A Brief Review of Laplace Transforms

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Introduction

- **Laplace Transformation is a very useful tool for analysis of a dynamic system in Frequency-Domain. This transformation helps to transform differential equations into the form of algebraic equations which is easier to manipulate.**

- **A time-domain signal *f(t)* which may represent a forcing function or the response of a system may be transformed into frequency domain by using the following transformation:**

$$L\left|f(t)\right| = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Where $s = \sigma + j\omega$, is a complex variable.

Example:

$$f(t) = A\,e^{-pt}\,u(t)$$

$$F(s) = \int_0^\infty Ae^{-pt}e^{-st}u(t)\,dt = A\int_0^\infty e^{-(s+p)t}\,dt$$

$$= -\frac{A}{s+p}e^{-(s+p)t}\Big|_{t=0}^{\infty} = \frac{A}{s+p}$$

Some useful Laplace Transforms

No.	$f(t), t \geq 0$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$1/s$
3	t	$1/s^2$
4	t^2	$2!/s^3$
5	e^{-at}	$1/(s+a)$
6	Sin at	$a/(s^2 + a^2)$
7	Cos at	$s/(s^2 + a^2)$
8	$(1-at)\, e^{-at}$	$s/(s+a)^2$

Laplace Transform of a Differential Equation

Applying the same principle on a differential equation one can obtained an algebraic equation. Consider a second order mechanical system represented by the following differential equation:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F\sin(\omega t)$$

Applying Laplace transform and assuming zero initial condition the above equation could be transformed as

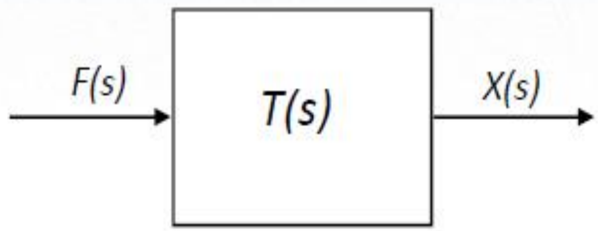
$$ms^2X(s)+csX(s)+kX(s)=\frac{F\omega}{s^2+\omega^2}$$

Denoting the right hand side of the above equation as *F(s)* , one can express the ratio of *F(s)* and frequency-domain response *X(s)* as

$$T(s)=\frac{X(s)}{\overline{F}(s)}=\frac{1}{ms^2+cs+k}$$

Transfer Function of a SDOF system

Here, we are considering a SDOF system with system parameters *m* **(mass),** *c* **(damping) and** *k* **(stiffness).**
T(s) **is also known as transfer function of the system. In a block diagram form this can be represented as**



$$T(s)=\frac{1}{ms^2+cs+k}$$

The response of a system in time domain could be obtained by carrying out inverse Laplace Transformation of the transfer function. The inverse Laplace Transform is written as

$$L^{-1}[T(s)]=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} T(s)e^{st}\,ds$$

Inverse Laplace Transform

However, the relationship stated is seldom used. If *T(s)* is rational, one commonly uses the method of partial fraction expansion. Consider a rational function *T(s)* expressed as:

$$T(s)=\frac{b_1 s^m + b_2 s^{m-1} + \ldots + b_{m+1}}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

Factoring the numerator and denominator polynomials one can also write

$$T(s)=K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

Poles and Zeros of the Transfer Function

- Corresponding to the numerator polynomial, z_i's are referred as the zeros of the transfer function while the roots of the numerator polynomial p_i's are known as the poles of the transfer function.
- Now, the transfer function *T(s)* may be expressed as

$$T(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}$$

- where,

$$C_i = (s-p_i)T(s) \Big|_{s=p_i}$$

- Finally, the response of the system may be expressed as

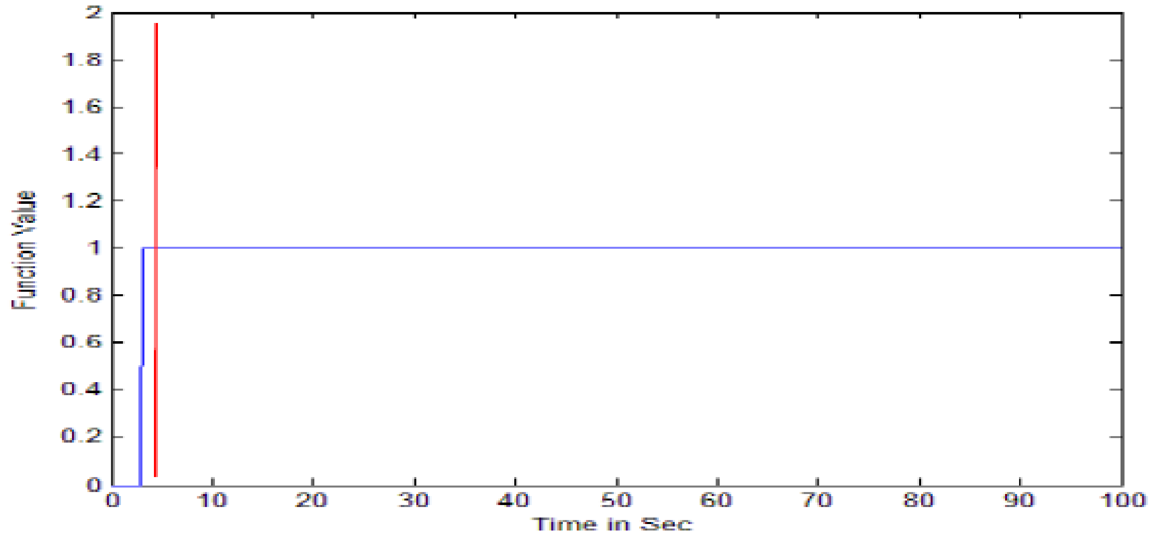
$$x(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

What if the roots are complex?

- Two ways to tackle –
  - Find out the complex roots and complex residue as usual
  - $\boldsymbol{Cos\,\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right); \; Sin\,\theta = \frac{1}{2j}\,\left(e^{j\theta} - e^{-j\theta}\right)}$
- Another way: Separate the complex pole part and write in the form: $\frac{As+B}{D(s)}$, Obtain A and B by balancing the coefficients.
- Now use the LT: $\boldsymbol{L[Ae^{-at}Cos(\omega t) + Be^{-at}Sin(\omega t)] = \frac{A(s+a)+B\,\omega}{(s+a)^2+\omega^2}}$

Examples:

A system is being excited by a step function at time $t = 3\ s$ and by a delta function at time $t = 5\ s$. Find out the Laplace Transform of these two functions by using the Transformation table. Now, find out the response of a SDOF system (mass 1kg, stiffness 10N/m and damping 1 N/m/s while subjected to the above excitations.



Steady State error of a System

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Steady state error - Introduction

- Steady state error refers to the long-term behavior of a dynamic system.
- The Type of a system is significant to predict the nature of this error.
- A system having no pole at the origin is referred as Type-0 system.
- Thus, Type-1, refers to one pole at the origin and so on.
- It will be shown in this lecture that, it is the type of a system which can directly determine whether a particular command will be followed by a system or not.
- We will consider three common commands: namely, step, ramp and parabolic ramp and find out the steady state response/error of a system to follow these commands.
- A closed loop control system shows remarkable performance in reducing the steady state error of a system.

Steady state error of a system

- Error in a system: E(s) = U(s) /(1+G(s))

$$\boxed{\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0}\frac{sU(s)}{1+G(s)}}$$

- For a step input

$$\boxed{e_{ss} = \frac{A}{1+G(0)}}$$

- Plant Transfer function G(s) is defined as

$$\boxed{G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^k\prod_{j=1}^{N}(s+p_j)}}$$

Error Constants

- **Position error constant** $$K_p = \lim_{s \to 0} G(s)$$

- **Steady state error of a step input of magnitude A is** $e_{ss} = A/(1+K_p)$

- **Steady state error will be zero for system with type greater than or equal to 1**

- For ramp input $$e_{ss} = \lim_{s \to 0} \frac{A}{sG(s)}$$

- Define velocity constant as $$K_v = \lim_{s \to 0} sG(s)$$

- Hence steady state error is A/K_v

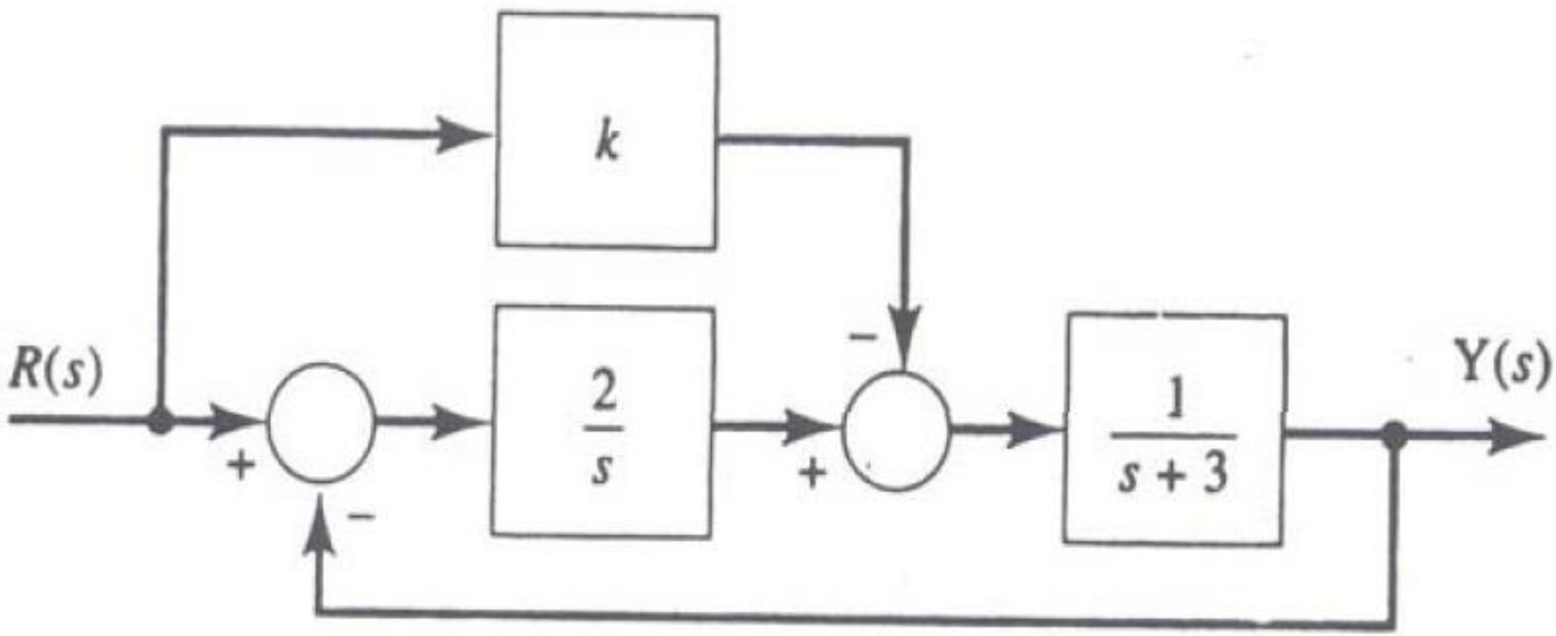
- Error will be zero for k greater than or equal to 2

Summary of Steady State Errors

Type	Step (A/s)	Ramp (A/s^2)	Parabolic Ramp (A/s^3)
0	$E_{ss} = A/(1+K_p)$	Inf	Inf
1	0	A/K_v	Inf
2	0	0	A/K_a

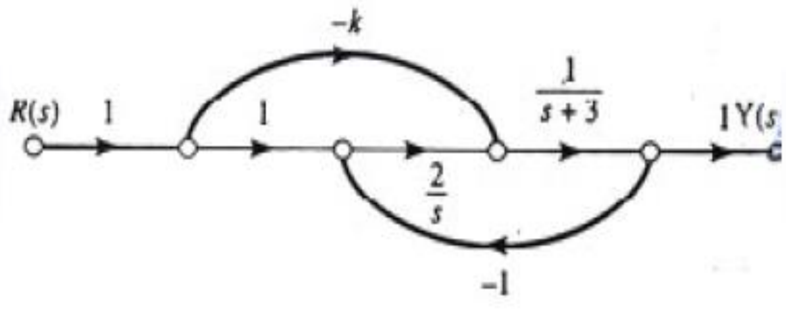
Design of Optimal systems

You can chose optimal gain such that the error constant could be minimized.
Consider the following system:



(a)

Optimized Gain



(b)

$$\boxed{T(s) = \frac{2 - ks}{s^2 + 3s + 2}}$$

This is a Type zero system. Hence, consider step input.

$$E(s) = \frac{A}{s}(1 - T_e(s)) = A\left(\frac{k+2}{s+1} - \frac{k+1}{s+2}\right)$$

$$J(k) = \int_0^{\infty} e^2(t)dt = \frac{A^2}{12}[k^2 + 6k + 11]$$

Note that the Performance index is a quadratic function of gain k, which can be minimized to obtain k.

$\boxed{dJ/dk=0 \text{ --- } k = -3}$

Stability of a Dynamic System

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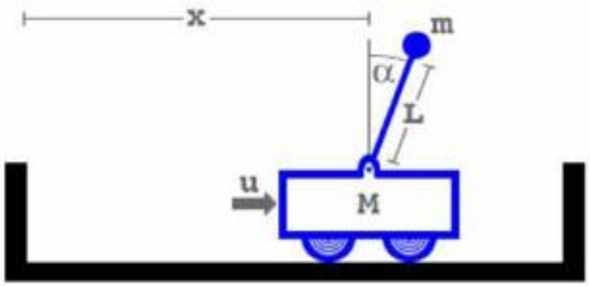
Introduction

- The response of a Dynamic System may become unbounded while subjected to a bounded input. Such systems are referred as unstable systems. One common example is an inverted pendulum on a rolling cart as shown below:



- A Control system could be designed such that by controlling the velocity of the rolling cart one can control the unstable response of the inverted pendulum.

- However, we need to first carry out a stability analysis of the system.

How to test the stability of a system

- A simple method to test the stability of a system is by checking the poles of the system transfer function.
- Consider a system which is represented by a generalized transfer function as follows:

$$T(s) = \frac{N(s)}{D(s)} = \frac{c_0 s^m + c_1 s^{m-1} + \dots + c_{m-1} s + c_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

- Now, equating the denominator polynomial D(s) to zero, one can obtain the characteristic equation for the system. The roots of this characteristic equations are the poles of the system.
- If you obtain one or more poles with positive real part then the system could be predicted to be an unstable system.
- However, it is often tedious to obtain the poles of a complex system before predicting stability condition of the system.

Routh's Test for Stability

For a characteristic polynomial D(s), the number of poles in the right-half plane may be determined without actually finding the roots by using the Routh Test.

The Routh array for the polynomial D(s) could be constructed as follows:

s^n	a_0	a_2	a_4	a_6 ...
s^{n-1}	a_1	a_3	a_5	a_7 ...
s^{n-2}	b_1	b_2	b_3	...
:	:	:	:	
s^0				

Routh's Theorem

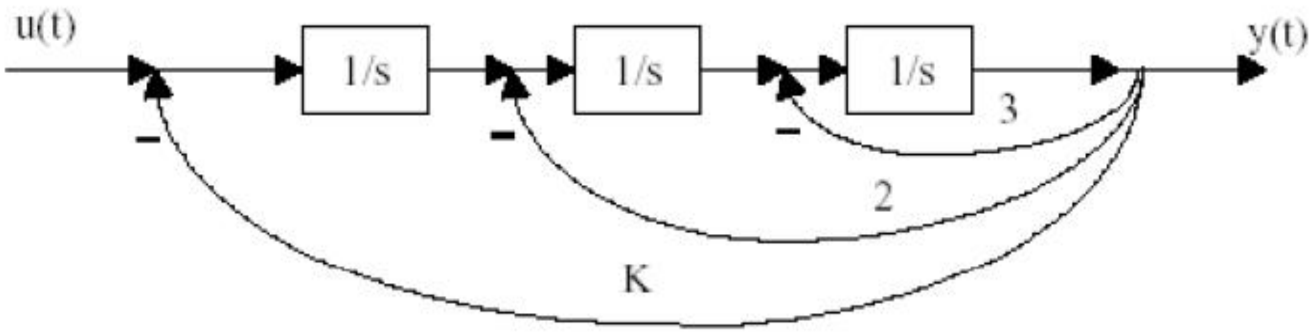
You may have observed that the first row of the Routh table consists of odd coefficients of D(s) starting from the first coefficient related to s^n . Again, the second row consists of the even coefficients starting from the second coefficient related to s^{n-1} .

The coefficients b_1 etc. For the third row could be computed as follows. The same pattern could be used for the subsequent rows.

$$b_1 = {-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}} \Big/ {a_1} \ , \quad b_2 = {-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}} \Big/ {a_1} \ , \quad b_3 = {-\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}} \Big/ {a_1} \ .$$

Routh's Theorem: The number of roots of the characteristic polynomial D(s) in the right-half plane equals the number of sign changes in the first column of the Routh Table.

Stability as a function of a parameter



1. Use Mason's rule to find out the determinant of the system
2. Numerator of the determinant will be the characteristic polynomial
3. Use Routh's test and obtain the array
4. Note the conditions for stability analysing the left coloumn

Unusual Case: Left Column Zero

- Consider $D(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$
- Note appearance of zero in the first column
- Rename the row as Row A
- Create row B from Row A by sliding the A row to left until you get a non-zero pivot
- The sign of the row is changed by $(-1)^n$ where n is the number of times this row is slided
- The new non-zero row is formed by adding A and B
- [Reference Benedir and Picinbond, IEEE Trans on Automatic Control, 1990]

Other approaches for zero left column

Alternate approaches:

Put a parameter, say ε instead of zero in the pivot, assuming it to be a very small positive number

Continue and find sign changes

OR

Write the polynomial in reverse order such that the roots of the reverse polynomial will be the reciprocal of the roots of the original polynomial and follow the same procedure

Assignment - Unusual Case: A zero row

Consider the polynomial D(s) as
$s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128$

Find out the stability of the system.

Hints: When you encounter a zero row, go back to the last row, construct the corresponding polynomial (even/odd).

Differentiate the polynomial and obtain the new non-zero row.

Root Locus Method – Part 1

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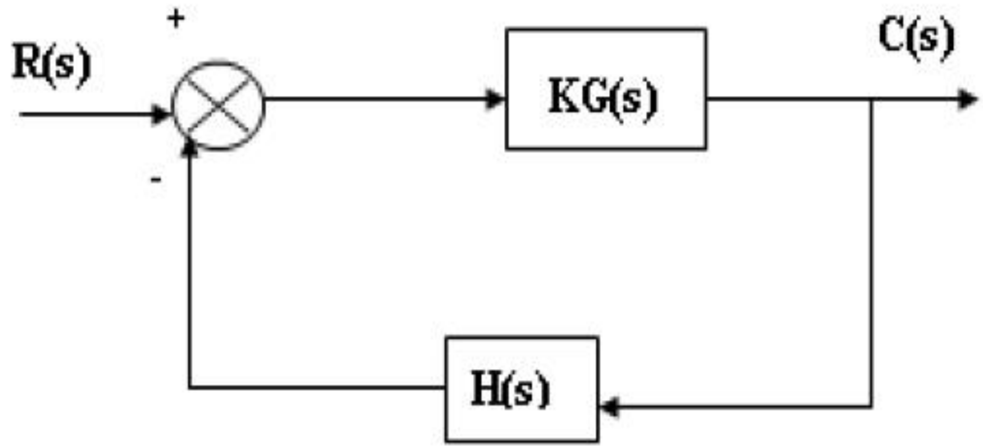
Recall Open and Closed Loop Transfer Function



- Open Loop Transfer Function: $KG(s)H(s)$

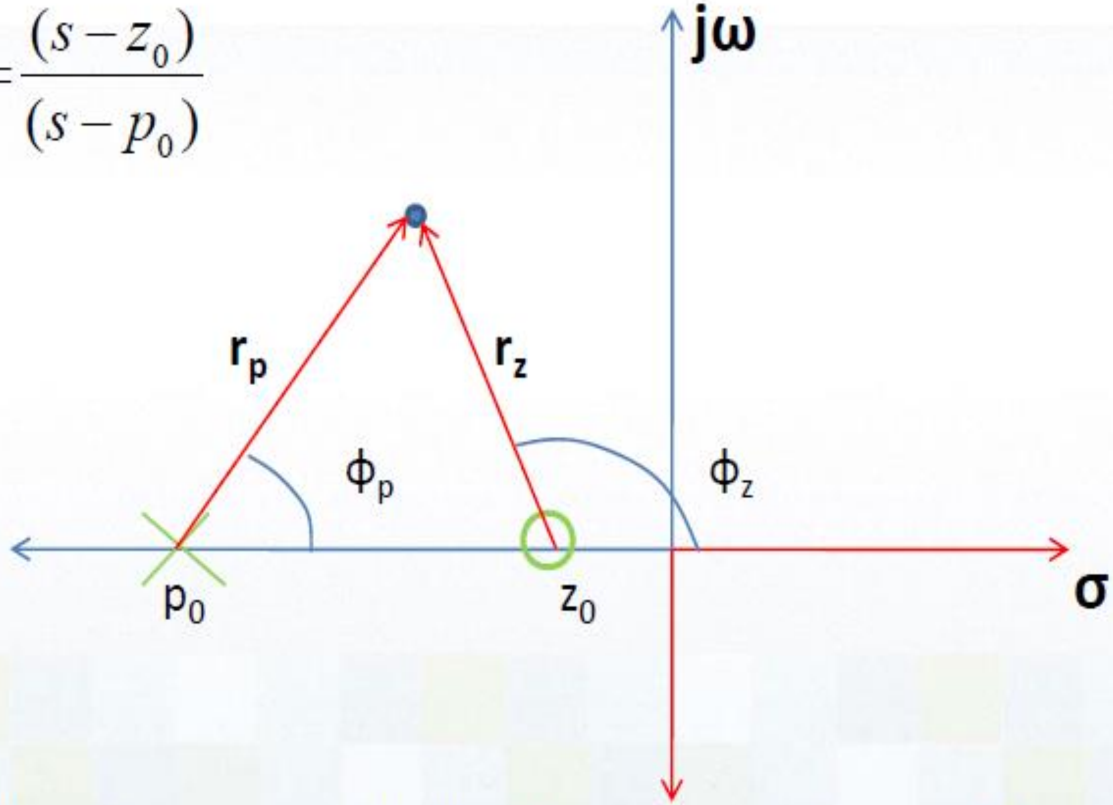
- Close Loop Transfer Function: $\dfrac{KG(s)}{1+KG(s)H(s)}$

✓ G(s) given, H(s) Given, Select the Gain K and vary until you receive the desired response.

Geometric Interpretation of Evaluating a Transfer Function

Consider a Transfer Function to be:

$$T(s) = \frac{(s - z_0)}{(s - p_0)}$$



The pole and zero are plotted in the s-plane as shown. The magnitude of the transfer function is the ratio of the zero and the pole length (**r_z / r_p**) and the phase of the transfer function is $\phi_Z - \phi_p$.

Evaluation of a Transfer Function

Consider a Transfer Function to be

$$F(s)=\frac{(s+1)}{s(s+2)}$$

- **Evaluate the function at a test point s = -3+j4, i.e. find out the magnitude and phase of the transfer function at the test point.**
- **Following the geometric technique or using simple complex algebra you may find out that :**
- **Zero length - $\sqrt{20}$, $\varphi_z = 116.6^0$**
- **Pole at Origin: Pole length 5, $\varphi_{p1} = 126.9^0$**
- **Pole at -2: Pole length $\sqrt{17}$, $\varphi_{p2} = 104.0^0$**
- **Magnitude of the transfer function:** $$F(s)=\frac{\sqrt{20}}{5\sqrt{17}}=0.217$$
- **Total Phase = 116.6-(126.9+104) = -124.6^0**

Objective of Root Locus

- Without Factorizing 1+KG(s)H(s) Every Time Can We Find Out The Location Of Closed Loop Poles and Comment On Stability?

- Answer: Yes, Get The Root Locus

Characteristic Equation of the closed loop system

$$1+KG(s)H(s)=0, \quad Or \quad \begin{aligned} 1+KF(s)&=0 \\ F(s)&=G(s)H(s) \end{aligned}$$

$$Or \quad F(s)=-\frac{1}{K}$$

$$Also \quad F(s)=\left|F(s)\right|e^{j\varphi_{F(s)}}, \left|F(s)\right|=\frac{1}{K}$$

The phase condition for F(s) is stated in the equation below which is further used for the root locus plot.

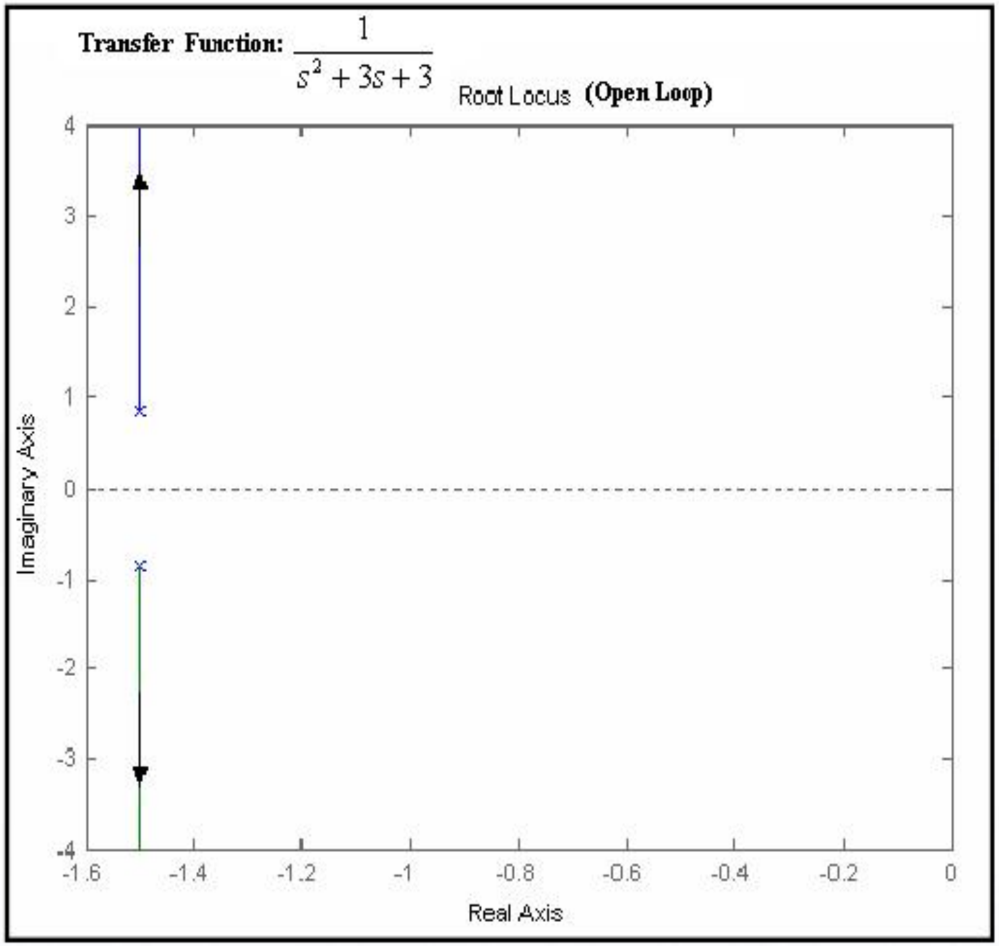
$$\boxed{\angle F(s) = (2k+1)180^0}$$

$$\text{If} \quad F(s) = \frac{\prod_{i=1}^{n}(s - zi)}{\prod_{i=1}^{n}(s - p_i)}$$

For any Values of F(s)

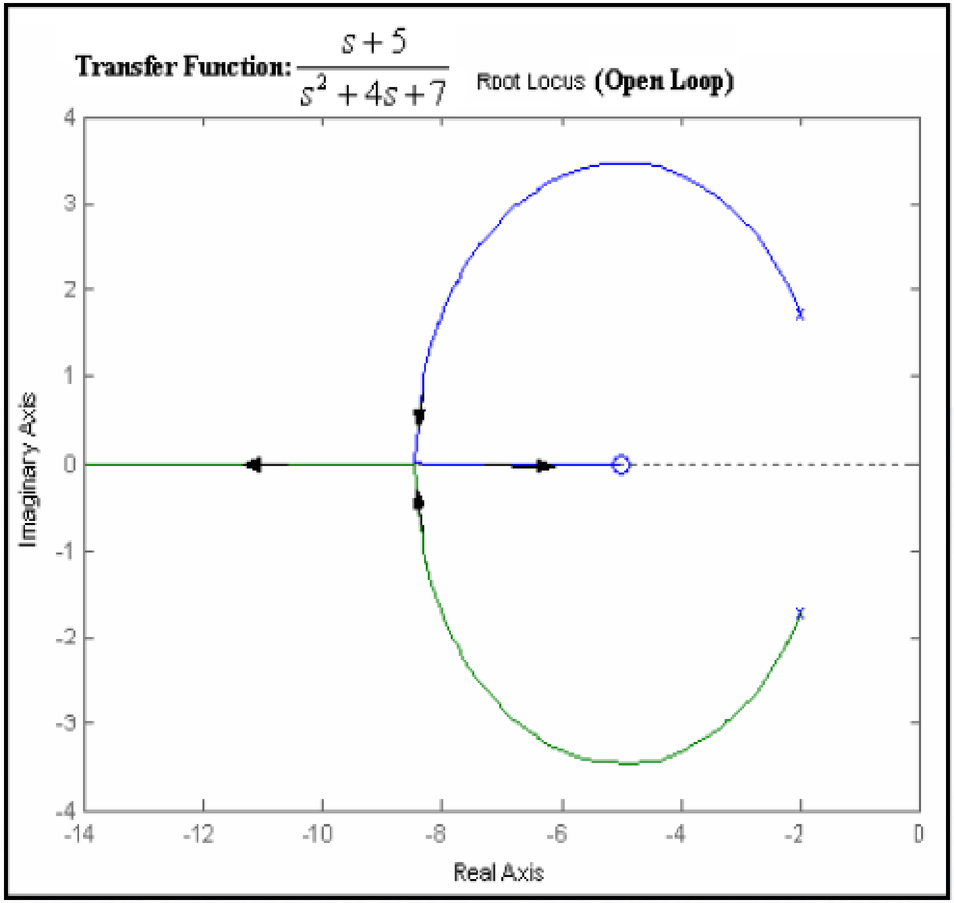
$$|F(s)| = \frac{\prod Zero \quad Lengths}{\prod Pole \quad Lenghts}$$

$$\varphi_{F(s)} = \sum Zero \quad angles - \sum Pole \quad angles$$



- ✓ Number of root loci – as many as open loop poles
- ✓Origin at poles

Root Locus of the Open Loop Transfer Function

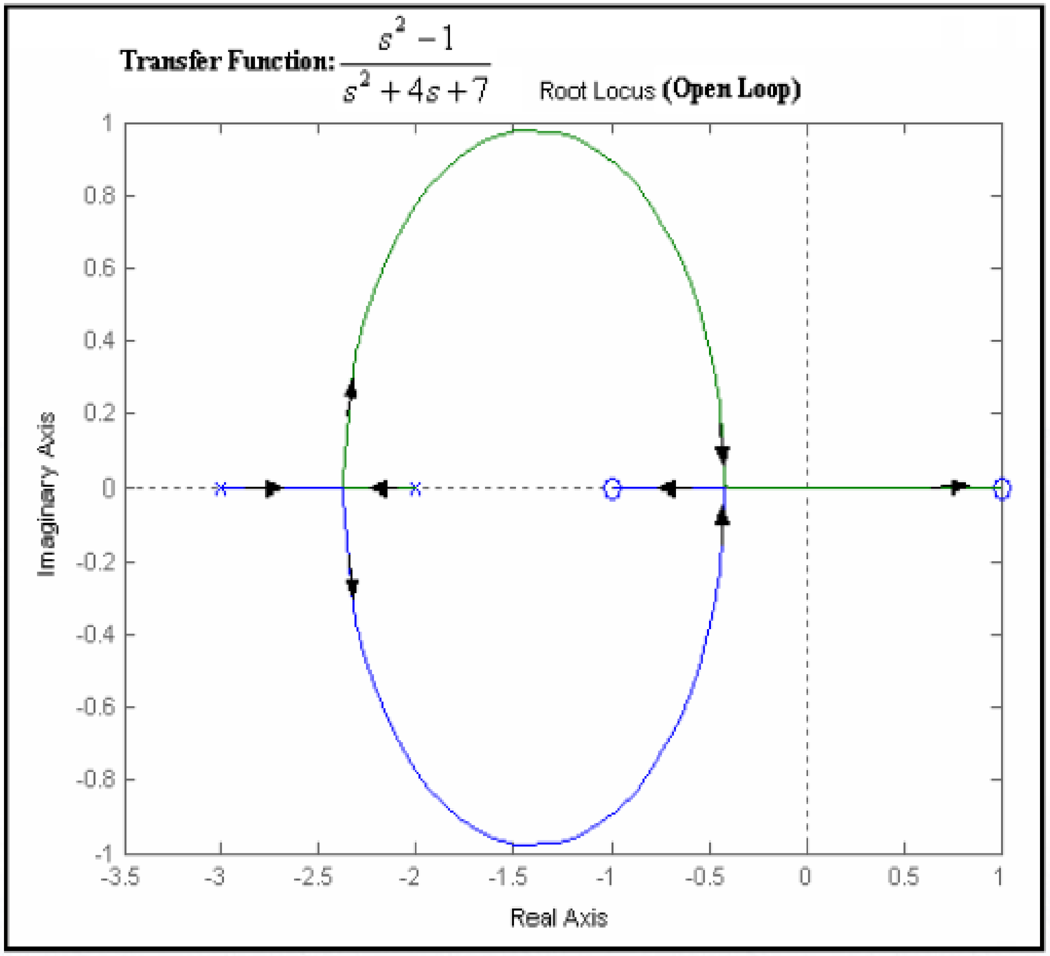


✓ Symmetry always about the real axis.

✓ Termination as $K \rightarrow \alpha$ m open loop poles to finite zeros

Of the open loop system, n-m number of poles approach zeroes at infinity.

Root Locus of the Open Loop Transfer Function



✓ Real axis Segment: - Root locus exists on the left of an odd number of real axis finite open loop poles and zeroes

Root Locus of the Open Loop Transfer Function

Special References for this lecture

- Feedback Control of Dynamic Systems, Frankline, Powell and Emami, Pearson
- *Control Systems Engineering* – Norman S Nise, John Wiley & Sons
- *Design of Feedback Control Systems* – Stefani, Shahian, Savant, Hostetter

  Oxford