Forward and Futures

- A futures contract is a legal agreement between a buyer and a seller in which:
  - The buyer agrees to take delivery of something at a specified price at the end of a designated period of time.
  - The seller agrees to make delivery of something at a specified price at the end of a designated period of time.

- Elements of a futures contract:
  - The price at which the parties agree to transact in the future is called the *futures price*.
  - The designated date at which the parities must transact is called *settlement date* or *delivery date*.
  - The "*something*" that the parties agree to exchange is called the *underlying*.

- Suppose a futures contract is traded on an exchange where the underlying to be bought or sold is asset XYZ, and the settlement is three months from now.

- Assume further that the investor A buys this futures contract, and investor B sells this futures contract and the price at which they agree to transact in the future is \$100.

- \$100 is the futures price.

- At the settlement date, B will deliver asset XYZ to A.
- Investor A will give $100 to B, the futures price.
- *Long position/Long futures*: When an investor takes a position in the market by buying a futures contract.

- *Short position/short futures*: When investor’s opening position is the sale of a futures contract.

- The buyer of a futures contract will realize a profit if the futures price increases; and seller of a futures contract will realize a profit if the futures price decreases.
- For example, suppose that one month after investor A and B take their positions in the futures contract, and the futures price of asset XYZ increases to $120.
- Investor A, the buyer of futures contract, could then sell the futures contract and realize the profit of $20.
- Effectively, at the settlement date, he has agreed to buy asset XYZ for $100 and has agreed to sell asset XYZ for $120.

- Investor B, the seller of futures contract will realize the loss of $20.
- If the futures price falls to $40 and investor B buys back the contract at $40, B realizes the profit of $60.
- Because B agreed to sell asset XYZ for $100 and now can buy it for $40.
- A would realize the loss of $60.

- It means that if the futures price decreases, the buyer of the futures contract realizes a loss while the seller of the futures contract realizes a profit.

- Liquidating the position
  - Futures contracts usually have settlement dates in the months of March, June, September or December.
  - Contracts are settled at a predetermined date during settlement month by the exchange.
  - On the specified date, investors have to settle the contract.
  - E.g. A agreeing to buy asset XYZ at $100, B agreeing to sell asset XYZ at $100.

- The exchange will determine a settlement price for the futures contract for that specified date.
- For example, if the exchange determines a settlement price of $130, then A has agreed to buy asset XYZ for $100 but can settle the position for $130, thereby realizing a profit of $30.
- B should realize the loss of $30.

- The contract with the closest settlement date is called the nearby futures contract.
- The next futures contract is the one that settles just after the nearby contract.
- The contract farthest away in time from settlement is called the most distant futures contract.

- An investor has two choices regarding the liquidation of the position.
  - First, the position can be liquidated prior to the settlement date
  - Then the investor has take offsetting position in the same contract.

  - For the buyers of a futures contract, this means selling the same number of identical futures contracts.
  - For the seller of a futures contract, this means buying the same number of identical futures contracts.
  - An identical contract means the contract for the same underlying and the same settlement date.
  - The alternative is to wait until the settlement date. At the time, the party purchasing a futures contract accepts delivery of the underlying.
  - The party that sells a futures contract liquidates the position by delivering the underlying at the agreed upon price.

- Open Interest: It measures the liquidity of a contract. It simply measures the number of contracts that have been entered into but not yet liquidated.

- Clearing House: A clearing house performs several functions
  - One function is to guarantee that the two parties to the transaction will perform.
  - Clearing house provides a financial strength and integrity to the buyers as well as sellers
  - Once the sell/buy is initiated, the clearing house interposes itself as the buyer for every sale and as the seller for every purchase.

- Margin Requirements
  - Initial Margin: the investor must deposit a minimum dollar amount per contract as specified by the exchange.
    - It is required as a deposit for the contract.
    - This amount can be in the form of risk free security or cash as collateral.
  - Investor’s equity: the amount in the initial margin account.
    - As the price of the futures contract fluctuates each trading day, the value of the investor’s equity in the position changes
    - At the end of each trading day, exchange determines the “ settlement price” for the futures contract.

- Settlement Price: It is that value the exchange considers to be the representative of trading at the end of the day.
  - The exchange uses the settlement price to mark to market the investor's position, so that any gain or loss from the position is quickly reflected in the investor's equity account.

- Maintenance Margin: is the minimum level by which an investors' equity position may fall as a result of unfavourable price movements before the investor is required to deposit the additional margin.

- Variation Margin: the additional margin deposited called variation margin, is an amount necessary to bring the equity in the account back to its initial margin levels.

- Mark-to-Market procedure
  - Suppose we have asset XYZ
  - Initial margin $7 per contract
  - Maintenance margin $4 per contract
  - Suppose the investor A buys 500 contracts at a futures price of $100 and investor B sells the same number of contracts at the same futures price.
  - The initial margin for both A and B is $3500 ($7*3500).
  - Now A & B must put up $3500 in cash or Treasury Bills or other acceptable collateral.
  - At this time, $3500 is the equity in the account.

- However, the maintenance margin for the two positions in $2000 (4*500)
- It means that the margin should not fall below $2000.
- If it does, then the party whose equity falls below the maintenance margin must put up additional margin which is the variation margin.
- Variation margin will come into play in two ways:
  - The variation margin must be in cash
  - The amount of variation margin required is the amount to bring the equity up to the initial margin, not the maintenance margin.

- Mark-to-Market Procedure
  - We assume that the following settlement prices at the end of four consecutive trading days after the transaction was entered into

Trading Day	Settlement Price
1	$99
2	97
3	98
4	95

- **At the end of Day 1**: A realizes the loss of $1 or $500 for the 500 contracts he bought
- A’s initial equity is $3500 is reduced by $500 to $3000.
- No action is taken by the clearing house, because A’s equity is above the maintenance margin of $2000.

- **At the end of second 2**: A looses further loss because futures price has fallen by $2, resulting in the additional reduction in his equity position by $1000.
- A’s equity is then $2000 after adjustment from $3500-$1500= $2000

- On Day 3, investor A realizes the profit of from the previous trading day of $1 per contract or $500.
- A’s equity increases to $2500.
- On Day 4, price further drops and it reaches upto $95. Investor A looses the money equivalent to $1500 and a reduction to A’s equity to $1000.
- Since A equity is now below the maintenance margin limit of $2000, A is required to put up additional of $2500 (variation margin) to bring the equity up to the initial margin of $3500.
- If Investor A cannot put up the variation margin, his position will be liquidated.
- His contract will be sold to Clearing house

- Let’s talk about investor’s B now:
  - Since investor has the futures contract, B benefits if the price of futures contract declines.
  - As a results, his/her equity increases at the end of the first 2 trading days
  - In fact, at the end of trading day 1, B realizes the profit of $500, which increases its equity to $3500+$500 = $4000.
  - B is entitled to remove the $500 profit and utilize these funds elsewhere.
  - Suppose B takes out $500 and as a result, his equity remains at $3500 at the end of trading day 1.
  - At the end of trading day 2, investor B realizes an additional profit of $1000 that he can withdraw.
  - At the end of trading day 3, loss of $500. This results in a reduction of his equity to $3000.
  - Finally on trading day 4, B realizes the profit of $1500, making his equity $4500. He can withdraw $1000.

- Leverage Aspect of Futures
  - In the futures market, the degree of leverage depends upon initial margin rate.
  - Suppose an exchange requires the initial margin of only 5%. It means that out of \$100 exposure, investor has to keep \$5 in the initial margin account.
  - Degree of leverage equals 1/0.05 = 20=1/margin rate
  - It means that investor A can purchase 20 futures contracts with his \$100 investment.

- Difference between Forward and Futures
  - A forward contract is an agreement for the futures delivery of the underlying at a specified price at the end of a designated period of time.
  - Futures contracts are standardized agreements as to the delivery date (or month) and quality of deliverable, and are traded on organized exchanges
  - Forwards are non standardized contracts
  - Forwards are over-the-counter instrument compared to futures which is an exchange traded contract.
  - Futures involve clearing house whereas forward involves counterparty risk

Forward	Futures
Non Standardized	Standardized
Traded OTC	Exchange traded
Counterparty risk	No counterparty risk
One specific delivery date	Multiple delivery dates
Settled only at the maturity of contract	Daily settled
Delivery or cash settled	Usually cash settled prior to maturity

- Basics of pricing of futures and forwards contracts
    - While, there are many models that have been proposed for valuing financial instruments that trade in the cash (spot) market, the valuation of all derivative models are based on arbitrage arguments.

- To understand the pricing model. We hypothesize six assumptions for a futures contract that has no initial and variation margin and for the underlying is asset U:
    1. The price of asset U in the cash market is $100
    2. There is a known cash flow for asset U over the life of the futures contract
    3. The cash flow for asset U is $8 per year paid quarterly ($2 per quarter)
    4. The next quarterly payment is exactly three months from now
    5. The futures contract requires delivery three months from now
    6. The current three-month interest rate at which funds can be lent or borrowed is 4% per year

- The objective is to determine what the futures price of this contract should be. To do so, suppose that the futures price in the market in \$105.
- Let’s check if that is the correct price
- We can check this by implementing following trading strategy
  - Sell the futures contract at \$105
  - Purchase asset U in the cash market (spot) for \$100.
  - Borrow \$100 for three months at 4% per year (pay \$1 per quarter)

  The purchase of the asset U is accomplished with the borrowed funds. Hence, this strategy does not involve any initial cash outlay

- At the end of three months, the following occurs
  - $2 is received from holding asset U
  - Asset U is delivered to settle the futures contract
  - The loan is repaid

- This strategy results in the following outcome:
  - From settlement of the futures contract:
  - Proceeds from sale of asset U to settle the futures contract = $105
  - Payment received from investing in asset U for three months = $2

Total proceeds	= $107

- From the loan:
  - Repayment of principal of loan = $100
  - Interest on loan (1% for three months) = $1
  - Total outlay = $101

  - Profit from the strategy = $107 - $101 = $6

  The profit of $6 from this strategy is guaranteed regardless of what the cash price of asset U is three months from now.

  - This strategy that resulted in the capturing of the arbitrage profit is referred to as a Cash-and-Carry trade.
  - The reason for this name is that implementation of the strategy involves borrowing cash to purchase the underlying and carrying that underlying to the settlement date of the futures contract.

- From cash and carry trade, we see that that the futures price cannot be $105.
- Suppose instead that the futures price is $95 rather $105
- Let’s try the following strategy to see if that price can be sustained in the market:
    - Buy the futures contract at $95
    - Sell (short) asset U for $100
    - Invest (lend) $100 for three months at 1% per year

    - We assume once again that in this strategy there is no initial margin and variation margin for the futures contract.
    - In addition, we assume that there is no cost to selling the asset short and lending the money.
    - Given these assumptions, there is no initial cash outlay for the strategy just as with the cash-and-carry trade.

- We take the long position in asset U
- Asset U is accepted for delivery
- Asset U is used to cover the short position in the cash market
- Payment is made of $2 to the lender of asset U as compensation for the quarterly payment
- Payment is received from the borrower of the load of $101 for principal and interest

- More specifically, the strategy produces the following at the end of three months:
  - From settlement of the futures contract:
    - Price paid for purchase of asset U to settle futures contract = \$95
    - Proceeds to lender of asset U to borrow the asset = \$2
    - Total outlay = 95+2 = \$97

From the loan:

Principle from loan = \$100

Interest earned on load (\$1 for three months) = \$1

Total proceeds = 100+1=101

Profit from the strategy = 101-97= \$4

- As with the cash-and-carry trade, the \$4 profit from this strategy is a riskless arbitrage profit.
- This strategy requires no initial cash outlay, but will generate a profit whatever the price of asset U is in the cash market at the settlement date
- In real world markets, this opportunity would lead arbitrageurs to buy the futures contract and short asset U.
- The implementation of this strategy would be to raise the futures price until the arbitrage profit disappeared.

- This strategy that is implemented to capture the arbitrage profit is known as a *reverse-cash-and-carry-trade.*
- *With this strategy, the underlying is sold short and the proceeds received from the short sale are invested.*

- *We can see that the futures price cannot be $95 or $105.*
- What is the theoretical futures price given the assumptions in our example.
- It can be shown that if the future price is $99, there is no opportunity for an arbitrage profit. That is, neither the cash-and-carry trade nor the reverse cash-cash-carry trade will generate an arbitrage profit.
- Theoretical futures price = Cash market price +(cash market price)*(Financing cost - cash yield)
- Financing cost = Interest rate to borrow funds
- Cash yield and cash yield is the payment received from investing in the asset as a percentage of the cash price.
- In our case, financing cost is 1% and cash yield is 2%.

- Since the cash price of the asset U is $100, the theoretical futures price is
  - $100+ ($100*(1%-2%) = $99

  - The futures price can be above or below the cash price depending on the different between the financing cost and cash yield.
  - The different between these rates is called the net financing cost.
  - A more commonly used term for the net financing cost is the cost of carry or simply carry.
  - Positive carry means that the cash yield exceeds the financing cost.
  - Negative carry means otherwise.

- Positive carry = Futures price will see at a discount to cash price
- Negative carry = Futures price will sell at a premium to cash price
- Zero = Futures price will be equal to the cash price
- At the settlement date of the futures contract, the futures price must equal the cash market price.

- **Contango, Backwardation and forward curve**
  - **Contango**
    - A market is said to be in contango if near term futures prices is higher than spot price, and distant futures is higher than near term futures prices.
    - A contango market results in upward sloping forward curve – longer maturity contracts trade at a high price compared to the shorter maturity contracts.

- Backwardation
  - In a backwardation market, the reverse happens.
  - The near term futures price is lesser than the spot price and a distant futures price is lesser than near term futures prices.
  - In case of backwardation forward contract, spot price is higher than futures contract.

- Forward curve
  - At times forward curve may exhibit both backwardation and contango.
  - Forward curve shows that even though there is no supply constraint in near future, but may likely to happen in distant future.

- Spot, futures and commodity basis
  - Spot and futures prices indicate whether a commodity contract is in contango or in backwardation. In a contango market, futures price is higher than the spot price while reverse is true for backwardation market.

  - During the life of a contract, i.e. from the contract origination date to contract maturity date, spot price relationship may change many times.

- A contract may start from contango and remain in contango until the maturity
- A contract may start from backwardation and remain in backwardation until the maturity.

- Difference in borrowing and lending rates and theoretical price
  - In the previous analysis, we assume that in both cases (cash-and-carry trade) and reverse-cash-carry trade), the borrowing and lending rates are equal.
  - Typically, the borrowing rates are higher than the lending rates
  - In this case, the theoretical futures price can further be modified and expressed as
  - Theoretical futures price based on borrowing rate =

    = Cash market price + (Cash market price)*(borrowing rate – cash yield)

  For the reverse cash-and-carry trade, it becomes

    = Theoretical futures price based on lending rate

    = Cash market price + (Cash market price)*(lending rate – cash yield)

- Assume that the borrowing rate is 6% per year, or 1.5% for three months, while, the lending rate is 4% per year, or 1% for three months.
- Using borrowing rate, the upper value for the theoretical futures price is $99.5 and using lending rate, the lower value for the theoretical futures price is $99.

- Pricing of forwards on investment assets
  - Forward price is dependent upon the cost of carry.
  - Cost of carry includes all cost an investor would incur if the investor would buy the asset today and carries it till the maturity minus any carrying return
  - Carrying cost includes financing cost, storage and insurance charges etc.
  - Carrying return is the dividend or bonus an investor would recover during the maturity period by virtue of owning the underlying asset.
  - Net carry cost is the difference between carrying cost and carrying return

- Cost of carry model assumes that the forward price equals to the cost of buying the asset today and carrying it till the maturity minus any carrying return.
- Forward price is the spot price plus the net carrying cost of the asset from today till the maturity of the contract.
  - Forward price = Spot price + net carrying cost till maturity of the contract

- We have three sets of equations to handle such scenarios
  - *Underlying asset does not provide any return/income*: $F_{(0,T)} = S_0 e^{RT}$ ….(1)
  - *Underlying asset provides known cash return/income*: $F_{(0,T)} = (S_0 - I)e^{RT}$.(2)
  - *Underlying asset provides known yield*: $F_{(0,T)} = (S_0)e^{(R-q)T}$ … … … … . . (3)

Where,

$F_{(0,T)}$ = Forward price of the contract on day) for a contract maturing on day T

S_0 = Spot price available of the contract on day 0

T = Time maturity in terms of years. T = 0.5 for six month forward contracts and T = 0.0833 for one month forward contract

R = continuously compounded risk free rate of return. Nominal rate (r) = ($e^{R/n}$-1)*n

*Continuous compounding rate (R) = n*ln(1+r/n)*

I = Present value of known income (dividend) the underlying asset provides during the life of the contract.

q = known yield the underlying asset provides during the life of the contract.

- Equations (1-3) give theoretical value of the contract $F_{theoretical}$.
- If the actual value of the contract in the futures market (F_{Actual}) is different than $F_{theoretical}$ then arbitrageurs undertake *cash-and-carry* or *reverse-cash-carry arbitrage* to benefit from the difference.
- This in turn will ensure that $F_{theoretical}$ does not deviate much with $F_{actual.}$

- Cash and carry and reverse cash and carry arbitrage

Factual > Ftheoretical		Factual < Ftheoretical	
Cash-and-carry-arbitrage		Reverse-cash-and-carry-arbitrage	
• On day 0	• On maturity day T	• On day 0	• On maturity day T
• Borrow S_0	• Deliver underlying	• Short sell underlying asset at S_0	• Receive S_0 and interest
• Buy underlying	• Receive Factual	• Lend S_0 for a period equal to the maturity of the forward contract	• Take delivery of the underlying and pay F_0
• Sell forward	• Return S_0 with interest	• Buy forward	• Deliver the underlying asset to make for the short sale

Example

- Suppose An investor would like to buy shares of PQR company after 4 months. The shares are quoted at $ 127 today. The investor can borrow money at nominal rate of 9.25% per annum.
  - What would be the forward price of the share if there is no dividend expected in these 4 months?
  - What would be the forward price of the share if there is Rs. 2.75 per share dividend after 2 months?

- Continuously compounded rate $(R) = n*ln(1+r/n)=1*ln(1+9.25/1)$= 8.85%
  - Forward price of the PQR company share without any income during the maturity
  - Maturity period = 4 months = 0.333 years
  - $F_{(0,T)} = S_0 e^{RT} = 127*e^{(8.55*0.3333)}$ = $130.80

Futures price of PQR Co. share with income during the maturity

Dividend = 2.75

Dividend date = 2 months = 0.1667 years

Present value of the dividend = $2.75/e^{(8.55*0.1667)= 2.711}$

Forward price = $F_{(0,T)} = (S_0 – I)e^{RT}$

$= 127- 2.711 *e^{(8.55*0.3333)} = 128.01$

Example

- An investor would like to buy forward contract on an equity index with 6 months maturity. The spot value of the index is 5430.15. Expected annual dividend yield is 3.20%. Find out what would be the value of index futures if the nominal interest rate is 7.5% with quarterly compounding.

- Continuous compounding rate (R) = n*ln(1+r/n)=4*ln(1+7.5%/4)
- Futures contract maturity period = 6 months = 0.5 years
- Futures price = $F_{(0,t)} = S_o e^{(R-q)T} = 5430.15 * e^{(7.43\%-3.20\%)*0.5}$ = 5639.91

Calculating arbitrage profit

- An investor would like to buy shares of ABC company after 6 months. The shares are quoting at Rs. 887 today. The investor can borrow money at nominal rate of 8.50% per annum. The six month forward quoting is at Rs. 945. Find out what would be arbitrage profit if arbitrage is possible.

- Continuous compounding rate = 8.16%, Maturity = 6 months = 0.5 years
- $F_{theoretical} = F_{(0,T)} = S_o e^{RT} = 887 * \mathrm{e}^{(8.16\% * 0.5)} =$ Rs. 924
- $\mathrm{F}_{\mathrm{actual}} = 945$.
- As $F_{actual} > F_{theoretical}$, cash-and-carry arbitrage is possible.

- On day 0
  - Investor borrows (S_0) = 887 for 6 months at 8.50%
  - Buy the underlying i.e. share of ABC Company
  - Sell forward at Factual = Rs. 945
- On maturity (after 4 months)
  - Deliver underlying share
  - Receive Rs. 945 for selling forward
  - Return Rs. 924 for borrowing Rs. 887 for 6 months
  - Profit = 945-924 = Rs. 21

- What happens when borrowing and lending rates are different?
  - Assuming the scenario that borrowing rate is higher than the lending rate
  - We know that in case of cash-and-carry model, cash in borrowed, whereas, in case of reverse-cash-and-carry model, investors' lend the cash.
  - $F_{Thoeretical}$ remains within a range.
  - This range is known as no arbitrage bound.

- With different borrowing and lending rates, equations (1-3) are modified as:

$S_0 e^{(Rlend)T} <= F_{(0,T)} <= S_0 e^{(Rborrow)T}$ .... (4)

$(S_0 - I) e^{(Rlend)T} <= F_{(0,T)} <= S_0 e^{(Rborrow)T}$ .... (5)

$(S_0) e^{(Rlend - q)T} <= F_{(0,T)} <= S_0 e^{(Rborrow-q)T}$ .... (6)

The right-side of the inequality condition relates to the cash and carry arbitrage while left-side relates to reverse-cash-and-carry arbitrage.

In other words, the upper bound price relates to cash-and-carry arbitrage while the lower bound relates o reverse cash-and-carry arbitrage.

- What happens when we incorporate the transaction costs associated with cash-and-carry and reverse cash-and-carry transactions have not been incorporated?

- With transactions costs, equation (4) can be modified as

$$S_0 e^{(Rlend)T} - TC_1 <= F_{(0,T)} <= S_0 e^{(Rborrow)T} + TC_2$$

Where,

TC_1 = Transaction costs associated with reverse cash-and-carry arbitrage trades of selling underlying spot asset and buying forward.

TC_2= Transaction costs associated with cash-and-carry arbitrage trades of buying underlying spot asset and selling forward/futures

- Different buying and lending rates
  - Spot price of ABC company’s share is Rs. 334. 3-month forward contract on XYZ company’s share is quoting at Rs. 347.
  - A retail trader can borrow money at 7.75% and can lend at 7% compounded annually. Transaction costs are 0.65% and 0.45% of the contract value in spot and forward market respectively paid at the time of trade.
  - Find out the no-arbitrage price bound. Is there any possibility for the retail trader to undertake arbitrage? If so what is the arbitrage profit.
  - An institutional investor can borrow and lend money at 7.50% and 7.25%, respectively. For institutional investors, the associated transaction costs for spot and forward market are 0.30% and 0.10%, respectively.
  - Find out the no-arbitrage bound price, and compare the no-arbitrage bound price for retail and institutional traders.

- Solution:

Factual = Rs. 347, So = Rs. 334, T = 0.25 years

For retail trader

Transaction cost (spot market) = 0.65% = 0.65%*334 = Rs. 2.17

R_{lend} (continuously compounded) = 6.77%

R_{borrow} (continuously compounded) = 7.46%

No-arbitrage bound price for retail trader

$$S_0 e^{(Rlend)T} - TC_1 <= F_{(0,T)} <= S_0 e^{(Rborrow)T} + TC_2$$

$$(334 * e^{(6.77\%)*0.25} - 2.17e^{(6.77\%*0.25)} <= F_0 <= (334 * e^{(7.46\%)*0.25} + 2.17e^{7.46\%*0.25})$$

$$= 337.49 <= F_0 <= 341.91$$

- No arbitrage bound price ranges between Rs. 337.49 and Rs. 341.91.
- As the actual forward price is Rs. 347, Factual > Ftheoretical
  - Cash-and-carry arbitrage will happen
- Transaction costs are also to be compounded as the trader is also losing out on transaction cost investment opportunity.

- Cash-and-carry profit calculation
  - On Day 0
    - Borrow Rs. 334 for 3 months at 7.46% compounded continuously
    - Buy spot asset at Rs. 334 on day 0
    - Pay transaction price for spot asset = 2.17 (0.65% of 334) borrowed at 7.46% compounded continuously
    - Pay transaction price of spot
    - Sell 3-month futures at Rs. 347
    - Pay transaction price of futures contract on Day 0: Rs. 1.55 (0.45% of 347) borrowed at 7.46% compounded continuously.

- On maturity day
  - Deliver asset and receive Rs. 347
  - Pay 344.07 [$(334+1.55+2.17)e^{(7.46\%*0.25)}$]
  - Make arbitrage profit of Rs. 2.92.

- No arbitrage bound for institutional traders
  - R_{lend} (continuously compounded) = 7.00%
  - R_{borrow} (continuously compounded) = 7.23%
  - Transaction cost (spot market) = 0.30%*334 = Rs. 1
  - No arbitrage bound price:
  - $S_0 e^{(Rlend)*T} - tc_1 <= F_{(0,T)} <= S_0 e^{(Rborrow)*T} + tc_2$

    $= 334e^{7\%*0.25} - 1*e^{7\%*0.25)} <= F_{(0,T)} <= 334e^{7.23\%*0.25} + 1.e^{7.23*0.25}$

    $= 338.87 <= F_0 <= 341.11$
  - In case of retail traders, the difference between the lower and upper bound no-arbitrage price range is Rs. 4.42, while, for institutional traders, it is 2.24.

  - It is clearly obvious that institutional traders get better opportunity to make arbitrage profit than retail traders.

- ***Pricing of forward contracts on investment asses like gold and silver***
  - Pricing of forward contracts on investment assets like gold and silver

When storage, insurance and other costs (U) in are associated with physical holding of assets	$F_{(0,T)} = (So+U)e^{RT}$
When storage, insurance and other costs (u) are expressed as a % to the underlying commodity price	$F_{(0,T)} = S_0 e^{(R+U)T}$

U = Present value of all costs including storage, insurance, etc.
u = Present value of all costs expressed as a percentage of the underlying spot price S0
T = Time to maturity in terms of years
R = Continuously compounded risk free rate of return

Example

- Spot price of gold (per 10 gram) is 10,550. The storage and insurance costs for 2 months is Rs. 275 for every 10 gm of gold. The investors can borrow/lend at 7.75% continuously compounded rate. If the two-month gold forward is trading either at:
  - A) Rs. 11, 230
  - B) Rs. 10, 453

  Find out how arbitragers will be able to make arbitrage profit.

- Spot price of the gold (per 10 gram) is Rs. 10,550 = Spot price

  R = 7.75%

  U = 275 (per 10 gram) paid at the beginning of the storage period

  T = 0.1667 (2 months)

$$F_{theoretical} = (S_0+U)*e^{RT} = (10{,}550+275)*e^{0.0775*0.1667} = 10965.75$$

CASE A: two months futures price of gold = 11,230 = F_0

CASE B: Two months futures price on gold = 10,453 = F_0

- Case A: Factual > F theoretical, cash and carry arbitrage will be undertaken by traders
- Case B: As Factual < F theoretical, reverse cash and carry arbitrage will happen

- CASE A: cash-and-carry-arbitrage
  - **On day 0**
    - Borrow Rs. 10,825 (purchase cost + storage cost)
    - Buy gold at Rs. 10,550
    - Pay storage cost Rs. 275
    - Sell forward at Rs. 11,230
  - **On maturity day**
    - Pay gold to forward country party and receive Rs. 11,230
    - Return Rs. 10,965.73 for borrowing Rs. 10, 825
    - Earn a net profit of Rs. 264.27

- Case B: Reverse cash-and-carry arbitrage
  - On day 0
    - Sell gold (from its own inventory or short sells)
    - Receive Rs. 10,550
    - Save Rs. 275 as storage cost
    - Invest Rs. 10,825 for 2 months
    - Buy forward at Rs. 10,453
  - On maturity day
    - Receive gold from long position
    - Pay Rs. 10,453 as payment for long period
    - Receive Rs. 10,965.75 from investment of Rs. 10, 825 for 2 months
    - Earn a net profit of 512.75

- Alternatively for B

So = 10,550 for 10 gram

R = 7.75%

u = 2.60%

T = 0.1667 (2 months)

$F_{theoretical} = S_0e^{(R+u)T} = 10{,}550*e^{(0.0775+0.026)*0.1667}$

= 10,733.60

- **Spread arbitrage**
  - Cash-and-carry and reverse cash-and-carry arbitrage can also happen among futures contracts with same underlying asset but of different maturity
  - This is known as spread arbitrage

  - Spread arbitrage ensures that the price differential between two futures contracts must be equal to the cost of carrying the underlying asset from nearby contract delivery date to the distant contract delivery date:

$$F_{(0,distant)} = F_{(0,nearby)} \times e^{CT}$$

T = Time difference between nearby and distant futures contract

C = Continuously compounded cost of carrying the asset that includes storage, insurance and interest costs

Example

- The price of gold futures contract with three months maturity (*Factual(0,3))* is 10,715. The cost of carrying (forward interest rate) gold from 3 months to 6 months is 9% per annum continuously compounded. If the market price for *Factual*(0,6) is
  - (a) 11,072
  - (b) 10,723 find out how arbitrage will take place?

- $F_{theoretical(0,6)} = F_{actual\ (0,3)} e^{CT}$
- $F_{thereotical(0,6)} = 10{,}715 * e^{0.09*0.25} = 10{,}958.82$
- If $F_{actual(0,6)} = F_{theoretical\ (0,6)}$, cash-and-carry arbitrage will be undertaken by the traders.
- If $F_{actual(0,6)} < F_{theoretical(0,6)}$, reverse cash and carry arbitrage would be undertaken by the traders

- Case A: Cash-and-carry transactions with Factual = Rs. 11,072
- On day 0
  - Go long in the nearby forward (3 months) at 10,715
  - Sell the distant forward
  - Contract to borrow Rs. 10,715 at 9% from t=3 to t=6

- On nearby contract maturity (t=3)
  - Borrow Rs. 10,715 as contracted at t=0
  - Pay Rs. 10,715 to the long forward country party
  - Take delivery of gold
  - Store gold

- On distant contract maturity (t=6)
  - Deliver gold as part of short forward contract
  - Receive Rs. 11,072
  - Pay Rs. 10,958,82 for borrowing 10,715
  - Net profit of Rs. 113.18

- Case B: Reverse cash and carry transactions with Factual = Rs. 10,723
  - *On day 0*
    - Go short in the nearby forward (3 months) at 10,715
    - Buy the distant forward at Rs. 10,723
    - Contract to lend Rs. 10,715 at 9% from t=3 and t=6
  - *On nearby contract maturity (t=3)*
    - Borrow gold for 3 months
    - Deliver gold for nearby short forward counterparty
    - Lend Rs. 10,715 for 3 months
  - *On distant contract maturity day t=6*
    - Receive 10,958.82 as lending receipt
    - Receive gold as part of long forward contract
    - Return gold to the lender
    - Pay Rs. 10,723 to long forward country party
    - Net profit of Rs. 235.82

Pricing of forwards for storable consumption commodities

- In case of consumable commodities like crude oil, copper, coal and agricultural products etc.
- Users of such commodities trade in commodity spot and forward/futures as they require these commodities to be either used in production process or for consumption
- Pricing of such commodities can be analyzed under two situations:
  - **WITH and WITHOUT SUPPLY CONSTRAINTS**

- **Without supply constraint**
  - If the commodity is available abundantly, i.e. there is no shortage currently or expected in future, then forward price will confirm to the cost-of-carry model as given below:

$$F_{(0,T)} = S_0 e^{(r+u)T}$$

Forward contracts are priced in a manner such that a trader can borrow S_0 for a period of *T* at a rate of *r* to buy the commodity in spot and store it for period *T* by incurring storage/insurance costs at a rate of *u*.

Cash and carry and reverse cash carry strategies are dependent upon actual and theoretical prices of forward

- **With supply constraints**
  - When commodities are in short supply even if, there is arbitrage opportunity, many users of such commodities may not be keen on taking arbitrage.
  - For certain commodity users, benefit from physical holding of assets outweighs the benefit from arbitrage profit.
  - For manufacturing companies, physical inventory helps in smoothing process

- Pricing of forwards for consumption commodities in case of supply restrictions is also undertaken using cost of carry but with some adjustments.
- This adjustment is known as “Convenience yield”
- Convenience yield arises from benefits of holding physical asset and such benefits are not available to forward/futures holders.
- Parties holding consumption commodities are not bothered about stocks outs, stoppage of production and sales.
- Convenience yield represents all these benefits.

- So while pricing consumption commodities, cost of carry model is adjusted to account for convenience yield.

$$F_{(0,T)} = S_0 e^{(R+u-Y)T}$$

where

R = continuously compounded risk free rate of return

u = present value of all costs, including storage, insurance, expressed as a percentage of the underlying spot price

Y = Convenience yield expressed as percentage of spot price S_0

T = time to maturity in terms of years.

- **Example**
  - Spot price of S_0 Soybean oil = Rs. 93, 500 (per 1000 kg), R = 7.75% per annum continuously compounded. u = 0.35% per 1000 kg per annum continuously compounded, T = 0.333 (4 months), $F_{(0,4)}$= Rs. 95,785 (per 1000 kg)

    a) Calculate the theoretical futures price based on cost of carry model

    b) Find out the convenience yield in absolute as well as percentage term?

- Theoretical price based on cost of carry model

$$a)F_{theoretical(0,T)} = S_0 e^{(R+u)T}$$

$$F_{theoretical(0,T)} = S_0 e^{(R+u)T} = 93550e^{(7.75\%+0.35\%)0.333} = 96,110.25$$

b) **Convenience yield**

The theoretical futures price = 96,110.25

Actual futures price = 95,785

Convenience yield = 96,110.25 – 95785 = 325.25

- Hence the convenience yield is 1.01% per annum continuously compounded

$$F_{(0,T)} = S_0 e^{(R+u-Y)T}$$

$$Y = (R+u) - \frac{1}{T}\ln\left(\frac{F_{(0,T)}}{S_0}\right)$$

$$= (7.75\% + 0.35\%) - \frac{1}{0.333}\ln\left(\frac{95785}{93550}\right) = 1.01\%$$

Example

Spot price of XYZ Company’s share is ₹ 334. 3-month forward contract on XYZ company’s share is quoting at ₹ 347. A retail trader can borrow money at 7.75% and can lend at 7% compounded annually. Transaction costs are 0.65% and 0.45% of the contract value in spot and forward market respectively paid at the time of trade.

Find out the non-arbitrage price bound. Is there any possibility for the retail trader to undertake arbitrage? If so what is the arbitrage profit?

Example

An institutional investor can borrow and lend money at 7.50% and 7.25%, respectively. For institutional investors the associated transaction costs for spot and forward market are 0.30% and 0.10%, respectively. Find out the no-arbitrage bound price, and compare the non-arbitrage bound price for retail and institutional traders.

Solution

F_{Actual} = 347, S_0 = 334, *T* = 0.25 years

For retails trader

Transaction cost (spot market) = 0.65% = 0.65%*334=2.17

R_{lend} (continuously compounded) = 6.77%

R_{borrow} (continuously compounded) = 7.46%

Solution

No-arbitrage bound price for retail trader

$$S_0 e^{(R_{lend})T} - TC_1 \leq F_{(0,T)} \leq S_0 e^{(R_{borrow})T} - TC_2$$

$$(334e^{(6.77\%)0.25} - 2.17e^{(6.77\%)0.25}) \leq F_{(0)} \leq (334e^{(7.46\%)0.25} - 2.17e^{(7.46\%)0.25})$$

$$= 337.49 \leq F_{(0)} \leq 341.91$$

Solution

As the actual forward price is 347, $F_{Actual} > F_{Theoretical}$, cash-and-carry arbitrage will happen.

Transaction costs are also to be compounded as the trader is also losing out transaction cost investment opportunity.

Solution

Cash and carry profit calculation

- Borrow Rs. 334 for 3 months at 7.46% compounded continuously
- Buy spot asset at Rs. 334 on day 0
- Pay transaction price for spot asset = Rs. 2.17 (0.65% of 334) borrowed at 7.46% compounded continuously
- Pay transaction price of spot
- Sell 3-month futures at Rs. 347

Solution

- Pay transaction price of futures contract on Day 0:
  - Rs. 1.55 (0.45% of Rs. 347) borrowed at 7.46% compounded continuously

- On maturity day
  - Deliver asset and receive Rs. 347
  - Pay Rs. 344.07 (334+1.55+2.17) $e^{(7.46\%*0.25)}$
  - Make arbitrage profit of Rs. 2.93

Solution

- **No arbitrage bound for institutional investor**

R_{lend} (continuously compounded) = 7.00%

R_{borrow} continuously compounded = 7.23%

Transaction cost (spot market) = 0.30% * 334 = Rs. 1.00

No arbitrage bound price

$$S_0 e^{(R_{lend})T} - tc_1 \leq F_{(0,T)} \leq S_0 e^{(R_{borrow})T} - tc_2$$

$$334e^{(7\%)0.25} - 1e^{(7\%)0.25} \leq F_{(0,T)} \leq 334e^{(7.23\%)0.25} - 1 * e^{(7.23\%)0.25}$$

Solution

= $$338.87 \leq F_{(0,T)} \leq 341.11$$

In case of retail traders, the difference between the lower and upper bound no-arbitrage price range is Rs. 4.42, while for institutional traders, it is Rs. 2.24.

It is clearly obvious that institutional traders get better opportunity to make arbitrage profit than retail traders.