An important result on differential equations and functionals

Consider a differential equation written as a linear transformation

$$Au = f,$$

where $A : \mathcal{D} \to H$ and $f \in H$. Here H denotes a Hilbert space.

Examples are

$$A = -\frac{d}{dx}\left(EA\frac{d}{dx}\right) \text{ or } A = \frac{d^2}{dx^2}\left(EI\frac{d^2}{dx^2}\right),$$

in $\Omega = [0, L]$. In these cases, $f \in H^0 = L_2(0, L)$ while $u \in C^2(0, L)$ in the first case and $\in C^4(0, L)$ in the second.

An operator is *self adjoint* or *symmetric* if for all u, v

$$(Au, v)_H = (u, Av)_H.$$

The operator A is *strictly positive* if for all $u \neq 0$

$$(Au, u)_H > 0.$$

Every bilinear form generates a *quadratic form* which is a functional quadratic in its arguments, as

$$B(u, u) = Q(u).$$

Theorem: If A is a strictly positive operator in $\mathcal{D}$, then for $f \in H$,

$$Au = f$$

has at most one solution in $\mathcal{D}$.

Theorem: Let A be a positive operator in $\mathcal{D}$ and $f \in H$. Let $Au = f$ have a solution $u_0 \in \mathcal{D}$. Then the quadratic functional

$$I(u) = \frac{1}{2}(Au, u)_H - (f, u)_H$$

assumes its minimal value in $\mathcal{D}$ for the element u_0. i.e.

$$I(u) \geq I(u_0),$$

except for $u = u_0$ when $I(u) = I(u_0)$.

The above theorem provides an important route to construct weak forms of problems governed by strong forms.

For proofs to the above see, Reddy (2002), *Energy principles and variational methods in applied mechanics*, John Wiley and sons.

Let us consider a differential equation in one variable that governs the transverse deformation $u(x)$ of a cable fixed at both ends and subjected to a transverse load $f(x)$. The tension in the cable is $a(x)$.

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for } 0 < x < L$$

with

$$u(0) = 0,\ u(L) = 0$$

Let us choose $f \in L_2(0, L)$ and $\mathcal{D}$ as the subset of H that contains functions that satisfy the end conditions and are differentiable upto the second order.

The operator

$$A = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)$$

is symmetric as

$$\begin{aligned}
(Au, v)_H &= \int_0^L \left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)\right] v dx \\
&= - a\frac{du}{dx}v\Big|_0^L + \int_0^L \frac{dv}{dx}\left(a\frac{du}{dx}\right) dx \\
&= \int_0^L \left[-\frac{d}{dx}\left(a\frac{dv}{dx}\right)\right] u dx = (u, Av)_H.
\end{aligned}$$

In the above use the fact that as $u, v \in \mathcal{D}$, $u(0) = u(L) = v(0) = v(L) = 0$. Thus the variational principle governing this problem is

$$\Pi(u) = \frac{1}{2}(Au, u)_H - (f, u)_H,$$

i.e.

$$\Pi(u) = \frac{1}{2}\int_)^L a(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^L f u dx.$$

Consider another equation, now in two variables:

$$\nabla^2\phi + c\phi + Q = 0$$

c and Q are functions of position only. The operator is:

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + c, f = -Q$$

The operator is self adjoint as

$$(\mathcal{L}\phi, \psi)_H = \int_V \psi \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right\} dV \text{ can be evaluated using}$$

$$\int_V \psi \frac{\partial^2 \phi}{\partial x^2} dV = \int_{\partial V} \psi \frac{\partial \phi}{\partial x} n_x dS - \int_V \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial x} dV$$

$$\text{Similarly} \quad \int_V \phi \frac{\partial^2 \psi}{\partial x^2} dV = \int_{\partial V} \phi \frac{\partial \psi}{\partial x} n_x dS - \int_V \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} dV$$

Thus,

$$(\mathcal{L}\phi, \psi)_H = (\mathcal{L}\psi, \phi)_H + \int_{\partial V} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS.$$

The boundary term is zero since at the boundary ∂V, either $\phi = \psi$ is specified or $\phi_{,n} = \psi_{,n}$ is specified. Thus $\mathcal{L}$ is self adjoint.

Thus the variational principle corresponsding to this equation becomes:

$$\Pi = \frac{1}{2}(\mathcal{L}\phi, \phi)_H - (Q, \phi)_H.$$

yielding

$$\int_V \left\{ \frac{1}{2}\phi \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + c\phi \right] + Q\phi \right\} dV$$

or, applying Gauss law,

$$\Pi = \int_V \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{2}c\phi^2 - Q\phi \right] dV + \text{boundary terms}$$

Equations governing bending of a Timoshenko beam with w being the trnsverse deflection and ϕ_x the rotation:

$$
\begin{aligned}
-\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]+c_f w &= q \\
-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right)+S\left(\frac{dw}{dx}+\phi_x\right) &= 0
\end{aligned}
$$

S shear stiffness, D bending stiffness, w transverse deflection, c_f foundation modulus.

Here,

$$
A=\begin{pmatrix} -S\frac{d^2}{dx^2}+c_f & -S\frac{d}{dx} \\ -S\frac{d}{dx} & -D\frac{d^2}{dx^2}+S \end{pmatrix}
$$

$$
f=\left\{\begin{array}{c} q \\ 0 \end{array}\right\} \text{ and u} = \left\{\begin{array}{c} w \\ \phi_x \end{array}\right\}
$$

so that the equations can be represented as

$$
Au=f.
$$

It is easily shown that under the following boundary conditions:

$$w(0) = \phi_x(0) = 0, \left[S\left(\frac{dw}{dx} + \phi_x\right)\right]_{x=L} = F_0, \left[D\frac{d\phi_x}{dx}\right]_{x=L} = M_0$$

A is self adjoint. Now,

$$\begin{aligned}
(Au, u)_H &= \int_0^L \left[-Sw\frac{d}{dx}\left(\frac{dw}{dx} + \phi_x\right) + S\phi\left(\frac{dw}{dx} + \phi_x\right) + c_f w^2 - D\phi_x \frac{d^2\phi_x}{dx^2}\right] dx \\
&= \int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + c_f w^2\right] dx \\
&\quad - \left[Sw\left(\frac{dw}{dx} + \phi_x\right) + D\phi_x\frac{d\phi_x}{dx}\right]\Bigg|_0^L
\end{aligned}$$

Thus, the variational principle is

$$\begin{aligned}
\Pi(w, \phi_x) &= \frac{1}{2}\int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + c_f w^2\right] dx - \int_0^L wq dx \\
&\quad - (wF_0 + \phi_x M_0)|_L
\end{aligned}$$

Assignment 2

Variational Methods

Pr.1: Using the Euler Lagrange equation, find the extremal of the following functional

$$\Pi[y] = \int_a^b \left[12xy + (y')^2\right] dx.$$

Pr. 2: Consider a functional consisting of several independent functions of *one* variable, i.e.

$$J[y_1, y_2, \dots y_n] = \int_a^b F(x, y_1, y_2 \dots y_n, y_1', \dots y_n') dx.$$

Using the definition of δJ, show that the stationary condition $\delta J = 0$ implies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\frac{\partial F}{\partial y_k'} = 0,$$

where, $k = 1, \dots n$. Now, derive the governing deq for

$$F = y_1 y_2^2 + y_1^2 y_2 + y_1' y_2'.$$

Pr.3: Show that minimisation of

$$
\begin{aligned}
J[\phi(x,y)] &= \int_A \left\{ \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + f\phi \right\} dxdy \\
J[\phi(x,t)] &= \int_A \left\{ \frac{1}{2}\phi_x^2 - \frac{1}{2c^2}\phi_t^2 \right\} dxdy,
\end{aligned}
$$

yield the 2-dimensional Poisson's equation and one dimensional wave equation respectively.

Pr. 4: A classical problem in the calculus of variations is the so-called *Brachistochrone problem.* Suppose that a particle of mass m is moving along some plane curve $y(x)$ under gravity. It starts from rest at point a and has to travel along the curve to a fixed point b. What is the shape of the curve so that the time taken to travel from a to b is a minimum? First show that the time of descent is

$$
T = \int_0^{x_b} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}},
$$

where, $a : (0,0)$ and $b : (x_b, y_b)$. The Euler Lagrange equation will yield the deq for $y(x)$. The solution is NOT a straight line!

Pr 5: Consider the solution to the pde

$$-\frac{\partial}{\partial x}\left(a_1\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_a\frac{\partial\phi}{\partial y}\right)+a_0\phi=f$$

in the 2-d domain V. Here a_i are constants. Show that a variational form of this problem exists and derive the variational statement. Further, derive the forms of the essential and natural boundary conditions necessary to solve this problem.

Pr 6: Consider a special case of the above variational problem where $a_1 = a_2 = k$ and $a_0 = 0$. Also, V is the rectangular domain $0 < x < 1, 0 < y < 1$. On the sides $x = 1$ and $y = 1$, $\phi = 0$. On the sides $x = 0$ and $y = 0$, $\partial\phi/\partial n = 0$. The function f is specified.

Consider a two parameter trial solution of the form

$$\hat{\phi}(x,y) = a_1\cos\frac{\pi}{2}x\cos\frac{3\pi}{2}y + a_2\cos\frac{3\pi}{2}x\cos\frac{3\pi}{2}y,$$

find the best choices for a_1 and a_2.

Pr. 7: Navier Stokes equation for two-dimensional flow of viscous, incompressible fluids is stated as:

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0
\end{aligned}
$$

in the domain V, with $u = u_0$ and $v = v_0$ on ∂V_u and

$$
\begin{aligned}
\nu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y\right) - \frac{1}{\rho}Pn_x &= \bar{t}_x \\
\nu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y\right) - \frac{1}{\rho}Pn_y &= \bar{t}_y
\end{aligned}
$$

on ∂V_t. The symbols have usual meanings, i.e u, v are components of velocity, P is the hydrostatic pressure, n_x, n_y are components of the normal to ∂V and ν is the viscosity. Find the weak form for this problem.

Pr. 8: Consider the deq

$$-\frac{d^2u}{dx^2} - u + x^2 = 0,$$

with $u(0) = 0, u'(1) = 1$. Consider approximating functions of the form $\phi_0 = a + bx$, $\phi_1 = a + bx + cx^2$ and $\phi_2 = a + bx + dx^3$. Using the Galerkin method solve the above problem.

Rayleigh Ritz method for solving problems cast in the weak form

In the Rayleigh-Ritz method, the approximation to the field variable is made as:

$$u^h(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0.$$

Here we assume that our problem requires us to solve for one field variable at a point. The approximation functions should satisfy

1. ϕ_0 must satisfy the essential boundary conditions so that the approximation u^h is admissible. Further, if homogeneous boundary conditions are present, ϕ_i must satisfy them.

2. The ϕ_i should be contonuous to the extent required by the variational problem

3. The set $\{\phi_i\}$'s must be linearly independent and *complete*. Completeness means that all terms upto the highest order term must be included. eg $\phi_1 = a_1 x$, $\phi_2 = a_2 x^2 + a_3 x^3$.

We know that the variational statement is of the form

$$I(u) = \frac{1}{2}(Au, u)_H - (f, u)_H.$$

We are interested particularly in cases where (Au, u) is a bilinear form $B(u, u)$ and $(f, u)_H$ is a linear form $l(u)$. Then the approximation to Π becomes:

$$\Pi = \frac{1}{2}B\left(\left\{\sum_{j=1}^{N} c_j\phi_j + \phi_0\right\}, \left\{\sum_{j=1}^{N} c_j\phi_j + \phi_0\right\}\right) - l(\sum_{j=1}^{N} c_j\phi_j + \phi_0)$$

and thus

$$\Pi = \Pi(c_1, c_2, \ldots c_N).$$

We need $\delta\Pi = 0$ so that

$$\frac{\partial \Pi}{\partial \boldsymbol{c}} \cdot \delta\boldsymbol{c} = 0,$$

which for arbitrary $\delta\boldsymbol{c}$ implies that

$$\frac{\partial \Pi}{\partial c_i} = 0 \quad i \in [1, N]$$

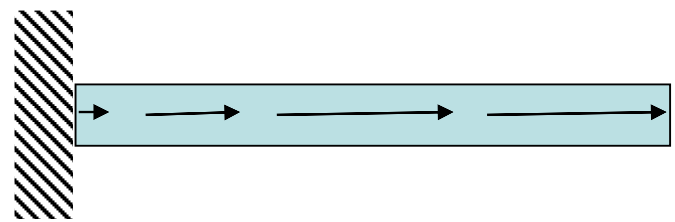
As an example, consider the problem of the uniaxial bar considered earlier with $f(x) = cx$ and without the end load.

$$\Pi = \int_0^L \frac{1}{2} E \left(\frac{du}{dx}\right)^2 A dx - \int_0^L ucxdx$$

Also, $u(0) = 0$.

Differential equation corresponding to the problem is obtained from the Euler Lagrange equation and is

$$AEu_{,xx} + cx = 0$$



Exact solution is easy to obtain

$$u = \frac{c}{6AE}(3L^2 x - x^3)$$

We will now solve it with the Rayleigh Ritz technique.

Inserting the approximation

$$u^h = \phi_0 + \sum_{i=1}^{N} c_i \phi_i,$$

we have

$$\frac{du^h}{dx} = \frac{d\phi_0}{dx} + \sum_{i=1}^{N} c_i \frac{d\phi_i}{dx}.$$

Further

$$\Pi(c_1, c_2 \ldots c_N) = \int_0^L \left[\frac{1}{2} EA \left(\frac{d\phi_0}{dx} + \sum_{i=1}^{N} c_i \frac{d\phi_i}{dx} \right)^2 - c \left(\phi_0 + \sum_{i=1}^{N} c_i \phi_i \right) x \right] dx.$$

so that

$$\frac{\partial \Pi}{\partial c_j} = \int_0^L \left[EA \frac{d\phi_j}{dx} \left(\frac{d\phi_0}{dx} + \sum_{i=1}^{N} c_i \frac{d\phi_i}{dx} \right) - c\phi_j x \right] dx = 0,$$

for $j \in [1, N]$

Each of the N simultaneous equations can be rewritten as

$$c_1 \int_0^L \frac{d\phi_j}{dx}\frac{d\phi_1}{dx}dx + c_2 \int_0^L \frac{d\phi_j}{dx}\frac{d\phi_2}{dx}dx + \ldots = -\int_0^L EA\frac{d\phi_j}{dx}\frac{d\phi_0}{dx}dx + \int_0^L c\phi_j x dx,$$

for $j \in [1, N]$, leading to a matrix equation of the form

$$a_{ij}c_j = b_j,$$

with

$$\begin{aligned} a_{ij} &= \int_0^L \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx \\ b_j &= -\int_0^L EA\frac{d\phi_j}{dx}\frac{d\phi_0}{dx}dx + \int_0^L c\phi_j x dx, \end{aligned}$$

First try: use just $u^h = a_1 x$

$$\Pi = \frac{AEL}{2}a_1^2 - \frac{cL^3}{3}a_1$$

$d\Pi/da_1$ yields $a_1 = cL^2/3AE$. Thus

$$u^h = \frac{cL^2}{3AE}x, \sigma = \frac{cL^2}{3A}$$

Second try: 2 term solution, i.e use $\partial \Pi / \partial a_1 = 0$ and $\partial \Pi / \partial a_2 = 0$

$$AEL \begin{pmatrix} 1 & L \\ L & 4L^2/3 \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{cL^2}{12} \begin{Bmatrix} 4 \\ 3L \end{Bmatrix}$$

This gives

$$u^h = \frac{cL}{12AE}(7Lx - 3x^2)$$

Using $u^h = a_1 x + a_2 x^2 + a_3 x^3$ gives

$$a_1 = \frac{cL^2}{2AE}, a_2 = 0, a_3 = -\frac{c}{6AE}$$

i.e the correct solution

Another method for creating weak forms from strong ones: Weighted residual technique

Again (in a 1-d example) assume for a differential equation

$$A\phi - b = 0$$

a solution of the form

$$\phi^h = \phi_0 + \sum_{j=1}^{N} c_j \phi_j$$

such that the *residual* is

$$R = A\phi^h - b \neq 0.$$

Then the *weighted residual* is defined as

$$\int_V \psi_i R(x, c_j) dx = 0$$

where ψ_j are *weight functions*. The fact that the corresponding residual and the weight function are orthogonal generates a weak form.

$$\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} = 0,$$

where $\boldsymbol{u}$ is a vector field variable, and $\boldsymbol{A}$ represents a set of pde's, assume

$$\boldsymbol{u}^h = \sum_{j=1}^{N} c_j \boldsymbol{u}_j + \boldsymbol{u}_0$$

we have

$$\int_V \boldsymbol{\psi}_i \cdot \boldsymbol{R}(\boldsymbol{x}, c_j) dV = 0$$

As the operator $\boldsymbol{A}$ is linear,

$$\boldsymbol{A}\boldsymbol{u}^h = \sum_{j=1}^{N} c_j \boldsymbol{A}\boldsymbol{u}_j + \boldsymbol{A}\boldsymbol{u}_0$$

and thus,

$$\sum_{j=1}^{N}\left[\int_V \boldsymbol{\psi}_i \cdot \boldsymbol{A}\boldsymbol{u}_j dV\right]\boldsymbol{c} = \int_V \boldsymbol{\psi}_i \cdot [\boldsymbol{b} - \boldsymbol{A}\boldsymbol{u}_0] dV$$

$\Rightarrow \boldsymbol{K}\boldsymbol{c} = \boldsymbol{F}$, where,

$$K_{ij} = \int_V \boldsymbol{\psi}_i \cdot \boldsymbol{A}\boldsymbol{u}_j dV, F_i = \int_V \boldsymbol{\psi}_i \cdot [\boldsymbol{b} - \boldsymbol{A}\boldsymbol{u}_0]\, dV$$

Various schemes can be used to choose ψ_i's.

$$\boldsymbol{\psi}_i \neq \boldsymbol{u}_i \quad \textit{Petrov-Galerkin scheme}$$

$$\boldsymbol{\psi}_i = \boldsymbol{u}_i \quad \textit{Galerkin scheme}$$

$$\boldsymbol{\psi}_i = \delta(\boldsymbol{x} - \boldsymbol{x}_i) \quad \textit{Collocation scheme}$$

$$\boldsymbol{\psi}_i = \boldsymbol{\mathcal{L}}\boldsymbol{u}_i \quad \textit{Least square scheme}$$

Weighted Residual techniques: an example

$$-\frac{d^2u}{dx^2} - u + x^2 = 0$$

with $u(0) = 0, u'(1) = 1$.

Assume

$$u^h = \sum c_j \phi_j + \phi_0$$

so that

$$\begin{aligned} \phi_j(0) &= 0, \phi_0(0) = 0 \\ \phi_0'(1) &= 1, \phi_j'(1) = 0 \end{aligned}$$

Assume $\phi_0 = a + bx \Rightarrow \phi_0(x) = x$

$\phi_1 = a + bx + cx \Rightarrow \phi_1(x) = x(2 - x)$

$\phi_2 = a + cx^2 + dx^3, \Rightarrow \phi_2(x) = x^2\,(1 - 2/3x)$

Thus

$$R(x) = c_1(2-2x+x^2)+c_2(-2+4x-x^2+\frac{2}{3}x^3)-x+x^3$$

1. **Petrov-Galerkin:** Let $\psi_1 = x, \psi_2 = x^2$ $\Rightarrow \int_0^1 xRdx = 0$, $\int_0^1 x^2Rdx = 0 \Rightarrow u^h = 1.302x - 0.173x^2 - 0.0147x^3$

2. **Galerkin:** $\psi_i = \phi_i \Rightarrow u^h = 1.289x - 0.1398x^2 - 0.00325x^3$

3. **Least Squares:** $\psi_i = \partial R/\partial c_i \Rightarrow u^h = 1.26x - 0.08x^2 - 0.003325x^3$

4. **Collocation:** $R(1/3) = 0, R(2/3) = 0 \Rightarrow u^h = 1.36x - 0.13x^2 - 0.00342x^3$