Heat conduction in 2-d

We will now demonstrate the versatility of the FE technique by formulating the numerical scheme for solving a steady state heat transfer problem. The governing equation in 2-d is:

$$\frac{\partial}{\partial x}\left(k_x T_{,x} + k_{xy} T_{,y}\right) + \frac{\partial}{\partial y}\left(k_{xy} T_{,x} + k_y T_{,y}\right) = 0,$$

in the absence of the heat generation term. The above equation assumes a 'constitutive equation' connecting heat flux $\boldsymbol{q}$ with the gradients of temperature (i.e. Fourier's law) as

$$\left\{ \begin{array}{c} q_x \\ q_y \end{array} \right\} = -\left(\begin{array}{cc} k_x & k_{xy} \\ k_{xy} & k_y \end{array} \right) \left\{ \begin{array}{c} T_{,x} \\ T_{,y} \end{array} \right\}$$

or

$$\boldsymbol{q} = -\boldsymbol{\kappa} \boldsymbol{\nabla} \boldsymbol{T}.$$

The flux normal to a boundary with normal $\boldsymbol{n}$ (flux is considered positive when directed into the body) is $q_n = -\boldsymbol{q} \cdot \boldsymbol{n}$, i.e.

$$q_n = (k_x T_{,x} + k_{xy} T_{,y}) n_x + (k_{xy} T_{,x} + k_y T_{,y}) n_y.$$

For the isotropic and homogeneous case (conductivity is a constant) , we have earlier derived the weak form as (in the absence of heat generation):

$$\Pi = \int_V \frac{1}{2} \left(T_{,x}^2 + T_{,y}^2 \right) k dx dy.$$

For the more general case posed here, we can use the Galerkin method to determine the weak form.

For our purpose, we take the following boundary conditions into consideration:

1. $T = T_s$ on some part of the boundary ∂V_1
2. $q_n = \bar{q}_n$ is specified on ∂V_2
3. $q_n = h(T - T_0)$ on some part ∂V_3.

Radiation boundary conditions are not considered. Also volumetric heat generation $Q = 0$.

To formulate the weak form over an element, discretise the tempearture as

$$T(x, y) = \sum N_I T_I = \boldsymbol{N}\boldsymbol{T}.$$

Also, define

$$\left\{ \begin{array}{c} T_{,x} \\ T_{,y} \end{array} \right\} = \left(\begin{array}{ccc} N_{1,x} & N_{2,x} & \cdots \\ N_{1,y} & N_{2,y} & \cdots \end{array} \right) \left\{ \begin{array}{c} T_1 \\ T_2 \\ \vdots \end{array} \right\} = \boldsymbol{B}\boldsymbol{T}.$$

So,

$$\boldsymbol{q} = -\boldsymbol{\kappa}\boldsymbol{B}\boldsymbol{T}.$$

To formulate the elemental equations set;

$$\int_V N_I \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right] dxdy = 0$$

for $I = 1, 2, \ldots$. Each term can be manipulated as (using divergence theorem)

$$\begin{aligned} \int_V N_I \frac{\partial q_x}{\partial x} dxdy &= \int_V \left[\frac{\partial}{\partial x} \left(N_I q_x \right) - \frac{\partial N_I}{\partial x} q_x \right] dxdy \\ &= \int_{\partial V} N_I q_x n_x dS - \int_V \frac{\partial N_I}{\partial x} q_x dxdy \end{aligned}$$

So, for each I we get:

$$\int_V \left[\frac{\partial N_I}{\partial x} q_x + \frac{\partial N_I}{\partial y} q_y \right] dxdy = \int_{\partial V_1} N_I q_n dS + \int_{\partial V_2} N_I \bar{q}_n dS + \int_{\partial V_3} N_I h (T - T_0) dS.$$

The equations can be expressed compactly as

$$\begin{aligned} \left[\int_V \boldsymbol{B}^T \boldsymbol{\kappa} \boldsymbol{B} dxdy + \int_{\partial V_3} h \boldsymbol{N}^T \boldsymbol{N} dS \right] \boldsymbol{T} &= -\int_{\partial V_1} \boldsymbol{N}^T \boldsymbol{q}^T \boldsymbol{n} dS \\ &- \int_{\partial V_2} \boldsymbol{N}^T \bar{q}_n dS + \int_{\partial V_3} \boldsymbol{N}^T T_0 dS. \end{aligned}$$

Here, we define:

$$
\begin{aligned}
\boldsymbol{K}_1 &= \int_V \boldsymbol{B}^T \boldsymbol{\kappa} \boldsymbol{B} dxdy \\
\boldsymbol{K}_2 &= \int_{\partial V_3} h \boldsymbol{N}^T \boldsymbol{N} dS \\
\boldsymbol{R}_1 &= -\int_{\partial V_1} \boldsymbol{N}^T q_n dS \\
\boldsymbol{R}_2 &= -\int_{\partial V_2} \boldsymbol{N}^T \bar{q}_n dS \\
\boldsymbol{R}_3 &= \int_{\partial V_3} \boldsymbol{N}^T T_0 dS
\end{aligned}
$$

which gives

$$
[\boldsymbol{K}_1 + \boldsymbol{K}_2] \boldsymbol{T} = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \boldsymbol{R}_3.
$$

Suppose we discretise a heat conduction problem with iso-p 4 noded quadrilateral elements. Thus

$$T(x,y) = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4,$$

and

$$\left\{ \begin{array}{c} T_{,x} \\ T_{,y} \end{array} \right\} = \boldsymbol{J}^{-1} \left(\begin{array}{cccc} N_{1,r} & N_{2,r} & N_{3,r} & N_{4,r} \\ N_{1,s} & N_{2,s} & N_{3,s} & N_{4,s} \end{array} \right) \left\{ \begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \end{array} \right\}$$

For an element with thickness t,

$$\boldsymbol{K}_1 = \int_{-1}^{1} \int_{-1}^{1} \boldsymbol{B}^T \boldsymbol{\kappa} \boldsymbol{B} t J dr ds.$$

The form of the stiffness $\boldsymbol{K}_2$ will depend on which edge has the conduction boundary condition. Let us suppose that it is applied on the edge $s = +1$.

In that case,

$$\boldsymbol{K}_2 = \int_1^1 \boldsymbol{N}^T\Big|_{s=1} \boldsymbol{N}|_{s=1}\, htJ_s dr = \frac{htL_s}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

where L_s is the length of the edge on which the conduction boundary condition is applied. Similarly,

$$\boldsymbol{R}_3 = \int_{-1}^1 \boldsymbol{N}^T hT_0 tJ_s dr = T_0 \frac{htL_s}{2}\left\{\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \end{array}\right\}.$$

Similar calculations will be necessary to determine $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ if they are applied.

If the heat conduction equation is to be solved in a cylindrical (r, θ, z) polar coordinate system, we have the gradient of $T(r, \theta, z)$ as:

$$\nabla T = \frac{\partial T}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial T}{\partial z}\boldsymbol{e}_z.$$

In an axisymmetric case, $\partial/\partial\theta = 0$, the heat conduction equation in the absence of heat generation and for isotropic homogeneous properties become:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0.$$

The derivation of the stiffness matrix for 2-d axisymmetric 4 noded quadrilateral elements (suitable for solving axisymmetric problems in solids of revolution) will follow similar lines as in the quadrilateral case. This is left as an exercise.

Viscous incompressible fluid flow

Two dimensional flow of a fluid which is viscous and incompressible is governed by the constitutive equations (*Newton's law of viscosity*) upto an undetermined pressure p:

$$\begin{aligned}
\sigma_{xx} &= 2\mu\frac{\partial u}{\partial x} - p \\
\sigma_{yy} &= 2\mu\frac{\partial v}{\partial y} - p \\
\sigma_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).
\end{aligned}$$

Here, $\mu \neq 0$ is the viscosity and $\boldsymbol{v} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2$ are the velocities in the x and y directions at any point in the domain. Inertial effects are neglected here for slow flows so that $\boldsymbol{v} \cdot \nabla\boldsymbol{v} \simeq 0$.

Using the constitutive relations, linear momentum and continuity equations in a flow domain V, we get (for steady flow and no body forces):

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]-\frac{\partial P}{\partial x} &= 0\\
\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}\right)-\frac{\partial P}{\partial y} &= 0\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0
\end{aligned}
$$

in V subject to boundary conditions:

$$
\begin{aligned}
\bar{t}_x &= \left(2\mu\frac{\partial u}{\partial x}-P\right)n_x+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_y\\
\bar{t}_y &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_x+\left(2\mu\frac{\partial v}{\partial y}-P\right)n_y
\end{aligned}
$$

on ∂V. Clearly the above are a set of three pde's in (u, v, P).

A variational principle can be framed from the above equations by multiplying the three governing equations by w_1, w_2, w_3 as:

$$\begin{aligned}
\int_V w_1 \left\{ \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] - \frac{\partial P}{\partial x}\right\} dV &= 0 \\
\int_V w_2 \left\{ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}\right) - \frac{\partial P}{\partial y}\right\} dV &= 0 \\
\int_V w_3 \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) dV &= 0,
\end{aligned}$$

which, with the specified boundary conditions become:

$$\begin{aligned}
\int_V \left[\left(\begin{array}{c} \frac{\partial w_1}{\partial x} \\ \frac{\partial w_2}{\partial y} \\ \frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x} \end{array} \right\}^T \boldsymbol{C} \left\{ \begin{array}{c} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \end{array} \right\} - P\left(\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}\right)\right] dV &= \int_{\partial V} (w_1\bar{t}_x + w_2\bar{t}_y) dS \\
\int_V w_3 \left\{ \begin{array}{c} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \end{array} \right\} dV &= 0
\end{aligned}$$

Here,

$$\boldsymbol{C} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that we need a FE approximation that is at least linear in the velocities and constant in P. Alternately, we can formulate a *constrained variational principle* based on the *penalty approach* or *Lagrange multiplier* approach. We will demonstrate the penalty approach to constrained variational problems here.

To that end, we will remove P from the equations and attempt to minimise

$$\Pi = \frac{1}{2}\int_V (\boldsymbol{D}\boldsymbol{v})^T \boldsymbol{C}\,(\boldsymbol{D}\boldsymbol{v})\,dV - \int_{\partial V} \boldsymbol{v}^T \bar{\boldsymbol{t}} dS + \frac{\gamma}{2}\int_V [G(\boldsymbol{v})]^2 dV, \text{ where,}$$

$$\boldsymbol{D} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix}, \boldsymbol{v} = \left\{ \begin{array}{c} u \\ v \end{array} \right\}.$$

Here, $G(\boldsymbol{v})$ is the constraint that the velocity field (u, v) satisfy the continuity equation such that

$$G(\boldsymbol{v}) : \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

such that when we make $\delta\Pi = 0$, the first two terms as well as the last one are minimised. Minimisation leads to:

$$\delta\Pi = \int_V (\boldsymbol{D}\delta\boldsymbol{v})^T \boldsymbol{C} (\boldsymbol{D}\boldsymbol{v})\, dV - \int_{\partial V} \delta\boldsymbol{v}^T \bar{\boldsymbol{t}} dS + \gamma \int_V \left(\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dV.$$

Here $\delta\boldsymbol{v}^T = \langle \delta u \;\; \delta v \rangle$.

Comparing with the previous variational formulation (setting $w_1 = \delta u, w_2 = \delta v$), it is easy to see that:

$$P = -\gamma \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

γ is called the *penalty parameter*. Note here that in the penalty approach, we are solving for (u, v) only and the pressure is obtained after the solution is done. in other words, we are treating pressure as a parameter that enforces the incompressibilty condition.

For the steady state case, the FE equations can easily be shown to be

$$[\boldsymbol{K}_v + \boldsymbol{K}_p]\,\boldsymbol{\Delta} = \boldsymbol{F},$$

where,

$$\begin{aligned}
\boldsymbol{K}_v &= \int_{V_e} \boldsymbol{B}_v^T \boldsymbol{C} \boldsymbol{B}_v dV \\
\boldsymbol{K}_p &= \int_{V_e} \gamma \boldsymbol{B}_p^T \boldsymbol{B}_p dV \\
\boldsymbol{F} &= \int_{\partial V_e} \boldsymbol{N}^T \bar{\boldsymbol{t}} dS.
\end{aligned}$$

Also $\boldsymbol{\Delta}^T = \langle u_1\ v_1\ u_2\ v_2 \ldots \rangle$ is the vector of nodal velocities. Also,

$$\boldsymbol{N} = \begin{pmatrix} N_1 & 0 & \ldots \\ 0 & N_1 & \ldots \end{pmatrix}, \boldsymbol{B}_v = \begin{pmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \ldots \\ 0 & \dfrac{\partial N_1}{\partial y} & \ldots \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \ldots \end{pmatrix},$$

$$\boldsymbol{B}_p = \begin{pmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_1}{\partial y} & \ldots \end{pmatrix}$$

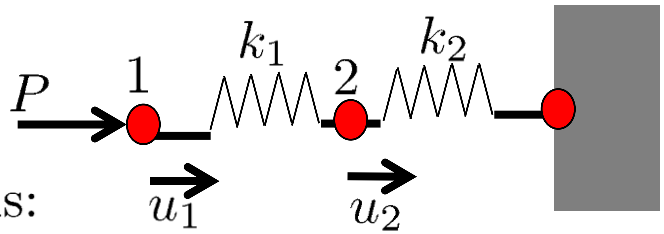
The solution to the FE equations depend on the choice of the penalty parameter γ. The larger the value of γ, higher is the penalty arising out of violation of the incompressibility constraint. But, if γ is so large that the penalty stiffness overwhelms the viscous stiffness, and $\boldsymbol{K}_p$ is non-singular, the solution tends to $\boldsymbol{\Delta} = 0$. Thus, for large γ, continuity will be exactly satisfied but momentum will be violated. The element is said to have *locked*.

It has been observed that under-integrating the penalty stiffness yields acceptable solutions to $\boldsymbol{\Delta}$. Thus, when 4 noded quadrilaterals are used, $\boldsymbol{K}_v$ is obtained by 2×2 quadrature but 1×1 is used for $\boldsymbol{K}_p$. This renders $\boldsymbol{K}_p$ singular but the overall stiffness is still invertible. This is known as *reduced integration* and prevents locking.

Element quality and related issues

How do we know that a FE analysis is giving results that are trustworthy? Let us look at some common situations that lead to errors in FE simulations.

- **Ill conditioned global stiffness matrix:** If the structure being analysed has a part that is far too flexible compared to others, the large rigid body displacements of the flexible part may overwhelm the accurate computation of displacements in the other members.



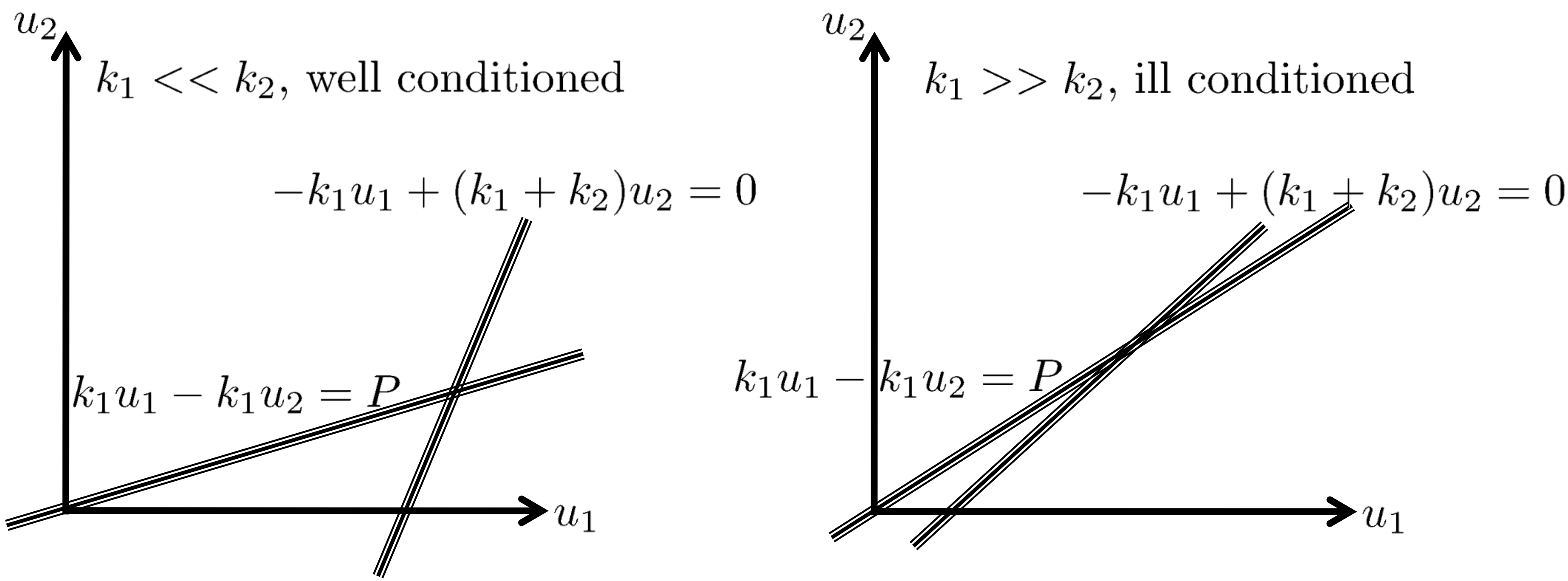
The assembled equation for the system shown is:

$$\begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left\{ \begin{array}{c} P \\ 0 \end{array} \right\}.$$

Consider a case where $k_1 >> k_2$. During Gauss elimination, the first equation will be multiplied by 1 and added to the second to yield,

$$[(k_1 + k_2) - k_1]\, u_2 = P,$$

i.e. $k_2 u_2 = P$, from which u_2 can be determined. Suppose that $k_1 = 1.000000$ and $k_2 = 4.444444 \times 10^{-6}$ and the computer you are using can store 7 digits after decimal. Then, the above equation yields $1.0000004 - 1.000000 = 4 \times 10^{-6}$, i.e. only a single significant digit will remain. If the computer stores 6 digits after decimal, the answer will be 0. This situation arises because the global stiffness is *ill conditioned.*



The *condition number* of the global stiffness matrix

$$C(\boldsymbol{K}) = \frac{\lambda_{max}}{\lambda_{min}},$$

gives an estimate of the number of digits that may be lost in solving $\boldsymbol{KU} = \boldsymbol{F}$, i.e. $C(\boldsymbol{K})$ provides a means of quantifying the *truncation error*. The idea is that $\boldsymbol{K}^{-1}$ is dominated by λ_{min} while $\boldsymbol{K}$ is dominated by λ_{max}.

Let d be the number of digits after decimal used to represent each number. The modified coefficients are then accurate to d_{acc} digits where

$$d_{acc} = d - d_{loss},$$

where,

$$d_{loss} \leq \log_{10} C(\boldsymbol{K}).$$

eg. id $d = 14$ and $C(\boldsymbol{K}) = 10^8$, then $d_{acc} \geq 6$, i.e a maximum of 6 digits may be lost from the displacements in the solution. Note that this is an upper bound estimate only. In real cases, the loss may be less.

Fried (AIAA Journal v28, 1990, 1322-1324) suggests a method for estimating the condition number as

$$c(\boldsymbol{K}) = b\left(\frac{h_{max}}{h_{min}}\right)^{2m-1} N_{els}^{2m/n}.$$

Here, $2m$ is the order of the pde being solved, n is the dimension of the system, h_{max}, h_{min} are the maximum and minimum node spacings in the mesh, N_{els} is the number of elements and b is a positive constant.

eg. for a beam problem, the highest derivative of the transverse displacement that enters the governing deq is 4, so that $2m/n = 4$. For an axially loaded bar it is 2, in plane stress ot strain it is $2/2 = 1$. So, in a beam, if the number of elements is doubled, $C(\boldsymbol{K})$ increases 16 times, if the elements are graded such that h_{max}/h_{min} is 10, condition number increases by 1000. If both are done, it will increase by a factor of 16000.

The quality of the FE system of equations can also be assessed by using the *residual*, i.e.

$$\Delta \boldsymbol{F} = \boldsymbol{F} - \boldsymbol{K}\boldsymbol{U},$$

after the displacements have been solved for. The ratio of the work done by the residuals to that by the actual applied forces is a measure of the error, i.e.

$$e = \frac{\boldsymbol{U}^T \boldsymbol{\Delta} \boldsymbol{F}}{\boldsymbol{U}^T \boldsymbol{F}}.$$

Note that while a large residual is an indication tat something is not right, a small residual may result even in ill-conditioned systems.

- **Discretisation error and rate of convergence** Consider the 1-d bar element govened by

$$u_{,xx} + \frac{q}{AE} = 0,$$

discretised by 2 noded linear elements. In any element bounded by nodes i and $i+1$, the error may be written as

$$e(x) = u(x) - \left[u_i\left(1 - \frac{x - x_i}{h_i}\right) + u_{i+1}\left(\frac{x - x_i}{h_i}\right)\right],$$

where, $h_i = x_{i+1} - x_i$ is the element size. We will assume that the values are exact at the nodes, i.e. $e(x_i) = e(x_{i+1}) = 0$. Using Rolle's theorem (which states that there must be a point z within the nodes where $e'(z) = 0$), it is easy to prove that

$$|e'(x)| \leq h_i \left(\mathrm{max}_{\mathrm{x_i} \leq \mathrm{x} \leq \mathrm{x_{i+1}}} |\mathrm{u}''(\mathrm{x})|\right).$$

To find the error in displacements, expand $e(x_i)$ about z,

$$e(x_i) = e(z) + (x_i - z)e'(z) + \frac{1}{2}(x_i - z)^2 e''(z).$$

So, given that $|z - x_i| \leq h_i/2$,

$$e(x) \leq \frac{1}{8}h_i^2 \left(\max_{x_i \leq x \leq x_{i+1}} |u''(x)|\right).$$

From this simple example we see that strain error is propertional to element size while displacement error is proprtional to h_i^2. Displacements are more accurate near nodes while strains are accurate in the interior of the element.

In general, if the shape functions contain complete polynomials of order p, $2m$ is the highest degree of the derivative of the field variable and h is a measure of element size, discretisation errors are

1. $O(h^{p+1})$ in representation of the field quantity
2. $O(h^{p+1-r})$ in representing the r-th derivative of the field quantity

eg. for 3 noded triangles and 4 noded quads, $p = 1$. So, in a plane stress/strain problem, if we change from 4-noded to 8-noded quads, the discretisation error changes from $O(h^2)$ to $O(h^3)$. For a 4-noded element, if mesh is halved, error reduces by 4 times.