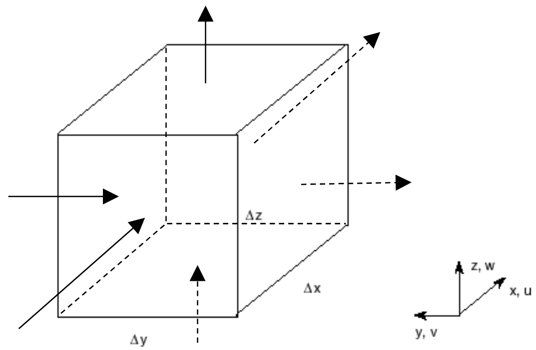
Equations of Continuity

Differential Mass Balance

Mass balance: $\begin{pmatrix}\text{Rate of}\\ \text{accumulation}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \text{mass in}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\ \text{mass out}\end{pmatrix}$



$$\begin{pmatrix}\text{Rate of mass}\\ \text{accumulation}\end{pmatrix} = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$$

$$\begin{pmatrix}\text{Rate of}\\ \text{mass in}\end{pmatrix} = (\rho u)_x \Delta y \Delta z + (\rho v)_y \Delta x \Delta z + (\rho w)_z \Delta x \Delta y$$

$$\begin{pmatrix}\text{Rate of}\\ \text{mass out}\end{pmatrix} = (\rho u)_{x+\Delta x} \Delta y \Delta z + (\rho v)_{y+\Delta y} \Delta x \Delta z + (\rho w)_{z+\Delta z} \Delta x \Delta y$$

Differential Mass Balance

Substituting:

$$\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z = \left[(\rho u)_x \Delta y \Delta z + (\rho v)_y \Delta x \Delta z + (\rho w)_z \Delta x \Delta y\right]$$
$$-\left[(\rho u)_{x+\Delta x} \Delta y \Delta z + (\rho v)_{y+\Delta y} \Delta x \Delta z + (\rho w)_{z+\Delta z} \Delta x \Delta y\right]$$

Rearranging:

$$\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z = \left[(\rho u)_x - (\rho u)_{x+\Delta x}\right]\Delta y \Delta z$$
$$+\left[(\rho v)_y - (\rho v)_{y+\Delta y}\right]\Delta x \Delta z$$
$$+\left[(\rho w)_z - (\rho w)_{z+\Delta z}\right]\Delta x \Delta y$$

Differential Equation of Continuity

Dividing everything by ΔV:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{(\rho u)_{x+\Delta x} - (\rho u)_x}{\Delta x} + \frac{(\rho v)_{y+\Delta y} - (\rho v)_y}{\Delta y} + \frac{(\rho w)_{z+\Delta z} - (\rho w)_z}{\Delta z} \right]$$

Taking the limit as Δx, Δy and $\Delta z \rightarrow 0$:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right]$$

Differential Equation of Continuity

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = -\nabla \cdot \boxed{\left(\rho \vec{V}\right)}$$

divergence of mass velocity vector $\rho \vec{V}$

Partial differentiation:

$$\frac{\partial \rho}{\partial t} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right)$$

Differential Equation of Continuity

Rearranging:

$$\boxed{\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

substantial time derivative

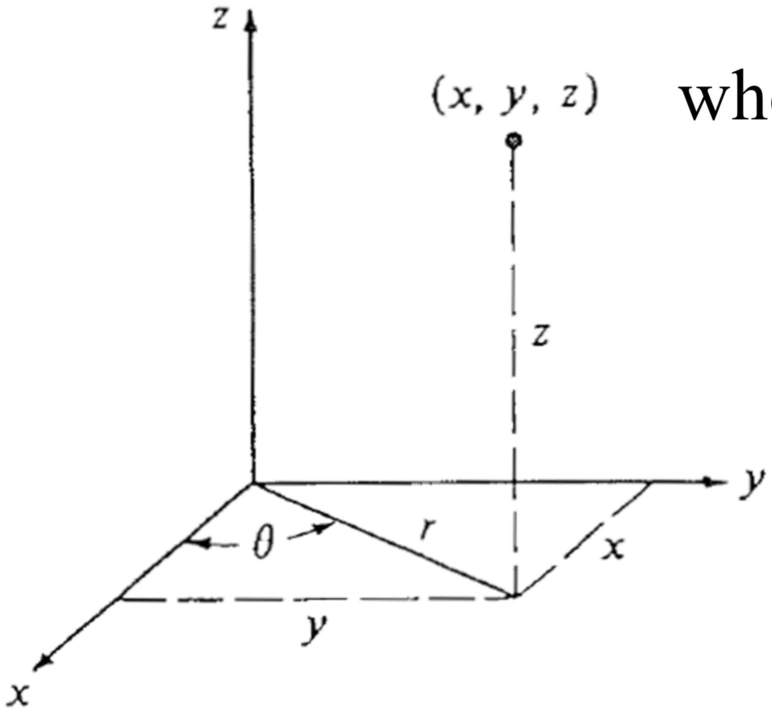
$$\boxed{\frac{D\rho}{Dt}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=-\rho\left(\nabla\cdot\vec{V}\right)}$$

If fluid is incompressible: $\nabla\cdot\vec{V}=0$

Differential Equation of Continuity

In cylindrical coordinates:

$$\frac{d\rho}{dt}+\frac{1}{r}\frac{\partial\left(\rho r v_r\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\rho v_\theta\right)}{\partial \theta}+\frac{\partial\left(\rho v_z\right)}{\partial z}=0$$



where $r=\sqrt{x^2+y^2},\ \theta=\tan^{-1}\left(\frac{y}{x}\right)$

If fluid is incompressible:

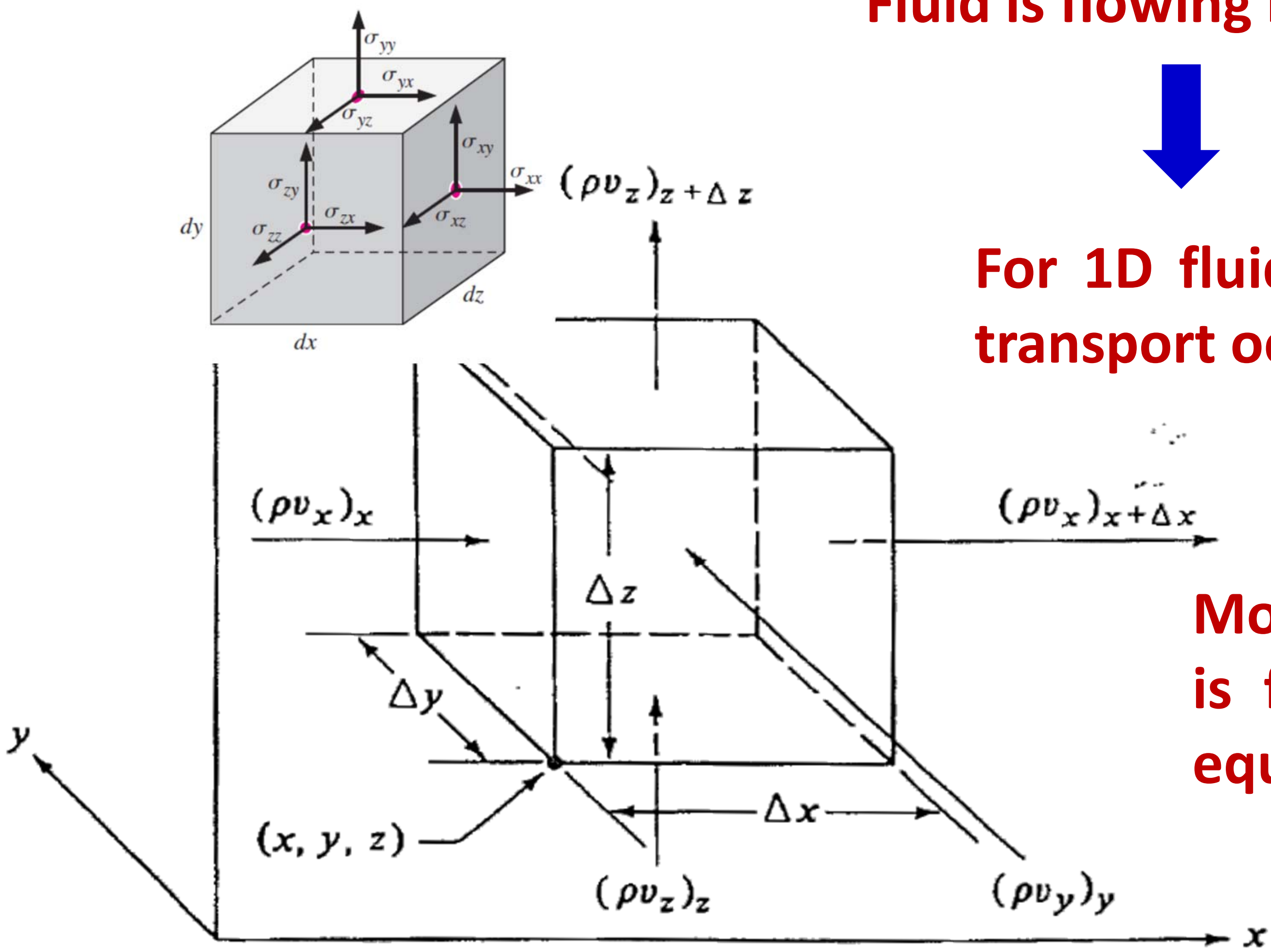
$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Differential Equations of Motion

Derivation Using an Infinitesimal Control Volume

Momentum Balance

Control Volume



Fluid is flowing in 3 directions

For 1D fluid flow, momentum transport occurs in 3 directions

Momentum transport is fully defined by 3 equations of motion

Momentum Balance

Consider the *x*-component of the momentum transport:

$$\begin{pmatrix}\text{Rate of}\\ \text{accumulation}\\ \text{of momentum}\end{pmatrix}_x = \boxed{\begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix}_x - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix}_x} + \begin{pmatrix}\text{Sum of forces}\\ \text{acting in}\\ \text{the system}\end{pmatrix}_x$$

$$\begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix}_x - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix}_x$$

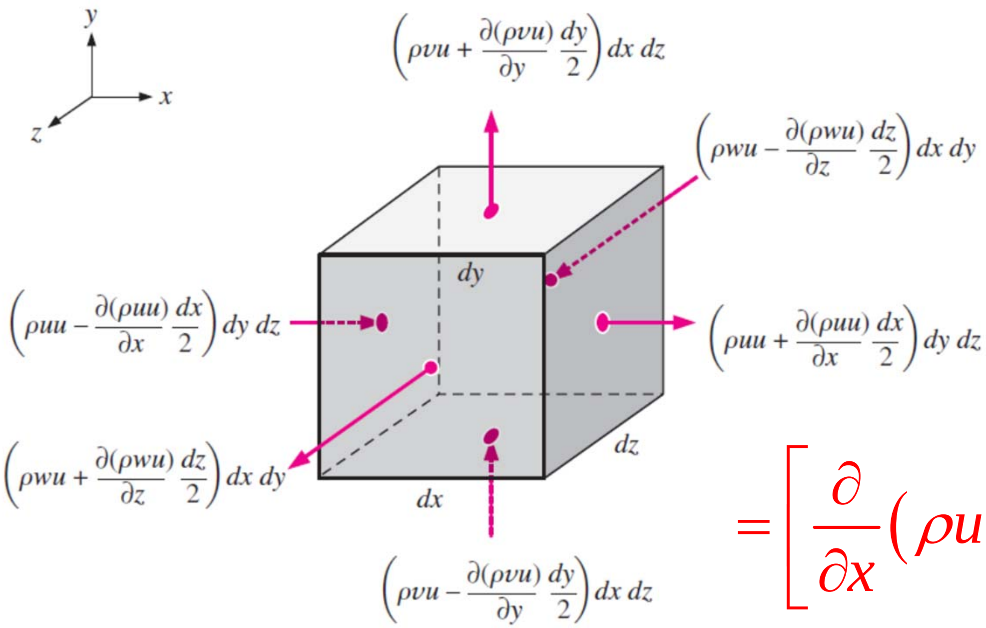
$$= \left[\begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix}_x - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix}_x\right]_{\text{convective}}$$

Momentum Balance

Due to convective transport:

$$\left[\begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix}_x - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix}_x\right]_{\text{convective}}$$

$$=\left[(\rho uu)_{x-\Delta x/2}-(\rho uu)_{x+\Delta x/2}\right]\Delta y\Delta z$$
$$+\left[(\rho vu)_{y-\Delta y/2}-(\rho vu)_{y+\Delta y/2}\right]\Delta x\Delta z$$
$$+\left[(\rho wu)_{z-\Delta z/2}-(\rho wu)_{z+\Delta z/2}\right]\Delta x\Delta y$$



$$=\left[\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)+\frac{\partial}{\partial z}(\rho wu)\right]\Delta x\Delta y\Delta z$$

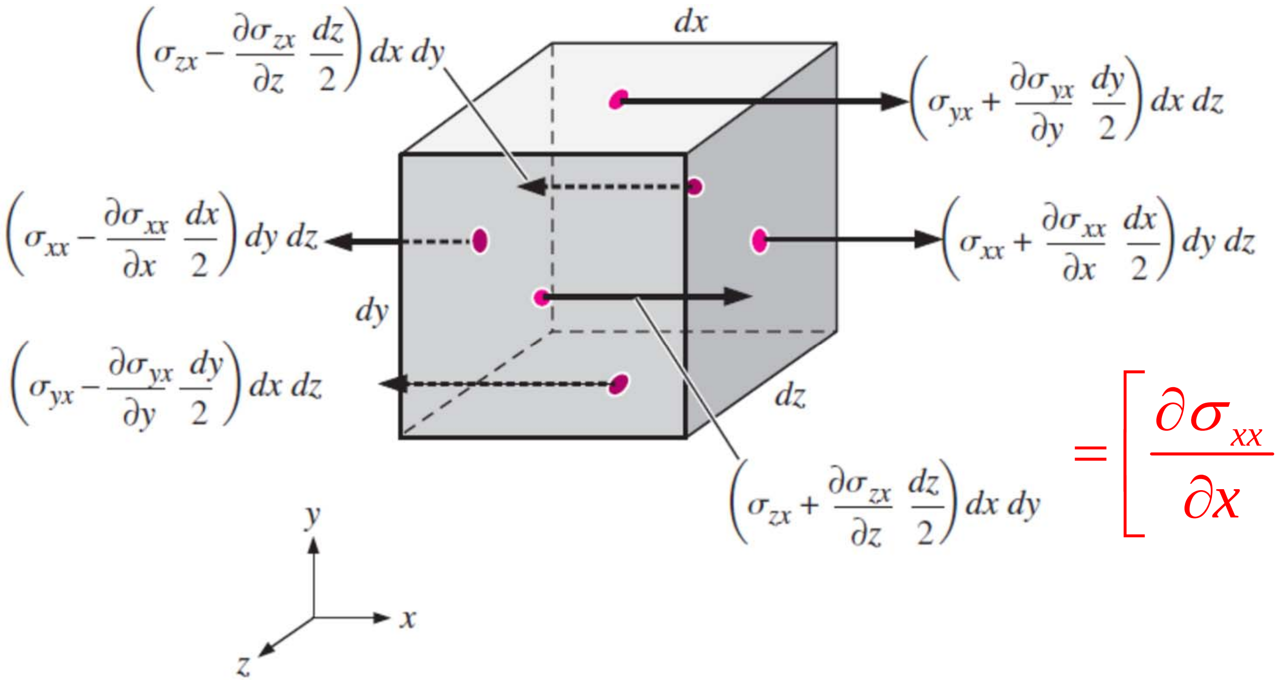
Origin at the centre of the cubic element

Momentum Balance

Surface Force: $\left(\sum F_s\right)_x$

$$\left(\sum F_s\right)_x = \left[\left(\sigma_{xx}\right)_{x+\Delta x/2} - \left(\sigma_{xx}\right)_{x-\Delta x/2}\right]\Delta y \Delta z$$
$$+\left[\left(\sigma_{yx}\right)_{y+\Delta y/2} - \left(\sigma_{yx}\right)_{y-\Delta y/2}\right]\Delta x \Delta z$$
$$+\left[\left(\sigma_{zx}\right)_{z+\Delta z/2} - \left(\sigma_{zx}\right)_{z-\Delta z/2}\right]\Delta x \Delta y$$

Origin at the centre of the cubic element



$$= \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right]\Delta x \Delta y \Delta z$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Momentum Balance

Consider the *x*-component of the momentum transport:

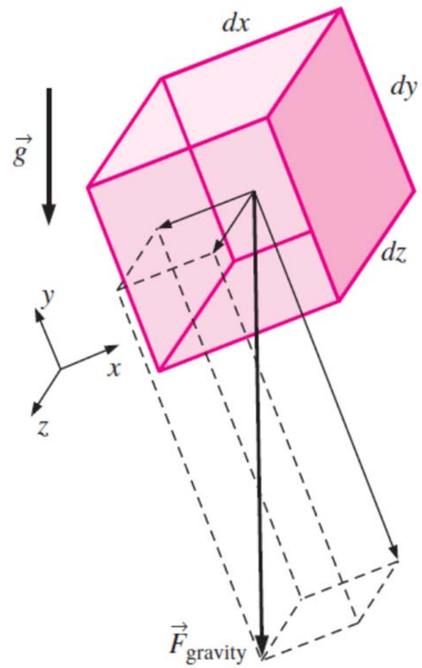
$$\begin{pmatrix}\text{Rate of}\\ \text{accumulation}\end{pmatrix}_x = \begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix}_x - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix}_x + \boxed{\begin{pmatrix}\text{Sum of forces}\\ \text{acting in}\\ \text{the system}\end{pmatrix}_x}$$

$$\begin{pmatrix}\text{Sum of forces}\\ \text{acting in}\\ \text{the system}\end{pmatrix}_x = \left(\sum F_B + \sum F_s\right)_x$$

$$= \text{Sum of body forces} + \text{Sum of surface forces}$$

$$\vec{g} = \hat{i}g_x + \hat{j}g_y + \hat{k}g_z$$

$$\left(\sum F_B\right)_x = \rho g_x \Delta x \Delta y \Delta z$$



Momentum Balance

Consider the *x*-component of the momentum transport:

$$\begin{pmatrix}\text{Rate of}\\ \text{accumulation}\end{pmatrix}_x = \begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix}_x - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix}_x + \begin{pmatrix}\text{Sum of forces}\\ \text{acting in}\\ \text{the system}\end{pmatrix}_x$$

$$\begin{pmatrix}\text{Rate of}\\ \text{accumulation}\end{pmatrix}_x = \frac{\partial(\rho u)}{\partial t}\Delta x \Delta y \Delta z$$

Differential Momentum Balance

Substituting:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\frac{\partial(\rho u)}{\partial t}\Delta x\Delta y\Delta z = \left[(\rho uu)_{x-\Delta x/2} - (\rho uu)_{x+\Delta x/2}\right]\Delta y\Delta z$$

$$+\left[(\rho vu)_{y-\Delta y/2} - (\rho vu)_{y+\Delta y/2}\right]\Delta x\Delta z$$

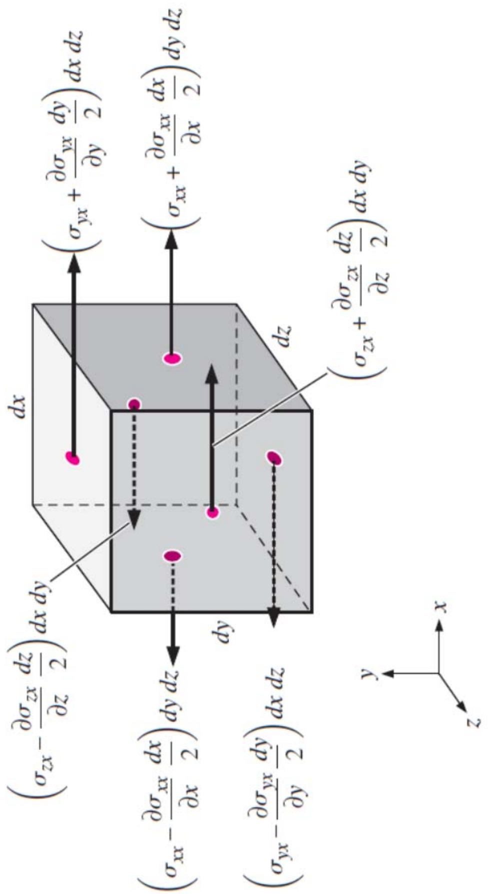
$$+\left[(\rho wu)_{z-\Delta z/2} - (\rho wu)_{z+\Delta z/2}\right]\Delta x\Delta y$$

$$+\left[(\tau_{xx})_{x+\Delta x/2} - (\tau_{xx})_{x-\Delta x/2}\right]\Delta y\Delta z$$ *positive x-direction*

$$+\left[(\tau_{yx})_{y+\Delta y/2} - (\tau_{yx})_{y-\Delta y/2}\right]\Delta x\Delta z$$

$$+\left[(\tau_{zx})_{z+\Delta z/2} - (\tau_{zx})_{z-\Delta z/2}\right]\Delta x\Delta y$$

$$+\left(p_{x-\Delta x/2} - p_{x+\Delta x/2}\right)\Delta y\Delta z + \rho g_x\Delta x\Delta y\Delta z$$



Differential Momentum Balance

Dividing everything by ΔV ($= \Delta x \Delta y \Delta z$):

$$\frac{\partial(\rho u)}{\partial t} = \frac{\left[(\rho uu)_{x-\Delta x/2} - (\rho uu)_{x+\Delta x/2}\right]}{\Delta x} + \frac{\left[(\rho vu)_{y-\Delta y/2} - (\rho vu)_{y+\Delta y/2}\right]}{\Delta y}$$

$$+ \frac{\left[(\rho wu)_{z-\Delta z/2} - (\rho wu)_{z+\Delta z/2}\right]}{\Delta z} + \frac{\left[(\tau_{xx})_{x+\Delta x/2} - (\tau_{xx})_{x-\Delta x/2}\right]}{\Delta x}$$

$$+ \frac{\left[(\tau_{yx})_{y+\Delta y/2} - (\tau_{yx})_{y-\Delta y/2}\right]}{\Delta y} + \frac{\left[(\tau_{zx})_{z+\Delta y/2} - (\tau_{zx})_{z-\Delta z/2}\right]}{\Delta z}$$

$$+ \frac{\left(p_{x-\Delta x/2} - p_{x+\Delta x/2}\right)}{\Delta x} + \rho g_x$$

Differential Equation of Motion

Taking the limit as Δx, Δy and $\Delta z \rightarrow 0$:

$$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial(\rho uu)}{\partial x} - \frac{\partial(\rho vu)}{\partial y} - \frac{\partial(\rho wu)}{\partial z}$$

$$+\frac{\partial(\tau_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z} - \frac{\partial p}{\partial x} + \rho g_x$$

Rearranging:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} =$$

$$+\frac{\partial(\tau_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z} - \frac{\partial p}{\partial x} + \rho g_x$$

Differential Momentum Balance

For the convective terms:

$$\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}+\frac{\partial(\rho wu)}{\partial z}$$

$$=\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+u\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]$$

For the accumulation term:

From continuity

$$\frac{\partial(\rho u)}{\partial t}=\rho\frac{\partial u}{\partial t}+u\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}.\left(\rho\vec{V}\right)=0 \quad \Rightarrow \frac{\partial \rho}{\partial t}+\rho\left(\vec{\nabla}.\vec{V}\right)+\left(\vec{V}.\vec{\nabla}\right)\rho=0$$

$$=\rho\frac{\partial u}{\partial t}-u\left[\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)\right]$$

Differential Equation of Motion

Substituting:

$$\rho\frac{\partial u}{\partial t}-u\cancel{\left[\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)\right]}$$

$$+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

$$+u\cancel{\left[\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}\right]}$$

$$=\left[\frac{\partial\left(\tau_{xx}\right)}{\partial x}+\frac{\partial\left(\tau_{yx}\right)}{\partial y}+\frac{\partial\left(\tau_{zx}\right)}{\partial z}\right]-\frac{\partial p}{\partial x}+\rho g_{x}$$

Differential Equation of Motion

Substituting:

$$\rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

$$=\left[\frac{\partial\left(\tau_{xx}\right)}{\partial x}+\frac{\partial\left(\tau_{yx}\right)}{\partial y}+\frac{\partial\left(\tau_{zx}\right)}{\partial z}\right]-\frac{\partial p}{\partial x}+\rho g_x$$

EQUATION OF MOTION FOR THE *x*-COMPONENT

Differential Equation of Motion

$$\rho\frac{\partial v}{\partial t}+\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)$$

$$=\left[\frac{\partial\left(\tau_{xy}\right)}{\partial x}+\frac{\partial\left(\tau_{yy}\right)}{\partial y}+\frac{\partial\left(\tau_{zy}\right)}{\partial z}\right]-\frac{\partial p}{\partial y}+\rho g_y$$

EQUATION OF MOTION FOR THE *y*-COMPONENT

Differential Equation of Motion

$$\rho \frac{\partial w}{\partial t} + \rho\left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$

$$= \left[\frac{\partial\left(\tau_{xz}\right)}{\partial x} + \frac{\partial\left(\tau_{yz}\right)}{\partial y} + \frac{\partial\left(\tau_{zz}\right)}{\partial z} \right] - \frac{\partial p}{\partial z} + \rho g_z$$

EQUATION OF MOTION FOR THE *z*-COMPONENT

Differential Equation of Motion

Substantial time derivatives:

$$\rho \frac{Du}{Dt} = \left[\frac{\partial(\tau_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z} \right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho \frac{Dv}{Dt} = \left[\frac{\partial(\tau_{xy})}{\partial x} + \frac{\partial(\tau_{yy})}{\partial y} + \frac{\partial(\tau_{zy})}{\partial z} \right] - \frac{\partial p}{\partial y} + \rho g_y$$

$$\rho \frac{Dw}{Dt} = \left[\frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} + \frac{\partial(\tau_{zz})}{\partial z} \right] - \frac{\partial p}{\partial z} + \rho g_z$$

Differential Equation of Motion

In vector-matrix notation:

$$\rho \frac{D}{Dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(\tau_{xx}\right)}{\partial x} & \frac{\partial\left(\tau_{yx}\right)}{\partial y} & \frac{\partial\left(\tau_{zx}\right)}{\partial z} \\ \frac{\partial\left(\tau_{xy}\right)}{\partial x} & \frac{\partial\left(\tau_{yy}\right)}{\partial y} & \frac{\partial\left(\tau_{zy}\right)}{\partial z} \\ \frac{\partial\left(\tau_{xz}\right)}{\partial x} & \frac{\partial\left(\tau_{yz}\right)}{\partial y} & \frac{\partial\left(\tau_{zz}\right)}{\partial z} \end{bmatrix} - \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} + \rho \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \left(\nabla \cdot \vec{\tau}\right) - \nabla p + \rho \vec{g}}$$

Differential Equation of Motion

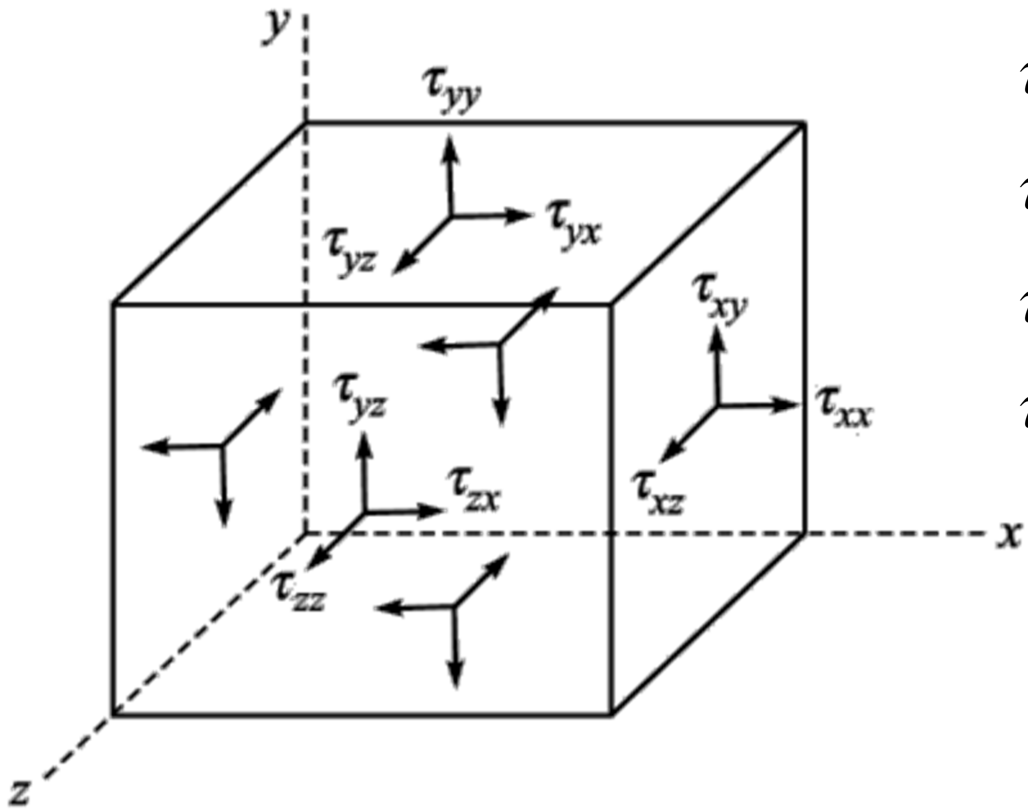
$$\rho \frac{D\vec{V}}{Dt} = \left(\nabla \cdot \vec{\tau}\right) - \nabla p + \rho \vec{g}$$

Cauchy momentum equation

- Equation of motion for a pure fluid
- Valid for any continuous medium (Eulerian)
- In order to determine velocity distributions, shear stress must be expressed in terms of velocity gradients and fluid properties (e.g. Newton’s law)

Cauchy Stress Tensor

Stress distribution:



$$\left.\begin{array}{l}\tau_{xx}\\ \tau_{yy}\\ \tau_{zz}\end{array}\right\}\text{ normal stresses}$$

$$\left.\begin{array}{l}\tau_{xy}=\tau_{yx}\\ \tau_{xz}=\tau_{zx}\\ \tau_{yz}=\tau_{zy}\end{array}\right\}\text{ shear stresses}$$

Cauchy Stress Tensor

Stokes relations (based on Stokes' hypothesis)

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{V}\right) \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\nabla \cdot \vec{V}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\nabla \cdot \vec{V}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

where $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ and $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$

Navier-Stokes Equations

Assumptions

1. **Newtonian fluid**
2. **Obeys Stokes' hypothesis**
3. **Continuum**
4. **Isotropic viscosity**
5. **Constant density**

Divergence of the stream velocity is zero (incompressible)

Navier-Stokes Equations

Applying the Stokes relations per component:

$$\frac{\partial\left(\tau_{xx}\right)}{\partial x}+\frac{\partial\left(\tau_{yx}\right)}{\partial y}+\frac{\partial\left(\tau_{zx}\right)}{\partial z}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial\left(\tau_{xy}\right)}{\partial x}+\frac{\partial\left(\tau_{yy}\right)}{\partial y}+\frac{\partial\left(\tau_{zy}\right)}{\partial z}=\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial\left(\tau_{xz}\right)}{\partial x}+\frac{\partial\left(\tau_{yz}\right)}{\partial y}+\frac{\partial\left(\tau_{zz}\right)}{\partial z}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Navier-Stokes Equations

Navier-Stokes equations in rectangular coordinates

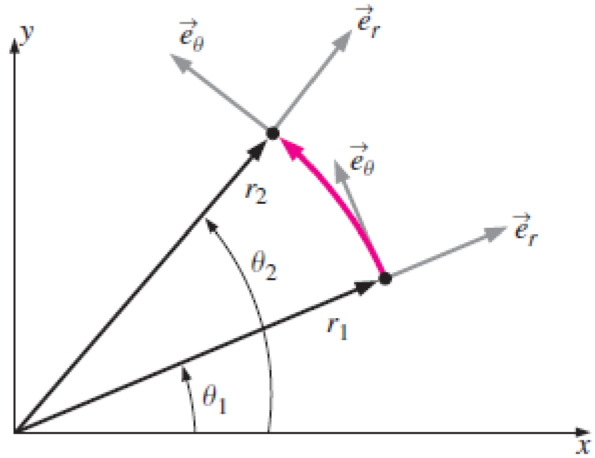
$$\rho \frac{Du}{Dt} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho \frac{Dv}{Dt} = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\partial p}{\partial y} + \rho g_y$$

$$\rho \frac{Dw}{Dt} = \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\partial p}{\partial z} + \rho g_z$$

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}}$$

Cylindrical Coordinates



$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 2\mu \frac{\partial u_r}{\partial r} & \mu\left[r \frac{\partial}{\partial_r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] & \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ \mu\left[r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] & 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) & 2\mu \frac{\partial u_z}{\partial z} \end{pmatrix}$$

Cylindrical Coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\left(\frac{\partial v_\theta}{\partial \theta}\right)+\frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\left(\frac{\partial v_\theta}{\partial \theta}\right)+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

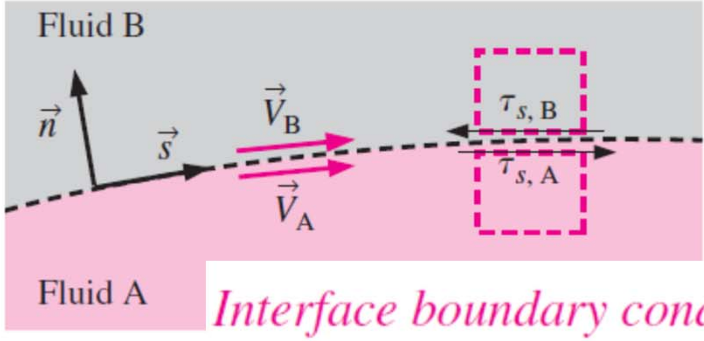
$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

Applications of Navier-Stokes Equations

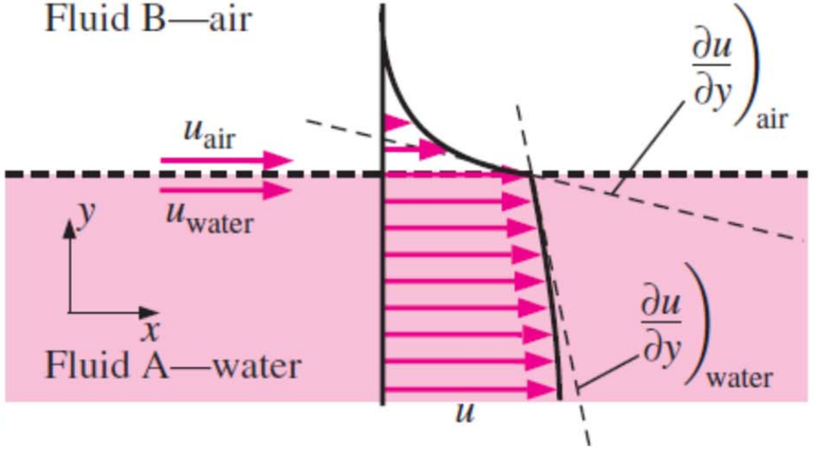
Exact Solutions of the Continuity and Navier–Stokes Equations

Step 1:	Set up the problem and geometry (sketches are helpful), identifying all relevant dimensions and parameters.
Step 2:	List all appropriate assumptions, approximations, simplifications, and boundary conditions.
Step 3:	Simplify the differential equations of motion (continuity and Navier–Stokes) as much as possible.
Step 4:	Integrate the equations, leading to one or more constants of integration.
Step 5:	Apply boundary conditions to solve for the constants of integration.
Step 6:	Verify your results.



Interface boundary conditions: $\vec{V}_A = \vec{V}_B$ and $\tau_{s,A} = \tau_{s,B}$

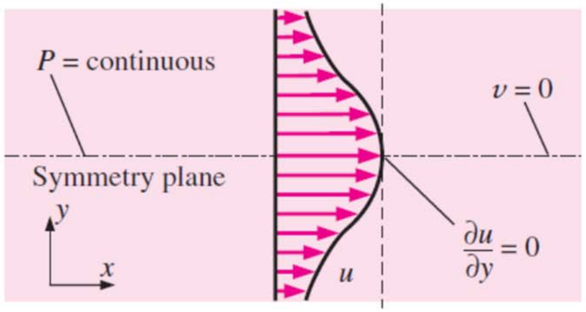
$$\mu_w = 50\mu_a$$



Boundary conditions at water–air interface:

$$u_{water} = u_{air} \quad \text{and} \quad \tau_{s,\,water} = \mu_{water}\left.\frac{\partial u}{\partial y}\right)_{water} = \tau_{s,\,air} = \mu_{air}\left.\frac{\partial u}{\partial y}\right)_{air}$$

Free-surface boundary conditions: $P_{liquid} = P_{gas}$ and $\tau_{s,\,liquid} \cong 0$



Symmetry boundary conditions

$$\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad v = 0$$

Euler Equation to Bernoulli Equation

The momentum equation for frictionless flow (Eq. 6.1) can be written (with $\vec{g}$ in the negative z direction) as

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{k} \qquad (1)$$

Equation (1) is a vector equation. It can be converted to a scalar equation by taking the dot product with $d\vec{s}$, where $d\vec{s}$ is an element of distance along a streamline. Thus

$$\frac{D\vec{V}}{Dt}\cdot d\vec{s} = \frac{DV}{Dt}ds = V\frac{\partial V}{\partial s}ds + \frac{\partial V}{\partial t}ds = -\frac{1}{\rho}\nabla p \cdot d\vec{s} - g\hat{k}\cdot d\vec{s} \qquad (2)$$

Examining the terms in Eq. (2) we note that

$$\frac{\partial V}{\partial s}ds = dV \qquad \text{(the change in } V \text{ along } s\text{)}$$

$$\nabla p \cdot d\vec{s} = dp \qquad \text{(the change in pressure along } s\text{)}$$

$$\hat{k}.d\vec{s} = dz \qquad \text{(the change in } z \text{ along } s\text{)}$$

Substituting into Eq. (2), we obtain

$$V\,dV + \frac{\partial V}{\partial t}\,ds = -\frac{dp}{\rho} - g\,dz \tag{3}$$

Integrating along a streamline from point 1 to point 2 yields

$$\int_1^2 \frac{dp}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) + \int_1^2 \frac{\partial V}{\partial t}\,ds = 0 \tag{4}$$

For incompressible flow, the density is constant. For this special case, Eq. (4) becomes

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t}\,ds \tag{5}$$