Some homework problems and discussion related to balancing and flywheels

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1. Basic background: least squares solution.

   If you have an overdetermined system of equations of the form $Ax = b$ where A has m rows and n columns with $m > n$; and b has m elements while x has only n elements, then the system is overdetermined and in general there is no exact solution. The *least squares* solution satisfies $A^T Ax = A^T b$. A Matlab example follows (note: text following % is a comment).

```
>> % m=3, n=2
>> A=rand(3,2), b=rand(3,1)
A =
   0.814723686393179   0.913375856139019
   0.905791937075619   0.632359246225410
   0.126986816293506   0.097540404999410
b =
   0.278498218867048
   0.546881519204984
   0.957506835434298
>> x=A\b    % Matlab does least squares automatically
x =
   1.289597327689021
  -0.820726191726098
>> x=inv(A'*A)*(A'*b)    % from the formula given above
x =
   1.289597327689020
  -0.820726191726099
>> [b,A*x]   % compare the two: not very good in this case
ans =
   0.278498218867048   0.301034000754048
   0.546881519204984   0.649113065537613
   0.957506835434298   0.083707893809255
```

   While solving large systems is laborious, I think solving up to about 5 equations in up to 5 unknowns is possible, and something like 3 equations in 2 unknowns is easy.

   Now suppose that the connecting rod of a hypothetical engine is laid out along the x axis, so that the center of the crank pin is at the origin, and the center of the (other pin) gudgeon pin or wrist pin is at $x = 14$ cm. The center of mass is found to be at $x = 6$ cm. The total mass is 110 grams, and the moment of inertia J (planar case) is 5000 gm cm^2. We wish to approximate the inertia properties of this connecting rod using two point masses m_A (at the origin or crank pin) and m_B (at the other pin). The equations we have (using grams and cm) are for total mass,

$$m_A + m_B = 110,$$

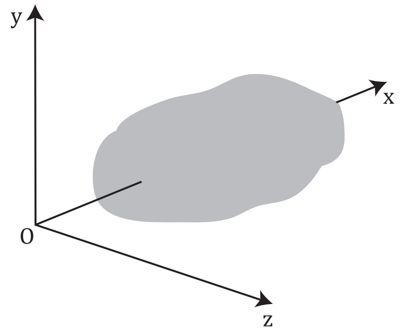

Figure 1: A rotor spins about the x-axis.

center of mass location

$$0 \cdot m_A + 14 \cdot m_B = 660,$$

and moment of inertia

$$6^2 \cdot m_A + 8^2 \cdot m_B = 5000.$$

The corresponding system of equations in matrix form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 14 \\ 36 & 64 \end{bmatrix} \left\{ \begin{array}{c} m_A \\ m_B \end{array} \right\} = \left\{ \begin{array}{r} 110 \\ 660 \\ 5000 \end{array} \right\}.$$

The corresponding least squares solution is $m_A = 55.14$ grams and $m_B = 47.11$ grams. Check for approximate satisfaction of the concerned equations, and learn how to write and solve these equations for different numerical input data.

An interesting issue that arises is this. Instead of grams and centimeters, if you use kilograms and meters, the least squares answer changes to (approximately) $m_A = 63$ grams and $m_B = 47$ grams. Why are the answers different? Which answer do you think is better? If the engine in question is to be used in a motorcycle, which answer would you recommend? If, by coincidence, an exact solution was possible, would choice of units matter?

2. Study the parallel axis theorem for computing moment of inertia matrices in 3D dynamics. In particular, if $\mathbf{I}_{\rm cm}$ is known, what is $\mathbf{I}$ about some other point for the same body, with coordinate axes parallel? This is background material from dynamics, and easy to derive.

   Now see figure 1. Suppose the mass of the rotor is 6 Kg, and its center of mass coordinates (in meters) are $(0.7, 0.013, 0.007)$. Further, its moment of inertia matrix about its center of mass, in Kg m^2, is

$$\mathbf{I}_{\rm cm} = \begin{bmatrix} 1.25 & 0.05 & -0.07 \\ 0.05 & 2.00 & -0.03 \\ -0.07 & -0.03 & 2.04 \end{bmatrix}.$$

   We will attempt a two-plane balancing calculation. We choose the two planes to be given by $x = 0.3$ and $x = 0.9$. Point masses m_1, m_2, m_3 and m_4 are to be added at the predetermined locations (0.3,0.1,0), (0.3,0,0.1), (0.9,0.1,0) and (0.9,0,0.1) respectively. What values of m_1, m_2, m_3 and m_4 are needed for balancing? What is the new total mass, center of mass location, and $\mathbf{I}_{\rm cm}$? How should negative solutions for the m_i be interpreted? What would happen if we attempted to balance the rotor by adding two or more masses, all at some suitable plane $x = x_c$?

3. Please do make sure you understand at least pages 196-198 of Ghosh and Mallik. Then consider the following (hypothetical) turning moment versus angle curve. Please note that in class I carelessly marked the range as 0 to 2π, but for a 4-stroke engine it should be 0 to 4π radians, or 0 to 720 degrees, as below. In the figure, successive portions of 180 degrees each are perfect half-sinusoids

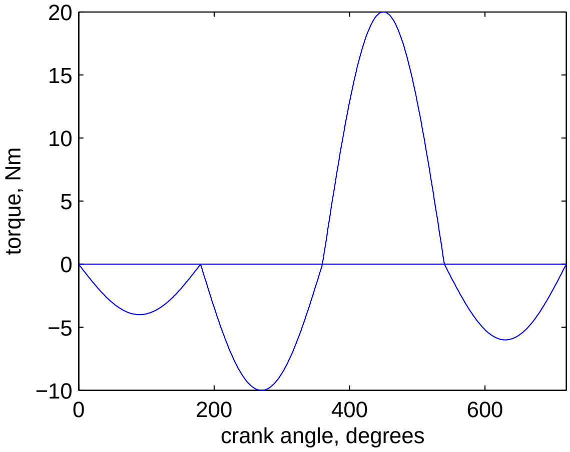

Figure 2: Hypothetical turning moment (torque) versus angle diagram.

for ease of calculation. The corresponding minima/maxima are at -4, -10, 20 and -6. If the mean crankshaft speed is 4000 RPM and the fluctuation is not to be more than 1%, what moment of inertia is needed for the flywheel? If the flywheel is made of steel, is a uniform disk, and has a thickness equal to 1/8 of its radius, then what are the dimensions of the flywheel? Its mass?