ASSIGNMENT II MSO 202 A

POWER SERIES, ANALYTIC FUNCTIONS, AND INTEGRATION

Exercise 0.1 : Does there exist a holomorphic function $f = u + iv$ on the complex plane such that $u(x, y) = x^2$ and $v(x, y) = y^2$?

Solution. If possible then $f = u+iv$ must satisfy the C-R equations, that is, $u_x = v_y$ and $u_y = -v_x$. This then implies that $2x = 2y$ or $x = y$. Hence $f = x^2 + iy^2$ does not satisfy the C-R equations everywhere. Hence the answer No.

Exercise 0.2 : Find the radius of convergence (for short, RoC) of the following power series:

(1) $\sum_{n=1}^{\infty} \frac{z^n}{n}$.
(2) $\sum_{n=1}^{\infty} z^{n!}$.
(3) $\sum_{n=1}^{\infty} n^{(-1)^n} z^n$.
(4) $\sum_{n=2}^{\infty} (\log n)^2 z^n$.
(5) $\sum_{n=2}^{\infty} a_n z^n$, where a_n is the number of prime numbers less than or equal to n.

Solution. Recall that RoC of $\sum_{n=0}^{\infty} a_n z^n$ is given by

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

(1) Here $a_n = 1/n$, and hence by Hadamard's formula, RoC is $R = 1$.
(2) Here $a_n = 1$ if $n = k!$ for some k, and $a_n = 0$ otherwise. Again by Hadamard's formula, RoC is $R = 1$.
(3) Note that $\frac{1}{n} \leqslant a_n \leqslant n$. Since $\sum_{n=0}^{\infty} \frac{z^n}{n}$ and $\sum_{n=0}^{\infty} nz^n$ have RoC equal to 1, RoC of $\sum_{n=0}^{\infty} n^{(-1)^n} z^n$ is also 1.
(4) Note that $1 \leqslant a_n \leqslant n$ for $n \geqslant e$, and hence one may argue as in (3).
(5) Note that $1 \leqslant a_n \leqslant n$ for $n \geqslant 2$, and hence one may argue as above.

Exercise 0.3 : Show that $f(z) = \frac{1}{1-z}$ defines an analytic function on the unit disc centered at 0, that is, for every $|a| < 1$, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ in some disc centered at a.

Solution. We must check that for every $|a| < 1$, $f(z)$ can be expanded as a absolutely convergent power series around a. For $|a| < 1$, note that

$$\frac{1}{1-z} = \frac{1}{1-a}\frac{1}{1-\frac{z-a}{1-a}} = \frac{1}{1-a}\sum_{n=0}^{\infty}\Big(\frac{1}{1-a}\Big)^n (z-a)^n,$$

which converges absolutely in the disc centered at a and of radius $|1-a|$.

Exercise 0.4 : Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial and let γ denote the unit circle with parametrization $z(t) = e^{it},\ 0 \leqslant t \leqslant 2\pi$. Show that

$$\int_\gamma (p(z) + p(1/z))dz = (2\pi i)a_1.$$

Solution. We have seen in the class that

$$\int_\gamma z^k = 0 \text{ for } k \neq -1, \text{ and } 2\pi i \text{ for } k = -1. \tag{0.1}$$

It follows that

$$\begin{aligned} \int_\gamma (p(z) + p(1/z))dz &= \int_\gamma \sum_{k=0}^{n} a_k z^k + \int_\gamma \sum_{k=0}^{n} a_k z^{-k} \\ &= \sum_{k=0}^{n} a_k \int_\gamma z^k + \sum_{k=0}^{n} a_k \int_\gamma z^{-k} \\ &= (2\pi i)a_1. \end{aligned}$$

Exercise 0.5 : Let γ be a circle of radius 2 centered at 0. Verify the following (*without* Cauchy Integral Formula):

(1) $\int_\gamma \frac{1}{z-1}dz = 2\pi i.$
(2) $\int_\gamma \frac{1}{z-3}dz = 0.$

Conclude that

$$\int_\gamma \frac{1}{(z-1)(z-3)}dz = -\pi i.$$

Solution. Let $z(t) = 2e^{it}$ $(0 \leqslant t \leqslant 2\pi)$ be a parametrization of γ.

(1) Note that $\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-z^{-1}} = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}}$ converges uniformly on $|z| = 2$. It follows from (0.1) that

$$\int_\gamma \frac{1}{z-1} dz = \sum_{k=0}^{\infty} \int_\gamma \frac{1}{z^{k+1}} dz = 2\pi i.$$

(2) Note that $\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = \sum_{k=0}^{\infty} \frac{z^k}{3^{k+1}}$ converges uniformly on $|z| = 2$. Once again, it follows from (0.1) that

$$\int_\gamma \frac{1}{z-3} dz = -\sum_{k=0}^{\infty} \int_\gamma \frac{z^k}{3^{k+1}} dz = 0.$$

The last part follows from $\frac{1}{(z-1)(z-3)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z-3}\right)$.

Exercise 0.6 : Let γ be the unit circle with following parametrizations:

$$\begin{aligned} z_1(t) &= e^{it} \ (0 \leq t \leq 2\pi), \\ z_2(t) &= e^{2it} \ (0 \leq t \leq 2\pi). \end{aligned}$$

Can you explain (with and without computations) why the integral of $\frac{1}{z}$ along the parametrizations z_1 and z_2 of the unit circle differ ?

Solution. Geometrically, the parametrization $z_1(t)$ travels once around the origin (in counter-clockwise direction) while $z_2(t)$ travels two times around 0. Hence the difference. Here is the mathematical justification. By (0.1), $\int_0^{2\pi} \frac{1}{z_1(t)} z_1'(t)dt = 2\pi i$. On the other hand,

$$\int_0^{2\pi} \frac{1}{z_2(t)} z_2'(t)dt = \int_0^{2\pi} e^{-2it} 2i dt = 4\pi i.$$

Exercise 0.7 : Let $\mathbb{D}$ be the unit disc centered at 0 and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Prove that if $\mathrm{Re}(f'(z)) > 0$ for all $z \in \mathbb{D}$ then f is injective.

Solution. Suppose that $f(b) = f(a)$ for some $a, b \in \mathbb{D}$. Let γ be the straight line joining a and b. Note that

$$\begin{aligned} 0 = f(b) - f(a) = \int_\gamma f'(z)dz &= \int_0^1 f'((1-t)a + tb)(((1-t)a + tb)'dt \\ &= (b-a)\int_0^1 f'((1-t)a + tb)dt. \end{aligned}$$

Since $\mathrm{Re}(f'(z)) > 0$ for all $z \in \mathbb{D}$, $\int_0^1 \mathrm{Re}(f'((1-t)a+tb))dt > 0$. In particular, $\int_0^1 f'((1-t)a+tb)dt \neq 0$. Hence $b = a$.