Canonical Form

MSO-203B

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Canonical Form for 2nd Order linear PDE

Definition

Consider the 2nd Order linear PDE:

$$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{1}$$

We have seen the existence of a C^1 diffeomorphic change of variable such that $Lu = g$ is tranformed into

$$\bar{L}(w) = Aw_{\theta\theta} + 2Bw_{\theta\eta} + Cw_{\eta\eta} + Dw_{\theta} + Ew_{\eta} + Fw = G$$

where,

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where,

$$\begin{aligned} A(\theta, \eta) &= a{\theta_x}^2 + 2b\theta_x\theta_y + c{\theta_y}^2 \\ B(\theta, \eta) &= a\theta_x\eta_x + b(\theta_x\eta_y + \eta_x\theta_y) + c\eta_y\theta_y \\ C(\theta, \eta) &= a{\eta_x}^2 + 2b\eta_x\eta_y + c{\eta_y}^2 \end{aligned}$$

Canonical Form-Elliptic Equation

Suppose equation (1) is Elliptic in Ω which means $b^2 - ac < 0$ at every point of Ω. We show the existence of a change of variable such that

$$a{\theta_x}^2 + 2b\theta_x\theta_y + c{\theta_y}^2 = a{\eta_x}^2 + 2b\eta_x\eta_y + c{\eta_y}^2 \tag{2}$$

$$B(\theta, \eta) = a\theta_x\eta_x + b(\theta_x\eta_y + \eta_x\theta_y) + c\eta_y\theta_y = 0 \tag{3}$$

Then the equation (1) reduced to $w_{\theta\theta} + w_{\eta\eta} + l(w) = h$.

Reduction Process

From equation (2) and multiplying $2i$ with the equation (3) we have,

$$a[{\theta_x}^2 - {\eta_x}^2] + 2b[\theta_x\theta_y - \eta_x\eta_y] + c[{\theta_y}^2 - {\eta_y}^2] = 0$$
$$a\theta_x(2i\eta_x) + b[\theta_x(2i\eta_y) + (2i\eta_x)\theta_y] + c\eta_y(2i\theta_y) = 0$$

General Framework

Reduction Process

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Definition

Define, $\phi = \theta + i\eta$ where $i^2 = -1$ and then one can write the above expression as

$$a{\phi_x}^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0 \qquad (4)$$

General Framework

Reduction Process

Note that equation (4) can be written as:

$$a[\phi_x - \mu_1\phi_y][\phi_x - \mu_2\phi_y] = 0$$

where,

$$\mu_1 = \frac{-b - i\sqrt{ac - b^2}}{a} \text{ and } \mu_2 = \frac{-b + i\sqrt{ac - b^2}}{a}$$

Note that these are the complex roots of $a\mu^2 + 2b\mu + c = 0$ and $\mu_1 = \bar{\mu_2}$.

General Framework

Reduction Process

Solving the characteristic equation one obtains two complex solutions. ϕ and γ such that they are complex conjugate of each other.
Choose $\theta = \mathbf{Re}\phi$ and $\eta = \mathbf{Im}\phi$.

Canonical Form

Define, $w(\theta, \eta) = u(x, y)$ and using change of variable we will have our canonical form.

An Example

Problem

Reduce the Tricomi equation

$$yu_{xx} + u_{yy} = 0$$

into canonical form

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Solution

Note that the problem is elliptic provided $y > 0$. If we define $\zeta = \theta + i\eta$ we have,

$$y\zeta_x^2 + \zeta_y^2 = 0 \tag{5}$$

An example

Solution

Note that equation (5) can be reduced to the two 1st order equations as follows:

$$(\zeta_y - i\sqrt{y}\zeta_x)(\zeta_y + i\sqrt{y}\zeta_x) = 0$$

An example

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Solution

Hence the complex family of characteristic is given by

$$\frac{2}{3}y^{\frac{3}{2}} + ix = C_1$$

and

$$\frac{2}{3}y^{\frac{3}{2}} - ix = C_2$$

.

An example

Choosing θ and η

Set $\theta = \frac{2}{3}y^{\frac{3}{2}}$ and $\eta = x$.

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Solution

Write, $u(x, y) = w(\theta, \eta)$ we get,

$$
\begin{aligned}
u_x &= w_\eta \\
u_y &= y^{\frac{1}{2}} w_\theta \\
u_{xx} &= w_{\theta\theta} \\
u_{yy} &= y w_{\theta\theta} + \frac{1}{2} y^{-\frac{1}{2}} w_\theta
\end{aligned}
$$

An example

Canonical Form

$$w_{\theta\theta} + w_{\eta\eta} + \frac{3}{\theta} w_{\theta} = 0$$