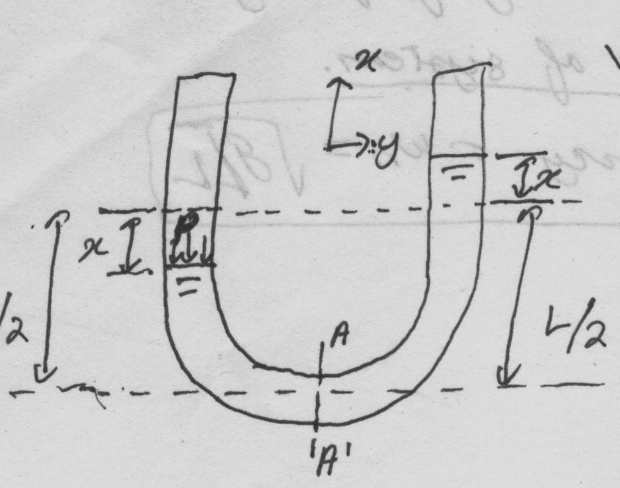
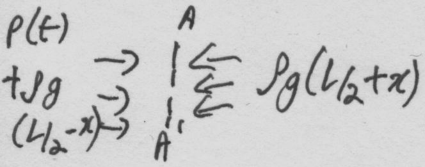
Q 2.4

- All quantities are in SI units.



- Pressure $p(t)$ is acting on the left side of tube $p(t) = p_0 \cos\omega t$
- ~~p_0~~ total length of fluid in tube $= L$
- density of fluid is ρ
- Area of cross section of uniform tube $= a$

- Pressure on the left side of section $AA' = p(t) + \rho g(L/2 - x)$
- " " " right side " " " $= \rho g(L/2 + x)$



- Net force ~~of~~ at section $AA' = \left[\rho g(L/2 + x) - p(t) - \rho g(L/2 - x)\right]a$

$$= \left[2\rho g x - p(t)\right]a \quad \left[\begin{array}{l}\text{This force}\\ \text{is acting}\\ \text{along } (-\hat{j})\end{array}\right]$$

This force is causing overall fluid to move with acceleration $\ddot{x}$.

This force is acting in the direction opposite to fluid motion

$$-(2\rho g x - p(t))a = (\rho a L)\ddot{x}$$

$$\Rightarrow \quad \rho a L \ddot{x} + \rho g a x = p(t)a$$

$$\Rightarrow \quad \ddot{x} + \frac{g}{L}x = \frac{p(t)}{\rho L}$$

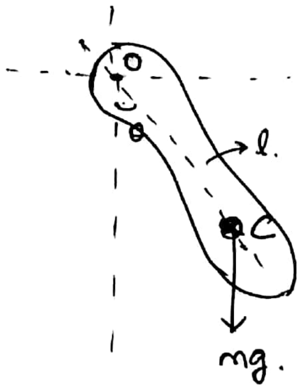
$$\Rightarrow \boxed{\ddot{x} + g/L\, x = \frac{p_0}{\rho L}\cos\omega t}$$

$\Leftarrow$ Differential equation of motion.

Resosance occurs when forcing frequency equals natural frequency of system.

So $\boxed{\text{resonance frequency} = \omega_n = \sqrt{g/L}}$

2.12



$mgl\sin\theta = I_0\alpha$. (Angular momentum balance)

for small values of θ. (slight disturbance)

$\Rightarrow mgl\theta = I_0\alpha$.

$\Rightarrow \omega^2 = \frac{mgl}{I_0}$; $\Rightarrow \omega = \sqrt{\frac{mgl}{I_0}}$.

also $I_0 = I_c + ml^2$.

$\Rightarrow \omega = \sqrt{\frac{mgl}{I_c + ml^2}}$.

$\Rightarrow \omega^2 I_c + \omega^2 ml^2 - mgl = 0$.

$\Rightarrow \omega = 6$ rad/s ; $m = 3\times10^{-3}$ kg ; $I_c = 0.432\times10^{-4}$ kg ; $g = 9.81$

$\Rightarrow 0.1080000h^2 + 0.0015552 - 0.02943h = 0$

$\Rightarrow h = 0.2007795$, 0.07172045

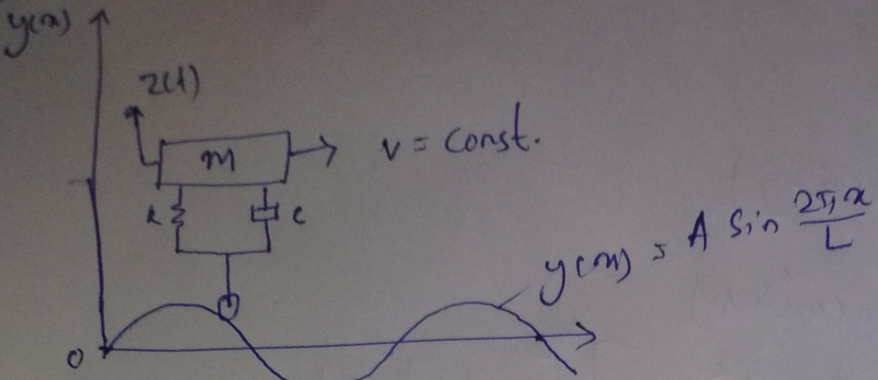
also value of length of simple pendulum (lumped mass)

$= \sqrt{\frac{g}{l}} = \omega^2 \Rightarrow$ $l = g/\omega^2$

$= \frac{9.81}{6\times6} = 0.2725$

$(h < l.)$

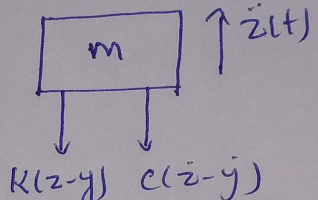
2.16



As v is constant

so. $y(t) = A\sin\frac{2\pi v t}{L}$, so. $\omega = \frac{2\pi v}{L}$

$$-k(z-y) - c(\dot{z}-\dot{y}) = m\ddot{z}$$



$$\Rightarrow m\ddot{z} + c\dot{z} + kz = ky + c\dot{y}$$

Taking Laplace transform on both sides

$$ms^2 z(s) + csZ(s) + kz(s) = csy(s) + ky(s)$$

$$\frac{Z}{Y} = \frac{sc+k}{s^2 m + sc + k.}$$

$$\left.\frac{Z}{Y}\right|_{\text{harmonic}} = \frac{1 + j\omega c/k}{1 - \frac{\omega^2 m}{k} + j\frac{c\omega}{k}}$$

$$\left(\frac{\omega c}{k} = 2\xi\frac{\omega}{\omega_n}\ , \text{ where } \xi = \text{damping ratio}, \ \omega_n = \sqrt{k/m} = \text{Natural frequency}\right)$$

Taking $\frac{\omega}{\omega_n} = r$.

$$\frac{Z}{Y} = \frac{1 + j2\xi r}{(1-r^2) + j2\xi r} = \frac{(1+j2\xi r)\{(1-r^2) - j2\xi r\}}{\{(1-r^2) + j2\xi r\}\{(1-r^2) - j2\xi r\}}$$

$$= \frac{(1-r^2) + (2\xi r)^2 - 2\xi r^3 j}{(1-r^2)^2 + (2\xi r)^2}$$

$$\left|\frac{Z}{Y}\right| = \left[\frac{1 + (2\xi r)^2}{(1-r^2)^2 + (2\xi r)^2}\right]^{1/2} = R \text{ (say)}$$

So. $Z = RY$

$$\& \ \phi = \tan^{-1}\frac{-2\xi r^3}{(1-r^2) + (2\xi r)^2}$$

$z(t) = RA\sin(\omega t + \phi)$ (Ans)

Force transmitted

$$F = k(z-y) + c(\dot{z}-\dot{y})$$

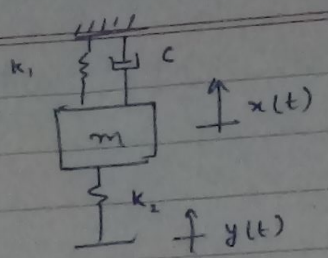
$$= -m\ddot{z}$$

$$= mRA\omega^2 \sin(\omega t + \phi)$$

$$F = kA r^2 \left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}\right]^{1/2} \sin(\omega t + \phi)$$

(Ans)

(2.21)



$$m\ddot{x} + k_1 x + k_2(x-y) + c\dot{x} = 0 \quad -(1)$$

given $\quad y(t) = B + \frac{At}{T} \qquad t = 0..T$

by using fourier series

$$y(t) = B + \frac{At}{T} = a_0 + \sum_{n=1}^{\infty}\left(a_n \sin\left(\frac{2n\pi t}{T}\right) + b_n \cos\left(\frac{2n\pi t}{T}\right)\right) \quad -(2)$$

integrating equation (2) from 0 to T

$$BT + \frac{A}{T}\frac{T^2}{2} = a_0 T + 0 + 0$$

$$\Rightarrow \quad a_0 = B + \frac{A}{2}$$

Now, $\int_0^T \left((2)\, dt \times \sin\left(\frac{2n\pi t}{T}\right)\right)$

$$\frac{A}{T}\int_0^T t\sin\left(\frac{2n\pi t}{T}\right)dt + \cancel{B\int_0^T \sin\left(\frac{2n\pi t}{T}\right)dt}^{\,0} = \int_0^T a_n \sin^2\left(\frac{2n\pi t}{T}\right)dt$$

(rest of the terms will be 0)

$$\frac{A}{T}\left(\left.\frac{-t\cos\left(\frac{2n\pi t}{T}\right)}{\frac{2n\pi}{T}}\right|_0^T - \cancel{\int_0^T \frac{\cos\left(\frac{2n\pi t}{T}\right)dt}{\frac{2n\pi}{T}}}^{\,0}\right) = a_n\frac{T}{2}$$

$$\frac{A}{\cancel{T}}\frac{-1}{\frac{2n\pi}{T}}\cdot T = a_n\frac{T}{2} \quad \Rightarrow \quad a_n = -\frac{A}{n\pi}$$

similarly integrating ② × $\cos\left(\frac{2\pi n t}{T}\right)$ will yield

$$b_n = 0$$

∴ putting in eq^n ① and rearranging

$$m\ddot{x} + (k_1+k_2)x + c\dot{x} = k_2\left(\frac{A}{2} + B - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi t}{T}\right)\right) \quad (3)$$

put $x(t) = \alpha_0 + \sum_{n=1}^{\infty}\left(\alpha_n \sin\left(\frac{2n\pi t}{T}\right) + \beta_n \cos\left(\frac{2n\pi t}{T}\right)\right)$

$$\dot{x}(t) = \sum_{n=1}^{\infty}\left(\frac{2n\pi\alpha_n}{T}\cos\left(\frac{2n\pi t}{T}\right) - \frac{2n\pi}{T}\beta_n \sin\left(\frac{2n\pi t}{T}\right)\right)$$

$$\ddot{x}(t) = -\sum_{n=1}^{\infty}\left(\frac{2n\pi}{T}\right)^2\left(\alpha_n \sin\left(\frac{2n\pi t}{T}\right) + \beta_n \cos\left(\frac{2n\pi t}{T}\right)\right)$$

put in eq ③ and compare coefficients.

~~coeff~~ constant term:

$$(k_1+k_2)\alpha_0 = k_2\left(\frac{A}{2} + B\right)$$

$$\Rightarrow \alpha_0 = \frac{k_2}{k_1+k_2}\left(B + \frac{A}{2}\right)$$

$\sin\left(\frac{2n\pi t}{T}\right)$ term's coefficients:

$$-m\left(\frac{2n\pi}{T}\right)^2\alpha_n + (k_1+k_2)\alpha_n - c\frac{2n\pi}{T}\beta_n = -\frac{A}{n\pi}$$

$$\alpha_n\left[k_1+k_2 - m \times \frac{4n^2\pi^2}{T^2}\right] - \beta_n\left(c\frac{2n\pi}{T}\right) = \frac{-A}{n\pi}$$

coefficient of $\cos\left(\frac{2n\pi t}{T}\right)$?

$$-m\left(\frac{2n\pi}{T}\right)^2\beta_n + (k_1+k_2)\beta_n + c\left(\frac{2n\pi}{T}\right)\alpha_n = 0$$

$$\alpha_n\left(\frac{2n\pi}{T}c\right) + \beta_n\left(k_1+k_2 - m\left(\frac{2n\pi}{T}\right)^2\right) = 0$$

solving the two equations and calling $\frac{2\pi}{T} = w$

$$\alpha_n = \frac{A(k_1+k_2 - mn^2w^2)}{n\pi(m^2n^4w^4 + c^2n^2w^2 - 2k_1mn^2w^2 - 2k_2mn^2w^2 + k_1^2+2k_1k_2+k_2^2)}$$

$$\beta_n = \frac{cwA}{\pi(m^2n^4w^4 + c^2n^2w^2 - 2k_1mn^2w^2 - 2k_2mn^2w^2 + k_1^2+2k_1k_2+k_2^2)}$$

Problem

Solve the ordinary Vibration Differential Equation using f(t) as shown in figure 1.

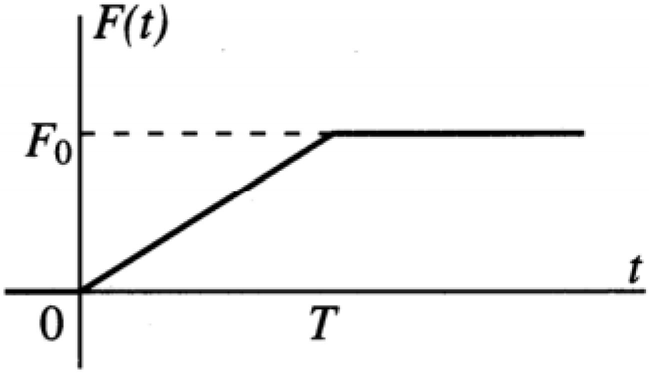


Figure 1: Problem

Solution

Equation The governing equation to be solved is

$$m\frac{\partial^2 \mathbf{x}}{\partial t^2} + c\frac{\partial \mathbf{x}}{\partial t} + k\mathbf{x} = f(t) \tag{1}$$

, with all the notations with usual meaning. This equation can also be written as

$$\ddot{\mathbf{x}} + 2\zeta\omega_n\dot{\mathbf{x}} + k\mathbf{x} = \frac{f(t)}{m} \tag{2}$$

where,

$$\zeta = \frac{c}{2\omega_n m}$$
$$\omega_n = \sqrt{\frac{k}{m}}$$

Now, from figure 1, the forcing function f(t) can be written as the superposition of ramp function (Please read about Ramp function in detail from any source) as

$$f(t) = \frac{F_0}{T}\left(r(t) - r(t-T)\right) \tag{3}$$

, where r(t) denotes unit ramp function given as

$$r(t) = tu(t)$$

Homogeneous Solution The above equation is a non homogeneous equation, so we need to compute solution as

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$$

For homogeneous equation, let solution be

$$\mathbf{x}_h = e^{\lambda t}$$

Putting it, the characteristic equation can be written as

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

This simple quadratic equation can be solved for values for λ, and following the same, the homogeneous solutions can be written as (assuming $\zeta < 1$, same can be done for all cases)

$$x_h(t) = Ce^{(-\zeta\omega_n \pm \iota\omega_d)t}$$

It can be reduced in the form of cos(ω_dt) and sin(ω_dt), which you are already familiar with. So, the final homogeneous solution can be written as

$$\mathbf{x}_h(t) = e^{-\zeta\omega_n t}\ \left(C_1 sin(\omega_d t) + C_2 cos(\omega_d t)\right) \tag{4}$$

The important part is particular solution of the equation 1. For particular solution, let's try solving a rather simpler problem to get a direct solution.

Simplification Let's try solving a simpler problem to reach solution of our case. The simpler problem can be posed as

$$\ddot{\mathbf{x}} + 2\zeta\omega_n\dot{\mathbf{x}} + k\mathbf{x} = \frac{r(t)}{m} \tag{5}$$

With the particular solution for above case, we can get the particular solution for any general cases like one in equation 1. Say, if for this simple case, we can represent solution as $x_p(t)$,then for our case i.e equation 1, the solution can be written as

$$\mathbf{x}_{pact} = \frac{F_0}{T}\left(\mathbf{x}_p(t) - \mathbf{x}_p(t-T)\right) \tag{6}$$

Back to obtaining particular solution for equation 5, considering that r(t) = t u(t), the particular solution using method of undetermined coefficients can be assumed as

$$\mathbf{x}_p = A_1 t + A_0$$

Putting it in equation 5, the equation can be processed as

$$\begin{aligned}
2\zeta\omega_n A_1 + \omega_n^2\left(A_1 t + A_0\right) &= \frac{t}{m} \\
\omega_n^2 A_1 t + \left(2\zeta\omega_n A_1 + \omega_n^2 A_0\right) &= \frac{t}{m} \\
A_1 &= \frac{1}{m\omega_n^2} \\
A_0 &= \frac{-2\zeta\omega_n A_1}{\omega_n^2} = \frac{-2\zeta}{m\omega_n^3}
\end{aligned}$$

So, the particular solution for equation 5 becomes

$$\begin{aligned}
\mathbf{x}_p(t) &= \frac{t}{m\omega_n^2} - \frac{2\zeta}{m\omega_n^3} \qquad (7)\\
&= \frac{1}{k}\left(t - \frac{2\zeta}{\omega_n}\right) \qquad (8)
\end{aligned}$$

Using discussion above, the particular solution for equation 1 can be written using equation 6.

Total Solution Using the Homogeneous and Particular solution obtained above total solution of the equation can be written as

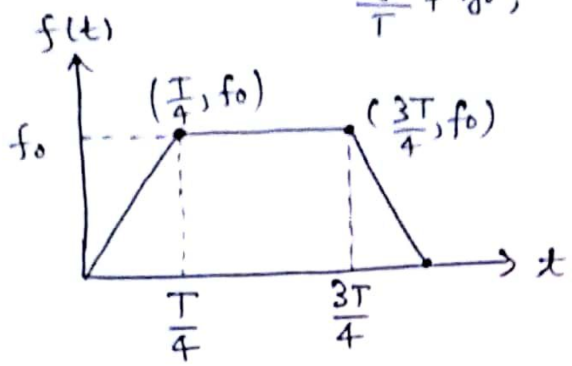
$$\mathbf{x} = e^{-\zeta\omega_n t}\ \left(C_1 sin(\omega_d t) + C_2 cos(\omega_d t)\right) + \frac{F_0}{T}\left(\mathbf{x}_p(t) - \mathbf{x}_p(t-T)\right) \tag{9}$$

, where $\mathbf{x}_p$(t) is given in equation 8.

C_1 and C_2 can be obtained using any initial condition.

2.28

Function $f(t) = \begin{cases} \frac{4f_0 t}{T} & ; \quad 0 < t < \frac{T}{4} \\ f_0 & ; \quad \frac{T}{4} \le t \le \frac{3T}{4} \\ -\frac{4f_0 t}{T} + 4f_0 & ; \quad \frac{3T}{4} < t < T \end{cases}$



Using fourier expansion,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Case I $\quad 0 < t < \frac{T}{4}$

$f(t) = -f(-t)$

odd func.

$\therefore \boxed{a_n = 0}$

Evaluate; $b_n = \frac{1}{T} \int_0^{\frac{T}{4}} \frac{4f_0 t}{T} \sin nt \, dt$

$= \frac{4f_0}{T^2} \int_0^{\frac{T}{4}} t \sin nt \, dt$

$$\boxed{b_n = \frac{4f_0}{T^2}\left[\frac{-T}{4n} \cos \frac{Tn}{4} + \frac{1}{n^2} \sin \frac{Tn}{4}\right]}$$

Case II $\quad \frac{T}{4} \le t \le \frac{3T}{4}$

$f(t) = f_0$

$a_0 = \frac{1}{T} \int_{T/4}^{3T/4} f_0 \, dt = \boxed{\frac{f_0}{2}}$

$a_n = \frac{1}{T} \int_{T/4}^{3T/4} f_0 \cos nt \, dt = \boxed{\frac{f_0}{nT}\left[\sin \frac{3nT}{4} - \sin \frac{nT}{4}\right]}$

$b_n = \frac{1}{T} \int_{T/4}^{3T/4} f_0 \sin nt \, dt = \boxed{\frac{f_0}{nT}\left[\cos \frac{nT}{4} - \cos \frac{3nT}{4}\right]}$

Case III $\frac{3T}{4} < t < T$

$f(t) =$ neither odd nor even

$$f(t) = -\frac{4f_0t}{T} + 4f_0$$

$$a_0 = \frac{1}{T}\int_{\frac{3T}{4}}^{T}\left(-\frac{4f_0t}{T} + 4f_0\right)dt$$

$$\boxed{a_0 = \frac{f_0}{8}}$$

$$a_n = \frac{1}{T}\int_{\frac{3T}{4}}^{T}\left(-\frac{4f_0t}{T} + 4f_0\right)\cos nt \, dt$$

$$= \frac{1}{T}\left\{-\frac{4f_0}{T}\left[\int_{\frac{3T}{4}}^{T} t\cos nt \, dt\right] + 4f_0\int_{\frac{3T}{4}}^{T}\cos nt \, dt\right\}$$

$$\boxed{a_n = \frac{1}{T}\left\{\frac{1}{n^2}\left[\cos nT - \cos\frac{3nT}{4}\right] - \frac{f_0}{n}\left[\sin\frac{3nT}{4}\right]\right\}}$$

$$b_n = \frac{1}{T}\int_{3T/4}^{T}\left(-\frac{4f_0 t}{T} + 4f_0\right)\sin nt\,dt$$

$$= \frac{1}{T}\left\{-\frac{4f_0}{T}\int_{3T/4}^{T} t\sin nt\,dt + 4f_0\int_{3T/4}^{T}\sin nt\,dt\right\}$$

$$= \frac{1}{T}\left\{-\frac{4f_0}{T}\left[\left[-\frac{t\cos nt}{n}\right]_{3T/4}^{T} + \int_{3T/4}^{T}\frac{\cos nt}{n}\,dt\right] + \frac{4f_0}{n}\left[-\cos nt\right]_{3T/4}^{T}\right\}$$

$$= \frac{1}{T}\left\{\frac{4f_0}{T}\left[\frac{T\cos nT}{n} - \frac{3T}{4n}\cos\frac{3nT}{4}\right] + \frac{1}{n^2}\left[\sin nT - \sin\frac{3nT}{4}\right] - \frac{4f_0}{n}\left[\cos nT - \cos\frac{3nT}{4}\right]\right\}$$

$$= \frac{1}{T}\left\{\cancel{\frac{4f_0\cos nT}{n}} - \frac{3f_0}{n}\cos\frac{3nT}{4} + \frac{1}{n^2}\left[\sin nT - \sin\frac{3nT}{4}\right] - \cancel{\frac{4f_0\cos nT}{n}} + \frac{4f_0}{n}\cos\frac{3nT}{4}\right\}$$

$$\boxed{= \frac{1}{4}\left\{\frac{f_0}{n}\cos\frac{3nT}{4} + \frac{1}{n^2}\left[\sin nT - \sin\frac{3nT}{4}\right]\right\}}$$

In this manner, one can obtain the particular integral of the differential eq^n

$$m\ddot{x} + c\dot{x} + kx = kf(t)$$

For homogenous soln. divide by m throughout,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}f(t)$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$$

where $\xi = \frac{c}{2\omega_n m}$, $\omega_n = \sqrt{\frac{k}{m}}$

Characteristic eq^n. $\quad m^2 + 2\xi\omega_n m + \omega_n^2 = 0$

$$m = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$m = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$m = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2}$$

Soln: $\boxed{x_h = e^{-\xi\omega_n t}\left(A\cos\omega_n\sqrt{1-\xi^2} + B\sin\omega_n\sqrt{1-\xi^2}\right)}$

Total soln. $= x_h + x_p$

multiply by $\frac{k}{m}$ in $f(t)$ and combine all the terms, that is your x_p.