1 a) $x^2 u_{xx} - 2xy\,u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0$.

Comparing with

$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$

we have,

$a = x^2,\ b = xy$ & $c = y^2$

$\therefore b^2 - ac = (xy)^2 - y^2 \cdot x^2 = 0$

$\therefore$ the Eqn is parabolic

b) $x^2 u_{xx} - 2xy\,u_{xy} + y^2 u_{yy} + x u_x + y u_y = u$.

Same as ⓐ since the nature of the PDE is determined by the sign of the discriminant and doesn't depend upon the lower-order term

$\therefore$ Parabolic Eqn.

c) $u_{xx} + x^2 u_{yy} = 0$

$b^2 - ac = 0 - 1 \cdot x^2 = -x^2 < 0 \ \forall x \in \mathbb{R}$

Hence the equation is elliptic.

d) $u_{tt} + x^2 u_{xt} = \sin(x)$.

$b^2 - ac = \left(\frac{x^2}{2}\right)^2 - 1 \cdot 0 = \frac{x^4}{4} > 0$

$\therefore$ The Eqn is Hyperbolic.

(2) Reduce $x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = 0$. (2)

Computing $\Delta = b^2 - ac = (-xy)^2 - x^2 \cdot y^2 = 0$

we deduce the eqn is parabolic $\Rightarrow$ the canonical form should be $\omega_{\eta\eta} + l(\omega) = g$.

The Eqn for the Characteristics is $\frac{dy}{dx} = -\frac{y}{x}$

[$\because$ We need to find a function η that is a soln of

$$a\eta_x^2 + 2b\eta_x \eta_y + c\eta_y^2 = \frac{1}{a}(a\eta_x + b\eta_y)^2 = 0.$$

or, η is a soln of the 1st order linear Eqn $a\eta_x + b\eta_y = 0$

ie η is constant along each Characteristics $\frac{dy}{dx} = \frac{b}{a}$].

$\therefore$ The Solution is $xy =$ constant $\therefore$ Choose $\theta(x,y) = xy$.

Define $\eta(x,y) = x$.

The choice is justified since $\begin{vmatrix} \theta_x & \theta_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x \neq 0$.

Let $v(\theta,\eta) = u(x,y)$.

Substituting the new coordinate we have,

$\cancel{x^2(y^2 v_{\eta\eta} + 2y v_{\theta\eta} + v_{\theta\theta}) - 2xy(v_\eta + xy v_{\eta\eta} + x v_{\xi\eta})} = 0$

$\cancel{\Rightarrow x^2 v_{\eta\eta}}$

$\Rightarrow x^2[y^2 v_{\theta\theta} + 2y v_{\theta\eta} + v_{\eta\eta}] - 2xy[v_\theta + xy v_{\theta\theta} + x v_{\theta\eta}] + y^2 \cdot x^2 v_{\theta\theta} = 0$.

$u_x = v_\theta \theta_x + v_\eta \eta_x$
$= y v_\theta + v_\eta$
$u_y = v_\theta \theta_y + v_\eta \eta_y$
$= x v_\theta + 0 = x v_\theta$.
$u_{xx} = y^2 v_{\theta\theta} + y v_{\theta\eta} + y v_{\theta\eta} + v_{\eta\eta}$
$u_{yy} = x[v_{\theta\eta} \eta_y + v_{\theta\theta} \theta_y]$
$= x^2 v_{\theta\theta}$.

$\Rightarrow$

$u_x = v_\theta \theta_x + v_\eta \eta_x = y v_\theta + v_\eta$ | $\theta_x = y$, $\theta_y = x$, $\eta_x = 1$, $\eta_y = 0$ (3)

$\Rightarrow u_{xx} = y[v_{\theta\theta}\theta_x + v_{\theta\eta}\eta_x] + [v_{\eta\theta}\theta_x + v_{\eta\eta}\eta_x]$

$= y^2 v_{\theta\theta} + 2y v_{\theta\eta} + v_{\eta\eta}$

$u_y = v_\theta \theta_y + v_\eta \eta_y = x v_\theta + 0 \cdot v_\eta = x v_\theta$.

$\Rightarrow u_{yy} = x[v_{\theta\theta}\theta_y + v_{\theta\eta}\eta_y]$ | $u_{yx} = x[v_{\theta\theta}\theta_x + v_{\theta\eta}\eta_x]$

$= x^2 v_{\theta\theta}$ | $= xy v_{\theta\theta} + x v_{\theta\eta}$

Now, since u satisfies the eqn

$x^2[y^2 v_{\theta\theta} + 2y v_{\theta\eta} + v_{\eta\eta}] - 2xy[xy v_{\theta\theta} + x v_{\theta\eta}] + y^2 x^2 v_{\theta\theta} = 0$

$\Rightarrow x^2 v_{\eta\eta} = 0$

$\Rightarrow v_{\eta\eta} = 0$ $[\because x \neq 0]$ is the required Canonical form.

(3) Find the solution of $u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0$

satisfying $u(0,y) = f(y)$ and $u_x(0,y) = g(y)$.

Soln:- $\Delta = b^2 - ac = (-\sin x)^2 - 1\cdot(-\cos^2 x) = 1 > 0$ - Hyperbolic.

the characteristic eqns are

$$\frac{dy_\pm}{dx} = \frac{b \pm \sqrt{b^2-ac}}{a} = \frac{-\sin x \pm 1}{1} = -\sin x \pm 1.$$

and their solutions are $y_\pm = \cos x \pm x + \text{constant}$

$\therefore$ the new independent variables are

$\theta(x,y) = y - \cos x + x$ and $\eta(x,y) = y - \cos x - x$.

Define $v(\theta,\eta) = u(x,y)$.

$\theta_x = 1+\sin x$

$\theta_y = 1$

$\eta_x = \sin x - 1$

$\eta_y = 1$

$\therefore u_x = v_\theta \theta_x + v_\eta \eta_x = v_\theta(1+\sin x) + v_\eta(\sin x - 1)$

$u_{xx} = \cos x \cdot v_\theta + (1+\sin x)[v_{\theta\theta}\theta_x + v_{\theta\eta}\eta_x]$

$+ \cos x \cdot v_\eta + (\sin x - 1)[v_{\eta\theta}\theta_x + v_{\eta\eta}\eta_x]$

$= \cos x\, v_\theta + (1+\sin x)^2 v_{\theta\theta} + (1+\sin^2 x) v_{\theta\eta} + \cos x \cdot v_\eta$

$+ (\sin^2 x - 1) v_{\eta\theta} + (\sin x - 1)^2 v_{\eta\eta}$

and $u_{xy} = (1+\sin x)[v_{\theta\theta}\theta_y + v_{\theta\eta}\eta_y] + (\sin x - 1)[v_{\theta\eta}\theta_y + v_{\eta\eta}\eta_y]$

$= (1+\sin x)v_{\theta\theta} + (1+\sin x)v_{\theta\eta} + (\sin x - 1)v_{\theta\eta} + (\sin x - 1)v_{\eta\eta}$.

$u_y = v_\theta\theta_y + v_\eta \eta_y$

$= v_\theta + v_\eta$

$u_{yy} = (v_{\theta\theta}\theta_y + v_{\theta\eta}\eta_y) + (v_{\eta\theta}\theta_y + v_{\eta\eta}\eta_y)$

$= v_{\theta\theta} + 2v_{\theta\eta} + v_{\eta\eta}$.

$\therefore$ u satisfies the equation we have,

$[(1+\sin x)^2 - 2\sin x(1+\sin x) - \cos^2 x]\, v_{\theta\theta} + [(\sin^2 x - 1) - 2\sin x(\sin x + 1)$

$- 2\cos^2 x - 2\sin x(\sin x - 1)]\, v_{\theta\eta} + [(\sin x - 1)^2 - 2\sin x(\sin x - 1) - \cos^2 x]\, v_{\eta\eta}$

$+ (\cos x - \cos x)v_\theta + (\cos x - \cos x)v_\eta = 0$.

$\Rightarrow v_{\theta\eta} = 0 \Rightarrow v(\theta,\eta) = \tilde{f}(\theta) + \tilde{g}(\eta)$ for an arbitrary $\tilde{f}$ & $\tilde{g} \in C^2$

$\therefore u(x,y) = \tilde{f}(x+y-\cos x) + \tilde{g}(y-x-\cos x)$

$\therefore u(x,y) = F(y+x-\cos x) + G(y-x-\cos x)$

$\Rightarrow f(y) = u(0,y) = F(y-1) + G(y-1)$ —(i)

$u_x(x,y) = 1\cdot F'(y+x-\cos x) - G'(y-x-\cos x)$

$\Rightarrow g(y) = u_x(0,y) = F'(y-1) - G'(y-1)$ —(ii)

Integrating (ii) we have,

$$\int_0^y g(s)\,ds = F(y-1) - F(-1) - G(y-1) + G(-1) \quad \text{—(iii)}$$

Adding (i) & (iii) we get,

$$2F(y-1) - F(-1) - G(-1) = f(y) + \int_0^y g(s)\,ds.$$

i.e. $F(y-1) = \frac{1}{2}\left[F(-1) + G(-1) + f(y) + \int_0^y g(s)\,ds\right]$

or, $F(x) = \frac{1}{2}\left[F(-1) + G(-1) + f(x+1) + \int_0^{x+1} g(s)\,ds\right]$. —(iv)

From (i), $G(y-1) = f(y) - \frac{1}{2}\left[F(-1) + G(-1) + f(y) + \int_0^y g(s)\,ds\right]$

$= -\frac{1}{2}\left[F(-1) + G(-1) - f(y) + \int_0^y g(s)\,ds\right]$

$\therefore G(x) = \frac{1}{2}\left[f(x+1) - F(-1) - G(-1) - \int_0^y g(s)\,ds\right]$. —(v)

$\therefore u(x,y) = F(y+x-\cos x) + G(y-x-\cos x)$

where F and G are given by (iv) & (v).

4. Reduce the eqn $u_{xx} + (1+y^2)^2 u_{yy} - 2y(1+y^2)u_y = 0$.

$\Delta = b^2 - ac = 0 - (1+y^2)^2 < 0$

– Elliptic.

$\therefore$ The Characteristics are $\frac{dy}{dx} = \pm \frac{i\sqrt{ac-b^2}}{a} = \pm i(1+y^2)$.

$\therefore \tan^{-1} y = \pm ix + c$ are the solutions.

$\therefore$ Choose, $\theta(x,y) = x$ & $\eta(x,y) = \tan^{-1} y$.

Define, $v(\theta,\eta) = u(x,y)$

$\therefore u_x = v_\theta \theta_x + v_\eta \eta_x = v_\theta$

and, $u_{xx} = v_{\theta\theta}\theta_x + v_{\theta\eta}\eta_x$

$= v_{\theta\theta}$

and, $u_{xy} = v_{\theta\theta}\theta_y + v_{\theta\eta}\eta_y$

$= v_{\theta\eta} \cdot \frac{1}{1+y^2}$.

$u_y = v_\theta \theta_y + v_\eta \eta_y$

$= v_\eta \cdot \frac{1}{1+y^2}$.

$u_{yy} = -\frac{2y v_\eta}{(1+y^2)^2} + \frac{1}{1+y^2}\left[v_{\eta\eta}\eta_y + v_{\eta\theta}\theta_y\right]$

$= -\frac{2y v_\eta}{(1+y^2)^2} + \frac{v_{\eta\eta}}{(1+y^2)^2}$

$\because u$ is the soln of the eqn we have,

$$v_{\theta\theta} + \cancel{(1+y^2)^2} \cdot \frac{-2y v_\eta + v_{\eta\eta}}{\cancel{(1+y^2)^2}} - 2y\cancel{(1+y^2)} \cdot \frac{v_\eta}{\cancel{1+y^2}} = 0.$$

$\Rightarrow v_{\theta\theta} + v_{\eta\eta} = 0$. which is the required canonical form.

(5) Prove that $u_{yy} - 4u_{xx} = 0$ has infinitely many solutions.

$\Delta = b^2 - ac = 0 + 4 > 0$ - Hyperbolic Eqn

The characteristics are given by $\frac{dy}{dx} = \pm\frac{2}{4} = \pm 1/2$

and the solutions are

$y = \pm 1/2\, x + c$.

Choose $\theta(x,y) = y + x/2$ and $\eta(x,y) = y - \frac{x}{2}$.

Define, $v(\theta, \eta) = u(x,y)$

$\therefore u_x = v_\theta \theta_x + v_\eta \eta_x = 1/2\, v_\theta + v_\eta \cdot (-\frac{1}{2}) = 1/2\, v_\theta - 1/2\, v_\eta$.

$u_y = v_\theta \theta_y + v_\eta \eta_y = v_\theta + v_\eta$.

$u_{xx} = 1/2\,[v_{\theta\theta}\theta_x + v_{\theta\eta}\eta_x] - 1/2\,[v_{\eta\theta}\theta_x + v_{\eta\eta}\eta_x]$

$= 1/2 \cdot 2\,[v_{\theta\theta} - v_{\theta\eta}] - 1/2 \cdot 2\,[v_{\eta\theta} - v_{\eta\eta}]$

$= 1/4\,(v_{\theta\theta} - 2v_{\theta\eta} + v_{\eta\eta})$

$u_{yy} = [v_{\theta\theta}\theta_y + v_{\theta\eta}\eta_y] + [v_{\eta\theta}\theta_y + v_{\eta\eta}\eta_y]$

$= v_{\theta\theta} + 2v_{\theta\eta} + v_{\eta\eta}$.

$\because$ u satisfies the eqn we have,

$v_{\theta\theta} + 2v_{\theta\eta} + v_{\eta\eta} - v_{\theta\theta} + 2v_{\theta\eta} - v_{\eta\eta} = 0$

or, $v_{\theta\eta} = 0 \Rightarrow v(\theta, \eta) = F(\theta) + G(\eta)$ for some $F, G \in C^2(\mathbb{R})$

$\therefore u(x,y) = F(y + \frac{x}{2}) + G(y - x/2)$.

$\because$ F and G are arbitrary C^2-fon the eqn admits infinitely many solutions.