Introduction to Fourier Series

MSO-203B

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Overview

Fourier Series and Applications:-

- Fourier Series.

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- Fourier Series.
- Fourier Sine and Cosine Series.

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Fourier Series and Applications:-

- Fourier Series.
- Fourier Sine and Cosine Series.
- Half Range Expansion.

Fourier Series

Question

Can a real valued periodic function be written in term of a infinite series of *sines* and *cosines* ?

Periodic Function

Definition

Let $L > 0$ be any real number. A function from $\mathbb{R} \to \mathbb{R}$ is called periodic of period L if we have $f(x+L) = f(x)$ for every $x \in \mathbb{R}$.

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Examples

- $\sin x$ and $\cos x$ are 2π periodic functions.
- $f(x) = \begin{cases} 1 \text{ if } x \in [n, n+\frac{1}{2}) \\ 0 \text{ if } x \in [n+\frac{1}{2}, n+1) \end{cases}$

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Remark

Note that if $f(x)$ is L periodic then $f(x + nL) = f(x)$ for every $x \in \mathbb{R}$.

Preliminary Lemmas

Definition

The system consists of 2π periodic functions given by $1, \cos x, \sin x, ..., \cos nx, \sin nx, ...$ is called an trigonometric system.

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Orthogonality of trigonometric system

The trigonometric system is orthogonal on $-\pi \leq x \leq \pi$ i.e, for any integer m and n we have,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = 0 \quad m \neq n$$
$$\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = 0 \quad m \neq n$$
$$\int_{-\pi}^{\pi} \sin nx \cos mx \; dx = 0 \quad m \neq n \text{ or } m = n.$$

Existence of Fourier Series

Fourier Series of a 2π periodic function

Given a piecewise continuous, 2π periodic function, there exists a representation of f as

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \tag{1}$$

where,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$
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a_0, a_n and b_n are the Fourier coefficients of f.

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Existence of a_0

Integrating both sides of (1) between π and $-\pi$

$$\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi}^{\pi} a_0\ dx + \sum_{n=1}^{\infty} [a_n \int_{-\pi}^{\pi} \cos nx + b_n \int_{-\pi}^{\pi} \sin nx]\ dx$$

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which implies

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Existence of Fourier Series

Existence of a_n

Multiplying both sides of (1) by $\cos mx$ for a fixed m and integrating both sides between π and $-\pi$

$$\int_{-\pi}^{\pi} f(x)\cos mx\ dx = \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi} \cos nx + b_n \int_{-\pi}^{\pi} \sin nx] \cos mx\ dx$$
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which implies $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx \ dx$, since $a_m \int_{-\pi}^{\pi} \cos^2 mx \ dx = a_m\pi$ and since the trigonometric system is orthogonal.

Existence of Fourier Series

Existence of b_n

Multiplying both sides of (1) by $\sin mx$ for a fixed m and integrating both sides between π and $-\pi$

$$\int_{-\pi}^{\pi} f(x) \sin mx \; dx = \sum_{n=1}^{\infty} [a_n \int_{-\pi}^{\pi} \cos nx + b_n \int_{-\pi}^{\pi} \sin nx] \sin mx \; dx$$
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which implies $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \; dx$, since $b_m \int_{-\pi}^{\pi} \cos^2 mx \; dx = b_m \pi$ and since the trigonometric system is orthogonal.

Convergence of Fourier Series

Convergence Theorem

Let $f(x)$ be periodic function with period 2π and piecewise continuous in the interval $-\pi \leq x \leq \pi$. Furthermore let $f(x)$ has a left hand derivative and a right hand derivative at each point at that interval. Then the Fourier Series (1) of $f(x)$ converges with sum $f(x)$ except for the points where f is discontinuous and for those points the sum of the series is the average of the left and right hand limits of $f(x)$.

Problem

Example 1

Consider the function $f(x) = \begin{cases} 0, \ -\pi \leq x < 0 \\ \pi, \ 0 \leq x \leq \pi. \end{cases}$ and $f(2\pi + x) = f(x)$.

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Solution

$a_0 = \frac{1}{2\pi}(\int_{-\pi}^{0} 0 \; dx + \int_0^\pi \pi \; dx) = \frac{\pi}{2}$.
$a_n = \frac{1}{\pi}\int_0^\pi \pi \cos(nx) \; dx = 0$ for $n \geq 1$.
$b_n = \frac{1}{\pi}\int_0^\pi \pi \sin(nx) \; dx = \frac{1}{n}(1 - \cos(n\pi)) = \frac{1}{n}(1 - (-1)^n)$.

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Conclusion

$f(x) \sim \frac{\pi}{2} + 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$ since $b_{2n} = 0$ and $b_{2n+1} = \frac{2}{2n+1}$.

Function of arbitrary period

Function of period $2L$

If $g(y)$ is a function of period 2π then using the change of variable $y = kx$ with k such that the old period $y = 2\pi$ gives for the new period $x = 2L$. Hence $k = \frac{\pi}{L}$ and the Fourier series of the function $f(x)$ of period $2L$ can be written by writing $g(y) = f(x)$ as

$$f(x) = a_0 + \sum_{1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

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$$f(x) = a_0 + \sum_{1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

with the Fourier coefficients as

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \quad n = 1, 2, ..$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx \quad n = 1, 2, ...$$

Representation of a function with arbitrary period

Function of Period 4

Consider the function $f(x) = x$ on $(-2, 2)$ and $f(x+4) = f(x)$.
Clearly it is function of period 4 and hence $L = 2$ and its Fourier series is given by

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$$a_0 = \frac{1}{2.2}\int_{-2}^{2} x\, dx = 0.$$
$$a_n = \frac{1}{2}\int_{-2}^{2} x\cos(\frac{n\pi x}{2})\, dx = 0$$
$$b_n = \frac{1}{2}\int_{-2}^{2} x\sin(\frac{n\pi x}{2})\, dx = -\frac{4}{n\pi}\cos(n\pi).$$

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Representation in terms of series

$$f(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi x}{2}).$$

Fourier Sine and Cosine Series

Fourier Cosine Series

The Fourier cosine series of an even function of period $2L$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

with coefficients $a_0 = \frac{1}{L}\int_0^L f(x)\, dx$ and $a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$

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Fourier Sine Series

The Fourier sine series of an odd function of period $2L$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,dx$$

with coefficients $b_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx$

Half Range Series

Question

Can we define the Fourier series expansion for a function defined on a finite interval say $[0, L]$?

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Solution

The Answer is Yes. It should be noted that the function is not periodic since we need an infinite interval for a function to be periodic.

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Can we define the Fourier series expansion for a function defined on a finite interval say $[0, L]$?

Solution

The Answer is Yes. It should be noted that the function is not periodic since we need an infinite interval for a function to be periodic.

The way out

Convert the non-periodic function to a periodic function and the most efficient way to do that is the Half Fourier Series.

Odd and Even Extensions

Odd Extensions

Let f be a function defined and integrable on $[0, \pi]$. Define the ODD extension as:

$$f_1(x) = \begin{cases} -f(-x), \ -\pi \leq x < 0 \\ f(x), \ 0 \leq x \leq \pi \end{cases}$$

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Even Extensions

Let f be a function defined and integrable on $[0, \pi]$. Define the EVEN extension as

$$f_(x) = \begin{cases} f(-x), \ -\pi \leq x < 0 \\ f(x), \ 0 \leq x \leq \pi \end{cases}$$

An Illustration

Example

Consider the function $f : (0, 2) \to \mathbb{R}$ as the following:

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < 1 \\ 2 \text{ if } 1 < x < 2 \end{cases}$$

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Fourier Even Extension

Given an even function of period $2L$ we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\, dx$$

Solution

Calculating a_0

$a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}[\int_0^1 1\,dx + \int_1^2 2\,dx] = \frac{3}{2}.$

Solution

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Calculating a_n

$a_n = \int_0^2 f(x)\cos(\frac{n\pi x}{2})\,dx = \int_0^1 \cos(\frac{n\pi x}{2})\,dx + \int_1^2 2\cos(\frac{n\pi x}{2})\,dx = -\frac{2}{n\pi}\sin(\frac{n\pi}{2})$

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Half range Fourier Cosine series

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\Big(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \ldots\Big)$$

The End