ASSIGNMENT III MSO 202 A

CAUCHY'S THEOREM, CAUCHY INTEGRAL FORMULAS, AND LIOUVILLE'S THEOREM

Exercise 0.1 : The aim of this exercise is to derive the following formula using Cauchy's Theorem:

$$\int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Verify the following:

(1) For $R > 0$, consider the closed curve γ (boundary of the sector at 0 of angle $\pi/4$) with parametrization

$$\gamma_1(t) = t,\ 0 \leqslant t \leqslant R, \quad \gamma_2(t) = Re^{it},\ 0 \leqslant t \leqslant \frac{\pi}{4}, \quad \gamma_3(t) = -te^{i\frac{\pi}{4}},\ -R \leqslant t \leqslant 0.$$

Then the integral of e^{iz^2} over γ equals 0.

(2) $\int_{\gamma_1} e^{-z^2}dz$ converges to $\int_0^\infty \cos(t^2)dt + i\int_0^\infty \sin(t^2)dt$ as $R \to \infty$.

(3) $\int_{\gamma_2} e^{iz^2}dz \to 0$ as $R \to \infty$ (Hint. Use $\sin(2t) \geqslant \frac{4t}{\pi}$ $(0 \leqslant t \leqslant \frac{\pi}{4})$).

(4) $\int_{\gamma_3} e^{iz^2}dz \to e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2}$ as $R \to \infty$ (Hint. Use $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$).

Exercise 0.2 : The aim of this exercise is to derive the following formula using Cauchy's Theorem:

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

Verify the following:

(1) Consider the indented semicircle γ (with $0 < r < R$) given by

$$\gamma_1(t) = t\ (-R \leqslant t \leqslant -r),\ \gamma_2(t) = re^{-it}\ (-\pi \leqslant t \leqslant 0),$$

$$\gamma_3(t) = t\ (r \leqslant t \leqslant R),\ \gamma_4(t) = Re^{it}\ (0 \leqslant t \leqslant \pi).$$

Then the integral of $f(z) = \frac{e^{iz}-1}{z}$ over γ is 0.

(2) $\int_{\gamma_1} \frac{e^{iz}-1}{z}dz \to \int_{-\infty}^0 \frac{e^{it}-1}{t}dt$ as $R \to \infty$ and $r \to 0$.

(3) $\int_{\gamma_2} \frac{e^{iz}-1}{z}dz \to 0$ as $r \to 0$.

(4) $\int_{\gamma_3} \frac{e^{iz}-1}{z}dz \to \int_0^\infty \frac{e^{it}-1}{t}dt$ as $R \to \infty$ and $r \to 0$.

(5) $\int_{\gamma_4} \frac{e^{iz}-1}{z}dz \to -i\pi$ as $R \to \infty$.

Exercise 0.3 : For $a > 0$, let γ be the circle $|z - ia| = a$. Whether $\int_\gamma \frac{1}{z^2+a^2} dz$ depends on a ? Justify your answer.

Exercise 0.4 : Compute the Taylor series of $\log z$ in the disc $|z-i| = \frac{1}{2}$.

Exercise 0.5 : Let f be entire and k a positive integer. If

$$|f(z)| \leqslant C|z^k| \ (z \in \mathbb{C})$$

for some $C > 0$ then show that f is a polynomial of degree at most k.

Exercise 0.6 : Let f be an entire function such that $|f(a)| \leqslant |f(z)|$ $(z \in \mathbb{C})$ for some $a \in \mathbb{C}$. Show that either $f(a) = 0$ or f is constant.

Exercise 0.7 : What are all entire functions f which satisfy $f(x) = e^{x^2}$ for all $x = 1, \frac{1}{2}, \frac{1}{3}, \cdots,$. Justify your answer.

Exercise 0.8 : Let f and g be two entire functions. Show that if $f(z)g(z) = 0$ for all $z \in \mathbb{C}$ then either $f(z) = 0$ for all $z \in \mathbb{C}$ or $g(z) = 0$ for all $z \in \mathbb{C}$.