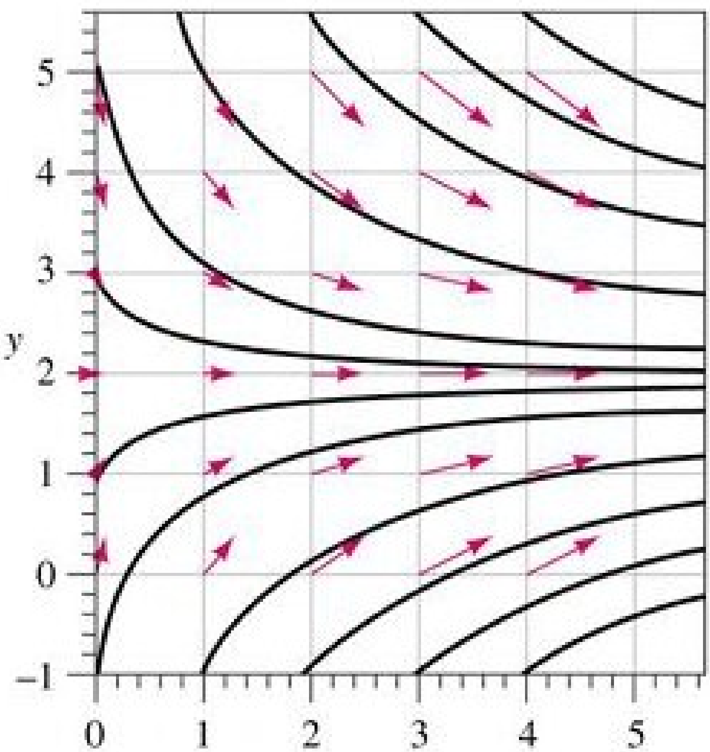
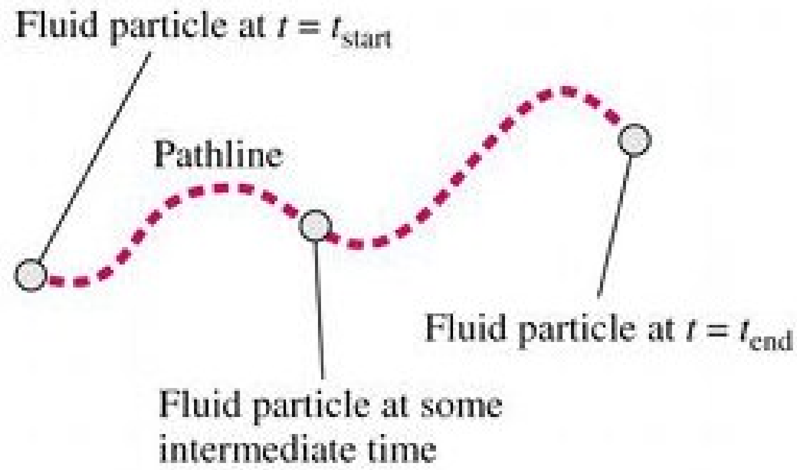
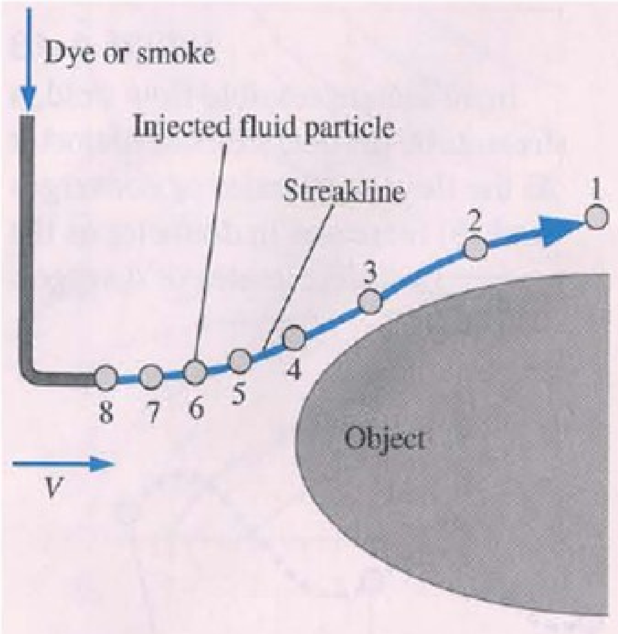
11 Flow Patterns: Streamlines, Streaklines, Pathlines

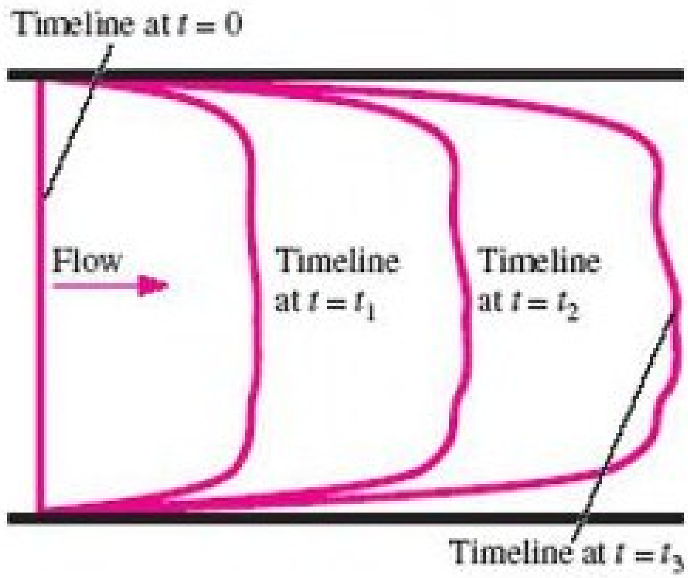




1.) A **Streamline** is a curve everywhere tangent to the local velocity vector at a given instant. **Instantaneous** lines; convinent to compute **mathematically**.

2.) A **Pathline** is the actual path traveled by an individual fluid particle over some time period. Generated as the **passage of time**; convinent to generate **experimentally**.





3.) A **Streakline** is the locus of particles that have earlier passed through a prescribed point. Generated as the **passage of time**; convinent to generate **experimentally**.

4.) A **Timeline** is a set of fluid particles that form a line at a given instant. **Instantaneous** lines; convinent to generate **experimentally**.

Note:

1. Streamlines, pathlines, and streaklines are identical in a steady flow.
2. For unsteady flow, streamline pattern changes with time, whereas pathlines and streaklines are generated as the passage of time.

11.1 Streamline

By definition we must have $\mathbf{V} \times d\mathbf{r} = 0$ which upon expansion yields the equation of the streamlines for a given time $t = t_1$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds \qquad s = \text{integration parameter}$$

So if (u, v, w) is known, integrate with respect to s for $t = t_1$ with initial condition (x_0, y_0, z_0, t_0) at $s = 0$ and then eliminate s.

11.2 Pathline

The pathline is defined by integration of the relationship between velocity and displacement.

$$\frac{dx}{dt} = u \quad \frac{dy}{dt} = v \quad \frac{dz}{dt} = w$$

Integrate u, v, w with respect to t using initial condition (x_0, y_0, z_0, t_0), then eliminate t.

11.3 Streakline

To find the streakline, use the integrated result for the pathline retaining time as a parameter. Now, find the integration constant which causes the pathline to pass through (x_0, y_0, z_0) for a sequence of times $\tau < t$. Then eliminate τ.

Example: an idealized velocity distribution is given by:

$$u = \frac{x}{1+t} \quad v = \frac{y}{1+2t} \quad w = 0$$

calculate and plot: 1) the streamlines 2) the pathlines 3) the streaklines which pass through (x_0, y_0, z_0) at $t = 0$.

1. First, note that since $w = 0$ there is no motion in the z direction and the flow is 2-D

$$\frac{dx}{ds} = u = \frac{x}{1+t} \qquad \frac{dy}{ds} = v = \frac{y}{1+2t}$$

$$x = C_1 \exp\left(\frac{s}{1+t}\right) \qquad y = C_2 \exp\left(\frac{s}{1+2t}\right)$$

$$s = 0 \ \text{ at } (x_0, y_0): \quad C_1 = x_0 \quad C_2 = y_0$$

and eliminating s:

$$s = (1+t)\ln\frac{x}{x_0} = (1+2t)\ln\frac{y}{y_0}$$

$$y = y_0\left(\frac{x}{x_0}\right)^n \text{ where } n = \frac{1+t}{1+2t}$$

This is the equation of the streamlines which pass through (x_0, y_0) for all times t.

$$t = 0, \quad \frac{y}{y_0} = \frac{x}{x_0}$$

$$t = \infty, \quad \frac{y}{y_0} = \left(\frac{x}{x_0}\right)^{1/2}$$

2. To find the pathlines we integrate

$$\frac{dx}{dt} = u = \frac{x}{1+t} \qquad \frac{dy}{dt} = v = \frac{y}{1+2t}$$

$$x = C_1(1+t) \qquad y = C_2(1+2t)^{1/2} \qquad \left(\int \frac{c}{ax+b}dx = \frac{c}{a}ln|ax+b| + C\right)$$

$$t = 0 \quad (x,y) = (x_0, y_0): \quad C_1 = x_0 \qquad C_2 = y_0$$

now eliminate t between the equations for (x,y)

$$y = y_0\left[1 + 2\left(\frac{x}{x_0} - 1\right)\right]^{1/2}$$

This is the pathline through (x_0, y_0) at $t = 0$ and does not coincide with the streamline at $t = 0$.

3. To find the streakline, we use the pathline equations to find the family of particles that have passed through the point (x_0, y_0) for all times $\tau < t$.

$$x = C_1(1+t) \qquad y = C_2(1+2t)^{1/2}$$

$$x_0 = C_1(1+\tau) \qquad y_0 = C_2(1+2\tau)^{1/2}$$

$$C_1 = \frac{x_0}{1+\tau} \qquad C_2 = \frac{y_0}{(1+2\tau)^{1/2}}$$

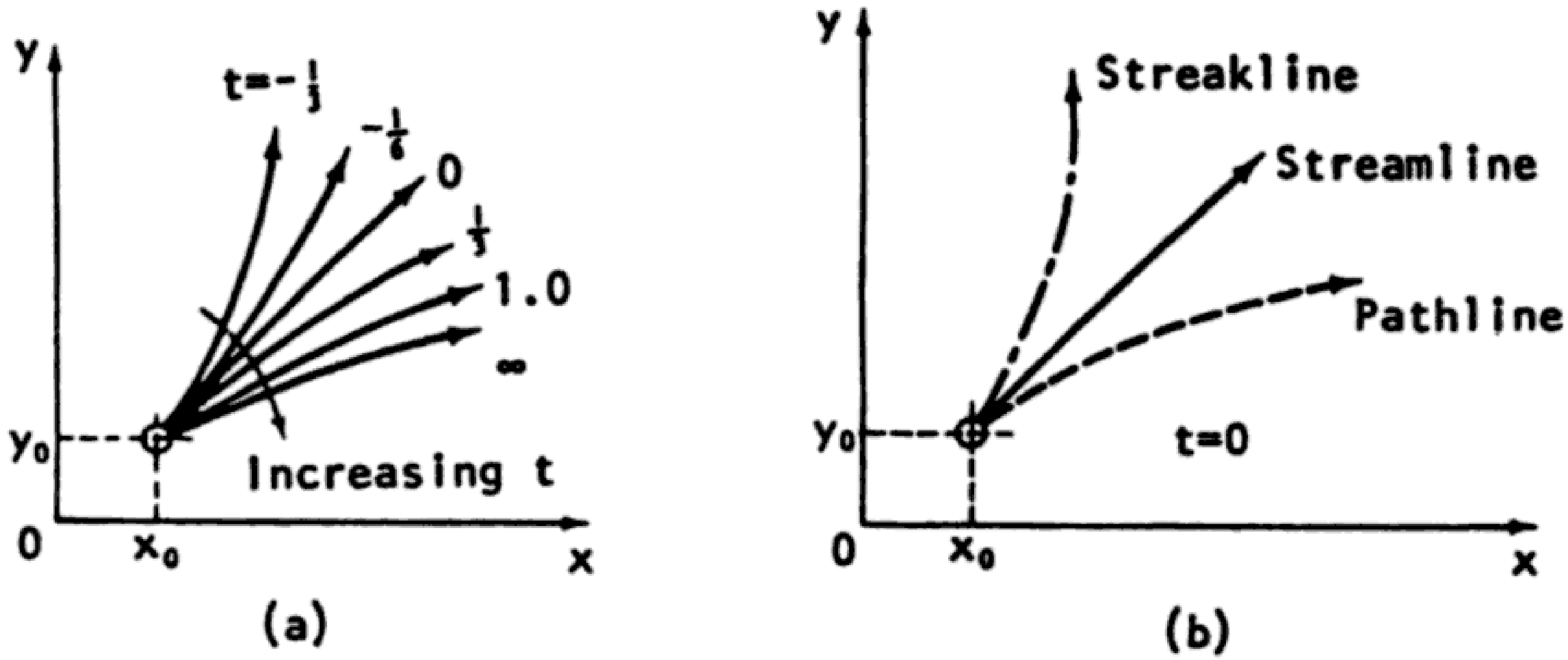
$$x = \frac{x_0}{1+\tau}(1+t) \qquad y = \frac{y_0}{(1+2\tau)^{1/2}}(1+2t)^{1/2}$$

$$\tau = (1+t)\frac{x_0}{x} - 1 = \frac{1}{2}\left[(1+2t)\left(\frac{y_0}{y}\right)^2 - 1\right]$$

$$\left(\frac{y_0}{y}\right)^2 = \frac{1+2t}{1+2\left[(1+t)\left(\frac{x_0}{x}\right)-1\right]}$$

$$t = 0: \quad \frac{y}{y_0} = \left[1 + 2\left(\frac{x_0}{x} - 1\right)\right]^{-1/2}$$

The streakline does not coincide with either the equivalent streamline or pathline.



(a) Streamlines through(x_0,y_0)as a function of time;(b) Streamline, Pathline and Streakline through (x_0,y_0) at time t=0.

Physically, the streakline reflects the streamline behavior before the specified time $t = 0$, while the pathline reflects the streamline behavior after $t = 0$.

11.4 Stream Function

The stream function is a powerful tool for 2D flows in which **V** is obtained by differentiation of a scalar Ψ which automatically satisfies the continuity equation.

Continuity equation: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

say: $u=\frac{\partial \Psi}{\partial y} \qquad v=-\frac{\partial \Psi}{\partial x}$

then: $\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \Psi}{\partial x}\right)=\frac{\partial^2 \Psi}{\partial x \partial y}-\frac{\partial^2 \Psi}{\partial x \partial y}=0$

$$\mathbf{V}=\frac{\partial \Psi}{\partial y}\mathbf{i}-\frac{\partial \Psi}{\partial x}\mathbf{j} \quad \Rightarrow \quad \nabla \times \mathbf{V}=\boldsymbol{\omega} \;\Rightarrow\; \omega_z=\omega=-\nabla^2 \Psi$$

2D vorticity transport equation: $\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\nu \nabla^2 \omega$

Replace u, v, ω:

$$\frac{\partial}{\partial t}(\nabla^2 \Psi)+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \Psi)-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \Psi)=\nu \nabla^4 \Psi \quad \left(\nabla^4 \Psi=\frac{\partial^4 \Psi}{\partial x^4}+2\frac{\partial^4 \Psi}{\partial x^2 \partial y^2}+\frac{\partial^4 \Psi}{\partial y^4}\right)$$

Steady flow: $\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \Psi)-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \Psi)=\nu \nabla^4 \Psi$

It is a single fourth-order scalar equation, which requires 4 boundary conditions

At infinity: $u=\partial \Psi/\partial y=U_\infty \qquad v=-\partial \Psi/\partial x=0$

On body: $\quad u=v=0=\partial \Psi/\partial y=-\partial \Psi/\partial x$

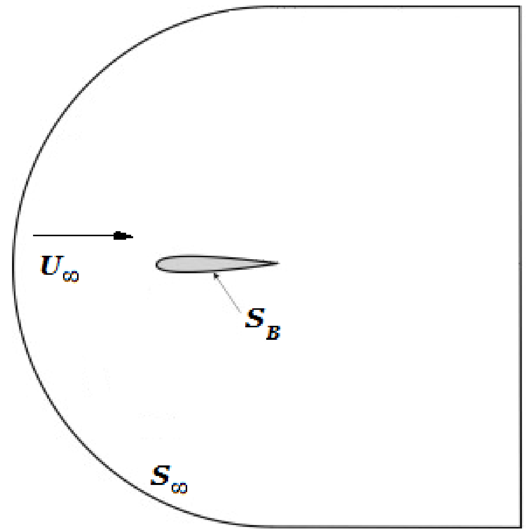
11.4.1 Irrotational Flow

$$\nabla \times \mathbf{V} = 0 \quad \Rightarrow \quad \nabla^2 \Psi = 0$$

Second-order linear Laplace equation

At infinity S_∞: $\Psi = U_\infty y + \text{const.}$

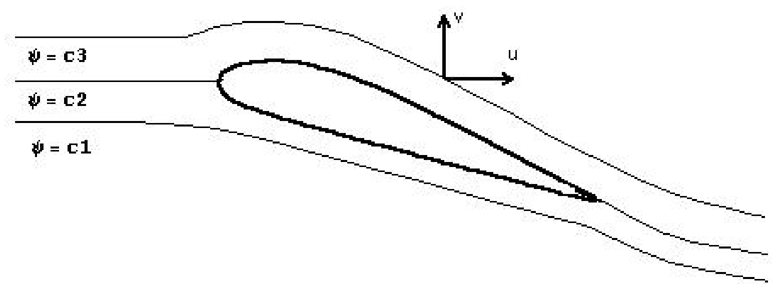
On body S_B: $\Psi = \text{const.}$



11.4.2 Geometric Interpretation of Ψ

Besides its importance mathematically Ψ also has important geometric significance.

$\Psi = \text{const.} = \text{streamline}$



Recall definition of a streamline:

$$\mathbf{V} \times d\mathbf{r} = 0$$

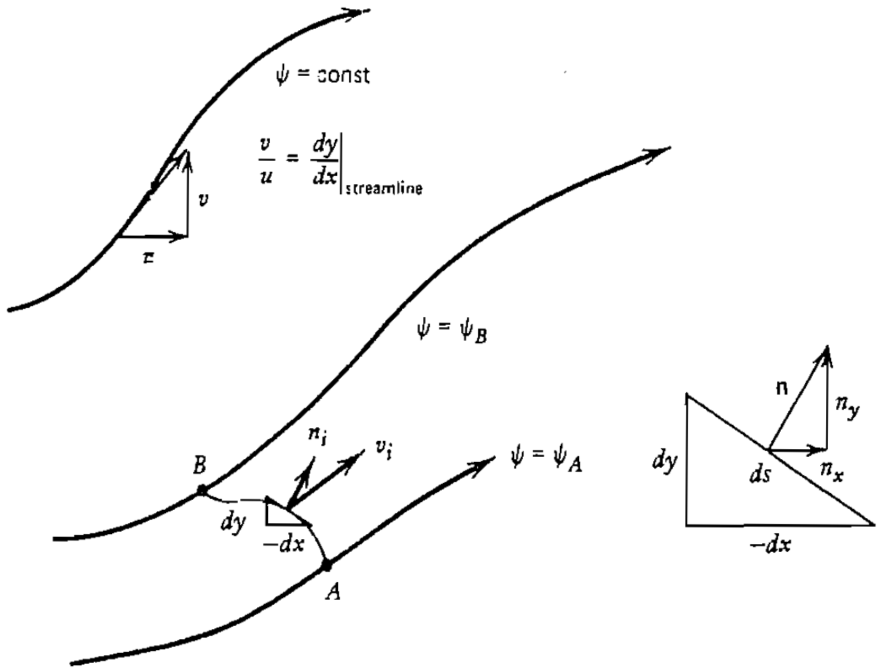
$$\frac{dx}{u} = \frac{dy}{v} \quad \Rightarrow \quad udy - vdx = 0$$

Compare with $d\Psi = \dfrac{\partial \Psi}{\partial x} dx + \dfrac{\partial \Psi}{\partial y} dy = -vdx + udy$

i.e., $d\Psi = 0$ along a streamline

Or $\Psi = \text{const.}$ along a streamline and curves of constant Ψ are the flow streamlines. If we know $\Psi(x, y)$ then we can plot $\Psi = \text{const.}$ curves to show streamlines.

11.4.3 Physical Interpretation



$$n_x = \frac{dy}{ds} \quad n_y = -\frac{dx}{ds}$$

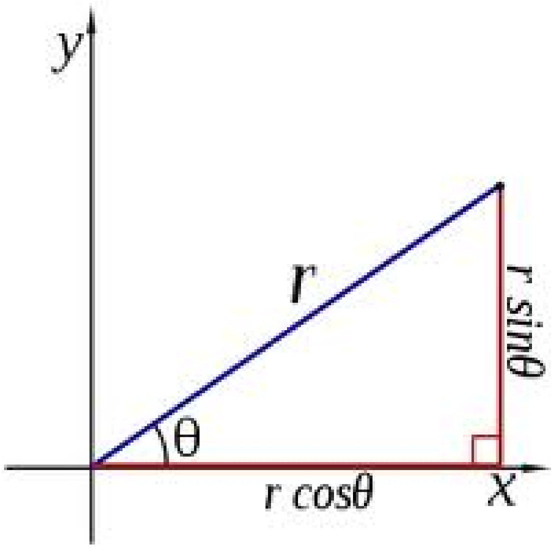
$$dQ = \mathbf{V}\cdot\mathbf{n}dA = \left(\frac{\partial\Psi}{\partial y}\mathbf{i} - \frac{\partial\Psi}{\partial x}\mathbf{j}\right)\cdot\left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right)d(s\cdot 1) = \frac{\partial\Psi}{\partial y}dy + \frac{\partial\Psi}{\partial x}dx = d\Psi$$

i.e. change in $d\Psi$ is the volume flux and along a streamline $dQ = 0$.

Consider flow between two streamlines

$$Q_{AB} = \int_A^B \mathbf{V}\cdot\mathbf{n}dA = \int_A^B d\Psi = \Psi_B - \Psi_A$$

11.4.4 Incompressible Plane Flow in Polar Coordinates



Continuity equation: $\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$

or $\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_\theta}{\partial \theta} = 0$

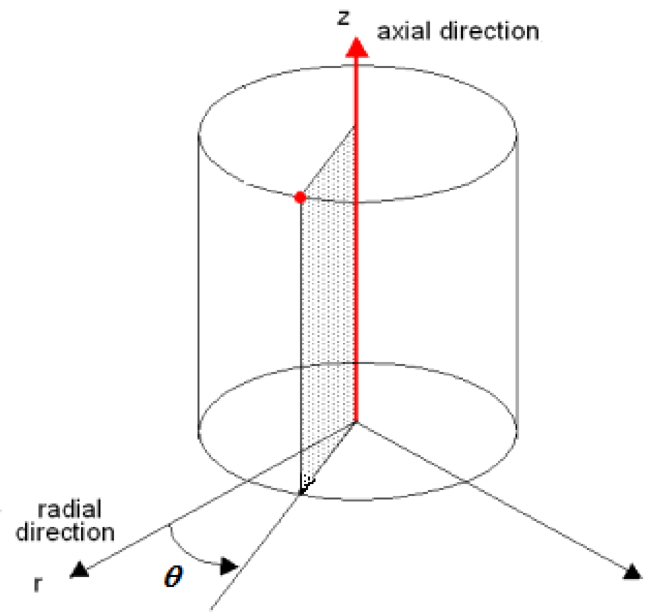
say: $v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}$ $\quad v_\theta = -\frac{\partial \Psi}{\partial r}$

then: $\frac{\partial}{\partial r}\left(r\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\partial \Psi}{\partial r}\right) = 0$

As before $d\Psi = 0$ along a streamline and

$dQ = d\Psi$ Volume flux = change in stream function

11.4.5 Incompressible Axisymmetric Flow



Continuity equation: $\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0$

say: $v_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$ $\quad v_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}$

then: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{-1}{r}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) = 0$

As before $d\Psi = 0$ along a streamline and

$dQ = d\Psi$ Volume flux = change in stream function

11.4.6 Generalization to Steady Plane Compressible Flow

In steady compressible flow, the continuity equation is

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

Define: $\rho u=\frac{\partial \Psi}{\partial y} \quad \rho v=-\frac{\partial \Psi}{\partial x}$

Streamline: $udy-vdx=0$

Compare with $\frac{1}{\rho}\frac{\partial \Psi}{\partial y}dy+\frac{1}{\rho}\frac{\partial \Psi}{\partial x}dx=0$

$$d\Psi=\frac{\partial \Psi}{\partial x}dx+\frac{\partial \Psi}{\partial y}dy \;\Rightarrow\; \frac{1}{\rho}(d\Psi)=0\,,$$

i.e., $d\Psi=0$ and $\Psi=\text{const.}$ is a streamline.

Now: $d\dot{m}=\rho(\mathbf{V}\cdot\mathbf{n})dA=d\Psi$

$$d\dot{m}_{AB}=\int_A^B \rho(\mathbf{V}\cdot\mathbf{n})dA=\Psi_B-\Psi_A$$

Change in Ψ is equivalent to the mass flux.