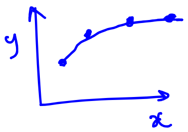
Function Approximation [Curve Fitting]

Interpolation by using polynomials



- Standard form by direct fit
  [Vandermonde matrix]
- Lagrange polynomials
- Newton's divided difference polynomials

All the above methods are applicable if the data points have arbitrary spacing

- For equal spacing - the Newton's divided difference form can be simplified

Errors in Interpolation

- Characteristics of errors
- Methods to reduce errors

INTERPOLATION ERRORS

- Let's compare Taylor series approximation with Newton's Divided Difference polynomial

Consider a continuous function $f(x)$ which is infinitely differentiable

Given x_0 $f(x_0)$

Find x $f(x)$

Taylor series

$$f(x) = f(x_0) + f'(x_0)(x-x_0)$$
$$+ \frac{f''(x_0)}{2!}(x-x_0)^2$$
$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$
$$+ \frac{f^n(x_0)}{n!}(x-x_0)^n + R_n$$

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)^{n+1} \qquad \xi \in (\underbrace{x_0, x}_{})$$

Newton's Divided Difference

$$P_n(x) = f(x_0)$$
$$+ f[x_1, x_0](x-x_0)$$
$$+ f[x_2, x_1, x_0](x-x_0)(x-x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x-x_0)(x-x_1)(x-x_2)$$
$$\vdots$$
$$+ f[x_n, x_{n-1}, \cdots x_0](x-x_0)(x-x_1)\cdots(x-x_{n-1})$$
$$+ \underline{R_n}$$

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

$\xi \rightarrow$ any value in the interval containing data points $(\underbrace{x_i \; i=0,1,\cdots n}_{})$ and $\underline{x}$.

Residual term in Newton's Divided Difference

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)$$

$$R_n \approx f[x_{n+1}, x_n, \ldots x_0](x-x_0)(x-x_1)\ldots(x-x_n)$$

n^{th} divided difference

$$R_n(x) = P_{n+1}(x) - P_n(x)$$

The magnitude of approximate error (R_n) is always less than the true error

Example

$x^* = 0.22$

→ after ordering

i	x_i	$y_i = f(x_i)$	First	Second	Third	Fourth
0	0	0.5	0.65	2.5	6.0	?
1	0.1	0.501	0.40	1.3		
2	0.2	0.516	0.01			
3	0.3	0.581				

First order

$$P_1(x) = f(x_0) + f[x_1, x_0](x-x_0)$$

$$= 0.529$$

$e_t(\%) = 1.07\%$

$$R_1(x) = f[x_2, x_1, x_0]\underbrace{(x-x_0)(x-x_1)}$$

$$= 2.5\,(0.22-0.2)(0.22-0.3)$$

$$= 0.04$$

$e_a(\%) = 0.75\%$

$P_2 = P_1(x) + \_\_(\,)$
$+ R_n$

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n) = \frac{f^{n+1}(\xi)}{(n+1)!}W_n(x)$$

Properties of the errors

1. What will be the errors at the data points x_i, $i=0,1,\ldots,n$

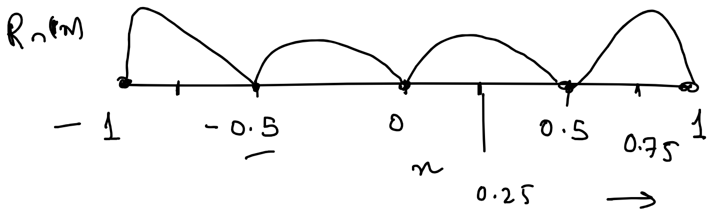
Zero

2. The errors are larger for the x's that are near the edges.

$$|W_n(x)| = |(x-x_0)(x-x_1)\cdots(x-x_n)|$$

$|W_n(x)|$ will be larger at the edges

Example.



$$|W_n(0.25)|$$
$$= |(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_4)|$$
$$= |(0.25+1)(0.25+0.5)(0.25-0)(0.25-0.5)(0.25-1)|$$
$$= |1.25 \times 0.75 \times 0.25 \times -0.25 \times -0.75|$$
$$= 0.0439$$

$$|W_n(0.75)|$$
$$= 1.75 \times 1.25 \times 0.75 \times 0.25 \times 0.25$$
$$= 0.1025$$

$$|W_n(0.75)| > |W_n(0.25)|$$

3. The errors are extremely large outside the data range

$$W_n(x) = \prod_{i=0}^{n} (x - x_i)$$

Methods to reduce interpolation errors when using polynomials

a. Selection of data points

b. Piecewise fitting of polynomials (splines)

A. Selection of data points

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!} \boxed{W_n(x)}$$

It is the property of function (data) we are trying to interpolate

Action

$$\text{minimize } \max \left| W_n(x) \right|$$

$$\text{minimize } \max \left| \prod_{i=1}^{n} (x - x_i) \right|$$

Select x_i's such that $W_n(x)$ is minimum

The solution of the optimization problem is given by Chebyshev points (nodes)

Chebyshev polynomial

- Orthogonal polynomial

$\rightarrow \quad x \in (-1, 1) \qquad W(x) = \frac{1}{\sqrt{1-x^2}}$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$\vdots$$
$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

The roots of the Chebyshev polynomial are Chebyshev points (nodes)

$$x_i = \cos\left(\frac{2i+1}{n} \frac{\pi}{2} \right)$$

$$i = 0, 1, \ldots, n-1$$

These x_i's minimize the maximum of $|W_n(x)|$

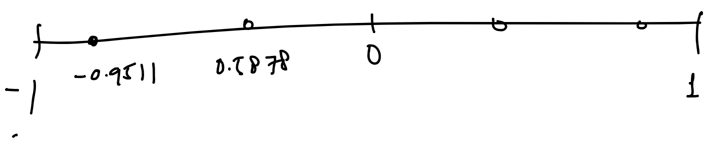
- Tchebycheff

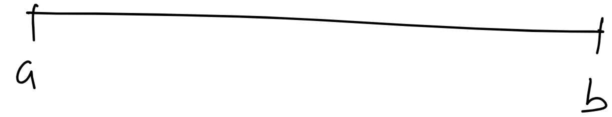
Example $x \in (-1, 1)$ $n = 5$

Chebyshev points

$$x_i = \cos\left(\frac{2i+1}{n}\frac{\pi}{2}\right)$$

i	x_i
0	$\cos\left(\frac{2\times 0+1}{5}\cdot\frac{\pi}{2}\right) = 0.9511$
1	0.5878
2	0
3	-0.5878
4	-0.9511





Linearly

Mapping the chebyshev points

$$x_i = \frac{a+b}{2} + \frac{a-b}{2}\cos\left(\frac{2i+1}{n}\frac{\pi}{2}\right)$$

$i = 0, 1, \ldots, n-1$