ASSIGNMENT V MSO 202 A

RESIDUE FORMULA, REMOVABLE SINGULARITIES, LAURENT SERIES

Exercise 0.1 : Show that $z = \pi/2$ is a simple pole of $\frac{\cos(z)}{(z-\pi/2)^2}$.

Solution. Consider the power series of $f(z) = \cos(z)$ around $z = \pi/2$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(z-\pi/2)^n = (-1)(z-\pi/2) + \sum_{n=3}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(z-\pi/2)^n.$$

Thus $\frac{f(z)}{(z-\pi/2)^2} = -\frac{1}{z-\pi/2} + \sum_{n=3}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(z-\pi/2)^{n-2}$, where

$$\sum_{n=3}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(z-\pi/2)^{n-2}$$

is a holomorphic function in a neighbourhood of $\pi/2$. Hence $\pi/2$ is a simple pole of $\frac{\cos(z)}{(z-\pi/2)^2}$.

Exercise 0.2 : Locate poles a of f and find residue $\mathrm{res}_a f$ of $f(z) = \frac{1}{1+z^4}$ at a.

Solution.

(1) For this, let us first solve $1 + z^4 = 0$. Taking modulus on both sides of $z^4 = -1$, we obtain $|z| = 1$. Thus $z = e^{i\theta}$, and hence $e^{4i\theta} = e^{i\pi}$. This forces $4\theta = \pi + 2\pi k$ for integer k. Thus $e^{i\theta} = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$.
(2) Note that $\frac{1}{f(z)} = (z - e^{i\frac{\pi}{4}})^{-1}h(z)$, where $h(z) = (z - e^{i\frac{3\pi}{4}})(z - e^{i\frac{5\pi}{4}})(z - e^{i\frac{7\pi}{4}})$ is non-zero in some neighborhood of $e^{i\frac{\pi}{4}}$. Thus $z = e^{i\frac{\pi}{4}}$ is a simple pole.
(3) Similar argument shows that $e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$ are simple poles of f.
(4) Recall that if a is a simple pole of f, then

$$\mathrm{res}_a f = \lim_{z \to a}(z-a)f(z). \tag{0.1}$$

Note that

$$\begin{aligned}
\operatorname{res}_{e^{i\frac{\pi}{4}}} f &= \lim_{z \to e^{i\frac{\pi}{4}}} (z - e^{i\frac{\pi}{4}}) f(z) = \lim_{z \to e^{i\frac{\pi}{4}}} \frac{1}{(z - e^{i\frac{3\pi}{4}})(z - e^{i\frac{5\pi}{4}})(z - e^{i\frac{7\pi}{4}})} \\
&= \frac{1}{(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}})(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}})(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}})} \\
&= \frac{e^{-i\frac{3\pi}{4}}}{4}.
\end{aligned}$$

Note further that

$$\begin{aligned}
\operatorname{res}_{e^{i\frac{3\pi}{4}}} f &= \lim_{z \to e^{i\frac{3\pi}{4}}} (z - e^{i\frac{3\pi}{4}}) f(z) = \lim_{z \to e^{i\frac{3\pi}{4}}} \frac{1}{(z - e^{i\frac{\pi}{4}})(z - e^{i\frac{5\pi}{4}})(z - e^{i\frac{7\pi}{4}})} \\
&= \frac{1}{(e^{i\frac{3\pi}{4}} - e^{i\frac{\pi}{4}})(e^{i\frac{3\pi}{4}} - e^{i\frac{5\pi}{4}})(e^{i\frac{3\pi}{4}} - e^{i\frac{7\pi}{4}})} \\
&= \frac{e^{-i\frac{\pi}{4}}}{4}.
\end{aligned}$$

Similarly, one can compute the residue at other poles.

Exercise 0.3 : The aim of this exercise is to prove that $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin \pi a}$ $(0 < a < 1)$ as an application of Residue Formula. For $0 < a < 1$, consider the function $f(z) = \frac{e^{az}}{1+e^z}$ and let γ_R denote the rectangular curve with parametrization

$$\gamma_1(t) = t \text{ for } -R \leqslant t \leqslant R, \quad \gamma_2(t) = R + it \text{ for } 0 \leqslant t \leqslant 2\pi,$$

$$\gamma_3(t) = -t + 2\pi i \text{ for } -R \leqslant t \leqslant R, \quad \gamma_4(t) = -R - it \text{ for } -2\pi \leqslant t \leqslant 0.$$

Verify the following:

(1) The only simple pole of f inside γ_R is at $a = \pi i$.
(2) The residue $\operatorname{res}_{\pi i} f$ of f at πi is equal to $-e^{a\pi i}$.
(3) $\sum_{j=1}^{4} \int_{\gamma_j} f(z) dz = -2\pi i e^{a\pi i}$.
(4) $\int_{\gamma_1} f(z) dz \to \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ as $R \to \infty$.
(5) $\int_{\gamma_3} f(z) dz \to -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ as $R \to \infty$.
(6) $\int_{\gamma_2} f(z) dz \to 0$ as $R \to \infty$ and $\int_{\gamma_4} f(z) dz \to 0$ as $R \to \infty$.

Solution.

(1) Note that $1 + e^z = 0$ if and only if $z = \pm \pi i$. Also, $\frac{d}{dz}(1 + e^z)|_{z=\pm\pi i} \neq 0$. Hence the only simple pole of $f(z)$ inside γ_R is at $a = \pi i$.

(2) By (0.1), $\text{res}_{\pi i} f = \lim_{z \to \pi i}(z - \pi i)\frac{e^{az}}{1+e^z}$. Note that $1 + e^z = e^{\pi i}(z - \pi i) + \frac{e^{\pi i}}{2}(z - \pi i)^2 + \cdots$. It follows that

$$\text{res}_{\pi i} f = -e^{a\pi i}.$$

(3) By Residue Formula, $\sum_{j=1}^4 \int_{\gamma_j} f(z)dz = -2\pi i e^{a\pi i}$.

(4) Note that $\int_{\gamma_1} f(z)dz = \int_{-R}^R \frac{e^{ax}}{1+e^x}dx \to \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx$ as $R \to \infty$.

(5) $\int_{\gamma_3} f(z)dz = \int_{-R}^R -\frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}dx \to -e^{2\pi i a}\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx$ as $R \to \infty$.

(6) Note that

$$|f(\gamma_2(t))| = |\frac{e^{a\gamma_2(t)}}{1+e^{\gamma_2(t)}}| \leq \frac{e^{aR}}{|1+e^{R+it}|} \leq \frac{e^{aR}}{e^R - 1} = \frac{1}{e^{R(1-a)} - e^{-aR}} \to 0$$

as $R \to \infty$ (since $0 < a < 1$). It follows that $|\int_{\gamma_2} f(z)dz| \to 0$ as $R \to \infty$. Similarly, one can check that $\int_{\gamma_4} f(z)dz \to 0$ as $R \to \infty$.

Exercise 0.4 : The aim of this exercise is to evaluate $\int_{-\infty}^\infty \frac{1}{1+x^4}dx$ as an application of Residue Formula. Let $f(z) = \frac{1}{1+z^4}$. Let γ_R be union of $[-R, R]$ and semicircle C_R:

$$z_1(t) = t \ (-R \leqslant t \leqslant R), \ z_2(t) = Re^{it} \ (0 \leqslant t \leqslant \pi).$$

(1) $e^{i\pi/4}, e^{3i\pi/4}$ are the only poles inside γ_R if $R > 1$.

(2) Compute $\text{res}_{e^{i\pi/4}} f$ and $\text{res}_{e^{i3\pi/4}} f$.

(3) $\int_{-R}^R \frac{1}{1+x^4}dx + \int_{C_R} f(z)dz = 2\pi i(\text{res}_{e^{i\pi/4}} f + \text{res}_{e^{i3\pi/4}} f)$.

(4) $\lim_{R\to\infty} \int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

Solution. We have already seen (1) and (2) in Exercise 0.2. By Residue Theorem, $\int_{-R}^R \frac{1}{1+x^4}dx + \int_{C_R} f(z)dz = 2\pi i(\text{res}_{e^{i\pi/4}} f + \text{res}_{e^{i3\pi/4}} f)$. Letting $R \to \infty$, we obtain

$$\int_{-\infty}^\infty \frac{1}{1+x^4}dx + \lim_{R\to\infty}\int_{C_R} f(z)dz = \pi i \frac{e^{-i\pi/4} + e^{-i3\pi/4}}{2}.$$

We claim that $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$. To see that,

$$\Big|\int_{C_R} f(z)dz\Big| \leqslant \int_0^\pi \Big|\frac{1}{1+R^4e^{4it}}\Big|Rdt \leqslant \int_0^\pi \Big|\frac{1}{R^4-1}\Big|Rdt = \pi\frac{R}{R^4-1} \to 0$$

as $R \to \infty$.

Exercise 0.5 : Show that any polynomial $p(z)$, the function $p(z)\sin(1/z)$ has essential singularity at 0.

Solution.

(1) It is clear from the Laurent series expansion of $\sin(1/z)$ that $\sin(1/z)$ has essential singularity at 0. This means that for infinitely many negative integers n,

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{\sin(1/z)}{z^{n+1}}dz \neq 0, \tag{0.2}$$

(2) Let $p(z) = b_0 + b_1 z + \cdots + b_k z^k$. The Laurent series expansion of $f(z) = p(z)\sin(1/z)$ is given by

$$\sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}dz$ for integers n and $0 < r < \infty$. Since

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\sum_{m=0}^{k} b_m \int_{|z|=r} \frac{\sin(1/z)}{z^{n+1-m}}dz,$$

by (0.2), $a_n \neq 0$ infinitely many negative integers n.

Exercise 0.6 : Write Laurent series of f around a and determine the type of the singularity at a (removable/pole/essential):

(1) $\frac{e^z}{(z-1)^3}$, $a = 1$;
(2) $(z-1)\cos(1/(z+2))$, $a = -2$;
(3) $(z - \sin z)/z^3$, $a = 0$.

Solution. Recall that the Laurent series expansion of $f(z)$ around z_0 is given by

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

where $a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}dz$ for integers n and $0 < r < \infty$.

(1) Note that $\frac{e^z}{(z-1)^3} = e\frac{e^{z-1}}{(z-1)^3} = e\sum_{n=0}^{\infty}\frac{(z-1)^{n-3}}{n!}$. Hence $a = 1$ is a pole of multiplicity 3.
(2) Note that $(z-1)\cos(1/(z+2)) = (z+2)\cos(1/(z+2)) - 3\cos(1/(z+2))$, and hence $a = -2$ is an essential singularity.
(3) Note that $z - \sin(z) = \frac{z^3}{3!} +$ terms of higher order. Hence $a = 0$ is a removable singularity.