Name: Roll no:

ME321A: Advanced Mechanics of Solids

End-Semester Examination, 17th November, 2017

Duration: 3 hours

Full Marks: 50

**Important note:**

- *Write the answer sought inside the box provided. Answers written elsewhere will not be counted.*
- *Nothing other than the specific answer must be written Within the box. All other calculations must be done outside the box.*
- *All calculations have to be shown. If workings are not shown in the space provided below each question, you will not be given any credit even if your answer is correct.*
- *You may use the coursenotes and books. No computers are allowed.*

Pr 1: Consider the triangular cantilever beam shown. The only load on the beam is due to its own weight.

(a). Draw the shear force and bending moment diagrams for the triangular beam. From these diagrams, figure out how the $\sigma_x$ stress varies with $x$. So, if $\sigma_x \sim x^\alpha$, then

$\alpha =$ 1.

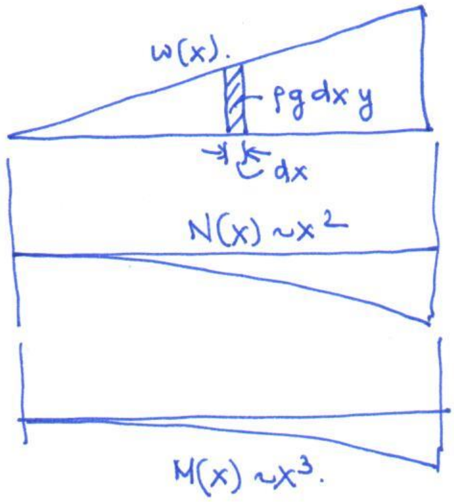
$I(x) \sim y^3 \sim x^3 \tan^3 \alpha$

$w(x) \sim \rho g x \tan\alpha$

Shear force $\frac{dN}{dx} = -w(x) \Rightarrow N \sim x^2$

Bending moment $\frac{dM}{dx} = N(x) \Rightarrow M \sim x^3$.

$\Rightarrow \sigma_{xx} \sim \frac{My}{I} \sim \frac{x^3 \cdot x \tan\alpha}{x^3} \sim x$.



(b). The body forces for loading by self weight can be written (in the coordinate system shown) as

| | |
|---|---|
| $b_x = 0$ | $b_y = -\rho g$ |

$$b_x = -\frac{\partial V}{\partial x} \qquad b_y = -\frac{\partial V}{\partial y}$$

$$\frac{\partial V}{\partial x} = 0 \qquad \rho g = \frac{\partial V}{\partial y} \Rightarrow V = \rho g y + C_2.$$

$$\Rightarrow V = C_1 + f(y)$$

$$\Rightarrow V = \rho g y.$$

(b). Convince your self that we can assume an Airy stress function of the form (no marks for thinking):

$$\phi = C_1x^2 + C_2y^2 + C_3xy + C_4x^3 + C_5x^2y + C_6xy^2 + C_7y^3$$

(c). Using the chosen Airy stress function, write the stresses:

$$\sigma_x = 2C_2 + 2C_6x + 6C_7y + \rho g y$$
$$\sigma_y = 2C_1 + 6C_4x + 2C_5y + \rho g y$$
$$\sigma_{xy} = -C_3 - 2C_5x - 2C_6y.$$

(d). The values of some of the quantities shown below are known at the boundaries from the given boundary conditions. Write the values of the quantities that are known at $y = 0$:

$\sigma_x =$ , $\sigma_y = 0$ , $\sigma_{xy} = 0$ , $u_x =$ , $u_y =$

(e). Complete the boundary conditions on $y = x \tan\alpha$.

$-\sigma_{xx} \sin\alpha + \sigma_{xy} \cos\alpha = 0$; $-\sigma_{xy} \sin\alpha + \sigma_{yy} \cos\alpha = 0$

(e). Write the boundary conditions in weak form on the surface $x = a$. Frame the boundary conditions using the weight per unit volume $\rho g$ and the geometry of the beam.

Also, $\int_0^{a\tan\alpha} \sigma_{xx}\, dy = 0$ is acceptable

$$\int_0^{a\tan\alpha} \boxed{\sigma_{xx}\, y}\, dy = \boxed{\frac{\rho g a^3 \tan\alpha}{6}}$$

$$\int_0^{a\tan\alpha} \boxed{\sigma_{xy}}\, dy = \boxed{\frac{\rho g a^2}{2}} \tan\alpha .$$

(f). The algebraic equations arising out of the boundary conditions on $y = 0$ are

$$C_3 + 2C_5 x = 0$$
$$2C_1 + 6C_4 x = 0.$$

As these have to be satisfied for all $x$,

$$C_1 = C_3 = C_5 = C_4 = 0.$$

(g). The algebraic equations arising out of the boundary conditions on $y = x \tan \alpha$ are

$$[2c_2 + 2c_6 x + 6c_7 y + \rho g y]\sin\alpha + [c_3 + 2c_5 x + 2c_6 y]\cos\alpha = 0$$

$$[c_3 + 2c_5 x + 2c_6 y]\sin\alpha + [2c_1 + 6c_4 x + 2c_5 y + \rho g y]\cos\alpha = 0$$

(h). The algebraic equations arising out of the boundary conditions on $x = a$ are

$$2C_1 + 3C_7(a\tan\alpha) = -\frac{\rho g}{2}(a\tan\alpha) + \frac{\rho g a}{\tan\alpha}$$

$$C_2 + 2C_7(a\tan\alpha) = \frac{\rho g a^2}{3a\tan\alpha} - \frac{\rho g(a\tan\alpha)}{3}.$$

(i). Find all the constants $C_1$ to $C_7$ in terms of $\rho, g, a$ and $\alpha$.

$C_1 = 0$ $C_2 = 0$ $C_3 = 0$ $C_4 = 0$ $C_5 = 0$ $C_6 = -\frac{\rho g}{2\tan\alpha}$ $C_7 = \frac{\rho g}{3\tan^2\alpha} - \frac{\rho g}{6}$

[Marks: $8 \times 2 = 16$]

Pr. 2: You have done experiments on the disc under diametral compression as shown in the figure. We need to find the stresses $\sigma_x, \sigma_y, \sigma_{xy}$ at any point on the disc. Plane stress can be assumed as the disc is thin. Also, we will use the polar coordinates $r_1, \theta_1$ and $r_2, \theta_2$ to identify a point on the disc. This problem can be solved in many ways. We will adopt a method based on superposition of three problems.

(a). Using the Michell table, find the state of stress at any point $(x, y)$ on a disc subjected to a radial traction $t = Te_r$. This is the first problem we need to solve for the superposition to work (Fig 2b).

$$\sigma_x = T$$
$$\sigma_y = T$$
$$\sigma_{xy} = 0$$

For a radial traction applied all around the circumference:

$$\sigma_{rr} = T, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad \text{at any } (r, \theta).$$

Transforming to the x-y system with

$$\underset{\sim}{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$$\Rightarrow \sigma_{x} = T, \quad \sigma_y = T, \quad \sigma_{xy} = 0.$$

(b). Using a singular solution that was developed in the class, find the stress at the point O in the domain shown (Fig 2c). The stresses must be in terms of $r_1, \theta_1$ and $P$. This is the second problem. We will use superscipt 2 to denote that.

$$\sigma_x^2 = -\frac{2P}{\pi r_1}\cos\theta_1 \sin^2\theta_1$$

$$\sigma_y^2 = -\frac{2P}{\pi r_1}\cos^3\theta_1$$

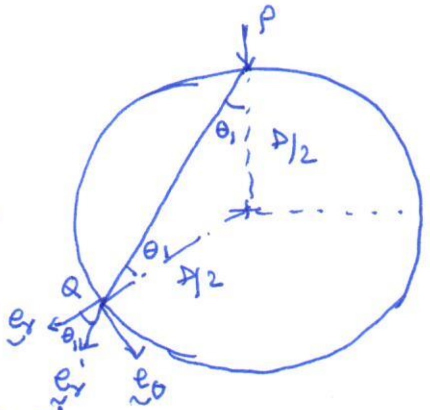
$$\sigma_{xy}^2 = -\frac{2P}{\pi r_1}\cos^2\theta_1 \sin\theta_1.$$

(c). Using the same singular solution, find the stress at the same point O. The stresses now must be in terms of $r_2, \theta_2$ and $P$ (Fig 2d). This is the third problem and we will use superscipt 3 to denote that.

$$\sigma_x^3 = -\frac{2P}{\pi r_2}\cos\theta_2\sin^2\theta_2$$
$$\sigma_y^3 = -\frac{2P}{\pi r_2}\cos^3\theta_2$$
$$\sigma_{xy}^3 = -\frac{2P}{\pi r_2}\cos^2\theta_2\sin\theta_2$$

(d). Imagine a cirular contour (of the size of the disc, i.e diameter $D$) as shown in the figures Fig 2c and 2d. What is the traction vector at point Q on the circle due to the loading shown in Fig. 2c? Write the answer in terms of $P$, $r_1$, $\theta_1$ and $D$.

$$t^1 = \boxed{-\frac{2P\cos^2\theta_1}{\pi D}}\, e_r + \boxed{-\frac{2P\cos\theta_1\sin\theta_1}{\pi D}}\, e_\theta$$




$$\sigma_r^2 = -\frac{2P\cos\theta_1}{\pi r_1}\Big|_Q = -\frac{2P}{\pi D} \quad \text{as } r_1 = 2\cdot\frac{D}{2}\cos\theta_1.$$

$$\underset{\sim}{\sigma}^2 = \sigma_r^2\, \underset{\sim}{e}_r' \otimes \underset{\sim}{e}_r'$$

$$\underset{\sim}{t}' = \underset{\sim}{\sigma}^2\cdot \underset{\sim}{n} = \underset{\sim}{\sigma}^2\cdot \underset{\sim}{e}_r' = -\frac{2P}{\pi D}\cos\theta_1\, \underset{\sim}{e}_r'$$

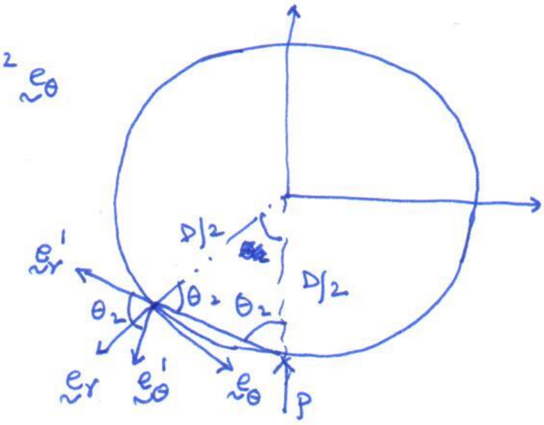
$$\underset{\sim}{t}'\cdot \underset{\sim}{e}_r = -\frac{2P}{\pi D}\cos^2\theta_1 \qquad \underset{\sim}{t}'\cdot \underset{\sim}{e}_\theta = -\frac{2P}{\pi D}\cos\theta_1\sin\theta_1 \qquad \text{as } \underset{\sim}{e}_r' = \cos\theta_1\, \underset{\sim}{e}_r + \sin\theta_1\, \underset{\sim}{e}_\theta.$$

(e). Similarly, for the problem 3, find the traction on the diameter of the dotted disc at point Q shown in Fig. 2d. Again, the answer should be in terms of $P$, $r_2$, $\theta_2$ and $D$.

$$t^2 = \boxed{-\frac{2P\sin^2\theta_1}{\pi D}}\, e_r + \boxed{-\frac{2P\sin\theta_1\cos\theta_1}{\pi D}}\, e_\theta$$

or $$-\frac{2P\cos^2\theta_2}{\pi D}\,\underset{\sim}{e_r} - \frac{2P\cos\theta_2\sin\theta_2}{\pi D}\,\underset{\sim}{e_\theta}$$

Note $\theta_1 + \theta_2 = \pi/2$.




(f). Show that at point P, the total traction due to $t^1$ and $t^2$ is purely radial and outwards. This means that if we apply an equal and opposite radial traction on the disc, we can make the dotted surface traction free. Since the outer surface of the original problem is traction free as well, superposing results will give us the solution to the stresses. Now, using the result from 2a. find the stresses at the point O (in terms of $P, D, r_1, \theta_1, r_2, \theta_2$) inside the disc that arise when the dotted circular curve is made traction free. This is our problem no 1.

$$\boxed{\begin{aligned} \sigma^1_x &= -2P/\pi D \\ \sigma^1_y &= -2P/\pi D \\ \sigma^1_{xy} &= 0. \end{aligned}}$$

(g). Now use superposition of problems 1, 2, and 3 to find the stress at any point O in the disc for case of diametral compression. The stresses should be in terms of $r_1, \theta_1$, $r_2, \theta_2$, $P$ and $D$.

$$
\begin{aligned}
\sigma_x &= -\frac{2P}{\pi r_1}\cos\theta_1\sin^2\theta_1 - \frac{2P}{\pi r_2}\cos\theta_2\sin^2\theta_2 - \frac{2P}{\pi D} \\
\sigma_y &= -\frac{2P}{\pi r_1}\cos^3\theta_1 - \frac{2P}{\pi r_2}\cos^3\theta_2 - \frac{2P}{\pi D} \\
\sigma_{xy} &= -\frac{2P}{\pi r_1}\cos^2\theta_1\sin\theta_1 - \frac{2P}{\pi r_2}\cos^2\theta_2\sin\theta_2
\end{aligned}
$$

[Marks: 2+2+2+3+2+2+2=15]

Pr. 3: We will solve the three dimensional version of the Kelvin problem that we did in the class. At the origin of a 3-d body a point force $P_i$ is applied in the $i$th direction. The point force, as discussed in the class, can be treated as a body force distribution such that $b_i = P_i\delta(x)$. Thus, we need to find the stresses from

$$\sigma_{ij,j} = -P_i \delta(x). \tag{1}$$

The cosntitutive equation for linear elasticity is

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}.$$

In this problem, an overbar denotes a Fourier transform.

(a). Taking Fourier transform of the equilibrium equation 1, complete the following result:

$$C_{ijkl}\bar{u}_k \boxed{p_l}\,\boxed{p_j} = \boxed{P_i}.$$

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \Rightarrow \sigma_{ij} = C_{ijkl}\,u_{k,l}.$$

$$\Rightarrow \sigma_{ij,j} = C_{ijkl}\,u_{k,lj}$$

$$\sim \sigma_{ij,j} = -P_i\delta(\underset{\sim}{x})$$

$$\Rightarrow C_{ijkl}\,u_{k,lj} = -P_i\,\delta(\underset{\sim}{x})$$

$$\Rightarrow C_{ijkl}\,p_l\,p_j\,\bar{u}_k = +P_i$$

(b). Assume now that we are working with an isotropic linear elastic material. The Lame's constants are $\lambda, \mu$ and their relations with $E, \nu$ have been given in the class. With this assumption, simplify and complete the left hand side of the equation you derived in 3(a):

$$\mu\left[\frac{1}{1-2\nu}\boxed{p_i}\,\boxed{p_k} + \delta_{ik}\boxed{p_m}\,\boxed{p_m}\right]\bar{u}_k = \boxed{P_i}.$$

$$C_{ijkl}\, p_l p_j = \left\{ \lambda \delta_{ij} \cdot \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right\} p_l p_j$$

or $$C_{ijkl}\, p_l p_j = (\lambda + \mu)\, p_i p_k + \mu\, \delta_{ik}\, p_m p_m$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\Rightarrow C_{ijkl}\, p_l p_j = \frac{E}{2(1+\nu)} \left\{ \frac{1}{1-2\nu}\, p_i p_k + \delta_{ik}\, p_m p_m \right\}.$$

$$= \mu \left\{ \frac{1}{1-2\nu}\, p_i p_k + p_m p_m\, \delta_{ik} \right\}.$$

(c): Use contraction in the equation you just derived to find:

$$\bar{u}_k = \frac{(1-2\nu)\boxed{P_k}}{2\mu(1-\nu)\boxed{p^2}}.$$

$$\mu\, (p_m p_m)\, \bar{u}_i + \frac{\mu}{1-2\nu}\, p_i p_k\, \bar{u}_k = P_i$$

By contracting $i$ & $k$,

$$\left\{ \mu p^2 + \frac{\mu}{1-2\nu}\, p^2 \right\} \bar{u}_k = P_k$$

$$\bar{u}_k = \frac{1-2\nu}{2\mu\,(1-\nu)\, p^2}\, P_k$$

(d). Use the results from 3(b) and (c) to complete the following equation (which is the inverse of the equation in 3(b). The procedure adopted here is exactly same as the one we used to get $S_{ijkl}$ from $c_{ijkl}$ in the isotropic case).

$$\bar{u}_i = \frac{1}{\mu}\left[\frac{\boxed{P_i}}{\boxed{p^2}} - \frac{\boxed{p_i}\,\boxed{p_k}\,\boxed{P_k}}{2(1-\nu)\boxed{p^4}}\right].$$

(e). Use the 3-d Fourier transform table to invert the equation in 3(d). This is the displacement Green's function for the 3-d Kelvin problem.

$$u_i = \frac{1}{4\pi\mu}\left[\frac{\boxed{P_i}}{\boxed{r}} - \frac{1}{4(1-\nu)}\left(\frac{\boxed{x_k}}{\boxed{r}}\right)_{,i}\right].$$

[Marks: 2+4+4+4+5=19]

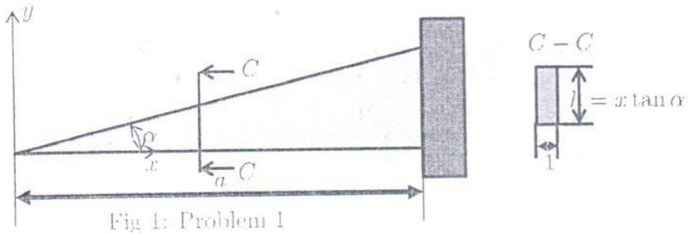


Fig 1: Problem 1

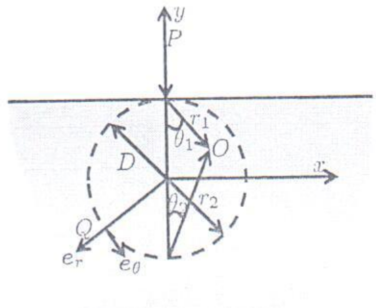


Fig 2c.

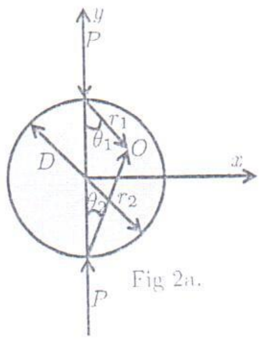


Fig 2a.

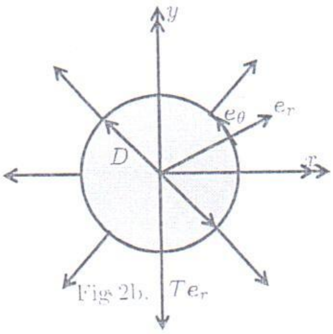


Fig 2b.

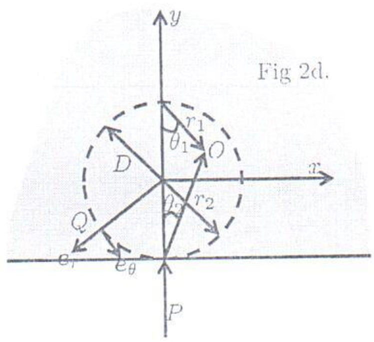


Fig 2d.