Recap :- Interpolation using polynomials

Forms - Standard form

Lagrange polynomials

Newton's divided difference

Errors - Given (x_i, y_i) $i = 0, 1, 2, \dots n$

and that the underlying function $f(x)$ is continuous and infinitely differentiable, the truncation error in interpolation by an nth order polynomial is given by

$$\left| f(x) - P_n(x) \right| = \left| \frac{\overbrace{f^{n+1}}^{(n+1)^{st}\text{ derivative}}(\xi)}{(n+1)!} \right| \left| (x-x_0)(x-x_1) \cdots (x-x_n) \right|$$

$$R_n = \left| \frac{f^{n+1}(\xi)}{(n+1)!} \right| \left| W_n(x) \right|$$

The error can be approximated by

$$R_n \simeq \underbrace{f[x_{n+1}, x_n, x_{n-1}, \dots x_0]}_{(n+1)^{th}\text{ order divided difference}} \left| W_n(x) \right|$$

- Can be easily estimated if Newton's DD is used for interpolation

- The error remains same for nth order polynomial fitted by using any form.

Characteristics of errors

- zero at the known points used in interpolation
- errors are large near the edges of the interpolation domain
  (Runge's phenomenon)
- extremely large errors outside the interpolation domain (extrapolation)

Approaches to reduce errors

1. Selection of interpolation points

   Chebyshev nodes (points)

   minimize maximum interpolation error

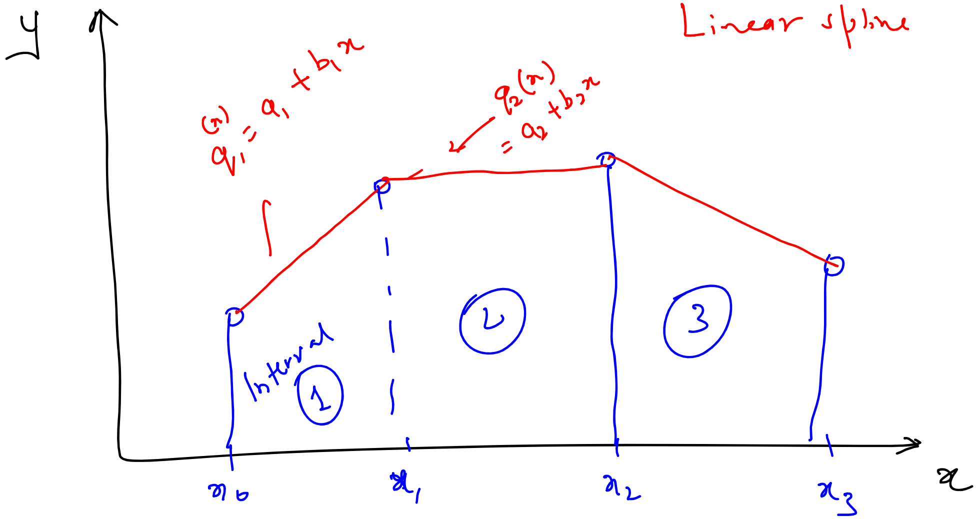
2. Piecewise fitting of polynomials

   (Splines)

   The concept originated from the drafting technique

   Given (x_i, y_i) $i = 0, 1, 2, \ldots n$ are ordered pairs,
   The spline is obtained by fitting a polynomial of an appropriate order between two points

Linear spline



$x_0 \le x_1 \le x_2 \le \cdots \le x_n$

Linear spline :- The simplest spline

$$\Rightarrow S(x) = \begin{cases} q_1(x) = a_1 + b_1 x & x_0 \le x \le x_1 \\ q_2(x) = a_2 + b_2 x & x_1 \le x \le x_2 \\ \vdots & \\ q_i(x) = a_i + b_i x & x_{i-1} \le x \le x_i \\ \vdots & \\ q_n(x) = a_n + b_n x & x_{n-1} \le x \le x_n \end{cases}$$

How many unknowns : $2n$

Conditions

$$2n \left\{ \begin{array}{ll} q_i(x_{i-1}) = y_{i-1} & i = 1, 2, \ldots n \\ q_i(x_i) = y_i & i = 1, 2, \ldots n \end{array} \right.$$

$$a_i + b_i x_{i-1} = y_{i-1}$$

$$a_i + b_i x_i = y_i$$

But usually this direct approach is not used

$$q_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i + \frac{x - x_i}{x_{i-1} - x_i} y_{i-1}$$

$i = 1, 2, \ldots n$

Example:

i	0	1	2	3
x_i	3.0	4.5	7.0	9.0
y_i	2.5	1.0	2.5	0.5

2. Quadratic Spline

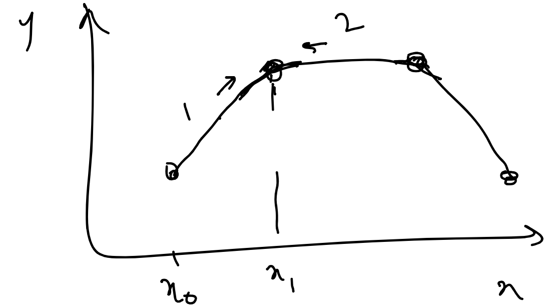
$$S(x) = \begin{cases} q_1(x) = \underline{a_1 + b_1 x + c_1 x^2} & x_0 \le x \le x_1 \\ \vdots \\ q_n(x) = a_n + b_n x + c_n x^2 & x_{n-1} \le x \le x_n \end{cases}$$

How many unknowns? $3n$

Conditions

(1) – $q_i(x_{i-1}) = y_{i-1}$ — $i = 1, 2, \dots n$

(2) – $q_i(x_i) = y_i$ — $i = 1, 2, \dots n$



(3) → C^1 continuity

$$q_i'(x_i) = q_{i+1}'(x_i) \qquad \underline{\underline{i = 1, 2, \dots n-1}}$$

$(n-1)$

$$q_1'(x_1) = q_2'(x_1)$$

$$\begin{cases} b_1 + 2c_1 x_1 = b_2 + 2c_2 x_1 \\ b_i + 2C_i x_i = b_{i+1} + 2c_{i+1} x_i \end{cases}$$

(4) $q_1''(x_0) = 0$

$c_1 = 0$

3. Cubic Spline

$$S(x) = \begin{cases} q_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3 & x_0 \le x \le x_1 \\ \vdots & \\ q_n(x) = a_n + b_n x + c_n x^2 + d_n x^3 & x_{n-1} \le x \le x_n \end{cases}$$

How many unknowns $4n$

(1) $q_i(x_{i-1}) = y_{i-1}$

(2) $q_i(x_i) = y_i$

$\}\ i = 1, 2, \ldots n$

(3) C^1 continuity

$q_i'(x_i) = q_{i+1}'(x_i) \quad i = 1, 2, \ldots n-1$

(4) C^2 continuity

$q_i''(x_i) = q_{i+1}''(x_i) \quad i = 1, 2, \ldots n-1$

$(4n-2)$ conditions

2 additional conditions (Types of splines)

(a) Natural cubic spline

$q_1''(x_0) = 0$ $2c_1 + 6d_1 x_0 = 0$

$q_n''(x_n) = 0$ $2c_n + 6d_n x_n = 0$

b) Not-a-knot cubic spline

$q_1'''(x_1) = q_2'''(x_1) \rightarrow d_1 = d_2$

$q_{n-1}'''(x_{n-1}) = q_n'''(x_{n-1}) \rightarrow d_{n-1} = d_n$

c) Clamped splines

$q'(x_0) = \beta_0$

$q^n(x_n) = \beta_n$

d) Cyclic or periodic spline

$$\left.\begin{aligned} q_1'(x_0) &= q_n'(x_n) \\ q_1''(x_0) &= q_n''(x_0) \end{aligned}\right\}$$

Requires - $y_0 = y_n$

Better approach for fitting cubic spline

The second derivative of $q_i(x)$ within each interval is a straight line

Unknowns the second derivative

$$q_i''(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} q''(x_i) + \frac{x - x_i}{x_{i-1} - x_i} q''(x_{i-1})$$

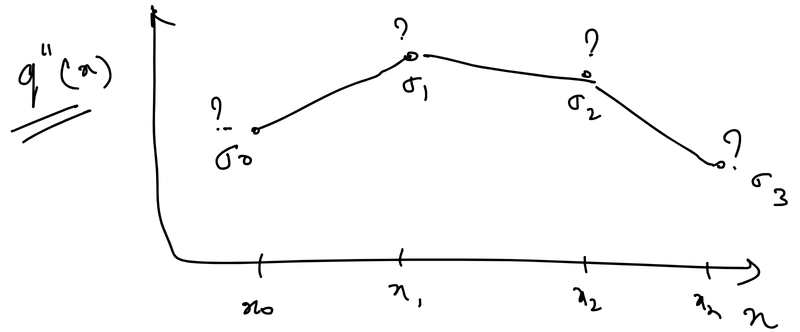
$x_{i-1} \le x \le x_i$

Define

$$q_i''(x_i) = q_{i+1}''(x_i) = \sigma_i$$

$$h_i = x_i - x_{i-1} \qquad q_1''(x_0) = \sigma_0$$

$$g_i = \frac{y_i - y_{i-1}}{h_i} \qquad q_n''(x_n) = \sigma_n$$



$$q_i''(x) = \frac{x - x_{i-1}}{h_i}\sigma_i - \frac{x - x_i}{h_i}\sigma_{i-1} \quad -(1)$$

$i = 1, 2, \ldots n$

Integrate twice

$$q_i'(x) = \frac{\sigma_i}{2h_i}(x - x_{i-1})^2 - \frac{\sigma_{i-1}}{h_i}(x - x_i)^2 + C_i - D_i \quad -(2)$$

$$q_i(x) = \frac{\sigma_i}{6h_i}(x - x_{i-1})^3 - \frac{\sigma_{i-1}}{6h_i}(x - x_i)^3 + C_i(x - x_{i-1}) - D_i(x - x_i) \quad -(3)$$

A_i B_i

$$q_i(x) = A_i(x - x_{i-1})^3 - B_i(x - x_i)^3 + C_i(x - x_{i-1}) - D_i(x - x_i)$$

$x_{i-1} \le x \le x_i$

Applying condition (1)

$$q_i(x_{i-1}) = y_{i-1}$$

$$y_{i-1} = -\frac{\sigma_{i-1}}{h_i}(x_{i-1}-x_i)^3 - D_i \overbrace{(x_{i-1}-x_i)}^{-h_i}$$

$$\Rightarrow \boxed{D_i = \frac{y_{i-1}}{h_i} - \frac{\sigma_{i-1}}{6} h_i}$$

Apply condition (2) $\quad q_i(x_i) = y_i$

$$\boxed{C_i = \frac{y_i}{h_i} - \frac{\sigma_i}{6} h_i}$$

Applying condition (3) $\quad q_i'(x_i) = q_{i+1}'(x_i)$

Using Eq. 2 and substituting C_i & D_i

$$\frac{\sigma_i}{2h_i} h_i^2 + \frac{y_i}{h_i} - \frac{\sigma_i}{6} h_i - \frac{y_{i-1}}{h_i} + \frac{\sigma_{i-1}}{6} h_i$$

$$= -\frac{\sigma_i}{2h_{i+1}} h_{i+1}^2 + \frac{y_{i+1}}{h_{i+1}} - \frac{\sigma_{i+1}}{6} h_{i+1} - \frac{y_i}{h_{i+1}} + \frac{\sigma_i h_{i+1}}{6}$$

$$\Rightarrow \sigma_{i-1}\frac{h_i}{6} + \sigma_i\left[\frac{h_i}{3} + \frac{h_{i+1}}{3}\right] + \sigma_{i+1}\frac{h_{i+1}}{6}$$

$$= \frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}$$

$\Rightarrow$

$$\sigma_{i-1} h_i + 2\sigma_i (h_i + h_{i+1}) + \sigma_{i+1} h_{i+1}$$
$$= 6 (g_{i+1} - g_i) \longrightarrow i = 1, 2, \dots n$$

$$\underbrace{\begin{bmatrix} \boxed{1 \quad \text{BCs}} & & & & & \\ h_1 & 2(h_1+h_2) & h_2 & - & - & - \\ & h_2 & 2(h_2+h_3) & h_3 & - & - \\ & & \ddots & & & \\ & & & \ddots & & \\ & & h_{n-1} & 2(h_{n-1}+h_n) & h_n & \\ & \boxed{\text{BCs}} & & & & \end{bmatrix}}_{n-1 \text{ equation}} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} = 6 \begin{bmatrix} g_2 - g_1 \\ g_3 - g_2 \\ \vdots \\ g_n - g_{n-1} \end{bmatrix}$$

n-1 equation

Natural spline