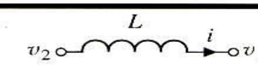
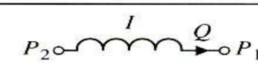
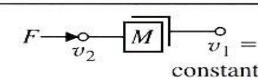
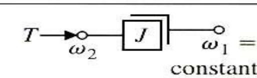
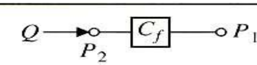
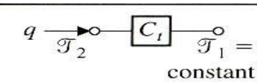
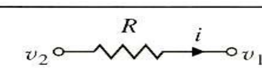
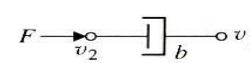
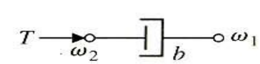
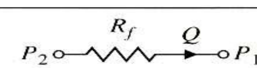
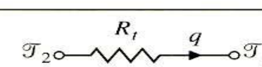
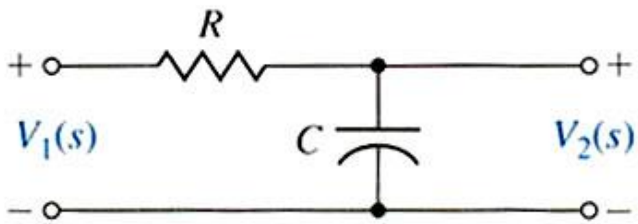
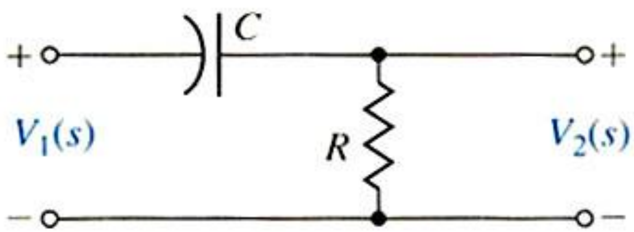
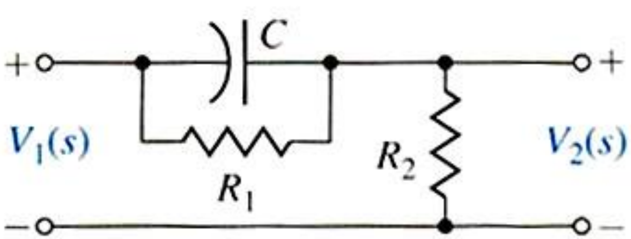
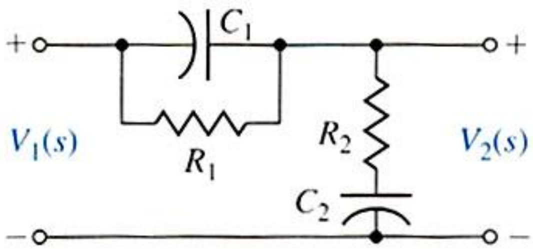
Electro-Mechanical Systems and their Representations

Summary of Describing Differential Equations for Ideal Elements

Type of Element	Physical Element	Describing Equation	Energy E or Power $\mathcal{P}$	Symbol
	Electrical inductance	$v_{21} = L \dfrac{di}{dt}$	$E = \dfrac{1}{2} Li^2$	
Inductive storage	Translational spring	$v_{21} = \dfrac{1}{k} \dfrac{dF}{dt}$	$E = \dfrac{1}{2} \dfrac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \dfrac{1}{k} \dfrac{dT}{dt}$	$E = \dfrac{1}{2} \dfrac{T^2}{k}$	
	Fluid inertia	$P_{21} = I \dfrac{dQ}{dt}$	$E = \dfrac{1}{2} IQ^2$	
	Electrical capacitance	$i = C \dfrac{dv_{21}}{dt}$	$E = \dfrac{1}{2} Cv_{21}^2$	
	Translational mass	$F = M \dfrac{dv_2}{dt}$	$E = \dfrac{1}{2} Mv_2^2$	
Capacitive storage	Rotational mass	$T = J \dfrac{d\omega_2}{dt}$	$E = \dfrac{1}{2} J\omega_2^2$	
	Fluid capacitance	$Q = C_f \dfrac{dP_{21}}{dt}$	$E = \dfrac{1}{2} C_f P_{21}^2$	
	Thermal capacitance	$q = C_t \dfrac{d\mathcal{T}_2}{dt}$	$E = C_t \mathcal{T}_2$	
	Electrical resistance	$i = \dfrac{1}{R} v_{21}$	$\mathcal{P} = \dfrac{1}{R} v_{21}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}^2$	
Energy dissipators	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}^2$	
	Fluid resistance	$Q = \dfrac{1}{R_f} P_{21}$	$\mathcal{P} = \dfrac{1}{R_f} P_{21}^2$	
	Thermal resistance	$q = \dfrac{1}{R_t} \mathcal{T}_{21}$	$\mathcal{P} = \dfrac{1}{R_t} \mathcal{T}_{21}$	

Element or System	$G(s)$
1. Integrating circuit, filter 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{1}{RCs + 1}$
2. Differentiating circuit 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{RCs}{RCs + 1}$
3. Differentiating circuit 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{s + 1/R_1 C}{s + (R_1 + R_2)/R_1 R_2 C}$

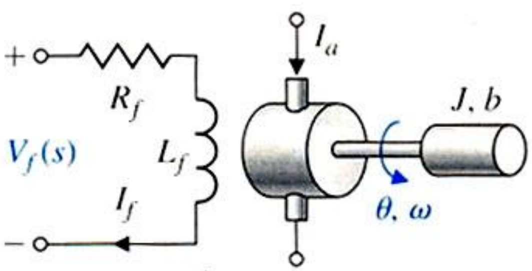
4. Lead-lag filter circuit



$$\frac{V_2(s)}{V_1(s)} = \frac{(1 + s\tau_a)(1 + s\tau_b)}{\tau_a\tau_b s^2 + (\tau_a + \tau_b + \tau_{ab})s + 1} = \frac{(1 + s\tau_a)(1 + s\tau_b)}{(1 + s\tau_1)(1 + s\tau_2)}$$

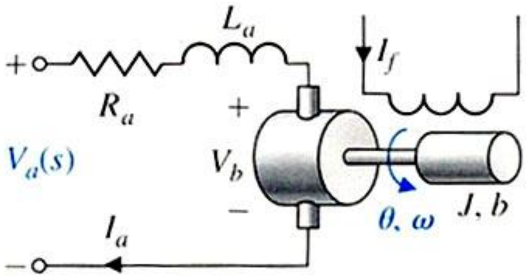
$$\tau_a = R_1 C_1$$
$$\tau_b = R_2 C_2$$
$$\tau_{ab} = R_1 C_2$$
$$\tau_1 \tau_2 = \tau_a \tau_b$$
$$\tau_1 + \tau_2 = \tau_a + \tau_b + \tau_{ab}$$

5. dc motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

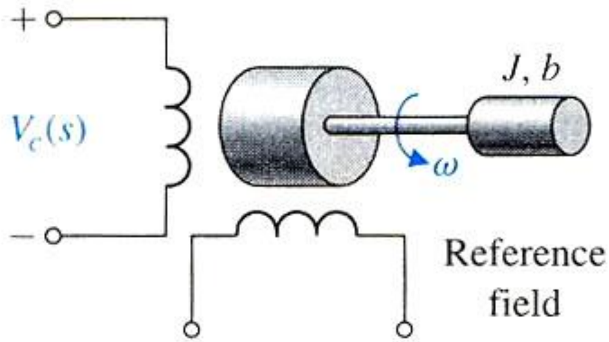
6. dc motor, armature-controlled, rotational actuator



$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$

Element or System	$G(s)$

7. ac motor, two-phase control field, rotational actuator

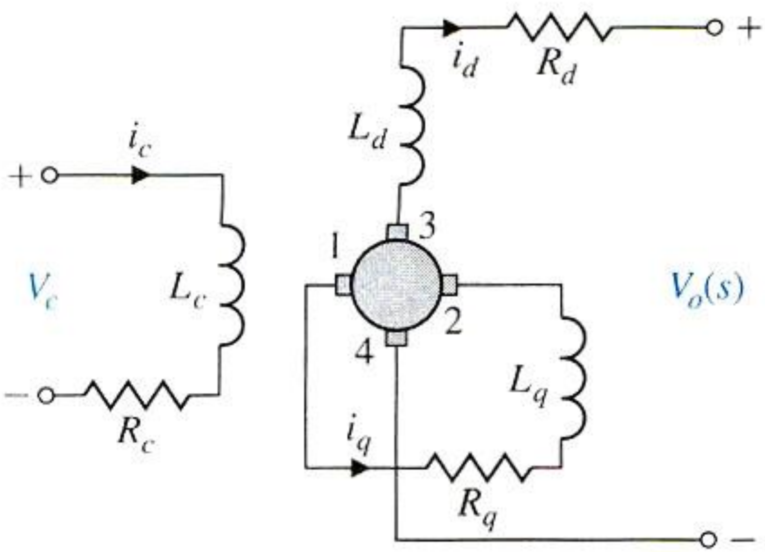


$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

8. Amplidyne, voltage and power amplifier



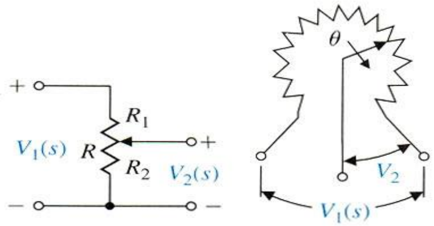
$$\frac{V_o(s)}{V_c(s)} = \frac{(K/R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$$

$$\tau_c = L_c/R_c, \quad \tau_q = L_q/R_q$$

For the unloaded case, $i_d \simeq 0$, $\tau_c \simeq \tau_q$,
$0.05\text{ s} < \tau_c < 0.5\text{ s}$
$V_{12} = V_q$, $V_{34} = V_d$

Element or System	***G*(*s*)**

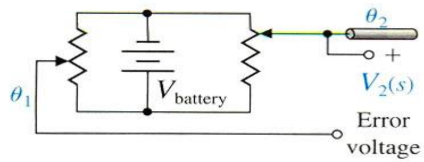
11. Potentiometer, voltage control



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

$$\frac{R_2}{R} = \frac{\theta}{\theta_{\text{max}}}$$

12. Potentiometer error detector bridge

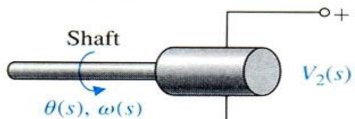


$$V_2(s) = k_s(\theta_1(s) - \theta_2(s))$$

$$V_2(s) = k_s\theta_{\text{error}}(s)$$

$$k_s = \frac{V_{\text{battery}}}{\theta_{\text{max}}}$$

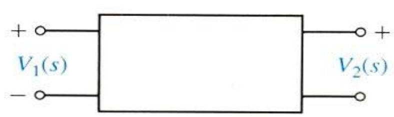
13. Tachometer, velocity sensor



$$V_2(s) = K_t\omega(s) = K_t s\theta(s);$$

$$K_t = \text{constant}$$

14. dc amplifier



$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s\tau + 1}$$

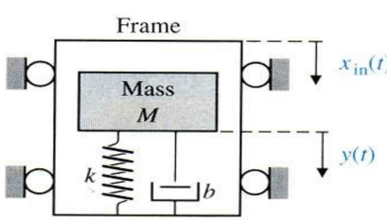
R_o = output resistance

C_o = output capacitance

$\tau = R_o C_o,\ \tau \ll 1$

and is often negligible for servomechanism amplifier

15. Accelerometer, acceleration sensor



$$x_o(t) = y(t) - x_{\text{in}}(t),$$

$$\frac{X_o(s)}{X_{\text{in}}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where $\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{\text{in}}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

14. dc amplifier

$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s\tau + 1}$$

R_o = output resistance

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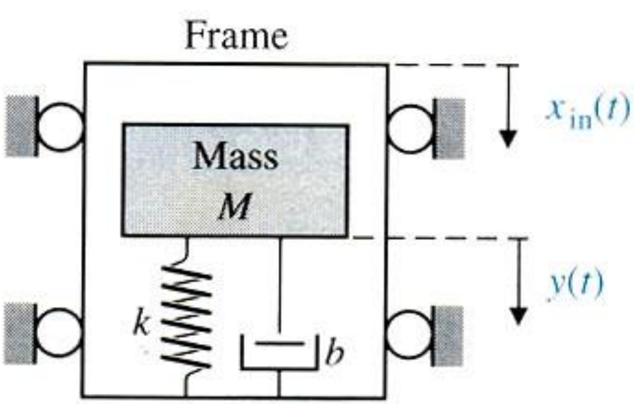

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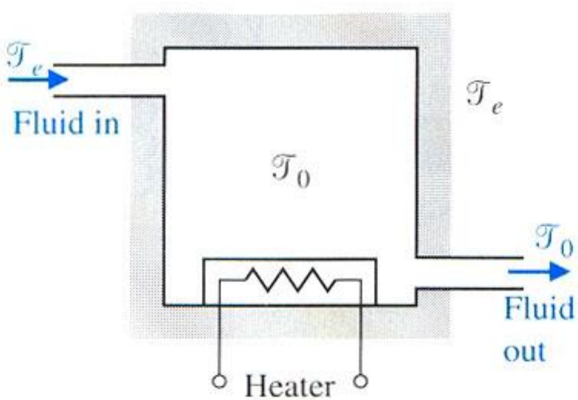
For low-frequency oscillations, where $\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Element or System	$G(s)$

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e =$ temperature difference due to thermal process

$C_t =$ thermal capacitance

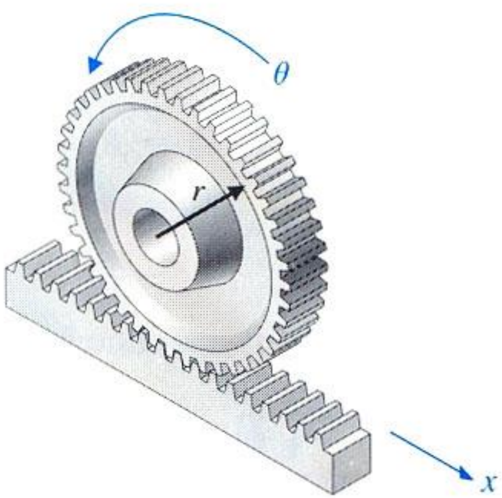
$Q =$ fluid flow rate $=$ constant

$S =$ specific heat of water

$R_t =$ thermal resistance of insulation

$q(s) =$ rate of heat flow of heating element

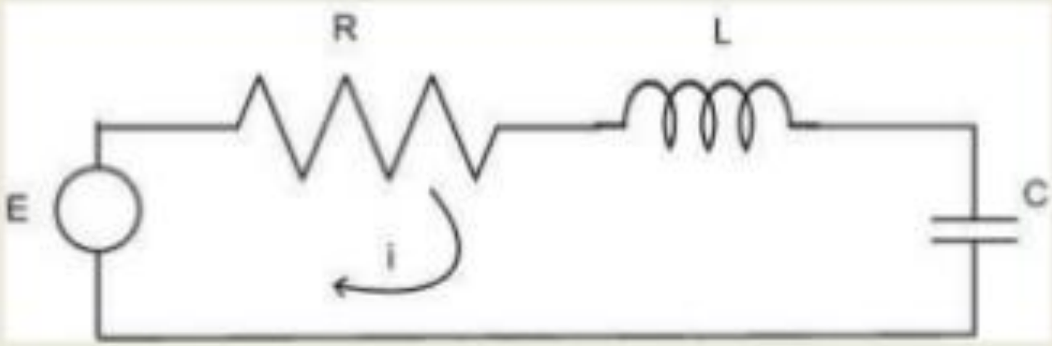
17. Rack and pinion



$x = r\theta$
converts radial motion
to linear motion

ANALOGOUS SYSTEM

Apply KVL in Series RLC Circuit



$$E = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \text{------} (1)$$

or

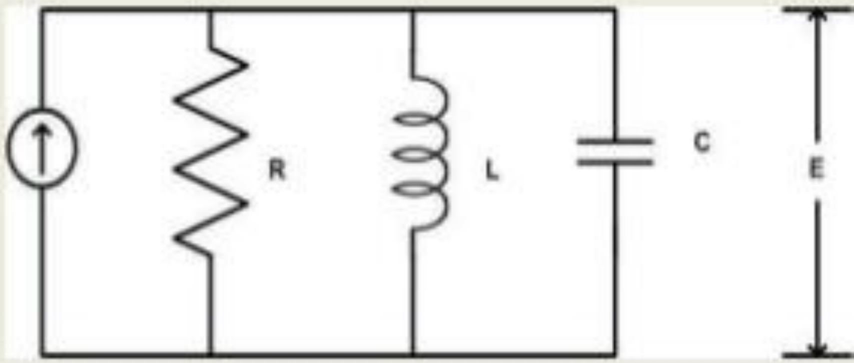
$$E = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q \text{-----} (2)$$

Consider parallel RLC circuit and apply KCL

$$I = \frac{E}{R} + \frac{1}{L}\int E dt + C\frac{dE}{dt} ------(3)$$

$$\phi = \int E dt, E = \frac{d\phi}{dt}$$

$$I = \frac{1}{R}(\frac{d\phi}{dt}) + \frac{1}{L}\phi + C\frac{d^2\phi}{dt^2} -----(4)$$



We know the equation of mechanical system

$$F(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) ------(5)$$

Compare equation(5) with equation(2)

FORCE –VOLTAGE ANALOGY (f-v)

S.NO.	TRANSLATIONAL SYSTEM	ELECTRICAL SYSTEM
1.	Force (F)	Voltage (E)
2.	Mass (M)	Inductance (L)
3.	Stiffness (K), Elastance (1/K)	Reciprocal of C, Capacitance (C)
4.	Damping coefficient (B)	Resistance (R)
5.	Displacement (x)	Charge (q)

Compare equation (5) with equation (4)

FORCE-CURRENT ANALOGY

S.NO.	TRANSLATIONAL SYSTEM	ELECTRICAL SYSTEM
1.	Force (F)	Current (I)
2.	Mass (M)	Capacitance (C)
3.	Damping coefficient (B)	Reciprocal of resistance (1/R) i.e conductance (G)
4.	Stiffness (K), Elastance (1/K)	Reciprocal of inductance (1/L)
5.	Displacement (x)	Flux linkage (ϕ)
6.	Velocity	Voltage (E)

For rotational system

$$T(t) = J\frac{d\omega(t)}{dt} + B\frac{d\theta(t)}{dt} + K\theta(t) ------(6)$$

Torque-voltage (T-V) analogy

Compare equation(2) with equation (6)

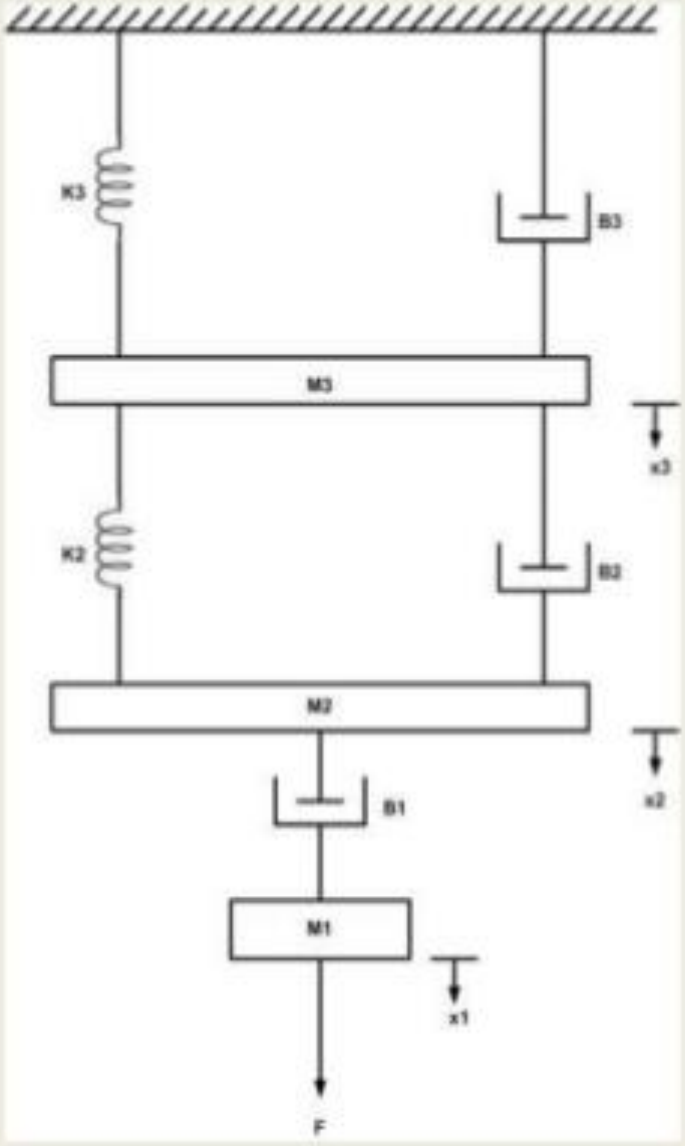
S.NO.	ROTATIONAL SYSTEM	ELECTRICAL SYSTEM
1.	Torque (T)	Voltage (E)
2.	Moment of inertia (J)	Inductance (L)
3.	Damping coefficient (B)	Resistance (R)
4.	Stiffness (K), Elastance (1/K)	Reciprocal of capacitance (1/C). Capacitance (C)
5.	Angular displacement (θ)	Charge (q)
6.	Angular velocity (ω)	Current (I)

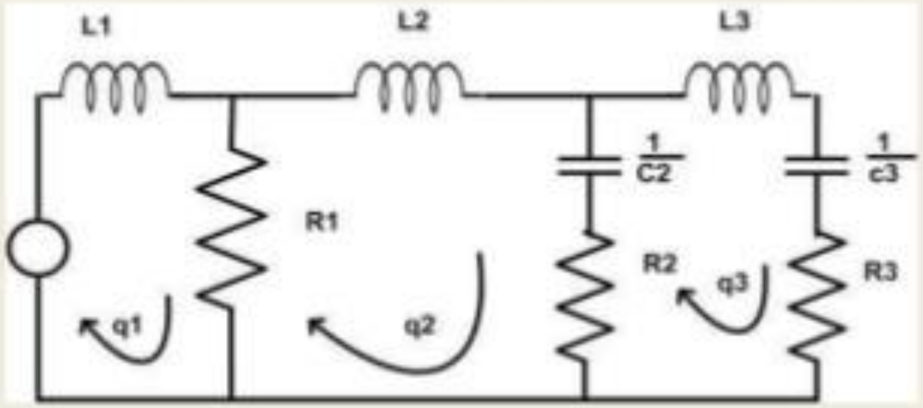
Compare equation (4) with equation (6)

TORQUE(T)-CURRENT (I) ANALOGY

S.NO.	ROTATIONAL SYSTEM	ELECTRICAL SYSTEM
1.	Torque (T)	Current (I)
2.	Moment of inertia (J)	Capacitance (C)
3.	Damping coefficient (B)	Reciprocal of resistance (R), conductance (G)
4.	Stiffness (K), Elastance (1/K)	Reciprocal of inductance (1/L)
5.	Angular displacement (ω)	Flux linkage (ϕ)

Draw the analogous electrical network of the given fig. using f-v analogy





Analogous Systems

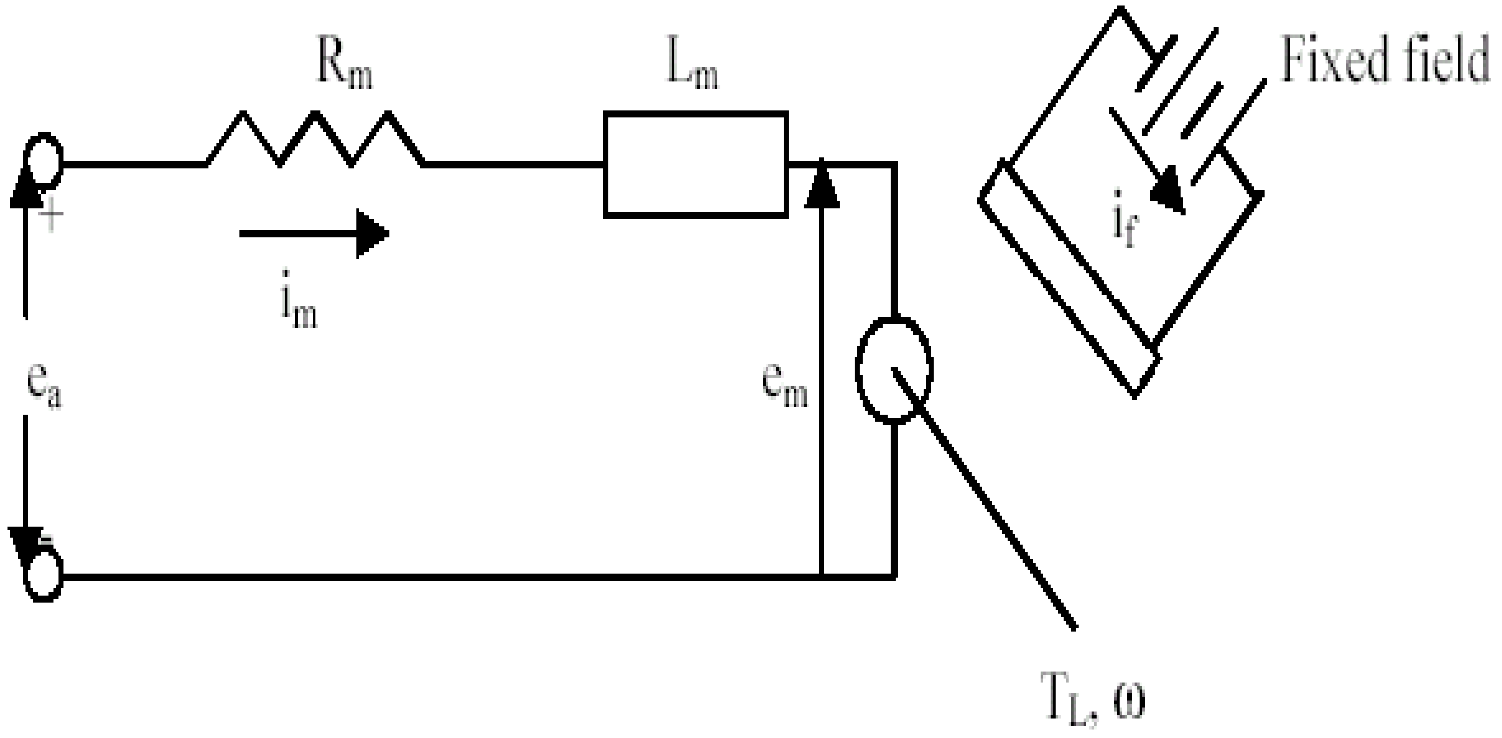
Topology-Preserving Set (book's analogy)

Intuitive Analogy Set

⇔ intuitive stretch ⇔ topology change

Description	Trans Mech	Rot Mech	Electrical	Thermal	Fluid	Trans Mech	Rot Mech	Description
"through" variable	f (force)	τ (torque)	i (current)	ϕ (heat flux)	q (flow)	v (velocity)	ω (angular velocity)	Motion
"across" variable	v (velocity)	ω (angular velocity)	v (voltage)	T, θ (temper-ature)	p (pressure)	f (force)	τ (torque)	Push (force)
Dissipative element	$v = \frac{1}{B} f$	$\omega = \frac{1}{B_r} \tau$	$v = iR$	$\theta = \phi R$	$p = qR$	$f = vB$	$\tau = \omega B_r$	Dissipative element
Dissipation	$f^2 \frac{1}{B} = \frac{v^2}{1/B}$	$\tau^2 \frac{1}{B} = \frac{\omega^2}{1/B}$	$i^2 R = v^2/R$	N/A	$q^2 R = p^2/R$	$v^2 B = f^2/B$	$\omega^2 B_r = \tau^2/B_r$	Dissipation
Through-variable storage element	$v = \frac{1}{K}\frac{df}{dt}$ or $\int v\,dt = \frac{1}{K} f$	$\omega = \frac{1}{K_r}\frac{d\tau}{dt}$ or $\int \omega\,dt = \frac{1}{K_r}\tau$	$v = L\frac{di}{dt}$	N/A	$p = I\frac{dq}{dt}$	$f = M\frac{dv}{dt}$ (one end must be "grounded")	$\tau = J\frac{d\omega}{dt}$ (one end must be "grounded")	Motion storage element
Energy	$E = \frac{1}{2}\frac{1}{K} f^2$	$E = \frac{1}{2}\frac{1}{K_r}\tau^2$	$E = \frac{1}{2} L i^2$		$E = \frac{1}{2} I q^2$	$E = \frac{1}{2} M v^2$	$E = \frac{1}{2} J \omega^2$	Energy
Impedance	Standard definition is at right		$V(s) = I(s) Ls$		$P(s) = Q(s) Is$	$F(s) = V(s) Ms$	$T(s) = \Omega(s) Js$	Impedance
Across-variable storage element	$f = M\frac{dv}{dt}$ (one end must be "grounded")	$\tau = J\frac{d\omega}{dt}$	$i = C\frac{dv}{dt}$	$\phi = C\frac{d\theta}{dt}$ (one end must be "grounded")	$q = C\frac{dp}{dt}$ (one end usually "grounded")	$v = \frac{1}{K}\frac{df}{dt}$ or $\int v\,dt = \frac{1}{K} f$	$\omega = \frac{1}{K_r}\frac{d\tau}{dt}$ or $\int \omega\,dt = \frac{1}{K_r}\tau$	Push (force) storage element
Energy	$E = \frac{1}{2} M v^2$	$E = \frac{1}{2} J \omega^2$	$E = \frac{1}{2} C v^2$	$E = CT$ (not analogous)	$E = \frac{1}{2} C p^2$	$E = \frac{1}{2}\frac{1}{K} f^2$	$E = \frac{1}{2}\frac{1}{K_r}\tau^2$	Energy
Impedance	The standard definition of mechanical impedance is the one on the right, based on the intuitive analogy.		$V(s) = I(s)\frac{1}{sC}$	$\Theta(s) = \Phi(s)\frac{1}{sC}$	$P(s) = Q(s)\frac{1}{sC}$	$F(s) = V(s)\frac{K}{s}$	$T(s) = \Omega(s)\frac{K_r}{s}$	Impedance

Modelling of an Armature Controlled DC Servomotor



- Torque developed by the motor T_m = K_t i_m

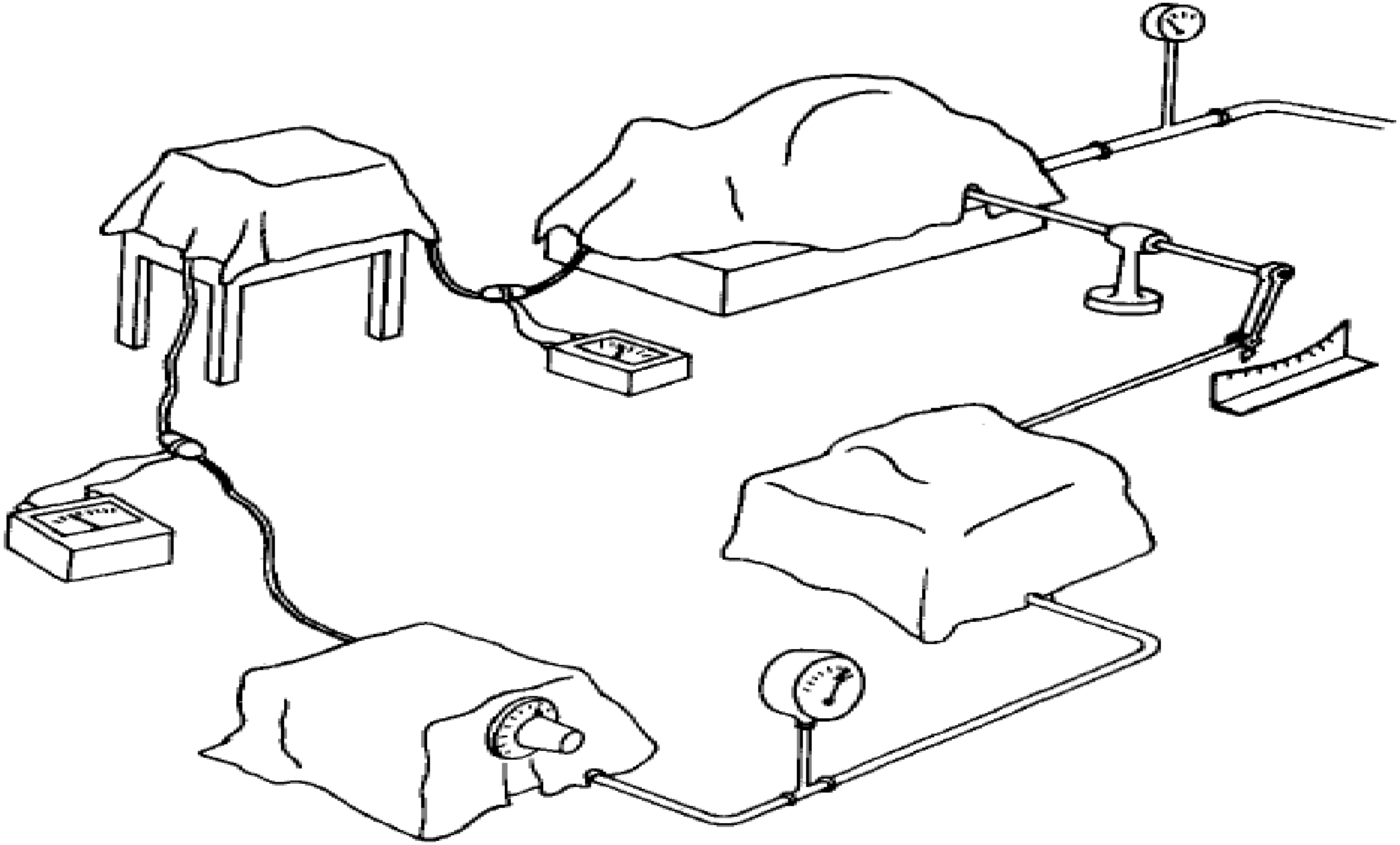
- Back emf $e_m = K_b\, d\Theta/dt$

- Kirchoff's Law: $e_a = L_m\, di_m/dt + R_m\, i_m + e_m$

- Force Balance: $T_m - T_l = J\, d^2\Theta/dt^2 + B\, d\Theta/dt$

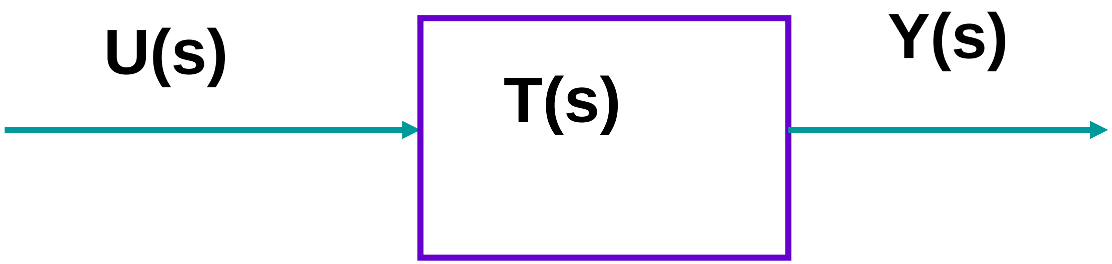
- *Take states as* $x_1 = \Theta$, $x_2 = d\Theta/dt$, $x_3 = i_m$, *Obtain EOM in state-space form*

Information Hiding!

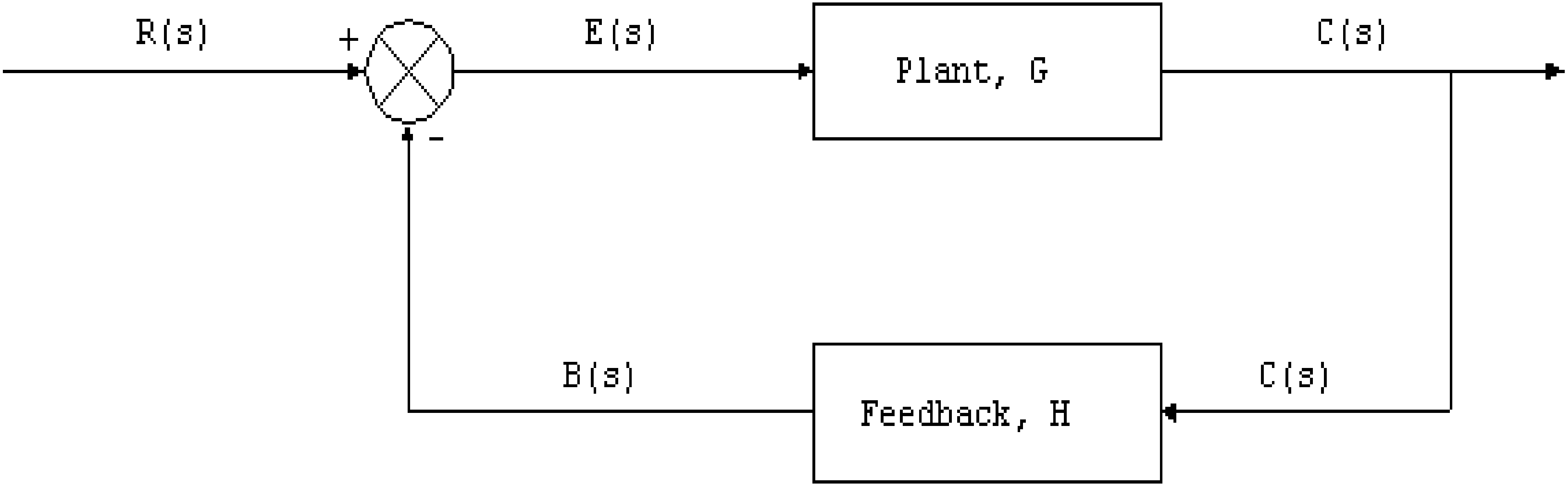


Yu Ho, Harvard

The Cause and Effect relationship between the input and outputs of a system could be represented graphically using Block Diagrams



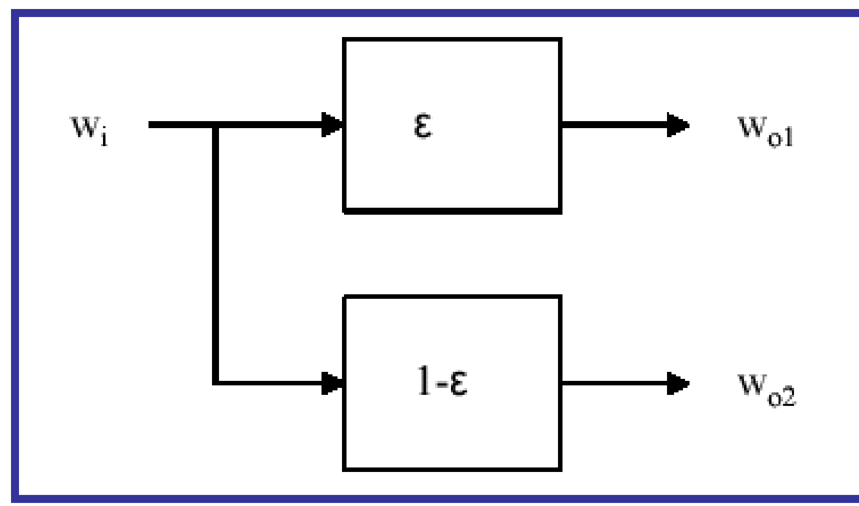
Block diagram of a Closed Loop System



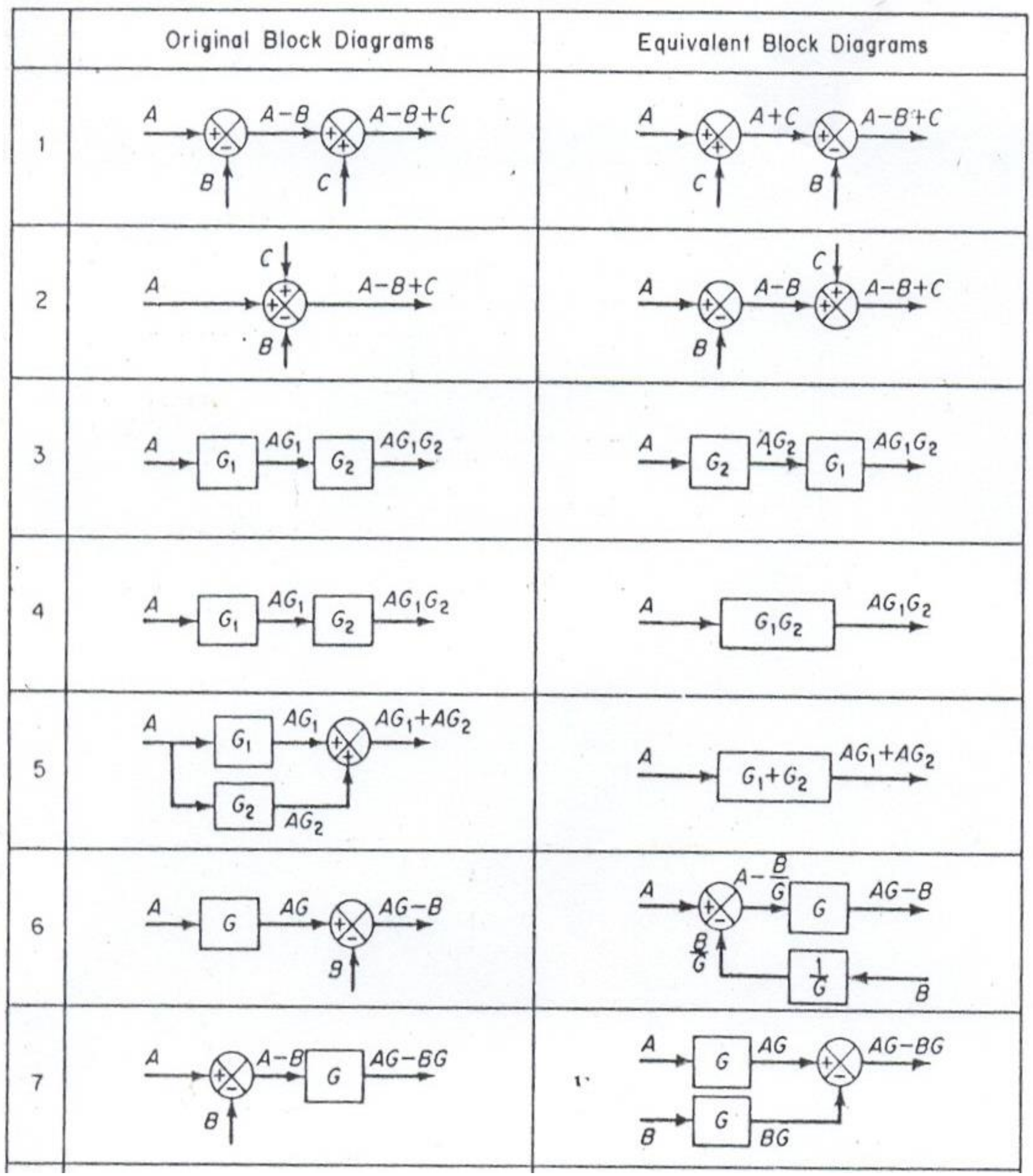
C/R = G/(1+GH) = (1/H) [1/(1+1/GH)]

Points to Remember

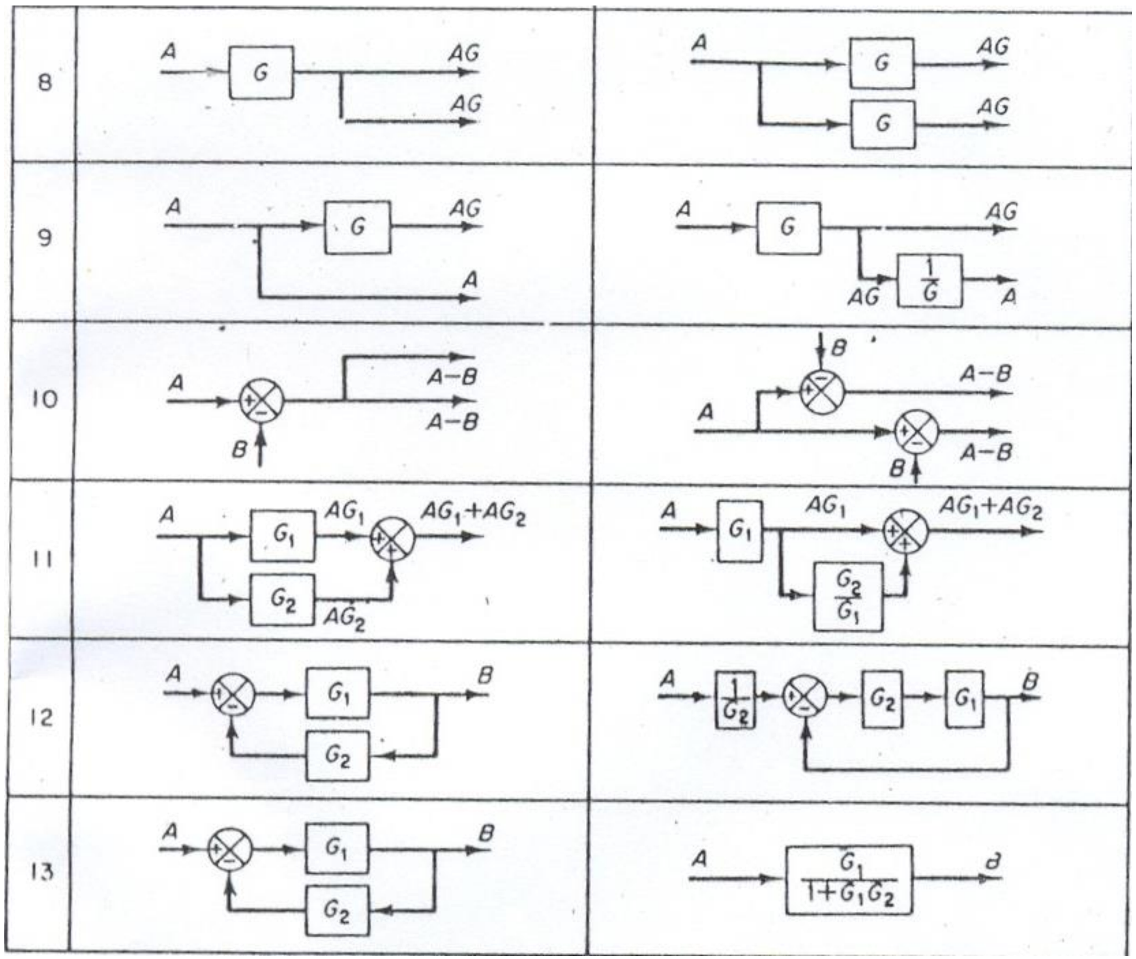
- Only one input and one output from a block
- Signals may be summed at explicit summing junction
- A single signal fed to multiple blocks does not imply the physical splitting of the signal



Block Diagram Algebra - I



Block Diagram Algebra - II



Advanced Block Diagrams: Simulink / MATLAB