Chapter 5

Uncertainty Analysis

5.1 INTRODUCTION

Whenever we plan a test or later report a test result, we need to know something about the quality of the results. Uncertainty analysis provides a methodical approach to estimating the quality of the results from an anticipated test or from a completed test. This chapter focuses on how to estimate the "$\pm$ what?" in a planned test or in a stated test result.

Suppose the competent dart thrower of Chapter 1 tossed several practice rounds of darts at a bull's-eye. This would give us a good idea of the thrower's tendencies. Then, let the thrower toss another round. Without looking, can you guess where the darts will hit? Test measurements that include systematic and random error components are much like this. We can calibrate a measurement system to get a good idea of its behavior and accuracy. However, from the calibration we can only estimate how well any measured value might estimate the actual "true" value in a subsequent measurement.

Errors are a property of the measurement. Measurement is the process of assigning a value to a physical variable based on a sampling from the population of that variable. Error causes a difference between the value assigned by measurement and the true value of the population of the variable. Measurement errors are introduced from various elements, for example, the individual instrument calibrations, the data set finite statistics, and the approach used. But because we do not know the true value and we only know the measured values, we do not know the exact values of errors. Instead, we draw from what we do know about the measurement to estimate a range of probable error. This estimate is an assigned value called the *uncertainty*. The uncertainty describes an interval about the measured value within which we suspect that the true value must fall with a stated probability. *Uncertainty analysis* is the process of identifying, quantifying, and combining the errors.

Uncertainty is a property of the result. The outcome of a measurement is a result, and the uncertainty quantifies the quality of that result. Uncertainty analysis provides a powerful design tool for evaluating different measurement systems and methods, designing a test plan, and reporting uncertainty. This chapter presents a systematic approach for identifying, quantifying, and combining the estimates of the errors in a measurement. While the chapter stresses the methodology of analyses, we emphasize the concomitant need for an equal application of critical thinking and professional judgment in applying the analyses. The quality of an uncertainty analysis depends on the engineer's knowledge of the test, the measured variables, the equipment, and the measurement procedures (1).

Errors are effects, and uncertainties are numbers. While errors are the effects that cause a measured value to differ from the true value, the uncertainty is an assigned numerical value that quantifies the probable range of these errors.

This chapter approaches uncertainty analysis as an evolution of information from test design through final data analysis. While the structure of the analysis remains the same at each step, the number of errors identified and their uncertainty values may change as more information becomes available. In fact, the uncertainty in the result may increase. There is no exact answer to an analysis, just the result from a reasonable approach using honest numbers. This is the nature of an uncertainty analysis.

There are two accepted professional documents on uncertainty analysis. The American National Standards Institute/American Society of Mechanical Engineers (ANSI/ASME) Power Test Codes (PTC) 19.1 Test Uncertainty (2) is the United States engineering test standard, and our approach favors that method. The International Organization on Standardization's "Guide to the Expression of Uncertainty in Measurement" (ISO GUM) (1) is an international metrology standard. The two differ in some terminology and how errors are cataloged. For example, PTC 19.1 refers to random and systematic errors, terms that classify errors by how they manifest themselves in the measurement. ISO GUM refers to type A and type B errors, terms that classify errors by how their uncertainties are estimated. These differences are real but they are not significant to the outcome. Once past the classifications, the two methods are quite similar. The important point is that the end outcome of an uncertainty analysis by either method will yield a similar result!

Upon completion of this chapter, the reader will be able to

- explain the relation between an error and an uncertainty,
- execute an appropriate uncertainty analysis regardless of the level and quantity of information available,
- explain the differences between systematic and random errors and treat their assigned uncertainties,
- analyze a test system and test approach from test design through data presentation to assign and propagate uncertainties, and
- propagate uncertainties to understand their impact on the final statement of a result.

5.2 MEASUREMENT ERRORS

In the discussion that follows, errors are grouped into two categories: systematic error and random error. We do not consider measurement blunders that result in obviously fallacious data—such data should be discarded.

Consider the repeated measurement of a variable under conditions that are expected to produce the same value of the measured variable. The relationship between the true value of the population and the measured data set, containing both systematic and random errors, can be illustrated as in Figure 5.1. The total error in a set of measurements obtained under seemingly fixed conditions can be described by the systematic errors and the random errors in those measurements. The systematic errors shift the sample mean away from the true mean by a fixed amount, and within a sample of many measurements, the random errors bring about a distribution of measured values about the sample mean. Even a so-called accurate measurement contains small amounts of systematic and random errors.

Measurement errors enter during all aspects of a test and obscure our ability to ascertain the information that we desire: the true value of the variable measured. If the result depends on more than one measured variable, these errors further propagate to the result. In Chapter 4, we stated that the best estimate of the true value sought in a measurement is provided by its sample mean value and

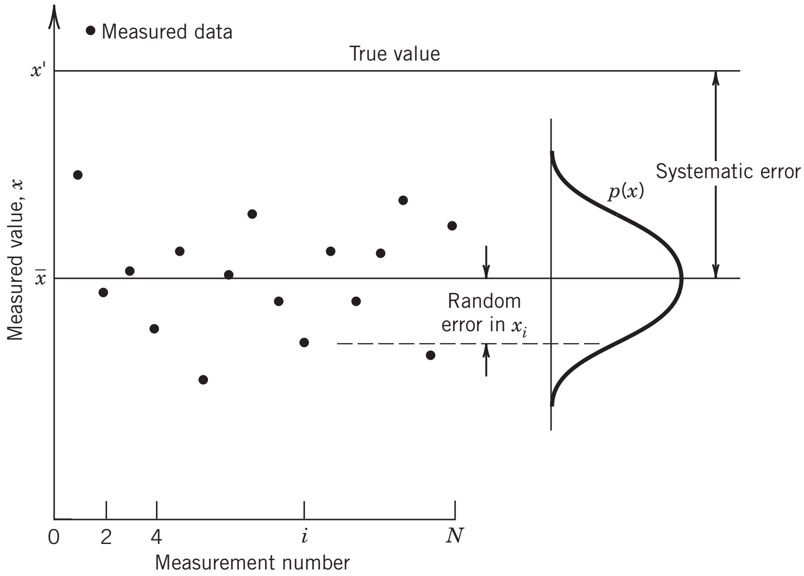

Figure 5.1 Distribution of errors on repeated measurements.

the uncertainty in that value,

$$x' = \overline{x} \pm u_x \quad (P\%) \tag{4.1}$$

But we considered only the random uncertainty due to the statistics of a measured data set. In this chapter, we extend this to uncertainty analysis so that the u_x term contains the uncertainties assigned to all known errors. Certain assumptions are implicit in an uncertainty analysis:

1. The test objectives are known and the measurement itself is a clearly defined process.
2. Any known corrections for systematic error have been applied to the data set, in which case the systematic uncertainty assigned is the uncertainty of the correction.
3. Except where stated otherwise, we assume a normal distribution of errors and reporting of uncertainties.
4. Unless stated otherwise, the errors are assumed to be independent (uncorrelated) of each other. But some errors are correlated, and we discuss how to handle these in Section 5.9.
5. The engineer has some "experience" with the system components.

In regards to item 5, by "experience" we mean that the engineer either has prior knowledge of what to expect from a system or can rely on the manufacturer's performance specifications or on information from the technical literature.

We might begin the design of an engineering test with an idea and some catalogs, and end the project after data have been obtained and analyzed. As with any part of the design process, the uncertainty analysis evolves as the design of the measurement system and process matures. We discuss uncertainty analysis for the following measurement situations: (1) design stage, where tests are planned but information is limited; (2) advanced stage or single measurement, where additional information about process control can be used to improve a design-stage uncertainty estimate; and (3) multiple measurements, where all available test information is combined to assess the uncertainty in a test result. The methods for situation 3 follow current engineering standards.

5.3 DESIGN-STAGE UNCERTAINTY ANALYSIS

Design-stage uncertainty analysis refers to an analysis performed in the formulation stage prior to a test. It provides only an estimate of the minimum uncertainty based on the instruments and method chosen. If this uncertainty value is too large, then alternate approaches will need to be found. So, it is useful for selecting instruments and selecting measurement techniques. At the test design stage, the measurement system and associated procedures may be but a concept. Often little may be known about the instruments, which in many cases might still be just pictures in a catalog. Major facilities may need to be built and equipment ordered with a considerable lead time. Uncertainty analysis at this time is used to assist in selecting equipment and test procedures based on their relative performance. In the design stage, distinguishing between systematic and random errors might be too difficult to be of concern. So for this initial discussion, consider only sources of error and their assigned uncertainty in general. A measurement system usually consists of sensors and instruments, each with their respective contributions to system uncertainty. We first discuss individual contributions to uncertainty.

Even when all errors are otherwise zero, a measured value must be affected by our ability to resolve the information provided by the instrument. This *zero-order uncertainty* of the instrument, u_0, assumes that the variation expected in the measured values will be only that amount due to instrument resolution and that all other aspects of the measurement are perfectly controlled. Essentially, u_0 is an estimate of the expected random uncertainty caused by the data scatter due to instrument resolution.

In lieu of any other information, assign a numerical value to u_0 of one-half of the analog instrument resolution[1] or to equal to its digital least count. This value will reasonably represent the uncertainty interval on either side of the reading with a probability of 95%. Then,

$$u_0 = \frac{1}{2}\text{resolution} = 1\,\text{LSD} \tag{5.1}$$

where LSD refers to the least significant digit of the readout.

Note that because we assume that the error has a normal distribution with its uncertainty applied equally to either side of the reading, we could write this as

$$u_0 = \pm\frac{1}{2}\text{ resolution (95\%)}$$

But unless specifically stated otherwise, the $\pm$ sign for the uncertainty will be assumed for any computed uncertainty value and applied only when writing the final uncertainty interval of a result.

The second piece of information that is usually available is the manufacturer's statement concerning instrument error. We can assign this stated value as the *instrument uncertainty*, u_c. Essentially, u_c is an estimate of the expected systematic uncertainty due to the instrument. If no probability level is provided with such information, a 95% level can be assumed.

Sometimes the instrument errors are delineated into parts, each part due to some contributing factor (Table 1.1). A probable estimate in u_c can be made by combining the uncertainties of known errors in some reasonable manner. An accepted approach of combining uncertainties is termed the *root-sum-squares* (RSS) method.

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[1] It is possible to assign a value for u_0 that differs from one-half the scale resolution. Discretion should be used. Instrument resolution is likely described by either a normal or a rectangular distribution, depending on the instrument.

Combining Elemental Errors: RSS Method

Each individual measurement error interacts with other errors to affect the uncertainty of a measurement. This is called *uncertainty propagation*. Each individual error is called an ''elemental error.'' For example, the sensitivity error and linearity error of a transducer are two elemental errors, and the numbers associated with these are their uncertainties. Consider a measurement of x that is subject to some K elements of error, each of uncertainty u_k, where $k = 1, 2, \ldots, K$. A realistic estimate of the uncertainty in the measured variable, u_x, due to these elemental errors can be computed using the RSS *method* to propagate the elemental uncertainties:

$$
\begin{aligned}
u_x &= \sqrt{u_1^2 + u_2^2 + \cdots + u_k^2} \\
&= \sqrt{\sum_{k=1}^{K} u_k^2} \quad (P\%)
\end{aligned}
\tag{5.2}
$$

The RSS method of combining uncertainties is based on the assumption that the square of an uncertainty is a measure of the variance (i.e., s^2) assigned to an error, and the propagation of these variances yields a probable estimate of the total uncertainty. Note that it is imperative to maintain consistency in the units of each uncertainty in Equation 5.2 and that each uncertainty term be assigned at the same probability level.

In test engineering, it is common to report final uncertainties at a 95% probability level ($P\% = 95\%$), and this is equivalent to assuming the probability covered by two standard deviations. When a probability level equivalent to a spread of one standard deviation is used, this uncertainty is called the ''standard'' uncertainty (1, 2). For a normal distribution, a standard uncertainty is a 68% probability level. Whatever level is used, consistency is important.

Design-Stage Uncertainty

The *design-stage uncertainty*, u_d, for an instrument or measurement method is an interval found by combining the instrument uncertainty with the zero-order uncertainty,

$$u_d = \sqrt{u_0^2 + u_c^2} \quad (P\%) \tag{5.3}$$

This procedure for estimating the design-stage uncertainty is outlined in Figure 5.2. The design-stage uncertainty for a test system is arrived at by combining each of the design-stage uncertainties for each component in the system using the RSS method while maintaining consistency of units and confidence levels.

Due to the limited information used, a design-stage uncertainty estimate is intended only as a guide for selecting equipment and procedures before a test, and is never used for reporting results. *If additional information about other measurement errors is known at the design stage, then their*

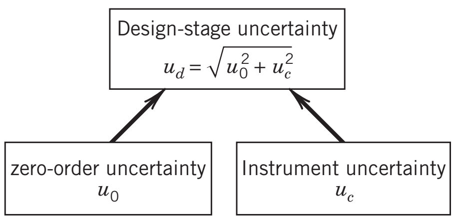

Figure 5.2 Design-stage uncertainty procedure in combining uncertainties.

uncertainties can and should be used to adjust Equation 5.3. So Equation 5.3 provides a minimum value for design stage uncertainty. In later sections of this chapter, we move towards more thorough uncertainty analyses.

Example 5.1

Consider the force measuring instrument described by the following catalog data. Provide an estimate of the uncertainty attributable to this instrument and the instrument design-stage uncertainty.

Resolution:	0.25 N
Range:	0 to 100 N
Linearity error:	within 0.20 N over range
Hysteresis error:	within 0.30 N over range

KNOWN Catalog specifications

ASSUMPTIONS Instrument uncertainty at 95% level; normal distribution

FIND u_c, u_d

SOLUTION We follow the procedure outlined in Figure 5.2. An estimate of the instrument uncertainty depends on the uncertainty assigned to each of the contributing elemental errors of linearity, e_1, and hysteresis, e_2, respectively assigned as

$$u_1 = 0.20\,\mathrm{N} \quad u_2 = 0.30\,\mathrm{N}$$

Then using Equation 5.2 with $K = 2$ yields

$$\begin{aligned} u_c &= \sqrt{(0.20)^2 + (0.30)^2} \\ &= 0.36\,\mathrm{N} \end{aligned}$$

The instrument resolution is given as 0.25 N, from which we assume $u_0 = 0.125$ N. From Equation 5.3, the design-stage uncertainty of this instrument would be

$$\begin{aligned} u_d &= \sqrt{u_0^2 + u_c^2} = \sqrt{(0.36)^2 + (0.125)^2} \\ &= \pm 0.38\,\mathrm{N} \quad (95\%) \end{aligned}$$

COMMENT The design-stage uncertainty for this instrument is simply an estimate based on the "experience" on hand, in this case the manufacturer's specifications. Additional information might justify modifying these numbers or including additional known elemental errors into the analysis.

Example 5.2

A voltmeter is used to measure the electrical output signal from a pressure transducer. The nominal pressure is expected to be about 3 psi (3 lb/in.$^2 = 0.2$ bar). Estimate the design-stage uncertainty in this combination. The following information is available:

Voltmeter	
Resolution:	10 μV
Accuracy:	within 0.001% of reading
Transducer	
Range:	±5 psi (∼±0.35 bar)
Sensitivity:	1 V/psi
Input power:	10 VDC ± 1%
Output:	±5 V
Linearity error:	within 2.5 mV/psi over range
Sensitivity error:	within 2 mV/psi over range
Resolution:	negligible

KNOWN Instrument specifications

ASSUMPTIONS Values at 95% probability; normal distribution of errors

FIND u_c for each device and u_d for the measurement system

SOLUTION The procedure in Figure 5.2 is used for both instruments to estimate the design-stage uncertainty in each. The resulting uncertainties are then combined using the RSS approximation to estimate the system u_d.

The uncertainty in the voltmeter at the design stage is given by Equation 5.3 as

$$(u_d)_E = \sqrt{(u_o)_E^2 + (u_c)_E^2}$$

From the information available,

$$(u_0)_E = 5\ \mu\text{V}$$

For a nominal pressure of 3 psi, we expect to measure an output of 3 V. Then,

$$(u_c)_E = (3\ \text{V} \times 0.00001) = 30\ \mu\text{V}$$

so that the design-stage uncertainty in the voltmeter is

$$(u_d)_E = 30.4\ \mu\text{V}$$

The uncertainty in the pressure transducer output at the design stage is also found using Equation 5.2. Assuming that we operate within the input power range specified, the instrument output uncertainty can be estimated by considering the uncertainty in each of the instrument elemental errors of linearity, e_1, and sensitivity, e_2:

$$\begin{aligned}(u_c)_p &= \sqrt{u_1^2 + u_2^2} \\ &= \sqrt{(2.5\ \text{mV/psi} \times 3\ \text{psi})^2 + (2\ \text{mV/psi} \times 3\ \text{psi})^2} \\ &= 9.61\ \text{mV}\end{aligned}$$

Since $(u_0) \approx 0$ V/psi, the design-stage uncertainty in the transducer in terms of indicated voltage is $(u_d)_p = 9.61$ mV.

Finally, u_d for the combined system is found by using the RSS method for the design-stage uncertainties of the two devices. The design-stage uncertainty in pressure as indicated by this measurement system is estimated to be

$$\begin{aligned} u_d &= \sqrt{(u_d)_E^2 + (u_d)_p^2} \\ &= \sqrt{(0.030\,\mathrm{mV})^2 + (9.61\,\mathrm{mV})^2} \\ &= \pm 9.61\,\mathrm{mV} \quad (95\%) \end{aligned}$$

But since the sensitivity is 1 V/psi, the uncertainty in pressure can be stated as

$$u_d = \pm 0.0096\,\mathrm{psi} \quad (95\%)$$

COMMENT Note that essentially all of the uncertainty is due to the transducer. Design-stage uncertainty analysis shows us that a better transducer, not a better voltmeter, is needed if we must improve the uncertainty in this measurement!

5.4 IDENTIFYING ERROR SOURCES

Design-stage uncertainty provides essential information to assess instrument selection and, to a limited degree, the measurement approach. But it does not address all of the possible errors that influence a measured result. Here we provide a helpful checklist of common errors. It is not necessary to classify error sources as we do here, but it is a good bookkeeping practice.

Consider the measurement process as consisting of three distinct stages: calibration, data acquisition, and data reduction. Errors that enter during each of these steps can be grouped under their respective error source heading: (1) calibration errors, (2) data-acquisition errors, and (3) data-reduction errors. Within each of these three *error source groups*, list the types of errors encountered. Such errors are the elemental errors of the measurement. Later, we will assign uncertainty values to each error. Do not become preoccupied with these groupings. Use them as a guide. If you place an error in an ''incorrect'' group, it is okay. The final uncertainty is not changed!

Calibration Errors

Calibration in itself does not eliminate system errors but it can help to quantify the uncertainty in the particular pieces of equipment used. Calibration errors include those elemental errors that enter the measuring system during its calibration. *Calibration errors* tend to enter through three sources: (1) the standard or reference value used in the calibration, (2) the instrument or system under calibration, and (3) the calibration process. For example, the laboratory standard used for calibration contains some inherent uncertainty, and this is passed along with the input value on which the calibration is based. Measuring system errors, such as linearity, repeatability, hysteresis, and so forth, contribute uncertainty. Depending on how the calibration is done, there can be a difference between the value supplied by the standard and the value actually sensed by the measuring system. These effects are built into the calibration data. In Table 5.1, we list the common elemental errors contributing to this error source group.

Table 5.1 Calibration Error Source Group

Element	Error Source[a]
1	Standard or reference value errors
2	Instrument or system errors
3	Calibration process errors
4	Calibration curve fit (or see Table 5.3)
etc.	

[a]Systematic error or random error in each element.

Data-Acquisition Errors

An error that arises during the act of measurement is listed as a data-acquisition error. These errors include sensor and instrument errors unaccounted for by calibration; uncontrolled variables, such as changes or unknowns in measurement system operating conditions; and sensor installation effects on the measured variable. In addition, the quality of the statistics of sampled variables is affected by sample size and assumed distributions, and any temporal and spatial variations, all contributing to uncertainty. We list some common elemental errors from this source in Table 5.2.

Data-Reduction Errors

Curve fits and correlations with their associated unknowns (Section 4.6) introduce data-reduction errors into test results. Also, truncation errors, interpolation errors, and errors from assumed models or functional relationships affect the quality of a result. We list elemental errors typical of this error source group in Table 5.3.

Table 5.2 Data-Acquisition Error Source Group

Element	Error Source[a]
1	Measurement system operating conditions
2	Sensor–transducer stage (instrument error)
3	Signal conditioning stage (instrument error)
4	Output stage (instrument error)
5	Process operating conditions
6	Sensor installation effects
7	Environmental effects
8	Spatial variation error
9	Temporal variation error
etc.	

[a]Systematic error or random error in each element.

Note: A total input-to-output measurement system calibration combines elements 2, 3, 4, and possibly 1 within this error source group.

Table 5.3 Data-Reduction Error Source Group

Element	Error Source[a]
1	Curve fit error
2	Truncation error
3	Modeling error
etc.	

[a]Systematic error or random error in each element.

5.5 SYSTEMATIC AND RANDOM ERRORS

Systematic Error

A systematic error[2] remains constant in repeated measurements under fixed operating conditions. A systematic error may cause either a high or a low offset in the estimate of the true value of the measured variable. Because its effect is constant, it can be difficult to estimate the value of a systematic error or in many cases even recognize its presence. Accordingly, an estimate of the range of systematic error is represented by an interval, defined as $\pm b$. The value b is the estimate of the *systematic standard uncertainty*. Its interval has a confidence level of one standard deviation, equivalent to a probability level of 68% for a normal distribution. The *systematic uncertainty* at any confidence level is given by $t_{\nu,P}b$, or simply tb. The interval defined by the systematic uncertainty at the 95% probability level is written as

$$\pm B = \pm 2b \quad (95\%) \tag{5.4}$$

which assigns a value of $t = 2$. This t value assumes large degrees of freedom in an assigned systematic uncertainty for which $t = 1.96$, which is rounded to 2 for convenience (2).

The reader has probably experienced systematic errors in measurements. Improperly using the floating tang at the end of a metal tape measure will offset the measurement, a systematic error. A more obvious example is reporting the barefoot height of a person based on a measurement taken while the person was wearing high-heeled shoes. In this case this systematic error, a data-acquisition error, is the height of the heels. But these errors are obvious!

Consider a home bathroom scale; does it have a systematic error? How might we assign an uncertainty to its indicated weight? Perhaps we can calibrate the scale using calibrated standard masses, account for local gravitational acceleration, and correct the output, thereby estimating the systematic error of the measurement (i.e., direct calibration against a local standard). Or perhaps we can compare it to a measurement taken in a physician's office or at the gym and compare each reading (i.e., a sort of interlaboratory comparison). Or perhaps we can carefully measure the person's volume displacement in water and compare the results to estimate differences (i.e., concomitant methodology). Or, we can use the specification provided by the manufacturer (i.e., experience). Without any of the above, what value would we assign? Would we even suspect a systematic error?

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[2] This error was called a "bias" error in engineering documents prior to the 1990s.

But let us think about this. The insidious aspect of systematic error has been revealed. Why doubt a measurement indication and suspect a systematic error? The mean value of the data set may be offset from some true value that we do not know. Figuratively speaking, there will be no shoe heels staring at us. Experience teaches us to think through each measurement carefully because systematic error is always present at some magnitude. We see that it is difficult to estimate systematic error without comparison, so a good design should include some means to estimate it. Various methodologies can be utilized: (1) calibration, (2) concomitant methodology, (3) interlaboratory comparisons, or (4) judgment/experience. When available, calibration using a suitable standard and method can reduce instrument systematic error to predictable intervals and estimate its associated uncertainty. A quality instrument may come with a certified calibration certificate. Concomitant methodology, which is using different methods of estimating the same thing, allows for comparing the results. Concomitant methods that depend on different physical measurement principles are preferable, as are methods that rely on calibrations that are independent of each other. In this regard, analytical methods could be used for comparison[3] or at least to estimate the range of systematic error due to influential sources such as environmental conditions, instrument response errors, and loading errors. Lastly, an elaborate but good approach is through interlaboratory comparisons of similar measurements, an excellent replication method. This approach introduces different instruments, facilities, and personnel into an otherwise similar measurement procedure. The variations in the results between facilities provide a statistical estimate of the systematic uncertainty (2).

In lieu of the above, a judgment value based on past experience may have to be assigned; these values are usually understood to be made at the 95% confidence level. For example, the value that first came to mind to you in the bathroom scale example above likely covered a 95% interval.

Note that calibration cannot eliminate systematic error, but it may reduce uncertainty. Consider the calibration of a temperature transducer against a National Institute of Standards and Technology (NIST) standard certified to be correct to within 0.01°C. If the calibration data show that the transducer output has a systematic offset of 0.2°C relative to the standard, then we would just correct all the data obtained with this transducer by 0.2°C. Simple enough, we correct it! But the standard itself still has an intrinsic systematic uncertainty of 0.01°C, and this uncertainty remains in the calibrated transducer. We would include any uncertainty in the correction value applied.

Random Error

When repeated measurements are made under fixed operating conditions, random errors manifest themselves as scatter of the measured data. Random error[4] is introduced through the repeatability and resolution of the measurement system components, calibration, and measurement procedure and technique; by the measured variable's own temporal and spatial variations; and by the variations in the process operating and environmental conditions from which measurements are taken.

The estimate of the probable range of a random error is given by its random uncertainty. The *random standard uncertainty,* $s_{\overline{x}}$, is defined by the interval given by $\pm s_{\overline{x}}$, where

$$s_{\overline{x}} = s_x / \sqrt{N} \tag{5.5}$$

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[3] Smith and Wenhofer (3) provide examples for determining jet engine thrust, and several complementary measurements are used with an energy balance to estimate the uncertainty assigned to the systematic error.

[4] This error was called a "precision" error in engineering documents prior to the 1990s.

with degrees of freedom $\nu = N - 1$ and assuming the errors are normally distributed.[5] The interval has a confidence level of one standard deviation, equivalent to a probability of 68% for a population of x having a normal distribution. The *random uncertainty* at a desired confidence level is defined by the interval $\pm t_{\nu,P} s_{\overline{x}}$, where t is found from Table 4.4.

5.6 UNCERTAINTY ANALYSIS: ERROR PROPAGATION

Suppose we want to determine how long it would take to fill a swimming pool from a garden hose. One way is to measure the time required to fill a bucket of known volume to estimate the flow rate from the garden hose. Armed with a measurement of the volume of the pool, we can calculate the time to fill the pool. Clearly, small errors in estimating the flow rate from the garden hose would translate into large differences in the time required to fill the pool! Here we are using measured values, the flow rate and volume, to estimate a result, the time required to fill the pool.

Very often in engineering, results are determined through a functional relationship with measured values. For example, we just calculated a flow rate above by measuring time, t, and bucket volume, $\forall$, since $Q = f(t, \forall) = \forall/t$. But how do uncertainties in either measured quantity contribute to uncertainty in flow rate? Is the uncertainty in Q more sensitive to uncertainty in volume or in time? More generally, how are uncertainties in variables propagated to a calculated result? We now explore these questions.

Propagation of Error

A general relationship between some dependent variable y and a measured variable x, that is, $y = f(x)$, is illustrated in Figure 5.3. Now suppose we measure x a number of times at some operating condition so as to establish its sample mean value and the uncertainty due to random error in this mean value, $t_{\nu,P} s_{\overline{x}}$, which for convenience we write simply as $ts_{\overline{x}}$. This implies that, neglecting other random and systematic errors, the true value for x lies somewhere within the interval $\overline{x} \pm ts_{\overline{x}}$. It is reasonable to assume that the true value of y, which is determined from the measured values of x, falls within the interval defined by

$$\overline{y} \pm \delta y = f(\overline{x} \pm ts_{\overline{x}}) \tag{5.6}$$

Expanding this as a Taylor series yields

$$\overline{y} \pm \delta y = f(\overline{x}) \pm \left[\left(\frac{dy}{dx}\right)_{x=\overline{x}} ts_{\overline{x}} + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{x=\overline{x}} (ts_{\overline{x}})^2 + \cdots \right] \tag{5.7}$$

By inspection, the mean value for y must be $f(\overline{x})$ so that the term in brackets estimates $\pm\delta y$. A linear approximation for δy can be made, which is valid when $ts_{\overline{x}}$ is small and neglects the higher order terms in Equation 5.7, as

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=\overline{x}} ts_{\overline{x}} \tag{5.8}$$

The derivative term, $(dy/dx)_{x=\overline{x}}$, defines the slope of a line that passes through the point specified by $\overline{x}$. For small deviations from the value of $\overline{x}$, this slope predicts an acceptable, approximate

[5] The estimate of standard uncertainty when estimated from a rectangular distribution (11) is $(b - a)/\sqrt{12}$, where b and a were defined in Table 4.2. The probability is about 58%.

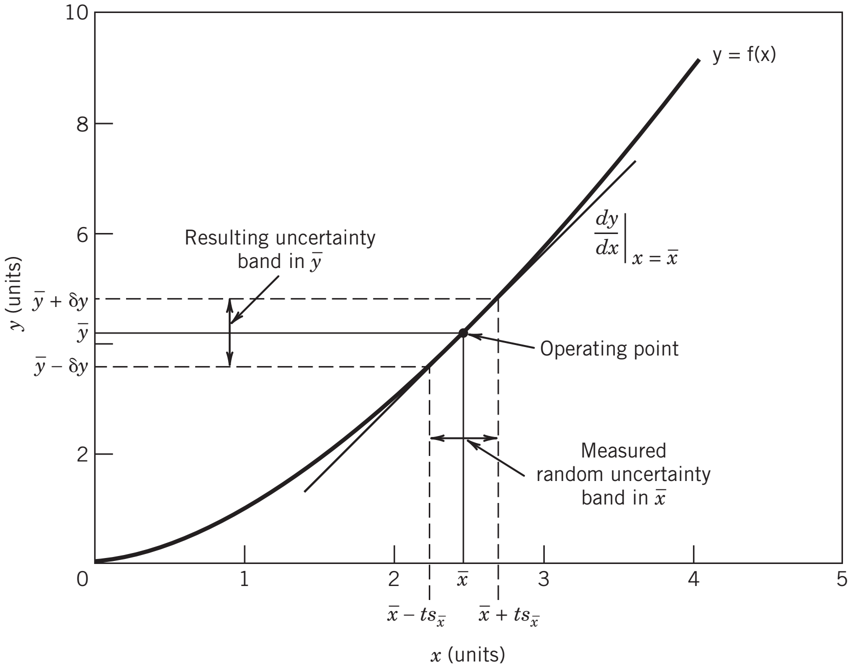

Figure 5.3 Relationship between a measured variable and a resultant calculated using the value of that variable.

relationship between $ts_{\overline{x}}$ and δy. The derivative term is a measure of the sensitivity of y to changes in x. Since the slope of the curve can be different for different values of x, it is important to evaluate the slope using a representative value of x. The width of the interval defined by $\pm ts_{\overline{x}}$ corresponds to $\pm\delta y$, within which we should expect the true value of y to lie. Figure 5.3 illustrates the concept that errors in a measured variable are propagated through to a resultant variable in a predictable way. In general, we apply this analysis to the errors that contribute to the uncertainty in x, written as u_x. The uncertainty in x is related to the uncertainty in the resultant y by

$$u_y = \left(\frac{dy}{dx}\right)_{x=\overline{x}} u_x \tag{5.9}$$

Compare the similarities between Equations 5.8 and 5.9 and in Figure 5.3.

This idea can be extended to multivariable relationships. Consider a result R, which is determined through some functional relationship between independent variables $x_1, x_2, \ldots, x_L$ defined by

$$R = f_1\{x_1, x_2, \ldots, x_L\} \tag{5.10}$$

where L is the number of independent variables involved. Each variable contains some measure of uncertainty that affects the result. The best estimate of the true mean value R' would be stated as

$$R' = \overline{R} \pm u_R \quad (\text{P\%}) \tag{5.11}$$

where the sample mean of R is found from

$$\overline{R} = f_1\{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_L\} \tag{5.12}$$

and the uncertainty in $\overline{R}$ is found from

$$u_R = f_1\{u_{\overline{x}_1}, u_{\overline{x}_2}, \ldots, u_{\overline{x}_L}\} \tag{5.13}$$

In Equation 5.13, each $u_{\overline{x}_i}$, $i = 1, 2, \ldots, L$ represents the uncertainty associated with the best estimate of x_1 and so forth through x_L. The value of u_R reflects the contributions of the individual uncertainties as they are propagated through to the result.

A general sensitivity index, θ_i, results from the Taylor series expansion, Equation 5.9, and the functional relation of Equation 5.10 and is given by

$$\theta_i = \frac{\partial R}{\partial x_{i_{x=\overline{x}}}} \quad i = 1, 2, \ldots, L \tag{5.14}$$

The sensitivity index relates how changes in each x_i affect R. Equation 5.14 can also be estimated numerically using finite differencing methods (5), which can be easily done within a spreadsheet or symbolic software package. The index is evaluated using either the mean values or, lacking these estimates, the expected nominal values of the variables.

The contribution of the uncertainty in x to the result R is estimated by the term $\theta_i u_{\overline{x}_i}$. The most probable estimate of u_R is generally accepted as that value given by the second power relation (4), which is the square root of the sum of the squares (RSS). The propagation of uncertainty in the variables to the result is by

$$u_R = \left[\sum_{i=1}^{L} (\theta_i u_{\overline{x}_i})^2\right]^{1/2} \tag{5.15}$$

Sequential Perturbation

A numerical approach can also be used to estimate the propagation of uncertainty through to a result that circumvents the direct differentiation of the functional relations (6). The approach is handy to reduce data already stored in discrete form.

The method uses a finite difference method to approximate the derivatives:

1. Based on measurements for the independent variables under some fixed operating condition, calculate a result R_o where $R_o = f(x_1, x_2, \ldots, x_L)$. This value fixes the operating point for the numerical approximation (e.g., see Fig. 5.3).
2. Increase the independent variables by their respective uncertainties and recalculate the result based on each of these new values. Call these values R_i^+. That is,

$$\begin{aligned} R_1^+ &= f(x_1 + u_{x1}, x_2, \ldots, x_L), \\ R_2^+ &= f(x_1, x_2 + u_{x2}, \ldots, x_L), \ldots \\ R_L^+ &= f(x_1, x_2, \ldots, x_L + u_{xL}), \end{aligned} \tag{5.16}$$

3. In a similar manner, decrease the independent variables by their respective uncertainties and recalculate the result based on each of these new values. Call these values R_i^-.
4. Calculate the differences δR_i^+ and δR_i^- for $i = 1, 2, \ldots, L$

$$\begin{aligned} \delta R_i^+ &= R_i^+ - R_o \\ \delta R_i^- &= R_i^- - R_o \end{aligned} \tag{5.17}$$

5. Evaluate the approximation of the uncertainty contribution from each variable,

$$\delta R_i = \frac{\delta R_i^+ - \delta R_i^-}{2} \approx \theta_i u_i \tag{5.18}$$

Then, the uncertainty in the result is

$$u_R = \left[\sum_{i=1}^{L} (\delta R_i)^2\right]^{1/2} \tag{5.19}$$

Equations 5.15 and 5.19 provide two methods for estimating the propagation of uncertainty to a result. In most cases, each equation yields nearly the identical result and the choice of method is left to the user. The method can also be used to estimate just the sensitivity index of Equation 5.14 (2). In this case, steps 2 and 3 would apply a small deviation value, typically 1% of the nominal value of the variable, used in place of the actual uncertainty to estimate the derivative (5).

We point out that sometimes either method may calculate unreasonable estimates of u_R. When this happens the cause can be traced to a sensitivity index that changes rapidly with small changes in the independent variable x_i coupled with a large value of the uncertainty u_{x_i}. This occurs when the operating point is close to an minima or maxima inflection in the functional relationship. In these situations, the engineer should examine the cause and extent of the variation in sensitivity and use a more accurate approximation for the sensitivity, including using the higher order terms in the Taylor series of Equation 5.7.

In subsequent sections, we develop methods to estimate the uncertainty values from available information.

Example 5.3

For a displacement transducer having the calibration curve, $y = KE$, estimate the uncertainty in displacement y for $E = 5.00$ V, if $K = 10.10$ mm/V with $u_K = \pm 0.10$ mm/V and $u_E = \pm 0.01$ V at 95% confidence.

KNOWN $y = KE$

$E = 5.00$ V $\quad u_E = 0.01$ V

$K = 10.10$ mm/V $\quad u_K = 0.10$ mm/V

FIND u_y

SOLUTION Based on Equations 5.12 and 5.13, respectively,

$$\bar{y} = f\left(\overline{E}, \overline{K}\right) \quad \text{and} \quad u_y = f(u_E, u_K)$$

From Equation 5.15, the uncertainty in the displacement at $y = KE$ is

$$u_y = \left[(\theta_E u_E)^2 + (\theta_K u_K)^2\right]^{1/2}$$

where the sensitivity indices are evaluated from Equation 5.14 as

$$\theta_E = \frac{\partial y}{\partial E} = K \quad \text{and} \quad \theta_K = \frac{\partial y}{\partial K} = E$$

or we can write Equation 5.15 as

$$u_y = \left[(Ku_E)^2 + (Eu_K)^2\right]^{1/2}$$

The operating point occurs at the nominal or the mean values of $E = 5.00$ V and $y = 50.50$ mm. With $E = 5.00$ V and $K = 10.10$ mm/V and substituting for u_E and u_K, evaluate u_y at its operating point:

$$u_y\big|_{y=50.5} = \left[(0.10)^2 + (0.50)^2\right]^{1/2} = 0.51 \text{ mm}$$

Alternatively, we can use sequential perturbation. The operating point for the perturbation is again $y = R_o = 50.5$ mm. Using Equations 5.16 through 5.18 gives

i	x_i	R_i^+	R_i^-	δR_i^+	δR_i^-	δR_i
1	E	50.60	50.40	0.10	−0.10	0.10
2	K	51.00	50.00	0.50	−0.50	0.50

Then, using Equation 5.19,

$$u_y\big|_{y=50.5} = \left[(0.10)^2 + (0.50)^2\right]^{1/2} = 0.51 \text{ mm}$$

The two methods give the identical result. We state the calculated displacement in the form of Equation 5.11 as

$$y' = 50.50 \pm 0.51 \text{ mm} \quad (95\%)$$

Monte Carlo Method

A Monte Carlo simulation provides another effective way to estimate the propagation of the uncertainties in the independent variables to the uncertainty in a result. As presented in Chapter 4, the outcomes of a converged Monte Carlo simulation are the statistics of the predicted population in a result R (i.e., $\overline{x}$, s_x, $s_{\overline{x}}$) from which we calculate the random uncertainty in the result.

Generally, convergence is claimed when the computed standard deviation no longer changes by 1% to 5%. As one test for convergence, we define a numerical tolerance Δ, as being one-half of the least significant digit of the estimated standard uncertainty ($s_{\overline{x}}$),

$$\Delta = \frac{1}{2} LSD \tag{5.20}$$

For example, if the estimate of $s_{\overline{x}}$ is 2 units, then $\Delta = 0.5$ units (i.e., $^1/_2$ of the 10^0 digit); if $s_{\overline{x}}$ is 0.2 units, then $\Delta = 0.05$ units. The Monte Carlo simulation is converged when $2s_x < \Delta$ (7).

5.7 ADVANCED-STAGE UNCERTAINTY ANALYSIS

In designing a measurement system, a pertinent question is, How would it affect the result if this particular aspect of the technique or equipment were changed? In design-stage uncertainty analysis, we only considered the errors due to a measurement system's resolution and estimated instrument calibration errors. But if additional information is available, we can get a better idea of the uncertainty in a measurement. So, an advanced-stage uncertainty analysis permits taking design-stage analysis further by considering procedural and test control errors that affect the measurement. We consider it as a method for a thorough uncertainty analysis when a large data set is not available. This is often the case in the early stages of a test program or for certain tests where repeating measurements may not be possible. Such an advanced-stage analysis, also known as *single-measurement uncertainty analysis* (3, 6), can be used: (1) in the advanced design stage of a test to estimate the expected uncertainty, beyond the initial design stage estimate; and (2) to report the results of a test program that involved measurements over a range of one or more parameters but with no or relatively few repeated measurements of the pertinent variables at each test condition. Essentially, the method assesses different aspects of the main test by quantifying potential errors though various well focused verification tests.

In this section, the goals are either to estimate the uncertainty in some measured value x or in some general result R through an estimation of the uncertainty in each of the factors that may affect x or R. We present a technique that uses a step-by-step approach for identifying and estimating the uncertainty associated with errors. We seek the combined value of the estimates at each step. We assume that the errors follow a normal distribution, but some errors might be better described by other distributions and these can be used. For example, calibration errors that specify only a range or operating condition errors that are related to establishing a controlled set point are well modeled by a rectangular distribution. The standard uncertainty for a rectangular distribution defined by the interval $(b - a)$ is given by

$$u_x = (b - a)/\sqrt{12} \tag{5.21}$$

which covers a 58% confidence level. Multiplying by 2 provides for two standard deviations coverage. If we assume that the errors propagate to the result with a normal distribution, then a coverage factor of 2 approximates the 95% confidence level for consistency.

Zero-Order Uncertainty

At zero-order uncertainty, all variables and parameters that affect the outcome of the measurement, including time, are assumed to be fixed except for the physical act of observation itself. Under such circumstances, any data scatter introduced upon repeated observations of the output value is the result of instrument resolution alone. The value u_0 estimates the extent of variation expected in the measured value when all influencing effects are controlled and is found using Equation 5.1. By itself, a zero-order uncertainty is inadequate for the reporting of test results.

Higher-Order Uncertainty

Higher-order uncertainty estimates consider the controllability of the test operating conditions and the variability of all measured variables. For example, at the first-order level, the effect of time as an extraneous variable in the measurement might be considered. That is, what would happen if we started the test, set the operating conditions, and sat back and watched? If a variation in the measured value is observed, then time is a factor in the test, presumably due to some extraneous influence affecting process control or simply inherent in the behavior of the variable being measured.

In practice, the uncertainty at this first level would be evaluated for each particular measured variable by operating the test facility at some single operating condition that would be within the range of conditions to be used during the actual tests. A set of data (say, $N \geq 30$) would be obtained under some set operating condition. The first-order uncertainty of our ability to estimate the true value of a measured value could be estimated as

$$u_1 = t_{\nu,P} s_{\overline{x}} \tag{5.22}$$

The uncertainty at u_1 includes the effects of resolution, u_o. So only when $u_1 = u_0$ is time not a factor in the test. In itself, the first-order uncertainty is inadequate for reporting of test results.

With each successive order, another factor identified as affecting the measured value is introduced into the analysis, thus giving a higher but more realistic estimate of the uncertainty. For example, at the second level it might be appropriate to assess the limits of the ability to duplicate the exact operating conditions and the consequences on the test outcome. Or perhaps spatial variations that affect the outcome are assessed, such as when a value from a point measurement is assigned to quantify a larger volume. These are a series of verification tests conducted to assess causality.

*N*th-Order Uncertainty

At the *N*th-order estimate, instrument calibration characteristics are entered into the scheme through the instrument uncertainty u_c. A practical estimate of the *N*th-order uncertainty u_N is given by

$$u_N = \left[u_c^2 + \left(\sum_1^{N-1} u_i^2 \right) \right]^{1/2} \quad (P\%) \tag{5.23}$$

Uncertainty estimates at the *N*th order allow for the direct comparison between results of similar tests obtained either using different instruments or at different test facilities. The procedure for a single-measurement analysis is outlined in Figure 5.4.

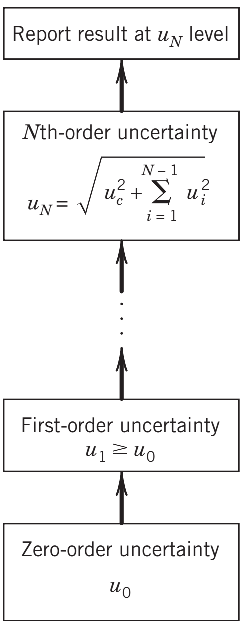

Figure 5.4 Advanced-stage and single-measurement uncertainty procedure in combining uncertainties.

Note that as a minimum, design-stage analysis includes only the effects found as a result of u_0 and u_c. It is those in-between levels that allow measurement procedure and control effects to be considered in the uncertainty analysis scheme. The Nth-order uncertainty estimate provides the uncertainty value sought in advanced design stage or in single-measurement analyses. It is an appropriate value to be used to report the results from single-measurement tests.

Example 5.4

As an exercise, obtain and examine a dial oven thermometer. How would you assess the zero- and first-order uncertainty in the measurement of the temperature of a kitchen oven using the device?

KNOWN Dial thermometer

ASSUMPTION Negligible systematic error in the instrument

FIND Estimate u_0 and u_1 in oven temperature

SOLUTION The zero-order uncertainty would be that contributed by the resolution error of the measurement system only. For example, most gauges of this type have a resolution of 10°C. So from Equation 5.1 we estimate

$$u_0 = 5^\circ\mathrm{C}$$

At the first order, the uncertainty would be affected by any time variation in the measured value (temperature) and variations in the operating conditions. If we placed the thermometer in the center of an oven, set the oven to some relevant temperature, allowed the oven to preheat to a steady condition, and then proceeded to record the temperature indicated by the thermometer at random intervals, we could estimate our ability to control the oven temperature. For J measurements, the first-order uncertainty of the measurement in this oven's mean temperature using this technique and this instrument would be, from Equation 5.22,

$$u_1 = t_{J-1,95} s_{\overline{T}}$$

COMMENT We could continue these verification tests. Let's suppose we are interested in how accurately we could set the oven mean temperature. Then the idea of setting the operating condition, the temperature, becomes important. We could estimate our ability to repeatedly set the oven to a desired mean temperature (time-averaged temperature). This could be done by changing the oven setting and then resetting the thermostat back to the original operating setting. If we were to repeat this sequence M times (reset thermostat, measure a set of data, reset thermostat, etc.), we could compute u_1 by

$$u_1 = t_{M(J-1),95} \langle s_{\overline{T}} \rangle$$

Note that now the variation in oven temperature with time at any setting is included in the pooled estimate. The effects of instrument calibration would enter at the Nth order through u_c. The uncertainty in oven temperature at some setting would be well approximated by the estimate from Equation 5.23:

$$u_N = \left(u_1^2 + u_c^2\right)^{1/2}$$

By inspection of u_0, u_c, and u_i, where $i = 1, 2, \ldots, N-1$, single-measurement analysis provides a capability to pinpoint those aspects of a test that contribute most to the overall uncertainty in the measurement, as well as a reasonable estimate of the uncertainty in a single measurement.

Example 5.5

A stopwatch is to be used to estimate the time between the start and end of an event. Event duration might range from several seconds to 10 minutes. Estimate the probable uncertainty in a time estimate using a hand-operated stopwatch that claims an accuracy of 1 min/month (95%) and a resolution of 0.01 s.

KNOWN $u_0 = 0.005\ \text{s} \quad (95\%)$

$u_c = 60\ \text{s/month} \quad (95\%\ \text{assumed})$

FIND u_d, u_N

SOLUTION The design-stage uncertainty gives an estimate of the suitability of an instrument for a measurement. At 60 s/month, the instrument accuracy works out to about 0.01 s/10 min of operation. This gives a design-stage uncertainty of

$$u_d = \left(u_o^2 + u_c^2\right)^{1/2} = \pm 0.01\ \text{s} \quad (95\%)$$

for an event lasting 10 minutes versus ±0.005 s (95%) for an event lasting 10 s. Note that instrument calibration error controls the longer duration measurement, whereas instrument resolution controls the short duration measurement.

But do instrument resolution and calibration error actually control the uncertainty in this measurement? The design-stage analysis does not include the data-acquisition error involved in the act of physically turning the watch on and off. But a first-order analysis might be run to estimate the uncertainty that enters through the procedure of using the watch. Suppose a typical trial run of 20 tries of simply turning a watch on and off suggests that the uncertainty in determining the duration of an occurrence is

$$u_1 = t_{\nu,p}\, s_{\bar{x}} = 0.05\ \text{s}$$

The uncertainty in measuring the duration of an event would then be better estimated by the *N*th-order uncertainty of Equation 5.23,

$$u_N = \left(u_1^2 + u_c^2\right)^{1/2} = \pm 0.05\ \text{s} \quad (95\%)$$

This estimate holds for periods of up to about two hours. Clearly, procedure controls the uncertainty, not the watch. This uncertainty estimate could be further improved by considering how well the operator can synchronize the watch action with the start and finish-line action.

Example 5.6

A flow meter can be calibrated by providing a known flow rate through the meter and measuring the meter output. One method of calibration with liquid systems is the use of a catch and time technique whereby a volume of liquid, after passing through the meter, is diverted to a tank for a measured period of time from which the flow rate volume/time, is computed. There are two procedures that can be used to determine the known flow rate Q in, say, ft^3/min:

1. The volume of liquid, $\forall$, caught in known time t can be measured.

   Suppose we arbitrarily set $t = 6$ s and assume that our available facilities can determine volume (at *N*th order) to 0.001 ft^3. Note: The chosen time value depends on how much liquid we can accommodate in the tank.

2. The time t required to collect 1 ft^3 of liquid can be measured.

Suppose we determine an Nth-order uncertainty in time of 0.15 s. In either case the same instruments are used. Determine which method is better to minimize uncertainty over a range of flow rates based on these preliminary estimates.

KNOWN $u_\forall = 0.001\ \text{ft}^3$

$u_t = 0.15\ \text{s}$

$Q = f(\forall, t) = \forall/t$

ASSUMPTIONS Flow diversion is instantaneous; 95% confidence levels.

FIND Preferred method

SOLUTION From the available information, the propagation of probable uncertainty to the result Q is estimated from Equation 5.15:

$$u_Q = \left[\left(\frac{\partial Q}{\partial \forall} u_\forall\right)^2 + \left(\frac{\partial Q}{\partial t} u_t\right)^2\right]^{1/2}$$

By dividing through by Q, we obtain the relative (fractional percent) uncertainty in flow rate u_Q/Q

$$\frac{u_Q}{Q} = \left[\left(\frac{u_\forall}{\forall}\right)^2 + \left(\frac{u_t}{t}\right)^2\right]^{1/2}$$

Representative values of Q are needed to solve this relation. Consider a range of flow rates, say, 1, 10, and 100 ft^3/min; the results are listed in the following table for both methods.

Q (ft^3/min)	t (s)	$\forall$ (ft^3)	$u_\forall/\forall$	u_t/t	$\pm u_Q/Q$
Method 1					
1	6	0.1	0.01	0.025	0.027
10	6	1.0	0.001	0.025	0.025
100	6	10.0	0.0001	0.025	0.025
Method 2					
1	60.0	1.0	0.001	0.003	0.003
10	6.0	1.0	0.001	0.025	0.025
100	0.6	1.0	0.001	0.250	0.250

In method 1, it is clear the uncertainty in time contributes the most to the relative uncertainty in Q, provided that the flow diversion is instantaneous. But in method 2, uncertainty in measuring either time or volume can contribute more to the uncertainty in Q, depending on the time sample length. The results for both methods are compared in Figure 5.5. For the conditions selected and preliminary uncertainty values used, method 2 would be a better procedure for flow rates up to 10 ft^3/min. At higher flow rates, method 1 would be better. However, the minimum uncertainty in method 1 is limited to 2.5%. The engineer may be able to reduce this uncertainty by improvements in the time measurement procedure.

COMMENT These results are without consideration of some other elemental errors that are present in the experimental procedure. For example, the diversion of the flow may not to occur instantaneously. A first-order uncertainty estimate could be used to estimate the added uncertainty

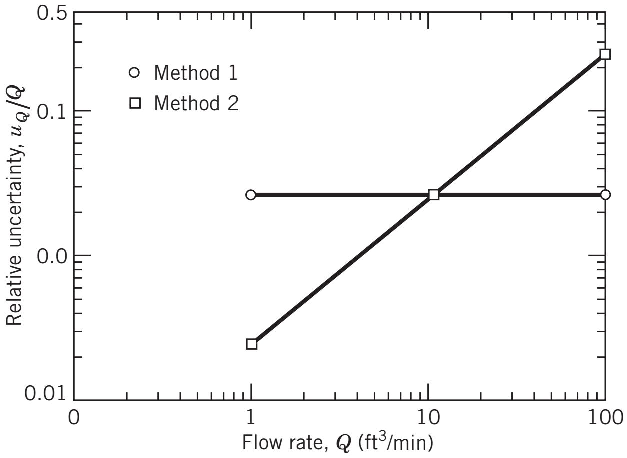

Figure 5.5 Uncertainty plot for the design analysis of Example 5.6.

to flow rate intrinsic to the time required to divert the liquid to and from the catch tank. If operator influence is a factor, it should be randomized in actual tests. Accordingly, the above calculations may be used as a guide in procedure selection.

Example 5.7

Repeat Example 5.6, using sequential perturbation for the operating conditions $\forall = 1$ ft^3 and $t = 6$ s.

SOLUTION The operating point is $R_o = Q = \forall/t = 0.1667$ ft^3/s. Then, solving Equations 5.16 to 5.18 gives:

i	x_i	R_i^+	R_i^-	δR_i^+	δR_i^+	δR_i
1	$\forall$	0.1668	0.1665	0.000167	−0.000167	0.000167
2	t	0.1626	0.1709	−0.00407	0.00423	0.00415

Applying Equation 5.19 gives the uncertainty about this operating point

$$u_Q = \left[0.00415^2 + 0.000167^2\right]^{1/2} = \pm 4.15 \times 10^{-3} \text{ ft}^3/\text{s} \quad (95\%)$$

or

$$u_Q/Q = \pm 0.025 \quad (95\%)$$

COMMENT These last two examples give similar results for the uncertainty aside from insignificant differences due to round-off in the computations.

5.8 MULTIPLE-MEASUREMENT UNCERTAINTY ANALYSIS

This section develops a method for estimating the uncertainty in the value assigned to a variable based on a set of measurements obtained under fixed operating conditions. The method parallels the uncertainty standards approved by professional societies and by NIST in the United States (2, 7) and is in harmony with international guidelines (1). The procedures assume that the errors follow a normal probability distribution, although the procedures are actually quite insensitive to deviations away from such behavior (4). Sufficient repetitions must be present in the measured data to assess random error; otherwise, some estimate of the magnitude of expected variation should be provided in the analysis.

Propagation of Elemental Errors

The procedures for multiple-measurement uncertainty analysis consist of the following steps:

- Identify the elemental errors in the measurement. As an aid, consider the errors in each of the three source groups (calibration, data acquisition, and data reduction).
- Estimate the magnitude of systematic and random error in each of the elemental errors.
- Calculate the uncertainty estimate for the result.

Considerable guidance can be obtained from Tables 5.1 to 5.3 for identifying the elemental errors. In multiple-measurement analysis, it is possible to divide the estimates for elemental errors into random and systematic uncertainties.

Consider the measurement of variable x, which is subject to elemental random errors each estimated by their random standard uncertainty, $(s_{\bar{x}})_k$, and systematic errors each estimated by their systematic standard uncertainty, $(b_{\bar{x}})_k$. Let subscript k, where $k = 1, 2, \ldots, K$, refer to each of up to K elements of error e_k. A method to estimate the uncertainty in x based on the uncertainties in each of the elemental random and systematic errors is given below and outlined in Figure 5.6.

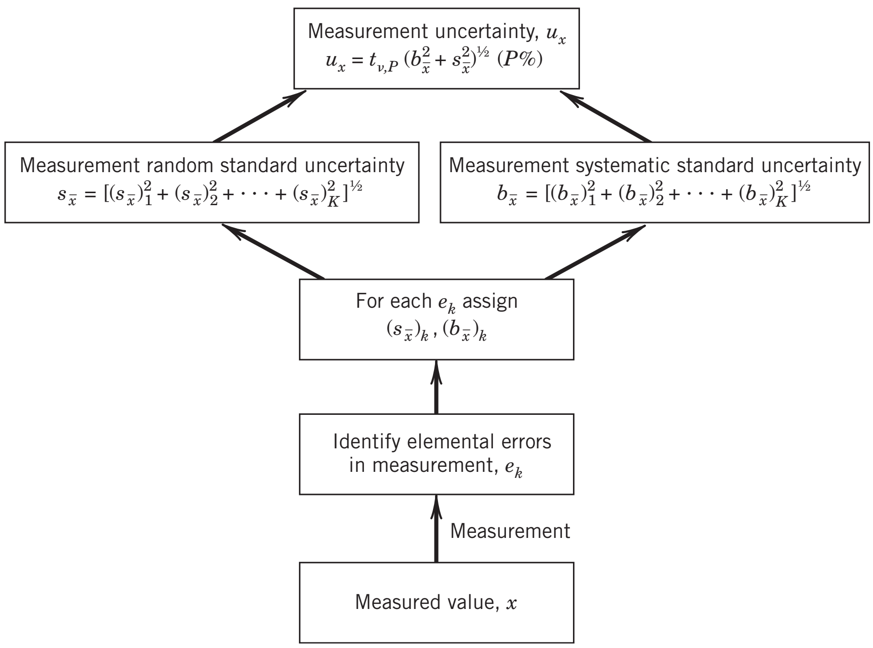

Figure 5.6 Multiple-measurement uncertainty procedure for combining uncertainties.

The *propagation of elemental random uncertainties* due to the K random errors in measuring x is given by the measurement random standard uncertainty, $s_{\bar{x}}$, as estimated by the RSS method of Equation 5.2:

$$s_{\bar{x}} = \left[(s_{\bar{x}})_1^2 + (s_{\bar{x}})_2^2 + \cdots + (s_{\bar{x}})_K^2 \right]^{1/2} \tag{5.24}$$

where each

$$(s_{\bar{x}})_k = s_{x_k} / \sqrt{N_k} \tag{5.25}$$

The *measurement random standard uncertainty* represents a basic measure of the uncertainty due to the known elemental errors affecting the variability of variable x at a one standard deviation confidence level. The degrees of freedom, ν, in the standard random uncertainty, is estimated using the Welch–Satterthwaite formula (2):

$$\nu = \frac{\left(\sum_{k=1}^{K} \left(s_{\bar{x}}^2\right)_k \right)^2}{\sum_{k=1}^{K} \left(\left(s_{\bar{x}}^4\right)_k / \nu_k \right)} \tag{5.26}$$

where k refers to each elemental error with its own $\nu_k = N_k - 1$.

The *propagation of elemental systematic uncertainties* due to K systematic errors in measuring x is treated in a similar manner. The measurement systematic standard uncertainty, $b_{\bar{x}}$, is given by

$$b_{\bar{x}} = \left[(b_{\bar{x}})_1^2 + (b_{\bar{x}})_2^2 + \cdots + (b_{\bar{x}})_K^2 \right]^{1/2} \tag{5.27}$$

The *measurement systematic standard uncertainty*, $b_{\bar{x}}$, represents a basic measure of the uncertainty due to the elemental systematic errors that affect the measurement of variable x at a one standard deviation confidence level.

The *combined standard uncertainty in x* is reported as a combination of the systematic standard uncertainty and random standard uncertainty in x at a one standard deviation confidence level,

$$u_x = \left(b_{\bar{x}}^2 + s_{\bar{x}}^2 \right)^{1/2} \tag{5.28}$$

For a normal distribution, this confidence interval is equivalent to a 68% probability level. A more general form of this equation extends it to include the total uncertainty at other confidence levels through the use of an appropriate t value as

$$u_x = t_{\nu,P} \left(b_{\bar{x}}^2 + s_{\bar{x}}^2 \right)^{1/2} \quad (P\%) \tag{5.29}$$

In this form, the equation is called the *expanded uncertainty* in x at the reported confidence level (1, 2). The degrees of freedom in Equation 5.29 is found by (2)

$$\nu = \frac{\left(\sum_{k=1}^{K} \left(s_{\bar{x}}^2\right)_k + \left(b_{\bar{x}}^2\right)_k \right)^2}{\sum_{k=1}^{K} \left(\left(s_{\bar{x}}^4\right)_k / \nu_k \right) + \sum_{k=1}^{K} \left(\left(b_{\bar{x}}^4\right)_k / \nu_k \right)} \tag{5.30}$$

The degrees of freedom in an uncertainty are calculated from measurements or prior information; otherwise the degrees of freedom may be assumed to be large ($\nu > 30$ in Table 4.4). Reference 1 provides guidance on assigning a value for ν associated with judgment estimates of uncertainty. When the degrees of freedoms in the systematic uncertainties are large, the second term in the denominator of Equation 5.30 is small. For a 95% confidence level with large degrees of freedom, t of 1.96 is assigned in Equation 5.29. This value can be conveniently rounded off to $t = 2$ (i.e., two standard deviations) (2).

Reference 1 prefers not to assign a probability level to an uncertainty statement but rather to associate a value of the one, two, or three standard deviations, such as we do above for the standard uncertainties. This accommodates different distribution functions and their respective probability coverage in the expanded uncertainty interval. This reflects a difference between the two standards (1, 2) and either approach is acceptable.

Example 5.8

Ten repeated measurements of force F are made over time under fixed operating conditions. The data are listed below. Estimate the random standard uncertainty due to the elemental error in estimating the true mean value of the force based on the limited data set.

N	F [N]	N	F [N]
1	123.2	6	119.8
2	115.6	7	117.5
3	117.1	8	120.6
4	125.7	9	118.8
5	121.1	10	121.9

KNOWN Measured data set $N = 10$

FIND Estimate $s_{\overline{F}}$

SOLUTION The mean value of the force based on this finite data set is computed to be $\overline{F} = 120.13$ N. A random error is associated with assigning the sample mean value as the true value of force because of the small data set. This error enters the measurement during data acquisition (Table 5.2). The random standard uncertainty in this elemental error is computed through the standard deviation of the means, Equation 5.5:

$$s_{\overline{F}} = \frac{s_{\mathrm{F}}}{\sqrt{N}} = \frac{3.04}{\sqrt{10}} = 0.96\,\mathrm{N} \quad \text{with} \quad \nu = 9$$

COMMENT (1) The uncertainty due to data scatter here could be classified as being due to a temporal variation error (i.e., variation with time under fixed conditions) as noted in Table 5.2. (2) Report the uncertainty values with the same decimal significant figures as the mean value (1).

Example 5.9

The force measuring device of Example 5.1 was used in acquiring the data set of Example 5.8. Estimate the systematic uncertainty in the measurement instrument.

KNOWN $B_1 = 0.20\,\text{N}$ $B_2 = 0.30\,\text{N}$ (95%)

ASSUMPTIONS Manufacturer specifications reliable at 95% probability

FIND B (95%)

SOLUTION Lacking information as to the statistics used to specify the uncertainty values associated with the two element errors, e_1 and e_2, we consider them as systematic uncertainties stated at 95% confidence level. The systematic standard uncertainties are found from Equation 5.4 as $b = B/2$,

$$b_1 = B_1/2 = 0.10\,\text{N} \quad b_2 = B_2/2 = 0.15\,\text{N}$$

The systematic standard uncertainty in the transducer is

$$b = \left(b_1^2 + b_2^2\right)^{1/2} = 0.18\,\text{N}$$

at one standard deviation confidence. The expanded systematic uncertainty with $t_{\nu,95} = 2$ is

$$B = 2\left(b_1^2 + b_2^2\right)^{1/2} = 0.36\,\text{N}$$

Elemental errors due to instruments are considered to be data-acquisition source errors (Table 5.2). If these uncertainties were found during a calibrations by the end-user, then we could list them as calibration source errors. In the end, as long as they are accounted for, the error source grouping assigned is not important.

COMMENT Note that the estimate for expanded systematic error due to the instrument is equal in value to the estimate of u_c in a design-stage analysis. The resolution error uncertainty u_0 used in Example 5.1 is now built into the data scatter and so included as part of the random uncertainty estimate in Example 5.8.

Example 5.10

In Examples 4.8 and 4.9, a set of measured data were used to generate a polynomial curve fit equation between variables x and y. If during a test, variable x were to be measured and y computed from x through the polynomial, estimate the random standard uncertainty due to the data-reduction error in the curve fit.

KNOWN Data set of Example 4.8
Polynomial fit of Example 4.9

ASSUMPTIONS Data fits curve $y = 1.04x + 0.02$.

FIND Random standard uncertainty in the curve fit

SOLUTION The curve fit was found to be given by $y = 1.04x + 0.02$ with a standard error of the fit, $s_{yx} = 0.16$ based on the approximation of Equation 4.38. The standard random uncertainty due to the curve fit, written as $(s_{\bar{x}})_1$ to be consistent with Table 5.3, is

$$(s_{\bar{x}})_1 = s_{yx}/\sqrt{N} = 0.072\,\text{V} \qquad \text{with} \quad \nu = 3$$

The expanded random uncertainty interval with $t_{3,95} = 3.18$ is $t_{3,95}(s_{\bar{x}})_1 = 0.23\,\text{V}$.

Example 5.11

The measurement of x contains three elemental random errors from data acquisition. Each elemental error is assigned a random standard uncertainty as follows:

$$\begin{aligned}(s_{\bar{x}})_1 &= 0.60\text{ units},\ \nu_1 = 29\\ (s_{\bar{x}})_2 &= 0.80\text{ units},\ \nu_2 = 9\\ (s_{\bar{x}})_3 &= 1.10\text{ units},\ \nu_3 = 19\end{aligned}$$

Assign the random standard uncertainty due to these data-acquisition errors.

KNOWN $(s_{\bar{x}})_k$ with $k = 1, 2, 3$

FIND $s_{\bar{x}}$

SOLUTION The random standard uncertainty due to data-acquisition errors is

$$s_{\bar{x}} = \left[(s_{\bar{x}})_1^2 + (s_{\bar{x}})_2^2 + (s_{\bar{x}})_3^2\right]^{1/2} = \left(0.60^2 + 0.80^2 + 1.10^2\right)^{1/2} = 1.49\text{ units}$$

at one standard deviation confidence and with degrees of freedom

$$\nu = \frac{\left(\displaystyle\sum_{k=1}^{3} (s_{\bar{x}})_k^2\right)^2}{\displaystyle\sum_{k=1}^{3} (s_{\bar{x}})_k^4/\nu_k} = \frac{\left(0.6^2 + 0.8^2 + 1.1^2\right)^2}{\left(0.6^4/29\right) + \left(0.8^4/9\right) + \left(1.1^4/19\right)} = 38.8 \approx 39$$

Example 5.12

In reporting the results of an experiment to measure stress in a loaded beam, an uncertainty analysis identifies three elemental errors in the stress measurement having the following values of uncertainty:

$$\begin{array}{lll}(b_{\bar{\sigma}})_1 = 0.5\text{ N/cm}^2 & (b_{\bar{\sigma}})_2 = 1.05\text{ N/cm}^2 & (b_{\bar{\sigma}})_3 = 0\text{ N/cm}^2\\ (s_{\bar{\sigma}})_1 = 4.6\text{ N/cm}^2,\ \nu_1 = 14 & (s_{\bar{\sigma}})_2 = 10.3\text{ N/cm}^2,\ \nu_2 = 37 & (s_{\bar{\sigma}})_3 = 1.2\text{ N/cm}^2,\ \nu_3 = 8\end{array}$$

where the degrees of freedom in the systematic uncertainies are very large. If the mean value of the stress in the measurement is $\bar{\sigma} = 223.4\text{ N/cm}^2$, determine the best estimate of the stress at a 95% confidence level, assuming all errors are accounted for.

KNOWN Experimental errors with assigned uncertainties

ASSUMPTIONS All elemental errors ($K = 3$) have been included.

FIND $\bar{\sigma} \pm u_\sigma$ (95%)

SOLUTION We seek values for the statement, $\sigma' = \bar{\sigma} \pm u_\sigma(95\%)$, given that $\bar{\sigma} = 223.4\text{ N/cm}^2$. The measurement random standard uncertainty is

$$s_{\bar{\sigma}} = \left[(s_{\bar{\sigma}})_1^2 + (s_{\bar{\sigma}})_2^2 + (s_{\bar{\sigma}})_3^2\right]^{1/2} = 11.3\text{ N/cm}^2$$

The measurement systematic standard uncertainty is

$$b_{\overline{\sigma}} = \left[(b_{\overline{\sigma}})_1^2 + (b_{\overline{\sigma}})_2^2 + (b_{\overline{\sigma}})_3^2\right]^{1/2} = 1.16\ \text{N/cm}^2$$

The combined standard uncertainty is

$$u_{\sigma} = \left(b_{\overline{\sigma}}^2 + s_{\overline{\sigma}}^2\right)^{1/2} = 11.4\ \text{N/cm}^2$$

The expanded uncertainty requires the combined degrees of freedom. The degrees of freedom is calculated to be

$$\nu = \frac{\left(\sum_{k=1}^{K} \left(s_{\overline{x}}^2\right)_k + \left(b_{\overline{x}}^2\right)_k\right)^2}{\sum_{k=1}^{K} \left(\left(s_{\overline{x}}^4\right)_k / \nu_k\right) + \sum_{k=1}^{K} \left(\left(b_{\overline{x}}^4\right)_k / \nu_k\right)} = 49$$

where the degrees of freedoms in the systematic uncertainties are assumed to be very large so that the second term in the denominator is essentially zero. Therefore, $t_{49,95} \sim 2$ and the expanded uncertainty is

$$\begin{aligned} u_{\sigma} &= 2\left[b_{\overline{\sigma}}^2 + s_{\overline{\sigma}}^2\right]^{1/2} = 2\left[\left(1.2\ \text{N/cm}^2\right)^2 + \left(11.3\ \text{N/cm}^2\right)^2\right]^{1/2} \\ &= 22.7\ \text{N/cm}^2 \quad (95\%) \end{aligned}$$

The best estimate of the stress measurement is

$$\sigma' = 223.4 \pm 22.7\ \text{N/cm}^2 \quad (95\%)$$

Propagation of Uncertainty to a Result

Now consider the result R, which is determined through the functional relationship between the measured independent variables x_i, $i = 1, 2, \ldots, L$ as defined by Equation 5.10. Again, L is the number of independent variables involved and each x_i has an associated systematic standard uncertainty $b_{\overline{x}}$, given by the measurement systematic standard uncertainty determined for that variable by Equation 5.27, and a measurement random standard uncertainty $s_{\overline{x}}$, determined using Equation 5.24. The best estimate of the true value R' is given as

$$R' = \overline{R} \pm u_R \quad (P\%) \tag{5.31}$$

where the mean value of the result is determined by

$$\overline{R} = f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_L) \tag{5.32}$$

and where the uncertainty in the result u_R is given by

$$u_R = f\left(b_{\overline{x}_1}, b_{\overline{x}_2}, \ldots, b_{\overline{x}_L};\ s_{\overline{x}_1}, s_{\overline{x}_2} \ldots, s_{\overline{x}_L}\right) \tag{5.33}$$

where subscripts x_1 through x_L refer to the measurement systematic uncertainties and measurement random uncertainties in each of these L variables.

The propagation of random uncertainty through the variables to the result gives the random standard uncertainty in the result

$$s_R = \left(\sum_{i=1}^{L} \left[\theta_i s_{\overline{x}_i} \right]^2 \right)^{1/2} \tag{5.34}$$

where θ_i is the sensitivity index as defined by Equation 5.14.

The degrees of freedom in the random uncertainty is estimated by the Welch–Satterthwaite formula

$$\nu_s = \frac{\left\{ \sum_{i=1}^{L} \left(\theta_i s_{\overline{x}_i} \right)^2 \right\}^2}{\sum_{i=1}^{L} \left\{ \left(\theta_i s_{\overline{x}_i} \right)^4 / \nu_{\overline{x}_i} \right\}} \tag{5.35}$$

By propagation of the systematic standard uncertainties of the variables, the systematic standard uncertainty in the result is

$$b_R = \left(\sum_{i=1}^{L} [\theta_i b_{x_i}]^2 \right)^{1/2} \tag{5.36}$$

The terms $\theta_i s_{\overline{x}_i}$ and $\theta_i b_{x_i}$ represent the individual contributions of the *i*th variable to the uncertainty in *R*. Comparing the magnitudes of each individual contribution identifies the relative importance of the uncertainty terms on the result.

The *combined standard uncertainty in the result*, written as u_R, is

$$u_R = \left[b_R^2 + {s_R}^2 \right]^{1/2} \tag{5.37}$$

with a confidence level of one standard deviation. The expanded uncertainty in the result is given as

$$u_R = t_{\nu,P} \left[b_R^2 + {s_R}^2 \right]^{1/2} \qquad (P\%) \tag{5.38}$$

The *t* value is used to provide a reasonable weight to the interval defined by the random uncertainty to achieve the desired confidence. If the degrees of freedom in each variable is large ($N \geq 30$), then a reasonable approximation is to take $t_{\nu,95} = 2$. When the degrees of freedom in each of the variables, x_i, is not the same, the Welch–Satterthwaite formula is used to estimate the degrees of freedom in the result expressed as

$$\nu_R = \frac{\left(\sum_{i=1}^{L} \left(\theta_i s_{\overline{x}_i} \right)^2 + \left(\theta_i b_{\overline{x}_i} \right)^2 \right)^2}{\sum_{i=1}^{L} \left(\left(\theta_i s_{\overline{x}_i} \right)^4 / \nu_{s_i} \right) + \sum_{i=1}^{L} \left(\left(\theta_i b_{\overline{x}} \right)^4 / \nu_{b_i} \right)} \tag{5.39}$$

When the individual degrees of freedom in each of the systematic uncertainties are very large, as is often the case, the second term in the denominator is essentially zero.

Example 5.13

The density of a gas, ρ, which follows the ideal gas equation of state, $\rho = p/RT$, is estimated through separate measurements of pressure p and temperature T. The gas is housed within a rigid impermeable vessel. The literature accompanying the pressure measurement system states an instrument uncertainty to within ±1% of the reading (95%), and that accompanying the temperature measuring system indicates ±0.6°R (95%) uncertainty. Twenty measurements of pressure, $N_p = 20$, and 10 measurements of temperature, $N_T = 10$, are made with the following statistical outcome:

$$\overline{p} = 2253.91 \text{ psfa} \quad s_p = 167.21 \text{ psfa}$$
$$\overline{T} = 560.4°\text{R} \quad s_T = 3.0°\text{R}$$

where psfa refers to lb/ft^2 absolute. Determine a best estimate of the density. The gas constant is $R = 54.7$ ft-lb/lb$_m$-°R.

KNOWN $\overline{p}$, s_p, $\overline{T}$, s_T

$\rho = p/RT$; $R = 54.7$ ft-lb/lb$_m$-°R

ASSUMPTIONS Gas behaves as an ideal gas.

FIND $\rho' = \overline{\rho} \pm u_\rho \quad (95\%)$

SOLUTION The measurement objective is to determine the density of an ideal gas through temperature and pressure measurements. The independent and dependent variables are related through the ideal gas law, $\rho = p/RT$. The mean value of density is

$$\overline{\rho} = \frac{\overline{p}}{R\overline{T}} = 0.0735 \text{ lb}_\text{m}/\text{ft}^3$$

The next step must be to identify and estimate the errors and determine how they contribute to the uncertainty in the mean value of density. Since no calibrations are performed and the gas is considered to behave as an ideal gas in an exact manner, the measured values of pressure and temperature are subject only to elemental errors within the data-acquisition error source group (Table 5.2): instrument errors and temporal variation errors.

The tabulated value of the gas constant is not without error. However, estimating the possible error in a tabulated value is sometimes difficult. According to Kestin (8), the systematic uncertainty in the gas constant is on the order of ±(0.33 J/kg-K)/(gas molecular weight) or ±0.06 (ft-lb/lb$_m$-°R)/(gas molecular weight). Since this yields a small value for a reasonable gas molecular weight, here we assume a zero (negligible) systematic error in the gas constant.

Consider the pressure measurement. The uncertainty assigned to the temporal variation (data scatter) error is based on the variation in the measured data obtained during presumably fixed operating conditions. The instrument error is assigned a systematic uncertainty based on the manufacturer's statement, which is assumed to be stated at 95% confidence:

$$\left(b_{\overline{p}}\right)_1 = \left(B_{\overline{p}}/2\right)_1 = \left(0.01 \times 2253.51/2\right) = 11.28 \text{ psfa} \qquad \left(s_{\overline{p}}\right)_1 = 0$$

where the subscript keeps track of the error identity. The temporal variation causes a random uncertainty in establishing the mean value of pressure and is calculated as

$$\left(s_{\overline{p}}\right)_2 = s_{\overline{p}} = \frac{s_p}{\sqrt{N}} = \frac{167.21 \text{ psfa}}{\sqrt{20}} = 37.39 \text{ psfa} \qquad \nu_{s_p} = 19$$

and assigning no systematic uncertainty to this error gives

$$\left(b_{\overline{p}}\right)_2 = 0$$

In a similar manner, we assign the uncertainty in the data-acquisition source error in temperature. The instrument errors are considered as systematic only and assigned a systematic uncertainty based on the manufacturer's statement as

$$\left(b_{\overline{T}}\right)_1 = \left(b_{\overline{T}}\right)_1/2 = 0.3^\circ\text{R} \qquad \left(s_{\overline{T}}\right)_1 = 0$$

The temporal variation causes a random uncertainty in establishing the mean value of temperature and this is calculated as

$$\left(s_{\overline{T}}\right)_2 = s_{\overline{T}} = \frac{s_T}{\sqrt{N}} = \frac{3.0^\circ\text{R}}{\sqrt{10}} = 0.9^\circ\text{R} \qquad v_{s_T} = 9$$

$$\left(b_{\overline{T}}\right)_2 = 0$$

The uncertainties in pressure and temperature due to the data-acquisition source errors are combined using the RSS method,

$$b_{\overline{p}} = \left[(11.28)^2 + (0)^2\right]^{1/2} = 11.28 \text{ psfa}$$

$$s_{\overline{p}} = \left[(0)^2 + (37.39)^2\right]^{1/2} = 37.39 \text{ psfa}$$

similarly,

$$b_{\overline{T}} = 0.3^\circ\text{R}$$

$$s_{\overline{T}} = 0.9^\circ\text{R}$$

with degrees of freedom in the random standard uncertainties to be

$$(v)_{s_p} = N_p - 1 = 19$$

$$(v)_{s_T} = 9$$

The degrees of freedom in the systematic standard uncertainties, v_{b_T} and v_{b_p}, are assumed large.

The systematic and random standard uncertainties propagate through to the result, calculated about the operating point as established by the mean values for temperature and pressure. That is,

$$\begin{aligned} s_{\overline{\rho}} &= \left[\left(\frac{\partial\rho}{\partial T}s_{\overline{T}}\right)^2 + \left(\frac{\partial\rho}{\partial p}s_{\overline{p}}\right)^2\right]^{1/2} \\ &= \left[\left(1.3\times10^{-4}\times0.9\right)^2 + \left(3.26\times10^{-5}\times37.39\right)^2\right]^{1/2} \\ &= 0.0012 \text{ lb}_\text{m}/\text{ft}^3 \end{aligned}$$

and

$$\begin{aligned} b_{\overline{\rho}} &= \left[\left(\frac{\partial\rho}{\partial T}b_{\overline{T}}\right)^2 + \left(\frac{\partial\rho}{\partial p}b_{\overline{p}}\right)^2\right]^{1/2} \\ &= \left[\left(1.3\times10^{-4}\times0.3\right)^2 + \left(3.26\times10^{-5}\times11.28\right)^2\right]^{1/2} \\ &= 0.0004 \text{ lb}_\text{m}/\text{ft}^3 \end{aligned}$$

The degrees of freedom in the density calculation is determined to be

$$\nu = \frac{\left[\left(\frac{\partial\rho}{\partial T}s_{\overline{T}}\right)^2 + \left(\frac{\partial\rho}{\partial p}s_{\overline{p}}\right)^2 + \left(\frac{\partial\rho}{\partial T}b_{\overline{T}}\right)^2 + \left(\frac{\partial\rho}{\partial T}b_{\overline{p}}\right)^2\right]^2}{\left[\left(\frac{\partial\rho}{\partial p}s_{\overline{p}}\right)^4\Big/\nu_{s_p} + \left(\frac{\partial\rho}{\partial T}s_{\overline{T}}\right)^4\Big/\nu_{s_T}\right] + \left[\left(\frac{\partial\rho}{\partial T}b_{\overline{T}}\right)^4\Big/\nu_{b_T} + \left(\frac{\partial\rho}{\partial p}b_{\overline{p}}\right)^4\Big/\nu_{b_p}\right]} = 23$$

The expanded uncertainty in the mean value of density, using $t_{23,95} = 2.06$, is

$$u_\rho = t_{23,95}\left[b_{\overline{\rho}}^2 + s_{\overline{\rho}}^2\right]^{1/2} = 2.06 \times \left[0.0004^2 + 0.0012^2\right]^{\frac{1}{2}}$$
$$= 0.0026\ \text{lb}_\text{m}/\text{ft}^3 \quad (95\%)$$

The best estimate of the density is reported as

$$\rho' = 0.0735 \pm 0.0026\ \text{lb}_\text{m}/\text{ft}^3 \quad (95\%)$$

This measurement of density has an uncertainty of about 3.4%.

COMMENT (1) We did not consider the uncertainty associated with our assumption of exact ideal gas behavior, a potential modeling error (see Table 5.3). (2) Note how pressure contributes more to either standard uncertainty than does temperature and that the systematic uncertainty is small compared to the random uncertainty. The uncertainty in density is best reduced by actions to reduce the effects of the random errors on the pressure measurements.

Example 5.14

Consider determining the mean diameter of a shaft using a hand-held micrometer. The shaft was manufactured on a lathe presumably to a constant diameter. Identify possible elements of error that can contribute to the uncertainty in the estimated mean diameter.

SOLUTION In the machining of the shaft, possible run-out during shaft rotation can bring about an eccentricity in the shaft cross-sectional diameter. Further, as the shaft is machined to size along its length, possible run-out along the shaft axis can bring about a difference in the machined diameter. To account for such deviations, the usual measurement procedure is to repeatedly measure the diameter at one location of the shaft, rotate the shaft, and repeatedly measure again. Then the micrometer is moved to a different location along the axis of the shaft and the above procedure repeated.

It is unusual to calibrate a micrometer within a working machine shop, although an occasional offset error check against accurate gauge blocks is normal procedure. Let us assume that the micrometer is used as is without calibration. Data-acquisition errors are introduced from at least several elements:

1. Since the micrometer is not calibrated, the reading during any measurement could be affected by a possible systematic error in the micrometer markings. This can be labeled as an uncertainty due to instrument error, b_1 (see Table 5.2). Experience shows that $2b_1$, is on the order of the resolution of the micrometer at 95% confidence.
2. The random uncertainty on repeated readings is affected by the resolution of the readout, eccentricity of the shaft, and the exact placement of the micrometer on any cross section along the shaft. It is not possible to separate these errors, so they are grouped as variation errors, with random standard uncertainty, s_2. This value can be discerned from the statistics of the measurements made at any cross section (replication).

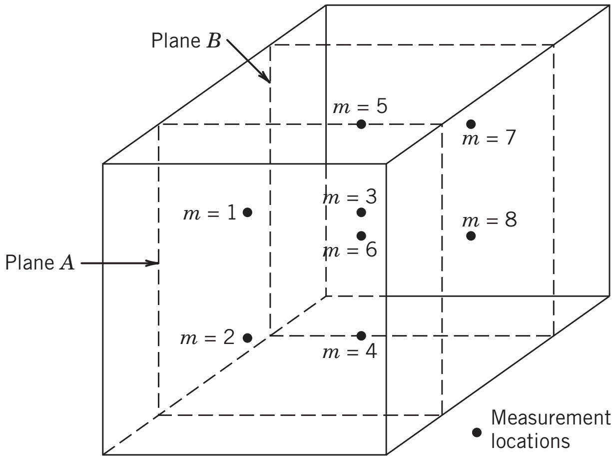

Figure 5.7 Measurement locations for Example 5.15.

3. Similarly, spatial variations in the diameter along its length introduce scatter between the statistical values for each cross section. Since this affects the overall mean diameter, its effect must be accounted for. Label this error as a spatial error with random standard uncertainty, s_3. It can be estimated by the pooled statistical standard deviation of the mean values at each measurement location (replication).

Example 5.15

The mean temperature in an oven is to be estimated by using the information obtained from a temperature probe. The manufacturer states an uncertainty of $\pm 0.6°$C (95%) for this probe. Determine the oven temperature.

The measurement process is defined such that the probe is to be positioned at several strategic locations within the oven. The measurement strategy is to make four measurements within each of two equally spaced cross-sectional planes of the oven. The positions are chosen so that each measurement location corresponds to the centroid of an area of value equal for all locations so that spatial variations in temperature throughout the oven are accounted for. The measurement locations within the cubical oven are shown in Figure 5.7. In this example, 10 measurements are made at each position, with the results given in Table 5.4. Assume that a temperature readout device with a resolution of 0.1°C is used.

KNOWN Data of Table 5.4

FIND $T' = \overline{T} \pm u_T \quad (95\%)$

Table 5.4 Example 5.15: Oven Temperature Data, $N = 10$

Location	$\overline{T}_m$	s_{T_m}	Location	$\overline{T}_m$	s_{T_m}
1	342.1	1.1	5	345.2	0.9
2	344.2	0.8	6	344.8	1.2
3	343.5	1.3	7	345.6	1.2
4	343.7	1.0	8	345.9	1.1

Note: $\overline{T}_m = \dfrac{1}{N}\sum_{n=1}^{N} T_{mn}; \quad s_{T_m} = \left[\dfrac{1}{N-1}\sum_{n=1}^{N}\left(T_{mn} - \overline{T}_m\right)^2\right]^{1/2}$

SOLUTION The mean values at each of the locations are averaged to yield a mean oven temperature using pooled averaging:

$$\langle \overline{T} \rangle = \frac{1}{8}\sum_{m=1}^{8} \overline{T}_m = 344.4^\circ\text{C}$$

Elemental errors in this test are found in the data-acquisition group (Table 5.2) and due to (1) the temperature probe system (instrument error), (2) spatial variation errors, and (3) temporal variation errors. First, consider the elemental error in the probe system. The manufacturer statement of $\pm 0.6^\circ$C is considered a systematic uncertainty at 95% confidence level. The standard uncertainties are assigned as

$$\left(b_{\overline{T}}\right)_1 = 0.6/2 = 0.3^\circ\text{C} \qquad \left(s_{\overline{T}}\right)_1 = 0$$

Consider next the spatial error contribution to the estimate of the mean temperature $\overline{T}$. This error arises from the spatial nonuniformity in the oven temperature. An estimate of spatial temperature distribution within the oven can be made by examining the mean temperatures at each measured location. These temperatures show that the oven is not uniform in temperature. The mean temperatures within the oven show a standard deviation of

$$s_T = \sqrt{\frac{\sum_{m=1}^{8}\left(\overline{T}_m - \langle \overline{T} \rangle\right)^2}{7}} = 1.26^\circ\text{C}$$

Thus, the random standard uncertainty of the oven mean temperature is found from $s_{\overline{T}}$ or

$$\left(s_{\overline{T}}\right)_2 = \frac{s_T}{\sqrt{8}} = 0.45^\circ\text{C}$$

with degrees of freedom, $\nu = 7$. We do not assign a systematic uncertainty to this error, so $\left(b_{\overline{T}}\right)_2 = 0$. One could reasonably argue that $\left(s_{\overline{T}}\right)_2$ represents a systematic error because it is an effect that would offset the final value of the result. It does not affect the final uncertainty, but the awareness of its effect is part of the usefulness of an uncertainty analysis! This is a case where it becomes the test engineer's decision.

Time variations in probe output during each of the 10 measurements at each location cause data scatter, as evidenced by the respective s_{T_m} values. Such time variations are caused by random local temperature variations as measured by the probe, probe resolution, and oven temperature control variations during fixed operating conditions. We have insufficient information to separate these, so they are estimated together as a single error. The pooled standard deviation is

$$\langle s_T \rangle = \sqrt{\frac{\sum_{m=1}^{8}\sum_{n=1}^{10}\left(\overline{T}_{mn} - \langle \overline{T} \rangle\right)^2}{M(N-1)}} = \sqrt{\frac{1}{M}\sum_{m=1}^{M} s_{T_m}^2} = 1.09\ ^\circ\text{C}$$

to give a random standard uncertainty of

$$\left(s_{\overline{T}}\right)_3 = \frac{\langle s_T \rangle}{\sqrt{80}} = 0.12^\circ\text{C}$$

with degrees of freedom, $\nu = 72$. We assign $\left(b_{\overline{T}}\right)_3 = 0$.

The measurement systematic standard uncertainty is

$$b_{\overline{T}} = \left[\left(b_{\overline{T}}\right)_1^2 + \left(b_{\overline{T}}\right)_2^2 + \left(b_{\overline{T}}\right)_3^2\right]^{1/2} = 0.3^\circ\text{C}$$

and the measurement random standard uncertainty is

$$s_{\overline{T}} = \left[\left(s_{\overline{T}}\right)_1^2 + \left(s_{\overline{T}}\right)_2^2 + \left(s_{\overline{T}}\right)_3^2\right]^{1/2} = 0.46°\text{C}$$

The degrees of freedom are found using Equation 5.30 giving $\nu = 17$, which assumes a large value for ν_{b_T}.

The combined standard uncertainty in the mean oven temperature is

$$u_T = \left[b_{\overline{T}}^2 + s_{\overline{T}}^2\right]^{1/2} = 0.55 \approx 0.6°\text{C}$$

with a confidence level of one standard deviation. Assigning $t_{17,95} = 2.11$, the best estimate of the mean oven temperature is

$$T' = \overline{T} \pm t_{\nu,P}\left(b_{\overline{T}}^2 + s_{\overline{T}}^2\right)^{1/2} = 344.4 \pm 1.2\,°\text{C} \quad (95\%)$$

5.9 CORRECTION FOR CORRELATED ERRORS

So far, we have assumed that all of the different elements of error in a test are independent from the others. If two errors are not independent, they are "correlated."

For example, when the same instrument is used to measure different variables, the instrument systematic errors between those variables are correlated. If multiple instruments are calibrated against the same standard, then the systematic uncertainty in the standard is passed to each instrument. Hence, these errors are correlated. The numerical effect of correlated errors on the uncertainty depends on the functional relationship between the variables and the magnitudes of the elemental systematic errors that are correlated. We now introduce a correction for treating the correlated systematic errors.

Consider the result R, which is determined through the functional relationship between the measured independent variables x_i, $i = 1, 2, \ldots, L$ where L is the number of independent variables involved. Each x_i is subject to elemental systematic errors with standard uncertainties, b_k, where $k = 1, 2, \ldots, K$, refer to each of up to any of K elements of error. Now allow that H of these K elemental errors are correlated between variables while the rest ($K - H$) are uncorrelated. When correlated errors are included, the systematic standard uncertainty in a result is estimated by

$$b_R = \left[\sum_{i=1}^{L}\left(\theta_i b_{\overline{x}_i}\right)^2 + 2\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\theta_i\theta_j b_{\overline{x}_i\overline{x}_j}\right]^{1/2} \tag{5.40}$$

where index j is a counter equal to $i + 1$ and with

$$\theta_i = \frac{\partial R}{\partial x_{i_{x=\overline{x}}}} \tag{5.14}$$

Equation 5.40 introduces the covariance, $b_{\overline{x}_i\overline{x}_j}$, to account for correlated errors and this is found from

$$b_{\overline{x}_i\overline{x}_j} = \sum_{h=1}^{H}\left(b_{\overline{x}_i}\right)_h\left(b_{\overline{x}_j}\right)_h \tag{5.41}$$

where H is the number of elemental errors that are correlated between variables x_i and x_j and h is a counter for each correlated error. Note that Equation 5.40 reduces to Equation 5.36 when no errors are correlated (i.e., when $H = 0$, $b_{\overline{x}_i\overline{x}_j} = 0$). References 2 and 9 discuss treatment of correlated systematic errors in extended detail.

Example 5.16

Suppose a result is a function of three variables, x_1, x_2, x_3. There are four systematic errors associated with both x_1 and x_2 and five associated with x_3. Only the first and second systematic elemental errors associated with the second and third variable (x_2 and x_3) are determined to be correlated because these errors arise from common sources. Express the systematic standard uncertainty in the result and find the covariance term.

SOLUTION Equation 5.40 is expanded to the form

$$b_R = \left[(\theta_1 b_{\overline{x}_1})^2 + (\theta_1 b_{\overline{x}_2})^2 + (\theta_3 b_{\overline{x}_3})^2 + 2\theta_1\theta_2 b_{\overline{x}_1\overline{x}_2} + 2\theta_1\theta_3 b_{\overline{x}_1\overline{x}_3} + 2\theta_2\theta_3 b_{\overline{x}_2\overline{x}_3}\right]^{1/2}$$

For the two correlated errors ($H = 2$) associated with variables 2 and 3 ($i = 2, j = 3$), the systematic uncertainty in the result reduces to

$$b_R = \left[(\theta_1 b_{\overline{x}_1})^2 + (\theta_2 b_{\overline{x}_2})^2 + (\theta_3 b_{\overline{x}_3})^2 + 2\theta_2\theta_3 b_{\overline{x}_2\overline{x}_3}\right]^{1/2}$$

where the covariance term is

$$b_{\overline{x}_2\overline{x}_3} = \sum_{h=1}^{2} (b_{\overline{x}_2})_h (b_{\overline{x}_3})_h = (b_{\overline{x}_2})_1 (b_{\overline{x}_3})_1 + (b_{\overline{x}_2})_2 (b_{\overline{x}_3})_2$$

Example 5.17

Suppose a result R is a function of two variables, X and Y, such that $R = X + Y$, and each variable having one elemental error. If $\overline{X} = 10.1$ V with $b_{\overline{X}} = 1.1$ V and $\overline{Y} = 12.2$ V with $b_{\overline{Y}} = 0.8$ V, estimate the systematic standard uncertainty in the result if the systematic errors are (1) uncorrelated and (2) correlated.

SOLUTION From the stated information, $\overline{R} = 10.1 + 12.2 = 22.3$ V. For the uncorrelated case, the systematic standard uncertainty is estimated as

$$b_{R_{\text{unc}}} = \left[(\theta_X b_{\overline{X}})^2 + (\theta_Y b_{\overline{Y}})^2\right]^{1/2} = \left[(1 \times 1.1)^2 + \left((1 \times 0.8)^2\right)^2\right]^{1/2} = 1.36 \text{ V}$$

For the correlated case, with $H = 1$ and $L = 2$, the uncertainty is estimated as

$$\begin{aligned} b_{R_{\text{cor}}} &= \left[(\theta_X b_{\overline{X}})^2 + (\theta_Y b_{\overline{Y}})^2 + 2\theta_X\theta_Y b_{\overline{X}\,\overline{Y}}\right]^{1/2} \\ &= \left[(1 \times 1.1)^2 + (1 \times 0.8)^2 + 2(1)(1)(1.1)(0.8)\right]^{1/2} = 3.12 \text{ V} \end{aligned}$$

which is stated at the one-standard deviation confidence level and where

$$b_{\overline{X}\,\overline{Y}} = \sum_{h=1}^{1} \left(b_{\overline{X}}\right)_1 \left(b_{\overline{Y}}\right)_1 = b_{\overline{X}} b_{\overline{Y}}$$

COMMENT In this case, the correlated systematic errors have a large impact on the systematic uncertainty. This is not always the case as it depends on the functional relationship itself. We leave it to the reader to show that if $R = X/Y$, then the covariance term in this problem would have little impact on the systematic standard uncertainty in the result.

There are situations where random errors can be correlated (10). In those cases, the random standard uncertainty in a result is estimated by

$$s_R = \left[\sum_{i=1}^{L} \left(\theta_i s_{\overline{x}_i}\right)^2 + 2 \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \theta_i \theta_j r_{\overline{x}_i \overline{x}_j} s_{\overline{x}_i} s_{\overline{x}_j} \right]^{1/2} \tag{5.42}$$

where $r_{x_i x_j}$ is the correlation coefficient between x_i and x_j as given by equation 4.41.

5.10 NONSYMMETRICAL SYSTEMATIC UNCERTAINTY INTERVAL

There are situations where the error must be bounded on one side or skewed about the measured mean value such that the systematic uncertainty is better described by a nonsymmetrical interval. To develop this case, we assume that the limits of possible systematic error are known, as is the mean of the measured data set $\overline{x}$. Let $\overline{x} + B_{\overline{x}}^{+}$ and $\overline{x} - B_{\overline{x}}^{-}$ be the upper and lower limits of the systematic uncertainty relative to the measured mean value (Fig. 5.8), with $B_{\overline{x}}^{-} = 2b_{\overline{x}}^{-}$ and $B_{\overline{x}}^{+} = 2b_{\overline{x}}^{+}$ where $b_{\overline{x}}^{-}$ and $b_{\overline{x}}^{+}$ are the lower and upper systematic standard uncertainty values. The modeling approach assumes that the errors are symmetric about some mean value of the error distribution but asymmetric about the measured mean value.

If we model the error distribution as a normal distribution, then we assume that the limits defined using $B_{\overline{x}}^{-}$ to $B_{\overline{x}}^{+}$ cover 95% of the error distribution. Define the offset between the mean of the

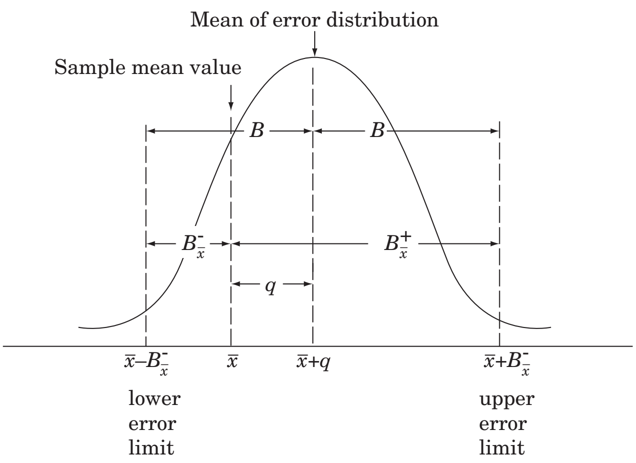

Figure 5.8 Relation between the measured mean value, the mean value of the distribution of errors, and the systematic uncertainties for treating nonsymmetrical uncertainties.

error distribution and the mean value of the measurement as

$$q = \frac{(\overline{x} + B^{+}_{\overline{x}}) + (\overline{x} - B^{-}_{\overline{x}})}{2} - \overline{x} = \frac{B^{+}_{\overline{x}} - B^{-}_{\overline{x}}}{2} = b^{+}_{\overline{x}} - b^{-}_{\overline{x}} \tag{5.43}$$

The systematic standard uncertainty has an average width,

$$b_{\overline{x}} = \frac{(\overline{x} + B^{+}_{\overline{x}}) - (\overline{x} - B^{-}_{\overline{x}})}{4} = \frac{(\overline{x} + b^{+}_{\overline{x}}) - (\overline{x} - b^{-}_{\overline{x}})}{2} \tag{5.44}$$

For stating the true value, we use the combined standard uncertainty

$$u_x = \sqrt{b^2_{\overline{x}} + s^2_{\overline{x}}} \tag{5.45}$$

to achieve the approximate confidence interval, $q \pm t_{\nu,P} u_x$. For large degrees of freedom, we can state

$$x' = \overline{x} + q - t_{\nu,P} u_x, \quad \overline{x} + q + t_{\nu,P} u_x \tag{5.46}$$

for which we choose an appropriate value for t, such as $t = 2$ for 95% confidence.

If we model the error distribution as a rectangular distribution, then we assume that the limits defined using $B^{-}_{\overline{x}}$ to $B^{+}_{\overline{x}}$ specify the limits of that distribution. The systematic standard uncertainty is then (see footnote 5)

$$b_{\overline{x}} = \frac{(\overline{x} + B^{+}_{\overline{x}}) - (\overline{x} - B^{-}_{\overline{x}})}{\sqrt{12}} \tag{5.47}$$

for use in Equations 5.45 and 5.46. Regardless of the assumed systematic error distribution, we assume that the uncertainties propagate normally.

A normal or rectangular distribution is not always the appropriate model for treating asymmetric systematic uncertainty. References 1 and 2 provide a number of scenarios to treat nonsymmetrical uncertainties including use of other error distributions. As an example, Monsch et al. (11) apply the above approach to estimating the nonsymmetrical systematic uncertainty in the induced drag on an aircraft wing in which the induced drag cannot be less than zero.

5.11 SUMMARY

Uncertainty analysis provides the "$\pm$ what" to a test result or the anticipated result from a proposed test plan. This chapter discussed the manner in which various errors can enter into a measurement with emphasis on their influences on the test result. Both random errors and systematic errors are considered. Random errors differ on repeated measurements, lead to data scatter, and can be quantified by statistical methods. Systematic errors remain constant in repeated measurements. Errors are quantified by uncertainties. Procedures for uncertainty analysis were introduced both as a means of estimating uncertainty propagation within a measurement and for the propagation of uncertainty among the independent measured variables that are used to determine a result through some functional relationship. We have discussed the role and use of uncertainty analysis in both the design of measurement systems, through selection of equipment and procedures, as well as in the interpretation of measured data. We have presented procedures for estimating uncertainty at various stages of a test. The procedures provide reasonable estimates of the uncertainty to be expected in measurements.