Chapter 3 - part 1
Modern Portfolio Theory

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed the theory of portfolio choice. This theory analyses how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced. (www.nobelprize.org)

Risk can be characterized in different ways:

- As a function of our ignorance (theory of errors)
  - if we were smart enough, risk would disappear
  - would only have (large) deterministic models
- As a function of frequency
  - may know how often event occurs
  - but not where in sequence
- As a function of complexity (algorithmic)
  - length of the shortest formula that computes a sequence
  - random sequence most, a constant least complex

2 ways to model future time and uncertain future variables:

1. Discretely

   enumerate (list) all possible:

   - points in time
   - outcomes of variables in each point with their probabilities

   example: binomial tree

2. Continuously

   use dynamic process with infinitesimal time steps

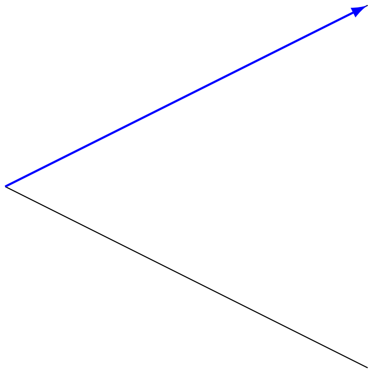
   - number of time steps $\to \infty$
   - probabilistic changes in variables (drawn from a distribution)

   example: geometric Brownian motion

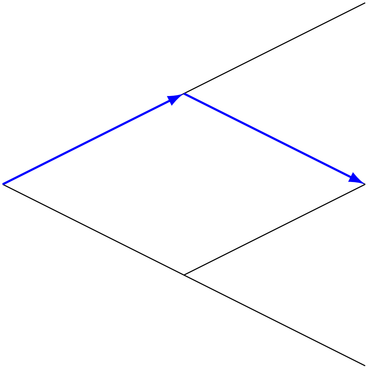
$$
\begin{array}{ccccc}
 & & & & 625 \\
 & & & \nearrow & \\
 & & 500 & & \\
 & \nearrow & & \searrow & \\
400 & & & & 400 \\
 & \searrow & & \nearrow & \\
 & & 320 & & \\
 & & & \searrow & \\
 & & & & 256 \\
t{=}0 & & t{=}1 & & t{=}2
\end{array}
$$

Binomial tree for a stock price

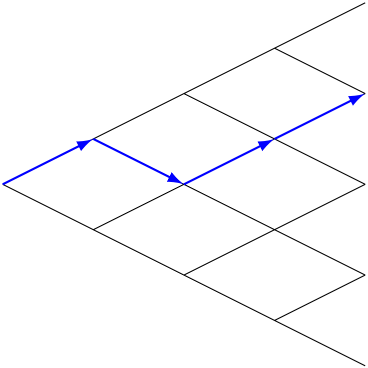
Each period, stock price can :
- go up with 25%, probability q
- go down with 20%, probability 1-q



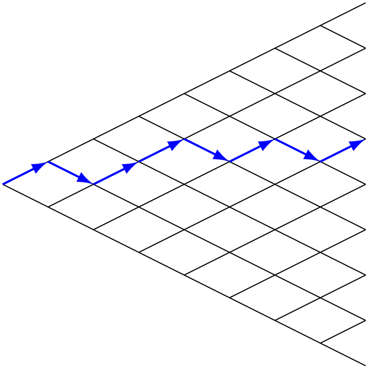
1 period of 1 year



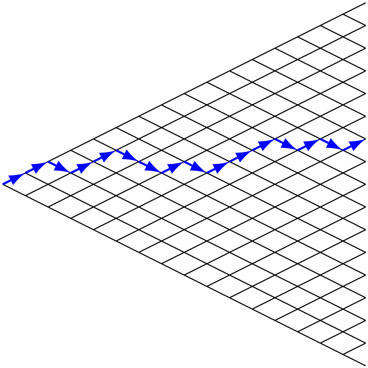
2 periods of 6 months



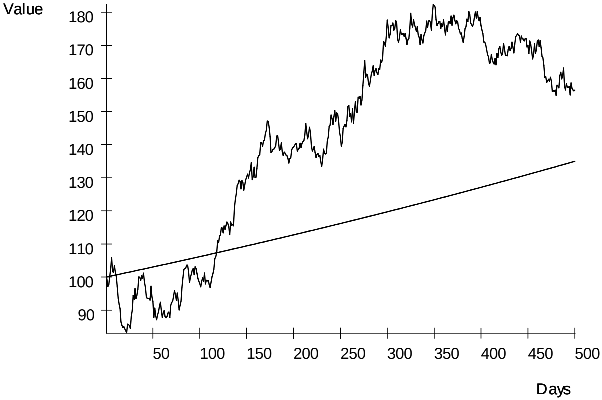
4 periods of 3 months



8 periods of 6 weeks



16 periods of 3 weeks



Sample path geometric Brownian motion, $\mu = .15$, $\sigma = .3$, t=500 days; smooth line is deterministic part of the motion.

Approaches can be combined to give a classification of models and techniques:

	Discrete time	Continuous time
Discrete var's	State preference theory, Binomial Option Pricing	Bankruptcy processes
Continuous var's	Portfolio theory, CAPM, Capital stucture	Black & Scholes Option pricing

What does risk mean for our choice problems?

- The results of our choices cannot be predicted with certainty
  - If we invest in shares of Apple today, we do not know the payoff next year
  - If we invest in all shares on London Stock Exchange, we do not know how much we will get back in 5 years
  - If we lend money to companies, by buying bonds, we do not know the real interest rate we will get

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- The results of some investments can be predicted (almost) with certainty
  - Do you know which?

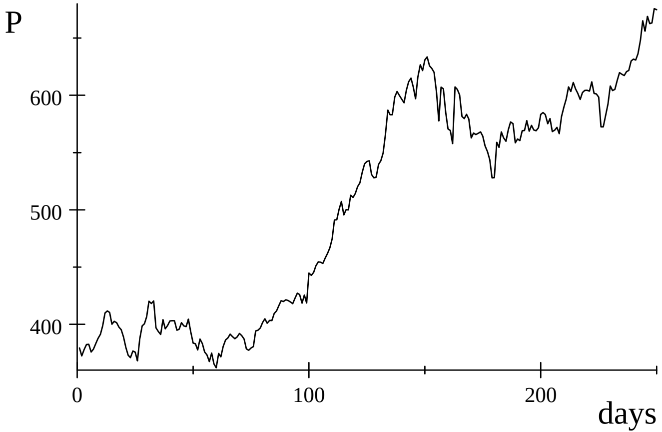
Risk of investments can be depicted in different ways

- We can look at prices of securities in financial markets
- We can translate prices (plus dividends) in returns:

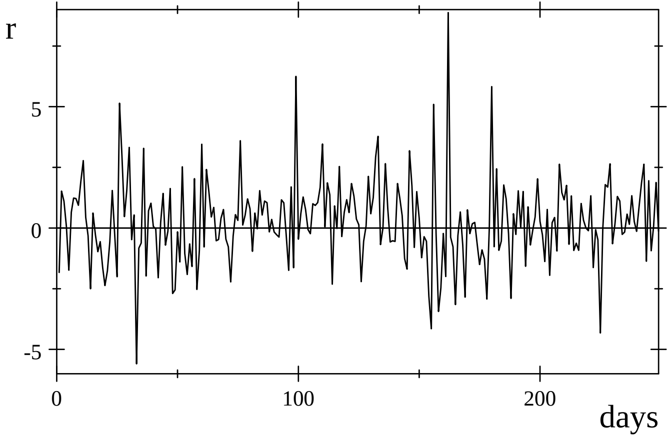
$$r_{it} = \frac{P_{i,t+1} - P_{it} + Div_{t+1}}{P_{it}}$$

- We can look at the distributional properties of returns
  - mean and variance, as in Markowitz' portfolio analysis
  - higher moments: skewness and curtosis

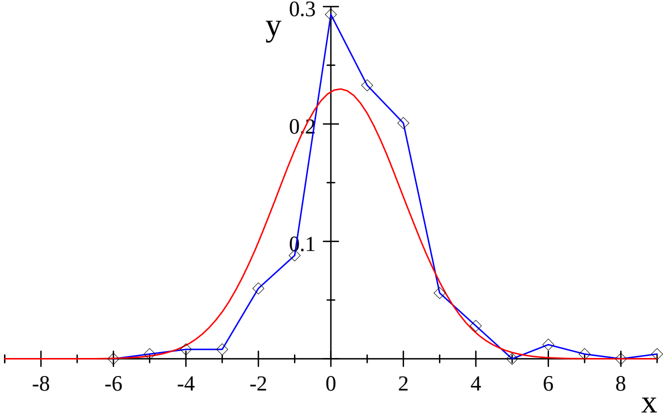
Illustrate with some actual data



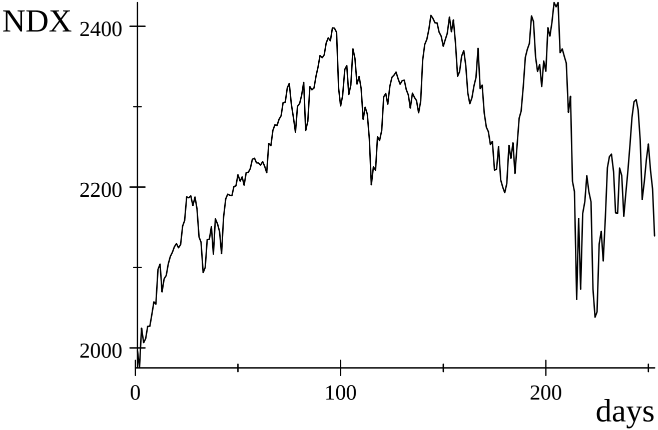
Daily closing prices Apple from 1 Sept. 2011 to 28 Aug. 2012



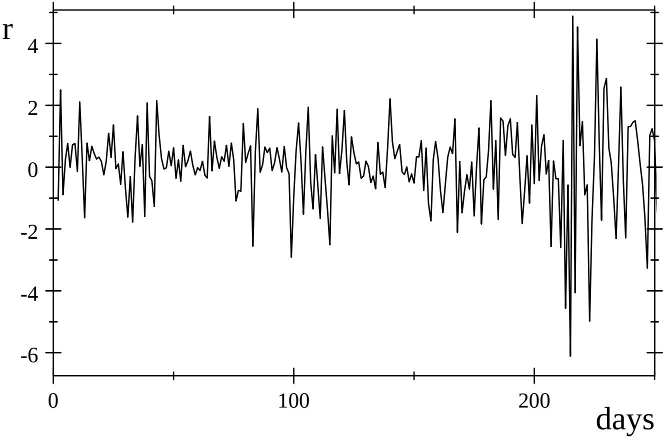
Daily returns Apple (%) from 1 Sept. 2011 to 28 Aug. 2012



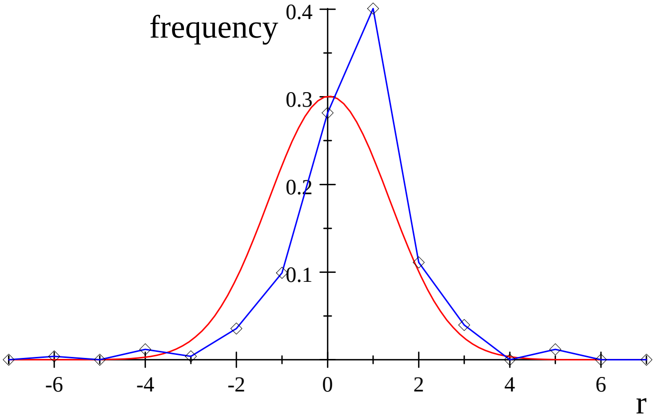
Frequency distribution of daily returns Apple (blue) and normal distribution with same mean and variance (red)



Nasdaq-100 index, daily closing prices, adjusted for dividends, for 253 trading days from 1 October 2010 to 30 September 2011



Daily returns Nasdaq-100 index for 252 days from 4 October 2010 to 30 September 2011



Frequency of daily returns Nasdaq-100 index over 252 days from 4 October 2010 to 30 September 2011

There are many quantitative risk measures, but:

Standard statistical measure of dispersion most often used: *Variance* or its square root *standard deviation*

- measures deviation from mean (historical) or expectation (forward looking)
- easily calculated, well known statistical properties
- also has disadvantages in financial analyses:
    - upward and downward deviations treated equally
    - ignores higher moments (skewness, kurtosis)
    - sometimes fails (e.g. in case of *stochastic dominance*)

We will use variance as risk measure, close to distributional properties

Calculating portfolio risk and return

- Diversification effect shows up in portfolio's variance
- demonstrate with simple numerical example
- illustrates the parallel, more general formulation of portfolio mean and variance

Asset returns in scenarios

Scenario:	1	2	3
Probability (π)	1/3	1/3	1/3
Return asset 1 (r_1)	.15	.09	.03
Return asset 2 (r_2)	.06	.06	.12

Expected asset returns, $E[r_i]$, are probability weighted sums over scenarios:

$$E[r_i] = \sum_{n=1}^{N} \pi_n r_{ni}$$

- assets are indexed i ($I = 2$)
- scenarios are indexed n ($N = 3$)
- π_n is the probability that scenario n will occur ($\sum_n \pi_n = 1$)

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In the numerical example:

$E[r_1] = 1/3 \times .15 + 1/3 \times .09 + 1/3 \times .03 = .09$
$E[r_2] = 1/3 \times .06 + 1/3 \times .06 + 1/3 \times .12 = .08.$

Asset variances are probability weighted sums of squared deviations from the expected returns:

$$\sigma_i^2 = \sum_{n=1}^{N} \pi_n (r_{ni} - E[r_i])^2$$

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In the numerical example:

$\sigma_1^2 = 1/3 \times (.15 - .09)^2 + 1/3 \times (.09 - .09)^2 + 1/3 \times (.03 - .09)^2 = 0.0024$

$\sigma_2^2 = 1/3 \times (.06 - .08)^2 + 1/3 \times (.06 - .08)^2 + 1/3 \times (.12 - .08)^2 = 0.0008.$

Now we combine equal parts of the assets in a portfolio

expected portfolio return is the weighted average of expected asset returns:

$$E[r_p] = \sum_{i=1}^{I} x_i E[r_i]$$

- where x_i are the asset weights ($\sum_i x_i = 1$)

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In the numerical example:

$\frac{1}{2} \times .09 + \frac{1}{2} \times .08 = .085$

Get same result by first calculating portfolio returns in scenarios:

$$\begin{aligned}
\tfrac{1}{2} \times .15 + \tfrac{1}{2} \times .06 &= 0.105 \\
\tfrac{1}{2} \times .09 + \tfrac{1}{2} \times .06 &= 0.075 \\
\tfrac{1}{2} \times .03 + \tfrac{1}{2} \times .12 &= 0.075
\end{aligned}$$

and then taking the expectation over scenarios:

$$1/3 \times .105 + 1/3 \times .075 + 1/3 \times .075 = 0.085$$

The variance of this portfolio return is:

$$\sigma_p^2 = 1/3 \times (.105 - .085)^2 + 1/3 \times (.075 - .085)^2 + 1/3 \times (.075 - .085)^2 = 0.0002$$

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- portfolio variance is *not* weighted average of asset variances
- would ignore correlation characteristics
- combining the 2 assets makes portfolio variance lower than any of asset variances (0.0024 and 0.0008)

Variance reducing effect of diversification can be shown by writing out the variance formula

Portfolio variance $= var(x_1 r_1 + x_2 r_2) = \sigma_p^2$
By definition:

$$\sigma_p^2 = \sum_{n=1}^{N} \pi_n \left[x_1 r_{n1} + x_2 r_{n2} - (x_1 E[r_1] + x_2 E[r_2])\right]^2$$

summation is over N scenarios.

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summation is over N scenarios. Rearranging terms:

$$\sigma_p^2 = \sum_{n=1}^{N} \pi_n \left[x_1 (r_{n1} - E[r_1]) + x_2 (r_{n2} - E[r_2])\right]^2$$

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Working out the square:

$$\sigma_p^2 = \sum_{n=1}^{N} \pi_n [x_1^2(r_{n1} - E[r_1])^2 + x_2^2(r_{n2} - E[r_2])^2 + 2x_1x_2(r_{n1} - E[r_1])(r_{n2} - E[r_2])]$$

rewriting gives 3 recognizable terms:

$$\sigma_p^2 = x_1^2 \underbrace{\sum_{n=1}^{N} \pi_n (r_{n1} - E[r_1])^2}_{\sigma_1^2} + x_2^2 \underbrace{\sum_{n=1}^{N} \pi_n (r_{n2} - E[r_2])^2}_{\sigma_2^2} +$$

$$2x_1 x_2 \underbrace{\sum_{n=1}^{N} \pi_n (r_{n1} - E[r_1])(r_{n2} - E[r_2])}_{\sigma_{1,2}}$$

portfolio variance is sum of asset variances plus covariances

$$\sigma_p^2 = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2x_1 x_2 \sigma_{1,2}$$

Covariance measures how assets move together through scenarios (or time):

$$\sigma_{ij} = \sum_{n=1}^{N} \pi_n (r_{ni} - E[r_i])(r_{nj} - E[r_j])$$

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$\sigma_{1,2} =$
$1/3 \times (.15 - .09)(.06 - .08) +$
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$1/3 \times (.03 - .09)(.12 - .08) = -0.0012.$

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How can covariance be negative while variance is always positive?

Filling in the numbers reproduces portfolio variance:

$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{1,2}$$

$$\sigma_p^2 = .5^2 \times .0024 + .5^2 \times .0008 + 2 \times .5 \times .5 \times -.0012$$

$$\sigma_p^2 = 0.0006 + 0.0002 - 0.0006 = .0002$$

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$$\sigma_p^2 = 0.0006 + 0.0002 - 0.0006 = .0002$$

Diversification effect: covariance term reduces σ_p^2:

- covariances can be small or negative
- number of covariance terms increases more rapidly with number of assets than variance terms
- becomes clear by writing portfolio variance as variance-covariance matrix

Portfolio variance as *variance-covariance matrix:*

$$
\begin{array}{ccc}
x_1^2\sigma_1^2 & x_1x_2\sigma_{1,2} & Asset1 \\
x_1x_2\sigma_{1,2} & x_2^2\sigma_2^2 & Asset2 \\
Asset1 & Asset2 & \Sigma = \sigma_p^2
\end{array}
$$

- main diagonal: covariances of asset returns with themselves, i.e. variances σ_1^2 and σ_2^2
- off-diagonal: covariances between assets
- portfolio variance sum of all cells: $\sigma_p^2 = \sum_{i=1}^{I} \sum_{j=1}^{I} x_i x_j \sigma_{ij}$
- with more assets, diversification effect becomes stronger:
    - with I assets, no. of cells=I^2
    - no. of variances=I, no. of covariances=I(I-1)

$$\begin{array}{ccc} x_1^2\sigma_1^2 & x_1x_2\sigma_{1,2} & Asset1 \\ x_1x_2\sigma_{1,2} & x_2^2\sigma_2^2 & Asset2 \\ Asset1 & Asset2 & \Sigma = \sigma_p^2 \end{array}$$

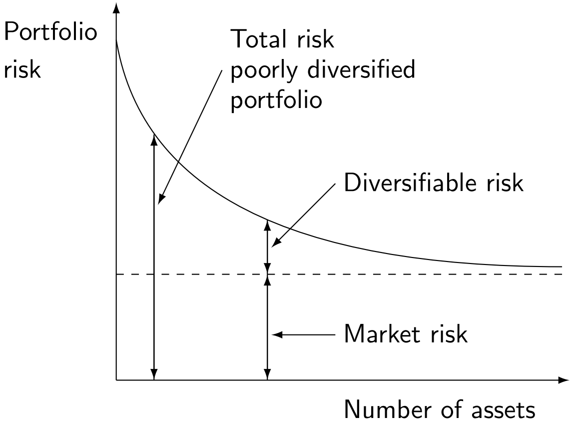
$$Assets: \quad 2 \quad Cells: \quad 4 \qquad var.'s: \quad 2 \quad covar.'s: \quad 2$$

$$\begin{array}{cccc}
x_1^2\sigma_1^2 & x_1x_2\sigma_{1,2} & x_1x_3\sigma_{1,3} & Asset1 \\
x_1x_2\sigma_{1,2} & x_2^2\sigma_2^2 & x_2x_3\sigma_{2,3} & Asset2 \\
x_1x_3\sigma_{1,3} & x_2x_3\sigma_{2,3} & x_3^2\sigma_3^2 & Asset3 \\
Asset1 & Asset2 & Asset3 & \Sigma = \sigma_p^2
\end{array}$$

$$Assets: \quad 3 \quad Cells: \quad 9 \qquad var.'s: \quad 3 \quad covar.'s: \quad 6$$

$$
\begin{array}{ccccc}
x_1^2\sigma_1^2 & x_1x_2\sigma_{1,2} & x_1x_3\sigma_{1,3} & x_1x_4\sigma_{1,4} & Asset1 \\
x_1x_2\sigma_{1,2} & x_2^2\sigma_2^2 & x_2x_3\sigma_{2,3} & x_2x_4\sigma_{2,4} & Asset2 \\
x_1x_3\sigma_{1,3} & x_2x_3\sigma_{2,3} & x_3^2\sigma_3^2 & x_3x_4\sigma_{3,4} & Asset3 \\
x_1x_4\sigma_{1,4} & x_2x_4\sigma_{2,4} & x_3x_4\sigma_{3,4} & x_4^2\sigma_4^2 & Asset4 \\
Asset1 & Asset2 & Asset3 & Asset4 & \Sigma = \sigma_p^2
\end{array}
$$

$$
Assets: \quad 4 \quad Cells: \quad 16 \quad var.'s: \quad 4 \quad covar.'s: \quad 12
$$



Diversification effect

Financial markets allow easy diversification:

- In USA several 1000s companies are listed
- In most European countries at least several 100s
- There are many mutual (investment) funds:
    - many hundreds or even thousands on most exchanges
    - allow diversification of small investment amounts
    - also small increases /decreases

Diversification is one of the very few 'free lunches' in finance

Big investors hold well diversified portfolios, so they are *not* sensitive to risk that disappears through diversification

- Risk that disappears is called unique, or unsystematic, or diversifiable risk
  - that is the risk engineers are concerned with
- Risk that remains is market risk, or systematic risk, or undiversifiable risk
  - that is the risk that counts in finance

Conclusion must be:

- *The risk of an investment is the risk in the context of a well diversified portfolio!*

The contribution of each stock to portfolio risk

- If risk = risk in well diversified portfolio
- risk of individual asset is not its variance
- but its contribution to portfolio risk
- taking covariance into account

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Measured as sum of row (column) entries in var-covar matrix

- e.g. for stock 1 in a 2 stock portfolio:

$$contr_1 = x_1^2\sigma_1^2 + x_1x_2\sigma_{1,2} = x_1\left[x_1\sigma_1^2 + x_2\sigma_{1,2}\right]$$

Manipulate a bit to get easy expression

Recall: variance is covariance with itself: $\sigma_1^2 = cov(r_1, r_1)$
so we can write:

$$contr_1 = x_1 \left[x_1 \sigma_1^2 + x_2 \sigma_{1,2}\right]$$

as:

$$contr_1 = x_1 \left[x_1 cov(r_1, r_1) + x_2 cov(r_1, r_2)\right]$$

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as:

$$contr_1 = x_1 \left[x_1 cov(r_1, r_1) + x_2 cov(r_1, r_2) \right]$$

We use the following properties of covariance:

$$cov(z_1, y) + cov(z_2, y) = cov(z_1 + z_2, y)$$
$$cov(c \times z, y) = c \times cov(z, y)$$

Using the second property

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'in reverse', we can write:

$$contr_1 = x_1 \left[x_1 cov(r_1, r_1) + x_2 cov(r_1, r_2) \right]$$

$$contr_1 = x_1 \left[cov(r_1, r_1 x_1) + cov(r_1, r_2 x_2) \right]$$

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and using the first property

$$cov(z_1, y) + cov(z_2, y) = cov(z_1 + z_2, y)$$

we can write

$$contr_1 = x_1 \left[cov(r_1, r_1 x_1 + r_2 x_2) \right]$$

since $r_1 x_1 + r_2 x_2 = r_p$, the portfolio return,

$$contr_1 = x_1 \left[cov(r_1, r_1 x_1 + r_2 x_2) \right]$$

is the same as:

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is the same as:

$$contr_1 = x_1 \left[cov(r_1, r_p)\right]$$

The relative contribution is the fraction of σ_p^2 :

$$\frac{contr_1}{\sigma_p^2} = \frac{x_1 \left[cov(r_1, r_p)\right]}{\sigma_p^2} = x_1 \frac{\sigma_{1p}}{\sigma_p^2}$$

Ratio σ_{1p}/σ_p^2 is defined as β_1, or in general notation:

$$\beta_i = \frac{\sigma_{ip}}{\sigma_p^2}$$

So relative contribution of asset i to portf. variance is:

$$\frac{contr_1}{\sigma_p^2} = x_i \beta_i$$

Risk of an asset expressed in a single variable β

- β measures only systematic risk
- not risk that disappears through diversification

Relation also interpreted other way around:

- β is sensitivity of stock returns for changes in portfolio returns
    - stocks with $\beta > 1$
      change more than proportionally with changes in portfolio returns
    - stocks with $\beta < 1$
      change less than proportionally

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Like variance, β is an objective measure:

- People who use the same data set
- will calculate the same βs
- but: not same as people's idea of risk (banks?)

More about β

- β add linearly (unlike variances):

$$\beta_p = \sum^{i} x_i \beta_i$$

- Company β also weighted average over:
  - projects:

$$\beta_{company} = x_1 \beta_{proj.1} + .. + x_n \beta_{proj.n}$$

  - capital categories:

$$\beta_{company} = x_E \beta_{equity} + x_D \beta_{debt}$$

  - or even fixed and variable costs
- Note: measuring risk as β is consequence of considering risk in context of portfolio, not result of a specific model as CAPM.

Covariance is often 'standardized' by standard deviations

- called correlation coefficient ρ :

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \times \sigma_j}$$

- correlation limited by -1 and +1 $(-1 \leq \rho \leq 1)$
- also written other way around: $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$

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- also written other way around: $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$

Applied to portfolio variance:

$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{1,2}$$
$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\rho_{1,2}\sigma_1\sigma_2$$

Maximum diversification if ρ is minimal (i.e.-1)

Illustrate diversification effect with numerical example:

- Take 4 stocks (1,2,3,4) in future scenarios
  - one pair perfectly positively correlated
  - one pair perfectly negatively correlated
  - one normal pair: low, positive correlation
- Stock 2,3,4 have same $E[r]$ and $\sigma^2(r)$
  only correlation with stock 1 differs
- Make portfolios of 2 stocks: 1,2 and 1,3 and 1,4
  - vary portfolio weights: 100%, 75%, 50%, 25%, 0%
  - weights $\geqslant 0$, so no *short selling*
  - calculate portfolio return and standard deviation
- Depict results in different ways

Stock returns in different future scenarios:

Scenario	Prob.(π)	r_1	r_2	r_3	r_4
1	.2	.125	.125	.225	.035
2	.2	.1	.075	.275	.2
3	.2	.15	.175	.175	.225
4	.2	.2	.275	.075	.2
5	.2	.175	.225	.125	.215
$E[r]$		.15	.175	.175	.175
$\sigma(r)$		.0354	.0707	.0707	.0706

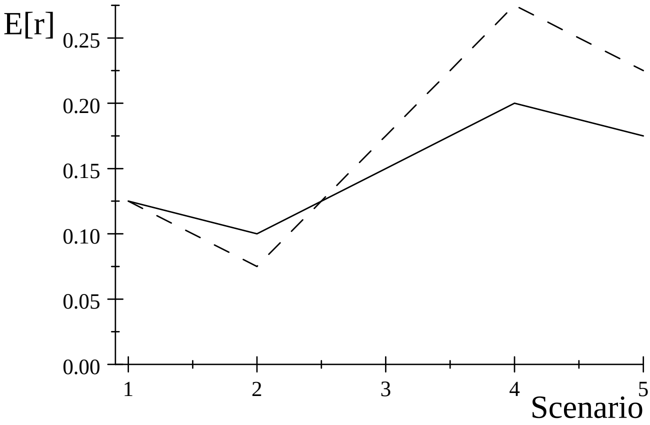
$E[r]$, $\sigma^2(r)$, covariances and correlations calculated as before

The relevant covariances and correlations are:

$$\begin{aligned} \sigma_{1,2} &= .0025 & \rho_{1,2} &= 1 \\ \sigma_{1,3} &= -.0025 & \rho_{1,3} &= -1 \\ \sigma_{1,4} &= .0009 & \rho_{1,4} &= .36 \end{aligned}$$

- Stock 2 and 3 are extreme cases with perfectly positive and negative correlation with stock 1
- Stock 4 is normal case

Next step: make portfolios of stock 1 and one other stock at the time, present portfolios in 5 different ways.
First stock 2:

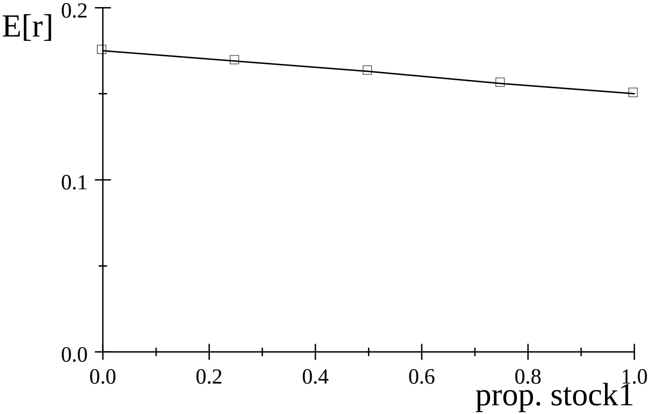


Returns stock 1 (solid) and stock 2 (dashed)

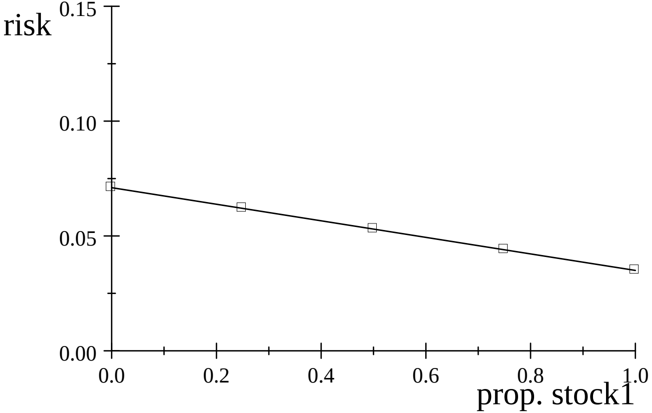
We make 5 portfolios with different proportions of the stocks:

x_1	x_2	$E[r_p]$	σ_p
1	0	.15	.035
.75	.25	.156	.044
.50	.50	.163	.053
.25	.75	.169	.062
0	1	.175	.071

- With perfectly positively correlated stocks there is no advantage of diversification (diversification is impossible).
- All combinations of stocks (portfolios) are straight line interpolations between the two stocks



Portfolios of stock 1 & 2



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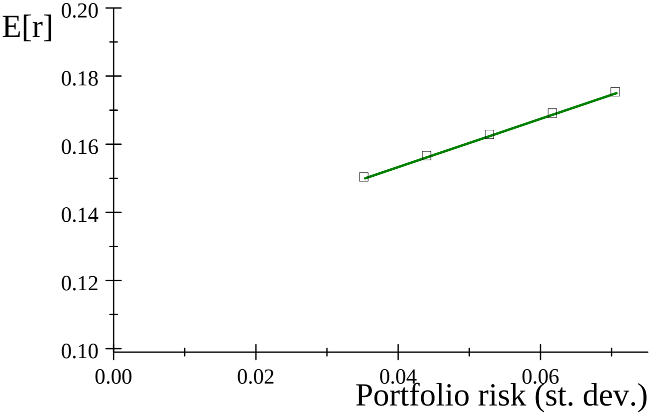
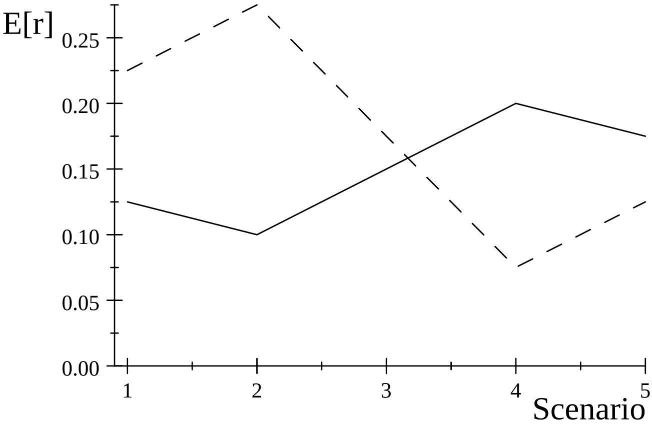


Expected portfolio return and standard deviation

Next, we repeat the procedure with stock 3:



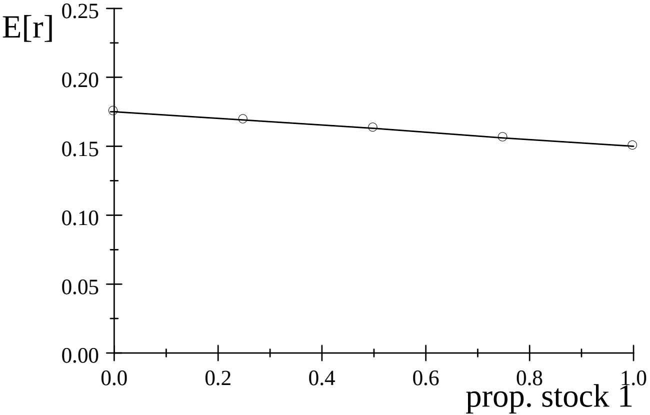
Returns stock 1 (solid) and stock 3 (dashed)

The portfolios are:

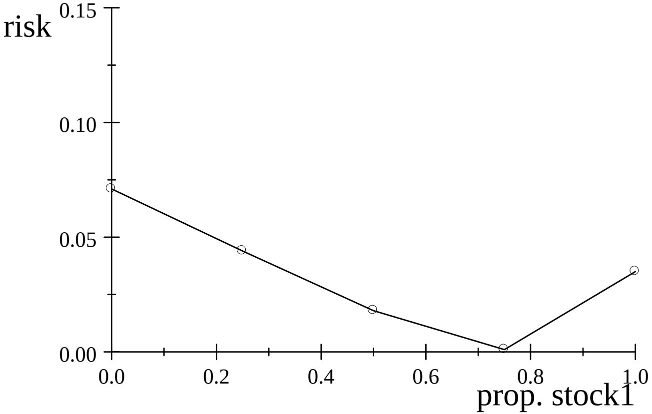
x_1	x_3	$E[r_p]$	σ_p
1	0	.15	.035
.75	.25	.156	.001
.50	.50	.163	.018
.25	.75	.169	.044
0	1	.175	.071

$\rho = -1$ gives large diversification effect:

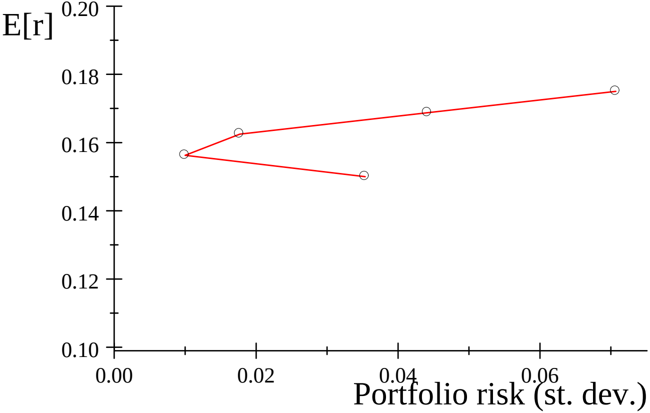
- portfolio return still straight line interpolation
- portfolio risk bent downwards, less risk
- In the extreme, no-risk portfolio can be made



Portfolios of stock 1 & 3

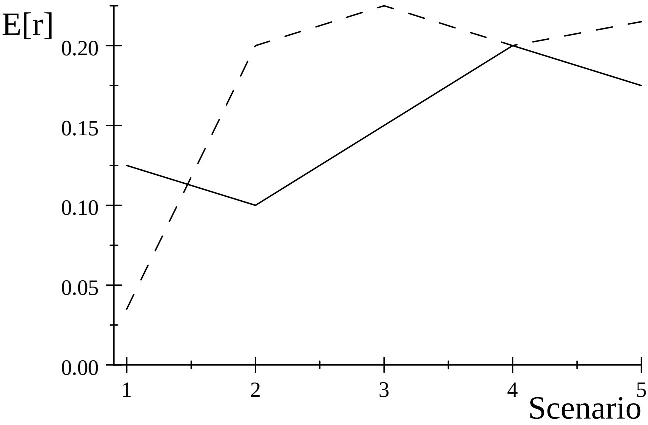


Portfolios of stock 1 & 3



Expected portfolio return and standard deviation

Finally, stock 4, the normal case:



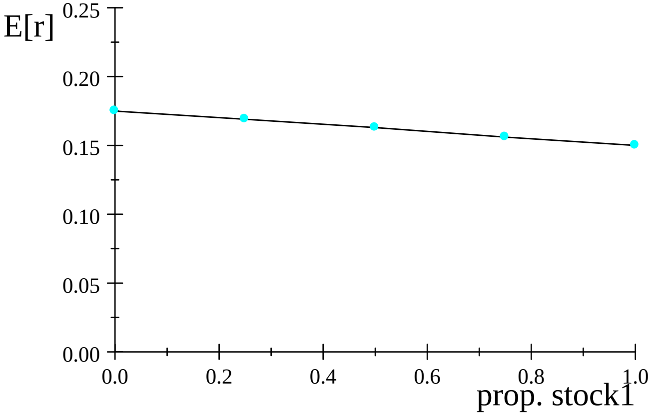
Returns stock 1 (solid) and stock 4 (dashed)

The portfolios:

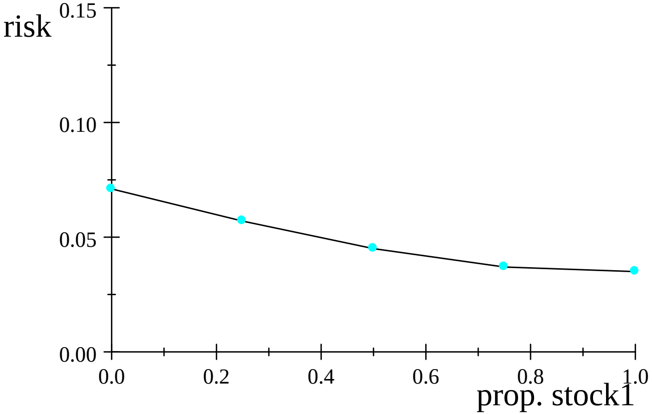
x_1	x_4	$E[r_p]$	σ_p
1	0	.15	.035
.75	.25	.156	.037
.50	.50	.163	.045
.25	.75	.169	.057
0	1	.175	.071

In the normal case of positive but imperfect correlation:

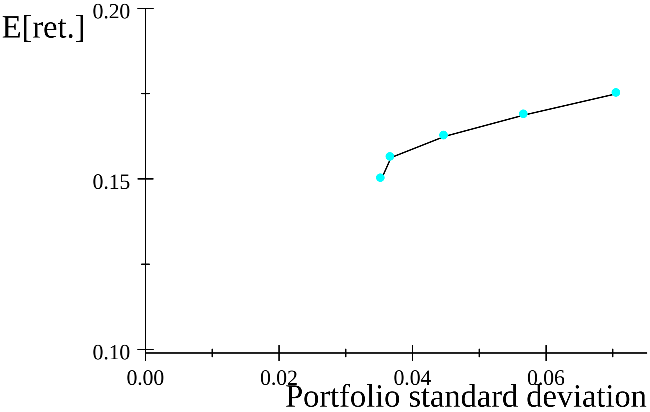
- portfolio variance is reduced but still present
- portfolio return again is a straight line interpolation
- portfolio risk bent downward, but to a much lesser degree



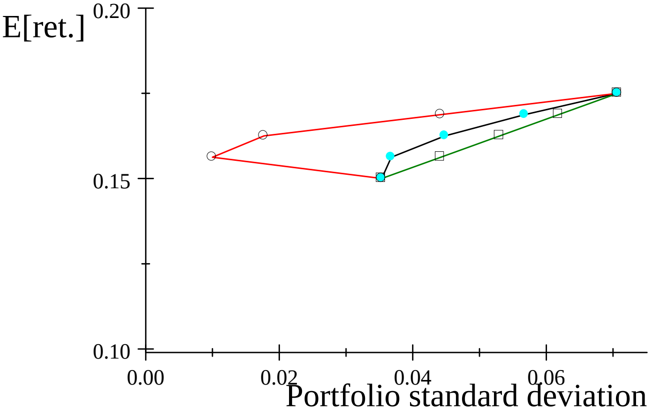
Portfolios of stock 1 & 4



Portfolios of stock 1 & 4



Expected portfolio return and standard deviation



Expected portfolio return and standard deviation

Lines from left to right: $\rho_{1,3} = -1, \;\; \rho_{1,4} = .36, \;\; \rho_{1,2} = 1$

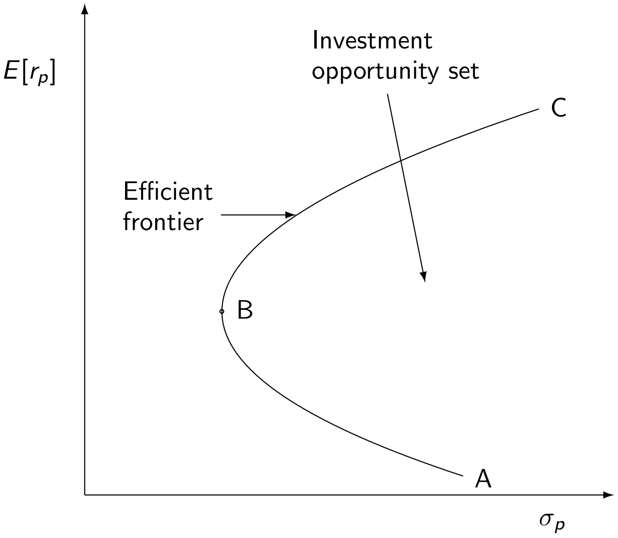
With more stocks and more combinations, picture remains the same:

- Negative correlations between assets (almost) do not occur
- Zero risk portfolios of risky assets are impossible
- Typical correlations are moderately positive
- Gives cone-like risk-return pictures (mean-variance characteristics)

Markowitz efficient portfolios

The setting:

- Collection of all possible combinations of investments is called the
  - *investment opportunity set or*
  - *investment universe*
- graphical representation
  - cone- or egg-shaped
  - also called *Markowitz bullet*



Investment universe and the efficient frontier

Not all opportunities will be chosen by rational investors:

- only those on the *efficient frontier* between
  - *minimum variance portfolio* B and
  - *maximum return portfolio* C

All other opportunities are inefficient:

- they can be replaced by an investment that
  - offers higher return for the same risk
  - or lower risk for the same return

We analyse portfolio selection first without, then with a financial market.

Investors choose portfolios:

- based on their preferences or risk aversion
- expressed in their indifference curves
- such that their utility is maximized (i.e. choice is on highest indifference curve)

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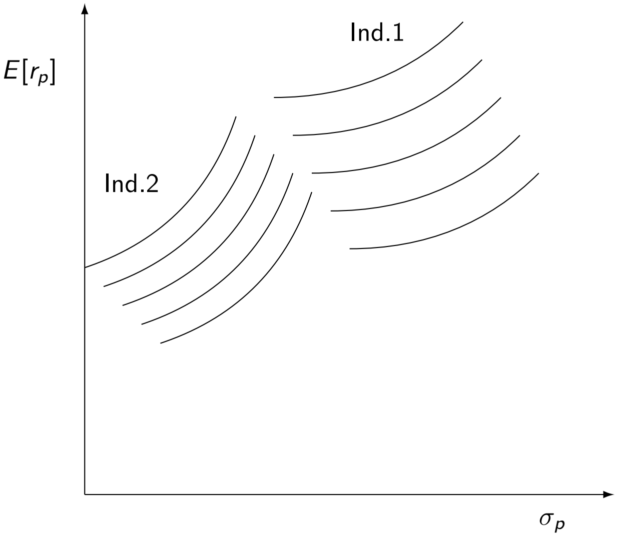
- What do indifference curves look like in a risk-return space?
- Which of the two individuals in the picture is more risk averse?
- In which direction increases utility?



Indifference curves in risk-return space

In this setting, portfolio selection is done in 2 steps:

1. the preferred risk return combination is chosen
    1. as the tangency point of the indifference curve and the efficient frontier
    2. individual preferences have to be known to make that decision!

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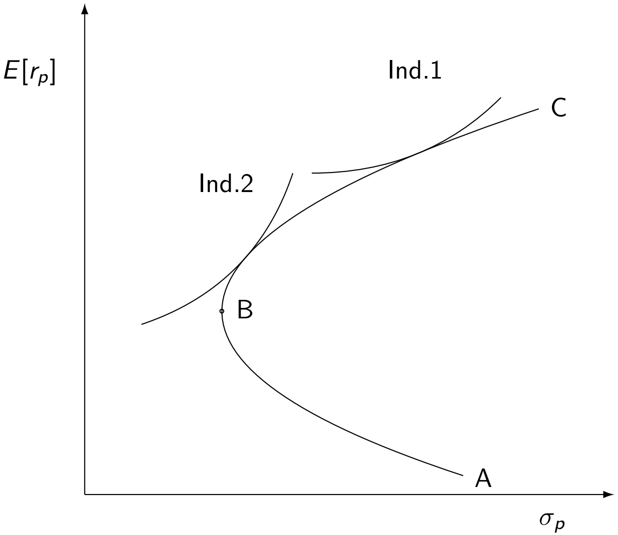
1. the preferred risk return combination is chosen
    1. as the tangency point of the indifference curve and the efficient frontier
    2. individual preferences have to be known to make that decision!
2. portfolio variance is minimized subject to the restrictions that
    1. the return is not less than the chosen return
    2. the portfolio weights sum to 1
    3. (the portfolio weights are positive, if no short sales are allowed)



Choices along the efficient frontier

Minimization can be done in different ways:

- analytically e.g. with Lagrange multipliers
- numerically

Banks used to provide this as an expensive service
Now you can do it at home with a spreadsheet

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Do you see a practical problem coming up?

- Number of covariances is $I(I-1)/2$, gets very large:
  - $I = 10 \Rightarrow I(I-1)/2 = 45$
  - $I = 100 \Rightarrow I(I-1)/2 = 4950$

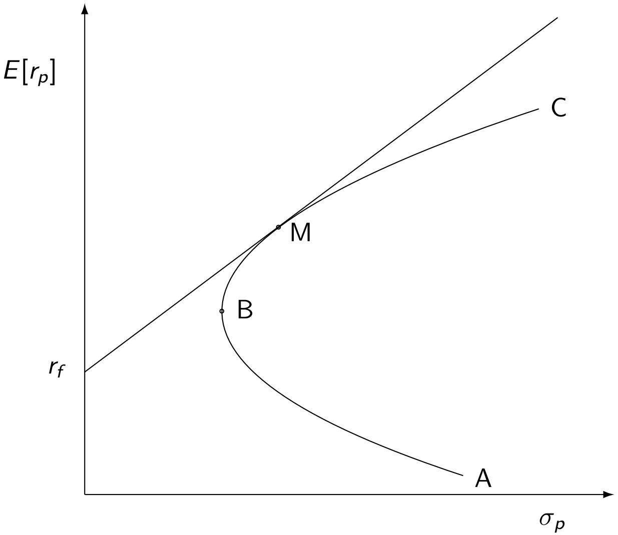
Pricing portfolios in equilibrium

We extend the analysis with a financial market (similar to Fisher's analysis) and market equilibrium

- Introduction of a financial (money) market
  - adds a new investment opportunity: risk free borrowing and lending
  - is also opportunity to move consumption back and forth in time

Looks trivial, but has profound effects

- changes the shape of the efficient frontier
- all investors want to hold combinations of risk free asset and tangency portfolio M (called *two-fund separation*)



The Capital Market Line

The straight line from r_f through portfolio M is called *Capital Market Line*

- offers higher exp. return than old efficient frontier BC
- investors will choose their optimal positions along it

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Notice that M is tangency point:

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- determined by r_f + returns, var-covar of risky assets
- not by investors' risk preferences

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- not by investors' risk preferences

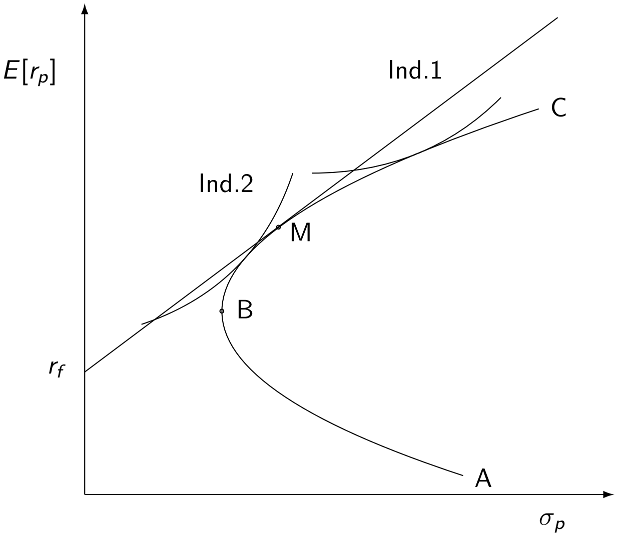
All investors will want to hold M $\Rightarrow$

- individual preferences expressed in proportion risk free investment

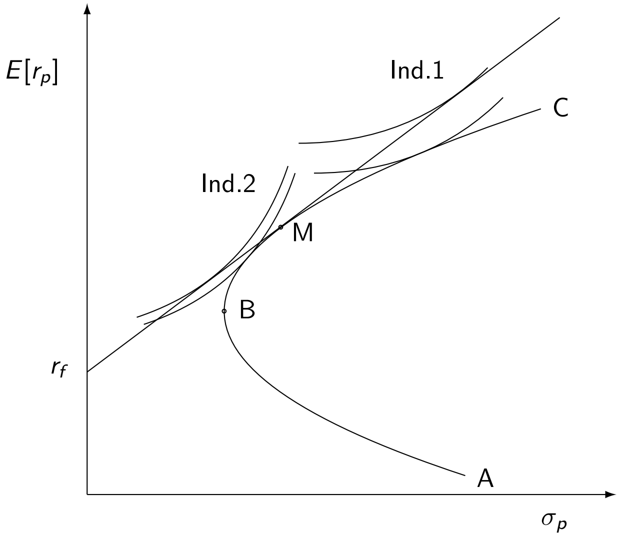
Market equilibrium requires:

- set of market clearing prices
- all assets must be held $\Rightarrow$ prices adjust so that excess demand/supply is zero
- includes risk free asset: risk free rate such that borrowing equals lending
- in tangency portfolio M:
    - all risky assets are held according to their market value weights
    - hence the name *market portfolio*
    - $\Rightarrow$ all investors hold risky assets in same proportions

Result: investors jump to higher indifference curves



Choices along the capital market line



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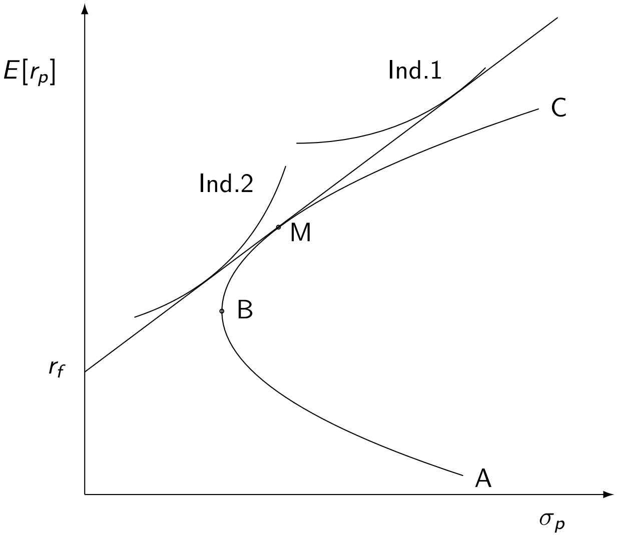

Choices along the capital market line

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How does Ind. 2 reach his optimal point on the CML between r_f and M?

- By investing a proportion of his money in the market portfolio and the rest in risk free lending

How does Ind. 1 reach her optimal point on the CML beyond M?

- By borrowing some amount risk free and investing *more than her money* in the market portfolio.
    - M is expected to earn more than r_f
    - if expectation is realized, difference $r_m - r_f$ is added to return, which will be $> r_m$
    - but if realized $r_m < r_f$, her return may be $< r_f$, risk is increased

Capital market line:

- equilibrium risk-return relation for *efficient* portfolios
- only valid when all risk comes from share of market portfolio M in any portfolio p

Expression for CML can easily be derived:

- invest x in M and $(1 - x)$ risk free
- this portfolio has expected return of:

$$E(r_p) = (1 - x)r_f + xE(r_m)$$

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$\sigma_p = x\sigma_m$ which means: $x = \frac{\sigma_p}{\sigma_m}$

Substituting this back in return relation eliminates x:

$$E(r_p) = (1 - \frac{\sigma_p}{\sigma_m})r_f + \frac{\sigma_p}{\sigma_m}E(r_m)$$

$$E(r_p) = r_f - \frac{\sigma_p}{\sigma_m}r_f + \frac{\sigma_p}{\sigma_m}E(r_m)$$

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p$$

- r_f = time value of money
- $\frac{E(r_m)-r_f}{\sigma_m}$ = price per unit of risk, the *market price of risk*
- σ_p = volume of risk

Capital market line is linear

- Intuition: in Markowitz' mean-variance model
  - return is function of a quadratic (σ_p^2)
  - marginal return (1^{st} derivative) will be linear
  - marginal risk-return trade-off is constant
- If marginal risk-return trade-off is constant
  - it is the same for all market participants
  - regardless of their attitudes to risk
    (shape of their indifference curves)
- By consequence, managers can use market price of risk
  - don't have to know preferences, risk attitude of shareholders
  - allows separation of ownership and management