Laplace Equation: Polar Coordinate

MSO-203B

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Introduction

Question

We know how to solve the Laplace equation given a rectangular domain using the seperation of variable.

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We know how to solve the Laplace equation given a rectangular domain using the seperation of variable.
Can we solve the Laplace equation is a disk $D \in \mathbb{R}^2$.

Answer

One can easily accomplies this using the polar coordinate.

Laplace equation in Polar coordinate

The Laplacian Equation in the (x, y) coordinate is given by

$$u_{xx} + u_{yy} = 0$$

The Cartesian coordinate can be represented as

$$x = r\cos\theta$$

and

$$y = r\sin\theta$$

Introduction

Laplace equation in Polar coordinate

Computing the partial derivatives of x, y w.rt. r and θ we have,

$$x_r = \cos\theta, \;\; y_r = \sin\theta$$
$$x_\theta = -r\sin\theta, \;\; y_\theta = r\cos\theta$$

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Computing u_r and u_θ

Using chain rule one has

$$u_r = u_x x_r + u_y y_r = \cos\theta u_x + \sin\theta u_y$$

and we also have,

$$u_\theta = u_x x_\theta + u_y y_\theta = -r\sin\theta u_x + r\cos\theta u_y$$

Laplacian in Polar coordinate

Computing u_{rr}

Again using chain rule on u_r we have

$$u_{rr} = \cos\theta \frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta \frac{\partial}{\partial r}\frac{\partial u}{\partial r}$$
$$= \cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right)$$
$$= \cos^2\theta\, u_{xx} + 2\sin\theta\cos\theta\, u_{xy} + \sin^2\theta\, u_{yy}$$

Computing $u_{\theta\theta}$

Using chain rule on u_θ we have,

$$u_{\theta\theta} = -r\cos\theta\, u_x - r\sin\theta \frac{\partial}{\partial\theta} u_x - r\sin\theta\, u_y + r\cos\theta \frac{\partial}{\partial\theta} u_y$$
$$= -r(\cos\theta\, u_x + \sin\theta\, u_y) + r^2\left(\sin^2\theta\, u_{xx} - 2\cos\theta\sin\theta\, u_{xy} + \cos^2\theta\, u_{yy}\right)$$

Laplacian in Polar coordinate

Final form

Hence one has,

$$\frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + \sin^2\theta u_{xx} - 2\cos\theta\sin\theta u_{xy} + \cos^2\theta u_{yy} = 0$$

and so one has

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r$$

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Laplace Equation in Polar form

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$$

Easy Application

Question

The function $u(x, y) = \frac{x}{x^2+y^2}$ is harmonic in $\mathbb{R}^2$.

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Answer

In Polar coordinates one has $u(r,\theta) = \frac{\cos\theta}{r}$.
Clearly $u_r = -\frac{1}{r^2}\cos\theta$ and $u_{rr} = \frac{2}{r^3}\cos\theta$.
Again $u_\theta = -\frac{\sin\theta}{r}$ and $u_{\theta\theta} = -\frac{\cos\theta}{r}$.
Hence

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$$

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Solving Laplace equation in unit disk

Question

Solve

$$\Delta u = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

subject to the condition

$$u(1, \theta) = \begin{cases} 1, & 0 \leq \theta \leq \pi \\ \sin^2\theta, & \pi \leq \theta \leq 2\pi \end{cases}$$

Solving Laplace equation in unit disk

Solution via Separation of Variable

We look for solution of the form $u(r,\theta) = R(r)\Theta(\theta)$. So one has

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

and hence,

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

.

Solution via Separation of Variable

Remark

To be a valid solution $u(r,\theta) = u(r,\theta + 2\pi)$ since $(r\cos\theta, r\sin\theta)$ is the same as $(r\cos(\theta + 2\pi), r\sin(\theta + 2\pi))$. But $\cos(\lambda\theta) = \cos(\lambda(\theta + 2\pi))$ and $\sin(\lambda\theta) = \sin(\lambda(\theta + 2\pi)) \implies \lambda \in \mathbb{Z}$

Solving the ODE's

From $\Theta'' + \lambda^2\Theta = 0$ we have $\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$
From $r^2R'' + rR' - \lambda^2 R = 0$ we get, $R(r) = Cr^{\lambda} + Dr^{-\lambda}$

The curious case of $\lambda = 0$

Solving $r^2R'' + rR' = 0$ one has $R(r) = C\ln|r| + D$

Solution via Separation of Variable

Solution

Putting together the value of $R(r)$ and $\Theta(\theta)$ we have,

$$u(x,y) = \begin{cases} (A_\lambda \cos(\lambda\theta) + B_\lambda \sin(\lambda\theta))(C_\lambda r^\lambda + D_\lambda r^{-\lambda}) & \text{when } \lambda \neq 0 \\ A_0 \ln|x| + B_0 & \text{when } \lambda = 0 \end{cases}$$

Viability of solution

Since $\ln|x|$ and r^{-n} both blows up near $r = 0$ we choose our solution to be

$$u_n(r,\theta) = \begin{cases} A_n \cos(n\theta) r^n + B_n \sin(n\theta) r^n & \text{when } n = 1,2,3.. \\ A_0 & \text{when } n = 0 \end{cases}$$

Solution via Separation of Variable

General Solution

$$u(r,\theta) = \sum_{n=0}^{\infty} (A_n \cos(n\theta) r^n + B_n \sin(n\theta) r^n)$$

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Incorporating the Boundary Conditions

We have that $u(1,\theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta))$

Using Fourier Series

Calculating the Fourier Coefficients

$A_0 = \frac{1}{2\pi}(\int_0^{\pi} 1 d\theta + \int_{\pi}^{2\pi} \sin^2\theta d\theta) = \frac{3}{4}$
$A_n = \frac{1}{\pi}(\int_0^{\pi} \cos(n\theta) d\theta + \int_{\pi}^{2\pi} \sin^2\theta \cos(n\theta) d\theta) = 0$ for $n \neq 2$ and $-\frac{1}{4}$ for $n = 2$
$B_n = \frac{1}{\pi}(\int_0^{\pi} \sin(n\theta) d\theta + \int_{\pi}^{2\pi} \sin^2\theta \sin(n\theta) d\theta) = \frac{1}{\pi}(\frac{2}{n} + \frac{4}{n(n^2-4)})$ when n is odd, zero otherwise.

Final Solution

$$u(r,\theta) = \frac{3}{4} - \frac{r^2}{4}\cos(2\theta) + \sum_{n=0}^{\infty}\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$