1. Recall that the Gauss Divergence Theorem says that for Ω - a closed bounded domain such that its boundary is a piecewise smooth orientable surface. Let $\vec{F}$ be a continuous vector field whose partial derivatives are continuous and differentiable. Then

$$\iiint_{\Omega} \operatorname{div} \vec{F} \, dV = \iint_{\partial\Omega} \vec{F} \cdot \vec{n} \, dA.$$

Using this prove the following :-

(i) Suppose $u \in C^1(\Omega)$. Then

$$\int_{\Omega} u_{x_i} \, dx = \int_{\partial\Omega} u \nu^i \, ds \quad (i = 1, 2, \dots, n)$$

(ii) If u and $v \in C^1(\Omega)$ then

$$\int_{\Omega} u_{x_i} v \, dx = -\int_{\Omega} u v_{x_i} \, dx + \int_{\partial\Omega} u v \nu_i \, ds.$$

(iii) $$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds.$$

(iv) $$\int_{\Omega} \nabla u \cdot \nabla v \, dx = -\int_{\Omega} u \Delta v \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u \, ds.$$

. . . . $\mathbb{R}^n$ as

(i) Assume,

$F(\hat{u}) = (0, 0, \ldots, u, 0, \ldots 0)$ where $\hat{u} = (u, u, \ldots, u, \ldots u)$.

$\uparrow$ ith component

$\therefore \operatorname{div} F = \frac{\partial}{\partial x_i}(u) = u_{x_i}$

Gauss Divergence.

$$\iiint_{\Omega} \operatorname{div} F \, dV = \iiint_{\Omega} u_{x_i} \, dx = \iint_{\partial\Omega} u \cdot \nu^i \, ds \quad \text{where } \hat{n} = (\nu_1, \nu_2, \ldots, \nu_i, \ldots, \nu_n)$$

(ii) Apply (i) to uv.

$$\Rightarrow \int_{\Omega} (u_{x_i} v + u v_{x_i}) \, dx = \int_{\partial\Omega} u v \nu_i \, ds.$$

$$\Rightarrow \int_{\Omega} u_{x_i} v \, dx = -\int_{\Omega} u v_{x_i} \, dx + \int_{\partial\Omega} u v \nu_i \, ds.$$

(iii) Use (ii) with u_{x_i} replaced by u and $v \equiv 1$ we see

$$\int_{\Omega} u_{x_i x_i} \, dx = \int_{\partial\Omega} u_{x_i} \nu^i \, ds.$$

Sum $i = 1, 2, \ldots, n$ and we have (iii).

(iv) Use (ii) with v_{x_i} replaced by v.

(2) If ϕ exists and is harmonic everywhere inside the closed curve C bounding the region R then prove that

$$\oint_C \frac{\partial \phi}{\partial n} \, ds = 0.$$

Proof :-

$$\oint_C \frac{\partial \phi}{\partial n} \, ds \underset{\text{Defn}}{=} \oint_C \nabla\phi \cdot n \, ds \underset{\left[\substack{\text{Gauss} \\ \text{Divergence} \\ \text{Theorem} \\ \text{(Corollary (iii))}}\right]}{=} \int_R \Delta \phi \, dx = 0.$$

(ϕ is harmonic)

3. Comment on the uniqueness of the problem :-

$$\left.\begin{array}{l} -\Delta u = f \text{ in } \Omega \\ u\big|_{\partial\Omega} = g \end{array}\right\} \text{---} (1)$$

without using maximum principle.

Proof :- Let u & $\tilde{u}$ solves (1)

Then $\omega :\equiv u - \tilde{u}$ solves the problem

$$-\Delta\omega = 0 \text{ in } \Omega$$

$$\omega\big|_{\partial\Omega} = 0 .$$

Now, $0 = -\int_{\Omega} \omega\Delta\omega \, dx \underset{\text{(Gauss Divergence)}}{=} \int_{\Omega} |\nabla\omega|^2$ (Integration by parts and taking into account that $\omega \equiv 0$ on $\partial\Omega$)

$\Rightarrow \int_{\Omega} |\nabla\omega|^2 = 0 \Rightarrow \nabla\omega \equiv 0.$

$\Rightarrow \omega$ is constant in Ω

$\because \omega = 0$ on $\partial\Omega \Rightarrow \omega = 0$ on Ω

$\Rightarrow u = \tilde{u}$ on Ω.

4. (Stability of Solution) :-

Let u_1 satisfies $-\Delta u_1 = f$ and $u_1 = h_1$ on $\partial\Omega$ and u_2 satisfies $-\Delta u_2 = f$ and $u_2 = h_2$ on $\partial\Omega$ then prove that

$$\max_{x\in\Omega} |u_2(x) - u_1(x)| \leq \max_{x\in\partial\Omega} |h_2(x) - h_1(x)|.$$

Soln :- Let $v = u_2 - u_1$.

$\Rightarrow v$ satisfies the following :-

$$-\Delta v = 0 \quad \text{in } \Omega$$

$$v\big|_{\partial\Omega} = h_2 - h_1.$$

Define, $M = \max\limits_{x \in \partial\Omega} |h_2 - h_1|$.

$\therefore \max\limits_{x \in \Omega} v(x) \leq \max\limits_{x \in \partial\Omega} (h_2 - h_1)(x) \leq M$. (Using the Maximum Principle for Harmonic functions).

Illy, $\min\limits_{x \in \Omega} v(x) \geq \min\limits_{x \in \partial\Omega} (h_2 - h_1)(x) \geq -M$. (")

$\Rightarrow \max\limits_{x \in \Omega} |v(x)| \leq M$

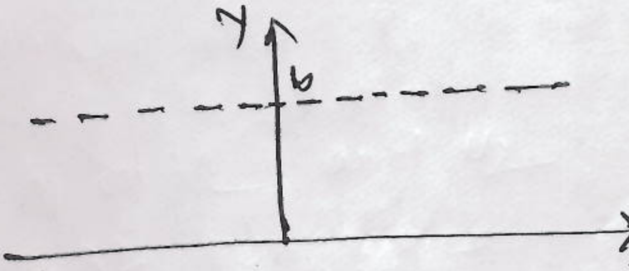
or, $\max\limits_{x \in \Omega} |u_2(x) - u_1(x)| \leq \max\limits_{x \in \partial\Omega} |h_2(x) - h_1(x)|$.

(4) Solve the Laplace Eqn :-

$$-\Delta u = 0 \quad \text{on } 0 < y < b \;\&\; x > 0.$$

$\&$ $u(x,0) = 0$; $u(x,b) + \alpha u_y(x,b) = 0$

$u(0,y) = f(y)$ where $\alpha > 0$.



Soln :- Using seperation of variable we have, $u(x,y) = X(x)\,Y(y)$

$Y'' + \lambda Y = 0$; $Y(0) = 0$ & $Y(b) + \alpha Y'(b) = 0$.

$X'' - \lambda X = 0$.

Solving the eigenvalue problem for y_{1}, it is easy to see that the problem doesn't admit a negative or zero eigenvalue.

$\therefore \lambda = \mu^2$ & $\mu > 0$ are the only eigenvalues.

$\because Y(0) = 0 \Rightarrow Y(y) = \sin \mu y$.

and $Y(b) + \gamma Y'(b) = 0 \Rightarrow \tan \mu b = -\gamma \mu$.

which has positive soln $\mu_1 < \mu_2 < \cdots$

$\therefore X_n = a_n e^{-\mu_n x} + b_n e^{\mu_n x}$

to make a viable soln we want $b_n \to 0$ [To make X_n - bdd]

$$\therefore u(x,y) = \sum_{n=1}^{\infty} a_n e^{-\mu_n x} \sin \mu_n y$$

$\because u(0,y) = f(y)$ we have,

$$f(y) = \sum_{n=1}^{\infty} a_n \sin \mu_n y$$

$$\Rightarrow a_n = \frac{\int_0^b f(y) \sin \mu_n y \, dy}{\int_0^b \sin^2 \mu_n y \, dy}.$$

(5) Solve the problem :-

$$\left.\begin{aligned} &u_{tt} = u_{xx} \; ; x \geq 0 \; \& \; t \geq 0 \\ &u(0,t) = 0 \; ; t \geq 0 \\ &u(x,0) = f(x) \; \& \; u_t(x,t) = g(x) \; ; x \geq 0. \end{aligned}\right\} - (*)$$

The boundary condition at infinity is that $u(x,t)$ is bdd as $x \to \infty$.

Soln:- We know that from D' Alembert Solution that

$$u(x,t) = \frac{1}{2}\left[f(x+t) - f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds.$$

is the solution of

$$u_{tt} = u_{xx} \quad \text{for } x \geq 0,\ t \geq 0$$

$$u(0,t) = 0\ ;\ t \geq 0$$

$$u(x,0) = f(x) \ \& \ u_t(x,0) = g(x).\quad ;\ x \in \mathbb{R}.$$

Let $\hat{f}(x)$ & $\hat{g}(x)$ be the odd extension of $f(x)$ and $g(x)$ ie

$$\tilde{f}(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases} \quad \& \quad \hat{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0. \end{cases}$$

Hence, $\tilde{u}(x,t) = \frac{1}{2}\left\{\hat{f}(x-t) + \hat{f}(x+t) + \int_{x-t}^{x+t} \hat{g}(s)\,ds\right\}$ is the

soln of $u_{tt} = u_{xx}\ ;\ -\infty < x < \infty\ ;\ t \geq 0$

$$u(x,0) = \hat{f}(x)\ \& \ u_t(x,0) = \hat{g}(x)\ ;\ -\infty < x < \infty.$$

Clearly, $\tilde{u}$ satisfies since

$$u_x = \frac{1}{2}\left\{\hat{f}'(x-t) + \hat{f}'(x+t) + \hat{g}(x+t) - \hat{g}(x-t)\right\}$$

$$u_y = \frac{1}{2}\left\{\hat{f}''(x-t) + \hat{f}''(x+t) + \hat{g}(x+t) - \hat{g}'(x-t)\right\}$$

$$u_t = \frac{1}{2}\left(\hat{f}'(x-t)(-1) + \hat{f}'(x+t) + \hat{g}(x+t) - \hat{g}(x-t)(-1)\right)$$

$$u_{tt} = \frac{1}{2}\left(\hat{f}''(x-t)(-1)^2 + \hat{f}''(x+t) + \hat{g}'(x+t) - \hat{g}'(x-t)(-1)^2\right].$$

$\therefore u_{xx} = u_{tt}$

Now for $x \geq 0$;

$u(x,0) = \hat{f}(x) = f(x)$.

& $u_t(x,0) = \hat{g}(x) = g(x)$.

Hence it satisfies the initial data.

Now, $u(0,t) = \frac{1}{2}\{-\hat{f}(t) + \hat{f}(t) + 0\} = 0$. [$\because \hat{f}$ is odd]

Thus we have our solu $\tilde{u}(x,t)\big|_{\substack{x \geq 0 \\ t \geq 0}}$,

or, $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$; $x \geq ct$.