ME 752

Chap 2: Single-Variable Optimization

2.1 ☆ Definition and Optimality Conditions

— Higher Order Analysis → 224-5

2.2 ☆ Bracketing: $f(x_1) > f(x_2) < f(x_3)$ for $x_1 < x_2 < x_3$

2.3 ☆ Gradient Based Methods

— Newton, Secant, Cubic and Quadratic Estimation → 226

2.4 ☆ Region Elimination Methods → 13-15

— Bisection and Interval Halving

— Fibonacci Search Method

— Golden Section Search Method

2.5 ☆ TWO Contexts of Univariate Optimization

- The complete problem itself
- The line-search "sub-problem"

2.6 ☆ Line Search → 16

- Exact line search (Accurate)
- Inexact line search (Inaccurate) → 17
  - → Termination conditions

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2.4 Region Elimination Methods

☆ Bisection and Interval Halving

Interval: [a, b]

{ If $f(x_1) < f(x_m)$
    { $b \leftarrow x_m$; break; }
If $f(x_2) < f(x_m)$
    { $a \leftarrow x_m$; break; }
$a \leftarrow x_1$; $b \leftarrow x_2$; }

Two new evaluations in every iteration.

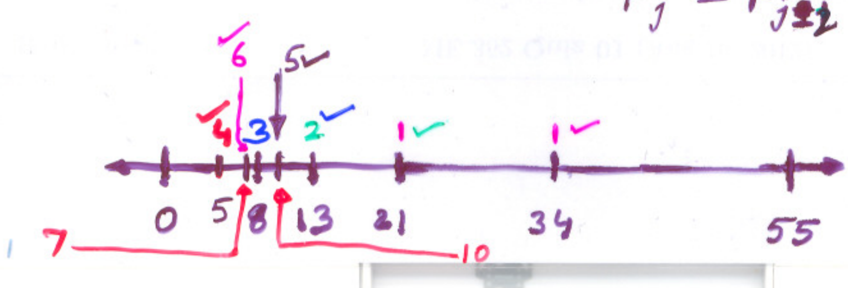
Interval reduces to **HALF** in every iteration.

Other (better) possibilities?

☆ Fibonacci Series

k	0	1	2	3	4	5	6	7	8	9
F_k	1	1	2	3	5	8	13	21	34	55

$F_j = F_{j-2} + F_{j-1}$



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2·4 Region Elimination (contd)

★ Fibonacci Search Method

$F_0 = 1, \; F_1 = 1, \quad F_j = F_{j-2} + F_{j-1}$

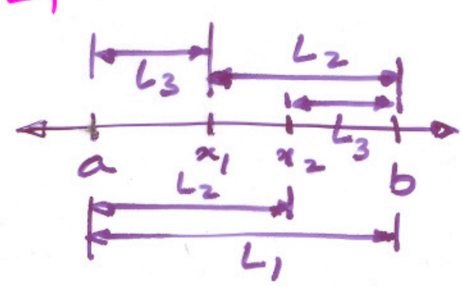
$$F_N = F_{N-2} + F_{N-1}$$

Starting: A unimodal bracket $[a, b]$ of length $L_1 = b - a$.

Iteration 1: $k = 2$

// Evaluate function f at
$x_1 = b - L_2$ and $x_2 = a + L_2$
where $L_2 = \frac{F_{N-1}}{F_N} L_1$

// If $f(x_1) > f(x_2)$
$a \leftarrow x_1$
Else $[$i.e. $f(x_2) > f(x_1)]$
$b \leftarrow x_2$



Iteration 2: $k = 3$

// With new $[a, b]$, evaluate at
$x_1 = b - L_3$ and $x_2 = a + L_3$
where $L_3 = \frac{F_{N-2}}{F_N} L_1 = \frac{F_{N-2}}{F_{N-1}} L_2$

Check: $a + L_3 = b - L_1 + \frac{F_{N-2}}{F_N} L_1$
$= b - \frac{F_N - F_{N-2}}{F_N} L_1 = b - \frac{F_{N-1}}{F_N} L_1 = b - L_2$

Iteration $(k-1)$: $L_k = \frac{F_{N-k+1}}{F_N} L_1$

Iteration $(N-1)$: $L_N = \frac{F_1}{F_N} L_1 = \frac{L_1}{F_N} = \epsilon$ (tolerance)

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2.4 Region Elimination (Contd)

★ Fibonacci Search : Algorithmic Steps

Given: Function f
Unimodal bracket $[a, b]$ of length L_1
Tolerance ϵ

// From $L_1 = b - a$, find $F_N > L_1/\epsilon$ & N.

// Evaluate f at $x_1 = b - L_2$ & $x_2 = a + L_2$

// Set $k = 2$

// If $f(x_1) > f(x_2)$
$\{ a \leftarrow x_1;\ x_1 \leftarrow x_2;\ x_2 \leftarrow a + L_{k+1};$
Evaluate $f(x_2) \}$

// Else $\{ b \leftarrow x_2;\ x_2 \leftarrow x_1;\ x_1 \leftarrow b - L_{k+1};$
Evaluate $f(x_1) \}$

// If $k = N$; STOP

// Update $k \leftarrow k+1$

★ Golden Section Search

Explore: With $N \to \infty$

$$\frac{F_{N-1}}{F_N} \approx \frac{F_{N-2}}{F_{N-1}} \approx \tau \Rightarrow \frac{F_{N-2}}{F_N} = \tau^2$$

$$\Rightarrow \tau^2 + \tau = 1 \Rightarrow \tau \approx 0.618$$

$$\epsilon = \tau^{N-1}(b-a) \Rightarrow \tau^{N-1} = \frac{\epsilon}{b-a}$$

$$\Rightarrow N = \log\left(\frac{\epsilon}{b-a}\right) / \log \tau + 1$$

N actually not ∞ !

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2-6 Line Search

For $f(x)$, $x \in R^n$,

after choosing d_k from x_k,

$$\phi(\alpha) = f(x) = f(x_k + \alpha d_k)$$

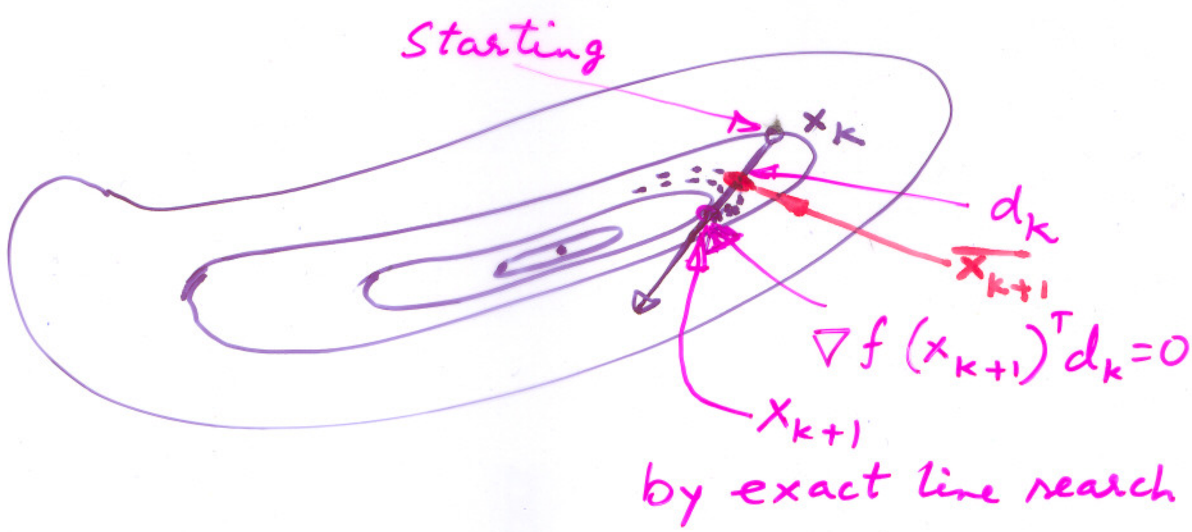
↳ single variable problem

$$\Rightarrow \phi(\alpha) \approx f(x_k) + \alpha [\nabla f(x_k)]^T d_k$$

1st order condition: $\phi'(\alpha) = 0$

$$\Rightarrow [\nabla f(x_k + \alpha d_k)]^T d_k = 0$$

or $\nabla f(x_{k+1})^T d_k = 0$



From x_{k+1}, a new line search will start.

? How much effort was spent in arriving at x_{k+1} exactly?

? How does it matter if the new line search starts at $\bar{x}_{k+1}$ instead?

2.6.3 Inexact Line Search

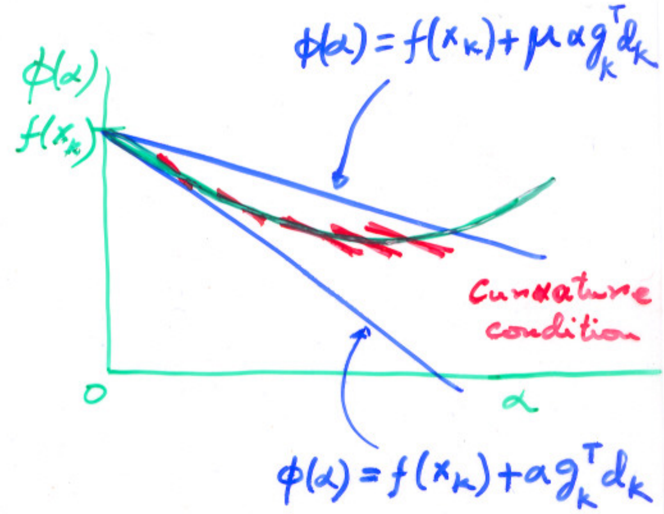
(Try enough, but not too much)

Notion of sufficient decrease:

$$\phi(\alpha) \le f(x_k) + \mu \alpha g_k^T d_k, \quad 0 < \mu < 1$$

$\boxed{\text{Armijo's rule}}$

Not sufficient alone



Notion of "NOT much hope" further:

$$\phi'(\alpha_k) \geq \eta \, \phi'(0), \quad \eta < 1$$

or, $[\nabla f(x_k + \alpha_k d_k)]^T d_k \geq \eta \, g_k^T d_k$,

$\boxed{\text{Curvature Condition}}$

Wolfe Conditions:

$$\phi(\alpha_k) \le f(x_k) + \mu \alpha_k g_k^T d_k$$

AND $[\nabla f(x_k + \alpha_k d_k)]^T d_k \geq \eta \, g_k^T d_k$,

for $\underline{0 < \mu < \eta < 1}$.

??

Caution: Inexact line search would NOT guarantee

$$d_k^T g_{k+1} = 0,$$

only exact line search will.