Department of Physics IIT-Kanpur

PHY103A/N **Mid Sem Examination** **Date: Feb. 19, 2015**

Time: 2 hours Max Marks: 80

Name : ____ Roll No. :___________________

Tutorial Section :T−

I **pledge my honour** as a gentleman/lady that during the examination I shall not resort to any unfair means, and will neither give nor receive assistance.

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(Signature)

- Your answer for each question must be in the **space provided**. **Your rough work elsewhere will not be graded**.
- There are total **four** questions and **19 numbered pages** in this booklet (including the cover page and the pages for the rough work).
- No other paper must be in your possession.
- No calculators or cell phones are allowed.
- You are free to use the formulae provided in page number 2.

	MARKS
$Q1$	
Q2	
Q3	
Q4	
TOTAL	

Useful Formulae

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}), \qquad \boldsymbol{\nabla}\times(f\mathbf{A})=f(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{A}\times(\boldsymbol{\nabla}f),\\
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}-(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\boldsymbol{\nabla}^2\mathbf{A}.
\end{aligned}$$

$$\begin{aligned}
\hat{\imath} &= \hat{r}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi, &\quad \hat{r}&=\hat{\imath}\sin\theta\cos\phi+\hat{\jmath}\sin\theta\sin\phi+\hat{k}\cos\theta\\
\hat{\jmath} &= \hat{r}\sin\theta\sin\phi+\hat{\theta}\cos\theta\sin\phi+\hat{\phi}\cos\phi, &\quad \hat{\theta}&=\hat{\imath}\cos\theta\cos\phi+\hat{\jmath}\cos\theta\sin\phi-\hat{k}\sin\theta\\
\hat{k} &= \hat{r}\cos\theta-\hat{\theta}\sin\theta, &\quad \hat{\phi}&=-\hat{\imath}\sin\phi+\hat{\jmath}\cos\phi.
\end{aligned}$$

The vector derivatives in cylindrical coordinates:

$$\begin{aligned}
\boldsymbol{\nabla}T &= \frac{\partial T}{\partial s}\hat{s}+\frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\phi}+\frac{\partial T}{\partial z}\hat{z}, \qquad \boldsymbol{\nabla}\cdot\mathbf{V}=\frac{1}{s}\frac{\partial(sV_s)}{\partial s}+\frac{1}{s}\frac{\partial V_\phi}{\partial\phi}+\frac{\partial V_z}{\partial z}\\
\boldsymbol{\nabla}\times\mathbf{V} &= \left[\frac{1}{s}\frac{\partial V_z}{\partial\phi}-\frac{\partial V_\phi}{\partial z}\right]\hat{s}+\left[\frac{\partial V_s}{\partial z}-\frac{\partial V_z}{\partial s}\right]\hat{\phi}+\frac{1}{s}\left[\frac{\partial(sV_\phi)}{\partial s}-\frac{\partial V_s}{\partial\phi}\right]\hat{z}\\
\nabla^2T &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2T}{\partial\phi^2}+\frac{\partial^2T}{\partial z^2}.
\end{aligned}$$

The vector derivatives in spherical coordinates:

$$\begin{aligned}
\boldsymbol{\nabla}T &= \frac{\partial T}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\phi},\\
\boldsymbol{\nabla}\cdot\mathbf{V} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2V_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta V_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V_\phi\\
\boldsymbol{\nabla}\times\mathbf{V} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta V_\phi)-\frac{\partial V_\theta}{\partial\phi}\right]\hat{r}+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi}-\frac{\partial}{\partial r}(rV_\phi)\right]\hat{\theta}+\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta)-\frac{\partial V_r}{\partial\theta}\right]\hat{\phi}\\
\nabla^2T &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2T}{\partial\phi^2}.
\end{aligned}$$

1. **(a)** A long solid cylinder of radius R_1 carrying uniform volume charge density ρ is placed coaxially inside a cylindrical shell of radius R_2 carrying an equal but opposite total charge uniformly distributed on its surface.

i) Calculate the electric field everywhere and sketch it neatly as a function of s. [**8**]

ii) Calculate the potential everywhere taking $V = 0$ on the outer cylinder. [**4**]

iii) Calculate the electrostatic energy per unit length. [**4**]

Answer: Choose the z axis to be the axis of the cylindrical object. It has cylindrical symmetry so that we can apply Gauss's law in the integral form. The symmetry dictates that the field will be pointing radially outward and its magnitude depends only on the distance from the z axis. Then $\mathbf{E} = E(s)\hat{s}$. Applying Gauss's law to a Gaussian cylinder of length L, coaxial with the z axis:

$$\oint \mathbf{E} \cdot d\mathbf{a} = 2\pi s L E(s) = \frac{Q_{en}}{\epsilon_0}.$$

For $vs < R_1$, the Gaussian cylinder is filled wit charge density ρ, $Q_{en} = \pi s^2 L \rho$.

$$E(s) = \frac{\rho s}{2\epsilon_0}.$$

For $R_1 < s < R_2$, $Q_{en} = \pi R_1^2 L \rho$,

$$E(s) = \frac{\rho a^2}{2\epsilon_0 s}.$$

For $s > R_2$, the enclosed charge is zero i.e. $Q_{en} = 0$. Hence $E(s) = 0$.

The potential can be calculated as

$$V(r) = V(r_0) - \int_{r_0}^{r} \mathbf{E} \cdot d\mathbf{r}.$$

This integral is path independent, so we can choose the path to be in the radial direction. For $s > R_2$, the potential is zero i.e. $V(s) = 0$ since the electric field is zero for $s > R_2$ implies the potential is constant and the $V(s = R_2) = 0$.

For $R_1 < s < R_2$,

$$V(s) = -\int_{R_2}^{s} \mathbf{E} \cdot d\mathbf{r} = \frac{\rho R_1^2}{2\epsilon_0} \int_{s}^{R_2} \frac{1}{s} ds = \frac{\rho R_1^2}{2\epsilon_0} \ln\left(\frac{R_2}{s}\right).$$

This is also valid at $s = R_1$, so

$$V(a) = \frac{\rho R_1^2}{2\epsilon_0} \ln\left(\frac{R_2}{R_1}\right).$$

For $s \le R_1$,

$$V(s) = -\int_{R_2}^{R_1} \mathbf{E}\cdot d\mathbf{r} - \int_{R_1}^{s} \mathbf{E}\cdot d\mathbf{r} = \frac{\rho R_1^2}{2\epsilon_0} \ln\left(\frac{R_2}{R_1}\right) + \frac{\rho}{2\epsilon_0}\int_s^{R_1} s\, ds$$
$$= \frac{\rho R_1^2}{2\epsilon_0} \ln\left(\frac{R_2}{R_1}\right) + \frac{\rho}{4\epsilon_0}(R_1^2 - s^2) = \frac{\rho R_1^2}{4\epsilon_0}\Big[2\ln\left(\frac{R_2}{R_1}\right) + 1 - \frac{s^2}{R_1^2}\Big]. \qquad (1)$$

The electrostatic energy is

$$W = \frac{\epsilon_0}{2}\int_V E^2 d\tau.$$

$$W = \frac{\epsilon_0}{2}\Big[\int_0^{R_1}\left(\frac{\rho s}{2\epsilon_0}\right)^2 2\pi s L ds + \int_{R_1}^{R_2}\left(\frac{\rho R_1^2}{2\epsilon_0 s}\right)^2 2\pi s L ds\Big]$$
$$\frac{W}{L} = \frac{\pi\rho^2}{4\epsilon_0}\Big(\int_0^{R_1} s^3 ds + \int_{R_1}^{R_2}\frac{R_1^4}{s}ds\Big] = \frac{\pi\rho^2 R_1^4}{16\epsilon_0}\Big[1 + 4\ln\left(\frac{R_2}{R_1}\right)\Big].$$

Alternatively, the energy can be calculated using

$$W = \frac{1}{2}\int \rho(\mathbf{r})V(\mathbf{r})d\tau.$$

Note that $V = 0$ on the outer cylinder, so we get contribution only by integrating over the inner cylinder.

$$\frac{W}{L} = \frac{\rho^2 R_1^2}{8\epsilon_0}\int_0^{R_1}\Big[2\ln\left(\frac{R_2}{R_1}\right) + 1 - \frac{s^2}{R_1^2}\Big]2\pi s ds = \frac{\pi\rho^2 R_1^4}{16\epsilon_0}\Big[1 + 4\ln\left(\frac{R_2}{R_1}\right)\Big].$$

1. (b) At a planar interface between two dielectric media with dielectric permittivities ϵ_1 and ϵ_2, the electric field in medium-1 makes an angle θ_1 with the normal to the interface and the electric field in medium-2 makes an angle θ_2. Find the relation between θ_1 and θ_2. [**4**] Boundary conditions on $\mathbf{D}$:

$$\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp$$
$$\epsilon_1 E_1 \cos\theta_1 = \epsilon_1 E_2 \cos\theta_2,$$

since there is no free charges at the interface. The parallel component of $\mathbf{E}$ is always continuous:

$$E_1 \sin\theta_1 = E_2 \sin\theta_2.$$

Finally we get

$$\frac{\epsilon_1}{\epsilon_2} = \frac{\tan\theta_1}{\tan\theta_2}.$$

2. **(a)** A sphere of radius R carries surface charge density $\sigma(\theta, \phi) = \sigma_0 \sin\theta \sin\phi$ on its surface. Calculate the dipole moments p_x, p_y and p_z. By analogy, write down the potential $V(r, \theta, \phi)$ for points inside the sphere. [**4+2**]

Answer: Here $p_x = 0$, $p_y = \sigma_0 \Omega$ and $p_z = 0$. Here $\Omega = 4\pi R^3/3$ is the volume of the sphere. This is an uniformly polarized object and hence the polarization is $\mathbf{P} = \sigma_0 \hat{\jmath}$. The potential inside the sphere is

$$V_{\text{in}} = \frac{P\,y}{3\epsilon_0} = \frac{\sigma_0\,y}{3\epsilon_0}.$$

2. (b) Two concentric spherical shells of radii R_1 and $R_2 < R_1$ carry surface charge density $\sigma_1 \cos\theta$ and $-\sigma_2 \cos\theta$, as shown in the figure.

i) For the potential outside ($r > R_1$) to be zero everywhere, determine the magnitude of σ_2 in terms of other quantities. [**4**]

ii) For the above value of σ_2, determine the potential inside ($r < R_2$). [**4**]

Answer: In this case, the polarization is $\mathbf{P} = \sigma \hat{k}$. The potential outside a uniformly polarized sphere of radius R and the polarization P is given by

$$V_{\text{out}} = \frac{PR^3}{3\epsilon_0} \frac{\cos\theta}{r^2}.$$

Using this result, one can show that the potential outside ($r > R_1$) will be zero if

$$\sigma_2 = \sigma_1 \left(\frac{R_1}{R_2}\right)^3.$$

The potential inside ($r < R_2$) is

$$V_{\text{in}}(r < R_2) = \frac{\sigma_1 r \cos\theta}{3\epsilon_0} + \frac{-\sigma_2 r \cos\theta}{3\epsilon_0} = \frac{\sigma_1 r \cos\theta}{3\epsilon_0} \left[\frac{R_2^3 - R_1^3}{R_2^3}\right].$$

2. (c) A point dipole $\mathbf{p}_1 = p_1 \hat{\imath}$ is fixed at the origin. Another dipole $\mathbf{p}_2$ is kept at the point $(a, a, 0)$ but it is free to take any orientation at that point. Find the orientation of $\mathbf{p}_2$ that will give the minimum potential energy for the pair of dipoles. [**6**]

Answer: Electric field due to $\mathbf{p}_1$ is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0 r^5} \left[3(\mathbf{p}_1 \cdot \mathbf{r})\mathbf{r} - r^2 \mathbf{p}_1\right] \\
&= \frac{1}{4\pi\epsilon_0 r^5} \left[3p_1 x(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) - p_1 r^2 \hat{\imath}\right] \\
&= \frac{p_1}{4\pi\epsilon_0 r^5} \left[(3x^2 - r^2)\hat{\imath} + 3xy\hat{\jmath} + 3xz\hat{k}\right]
\end{aligned}$$

The fielad at $(a, a, 0)$ is

$$\mathbf{E}(a,a,0) = \frac{p_1}{4\pi\epsilon_0(2a^2)^{5/2}}(3a^2 - 2a^2)\hat{\imath} + 3a^2\hat{\jmath}) = \frac{p_1 a^2}{4\pi\epsilon_0(2a^2)^{5/2}}(\hat{\imath} + 3\hat{\jmath}) = \frac{p_1}{16\sqrt{2}\pi\epsilon_0 a^3}(\hat{\imath} + 3\hat{\jmath}).$$

The potential energy will be minimum when $\mathbf{E}$ and $\mathbf{p}_2$ are parallel. The second dipole will be oriented along $\hat{n} = \hat{\imath} + 3\hat{\jmath}$ direction.

3. **(a)** A straight wire of length L is placed along the z axis centered at the origin. The wire carries a total charge Q which is distributed uniformly. Calculate **explicitly** all components of the quadrupole moments for this charge distribution. Write down an expression for the potential $V(r, \theta)$ due to the quadrupole moments for large r. Here, r and θ are the spherical coordinates. [**6+4**]

Answer: The quadrupole moments can be calculated using the following formula:

$$Q_{ij} = \int \rho\Big[3r_i r_j - r^2\delta_{ij}\Big]d\tau.$$

$$Q_{xx} = Q_{yy} = \int \lambda(3 \times 0 - z^2)dz = -\frac{\lambda L^3}{12}.$$

$$Q_{zz} = \int \lambda(3z^2 - z^2)dz = \frac{\lambda L^3}{6}.$$

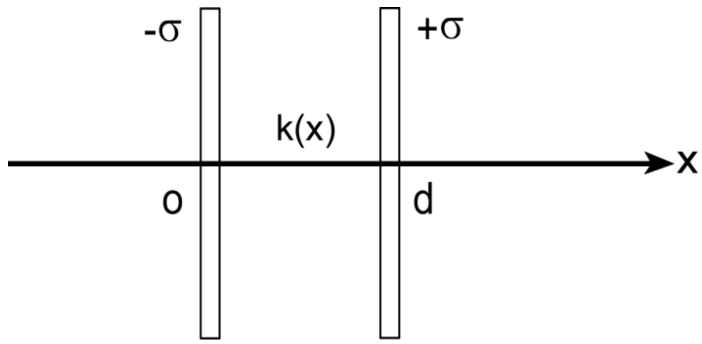
Similarly, one can show that $Q_{xy} = 0, Q_{xz} = 0, Q_{yz} = 0$. The quadrupolar contribution to the potential is

$$\begin{aligned}
V_q(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\Big[\frac{1}{2r^5}(x^2Q_{xx} + y^2Q_{yy} + z^2Q_{zz} + 2xyQ_{xy} + 2xzQ_{xz} + 2yzQ_{yz}\Big] \\
&= \frac{1}{4\pi\epsilon_0}\Big[\frac{1}{2r^5}(x^2Q_{xx} + y^2Q_{yy} + z^2Q_{zz})\Big] = \frac{1}{4\pi\epsilon_0}\Big[\frac{Q_0}{2r^5}[-x^2 - y^2 + 2z^2]\Big] \\
&= \frac{1}{4\pi\epsilon_0}\Big[\frac{Q_0}{2r^5}[-r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) + 2r^2\cos^2\theta]\Big] \\
&= \frac{1}{4\pi\epsilon_0}\Big[\frac{\lambda L^3}{12r^3}(3\cos^2\theta - 1)\Big],
\end{aligned}$$

where $Q_0 = \lambda L^3/(12)$.

3. (b) A parallel plate capacitor has charge densities $\pm\sigma$ on its plates which are separated by a distance d. The space between the capacitor plates is filled with a linear but inhomogeneous dielectric. The relative permitivity is given by $k(x) = (1 + x/d)$.

i) Calculate the electric displacement vector **D**, electric field **E** and polarization vector **P** as functions of x. [**3**]
ii) Calculate the potential difference between the plates. [**2**]
iii) Calculate the bound charge densities everywhere. [**5**]



Answer: The dielectric is linear so

$$D = \epsilon_0 k E = \epsilon_0(1 + x/d)E.$$

The dielectric does not affect the free charges, the displacement vector **D** remains same as it would be in the absence of the dielectric. Thus $\mathbf{D} = -\hat{\imath}\sigma$. The electric field is given by

$$\mathbf{E} = -\hat{\imath}\frac{\sigma}{\epsilon_0}\frac{d}{x+d}.$$

Difference between the plates:

$$V = -\int_d^0 E dx = \int_0^d E dx = \frac{\sigma d}{\epsilon_0}\int_0^d \frac{dx}{x+d} = \frac{\sigma d}{\epsilon_0}\ln 2.$$

The polarization is given by

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = -\frac{\sigma x}{x+d}\hat{\imath}.$$

The bound charges are:

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\sigma\frac{d}{dx}\Big[-\frac{x}{x+d}\Big] = \frac{\sigma d}{(x+d)^2}.$$

The bound surface charge density on the dielectric adjacent to the negative plate is

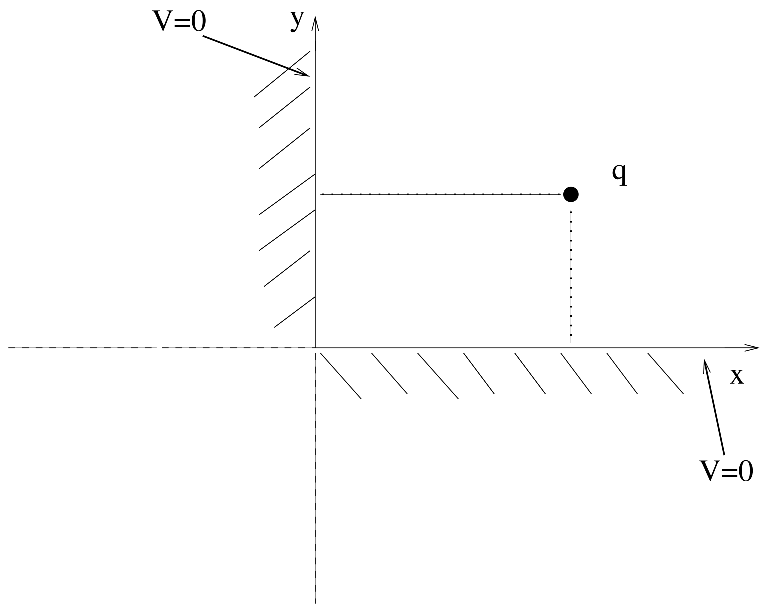
$$\sigma_b = \mathbf{P}\cdot\hat{n}\Big|_{x=0} = \mathbf{P}\cdot(-\hat{\imath})\Big|_{x=0} = 0.$$

and the bound surface charge density on the dielectric adjacent to the positive plate is

$$\sigma_b = \mathbf{P} \cdot \hat{n}\Big|_{x=d} = \mathbf{P} \cdot (\hat{\imath})\Big|_{x=d} = -\frac{\sigma}{2}.$$

Note that the total bound charge is zero.

4. **(a)** Two semi-infinite grounded conducting planes meet at right angle. In the region between them, there is a point charge q located at the point $(a, b, 0)$ as shown in the figure. For this configuration, find out the locations and the values of the image charges. Find the force on the point charge q. [**4+2**]
(Hint: You may consider image of image.)



Answer: We have to keep three imgae charges: imgae charge $-q$ at $(-a, b, 0)$, imgae charge $-q$ at $(a, -b, 0)$ and imgae charge q at $(-a, -b, 0)$.

The force acting on $+q$ is

$$\mathbf{F} = \frac{q^2}{4\pi\epsilon_0}\Big[-\frac{\hat{\imath}}{4b^2} - \frac{\hat{\jmath}}{4a^2} + \frac{a\hat{\imath} + b\hat{\jmath})}{4(a^2 + b^2)^{3/2}}\Big].$$

4. (b) A metal sphere of radius R_1 carries a charge Q. It is surrounded up to radius R_2 by a dielectric material of permittivity ϵ. Find $\mathbf{D}, \mathbf{E}$ and $\mathbf{P}$ everywhere. [**6**]

Answer: For $r < R_1$: As the sphere is metallic, the free charge Q will be distributed on its surface. For a spherical surface inside, $\mathbf{D} \cdot d\mathbf{r} = Q_{\text{free}} = 0$, it gives $\mathbf{D} = 0$. Field

$\mathbf{E}$ inside a conductor is zero. Hence $\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = 0$.

For $R_1 < r < R_2$: For a Gaussian surface in the region, the free charge enclosed is Q. Hence $D 4\pi r^2 = Q$.

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{r},$$

$$\mathbf{E} = \frac{Q}{4\pi \epsilon r^2}\hat{r}.$$

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \frac{Q}{4\pi r^2}(1 - \frac{\epsilon_0}{\epsilon})\hat{r}.$$

For $r > R_2$: For a Gaussian surface in this region, $Q_{\text{free}} = Q$.

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{r},$$

$$\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\hat{r}.$$

There is no material, $\mathbf{P} = 0$.

4. (c) The surface current density in xy plane is given as

$$K(x,y) = K_1\Big[\frac{x^2}{a^2}\hat{i} + \frac{y^2}{b^2}\hat{j}\Big].$$

Calculate the total current flowing out of the boundary of the first quadrant of the unit circle centered at the origin. [**8**]

Answer: Note that the the current is not flowing out of the x and y axes. The current is flowing out of the curved line only. We know $d\mathbf{r} = \hat{s}ds + \hat{\phi}sd\phi = \hat{\phi}d\phi$ since radius is one.

$$\begin{aligned}
\mathbf{K} \times d\mathbf{r} &= K_0\Big[\frac{s^2\cos^2\phi}{a^2}\hat{i} + \frac{s^2\sin^2\phi}{b^2}\hat{j}\Big] \times \hat{\phi}d\phi \\
&= K_0\Big[\frac{s^2\cos^2\phi}{a^2}\hat{i} + \frac{s^2\sin^2\phi}{b^2}\hat{j}\Big] \times [-\sin\phi\hat{i} + \cos\phi\hat{j}]d\phi \\
&= K_0\Big[\frac{s^2\cos^3\phi}{a^2} + \frac{s^2\sin^3\phi}{b^2}\Big]d\phi\hat{k}.
\end{aligned}$$

The current flowing out of the first quadrant of a unit circle:

$$\int_0^{\pi/2} |\mathbf{K} \times d\mathbf{r}| = K_0 \int_0^{\pi/2} \Big[\frac{\cos^3\phi}{a^2} + \frac{\sin^3\phi}{b^2}\Big]d\phi = \frac{2K_0}{3}\Big[\frac{1}{a^2} + \frac{1}{b^2}\Big].$$

ROUGH WORKS

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