ASSIGNMENT VI MSO 202 A

ROUCHÉ'S THEOREM AND MÖBIUS TRANSFORMATIONS

Exercises 0.2-0.4 relies on the following theorem:

Theorem 0.1 (Rouché's Theorem)**.** *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside the circle C.*

Exercise 0.2 : Let a, b, c be positive real numbers such that $a + c < b$. Consider the polynomial $p(z) = az^7 + bz^3 + c$. Then $p(z)$ has exactly 3 zeros in the open unit disc $\mathbb{D}$.

Exercise 0.3 : Show that the functional equation $\lambda = z + e^{-z}$ $(\lambda > 1)$ has exactly one (real) solution in the right half plane.

Exercise 0.4 : Find the number of zeros of $3e^z - z$ in the closed unit disc centered at the origin.

Exercise 0.5 : If γ is a line and $f(z) = \frac{1}{z}$ then $f(\gamma)$ is a circle or line.

Exercise 0.6 : Show that for any $z_1, z_2 \in \mathbb{D}$, there exists a bijective holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_1) = z_2$.

Exercise 0.7 : Consider the Möbius transformation $f(z) = \frac{az+b}{cz+d}$ for real numbers a, b, c, d. Show that if $ad - bc > 0$ then f maps the open upper half plane $\mathbb{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ onto itself.

Remark 0.8 : It turns out that any bijective, holomorphic function which maps $\mathbb{U}$ onto itself is given by $f(z) = \frac{az+b}{cz+d}$ for real numbers a, b, c, d such that $ad - bc > 0$.