ASSIGNMENT I MSO 202 A

COMPLEX NUMBERS, HOLOMORPHICITY, AND C-R EQUATIONS

Exercise 0.1 : Verify the following for all complex numbers z and w:

(1) $|z + w| \leq |z| + |w|$.
(2) $||z| - |w|| \leq |z + w|$.

Solution. (1) Let Re(z) denotes the real part of the complex number z. Note that $(z + w)\overline{(z + w)} = |z + w|^2 \leq (|z| + |w|)^2$. However,

$$\begin{aligned}(z + w)\overline{(z + w)} &= (z + w)(\overline{z} + \overline{w}) = |z|^2 + z\overline{w} + \overline{z}w + |w|^2 \\ &= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2.\end{aligned}$$

But $\mathrm{Re}(z\overline{w}) \leq |z||w|$.

(2) The argument is similar to one given in (1).

Exercise 0.2 : Let $z, w \in \mathbb{C}$ belong to the upper half plane. Show that the distance between z and w is at most the distance between z and $\overline{w}$.

Solution. Let Im(z) denotes the imaginary part of the complex number z. We need to verify that $|z-w| \leq |z-\overline{w}|$ if $\mathrm{Im}(z), \mathrm{Im}(w) \geq 0$. Note that

$$|z-w|^2 = \mathrm{Re}(z-w)^2+\mathrm{Im}(z-w)^2, \ |z-\overline{w}|^2 = \mathrm{Re}(z-\overline{w})^2+\mathrm{Im}(z-\overline{w})^2.$$

However, $\mathrm{Re}(z - w) = \mathrm{Re}(z - \overline{w})$ and $\mathrm{Im}(z - \overline{w}) = \mathrm{Im}(z + w)$. Hence $|z - w| \leq |z - \overline{w}|$ holds provided

$$|\mathrm{Im}(z) - \mathrm{Im}(w)| = |\mathrm{Im}(z - w)| \leq |\mathrm{Im}(z + w)| = |\mathrm{Im}(z) + \mathrm{Im}(w)|,$$

which holds trivially if $\mathrm{Im}(z), \mathrm{Im}(w) \geq 0$.

Recall the De Moivers formula: If $z = r(\cos(\theta) + i\sin(\theta))$ then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Exercise 0.3 : Find all complex numbers z such that $z^3 + 1 = 0$.

Solution. Note that $z^3 = -1$, and hence $|z| = 1$. By De Moivers formula,

$$z^3 = \cos(3\theta) + i\sin(3\theta) = -1.$$

In particular, $\cos(3\theta) = -1$ and $\sin(3\theta) = 0$. The possible choices for θ are $\theta = \frac{\pi+2k\pi}{3}$ for integers k. This yields the desired solutions

$$\cos(\pi/3) + i\sin(\pi/3),\ \cos(\pi) + i\sin(\pi),\ \cos(5\pi/3) + i\sin(5\pi/3).$$

A map f from $\mathbb{C}$ is $\underline{\mathbb{R}\text{-linear}}$ if $f(z+w) = f(z) + f(w)$ and $f(a\ z) = a\, f(z)$ for all $z, w \in \mathbb{C}$ and $a \in \mathbb{R}$. A map f from $\mathbb{C}$ is $\underline{\mathbb{C}\text{-linear}}$ if $f(z+w) = f(z) + f(w)$ and $f(a\ z) = a\, f(z)$ for all $z, w \in \mathbb{C}$ and $a \in \mathbb{C}$.

Exercise 0.4 : For given scalars $a, b \in \mathbb{C}$, show that $f(z) = az + b\overline{z}$ is always $\mathbb{R}$-linear, where $\overline{z} = x - iy$ for $z = x + iy$. Verify further that f is $\mathbb{C}$-linear if and only if $b = 0$.

Solution. The first part follows from $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{az} = a\overline{z}$ for $z, w \in \mathbb{C}$ and $a \in \mathbb{R}$. To see the second part, note that if $f(i \cdot 1) = if(1)$ then $ai - bi = ai + bi$, which implies that $b = 0$. If $b = 0$ then clearly $f(z) = az$ is $\mathbb{C}$-linear.

Exercise 0.5 : Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that

$$f(a\, z) = a\, f(z) \text{ for all } z, w, a \in \mathbb{C}. \tag{0.1}$$

Show that there exists $\alpha \in \mathbb{C}$ such that $f(z) = \alpha\, z$ for all $z \in \mathbb{C}$.

Solution. Apply (0.1) to $a = w$ and $z = 1$ to conclude that

$$f(w) = f(w \cdot 1) = wf(1) \text{ for any } w \in \mathbb{C}.$$

Hence one may take $\alpha = f(1)$.

Exercise 0.6 : Show that a holomorphic function $f = u + iv : \mathbb{C} \to \mathbb{C}$ is constant if $\overline{f} = u - iv$ is holomorphic.

Solution. Suppose that $\overline{f}$ is holomorphic. By C-R equations, $u_x = (-v)_y$ and $u_y = -(-v)_x$, that is, $u_x = -v_y$ and $u_y = v_x$. But since f is holomorphic, we also have $u_x = v_y$ and $u_y = -v_x$. As a consequence, we obtain $u_x = u_y = 0 = v_x = v_y$, and hence u and v are constant.

Exercise 0.7 : Show that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant if the range of f is contained in a circle.

Solution. Suppose the range of f is contained in a circle $|z-z_0| = R$. Then the range of the holomorphic function $g = \frac{1}{R}(f - z_0)$ is contained in the unit circle centered at 0. Suppose $g = u + iv$. Then $u^2 + v^2 = 1$. But then

$$2uu_x + 2vv_x = 0 = 2uu_y + 2vv_y.$$

By C-R equations, $uu_x - vu_y = 0 = uu_y + vu_x$. This gives the system

$$\begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

Since u and v can not be 0 simultaneousely (as $u^2 + v^2 = 1$), the determinant must be 0, and hence $u_x^2 + u_y^2 = 0$. We have seen in the class that $g' = u_x^2 + u_y^2$. This shows that $g' = 0$ or equivalently g is constant.

Exercise 0.8 : Show that a a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant if the range of f is contained in a parabola.

Solution. Without loss of generality, assume that the range of f is contained in the parabola $(Y - k)^2 = p(X - h)$ for real constants p, h and k with $p \neq 0$. But then $g = f - (h + ik)$ is also holomorphic and satisfies $v^2 = pu$, where $g = u + iv$. Since $2vv_x = pu_x,\ 2vv_y = pu_y$, by C-R equations,

$$-2vu_y = pu_x,\ 2vu_x = pu_y,$$

which can be solved to obtain $-2v\frac{(2vu_x)}{p} = pu_x$, that is, $-4v^2u_x = p^2u_x$. Thus we have $(p^2 + 4v^2)u_x$ implying $u_x = 0$ and hence $u_y = 0$. This proves that u is constant. It can be seen similarly that v is constant.