force vector can be written as

$$\boldsymbol{f} = \boldsymbol{T}\boldsymbol{F}$$

Now we can easily derive the relationship between the global and local stiffness matrices.

$$\begin{aligned}\boldsymbol{f} &= \boldsymbol{K}_L \boldsymbol{u} \\ \boldsymbol{T}\boldsymbol{F} &= \boldsymbol{K}_L \boldsymbol{T}\boldsymbol{U} \\ \boldsymbol{F} &= \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}\boldsymbol{U}\end{aligned}$$

Thus, the global equilibrium for the generic element can be written as:

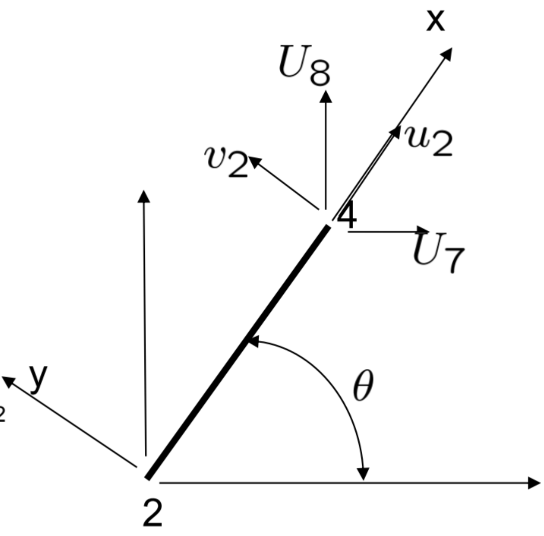
$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

where,

$$\boldsymbol{K} = \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}$$

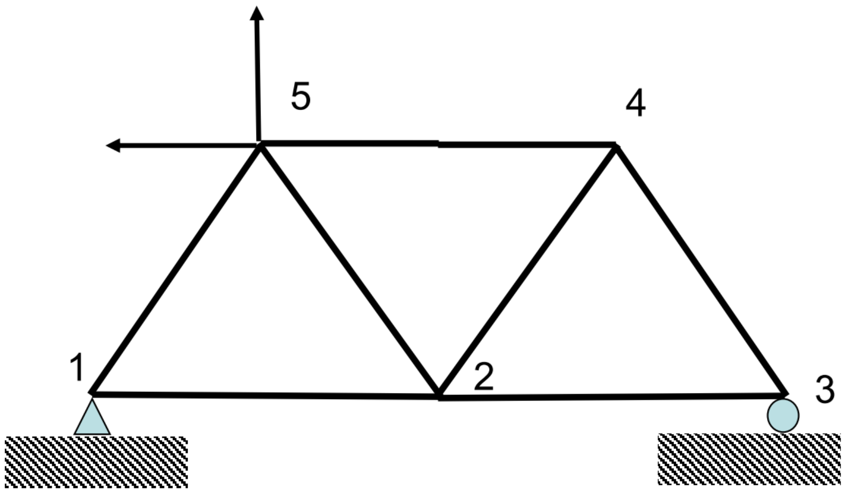
Assembly procedure

$$
\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \boldsymbol{T}^T
\begin{array}{c} \begin{array}{cccc} \scriptstyle 3/1 & \scriptstyle 4/2 & \scriptstyle 7/3 & \scriptstyle 8/4 \end{array} \\
\begin{pmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
\boldsymbol{T} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}
\begin{array}{l} \scriptstyle 3/1 \\ \scriptstyle 4/2 \\ \scriptstyle 7/3 \\ \scriptstyle 8/4 \end{array}
$$

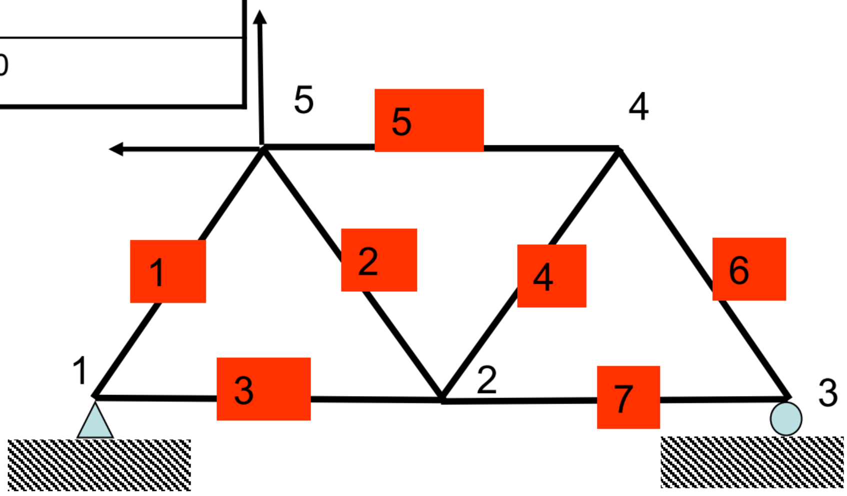


Eg: 2,2 → 4,4

3,1 → 7,3



Elem no	Local dof	Destination
1	1	1
	2	2
	3	9
	4	10
2	1	3
	2	4
	3	9
	4	10



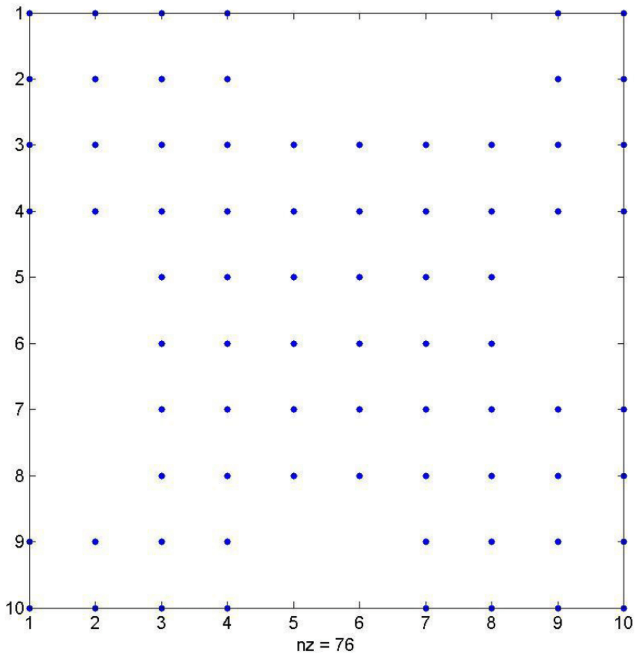
After assembling elements 1 and 2

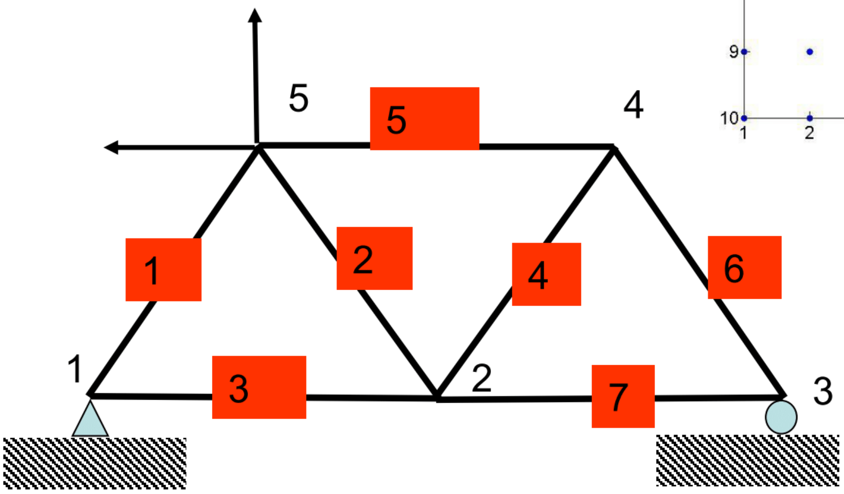
$$
\begin{pmatrix}
K^1_{11} & K^1_{12} & 0 & 0 & 0 & 0 & 0 & 0 & K^1_{13} & K^1_{14} \\
K^1_{21} & K^1_{22} & 0 & 0 & 0 & 0 & 0 & 0 & K^1_{23} & K^1_{24} \\
0 & 0 & K^2_{11} & K^2_{12} & 0 & 0 & 0 & 0 & K^2_{13} & K^2_{14} \\
0 & 0 & K^2_{21} & K^2_{22} & 0 & 0 & 0 & 0 & K^2_{23} & K^2_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
K^1_{31} & K^1_{32} & K^2_{31} & K^2_{32} & 0 & 0 & 0 & 0 & K^1_{33} + K^2_{33} & K^1_{34} + K^2_{34} \\
K^1_{41} & K^1_{42} & K^2_{41} & K^2_{42} & 0 & 0 & 0 & 0 & K^1_{43} + K^2_{43} & K^1_{44} + K^2_{44}
\end{pmatrix}
$$

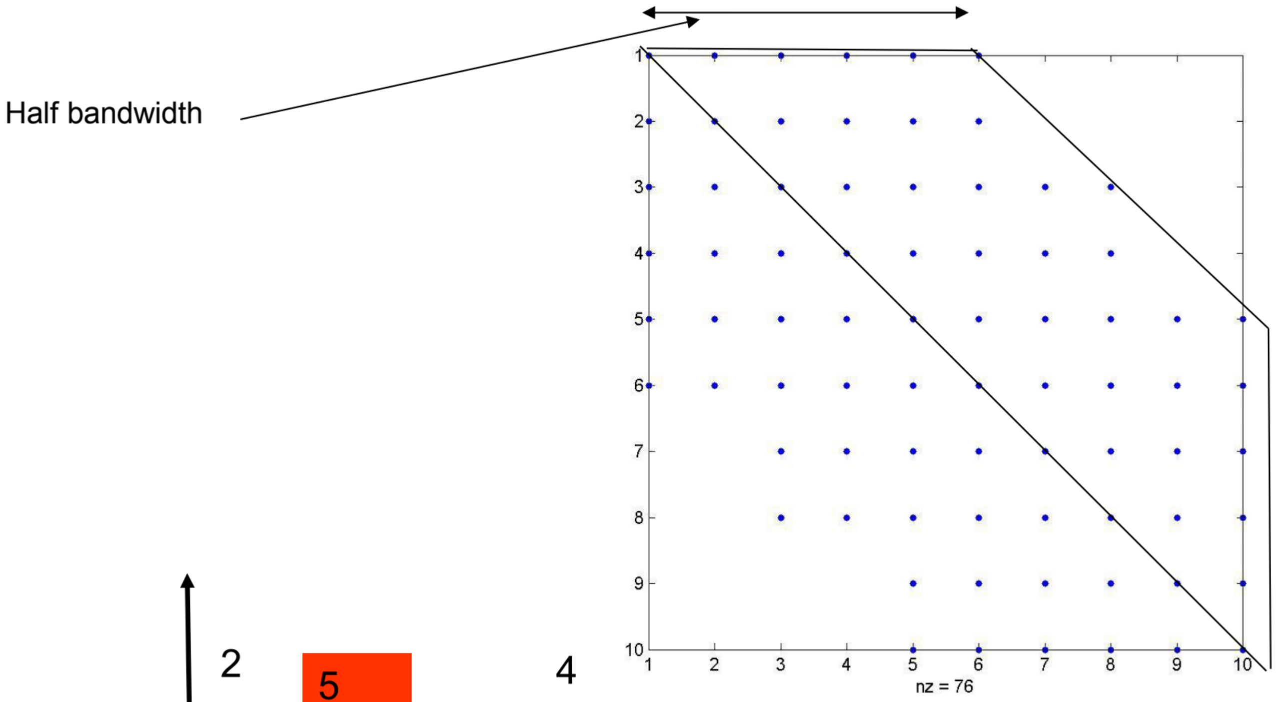
X+X	X+X	X	X					X	X
X+X	X+X	X	X					X	X
X	X	X+X+X	X+X+X			X	X	X	X
X	X	X+X+X	X+X+X			X	X	X	X
		X	X	X+X	X+X	X	X		
		X	X	X+X	X+X	X	X		
		X+X	X+X	X+X	X+X	X+X+X	X+X+X	X	X
		X+X	X+X	X+X	X+X	X+X+X	X+X+X	X	X
X	X	X	X			X	X	X+X+X	X+X+X
X	X	X	X			X	X	X+X+X	X+X+X

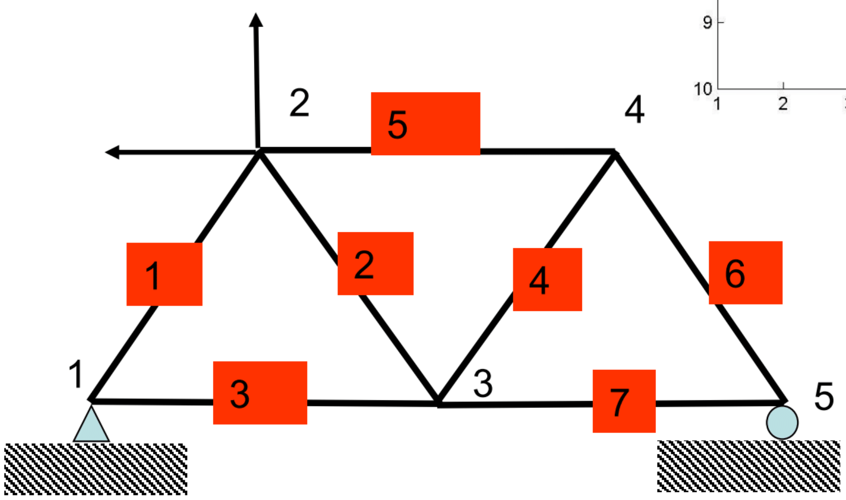
Stiffness matrix is symmetric, diagonally dominant, sparse and banded.

Sparsity pattern






Half bandwidth
1
2
3
4
5
6
7
8
9
10
nz = 76


1
2
3
4
5
6
7

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

Global displacement vector

Global force vector

Global stiffness matrix

Notes:

Global stiffness matrix is singular i.e. it has zero eigenvalues

Hence it cannot be inverted!

Boundary conditions

Force specified: eg. dof 9 and 10 in our example

Displacement specified: eg. dof 1,2,and 6 in our example

Both forces and displacements cannot be specified at the same dof.

$$\boldsymbol{F} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_1 \\ F_2 \end{array} \right\} \qquad \boldsymbol{U} = \left\{ \begin{array}{c} 0 \\ 0 \\ U_3 \\ U_4 \\ U_5 \\ 0 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \end{array} \right\}$$

Naïve approach for imposing displacement boundary conditions

$$\begin{pmatrix} L & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} & K_{19} & K_{1,10} \\ K_{21} & L & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} & K_{29} & K_{2,10} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} & K_{39} & K_{3,10} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & K_{49} & K_{4,10} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} & K_{5,10} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & L & K_{67} & K_{68} & K_{69} & K_{6,10} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} & K_{7,10} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & K_{89} & K_{8,10} \\ K_{91} & K_{92} & K_{93} & K_{94} & K_{95} & K_{96} & K_{97} & K_{98} & K_{99} & K_{9,10} \\ K_{10,1} & K_{10,2} & K_{10,3} & K_{10,4} & K_{10,5} & K_{10,6} & K_{10,7} & K_{10,8} & K_{10,9} & K_{10,10} \end{pmatrix}$$

L=a very large number. Also replace the corresponding dofs in the rhs vector by LXspecified displacement value

$$U_1 = L\delta_1 - \frac{K_{12}U_2 + \ldots K_{1,10}U_{10}}{L} \simeq \delta_1$$

The “proper” way of imposing displacement constraints

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= f_1 \\
a_2 x + b_2 y + c_2 z &= f_2 \\
a_3 x + b_3 y + c_3 z &= f_3
\end{aligned}
$$

Suppose $y = \delta$ (known).

$$
\begin{aligned}
a_1 x + c_1 z &= f_1 - b_1 \delta \\
\\
a_3 x + c_3 z &= f_3 - b_3 \delta
\end{aligned}
$$

$$
\begin{pmatrix} a_1 & 0 & c_1 \\ 0 & 1 & 0 \\ a_3 & 0 & c_3 \end{pmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \delta \\ f_3 \end{Bmatrix} - \delta \begin{Bmatrix} b_1 \\ 0 \\ b_3 \end{Bmatrix}
$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_6 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_6 \\ \vdots \end{Bmatrix} - \delta_1 \begin{Bmatrix} 0 \\ 0 \\ K_{15} \\ 0 \\ \vdots \end{Bmatrix} - \delta_2 \begin{Bmatrix} 0 \\ \vdots \\ K_{25} \\ 0 \\ \vdots \end{Bmatrix} - \delta_6 \begin{Bmatrix} 0 \\ \vdots \\ K_{65} \\ 0 \\ \vdots \end{Bmatrix}$$

Suppose dof k is specified

Traspose negative of the specified value X kth column to the right

Replace k th row and columns in the stiffness matrix to zero

Replace K(k,k) by 1

Set F(k)=specified value

Repeat above steps for all specified dofs.