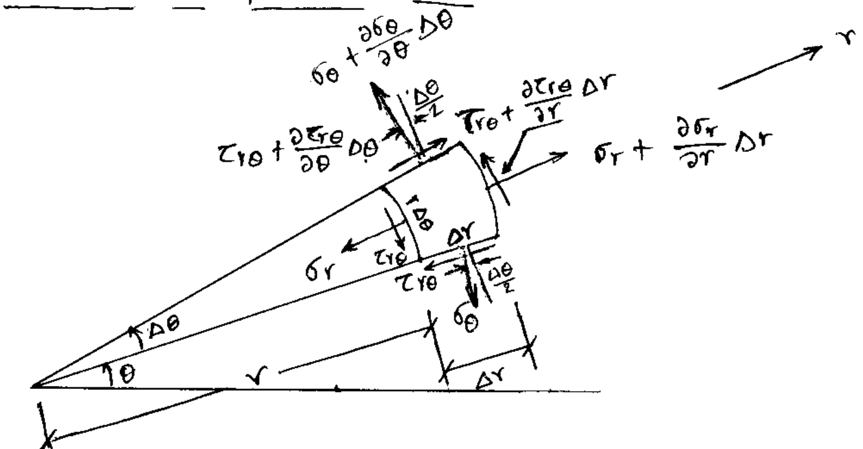
Solutions to H/W Problems

Chapter 4

Solution to problem 4.3:



i) $\tau_{\theta r} = \tau_{r\theta}$

ii) unit thickness in z-direction

iii) $\sin\frac{\Delta\theta}{2} \cong \frac{\Delta\theta}{2}$; $\cos\frac{\Delta\theta}{2} \cong 1$.

$$\Sigma F_r = 0 \Rightarrow -\sigma_r(r\Delta\theta) + \left(\sigma_r + \Delta r\frac{\partial \sigma_r}{\partial r}\right)(r+\Delta r)\Delta\theta - \sigma_\theta \sin\frac{\Delta\theta}{2}(\Delta r)$$

$$-\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}\Delta\theta\right)\sin\frac{\Delta\theta}{2}\cdot(\Delta r) - \tau_{r\theta}\cos\frac{\Delta\theta}{2}\cdot(\Delta r) + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}\Delta\theta\right)\cos\frac{\Delta\theta}{2}\cdot(\Delta r) = 0$$

$$\Rightarrow \sigma_r \Delta r\Delta\theta + \frac{\partial \sigma_r}{\partial r} r\Delta r\Delta\theta + \frac{\partial \sigma_r}{\partial r}(\Delta r)^2\Delta\theta - 2\sigma_\theta\frac{\Delta\theta\Delta r}{2}$$

$$-\frac{\partial \sigma_\theta}{\partial \theta}\frac{(\Delta\theta)^2\cdot\Delta r}{2} + \frac{\partial \tau_{r\theta}}{\partial \theta}\Delta\theta\Delta r = 0.$$

$$\Rightarrow \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r}\left(\frac{\partial \sigma_r}{\partial r}\Delta r - \frac{\partial \sigma_\theta}{\partial \theta}\frac{\Delta\theta}{2}\right) = 0.$$

$$\Rightarrow \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

by taking the limit as $\Delta r \to 0$, $\Delta\theta \to 0$

(Problem 4.9 contd.)

$\Sigma F_\theta = 0 \Rightarrow$

$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}\Delta\theta\right)\cos\frac{\Delta\theta}{2}(\Delta r) - \sigma_\theta \cos\frac{\Delta\theta}{2}\Delta r + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial\theta}\Delta\theta\right)\sin\frac{\Delta\theta}{2}\Delta r$$

$$+ \tau_{r\theta}\sin\frac{\Delta\theta}{2}(\Delta r) - \tau_{r\theta}(r\Delta\theta) + \left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial r}\Delta r\right)(r+\Delta r)(\Delta\theta) = 0.$$

$$\Rightarrow \frac{\partial\sigma_\theta}{\partial\theta}\Delta\theta\Delta r + 2\tau_{r\theta}\frac{\Delta\theta}{2}\Delta r + \frac{\partial\tau_{r\theta}}{\partial\theta}(\Delta\theta)^2\frac{\Delta r}{2} + \tau_{r\theta}(\Delta r\Delta\theta)$$

$$+ \frac{\partial\tau_{r\theta}}{\partial r} r\,\Delta r\Delta\theta + \frac{\partial\tau_{r\theta}}{\partial r}(\Delta r)^2\Delta\theta = 0.$$

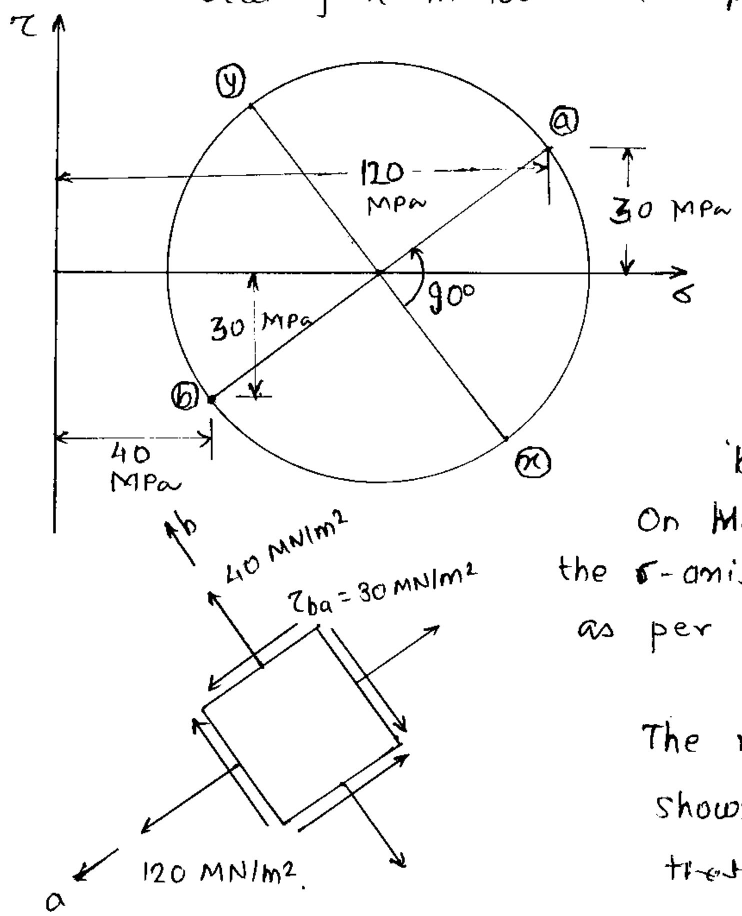
$$\Rightarrow \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} + \frac{1}{r}\left(\frac{\partial\tau_{r\theta}}{\partial\theta}\frac{\Delta\theta}{2} + \frac{\partial\tau_{r\theta}}{\partial r}\Delta r\right) = 0$$

$$\Rightarrow \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} = 0.$$

by taking the limit as $\Delta r \to 0,\ \Delta\theta \to 0$

Solution to problem 4.5

Reversing the direction of the 'a' axis is equivalent to rotating it in 180° in the physical plane.

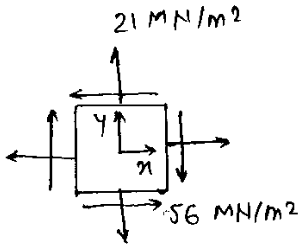


This corresponds to a 2×180 = 360° rotation in Mohr's circle, which leaves the point a unchanged.

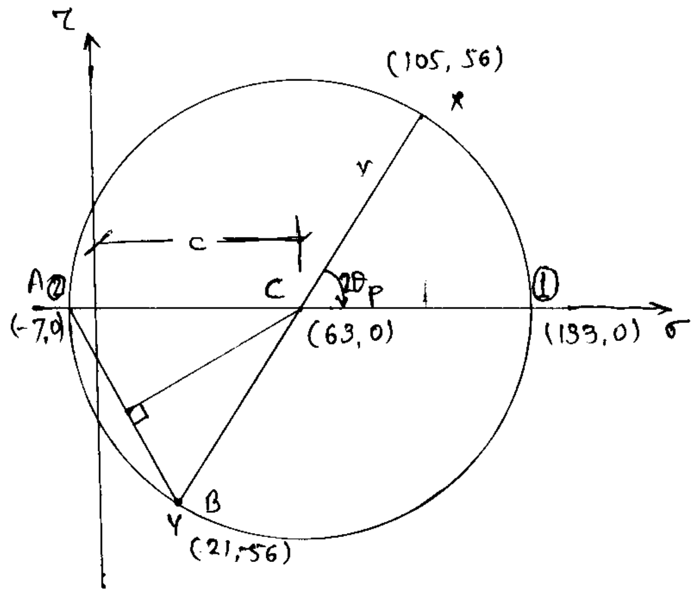
Reversed a axis ⇒ 'b' is now ~~clockwise~~ x-like axis. On Mohr's circle, "b" lies below the σ-axis. Therefor τ_{ba} is positive as per our sign convention

The resulting stress picture is shown. It is identical to that shown in Fig 4.18 (c) (Especially the direction of the shear stress)

Solution to problem 4.8

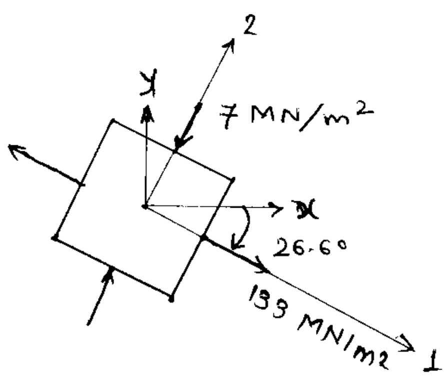


minimum principal stress
$= -7$ MN/m².



Point A corresponds to the lowest principle stress. It is located by laying at $\sigma = -7$, $\tau = 0$. Point B corresponding to 'y' is located as usual (note that τ_{xy} is negative).

Join AB. Perpendicular bisector of AB intersects σ-axis at centre C.



Thus

$\theta_p = 26.6°$

$\sigma_1 = 133$ MN/m2

$\sigma_x = 105$ MN/m2.

Calculation Details

- $(AC)^2 = (BC)^2 \Rightarrow [c-(-7)]^2 = (c-21)^2 + [0-(-56)]^2 \Rightarrow c = 63$ MPa
- $r = 63-(-7) = 70$ MPa, $\sigma_1 = c + r = 63 + 70 = 133$ MPa
- $2\theta_p = \sin^{-1}\frac{\tau_{xy}}{r} = \sin^{-1}\frac{-56}{70} \Rightarrow 2\theta_p = -53.1$ (-ve sign means CW)
- $\sigma_x = c + r\cos|2\theta_p| = 63 + 70\cos(53.1) = 105$ MPa

Solution to problem 4.11.

From problem 4.10.

$$\left.\begin{aligned} \sigma_r &= 0 \\ \sigma_\theta &= \frac{pr}{t} \\ \sigma_z &= \frac{F}{2\pi r t}. \end{aligned}\right\} \quad \left(\begin{array}{c}\text{principal}\\ \text{stresses}\end{array}\right)$$

Now $\tau_{max} = \frac{1}{2} \left| \sigma_{max} - \sigma_{min} \right|$.

a) $\tau_{max} = \sigma_{max}$

Case 1: if $F > 0$

then $\sigma_z > 0$. and $\tau_{max} = \frac{1}{2}\sigma_z$ or $\frac{1}{2}\sigma_\theta$ whichever is greater. Also $\sigma_{max} = \sigma_\theta$ or σ_z whichever is greater. Hence τ_{max} cannot possibly be equal to σ_{max}

Case 2: $F < 0$:

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_z) = \frac{1}{2}\left(\frac{pr}{t} - \frac{F}{2\pi r t}\right)$$

& $\sigma_{max} = \sigma_\theta = \frac{pr}{t}$.

$$\therefore \frac{1}{2}\left(\frac{pr}{t} - \frac{F}{2\pi r t}\right) = \frac{pr}{t}. \Rightarrow \quad F = -2\pi r^2 p.$$

b) $\tau_{max} = \frac{1}{2}\sigma_{max}$.

Case 1: $F > 0$, $\sigma_z > 0$ then τ_{max} is always $\frac{1}{2}\sigma_{max}$.

Case 2: $F < 0$, $\sigma_z < 0$

then $\frac{1}{2}\left(\frac{pr}{t} - \frac{F}{2\pi r t}\right) = \frac{1}{2}\left(\frac{pr}{t}\right)$

$$\Rightarrow F = 0.$$

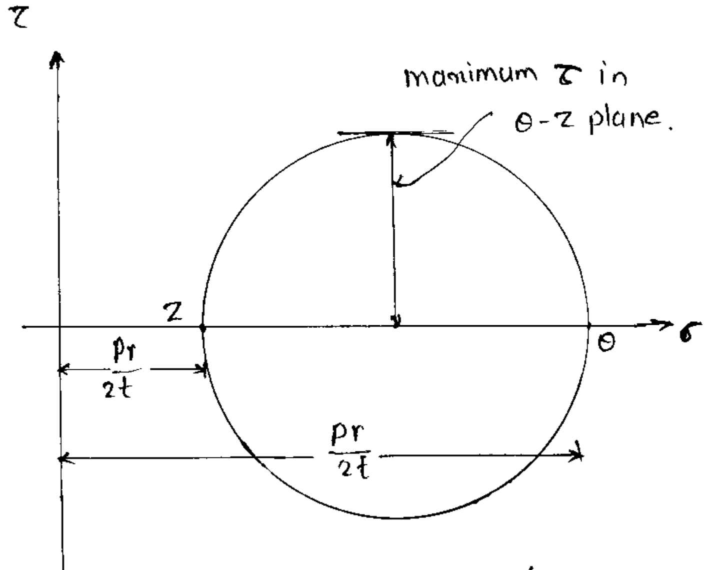
$\therefore F \geq 0$.

Solution to problem 4.13.

The state of stress is $\tau_{\theta z} = 0$,

$$\sigma_\theta = \frac{pr}{t},$$

$$\sigma_z = \frac{pr}{2t}.$$



$$(\tau_{max})_{\theta z} = \frac{1}{2}(\sigma_\theta - \sigma_z) = \frac{pr}{4t}.$$

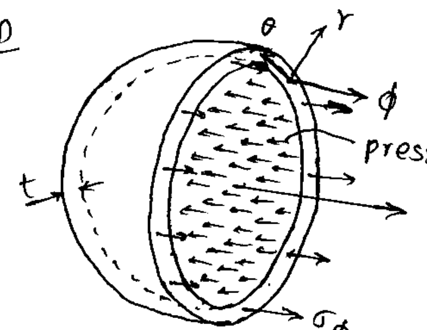
$$(\sigma_{max})_{\theta z} = \sigma_\theta = \frac{pr}{t}.$$

$$\therefore (\tau_{max})_{\theta z} = \frac{1}{4}(\sigma_{max})_{\theta z}.$$

[Note: $(\tau_{max})_{\theta z}$ is not the maximum shear stress].

Solution to problem 4.14

- By symmetry it can be shown that the shear stresses $\tau_{r\theta}$, $\tau_{\theta\phi}$ and $\tau_{\phi r}$ are zero.
- F.B.D



r = radius of sphere.

Hemisphere + fluid in it.

Equilibrium:

$$\Sigma F_x = 0 \Rightarrow -p \times \pi r^2 + \sigma_\phi (2\pi r t) = 0.$$

$$\therefore \boxed{\sigma_\phi = \frac{pr}{2t}}.$$

Similarly $\boxed{\sigma_\theta = \frac{pr}{2t}}$

- Inside surface $\sigma_r = -p$.

  Outside surface $\sigma_r = 0$.

  Hence it is reasonable to assume $-p \le \sigma_r \le 0$ everywhere.

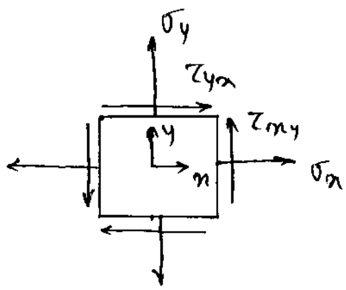
  Since $r/t \gg 1$, $\sigma_r \ll \sigma_\theta, \sigma_\phi$ and can be neglected.

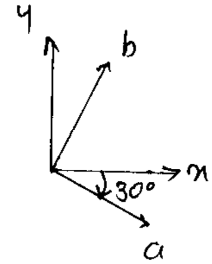
  $\therefore \boxed{\sigma_r \simeq 0}$.

  Since $\tau_{r\theta}$ and $\tau_{r\phi}$ are also zero, it is a plane stress problem in $\theta-\phi$ plane.

- In $\theta-\phi$ plane, $\tau_{\theta\phi}$ is zero. Therefore, σ_θ and σ_ϕ are the principal stresses
- When the problem is considered as a 3-D, σ_r, σ_θ and σ_ϕ are the principal stresses

Solution to problem 4.15





Given: $\sigma_x = -15\ MN/m^2$

$\tau_{xy} = 0$

max. tensile stress $= 75 MN/m^2$.

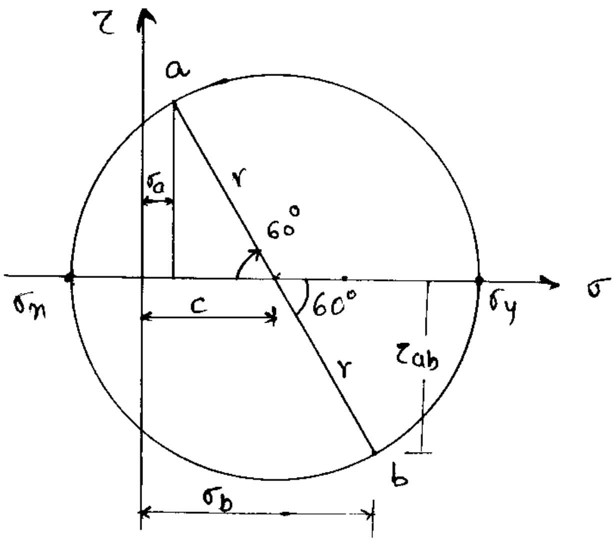
To find: stress components on faces perpendicular to a, b axes.

To sketch: Stress components on faces ⊥ to a, b axes.

$\tau_{yx} = \tau_{xy} = 0$.

∴ stresses on x and y planes are principal stresses.

Since max. normal stress = 75 MN/m² $\sigma_y = 75\ MN/m^2$.



$$c = \frac{\sigma_x + \sigma_y}{2} = \frac{-15+75}{2} = 30\ MN/m^2.$$

$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \left|\frac{\sigma_x - \sigma_y}{2}\right|$$

$$= 45\ MN/m^2.$$

$$\sigma_b = c + r\cos 60$$

$$= 30 + 45 \cdot \tfrac{1}{2} = 52.5\ MN/m^2.$$

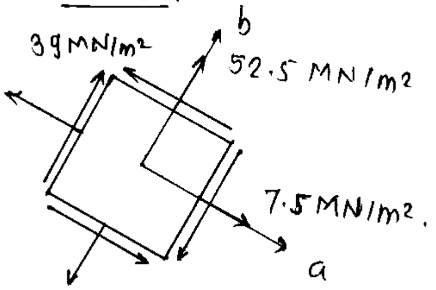
$$\sigma_a = c - r\cos 60$$

$$= 30 - 45 \cdot \tfrac{1}{2} = 7.5\ MN/m^2.$$

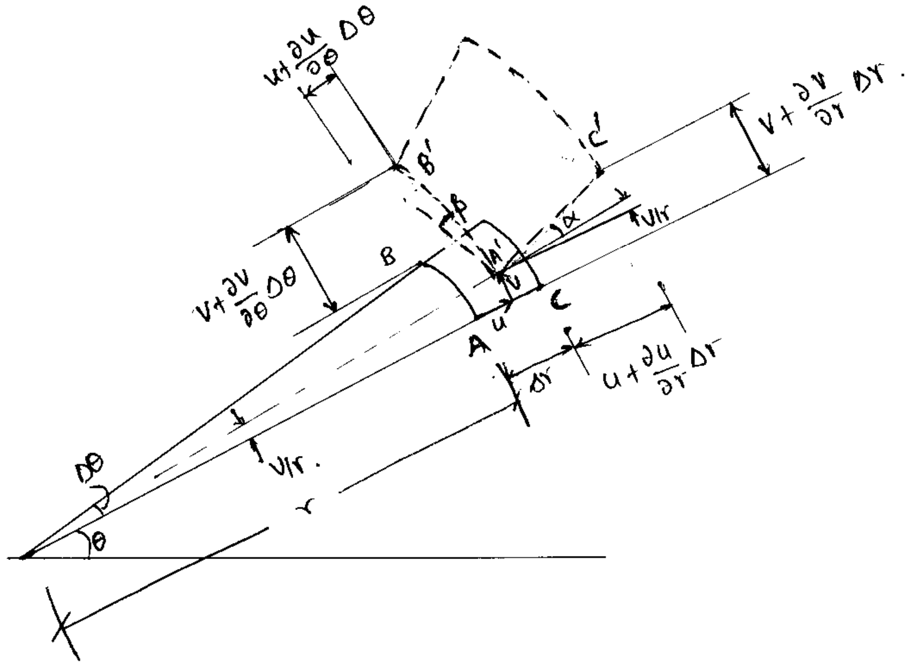
$$\tau_{ab} = r\sin 60 = 39\ MN/m^2.$$

(Since it is above σ-axis at a, it is negative)

Sketch:



Solution to problem 4.18



$$\varepsilon_{rr} = \lim_{\Delta r\to 0} \frac{A'C' - AC}{AC} = \frac{\left[\Delta r + \left(u + \frac{\partial u}{\partial r}\Delta r\right) - u\right] - \Delta r}{\Delta r} = \frac{\partial u}{\partial r}.$$

$$\varepsilon_{\theta\theta} = \lim_{\Delta\theta\to 0} \frac{A'B' - AB}{AB} = \frac{\left[(r+u)\Delta\theta + \left(v + \frac{\partial v}{\partial\theta}\Delta\theta\right) - v\right] - r\Delta\theta}{r\Delta\theta}$$

$$= \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{v}.$$

$$\gamma_{r\theta} = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}} \left(\angle BAC - \angle B'A'C'\right) = \frac{\pi}{2} - \left[\frac{\pi}{2} - (\beta+\alpha)\right]$$

$$= \beta + \alpha$$

$$= \frac{u + \frac{\partial u}{\partial\theta}\Delta\theta - u}{r\Delta\theta} + \left[\frac{v + \frac{\partial v}{\partial r}\Delta r - v}{\Delta r} - \frac{v}{r}\right]$$

$$= \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial\theta}.$$

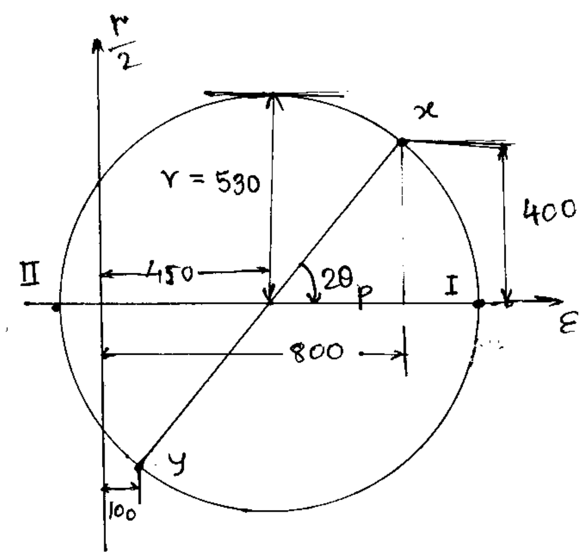
Solution to problem 4.20.

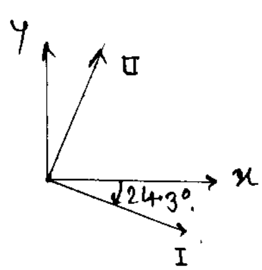
$\varepsilon_x = 800 \times 10^{-6}$

$\varepsilon_y = 100 \times 10^{-6}$

$\gamma_{xy} = -800 \times 10^{-6}$.

Mohr's Circle:





$$c = \frac{\varepsilon_x + \varepsilon_y}{2} = \frac{(800+100) \times 10^{-6}}{2}$$

$$= 450 \times 10^{-6}$$

$$r = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \sqrt{\left(\frac{800-100}{2}\right)^2 + (-400)^2} \times 10^{-6}$$

$$\simeq 530 \times 10^{-6}$$

$$\theta_p = \frac{1}{2} \tan^{-1}\left(\frac{-400}{350}\right)$$

$$= -24.3^\circ \quad (- \text{ sign means cw})$$

$$\varepsilon_{I} = (450 + 530) \times 10^{-6}$$

$$= 980 \times 10^{-6}$$

$$\varepsilon_{II} = [450 - 530] \times 10^{-6}$$

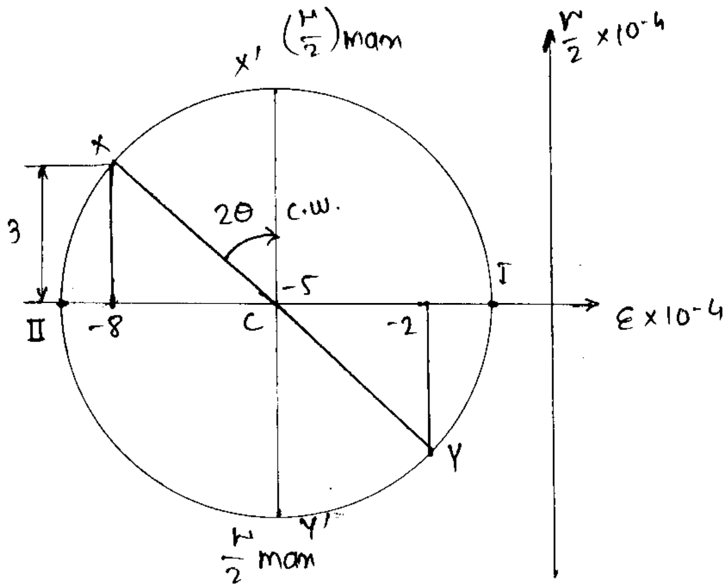
$$= -80 \times 10^{-6}.$$

Solution to problem 4.23.

$\varepsilon_{xx} = -800 \times 10^{-6}$

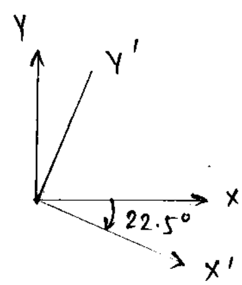
$\varepsilon_{yy} = -200 \times 10^{-6}$

$\gamma_{xy} = -600 \times 10^{-6}$



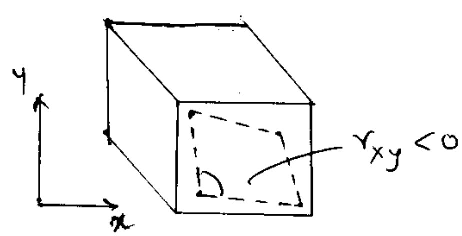
$$C = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} = -\frac{8+2}{2} \times 10^{-4}$$
$$= -5 \times 10^{-4}$$

$$\tan 2\theta = \frac{|\varepsilon_{xx} - C|}{\left|\gamma_{xy}/2\right|}$$
$$= \frac{|-8-(-5)| \times 10^{-4}}{3 \times 10^{-4}}$$
$$= 1$$
$$\therefore \theta = \frac{45}{2}$$
$$= 22.5 \text{ c.w.}$$



$x'y' \equiv$ axes associated with γ_{max}.

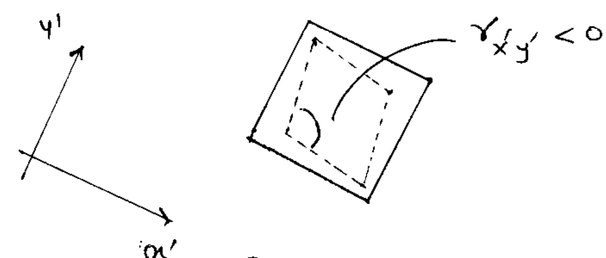
Deformed shape: —— : undeformed; - - - : deformed



$\varepsilon_{xx} < 0$

$\varepsilon_{yy} < 0$

$\gamma_{xy} < 0$



$\varepsilon_{x'x'} < 0$

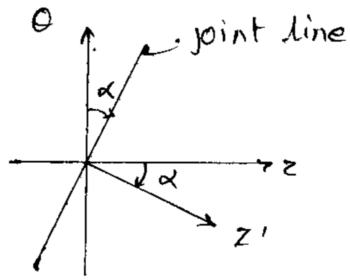
$\varepsilon_{y'y'} < 0$

$\gamma_{x'y'} < 0$.

Solution to problem 4.25

From problem 4.12 we get $\sigma_\theta = \frac{pr}{t}$, $\sigma_z = \frac{pr}{2t}$

$\tau_{\theta z} = 0.$



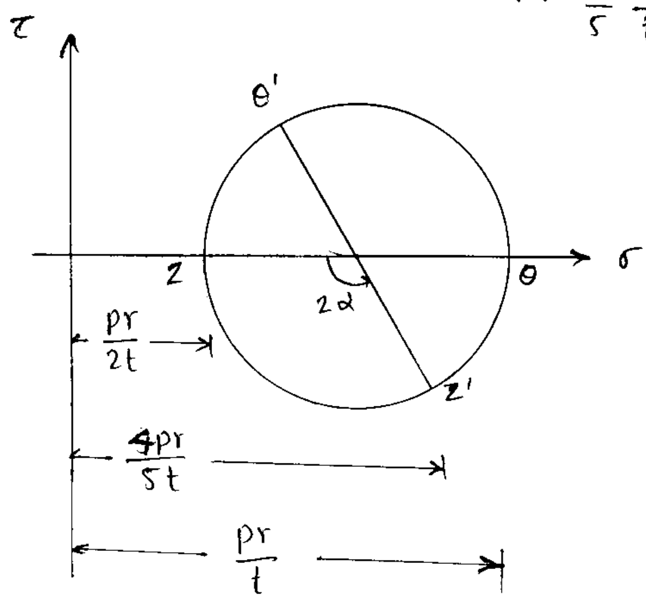
$$\sigma_{z'} = \frac{\sigma_z + \sigma_\theta}{2} + \frac{\sigma_z - \sigma_\theta}{2}\cos 2\alpha + \cancel{\tau_{\theta z}}^{\,0} \sin 2\alpha$$

$$\sigma_{z'} = \frac{4pr}{5t} \quad (\sigma_{z'} = 0.8\,\sigma_\theta \text{ given})$$

$$\therefore \frac{4}{5}\frac{pr}{t} = \frac{3}{4}\frac{pr}{t} - \frac{1}{4}\frac{pr}{t}\cos 2\alpha .$$

$$\therefore \cos 2\alpha = -\frac{1}{5}.$$

$$\alpha = -50.76^\circ . \quad \left(\begin{matrix}\text{- ve sign}\\ \text{means cw}\end{matrix}\right)$$



This is shown in the neighbouring Mohr's circle

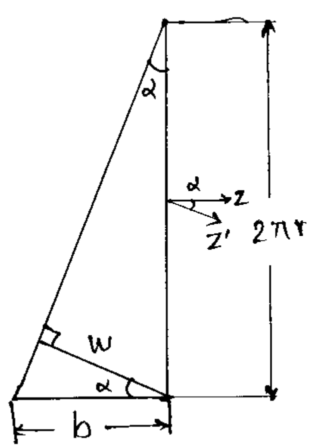
Let,

b = axial distance travelled in 1 revolution.

Then

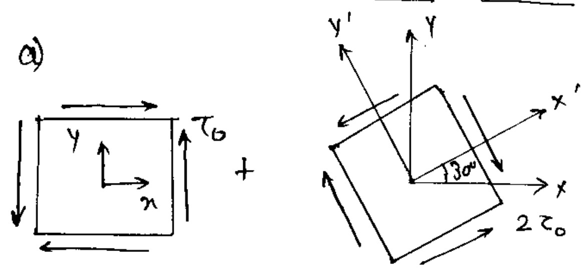
$$\sin\alpha = \frac{w}{2\pi r} \Rightarrow$$

$$w = 2\pi r \sin\alpha = 4.866r$$

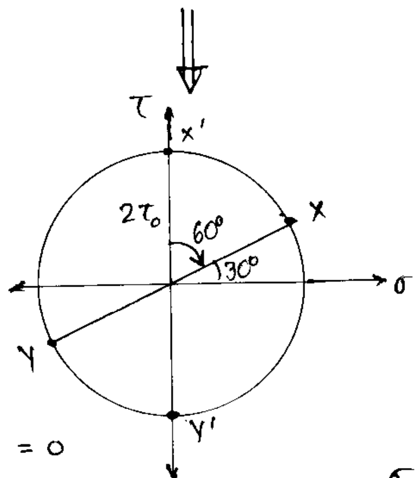


Solution to problem 4.26

a)



x : 30° c.w. from x'.



$c = 0$

$r = 2\tau_0$

$\sigma_x = 2\tau_0 \cos 30 = \sqrt{3}\,\tau_0$

$\sigma_y = -2\tau_0 \cos 30 = -\sqrt{3}\,\tau_0$

$\tau_{xy} = -2\tau_0 \sin 30 = -\tau_0$.

(-ve sign according to sign convention

Total state of stress:

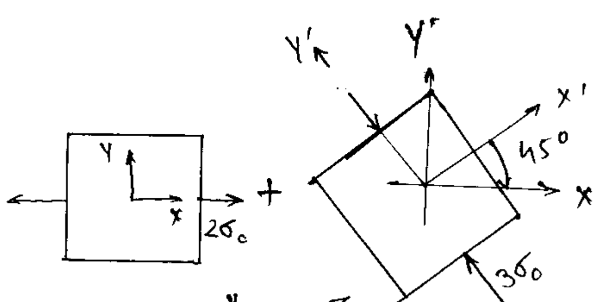
$\sigma_x = 0 + \sqrt{3}\,\tau_0 = \sqrt{3}\,\tau_0$

$\sigma_y = 0 - \sqrt{3}\,\tau_0 = -\sqrt{3}\,\tau_0$.

$\tau_{xy} = \tau_0 - \tau_0 = 0$.

Since $\tau_{xy} = 0$, x and y are principal stress directions

b)



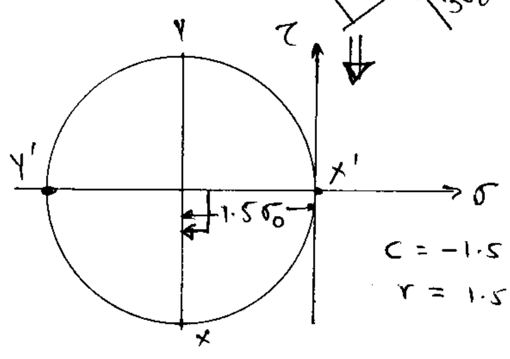
x : 45° c.w. from x'.

$\sigma_x = -1.5\sigma_0$

$\sigma_y = -1.5\sigma_0$

$\tau_{xy} = 1.5\sigma_0$

(+ve sign according to the sign convention.)



$c = -1.5\sigma_0$

$r = 1.5\sigma_0$

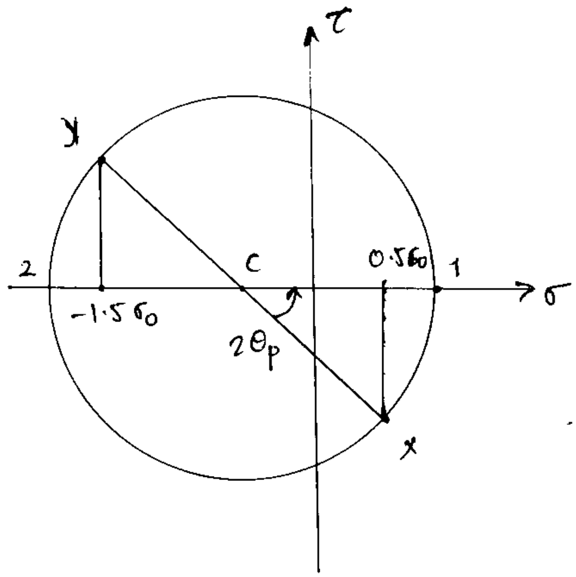
(Problem 4.26 contd.)

Total state of stress:

$\sigma_x = 2\sigma_0 - 1.5\sigma_0 = 0.5\sigma_0$

$\sigma_y = 0 - 1.5\sigma_0 = -1.5\sigma_0$

$\tau_{xy} = 0 + 1.5\sigma_0 = 1.5\sigma_0$

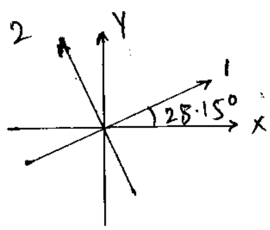


$$2\theta_p = \tan^{-1}\frac{\tau_{xy}}{[(\sigma_x - \sigma_y)/2]}$$

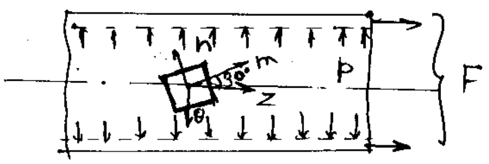
$$= \tan^{-1}\left(\frac{1.5\sigma_0}{\sigma_0}\right)$$

$$= 56.3°.$$

$\therefore\ \theta_p = 28.15°$, ccw from x.



Solution to problem 4.27



$r = 10''$ $\quad t = 0.1''$

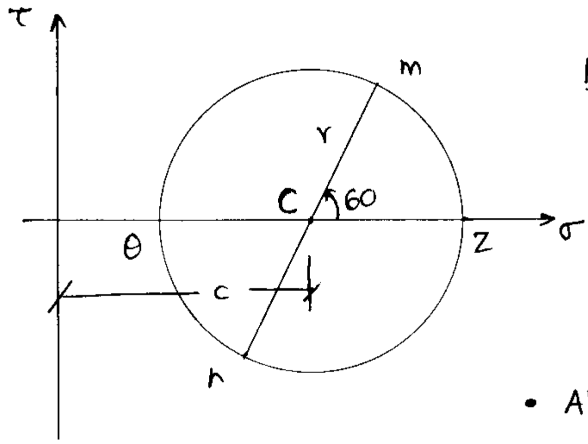
To find: p & F

Such that

a) $\sigma_m = 15\,000$ psi, $\sigma_n = 5000$ psi

b) $\sigma_m = 15000$ psi, $\sigma_n = 15000$ psi

θ and z are principal axes. Further σ_θ has to be less than σ_z as $\sigma_m > \sigma_n$.



Note: • $c = \dfrac{\sigma_m + \sigma_n}{2}$.

• $\sigma_m = c + r\cos 60$

$\Rightarrow r = \dfrac{\sigma_m - c}{\cos 60}$.

• $\sigma_z = c + r$, $\quad \sigma_\theta = c - r$.

• Also $\quad \sigma_z = \dfrac{F}{2\pi r t}$, $\quad \sigma_\theta = \dfrac{pr}{t}$.

$\therefore \quad F = \sigma_z \cdot 2\pi r t$

$p = \sigma_\theta \, (t/r)$

(a) $C = \dfrac{15000 + 5000}{2} = 10000$ psi.

$r = \dfrac{\sigma_m - c}{\cos 60} = \dfrac{15000 - 10000}{1/2} = 10000$ psi $\Big\} \Rightarrow \sigma_z = 20000$ psi, $\sigma_\theta = 0$ psi.

$\Rightarrow F = 2\pi \times 10 \times 0.1 \times 20{,}000 = 1.26 \times 10^5$ lb. and $p = 0$.

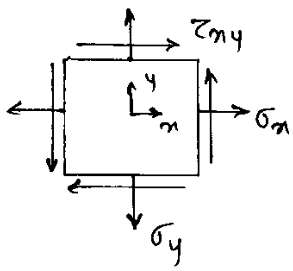
(b) $C = \dfrac{15000 + 15000}{2} = 15000$ psi.

$r = \dfrac{\sigma_m - c}{\cos 60} = 0$ $\Big\} \Rightarrow \sigma_z = 15{,}000$ psi. $\sigma_\theta = 15{,}000$ psi.

$\Rightarrow F = 2\pi \times 10 \times 0.1 \times 15000 = 0.94 \times 10^5$ lb.

$p = 0.1/10 \times 15000 = 150$ lb/in² $= 150$ psi.

Solution to problem 4.28.



Given: $\sigma_1 = 20\,MPa$

$\sigma_2 = -45\,MPa$

$\tau_{xy} = 20\,MPa$.

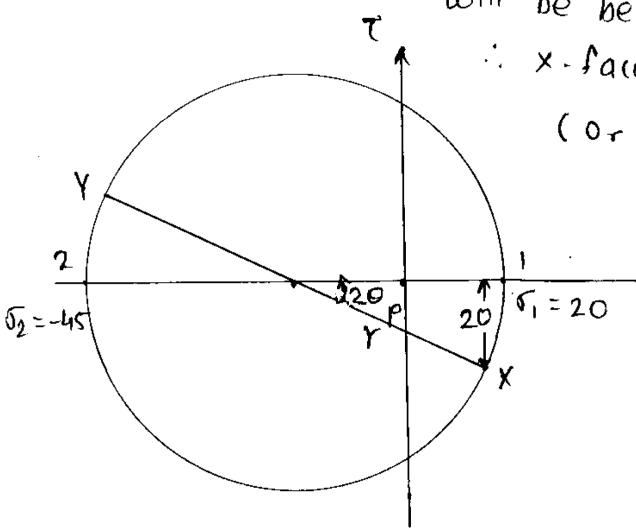
To find: σ_x, σ_y.

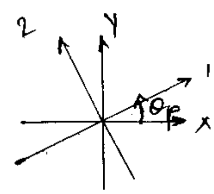
To sketch: directions and magnitude.

Mohr's Circle: Since τ_{xy} is c.c.w on x-face, point x will be below σ-axis on mohr's circle.

$\therefore$ x-face is clockwise from 1-face.

(or 1-face is ccw from x-face)





$$r = \frac{\sigma_1 - \sigma_2}{2} = \frac{20-(-45)}{2} = 32.5\,MPa.$$

$\tau_{xy} = r\sin 2\theta_p$

$\therefore\ 20 = 32.5 \sin 2\theta_p$

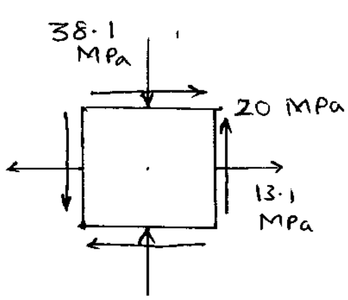
$\Rightarrow 2\theta_p = 38^\circ$ or $\theta_p = 19^\circ$.

$$C = \frac{\sigma_1+\sigma_2}{2} = \frac{20-45}{2} = -12.5\,MPa.$$

$\sigma_x = C + r\cos 2\theta = -12.5 + 32.5\cos 38 = 13.1\,MPa.$

$\sigma_y = C - r\cos 2\theta = -12.5 - 32.5(\cos 38) = -38.1\,MPa.$

Sketch:



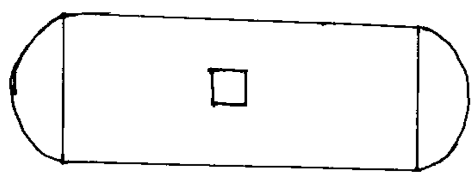
Therefore,

$\sigma_x = 13.1\,MPa$.

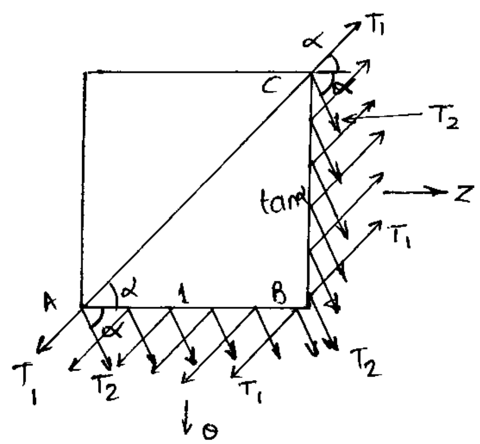
$\sigma_y = -38.1\,MPa$.

$\tau_{xy} = 20\,MPa$.

Solution to problem 4.32 :



Thickness = t



- Element sides are chosen so as to have the same no. of fibres on each side.
  If AB = 1, then BC = tanα.
- FBD of a small element in θ - z plane is shown.
- The resin doesn't carry any load. Thus σ_θ and σ_z are obtained by deviding the net forces carried by the glass fibres by the appropriate areas.
- T_1 and T_2 are forces carried by fibres. Further, they are equal.
- $\sigma_\theta = \frac{pr}{t}$, $\sigma_z = \frac{pr}{2t}$.
- n be the no. of fibres on each face

On face AB :

$$\frac{nT_1 \sin\alpha}{1 \cdot t} + \frac{nT_2 \sin\alpha}{1 \cdot t} = \sigma_\theta \quad \text{— i)}$$

On face BC :

$$\frac{nT_1 \cos\alpha}{\tan\alpha \cdot t} + \frac{nT_2 \cos\alpha}{\tan\alpha \cdot t} = \sigma_z \quad \text{— ii)}$$

For given α, T_1 and T_2 can be found as follows.

Given $T_1 = T_2 = T$. Then

$$\frac{2nT \sin\alpha}{t} = \frac{pr}{t} \quad \text{—— iii)}$$

$$\frac{2nT \cos\alpha}{t \cdot \tan\alpha} = \frac{pr}{2t} \quad \text{—— iv)}$$

equations iii) and iv) give :

$$\frac{\sin\alpha \tan\alpha}{\cos\alpha} = 2 \Rightarrow \tan^2\alpha = 2 \Rightarrow \alpha = 54.74^\circ.$$