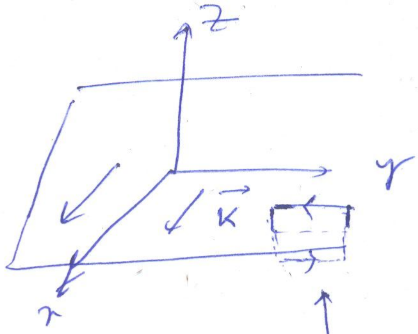
P.S #9

[DpC]



1. (a)

Magnetic field due to the current $\vec{K} = K\hat{x}$

$$\vec{B} = \begin{cases} -\frac{\mu_0}{2} K \hat{j} & \text{for } z > 0 \\ \frac{\mu_0}{2} K \hat{j} & \text{for } z < 0 \end{cases}$$

⇐ consider an Amperean loop to find $\vec{B}$.

$\vec{A}$ is $\parallel$ to $\vec{K}$ and depends only on z (by symmetry)

$$\Rightarrow \vec{A} = A(z)\,\hat{i}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ A(z) & 0 & 0 \end{vmatrix} = \hat{j}\,\partial_z A(z)$$

$$\Rightarrow \hat{j}\,\frac{\partial}{\partial z} A(z) \begin{cases} = -\frac{\mu_0 K}{2}\hat{j} & \text{for } z > 0 \\ = \frac{\mu_0 K}{2}\hat{j} & \text{for } z < 0 \end{cases}$$

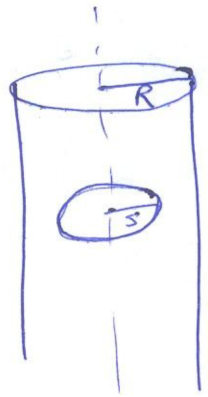
$$\Rightarrow A(z) = \begin{cases} -\frac{\mu_0}{2} K z & \text{for } z > 0 \\ \frac{\mu_0}{2} K z & \text{for } z < 0 \end{cases}$$

$$\Rightarrow \boxed{\vec{A} = -\frac{\mu_0}{2} K |z|\,\hat{i}}$$

check: $\left.\begin{array}{l} \vec{\nabla} \times \vec{A} = \vec{B} \\ \& \ \vec{\nabla} \cdot \vec{A} = 0 \end{array}\right\}$

1 (b) For an infinitely long solenoid $\begin{cases} \vec{B} = \mu_0 K & \text{inside} \\ = 0 & \text{outside} \end{cases}$

$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$$
$$= \int \vec{B}\cdot d\vec{a} = \Phi = \text{flux}$$



$\vec{A}$ is in the direction of $\vec{K}$

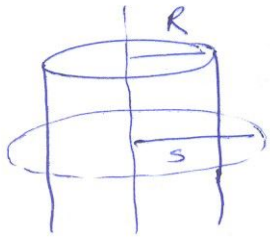
i.e. $\vec{A} = A\,\hat{\phi}$

$s<R$

$$\oint \vec{A}\cdot d\vec{l} = A\cdot 2\pi s = \int \vec{B}\cdot d\vec{a}$$
$$= (\mu_0 K\cdot \pi s^2)$$

$$\Rightarrow A = \frac{\mu_0 K s}{2}$$

$$\Rightarrow \vec{A} = \frac{1}{2}\mu_0 K s\,\hat{\phi} \quad \text{for } s<R$$



for $s>R$

$$\oint \vec{A}\cdot d\vec{l} = A\cdot 2\pi s = \int \vec{B}\cdot d\vec{a}$$
$$= \mu_0 K\cdot \pi R^2$$

$$\Rightarrow A = \frac{\mu_0 K R^2}{2s}$$

$$\Rightarrow \vec{A} = \frac{\mu_0 K R^2}{2s}\,\hat{\phi} \quad \text{for } s>R$$

Check: $\vec{\nabla}\times\vec{A} = B$

& $\vec{\nabla}\cdot\vec{A} = 0$

2.

$$\vec{A} = \begin{cases} \mu_0 k s \,\hat{z} & s \leq R \\ \mu_0 k R \ln s \,\hat{z} & s > R \end{cases}$$ in cylindrical coordinates

$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \begin{cases} -\mu_0 k \,\hat{\phi} & \text{for } s \leq R \\ -\frac{\mu_0 k R}{s}\hat{\phi} & \text{for } s > R \end{cases}$$

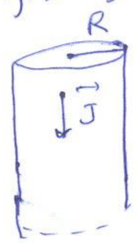
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{1}{s}\frac{\partial}{\partial s}(s B_\phi)\,\hat{z}$$

$$= \begin{cases} \frac{1}{s}\frac{\partial}{\partial s}(-\mu_0 k s)\,\hat{z} = -\frac{\mu_0 k}{s}\hat{z} & \text{for } s \leq R \\ \frac{1}{s}\frac{\partial}{\partial s}\left(-\cancel{s}\cdot\frac{\mu_0 k R}{\cancel{s}}\right)\hat{z} = 0 & \text{For } s \geq R \end{cases}$$

$$\Rightarrow \begin{cases} \mu_0 \vec{J} = -\frac{\mu_0 k}{s}\hat{z} \Rightarrow \vec{J} = -\frac{k}{s}\hat{z} & \text{for } s \leq R \\ \mu_0 \vec{J} = 0 \Rightarrow \vec{J} = 0 & \text{for } s > R \end{cases}$$

$\rightarrow$ long wire of radius R carries a current of density $\vec{J} = -\frac{k}{s}\hat{z}$



Another method:

$$\vec{\nabla}\cdot\vec{A} = 0$$

Ampere's law

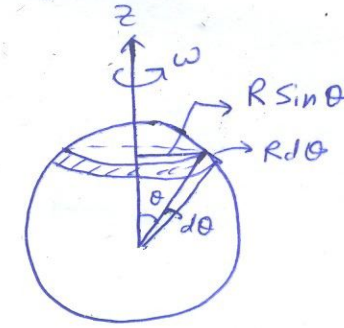
$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \Rightarrow \nabla^2 A_z = -\mu_0 J_z \quad \& \quad A_\phi = A_s = 0 \Rightarrow J_\phi = J_s = 0$$

In cylindrical coordinate $\nabla^2 A_z = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_z}{\partial s}\right)$

$$= \begin{cases} \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}(\mu_0 k s)\right) & \text{for } s \leq R \\ \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}(\mu_0 k R \ln s)\right) & \text{for } s > R \end{cases}$$

$$= \begin{cases} \frac{1}{s}\mu_0 k & s \leq R \\ \frac{1}{s}\frac{\partial}{\partial s}\left(\cancel{s}\frac{\mu_0 k R}{\cancel{s}}\right) = 0 & s \geq R \end{cases}$$

$$\Rightarrow \vec{J} = \begin{cases} -\frac{k}{s}\hat{z} & s \leq R \\ 0 & s > R \end{cases}$$



3. Charge on the ring

$$dq = \sigma . (2\pi R \sin\theta) R d\theta$$

Time for one revolution $T = \frac{2\pi}{\omega}$

Current in the ring $I = \frac{dq}{T} = \frac{\sigma\, 2\pi R^2 \sin\theta d\theta}{(2\pi/\omega)}$

$$= \sigma\omega R^2 \sin\theta d\theta$$

Surface current density
$\vec{K} = \sigma \vec{v}$
$\vec{v} = \vec{\omega} \times \vec{r}$
$= \omega R \sin\theta \hat{\varphi}$

$I = K . dl = K(R d\theta)$
$= \omega\sigma R^2 \sin\theta d\theta$

Magnetic moment of the ring

$$d\vec{m} = I\pi (R\sin\theta)^2 \hat{k}$$

$$= \sigma\omega R^4 \pi \sin^3\theta \, d\theta \, \hat{k}$$

Total dipole moment of the sphere

$$\vec{m} = \int d\vec{m} = \pi R^4 \sigma\omega \int_0^\pi \sin^3\theta \, d\theta \, \hat{k} = \underline{\underline{\frac{4\pi}{3} R^4 \sigma\omega}} \; \hat{k}$$

$$= \underbrace{\left(\frac{4}{3}\pi R^3\right)}_{\text{volume}} \sigma \underbrace{(R\omega)}_{\text{velocity at } \theta = \pi/2} \hat{k}$$

Magnetic field at a point (r, θ, φ) with $r > R$ due to $\vec{m}$:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{m}{r^3} (2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta})$$

$$= \frac{\mu_0 \sigma\omega R^4}{3 r^3} (2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta})$$

Vector potential $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi} = \frac{\mu_0 \sigma\omega R^4}{3 r^2} \sin\theta \, \hat{\varphi}$

→ results are same as derived in Ex. 5.11 in Griffiths in a completely different method.

<u>Note</u>: we <u>cannot</u> apply this formula (for dipole) to obtain the magnetic field at a point inside the <u>sphere (i.e. for $r < R$)</u>.

4. First bring $\vec{m}_2$ from ∞ to the distance r, keeping it perpendicular to $\vec{m}_1$

Since $\vec{B}_\theta$ due to $\vec{m}_1$ on the axis

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{m_1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)\Big|_{\theta=0}$$

$$= \frac{\mu_0}{4\pi}\frac{2m_1}{r^3}\hat{r}$$

& $\vec{m}_2$ is perpendicular to $\vec{B}$, force on $m_2 \Rightarrow \vec{F} = \vec{\nabla}(\vec{m}_2\cdot\vec{B}) = 0$

$\Rightarrow$ work done $= 0$.

Now rotate m_1 to the angle θ_1

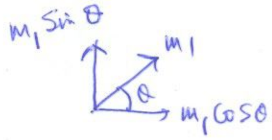
$\vec{B}_2$ = field due to $\vec{m}_2$ at the position of $\vec{m}_1$

$$= \frac{\mu_0}{4\pi}\frac{m_2}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)\Big|_{\theta=\pi/2} = \frac{\mu_0}{4\pi}\frac{m_1}{r^3}\hat{\theta}$$

When $\vec{m}_1$ is at an angle θ with $\vec{r}$, the torque on m_1.

$$\vec{N}_1 = \vec{m}_1 \times \vec{B}_2$$

$$= m_1\left(\cos\theta\,\hat{r} + \sin\theta\,(-\hat{\theta})\right) \times \frac{\mu_0}{4\pi}\frac{m_2}{r^3}\hat{\theta}$$



$$= \frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\cos\theta\,\hat{\phi} \quad [\because \hat{r}\times\hat{\theta} = \hat{\phi}]$$

$\Rightarrow$ work done to rotate m_1 to the angle θ_1

$$W_1 = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\int_{\theta=0}^{\theta_1}\cos\theta\, d\theta = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\sin\theta_1$$

Now rotate $\vec{m}_2$ to the desired orientation

Field at m_2 due to m_1

$$\vec{B}_1 = \frac{\mu_0}{4\pi}m_1\left(2\cos\theta_1\hat{r} + \sin\theta_1\,\hat{\theta}\right)$$

If $\vec{m}_2$ is at an angle α, then torque $\vec{N}_2 = \vec{m}_2 \times \vec{B}_1$

$$\cancel{\Rightarrow \vec{N}_2 = \frac{\mu_0 m_1}{4\pi r^3}\left[m_2 \left(\cos\alpha\, \hat{r} + \sin\alpha\,(-\hat{\theta})\right)\right] \times \left[2\cos\theta_1\, \hat{r} + \sin\theta_1\, \hat{\theta}\right]}$$

$$\vec{N}_2 = \left(m_2 \cos\alpha\, \hat{r} + m_2 \sin\alpha\,(-\hat{\theta})\right) \times \frac{\mu_0 m_1}{4\pi r^3}\left(2\cos\theta_1\, \hat{r} + \sin\theta_1\, \hat{\theta}\right)$$

$$= \frac{\mu_0}{4\pi r^3} m_1 m_2 \left(\cos\alpha \sin\theta_1\, \hat{\phi} + 2\sin\alpha \cos\theta_1\, \hat{\phi}\right)$$

$\therefore$ Work done to rotate m_2 from $\alpha = \pi/2$ to $\alpha = \theta_2$

$$W_2 = \frac{\mu_0 m_1 m_2}{4\pi r^3} \int_{\alpha=\pi/2}^{\theta_2} \left(2\sin\alpha \cos\theta_1 + \cos\alpha \sin\theta_1\right) d\alpha$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[-2\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 - \sin\theta_1\right]$$

$\therefore$ Total work done:

$$W = W_1 + W_2 = \frac{\mu_0 m_1 m_2}{4\pi r^3} \sin\theta_1 + \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\sin\theta_1 \sin\theta_2 - \sin\theta_1 - 2\cos\theta_1 \cos\theta_2\right]$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2\right]$$

work done due to orientation only.