Introduction to PDE

MSO-203B

Indian Institute of Technology, Kanpur

kaushik@iitk.ac.in

September 9, 2016

Overview

Basic Notions of PDE:-

- Notations.

Overview

Basic Notions of PDE:-

- Notations.
- Definitions and Examples.

Overview

Basic Notions of PDE:-

- Notations.
- Definitions and Examples.
- Classification of PDE and Examples

Notations

Derivatives

Given $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}$, we define the following:

- $u_{x_i} := \frac{\partial u}{\partial x_i} := \lim\limits_{h \to 0} \frac{u(x+he_i)-u(x)}{h}$ (provided the limit exists) is the partial derivative of u in x_i direction.

Notations

Derivatives

Given $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}$, we define the following:

- $u_{x_i} := \frac{\partial u}{\partial x_i} := \lim\limits_{h \to 0} \frac{u(x+he_i)-u(x)}{h}$ (provided the limit exists) is the partial derivative of u in x_i direction.
- $\nabla u = (u_{x_1}, ..., u_{x_n})$ is the Gradient Vector.

Notations

Derivatives

Given $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}$, we define the following:

- $u_{x_i} := \frac{\partial u}{\partial x_i} := \lim\limits_{h \to 0} \frac{u(x+he_i)-u(x)}{h}$ (provided the limit exists) is the partial derivative of u in x_i direction.
- $\nabla u = (u_{x_1}, ..., u_{x_n})$ is the Gradient Vector.
- If u is differentiable then $Du(x)$ is identified as $\nabla u(x)$.

Notations

Multi-Index Notation

- $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ for $\alpha_i \in \mathbb{N}$ is called a multiindex of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

Notations

Multi-Index Notation

- $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ for $\alpha_i \in \mathbb{N}$ is called a multiindex of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.
- Given a multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} .... \partial x_n^{\alpha_n}}$$

Notations

Multi-Index Notation

- $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ for $\alpha_i \in \mathbb{N}$ is called a multiindex of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.
- Given a multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} .... \partial x_n^{\alpha_n}}$$

- For $k \in \mathbb{N} \cup \{0\}$ define

$$D^k u(x) := \{D^\alpha u : |\alpha| = k\}$$

the set of all partial derivatives of order k and moreover $D^k u(x)$ can be regarded as a point in $\mathbb{R}^{n^k}$.

Notations

Space of Functions

- $C(\Omega)$ is the set of all continuous functions on Ω.

Notations

Space of Functions

- $C(\Omega)$ is the set of all continuous functions on Ω.
- $C(\bar{\Omega})$ is the set of all uniformly continuous functions on bounded subsets of Ω.

Notations

Space of Functions

- $C(\Omega)$ is the set of all continuous functions on Ω.
- $C(\bar{\Omega})$ is the set of all uniformly continuous functions on bounded subsets of Ω.
- $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on Ω.

Notations

Space of Functions

- $C(\Omega)$ is the set of all continuous functions on Ω.
- $C(\bar{\Omega})$ is the set of all uniformly continuous functions on bounded subsets of Ω.
- $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on Ω.
- $C^\infty(\Omega) := \cap_{k\in\mathbb{N}\cup\{0\}} C^k(\Omega)$.

Partial Differential Equation

Definition of a PDE

Given an unknown function $u : \Omega \to \mathbb{R}$ and $k \in \mathbb{N}$, an expression of the form

$$F(D^k u(x), D^{k-1} u(x), ..., Du(x), u(x), x) = 0 \quad \text{for} \quad x \in \Omega$$

is called an k-th order PDE.

Definition of a PDE

Given an unknown function $u : \Omega \to \mathbb{R}$ and $k \in \mathbb{N}$, an expression of the form

$$F(D^k u(x), D^{k-1} u(x), ..., Du(x), u(x), x) = 0 \;\text{ for }\; x \in \Omega$$

is called an k-th order PDE.

Assumptions

- Here $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times .. \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$
- Ω is an open subset of $\mathbb{R}^n$.

Classification of PDE

Definitions

A PDE is called linear (in u) if it has a form

$$\sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u = f(x)$$

for a given function a_α ($|\alpha| \leq k$) and f. This PDE is homogeneous if $f = 0$.

Classification of PDE

Definitions

A PDE is called linear (in u) if it has a form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

for a given function a_α ($|\alpha| \leq k$) and f. This PDE is homogeneous if $f = 0$.

Examples

The Laplacian Operator $\Delta u := u_{x_1 x_1} + u_{x_2 x_2}$ is linear PDE.

Classification of PDE

Definitions

A PDE is called Semilinear (in u) if it has a form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, .., Du, u, x) = f(x)$$

for a given function a_α ($|\alpha| = k$), a_0 and f.

Classification of PDE

Definitions

A PDE is called Semilinear (in u) if it has a form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, .., Du, u, x) = f(x)$$

for a given function a_α ($|\alpha| = k$), a_0 and f.

Examples

The Operator $Lu := x^2 \Delta u + u^2$ is semi-linear PDE.

Classification of PDE

Definitions

A PDE is called Quasilinear (in u) if it has a form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, .., Du, u, x)D^\alpha u + a_0(D^{k-1}u, .., Du, u, x) = f(x)$$

for a given function a_α ($|\alpha| = k$), a_0 and f.

Classification of PDE

Definitions

A PDE is called Quasilinear (in u) if it has a form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, .., Du, u, x)D^\alpha u + a_0(D^{k-1}u, .., Du, u, x) = f(x)$$

for a given function a_α ($|\alpha| = k$), a_0 and f.

Examples

The Operator $Lu := |\nabla u|^2 \Delta u + u^3$ is a quasilinear PDE.

Definitions

A PDE is called Nonlinear (in u) if it depends nonlinearly upon the highest order term.

Classification of PDE

Definitions

A PDE is called Nonlinear (in u) if it depends nonlinearly upon the highest order term.

Examples

The Operator $Lu := \det(D^2u)$ is a fully nonlinear PDE.

Method of Separation of Variable

We are interested to solve the problem

$$-\Delta u = 0 \text{ in } \Omega = (0,1) \times (0,1)$$
$$u(x,0) = u(0,y) = u(1,y) = 0 \text{ and } u(x,1) = f(x)$$

where f is any function to be specified later.

Laplace Equation

Method of Separation of Variable

Let us assume that $u(x, y) = X(x)Y(y)$ is a solution of the equation $\Delta u = 0$. Hence one has

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Method of Separation of Variable

Let us assume that $u(x,y) = X(x)Y(y)$ is a solution of the equation $\Delta u = 0$. Hence one has

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Solution without boundary conditions for $\lambda = \mu^2$

Separately solving for X and Y we have,

$$u(x,y) = (A\cosh \mu x + B\sinh \mu x)(C\cos \mu y + D\sin \mu y)$$

Putting the boundary condition $u(0, y) = 0$

$$0 = u(0, y) = A(C \sin \mu y + D \cos \mu y)$$

which imply $A = 0$ and so, $u(x, y) = B \sinh \mu x(C \cos \mu y + D \sin \mu y)$

Putting the boundary condition $u(0, y) = 0$

$$0 = u(0, y) = A(C \sin \mu y + D \cos \mu y)$$

which imply $A = 0$ and so, $u(x, y) = B \sinh \mu x (C \cos \mu y + D \sin \mu y)$

Putting the boundary condition $u(x, 0) = 0$

$$0 = u(x, 0) = BC \sinh \mu x$$

which imply $C = 0$ and so, $u(x, y) = BD \sinh \mu x \sin \mu y$

Putting the boundary condition $u(1, y) = 0$

$$0 = u(1, y) = BD \sinh \mu \sin \mu y$$

which imply $B = 0$ or $D = 0$ since $\sinh \mu \neq 0$, hence $u(x, y) = 0$

Laplace Equation

Putting the boundary condition $u(1, y) = 0$

$$0 = u(1, y) = BD \sinh \mu \sin \mu y$$

which imply $B = 0$ or $D = 0$ since $\sinh \mu \neq 0$, hence $u(x, y) = 0$

Conclusion

There are no positive eigenvalues.

Solution without boundary conditions for $\lambda = 0$

Separately solving for X and Y we have,

$$u(x, y) = (Ax + B)(Cy + D)$$

Solution without boundary conditions for $\lambda = 0$

Separately solving for X and Y we have,

$$u(x, y) = (Ax + B)(Cy + D)$$

Putting the boundary conditions $u(x, 0) = 0$

$$0 = u(x, 0) = (Ax + B)D$$

which implies $D = 0$ and hence $u(x, y) = Cy(Ax + B)$

Laplace Equation

Putting the boundary value $u(0, y) = 0$

$$0 = u(0, y) = BCy$$

which imply that $B = 0$ and hence $u(x, y) = ACxy$

Laplace Equation

Putting the boundary value $u(0, y) = 0$

$$0 = u(0, y) = BCy$$

which imply that $B = 0$ and hence $u(x, y) = ACxy$

Putting the boundary value $u(1, y) = 0$

$$0 = u(1, y) = ACy$$

which imply that $A = 0$ or $C = 0$ and hence $u(x, y) = 0$ in any case.

Putting the boundary value $u(0, y) = 0$

$$0 = u(0, y) = BCy$$

which imply that $B = 0$ and hence $u(x, y) = ACxy$

Putting the boundary value $u(1, y) = 0$

$$0 = u(1, y) = ACy$$

which imply that $A = 0$ or $C = 0$ and hence $u(x, y) = 0$ in any case.

Conclusion

0 is not an eigenvalue.

Laplace Equation

Solution without boundary conditions for $\lambda = -\mu^2$

Separately solving for X and Y we have,

$$u(x,y) = (A\cos\mu x + B\sin\mu x)(C\cosh\mu y + D\sinh\mu y)$$

Laplace Equation

Solution without boundary conditions for $\lambda = -\mu^2$

Separately solving for X and Y we have,

$$u(x,y) = (A\cos\mu x + B\sin\mu x)(C\cosh\mu y + D\sinh\mu y)$$

Putting the boundary condition $u(0,y) = 0$

$$0 = u(0,y) = A(C\cosh\mu y + D\sinh\mu y)$$

which implies $A = 0$ and so, $u(x,y) = B\sin\mu x(C\cosh\mu y + D\sinh\mu y)$.

Laplace Equation

Putting the boundary condition $u(x,0) = 0$

$$0 = u(x,0) = CB \sin \mu x$$

which implies $C = 0$ and hence $u(x,y) = BD \sinh \mu y \sin \mu x$

Putting the boundary condition $u(1,y) = 0$

$$0 = u(1,y) = BD \sinh \mu y \sin \mu$$

So, $B \neq 0$ and $D \neq 0$ then $u(x,y) = BD \sin n\pi x \sinh n\pi y$

Laplace Equation

Superposition

The required solution without the condition $u(x,1) = f(x)$ is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi y$$

Laplace Equation

Superposition

The required solution without the condition $u(x, 1) = f(x)$ is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi y$$

Setting a specific $f(x)$

When $f(x) = \sin \pi$ then we have

$$\sin \pi = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi$$

Laplace Equation

Superposition

The required solution without the condition $u(x,1) = f(x)$ is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi y$$

Setting a specific $f(x)$

When $f(x) = \sin \pi$ then we have

$$\sin \pi = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi$$

Required Solution

$$u(x,y) = \frac{1}{\sinh \pi} \sin \pi x \sinh \pi y$$

The End