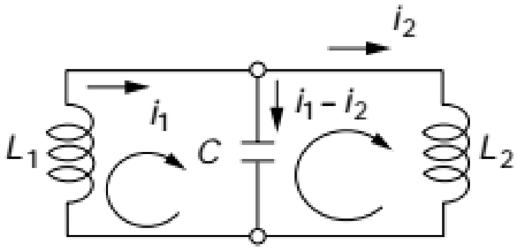
Solved Questions: ME772A

Q1. Derive the mathematical model of the electrical system shown in figure below using Kirchhoff's laws and the energy method. Determine the natural frequencies and the modes (eigenvectors) of this system for L_1 = 0.5H, L_2 = 0.3H, and C = 0.02F.



Ans: Application of KVL to the electrical circuit generates the equations

$$\begin{cases} L_1 \dfrac{di_1(t)}{dt} + \dfrac{1}{C}\int [i_1(t) - i_2(t)]\, dt = 0 \\ L_2 \dfrac{di_2(t)}{dt} - \dfrac{1}{C}\int [i_1(t) - i_2(t)]\, dt = 0 \end{cases}$$

which can be written in charge form as

$$\begin{cases} L_1 \ddot{q}_1 + \dfrac{1}{C}(q_1 - q_2) = 0 \\ L_2 \ddot{q}_2 + \dfrac{1}{C}(q_2 - q_1) = 0 \end{cases} \qquad \boxed{(1)}$$

The energy collected by the three electrical components of the circuit is

$$E = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}\frac{(q_1 - q_2)^2}{C} + \frac{1}{2}L_2\dot{q}_2^2 \qquad \boxed{(2)}$$

The energy being constant, its time derivative is zero, which leads to

$$\dot{q}_1\left[L_1\ddot{q}_1 + \frac{1}{C}(q_1 - q_2)\right] + \dot{q}_2\left[L_2\ddot{q}_2 + \frac{1}{C}(q_2 - q_1)\right] = 0 \qquad \boxed{(3)}$$

Equation (3) has to be valid at all times, but the charge rates cannot be zero at all times; therefore, compliance with the condition of Eq. (3) results in Eqs. (1), which have been obtained by means of Kirchhoff's voltage law: They represent the mathematical model of the conservative electrical system.

The free response of a conservative system requires solution of the harmonic type:

$$\begin{cases} q_1 = Q_1 \sin(\omega t) \\ q_2 = Q_2 \sin(\omega t) \end{cases} \qquad \boxed{(4)}$$

Substitution of Eqs. (4) into Eqs. (3) yields the following algebraic equations

$$\begin{cases} \left(\frac{1}{C} - \omega^2 L_1\right)Q_1 - \frac{1}{C}Q_2 = 0 \\ -\frac{1}{C}Q_1 + \left(\frac{1}{C} - \omega^2 L_2\right)Q_2 = 0 \end{cases} \quad \boxed{(5)}$$

For the equations system (5) to have nontrivial solutions in the amplitudes Q_1 and Q_2, the determinant of the system needs to be zero:

$$\begin{vmatrix} \frac{1}{C} - \omega^2 L_1 & -\frac{1}{C} \\ -\frac{1}{C} & \frac{1}{C} - \omega^2 L_2 \end{vmatrix} = 0 \quad \boxed{(6)}$$

which produces the following algebraic equation (characteristic equation) in ω:

$$\omega^2\left[\omega^2 L_1 L_2 - \frac{1}{C}(L_1 + L_2)\right] = 0$$

whose solution consists of the natural frequencies

$$\begin{cases} \omega_{n1} = 0 \\ \omega_{n2} = \sqrt{\frac{L_1 + L_2}{CL_1 L_2}} \end{cases} \quad \boxed{(7)}$$

The numerical value of the nonzero natural frequency of above equation is $\omega_{n,2} = 16.33$ rad/s.

Similar to mechanical systems, modes and eigenvectors can be expressed for electrical systems. The following ratio is obtained from Eqs. (5):

$$\frac{Q_1}{Q_2} = \frac{1}{1 - \omega^2 L_1 C}$$

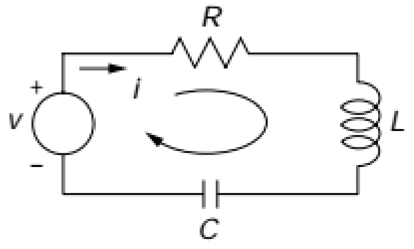
which, for the nontrivial natural frequency of Eq. (7), becomes

$$\left(\frac{Q_1}{Q_2}\right)_{\omega = \sqrt{\frac{L_1 + L_2}{CL_1 L_2}}} = -\frac{L_2}{L_1} = -0.6 \quad \boxed{(8)}$$

Equation (8) indicates that the mode consists of two charge amplitudes that have opposite signs (which means the corresponding currents circulate in opposite directions at $\omega_{n,2}$). The magnitude of Q_2 is larger than that of Q_1. One eigenvector, corresponding to the nonzero natural frequency and the ratio of Eq. (8), is obtained by considering that $Q_2 = 1$, for instance, which leads to

$$\{Q^{(1)}\} = \begin{Bmatrix} -\frac{L_2}{L_1} \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.6 \\ 1 \end{Bmatrix}$$

Q2. Derive a state-space model of the single-mesh electrical circuit sketched in below Figure.



Ans: The following differential equation is the mathematical model of the electrical system

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = v$$

where L is the inductance, R is the resistance, C is the capacitance, q is the charge, and v is the source voltage. Let us drop the time variable and select the variables x_1 and x_2 as

$$\begin{cases} x_1 = q \\ x_2 = \dot{q} \end{cases}$$

to be the state variables of the state vector

$$\{x\} = \{x_1 \quad x_2\}^t$$

In Eqs. above, the variable t (time) has been dropped to simplify notation but all variables (q, v, x_1, and x_2) are functions of time. Thus,

$$\dot{x}_1 = x_2$$

The differential equation can be written in the form

$$\ddot{q} = -\frac{1}{LC}q - \frac{R}{L}\dot{q} + \frac{1}{L}v$$

which can be formulated using the state variables as

$$\dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{L}v$$

In vector-matrix form as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dfrac{1}{L} \end{Bmatrix} u$$

Comparison above Eq. with the standard state equation

$$\{\dot{x}(t)\} = [A]\{x(t)\} + [B]\{u(t)\}$$

$$[A] = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}; [B] = \begin{Bmatrix} 0 \\ \frac{1}{L} \end{Bmatrix}$$

If the charge is the output, which can be denoted formally as **y = q**, then the first Eq. becomes

$$y = x_1$$

It can be written in vector form as

$$y = \{1 \;\; 0\}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 0 \times u$$

Comparison of this equation with standard Eq.

$$\{y(t)\} = [C]\{x(t)\} + [D]\{u(t)\}$$

It indicates that, indeed, above eq. represents the output equation, and its defining matrices are

$$[C] = \{1 \;\; 0\}; \;\; [D] = 0$$

Thus, these two below eq. form the state space model of the electrical system

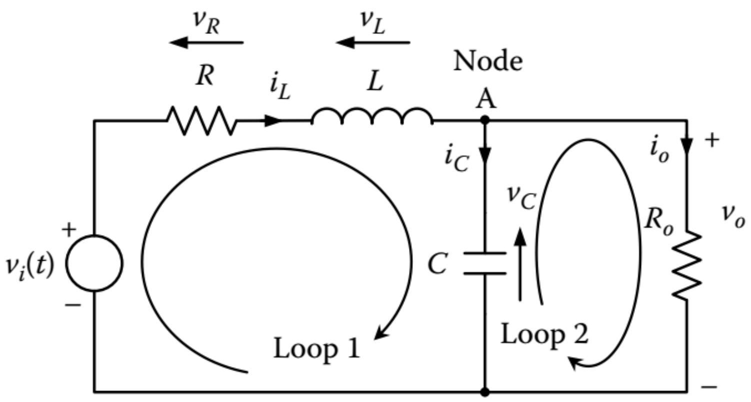
$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{L} \end{Bmatrix} u$$

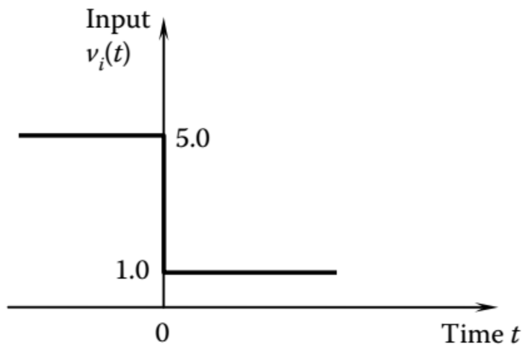
$$y = \{1 \;\; 0\}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 0 \times u$$

Q3. The circuit shown below consists of an inductor L, a capacitor C, and two resistors R and R_o. The input is the voltage $v_i(t)$ and the output is the voltage v_o across the resistor R_o.

a) Obtain a complete state-space model for the system.
b) Obtain an input–output differential equation for the system.
c) Obtain expressions for undamped natural frequency and the damping ratio of the system.
d) The system starts at steady state with an input of 5 V (for all t < 0). Then suddenly, the input is dropped to 1 V (for all t > 0), which corresponds to a step input as shown in figure below.

For R = R_o = 1 Ω, L = 1 H, and C = 1 F, what are the initial conditions of the system and their derivatives at both t= 0^- and t= 0^+? What are the final (steady state) values of the state variables and the output variable? Sketch the nature of the system response.





Solution

(a)
State variables:
Current through independent inductors (i_L); Voltage across independent capacitors (v_C)
Constitutive equations:

$$v_L = L\frac{di_L}{dt}; \quad i_C = C\frac{dv_C}{dt}; \quad v_R = Ri_L; \quad v_o = Ri_o$$

First two equations are for independent energy storage elements, and they form the state-space shell.
Continuity equation:
Node A (Kirchhoff's current law):

$$i_L - i_C - i_o = 0$$

Compatibility equations:
Loop 1 (Kirchhoff's voltage law): $v_i - v_R - v_L - v_C = 0$
Loop 2 (Kirchhoff's voltage law): $v_C - v_o = 0$
Eliminate auxiliary variables. We have the state equations:

$$L\frac{di_L}{dt} = v_L = v_i - v_R - v_C = v_i - Ri_L - v_C$$

$$C\frac{dv_C}{dt} = i_C = i_L - i_o = i_L - \frac{v_o}{R_o} = i_L - \frac{v_C}{R_o}$$

State equations:

$$\frac{di_L}{dt} = \frac{1}{L}[-Ri_L - v_C + v_i] \qquad \text{(i)}$$

$$\frac{dv_C}{dt} = \frac{1}{C}\left[i_L - \frac{v_C}{R_o}\right] \qquad \text{(ii)}$$

Output equation: $v_o = v_C$

Vector-matrix representation: $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$; $\boldsymbol{y} = \boldsymbol{Cx}$
where

$$\text{System matrix } \boldsymbol{A} = \begin{bmatrix} -R/L & -1/L \\ 1/C & -1/(R_oC) \end{bmatrix}; \quad \text{Input gain matrix } \boldsymbol{B} = \begin{bmatrix} 1/L \\ 0 \end{bmatrix};$$

$$\text{Measurement gain matrix } \boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad \text{State vector} = \boldsymbol{x} = \begin{bmatrix} i_L \\ v_C \end{bmatrix};$$

$$\text{Input} = \boldsymbol{u} = [v_i]; \quad \text{Output} = \boldsymbol{y} = [v_o]$$

(b)
From (ii):

$$i_L = C\frac{dv_C}{dt} + \frac{v_C}{R_o}$$

Substitute in (i) for i_L:

$$L\frac{d}{dt}\left(C\frac{dv_C}{dt} + \frac{v_C}{R_o}\right) = -R\left(C\frac{dv_C}{dt} + \frac{v_C}{R_o}\right) - v_C + v_i$$

This simplifies to the input–output differential equation (since $v_o = v_C$)

$$LC\frac{d^2v_o}{dt^2} + \left(\frac{L}{R_o} + RC\right)\frac{dv_o}{dt} + \left(\frac{R}{R_o} + 1\right)v_o = v_i \qquad \text{(iii)}$$

(c)
The input–output differential equation is of the form

$$\frac{d^2v_o}{dt^2} + 2\zeta\omega_n\frac{dv_o}{dt} + \omega_n^2 v_o = \frac{1}{LC}v_i$$

Hence,

$$\text{Natural frequency } \omega_n = \sqrt{\frac{1}{LC}\left(\frac{R}{R_o} + 1\right)} \qquad \text{(iv)}$$

$$\text{Damping ratio } \zeta = \frac{1}{2\sqrt{LC\left(\frac{R}{R_o} + 1\right)}}\left(\frac{L}{R_o} + RC\right) \qquad \text{(v)}$$

Note: $1/LC$ has units of (frequency)^2.
RC and L/R_o have units of "time" (i.e., time constant).

(d)
Initial conditions:
For $t < 0$ (initial steady state): $di_L/dt = 0$; $dv_C/dt = 0$

Hence,

$$\text{(i):}\quad \frac{di_L(0^-)}{dt} = 0 = \frac{1}{L}\left[-Ri_L(0^-) - v_C(0^-) + v_i(0^-)\right]$$

$$\text{(ii):}\quad \frac{dv_C(0^-)}{dt} = 0 = \frac{1}{C}\left[i_L(0^-) - \frac{v_C(0^-)}{R_o}\right]$$

Substitute the given parameter values $R = R_o = 1\,\Omega$, $L = 1\,\text{H}$, and $C = 1\,\text{F}$, and the input $v_i(0^-) = 5.0$:

$$-i_L(0^-) - v_C(0^-) + 5 = 0$$

$$i_L(0^-) - v_C(0^-) = 0$$

We get

$$i_L(0^-) = 2.5\text{ A},\ v_C(0^-) = 2.5\,\text{V}$$

State variables cannot undergo step changes (because that violates the corresponding physical laws—constitutive equations). Specifically:
Inductor cannot have a step change in current (needs infinite voltage).
Capacitor cannot have a step change in voltage (needs infinite current).
Hence,

$$i_L(0^+) = i_L(0^-) = 2.5\text{ A}$$

$$v_C(0^+) = v_C(0^-) = 2.5\text{ V}$$

Note: Since $v_i(0^+) = 1.0$

$$\text{(i):}\quad \frac{di_L(0^+)}{dt} = -i_L(0^+) - v_C(0^+) + 1.0 = -2.5 - 2.5 + 1.0 = -4.0\text{ A/s} \neq 0$$

$$\text{(ii):}\quad \frac{dv_C(0^+)}{dt} = i_L(0^+) - v_C(0^+) = 2.5 - 2.5 = 0.0\text{ V/s}$$

Final values:
As $t \to \infty$ (at final steady state)

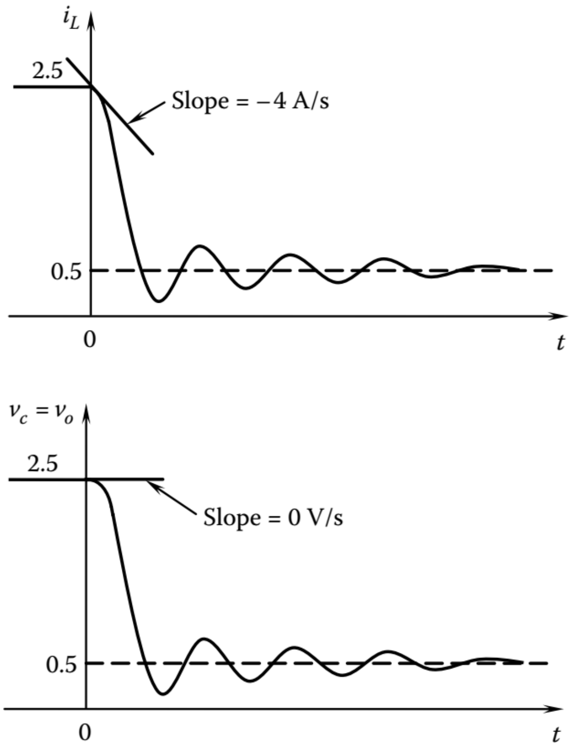
$$\frac{di_L}{dt} = 0$$

$$\frac{dv_C}{dt} = 0$$

and $v_i = 1.0$
Substitute:

$$\text{(i):}\quad \frac{di_L(\infty)}{dt} = 0 = -i_L(\infty) - v_C(\infty) + 1.0$$

$$\text{(ii):}\quad \frac{dv_C(\infty)}{dt} = 0 = i_L(\infty) - v_C(\infty)$$



Responses of the state variables

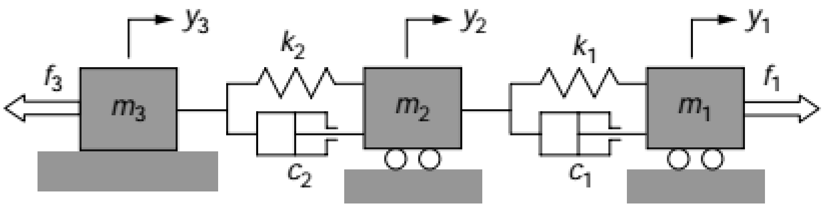
Solution: $i_L(\infty) = 0.5$ A, $v_C(\infty) = 0.5$ V
For the given parameter values,

$$\text{(iii):} \quad \frac{d^2 v_o}{dt^2} + 2\frac{dv_o}{dt} + 2v_o = 1$$

Hence, $\omega_n = \sqrt{2}$ and $2\zeta\omega_n = 2$, or, $\zeta = 1/\sqrt{2}$.

This is an under-damped system, producing an oscillatory response as a result. The nature of the responses of the two state variables is shown in above Figure. *Note*: Output $v_o = v_C$.

Q4. Consider the mechanical system shown below, where f_1 is the actuation force and f_3 is a friction force. Determine a state space model for this system when the output vector is formed of the displacements y_1, y_2, and y_3.



Ans: Newton's second law of motion applied to the three bodies results in

$$\begin{cases} m_1\ddot{y}_1 = f_1 - c_1(\dot{y}_1 - \dot{y}_2) - k_1(y_1 - y_2) \\ m_2\ddot{y}_2 = -c_1(\dot{y}_2 - \dot{y}_1) - k_1(y_2 - y_1) - c_2(\dot{y}_2 - \dot{y}_3) - k_2(y_2 - y_3) \\ m_3\ddot{y}_3 = -f_3 - c_2(\dot{y}_3 - \dot{y}_2) - k_2(y_3 - y_2) \end{cases} \qquad \boxed{(1)}$$

The mathematical model of this mechanical system consists of three second-order differential equations. As a consequence, we need $3 \times 2 = 6$ state variables, which are selected as

$$x_1 = y_1;\ \ x_2 = \dot{y}_1;\ \ x_3 = y_2;\ \ x_4 = \dot{y}_2;\ \ x_5 = y_3;\ \ x_6 = \dot{y}_3$$

Thus, it indicates the following state variable connections

$$\dot{x}_1 = x_2;\ \ \dot{x}_3 = x_4;\ \ \dot{x}_5 = x_6 \qquad \boxed{(2)}$$

Thus, eq. (1) can be written as:

$$\begin{cases} \ddot{y}_1 = -\dfrac{k_1}{m_1}y_1 - \dfrac{c_1}{m_1}\dot{y}_1 + \dfrac{k_1}{m_1}y_2 + \dfrac{c_1}{m_1}\dot{y}_2 + \dfrac{1}{m_1}f_1 \\ \ddot{y}_2 = \dfrac{k_1}{m_2}y_1 + \dfrac{c_1}{m_2}\dot{y}_1 - \dfrac{k_1 + k_2}{m_2}y_2 - \dfrac{c_1 + c_2}{m_2}\dot{y}_2 + \dfrac{k_2}{m_2}y_3 + \dfrac{c_2}{m_2}\dot{y}_3 \\ \ddot{y}_3 = \dfrac{k_2}{m_3}y_2 + \dfrac{c_2}{m_3}\dot{y}_2 - \dfrac{k_2}{m_3}y_3 - \dfrac{c_2}{m_3}\dot{y}_3 - \dfrac{1}{m_3}f_3 \end{cases} \qquad \boxed{(3)}$$

Collecting Eqs. (2) and (3) into a vector-matrix form results in the state equation

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1}{m_1} & \frac{k_1}{m_1} & \frac{c_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_2} & \frac{c_1}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{c_1+c_2}{m_2} & \frac{k_2}{m_2} & \frac{c_2}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_2}{m_3} & \frac{c_2}{m_3} & -\frac{k_2}{m_3} & -\frac{c_2}{m_3} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{m_3} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \boxed{\mathbf{(4)}}$$

where $u_1 = f_1$ and $u_2 = f_3$. The matrix multiplying the state vector in Eq. (4) is [A] and the one multiplying the input vector is [B].

$$\{\dot{x}(t)\} = [A]\{x(t)\} + [B]\{u(t)\}$$

The output vector $\{y\} = \{y_1\ y_2\ y_3\}'$ is connected to the state and input vectors as

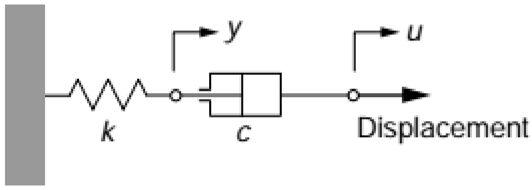
$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Compare with standard form

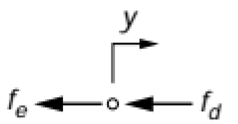
$$\{y(t)\} = [C]\{x(t)\} + [D]\{u(t)\}$$

The matrix multiplying the state vector is [C] and the matrix multiplying the input vector is [D].

Q5. Obtain the state space model for the mechanical system shown in Figure and transform the resulting model into a transfer function model.



Ans: The free body diagram is



Newton's second law of motion corresponding to the free-body diagram results in

$$0 = -f_d - f_e$$

where f_d and f_e are the damping and elastic forces, defned as

$$\begin{cases} f_d = c(\dot{y} - \dot{u}) \\ f_e = ky \end{cases}$$

Combining above equations, we get

$$c\dot{y} + ky = c\dot{u}$$

The mechanical system is SISO and its mathematical model consists of a first-order differential equation. As a result, a single state variable is needed. Because the input function appears as a time derivative, it is necessary to apply the Laplace transform to this above Eq., which results in

$$\frac{Y(s)}{U(s)} = \frac{s}{s + \frac{k}{c}} \qquad \boxed{\textbf{(a)}}$$

Using the intermediate function $Z(s)$

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{U(s)} \times \frac{Y(s)}{Z(s)} = \frac{1}{s + \frac{k}{c}} \times s$$

which indicates the following selection needs to be made:

$$\begin{cases} \dfrac{Z(s)}{U(s)} = \dfrac{1}{s + \frac{k}{c}} \\ \dfrac{Y(s)}{Z(s)} = s \end{cases} \qquad \boxed{\textbf{(1)}}$$

Cross-multiplication in the first Eq. above and application of the inverse Laplace transform to the resulting equation yields

$$c\dot{z} + kz = cu$$

This Equation no longer contains input derivatives; therefore, we can choose x = z as the state variable. As a consequence, it becomes

which is the state equation with $A = -k/c$ and $B = 1$ $$\dot{x} = -\frac{k}{c}x + u \qquad \boxed{(2)}$$

Cross-multiplication in the second Eq. of (1) followed by inverse Laplace transformation results in

$$y = \dot{z} = \dot{x} \qquad \boxed{(3)}$$

Combining (2) and (3) yields,

$$y = -\frac{k}{c}x + u \qquad \boxed{(4)}$$

with $C = -k/c$ and $D = 1$.

The transfer function corresponding to the state space model defined by the state Eq. (2) and output Eq. (4) is the one given in Eq. (a). This can readily be verified using the known equation

$G(s) = C\,(sI - A)^{-1}\,B + D.$

Q6. Find State Transition Matrix of a 2nd Order System or in other words find Φ(s) and φ(t) if $\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-3} \end{pmatrix}$

Ans: The state transition matrix is an important part of both the zero input and the zero state solutions of systems represented in state space. The state transition matrix in the Laplace Domain, Φ(s), is defined as

$\Phi(s) = (sI - A)^{-1}$

$$= \left(s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \right)^{-1}$$

$$= \left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1}$$

The inverse of a 2×2 matrix is given here.

$$\mathbf{\Phi}(s) = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} = \frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{s(s+3)+2}$$

$$= \frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{s^2+3s+2}$$

To find $\boldsymbol{\varphi}(t)$ we must take the inverse Laplace Transform of every term in the matrix

$$\varphi(t) = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \mathcal{L}^{-1}\left(\frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{s^2+3s+2} \right) = \mathcal{L}^{-1}\left(\frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{(s+1)(s+2)} \right)$$

$$= \mathcal{L}^{-1}\begin{pmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{pmatrix}$$

We now must perform a partial fraction expansion of each term, and solve

$$\boldsymbol{\varphi}(t) = \mathcal{L}^{-1}\begin{pmatrix} \frac{2}{(s+1)} + \frac{-1}{(s+2)} & \frac{1}{(s+1)} + \frac{-1}{(s+2)} \\ \frac{-2}{(s+1)} + \frac{2}{(s+2)} & \frac{-1}{(s+1)} + \frac{2}{(s+2)} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$