Left over Topic

Eigen value estimation by using Similarity Transformation

Motivation - Transform the matrix A into either a diagonal or an upper triangular matrix, where eigen values can be easily read.

Similarity Transformation

For a given $A_{n \times n}$, the similarity transformation

$$B = M^{-1} A M$$

where M is any invertible matrix

- The transformation does not change eigen values i.e. A and B will have same eigen values

How to select M ?

The QR method

Q [Orthogonal Matrix] : A square matrix whose columns are orthonormal

$$Q = [q_1 \; q_2 \; \ldots \; q_n]$$

$$q_i^T q_j = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j$$

Also, $Q^T Q = Q Q^T = I$

$$Q^T = Q^{-1}$$

R [Upper triangular Matrix]

QR algorithm for finding Eigen Values

$A_0 = A$

(i) $A_k = Q_k R_k$ ←

(ii) $A_{k+1} = R_k Q_k$

(iii) Repeat (i) and (ii) until convergence

⇒ A_k's have same eigen values as A

⇒ After sufficient iterations, i.e. $k \to \infty$

$A_k \to$ diagonal matrix if A was symmetric

$\to$ upper triangular matrix otherwise

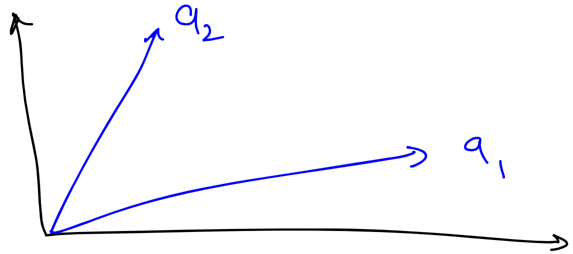
→ Schur's Lemma

QR decomposition by using Gram-Schmidt process

Given two independent vectors a_1 and a_2, the Gram-Schmidt process transform these vectors such that

(i) they are perpendicular to one another

(ii) they have unit length.



$A = [a_1 \quad a_2]$

$Q = [q_1 \quad q_2]$ $\qquad q_1 \perp q_2$

1. $q_1 = \dfrac{a_1}{\|a_1\|}$

2. Make a_2 $\perp^r$ to q_1 and normalize

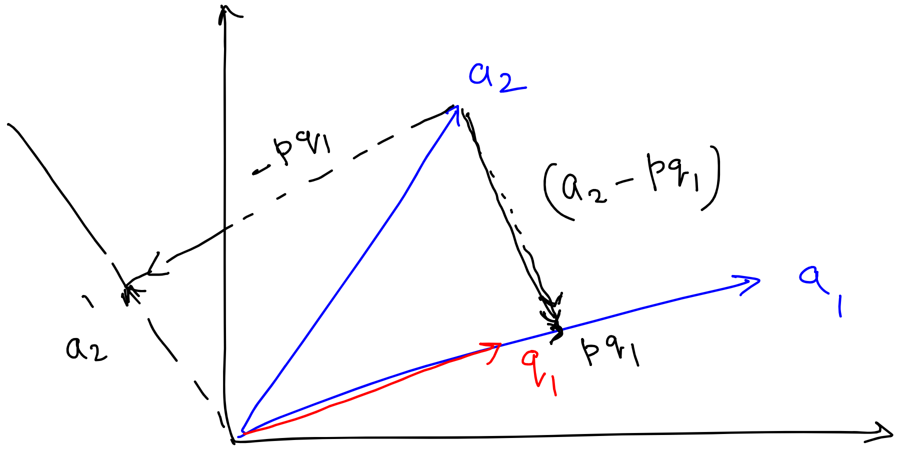
$$(a_2 - p q_1) \perp q_1$$

$$q_1^T (a_2 - p q_1) = 0$$

$$\Rightarrow \quad p = \frac{q_1^T a_2}{q_1^T q_1} = q_1^T a_2$$

$$a_2' = a_2 - (q_1^T a_2)\, q_1$$

$$q_2 = a_2' / \|a_2\|$$



In general, the Gram Schmidt process starts with independent vectors $A = [a_1 \ a_2 \ \cdots \ a_n]$ and ends with orthonormal vectors $Q = [q_1 \ q_2 \ \cdots \ q_n]$. At jth step

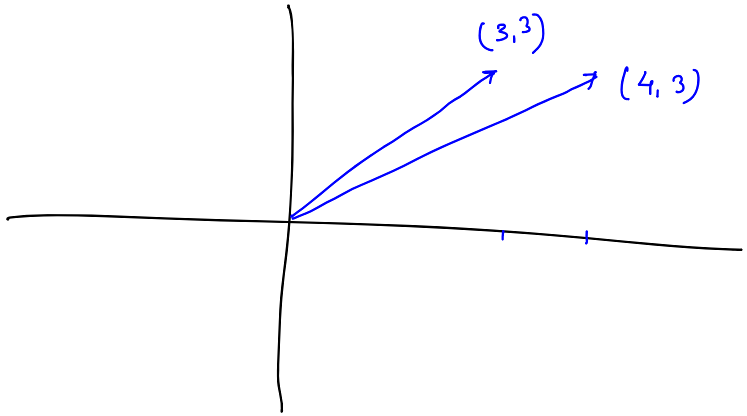
$$a_j' = a_j - (q_1^T a_j)\, q_1 - (q_2^T a_j)\, q_2 - \cdots - (q_{j-1}^T a_j)\, q_{j-1}$$

$$q_j = a_j' / \|a_j'\|$$

$$A = Q R$$

$$\begin{bmatrix} a_1 & a_2 & a_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & \cdots & q_1^T a_n \\ & q_2^T a_2 & \cdots & q_2^T a_n \\ & & \ddots & \\ & & & q_n^T a_n \end{bmatrix}$$

$$A \qquad Q \qquad R$$



Example

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix}$$

1. $$q_1 = \frac{a_1}{\|a_1\|} = \frac{[4 \quad 3]^T}{5} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

2. $$a_2' = a_2 - (q_1^T a_2)\, q_1$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \left(\underbrace{\begin{bmatrix} 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}} \right) \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$q_2' = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - 4.2 \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 3.36 \\ 2.52 \end{bmatrix} = \begin{bmatrix} -0.36 \\ 0.48 \end{bmatrix}$$

$$q_2 = \frac{q_2'}{\|a_2\|} = \frac{[-0.36 \quad 0.48]^T}{0.6} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$$

$$R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix} = \begin{bmatrix} 5 & 4.2 \\ & 0.6 \end{bmatrix}$$

Example - Eigen values by using QR

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix}$$

1. $A_0 = A = \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix}$

2. $A_0 = Q_0 R_0$

$$= \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 5 & 4.2 \\ 0 & 0.6 \end{bmatrix}$$

3. $A_1 = R_0 Q_0$

$$= \begin{bmatrix} 5 & 4.2 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 6.52 & 0.36 \\ 0.36 & 0.48 \end{bmatrix}$$

4. $A_1 = Q_1 R_1$

$$= \begin{bmatrix} 0.9985 & -0.0551 \\ 0.0551 & 0.9885 \end{bmatrix} \begin{bmatrix} 6.5299 & 0.3859 \\ 0 & 0.4594 \end{bmatrix}$$

5. $A_2 = R_1 Q_1$

$$= \begin{bmatrix} 6.5413 & 0.0253 \\ 0.0253 & 0.4587 \end{bmatrix}$$

$\varepsilon_a(\%) = 4.64\%$

6. $A_2 = Q_2 R_2$

7. $A_3 = R_2 Q_2 = \begin{bmatrix} 6.5414 & 0.0018 \\ 0.0018 & 0.4586 \end{bmatrix}$

$\varepsilon_a(\%) = \underline{\underline{0.023}}$

eigen values $[\, 6.5414 \quad 0.4586 \,]$

Convergence of Jacobi / Gauss - Seidel

Jacobi

$$Ax = b$$

$$\Rightarrow \boxed{X_{k+1} = SX_k + B} \quad —(1)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_2 - \frac{a_{23}}{a_{22}} x_3$$

$$x_3 = --$$

$$S = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 / a_{11} \\ b_2 / a_{22} \\ b_3 / a_{33} \end{bmatrix}$$

At convergence, if X is the true solution

$$\underline{X} = S\underline{X} + B \quad —(2)$$

(2) - (1)

$$(X - X_{k+1}) = S(X - X_k)$$

$$e_{k+1} = S e_k$$

$$e_{k+1} = S\, e_k$$

$$\Rightarrow \boxed{e_k = S^k e_0}$$

For algorithm to converge

$$\lim_{k \to \infty} e_k = 0$$

If S has independent eigen vectors $v_1\ v_2\ \ldots\ v_n$ and corresponding eigen values are $\lambda_1\ \lambda_2\ \ldots\ \lambda_n$

Then

$$e_0 = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$$

$$= \sum_{i=1}^{n} c_i v_i$$

$$e_k = S^k e_0$$

$$\boxed{e_k = \sum_{i=1}^{n} c_i \lambda_i^k v_i}$$

For $e_k = 0$ as $k \to \infty$, possible only if the magnitude of the largest eigen value is less than 1.0

$$\rho(S) < 1 \quad \text{— necessary condition}$$

But since eigen values are relatively expensive to estimate

$$\boxed{\rho(S) < \|S\|}$$

$$\|S\| < 1 \quad \text{— sufficient condition}$$

$$\|S\|_\infty \left[\underline{|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad i = 1, 2 \ldots n} \right]$$