Dynamics

I Course Overview and Basic Ideas

27/7/2016 (1)

☆ Course Outline

☆ Dynamics
- Very little and very important content from fundamentals PHYSICS
- A lot of applications in varied situations
- Problem solving and practice crucial
  - Deep-seated intuitive prejudices

☆ In the Background
- Geometry
- Vector Algebra
- Statics

☆ Statics lurking behind Dynamics
Reasonable assumptions

1/8/2016 (2)

Example 1: (a) Forces on a truss member
(b) Forces on a connecting rod

Example 2: (a) Tension in the string/rope around a pulley
(b) Tension in the rope/belt in the case of an anchor or belt-drive

Example 3: Point of application of friction force on a block on table
(a) in ordinary contact,
(b) at the point of tipping.

☆ Free-Body Diagrams

☆ Units

2/8 Tutorial
Statics 2/76, 2/83, 2/144

II Kinematics of Particles

☆ Rectilinear Motion

Def: $v = \frac{ds}{dt} = \dot{s}$, $a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = \ddot{s}$

$\Rightarrow ds = v\,dt$, $dv = a\,dt$, AND $v\,dv = a\,ds$.

$\Rightarrow s_2 - s_1 = \int_{t_1}^{t_2} v\,dt$, $v_2 - v_1 = \int_{t_1}^{t_2} a\,dt$,

AND $v_2^2 - v_1^2 = 2\int_{s_1}^{s_2} a\,ds$

Cases:

(i) Constant Acceleration

$v = v_0 + at$, $s = s_0 + v_0 t + \frac{1}{2}at^2$, $v^2 = v_0^2 + 2as$.

(ii) Acceleration $a = f(t)$

$v = v_0 + \int_0^t f(t)\,dt$, $s = s_0 + \int_0^t v\,dt$.

(iii) Acceleration $a = f(v)$

$t = \int_0^t dt = \int_{v_0}^{v} \frac{dv}{f(v)} = g(v)$

Hopefully,

$v = g^{-1}(t) = h(t)$ AND $s = s_0 + \int_0^t h(t)\,dt$

~~OR~~ $\cancel{s = \int_{s_0}^{s} ds = \int_{v_0}^{v} \frac{v\,dv}{f(v)}}$

OR, $s = s_0 + \int_{s_0}^{s} ds = s_0 + \int_{v_0}^{v} \frac{v\,dv}{f(v)} = p(v)$

Hopefully,

$v = p^{-1}(s) = q(s)$ AND $t = \int_{s_0}^{s} \frac{ds}{q(s)} = r(s)$

and again hopefully $s = r^{-1}(t)$.

(iv) Acceleration $a = f(s)$

$v^2 = v_0^2 + 2\int_{s_0}^{s} f(s)\,ds \Rightarrow v = q(s)$

AND $t = \int_{s_0}^{s} \frac{ds}{q(s)} = r(s) \Rightarrow s = r^{-1}(t)$.

II (Contd) 3/8/2016 (3)

Assignment 1: 2/48, 2/76, 2/98, 2/129, 2/139, 2/151 and ANY ONE of 2/256 and 2/258 (due 8/8).

☆ Plane Curvilinear Motion

Position $\mathbf{r}$, Velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = v\mathbf{e}_t$, $v = |\mathbf{v}| = \frac{ds}{dt} = \dot{s}$; where s = arc-length (of trajectory).

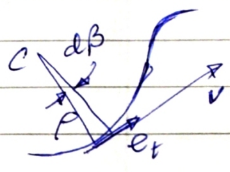
Acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$. $[a = |\mathbf{a}| \neq \dot{v}]$

☆ Plane Motion in Cartesian Coordinates (x-y)

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad \mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}, \quad \mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j};$$
$$v = \sqrt{v_x^2 + v_y^2}, \quad \tan\psi = v_y/v_x, \quad a = \sqrt{a_x^2 + a_y^2} \text{ etc.}$$

→ Projectile motion a case in which this representation (itself) is very convenient. [There are many.]

☆ Plane Motion in Tangential and Normal Coordinates (t-n)



$$\mathbf{v} = v\mathbf{e}_t = \rho\dot{\beta}\mathbf{e}_t$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{e}_t) = \dot{v}\mathbf{e}_t + v\dot{\mathbf{e}}_t$$
$$= \dot{v}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{d\beta}\frac{d\beta}{dt} = \dot{v}\mathbf{e}_t + v\dot{\beta}\mathbf{e}_n$$
$$a_t = \dot{v} = \ddot{s}, \quad a_n = v\dot{\beta} = \frac{v^2}{\rho} = \rho\dot{\beta}^2, \quad a = \sqrt{a_n^2 + a_t^2}.$$

→ Circular motion is an important case where this representation is most convenient.

☆ Plane Motion in Polar Coordinates (r-θ)

$$\mathbf{r} = r\mathbf{e}_r, \quad \mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$
$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

II (Contd)

$$a = \dot{v} = \ddot{r}\, e_r + \dot{r}\, \dot{e}_r + \dot{r}\dot{\theta}\, e_\theta + r\ddot{\theta}\, e_\theta + r\dot{\theta}\, \dot{e}_\theta$$
$$= (\ddot{r} - r\dot{\theta}^2)\, e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\, e_\theta$$
$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

Velocity changes:

Magnitude of $v_r = \dot{r} \longrightarrow \ddot{r}\, e_r$

Direction of $v_r = \dot{r} \longrightarrow \dot{r}\dot{\theta}\, e_\theta$

Magnitude of $v_\theta = r\dot{\theta} \longrightarrow (r\ddot{\theta} + \dot{r}\dot{\theta})\, e_\theta$

Direction of $v_\theta = r\dot{\theta} \longrightarrow r\dot{\theta}^2 (-e_r)$

→ Analysis of circular motion is equally convenient in this representation.

8/8/2016 (4)

// These representations and formulations can handle many situations in which all points on all bodies of interest move in parallel plane, i.e. every point moves in its own plane.

Further, in many problems, rigid bodies and their rotations also may be involved; but these methods still suffice if the interest is on specific points.

☆ Motion in (3-d) Space

Rectangular Coordinates (x-y-z)

$$R = x\,i + y\,j + z\,k \qquad [= r + zk]$$
$$v = \dot{R} = \dot{x}\,i + \dot{y}\,j + \dot{z}\,k$$
$$a = \dot{v} = \ddot{R} = \ddot{x}\,i + \ddot{y}\,j + \ddot{z}\,k$$

II (Contd)

Cylindrical Coordinates $(r-\theta-z)$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z \equiv z.$$

$$R = r\,e_r + z\,k$$

$$v = \dot{r}\,e_r + r\dot{\theta}\,e_\theta + \dot{z}\,k, \quad v = \sqrt{\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2}$$

$$a = (\ddot{r} - r\dot{\theta}^2)\,e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,e_\theta + \ddot{z}\,k,$$

$$a = |a| = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2 + \ddot{z}^2}$$

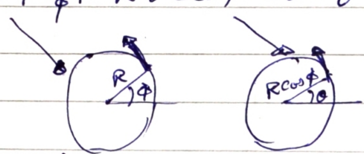
OMITTING

Spherical Coordinates $(R-\phi-\theta)$

$$x = r\cos\theta = R\cos\phi\cos\theta, \quad y = r\sin\theta = R\cos\phi\sin\theta, \quad z = R\sin\phi$$

$\phi \equiv$ latitude $\in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta =$ longitude $\in [-\pi, \pi)$ or $[0, 2\pi)$

$$v = v_R\hat{e}_R + v_\phi\hat{e}_\phi + v_\theta\hat{e}_\theta = \dot{R}\,\hat{e}_R + R\dot{\phi}\,\hat{e}_\phi + R\dot{\theta}\cos\phi\,\hat{e}_\theta$$



Meridian plane

$$a_R = \ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi$$

$$a_\phi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin\phi\cos\phi$$

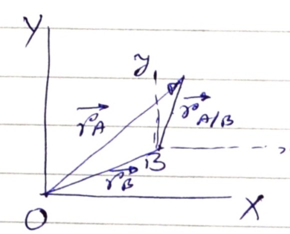
$$a_\theta = \frac{\cos\phi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\phi}\sin\phi.$$

☆ Relative Motion (Translating ~~Axes~~ Frames)

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$



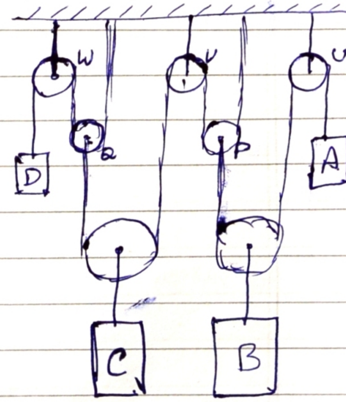
If $\mathbf{a}_B = 0$, then frame B is an inertial frame.

Any frame of reference translating with constant absolute velocity is as good as a fixed reference.

II (Contd)

Constrained Motion of connected particles

Ex. 1 : Given the motion of A and D, can you determine the motion of B and C?



No. Why?

Degrees of Freedom: Number of independent coordinates needed to specify the configuration of the system completely.

→ Assign notations for coordinates to describe the position of each particle (body) of interest.

y_A, y_B, y_C, y_D. Count: $n = 4$.

→ Impose all the constraints (m of them) on them to get relationships among them. At this stage, if needed, bring in additional particles (bodies) and their coordinates, e.g. y_P, y_Q in this case. Get n'.

$y_A + y_B + (y_B - y_P) = C_1$ or, $y_A + 2y_B - y_P = C_1$

$y_P + y_P + y_C + (y_C - y_Q) = C_2$ or, $2y_P + 2y_C - y_Q = C_2$

$y_Q + y_Q + y_D = C_3$ or, $2y_Q + y_D = C_3$

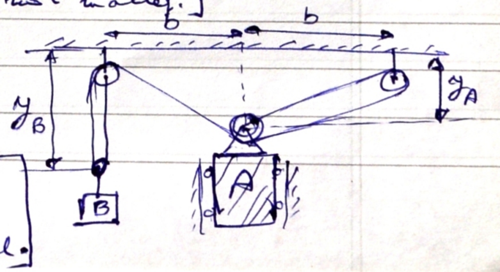
→ Number of degrees of freedom: $f = n' - m$.

→ If desirable, eliminate additional coordinates: $4y_A + 8y_B + 4y_C + y_D = C$

→ Hence, velocity and acceleration constraints.

[Pulley diameters did not matter.]

Ex. 2 Velocity ratio v_B / v_A?



$$2y_B + 3\sqrt{y_A^2 + b^2} = \text{Const.}$$

$$\Rightarrow 2v_B + \frac{3y_A}{\sqrt{y_A^2+b^2}} v_A = 0$$

$$\Rightarrow \frac{v_B}{v_A} = -\frac{3y_A}{\sqrt{y_A^2+b^2}}$$

[Pulley diameters neglected as small.]

10/8/2016 : Cancelled

~~5~~ 17/8/2016 (5)

III Kinetics of Particles

☆ Newton's Second Law ... and Third Law

$$\Sigma \vec{F} = m \vec{a}$$

☆ Two types of problems

- Particle motion, some forces known;
  Some other forces to be determined.
  May arise
  - when we want to work out 'input forces' needed to give rise to a 'desired motion',
  - when the details of motion is available from observation and we investigate the cause.

  Relatively straightforward to solve
  ↳ just by applying equations of motion
  Essentially a problem of a single instance
  → even if repeated many times.

- All forces known,
  Resulting motion to be determined.
  → Prediction of the response of a system.
  (Simulation)
  Differential equations are to be solved.
  → Initial conditions to be needed.
  Difficult problem — over a duration typically.
  Equations of motion give (algebraically) just acceleration : to know motion integration over time required
  ↳ Intermediate accuracy → Final accuracy
  Complete initial condition : Position and Velocity
  Approximation involved in modelling as well as solution

~~Lecture~~ III (Contd)

☆ Unconstrained Motion: 3-dof motion of a particle in 3-d space.

☆ Constrained Motion: In case of connected/guided motion
↳ Constraining agency gives forces.
These force (known/unknown) are to be accounted for.
Analysis involves identification of degrees of freedom in which
- motion is restrained
- motion is free

☆ Free-Body Diagram
- Isolate the particle/body
- Replace all other bodies by the forces they apply on this particle/body
- Show directions of forces appropriately, typically also showing a frame of reference.
- Do not duplicate forces
- Do not show acceleration in the same diagram.

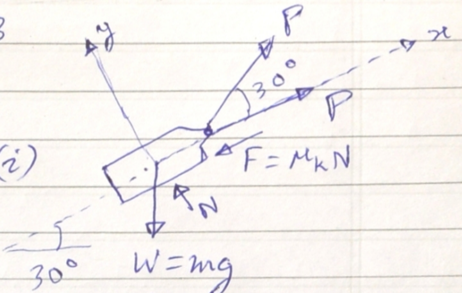
☆ Examples
Prob 3/13: Later

☆ Solution Methodology
- From FBD, collect components and apply
  $\Sigma F_x = m a_x$, $\Sigma F_y = m a_y$ etc
  or, $\Sigma F_n = m a_n$, $\Sigma F_t = m a_t$ ~~etc~~
  or, $\Sigma F_r = m a_r$, $\Sigma F_\theta = m a_\theta$
  for each body under consideration.
- Examine number of unknowns and number of independent equations
- Determine solvability/indeterminacy/conflict.
- Look for efficient solution tactics/other issues.

III (Contd)

22/8/2016 (6)

Example Problems: 3/13 and 3/43



Ex. 3/13

$\sum F_y = 0$:

$N - mg\cos 30^\circ + P\sin 30^\circ = 0 \quad$ (i)

$\sum F_x = ma$:

$P(1+\cos 30^\circ) - \mu_k N - mg\sin 30^\circ = ma \quad$ (ii)

Solving (i) and (ii), $P = 227\,N$.

Using $\mu_k = 0.25$, Eqn (i) + 4 × Eqn (ii):

$P[\sin 30^\circ + 4(1+\cos 30^\circ)]$

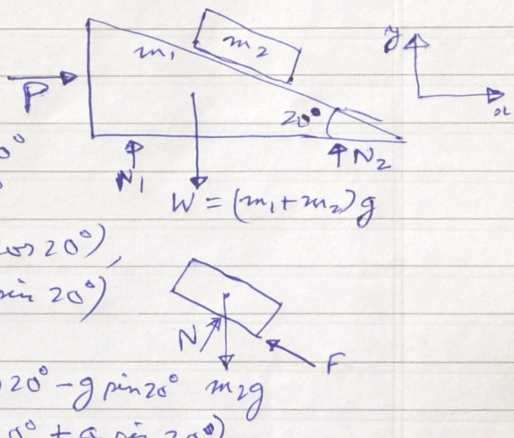
$= m[4a + g(\cos 30^\circ + 4\sin 30^\circ)]$

or, $7.964\,P = 50[8 + 9.81 \times 2.866] = 1806\,N$

$\Rightarrow P = 227\,N$

Ex. 3/43

Approach I:



$\sum F_x = Ma \Rightarrow P = (m_1+m_2)a$.

For mass m_2, assuming no slip,

$N - m_2 g\cos 20^\circ = m_2 a\sin 20^\circ$

$m_2 g\sin 20^\circ - F = m_2 a\cos 20^\circ$

Hence, $F = m_2(g\sin 20^\circ - a\cos 20^\circ)$,

$N = m_2(g\cos 20^\circ + a\sin 20^\circ)$

For no slip, $|F| \le \mu_s N$

or, $-\mu_s(g\cos 20^\circ + a\sin 20^\circ) \le a\cos 20^\circ - g\sin 20^\circ$

$\le \mu_s(g\cos 20^\circ + a\sin 20^\circ)$

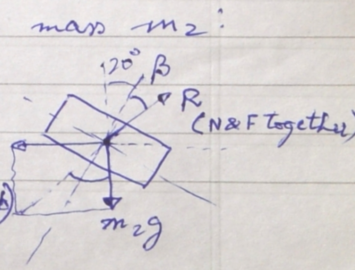
$\Rightarrow g(\sin 20^\circ - \mu_s\cos 20^\circ) \le a(\cos 20^\circ + \mu_s\sin 20^\circ)$

& $a(\cos 20^\circ - \mu_s\sin 20^\circ) \le g(\sin 20^\circ + \mu_s\cos 20^\circ)$

or, $g\tan(20^\circ - \phi) \le a \le g\tan(20^\circ + \phi) \Rightarrow$ Limits on P.

Approach II: D'Alembert's principle on mass m_2:

$$\tan(20^\circ + \beta) = \frac{a}{g} = \frac{P}{(m_1+m_2)g}$$



For no slip, $-\phi \le \beta \le \phi$

$\Rightarrow (m_1+m_2)g\tan(20^\circ - \phi) \le P \le (m_1+m_2)g\tan(20^\circ + \phi)$

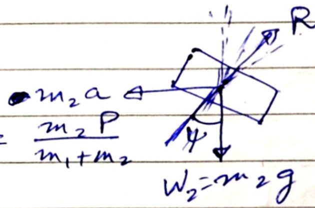
III (Contd) 24/8/2016 (T)

Example Problem: 3/43 Alternative ~~(D'Alembert)~~ Method
(Using D'Alembert's principle)

$$\Sigma \vec{F} = m\vec{a} \quad \equiv \quad \Sigma \vec{F} - m\vec{a} = 0.$$

★ Including INERTIA FORCE $= -m\vec{a}$,
a dynamics problem converted to
a problem of 'dynamic' equilibrium.

$$\theta - \phi \le \psi \le \theta + \phi$$



$\theta = 20°$
$\phi = \tan^{-1}(0.3)$
$= 16.7°$

$$\tan\psi = \frac{m_2 P}{m_1 + m_2} \Big/ m_2 g = \frac{P}{(m_1+m_2)g} = \frac{m_2 P}{m_1+m_2}$$

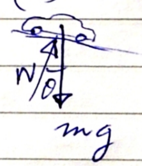
Hence

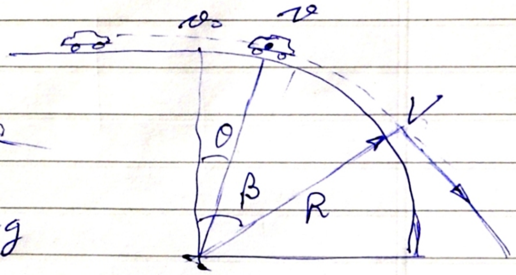
$$\tan(\theta - \phi) \le \frac{P}{(m_1+m_2)g} \le \tan(\theta + \phi)$$

$$\tan 3.3° \le \frac{P}{(m_1+m_2)g} \le \tan(36.7°)$$

Example Problem: 3/93

$$mg\cos\theta - N = m\frac{v^2}{R} \quad \text{---(i)}$$





$$mg\sin\theta = m\frac{dv}{dt}$$

or, $$\frac{v}{R}\frac{dv}{d\theta} = g\sin\theta$$

$$\int_{v_0}^{V} v\,dv = gR\int_0^{\beta}\sin\theta\,d\theta$$

$$V^2 = v_0^2 + 2gR(1-\cos\beta)$$

In eqn (i),

when $\theta = \beta$, $v = V$, $N = 0$.

$$\frac{V^2}{R} = g\cos\beta$$

Hence, $$gR\cos\beta = v_0^2 + 2gR(1-\cos\beta)$$

or, $$3gR\cos\beta = v_0^2 + 2gR$$

or, $$\cos\beta = \frac{2}{3} + \frac{v_0^2}{3gR}.$$

III (Contd)

Kinematics : $v\,dv = a\,ds$ & $dv = a\,dt$

Kinetics : $a = \Sigma F/m$

Inserting, we will get

$$m v\,dv = (\Sigma F)\,ds \quad \& \quad m\,dv = (\Sigma F)dt$$

leading to Work-Energy & Impulse-Momentum

★ Work and Kinetic Energy

$$dU = \vec{F}\cdot d\vec{r} = F_t\,ds$$

$$U = \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 (F_x dx + F_y dy + F_z dz) = \int_{s_1}^{s_2} F_t\,ds$$

Work done by a spring force $-kx$ is

$$U_{1-2} = \int_1^2 (-kx)\,dx = \frac{1}{2}k(x_1^2 - x_2^2)$$

Work done by weight for the case of small altitude change:

$$U_{1-2} = \int_1^2 (-mg)\,dy = mg(y_1 - y_2)$$

Work done by weight for large altitude change:

$$U_{1-2} = \int_1^2 \left(-\frac{G m_e m}{r^2}\right)\hat{e}_r\cdot d\vec{r} = -G m_e m \int_{r_1}^{r_2} \frac{dr}{r^2}$$

$$= G m_e m \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad \text{or} \quad mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

where g = accel due to gravity at earth's surface, $r = R$.

$$g = \frac{G m_e}{R^2}$$

29/8/2016 (8)

• Using $\Sigma\vec{F} = m\vec{a}$ in

$$U_{1-2} = \int_1^2 (\Sigma\vec{F})\cdot d\vec{r} = \int_1^2 F_t\,ds,$$

$$U_{1-2} = m\int_1^2 \vec{a}\cdot d\vec{r} = m\int_1^2 \left(\frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n\right)\cdot(ds\,\hat{e}_t)$$

$$= m\int_1^2 \frac{dv}{dt}\,ds = m\int_1^2 v\,dv = \frac{1}{2}m(v_2^2 - v_1^2)$$

III (Contd)

Principle of work and kinetic energy:

Kinetic energy $T = \frac{1}{2} m v^2$ (Def)

$$U_{1-2} = T_2 - T_1 = \Delta T$$

or $T_1 + \Delta T = T_2$ or $T_1 + U_{1-2} = T_2$

Work-energy method avoids

- → unnecessary computation of acceleration,
- → involvement of forces doing no work, and
- → involvement of equal and opposite forces between two particles (bodies) if the point(s) of mutual interaction under go(es) the same motion.

Power

$$P = \frac{dU}{dt} = \vec{F} \cdot d\vec{r}/dt = \vec{F} \cdot \vec{v}$$

Efficiency of a Machine

$$e_m = \frac{P_{out}}{P_{in}}$$

★ Potential Energy

Conservative force field:

Whenever a force $\vec{F}(\vec{r})$

or $\vec{F}(x, y, z) = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

is such that

$$U_{1-2} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 (F_x\, dx + F_y\, dy + F_z\, dz)$$

depends only on the end-points $\vec{r}_1$ and $\vec{r}_2$ and not at all on the path,

we call it a CONSERVATIVE FORCE.

III (Contd)

This would happen when

$$\vec{F}\cdot d\vec{r} = F_x dx + F_y dy + F_z dz$$

is an exact (perfect) differential,

i.e. differential of some scalar function $-V(x,y,z)$, such that $\vec{F} = -\nabla V$,

in which case, $U_{1-2} = -\int_1^2 dV = V_1 - V_2$

$$= V(x_1,y_1,z_1) - V(x_2,y_2,z_2)$$

$V(x,y,z)$: Potential energy.

→ Gravitational p.e. (small altitude change):

$$V_g = \int_{y_0} mg\,dy = mg(y-y_0) = mgh$$

→ Gravitational p.e. (large altitude change):

$$\Delta V_g = \int_{r_1}^{r_2} \frac{mgR^2}{r^2}\,dr = mgR^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = V_{g_2} - V_{g_1}$$

With $r_1 = \infty$, $r_2 = r$, $V_g = -\dfrac{mgR^2}{r}$.

→ Similarly, electrical p.e.

→ Elastic potential energy : $V_e = \frac{1}{2}kx^2$ (spring)

Work done by a conservative force

= Loss in the 'corresponding' potential energy

$= -\Delta V$

Work-energy equation:

$$U_{1-2} = U'_{1-2} - \Delta V = \Delta T$$

$$\text{or,}\quad U'_{1-2} = \Delta T + \Delta V = (T_2+V_2) - (T_1+V_1)$$

Here, U'_{1-2} = Work done by forces which are NOT incorporated in potential energy V.

III (Contd) 31/8/2016 (9)

☆ Linear Impulse and Linear Momentum

$$\Sigma \vec{F} = m\dot{\vec{v}} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{G})$$

$\vec{G} = m\vec{v}$: Linear momentum

Componentwise,

$$\Sigma F_x = \dot{G}_x \quad etc$$

$$\int_{t_1}^{t_2} \Sigma \vec{F}\, dt = \vec{G}_2 - \vec{G}_1 = \Delta\vec{G} \quad \text{or} \quad \vec{G}_1 + \underbrace{\int_{t_1}^{t_2} \Sigma \vec{F}\, dt}_{\text{Linear Impulse}} = \vec{G}_2$$

Linear impulse of a force: $\int_{t_1}^{t_2} \vec{F}\, dt$

Impulsive force:

A force with high impulse over a very small period

↳ Often motion taking place over the small period $\Delta t = t_2 - t_1$ is of little consequence

AND Exact variation of $\vec{F}$ with time is neither available nor important

In such cases, a representation constant $\vec{F}$ is often used to simplify it to $\vec{F}\Delta t$.

Conservation of Linear Momentum:

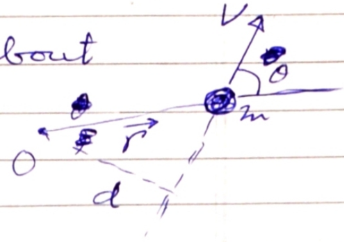
→ If resultant force on a ~~particula~~ particle (in a direction) is zero, then its momentum (in that direction) is conserved.

$\Sigma \vec{F} = 0 \Rightarrow \vec{G}_1 = \vec{G}_2$ or $\Sigma F_x = 0 \Rightarrow G_{1x} = G_{2x}$.

→ If $\vec{F}$ and $-\vec{F}$, the forces of interaction between two particles, are the only unbalanced forces on them, then their TOTAL MOMENTUM remains conserved.

III (Contd)

☆ Angular Impulse and Angular Momentum



Angular momentum of a particle about a point O

= Moment of ~~the~~ its (linear) momentum about the point

$$\vec{H}_o = \vec{r} \times m\vec{v}$$
$$= m v r \sin\theta \, \hat{n} = m v d \, \hat{n}$$
$$= m \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

Rate of change of angular momentum

Since $\vec{M}_o = \vec{r} \times \vec{F}$,

$$\Sigma \vec{M}_o = \vec{r} \times \Sigma \vec{F} = \vec{r} \times m\dot{\vec{v}}$$
$$= \vec{v} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}}$$
$$= \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}} = \frac{d}{dt}\left[\vec{r} \times m\vec{v}\right]$$
$$= \dot{H}_o$$

$\boxed{\Sigma \vec{M}_o = \dot{\vec{H}}_o}$ vs $\boxed{\Sigma \vec{F} = \dot{\vec{P}}}$

Component-wise equality. No new PHYSICS here!

Angular Impulse-Momentum principle

Integrating,

$$\int_{t_1}^{t_2} \Sigma \vec{M}_o \, dt = (\vec{H}_o)_2 - (\vec{H}_o)_1 = \Delta \vec{H}_o$$

or,

$$(\vec{H}_o)_1 + \int_{t_1}^{t_2} \Sigma \vec{M}_o \, dt = (\vec{H}_o)_2$$

⇒ Conservation of angular momentum.

Plane motion applications: Only ONE non-trivial equation!

III (Contd)

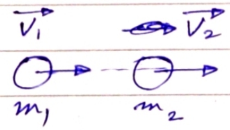
☆ Impact (or Collision)

- Impulsive forces exchanged between short time interval
- Deformation followed by (possibly) partial or complete restoration of shape (restitution)
- Accompanying changes in momenta of individual particles

Direct Central Impact

$$m_1 \vec{V_1} + m_2 \vec{V_2} = m_1 \vec{V_1}' + m_2 \vec{V_2}'$$

or $$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$



$\underline{v_1 > v_2}$

Coefficient of restitution

e = Ratio of restoration impulse magnitude to the deformation impulse magnitude

Consider first body:

$$e = \frac{\int_{t_0}^{T} F_r \, dt}{\int_0^{t_0} F_d \, dt} = \frac{m_1 (v_1' - v_0)}{m_1 (v_0 - v_1)} \qquad v_1' \leq v_0 < v_1$$

Consider second body: $$e = \frac{v_2' - v_0}{v_0 - v_2} \qquad v_2' \geq v_0 > v_2$$

That is,

$$e = -\frac{v_2' - v_1'}{v_2 - v_1}, \qquad 0 \leq e \leq 1,$$

or, $$v_2' - v_1' = -e(v_2 - v_1).$$

0: Completely inelastic

1: Perfectly elastic (mild impact, elastic bodies)

e = numerical value of the ratio of the 'relative' velocity after the impat and before.

Compensation
3/9/2016 (10)

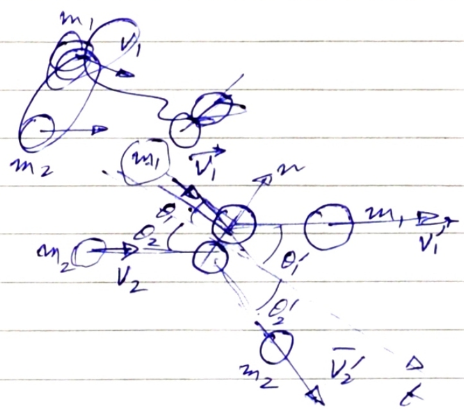
III (Contd)

Loss of 'mechanical' energy during impact
→ deformation → heat, sound, light

$$\left.\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2' \\ \& \quad v_2' - v_1' &= -e(v_2 - v_1) \end{aligned}\right\} \Rightarrow$$

$$\Delta T = -\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)(v_1 - v_2)^2$$

Oblique Central Impact



$(v_1)_t = v_1 \cos\theta_1$, $(v_1)_n = v_1 \sin\theta_1$

$(v_2)_t = v_2 \cos\theta_2$, $(v_2)_n = v_2 \sin\theta_2$

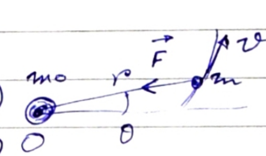
Velocities after the impact:

$$\left.\begin{aligned} (v_1')_t &= (v_1)_t \\ (v_2')_t &= (v_2)_t \end{aligned}\right\}$$

$$\left.\begin{aligned} m_1 (v_1)_n + m_2 (v_2)_n &= m_1 (v_1')_n + m_2 (v_2')_n \\ \text{and} \quad (v_2')_n - (v_1')_n &= -e\{(v_2)_n - (v_1)_n\} \end{aligned}\right\}$$

☆ Central Force Motion

→ Motion of a particle due to a force towards a fixed centre of attraction



$$\vec{F} = -\frac{G m_0 m}{r^2}\hat{e}_r$$

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{G m_0 m}{r^2} \quad ---(i)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad ----(ii)$$

Eqn. (ii) $\Rightarrow \cancel{m}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = 0$

~~Integrating~~ or, $\cancel{m}\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow \cancel{m} r^2\dot{\theta} = \cancel{m} h$

III (Contd)

$$\cancel{m} r^2 \dot\theta = \cancel{m} h = \text{constant} \quad \text{-----} \quad (iii)$$

$$\vec{H}_o = \vec{r} \times m\vec{v}$$
$$= r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot\theta\hat{e}_\theta)$$
$$= m r^2 \dot\theta \, \hat{e}_z$$
$$H_o = m r^2 \dot\theta$$

~~No force along z-direction~~

$$\frac{d}{dt}(\vec{H}_o) = \vec{r} \times \vec{F} = 0 \Rightarrow$$ Angular momentum $\vec{H}_o$ is constant. (iii)

From Eqn (iii), area swept swept through angle $d\theta$ is

$$dA = \frac{1}{2} r^2 d\theta$$
$$\Rightarrow \dot{A} = \frac{1}{2} r^2 \dot\theta = h/2 \quad (\text{const}),$$

i.e. equal areas are swept over equal intervals of time [Kepler's 2nd law.]

From Eqn. (i), we can find out the trajectory, i.e. equation between r and θ, by eliminating time.

Let $r = \frac{1}{u}$, $\dot{r} = -\frac{1}{u^2}\dot{u}$ or, $u^2\dot{r} = -\dot{u}$

Since $\dot\theta = h/r^2 = hu^2$,

$$\frac{u^2\dot{r}}{hu^2} = -\frac{\dot{u}}{\dot\theta} \quad \text{or,} \quad \dot{r} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot\theta = -h\frac{d^2u}{d\theta^2}(hu^2) = -h^2u^2\frac{d^2u}{d\theta^2}$$

Using $r = \frac{1}{u}$, $\dot{r} = -h\frac{du}{d\theta}$ and $\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}$ in Eqn(i),

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(hu^2)^2 = -Gm_o u^2$$

$$\text{or,} \quad \boxed{\frac{d^2u}{d\theta^2} + u = \frac{Gm_o}{h^2}}$$

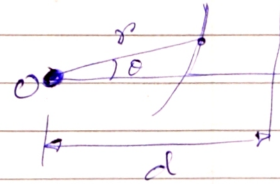
Solution: $u = \frac{1}{r} = C\cos(\theta + \delta) + \frac{Gm_o}{h^2}$

By choice of x-axis ($\theta = 0$) at minimum r, $\delta = 0$.

That is $\frac{1}{r} = C\cos\theta + \frac{Gm_o}{h^2}$.

III (Contd) 5/9/2016 (11)

Conic Section:



$$\frac{\text{Distance from focus}}{\text{Distance from Directrix}} = e \ (\text{const})$$

or, $\dfrac{r}{d - r\cos\theta} = e$ or, $r(1 + e\cos\theta) = ed$

or, $\dfrac{1}{r} = \dfrac{\cos\theta}{d} + \dfrac{1}{ed}$

Comparing with $\dfrac{1}{r} = C\cos\theta + \dfrac{Gm_0}{h^2}$,

$d = \dfrac{1}{C}$ and $e = Ch^2/Gm_0$.

As $e<1$, $e=1$ or $e>1$, the orbit is elliptic, parabolic or hyperbolic.

Parabolic/Hyperbolic orbits are open, i.e. non-periodic.

For elliptic orbits, further,

r is minimum at $\theta = 0$, $r_{min} = \dfrac{ed}{1+e}$;

r is maximum at $\theta = \pi$, $r_{max} = \dfrac{ed}{1-e}$.

$r_{min} + r_{max} = \dfrac{2ed}{1-e^2} = 2a$, a being major radius.

and $r_{max} - r_{min} = \dfrac{2e^2d}{1-e^2} = 2ae$.

$r_{min} = a(1-e)$, $r_{max} = a(1+e)$.

Planets move in elliptic orbit around the sun at a focus [Kepler's 1st law.]

Period for the elliptic orbit:

$$T = \frac{A}{\dot{A}} = \frac{\pi ab}{\frac{1}{2}r^2\dot{\theta}}$$ [where $b = a\sqrt{1-e^2}$

$$= \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{ed\,Gm_0}} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{a(1-e^2)Gm_0}}$$

$$= \frac{2\pi}{\sqrt{Gm_0}} a^{3/2}$$

Square of the period of motion is proportional to the cube of semimajor axis of the orbit.

[Kepler's 2nd law.]

III (Contd)

The Conquest of Human Science over ignorance & bigotry

Nikolaus Copernicus	Tycho Brahe ↔	Johannes Kepler	Galileo Galilei
1473-1543	1546-1601	1571-1630	1564-1642
(Poland)	(Denmark)	(Germany)	(Italy)

Sir Isaac Newton
1642-1726 (England)

d'Alembert, Euler, Lagrange, Laplace, Hamilton
and others

Energy Analysis

$$E = T + V = \frac{1}{2}mv^2 - \frac{Gm_0 m}{r} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{Gm_0 m}{r}$$

Conservative system: E is same at perigee ($\theta = 0$, $r = r_{min}$, $\dot{r} = 0$)

$$\frac{1}{r} = C + \frac{Gm_0}{h^2}$$

$$\frac{2E}{m} = \frac{h^2}{r^2} - \frac{2Gm_0}{r}$$

$$= \left(hC + \frac{Gm_0}{h}\right)^2 - 2Gm_0\left(C + \frac{Gm_0}{h^2}\right)$$

$$= h^2C^2 + \frac{G^2m_0^2}{h^2} + 2CGm_0 - 2GM_0C - \frac{2G^2m_0^2}{h^2}$$

$$= h^2C^2 - \frac{G^2m_0^2}{h^2} \qquad \left[e = \frac{Ch^2}{Gm_0}\right]$$

$$= \left(\frac{eGm_0}{h}\right)^2 - \left(\frac{Gm_0}{h}\right)^2$$

or, $$e = \sqrt{1 + \frac{2Eh^2}{mG^2m_0^2}} \quad \begin{cases} E > 0, & \text{hyperbolic orbit} \\ E = 0, & \text{parabolic orbit} \\ E < 0, & \text{elliptic orbit} \end{cases}$$

Velocity:

$$\frac{1}{2}mv^2 - \frac{Gm_0 m}{r} = E = -\frac{m}{2}(1-e^2)\frac{G^2m_0^2}{h^2}$$

Using $e = \frac{Ch^2}{Gm_0}$, $ed = \frac{h^2}{Gm_0}$, $\frac{1}{C} = d = \frac{a(1-e^2)}{e}$,

$$E = -\frac{Gm_0 m}{2a} \quad \text{and} \quad v = 2Gm_0\left(\frac{1}{r} - \frac{1}{2a}\right)$$

$\boxed{Gm_0 = gR^2}$ may be used.

IV Kinetics of Systems of Particles

7/9/2016 (12)

☆ Generalized Newton's Second Law

Consider ~~a number of~~ n particles. ~~Let~~

On i-th particle,

$\vec{F}_{i,1}, \vec{F}_{i,2}, \ldots \vec{F}_{i,n_i}$: Forces ~~external~~ on m_i from agencies external to this system.

$\vec{f}_{i,j}$: Forces on m_i from m_j, for several j's, may be.

Then,

$$\sum_k \vec{F}_{i,k} + \sum_j \vec{f}_{ij} = m_i \ddot{\vec{r}}_i$$

Since $\vec{f}_{ij} = -\vec{f}_{ji}$ (3rd law)

$$\sum_i \left(\sum_k \vec{F}_{i,k} + \sum_j \vec{f}_{i,j} \right) = \sum_i \left(\sum_k \vec{F}_{i,k} + 0 \right) = \sum \vec{F}$$

From definition of centre of mass,

$$m \bar{\vec{r}} = \sum m_i \vec{r}_i, \qquad \left[m = \sum m_i \right]$$

$$m \ddot{\bar{\vec{r}}} = \sum m_i \ddot{\vec{r}}_i$$

Hence

$$\boxed{\sum \vec{F} = m \ddot{\bar{\vec{r}}}}$$

Resultant of EXTERNAL forces equal to total mass multiplied with ACCELERATION OF MASS CENTRE.

[For constant-mass systems, i.e. isolated systems]

→ Can use component-wise scalar equations.

☆ ~~Note-Eqn~~ Note: We are using a Newtonian frame of reference, one which is fixed or moving with constant velocity, neither rotating ~~nor~~ nor accelerating.

IV (Contd)

★ Work-Energy

$$U_{1-2} = \Delta T = T_2 - T_1$$

where U_{1-2} = work done ~~by~~ on all the particles in the system by the external forces.

or, $$U'_{1-2} = \Delta T + \Delta V$$

where U'_{1-2} = the work done by those external forces ~~whose~~ the contributions of which are NOT incorporated in ΔV.

$$\Delta V = \Delta V_g + \Delta V_e + \cdots\cdots$$

= change in potential energy

= work done on the system by several CONSERVATIVE forces.

However, kinetic energy of a system of particles

$$T = \sum T_i = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2}\sum (m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \frac{1}{2}\sum m_i (\vec{\bar{v}} + \dot{\vec{\rho}}_i)\cdot(\vec{\bar{v}} + \dot{\vec{\rho}}_i)$$

$$\cancel{= \frac{1}{2}\sum m_i (\dot{\vec{\rho}}\cdot\dot{\vec{\rho}}) + \frac{1}{2}\sum m_i (\dot{\vec{\rho}}_i \cdot \dot{\vec{\rho}}_i)}$$

$$\cancel{+ \frac{1}{2}\sum m_i \dot{\vec{\rho}}_i \quad + \sum m_i (\dot{\vec{\rho}} \cdot \rho_i}$$

$$= \frac{1}{2}\left(\sum m_i\right)\bar{v}^2 + \frac{1}{2}\sum m_i |\dot{\rho}_i|^2 + \frac{1}{2}(m_i \vec{\bar{v}} \cdot \dot{\vec{\rho}}_i)$$

$$= \frac{1}{2} m \bar{v}^2 + \sum \frac{1}{2} m_i |\dot{\rho}_i|^2 + 0$$

i.e. kinetic energy of mass-centre translation

+ kinetic energy due to motion of individual particles relative to the mass-centre

IV (Contd)

☆ Impulse - Momentum

Linear Momentum

$$\vec{G} = \sum m_i (\vec{\bar{v}} + \dot{\vec{\rho}}_i) = (\sum m_i)\vec{\bar{v}} + \underbrace{(\sum m_i \dot{\vec{\rho}}_i)}^{\text{ZERO}}\vec{v}$$

$$= m\vec{\bar{v}}$$

and hence $\sum \vec{F} = \frac{d}{dt}(\vec{G})$ [as usual]

Angular Momentum (about a point)

~~Position of i-th mass from point P:~~ $\cancel{\vec{\rho}_i = \vec{r}_i - \vec{r}_P}$, $\cancel{\vec{\rho} = \vec{\rho} - \vec{r}_P}$

Angular momentum about P

$$H_P = \sum \vec{\rho}_i \times m_i \dot{\vec{r}}_i \cancel{= (\sum(\vec{\rho} + \vec{\rho}_i) \times m_i \dot{\vec{r}}_i}$$

If P is a fixed point O;

$$\vec{H}_O = \sum \vec{r}_i \times m_i \dot{\vec{r}}_i \quad \text{and} \quad \boxed{\sum \vec{M}_O = \frac{d\vec{H}_O}{dt} \ \text{(as usual)}}$$

Even if P is ~~fixed~~ not fixed, but mass-centre G,

$$\vec{H}_G = \sum \vec{\rho}_i \times m_i (\dot{\bar{\vec{r}}} + \dot{\vec{\rho}}_i) = \sum (m_i \vec{\rho}_i) \times \dot{\bar{\vec{r}}} + \sum (\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i)$$

$$= 0 + \sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

and $\boxed{\sum \vec{M}_G = \frac{d}{dt}(\vec{H}_G) \ \text{(as usual)}}$

But if P is a general point, then

$$H_P = \sum \vec{\rho}_i \times m_i \dot{\vec{r}}_i = \sum(\vec{\bar{\rho}} + \vec{\rho}_i) \times m_i \dot{\vec{r}}_i$$

$$= \sum \vec{\rho}_i \times m_i \dot{\vec{r}}_i + \vec{\bar{\rho}} \times \sum m \vec{v}_i = \vec{H}_G + \vec{\bar{\rho}} \times m\vec{\bar{v}}$$

and $\sum M_P = \sum \vec{M}_G + \bar{\rho} \times \sum \vec{F}$

or, $\sum M_P = \frac{d}{dt}(\vec{H}_G) + \bar{\rho} \times m\vec{\bar{a}}$ (not usual)

IV (Contd) OMITTED

☆ Conservation of Energy and Momentum

Dynamics

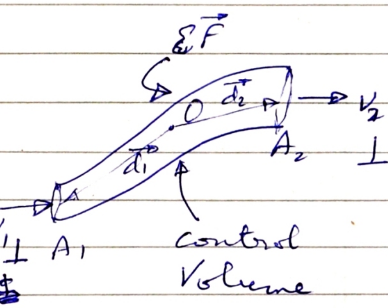
IV (Contd)

19/9/2016 (13)

- ☆ Steady Mass Flow
- ☆ Variable Mass Helicopter problem, ~~Solved 4/4~~ 4/87, Solved 4/9

ABHISHEK (19/8/2016)

* Some videos (helicopters)
* Steady Mass Flow



$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

$\Sigma \vec{F} =$ { Reaction due to support; Forces at A_1 & A_2 due to static pressure; Weight }

$$\Sigma \vec{F} = \frac{d}{dt}(\vec{G}) \quad , \quad \Delta m_1 = \Delta m_2 = \Delta m \quad (\text{Steady})$$

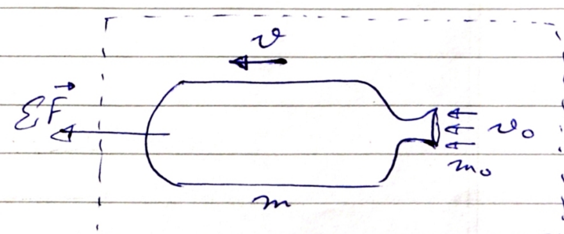
$$\Delta \vec{G} = (\Delta m)(\vec{v_2} - \vec{v_1}) \Rightarrow \frac{d}{dt}(\vec{G}) = \frac{dm}{dt}(\vec{v_2} - \vec{v_1})$$

Hence, $\boxed{\Sigma \vec{F} = m' \vec{\Delta v}}$

Next, $\Sigma \vec{M_o} = \dot{\vec{H_o}} = m'(\vec{d_2} \times \vec{v_2} - \vec{d_1} \times \vec{v_1})$

Example (Helicopter)

* Variable Mass System



$$\Sigma \vec{F} = \dot{G}$$
$$= \frac{d}{dt}(mv + m_0 v_0)$$
$$= \dot{m} v + m\dot{v} + \dot{m}_0 v_0 + m_0 \dot{v}_0$$
$$= m\dot{v} + \dot{m}(v - v_0)$$
$$= m\dot{v} + \dot{m} u$$
or, $m\dot{v} - \dot{m}_0 u$

$\dot{m} = -\dot{m}_0$
$\dot{v}_0 = 0$

$u = v - v_0$, rel. vel. of rocket w.r.t. expelled mass.

Rocket (See book for fig.): $PA - mg - R$

Example (Aircraft) 4/187

V Plane Kinematics of Rigid Bodies

21/9/2016 (14)

Particles Vs Rigid Bodies

Plane Motion
- Rectilinear translation
- Curvilinear translation
- Fixed-axis rotation
- General plane motion

★ Rotation

If a line on the planar (laminar) body makes an angle θ with a line fixed in the plane, then ~~ang~~ angular velocity

$$\omega = \dot{\theta}$$

and angular acceleration

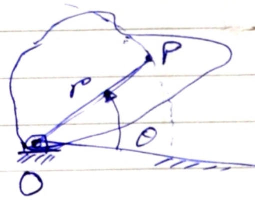
$$\alpha = \dot{\omega} = \ddot{\theta}$$

$$\begin{cases} \omega = \omega_0 + \alpha t \\ \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \end{cases}$$

Fixed-axis rotation

$s = r\theta$, $\quad v = r\omega$ $\quad$ [$r = OP$ distance

$|a_n| = r\omega^2 = \frac{v^2}{r}$, $a_t = r\alpha$



In vector notation,

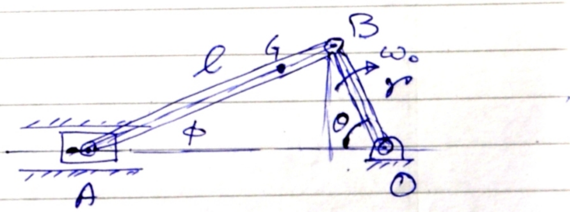
$\vec{v} = \vec{\omega} \times \vec{r}$ $\quad$ [$\vec{r}$: position vector

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\alpha} \times \vec{r}$$

Example Problem 5/58

Crank motion : ω_0 constant

ω_{AB} : ? $\quad \alpha_{AB}$: ?



Ans. $\omega_{AB} = \frac{r\omega_0}{l} \frac{\cos\theta}{\sqrt{1 - \frac{r^2}{l^2}\sin^2\theta}}$

$$\simeq \frac{r\omega_0 \cos\theta}{l}\left[1 + \frac{r^2\sin^2\theta}{2l^2}\right]$$

$$\alpha_{AB} = \frac{r\omega_0^2 \sin\theta}{l} \frac{(r/l)^2 - 1}{\left(1 - \frac{r^2}{l^2}\sin^2\theta\right)^{3/2}} \simeq -\frac{r\omega_0^2 \sin\theta}{l}\left[1 + \left(\frac{3}{2}\sin^2\theta - 1\right)\frac{r^2}{l^2}\right]$$

Dynamics

V (Contd)

☆ Relative Velocity

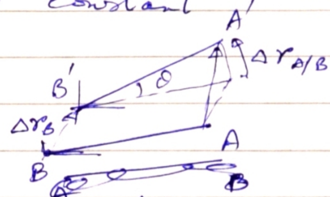
Recall $\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$, $\vec{V}_A = \vec{V}_B + \vec{V}_{A/B}$

for two points A & B.

If A and B are two points on the SAME rigid body,

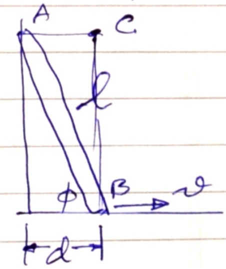
then $|\vec{r}_{A/B}| = \cancel{\text{constant}}\ AB$ constant

$\vec{V}_{A/B} = \cancel{AB}\ \vec{\omega} \times \vec{r}_{A/B}$



Example

Ladder sliding, ω : ?



Method I

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$
$$= v\hat{i} + \omega\hat{k} \times (-d\,\hat{i} + h\hat{j})$$
$$= (v - \omega h)\hat{i} - \omega d\hat{j} = -\omega d\hat{j}$$

$$\omega = \frac{v}{h} = \frac{v}{\sqrt{l^2 - d^2}}$$

Method II

$$d = l\cos\phi$$
$$\Rightarrow v = -l\sin\phi\,\dot{\phi} \Rightarrow \omega = \frac{v}{l\sin\phi} = \frac{v}{\sqrt{l^2 - d^2}}$$

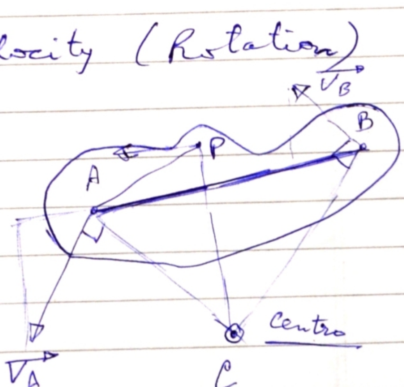
Method III

Find C. $v = \omega h \Rightarrow \omega = v/h$.

☆ Instantaneous Centre of Zero Velocity (Rotation)

$\vec{V}_A = \vec{\omega} \times \vec{CA}$, $\vec{V}_B = \vec{\omega} \times \vec{CB}$

$$\vec{V}_P = \vec{V}_A + \cancel{\vec{\omega} \times \vec{AP}}\ \vec{V}_{P/A}$$
$$= \vec{V}_A + \vec{\omega} \times \vec{AP}$$
$$= \vec{\omega} \times \vec{CA} + \vec{\omega} \times \vec{AP}$$
$$= \vec{\omega} \times (\vec{CA} + \vec{AP})$$
$$= \vec{\omega} \times \vec{CP}$$



IV (Contd) 26/9/2016 (15)

Velocity (Motion)	Force System
$\vec{\omega}\times\vec{r},\ \vec{\alpha}\times\vec{r}$	$\vec{r}\times\vec{F},\ \vec{r}\times m\vec{v}$
Instantaneous rotation about a point	Force along a line

✩ Relative Acceleration

$$\vec{V_A} = \vec{V_B} + \vec{V_{A/B}} \Rightarrow \vec{a_A} = \vec{a_B} + \vec{a_{A/B}}$$

$$\vec{a_A} = \vec{a_B} + (\vec{a_{A/B}})_n + (\vec{a_{A/B}})_t$$

$(\vec{a_{A/B}})_n : v_{A/B}^2/r = r\omega^2$ from A to B. $[r = AB]$

$(\vec{a_{A/B}})_t : \dot{v}_{A/B} = r\alpha$ perp AB

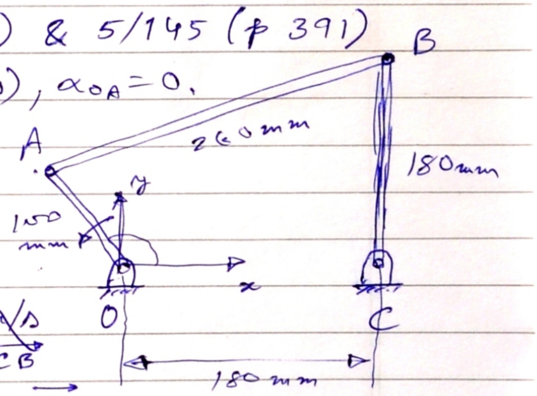
$$(\vec{a_{A/B}})_n = \vec{\omega}\times(\vec{\omega}\times\vec{r}), \quad (\vec{a_{A/B}})_t = \vec{\alpha}\times\vec{r}$$

Example problem 5/88 (p 369) & 5/145 (p 391)

$\omega_{OA} = \omega_0 = 10$ rad/s (ccw), $\alpha_{OA} = 0$.

$\vec{OA} = -60\hat{i} + 80\hat{j}$ (mm)

ω_{AB} : ? α_{AB} : ?



Soln. $\vec{AB} = 240\hat{i} + 100\hat{j}$ (mm)

~~$\vec{V_A} = \vec{\omega_0}\times\vec{OA} = (\hat{i} + 2.4\hat{j})$ m/s~~

~~$\vec{V_B} = \vec{V_A} + \vec{\omega_{AB}}\times\vec{AB} = \vec{\omega_{BC}}\times\vec{CB}$~~

$\vec{V_B} = \vec{V_A} + \vec{\omega_{AB}}\times\vec{AB} = \vec{\omega_{AC}}\times\vec{CB}$

or, $\vec{\omega_0}\times\vec{OA} + \vec{\omega_{AB}}\times\vec{AB} = \vec{\omega_{BC}}\times\vec{CB}$

or, $-800\hat{i} - 600\hat{j} - 100\omega_{AB}\hat{i} + 240\omega_{AB}\hat{j} = -180\omega_{BC}\hat{i}$

$\Rightarrow \omega_{AB} = \frac{600}{240} = 2.5$ rad/s, $\omega_{BC} = \frac{1050}{180} = 5.8$ rad/s

$\vec{a_B} = \vec{a_A} - \omega_{AB}^2\vec{AB} + \vec{\alpha_{AB}}\times\vec{AB} = -\omega_{BC}^2\vec{CB} + \vec{\alpha_{BC}}\times\vec{CB}$

$-\omega_0^2\vec{OA} - \omega_{AB}^2\vec{AB} + \vec{\alpha_{AB}}\times\vec{AB} = -\omega_{BC}^2\vec{CB} + \vec{\alpha_{BC}}\times\vec{CB}$

or, $-100(-60\hat{i} + 80\hat{j}) - (2.5)^2(240\hat{i} + 100\hat{j}) + \alpha_{AB}(-100\hat{i} + 240\hat{j})$

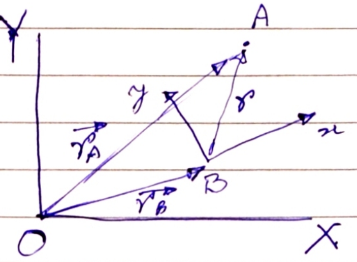
$= -(5.8)^2 180\hat{j} - \alpha_{BC}180\hat{i}$

Equating $\hat{j}$ terms, $\alpha_{AB} = 10.42$ rad/s^2 $\quad \frac{8625 - 6200}{240}$

V (Contd) 28/9/2016 (16)

✲ Motion Relative to Rotating Axes

$$\vec{r}_A = \vec{r}_B + \vec{r} = \vec{r}_B + (x\hat{i} + y\hat{j})$$



Motion of A

Motion of B ($\dot{\vec{r}}_B$)

Motion of A as perceived by B

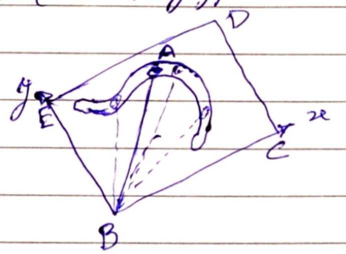
$$V_{rel} = \dot{x}\hat{i} + \dot{y}\hat{j} = \left(\frac{d\vec{r}}{dt}\right)_{xy}$$

Discrepancy in B's perception due to its rotation

$$x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt}$$
$$= x(\omega\hat{j}) + y(-\omega\hat{i})$$
$$= x(\vec{\omega}\times\hat{i}) + y(\vec{\omega}\times\hat{j})$$
$$= \vec{\omega}\times(x\hat{i} + y\hat{j}) = \vec{\omega}\times\vec{r}$$

Thus,

$$\dot{\vec{r}}_A = \underbrace{\dot{\vec{r}}_B + \vec{\omega}\times\vec{r}} + \vec{V}_{rel}$$



$\vec{V}_{rel}$ = Velocity of the particle A relative to the point P of the board BCDE coincident (instantaneously) with A

$\vec{\omega}\times\vec{r}$ = Velocity of that coincident point P due to rigid body rotation of board BCDE

$\dot{r}_B$ = Velocity of that point P due to translation of BCDE.

$$\vec{V}_A = \underbrace{\vec{V}_B + \vec{\omega}\times\vec{r}}_{\vec{V}_P} + \vec{V}_{rel}$$

$$\vec{V}_{A/B} = \left(\frac{d\vec{r}}{dt}\right)_{XY}$$

$$\vec{V}_{A/P} = \left(\frac{d\vec{r}}{dt}\right)_{xy}$$

$$\left(\frac{d\vec{q}}{dt}\right)_{XY} = \left(\frac{d\vec{q}}{dt}\right)_{xy} + \vec{\omega}\times\vec{q}$$

V (Contd)

$$\vec{V}_A = \vec{V}_B + \vec{\omega}\times\vec{r} + \vec{V}_{rel}$$

$$\vec{a}_A = \frac{d}{dt}(\vec{V}_B) + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} + \frac{d}{dt}(\vec{V}_{rel})$$

$$= \vec{a}_B + \vec{\alpha}\times\vec{r} + \vec{\omega}\times\left[\vec{\omega}\times\vec{r} + \vec{V}_{rel}\right] + \left[\vec{\omega}\times\vec{V}_{rel} + \vec{a}_{rel}\right]$$

$$= \vec{a}_B + \underbrace{\vec{\alpha}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\vec{a}_{P/B}} + \underbrace{2\vec{\omega}\times\vec{V}_{rel} + \vec{a}_{rel}}_{\vec{a}_{A/P}}$$

$$\underbrace{\vec{a}_B + \vec{a}_{P/B}}_{\vec{a}_P} \qquad \underbrace{\vec{a}_{P/B} + \vec{a}_{A/P}}_{\vec{a}_{A/B}}$$

3/10/2016 (17)

Example problem: Ghosh & Mallik 2.10 (p 81-84)

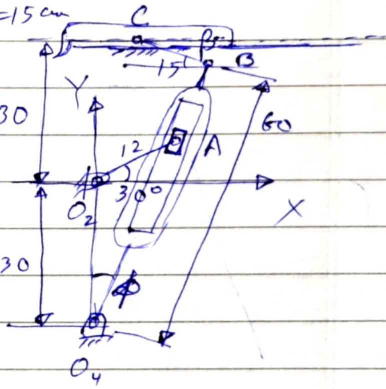
← return
→ cut

$O_2A = 12\,cm$, $O_2O_4 = 30\,cm$, $O_4B = 60\,cm$, $BC = 15\,cm$

line of C : 30 cm above O_2.

$\omega_{O_2A} = 30$ rpm ccw const.

$= \pi$ rad/s



Position

$O_4A \sin\phi = O_2A \cos 30^\circ$

$O_4A \cos\phi = O_2O_4 + O_2A \sin 30^\circ$

$\Rightarrow O_4A ✓, \phi ✓$

$O_4B\cos\phi + BC\sin\beta = 60 \Rightarrow \beta ✓$

$x_c = O_4B\sin\alpha - BC\cos\beta$

Velocity

$\vec{V}_A = \omega_{O_2A}\, O_2A\, (-\sin 30^\circ\,\hat{i} + \cos 30^\circ\,\hat{j})$

$\vec{V}_A = \omega_{O_4B}\, O_4A\, (-\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$
$+ V_{rel}(\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$

$\Rightarrow \omega_{O_4B} ✓, V_{rel} ✓$

V (Contd)

$$\vec{V_B} = \omega_{O_4B}\, O_4B\,(-\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$

$$\vec{V_C} = \vec{V_B} + \omega_{BC}\, BC\,(-\sin\beta\,\hat{i} - \cos\beta\,\hat{j}) = V_C\,\hat{i}$$

$$\Rightarrow \omega_{BC}\checkmark,\quad V_C\checkmark \qquad \boxed{V_C^{\text{(i)}}}\ \checkmark$$

Acceleration

$$\vec{a_A} = \omega_{O_2A}^2\, O_2A\,(-\cos 30^\circ\,\hat{i} - \sin 30^\circ\,\hat{j}) + 0$$

$$\vec{a_A} = \omega_{O_4B}^2\, O_4A\,(-\sin\phi\,\hat{i} - \cos\phi\,\hat{j}) + \alpha_{O_4B}\, O_4A\,(-\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$
$$+ 2\,\omega_{O_4B}\, V_{rel}\,(-\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$
$$+ a_{rel}\,(\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$$\Rightarrow \alpha_{O_4B}\checkmark,\quad a_{rel}\checkmark$$

$$\vec{a_B} = \omega_{O_4B}^2\, O_4B\,(-\sin\phi\,\hat{i} - \cos\phi\,\hat{j}) + \alpha_{O_4B}\, O_4B\,(-\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$

$$\vec{a_C} = \vec{a_B} + \omega_{BC}^2\, BC\,(\cos\beta\,\hat{i} - \sin\beta\,\hat{j}) + \alpha_{BC}\, BC\,(-\sin\beta\,\hat{i} - \cos\beta\,\hat{j})$$
$$= a_C\,\hat{i}$$

$$\Rightarrow \alpha_{BC}\checkmark,\quad a_C\checkmark \qquad \boxed{a_C^{\text{(ii)}}}\ \checkmark$$

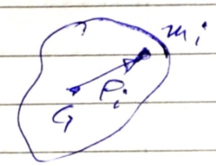
VI Plane Kinetics of Rigid Bodies

Rigid body : thin slab/lamina,
Motion : in the plane, containing mass centre
Forces : Projection in the plane of motion
Moments : in the plane of motion, i.e. axis $\perp$ plane of motion

☆ General Equations of Motion

$$\Sigma \vec{F} = m\,\vec{\bar{a}}, \qquad \vec{\bar{a}} : \text{acceleration of mass centre}$$

$$\Sigma \vec{M}_G = \dot{\vec{H}}_G$$



$$\vec{H}_G = \Sigma \vec{\rho}_i \times m_i \dot{\vec{r}}_i = \Sigma \vec{\rho}_i \times m_i (\dot{\vec{\bar{r}}} + \dot{\vec{\rho}}_i)$$

$$= \underbrace{\Sigma (m_i \vec{\rho}_i)}_{\vec{0}} \times \dot{\vec{\bar{r}}} + \Sigma \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

Since $\dot{\vec{\rho}}_i = \omega \times \rho_i$,

$$\vec{H}_G = \Sigma \vec{\rho}_i \times m_i (\vec{\omega} \times \vec{\rho}_i) = \Sigma m_i \rho_i^2 \vec{\omega} - 0$$
$$= \bar{I}\,\vec{\omega}$$

Hence, $\Sigma \vec{M}_G = \dot{H}_G = \bar{I}\,\dot{\vec{\omega}} = \bar{I}\,\vec{\alpha}$

Thus, in all,
$$\boxed{\begin{aligned} \Sigma \vec{F} &= m\,\vec{\bar{a}} \\ \Sigma \vec{M}_G &= \bar{I}\,\vec{\alpha} \end{aligned}}$$

About fixed point: $\vec{H}_O = \Sigma \vec{r}_i \times m_i \vec{v}_i$

$$\dot{\vec{H}}_O = \underbrace{\Sigma \vec{v}_i \times m_i \vec{v}_i}_{0} + \Sigma r_i \times m_i \dot{\vec{v}}_i$$

$$= \Sigma r_i \times m_i \vec{a}_i = \Sigma \vec{r}_i \times m_i (-\omega^2 \vec{r}_i + \vec{\alpha} \times \vec{r}_i)$$
$$= \Sigma \vec{r}_i \times m_i (\vec{\alpha} \times \vec{r}_i)$$
$$= (\Sigma m_i r_i^2)\,\vec{\alpha} + 0$$

or, $\Sigma \vec{M}_O = \dot{\vec{H}}_O = I_O \alpha$.

VI (Contd) 5/10/2016 (18)

About an arbitrary point (P),

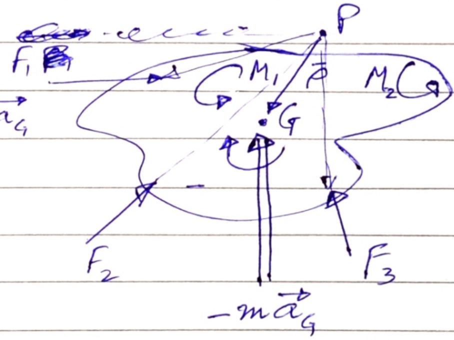
$$\Sigma M_P = \frac{d}{dt}\overline{H}_G + \vec{\bar{\rho}} \times m\vec{\bar{a}} = \bar{I}\alpha + m\bar{a}d$$

where $\vec{\bar{\rho}} = \vec{PG}$, $\vec{\bar{a}} = \vec{a}_G$

In the D'Alembert picture, it makes immediate sense.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + (-m\vec{a}_G) = 0$$

$$\vec{\rho}_1 \times \vec{F}_1 + \vec{\rho}_2 \times \vec{F}_2 + \vec{\rho}_3 \times \vec{F}_3 - \vec{\bar{\rho}} \times m\vec{a}_G + \vec{M}_1 + \vec{M}_2 + (-\bar{I}\vec{\alpha}) = 0$$



Constrained Motion: Kinematic constraints need to be taken into account BEFORE considering EoM.

System of interconnected bodies

$$\Sigma\vec{F} = \Sigma m_i \vec{\bar{a}}_i$$

$$\& \quad \Sigma M_P = \Sigma \bar{I}_i \vec{\alpha}_i + \Sigma \vec{\bar{\rho}}_i \times m_i \vec{\bar{a}}_i$$

Procedure on solving problems on rigid body dynamics

1. Kinematic Analysis
2. Diagrams
3. Equations of Motion
4. Solution
5. Post-processing and Interpretation

~~Examples:~~

~~☆ Translation $\Sigma\vec{F} = m\vec{\bar{a}}$, $\Sigma\vec{M}_G = \bar{I}\vec{\alpha} = 0$ or $\Sigma\vec{M}_P = \vec{\bar{\rho}} \times m\vec{\bar{a}}$~~

VI (Contd) 17/10/2016 (19)

☆ Translation : $\Sigma \vec{F} = m\vec{a}_G$, $\Sigma \vec{M}_P = \vec{\rho}_G \times m\,\vec{a}_G$

$P \equiv G : \vec{\rho}_G = 0$ $\quad P \equiv O : \vec{a}_G = 0$

Ex. Problem 6/11 (p 434)

☆ Fixed-Axis Rotation : $\Sigma \vec{M}_O = I_O \alpha$

Ex. Problem 6/57 (p 449)

19/10/2016 (20)

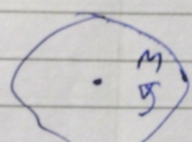
* General Plane Motion : $\Sigma \vec{F} = m\vec{a}_G$, $\Sigma \vec{M}_P = I\vec{\alpha} + \vec{\rho}_G \times m\vec{a}_G$

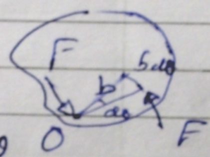
Ex. Problem 6/89 (p 464)

☆ Work-Energy Relations

Work done by moment M through angular displacement $d\theta$: $dU = M\,d\theta$ [just like $\vec{F}\cdot d\vec{r}$]

Picturing the moment as produced by two opp forces of equal magnitude, one at a fixed point and the other 'b' distance apart.





$dU = \vec{F}\cdot d\vec{r} = F(b\,d\theta) = (Fb)\,d\theta = M\,d\theta$

Kinetic Energy

$\vec{v}$: Velocity of CM

$\vec{\omega}$: Ang. vel

$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$ How?

$T = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \cancel{= \frac{1}{2}\sum m_i (\vec{v} + \vec{\omega}\times\vec{\rho}_i)}$

$= \frac{1}{2} \sum m_i (\vec{v} + \vec{\omega} \times \vec{\rho}_i)\cdot(\vec{v} + \vec{\omega} \times \vec{\rho}_i)$

VI (Contd)

$$T = \frac{1}{2}\sum m_i \bar{v}^2 + \frac{1}{2}\sum m_i\left[\omega^2\rho_i^2 - (\omega\cdot\rho_i)^2\right] \xrightarrow{} 0$$
$$+ \sum (m_i \vec{\bar{v}}\cdot\vec{\omega}\times\rho_i)$$

$$= \frac{1}{2}(\sum m_i)\bar{v}^2 + \frac{1}{2}\left(\sum m_i\rho_i^2\right)\omega^2 + \vec{\bar{v}}\cdot\vec{\omega}\times\left(\sum m_i\vec{\rho_i}\right)$$

$$= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 + 0$$

24/10/2016 (21)

Corollary or Similar Deduction

(a) Trans : $T = \frac{1}{2}m\bar{v}^2$

(b) Rot about G : $T = \frac{1}{2}\bar{I}\omega^2$

(c) Rot fixed point O : $T = \frac{1}{2}I_O\omega^2$

(d) In terms of I.C. : $T = \frac{1}{2}I_C\omega^2$

Work-Energy Equation

$$T_1 + U_{1-2} = T_2$$

or, $T_1 + V_1 + U'_{1-2} = T_2 + V_2$

Power $P = \frac{dU}{dt} = \vec{F}\cdot\vec{v} + M\omega$

$$\begin{cases} dU' = dT + dV \\ P = \frac{dU'}{dt} = \frac{dT}{dt} + \frac{dV}{dt} \end{cases}$$

$$\frac{dT}{dt} = \frac{d}{dt}\left[\frac{1}{2}m\vec{\bar{v}}\cdot\vec{\bar{v}} + \frac{1}{2}\bar{I}\omega^2\right]$$
$$= m\vec{\bar{a}}\cdot\vec{\bar{v}} + \bar{I}\omega\alpha$$
$$= \vec{R}\cdot\vec{\bar{v}} + \bar{M}\alpha$$

~~Examples~~

Example: Problem 6/135 (p 483)

Acceleration from Work-Energy Principle

For a system of inter-connected bodies,

$$dU' = dT + dV$$

Here,

$$dT = d\left[\sum \tfrac{1}{2} m_i \bar{v}_i^2 + \sum \tfrac{1}{2} \bar{I}_i \omega_i^2\right]$$

$$= \sum m_i \bar{v}_i \, d\bar{v}_i + \sum \bar{I}_i \omega_i \, d\omega_i$$

$$= \sum_{m_i} m_i \vec{\bar{a}}_i \cdot d\vec{\bar{s}}_i + \sum \bar{I}_i \alpha_i \, d\theta_i$$

$$= \sum \vec{R}_i \cdot d\bar{s}_i + \sum \vec{M}_{G_i} \cdot d\vec{\theta}_i$$

$$dV = d\left(\sum m_i g h_i + \sum \tfrac{1}{2} k_j x_j^2\right)$$

$$= \sum m_i g \, dh_i + \sum k_j x_j \, dx_j$$

[including gravitational potential energy and that in all the springs in V]

Then, differential work done by the non-conservative forces (that do not contribute to V)

$$dU' = dT + dV$$

$$= \sum m_i \vec{\bar{a}}_i \cdot d\bar{s}_i + \sum \bar{I}_i \alpha_i \, d\theta_i + \sum m_i g \, dh_i + \sum k_j x_j \, dx_j$$

Example: Problem 6/159 (p 494) 26/10/2016 (TD)

☆ Impulse-Momentum Equations

Linear Momentum

$$G = m\vec{v}, \quad \Sigma F = \dot{G}, \quad \vec{G}_1 + \int_{t_1}^{t_2} \Sigma \vec{F}\, dt = \vec{G}_2.$$

Angular Momentum

$$H_G = I\omega, \quad \Sigma M_G = \frac{dH_G}{dt},$$

$$(H_G)_1 + \int_{t_1}^{t_2} \Sigma M_G\, dt = (H_G)_2$$

$$H_O = I\omega + \vec{r} \times \vec{G} = I_O\, \omega$$

$$\Sigma M_O = \frac{dH_O}{dt}, \quad (H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O\, dt = (H_O)_2$$

→ System of inter-connected rigid bodies

→ Conservation of linear/angular momentum

Example: Prob 6/189

VII Introduction to Three-Dimensional Dynamics of Rigid Bodies

31/10/2016 (23)

Kinematics

☆ Translation

→ Every line AB in the body remains parallel to its original position.

→ All points move in
- parallel straight lines (rectilinear transl)

OR
- congruent curves shifted from one another (curvilinear translation)

☆ ~~Fixed-Axis Rotation~~ $\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$, $\vec{v}_A = \vec{v}_B$, $\vec{a}_A = \vec{a}_B$

☆ Fixed-Axis Rotation

$$\vec{v} = \vec{\omega} \times \vec{r}, \qquad \vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$= \vec{\alpha} \times \vec{r} + [\underbrace{(\vec{\omega} \cdot \vec{r})}_{?}\vec{\omega} - \omega^2 \vec{r}]$$

☆ Parallel-Plane Motion

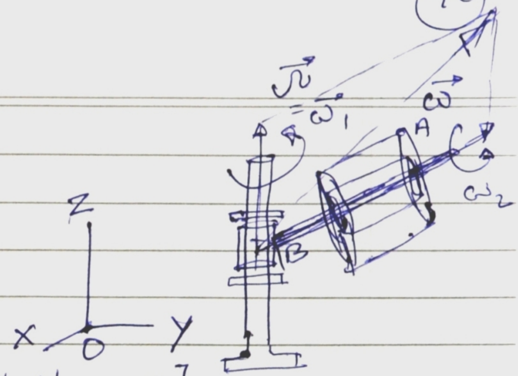
→ Similar to general plane motion

☆ Rotation about a Fixed Point

→ Rotations, in general, cannot be represented as vectors.

→ Infinitesimal rotations, however, follow rules of vector algebra and, hence, are vectors.

↳ Angular velocities and accelerations are proper vectors.



VII (Contd)

→ Instantaneous Axis of rotation

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

$$\dot{\vec{\omega}} = \vec{\alpha} = \vec{\omega}_1 \times \vec{\omega} \qquad [\text{for } |\vec{\omega}| \text{ const}]$$

Sometimes denoted by $\vec{\Omega} \times \vec{\omega}$ $\quad [\vec{\omega}_1 = \vec{\Omega}]$

$$\vec{v} = \vec{\omega} \times \vec{r}$$
$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

☆ General Motion

→ Translating Reference Axes (Body may rotate, frame does not)

$$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B}, \qquad \vec{a}_A = \vec{a}_B + \vec{a}_{A/B}$$

or, $\vec{v}_A = \vec{v}_B + \vec{\omega} \times r_{A/B}$, $\quad \vec{a}_A = \vec{a}_B + \vec{\alpha} \times r_{A/B} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

→ General (rotating as well as translating reference frame)

Denote $\vec{\Omega}$: Ang. vel of the frame
$\vec{\omega}$: Ang. vel of the body.

$\dfrac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i}$, etc. $\quad$ [Consider the top right example & let it translate as well.]

$$\vec{v}_A = \vec{v}_B + \vec{\Omega} \times \vec{r}_{A/B} + \vec{v}_{rel}$$
$$\vec{a}_A = \vec{a}_B + \dot{\vec{\Omega}} \times \vec{r}_{A/B} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{A/B}) + 2\vec{\Omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

$$\underbrace{\left(\frac{d\,[\text{A Vector}]}{dt}\right)_{XYZ}}_{\text{Absolute}} = \underbrace{\left(\frac{d\,[\text{The Vector}]}{dt}\right)_{xyz}}_{\text{In frame}} + \vec{\Omega} \times \begin{bmatrix}\text{The} \\ \text{Vector}\end{bmatrix}$$

Abs. Ang. Vel. of the frame.

Dynamics

VII (Contd) 2/11/2016 (24)

* Angular Momentum

For a rigid body, $\vec{v_i} = \vec{v} + \vec{\omega} \times \vec{\rho_i}$ and hence

$$\vec{H_G} = \sum \vec{\rho_i} \times m_i (\vec{v} + \vec{\omega} \times \vec{\rho_i})$$
$$= \left(\sum m_i \rho_i\right) \times \vec{v} + \sum \rho_i \times m_i (\omega \times \rho_i)$$
$$= 0 + \left[\sum (m_i \vec{\rho_i} \cdot \vec{\rho_i})\right] \vec{\omega} - \sum m_i (\vec{\rho_i} \cdot \vec{\omega}) \vec{\rho_i}$$

Writing $\vec{\rho_i} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$\cancel{\vec{H_G} = m_i [(y^2+z^2)\omega_x}$

$$\vec{H_G} = \sum m_i \Big[\{(y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z\}\hat{i}$$
$$+ \{-yx\,\omega_x + (z^2+x^2)\omega_y - yz\,\omega_z\}\hat{j}$$
$$+ \{-zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\omega_z\}\hat{k}\Big]$$

Defining $I_{xx} = \sum m_i (y^2+z^2)$ or $\int (y^2+z^2)\,dm$,
$I_{yy} = \sum m_i (z^2+x^2)$ or $\int (z^2+x^2)\,dm$,
$I_{zz} = \sum m_i (x^2+y^2)$ or $\int (x^2+y^2)\,dm$,
$I_{xy} = I_{yx} = \sum m_i xy$ or $\int xy\,dm$,
$I_{yz} = I_{zy} = \sum m_i yz$ or $\int yz\,dm$,
$I_{zx} = I_{xz} = \sum m_i zx$ or $\int zx\,dm$;

$$\vec{H_G} = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\hat{i}$$
$$+ (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\hat{j}$$
$$+ (-I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\hat{k}$$
$$\equiv H_x \hat{i} + H_y \hat{j} + H_z \hat{k}$$

Similar will be found the case of $\vec{H_O}$, angular momentum about a fixed point.

For any other (moving) point, say P,

$$\vec{H_P} = \vec{H_G} + \vec{r} \times \vec{G} = \vec{H_G} + \vec{r} \times m\vec{v_G}$$

VII (Contd)

→ Moments of inertia, Products of inertia and the Inertia Matrix or Inertia Tensor

$$\{I\} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Gross up in Meriam and Kraige.

Starting straight from

$$\vec{H}_G = \sum m_i \left[(\vec{\rho}_i \cdot \vec{\rho}_i)\, \vec{\omega} - (\vec{\rho}_i \cdot \vec{\omega})\, \rho_i \right],$$

we have
$$\vec{H}_G = \sum m_i \left[\rho_i^T \rho_i\, \vec{\omega} - (\rho_i \rho_i^T)\, \vec{\omega} \right]$$
$$= \left[\sum m_i \left(\rho_i^T \rho_i E_3 - \rho_i \rho_i^T \right) \right] \vec{\omega}$$
$$= \{I\}\, \vec{\omega}$$

Inertia tensor (matrix) is symmetrical, can always be diagonalized by rotation of axes.

In the newly oriented (principal) axes,

$$I' = Q^T I Q = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

and $\vec{H}_G = I_{11}\omega_1 \hat{e}_1 + I_{22}\omega_2 \hat{e}_2 + I_{33}\omega_3 \hat{e}_3$

[This is always possible, but not always done.]

VII (Contd) 5/11/2016 (25) (Sat, Extra Class)

☆ Kinetic Energy

$$T = \frac{1}{2} m \bar{v}^2 + \sum \frac{1}{2} m_i \left|\dot{\vec{\rho}}_i\right|^2$$

$$\frac{1}{2} m_i \left|\dot{\vec{\rho}}_i\right|^2 = \frac{1}{2} m_i (\vec{\omega} \times \vec{\rho}_i) \cdot (\vec{\omega} \times \vec{\rho}_i)$$
$$= \frac{1}{2} \cancel{m_i}\, \vec{\omega} \cdot \left[\vec{\rho}_i \times m_i(\vec{\omega} \times \vec{\rho}_i)\right]$$

Hence $T = \frac{1}{2} \vec{v}_G \cdot \vec{G} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_G$

$$= \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \vec{\omega} \cdot \bar{I} \vec{\omega}$$

$$= \frac{1}{2} m \vec{v}^T \vec{v} + \frac{1}{2} \vec{\omega}^T \bar{I} \vec{\omega}$$

$$= \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \left(\bar{I}_{xx} \omega_x^2 + \bar{I}_{yy} \omega_y^2 + \bar{I}_{zz} \omega_z^2\right)$$
$$- \left(\bar{I}_{xy} \omega_x \omega_y + \bar{I}_{yz} \omega_y \omega_z + \bar{I}_{zx} \omega_z \omega_x\right)$$

→ Rotation about fixed point O:

$$T = \frac{1}{2} \vec{\omega} \cdot I_0 \vec{\omega}$$

☆ Momentum and Energy Equations of Motion

$$\sum \vec{F} = \frac{d\vec{G}}{dt} = m \vec{a}_G$$

$$\& \quad \sum \vec{M}_G = \left(\frac{d\vec{H}}{dt}\right)_{XYZ} = \left(\frac{d\vec{H}}{dt}\right)_{xyz} + \vec{\Omega} \times \vec{H}$$

$$= \dot{H}_x \hat{i} + \dot{H}_y \hat{j} + \dot{H}_z \hat{k} + \vec{\Omega} \times \vec{H}$$

$$= (\dot{H}_x + \Omega_y H_z - \Omega_z H_y)\, \hat{i}$$
$$+ (\dot{H}_y + \Omega_z H_x - \Omega_x H_z)\, \hat{j}$$
$$+ (\dot{H}_z + \Omega_x H_y - \Omega_y H_x)\, \hat{k}$$

VII (Contd)

If frame of reference (xyz) is attached to the body itself, then $\vec{\Omega} = \vec{\omega}$,

$$\boxed{\begin{aligned} &\Sigma \vec{F} = m\,\vec{a}_G \\ &\Sigma \vec{M}_G = \left[\frac{d}{dt}(\bar{I}\,\vec{\omega})\right]_{xyz} = \bar{I}\,\vec{\alpha} + \vec{\omega} \times \bar{I}\,\vec{\omega} \\ &\text{Newton-Euler Dynamic Formulation} \end{aligned}}$$

Energy Equations: $U'_{1-2} = \Delta T + \Delta V$ OR $U_{1-2} = \Delta T$

$$\left.\begin{aligned} \int_1^2 \Sigma \vec{F} \cdot \vec{ds} &= \frac{1}{2} m (v_2^2 - v_1^2) \\ \int_1^2 \Sigma \vec{M}_G \cdot \vec{\omega}\, dt &= \left[\frac{1}{2}\vec{\omega} \cdot \bar{I}\,\vec{\omega}\right]_1^2 \end{aligned}\right\} + : \Delta T$$

☆ Lagrangian (or Euler-Lagrange) formulation of Dynamics

→ Inter-connected bodies

$$\mathcal{L}(q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) = KE - PE$$

$$= T - V$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \; ; \quad j = 1, 2, \ldots n \text{ (DoF)}$$

☆ Hamilton's Principle of Least Action

A system evolves along a trajectory $q(t)$ for which the action

$$S = \int_{t_1}^{t_2} \mathcal{L}\,[q(t), \dot{q}(t), t]\, dt$$

is a minimum (stationary).

VII (Contd) 7/11/2016 (26)

☆ Parallel-Plane Motion

Every line normal to the plane of motion (say x-y plane) remains ~~parallel~~ so at all times.

$$\Sigma \vec{F} = m\vec{a} = m(a_x \hat{i} + a_y \hat{j})$$

and $\vec{\omega} = \omega_z \hat{k} = \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}$

Angular momentum $\vec{H} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} = \begin{bmatrix} -I_{xz} \\ -I_{yz} \\ I_{zz} \end{bmatrix} \omega_z$

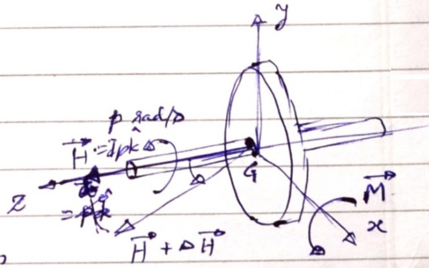
and

$$\Sigma \vec{M} = I\dot{\vec{\omega}} + \vec{\omega} \times I\vec{\omega} \quad \text{[Frame attached to body]}$$

$$= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} \times \begin{bmatrix} -I_{xz}\omega_z \\ -I_{yz}\omega_z \\ I_{zz}\omega_z \end{bmatrix}$$

$$= \begin{bmatrix} -I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2 \\ -I_{yz}\dot{\omega}_z - I_{xz}\omega_z^2 \\ I_{zz}\dot{\omega}_z \end{bmatrix} \equiv (-I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2)\hat{i} + (-I_{yz}\dot{\omega}_z - I_{xz}\omega_z^2)\hat{j} + I_{zz}\dot{\omega}_z\hat{k}$$

☆ Gyroscopic Motion



Rotor spinning about its z-axis, axis of symmetry.

SMALL Moment $\vec{M}$ applied about its x-axis

$$\vec{\omega} = p\hat{k}, \quad \vec{H} = Ip\hat{k} \quad \text{[neglecting]}$$

$$d\vec{H} = \vec{M}dt = Mdt\,\hat{i}$$

$$d\psi = \frac{M\,dt}{Ip} \Rightarrow \frac{d\psi}{dt} = \dot{\psi} = \frac{M}{Ip} \quad \& \quad \vec{M} = \vec{\Omega} \times Ip\hat{k} = \dot{\psi}\hat{j} \times Ip\hat{k}$$

VII (Contd)

If the rotor were NOT spinning, M_x would produce an angular acceleration about x-axis and cause it to turn about x-axis [The way a central force would attract a STATIONARY object to itself.]

Since the rotor IS spinning with ω_z, M_x would NOT cause to turn about x-axis because that would need a non-zero dH_y, it would simply provide dH_x and turn the Ang. Mom. vector and Ang. vel. vector [The way a central force acting on a moving object with transverse velocity provides the centripetal acceleration to turn its direction.]

If an active moment is NOT supplied, but the spin axis is forced to precess by mechanical connection (chassis takes a turn), then a gyroscopic couple would arise as reaction.

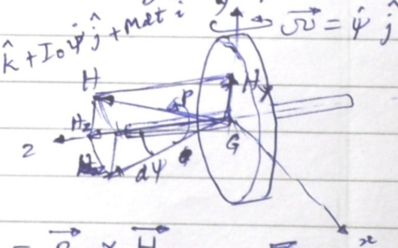
In the above, $\dot{\psi} << p$, M_x was small.

If not so, as M_x continues to act and the rotor continues to precess, $\vec{H}$ will have a y-component too.

$$\vec{H} = H_z \hat{k} + H_y \hat{j}$$

$$\vec{H} + d\vec{H} = H_z \hat{k} + H_y \hat{j} + dH \hat{i} = Ip\hat{k} + I\omega\dot{\psi}\hat{j} + Mdt\,\hat{i}$$

$$dH = Mdt, \quad d\psi = \frac{Mdt}{Ip}$$



and

$$M = Ip\dot{\psi} = \vec{\Omega} \times H_z \hat{k} = \vec{\Omega} \times \vec{H}$$

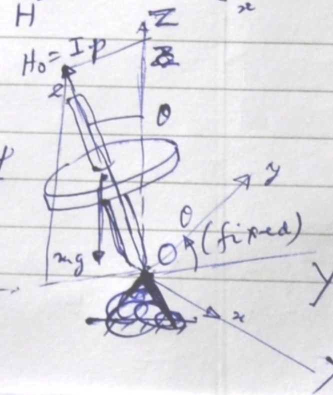
Steady precession of a symmetrical top:

$$M_O = mg\,\bar{r}\sin\theta = \frac{dH_O}{dt} = \frac{Ip\sin\theta \cdot d\psi}{dt}$$

$$= I\Omega p\sin\theta \Rightarrow \Omega = \frac{mg\bar{r}}{Ip} = \frac{g\bar{r}}{k^2 p}$$

If right initial θ, then steady precession.

Else, NUTATION: oscillation of θ.



VII (Contd)

A little more generalization

- General Moment $\Sigma\vec{M} = \Sigma M_x\hat{i} + \Sigma M_y\hat{j} + \Sigma M_z\hat{k}$
- Fixed frame XYZ with Z vertical, origin at G/O.
- Precessing frame xyz with z along spin axis, origin same.
- Rotation about common origin G/O.
- Spin axis along axis of symmetry of rotor

$$I_{zz} = I, \quad I_{xx} = I_{yy} = I_0, \quad I_{xy} = I_{yz} = I_{xz} = 0.$$

Angular velocity of frame:

$$\vec{\Omega} = \Omega_x\hat{i} + \Omega_y\hat{j} + \Omega_z\hat{k}$$
$$= \dot{\theta}\hat{i} + \dot{\psi}\sin\theta\,\hat{j} + \dot{\psi}\cos\theta\,\hat{k}$$

Angular velocity of rotor: $\vec{\omega} = \dot{\theta}\hat{i} + \dot{\psi}\sin\theta\,\hat{j} + (\dot{\psi}\cos\theta + p)\hat{k}$

- x axis: line of intersection of X-Y (horizontal) plane and plane perpendicular to spin (z) axis

Angular momentum:

$$\vec{H} = H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$$
$$= I\vec{\omega} = \begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta + p \end{bmatrix} = \begin{bmatrix} I_0\dot{\theta} \\ I_0\dot{\psi}\sin\theta \\ I(\dot{\psi}\cos\theta + p) \end{bmatrix}$$
$$= I_0\dot{\theta}\hat{i} + I_0\dot{\psi}\sin\theta\,\hat{j} + I(\dot{\psi}\cos\theta + p)\hat{k}$$

Moment and Equation of Motion:

$$\Sigma\vec{M} = I\dot{\vec{\omega}} + \vec{\Omega} \times I\vec{\omega}$$
$$= \begin{bmatrix} I_0\ddot{\theta} \\ I_0\ddot{\psi}\sin\theta + I_0\dot{\psi}\dot{\theta}\cos\theta \\ I\frac{d}{dt}(\dot{\psi}\cos\theta + p) \end{bmatrix} + \begin{bmatrix} -I_0\dot{\psi}^2\sin\theta\cos\theta + I\dot{\psi}(\dot{\psi}\cos\theta + p)\sin\theta \\ I_0\dot{\psi}\dot{\theta}\cos\theta - I\dot{\theta}(\dot{\psi}\cos\theta + p) \\ 0 \end{bmatrix}$$

= Just add.

VII (Contd) 9/11/2016 (27)

Special Cases:

$$\Sigma M_x = I_0(\ddot{\theta} - \dot{\psi}^2 \sin\theta\cos\theta) + I\dot{\psi}(\dot{\psi}\cos\theta + p)\sin\theta$$
$$\Sigma M_y = I_0(\ddot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}\cos\theta) - I\dot{\theta}(\dot{\psi}\cos\theta + p)$$
$$\Sigma M_z = I\frac{d}{dt}(\dot{\psi}\cos\theta + p)$$

(i) Steady-State Precession

$\dot{\psi}$ constant, $\ddot{\psi} = 0$;
θ constant, $\dot{\theta} = 0$, $\ddot{\theta} = 0$;
p constant, $\dot{p} = 0$.

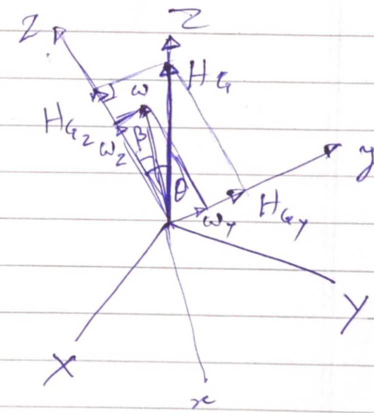
$$\Sigma M_x = \dot{\psi}\sin\theta\left[I(\dot{\psi}\cos\theta + p) - I_0\dot{\psi}\cos\theta\right]$$
$$\Sigma M_y = \Sigma M_z = 0.$$

(ii) Steady Precession with Zero Moment

$\tan\theta = \dfrac{H_{Gy}}{H_{Gz}}$, $\tan\beta = \dfrac{\omega_y}{\omega_z}$

& $H_{Gy} = I_0\omega_y$, $I_{Gz} = I\omega_z$

$\Rightarrow \tan\theta = \dfrac{I_0}{I}\tan\beta$



$\Sigma M_x = 0 \Rightarrow \dot{\psi} = \dfrac{Ip}{(I_0 - I)\cos\theta}$

$I_0 > I$: $\beta < \theta$: direct precession : Body cone rolls over Space cone EXTERNALLY

$I < I_0$: $\beta > \theta$: retrograde precession : Body cone rolls over Space cone INTERNALLY

$I = I_0$: $\beta = \theta$: No precession : z & Z axes coincide. (Spherical?) Body simply spins with $\omega_z = p$.

— THE END —
— For the time being —