ASSIGNMENT I MSO 202 A

COMPLEX NUMBERS, HOLOMORPHICITY, AND C-R EQUATIONS

Exercise 0.1 : Verify the following for all complex numbers z and w:

(1) $|z+w| \leq |z| + |w|$.
(2) $||z| - |w|| \leq |z+w|$.

Exercise 0.2 : Let $z, w \in \mathbb{C}$ belong to the upper half plane. Show that the distance between z and w is at most the distance between z and $\overline{w}$.

Recall the De Moivers formula: If $z = r(\cos(\theta) + i\sin(\theta))$ then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Exercise 0.3 : Find all complex numbers z such that $z^3 + 1 = 0$.

A map f from $\mathbb{C}$ is $\underline{\mathbb{R}\text{-linear}}$ if $f(z+w) = f(z)+f(w)$ and $f(a\,z) = a\,f(z)$ for all $z, w \in \mathbb{C}$ and $a \in \mathbb{R}$. A map f from $\mathbb{C}$ is $\underline{\mathbb{C}\text{-linear}}$ if $f(z+w) = f(z) + f(w)$ and $f(a\,z) = a\,f(z)$ for all $z, w \in \mathbb{C}$ and $a \in \mathbb{C}$.

Exercise 0.4 : For given scalars $a, b \in \mathbb{C}$, show that $f(z) = az + b\overline{z}$ is always $\mathbb{R}$-linear, where $\overline{z} = x - iy$ for $z = x + iy$. Verify further that f is $\mathbb{C}$-linear if and only if $b = 0$.

Exercise 0.5 : Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that

$$f(a\,z) = a\,f(z) \text{ for all } z, w, a \in \mathbb{C}. \tag{0.1}$$

Show that there exists $\alpha \in \mathbb{C}$ such that $f(z) = \alpha\, z$ for all $z \in \mathbb{C}$.

Exercise 0.6 : Show that a holomorphic function $f = u + iv : \mathbb{C} \to \mathbb{C}$ is constant if $\overline{f} = u - iv$ is holomorphic.

Exercise 0.7 : Show that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant if the range of f is contained in a circle.

Exercise 0.8 : Show that a a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant if the range of f is contained in a parabola.