This is *also* the *exact* potential. *Conclusion:* all multiple moments of this distribution (except the dipole) are exactly zero.

Problem 3.29

Using Eq. 3.94 with $r' = d/2$:

$$\frac{1}{\imath_+} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n P_n(\cos\theta);$$

for $\imath_-$, we let $\theta \to 180^\circ + \theta$, so $\cos\theta \to -\cos\theta$:

$$\frac{1}{\imath_-} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n P_n(-\cos\theta).$$

But $P_n(-x) = (-1)^n P_n(x)$, so

$$V = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\imath_+} - \frac{1}{\imath_-}\right) = \frac{1}{4\pi\epsilon_0}q\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n [P_n(\cos\theta) - P_n(-\cos\theta)] = \frac{2q}{4\pi\epsilon_0 r}\sum_{n=1,3,5,\ldots}\left(\frac{d}{2r}\right)^n P_n(\cos\theta).$$

Therefore

$$V_{\text{dip}} = \frac{2q}{4\pi\epsilon_0}\frac{1}{r}\frac{d}{2r}P_1(\cos\theta) = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}, \quad \text{while} \quad \boxed{V_{\text{quad}} = 0.}$$

$$V_{\text{oct}} = \frac{2q}{4\pi\epsilon_0 r}\left(\frac{d}{2r}\right)^3 P_3(\cos\theta) = \frac{2q}{4\pi\epsilon_0}\frac{d^3}{8r^4}\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) = \boxed{\frac{qd^3}{4\pi\epsilon_0}\frac{1}{8r^4}\left(5\cos^3\theta - 3\cos\theta\right).}$$

Problem 3.30

(a) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right] = \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right].}$

(b) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right].}$

(c) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{y}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$ (from Eq. 1.64, $\hat{\mathbf{y}}\cdot\hat{\mathbf{r}} = \sin\theta\sin\phi$).

Problem 3.31

(a) This point is at $r = a$, $\theta = \frac{\pi}{2}$, $\phi = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\epsilon_0 a^3}(-\hat{\mathbf{z}})$; $\mathbf{F} = q\mathbf{E} = \boxed{-\frac{pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.}$

(b) Here $r = a$, $\theta = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}(2\hat{\mathbf{r}}) = \frac{2p}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.$ $\boxed{\mathbf{F} = \frac{2pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.}$

(c) $V = q\left[V(0,0,a) - V(a,0,0)\right] = \frac{qp}{4\pi\epsilon_0 a^2}\left[\cos(0) - \cos\left(\frac{\pi}{2}\right)\right] = \boxed{\frac{pq}{4\pi\epsilon_0 a^2}.}$

Problem 3.32

$Q = -q$, so $V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r}$; $\mathbf{p} = qa\,\hat{\mathbf{z}}$, so $V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}$. Therefore

$$\boxed{V(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right).} \quad \boxed{\mathbf{E}(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\hat{\mathbf{r}} + \frac{a}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right].}$$

Problem 3.33

$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = p\cos\theta\,\hat{\mathbf{r}} - p\sin\theta\,\hat{\boldsymbol{\theta}}$ (Fig. 3.36). So $3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}} = 2p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$. So Eq. 3.104 $\equiv$ Eq. 3.103. ✓

Problem 3.34

At height x above the plane, the force on q is given by Eq. 3.12: $F = -\dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{4z^2} = m\dfrac{d^2x}{dt^2}$; $\quad \dfrac{d^2x}{dt^2} = -A/x^2$, where $A \equiv \dfrac{q^2}{16\pi\epsilon_0 m}$. Multiply by $v = \dfrac{dx}{dt}$: $\quad v\dfrac{dv}{dt} = -\dfrac{A}{x^2}\dfrac{dx}{dt} \Rightarrow \dfrac{d}{dt}\left(\dfrac{1}{2}v^2\right) = \dfrac{d}{dt}\left(\dfrac{A}{x}\right) \Rightarrow \dfrac{1}{2}v^2 = \dfrac{A}{x} +$ constant. But $v = 0$ when $x = d$, so constant $= -A/d$, and hence $v^2 = 2A\left(\dfrac{1}{x} - \dfrac{1}{d}\right)$; $\quad -\dfrac{dx}{dt} = \sqrt{2A}\sqrt{\dfrac{1}{x} - \dfrac{1}{d}} = \sqrt{\dfrac{2A}{d}}\sqrt{\dfrac{d-x}{x}}$.

$$\int_d^0 \frac{\sqrt{x}}{\sqrt{d-x}}\,dx = -\sqrt{\frac{2A}{d}}\int_0^t dt = -\sqrt{\frac{2A}{d}}\,t.$$

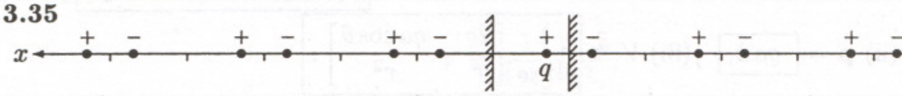
This integral can also be integrated directly. Let $x = u^2$; $dx = 2u\,du$.

$$\int_d^0 \frac{\sqrt{x}}{\sqrt{d-x}}\,dx = 2\int_{\sqrt{d}}^0 \frac{u^2}{\sqrt{d-u^2}}\,du = 2\left\{-\frac{u}{2}\sqrt{d-u^2} + \frac{d}{2}\sin^{-1}\left(\frac{u}{\sqrt{d}}\right)\right\}\Bigg|_{\sqrt{d}}^0 = -d\sin^{-1}(1) = -d\frac{\pi}{2}.$$

Therefore

$$t = \sqrt{\frac{d}{2A}}\frac{\pi d}{2} = \sqrt{\frac{\pi^2 d^2}{4}\frac{d}{2q^2}16\pi\epsilon_0 m} = \boxed{\sqrt{\frac{2\pi^3 d^3 \epsilon_0 m}{q^2}}.}$$

Problem 3.35



The image configuration is shown in the figure; the positive image charge forces cancel in pairs. The net force of the negative image charges is:

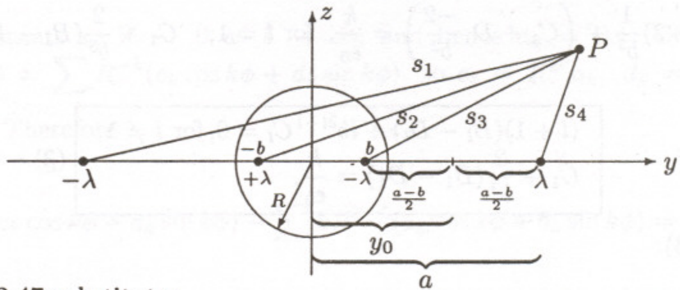
$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}q^2\left\{\frac{1}{[2(a-x)]^2} + \frac{1}{[2a+2(a-x)]^2} + \frac{1}{[4a+2(a-x)]^2} + \dots\right. \\ &\quad \left. -\frac{1}{(2x)^2} - \frac{1}{(2a+2x)^2} - \frac{1}{(4a+2x)^2} - \dots\right\} \\ &= \boxed{\frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left\{\left[\frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \frac{1}{(3a-x)^2} + \dots\right] - \left[\frac{1}{x^2} + \frac{1}{(a+x)^2} + \frac{1}{(2a+x)^2} + \dots\right]\right\}.} \end{aligned}$$

When $a \to \infty$ (i.e. $a \gg x$) only the $\dfrac{1}{x^2}$ term survives: $F = -\dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{(2x)^2}$ ✓ (same as for only *one* plane—Eq. 3.12). When $x = a/2$,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left\{\left[\frac{1}{(a/2)^2} + \frac{1}{(3a/2)^2} + \frac{1}{(5a/2)^2} + \dots\right] - \left[\frac{1}{(a/2)^2} + \frac{1}{(3a/2)^2} + \frac{1}{(5a/2)^2} + \dots\right]\right\} = 0. ✓$$

Problem 3.36

Following Prob. 2.47, we place image line charges $-\lambda$ at $y = b$ and $+\lambda$ at $y = -b$ (here y is the horizontal axis, z vertical).



In the solution to Prob. 2.47 substitute:

$$a \to \frac{a-b}{2},\ y_0 \to \frac{a+b}{2} \text{ so } \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 - R^2 \Rightarrow b = \frac{R^2}{a}.$$

$$\begin{aligned} V &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{s_3^2}{s_4^2}\right) + \ln\left(\frac{s_1^2}{s_2^2}\right)\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_1^2 s_3^2}{s_4^2 s_2^2}\right) \\ &= \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{[(y+a)^2+z^2][(y-b)^2+z^2]}{[(y-a)^2+z^2][(y+b)^2+z^2]}\right\}, \quad \text{or, using } y = s\cos\phi,\ z = s\sin\phi, \\ &= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(s^2+a^2+2as\cos\phi)[(as/R)^2+R^2-2as\cos\phi]}{(a^2+a^2-2as\cos\phi)[(as/R)^2+R^2+2as\cos\phi]}\right\}}. \end{aligned}$$

Problem 3.37

Since the configuration is azimuthally symmetric, $V(r,\theta) = \sum\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$.

(a) $r > b$: $A_l = 0$ for all l, since $V \to 0$ at ∞. Therefore $V(r,\theta) = \sum \frac{B_l}{r^{l+1}}P_l(\cos\theta)$.

$a < r < b:\quad V(r,\theta) = \sum\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos\theta). \quad r < a:\quad V(r,\theta) = V_0.$

We need to determine B_l, C_l, D_l, and V_0. To do this, invoke boundary conditions as follows: (i) V is continuous at a, (ii) V is continuous at b, (iii) $\Delta\left(\frac{\partial V}{\partial r}\right) = -\frac{1}{\epsilon_0}\sigma(\theta)$ at b.

(ii) $\Rightarrow \sum \frac{B_l}{b^{l+1}}P_l(\cos\theta) = \sum\left(C_l b^l + \frac{D_l}{b^{l+1}}\right)P_l(\cos\theta);\quad \frac{B_l}{b^{l+1}} = C_l b^l + \frac{D_l}{b^{l+1}} \Rightarrow \boxed{B_l = b^{2l+1}C_l + D_l.}$ (1)

(i) $\Rightarrow \sum\left(C_l a^l + \frac{D_l}{a^{l+1}}\right)P_l(\cos\theta) = V_0;\ \left\{\begin{array}{ll} C_l a^l + \frac{D_l}{a^{l+1}} = 0, & \text{if } l \neq 0, \\ C_0 a^0 + \frac{D_0}{a^1} = V_0, & \text{if } l = 0; \end{array}\right\}\ \boxed{\begin{array}{l} D_l = -a^{2l+1}C_l,\ l \neq 0, \\ D_0 = aV_0 - aC_0. \end{array}}$ (2)

Putting (2) into (1) gives $B_l = b^{2l+1}C_l - a^{2l+1}C_l,\ l \neq 0,\quad B_0 = bC_0 + aV_0 - aC_0$. Therefore

$$\boxed{\begin{array}{l} B_l = \left(b^{2l+1} - a^{2l+1}\right)C_l,\ l \neq 0, \\ B_0 = (b-a)C_0 + aV_0. \end{array}} \quad (1')$$

(iii) $\Rightarrow \sum B_l[-(l+1)]\frac{1}{b^{l+2}}P_l(\cos\theta) - \sum\left(C_l l b^{l-1} + D_l\frac{-(l+1)}{b^{l+2}}\right)P_l(\cos\theta) = \frac{-k}{\epsilon_0}P_1(\cos\theta)$. So

$$-\frac{(l+1)}{b^{l+2}}B_l - \left(C_l l b^{l-1} + D_l\frac{-(l+1)}{b^{l+2}}\right) = 0, \text{ if } l \neq 1;$$

or

$$-(l+1)B_l - lC_l b^{2l+1} + (l+1)D_l = 0; \quad (l+1)(B_l - D_l) = -lb^{2l+1}C_l.$$

$$B_1(+2)\frac{1}{b^2} + \left(C_1 + D_1\frac{-2}{b^2}\right) = \frac{k}{\epsilon_0}, \text{ for } l = 1; \quad C_1 + \frac{2}{b^3}(B_1 - D_1) = k.$$

Therefore

$$\boxed{\begin{array}{l}(l+1)(B_l - D_l) + lb^{2l+1}C_l = 0, \text{for } l \neq 1,\\ C_1 + \frac{2}{b^3}(B_1 - D_1) = \frac{k}{\epsilon_0}.\end{array}} \quad (3)$$

Plug (2) and (1′) into (3):

For $l \neq 0$ or 1:

$(l+1)\left[\left(b^{2l+1} - a^{2l+1}\right)C_l + a^{2l+1}C_l\right] + lb^{2l+1}C_l = 0; \quad (l+1)b^{2l+1}C_l + lb^{2l+1}C_l = 0;\ (2l+1)C_l = 0 \Rightarrow C_l = 0.$

Therefore (1′) and (2) $\Rightarrow \boxed{B_l = C_l = D_l = 0 \text{ for } l > 1.}$

For $l = 1$: $C_1 + \frac{2}{b^3}\left[\left(b^3 - a^3\right)C_1 + a^3C_1\right] = k; \quad C_1 + 2C_1 = k \Rightarrow \boxed{C_1 = k/3\epsilon_0;}$ $D_1 = -a^3C_1 \Rightarrow$ $\boxed{D_1 = -a^3k/3\epsilon_0;}$ $\quad B_1 = \left(b^3 - a^3\right)C_1 \Rightarrow \boxed{B_1 = \left(b^3 - a^3\right)k/3\epsilon_0.}$

For $l = 0$: $B_0 - D_0 = 0 \Rightarrow B_0 = D_0 \Rightarrow (b-a)C_0 + aV_0 = aV_0 - aC_0$, so $bC_0 = 0 \Rightarrow \boxed{C_0 = 0;\ D_0 = aV_0 = B_0.}$

Conclusion: $\boxed{V(r,\theta) = \frac{aV_0}{r} + \frac{\left(b^3 - a^3\right)k}{3r^2\epsilon_0}\cos\theta, \quad r \geq b.}$ $\boxed{V(r,\theta) = \frac{aV_0}{r} + \frac{k}{3\epsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta,}$ $\quad a \leq r \leq b.$

(b) $\sigma_i(\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_a = -\epsilon_0\left[-\frac{aV_0}{a^2} + \frac{k}{3\epsilon_0}\left(1 + 2\frac{a^3}{a^3}\right)\cos\theta\right] = -\epsilon_0\left(-\frac{V_0}{a} + \frac{k}{\epsilon_0}\cos\theta\right) = \boxed{-k\cos\theta + V_0\frac{\epsilon_0}{a}.}$

(c) $q_i = \int \sigma_i\, da = \frac{V_0\epsilon_0}{a}4\pi a^2 = \boxed{4\pi a\epsilon_0 V_0 = Q_{\text{tot}}.}$ At large r: $V \approx \frac{aV_0}{r} \stackrel{?}{=} \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = \frac{1}{4\pi\epsilon_0}\frac{4\pi a\epsilon_0 V_0}{r} = \frac{aV_0}{r}.$ ✓

Problem 3.38

Use multipole expansion (Eq. 3.95): $\rho\, d\tau \to \lambda\, dz = \frac{Q}{2a}\, dz$, and $r' \to z$:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{-a}^{a} z^n P_n(\cos\theta)\frac{Q}{2a}\, dz.$$

The integral is

$$\frac{Q}{2a}P_n(\cos\theta)\int_{-a}^{a} z^n\, dz = \frac{Q}{2a}P_n(\cos\theta)\left.\frac{z^{n+1}}{n+1}\right|_{-a}^{a} = \frac{Q}{2a}P_n(\cos\theta)\frac{2a^{n+1}}{n+1} \quad \text{for } n \text{ even, zero for } n \text{ odd.}$$

Therefore

$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0,2,4,\ldots}\left[\frac{1}{n+1}\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right]. \quad \text{qed}$$

Problem 3.39

Use separation of variables in cylindrical coordinates (Prob. 3.23):

$$V(s,\phi) = a_0 + b_0\ln s + \sum_{k=1}^{\infty}\left[s^k\left(a_k\cos k\phi + b_k\sin k\phi\right) + s^{-k}\left(c_k\cos k\phi + d_k\sin k\phi\right)\right].$$

$s < R$: $\quad V(s,\phi) = \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi)$ $\quad$ ($\ln s$ and s^{-k} blow up at $s = 0$);
$s > R$: $\quad V(s,\phi) = \sum_{k=1}^{\infty} s^{-k} (c_k \cos k\phi + d_k \sin k\phi)$ $\quad$ ($\ln s$ and s^k blow up as $s \to \infty$).

(We may as well pick constants so $V \to 0$ as $s \to \infty$, and hence $a_0 = 0$.) Continuity at $s = R \Rightarrow$ $\sum R^k (a_k \cos k\phi + b_k \sin k\phi) = \sum R^{-k}(c_k \cos k\phi + d_k \sin k\phi)$, so $c_k = R^{2k} a_k$, $d_k = R^{2k} b_k$. Eq. 2.36 says: $\left.\frac{\partial V}{\partial s}\right|_{R^+} - \left.\frac{\partial V}{\partial s}\right|_{R^-} = -\frac{1}{\epsilon_0}\sigma$. Therefore

$$\sum \frac{-k}{R^{k+1}}(c_k \cos k\phi + d_k \sin k\phi) - \sum k R^{k-1}(a_k \cos k\phi + b_k \sin k\phi) = -\frac{1}{\epsilon_0}\sigma,$$

or:

$$\sum 2kR^{k-1}(a_k \cos k\phi + b_k \sin k\phi) = \left\{ \begin{array}{ll} \sigma_0/\epsilon_0 & (0 < \phi < \pi) \\ -\sigma_0/\epsilon_0 & (\pi < \phi < 2\pi) \end{array} \right\}.$$

Fourier's trick: multiply by $(\cos l\phi)\, d\phi$ and integrate from 0 to 2π, using

$$\int_0^{2\pi} \sin k\phi \cos l\phi \, d\phi = 0; \quad \int_0^{2\pi} \cos\, k\phi \cos l\phi \, d\phi = \left\{ \begin{array}{ll} 0, & k \neq l \\ \pi, & k = l \end{array} \right\}.$$

Then

$$2lR^{l-1}\pi a_l = \frac{\sigma_0}{\epsilon_0}\left[\int_0^{\pi} \cos l\phi \, d\phi - \int_{\pi}^{2\pi} \cos l\phi \, d\phi\right] = \frac{\sigma_0}{\epsilon_0}\left\{ \left.\frac{\sin l\phi}{l}\right|_0^{\pi} - \left.\frac{\sin l\phi}{l}\right|_{\pi}^{2\pi} \right\} = 0; \quad a_l = 0.$$

Multiply by $(\sin l\phi)\, d\phi$ and integrate, using $\int_0^{2\pi} \sin\, k\phi \sin l\phi \, d\phi = \left\{ \begin{array}{ll} 0, & k \neq l \\ \pi, & k = l \end{array} \right\}$:

$$\begin{aligned} 2lR^{l-1}\pi b_l &= \frac{\sigma_0}{\epsilon_0}\left[\int_0^{\pi} \sin l\phi \, d\phi - \int_{\pi}^{2\pi} \sin l\phi \, d\phi\right] = \frac{\sigma_0}{\epsilon_0}\left\{ -\left.\frac{\cos l\phi}{l}\right|_0^{\pi} + \left.\frac{\cos l\phi}{l}\right|_{\pi}^{2\pi} \right\} = \frac{\sigma_0}{l\epsilon_0}(2 - 2\cos l\pi) \\ &= \left\{ \begin{array}{ll} 0, & \text{if } l \text{ is even} \\ 4\sigma_0/l\epsilon_0, & \text{if } l \text{ is odd} \end{array} \right\} \Rightarrow b_l = \left\{ \begin{array}{ll} 0, & \text{if } l \text{ is even} \\ 2\sigma_0/\pi\epsilon_0 l^2 R^{l-1}, & \text{if } l \text{ is odd} \end{array} \right\}. \end{aligned}$$

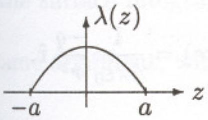
Conclusion:

$$\boxed{V(s,\phi) = \frac{2\sigma_0 R}{\pi\epsilon_0} \sum_{k=1,3,5,\ldots} \frac{1}{k^2} \sin k\phi \left\{ \begin{array}{ll} (s/R)^k & (s < R) \\ (R/s)^k & (s > R) \end{array} \right\}.}$$

Problem 3.40

Use Eq. 3.95, in the form $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{r^{n+1}} I_n; \quad I_n = \int_{-a}^{a} z^n \lambda(z)\, dz.$

(a) $I_0 = k\int_{-a}^{a} \cos\left(\frac{\pi z}{2a}\right) dz = k\left[\frac{2a}{\pi}\sin\left(\frac{\pi z}{2a}\right)\right]\Big|_{-a}^{a} = \frac{2ak}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{4ak}{\pi}$. Therefore:



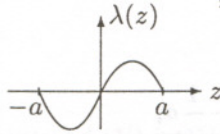
$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(\frac{4ak}{\pi}\right)\frac{1}{r}.} \quad \text{(Monopole.)}$$

(b)

$$I_0 = 0.$$

$$I_1 = k\int_{-a}^{a} z\sin(\pi z/a)\,dz = k\left\{\left(\frac{a}{\pi}\right)^2\sin\left(\frac{\pi z}{a}\right) - \frac{az}{\pi}\cos\left(\frac{\pi z}{a}\right)\right\}\Bigg|_{-a}^{a}$$

$$= k\left\{\left(\frac{a}{\pi}\right)^2[\sin(\pi)-\sin(-\pi)] - \frac{a^2}{\pi}\cos(\pi) - \frac{a^2}{\pi}\cos(-\pi)\right\} = k\frac{2a^2}{\pi};$$



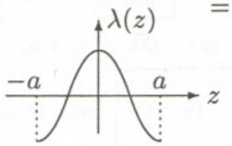
$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(\frac{2a^2k}{\pi}\right)\frac{1}{r^2}\cos\theta.} \quad \text{(Dipole.)}$$

(c)

$$I_0 = I_1 = 0.$$

$$I_2 = k\int_{-a}^{a} z^2\cos\left(\frac{\pi z}{a}\right)dz = k\left\{\frac{2z\cos(\pi z/a)}{(\pi/a)^2} + \frac{(\pi z/a)^2-2}{(\pi/a)^3}\sin\left(\frac{\pi z}{a}\right)\right\}\Bigg|_{-a}^{a}$$

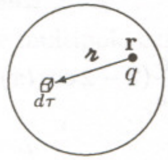
$$= 2k\left(\frac{a}{\pi}\right)^2[a\cos(\pi)+a\cos(-\pi)] = -\frac{4a^3k}{\pi^2}.$$



$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(-\frac{4a^3k}{\pi^2}\right)\frac{1}{2r^3}\left(3\cos^2\theta - 1\right).} \quad \text{(Quadrupole.)}$$

Problem 3.41

(a) The average field due to a point charge q at $\mathbf{r}$ is



$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi\epsilon_0 R^3\right)}\int \mathbf{E}\,d\tau, \quad \text{where } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}},$$

$$\text{so } \mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi\epsilon_0 R^3\right)}\frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\,d\tau.$$

(Here $\mathbf{r}$ is the source point, $d\tau$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $\mathbf{r}$ to $d\tau$.) The field at $\mathbf{r}$ due to uniform charge ρ over the sphere is $\mathbf{E}_s = \frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\,d\tau$. This time $d\tau$ is the source point and $\mathbf{r}$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $d\tau$ to $\mathbf{r}$, and hence carries the opposite sign. So with $\rho = -q/\left(\frac{4}{3}\pi R^3\right)$, the two expressions agree: $\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho$.

(b) From Prob. 2.12:

$$\mathbf{E}_\rho = \frac{1}{3\epsilon_0}\rho\,\mathbf{r} = -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{R^3} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}.$$

(c) If there are many charges inside the sphere, $\mathbf{E}_{\text{ave}}$ is the sum of the individual averages, and $\mathbf{p}_{\text{tot}}$ is the sum of the individual dipole moments. So $\mathbf{E}_{\text{ave}} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}$. qed

(d) The same argument, only with q placed at $\mathbf{r}$ *outside* the sphere, gives

$$\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{4}{3}\pi R^3\rho\right)}{r^2}\hat{\mathbf{r}} \quad \text{(field at } \mathbf{r} \text{ due to uniformly charged sphere)} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\hat{\mathbf{r}}.$$

But this is precisely the field produced by q (at $\mathbf{r}$) at the *center* of the sphere. So the average field (over the sphere) due to a point charge *outside* the sphere is the same as the field that same charge produces at the center. And by superposition, this holds for any *collection* of exterior charges.

Problem 3.42

(a)

$$\begin{aligned}
\mathbf{E}_{\text{dip}} &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})\right. \\
&\quad \left. + \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})\right] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left[3\sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + 3\sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \underbrace{(2\cos^2\theta - \sin^2\theta)}_{=3\cos^2\theta-1}\,\hat{\mathbf{z}}\right]. \\
\mathbf{E}_{\text{ave}} &= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{E}_{\text{dip}}\,d\tau \\
&= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)\int \frac{1}{r^3}\left[3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + (3\cos^2\theta - 1)\,\hat{\mathbf{z}}\right] r^2\sin\theta\,dr\,d\theta\,d\phi.
\end{aligned}$$

But $\int_0^{2\pi}\cos\phi\,d\phi = \int_0^{2\pi}\sin\phi\,d\phi = 0$, so the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms drop out, and $\int_0^{2\pi} d\phi = 2\pi$, so

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)2\pi\int_0^R \frac{1}{r}\,dr\;\underbrace{\int_0^\pi (3\cos^2\theta - 1)\sin\theta\,d\theta}_{(-\cos^3\theta+\cos\theta)|_0^\pi=1-1+1-1=0}.$$

Evidently $\boxed{\mathbf{E}_{\text{ave}} = 0,}$ which contradicts the result of Prob. 3.41. [Note, however, that the r integral, $\int_0^R \frac{1}{r}\,dr$, blows up, since $\ln r \to -\infty$ as $r \to 0$. If, as suggested, we truncate the r integral at $r = \epsilon$, then it is finite, and the θ integral gives $\mathbf{E}_{\text{ave}} = 0$.]

(b) We want $\mathbf{E}$ within the ϵ-sphere to be a delta function: $\mathbf{E} = \mathbf{A}\delta^3(\mathbf{r})$, with $\mathbf{A}$ selected so that the *average* field is consistent with the general theorem in Prob. 3.41:

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{A}\delta^3(\mathbf{r})\,d\tau = \frac{\mathbf{A}}{\left(\frac{4}{3}\pi R^3\right)} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3} \Rightarrow \mathbf{A} = -\frac{\mathbf{p}}{3\epsilon_0}, \text{ and hence } \boxed{\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}).}$$

Problem 3.43

(a) $I = \int(\nabla V_1)\cdot(\nabla V_2)\,d\tau$. But $\nabla\cdot(V_1\nabla V_2) = (\nabla V_1)\cdot(\nabla V_2) + V_1(\nabla^2 V_2)$, so

$$I = \int \nabla\cdot(V_1\nabla V_2)\,d\tau - \int V_1(\nabla^2 V_2) = \oint_S V_1(\nabla V_2)\cdot d\mathbf{a} + \frac{1}{\epsilon_0}\int V_1\rho_2\,d\tau.$$

But the surface integral is over a huge sphere "at infinity", where V_1 and $V_2 \to 0$. So $I = \frac{1}{\epsilon_0}\int V_1\rho_2\,d\tau$. By the same argument, with 1 and 2 reversed, $I = \frac{1}{\epsilon_0}\int V_2\rho_1\,d\tau$. So $\int V_1\rho_2\,d\tau = \int V_2\rho_1\,d\tau$. qed

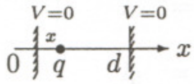
(b) $\left\{ \begin{array}{l} \textit{Situation (1)}: Q_a = \int_a \rho_1 \, d\tau = Q; \; Q_b = \int_b \rho_1 \, d\tau = 0; \; V_{1b} \equiv V_{ab}. \\ \textit{Situation (2)}: Q_a = \int_a \rho_2 \, d\tau = 0; \; Q_b = \int_b \rho_2 \, d\tau = Q; \; V_{2a} \equiv V_{ba}. \end{array} \right\}$

$$\left\{ \begin{array}{l} \int V_1 \rho_2 \, d\tau = V_{1a} \int_a \rho_2 \, d\tau + V_{1b} \int_b \rho_2 \, d\tau = V_{ab} Q. \\ \int V_2 \rho_1 \, d\tau = V_{2a} \int_a \rho_1 \, d\tau + V_{2b} \int_b \rho_1 \, d\tau = V_{ba} Q. \end{array} \right\}$$

Green's reciprocity theorem says $QV_{ab} = QV_{ba}$, so $V_{ab} = V_{ba}$. qed

Problem 3.44

(a) *Situation (1):* actual. *Situation (2):* right plate at V_0, left plate at $V = 0$, no charge at x.



$$\int V_1 \rho_2 \, d\tau = V_{l_1} Q_{l_2} + V_{x_1} Q_{x_2} + V_{r_1} Q_{r_2}.$$

But $V_{l_1} = V_{r_1} = 0$ and $Q_{x_2} = 0$, so $\int V_1 \rho_2 \, d\tau = 0$.

$$\int V_2 \rho_1 \, d\tau = V_{l_2} Q_{l_1} + V_{x_2} Q_{x_1} + V_{r_2} Q_{r_1}.$$

But $V_{l_2} = 0$ $Q_{x_1} = q$, $V_{r_2} = V_0$, $Q_{r_1} = Q_2$, and $V_{x_2} = V_0(x/d)$. So $0 = V_0(x/d)q + V_0 Q_2$, and hence

$$\boxed{Q_2 = -qx/d.}$$

Situation (1): actual. *Situation (2):* left plate at V_0, right plate at $V = 0$, no charge at x.

$$\int V_1 \rho_2 \, d\tau = 0 = \int V_2 \rho_1 \, d\tau = V_{l_2} Q_{l_1} + V_{x_2} Q_{x_1} + V_{r_2} Q_{r_1} = V_0 Q_1 + q V_{x_2} + 0.$$

But $V_{x_2} = V_0 \left(1 - \frac{x}{d}\right)$, so

$$\boxed{Q_1 = -q(1 - x/d).}$$

(b) *Situation (1):* actual. *Situation (2):* inner sphere at V_0, outer sphere at zero, no charge at r.

$$\int V_1 \rho_2 \, d\tau = V_{a_1} Q_{a_2} + V_{r_1} Q_{r_2} + V_{b_1} Q_{b_2}.$$

But $V_{a_1} = V_{b_1} = 0$, $Q_{r_2} = 0$. So $\int V_1 \rho_2 \, d\tau = 0$.

$$\int V_2 \rho_1 \, d\tau = V_{a_2} Q_{a_1} + V_{r_2} Q_{r_1} + V_{b_2} Q_{b_1} = Q_a V_0 + q V_{r_2} + 0.$$

But V_{r_2} is the potential at r in configuration 2: $V(r) = A + B/r$, with $V(a) = V_0 \Rightarrow A + B/a = V_0$, or $aA + B = aV_0$, and $V(b) = 0 \Rightarrow A + B/b = 0$, or $bA + B = 0$. Subtract: $(b-a)A = -aV_0 \Rightarrow A = -aV_0/(b-a)$; $B\left(\frac{1}{a} - \frac{1}{b}\right) = V_0 = B\frac{(b-a)}{ab} \Rightarrow B = abV_0/(b-a)$. So $V(r) = \frac{aV_0}{(b-a)}\left(\frac{b}{r} - 1\right)$. Therefore

$$Q_a V_0 + q\frac{aV_0}{(b-a)}\left(\frac{b}{r} - 1\right) = 0; \quad \boxed{Q_a = -\frac{qa}{(b-a)}\left(\frac{b}{r} - 1\right).}$$

Now let *Situation (2)* be: inner sphere at zero, outer at V_0, no charge at r.

$$\int V_1\rho_2\,d\tau = 0 = \int V_2\rho_1\,d\tau = V_{a_2}Q_{a_1} + V_{r_2}Q_{r_1} + V_{b_2}Q_{b_1} = 0 + qV_{r_2} + Q_bV_0.$$

This time isplaystyle$V(r) = A + \frac{B}{r}$ with $V(a) = 0 \Rightarrow A + B/a = 0$; $V(b) = V_0 \Rightarrow A + B/b = V_0$, so $V(r) = \frac{bV_0}{(b-a)}\left(1 - \frac{a}{r}\right)$. Therefore, $q\frac{bV_0}{(b-a)}\left(1-\frac{a}{r}\right) + Q_bV_0 = 0$; $\boxed{Q_b = -\frac{qb}{(b-a)}\left(1-\frac{a}{r}\right).}$

Problem 3.45

(a) $\frac{1}{2}\sum_{i,j=1}^{3}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_jQ_{ij} = \frac{1}{2}\int\left\{3\sum_{i=1}^{3}\hat{\mathbf{r}}_ir_i'\sum_{j=1}^{3}\hat{\mathbf{r}}_jr_j' - (r')^2\sum_{i,j}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\delta_{ij}\right\}\rho\,d\tau'$

But $\sum_{i=1}^{3}\hat{\mathbf{r}}_ir_i' = \hat{\mathbf{r}}\cdot\mathbf{r}' = r'\cos\theta' = \sum_{j=1}^{3}\hat{\mathbf{r}}_jr_j'$; $\sum_{i,j}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\delta_{ij} = \sum\hat{\mathbf{r}}_j\hat{\mathbf{r}}_j = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$. So

$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int\frac{1}{2}\left(r'^2\cos^2\theta' - r'^2\right)\rho\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r'^2P_2(\cos\theta')\rho\,d\tau'$ (the $n = 2$ term in Eq. 3.95).

(b) Because $x^2 = y^2 = (a/2)^2$ for all four charges, $Q_{xx} = Q_{yy} = \left[3(a/2)^2 - (\sqrt{2}a/2)^2\right](q - q - q + q) = 0$. Because $z = 0$ for all four charges, $Q_{zz} = -(\sqrt{2}a/2)^2(q - q - q + q) = 0$ and $Q_{xz} = Q_{yz} = Q_{zx} = Q_{zy} = 0$. This leaves only

$$Q_{xy} = Q_{yx} = 3\left[\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)q + \left(\frac{a}{2}\right)\left(-\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)q\right] = \boxed{3a^2q.}$$

(c)

$$\begin{aligned}
\overline{Q}_{ij} &= \int\left[3(r_i - d_i)(r_j - d_j) - (\mathbf{r} - \mathbf{d})^2\delta_{ij}\right]\rho\,d\tau \quad \text{(I'll drop the primes, for simplicity.)}\\
&= \int\left[3r_ir_j - r^2\delta_{ij}\right]\rho\,d\tau - 3d_i\int r_j\rho\,d\tau - 3d_j\int r_i\rho\,d\tau + 3d_id_j\int\rho\,d\tau + 2\mathbf{d}\cdot\int\mathbf{r}\rho\,d\tau\,\delta_{ij}\\
&\quad - d^2\delta_{ij}\int\rho\,d\tau = Q_{ij} - 3(d_ip_j + d_jp_i) + 3d_id_jQ + 2\delta_{ij}\mathbf{d}\cdot\mathbf{p} - d^2\delta_{ij}Q.
\end{aligned}$$

So if $\mathbf{p} = 0$ and $Q = 0$ then $\overline{Q}_{ij} = Q_{ij}$. qed

(d) Eq. 3.95 with $n = 3$:

$$V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int(r')^3P_3(\cos\theta')\rho\,d\tau'; \quad P_3(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right).$$

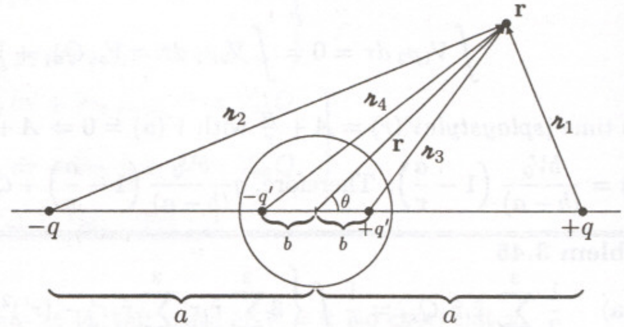
$$\boxed{V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{1}{2}\sum_{i,j,k}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\hat{\mathbf{r}}_kQ_{ijk}\right)}{r^4}},$$

Define the "octopole moment" as

$$\boxed{Q_{ijk} \equiv \int\left(5r_i'r_j'r_k' - (r')^2(r_i'\delta_{jk} + r_j'\delta_{ik} + r_k'\delta_{ij})\right)\rho(\mathbf{r}')\,d\tau'.}$$

Problem 3.46

$$V = \frac{1}{4\pi\epsilon_0}\left\{q\left(\frac{1}{\mathscr{r}_1} - \frac{1}{\mathscr{r}_2}\right) + q'\left(\frac{1}{\mathscr{r}_3} - \frac{1}{\mathscr{r}_4}\right)\right\}$$



$$\begin{aligned}
\mathscr{r}_1 &= \sqrt{r^2 + a^2 - 2ra\cos\theta},\\
\mathscr{r}_2 &= \sqrt{r^2 + a^2 + 2ra\cos\theta},\\
\mathscr{r}_3 &= \sqrt{r^2 + b^2 - 2rb\cos\theta},\\
\mathscr{r}_4 &= \sqrt{r^2 + b^2 + 2rb\cos\theta}.
\end{aligned}$$

Expanding as in Ex. 3.10: $\left(\frac{1}{\mathscr{r}_1} - \frac{1}{\mathscr{r}_2}\right) \cong \frac{2r}{a^2}\cos\theta$ (we want $a \gg r$, not $r \gg a$, this time).

$$\begin{aligned}
\left(\frac{1}{\mathscr{r}_3} - \frac{1}{\mathscr{r}_4}\right) &\cong \frac{2b}{r^2}\cos\theta \text{ (here we want } b \ll r, \text{ because } b = R^2/a, \text{ Eq. 3.16)}\\
&= \frac{2}{a}\frac{R^2}{r^2}\cos\theta.
\end{aligned}$$

But $q' = -\frac{R}{a}q$ (Eq. 3.15), so

$$V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left[q\frac{2r}{a^2}\cos\theta - \frac{R}{a}q\frac{2}{a}\frac{R^2}{r^2}\cos\theta\right] = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{a^2}\right)\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

Set $E_0 = -\frac{1}{4\pi\epsilon_0}\frac{2q}{a^2}$ (field in the vicinity of the sphere produced by $\pm q$):

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta} \quad \text{(agrees with Eq. 3.76).}$$

Problem 3.47

The boundary conditions are

$$\left.\begin{array}{ll}
\text{(i)} & V = 0 \text{ when } y = 0,\\
\text{(ii)} & V = V_0 \text{ when } y = a,\\
\text{(iii)} & V = 0 \text{ when } x = b,\\
\text{(iv)} & V = 0 \text{ when } x = -b.
\end{array}\right\}$$

Go back to Eq. 3.26 and examine the case $k = 0$: $d^2X/dx^2 = d^2Y/dy^2 = 0$, so $X(x) = Ax + B$, $Y(y) = Cy + D$. But this configuration is symmetric in x, so $A = 0$, and hence the $k = 0$ solution is $V(x,y) = Cy + D$. Pick $D = 0$, $C = V_0/a$, and subtract off this part:

$$V(x,y) = V_0\frac{y}{a} + \bar{V}(x,y).$$

The *remainder* ($\bar{V}(x,y)$) satisfies boundary conditions similar to Ex. 3.4:

$$\left.\begin{array}{ll}
\text{(i)} & \bar{V} = 0 \text{ when } y = 0,\\
\text{(ii)} & \bar{V} = 0 \text{ when } y = a,\\
\text{(iii)} & \bar{V} = -V_0(y/a) \text{ when } x = b,\\
\text{(iv)} & \bar{V} = -V_0(y/a) \text{ when } x = -b.
\end{array}\right\}$$

(The point of peeling off $V_0(y/a)$ was to recover (ii), on which the constraint $k = n\pi/a$ depends.)
The solution (following Ex. 3.4) is

$$\bar{V}(x,y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\sin(n\pi y/a),$$

and it remains to fit condition (iii):

$$\bar{V}(b,y) = \sum C_n \cosh(n\pi b/a)\sin(n\pi y/a) = -V_0(y/a).$$

Invoke Fourier's trick:

$$\sum C_n \cosh(n\pi b/a)\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = -\frac{V_0}{a}\int_0^a y\sin(n'\pi y/a)\,dy,$$

$$\frac{a}{2}C_n\cosh(n\pi b/a) = -\frac{V_0}{a}\int_0^a y\sin(n\pi y/a)\,dy.$$

$$\begin{aligned} C_n &= -\frac{2V_0}{a^2\cosh(n\pi b/a)}\left[\left(\frac{a}{n\pi}\right)^2\sin(n\pi y/a) - \left(\frac{ay}{n\pi}\right)\cos(n\pi y/a)\right]\Bigg|_0^a \\ &= \frac{2V_0}{a^2\cosh(n\pi b/a)}\left(\frac{a^2}{n\pi}\right)\cos(n\pi) = \frac{2V_0}{n\pi}\frac{(-1)^n}{\cosh(n\pi b/a)}. \end{aligned}$$

$$V(x,y) = \boxed{V_0\left[\frac{y}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)\right].}$$

Problem 3.48

(a) Using Prob. 3.14b (with $b = a$):

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n\text{ odd}}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi)}.$$

$$\begin{aligned} \sigma(y) &= -\epsilon_0\frac{\partial V}{\partial x}\bigg|_{x=0} = -\epsilon_0\frac{4V_0}{\pi}\sum_{n\text{ odd}}\left(\frac{n\pi}{a}\right)\frac{\cosh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi)}\bigg|_{x=0} \\ &= -\frac{4\epsilon_0 V_0}{a}\sum_{n\text{ odd}}\frac{\sin(n\pi y/a)}{\sinh(n\pi)}. \\ \lambda &= \int_0^a \sigma(y)\,dy = -\frac{4\epsilon_0 V_0}{a}\sum_{n\text{ odd}}\frac{1}{\sinh(n\pi)}\int_0^a \sin(n\pi y/a)\,dy. \end{aligned}$$

$$\text{But } \int_0^a \sin(n\pi y/a)\,dy = -\frac{a}{n\pi}\cos(n\pi y/a)\Big|_0^a = \frac{a}{n\pi}[1-\cos(n\pi)] = \frac{2a}{n\pi}(\text{since } n \text{ is odd}).$$

$$= -\frac{8\epsilon_0 V_0}{\pi}\sum_{n\text{ odd}}\frac{1}{n\sinh(n\pi)} = \boxed{-\frac{\epsilon_0 V_0}{\pi}\ln 2.}$$

[I have not found a way to sum this series analytically. Mathematica gives the numerical value 0.0866434, which agrees precisely with $\ln 2/8$.]

Using Prob. 3.47 (with $b = a/2$):

$$V(x,y) = V_0 \left[\frac{y}{a} + \frac{2}{\pi} \sum_n \frac{(-1)^n \cosh(n\pi x/a) \sin(n\pi y/a)}{n \cosh(n\pi/2)} \right].$$

$$\begin{aligned}
\sigma(x) &= -\epsilon_0 \frac{\partial V}{\partial y}\Big|_{y=0} = -\epsilon_0 V_0 \left[\frac{1}{a} + \frac{2}{\pi} \sum_n \left(\frac{n\pi}{a}\right) \frac{(-1)^n \cosh(n\pi x/a) \cos(n\pi y/a)}{n \cosh(n\pi/2)} \right]\Big|_{y=0} \\
&= -\epsilon_0 V_0 \left[\frac{1}{a} + \frac{2}{a} \sum_n \frac{(-1)^n \cosh(n\pi x/a)}{\cosh(n\pi/2)} \right] = -\frac{\epsilon_0 V_0}{a} \left[1 + 2 \sum_n \frac{(-1)^n \cosh(n\pi x/a)}{\cosh(n\pi/2)} \right]. \\
\lambda &= \int_{-a/2}^{a/2} \sigma(x)\, dx = -\frac{\epsilon_0 V_0}{a} \left[a + 2 \sum_n \frac{(-1)^n}{\cosh(n\pi/2)} \int_{-a/2}^{a/2} \cosh(n\pi x/a)\, dx \right].
\end{aligned}$$

$$\text{But } \int_{-a/2}^{a/2} \cosh(n\pi x/a)\, dx = \frac{a}{n\pi} \sinh(n\pi x/a)\Big|_{-a/2}^{a/2} = \frac{2a}{n\pi} \sinh(n\pi/2).$$

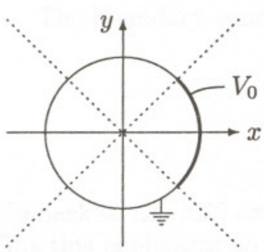
$$\begin{aligned}
&= -\frac{\epsilon_0 V_0}{a} \left[a + \frac{4a}{\pi} \sum_n \frac{(-1)^n \tanh(n\pi/2)}{n} \right] = -\epsilon_0 V_0 \left[1 + \frac{4}{\pi} \sum_n \frac{(-1)^n \tanh(n\pi/2)}{n} \right] \\
&= \boxed{-\frac{\epsilon_0 V_0}{\pi} \ln 2.}
\end{aligned}$$

[Again, I have not found a way to sum this series analytically. The numerical value is -0.612111, which agrees with the expected value $(\ln 2 - \pi)/4$.]

(b) From Prob. 3.23:

$$V(s,\phi) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty} \left(a_k s^k + b_k \frac{1}{s^k} \right) [c_k \cos(k\phi) + d_k \sin(k\phi)].$$

In the interior ($s < R$) b_0 and b_k must be zero ($\ln s$ and $1/s$ blow up at the origin). Symmetry $\Rightarrow d_k = 0$. So



$$V(s,\phi) = a_0 + \sum_{k=1}^{\infty} a_k s^k \cos(k\phi).$$

At the surface:

$$V(R,\phi) = \sum_{k=0} a_k R^k \cos(k\phi) = \begin{cases} V_0, & \text{if } -\pi/4 < \phi < \pi/4, \\ 0, & \text{otherwise.} \end{cases}$$

Fourier's trick: multiply by $\cos(k'\phi)$ and integrate from $-\pi$ to π:

$$\sum_{k=0}^{\infty} a_k R^k \int_{-\pi}^{\pi} \cos(k\phi)\cos(k'\phi)\, d\phi = V_0 \int_{-\pi/4}^{\pi/4} \cos(k'\phi)\, d\phi = \begin{cases} V_0 \sin(k'\phi)/k' \Big|_{-\pi/4}^{\pi/4} = (V_0/k')\sin(k'\pi/4), & \text{if } k' \neq 0, \\ V_0 \pi/2, \text{ if } k' = 0. \end{cases}$$

But

$$\int_{-\pi}^{\pi} \cos(k\phi)\cos(k'\phi)\, d\phi = \begin{cases} 0, & \text{if } k \neq k' \\ 2\pi, & \text{if } k = k' = 0, \\ \pi, & \text{if } k = k' \neq 0. \end{cases}$$

So $2\pi a_0 = V_0\pi/2 \Rightarrow a_0 = V_0/4$; $\pi a_k R^k = (2V_0/k)\sin(k\pi/4) \Rightarrow a_k = (2V_0/\pi k R^k)\sin(k\pi/4)$ $(k \neq 0)$; hence

$$V(s,\phi) = V_0\left[\frac{1}{4} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)}{k}\left(\frac{s}{R}\right)^k\cos(k\phi)\right].$$

Using Eq. 2.49, and noting that in this case $\hat{\mathbf{n}} = -\hat{\mathbf{s}}$:

$$\sigma(\phi) = \epsilon_0\frac{\partial V}{\partial s}\bigg|_{s=R} = \epsilon_0 V_0\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)}{kR^k}ks^{k-1}\cos(k\phi)\bigg|_{s=R} = \frac{2\epsilon_0 V_0}{\pi R}\sum_{k=1}^{\infty}\sin(k\pi/4)\cos(k\phi).$$

We want the net (line) charge on the segment opposite to V_0 ($-\pi < \phi < -3\pi/4$ and $3\pi/4 < \phi < \pi$):

$$\begin{aligned}\lambda &= \int\sigma(\phi)R\,d\phi = 2R\int_{3\pi/4}^{\pi}\sigma(\phi)\,d\phi = \frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\sin(k\pi/4)\int_{3\pi/4}^{\pi}\cos(k\phi)\,d\phi \\ &= \frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\sin(k\pi/4)\left[\frac{\sin(k\phi)}{k}\bigg|_{3\pi/4}^{\pi}\right] = -\frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)\sin(3k\pi/4)}{k}.\end{aligned}$$

k	$\sin(k\pi/4)$	$\sin(3k\pi/4)$	product
1	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2
2	1	-1	-1
3	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2
4	0	0	0
5	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1/2
6	-1	1	-1
7	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1/2
8	0	0	0

$$\lambda = -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2}\sum_{1,3,5\ldots}\frac{1}{k} - \sum_{2,6,10,\ldots}\frac{1}{k}\right] = -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2}\sum_{1,3,5\ldots}\frac{1}{k} - \frac{1}{2}\sum_{1,3,5,\ldots}\frac{1}{k}\right] = 0.$$

Ouch! What went wrong? The problem is that the series $\sum(1/k)$ is divergent, so the "subtraction" $\infty - \infty$ is suspect. One way to avoid this is to go back to $V(s,\phi)$, calculate $\epsilon_0(\partial V/\partial s)$ at $s \neq R$, and save the limit

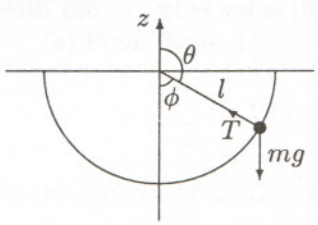
$s \to R$ until the end:

$$\begin{aligned}
\sigma(\phi, s) &\equiv \epsilon_0 \frac{\partial V}{\partial s} = \frac{2\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi/4)}{k} \frac{k s^{k-1}}{R^k} \cos(k\phi) \\
&= \frac{2\epsilon_0 V_0}{\pi R} \sum_{k=1}^{\infty} x^{k-1} \sin(k\pi/4)\cos(k\phi) \quad \text{(where } x \equiv s/R \to 1 \text{ at the end)}. \\
\lambda(x) &\equiv \sigma(\phi, s) R \, d\phi = -\frac{4\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} x^{k-1} \sin(k\pi/4)\sin(3k\pi/4) \\
&= -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2x}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) - \frac{1}{x}\left(\frac{x^2}{2} + \frac{x^6}{6} + \frac{x^{10}}{10} + \cdots\right)\right] \\
&= -\frac{2\epsilon_0 V_0}{\pi x}\left[\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) - \left(x^2 + \frac{x^6}{3} + \frac{x^{10}}{5} + \cdots\right)\right].
\end{aligned}$$

$$\text{But (see math tables)} : \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

$$\begin{aligned}
&= -\frac{2\epsilon_0 V_0}{\pi x}\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+x^2}{1-x^2}\right)\right] = -\frac{\epsilon_0 V_0}{\pi x}\ln\left[\left(\frac{1+x}{1-x}\right)\left(\frac{1+x^2}{1-x^2}\right)\right] \\
&= -\frac{\epsilon_0 V_0}{\pi x}\ln\left[\frac{(1+x)^2}{1+x^2}\right]; \quad \lambda = \lim_{x\to 1}\lambda(x) = \boxed{\frac{-\epsilon_0 V_0}{\pi}\ln 2.}
\end{aligned}$$

Problem 3.49



$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

Now consider the pendulum: $\mathbf{F} = -mg\,\hat{\mathbf{z}} - T\,\hat{\mathbf{r}}$, where $T - mg\cos\phi = mv^2/l$ and (by conservation of energy) $mgl\cos\phi = (1/2)mv^2 \Rightarrow v^2 = 2gl\cos\phi$ (assuming it started from rest at $\phi = 90°$, as stipulated). But $\cos\phi = -\cos\theta$, so $T = mg(-\cos\theta) + (m/l)(-2gl\cos\theta) = -3mg\cos\theta$, and hence

$$\mathbf{F} = -mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) + 3mg\cos\theta\,\hat{\mathbf{r}} = mg(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

This total force is such as to keep the pendulum on a circular arc, and it is identical to the force on q in the field of a dipole, with $mg \leftrightarrow qp/4\pi\epsilon_0 l^3$. Evidently q also executes semicircular motion, as though it were on a tether of fixed length l.

Chapter 4

Electrostatic Fields in Matter

Problem 4.1

$E = V/x = 500/10^{-3} = 5\times10^5$. Table 4.1: $\alpha/4\pi\epsilon_0 = 0.66\times10^{-30}$, so $\alpha = 4\pi(8.85\times10^{-12})(0.66\times10^{-30}) = 7.34\times10^{-41}$. $p = \alpha E = ed \Rightarrow d = \alpha E/e = (7.34\times10^{-41})(5\times10^5)/(1.6\times10^{-19}) = 2.29\times10^{-16}$ m. $d/R = (2.29\times10^{-16})/(0.5\times10^{-10}) = \boxed{4.6\times10^{-6}.}$ To ionize, say $d = R$. Then $R = \alpha E/e = \alpha V/ex \Rightarrow V = Rex/\alpha = (0.5\times10^{-10})(1.6\times10^{-19})(10^{-3})/(7.34\times10^{-41}) = \boxed{10^8\text{ V.}}$

Problem 4.2

First find the field, at radius r, using Gauss' law: $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}}$, or $E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}Q_{\text{enc}}$.

$$\begin{aligned} Q_{\text{enc}} &= \int_0^r \rho\, d\tau = \frac{4\pi q}{\pi a^3}\int_0^r e^{-2\bar{r}/a}\bar{r}^2\, d\bar{r} = \frac{4q}{a^3}\left[-\frac{a}{2}e^{-2\bar{r}/a}\left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2}\right)\right]\Bigg|_0^r \\ &= -\frac{2q}{a^2}\left[e^{-2r/a}\left(r^2 + ar + \frac{a^2}{2}\right) - \frac{a^2}{2}\right] = q\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]. \end{aligned}$$

[*Note:* $Q_{\text{enc}}(r\to\infty) = q$.] So the field of the electron cloud is $E_e = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$. The proton will be shifted from $r = 0$ to the point d where $E_e = E$ (the external field):

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right].$$

Expanding in powers of (d/a):

$$\begin{aligned} e^{-2d/a} &= 1 - \left(\frac{2d}{a}\right) + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{3!}\left(\frac{2d}{a}\right)^3 + \cdots = 1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots \\ 1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) &= 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) \\ &= \cancel{1} - \cancel{1} - \cancel{2\frac{d}{a}} - \cancel{2\frac{d^2}{a^2}} + \cancel{2\frac{d}{a}} + \cancel{4\frac{d^2}{a^2}} + \cancel{4\frac{d^3}{a^3}} - \cancel{2\frac{d^2}{a^2}} - \cancel{4\frac{d^3}{a^3}} + \frac{4}{3}\frac{d^3}{a^3} + \cdots \\ &= \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{higher order terms.} \end{aligned}$$

$$E=\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right)=\frac{1}{4\pi\epsilon_0}\frac{4}{3a^3}(qd)=\frac{1}{3\pi\epsilon_0 a^3}p. \quad \boxed{\alpha=3\pi\epsilon_0 a^3.}$$

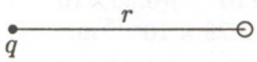
[Not so different from the *uniform* sphere model of Ex. 4.1 (see Eq. 4.2). Note that this result predicts $\frac{1}{4\pi\epsilon_0}\alpha=\frac{3}{4}a^3=\frac{3}{4}\left(0.5\times10^{-10}\right)^3=0.09\times10^{-30}\,\mathrm{m}^3$, compared with an experimental value (Table 4.1) of $0.66\times10^{-30}\,\mathrm{m}^3$. Ironically the "classical" formula (Eq. 4.2) is slightly *closer* to the empirical value.]

Problem 4.3

$\rho(r)=Ar$. Electric field (by Gauss's Law): $\oint \mathbf{E}\cdot d\mathbf{a}=E\left(4\pi r^2\right)=\frac{1}{\epsilon_0}Q_{\rm enc}=\frac{1}{\epsilon_0}\int_0^r A\bar{r}\,4\pi\bar{r}^2\,d\bar{r}$, or $E=\dfrac{1}{4\pi r^2}\dfrac{4\pi A}{\epsilon_0}\dfrac{r^4}{4}=\dfrac{Ar^2}{4\epsilon_0}$. This "internal" field balances the external field $\mathbf{E}$ when nucleus is "off-center" an amount d: $ad^2/4\epsilon_0=E \Rightarrow d=\sqrt{4\epsilon_0 E/A}$. So the induced dipole moment is $p=ed=2e\sqrt{\epsilon_0/A}\sqrt{E}$. Evidently $\boxed{p \text{ is proportional to } E^{1/2}.}$

For Eq. 4.1 to hold in the weak-field limit, *E must be proportional to r*, for small r, which means that ρ must go to a constant (not zero) at the origin: $\boxed{\rho(0)\neq 0}$ (nor infinite).

Problem 4.4



Field of q: $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$. Induced dipole moment of atom: $\mathbf{p}=\alpha\mathbf{E}=\frac{\alpha q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$.

Field of this dipole, at location of q ($\theta=\pi$, in Eq. 3.103): $E=\dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left(\dfrac{2\alpha q}{4\pi\epsilon_0 r^2}\right)$ (to the right).

Force on q due to this field: $\boxed{F=2\alpha\left(\dfrac{q}{4\pi\epsilon_0}\right)^2\dfrac{1}{r^3}}$ (attractive).

Problem 4.5

Field of $\mathbf{p}_1$ at $\mathbf{p}_2$ ($\theta=\pi/2$ in Eq. 3.103): $\mathbf{E}_1=\dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}}$ (points *down*).

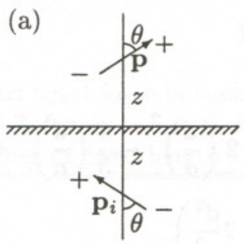
Torque on $\mathbf{p}_2$: $\mathbf{N}_2=\mathbf{p}_2\times\mathbf{E}_1=p_2E_1\sin 90^\circ=p_2E_1=\boxed{\dfrac{p_1p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Field of $\mathbf{p}_2$ at $\mathbf{p}_1$ ($\theta=\pi$ in Eq. 3.103): $\mathbf{E}_2=\dfrac{p_2}{4\pi\epsilon_0 r^3}(-2\,\hat{\mathbf{r}})$ (points to the *right*).

Torque on $\mathbf{p}_1$: $\mathbf{N}_1=\mathbf{p}_1\times\mathbf{E}_2=\boxed{\dfrac{2p_1p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Problem 4.6

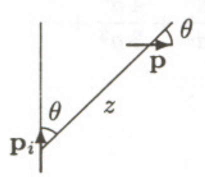
(a)



Use image dipole as shown in Fig. (a). Redraw, placing $\mathbf{p}_i$ at the origin, Fig. (b).

$$\mathbf{E}_i=\frac{p}{4\pi\epsilon_0(2z)^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}});\qquad \mathbf{p}=p\cos\theta\,\hat{\mathbf{r}}+p\sin\theta\,\hat{\boldsymbol{\theta}}.$$

(b)

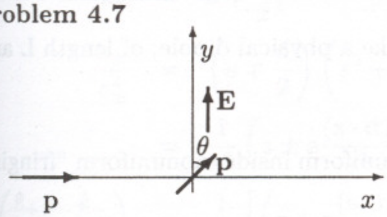


$$\begin{aligned}\mathbf{N}&=\mathbf{p}\times\mathbf{E}_i=\frac{p^2}{4\pi\epsilon_0(2z)^3}\left[(\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})\times(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})\right]\\&=\frac{p^2}{4\pi\epsilon_0(2z)^3}\left[\cos\theta\sin\theta\,\hat{\boldsymbol{\phi}}+2\sin\theta\cos\theta(-\hat{\boldsymbol{\phi}})\right]\\&=\frac{p^2\sin\theta\cos\theta}{4\pi\epsilon_0(2z)^3}(-\hat{\boldsymbol{\phi}})\ \text{(out of the page).}\end{aligned}$$

But $\sin\theta\cos\theta = (1/2)\sin 2\theta$, so $\boxed{N = \dfrac{p^2 \sin 2\theta}{4\pi\epsilon_0(16z^3)}}$ (out of the page).

For $0 < \theta < \pi/2$, $\mathbf{N}$ tends to rotate $\mathbf{p}$ counterclockwise; for $\pi/2 < \theta < \pi$, $\mathbf{N}$ rotates $\mathbf{p}$ clockwise. Thus the stable orientation is perpendicular to the surface—either ↑ or ↓.

Problem 4.7



Say the field is uniform and points in the y direction. First slide $\mathbf{p}$ in from infinity along the x axis—this takes no work, since $\mathbf{F}$ is $\perp d\mathbf{l}$. (If $\mathbf{E}$ is *not* uniform, slide $\mathbf{p}$ in along a trajectory $\perp$ the field.) Now rotate (counterclockwise) into final position. The torque exerted by $\mathbf{E}$ is $\mathbf{N} = \mathbf{p}\times\mathbf{E} = pE\sin\theta\,\hat{\mathbf{z}}$. The torque *we* exert is $N = pE\sin\theta$ *clockwise*, and $d\theta$ is *counterclockwise*, so the net work done by us is *negative*:

$U = \int_{\pi/2}^{\theta} pE\sin\bar\theta\, d\bar\theta = pE\left(-\cos\bar\theta\right)\Big|_{\pi/2}^{\theta} = -pE\left(\cos\theta - \cos\frac{\pi}{2}\right) = -pE\cos\theta = -\mathbf{p}\cdot\mathbf{E}.$ qed

Problem 4.8

$U = -\mathbf{p}_1\cdot\mathbf{E}_2$, but $\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}_2\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_2\right]$. So $U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{\mathbf{r}})(\mathbf{p}_2\cdot\hat{\mathbf{r}})\right]$. qed

Problem 4.9

(a) $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$ (Eq. 4.5); $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\,\hat{\mathbf{r}} = \dfrac{q}{4\pi\epsilon_0}\dfrac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$.

$$
\begin{aligned}
F_x &= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{p_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right] + p_y\left[-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right]\right. \\
&\quad \left. + p_z\left[-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right]\right\} = \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x .
\end{aligned}
$$

$$
\mathbf{F} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\right].}
$$

(b) $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left\{3\left[\mathbf{p}\cdot(-\hat{\mathbf{r}})\right](-\hat{\mathbf{r}}) - \mathbf{p}\right\} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right]$. (This is from Eq. 3.104; the minus signs are because $\mathbf{r}$ points *toward* $\mathbf{p}$, in this problem.)

$$
\mathbf{F} = q\mathbf{E} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right].}
$$

[Note that the forces are equal and opposite, as you would expect from Newton's third law.]

Problem 4.10

(a) $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \boxed{kR;}$ $\rho_b = -\nabla\cdot\mathbf{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 kr) = -\dfrac{1}{r^2}3kr^2 = \boxed{-3k.}$

(b) For $r < R$, $\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\,\hat{\mathbf{r}}$ (Prob. 2.12), so $\mathbf{E} = \boxed{-(k/\epsilon_0)\,\mathbf{r}.}$

For $r > R$, same as if all charge at center; but $Q_{\text{tot}} = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3) = 0$, so $\boxed{\mathbf{E} = 0.}$

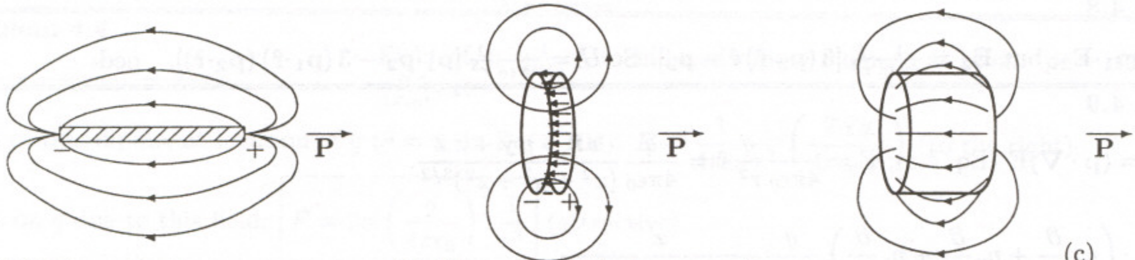
Problem 4.11

$\rho_b = 0$; $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \pm P$ (plus sign at one end—the one $\mathbf{P}$ points *toward*; minus sign at the other—the one $\mathbf{P}$ points *away* from).

(i) $L \gg a$. Then the ends look like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$. See Fig. (a).

(ii) $L \ll a$. Then it's like a circular parallel-plate capacitor. Field is nearly uniform inside; nonuniform "fringing field" at the edges. See Fig. (b).

(iii) $L \approx a$. See Fig. (c).



(a) Like a dipole (b) Like a parallel-plate capacitor (c)

Problem 4.12

$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau = \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau\right\}$. But the term in curly brackets is precisely the *field* of a uniformly *charged* sphere, divided by ρ. The integral was done explicitly in Prob. 2.7 and 2.8:

$$\frac{1}{4\pi\epsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau = \frac{1}{\rho}\left\{\begin{array}{ll}\frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{r^2}\,\hat{\mathbf{r}}, & (r > R),\\ \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{R^3}\,\mathbf{r}, & (r < R).\end{array}\right\}\quad \text{So } V(r,\theta) = \left\{\begin{array}{ll}\frac{R^3}{3\epsilon_0 r^2}\mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{R^3 P\cos\theta}{3\epsilon_0 r^2}}, & (r > R),\\ \frac{1}{3\epsilon_0}\mathbf{P}\cdot\mathbf{r} = \boxed{\frac{Pr\cos\theta}{3\epsilon_0}}, & (r < R).\end{array}\right\}$$

Problem 4.13

Think of it as two cylinders of opposite uniform charge density $\pm\rho$. *Inside,* the field at a distance s from the axis of a uniformly charge cylinder is given by Gauss's law: $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E} = (\rho/2\epsilon_0)\mathbf{s}$. For *two* such cylinders, one plus and one minus, the net field (inside) is $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = (\rho/2\epsilon_0)(\mathbf{s}_+ - \mathbf{s}_-)$. But $\mathbf{s}_+ - \mathbf{s}_- = -\mathbf{d}$, so $\mathbf{E} = \boxed{-\rho\mathbf{d}/(2\epsilon_0)},$ where $\mathbf{d}$ is the vector from the negative axis to positive axis. In this case the total dipole moment of a chunk of length ℓ is $\mathbf{P}\left(\pi a^2\ell\right) = \left(\rho\pi a^2\ell\right)\mathbf{d}$. So $\rho\mathbf{d} = \mathbf{P}$, and $\boxed{\mathbf{E} = -\mathbf{P}/(2\epsilon_0)},$ for $s < a$.

Outside, Gauss's law gives $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi a^2\ell \Rightarrow \mathbf{E} = \frac{\rho a^2}{2\epsilon_0}\frac{\hat{\mathbf{s}}}{s}$, for *one* cylinder. For the combination, $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right)$, where

$$\begin{aligned}
\mathbf{s}_\pm &= \mathbf{s} \mp \frac{\mathbf{d}}{2};\\
\frac{\mathbf{s}_\pm}{s_\pm^2} &= \left(\mathbf{s}\mp\frac{\mathbf{d}}{2}\right)\left(s^2+\frac{d^2}{4}\mp\mathbf{s}\cdot\mathbf{d}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s}\mp\frac{\mathbf{d}}{2}\right)\left(1\mp\frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s}\mp\frac{\mathbf{d}}{2}\right)\left(1\pm\frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)\\
&= \frac{1}{s^2}\left(\mathbf{s}\pm\mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2}\mp\frac{\mathbf{d}}{2}\right) \text{ (keeping only 1st order terms in } \mathbf{d}).\\
\left(\frac{\hat{\mathbf{s}}_+}{s_+}-\frac{\hat{\mathbf{s}}_-}{s_-}\right) &= \frac{1}{s^2}\left[\left(\mathbf{s}+\mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2}-\frac{\mathbf{d}}{2}\right)-\left(\mathbf{s}-\mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2}+\frac{\mathbf{s}}{2}\right)\right] = \frac{1}{s^2}\left(2\frac{\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^2}-\mathbf{d}\right).
\end{aligned}$$

$$\boxed{\mathbf{E}(\mathbf{s}) = \frac{a^2}{2\epsilon_0}\frac{1}{s^2}\left[2(\mathbf{P}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}}-\mathbf{P}\right],} \quad \text{for } s > a.$$

Problem 4.14

Total charge on the dielectric is $Q_{\text{tot}} = \oint_S \sigma_b\, da + \int_{\mathcal{V}} \rho_b\, d\tau = \oint_S \mathbf{P}\cdot d\mathbf{a} - \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{P}\, d\tau$. But the divergence theorem says $\oint_S \mathbf{P}\cdot d\mathbf{a} = \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{P}\, d\tau$, so $Q_{\text{enc}} = 0$. qed

Problem 4.15

(a) $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \left\{\begin{array}{ll} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r=b),\\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r=a).\end{array}\right\}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\,\hat{\mathbf{r}}$. For $r < a$, $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E}=0.}$ For $r > b$, $Q_{\text{enc}} = 0$ (Prob. 4.14), so $\boxed{\mathbf{E}=0.}$

For $a < r < b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right)4\pi\bar{r}^2 d\bar{r} = -4\pi ka - 4\pi k(r-a) = -4\pi kr$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = 0$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0 \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so

$\boxed{\mathbf{E} = 0 \text{ (for } r<a \text{ and } r>b);}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a<r<b).}$

Problem 4.16

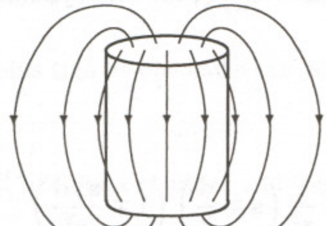
(a) Same as $\mathbf{E}_0$ minus the field at the center of a sphere with uniform polarization $\mathbf{P}$. The latter (Eq. 4.14) is $-\mathbf{P}/3\epsilon_0$. So $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}.}$ $\quad \mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}.}$

(b) Same as $\mathbf{E}_0$ minus the field of $\pm$ charges at the two ends of the "needle"—but these are small, and far away, so $\boxed{\mathbf{E} = \mathbf{E}_0.}$ $\quad \mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \mathbf{P}.}$

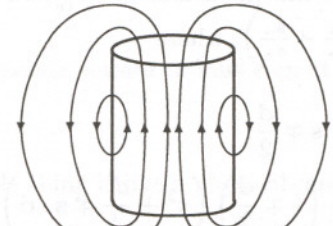
(c) Same as $\mathbf{E}_0$ minus the field of a parallel-plate capacitor with upper plate at $\sigma = P$. The latter is $-(1/\epsilon_0)P$, so $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{\epsilon_0}\mathbf{P}.}$ $\quad \mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0.}$

Problem 4.17

P
(uniform)



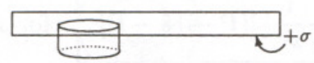
E
(field of two circular plates)



D
(same as **E** *outside*, but lines *continuous*, since $\nabla \cdot \mathbf{D} = 0$)

Problem 4.18

(a) Apply $\int \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$ to the gaussian surface shown. $DA = \sigma A \Rightarrow \boxed{D = \sigma.}$ (*Note:* $\mathbf{D} = 0$ inside the metal plate.) This is true in both slabs; $\mathbf{D}$ points *down*.



(b) $\mathbf{D} = \epsilon \mathbf{E} \Rightarrow E = \sigma/\epsilon_1$ in slab 1, $E = \sigma/\epsilon_2$ in slab 2. But $\epsilon = \epsilon_0 \epsilon_r$, so $\epsilon_1 = 2\epsilon_0$; $\epsilon_2 = \frac{3}{2}\epsilon_0$. $\boxed{E_1 = \sigma/2\epsilon_0,}$ $\boxed{E_2 = 2\sigma/3\epsilon_0.}$

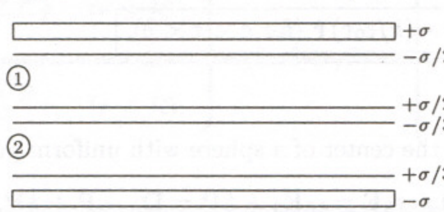
(c) $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, so $P = \epsilon_0 \chi_e d/(\epsilon_0 \epsilon_r) = (\chi_e/\epsilon_r)\sigma$; $\chi_e = \epsilon_r - 1 \Rightarrow P = (1 - \epsilon_r^{-1})\sigma$. $\boxed{P_1 = \sigma/2,}$ $\boxed{P_2 = \sigma/3.}$

(d) $V = E_1 a + E_2 a = (\sigma a/6\epsilon_0)(3 + 4) = \boxed{7\sigma a/6\epsilon_0.}$

(e) $\rho_b = 0$; $\boxed{\begin{array}{l} \sigma_b = +P_1 \text{ at bottom of slab (1)} = \sigma/2, \\ \sigma_b = -P_1 \text{ at top of slab (1)} = -\sigma/2; \end{array}}$ $\boxed{\begin{array}{l} \sigma_b = +P_2 \text{ at bottom of slab (2)} = \sigma/3, \\ \sigma_b = -P_2 \text{ at top of slab (2)} = -\sigma/3. \end{array}}$

(f) In slab 1: $\left\{ \begin{array}{l} \text{total surface charge } above\text{: } \sigma - (\sigma/2) = \sigma/2, \\ \text{total surface charge } below\text{: } (\sigma/2) - (\sigma/3) + (\sigma/3) - \sigma = -\sigma/2, \end{array} \right\} \Longrightarrow E_1 = \dfrac{\sigma}{2\epsilon_0}. \checkmark$

In slab 2: $\left\{ \begin{array}{l} \text{total surface charge } above\text{: } \sigma - (\sigma/2) + (\sigma/2) - (\sigma/3) = 2\sigma/3, \\ \text{total surface charge } below\text{: } (\sigma/3) - \sigma = -2\sigma/3, \end{array} \right\} \Longrightarrow E_2 = \dfrac{2\sigma}{3\epsilon_0}. \checkmark$



Problem 4.19

With no dielectric, $C_0 = A\epsilon_0/d$ (Eq. 2.54).

In configuration (a), with $+\sigma$ on upper plate, $-\sigma$ on lower, $D = \sigma$ between the plates.
$E = \sigma/\epsilon_0$ (in air) and $E = \sigma/\epsilon$ (in dielectric). So $V = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Qd}{2\epsilon_0 A}\left(1 + \frac{\epsilon_0}{\epsilon}\right)$.

$C_a = \frac{Q}{V} = \frac{\epsilon_0 A}{d}\left(\frac{2}{1+1/\epsilon_r}\right) \Longrightarrow \boxed{\dfrac{C_a}{C_0} = \dfrac{2\epsilon_r}{1+\epsilon_r}.}$

In configuration (b), with potential difference V: $E = V/d$, so $\sigma = \epsilon_0 E = \epsilon_0 V/d$ (in air).

$P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ (in dielectric), so $\sigma_b = -\epsilon_0\chi_e V/d$ (at top surface of dielectric).
$\sigma_{\text{tot}} = \epsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \epsilon_0\chi_e V/d$, so $\sigma_f = \epsilon_0 V(1+\chi_e)/d = \epsilon_0\epsilon_r V/d$ (on top plate above dielectric).

$$\Longrightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right). \quad \boxed{\frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}.}$$

[Which is greater? $\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1+\epsilon_r)^2 - 4\epsilon_r}{2(1+\epsilon_r)} = \frac{1+2\epsilon_r+4\epsilon_r^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0$. So $C_b > C_a$.]

If the x axis points *down*:

	$\mathbf{E}$	$\mathbf{D}$	$\mathbf{P}$	σ_b (top surface)	σ_f (top plate)
(a) air	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0	0	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}$
(a) dielectric	$\frac{2}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$	—
(b) air	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0	0	$\frac{\epsilon_0 V}{d}$ (left)
(b) dielectric	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$(\epsilon_r - 1)\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$-(\epsilon_r-1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (right)

Problem 4.20

$\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow D4\pi r^2 = \rho\frac{4}{3}\pi r^3 \Rightarrow D = \frac{1}{3}\rho r \Rightarrow \mathbf{E} = (\rho r/3\epsilon)\,\hat{\mathbf{r}}$, for $r < R$; $D4\pi r^2 = \rho\frac{4}{3}\pi R^3 \Rightarrow D = \rho R^3/3r^2 \Rightarrow \mathbf{E} = (\rho R^3/3\epsilon_0 r^2)\,\hat{\mathbf{r}}$, for $r > R$.

$$V = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = \frac{\rho R^3}{3\epsilon_0}\frac{1}{r}\bigg|_\infty^R - \frac{\rho}{3\epsilon}\int_R^0 r\,dr = \frac{\rho R^2}{3\epsilon_0} + \frac{\rho}{3\epsilon}\frac{R^2}{2} = \boxed{\frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right).}$$

Problem 4.21

Let Q be the charge on a length ℓ of the inner conductor.

$$\oint \mathbf{D}\cdot d\mathbf{a} = D2\pi s\ell = Q \Rightarrow D = \frac{Q}{2\pi s\ell};\quad E = \frac{Q}{2\pi\epsilon_0 s\ell}\ (a < s < b),\quad E = \frac{Q}{2\pi\epsilon s\ell}\ (b < r < c).$$

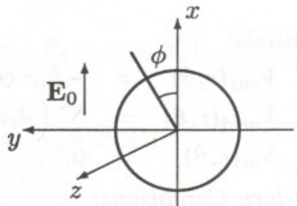
$$V = -\int_c^a \mathbf{E}\cdot d\mathbf{l} = \int_a^b \left(\frac{Q}{2\pi\epsilon_0\ell}\right)\frac{ds}{s} + \int_b^c \left(\frac{Q}{2\pi\epsilon\ell}\right)\frac{ds}{s} = \frac{Q}{2\pi\epsilon_0\ell}\left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon}\ln\left(\frac{c}{b}\right)\right].$$

$$\frac{C}{\ell} = \frac{Q}{V\ell} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a) + (1/\epsilon_r)\ln(c/b)}.}$$

Problem 4.22

Same method as Ex. 4.7: solve Laplace's equation for $V_{\text{in}}(s,\phi)$ $(s < a)$ and $V_{\text{out}}(s,\phi)$ $(s > a)$, subject to the boundary conditions

$$\begin{cases} \text{(i)}\ V_{\text{in}} = V_{\text{out}} & \text{at } s = a, \\ \text{(ii)}\ \epsilon\frac{\partial V_{\text{in}}}{\partial s} = \epsilon_0\frac{\partial V_{\text{out}}}{\partial s} & \text{at } s = a, \\ \text{(iii)}\ V_{\text{out}} \to -E_0 s\cos\phi & \text{for } s \gg a. \end{cases}$$



From Prob. 3.23 (invoking boundary condition (iii)):

$$V_{\text{in}}(s,\phi) = \sum_{k=1}^{\infty} s^k(a_k\cos k\phi + b_k\sin k\phi),\quad V_{\text{out}}(s,\phi) = -E_0 s\cos\phi + \sum_{k=1}^{\infty} s^{-k}(c_k\cos k\phi + d_k\sin k\phi).$$

(I eliminated the constant terms by setting $V = 0$ on the yz plane.) Condition (i) says

$$\sum a^k(a_k \cos k\phi + b_k \sin k\phi) = -E_0 s \cos\phi + \sum a^{-k}(c_k \cos k\phi + d_k \sin k\phi),$$

while (ii) says

$$\epsilon_r \sum k a^{k-1}(a_k \cos k\phi + b_k \sin k\phi) = -E_0 \cos\phi - \sum k a^{-k-1}(c_k \cos k\phi + d_k \sin k\phi).$$

Evidently $b_k = d_k = 0$ for all k, $a_k = c_k = 0$ unless $k = 1$, whereas for $k = 1$,

$$a a_1 = -E_0 a + a^{-1} c_1, \quad \epsilon_r a_1 = -E_0 - a^{-2} c_1.$$

Solving for a_1,

$$a_1 = -\frac{E_0}{(1+\chi_e/2)}, \quad \text{so } V_{\text{in}}(s,\phi) = -\frac{E_0}{(1+\chi_e/2)} s\cos\phi = -\frac{E_0}{(1+\chi_e/2)} x,$$

and hence $\mathbf{E}_{\text{in}}(s,\phi) = -\dfrac{\partial V_{\text{in}}}{\partial x}\,\hat{\mathbf{x}} = \boxed{\dfrac{\mathbf{E}_0}{(1+\chi_e/2)}}$. As in the spherical case (Ex. 4.7), the field inside is *uniform*.

Problem 4.23

$$\mathbf{P}_0 = \epsilon_0\chi_e\mathbf{E}_0;\ \mathbf{E}_1 = -\frac{1}{3\epsilon_0}\mathbf{P}_0 = -\frac{\chi_e}{3}\mathbf{E}_0;\ \mathbf{P}_1 = \epsilon_0\chi_e\mathbf{E}_1 = -\frac{\epsilon_0\chi_e^2}{3}\mathbf{E}_0;\ \mathbf{E}_2 = -\frac{1}{3\epsilon_0}\mathbf{P}_1 = \frac{\chi_e^2}{9}\mathbf{E}_0;\ \ldots.$$

Evidently $\mathbf{E}_n = \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0$, so

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots = \left[\sum_{n=0}^{\infty}\left(-\frac{\chi_e}{3}\right)^n\right]\mathbf{E}_0.$$

The geometric series can be summed explicitly:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{so} \quad \boxed{\mathbf{E} = \frac{1}{(1+\chi_e/3)}\mathbf{E}_0,}$$

which agrees with Eq. 4.49. [Curiously, this method formally requires that $\chi_e < 3$ (else the infinite series diverges), yet the *result* is subject to no such restriction, since we can also get it by the method of Ex. 4.7.]

Problem 4.24

Potentials:

$$\begin{cases} V_{\text{out}}(r,\theta) &= -E_0 r\cos\theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r > b); \\ V_{\text{med}}(r,\theta) &= \sum \left(A_l r^l + \frac{\bar{B}_l}{r^{l+1}}\right) P_l(\cos\theta), & (a < r < b); \\ V_{\text{in}}(r,\theta) &= 0, & (r < a). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)}\ V_{\text{out}} &= V_{\text{med}}, & (r = b); \\ \text{(ii)}\ \epsilon\frac{\partial V_{\text{med}}}{\partial r} &= \epsilon_0\frac{\partial V_{\text{out}}}{\partial r}, & (r = b); \\ \text{(iii)}\ V_{\text{med}} &= 0, & (r = a). \end{cases}$$

$$\text{(i)} \quad \Rightarrow \quad -E_0 b\cos\theta + \sum \frac{B_l}{b^{l+1}} P_l(\cos\theta) = \sum \left(A_l b^l + \frac{\bar{B}_l}{b^{l+1}}\right) P_l(\cos\theta);$$

$$\text{(ii)} \quad \Rightarrow \quad \epsilon_r \sum \left[l A_l b^{l-1} - (l+1)\frac{\bar{B}_l}{b^{l+2}}\right] P_l(\cos\theta) = -E_0\cos\theta - \sum (l+1)\frac{B_l}{b^{l+2}} P_l(\cos\theta);$$

$$\text{(iii)} \quad \Rightarrow \quad A_l a^l + \frac{\bar{B}_l}{a^{l+1}} = 0 \;\Rightarrow\; \bar{B}_l = -a^{2l+1} A_l.$$

For $l \neq 1$:

$$\text{(i)} \quad \frac{B_l}{b^{l+1}} = \left(A_l b^l - \frac{a^{2l+1}A_l}{b^{l+1}}\right) \;\Rightarrow\; B_l = A_l\left(b^{2l+1} - a^{2l+1}\right);$$

$$\text{(ii)} \quad \epsilon_r\left[l A_l b^{l-1} + (l+1)\frac{a^{2l+1}A_l}{b^{l+2}}\right] = -(l+1)\frac{B_l}{b^{l+2}} \;\Rightarrow\; B_l = -\epsilon_r A_l\left[\left(\frac{l}{l+1}\right)b^{2l+1} + a^{2l+1}\right] \;\Rightarrow\; A_l = B_l = 0.$$

For $l = 1$:

$$\text{(i)} \quad -E_0 b + \frac{B_1}{b^2} = A_1 b - \frac{a^3 A_1}{b^2} \;\Rightarrow\; B_1 - E_0 b^3 = A_1 2\left(b^3 - a^3\right);$$

$$\text{(ii)} \quad \epsilon_r\left(A_1 + 2\frac{a^3 A_1}{b^3}\right) = -E_0 - 2\frac{B_1}{b^3} \;\Rightarrow\; -2B_1 - E_0 b^3 = \epsilon_r A_1\left(b^3 + 2a^3\right).$$

So $-3E_0 b^3 = A_1\left[2\left(b^3 - a^3\right) + \epsilon_r\left(b^3 + 2a^3\right)\right]$; $\quad A_1 = \dfrac{-3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}$.

$$V_{\text{med}}(r,\theta) = \frac{-3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}\left(r - \frac{a^3}{r^2}\right)\cos\theta,$$

$$\mathbf{E}(r,\theta) = -\nabla V_{\text{med}} = \boxed{\frac{3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}\left\{\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right\}.}$$

Problem 4.25

There are four charges involved: (i) q, (ii) polarization charge surrounding q, (iii) surface charge (σ_b) on the top surface of the lower dielectric, (iv) surface charge (σ_b') on the lower surface of the upper dielectric. In view of Eq. 4.39, the bound charge (ii) is $q_p = -q(\chi_e'/(1+\chi_e'))$, so the *total* (point) charge at $(0,0,d)$ is $q_t = q + q_p = q/(1+\chi_e') = q/\epsilon_r'$. As in Ex. 4.8,

$$\text{(a)}\ \sigma_b = \epsilon_0\chi_e\left[\frac{-1}{4\pi\epsilon_0}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0} - \frac{\sigma_b'}{2\epsilon_0}\right] \quad (\text{here } \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = +P_z = \epsilon_0\chi_e E_z);$$

$$\text{(b)}\ \sigma_b' = \epsilon_0\chi_e'\left[\frac{1}{4\pi\epsilon_0}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0} - \frac{\sigma_b'}{2\epsilon_0}\right] \quad (\text{here } \sigma_b = -P_z = -\epsilon_0\chi_e' E_z).$$

Solve for σ_b, σ_b': first divide by χ_e and χ_e' (respectively) and subtract:

$$\frac{\sigma_b'}{\chi_e'} - \frac{\sigma_b}{\chi_e} = \frac{1}{2\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}} \;\Rightarrow\; \sigma_b' = \chi_e'\left[\frac{\sigma_b}{\chi_e} + \frac{1}{2\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}}\right].$$

Plug this into (a) and solve for σ_b, using $\epsilon'_r = 1 + \chi'_e$:

$$\sigma_b = \frac{-1}{4\pi}\frac{qd/\epsilon'_r}{(r^2+d^2)^{\frac{3}{2}}}\chi_e(1+\chi'_e) - \frac{\sigma_b}{2}(\chi_e+\chi'_e), \text{ so } \boxed{\sigma_b = \frac{-1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{\chi_e}{[1+(\chi_e+\chi'_e)/2]}};$$

$$\sigma'_b = \chi'_e\left\{\frac{-1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{1}{[1+(\chi_e+\chi'_e)/2]} + \frac{1}{2\pi}\frac{qd/\epsilon'_r}{(r^2+d^2)^{\frac{3}{2}}}\right\}, \text{ so } \boxed{\sigma'_b = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{\epsilon_r\chi'_e/\epsilon'_r}{[1+(\chi_e+\chi'_e)/2]}}.$$

The *total* bound surface charge is $\sigma_t = \sigma_b + \sigma'_b = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{(\chi'_e-\chi_e)}{\epsilon'_r[1+(\chi_e+\chi'_e)/2]}$ (which vanishes, as it should, when $\chi'_e = \chi_e$). The total bound charge is (compare Eq. 4.51):

$$q_t = \frac{(\chi'_e-\chi_e)q}{2\epsilon'_r[1+(\chi_e+\chi'_e)/2]} = \boxed{\left(\frac{\epsilon'_r-\epsilon_r}{\epsilon'_r+\epsilon_r}\right)\frac{q}{\epsilon'_r}}, \text{ and hence}$$

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q/\epsilon'_r}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_t}{\sqrt{x^2+y^2+(z+d)^2}}\right\}} \text{ (for } z>0).$$

Meanwhile, since $\frac{q}{\epsilon'_r} + q_t = \frac{q}{\epsilon'_r}\left[1+\frac{\epsilon'_r-\epsilon_r}{\epsilon'_r+\epsilon_r}\right] = \frac{2q}{\epsilon'_r+\epsilon_r}$, $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{[2q/(\epsilon'_r+\epsilon_r)]}{\sqrt{x^2+y^2+(z-d)^2}}}$ (for $z<0$).

Problem 4.26

From Ex. 4.5:

$$\mathbf{D} = \begin{Bmatrix} 0, & (r<a) \\ \dfrac{Q}{4\pi r^2}\hat{\mathbf{r}}, & (r>a)\end{Bmatrix}, \quad \mathbf{E} = \begin{Bmatrix} 0, & (r<a) \\ \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & (a<r<b) \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & (r>b)\end{Bmatrix}.$$

$$\begin{aligned} W &= \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau = \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left\{\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2 dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}dr\right\} = \frac{Q^2}{8\pi}\left\{\frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty\right\} \\ &= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\} = \boxed{\frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)} \end{aligned}$$

Problem 4.27

Using Eq. 4.55: $W = \frac{\epsilon_0}{2}\int E^2\,d\tau$. From Ex. 4.2 and Eq. 3.103,

$$\mathbf{E} = \left\{ \begin{array}{ll} \dfrac{-1}{3\epsilon_0} P\,\hat{\mathbf{z}}, & (r < R) \\ \dfrac{R^3 P}{3\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), & (r > R) \end{array} \right\}, \quad \text{so}$$

$$W_{r<R} = \frac{\epsilon_0}{2}\left(\frac{P}{3\epsilon_0}\right)^2 \frac{4}{3}\pi R^3 = \frac{2\pi}{27}\frac{P^2 R^3}{\epsilon_0}.$$

$$\begin{aligned} W_{r>R} &= \frac{\epsilon_0}{2}\left(\frac{R^3 P}{3\epsilon_0}\right)^2 \int \frac{1}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) r^2 \sin\theta\,dr\,d\theta\,d\phi \\ &= \frac{(R^3P)^2}{18\epsilon_0} 2\pi \int_0^{\pi} (1 + 3\cos^2\theta)\sin\theta\,d\theta \int_R^{\infty} \frac{1}{r^4}\,dr = \frac{\pi (R^3 P)^2}{9\epsilon_0}\left(-\cos\theta - \cos^3\theta\right)\Big|_0^{\pi} \left(-\frac{1}{3r^3}\right)\Bigg|_R^{\infty} \\ &= \frac{\pi (R^3 P)^2}{9\epsilon_0}\left(\frac{4}{3R^3}\right) = \frac{4\pi R^3 P^2}{27\epsilon_0}. \end{aligned}$$

$$W_{\text{tot}} = \boxed{\frac{2\pi R^3 P^2}{9\epsilon_0}}.$$

This is the correct electrostatic energy of the configuration, but it is not the "total work necessary to assemble the system," because it leaves out the mechanical energy involved in polarizing the molecules.

Using Eq. 4.58: $W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau$. For $r < R$, $\mathbf{D} = \epsilon_0 \mathbf{E}$, so this contribution is the same as before. For $r < R$, $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = -\frac{1}{3}\mathbf{P} + \mathbf{P} = \frac{2}{3}\mathbf{P} = -2\epsilon_0\mathbf{E}$, so $\frac{1}{2}\mathbf{D}\cdot\mathbf{E} = -2\frac{\epsilon_0}{2}E^2$, and this contribution is now $(-2)\left(\frac{2\pi}{27}\frac{P^2R^3}{\epsilon_0}\right) = -\frac{4\pi}{27}\frac{R^3P^2}{\epsilon_0}$, exactly cancelling the exterior term. *Conclusion:* $\boxed{W_{\text{tot}} = 0.}$ This is not surprising, since the derivation in Sect. 4.4.3 calculates the work done on the *free* charge, and in this problem there is no free charge in sight. Since this is a nonlinear dielectric, however, the result cannot be interpreted as the "work necessary to assemble the configuration"—the latter would depend entirely on *how* you assemble it.

Problem 4.28

First find the capacitance, as a function of h:

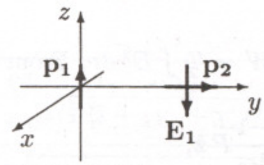
$$\left. \begin{array}{l} \textit{Air part: } E = \frac{2\lambda}{4\pi\epsilon_0 s} \Longrightarrow V = \frac{2\lambda}{4\pi\epsilon_0}\ln(b/a), \\ \textit{Oil part: } D = \frac{2\lambda'}{4\pi s} \Longrightarrow E = \frac{2\lambda'}{4\pi\epsilon s} \Longrightarrow V = \frac{2\lambda'}{4\pi\epsilon}\ln(b/a), \end{array} \right\} \Longrightarrow \frac{\lambda}{\epsilon_0} = \frac{\lambda'}{\epsilon};\ \lambda' = \frac{\epsilon}{\epsilon_0}\lambda = \epsilon_r\lambda.$$

$Q = \lambda' h + \lambda(\ell - h) = \epsilon_r\lambda h - \lambda h + \lambda\ell = \lambda[(\epsilon_r - 1)h + \ell] = \lambda(\chi_e h + \ell)$, where ℓ is the total height.

$$C = \frac{Q}{V} = \frac{\lambda(\chi_e h + \ell)}{2\lambda\ln(b/a)}4\pi\epsilon_0 = 2\pi\epsilon_0\frac{(\chi_e h + \ell)}{\ln(b/a)}.$$

$$\left. \begin{array}{l} \text{The net upward force is given by Eq. 4.64: } F = \frac{1}{2}V^2\frac{dC}{dh} = \frac{1}{2}V^2\frac{2\pi\epsilon_0\chi_e}{\ln(b/a)}. \\ \text{The gravitational force } \textit{down} \text{ is } F = mg = \rho\pi(b^2 - a^2)gh. \end{array} \right\} \boxed{h = \frac{\epsilon_0\chi_e V^2}{\rho(b^2 - a^2)g\ln(b/a)}.}$$

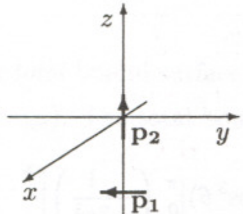
Problem 4.29



(a) Eq. 4.5 $\Rightarrow \mathbf{F}_2 = (\mathbf{p}_2 \cdot \boldsymbol{\nabla})\,\mathbf{E}_1 = p_2 \dfrac{\partial}{\partial y}(\mathbf{E}_1)$;

Eq. 3.103 $\Rightarrow \mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}} = -\dfrac{p_1}{4\pi\epsilon_0 y^3}\,\hat{\mathbf{z}}$. Therefore

$$\mathbf{F}_2 = -\frac{p_1 p_2}{4\pi\epsilon_0}\left[\frac{d}{dy}\left(\frac{1}{y^3}\right)\right]\hat{\mathbf{z}} = \frac{3p_1p_2}{4\pi\epsilon_0 y^4}\,\hat{\mathbf{z}},\ \text{or}\ \boxed{\mathbf{F}_2 = \frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}}\ \text{(upward)}.$$



To calculate $\mathbf{F}_1$, put $\mathbf{p}_2$ at the origin, pointing in the z direction; then $\mathbf{p}_1$ is at $-r\,\hat{\mathbf{z}}$, and it points in the $-\hat{\mathbf{y}}$ direction. So $\mathbf{F}_1 = (\mathbf{p}_1 \cdot \boldsymbol{\nabla})\,\mathbf{E}_2 = -p_1 \left.\dfrac{\partial \mathbf{E}_2}{\partial y}\right|_{x=y=0,\,z=-r}$; we need $\mathbf{E}_2$ as a function of x, y, and z.

From Eq. 3.104: $\mathbf{E}_2 = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[\dfrac{3(\mathbf{p}_2\cdot\mathbf{r})\mathbf{r}}{r^2} - \mathbf{p}\right]$, where $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{p}_2 = -p_2\,\hat{\mathbf{y}}$, and hence $\mathbf{p}_2\cdot\mathbf{r} = -p_2 y$.

$$\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0}\left[\frac{-3y(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) + (x^2+y^2+z^2)\,\hat{\mathbf{y}}}{(x^2+y^2+z^2)^{5/2}}\right] = \frac{p_2}{4\pi\epsilon_0}\left[\frac{-3xy\,\hat{\mathbf{x}} + (x^2-2y^2+z^2)\,\hat{\mathbf{y}} - 3yz\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$\frac{\partial \mathbf{E}_2}{\partial y} = \frac{p_2}{4\pi\epsilon_0}\left\{-\frac{5}{2}\frac{1}{r^7}2y[-3xy\,\hat{\mathbf{x}} + (x^2-2y^2+z^2)\,\hat{\mathbf{y}} - 3yz\,\hat{\mathbf{z}}] + \frac{1}{r^5}(-3x\,\hat{\mathbf{x}} - 4y\,\hat{\mathbf{y}} - 3z\,\hat{\mathbf{z}})\right\};$$

$$\left.\frac{\partial \mathbf{E}_2}{\partial y}\right|_{(0,0)} = \frac{p_2}{4\pi\epsilon_0}\frac{-3z}{r^5}\,\hat{\mathbf{z}};\quad \mathbf{F}_1 = -p_1\left(\frac{p_2}{4\pi\epsilon_0}\frac{3r}{r^5}\,\hat{\mathbf{z}}\right) = \boxed{-\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}.}$$

These results *are* consistent with Newton's third law: $\mathbf{F}_1 = -\mathbf{F}_2$.

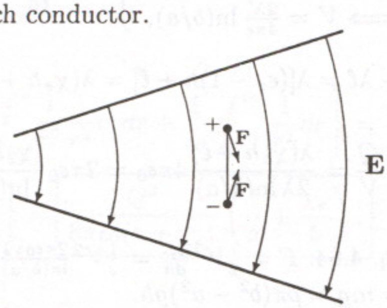
(b) From page 165, $\mathbf{N}_2 = (\mathbf{p}_2 \times \mathbf{E}_1) + (\mathbf{r} \times \mathbf{F}_2)$. The first term was calculated in Prob. 4.5; the second we get from (a), using $\mathbf{r} = r\,\hat{\mathbf{y}}$:

$$\mathbf{p}_2 \times \mathbf{E}_1 = \frac{p_1p_2}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{x}});\quad \mathbf{r}\times\mathbf{F}_2 = (r\,\hat{\mathbf{y}}) \times \left(\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}\right) = \frac{3p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}};\ \text{so}\ \boxed{\mathbf{N}_2 = \frac{2p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}}.}$$

This is equal and opposite to the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$, with respect to the center of $\mathbf{p}_1$ (see Prob. 4.5).

Problem 4.30

Net force is $\boxed{\text{to the right}}$ (see diagram). Note that the field lines must bulge to the right, as shown, because $\mathbf{E}$ is perpendicular to the surface of each conductor.



Problem 4.31

$\mathbf{P} = k\mathbf{r} = k(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \Longrightarrow \rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -k(1+1+1) = \boxed{-3k.}$

Total volume bound charge: $\boxed{Q_{\text{vol}} = -3ka^3.}$

$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. At top surface, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $z = a/2$; so $\sigma_b = ka/2$. Clearly, $\boxed{\sigma_b = ka/2}$ on all six surfaces.

Total surface bound charge: $\boxed{Q_{\text{surf}} = 6(ka/2)a^2 = 3ka^3.}$ Total bound charge is zero. ✓

Problem 4.32

$$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow \mathbf{D} = \frac{q}{4\pi r^2}\,\hat{\mathbf{r}};\ \mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \boxed{\frac{q}{4\pi\epsilon_0(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2};}\ \mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2}.}$$

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) = \boxed{-q\frac{\chi_e}{1+\chi_e}\delta^3(\mathbf{r})}\ \text{(Eq. 1.99)};\ \sigma_b = \mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)R^2};}$$

$Q_{\text{surf}} = \sigma_b(4\pi R^2) = \boxed{q\dfrac{\chi_e}{1+\chi_e}.}$ The compensating negative charge is at the center:

$$\int \rho_b\,d\tau = -\frac{q\chi_e}{1+\chi_e}\int \delta^3(\mathbf{r})\,d\tau = -q\frac{\chi_e}{1+\chi_e}.$$

Problem 4.33

$\mathbf{E}^{\parallel}$ is continuous (Eq. 4.29); $D_\perp$ is continuous (Eq. 4.26, with $\sigma_f = 0$). So $E_{x_1} = E_{x_2}$, $D_{y_1} = D_{y_2} \Rightarrow \epsilon_1 E_{y_1} = \epsilon_2 E_{y_2}$, and hence

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{x_2}/E_{y_2}}{E_{x_1}/E_{y_1}} = \frac{E_{y_1}}{E_{y_2}} = \frac{\epsilon_2}{\epsilon_1}.\quad \text{qed}$$

If 1 is air and 2 is dielectric, $\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_0 > 1$, and the field lines bend *away* from the normal. This is the opposite of light rays, so a convex "lens" would *defocus* the field lines.

Problem 4.34

In view of Eq. 4.39, the *net* dipole moment at the center is $\mathbf{p}' = \mathbf{p} - \frac{\chi_e}{1+\chi_e}\mathbf{p} = \frac{1}{1+\chi_e}\mathbf{p} = \frac{1}{\epsilon_r}\mathbf{p}$. We want the potential produced by $\mathbf{p}'$ (at the center) and σ_b (at R). Use separation of variables:

$$\left\{\begin{array}{lll} \textit{Outside:} & V(r,\theta) = \displaystyle\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & \text{(Eq. 3.72)} \\ \textit{Inside:} & V(r,\theta) = \displaystyle\frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) & \text{(Eqs. 3.66, 3.102)} \end{array}\right\}.$$

$$V \text{ continuous at } R \Rightarrow \left\{\begin{array}{ll} \dfrac{B_l}{R^{l+1}} = A_l R^l, & \text{or}\quad B_l = R^{2l+1}A_l\ (l \neq 1) \\ \dfrac{B_1}{R^2} = \dfrac{1}{4\pi\epsilon_0}\dfrac{p}{\epsilon_r R^2} + A_1 R, & \text{or}\quad B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r} + A_1 R^3 \end{array}\right\}.$$

$$\begin{aligned} \left.\frac{\partial V}{\partial r}\right|_{R+} - \left.\frac{\partial V}{\partial r}\right|_{R-} &= -\sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3} - \sum l A_l R^{l-1}P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_b \\ &= -\frac{1}{\epsilon_0}\mathbf{P}\cdot\hat{\mathbf{r}} = -\frac{1}{\epsilon_0}(\epsilon_0\chi_e\mathbf{E}\cdot\hat{\mathbf{r}}) = \chi_e\left.\frac{\partial V}{\partial r}\right|_{R-} = \chi_e\left\{-\frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3} + \sum l A_l R^{l-1}P_l(\cos\theta)\right\}. \end{aligned}$$

$$-(l+1)\frac{B_l}{R^{l+2}} - lA_lR^{l-1} = \chi_e lA_lR^{l-1}\ (l \neq 1);\ \text{or}\ -(2l+1)A_lR^{l-1} = \chi_e lA_lR^{l-1}\ \Rightarrow A_l = 0\ (\ell \neq 1).$$

For $l = 1$: $-2\frac{B_1}{R^3} + \frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3} - A_1 = \chi_e\left(-\frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3} + A_1\right) - B_1 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1R^3}{2}$;

$$-\frac{p}{4\pi\epsilon_0\epsilon_r} - A_1R^3 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1R^3}{2} \Rightarrow \frac{A_1R^3}{2}(3+\chi_e) = \frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r}.$$

$$\Rightarrow A_1 = \frac{1}{4\pi\epsilon_0}\frac{2\chi_e p}{R^3\epsilon_r(3+\chi_e)} = \frac{1}{4\pi\epsilon_0}\frac{2(\epsilon_r-1)p}{R^3\epsilon_r(\epsilon_r+2)};\quad B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r}\left[1 + \frac{2(\epsilon_r-1)}{(\epsilon_r+2)}\right] = \frac{p}{4\pi\epsilon_0\epsilon_r}\frac{3\epsilon_r}{\epsilon_r+2}.$$

$$\boxed{V(r,\theta) = \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)\left(\frac{3}{\epsilon_r+2}\right)\ (r \geq R).}$$

Meanwhile, for $r \leq R$, $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \frac{1}{4\pi\epsilon_0}\frac{pr\cos\theta}{R^3}\frac{2(\epsilon_r-1)}{\epsilon_r(\epsilon_r+2)}$

$$= \boxed{\frac{p\cos\theta}{4\pi\epsilon_0 r^2\epsilon_r}\left[1 + 2\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)\frac{r^3}{R^3}\right]\ (r \leq R).}$$

Problem 4.35

Given two solutions, V_1 (and $\mathbf{E}_1 = -\nabla V_1$, $\mathbf{D}_1 = \epsilon\mathbf{E}_1$) and V_2 ($\mathbf{E}_2 = -\nabla V_2$, $\mathbf{D}_2 = \epsilon\mathbf{E}_2$), define $V_3 \equiv V_2 - V_1$ ($\mathbf{E}_3 = \mathbf{E}_2 - \mathbf{E}_1$, $\mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1$).

$\int_{\mathcal{V}} \nabla\cdot(V_3\mathbf{D}_3)\,d\tau = \int_{\mathcal{S}} V_3\mathbf{D}_3\cdot d\mathbf{a} = 0$, ($V_3 = 0$ on $\mathcal{S}$), so $\int(\nabla V_3)\cdot\mathbf{D}_3\,d\tau + \int V_3(\nabla\cdot\mathbf{D}_3)\,d\tau = 0$.
But $\nabla\cdot\mathbf{D}_3 = \nabla\cdot\mathbf{D}_2 - \nabla\cdot\mathbf{D}_1 = \rho_f - \rho_f = 0$, and $\nabla V_3 = \nabla V_2 - \nabla V_1 = -\mathbf{E}_2 + \mathbf{E}_1 = -\mathbf{E}_3$, so $\int \mathbf{E}_3\cdot\mathbf{D}_3\,d\tau = 0$. But $\mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1 = \epsilon\mathbf{E}_2 - \epsilon\mathbf{E}_1 = \epsilon\mathbf{E}_3$, so $\int \epsilon(E_3)^2\,d\tau = 0$. But $\epsilon > 0$, so $\mathbf{E}_3 = 0$, so $V_2 - V_1 =$ constant. But at surface, $V_2 = V_1$, so $V_2 = V_1$ everywhere. qed

Problem 4.36

(a) Proposed potential: $\boxed{V(r) = V_0\frac{R}{r}.}$ If so, then $\boxed{\mathbf{E} = -\nabla V = V_0\frac{R}{r^2}\,\hat{\mathbf{r}},}$ in which case $\boxed{\mathbf{P} = \epsilon_0\chi_e V_0\frac{R}{r^2}\,\hat{\mathbf{r}},}$ in the region $z < 0$. ($\mathbf{P} = 0$ for $z > 0$, of course.) Then $\sigma_b = \epsilon_0\chi_e V_0\frac{R}{R^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}) = \boxed{-\frac{\epsilon_0\chi_e V_0}{R}.}$ (*Note:* $\hat{\mathbf{n}}$ points *out* of dielectric $\Rightarrow \hat{\mathbf{n}} = -\hat{\mathbf{r}}$.) This σ_b is on the surface at $r = R$. The flat surface $z = 0$ carries no bound charge, since $\hat{\mathbf{n}} = \hat{\mathbf{z}} \perp \hat{\mathbf{r}}$. Nor is there any volume bound charge (Eq. 4.39). If V is to have the required spherical symmetry, the *net* charge must be uniform:

$\sigma_{\text{tot}}4\pi R^2 = Q_{\text{tot}} = 4\pi\epsilon_0 RV_0$ (since $V_0 = Q_{\text{tot}}/4\pi\epsilon_0 R$), so $\sigma_{\text{tot}} = \epsilon_0V_0/R$. Therefore

$$\boxed{\sigma_f = \left\{\begin{array}{ll}(\epsilon_0V_0/R), & \text{on northern hemisphere}\\ (\epsilon_0V_0/R)(1+\chi_e), & \text{on southern hemisphere}\end{array}\right\}.}$$

(b) By construction, $\sigma_{\text{tot}} = \sigma_b + \sigma_f = \epsilon_0V_0/R$ is uniform (on the northern hemisphere $\sigma_b = 0$, $\sigma_f = \epsilon_0V_0/R$; on the southern hemisphere $\sigma_b = -\epsilon_0\chi_eV_0/R$, so $\sigma_f = \epsilon V_0/R$). The potential of a uniformly charged sphere is

$$V_0 = \frac{Q_{\text{tot}}}{4\pi\epsilon_0 r} = \frac{\sigma_{\text{tot}}(4\pi R^2)}{4\pi\epsilon_0 r} = \frac{\epsilon_0V_0}{R}\frac{R^2}{\epsilon_0 r} = V_0\frac{R}{r}.\quad \checkmark$$

(c) Since everything is consistent, and the boundary conditions ($V = V_0$ at $r = R$, $V \to 0$ at ∞) are met, Prob. 4.35 guarantees that this is *the* solution.

(d) Figure (b) works the same way, but Fig. (a) does *not:* on the flat surface, $\mathbf{P}$ is *not* perpendicular to $\hat{\mathbf{n}}$, so we'd get bound charge on this surface, spoiling the symmetry.

Problem 4.37

$\mathbf{E}_{\text{ext}} = \dfrac{\lambda}{2\pi\epsilon_0 s}\,\hat{\mathbf{s}}$. Since the sphere is tiny, this is essentially constant, and hence $\mathbf{P} = \dfrac{\epsilon_0\chi_e}{1+\chi_e/3}\mathbf{E}_{\text{ext}}$ (Ex. 4.7).

$$\begin{aligned}\mathbf{F} &= \int\left(\frac{\epsilon_0\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\frac{d}{ds}\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\hat{\mathbf{s}}\,d\tau = \left(\frac{\epsilon_0\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\epsilon_0}\right)^2\left(\frac{1}{s}\right)\left(\frac{-1}{s^2}\right)\hat{\mathbf{s}}\int d\tau \\ &= \frac{-\chi_e}{1+\chi_e/3}\left(\frac{\lambda^2}{4\pi^2\epsilon_0}\right)\frac{1}{s^3}\frac{4}{3}\pi R^3\,\hat{\mathbf{s}} = \boxed{-\left(\frac{\chi_e}{3+\chi_e}\right)\frac{\lambda^2 R^3}{\pi\epsilon_0 s^3}\,\hat{\mathbf{s}}.}\end{aligned}$$

Problem 4.38

The density of atoms is $N = \frac{1}{(4/3)\pi R^3}$. The macroscopic field $\mathbf{E}$ is $\mathbf{E}_{\text{self}} + \mathbf{E}_{\text{else}}$, where $\mathbf{E}_{\text{self}}$ is the average field over the sphere due to the atom itself.

$$\mathbf{p} = \alpha\mathbf{E}_{\text{else}} \Rightarrow \mathbf{P} = N\alpha\mathbf{E}_{\text{else}}.$$

[Actually, it is the field at the *center*, not the average over the sphere, that belongs here, but the two are in fact equal, as we found in Prob. 3.41d.] Now

$$\mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$$

(Eq. 3.105), so

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\mathbf{E}_{\text{else}} + \mathbf{E}_{\text{else}} = \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\mathbf{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}.$$

So

$$\mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}\mathbf{E} = \epsilon_0\chi_e\mathbf{E},$$

and hence

$$\chi_e = \frac{N\alpha/\epsilon_0}{(1 - N\alpha/3\epsilon_0)}.$$

Solving for α:

$$\chi_e - \frac{N\alpha}{3\epsilon_0}\chi_e = \frac{N\alpha}{\epsilon_0} \Rightarrow \frac{N\alpha}{\epsilon_0}\left(1 + \frac{\chi_e}{3}\right) = \chi_e,$$

or

$$\alpha = \frac{\epsilon_0}{N}\frac{\chi_e}{(1+\chi_e/3)} = \frac{3\epsilon_0}{N}\frac{\chi_e}{(3+\chi_e}. \quad \text{But } \chi_e = \epsilon_r - 1, \text{ so } \alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right). \quad \text{qed}$$

Problem 4.39

For an ideal gas, $N =$ Avagadro's number/22.4 liters $= (6.02\times10^{23})/(22.4\times10^{-3}) = 2.7\times10^{25}$. $N\alpha/\epsilon_0 = (2.7\times10^{25})(4\pi\epsilon_0\times10^{-30})\beta/\epsilon_0 = 3.4\times10^{-4}\beta$, where β is the number listed in Table 4.1.

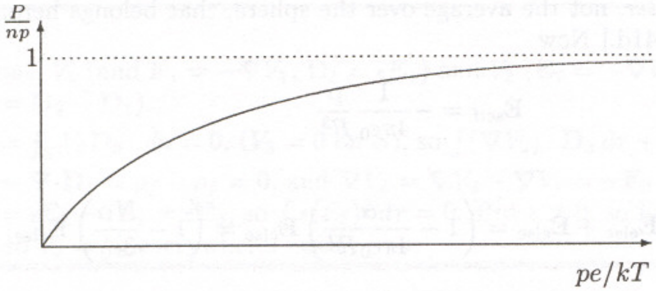
$$\left.\begin{array}{ll}\text{H:} & \beta = 0.667, \quad N\alpha/\epsilon_0 = (3.4\times10^{-4})(0.67) = 2.3\times10^{-4}, \quad \chi_e = 2.5\times10^{-4} \\ \text{He:} & \beta = 0.205, \quad N\alpha/\epsilon_0 = (3.4\times10^{-4})(0.21) = 7.1\times10^{-5}, \quad \chi_e = 6.5\times10^{-5} \\ \text{Ne:} & \beta = 0.396, \quad N\alpha/\epsilon_0 = (3.4\times10^{-4})(0.40) = 1.4\times10^{-4}, \quad \chi_e = 1.3\times10^{-4} \\ \text{Ar:} & \beta = 1.64, \quad N\alpha/\epsilon_0 = (3.4\times10^{-4})(1.64) = 5.6\times10^{-4}, \quad \chi_e = 5.2\times10^{-4}\end{array}\right\}\ \text{agreement is quite good.}$$

Problem 4.40

(a)
$$\langle u\rangle = \frac{\int_{-pE}^{pE} ue^{-u/kT}\,du}{\int_{-pE}^{pE} e^{-u/kT}\,du} = \frac{(kT)^2e^{-u/kT}\left[-(u/kT)-1\right]\Big|_{-pE}^{pE}}{-kTe^{-u/kT}\Big|_{-pE}^{pE}}$$
$$= kT\left\{\frac{\left[e^{-pE/kT}-e^{pE/kT}\right]+\left[(pE/kT)e^{-pE/kT}+(pE/kT)e^{pE/kT}\right]}{e^{-pE/kT}-e^{pE/kT}}\right\}$$
$$= kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth\left(\frac{pE}{kT}\right).$$

$$\mathbf{P} = N\langle\mathbf{p}\rangle;\ \mathbf{p} = \langle p\cos\theta\rangle\hat{\mathbf{E}} = \langle\mathbf{P}\cdot\mathbf{E}\rangle(\hat{\mathbf{E}}/E) = -\langle u\rangle(\hat{\mathbf{E}}/E);\ P = Np\frac{-\langle u\rangle}{pE} = \boxed{Np\left\{\coth\left(\frac{pE}{kT}\right)-\frac{kT}{pE}\right\}}.$$

Let $y \equiv P/Np$, $x \equiv pE/kT$. Then $y = \coth x - 1/x$. As $x \to 0$, $y = \left(\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\cdots\right)-\frac{1}{x} = \frac{x}{3}-\frac{x^3}{45}+\cdots \to 0$, so the graph starts at the origin, with an initial slope of $1/3$. As $x\to\infty$, $y \to \coth(\infty) = 1$, so the graph goes asymptotically to $y = 1$ (see Figure).



(b) For small x, $y \approx \frac{1}{3}x$, so $\frac{P}{Np} \approx \frac{pE}{3kT}$, or $P \approx \frac{Np^2}{3kT}E = \epsilon_0\chi_e E \Rightarrow P$ is proportional to E, and $\boxed{\chi_e = \frac{Np^2}{3\epsilon_0 kT}}$.

For water at $20° = 293\,\mathrm{K}$, $p = 6.1\times10^{-30}\,\mathrm{C\,m}$; $N = \frac{\text{molecules}}{\text{volume}} = \frac{\text{molecules}}{\text{mole}}\times\frac{\text{moles}}{\text{gram}}\times\frac{\text{grams}}{\text{volume}}$.
$N = (6.0\times10^{23})\times\left(\frac{1}{18}\right)\times(10^6) = 0.33\times10^{29}$; $\chi_e = \frac{(0.33\times10^{29})(6.1\times10^{-30})^2}{(3)(8.85\times10^{-12})(1.38\times10^{-23})(293)} = \boxed{12.}$ Table 4.2 gives an experimental value of 79, so it's pretty far off.

For water vapor at $100° = 373\,\mathrm{K}$, treated as an ideal gas, $\frac{\text{volume}}{\text{mole}} = (22.4\times10^{-3})\times\left(\frac{373}{293}\right) = 2.85\times10^{-2}\ \mathrm{m}^3$.

$$N = \frac{6.0\times10^{23}}{2.85\times10^{-2}} = 2.11\times10^{25};\quad \chi_e = \frac{(2.11\times10^{25})(6.1\times10^{-30})^2}{(3)(8.85\times10^{-12})(1.38\times10^{-23})(373)} = \boxed{5.7\times10^{-3}.}$$

Table 4.2 gives 5.9×10^{-3}, so this time the agreement is quite good.

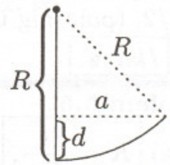
Chapter 5

Magnetostatics

Problem 5.1

Since $\mathbf{v} \times \mathbf{B}$ points upward, and that is also the direction of the force, q must be $\boxed{\text{positive.}}$ To find R, in terms of a and d, use the pythagorean theorem:

$$(R-d)^2 + a^2 = R^2 \;\Rightarrow\; R^2 - 2Rd + d^2 + a^2 = R^2 \;\Rightarrow R = \frac{a^2+d^2}{2d}.$$



The cyclotron formula then gives

$$p = qBR = \boxed{qB\frac{(a^2+d^2)}{2d}}.$$

Problem 5.2

The general solution is (Eq. 5.6):

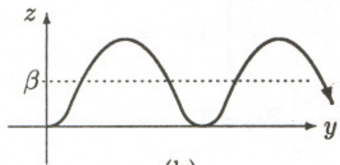
$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3; \quad z(t) = C_2\cos(\omega t) - C_1\sin(\omega t) + C_4.$$

(a) $y(0) = z(0) = 0$; $\dot{y}(0) = E/B$; $\dot{z}(0) = 0$. Use these to determine C_1, C_2, C_3, and C_4.
$y(0) = 0 \Rightarrow C_1 + C_3 = 0$; $\dot{y}(0) = \omega C_2 + E/B = E/B \Rightarrow C_2 = 0$; $z(0) = 0 \Rightarrow C_2 + C_4 = 0 \Rightarrow C_4 = 0$; $\dot{z}(0) = 0 \Rightarrow C_1 = 0$, and hence also $C_3 = 0$. So $\boxed{y(t) = Et/B;\ z(t) = 0.}$ Does this make sense? The magnetic force is $q(\mathbf{v}\times\mathbf{B}) = -q(E/B)B\,\hat{\mathbf{z}} = -q\mathbf{E}$, which exactly cancels the electric force; since there is no net force, the particle moves in a straight line at constant speed. ✓

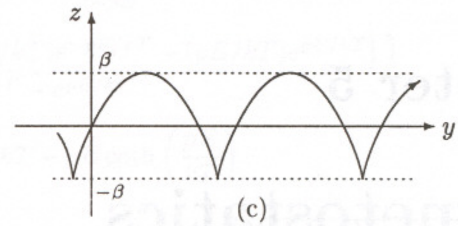
(b) Assuming it starts from the origin, so $C_3 = -C_1$, $C_4 = -C_2$, we have $\dot{z}(0) = 0 \Rightarrow C_1 = 0 \Rightarrow C_3 = 0$; $\dot{y}(0) = \frac{E}{2B} \Rightarrow C_2\omega + \frac{E}{B} = \frac{E}{2B} \Rightarrow C_2 = -\frac{E}{2\omega B} = -C_4$; $y(t) = -\frac{E}{2\omega B}\sin(\omega t) + \frac{E}{B}t$;
$z(t) = -\frac{E}{2\omega B}\cos(\omega t) + \frac{E}{2\omega B}$, or $\boxed{y(t) = \frac{E}{2\omega B}\left[2\omega t - \sin(\omega t)\right];\quad z(t) = \frac{E}{2\omega B}\left[1 - \cos(\omega t)\right].}$ Let $\beta \equiv E/2\omega B$.
Then $y(t) = \beta\left[2\omega t - \sin(\omega t)\right]$; $z(t) = \beta\left[1-\cos(\omega t)\right]$; $(y - 2\beta\omega t) = -\beta\sin(\omega t)$, $(z-\beta) = -\beta\cos(\omega t) \Rightarrow$ $(y-2\beta\omega t)^2 + (z-\beta)^2 = \beta^2$. This is a circle of radius β whose center moves to the right at constant speed: $y_0 = 2\beta\omega t$; $z_0 = \beta$.

(c) $\dot{z}(0) = \dot{y}(0) = \frac{E}{B} \Rightarrow -C_1\omega = \frac{E}{B} \Rightarrow C_1 = -C_3 = -\frac{E}{\omega B}$; $C_2\omega + \frac{E}{B} = \frac{E}{B} \Rightarrow C_2 = C_4 = 0.$

$y(t) = -\dfrac{E}{\omega B}\cos(\omega t) + \dfrac{E}{B}t + \dfrac{E}{\omega B}$; $z(t) = \dfrac{E}{\omega B}\sin(\omega t)$. $\boxed{y(t) = \dfrac{E}{\omega B}\left[1 + \omega t - \cos(\omega t)\right];\ z(t) = \dfrac{E}{\omega B}\sin(\omega t).}$
Let $\beta \equiv E/\omega B$; then $[y - \beta(1+\omega t)] = -\beta\cos(\omega t)$, $z = \beta\sin(\omega t)$; $[y-\beta(1+\omega t)]^2 + z^2 = \beta^2$. This is a circle of radius β whose center is at $y_0 = \beta(1+\omega t)$, $z_0 = 0$.



(b)



(c)

Problem 5.3

(a) From Eq. 5.2, $\mathbf{F} = q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] = 0 \Rightarrow E = vB \Rightarrow \boxed{v = \dfrac{E}{B}.}$

(b) From Eq. 5.3, $mv = qBR \Rightarrow \dfrac{q}{m} = \dfrac{v}{BR} = \boxed{\dfrac{E}{B^2R}.}$

Problem 5.4

Suppose I flows counterclockwise (if not, change the sign of the answer). The force on the left side (toward the left) cancels the force on the right side (toward the right); the force on the top is $IaB = Iak(a/2) = Ika^2/2$, (pointing upward), and the force on the bottom is $IaB = -Ika^2/2$ (also upward). So the net force is $\mathbf{F} = \boxed{Ika^2\,\hat{\mathbf{z}}.}$

Problem 5.5

(a) $\boxed{K = \dfrac{I}{2\pi a},}$ because the length-perpendicular-to-flow is the circumference.

(b) $J = \dfrac{\alpha}{s} \Rightarrow I = \displaystyle\int J\,da = \alpha\int \frac{1}{s}s\,ds\,d\phi = 2\pi\alpha\int ds = 2\pi\alpha a \Rightarrow \alpha = \frac{I}{2\pi a}$; $J = \boxed{\dfrac{I}{2\pi a s}.}$

Problem 5.6

(a) $v = \omega r$, so $\boxed{K = \sigma\omega r.}$ (b) $\mathbf{v} = \omega r\sin\theta\,\hat{\boldsymbol{\phi}} \Rightarrow \boxed{\mathbf{J} = \rho\omega r\sin\theta\,\hat{\boldsymbol{\phi}},}$ where $\rho \equiv Q/(4/3)\pi R^3$.

Problem 5.7

$\dfrac{d\mathbf{p}}{dt} = \dfrac{d}{dt}\displaystyle\int_{\mathcal{V}} \rho\mathbf{r}\,d\tau = \int\left(\frac{\partial\rho}{\partial t}\right)\mathbf{r}\,d\tau = -\int(\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}\,d\tau$ (by the continuity equation). Now product rule #5 says $\boldsymbol{\nabla}\cdot(x\mathbf{J}) = x(\boldsymbol{\nabla}\cdot\mathbf{J}) + \mathbf{J}\cdot(\boldsymbol{\nabla}x)$. But $\boldsymbol{\nabla}x = \hat{\mathbf{x}}$, so $\boldsymbol{\nabla}\cdot(x\mathbf{J}) = x(\boldsymbol{\nabla}\cdot\mathbf{J}) + J_x$. Thus $\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})x\,d\tau = \displaystyle\int_{\mathcal{V}}\boldsymbol{\nabla}\cdot(x\mathbf{J})\,d\tau - \int_{\mathcal{V}} J_x\,d\tau$. The first term is $\int_{\mathcal{S}} x\mathbf{J}\cdot d\mathbf{a}$ (by the divergence theorem), and since $\mathbf{J}$ is entirely *inside* $\mathcal{V}$, it is zero on the surface $\mathcal{S}$. Therefore $\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})x\,d\tau = -\int_{\mathcal{V}} J_x\,d\tau$, or, combining this with the y and z components, $\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}\,d\tau = -\int_{\mathcal{V}}\mathbf{J}\,d\tau$. Or, referring back to the first line, $\dfrac{d\mathbf{p}}{dt} = \displaystyle\int\mathbf{J}\,d\tau$. qed

Problem 5.8

(a) Use Eq. 5.35, with $z = R, \theta_2 = -\theta_1 = 45°$, and four sides: $B = \boxed{\dfrac{\sqrt{2}\mu_0 I}{\pi R}.}$

(b) $z = R$, $\theta_2 = -\theta_1 = \dfrac{\pi}{n}$, and n sides: $B = \boxed{\dfrac{n\mu_0 I}{2\pi R}\sin(\pi/n).}$

(c) For small θ, $\sin\theta \approx \theta$. So as $n \to \infty$, $B \to \dfrac{n\mu_0 I}{2\pi R}\left(\dfrac{\pi}{n}\right) = \boxed{\dfrac{\mu_0 I}{2R}}$ (same as Eq. 5.38, with $z = 0$).

Problem 5.9

(a) The straight segments produce no field at P. The two quarter-circles give $B = \boxed{\dfrac{\mu_0 I}{8}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)}$ (out).

(b) The two half-lines are the same as one infinite line: $\dfrac{\mu_0 I}{2\pi R}$; the half-circle contributes $\dfrac{\mu_0 I}{4R}$.

So $B = \boxed{\dfrac{\mu_0 I}{4R}\left(1 + \dfrac{2}{\pi}\right)}$ (into the page).

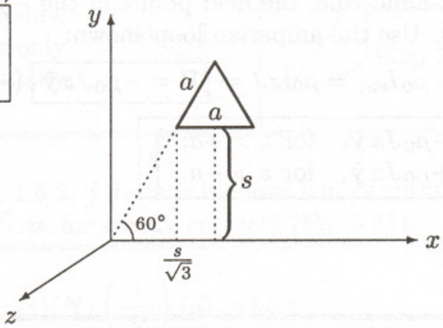
Problem 5.10

(a) The forces on the two sides cancel. At the bottom, $B = \dfrac{\mu_0 I}{2\pi s} \Rightarrow F = \left(\dfrac{\mu_0 I}{2\pi s}\right) Ia = \dfrac{\mu_0 I^2 a}{2\pi s}$ (up). At the top, $B = \dfrac{\mu_0 I}{2\pi(s+a)} \Rightarrow F = \dfrac{\mu_0 I^2 a}{2\pi(s+a)}$ (down). The net force is $\boxed{\dfrac{\mu_0 I^2 a^2}{2\pi s(s+a)}}$ (up).

(b) The force on the bottom is the same as before, $\mu_0 I^2/2\pi$ (up). On the left side, $\mathbf{B} = \dfrac{\mu_0 I}{2\pi y}\,\hat{\mathbf{z}}$;

$d\mathbf{F} = I(d\mathbf{l} \times \mathbf{B}) = I(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \times \left(\dfrac{\mu_0 I}{2\pi y}\,\hat{\mathbf{z}}\right) = \dfrac{\mu_0 I^2}{2\pi y}(-dx\,\hat{\mathbf{y}} + dy\,\hat{\mathbf{x}})$. But the x component cancels the corresponding term from the right side, and $F_y = -\dfrac{\mu_0 I^2}{2\pi}\displaystyle\int_{s/\sqrt{3}}^{(s/\sqrt{3}+a/2)} \frac{1}{y}\,dx$. Here $y = \sqrt{3}x$, so

$F_y = -\dfrac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(\dfrac{s/\sqrt{3}+a/2}{s/\sqrt{3}}\right) = -\dfrac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(1 + \dfrac{\sqrt{3}a}{2s}\right)$. The force on the right side is the same, so the net force on the triangle is $\boxed{\dfrac{\mu_0 I^2}{2\pi}\left[1 - \dfrac{2}{\sqrt{3}}\ln\left(1 + \dfrac{\sqrt{3}a}{2s}\right)\right]}$.



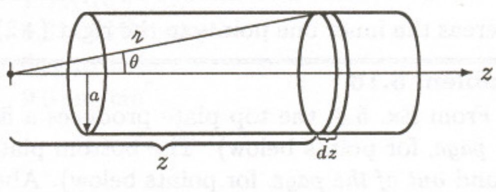
Problem 5.11

Use Eq. 5.38 for a ring of width dz, with $I \to nI\,dz$:

$B = \dfrac{\mu_0 nI}{2}\displaystyle\int \frac{a^2}{(a^2+z^2)^{3/2}}\,dz$. But $z = a\cot\theta$,

so $dz = -\dfrac{a}{\sin^2\theta}\,d\theta$, and $\dfrac{1}{(a^2+z^2)^{3/2}} = \dfrac{\sin^3\theta}{a^3}$.

So



$$B = \frac{\mu_0 nI}{2}\int \frac{a^2\sin^3\theta}{a^3\sin^2\theta}(-a\,d\theta) = -\frac{\mu_0 nI}{2}\int \sin\theta\,d\theta = \frac{\mu_0 nI}{2}\cos\theta\Big|_{\theta_1}^{\theta_2} = \boxed{\frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1)}.$$

For an infinite solenoid, $\theta_2 = 0$, $\theta_1 = \pi$, so $(\cos\theta_2 - \cos\theta_1) = 1 - (-1) = 2$, and $B = \boxed{\mu_0 nI}$. ✓

Problem 5.12

Magnetic attraction per unit length (Eqs. 5.37 and 5.13): $f_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$.

Electric field of one wire (Eq. 2.9): $E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}$. Electric repulsion per unit length on the other wire: $f_e = \frac{1}{2\pi\epsilon_0}\frac{\lambda^2}{d}$. They balance when $\mu_0 v^2 = \frac{1}{\epsilon_0}$, or $\boxed{v = \frac{1}{\sqrt{\epsilon_0\mu_0}}}$. Putting in the numbers,

$v = \frac{1}{\sqrt{(8.85\times 10^{-12})(4\pi\times 10^{-7})}} = \boxed{3.00\times 10^8\ \mathrm{m/s}.}$ This is precisely the *speed of light(!)*, so in fact you could *never* get the wires going fast enough; the electric force always dominates.

Problem 5.13

(a) $\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{\text{enc}} \Rightarrow \boxed{\mathbf{B} = \left\{\begin{array}{ll} 0, & \text{for } s < a; \\ \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

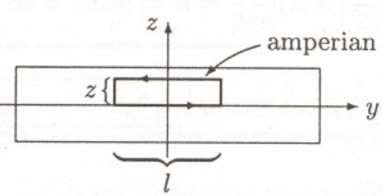
(b) $J = ks$; $I = \int_0^a J\,da = \int_0^a ks(2\pi s)\,ds = \frac{2\pi k a^3}{3} \Rightarrow k = \frac{3I}{2\pi a^3}$. $I_{\text{enc}} = \int_0^s J\,da = \int_0^s k\bar{s}(2\pi\bar{s})\,d\bar{s} = \frac{2\pi k s^3}{3} = I\frac{s^3}{a^3}$, for $s < a$; $I_{\text{enc}} = I$, for $s > a$. So $\boxed{\mathbf{B} = \left\{\begin{array}{ll} \frac{\mu_0 I s^2}{2\pi a^3}\,\hat{\boldsymbol{\phi}}, & \text{for } s < a; \\ \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

Problem 5.14

By the right-hand-rule, the field points in the $-\hat{\mathbf{y}}$ direction for $z > 0$, and in the $+\hat{\mathbf{y}}$ direction for $z < 0$. At $z = 0, B = 0$. Use the amperian loop shown:

$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0 l z J \Rightarrow \boxed{\mathbf{B} = -\mu_0 J z\,\hat{\mathbf{y}}}$ $(-a < z < a)$. If $z > a, I_{\text{enc}} = \mu_0 l a J$,

so $\boxed{\mathbf{B} = \left\{\begin{array}{ll} -\mu_0 J a\,\hat{\mathbf{y}}, & \text{for } z > +a; \\ +\mu_0 J a\,\hat{\mathbf{y}}, & \text{for } z > -a. \end{array}\right\}}$



Problem 5.15

The field inside a solenoid is $\mu_0 n I$, and outside it is zero. The outer solenoid's field points to the left $(-\hat{\mathbf{z}})$, whereas the inner one points to the right $(+\hat{\mathbf{z}})$. So: (i) $\boxed{\mathbf{B} = \mu_0 I(n_1 - n_2)\,\hat{\mathbf{z}},}$ (ii) $\boxed{\mathbf{B} = -\mu_0 I n_2\,\hat{\mathbf{z}},}$ (iii) $\boxed{\mathbf{B} = 0.}$

Problem 5.16

From Ex. 5.8, the top plate produces a field $\mu_0 K/2$ (aiming *out of the page*, for points above it, and *into the page*, for points below). The bottom plate produces a field $\mu_0 K/2$ (aiming *into the page*, for points above it, and *out of the page*, for points below). Above and below *both* plates the two fields cancel; *between* the plates they add up to $\mu_0 K$, pointing *in*.

(a) $\boxed{B = \mu_0 \sigma v \text{ (in)}}$ betweem the plates, $\boxed{B = 0}$ elsewhere.

(b) The Lorentz force law says $\mathbf{F} = \int(\mathbf{K}\times\mathbf{B})\,da$, so the force *per unit area* is $\mathbf{f} = \mathbf{K}\times\mathbf{B}$. Here $K = \sigma v$, to the right, and $\mathbf{B}$ (the field of the lower plate) is $\mu_0\sigma v/2$, into the page. So $\boxed{f_m = \mu_0\sigma^2 v^2/2 \text{ (up)}.}$

(c) The electric field of the lower plate is $\sigma/2\epsilon_0$; the electric force per unit area on the upper plate is $\boxed{f_e = \sigma^2/2\epsilon_0 \text{ (down)}.}$ They balance if $\mu_0 v^2 = 1/\epsilon_0$, or $\boxed{v = 1/\sqrt{\epsilon_0\mu_0} = c}$ (the speed of light), as in Prob. 5.12.

Problem 5.17

We might as well orient the axes so the field point $\mathbf{r}$ lies on the y axis: $\mathbf{r} = (0, y, 0)$. Consider a source point at (x', y', z') on loop #1:

$$\boldsymbol{\mathscr{r}} = -x'\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} - z'\,\hat{\mathbf{z}};\;\; d\mathbf{l}' = dx'\,\hat{\mathbf{x}} + dy'\,\hat{\mathbf{y}};$$

$$d\mathbf{l}' \times \boldsymbol{\mathscr{r}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ dx' & dy' & 0 \\ -x' & (y-y') & -z' \end{vmatrix} = (-z'\,dy')\,\hat{\mathbf{x}} + (z'\,dx')\,\hat{\mathbf{y}} + [(y-y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}.$$

$$d\mathbf{B}_1 = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}' \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi}\frac{(-z'\,dy')\,\hat{\mathbf{x}} + (z'\,dx')\,\hat{\mathbf{y}} + [(y-y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}}{[(x')^2 + (y-y')^2 + (z')^2]^{3/2}}.$$

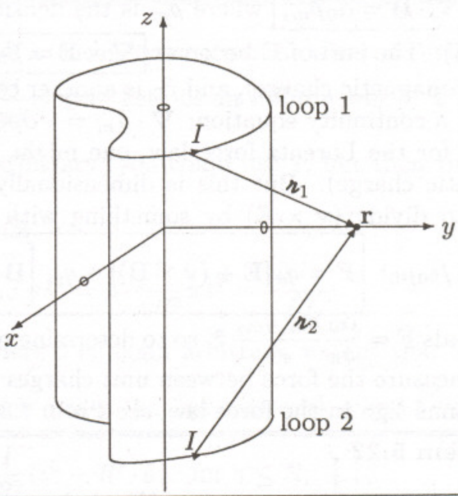
Now consider the symmetrically placed source element on loop #2, at $(x', y', -z')$. Since z' changes sign, while everything else is the same, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components from $d\mathbf{B}_1$ and $d\mathbf{B}_2$ cancel, leaving only a $\hat{\mathbf{z}}$ component. qed

With this, Ampére's law yields immediately:

$$\boxed{\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid;} \\ 0, & \text{outside} \end{cases}}$$

(the same as for a circular solenoid—Ex. 5.9).

For the toroid, $N/2\pi s = n$ (the number of turns per unit length), so Eq. 5.58 yields $B = \mu_0 n I$ inside, and zero outside, consistent with the solenoid. [*Note:* $N/2\pi s = n$ applies only if the toroid is large in circumference, so that s is essentially constant over the cross-section.]



Problem 5.18

$\boxed{\text{It doesn't matter.}}$ According to Theorem 2, in Sect. 1.6.2, $\int \mathbf{J}\cdot d\mathbf{a}$ is independent of surface, for any given boundary line, provided that $\mathbf{J}$ is divergenceless, which it *is*, for steady currents (Eq. 5.31).

Problem 5.19

(a) $\rho = \dfrac{\text{charge}}{\text{volume}} = \dfrac{\text{charge}}{\text{atom}} \cdot \dfrac{\text{atoms}}{\text{mole}} \cdot \dfrac{\text{moles}}{\text{gram}} \cdot \dfrac{\text{grams}}{\text{volume}} = (e)(N)\left(\dfrac{1}{M}\right)(d)$, where

$$\begin{aligned} e &= \text{charge of electron} &&= 1.6\times10^{-19}\,\text{C}, \\ N &= \text{Avogadro's number} &&= 6.0\times10^{23}\,\text{mole}, \\ M &= \text{atomic mass of copper} &&= 64\,\text{gm/mole}, \\ d &= \text{density of copper} &&= 9.0\,\text{gm/cm}^3. \end{aligned}$$

$$\rho = (1.6\times10^{-19})(6.0\times10^{23})\left(\frac{9.0}{64}\right) = \boxed{1.4\times10^4\,\text{C/cm}^3.}$$

(b) $J = \dfrac{I}{\pi s^2} = \rho v \Rightarrow v = \dfrac{I}{\pi s^2 \rho} = \dfrac{1}{\pi(2.5\times10^{-3})(1.4\times10^4)} = \boxed{9.1\times10^{-3}\,\text{cm/s},}$ or about 33 cm/hr. This is astonishingly small—literally slower than a snail's pace.

(c) From Eq. 5.37, $f_m = \dfrac{\mu_0}{2\pi}\left(\dfrac{I_1 I_2}{d}\right) = \dfrac{(4\pi\times10^{-7})}{2\pi} = \boxed{2\times10^{-7}\,\text{N/cm}.}$

(d) $E = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda}{d}$; $\quad f_e = \dfrac{1}{2\pi\epsilon_0}\left(\dfrac{\lambda_1\lambda_2}{d}\right) = \dfrac{1}{v^2}\dfrac{1}{2\pi\epsilon_0}\left(\dfrac{I_1 I_2}{d}\right) = \left(\dfrac{c^2}{v^2}\right)\dfrac{\mu_0}{2\pi}\left(\dfrac{I_1 I_2}{d}\right) = \dfrac{c^2}{v^2} f_m$, where $c \equiv 1/\sqrt{\epsilon_0\mu_0} = 3.00 \times 10^8$ m/s. Here $\dfrac{f_e}{f_m} = \dfrac{c^2}{v^2} = \left(\dfrac{3.0\times 10^{10}}{9.1\times 10^{-3}}\right)^2 = \boxed{1.1\times 10^{25}}.$

$f_e = (1.1\times 10^{25})(2\times 10^{-7}) = \boxed{2\times 10^{18}\ \text{N/cm.}}$

Problem 5.20

Ampére's law says $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$. Together with the continuity equation (5.29) this gives $\nabla\cdot(\nabla\times\mathbf{B}) = \mu_0\nabla\cdot\mathbf{J} = -\mu_0\partial\rho/\partial t$, which is *inconsistent* with div(curl)=0 unless ρ is constant (magnetostatics). The other Maxwell equations are OK: $\nabla\times\mathbf{E} = 0 \Rightarrow \nabla\cdot(\nabla\times\mathbf{E}) = 0$ (✓), and as for the two divergence equations, there is no relevant vanishing second derivative (the other one is curl(grad), which doesn't involve the divergence).

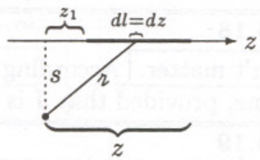
Problem 5.21

At this stage I'd expect no changes in Gauss's law or Ampére's law. The divergence of $\mathbf{B}$ would take the form $\boxed{\nabla\cdot\mathbf{B} = \alpha_0\rho_m,}$ where ρ_m is the density of magnetic charge, and α_0 is some constant (analogous to ϵ_0 and μ_0). The curl of $\mathbf{E}$ becomes $\boxed{\nabla\times\mathbf{E} = \beta_0\mathbf{J}_m,}$ where $\mathbf{J}_m$ is the magnetic current density (representing the flow of magnetic charge), and β_0 is another constant. Presumably magnetic charge is conserved, so ρ_m and $\mathbf{J}_m$ satisfy a continuity equation: $\nabla\cdot\mathbf{J}_m = -\partial\rho_m/\partial t$.

As for the Lorentz force law, one might guess something of the form $q_m[\mathbf{B} + (\mathbf{v}\times\mathbf{E})]$ (where q_m is the magnetic charge). But this is dimensionally impossible, since E has the same units as vB. Evidently we need to divide $(\mathbf{v}\times\mathbf{E})$ by something with the dimensions of velocity-squared. The natural candidate is $c^2 = 1/\epsilon_0\mu_0$: $\boxed{\mathbf{F} = q_e[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] + q_m\left[\mathbf{B} - \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})\right]}$. In this form the magnetic analog to Coulomb's law reads $\mathbf{F} = \dfrac{\alpha_0}{4\pi}\dfrac{q_{m_1}q_{m_2}}{r^2}\hat{\mathbf{r}}$, so to determine α_0 we would first introduce (arbitrarily) a unit of magnetic charge, then measure the force between unit charges at a given separation. [For further details, and an explanation of the minus sign in the force law, see Prob. 7.35.]

Problem 5.22

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{I\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{z_1}^{z_2}\frac{dz}{\sqrt{z^2+s^2}}$$



$$= \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]\Big|_{z_1}^{z_2} = \boxed{\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{z_1+\sqrt{(z_1)^2+s^2}}\right]\hat{\mathbf{z}}}$$

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{(z_2)^2+s^2}}\frac{s}{\sqrt{(z_2)^2+s^2}} - \frac{1}{z_1+\sqrt{(z_1)^2+s^2}}\frac{s}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{(z_2)^2+s^2}}{(z_2)^2-[(z_2)^2+s^2]}\frac{1}{\sqrt{(z_2)^2+s^2}} - \frac{z_1-\sqrt{(z_1)^2+s^2}}{z_1^2-[(z_1)^2+s^2]}\frac{1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2+s^2}} + 1\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - \frac{z_1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}},
\end{aligned}$$

or, since $\sin\theta_1 = \dfrac{z_1}{\sqrt{(z_1)^2+s^2}}$ and $\sin\theta_2 = \dfrac{z_2}{\sqrt{(z_2)^2+s^2}}$,

$$= \boxed{\frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\,\hat{\boldsymbol{\phi}}}\ \text{(as in Eq. 5.35).}$$

Problem 5.23

$A_\phi = k \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\,\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}}$; $\mathbf{J} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\boldsymbol{\phi}} = \boxed{\frac{k}{\mu_0 s^2}\,\hat{\boldsymbol{\phi}}.}$

Problem 5.24

$\nabla \cdot \mathbf{A} = -\frac{1}{2}\nabla \cdot (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}[\mathbf{B} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{B})] = 0$, since $\nabla \times \mathbf{B} = 0$ ($\mathbf{B}$ is uniform) and $\nabla \times \mathbf{r} = 0$ (Prob. 1.62). $\nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}[(\mathbf{B} \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{r}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{r})]$. But $(\mathbf{r} \cdot \nabla)\mathbf{B} = 0$ and $\nabla \cdot \mathbf{B} = 0$ (since $\mathbf{B}$ is uniform), and $\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$. Finally, $(\mathbf{B} \cdot \nabla)\mathbf{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}} = \mathbf{B}$. So $\nabla \times \mathbf{A} = -\frac{1}{2}(\mathbf{B} - 3\mathbf{B}) = \mathbf{B}$. qed

Problem 5.25

(a) $\mathbf{A}$ points in the same direction as $\mathbf{I}$, and is a function only of s (the distance from the wire). In cylindrical coordinates, then, $\mathbf{A} = A(s)\,\hat{\mathbf{z}}$, so $\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A}{\partial s}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$ (the field of an infinite wire). Therefore $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi s}$, and $\boxed{\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln(s/a)\,\hat{\mathbf{z}}}$ (the constant a is arbitrary; you could use 1, but then the units look fishy). $\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = 0.$ ✓ $\nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}} = \mathbf{B}.$ ✓

(b) Here Ampére's law gives $\oint \mathbf{B} \cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0 J\,\pi s^2 = \mu_0 \frac{I}{\pi R^2}\pi s^2 = \frac{\mu_0 I s^2}{R^2}$.

$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{Is}{R^2}\,\hat{\boldsymbol{\phi}}$. $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi}\frac{s}{R^2} \Rightarrow \mathbf{A} = -\frac{\mu_0 I}{4\pi R^2}(s^2 - b^2)\,\hat{\mathbf{z}}$. Here b is again arbitrary, except that since $\mathbf{A}$ must be continuous at R, $-\frac{\mu_0 I}{2\pi}\ln(R/a) = -\frac{\mu_0 I}{4\pi R^2}(R^2 - b^2)$, which means that we must pick a and b such that $2\ln(R/b) = 1 - (b/R)^2$. I'll use $a = b = R$. Then

$$\boxed{\mathbf{A} = \begin{cases} -\dfrac{\mu_0 I}{4\pi R^2}(s^2 - R^2)\,\hat{\mathbf{z}}, & \text{for } s \le R; \\ -\dfrac{\mu_0 I}{2\pi}\ln(s/R)\,\hat{\mathbf{z}}, & \text{for } s \ge R. \end{cases}}$$

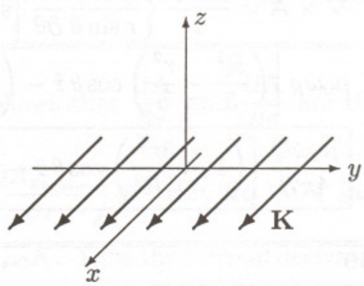
Problem 5.26

$\mathbf{K} = K\,\hat{\mathbf{x}} \Rightarrow \mathbf{B} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}$ (plus for $z < 0$, minus for $z > 0$).

$\mathbf{A}$ is parallel to $\mathbf{K}$, and depends only on z, so $\mathbf{A} = A(z)\,\hat{\mathbf{x}}$.

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A(z) & 0 & 0 \end{vmatrix} = \frac{\partial A}{\partial z}\,\hat{\mathbf{y}} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}.$$

$\boxed{\mathbf{A} = -\frac{\mu_0 K}{2}|z|\,\hat{\mathbf{x}}}$ will do the job—or this plus any constant.



Problem 5.27

(a) $\nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{\mathbf{J}}{\mathscr{r}}\right) d\tau'$. $\nabla \cdot \left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla \cdot \mathbf{J}) + \mathbf{J} \cdot \nabla\left(\frac{1}{\mathscr{r}}\right)$. But the first term is zero, because $\mathbf{J}(\mathbf{r}')$ is a function of the *source* coordinates, not the *field* coordinates. And since $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$, $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right)$. So

$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\mathbf{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$. But $\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J})+\mathbf{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$, and $\nabla'\cdot\mathbf{J} = 0$ in magnetostatics (Eq. 5.31). So $\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$, and hence, by the divergence theorem, $\nabla\cdot\mathbf{A} = -\frac{\mu_0}{4\pi}\int\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau' = -\frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}'$, where the integral is now over the *surface* surrounding all the currents. But $\mathbf{J} = 0$ on this surface, so $\nabla\cdot\mathbf{A} = 0$. ✓

(b) $\nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau' = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}(\nabla\times\mathbf{J}) - \mathbf{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$. But $\nabla\times\mathbf{J} = 0$ (since $\mathbf{J}$ is not a function of $\mathbf{r}$), and $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ (Eq. 1.101), so $\nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \mathbf{B}$. ✓

(c) $\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau'$. But $\nabla^2\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \mathbf{J}\nabla^2\left(\frac{1}{\mathscr{r}}\right)$ (once again, $\mathbf{J}$ is a *constant*, as far as differentiation with respect to $\mathbf{r}$ is concerned), and $\nabla^2\left(\frac{1}{\mathscr{r}}\right) = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$ (Eq. 1.102).

So $\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{r}')\left[-4\pi\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau' = -\mu_0\mathbf{J}(\mathbf{r})$. ✓

Problem 5.28

$\mu_0 I = \oint\mathbf{B}\cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}}\nabla U\cdot d\mathbf{l} = -[U(\mathbf{b}) - U(\mathbf{a})]$ (by the gradient theorem), so $U(\mathbf{b}) \neq U(\mathbf{a})$. qed

For an infinite straight wire, $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\boxed{U = -\frac{\mu_0 I\phi}{2\pi}}$ would do the job, in the sense that $-\nabla U = \frac{\mu_0 I}{2\pi}\nabla(\phi) = \frac{\mu_0 I}{2\pi}\frac{1}{s}\frac{\partial\phi}{\partial\phi}\hat{\boldsymbol{\phi}} = \mathbf{B}$. But when ϕ advances by 2π, this function does *not* return to its initial value; it works (say) for $0 \le \phi < 2\pi$, but at 2π it "jumps" back to zero.

Problem 5.29

Use Eq. 5.67, with $R \to \bar{r}$ and $\sigma \to \rho\, d\bar{r}$:

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}\int_0^r\bar{r}^4\,d\bar{r} + \frac{\mu_0\omega\rho}{3}r\sin\theta\,\hat{\boldsymbol{\phi}}\int_r^R\bar{r}\,d\bar{r}\\
&= \left(\frac{\mu_0\omega\rho}{3}\right)\sin\theta\left[\frac{1}{r^2}\left(\frac{r^5}{5}\right) + \frac{r}{2}(R^2 - r^2)\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0\omega\rho}{2}r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\hat{\boldsymbol{\phi}}.\\
\mathbf{B} &= \nabla\times\mathbf{A} = \frac{\mu_0\omega\rho}{2}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left[r^2\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right]\hat{\boldsymbol{\theta}}\right\}\\
&= \mu_0\omega\rho\left[\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\cos\theta\,\hat{\mathbf{r}} - \left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right]. \text{ But } \rho = \frac{Q}{(4/3)\pi R^3}, \text{ so}\\
&= \boxed{\frac{\mu_0\omega Q}{4\pi R}\left[\left(1 - \frac{3r^2}{5R^2}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1 - \frac{6r^2}{5R^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right]}.
\end{aligned}$$

Problem 5.30

(a)
$$\left\{\begin{aligned}
-\frac{\partial W_z}{\partial x} &= F_y \Rightarrow W_z(x,y,z) = -\int_0^x F_y(x',y,z)\,dx' + C_1(y,z).\\
\frac{\partial W_y}{\partial x} &= F_z \Rightarrow W_y(x,y,z) = +\int_0^x F_z(x',y,z)\,dx' + C_2(y,z).
\end{aligned}\right\}$$

These satisfy (ii) and (iii), for *any* C_1 and C_2; it remains to choose these functions so as to satisfy (i):

$-\int_0^x \frac{\partial F_y(x',y,z)}{\partial y}\,dx' + \frac{\partial C_1}{\partial y} - \int_0^x \frac{\partial F_z(x',y,z)}{\partial z}\,dx' - \frac{\partial C_2}{\partial z} = F_x(x,y,z)$. But $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$, so $\int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}\,dx' + \frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = F_x(x,y,z)$. Now $\int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}\,dx' = F_x(x,y,z) - F_x(0,y,z)$, so $\frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = F_x(0,y,z)$. We may as well pick $C_2 = 0$, $C_1(y,z) = \int_0^y F_x(0,y',z)\,dy'$, and we're done, with

$$W_x = 0;\quad W_y = \int_0^x F_z(x',y,z)\,dx';\quad W_z = \int_0^y F_x(0,y',z)\,dy' - \int_0^x F_y(x',y,z)\,dx'.$$

(b) $\nabla\times\mathbf{W} = \left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y}\right)\hat{\mathbf{z}}$

$= \left[F_x(0,y,z) - \int_0^x \frac{\partial F_y(x',y,z)}{\partial y}\,dx' - \int_0^x \frac{\partial F_z(x',y,z)}{\partial z}\,dx'\right]\hat{\mathbf{x}} + [0 + F_y(x,y,z)]\,\hat{\mathbf{y}} + [F_z(x,y,z) - 0]\,\hat{\mathbf{z}}$.

But $\nabla\cdot\mathbf{F} = 0$, so the $\hat{\mathbf{x}}$ term is $\left[F_x(0,y,z) + \int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}\,dx'\right] = F_x(0,y,z) + F_x(x,y,z) - F_x(0,y,z)$, so $\nabla\times\mathbf{W} = \mathbf{F}$. ✓

$$\nabla\cdot\mathbf{W} = \frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z} = 0 + \int_0^x \frac{\partial F_z(x',y,z)}{\partial y}\,dx' + \int_0^y \frac{\partial F_x(0,y',z)}{\partial z}\,dy' - \int_0^x \frac{\partial F_y(x',y,z)}{\partial z}\,dx' \neq 0,$$

in general.

(c) $W_y = \int_0^x x'\,dx' = \frac{x^2}{2}$; $W_z = \int_0^y y'\,dy' - \int_0^x z\,dx' = \frac{y^2}{2} - zx$.

$$\boxed{\mathbf{W} = \frac{x^2}{2}\,\hat{\mathbf{y}} + \left(\frac{y^2}{2} - zx\right)\hat{\mathbf{z}}.}\qquad \nabla\times\mathbf{W} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^2/2 & (y^2/2 - zx) \end{vmatrix} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}} = \mathbf{F}.\ ✓$$

Problem 5.31

(a) At the surface of the solenoid, $\mathbf{B}_{\text{above}} = 0$, $\mathbf{B}_{\text{below}} = \mu_0 nI\,\hat{\mathbf{z}} = \mu_0 K\,\hat{\mathbf{z}}$; $\hat{\mathbf{n}} = \hat{\mathbf{s}}$; so $\mathbf{K}\times\hat{\mathbf{n}} = -K\,\hat{\mathbf{z}}$. Evidently Eq. 5.74 holds. ✓

(b) In Eq. 5.67, both expressions reduce to $(\mu_0 R^2\omega\sigma/3)\sin\theta\,\hat{\boldsymbol{\phi}}$ at the surface, so Eq. 5.75 is satisfied. $\left.\frac{\partial\mathbf{A}}{\partial r}\right|_{R+} = \left.\frac{\mu_0 R^4\omega\sigma}{3}\left(-\frac{2\sin\theta}{r^3}\right)\hat{\boldsymbol{\phi}}\right|_R = -\frac{2\mu_0 R\omega\sigma}{3}\sin\theta\,\hat{\boldsymbol{\phi}}$; $\left.\frac{\partial\mathbf{A}}{\partial r}\right|_{R-} = \frac{\mu_0 R\omega\sigma}{3}\sin\theta\,\hat{\boldsymbol{\phi}}$. So the left side of Eq. 5.76 is $-\mu_0 R\omega\sigma\sin\theta\,\hat{\boldsymbol{\phi}}$. Meanwhile $\mathbf{K} = \sigma\mathbf{v} = \sigma(\boldsymbol{\omega}\times\mathbf{r}) = \sigma\omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, so the right side of Eq. 5.76 is $-\mu_0\sigma\omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, and the equation is satisfied.

Problem 5.32

Because $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$ at every point on the surface, it follows that $\frac{\partial\mathbf{A}}{\partial x}$ and $\frac{\partial\mathbf{A}}{\partial y}$ are the same above and below; any discontinuity is confined to the normal derivative.

$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \left(-\frac{\partial A_{y_{\text{above}}}}{\partial z} + \frac{\partial A_{y_{\text{below}}}}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_{x_{\text{above}}}}{\partial z} - \frac{\partial A_{x_{\text{below}}}}{\partial z}\right)\hat{\mathbf{y}}$. But Eq. 5.74 says this equals $\mu_0 K(-\hat{\mathbf{y}})$. So $\frac{\partial A_{y_{\text{above}}}}{\partial z} = \frac{\partial A_{y_{\text{below}}}}{\partial z}$, and $\frac{\partial A_{x_{\text{above}}}}{\partial z} - \frac{\partial A_{x_{\text{below}}}}{\partial z} = -\mu_0 K$. Thus the *normal* derivative of the *component of* $\mathbf{A}$ *parallel to* $\mathbf{K}$ suffers a discontinuity $-\mu_0 K$, or, more compactly: $\boxed{\frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}.}$

Problem 5.33

(Same idea as Prob. 3.33.) Write $\mathbf{m} = (\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{m}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = m\cos\theta\,\hat{\mathbf{r}} - m\sin\theta\,\hat{\boldsymbol{\theta}}$ (Fig. 5.54). Then $3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m} = 3m\cos\theta\,\hat{\mathbf{r}} - m\cos\theta\,\hat{\mathbf{r}} + m\sin\theta\,\hat{\boldsymbol{\theta}} = 2m\cos\theta\,\hat{\mathbf{r}} + m\sin\theta\,\hat{\boldsymbol{\theta}}$, and Eq. 5.87 ⇔ Eq. 5.86. qed

Problem 5.34

(a) $\mathbf{m} = I\mathbf{a} = \boxed{I\pi R^2\,\hat{\mathbf{z}}.}$

(b) $\mathbf{B} \approx \boxed{\dfrac{\mu_0}{4\pi}\dfrac{I\pi R^2}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right).}$

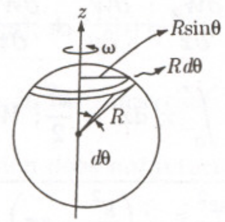
(c) On the z axis, $\theta = 0$, $r = z$, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ (for $z > 0$), so $\boxed{\mathbf{B} \approx \dfrac{\mu_0 I R^2}{2z^3}\,\hat{\mathbf{z}}}$ (for $z < 0$, $\theta = \pi$, $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$, so the field is the same, with $|z|^3$ in place of z^3). The exact answer (Eq. 5.38) reduces (for $z \gg R$) to $B \approx \mu_0 I R^2/2|z|^3$, so they agree.

Problem 5.35

For a ring, $m = I\pi r^2$. Here $I \to \sigma v\,dr = \sigma\omega r\,dr$, so $m = \int_0^R \pi r^2 \sigma\omega r\,dr = \boxed{\pi\sigma\omega R^4/4.}$

Problem 5.36

The total charge on the shaded ring is $dq = \sigma(2\pi R\sin\theta)R\,d\theta$. The time for one revolution is $dt = 2\pi/\omega$. So the current in the ring is $I = \dfrac{dq}{dt} = \sigma\omega R^2\sin\theta\,d\theta$. The area of the ring is $\pi(R\sin\theta)^2$, so the magnetic moment of the ring is $dm = (\sigma\omega R^2\sin\theta\,d\theta)\pi R^2\sin^2\theta$, and the total dipole moment of the shell is



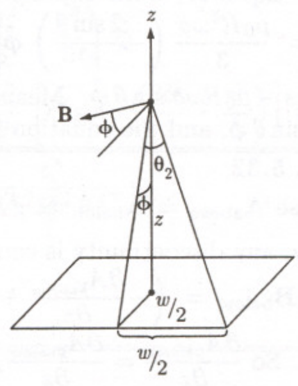
$m = \sigma\omega\pi R^4\int_0^\pi \sin^3\theta\,d\theta = (4/3)\sigma\omega\pi R^4$, or $\boxed{\mathbf{m} = \dfrac{4\pi}{3}\sigma\omega R^4\,\hat{\mathbf{z}}.}$

The dipole term in the multipole expansion for $\mathbf{A}$ is therefore $\mathbf{A}_{\text{dip}} = \dfrac{\mu_0}{4\pi}\dfrac{4\pi}{3}\sigma\omega R^4\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}} = \dfrac{\mu_0\sigma\omega R^4}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}$, which is also the *exact* potential (Eq. 5.67); evidently a spinning sphere produces a perfect dipole field, with no higher multipole contributions.

Problem 5.37

The field of one side is given by Eq. 5.35, with $s \to \sqrt{z^2 + (w/2)^2}$ and $\sin\theta_2 = -\sin\theta_1 = \dfrac{(w/2)}{\sqrt{z^2 + w^2/2}}$;

$B = \dfrac{\mu_0 I}{4\pi}\dfrac{w}{\sqrt{z^2 + (w^2/4)}\sqrt{z^2 + (w^2/2)}}$. To pick off the vertical component, multiply by $\sin\phi = \dfrac{(w/2)}{\sqrt{z^2 + (w/2)^2}}$; for all four sides, multiply by 4: $\boxed{\mathbf{B} = \dfrac{\mu_0 I}{2\pi}\dfrac{w^2}{(z^2 + w^2/4)\sqrt{z^2 + w^2/2}}\,\hat{\mathbf{z}}.}$ For $z \gg w$, $\mathbf{B} \approx \dfrac{\mu_0 I w^2}{2\pi z^3}\,\hat{\mathbf{z}}$. The field of a dipole $\boxed{m = Iw^2,}$ for points on the z axis (Eq. 5.86, with $r \to z$, $\hat{\mathbf{r}} \to \hat{\mathbf{z}}$, $\theta = 0$) is $\mathbf{B} = \dfrac{\mu_0}{2\pi}\dfrac{m}{z^3}\,\hat{\mathbf{z}}$. ✓



Problem 5.38

The mobile charges *do* pull in toward the axis, but the resulting concentration of (negative) charge sets up an *electric* field that repels away further accumulation. Equilibrium is reached when the electric repulsion on a mobile charge q balances the magnetic attraction: $\mathbf{F} = q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] = 0 \Rightarrow \mathbf{E} = -(\mathbf{v}\times\mathbf{B})$. Say the current is in the z direction: $\mathbf{J} = \rho_- v\,\hat{\mathbf{z}}$ (where ρ_- and v are both negative).

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 J\pi s^2 \Rightarrow \mathbf{B} = \frac{\mu_0\rho_- vs}{2}\,\hat{\boldsymbol{\phi}};$$

$$\int \mathbf{E}\cdot d\mathbf{a} = E\,2\pi sl = \frac{1}{\epsilon_0}(\rho_+ + \rho_-)\pi s^2 l \Rightarrow \mathbf{E} = \frac{1}{2\epsilon_0}(\rho_+ + \rho_-)s\,\hat{\mathbf{s}}.$$

$$\frac{1}{2\epsilon_0}(\rho_+ + \rho_-)s\,\hat{\mathbf{s}} = -\left[(v\,\hat{\mathbf{z}})\times\left(\frac{\mu_0\rho_- vs}{2}\,\hat{\boldsymbol{\phi}}\right)\right] = \frac{\mu_0}{2}\rho_- v^2 s\,\hat{\mathbf{s}} \Rightarrow \rho_+ + \rho_- = \rho_-(\epsilon_0\mu_0 v^2) = \rho_-\left(\frac{v^2}{c^2}\right).$$

Evidently $\rho_+ = -\rho_-\left(1-\frac{v^2}{c^2}\right) = \frac{\rho_-}{\gamma^2}$, or $\rho_- = -\gamma^2\rho_+$. In this naive model, the mobile negative charges fill a smaller inner cylinder, leaving a shell of positive (stationary) charge at the outside. But since $v \ll c$, the effect is extremely small.

Problem 5.39

(a) If *positive* charges flow to the *right*, they are deflected $\boxed{\text{down,}}$ and the bottom plate acquires a *positive* charge.

(b) $qvB = qE \Rightarrow E = vB \Rightarrow V = Et = \boxed{vBt,}$ with the *bottom* at higher potential.

(c) If *negative* charges flow to the *left*, they are *also* deflected down, and the bottom plate acquires a *negative* charge. The potential difference is still the same, but this time the *top* plate is at the higher potential.

Problem 5.40

From Eq. 5.17, $\mathbf{F} = I\int(d\mathbf{l}\times\mathbf{B})$. But $\mathbf{B}$ is constant, in this case, so it comes outside the integral: $\mathbf{F} = I\left(\int d\mathbf{l}\right)\times\mathbf{B}$, and $\int d\mathbf{l} = \mathbf{w}$, the vector displacement from the point at which the wire first enters the field to the point where it leaves. Since $\mathbf{w}$ and $\mathbf{B}$ are perpendicular, $F = IBw$, and $\mathbf{F}$ is perpendicular to $\mathbf{w}$.

Problem 5.41

The angular momentum acquired by the particle as it moves out from the center to the edge is

$$\mathbf{L} = \int\frac{d\mathbf{L}}{dt}\,dt = \int \mathbf{N}\,dt = \int(\mathbf{r}\times\mathbf{F})\,dt = \int \mathbf{r}\times q(\mathbf{v}\times\mathbf{B})\,dt = q\int\mathbf{r}\times(d\mathbf{l}\times\mathbf{B}) = q\left[\int(\mathbf{r}\cdot\mathbf{B})\,d\mathbf{l} - \int\mathbf{B}(\mathbf{r}\cdot d\mathbf{l})\right].$$

But $\mathbf{r}$ is perpendicular to $\mathbf{B}$, so $\mathbf{r}\cdot\mathbf{B} = 0$, and $\mathbf{r}\cdot d\mathbf{l} = \mathbf{r}\cdot d\mathbf{r} = \frac{1}{2}d(\mathbf{r}\cdot\mathbf{r}) = \frac{1}{2}d(r^2) = r\,dr = (1/2\pi)(2\pi r\,dr)$. So $\mathbf{L} = -\frac{q}{2\pi}\int_0^R \mathbf{B}\,2\pi r\,dr = -\frac{q}{2\pi}\int\mathbf{B}\,da$. It follows that $\boxed{L = -\frac{q}{2\pi}\Phi,}$ where $\Phi = \int B\,da$ is the total flux. In particular, if $\Phi = 0$, then $L = 0$, and the charge emerges with zero angular momentum, which means it is going along a radial line. qed

Problem 5.42

From Eq. 5.24, $\mathbf{F} = \int(\mathbf{K}\times\mathbf{B}_{\text{ave}})\,da$. Here $\mathbf{K} = \sigma\mathbf{v}$, $\mathbf{v} = \omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, $da = R^2\sin\theta\,d\theta\,d\phi$, and $\mathbf{B}_{\text{ave}} = \frac{1}{2}(\mathbf{B}_{\text{in}} + \mathbf{B}_{\text{out}})$. From Eq. 5.68,

$$\begin{aligned}
\mathbf{B}_{\text{in}} &= \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\omega(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}). \text{ From Eq. 5.67,}\\
\mathbf{B}_{\text{out}} &= \boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\times\left(\frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}\right) = \frac{\mu_0 R^4\omega\sigma}{3}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}}\right]\\
&= \frac{\mu_0 R^4\omega\sigma}{3r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{\mu_0 R\omega\sigma}{3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \text{ (since } r = R).\\
\mathbf{B}_{\text{ave}} &= \frac{\mu_0 R\omega\sigma}{6}(4\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}).\\
\mathbf{K}\times\mathbf{B}_{\text{ave}} &= (\sigma\omega R\sin\theta)\left(\frac{\mu_0 R\omega\sigma}{6}\right)\left[\hat{\boldsymbol{\phi}}\times(4\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})\right] = \frac{\mu_0}{6}(\sigma\omega R)^2(4\cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\mathbf{r}})\sin\theta.
\end{aligned}$$

Picking out the z component of $\hat{\boldsymbol{\theta}}$ (namely, $-\sin\theta$) and of $\hat{\mathbf{r}}$ (namely, $\cos\theta$), we have $(\mathbf{K}\times\mathbf{B}_{\text{ave}})_z = -\frac{\mu_0}{2}(\sigma\omega R)^2\sin^2\theta\cos\theta$, so

$$F_z = -\frac{\mu_0}{2}(\sigma\omega R)^2 R^2\int\sin^3\theta\cos\theta\,d\theta\,d\phi = -\frac{\mu_0}{2}(\sigma\omega R^2)^2\,2\pi\left(\frac{\sin^4\theta}{4}\right)\Big|_0^{\pi/2}, \text{ or } \boxed{\mathbf{F} = -\frac{\mu_0\pi}{4}(\sigma\omega R^2)^2\,\hat{\mathbf{z}}.}$$

Problem 5.43

(a) $\mathbf{F} = m\mathbf{a} = q_e(\mathbf{v}\times\mathbf{B}) = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^2}(\mathbf{v}\times\hat{\mathbf{r}})$; $\boxed{\mathbf{a} = \frac{\mu_0}{4\pi}\frac{q_e q_m}{mr^3}(\mathbf{v}\times\mathbf{r}).}$

(b) Because $\mathbf{a}\perp\mathbf{v}$, $\mathbf{a}\cdot\mathbf{v} = 0$. But $\mathbf{a}\cdot\mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}\frac{d}{dt}(v^2) = v\frac{dv}{dt}$. So $\frac{dv}{dt} = 0$. qed

(c) $\frac{d\mathbf{Q}}{dt} = m(\mathbf{v}\times\mathbf{v}) + m(\mathbf{r}\times\mathbf{a}) - \frac{\mu_0 q_e q_m}{4\pi}\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = 0 + \frac{\mu_0 q_e q_m}{4\pi r^3}[\mathbf{r}\times(\mathbf{v}\times\mathbf{r})] - \frac{\mu_0 q_e q_m}{4\pi}\left(\frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right)$

$$= \frac{\mu_0 q_e q_m}{4\pi}\left\{\frac{1}{r^3}[r^2\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\mathbf{r}] - \frac{\mathbf{v}}{r} + \frac{\mathbf{r}}{r^2}\frac{d}{dt}(\sqrt{\mathbf{r}\cdot\mathbf{r}})\right\} = \frac{\mu_0 q_e q_m}{4\pi}\left[\frac{\mathbf{v}}{r} - \frac{(\hat{\mathbf{r}}\cdot\mathbf{v})}{r}\hat{\mathbf{r}} - \frac{\mathbf{v}}{r} + \frac{\hat{\mathbf{r}}}{2r}\frac{2(\mathbf{r}\cdot\mathbf{v})}{r}\right] = 0.\ \checkmark$$

(d) (i) $\mathbf{Q}\cdot\hat{\boldsymbol{\phi}} = Q(\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}}) = m(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\phi}} - \frac{\mu_0 q_e q_m}{4\pi}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}})$. But $\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = 0$, so $(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\phi}} = 0$. But $\mathbf{r} = r\,\hat{\mathbf{r}}$, and $\mathbf{v} = \frac{d\mathbf{l}}{dt} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\,\hat{\boldsymbol{\phi}}$ (where dots denote differentiation with respect to time), so

$$\mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & r\sin\theta\dot{\phi} \end{vmatrix} = (-r^2\sin\theta\dot{\phi})\,\hat{\boldsymbol{\theta}} + (r^2\dot{\theta})\,\hat{\boldsymbol{\phi}}.$$

Therefore $(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\phi}} = r^2\dot{\theta} = 0$, so θ is constant. qed

(ii) $\mathbf{Q}\cdot\hat{\mathbf{r}} = Q(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}) = m(\mathbf{r}\times\mathbf{v})\cdot\hat{\mathbf{r}} - \frac{\mu_0 q_e q_m}{4\pi}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})$. But $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$, and $(\mathbf{r}\times\mathbf{v})\perp\mathbf{r} \Rightarrow (\mathbf{r}\times\mathbf{v})\cdot\hat{\mathbf{r}} = 0$, so $Q\cos\theta = -\frac{\mu_0 q_e q_m}{4\pi}$, or $Q = -\frac{\mu_0 q_e q_m}{4\pi\cos\theta}$. And since θ is constant, so too is Q. qed

(iii) $\mathbf{Q}\cdot\hat{\boldsymbol{\theta}} = Q(\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\theta}}) = m(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\theta}} - \frac{\mu_0 q_e q_m}{4\pi}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}})$. But $\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\theta}} = -\sin\theta$, $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = 0$, and $(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\theta}} = -r^2\sin\theta\dot{\phi}$ (from (i)), so $-Q\sin\theta = -mr^2\sin\theta\dot{\phi} \Rightarrow \dot{\phi} = \frac{Q}{mr^2} = \frac{k}{r^2}$, with $\boxed{k \equiv \frac{Q}{m} = -\frac{\mu_0 q_e q_m}{4\pi m\cos\theta}.}$

(e) $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$, but $\dot{\theta} = 0$ and $\dot{\phi} = \frac{k}{r^2}$, so $\dot{r}^2 = v^2 - r^2\sin^2\theta\frac{k^2}{r^4} = v^2 - \frac{k^2\sin^2\theta}{r^2}$.

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{v^2 - (k\sin\theta/r)^2}{(k^2/r^4)} = r^2\left[\left(\frac{vr}{k}\right)^2 - \sin^2\theta\right]; \quad \boxed{\frac{dr}{d\phi} = r\sqrt{\left(\frac{vr}{k}\right)^2 - \sin^2\theta.}}$$

(f) $\displaystyle\int \frac{dr}{r\sqrt{(vr/k)^2 - \sin^2\theta}} = \int d\phi \Rightarrow \phi - \phi_0 = \frac{1}{\sin\theta}\sec^{-1}\left(\frac{vr}{k\sin\theta}\right)$; $\sec[(\phi-\phi_0)\sin\theta] = \dfrac{vr}{k\sin\theta}$, or

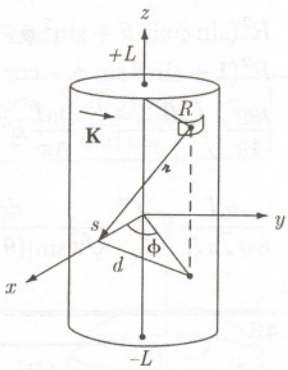
$\boxed{r(\phi) = \dfrac{A}{\cos[(\phi-\phi_0)\sin\theta]},}$ where $A \equiv -\dfrac{\mu_0 q_e q_m \tan\theta}{4\pi m v}$.

Problem 5.44

Put the field point on the x axis, so $\mathbf{r} = (s, 0, 0)$. Then $\mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{(\mathbf{K}\times\hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2}\,da$; $da = R\,d\phi\,dz$; $\mathbf{K} = K\hat{\boldsymbol{\phi}} = K(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})$; $\boldsymbol{\mathscr{r}} = (s - R\cos\phi)\,\hat{\mathbf{x}} - R\sin\phi\,\hat{\mathbf{y}} - z\,\hat{\mathbf{z}}$.

$$\mathbf{K}\times\boldsymbol{\mathscr{r}} = K\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\sin\phi & \cos\phi & 0 \\ (s - R\cos\phi) & (-R\sin\phi) & (-z)\end{vmatrix} =$$

$K[(-z\cos\phi)\,\hat{\mathbf{x}} + (-z\sin\phi)\,\hat{\mathbf{y}} + (R - s\cos\phi)\,\hat{\mathbf{z}}]$; $\mathscr{r}^2 = z^2 + R^2 + s^2 - 2Rs\cos\phi$. The x and y components integrate to zero (z integrand is odd, as in Prob. 5.17).



$$\begin{aligned}
B_z &= \frac{\mu_0}{4\pi}KR\int \frac{(R - s\cos\phi)}{(z^2 + R^2 + s^2 - 2Rs\cos\phi)^{3/2}}\,d\phi\,dz \\
&= \frac{\mu_0 KR}{4\pi}\int_0^{2\pi}(R - s\cos\phi)\left\{\int_{-\infty}^{\infty}\frac{dz}{(z^2+d^2)^{3/2}}\right\}d\phi,
\end{aligned}$$

where $d^2 \equiv R^2 + s^2 - 2Rs\cos\phi$. Now $\displaystyle\int_{-\infty}^{\infty}\frac{dz}{(z^2+d^2)^{3/2}} = \frac{2z}{d^2\sqrt{z^2+d^2}}\bigg|_0^{\infty} = \frac{2}{d^2}$.

$$\begin{aligned}
&= \frac{\mu_0 KR}{2\pi}\int_0^{2\pi}\frac{(R - s\cos\phi)}{(R^2 + s^2 - 2Rs\cos\phi)}\,d\phi; \; (R - s\cos\phi) = \frac{1}{2R}\left[(R^2 - s^2) + (R^2 + s^2 - 2Rs\cos\phi)\right]. \\
&= \frac{\mu_0 K}{4\pi}\left[(R^2 - s^2)\int_0^{2\pi}\frac{d\phi}{(R^2 + s^2 - 2Rs\cos\phi)} + \int_0^{2\pi}d\phi\right].
\end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi}\frac{d\phi}{a + b\cos\phi} &= 2\int_0^{\pi}\frac{d\phi}{a + b\cos\phi} = \frac{4}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(\phi/2)}{a+b}\right]\bigg|_0^{\pi} \\
&= \frac{4}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(\pi/2)}{a+b}\right] = \frac{4}{\sqrt{a^2-b^2}}\left(\frac{\pi}{2}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}.
\end{aligned}$$

Here $a = R^2 + s^2$, $b = -2Rs$, so $a^2 - b^2 = R^4 + 2R^2s^2 + s^4 - 4R^2s^2 = R^4 - 2R^2s^2 + s^4 = (R^2 - s^2)^2$; $\sqrt{a^2 - b^2} = |R^2 - d^2|$.

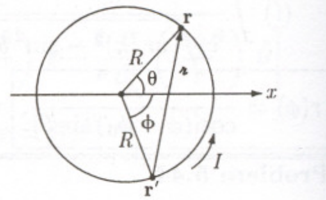
$$B_z = \frac{\mu_0 K}{4\pi}\left[\frac{(R^2 - s^2)}{|R^2 - s^2|}2\pi + 2\pi\right] = \frac{\mu_0 K}{2}\left(\frac{R^2 - s^2}{|R^2 - s^2|} + 1\right).$$

Inside the solenoid, $s < R$, so $B_z = \dfrac{\mu_0 K}{2}(1+1) = \mu_0 K$. *Outside* the solenoid, $s > R$, so $B_z = \dfrac{\mu_0 K}{2}(-1+1) = 0$.

Here $K = nI$, so $\boxed{\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}(\text{inside}),\ \text{and}\ 0(\text{outside})}$ (as we found more easily using Ampére's law, in Ex. 5.9).

Problem 5.45

Let the source point be $\mathbf{r}' = R\cos\phi\,\hat{\mathbf{x}} - R\sin\phi\,\hat{\mathbf{y}}$, and the field point be $\mathbf{r} = R\cos\theta\,\hat{\mathbf{x}} + R\sin\theta\,\hat{\mathbf{y}}$; then $\boldsymbol{\mathscr{r}} = R\left[(\cos\theta - \cos\phi)\,\hat{\mathbf{x}} + (\sin\theta + \sin\phi)\,\hat{\mathbf{y}}\right]$ and $d\mathbf{l} = R\sin\phi\,d\phi\,\hat{\mathbf{x}} + R\cos\phi\,d\phi\,\hat{\mathbf{y}} = R\,d\phi(\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})$.



$$
\begin{aligned}
d\mathbf{l}\times\boldsymbol{\mathscr{r}} &= R^2\,d\phi\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \sin\phi & \cos\phi & 0\\ (\cos\theta-\cos\phi) & (\sin\theta+\sin\phi) & 0\end{vmatrix}\\
&= R^2(\sin\phi\sin\theta + \sin^2\phi - \cos\theta\cos\phi + \cos^2\phi)\,d\phi\,\hat{\mathbf{z}}\\
&= R^2(1+\sin\theta\sin\phi - \cos\theta\cos\phi)\,d\phi\,\hat{\mathbf{z}} = R^2\left[1-\cos(\theta+\phi)\right]\,d\phi\,\hat{\mathbf{z}}.\\
\mathbf{B} &= \frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi}R^2\,\hat{\mathbf{z}}\int_0^\pi \frac{[1-\cos(\theta+\phi)]}{[2R^2-2R^2\cos(\theta+\phi)]^{3/2}}\,d\phi = \frac{\mu_0 I R^2}{4\pi(2R^2)^{3/2}}\,\hat{\mathbf{z}}\int_0^\pi\frac{d\phi}{\sqrt{1-\cos(\theta+\phi)}}\\
&= \frac{\mu_0 I}{8\sqrt{2}\pi R}\,\hat{\mathbf{z}}\int_0^\pi\frac{d\phi}{\sqrt{2}\sin[(\theta+\phi)/2]} = \frac{\mu_0 I}{16\pi R}\,\hat{\mathbf{z}}\left\{2\ln\left[\tan\left(\frac{\theta+\phi}{4}\right)\right]\right\}\Bigg|_0^\pi = \boxed{\frac{\mu_0 I}{8\pi R}\ln\left[\frac{\tan\left(\frac{\theta+\pi}{4}\right)}{\tan\left(\frac{\theta}{4}\right)}\right]\,\hat{\mathbf{z}}.}
\end{aligned}
$$

Problem 5.46

(a) From Eq. 5.38, $\boxed{\mathbf{B} = \frac{\mu_0 I R^2}{2}\left\{\frac{1}{[R^2+(d/2+z)^2]^{3/2}} + \frac{1}{[R^2+(d/2-z)^2]^{3/2}}\right\}.}$

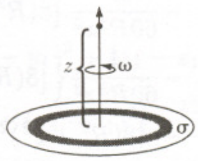
$$
\begin{aligned}
\frac{\partial B}{\partial z} &= \frac{\mu_0 I R^2}{2}\left\{\frac{(-3/2)2(d/2+z)}{[R^2+(d/2+z)^2]^{5/2}} + \frac{(-3/2)2(d/2-z)(-1)}{[R^2+(d/2-z)^2]^{5/2}}\right\}\\
&= \frac{3\mu_0 I R^2}{2}\left\{\frac{-(d/2+z)}{[R^2+(d/2+z)^2]^{5/2}} + \frac{(d/2-z)}{[R^2+(d/2-z)^2]^{5/2}}\right\}.\\
\left.\frac{\partial B}{\partial z}\right|_{z=0} &= \frac{3\mu_0 I R^2}{2}\left\{\frac{-d/2}{[R^2+(d/2)^2]^{5/2}} + \frac{d/2}{[R^2+(d/2)^2]^{5/2}}\right\} = 0.\ \checkmark
\end{aligned}
$$

(b) Differentiating again:

$$
\begin{aligned}
\frac{\partial^2 B}{\partial z^2} &= \frac{3\mu_0 I R^2}{2}\left\{\frac{-1}{[R^2+(d/2+z)^2]^{5/2}} + \frac{-(d/2+z)(-5/2)2(d/2+z)}{[R^2+(d/2+z)^2]^{7/2}}\right.\\
&\quad\left. + \frac{-1}{[R^2+(d/2-z)^2]^{5/2}} + \frac{(d/2-z)(-5/2)2(d/2-z)(-1)}{[R^2+(d/2-z)^2]^{7/2}}\right\}.\\
\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} &= \frac{3\mu_0 I R^2}{2}\left\{\frac{-2}{[R^2+(d/2)^2]^{5/2}} + \frac{2(5/2)2(d/2)^2 2}{[R^2+(d/2)^2]^{7/2}}\right\} = \frac{3\mu_0 I R^2}{[R^2+(d/2)^2]^{7/2}}\left(-R^2 - \frac{d^2}{4} + \frac{5d^2}{4}\right)\\
&= \frac{3\mu_0 I R^2}{[R^2+(d/2)^2]^{7/2}}\left(d^2 - R^2\right).\ \text{ Zero if } \boxed{d = R,} \text{ in which case}\\
B(0) &= \frac{\mu_0 I R^2}{2}\left\{\frac{1}{[R^2+(R/2)^2]^{3/2}} + \frac{1}{[R^2+(R/2)^2]^{3/2}}\right\} = \mu_0 I R^2\frac{1}{(5R^2/4)^{3/2}} = \boxed{\frac{8\mu_0 I}{5^{3/2}R}.}
\end{aligned}
$$

Problem 5.47

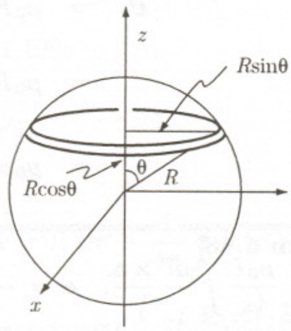
(a) The total charge on the shaded ring is $dq = \sigma(2\pi r)\,dr$. The time for one revolution is $dt = 2\pi/\omega$. So the current in the ring is $I = \frac{dq}{dt} = \sigma\omega r\,dr$. From Eq. 5.38, the magnetic field of this ring (for points on the axis) is $d\mathbf{B} = \frac{\mu_0}{2}\sigma\omega r\frac{r^2}{(r^2+z^2)^{3/2}}\,dr\,\hat{\mathbf{z}}$, and the total field of the disk is



$$\mathbf{B} = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3\,dr}{(r^2+z^2)^{3/2}}\,\hat{\mathbf{z}}. \text{ Let } u \equiv r^2, \text{ so } du = 2r\,dr. \text{ Then}$$

$$= \frac{\mu_0\sigma\omega}{4}\int_0^{R^2}\frac{u\,du}{(u+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{4}\left[2\left(\frac{u+2z^2}{\sqrt{u+z^2}}\right)\right]\Bigg|_0^{R^2} = \boxed{\frac{\mu_0\sigma\omega}{2}\left[\frac{(R^2+2z^2)}{\sqrt{R^2+z^2}}-2z\right]\hat{\mathbf{z}}.}$$

(b) Slice the sphere into slabs of thickness t, and use (a). Here $t = |d(R\cos\theta)| = R\sin\theta\,d\theta$; $\sigma \to \rho t = \rho R\sin\theta\,d\theta$; $R \to R\sin\theta$; $z \to z - R\cos\theta$. First rewrite the term in square brackets:



$$\left[\frac{(R^2+2z^2)}{\sqrt{R^2+z^2}}-2z\right] = \frac{2(R^2+z^2)}{\sqrt{R^2+z^2}} - \frac{R^2}{\sqrt{R^2+z^2}} - 2z$$

$$= 2\left[\sqrt{R^2+z^2} - \frac{R^2/2}{\sqrt{R^2+z^2}} - z\right].$$

But $R^2+z^2 \to R^2\sin^2\theta + (z^2 - 2Rz\cos\theta + R^2\cos^2\theta) = R^2 + z^2 - 2Rz\cos\theta$. So

$$B_z = \frac{\mu_0\rho R\omega}{2}2\int_0^\pi \sin\theta\,d\theta\left[\sqrt{R^2+z^2-2Rz\cos\theta} - \frac{(R^2/2)\sin^2\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}} - (z - R\cos\theta)\right].$$

Let $u \equiv \cos\theta$, so $du = -\sin\theta\,d\theta$; $\theta : 0 \to \pi \Rightarrow u : 1 \to -1$; $\sin^2\theta = 1 - u^2$.

$$= \mu_0\rho R\omega\int_{-1}^1\left[\sqrt{R^2+z^2-2Rzu} - \frac{(R^2/2)(1-u^2)}{\sqrt{R^2+z^2-2Rzu}} - z + Ru\right]du$$

$$= \mu_0\rho R\omega\left[I_1 - \frac{R^2}{2}(I_2 - I_3) - I_4 + I_5\right].$$

$$I_1 = \int_{-1}^1\sqrt{R^2+z^2-2Rzu}\,du = -\frac{1}{3Rz}\left(R^2+z^2-2Rzu\right)^{3/2}\Big|_{-1}^1$$

$$= -\frac{1}{3Rz}\left[\left(R^2+z^2-2Rz\right)^{3/2} - \left(R^2+z^2+2Rz\right)^{3/2}\right] = -\frac{1}{3Rz}\left[(z-R)^3 - (z+R)^3\right]$$

$$= -\frac{1}{3Rz}\left(z^3 - 3z^2R + 3zR^2 - R^3 - z^3 - 3z^2R - 3zR^2 - R^3\right) = \frac{2}{3z}(3z^2 + R^2).$$

$$I_2 = \int_{-1}^1\frac{1}{\sqrt{R^2+z^2-2Rzu}}\,du = -\frac{1}{Rz}\sqrt{R^2+z^2-2Rzu}\Big|_{-1}^1 = -\frac{1}{Rz}\left[(z-R)-(z+R)\right] = \frac{2}{z}.$$

$$\begin{aligned}
I_3 &= \int_{-1}^{1} \frac{u^2}{\sqrt{R^2+z^2-2Rzu}}\,du \\
&= -\frac{1}{60R^3z^3}\left[8(R^2+z^2)^2+4(R^2+z^2)2Rzu+3(2Rz)^2u^2\right]\sqrt{R^2+z^2-2Rzu}\Big|_{-1}^{1} \\
&= -\frac{1}{60R^3z^3}\Big\{\left[8(R^2+z^2)^2+8Rz(R^2+z^2)+12R^2z^2\right](z-R) \\
&\quad -\left[8(R^2+z^2)^2-8Rz(R^2+z^2)+12R^2z^2\right](z+R)\Big\} \\
&= -\frac{1}{60R^3z^3}\left\{z\left[16Rz(R^2+z^2)\right]-R\left[16(R^2+z^2)^2+24R^2z^2\right]\right\} \\
&= -\frac{1}{60R^3z^3}16R\left(R^2z^2+z^4-R^4-2R^2z^2-z^4-\frac{3}{2}R^2z^2\right) \\
&= -\frac{4}{15R^2z^3}\left(-\frac{5}{2}R^2z^2-R^4\right)=\frac{4}{15z^3}\left(R^2+\frac{5}{2}z^2\right). \quad I_4 = z\int_{-1}^{1} du = 2z; \quad I_5 = R\int_{-1}^{1} u\,du = 0.
\end{aligned}$$

$$\begin{aligned}
B_z &= \mu_0 R\rho\omega\left[\frac{2}{3z}(3z^2+R^2)-\frac{R^2}{2}\frac{2}{z}+\frac{R^2}{2}\frac{4}{15z^3}\left(R^2+\frac{5}{2}z^2\right)-2z\right] \\
&= \mu_0 R\rho\omega\left(2z+\frac{2R^2}{3z}-\frac{R^2}{z}+\frac{2R^4}{15z^3}+\frac{R^2}{3z}-2z\right) \\
&= \mu_0\rho\omega\frac{2R^5}{15z^3}. \quad \text{But } \rho=\frac{Q}{(4/3)\pi R^3}, \text{ so } \boxed{\mathbf{B}=\frac{\mu_0 Q\omega R^2}{10\pi z^3}\,\hat{\mathbf{z}}.}
\end{aligned}$$

Problem 5.48

$\mathbf{B}=\dfrac{\mu_0 I}{4\pi}\displaystyle\int\frac{d\mathbf{l}'\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}$. $\boldsymbol{\mathscr{r}} = -R\cos\phi\,\hat{\mathbf{x}}+(y-R\sin\phi)\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}$. (For simplicity I'll drop the prime on ϕ.) $\mathscr{r}^2=R^2\cos^2\phi+y^2-2Ry\sin\phi+R^2\sin^2\phi+z^2=R^2+y^2+z^2-2Ry\sin\phi$. The source coordinates (x',y',z') satisfy $x'=R\cos\phi\Rightarrow dx'=-R\sin\phi\,d\phi$; $y'=R\sin\phi\Rightarrow dy'=R\cos\phi\,d\phi$; $z'=0\Rightarrow dz'=0$. So $d\mathbf{l}'=-R\sin\phi\,d\phi\,\hat{\mathbf{x}}+R\cos\phi\,d\phi\,\hat{\mathbf{y}}$.

$$d\mathbf{l}'\times\boldsymbol{\mathscr{r}}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ -R\sin\phi\,d\phi & R\cos\phi\,d\phi & 0\\ -R\cos\phi & (y-R\sin\phi) & z\end{vmatrix}=(Rz\cos\phi\,d\phi)\,\hat{\mathbf{x}}+(Rz\sin\phi\,d\phi)\,\hat{\mathbf{y}}+(-Ry\sin\phi\,d\phi+R^2\,d\phi)\,\hat{\mathbf{z}}.$$

$$B_x=\frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\cos\phi\,d\phi}{(R^2+y^2+z^2-2Ry\sin\phi)^{3/2}}=\frac{\mu_0 IRz}{4\pi}\frac{1}{Ry}\frac{1}{\sqrt{R^2+y^2+z^2-2Ry\sin\phi}}\Bigg|_0^{2\pi}=0,$$

since $\sin\phi=0$ at both limits. The y and z components are elliptic integrals, and cannot be expressed in terms of elementary functions.

$$\boxed{B_x=0;\quad B_y=\frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\sin\phi\,d\phi}{(R^2+y^2+z^2-2Ry\sin\phi)^{3/2}};\,B_z=\frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{(R-y\sin\phi)\,d\phi}{(R^2+y^2+z^2-2Ry\sin\phi)^{3/2}}.}$$

Problem 5.49

From the Biot-Savart law, the field of loop #1 is $\mathbf{B}=\dfrac{\mu_0 I_1}{4\pi}\displaystyle\oint_1\frac{d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$; the force on loop #2 is

$\mathbf{F}=I_2\displaystyle\oint_2 d\mathbf{l}_2\times\mathbf{B}=\frac{\mu_0}{4\pi}I_1I_2\oint_1\oint_2\frac{d\mathbf{l}_2\times(d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2}$. Now $d\mathbf{l}_2\times(d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}})=d\mathbf{l}_1(d\mathbf{l}_2\cdot\hat{\boldsymbol{\mathscr{r}}})-\hat{\boldsymbol{\mathscr{r}}}(d\mathbf{l}_1\cdot d\mathbf{l}_2)$, so

$$\mathbf{F} = -\frac{\mu_0}{4\pi} I_1 I_2 \left\{ \oint \oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} (d\mathbf{l}_1 \cdot d\mathbf{l}_2) - \oint d\mathbf{l}_1 \oint \frac{(d\mathbf{l}_2 \cdot \hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2} \right\}$$

The first term is what we want. It remains to show that the second term is zero:

$\boldsymbol{\mathscr{r}} = (x_2 - x_1)\,\hat{\mathbf{x}} + (y_2 - y_1)\,\hat{\mathbf{y}} + (z_2 - z_1)\,\hat{\mathbf{z}}$, so $\boldsymbol{\nabla}_2(1/\mathscr{r}) = \frac{\partial}{\partial x_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\,\hat{\mathbf{x}}$ $+ \frac{\partial}{\partial y_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\,\hat{\mathbf{z}}$ $= -\frac{(x_2 - x_1)}{\mathscr{r}^3}\,\hat{\mathbf{x}} - \frac{(y_2 - y_1)}{\mathscr{r}^3}\,\hat{\mathbf{y}} - \frac{(z_2 - z_1)}{\mathscr{r}^3}\,\hat{\mathbf{z}} = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$. So $\oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot d\mathbf{l}_2 = -\oint \boldsymbol{\nabla}_2\left(\frac{1}{\mathscr{r}}\right) \cdot d\mathbf{l}_2 = 0$ (by Corollary 2 in Sect. 1.3.3). qed

Problem 5.50

Poisson's equation (Eq. 2.24) says $\nabla^2 V = -\frac{1}{\epsilon_0}\rho$. For *dielectrics* (with no *free* charge), $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ (Eq. 4.12), and the resulting potential is $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$. In *general*, $\rho = \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E}$ (Gauss's law), so the analogy is $\mathbf{P} \to -\epsilon_0\mathbf{E}$, and hence $V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{\mathbf{E}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$. qed

[There are many other ways to obtain this result. For example, using Eq. 1.100:

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = -\boldsymbol{\nabla}'\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}) = 4\pi\delta^3(\mathbf{r} - \mathbf{r}'),$$

$$V(\mathbf{r}) = \int V(\mathbf{r}')\delta^3(\mathbf{r} - \mathbf{r}')\,d\tau' = -\frac{1}{4\pi}\int V(\mathbf{r}')\boldsymbol{\nabla}'\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)d\tau' = \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\left[\boldsymbol{\nabla}' V(\mathbf{r}')\right]d\tau' - \frac{1}{4\pi}\oint V(\mathbf{r}')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot d\mathbf{a}'$$

(Eq. 1.59). But $\boldsymbol{\nabla}' V(\mathbf{r}') = -\mathbf{E}(\mathbf{r}')$, and the surface integral $\to 0$ at ∞, so $V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{\mathbf{E}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$, as before. You can also *check* the result, by computing its gradient—but it's not easy.]

Problem 5.51

(a) For uniform $\mathbf{B}$, $\int_0^{\mathbf{r}}(\mathbf{B}\times d\mathbf{l}) = \mathbf{B}\times\int_0^{\mathbf{r}} d\mathbf{l} = \boxed{\mathbf{B}\times\mathbf{r}} \neq \mathbf{A} = -\frac{1}{2}(\mathbf{B}\times\mathbf{r})$.

(b) $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$, so $\oint \mathbf{B}\times d\mathbf{l} = \left(\frac{\mu_0 I}{2\pi a}\,\hat{\mathbf{s}} - \frac{\mu_0 I}{2\pi b}\,\hat{\mathbf{s}}\right)w = \boxed{\frac{\mu_0 I w}{2\pi}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{s}} \neq 0.}$

(c) $\mathbf{A} = -\mathbf{r}\times\mathbf{B}\int_0^1 \lambda\,d\lambda = \boxed{-\frac{1}{2}(\mathbf{r}\times\mathbf{B}).}$

(d) $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$; $\mathbf{B}(\lambda\mathbf{r}) = \frac{\mu_0 I}{2\pi\lambda s}\,\hat{\boldsymbol{\phi}}$; $\mathbf{A} = -\frac{\mu_0 I}{2\pi s}(\mathbf{r}\times\hat{\boldsymbol{\phi}})\int_0^1 \lambda\frac{1}{\lambda}\,d\lambda = -\frac{\mu_0 I}{2\pi s}(\mathbf{r}\times\hat{\boldsymbol{\phi}})$. But $\mathbf{r}$ here is the vector from the origin—in cylindrical coordinates $\mathbf{r} = s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}}$. So $\mathbf{A} = -\frac{\mu_0 I}{2\pi s}\left[s(\hat{\mathbf{s}}\times\hat{\boldsymbol{\phi}}) + z(\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}})\right]$, and $(\hat{\mathbf{s}}\times\hat{\boldsymbol{\phi}}) = \hat{\mathbf{z}}$, $(\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}}) = -\hat{\mathbf{s}}$. So $\boxed{\mathbf{A} = \frac{\mu_0 I}{2\pi s}(z\,\hat{\mathbf{s}} - s\,\hat{\mathbf{z}}.}$

The examples in (c) and (d) happen to be divergenceless, but this is not the case in general. For (letting $\mathbf{L} \equiv \int_0^1 \lambda\mathbf{B}(\lambda\mathbf{r})\,d\lambda$, for short) $\boldsymbol{\nabla}\cdot\mathbf{A} = -\boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{L}) = -[\mathbf{L}\cdot(\boldsymbol{\nabla}\times\mathbf{r}) - \mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{L})] = \mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{L})$, and $\boldsymbol{\nabla}\times\mathbf{L} = \int_0^1 \lambda[\boldsymbol{\nabla}\times\mathbf{B}(\lambda\mathbf{r})]\,d\lambda = \int_0^1 \lambda^2[\boldsymbol{\nabla}_\lambda\times\mathbf{B}(\lambda\mathbf{r})]\,d\lambda = \mu_0\int_0^1 \lambda^2\mathbf{J}(\lambda\mathbf{r})\,d\lambda$, so $\boldsymbol{\nabla}\cdot\mathbf{A} = \mu_0\mathbf{r}\cdot\int_0^1 \lambda^2\mathbf{J}(\lambda\mathbf{r})\,d\lambda$, and it vanishes in regions where $\mathbf{J} = 0$ (which is why the examples in (c) and (d) were divergenceless). To construct an explicit counterexample, we need the field at a point where $\mathbf{J} \neq 0$—say, *inside* a wire with uniform current.

Here Ampére's law gives $B\, 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 J\pi s^2 \Rightarrow \mathbf{B} = \frac{\mu_0 J}{2} s\,\hat{\boldsymbol{\phi}}$, so

$$\begin{aligned}
\mathbf{A} &= -\mathbf{r}\times\int_0^1 \lambda\left(\frac{\mu_0 J}{2}\right)\lambda s\,\hat{\boldsymbol{\phi}}\,d\lambda = -\frac{\mu_0 J}{6}s(\mathbf{r}\times\hat{\boldsymbol{\phi}}) = \frac{\mu_0 J s}{6}(z\,\hat{\mathbf{s}} - s\hat{\mathbf{z}}).\\
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{\mu_0 J}{6}\left[\frac{1}{s}\frac{\partial}{\partial s}(s^2 z) + \frac{\partial}{\partial z}(-s^2)\right] = \frac{\mu_0 J}{6}\left(\frac{1}{s}2sz\right) = \frac{\mu_0 J z}{3} \neq 0.
\end{aligned}$$

Conclusion: $\boxed{\text{(ii) does } not \text{ automatically yield } \boldsymbol{\nabla}\cdot\mathbf{A}=0.}$

Problem 5.52

(a) Exploit the analogy with the electrical case:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}] \quad \text{(Eq. 3.104)} = -\boldsymbol{\nabla}V, \quad \text{with } V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \quad \text{(Eq. 3.102)}.\\
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}] \quad \text{(Eq. 5.87)} = -\boldsymbol{\nabla}U, \text{ (Eq. 5.65)}.
\end{aligned}$$

Evidently the prescription is $\mathbf{p}/\epsilon_0 \to \mu_0\mathbf{m}$: $\boxed{U(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{r^2}.}$

(b) Comparing Eqs. 5.67 and 5.85, the dipole moment of the shell is $\mathbf{m} = (4\pi/3)\omega\sigma R^4\,\hat{\mathbf{z}}$ (which we also got in Prob. 5.36). Using the result of (a), then, $\boxed{U(\mathbf{r}) = \frac{\mu_0\omega\sigma R^4}{3}\frac{\cos\theta}{r^2}}$ for $r > R$.

Inside the shell, the field is uniform (Eq. 5.38): $\mathbf{B} = \frac{2}{3}\mu_0\sigma\omega R\,\hat{\mathbf{z}}$, so $U(\mathbf{r}) = -\frac{2}{3}\mu_0\sigma\omega R z + \text{constant}$. We may as well pick the constant to be zero, so $\boxed{U(\mathbf{r}) = -\frac{2}{3}\mu_0\sigma\omega R r\cos\theta}$ for $r < R$.

[Notice that $U(\mathbf{r})$ is *not continuous* at the surface ($r = R$): $U_{\text{in}}(R) = -\frac{2}{3}\mu_0\sigma\omega R^2\cos\theta \neq U_{\text{out}}(R) = \frac{1}{3}\mu_0\sigma\omega R^2\cos\theta$. As I warned you on p. 236: if you insist on using magnetic scalar potentials, keep away from places where there is current!]

(c)

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0\omega Q}{4\pi R}\left[\left(1-\frac{3r^2}{5R^2}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1-\frac{6r^2}{5R^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] = -\boldsymbol{\nabla}U = -\frac{\partial U}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial U}{\partial\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}\hat{\boldsymbol{\phi}}.\\
\frac{\partial U}{\partial\phi} &= 0 \Rightarrow U(r,\theta,\phi) = U(r,\theta).\\
\frac{1}{r}\frac{\partial U}{\partial\theta} &= \left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{6r^2}{5R^2}\right)\sin\theta \Rightarrow U(r,\theta) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{6r^2}{5R^2}\right)r\cos\theta + f(r).\\
\frac{\partial U}{\partial r} &= -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{3r^2}{5R^2}\right)\cos\theta \Rightarrow U(r,\theta) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(r-\frac{r^3}{5R^2}\right)\cos\theta + g(\theta).
\end{aligned}$$

Equating the two expressions:

$$-\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{6r^2}{5R^2}\right)r\cos\theta + f(r) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{r^2}{5R^2}\right)r\cos\theta + g(\theta),$$

or

$$\left(\frac{\mu_0\omega Q}{4\pi R^3}\right)r^3\cos\theta + f(r) = g(\theta).$$

But there is no way to write $r^3 \cos\theta$ as the sum of a function of θ and a function of r, so we're stuck. The reason is that you can't *have* a scalar magnetic potential in a region where the current is nonzero.

Problem 5.53

(a) $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, and $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = \mathbf{B}$ $\Rightarrow \mathbf{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}}{\mathscr{r}}\, d\tau'$, so

$\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = \mathbf{B}$, and $\nabla \cdot \mathbf{W} = 0$ (we'll choose it so), $\nabla \times \mathbf{W} = \mathbf{A}$ $\Rightarrow$ $\boxed{\mathbf{W} = \dfrac{1}{4\pi}\displaystyle\int \frac{\mathbf{B}}{\mathscr{r}}\, d\tau'.}$

(b) $\mathbf{W}$ will be proportional to $\mathbf{B}$ and to two factors of $\mathbf{r}$ (since differentiating *twice* must recover $\mathbf{B}$), so I'll try something of the form $\mathbf{W} = \alpha \mathbf{r}(\mathbf{r}\cdot\mathbf{B}) + \beta r^2 \mathbf{B}$, and see if I can pick the constants α and β in such a way that $\nabla \cdot \mathbf{W} = 0$ and $\nabla \times \mathbf{W} = \mathbf{A}$.

$\nabla \cdot \mathbf{W} = \alpha\left[(\mathbf{r}\cdot\mathbf{B})(\nabla\cdot\mathbf{r}) + \mathbf{r}\cdot\nabla(\mathbf{r}\cdot\mathbf{B})\right] + \beta\left[r^2(\nabla\cdot\mathbf{B}) + \mathbf{B}\cdot\nabla(r^2)\right]$. $\nabla\mathbf{r} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 1+1+1 = 3;$

$\nabla(\mathbf{r}\cdot\mathbf{B}) = \mathbf{r}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{r}) + (\mathbf{r}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{r}$; but $\mathbf{B}$ is constant, so all derivatives of $\mathbf{B}$ vanish, and $\nabla\times\mathbf{r} = 0$ (Prob. 1.62), so

$$\nabla(\mathbf{r}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}} = \mathbf{B};$$

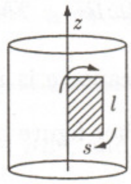
$$\nabla(r^2) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2) = 2x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 2z\,\hat{\mathbf{z}} = 2\mathbf{r}.$$ So

$\nabla\cdot\mathbf{W} = \alpha\left[3(\mathbf{r}\cdot\mathbf{B}) + (\mathbf{r}\cdot\mathbf{B})\right] + \beta\left[0 + 2(\mathbf{r}\cdot\mathbf{B})\right] = 2(\mathbf{r}\cdot\mathbf{B})(2\alpha+\beta)$, which is zero if $2\alpha + \beta = 0$.

$$\nabla\times\mathbf{W} = \alpha\left[(\mathbf{r}\cdot\mathbf{B})(\nabla\times\mathbf{r}) - \mathbf{r}\times\nabla(\mathbf{r}\cdot\mathbf{B})\right] + \beta\left[r^2(\nabla\times\mathbf{B}) - \mathbf{B}\times\nabla(r^2)\right] = \alpha\left[0 - (\mathbf{r}\times\mathbf{B})\right] + \beta\left[0 - 2(\mathbf{B}\times\mathbf{r})\right]$$

$= -(\mathbf{r}\times\mathbf{B})(\alpha - 2\beta) = -\dfrac{1}{2}(\mathbf{r}\times\mathbf{B})$ (Prob. 5.24). So we want $\alpha - 2\beta = 1/2$. Evidently $\alpha - 2(-2\alpha) = 5\alpha = 1/2$, or $\alpha = 1/10$; $\beta = -2\alpha = -1/5$. *Conclusion:* $\boxed{\mathbf{W} = \dfrac{1}{10}\left[\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) - 2r^2\mathbf{B}\right].}$ (But this is certainly not unique.)

(c) $\nabla\times\mathbf{W} = \mathbf{A} \Rightarrow \int(\nabla\times\mathbf{W})\cdot d\mathbf{a} = \int \mathbf{A}\cdot d\mathbf{a}$. Or $\oint \mathbf{W}\cdot d\mathbf{l} = \int \mathbf{A}\cdot d\mathbf{a}$. Integrate around the amperian loop shown, taking W to point parallel to the axis, and choosing $\mathbf{W} = 0$ on the axis:



$$-Wl = \int_0^s \left(\frac{\mu_0 n I}{2}\right) l\bar{s}\, d\bar{s} = \frac{\mu_0 n I}{2}\frac{s^2 l}{2}$$ (using Eq. 5.70 for $\mathbf{A}$).

$\boxed{\mathbf{W} = -\dfrac{\mu_0 n I s^2}{4}\,\hat{\mathbf{z}}}$ $(s < R)$.

For $s > R$, $-Wl = \dfrac{\mu_0 n I R^2 l}{4} + \displaystyle\int_R^s \left(\frac{\mu_0 n I}{2}\right)\frac{R^2}{\bar{s}}\, l\, d\bar{s} = \frac{\mu_0 n I R^2 l}{4} + \frac{\mu_0 n I R^2 l}{2}\ln(s/R);$

$$\boxed{\mathbf{W} = -\frac{\mu_0 n I R^2}{4}\left[1 + 2\ln(s/R)\right]\hat{\mathbf{z}}} \quad (s > R).$$

Problem 5.54

Apply the divergence theorem to the function $[\mathbf{U}\times(\nabla\times\mathbf{V})]$, noting (from the product rule) that $\nabla\cdot[\mathbf{U}\times(\nabla\times\mathbf{V})] = (\nabla\times\mathbf{V})\cdot(\nabla\times\mathbf{U}) - \mathbf{U}\cdot[\nabla\times(\nabla\times\mathbf{V})]$:

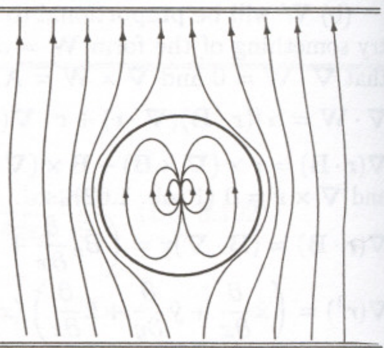
$$\int \nabla\cdot[\mathbf{U}\times(\nabla\times\mathbf{V})]\, d\tau = \int \{(\nabla\times\mathbf{V})\cdot(\nabla\times\mathbf{U}) - \mathbf{U}\cdot[\nabla\times(\nabla\times\mathbf{V})]\}\, d\tau = \oint [\mathbf{U}\times(\nabla\times\mathbf{V})]\cdot d\mathbf{a}.$$

As always, suppose we have *two* solutions, $\mathbf{B}_1$ (and $\mathbf{A}_1$) and $\mathbf{B}_2$ (and $\mathbf{A}_2$). Define $\mathbf{B}_3 \equiv \mathbf{B}_2 - \mathbf{B}_1$ (and $\mathbf{A}_3 \equiv \mathbf{A}_2 - \mathbf{A}_1$), so that $\nabla\times\mathbf{A}_3 = \mathbf{B}_3$ and $\nabla\times\mathbf{B}_3 = \nabla\times\mathbf{B}_1 - \nabla\times\mathbf{B}_2 = \mu_0\mathbf{J} - \mu_0\mathbf{J} = 0$. Set $\mathbf{U} = \mathbf{V} = \mathbf{A}_3$ in the above identity:

$$\int\{(\nabla\times\mathbf{A}_3)\cdot(\nabla\times\mathbf{A}_3)-\mathbf{A}_3\cdot[\nabla\times(\nabla\times\mathbf{A}_3)]\}\,d\tau=\int\{(\mathbf{B}_3)\cdot(\mathbf{B}_3)-\mathbf{A}_3\cdot[\nabla\times\mathbf{B}_3]\}\,d\tau=\int(B_3)^2\,d\tau$$

$=\oint[\mathbf{A}_3\times(\nabla\times\mathbf{A}_3)]\cdot d\mathbf{a}=\oint(\mathbf{A}_3\times\mathbf{B}_3)\cdot d\mathbf{a}$. But either $\mathbf{A}$ is specified (in which case $\mathbf{A}_3=0$), or else $\mathbf{B}$ is specified (in which case $\mathbf{B}_3=0$), at the surface. In either case $\oint(\mathbf{A}_3\times\mathbf{B}_3)\cdot d\mathbf{a}=0$. So $\int(B_3)^2\,d\tau=0$, and hence $\mathbf{B}_1=\mathbf{B}_2$. qed

Problem 5.55



From Eq. 5.86, $\mathbf{B}_{\text{tot}}=B_0\,\hat{\mathbf{z}}-\dfrac{\mu_0 m_0}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})$. Therefore $\mathbf{B}\cdot\hat{\mathbf{r}}=B_0(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})-\dfrac{\mu_0 m_0}{4\pi r^3}2\cos\theta=\left(B_0-\dfrac{\mu_0 m_0}{2\pi r^3}\right)\cos\theta$.
This is zero, for all θ, when $r=R$, given by $B_0=\dfrac{\mu_0 m_0}{2\pi R^3}$, or

$\boxed{R=\left(\dfrac{\mu_0 m_0}{2\pi B_0}\right)^{1/3}}$. Evidently no field lines cross this sphere.

Problem 5.56

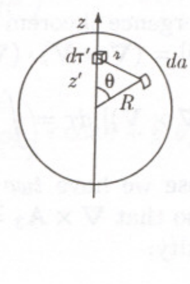
(a) $I=\dfrac{Q}{(2\pi/\omega)}=\dfrac{Q\omega}{2\pi}$; $a=\pi R^2$; $\mathbf{m}=\dfrac{Q\omega}{2\pi}\pi R^2\,\hat{\mathbf{z}}=\dfrac{Q}{2}\omega R^2\,\hat{\mathbf{z}}$. $L=RMv=M\omega R^2$; $\mathbf{L}=M\omega R^2\,\hat{\mathbf{z}}$.

$\dfrac{m}{L}=\dfrac{Q}{2}\dfrac{\omega R^2}{M\omega R^2}=\dfrac{Q}{2M}$. $\boxed{\mathbf{m}=\left(\dfrac{Q}{2M}\right)\mathbf{L},}$ and the gyromagnetic ratio is $\boxed{g=\dfrac{Q}{2M}.}$

(b) Because g is *independent of* R, the same ratio applies to all "donuts", and hence to the entire sphere (or any other figure of revolution): $\boxed{g=\dfrac{Q}{2M}.}$

(c) $m=\dfrac{e}{2m}\dfrac{\hbar}{2}=\dfrac{e\hbar}{4m}=\dfrac{(1.60\times10^{-19})(1.05\times10^{-34})}{4(9.11\times10^{-31})}=\boxed{4.61\times10^{-24}\,\mathrm{A\,m^2}.}$

Problem 5.57



(a) $\mathbf{B}_{\text{ave}}=\dfrac{1}{(3/4)\pi R^3}\displaystyle\int\mathbf{B}\,d\tau=\dfrac{3}{4\pi R^3}\int(\nabla\times\mathbf{A})\,d\tau=$
$-\dfrac{3}{4\pi R^3}\displaystyle\oint\mathbf{A}\times d\mathbf{a}=-\dfrac{3}{4\pi R^3}\dfrac{\mu_0}{4\pi}\oint\left\{\int\dfrac{\mathbf{J}}{\scriptstyle{\mathcal{R}}}\,d\tau'\right\}\times d\mathbf{a}=$
$-\dfrac{3\mu_0}{(4\pi)^2R^3}\displaystyle\int\mathbf{J}\times\left\{\oint\dfrac{1}{\scriptstyle{\mathcal{R}}}\,d\mathbf{a}\right\}d\tau'$. Note that $\mathbf{J}$ depends on the source point $\mathbf{r}'$, not on the field point $\mathbf{r}$. To do the surface integral, choose the (x,y,z) coordinates so that $\mathbf{r}'$ lies on the z axis (see diagram). Then $\scriptstyle{\mathcal{R}}$ $=\sqrt{R^2+(z')^2-2Rz'\cos\theta}$, while $d\mathbf{a}=R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$. By symmetry, the x and y components must integrate to zero; since the z component of $\hat{\mathbf{r}}$ is $\cos\theta$, we have

$$\begin{aligned}
\oint \frac{1}{\mathfrak{r}}\, d\mathbf{a} &= \hat{\mathbf{z}} \int \frac{\cos\theta}{\sqrt{R^2+(z')^2-2Rz'\cos\theta}} R^2 \sin\theta\, d\theta\, d\phi = 2\pi R^2\, \hat{\mathbf{z}} \int_0^\pi \frac{\cos\theta\sin\theta}{\sqrt{R^2+(z')^2-2Rz'\cos\theta}}\, d\theta.
\end{aligned}$$

Let $u \equiv \cos\theta$, so $du = -\sin\theta\, d\theta$.

$$\begin{aligned}
&= 2\pi R^2\, \hat{\mathbf{z}} \int_{-1}^{1} \frac{u}{\sqrt{R^2+(z')^2-2Rz'u}}\, du \\
&= 2\pi R^2\, \hat{\mathbf{z}} \left\{ -\frac{2\left[2(R^2+(z')^2)+2Rz'u\right]}{3(2Rz')^2} \sqrt{R^2+(z')^2-2Rz'u} \right\}\Bigg|_{-1}^{1} \\
&= -\frac{2\pi R^2\, \hat{\mathbf{z}}}{3(Rz')^2} \left\{ \left[R^2+(z')^2+Rz'\right]\sqrt{R^2+(z')^2-2Rz'} - \left[R^2+(z')^2-Rz'\right]\sqrt{R^2+(z')^2+2Rz'} \right\} \\
&= -\left[\frac{2\pi}{3(z')^2}\hat{\mathbf{z}}\right] \left\{ \left[R^2+(z')^2+Rz'\right]|R-z'| - \left[R^2+(z')^2-Rz'\right](R+z') \right\} \\
&= \left\{ \begin{array}{ll} \dfrac{4\pi}{3} z'\, \hat{\mathbf{z}} = \dfrac{4\pi}{3}\mathbf{r}', & (r'<R); \\[2ex] \dfrac{4\pi R^3}{3(z')^2}\hat{\mathbf{z}} = \dfrac{4\pi}{3}\dfrac{R^3}{(r')^3}\mathbf{r}', & (r'>R). \end{array} \right.
\end{aligned}$$

For now we want $r' < R$, so $\mathbf{B}_{\text{ave}} = -\dfrac{3\mu_0}{(4\pi)^2 R^3}\dfrac{4\pi}{3}\displaystyle\int (\mathbf{J}\times\mathbf{r}')\, d\tau' = -\dfrac{\mu_0}{4\pi R^3}\displaystyle\int (\mathbf{J}\times\mathbf{r}')\, d\tau'$. Now $\mathbf{m} = \frac{1}{2}\int(\mathbf{r}\times\mathbf{J})\, d\tau$ (Eq. 5.91), so $\mathbf{B}_{\text{ave}} = \dfrac{\mu_0}{4\pi}\dfrac{2\mathbf{m}}{R^3}$. qed

(b) This time $r' > R$, so $\mathbf{B}_{\text{ave}} = -\dfrac{3\mu_0}{(4\pi)^2 R^3}\dfrac{4\pi}{3}R^3 \displaystyle\int \left(\mathbf{J}\times\frac{\mathbf{r}'}{(r')^3}\right) d\tau' = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}\times\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\, d\tau'$, where $\boldsymbol{\mathfrak{r}}$ now goes from the source point to the *center* ($\boldsymbol{\mathfrak{r}} = -\mathbf{r}'$). Thus $\mathbf{B}_{\text{ave}} = \mathbf{B}_{\text{cen}}$. qed

Problem 5.58

(a) Problem 5.51 gives the dipole moment of a shell: $\mathbf{m} = \dfrac{4\pi}{3}\sigma\omega R^4\, \hat{\mathbf{z}}$. Let $R \to r, \sigma \to \rho\, dr$, and integrate:

$$\mathbf{m} = \frac{4\pi}{3}\omega\rho\, \hat{\mathbf{z}} \int_0^R r^4\, dr = \frac{4\pi}{3}\omega\rho\frac{R^5}{5}\hat{\mathbf{z}}. \quad \text{But } \rho = \frac{Q}{(4/3)\pi R^3}, \text{ so } \boxed{\mathbf{m} = \frac{1}{5}Q\omega R^2\, \hat{\mathbf{z}}.}$$

(b) $\mathbf{B}_{\text{ave}} = \dfrac{\mu_0}{4\pi}\dfrac{2\mathbf{m}}{R^3} = \boxed{\dfrac{\mu_0}{4\pi}\dfrac{2Q\omega}{5R}\hat{\mathbf{z}}.}$

(c) $\mathbf{A} \cong \dfrac{\mu_0}{4\pi}\dfrac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} = \boxed{\dfrac{\mu_0}{4\pi}\dfrac{Q\omega R^2}{5}\dfrac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}.}$

(d) Use Eq. 5.67, with $R \to \bar{r}, \sigma \to \rho\, d\bar{r}$, and integrate:

$$\mathbf{A} = \frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \int_0^R \bar{r}^4\, d\bar{r} = \frac{\mu_0\omega}{3}\frac{3Q}{4\pi R^3}\frac{\sin\theta}{r^2}\frac{R^5}{5}\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0}{4\pi}\frac{Q\omega R^2}{5}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}.}$$

This is *identical* to (c); evidently the field is *pure* dipole, for points outside the sphere.

(e) According to Prob. 5.29, the field is $\mathbf{B} = \dfrac{\mu_0\omega Q}{4\pi R}\left[\left(1-\dfrac{3r^2}{5R^2}\right)\cos\theta\, \hat{\mathbf{r}} - \left(1-\dfrac{6r^2}{5R^2}\right)\sin\theta\, \hat{\boldsymbol{\theta}}\right]$. The average obviously points in the z direction, so take the z component of $\hat{\mathbf{r}}$ $(\cos\theta)$ and $\hat{\boldsymbol{\theta}}$ $(-\sin\theta)$:

$$\begin{aligned}
B_{\text{ave}} &= \frac{\mu_0\omega Q}{4\pi R}\frac{1}{(4/3)\pi R^3}\int\left[\left(1-\frac{3r^2}{5R^2}\right)\cos^2\theta+\left(1-\frac{6r^2}{5R^2}\right)\sin^2\theta\right]r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{3\mu_0\omega Q}{(4\pi R^2)^2}2\pi\int_0^\pi\left[\left(\frac{r^3}{3}-\frac{3}{5}\frac{R^5}{5R^2}\right)\cos^2\theta+\left(\frac{R^3}{3}-\frac{6}{5}\frac{R^5}{5R^2}\right)\sin^2\theta\right]\sin\theta\,d\theta \\
&= \frac{3\mu_0\omega Q}{8\pi R^4}R^3\int_0^\pi\left(\frac{16}{75}\cos^2\theta+\frac{7}{75}\sin^2\theta\right)\sin\theta\,d\theta=\frac{3\mu_0\omega Q}{8\pi R}\frac{1}{75}\int_0^\pi(7+9\cos^2\theta)\sin\theta\,d\theta \\
&= \frac{\mu_0\omega Q}{200\pi R}\left(-7\cos\theta-3\cos^3\theta\right)\Big|_0^\pi=\frac{\mu_0\omega Q}{200\pi R}(20)=\frac{\mu_0\omega Q}{10\pi R}\text{ (same as (b)).} \checkmark
\end{aligned}$$

Problem 5.59

The issue (and the integral) is identical to the one in Prob. 3.42. The resolution (as before) is to regard Eq. 5.87 as correct outside an infinitesimal sphere centered at the dipole. *Inside* this sphere the field is a delta-function, $\mathbf{A}\delta^3(\mathbf{r})$, with $\mathbf{A}$ selected so as to make the average field consistent with Prob. 5.57:

$$\mathbf{B}_{\text{ave}}=\frac{1}{(4/3)\pi R^3}\int\mathbf{A}\delta^3(\mathbf{r})\,d\tau=\frac{3}{4\pi R^3}\mathbf{A}=\frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}\Rightarrow\mathbf{A}=\frac{2\mu_0\mathbf{m}}{3}.\text{ The added term is }\boxed{\frac{2\mu_0}{3}\mathbf{m}\delta^3(\mathbf{r}).}$$

Problem 5.60

(a) $I\,d\mathbf{l}\to\mathbf{J}\,d\tau$, so $\boxed{\mathbf{A}=\frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int(r')^nP_n(\cos\theta)\mathbf{J}\,d\tau.}$

(b) $\mathbf{A}_{\text{mon}}=\frac{\mu_0}{4\pi r}\int\mathbf{J}\,d\tau=\frac{\mu_0}{4\pi r}\frac{d\mathbf{p}}{dt}$ (Prob. 5.7), where $\mathbf{p}$ is the total electric dipole moment. In magnetostatics, $\mathbf{p}$ is constant, so $d\mathbf{p}/dt=0$, and hence $\mathbf{A}_{\text{mon}}=0$. qed

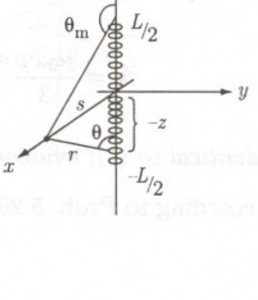
(c) $\mathbf{m}=I\mathbf{a}=\frac{1}{2}I\oint(\mathbf{r}\times d\mathbf{l})\to\mathbf{m}=\frac{1}{2}\int(\mathbf{r}\times\mathbf{J})\,d\tau$. qed

Problem 5.61

For a dipole at the origin and a field point in the xz plane ($\phi=0$), we have

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})=\frac{\mu_0}{4\pi}\frac{m}{r^3}[2\cos\theta(\sin\theta\,\hat{\mathbf{x}}+\cos\theta\,\hat{\mathbf{z}})+\sin\theta(\cos\theta\,\hat{\mathbf{x}}-\sin\theta\,\hat{\mathbf{z}})] \\
&= \frac{\mu_0}{4\pi}\frac{m}{r^3}[3\sin\theta\cos\theta\,\hat{\mathbf{x}}+(2\cos^2\theta-\sin^2\theta)\,\hat{\mathbf{z}}].
\end{aligned}$$

Here we have a *stack* of such dipoles, running from $z=-L/2$ to $z=+L/2$. Put the field point at s on the x axis. The $\hat{\mathbf{x}}$ components cancel (because of symmetrically placed dipoles above and below $z=0$), leaving $\mathbf{B}=\frac{\mu_0}{4\pi}2\mathcal{M}\,\hat{\mathbf{z}}\int_0^{L/2}\frac{(3\cos^2\theta-1)}{r^3}\,dz$, where $\mathcal{M}$ is the dipole moment per unit length: $m=I\pi R^2=(\sigma vh)\pi R^2=\sigma\omega R\pi R^2h\Rightarrow\mathcal{M}=\frac{m}{h}=\pi\sigma\omega R^3$. Now $\sin\theta=\frac{s}{r}$, so $\frac{1}{r^3}=\frac{\sin^3\theta}{s^3}$; $z=-s\cot\theta\Rightarrow dz=\frac{s}{\sin^2\theta}\,d\theta$. Therefore



$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{2\pi}(\pi\sigma\omega R^3)\,\hat{\mathbf{z}}\int_{\pi/2}^{\theta_m}(3\cos^2\theta-1)\frac{\sin^3\theta}{s^3}\,\frac{s}{\sin^2\theta}\,d\theta = \frac{\mu_0\sigma\omega R^3}{2s^2}\,\hat{\mathbf{z}}\int_{\pi/2}^{\theta_m}(3\cos^2\theta-1)\sin\theta\,d\theta \\
&= \frac{\mu_0\sigma\omega R^3}{2s^2}\,\hat{\mathbf{z}}\left(-\cos^3\theta+\cos\theta\right)\Big|_{\pi/2}^{\theta_m} = \frac{\mu_0\sigma\omega R^3}{2s^2}\cos\theta_m\left(1-\cos^2\theta_m\right)\,\hat{\mathbf{z}} = \frac{\mu_0\sigma\omega R^3}{2s^2}\cos\theta_m\sin^2\theta_m\,\hat{\mathbf{z}}.
\end{aligned}$$

But $\sin\theta_m = \dfrac{s}{\sqrt{s^2+(L/2)^2}}$, and $\cos\theta_m = \dfrac{-(L/2)}{\sqrt{s^2+(L/2)^2}}$, so $\boxed{\mathbf{B} = -\dfrac{\mu_0\sigma\omega R^3 L}{4[s^2+(L/2)^2]^{3/2}}\,\hat{\mathbf{z}}.}$

Chapter 6

Magnetostatic Fields in Matter

Problem 6.1

$$\mathbf{N} = \mathbf{m}_2 \times \mathbf{B}_1;\ \mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1\right];\ \hat{\mathbf{r}} = \hat{\mathbf{y}};\mathbf{m}_1 = m_1\hat{\mathbf{z}};\ \mathbf{m}_2 = m_2\hat{\mathbf{y}}.\quad \mathbf{B}_1 = -\frac{\mu_0}{4\pi}\frac{m_1}{r^3}\hat{\mathbf{z}}.$$

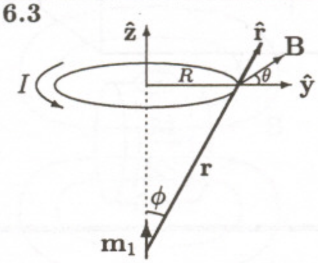
$\mathbf{N} = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}(\hat{\mathbf{y}}\times\hat{\mathbf{z}}) = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\hat{\mathbf{x}}$. Here $m_1 = \pi a^2 I$, $m_2 = b^2 I$. So $\boxed{\mathbf{N} = -\frac{\mu_0}{4}\frac{(abI)^2}{r^3}\hat{\mathbf{x}}.}$ Final orientation : $\boxed{\text{downward}}$ $(-\hat{\mathbf{z}})$.

Problem 6.2

$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}$; $d\mathbf{N} = \mathbf{r} \times d\mathbf{F} = I\,\mathbf{r} \times (d\mathbf{l} \times \mathbf{B})$. Now (Prob. 1.6): $\mathbf{r} \times (d\mathbf{l} \times \mathbf{B}) + d\mathbf{l} \times (\mathbf{B} \times \mathbf{r}) + \mathbf{B} \times (\mathbf{r} \times d\mathbf{l}) = 0$. But $d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] = d\mathbf{r} \times (\mathbf{r} \times \mathbf{B}) + \mathbf{r} \times (d\mathbf{r} \times \mathbf{B})$ (since $\mathbf{B}$ is constant), and $d\mathbf{r} = d\mathbf{l}$, so $d\mathbf{l} \times (\mathbf{B} \times \mathbf{r}) = \mathbf{r} \times (d\mathbf{l} \times \mathbf{B}) - d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})]$. Hence $2\mathbf{r} \times (d\mathbf{l} \times \mathbf{B}) = d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times (\mathbf{r} \times d\mathbf{l})$. $d\mathbf{N} = \frac{1}{2}I\left\{d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times (\mathbf{r} \times d\mathbf{l})\right\}$. $\therefore \mathbf{N} = \frac{1}{2}I\left\{\oint d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times \oint(\mathbf{r} \times d\mathbf{l})\right\}$. But the first term is zero ($\oint d(\cdots) = 0$), and the second integral is $2\mathbf{a}$ (Eq. 1.107). So $\mathbf{N} = -I(\mathbf{B} \times \mathbf{a}) = \mathbf{m} \times \mathbf{B}$. qed

Problem 6.3

(a)



According to Eq. 6.2, $F = 2\pi IRB\cos\theta$. But $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1]}{r^3}$, and $B\cos\theta = \mathbf{B}\cdot\hat{\mathbf{y}}$, so $B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}) - (\mathbf{m}_1\cdot\hat{\mathbf{y}})\right]$. But $\mathbf{m}_1\cdot\hat{\mathbf{y}} = 0$ and $\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\phi$, while $\mathbf{m}_1\cdot\hat{\mathbf{r}} = m_1\cos\theta$. $\therefore\ B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$.

$F = 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$. Now $\sin\phi = \frac{R}{r}$, $\cos\phi = \sqrt{r^2 - R^2}/r$, so $F = 3\frac{\mu_0}{2}m_1 IR^2\frac{\sqrt{r^2-R^2}}{r^5}$.

But $IR^2\pi = m_2$, so $F = \frac{3\mu_0}{2\pi}m_1 m_2\frac{\sqrt{r^2-R^2}}{r^5}$, while for a dipole, $R \ll r$, so $\boxed{F = \frac{3\mu_0}{2\pi}\frac{m_1 m_2}{r^4}.}$

(b) $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}_2\cdot\mathbf{B}) = (\mathbf{m}_2\cdot\boldsymbol{\nabla})\mathbf{B} = \left(m_2\frac{d}{dz}\right)\left[\frac{\mu_0}{4\pi}\frac{1}{z^3}\underbrace{(3(\mathbf{m}_1\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{m}_1)}_{2\mathbf{m}_1}\right] = \frac{\mu_0}{2\pi}m_1 m_2\hat{\mathbf{z}}\underbrace{\frac{d}{dz}\left(\frac{1}{z^3}\right)}_{-3\frac{1}{z^4}}$,

or, since $z = r$: $\boxed{\mathbf{F} = -\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{r^4}\hat{\mathbf{z}}.}$

Problem 6.4

$$d\mathbf{F} = I\left\{(dy\,\hat{\mathbf{y}}) \times \mathbf{B}(0,y,0) + (dz\,\hat{\mathbf{z}}) \times \mathbf{B}(0,\epsilon,z) - (dy\,\hat{\mathbf{y}}) \times \mathbf{B}(0,y,\epsilon) - (dz\,\hat{\mathbf{z}}) \times \mathbf{B}(0,0,z)\right\}$$

$$= I\Big\{-(dy\,\hat{\mathbf{y}}) \times \underbrace{[\mathbf{B}(0,y,\epsilon) - \mathbf{B}(0,y,0)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial z}} + (dz\,\hat{\mathbf{z}}) \times \underbrace{[\mathbf{B}(0,\epsilon,z) - \mathbf{B}(0,0,z)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial y}}\Big\}$$

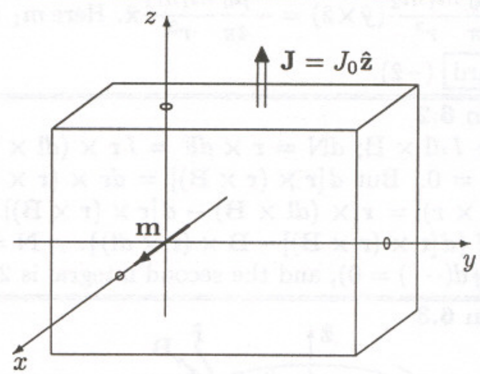
$$\Rightarrow I\epsilon^2\left\{\hat{\mathbf{z}} \times \frac{\partial\mathbf{B}}{\partial y} - \hat{\mathbf{y}} \times \frac{\partial\mathbf{B}}{\partial z}\right\}. \quad \left[\text{Note that } \int dy\, \frac{\partial\mathbf{B}}{\partial z}\Big|_{0,y,0} \approx \epsilon\, \frac{\partial\mathbf{B}}{\partial z}\Big|_{0,0,0} \text{ and } \int dz\, \frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,z} \approx \epsilon\, \frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,0}.\right]$$

$$\mathbf{F} = m\left\{\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \end{vmatrix} - \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 0 \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{vmatrix}\right\} = m\left\{\hat{\mathbf{y}}\frac{\partial B_x}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_y}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_z}{\partial z} - \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right\}$$

$$= m\left[\hat{\mathbf{x}}\frac{\partial B_x}{\partial x} + \hat{\mathbf{y}}\frac{\partial B_x}{\partial y} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right] \quad \left(\text{using } \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \text{ to write } \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x}\right).$$

But $\mathbf{m}\cdot\mathbf{B} = mB_x$ (since $\mathbf{m} = m\hat{\mathbf{x}}$, here), so $\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) = m\boldsymbol{\nabla}(B_x) = m\left(\frac{\partial B_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right)$.
Therefore $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})$. qed

Problem 6.5



(a) $\mathbf{B} = \mu_0 J_0 x\hat{\mathbf{y}}$ (Prob. 5.14).
$\mathbf{m}\cdot\mathbf{B} = 0$, so Eq. 6.3 says $\boxed{\mathbf{F} = 0.}$

(b) $\mathbf{m}\cdot\mathbf{B} = m_0\mu_0 J_0 x$, so $\boxed{\mathbf{F} = m_0\mu_0 J_0\hat{\mathbf{x}}.}$

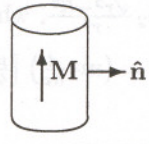
(c) Use product rule #4: $\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E})$
$= \mathbf{p}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{p}) + (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E} + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{p}$.
But $\mathbf{p}$ does not depend on (x,y,z), so the second and fourth terms vanish, and $\boldsymbol{\nabla}\times\mathbf{E} = 0$, so the first term is zero. Hence $\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}) = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$. qed

This argument does *not* apply to the magnetic analog, since $\boldsymbol{\nabla}\times\mathbf{B} \neq 0$. In fact, $\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B} + \mu_0(\mathbf{m}\times\mathbf{J})$.
$(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}_a = m_0\frac{\partial}{\partial x}(\mathbf{B}) = m_0\mu_0 J_0\hat{\mathbf{y}}$, $(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}_b = m_0\frac{\partial}{\partial y}(\mu_0 J_0 x\hat{\mathbf{y}}) = 0$.

Problem 6.6

Aluminum, copper, copper chloride, and sodium all have an *odd* number of electrons, so we expect them to be paramagnetic. The rest (having an even number) should be diamagnetic.

Problem 6.7



$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = 0$; $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$.
The field is that of a surface current $\mathbf{K}_b = M\hat{\boldsymbol{\phi}}$, but that's just a solenoid, so the field $\boxed{\text{outside is zero,}}$ and inside $B = \mu_0 K_b = \mu_0 M$. Moreover, it points upward (in the drawing), so $\boxed{\mathbf{B} = \mu_0\mathbf{M}.}$

Problem 6.8

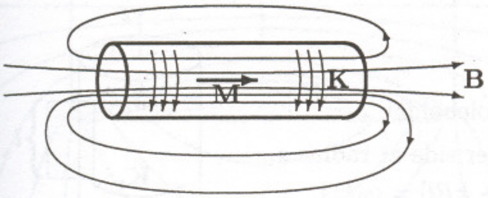
$$\nabla\times\mathbf{M}=\mathbf{J}_b=\frac{1}{s}\frac{\partial}{\partial s}(s\,ks^2)\hat{\mathbf{z}}=\frac{1}{s}(3ks^2)\hat{\mathbf{z}}=3ks\hat{\mathbf{z}},\quad \mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}=ks^2(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{s}})=-kR^2\hat{\mathbf{z}}.$$

So the bound current flows up the cylinder, and returns down the surface. [Incidentally, the *total* current should be zero ... *is* it? *Yes*, for $\int J_b\,da=\int_0^R(3ks)(2\pi s\,ds)=2\pi kR^3$, while $\int K_b\,dl=(-kR^2)(2\pi R)=-2\pi kR^3$.] Since these currents have cylindrical symmetry, we can get the field by Ampère's law:

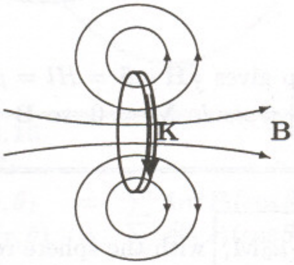
$$B\cdot 2\pi s=\mu_0 I_{\text{enc}}=\mu_0\int_0^s J_b\,da=2\pi k\mu_0 s^3\Rightarrow \boxed{\mathbf{B}=\mu_0 ks^2\hat{\boldsymbol{\phi}}}=\mu_0\mathbf{M}.$$

Outside the cylinder $I_{\text{enc}}=0$, so $\boxed{\mathbf{B}=0.}$

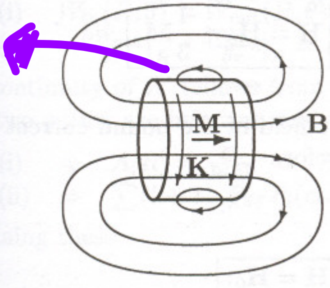
Problem 6.9



$\boxed{\mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}=M\hat{\boldsymbol{\phi}}.}$
(Essentially a long solenoid)



(Essentially a physical dipole)



(Intermediate case)

[The external fields are the same as in the electrical case; the *internal* fields (inside the bar) are completely different—in fact, opposite in direction.]

Problem 6.10

$K_b=M$, so the field inside a *complete* ring would be $\mu_0 M$. The field of a square loop, at the center, is given by Prob. 5.8: $B_{\text{sq}}=\sqrt{2}\,\mu_0 I/\pi R$. Here $I=Mw$, and $R=a/2$, so

$$B_{\text{sq}}=\frac{\sqrt{2}\,\mu_0 Mw}{\pi(a/2)}=\frac{2\sqrt{2}\,\mu_0 Mw}{\pi a};\quad \text{net field in gap}:\ \boxed{\mathbf{B}=\mu_0\mathbf{M}\left(1-\frac{2\sqrt{2}\,w}{\pi a}\right).}$$

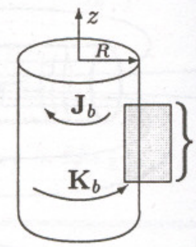
Problem 6.11

As in Sec. 4.2.3, we want the average of $\mathbf{B} = \mathbf{B}_{\text{out}} + \mathbf{B}_{\text{in}}$, where $\mathbf{B}_{\text{out}}$ is due to molecules *outside* a small sphere around point P, and $\mathbf{B}_{\text{in}}$ is due to molecules *inside* the sphere. The average of $\mathbf{B}_{\text{out}}$ is same as field at center (Prob. 5.57b), and for this it is OK to use Eq. 6.10, since the center is "far" from all the molecules in question:

$$\mathbf{A}_{\text{out}} = \frac{\mu_0}{4\pi} \int_{\text{outside}} \frac{\mathbf{M} \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau$$

The average of $\mathbf{B}_{\text{in}}$ is $\frac{\mu_0}{4\pi}\left(\frac{2\mathbf{m}}{R^3}\right)$—Eq. 5.89—where $\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M}$. Thus the average $\mathbf{B}_{\text{in}}$ is $2\mu_0\mathbf{M}/3$. But what is *left out* of the integral $\mathbf{A}_{\text{out}}$ is the contribution of a uniformly magnetized sphere, to wit: $2\mu_0\mathbf{M}/3$ (Eq. 6.16), and this is precisely what $\mathbf{B}_{\text{in}}$ puts back in. So we'll get the correct macroscopic field using Eq. 6.10. qed

Problem 6.12



(a) $\mathbf{M} = ks\hat{\mathbf{z}}$; $\mathbf{J}_b = \nabla\times\mathbf{M} = -k\hat{\boldsymbol{\phi}}$; $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = kR\hat{\boldsymbol{\phi}}$.

$\mathbf{B}$ is in the z direction (this is essentially a superposition of solenoids). So $\boxed{\mathbf{B} = 0 \text{ outside.}}$ Use the amperian loop shown (shaded)—inner side at radius s:

$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0\left[\int J_b\, da + K_b l\right] = \mu_0\left[-kl(R-s) + kRl\right] = \mu_0 kls.$

$\therefore \boxed{\mathbf{B} = \mu_0 ks\hat{\mathbf{z}} \text{ inside.}}$

(b) By symmetry, $\mathbf{H}$ points in the z direction. That same amperian loop gives $\oint \mathbf{H}\cdot d\mathbf{l} = Hl = \mu_0 I_{f_{\text{enc}}} = 0$, since there *is* no free current here. So $\boxed{\mathbf{H} = 0}$, and hence $\boxed{\mathbf{B} = \mu_0\mathbf{M}.}$ *Outside* $\mathbf{M} = 0$, so $\mathbf{B} = 0$; *inside* $\mathbf{M} = ks\hat{\mathbf{z}}$, so $\mathbf{B} = \mu_0 ks\hat{\mathbf{z}}$.

Problem 6.13

(a) The field of a magnetized sphere is $\frac{2}{3}\mu_0\mathbf{M}$ (Eq. 6.16), so $\boxed{\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M},}$ with the sphere removed.

In the cavity, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$, so $\mathbf{H} = \frac{1}{\mu_0}\left(\mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}\right) = \mathbf{H}_0 + \mathbf{M} - \frac{2}{3}\mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0 + \frac{1}{3}\mathbf{M}.}$

(b)

The field inside a long solenoid is $\mu_0 K$. Here $K = M$, so the field of the bound current on the inside surface of the cavity is $\mu_0 M$, pointing *down*. Therefore

$$\boxed{\mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M};}$$

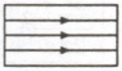
$$\mathbf{H} = \frac{1}{\mu_0}(\mathbf{B}_0 - \mu_0\mathbf{M}) = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0.}$$

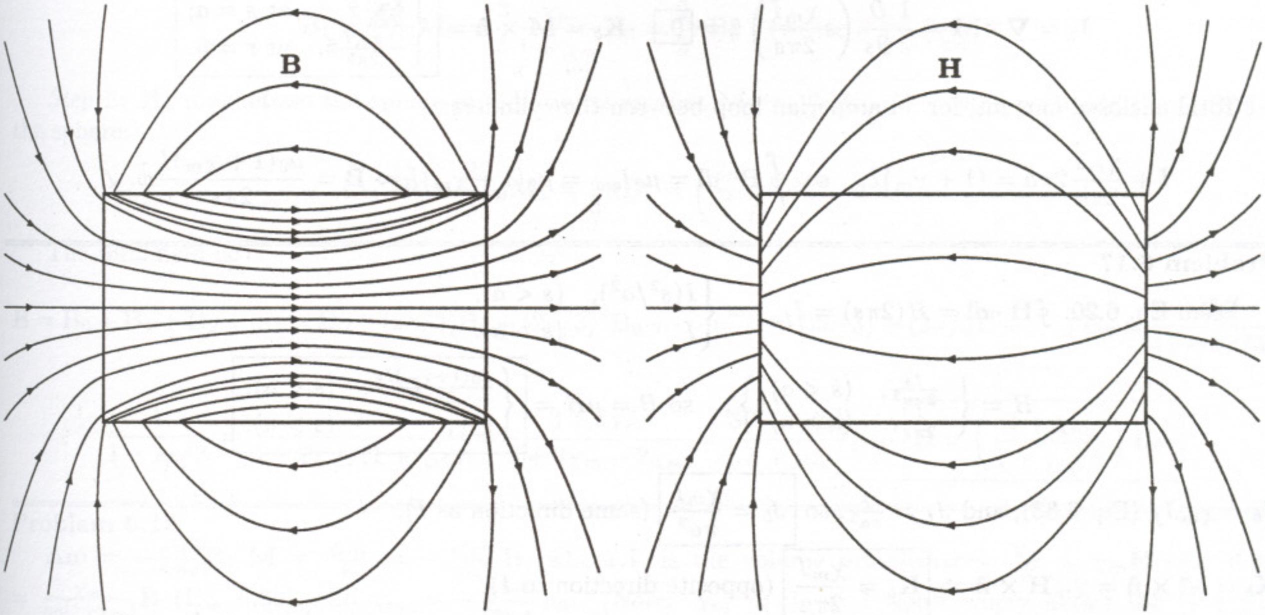
(c)

This time the bound currents are small, and far away from the center, so $\boxed{\mathbf{B} = \mathbf{B}_0,}$ while $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 = \mathbf{H}_0 + \mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0 + \mathbf{M}.}$

[*Comment*: In the wafer, $\mathbf{B}$ is the field in the medium; in the needle, $\mathbf{H}$ is the $\mathbf{H}$ in the medium; in the sphere (intermediate case) both $\mathbf{B}$ and $\mathbf{H}$ are modified.]

Problem 6.14

M:  ; **B** is the same as the field of a short solenoid; $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$.



Problem 6.15

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) &= \sum A_l r^l P_l(\cos\theta), \quad (r < R); \\ W_{\text{out}}(r,\theta) &= \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), \quad (r > R). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta), \\ \text{(ii)} & -\frac{\partial W_{\text{out}}}{\partial r}\Big|_R + \frac{\partial W_{\text{in}}}{\partial r}\Big|_R = M^{\perp} = M\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = M\cos\theta. \end{cases}$$

(The continuity of W follows from the gradient theorem: $W(\mathbf{b}) - W(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \nabla W \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{H}\cdot d\mathbf{l}$; if the two points are infinitesimally separated, this last integral $\to 0$.)

$$\begin{cases} \text{(i)} & \Rightarrow \quad A_l R^l = \frac{B_l}{R^{l+1}} \quad \Rightarrow \quad B_l = R^{2l+1} A_l, \\ \text{(ii)} & \Rightarrow \quad \sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) + \sum l A_l R^{l-2} P_l(\cos\theta) = M\cos\theta. \end{cases}$$

Combining these:

$$\sum (2l+1) R^{l-1} A_l P_l(\cos\theta) = M\cos\theta, \text{ so } A_l = 0 \ (l \neq 1), \text{ and } 3A_1 = M \Rightarrow A_1 = \frac{M}{3}.$$

Thus $W_{\text{in}}(r,\theta) = \dfrac{M}{3} r\cos\theta = \dfrac{M}{3} z$, and hence $\mathbf{H}_{\text{in}} = -\nabla W_{\text{in}} = -\dfrac{M}{3}\hat{\mathbf{z}} = -\dfrac{1}{3}\mathbf{M}$, so

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0\left(-\frac{1}{3}\mathbf{M} + \mathbf{M}\right) = \boxed{\frac{2}{3}\mu_0\mathbf{M}.} \checkmark$$

Problem 6.16

$\oint \mathbf{H}\cdot d\mathbf{l} = I_{f_{enc}} = I$, so $\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \boxed{\mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\boldsymbol{\phi}}.}$ $\mathbf{M} = \chi_m\mathbf{H} = \boxed{\frac{\chi_m I}{2\pi s}\hat{\boldsymbol{\phi}}.}$

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\hat{\mathbf{z}} = \boxed{0.}\quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \boxed{\begin{cases}\frac{\chi_m I}{2\pi a}\hat{\mathbf{z}}, & \text{at } s=a;\\ -\frac{\chi_m I}{2\pi b}\hat{\mathbf{z}}, & \text{at } r=b.\end{cases}}$$

Total enclosed current, for an amperian loop between the cylinders:

$$I + \frac{\chi_m I}{2\pi a}2\pi a = (1+\chi_m)I, \quad \text{so} \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0(1+\chi_m)I \Rightarrow \mathbf{B} = \frac{\mu_0(1+\chi_m)I}{2\pi s}\hat{\boldsymbol{\phi}}. \checkmark$$

Problem 6.17

From Eq. 6.20: $\oint \mathbf{H}\cdot d\mathbf{l} = H(2\pi s) = I_{f_{enc}} = \begin{cases} I(s^2/a^2), & (s<a);\\ I & (s>a).\end{cases}$

$$H = \left\{\begin{array}{ll}\frac{Is}{2\pi a^2}, & (s<a)\\ \frac{I}{2\pi s}, & (s>a)\end{array}\right\}, \quad \text{so } B = \mu H = \boxed{\begin{cases}\frac{\mu_0(1+\chi_m)Is}{2\pi a^2}, & (s<a);\\ \frac{\mu_0 I}{2\pi s}, & (s>a).\end{cases}}$$

$\mathbf{J}_b = \chi_m \mathbf{J}_f$ (Eq. 6.33), and $J_f = \frac{I}{\pi a^2}$, so $\boxed{J_b = \frac{\chi_m I}{\pi a^2}}$ (same direction as I).

$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \chi_m\mathbf{H}\times\hat{\mathbf{n}} \Rightarrow \boxed{\mathbf{K}_b = \frac{\chi_m I}{2\pi a}}$ (opposite direction to I).

$I_b = J_b(\pi a^2) + K_b(2\pi a) = \chi_m I - \chi_m I = \boxed{0}$ (as it *should* be, of course).

Problem 6.18

By the method of Prob. 6.15:

For large r, we want $\mathbf{B}(r,\theta) \to \mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, so $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} \to \frac{1}{\mu_0}B_0\,\hat{\mathbf{z}}$, and hence $W \to -\frac{1}{\mu_0}B_0 z = -\frac{1}{\mu_0}B_0 r\cos\theta$.

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) &= \sum A_l r^l P_l(\cos\theta), & (r<R);\\ W_{\text{out}}(r,\theta) &= -\frac{1}{\mu_0}B_0 r\cos\theta + \sum \frac{B_l}{r^{l+1}}P_l(\cos\theta), & (r>R).\end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta),\\ \text{(ii)} & -\mu_0\frac{\partial W_{\text{out}}}{\partial r}\Big|_R + \mu\frac{\partial W_{\text{in}}}{\partial r}\Big|_R = 0.\end{cases}$$

(The latter follows from Eq. 6.26.)

$$\text{(ii)} \Rightarrow \mu_0\left[\frac{1}{\mu_0}B_0\cos\theta + \sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta)\right] + \mu\sum l A_l R^{l-1}P_l(\cos\theta) = 0.$$

For $l\neq 1$, (i) $\Rightarrow B_l = R^{2l+1}A_l$, so $[\mu_0(l+1)+\mu l]A_l R^{l-1} = 0$, and hence $A_l = 0$.

For $l=1$, (i) $\Rightarrow A_1 R = -\frac{1}{\mu_0}B_0 R + B_1/R^2$, and (ii) $\Rightarrow B_0 + 2\mu_0 B_1/R^3 + \mu A_1 = 0$, so $A_1 = -3B_0/(2\mu_0+\mu)$.

$$W_{\text{in}}(r,\theta) = -\frac{3B_0}{(2\mu_0+\mu)}r\cos\theta = -\frac{3B_0 z}{(2\mu_0+\mu)}. \quad \mathbf{H}_{\text{in}} = -\boldsymbol{\nabla}W_{\text{in}} = \frac{3B_0}{(2\mu_0+\mu)}\hat{\mathbf{z}} = \frac{3\mathbf{B}_0}{(2\mu_0+\mu)}.$$

$$\mathbf{B} = \mu\mathbf{H} = \frac{3\mu\mathbf{B}_0}{(2\mu_0+\mu)} = \boxed{\left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0.}$$

By the method of Prob. 4.23:

Step 1: $\mathbf{B}_0$ magnetizes the sphere: $\mathbf{M}_0 = \chi_m \mathbf{H}_0 = \frac{\chi_m}{\mu_0(1+\chi_m)}\mathbf{B}_0$. This magnetization sets up a field within the sphere given by Eq. 6.16:

$$\mathbf{B}_1 = \frac{2}{3}\mu_0 \mathbf{M}_0 = \frac{2}{3}\frac{\chi_m}{1+\chi_m}\mathbf{B}_0 = \frac{2}{3}\kappa \mathbf{B}_0 \quad (\text{where } \kappa \equiv \tfrac{\chi_m}{1+\chi_m}).$$

Step 2: $\mathbf{B}_1$ magnetizes the sphere an additional amount $\mathbf{M}_1 = \frac{\kappa}{\mu_0}\mathbf{B}_1$. This sets up an additional field in the sphere:

$$\mathbf{B}_2 = \frac{2}{3}\mu_0 \mathbf{M}_1 = \frac{2}{3}\kappa \mathbf{B}_1 = \left(\frac{2\kappa}{3}\right)^2 \mathbf{B}_0, \quad \text{etc.}$$

The *total* field is:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2 + \cdots = \mathbf{B}_0 + (2\kappa/3)\mathbf{B}_0 + (2\kappa/3)^2\mathbf{B}_0 + \cdots = \left[1 + (2\kappa/3) + (2\kappa/3)^2 + \cdots\right]\mathbf{B}_0 = \frac{\mathbf{B}_0}{(1-2\kappa/3)}.$$

$$\frac{1}{1-2\kappa/3} = \frac{3}{3-2\chi_m/(1+\chi_m)} = \frac{3+3\chi_m}{3+3\chi_m-2\chi_m} = \frac{3(1+\chi_m)}{3+\chi_m}, \text{ so } \boxed{\mathbf{B} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0.}$$

Problem 6.19

$\Delta\mathbf{m} = -\frac{e^2r^2}{4m_e}\mathbf{B}$; $\mathbf{M} = \frac{\Delta\mathbf{m}}{V} = -\frac{e^2r^2}{4m_eV}\mathbf{B}$, where V is the volume per electron. $\mathbf{M} = \chi_m\mathbf{H}$ (Eq. 6.29) $= \frac{\chi_m}{\mu_0(1+\chi_m)}\mathbf{B}$ (Eq. 6.30). So $\chi_m = -\frac{e^2r^2}{4m_eV}\mu_0$. [*Note*: $\chi_m \ll 1$, so I won't worry about the $(1+\chi_m)$ term; for the same reason we need not distinguish $\mathbf{B}$ from $\mathbf{B}_{\text{else}}$, as we did in deriving the Clausius-Mossotti equation in Prob. 4.38.] Let's say $V = \frac{4}{3}\pi r^3$. Then $\chi_m = -\frac{\mu_0}{4\pi}\left(\frac{3e^2}{4m_er}\right)$. I'll use 1 Å$= 10^{-10}$ m for r. Then $\chi_m = -(10^{-7})\left(\frac{3(1.6\times10^{-19})^2}{4(9.1\times10^{-31})(10^{-10})}\right) = \boxed{-2\times10^{-5},}$ which is not bad—Table 6.1 says $\chi_m = -1\times10^{-5}$. However, I used only *one electron* per atom (copper has 29) and a very crude value for r. Since the orbital radius is smaller for the inner electrons, they count for less ($\Delta m \sim r^2$). I have also neglected competing paramagnetic effects. But never mind ... this is in the right ball park.

Problem 6.20

Place the object in a region of zero magnetic field, and heat it above the Curie point—or simply drop it on a hard surface. If it's delicate (a watch, say), place it between the poles of an electromagnet, and magnetize it back and forth many times; each time you reverse the direction, reduce the field slightly.

Problem 6.21

(a) Identical to Prob. 4.7, only starting with Eqs. 6.1 and 6.3 instead of Eqs. 4.4 and 4.5.

(b) Identical to Prob. 4.8, but starting with Eq. 5.87 instead of 3.104.

(c) $U = -\frac{\mu_0}{4\pi}\frac{1}{r^3}[3\cos\theta_1\cos\theta_2 - \cos(\theta_2-\theta_1)]m_1m_2$. Or, using $\cos(\theta_2-\theta_1) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$,

$$\boxed{U = \frac{\mu_0}{4\pi}\frac{m_1m_2}{r^3}(\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2).}$$

Stable position occurs at minimum energy: $\frac{\partial U}{\partial\theta_1} = \frac{\partial U}{\partial\theta_2} = 0$

$$\begin{cases} \frac{\partial U}{\partial\theta_1} = \frac{\mu_0m_1m_2}{4\pi r^3}(\cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2) = 0 \Rightarrow 2\sin\theta_1\cos\theta_2 = -\cos\theta_1\sin\theta_2; \\ \frac{\partial U}{\partial\theta_2} = \frac{\mu_0m_1m_2}{4\pi r^3}(\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2) = 0 \Rightarrow 2\sin\theta_1\cos\theta_2 = -4\cos\theta_1\sin\theta_2. \end{cases}$$

$$\text{Thus } \sin\theta_1\cos\theta_2 = \sin\theta_2\cos\theta_1 = 0. \begin{cases} \text{Either } \sin\theta_1 = \sin\theta_2 = 0: & \overset{①}{\longrightarrow\longrightarrow} \text{ or } \overset{②}{\longrightarrow\longleftarrow} \\ \text{or } \quad \cos\theta_1 = \cos\theta_2 = 0: & \underset{③}{\uparrow\,\uparrow} \text{ or } \underset{④}{\uparrow\,\downarrow} \end{cases}$$

Which of these is the *stable* minimum? Certainly not ② or ③—for these $\mathbf{m}_2$ is not parallel to $\mathbf{B}_1$, whereas we know $\mathbf{m}_2$ will line up along $\mathbf{B}_1$. It remains to compare ① (with $\theta_1 = \theta_2 = 0$) and ④ (with $\theta_1 = \pi/2, \theta_2 = -\pi/2$): $U_1 = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-2)$; $U_2 = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-1)$. U_1 is the lower energy, hence the more stable configuration.

$\boxed{\textit{Conclusion}\text{: They line up parallel, along the line joining them: } \longrightarrow \ \longrightarrow}$

(d) They'd line up the same way: $\longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow$

Problem 6.22

$$\mathbf{F} = I\oint d\mathbf{l}\times\mathbf{B} = I\left(\oint d\mathbf{l}\right)\times\mathbf{B}_0 + I\oint d\mathbf{l}\times[(\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{B}_0] - I\left(\oint d\mathbf{l}\right)\times[(\mathbf{r}_0\cdot\boldsymbol{\nabla}_0)\mathbf{B}_0] = I\oint d\mathbf{l}\times[(\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{B}_0]$$

(because $\oint d\mathbf{l} = 0$). Now

$$(d\mathbf{l}\times\mathbf{B}_0)_i = \sum_{j,k}\epsilon_{ijk}\,dl_j(B_0)_k, \quad \text{and } (\mathbf{r}\cdot\boldsymbol{\nabla}_0) = \sum_l r_l(\nabla_0)_l, \text{ so}$$

$$\begin{aligned} F_i &= I\sum_{j,k,l}\epsilon_{ijk}\left[\oint r_l\,dl_j\right][(\nabla_0)_l(B_0)_k] \quad \left\{\text{Lemma 1: } \oint r_l\,dl_j = \sum_m \epsilon_{ljm}a_m \text{ (proof below).}\right\} \\ &= I\sum_{j,k,l,m}\epsilon_{ijk}\epsilon_{ljm}a_m(\nabla_0)_l(B_0)_k \quad \left\{\text{Lemma 2: } \sum_j \epsilon_{ijk}\epsilon_{ljm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl} \text{ (proof below).}\right\} \\ &= I\sum_{k,l,m}(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\,a_m(\nabla_0)_l(B_0)_k = I\sum_k[a_k(\nabla_0)_i(B_0)_k - a_i(\nabla_0)_k(B_0)_k] \\ &= I\,[(\nabla_0)_i(\mathbf{a}\cdot\mathbf{B}_0) - a_i(\boldsymbol{\nabla}_0\cdot\mathbf{B}_0)]. \end{aligned}$$

But $\boldsymbol{\nabla}_0\cdot\mathbf{B}_0 = 0$ (Eq. 5.48), and $\mathbf{m} = I\mathbf{a}$ (Eq. 5.84), so $\mathbf{F} = \boldsymbol{\nabla}_0(\mathbf{m}\cdot\mathbf{B}_0)$ (the subscript just reminds us to take the derivatives at the point where $\mathbf{m}$ is located). qed

Proof of Lemma 1:
Eq. 1.108 says $\oint(\mathbf{c}\cdot\mathbf{r})\,d\mathbf{l} = \mathbf{a}\times\mathbf{c} = -\mathbf{c}\times\mathbf{a}$. The jth component is $\sum_p\oint c_p r_p\,dl_j = -\sum_{p,m}\epsilon_{jpm}c_p a_m$. Pick $c_p = \delta_{pl}$ (i.e. 1 for the lth component, zero for the others). Then $\oint r_l\,dl_j = -\sum_m\epsilon_{jlm}a_m = \sum_m\epsilon_{ljm}a_m$. qed

Proof of Lemma 2:
$\epsilon_{ijk}\epsilon_{ljm} = 0$ unless ijk and ljm are both permutations of 123. In particular, i must either be l or m, and k must be the other, so

$$\sum_j \epsilon_{ijk}\epsilon_{ljm} = A\delta_{il}\delta_{km} + B\delta_{im}\delta_{kl}.$$

To determine the constant A, pick $i = l = 1$, $k = m = 3$; the only contribution comes from $j = 2$:

$$\epsilon_{123}\epsilon_{123} = 1 = A\delta_{11}\delta_{33} + B\delta_{13}\delta_{31} = A \Rightarrow A = 1.$$

To determine B, pick $i = m = 1$, $k = l = 3$:

$$\epsilon_{123}\epsilon_{321} = -1 = A\delta_{13}\delta_{31} + B\delta_{11}\delta_{33} = B \Rightarrow B = -1.$$

So

$$\sum_j \epsilon_{ijk}\epsilon_{ljm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl}. \quad \text{qed}$$

Problem 6.23

(a) The *electric* field inside a uniformly *polarized* sphere, $\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}$ (Eq. 4.14) translates to $\mathbf{H} = -\frac{1}{3\mu_0}(\mu_0\mathbf{M}) = -\frac{1}{3}\mathbf{M}$. But $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$. So the *magnetic* field inside a uniformly *magnetized* sphere is $\mathbf{B} = \mu_0(-\frac{1}{3}\mathbf{M}+\mathbf{M}) = \boxed{\frac{2}{3}\mu_0\mathbf{M}}$ (same as Eq. 6.16).

(b) The *electric* field inside a sphere of linear *dielectric* in an otherwise uniform *electric* field is $\mathbf{E} = \frac{1}{1+\chi_e/3}\mathbf{E}_0$ (Eq. 4.49). Now χ_e translates to χ_m, for then Eq. 4.30 ($\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$) goes to $\mu_0\mathbf{M} = \mu_0\chi_m\mathbf{H}$, or $\mathbf{M} = \chi_m\mathbf{H}$ (Eq. 6.29). So Eq. 4.49 $\Rightarrow \mathbf{H} = \frac{1}{1+\chi_m/3}\mathbf{H}_0$. But $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$, and $\mathbf{B}_0 = \mu_0\mathbf{H}_0$ (Eqs. 6.31 and 6.32), so the *magnetic* field inside a sphere of linear *magnetic* material in an otherwise uniform *magnetic* field is

$$\frac{\mathbf{B}}{\mu_0(1+\chi_m)} = \frac{1}{(1+\chi_m/3)}\frac{\mathbf{B}_0}{\mu_0}, \text{ or } \boxed{\mathbf{B} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0} \text{ (as in Prob. 6.18).}$$

(c) The average *electric* field over a sphere, due to charges within, is $\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$. Let's pretend the charges are all due to the frozen-in polarization of some medium (whatever ρ might be, we can solve $\nabla\cdot\mathbf{P} = -\rho$ to find the appropriate $\mathbf{P}$). In this case there are *no* free charges, and $\mathbf{p} = \int \mathbf{P}\,d\tau$, so $\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\int \mathbf{P}\,d\tau$, which translates to

$$\mathbf{H}_{\text{ave}} = -\frac{1}{4\pi\mu_0}\frac{1}{R^3}\int \mu_0\mathbf{M}\,d\tau = -\frac{1}{4\pi R^3}\mathbf{m}.$$

But $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, so $\mathbf{B}_{\text{ave}} = -\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{R^3} + \mu_0\mathbf{M}_{\text{ave}}$, and $\mathbf{M}_{\text{ave}} = \frac{\mathbf{m}}{\frac{4}{3}\pi R^3}$, so $\boxed{\mathbf{B}_{\text{ave}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}}$, in agreement with Eq. 5.89. (We must assume for this argument that all the currents are *bound*, but again it doesn't really matter, since we can model any current configuration by an appropriate frozen-in magnetization. See G. H. Goedecke, *Am. J. Phys.* **66**, 1010 (1998).)

Problem 6.24

$$\begin{array}{llll} \text{Eq. 2.15}: & \mathbf{E} & = & \rho\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'\right\} & \text{(for uniform charge density);} \\ \text{Eq. 4.9}: & V & = & \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'\right\} & \text{(for uniform polarization);} \\ \text{Eq. 6.11}: & \mathbf{A} & = & \mu_0\epsilon_0\mathbf{M}\times\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'\right\} & \text{(for uniform magnetization).} \end{array}$$

For a uniformly charged sphere (radius R): $\left\{\begin{array}{lll} \mathbf{E}_{\text{in}} & = & \rho\left(\frac{1}{3\epsilon_0}\mathbf{r}\right) \quad \text{(Prob. 2.12),} \\ \mathbf{E}_{\text{out}} & = & \rho\left(\frac{1}{3\epsilon_0}\frac{R^3}{r^2}\hat{\mathbf{r}}\right) \quad \text{(Ex. 2.2).} \end{array}\right.$

So the scalar potential of a uniformly polarized sphere is: $\left\{\begin{array}{lll} V_{\text{in}} & = & \frac{1}{3\epsilon_0}(\mathbf{P}\cdot\mathbf{r}), \\ V_{\text{out}} & = & \frac{1}{3\epsilon_0}\frac{R^3}{r^2}(\mathbf{P}\cdot\hat{\mathbf{r}}), \end{array}\right.$

and the vector potential of a uniformly magnetized sphere is: $\left\{\begin{array}{lll} \mathbf{A}_{\text{in}} & = & \frac{\mu_0}{3}(\mathbf{M}\times\mathbf{r}), \\ \mathbf{A}_{\text{out}} & = & \frac{\mu_0}{3}\frac{R^3}{r^2}(\mathbf{M}\times\hat{\mathbf{r}}), \end{array}\right.$

(confirming the results of Ex. 4.2 and of Exs. 6.1 and 5.11).

Problem 6.25

(a) $\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{2m}{z^3}\hat{\mathbf{z}}$ (Eq. 5.86, with $\theta = 0$). So $\mathbf{m}_2\cdot\mathbf{B}_1 = -\frac{\mu_0}{2\pi}\frac{m^2}{z^3}$. $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$ (Eq. 6.3) $\Rightarrow$ $\mathbf{F} = \frac{\partial}{\partial z}\left[-\frac{\mu_0}{2\pi}\frac{m^2}{z^3}\right]\hat{\mathbf{z}} = \frac{3\mu_0 m^2}{2\pi z^4}\hat{\mathbf{z}}$. This is the magnetic force *upward* (on the upper magnet); it balances the gravitational force downward ($-m_d g\hat{\mathbf{z}}$):

$$\frac{3\mu_0 m^2}{2\pi z^4} - m_d g = 0 \Rightarrow \boxed{z = \left[\frac{3\mu_0 m^2}{2\pi m_d g}\right]^{1/4}}.$$

(b) The middle magnet is repelled *upward* by lower magnet and *downward* by upper magnet:

$$\frac{3\mu_0 m^2}{2\pi x^4} - \frac{3\mu_0 m^2}{2\pi y^4} - m_d g = 0.$$

The top magnet is repelled *upward* by middle magnet, and attracted *downward* by lower magnet:

$$\frac{3\mu_0 m^2}{2\pi y^4} - \frac{3\mu_0 m^2}{2\pi (x+y)^4} - m_d g = 0.$$

Subtracting: $\frac{3\mu_0 m^2}{2\pi}\left[\frac{1}{x^4} - \frac{1}{y^4} - \frac{1}{y^4} + \frac{1}{(x+y)^4}\right] - m_d g + m_d g = 0$, or $\frac{1}{x^4} - \frac{2}{y^4} + \frac{1}{(x+y)^4} = 0$, so: $2 = \frac{1}{(x/y)^4} + \frac{1}{(x/y+1)^4}$. Let $\alpha \equiv x/y$; then $2 = \frac{1}{\alpha^4} + \frac{1}{(\alpha+1)^4}$. Mathematica gives the numerical solution $\alpha = \boxed{x/y = 0.850115\ldots}$

Problem 6.26

At the interface, the perpendicular component of $\mathbf{B}$ is continuous (Eq. 6.26), and the parallel component of $\mathbf{H}$ is continuous (Eq. 6.25 with $\mathbf{K}_f = 0$). So $B_1^\perp = B_2^\perp$, $\mathbf{H}_1^\parallel = \mathbf{H}_2^\parallel$. But $\mathbf{B} = \mu\mathbf{H}$ (Eq. 6.31), so $\frac{1}{\mu_1}\mathbf{B}_1^\parallel = \frac{1}{\mu_2}\mathbf{B}_2^\parallel$. Now $\tan\theta_1 = B_1^\parallel/B_1^\perp$, and $\tan\theta_2 = B_2^\parallel/B_2^\perp$, so

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{B_2^\parallel}{B_2^\perp}\frac{B_1^\perp}{B_1^\parallel} = \frac{B_2^\parallel}{B_1^\parallel} = \frac{\mu_2}{\mu_1}$$

(the same form, though for different reasons, as Eq. 4.68).

Problem 6.27

In view of Eq. 6.33, there is a *bound* dipole at the center: $\mathbf{m}_b = \chi_m \mathbf{m}$. So the *net* dipole moment at the center is $\mathbf{m}_{\text{center}} = \mathbf{m} + \mathbf{m}_b = (1+\chi_m)\mathbf{m} = \frac{\mu}{\mu_0}\mathbf{m}$. This produces a field given by Eq. 5.87:

$$\mathbf{B}_{\substack{\text{center}\\ \text{dipole}}} = \frac{\mu}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right].$$

This accounts for the *first* term in the field. The remainder must be due to the bound surface current ($\mathbf{K}_b$) at $r = R$ (since there can be no volume bound current, according to Eq. 6.33). Let us make an educated guess (based either on the answer provided or on the analogous electrical Prob. 4.34) that the field due to the surface bound current is (for interior points) of the form $\mathbf{B}_{\substack{\text{surface}\\ \text{current}}} = A\mathbf{m}$ (i.e. a constant, proportional to $\mathbf{m}$). In that case the magnetization will be:

$$\mathbf{M} = \chi_m \mathbf{H} = \frac{\chi_m}{\mu}\mathbf{B} = \frac{\chi_m}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{\chi_m}{\mu}A\mathbf{m}.$$

This will produce bound currents $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = 0$, as it should, for $0 < r < R$ (no need to *calculate* this curl—the second term is *constant*, and the first is essentially the field of a dipole, which we know is curl-less, except at $r = 0$), and

$$\mathbf{K}_b = \mathbf{M}(R)\times\hat{\mathbf{r}} = \frac{\chi_m}{4\pi R^3}(-\mathbf{m}\times\hat{\mathbf{r}}) + \frac{\chi_m A}{\mu}(\mathbf{m}\times\hat{\mathbf{r}}) = \chi_m m\left(-\frac{1}{4\pi R^3} + \frac{A}{\mu}\right)\sin\theta\,\hat{\boldsymbol{\phi}}.$$

But this is exactly the surface current produced by a spinning sphere: $\mathbf{K} = \sigma\mathbf{v} = \sigma\omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, with $(\sigma\omega R) \leftrightarrow \chi_m m\left(\frac{A}{\mu} - \frac{1}{4\pi R^3}\right)$. So the field it produces (for points inside) is (Eq. 5.68):

$$\mathbf{B}_{\substack{\text{surface}\\ \text{current}}} = \frac{2}{3}\mu_0(\sigma\omega R) = \frac{2}{3}\mu_0\chi_m\mathbf{m}\left(\frac{A}{\mu} - \frac{1}{4\pi R^3}\right).$$

Everything is consistent, therefore, provided $A = \frac{2}{3}\mu_0\chi_m\left(\frac{A}{\mu} - \frac{1}{4\pi R^3}\right)$, or $A\left(1 - \frac{2\mu_0}{3\mu}\chi_m\right) = -\frac{2}{3}\frac{\mu_0\chi_m}{4\pi R^3}$. But $\chi_m = \left(\frac{\mu}{\mu_0}\right) - 1$, so $A\left(1 - \frac{2}{3} + \frac{2}{3}\frac{\mu_0}{\mu}\right) = -\frac{2}{3}\frac{(\mu-\mu_0)}{4\pi R^3}$, or $A\left(1 + \frac{2\mu_0}{\mu}\right) = 2\frac{(\mu_0-\mu)}{4\pi R^3}$; $A = \frac{\mu}{4\pi}\frac{2(\mu_0-\mu)}{R^3(2\mu_0+\mu)}$, and hence

$$\mathbf{B} = \frac{\mu}{4\pi}\left\{\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2(\mu_0-\mu)\mathbf{m}}{R^3(2\mu_0+\mu)}\right\}. \quad \text{qed}$$

The *exterior* field is that of the central dipole plus that of the surface current, which, according to Prob. 5.36, is *also* a perfect dipole field, of dipole moment

$$\mathbf{m}_{\substack{\text{surface}\\\text{current}}} = \frac{4}{3}\pi R^3(\sigma\omega R) = \frac{4}{3}\pi R^3\left(\frac{3}{2\mu_0}\mathbf{B}_{\substack{\text{surface}\\\text{current}}}\right) = \frac{2\pi R^3}{\mu_0}\frac{\mu}{4\pi}\frac{2(\mu_0-\mu)\mathbf{m}}{R^3(2\mu_0+\mu)} = \frac{\mu(\mu_0-\mu)\mathbf{m}}{\mu_0(2\mu_0+\mu)}.$$

So the *total* dipole moment is:

$$\mathbf{m}_{\text{tot}} = \frac{\mu}{\mu_0}\mathbf{m} + \frac{\mu}{\mu_0}\mathbf{m}\frac{(\mu_0-\mu)}{(2\mu_0+\mu)} = \frac{3\mu\mathbf{m}}{(2\mu_0+\mu)},$$

and hence the field (for $r > R$) is

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3\mu}{2\mu_0+\mu}\right)\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}].}$$

Problem 6.28

The problem is that the field inside a *cavity* is not the same as the field in the material itself.

(a) *Ampére type.* The field deep inside the magnet is that of a long solenoid, $\mathbf{B}_0 \approx \mu_0\mathbf{M}$. From Prob. 6.13:

$$\begin{cases} \text{Sphere}: & \mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M} = \frac{1}{3}\mu_0\mathbf{M}; \\ \text{Needle}: & \mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M} = 0; \\ \text{Wafer}: & \mathbf{B} = \mu_0\mathbf{M}. \end{cases}$$

(b) *Gilbert type.* This is analogous to the *electric* case. The field at the center is approximately that midway between two distant point charges, $\mathbf{B}_0 \approx 0$. From Prob. 4.16 (with $\mathbf{E} \to \mathbf{B}$, $1/\epsilon_0 \to \mu_0$, $\mathbf{P} \to \mathbf{M}$):

$$\begin{cases} \text{Sphere}: & \mathbf{B} = \mathbf{B}_0 + \frac{\mu_0}{3}\mathbf{M} = \frac{1}{3}\mu_0\mathbf{M}; \\ \text{Needle}: & \mathbf{B} = \mathbf{B}_0 = 0; \\ \text{Wafer}: & \mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} = \mu_0\mathbf{M}. \end{cases}$$

In the cavities, then, the fields are the *same* for the two models, and this will be no test at all. $\boxed{\text{Yes.}}$ Fund it with \$1 M from the Office of Alternative Medicine.

Chapter 7

Electrodynamics

Problem 7.1

(a) Let Q be the charge on the inner shell. Then $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$ in the space between them, and $(V_a - V_b) = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{4\pi\epsilon_0}Q\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$.

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\frac{Q}{\epsilon_0} = \frac{\sigma}{\epsilon_0}\frac{4\pi\epsilon_0(V_a - V_b)}{(1/a - 1/b)} = \boxed{4\pi\sigma\frac{(V_a - V_b)}{(1/a - 1/b)}}.$$

(b) $R = \dfrac{V_a - V_b}{I} = \boxed{\dfrac{1}{4\pi\sigma}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}.$

(c) For large b ($b \gg a$), the second term is negligible, and $R = 1/4\pi\sigma a$. Essentially all of the resistance is in the region right around the inner sphere. Successive shells, as you go out, contribute less and less, because the cross-sectional area ($4\pi r^2$) gets larger and larger. For the two submerged spheres, $R = \frac{2}{4\pi\sigma a} = \frac{1}{2\pi\sigma a}$ (one R as the current leaves the first, one R as it converges on the second). Therefore $I = V/R = \boxed{2\pi\sigma aV.}$

Problem 7.2

(a) $V = Q/C = IR$. Because positive I means the charge on the capacitor is *decreasing*, $\dfrac{dQ}{dt} = -I = -\dfrac{1}{RC}Q$, so $Q(t) = Q_0e^{-t/RC}$. But $Q_0 = Q(0) = CV_0$, so $\boxed{Q(t) = CV_0e^{-t/RC}.}$

Hence $I(t) = -\dfrac{dQ}{dt} = CV_0\dfrac{1}{RC}e^{-t/RC} = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(b) $W = \boxed{\frac{1}{2}cV_0^2.}$ The energy delivered to the resistor is $\displaystyle\int_0^\infty P\,dt = \int_0^\infty I^2R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-2t/RC}\right)\Big|_0^\infty = \frac{1}{2}CV_0^2.$ ✓

(c) $V_0 = Q/C + IR$. This time positive I means Q is *increasing*: $\dfrac{dQ}{dt} = I = \dfrac{1}{RC}(CV_0 - Q) \Rightarrow \dfrac{dQ}{Q - CV_0} = -\dfrac{1}{RC}dt \Rightarrow \ln(Q - CV_0) = -\dfrac{1}{RC}t + \text{constant} \Rightarrow Q(t) = CV_0 + ke^{-t/RC}$. But $Q(0) = 0 \Rightarrow k = -CV_0$, so $\boxed{Q(t) = CV_0\left(1 - e^{-t/RC}\right).}$ $I(t) = \dfrac{dQ}{dt} = CV_0\left(\dfrac{1}{RC}e^{-t/RC}\right) = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(d) Energy from battery: $\int_0^\infty V_0 I\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}dt = \frac{V_0^2}{R}\left(-RCe^{-t/RC}\right)\Big|_0^\infty = \frac{V_0^2}{R}RC = \boxed{CV_0^2.}$

Since $I(t)$ is the same as in (a), the energy delivered to the resistor is again $\boxed{\frac{1}{2}CV_0^2.}$ The final energy in the capacitor is also $\boxed{\frac{1}{2}CV_0^2,}$ so $\boxed{\text{half}}$ the energy from the battery goes to the capacitor, and the other half to the resistor.

Problem 7.3

(a) $I = \int \mathbf{J}\cdot d\mathbf{a}$, where the integral is taken over a surface enclosing the positively charged conductor. But $\mathbf{J} = \sigma\mathbf{E}$, and Gauss's law says $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q$, so $I = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}Q$. But $Q = CV$, and $V = IR$, so $I = \frac{\sigma}{\epsilon_0}CIR$, or $\boxed{R = \frac{\epsilon_0}{\sigma C}.}$ qed

(b) $Q = CV = CIR \Rightarrow \frac{dQ}{dt} = -I = -\frac{1}{RC}Q \Rightarrow \boxed{Q(t) = Q_0 e^{-t/RC}}$, or, since $V = Q/C$, $V(t) = V_0 e^{-t/RC}$. The time constant is $\tau = RC = \boxed{\epsilon_0/\sigma.}$

Problem 7.4

$I = J(s)\,2\pi sL \Rightarrow J(s) = I/2\pi sL.\quad E = J/\sigma = I/2\pi s\sigma L = I/2\pi kL.$

$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\frac{I}{2\pi kL}(a-b).$ So $\boxed{R = \frac{b-a}{2\pi kL}.}$

Problem 7.5

$$I = \frac{\mathcal{E}}{r+R};\; P = I^2R = \frac{\mathcal{E}^2R}{(r+R)^2};\; \frac{dP}{dR} = \mathcal{E}^2\left[\frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3}\right] = 0 \Rightarrow r+R = 2R \Rightarrow \boxed{R = r.}$$

Problem 7.6

$\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = \boxed{\text{zero}}$ for *all* electrostatic fields. It *looks* as though $\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = (\sigma/\epsilon_0)h$, as would indeed be the case if the field were really just σ/ϵ_0 inside and zero outside. But in fact there is always a "fringing field" at the edges (Fig. 4.31), and this is evidently just right to kill off the contribution from the left end of the loop. The current is $\boxed{\text{zero.}}$

Problem 7.7

(a) $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv$; $\mathcal{E} = IR \Rightarrow \boxed{I = \frac{Blv}{R}.}$ (Never mind the minus sign—it just tells you the *direction* of flow: $(\mathbf{v}\times\mathbf{B})$ is *upward*, in the bar, so *downward* through the resistor.)

(b) $F = IlB = \boxed{\frac{B^2l^2v}{R},}$ to the $\boxed{\text{left.}}$

(c) $F = ma = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \Rightarrow \frac{dv}{dt} = -(\frac{B^2l^2}{Rm})v \Rightarrow \boxed{v = v_0e^{-\frac{B^2l^2}{mR}t}.}$

(d) The energy goes into heat in the resistor. The power delivered to resistor is I^2R, so

$$\frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R^2}R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t},\text{ where } \alpha \equiv \frac{B^2l^2}{mR};\quad \frac{dW}{dt} = \alpha m v_0^2 e^{-2\alpha t}.$$

The total energy delivered to the resistor is $W = \alpha m v_0^2\int_0^\infty e^{-2\alpha t}dt = \alpha m v_0^2\left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^\infty = \alpha m v_0^2\frac{1}{2\alpha} = \frac{1}{2}mv_0^2.$ ✓

Problem 7.8

(a) The field of long wire is $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$, so $\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0 I}{2\pi}\int\limits_s^{s+a} \frac{1}{s}(a\,ds) = \boxed{\frac{\mu_0 I a}{2\pi}\ln\left(\frac{s+a}{s}\right).}$

(b) $\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi}\frac{d}{dt}\ln\left(\frac{s+a}{s}\right)$, and $\frac{ds}{dt} = v$, so $-\frac{\mu_0 I a}{2\pi}\left(\frac{1}{s+a}\frac{ds}{dt} - \frac{1}{s}\frac{ds}{dt}\right) = \boxed{\frac{\mu_0 I a^2 v}{2\pi s(s+a)}.}$

The field points *out* of the page, so the force on a charge in the nearby side of the square is *to the right*. In the far side it's also to the right, but here the field is weaker, so the current flows $\boxed{\text{counterclockwise.}}$

(c) This time the flux is *constant*, so $\boxed{\mathcal{E} = 0.}$

Problem 7.9

Since $\nabla\cdot\mathbf{B} = 0$, Theorem 2(c) (Sect. 1.6.2) guarantees that $\int \mathbf{B}\cdot d\mathbf{a}$ is the same for *all* surfaces with a given boundary line.

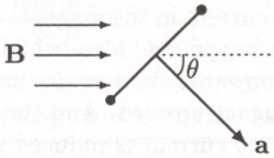
Problem 7.10

$\Phi = \mathbf{B}\cdot\mathbf{a} = Ba^2\cos\theta$

Here $\theta = \omega t$, so

$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba^2(-\sin\omega t)\omega;$

$\boxed{\mathcal{E} = B\omega a^2 \sin\omega t.}$



(view from above)

Problem 7.11

$\mathcal{E} = Blv = IR \Rightarrow I = \frac{Bl}{R}v \Rightarrow$ upward magnetic force $= IlB = \frac{B^2l^2}{R}v$. This opposes the gravitational force downward:

$$mg - \frac{B^2l^2}{R}v = m\frac{dv}{dt};\quad \frac{dv}{dt} = g - \alpha v,\ \text{where } \alpha \equiv \frac{B^2l^2}{mR}.\quad g - \alpha v_t = 0 \Rightarrow v_t = \frac{g}{\alpha} = \boxed{\frac{mgR}{B^2l^2}.}$$

$$\frac{dv}{g-\alpha v} = dt \Rightarrow -\frac{1}{\alpha}\ln(g-\alpha v) = t + \text{const.} \Rightarrow g - \alpha v = Ae^{-\alpha t};\ \text{at } t=0,\ v=0,\ \text{so } A = g.$$

$$\alpha v = g(1-e^{-\alpha t});\quad v = \frac{g}{\alpha}(1-e^{-\alpha t}) = \boxed{v_t(1-e^{-\alpha t}).}$$

At 90% of terminal velocity, $v/v_t = 0.9 = 1 - e^{-\alpha t} \Rightarrow e^{-\alpha t} = 1 - 0.9 = 0.1$; $\ln(0.1) = -\alpha t$; $\ln 10 = \alpha t$; $t = \frac{1}{\alpha}\ln 10$, or $\boxed{t_{90\%} = \frac{v_t}{g}\ln 10.}$

Now the numbers: $m = 4\eta A l$, where η is the mass density of aluminum, A is the cross-sectional area, and l is the length of a side. $R = 4l/A\sigma$, where σ is the conductivity of aluminum. So

$$v_t = \frac{4\eta A l g 4l}{A\sigma B^2 l^2} = \frac{16\eta g}{\sigma B^2} = \frac{16 g\eta\rho}{B^2},\ \text{and}\ \left\{\begin{array}{l}\rho = 2.8\times10^{-8}\,\Omega\,\text{m}\\ g = 9.8\,\text{m/s}^2\\ \eta = 2.7\times10^3\,\text{kg/m}^3\\ B = 1\,\text{T}\end{array}\right\}.$$

So $v_t = \frac{(16)(9.8)(2.7\times10^3)(2.8\times10^{-8})}{1} = \boxed{1.2\text{ cm/s};}\ t_{90\%} = \frac{1.2\times10^{-2}}{9.8}\ln(10) = \boxed{2.8\text{ ms.}}$

$\boxed{\text{If the loop were cut, it would fall freely, with acceleration } g.}$

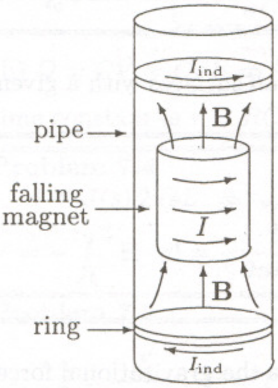
Problem 7.12

$$\Phi = \pi\left(\frac{a}{2}\right)^2 B = \frac{\pi a^2}{4} B_0 \cos(\omega t);\ \mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi a^2}{4} B_0 \omega \sin(\omega t).\quad I(t) = \frac{\mathcal{E}}{R} = \boxed{\frac{\pi a^2 \omega}{4R} B_0 \sin(\omega t).}$$

Problem 7.13

$$\Phi = \int B\, dx\, dy = kt^2 \int_0^a dx \int_0^a y^3\, dy = \frac{1}{4} kt^2 a^5.\quad \mathcal{E} = -\frac{d\Phi}{dt} = -\boxed{\tfrac{1}{2} kta^5.}$$

Problem 7.14



Suppose the current (I) in the magnet flows counterclockwise (viewed from above), as shown, so its field, near the ends, points *upward*. A ring of pipe *below* the magnet experiences an increasing upward flux, as the magnet approaches, and hence (by Lenz's law) a current (I_{ind}) will be induced in it such as to produce a *downward* flux. Thus I_{ind} must flow *clockwise*, which is *opposite* to the current in the magnet. Since opposite currents repel, the force on the magnet is *upward*. Meanwhile, a ring *above* the magnet experiences a *decreasing* (upward) flux, so *its* induced current is *parallel* to I, and it *attracts* the magnet upward. And the flux through rings *next to* the magnet is constant, so *no* current is induced in them. *Conclusion:* the delay is due to forces exerted on the magnet by induced eddy currents in the pipe.

Problem 7.15

In the quasistatic approximation, $\mathbf{B} = \begin{cases} \mu_0 n I\, \hat{\mathbf{z}}, & (s < a); \\ 0, & (s > a). \end{cases}$

Inside: for an "amperian loop" of radius $s < a$,

$$\Phi = B\pi s^2 = \mu_0 n I \pi s^2;\ \oint \mathbf{E}\cdot d\mathbf{l} = E\, 2\pi s = -\frac{d\Phi}{dt} = -\mu_0 n \pi s^2 \frac{dI}{dt};\quad \boxed{\mathbf{E} = -\frac{\mu_0 n s}{2}\frac{dI}{dt}\,\hat{\boldsymbol{\phi}}.}$$

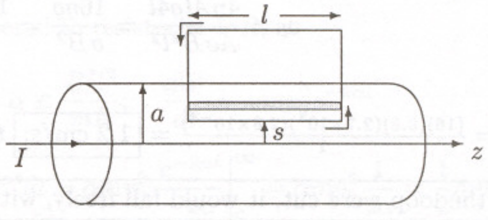
Outside: for an "amperian loop" of radius $s > a$:

$$\Phi = B\pi a^2 = \mu_0 n I \pi a^2;\ E\, 2\pi s = -\mu_0 n \pi a^2 \frac{dI}{dt};\quad \boxed{\mathbf{E} = -\frac{\mu_0 n a^2}{2s}\frac{dI}{dt}\,\hat{\boldsymbol{\phi}}.}$$

Problem 7.16

(a) The magnetic field (in the quasistatic approximation) is "circumferential". This is analogous to the current in a solenoid, and hence the field is $\boxed{\text{longitudinal.}}$

(b) Use the "amperian loop" shown.
Outside, $\mathbf{B} = 0$, so here $\mathbf{E} = 0$ (like $\mathbf{B}$ outside a solenoid).
So $\oint \mathbf{E}\cdot d\mathbf{l} = El = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'} l\, ds'$
$\therefore E = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\left(\frac{a}{s}\right)$. But $\frac{dI}{dt} = -I_0\omega \sin\omega t$,



so $\boxed{\mathbf{E} = \frac{\mu_0 I_0 \omega}{2\pi}\sin(\omega t)\ln\left(\frac{a}{s}\right)\hat{\mathbf{z}}.}$

Problem 7.17

(a) The field inside the solenoid is $B = \mu_0 nI$. So $\Phi = \pi a^2 \mu_0 nI \Rightarrow \mathcal{E} = -\pi a^2 \mu_0 n (dI/dt)$.

In magnitude, then, $\mathcal{E} = \pi a^2 \mu_0 nk$. Now $\mathcal{E} = I_r R$, so $\boxed{I_{\text{resistor}} = \dfrac{\pi a^2 \mu_0 nk}{R}.}$

$\mathbf{B}$ is to the right and increasing, so the field of the loop is to the *left*, so the current is counterclockwise, or $\boxed{\text{to the right,}}$ through the resistor.

(b) $\Delta\Phi = 2\pi a^2 \mu_0 nI$; $I = \dfrac{dQ}{dt} = \dfrac{\mathcal{E}}{R} = -\dfrac{1}{R}\dfrac{d\Phi}{dt} \Rightarrow \Delta Q = \dfrac{1}{R}\Delta\Phi$, in magnitude. So $\boxed{\Delta Q = \dfrac{2\pi a^2 \mu_0 nI}{R}.}$

Problem 7.18

$$\Phi = \int \mathbf{B}\cdot d\mathbf{a};\ \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}};\ \Phi = \frac{\mu_0 I a}{2\pi}\int_a^{2a}\frac{ds}{s} = \frac{\mu_0 I a \ln 2}{2\pi};\ \mathcal{E} = I_{\text{loop}} R = \frac{dQ}{dt}R = -\frac{d\Phi}{dt} = -\frac{\mu_0 a \ln 2}{2\pi}\frac{dI}{dt}.$$

$$dQ = -\frac{\mu_0 a \ln 2}{2\pi R}\,dI \Rightarrow \boxed{Q = \frac{I\mu_0 a \ln 2}{2\pi R}.}$$

The field of the wire, at the square loop, is *out of the page*, and *decreasing*, so the field of the induced current must point out of page, within the loop, and hence the induced current flows $\boxed{\text{counterclockwise.}}$

Problem 7.19

In the quasistatic approximation, $\mathbf{B} = \begin{cases} \frac{\mu_0 NI}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{(inside toroid)};\\ 0, & \text{(outside toroid)} \end{cases}$

(Eq. 5.58). The flux around the toroid is therefore

$$\Phi = \frac{\mu_0 NI}{2\pi}\int_a^{a+w}\frac{1}{s}h\,ds = \frac{\mu_0 NIh}{2\pi}\ln\left(1+\frac{w}{a}\right) \approx \frac{\mu_0 Nhw}{2\pi a}I.\quad \frac{d\Phi}{dt} = \frac{\mu_0 Nhw}{2\pi a}\frac{dI}{dt} = \frac{\mu_0 Nhwk}{2\pi a}.$$

The electric field is the same as the *magnetic* field of a circular current (Eq. 5.38):

$$\mathbf{B} = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}},$$

with (Eq. 7.18)

$$I \to -\frac{1}{\mu_0}\frac{d\Phi}{dt} = -\frac{Nhwk}{2\pi a}.\quad \text{So } \mathbf{E} = \frac{\mu_0}{2}\left(-\frac{Nhwk}{2\pi a}\right)\frac{a^2}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}} = \boxed{-\frac{\mu_0}{4\pi}\frac{Nhwka}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}}.}$$

Problem 7.20

(a) From Eq. 5.38, the field (on the axis) is $\mathbf{B} = \frac{\mu_0 I}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\hat{\mathbf{z}}$, so the flux through the little loop (area πa^2) is $\boxed{\Phi = \dfrac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

(b) The field (Eq. 5.86) is $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$, where $m = I\pi a^2$. Integrating over the spherical "cap" (bounded by the big loop and centered at the little loop):

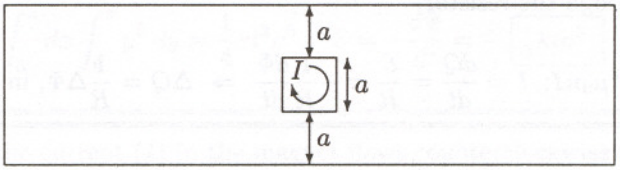
$$\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\int(2\cos\theta)(r^2\sin\theta\,d\theta\,d\phi) = \frac{\mu_0 I a^2}{2r}2\pi\int_0^{\bar\theta}\cos\theta\sin\theta\,d\theta$$

where $r = \sqrt{b^2 + z^2}$ and $\sin\bar{\theta} = b/r$. Evidently $\Phi = \frac{\mu_0 I \pi a^2}{r} \left.\frac{\sin^2\theta}{2}\right|_0^{\bar{\theta}} = \boxed{\frac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{3/2}},}$ the same as in (a)!!

(c) Dividing off I ($\Phi_1 = M_{12} I_2$, $\Phi_2 = M_{21} I_1$): $\boxed{M_{12} = M_{21} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

Problem 7.21

$$\mathcal{E} = -\frac{d\Phi}{dt} = -M\frac{dI}{dt} = -Mk.$$



It's hard to calculate M using a current in the little loop, so, exploiting the equality of the mutual inductances, I'll find the flux through the *little* loop when a current I flows in the *big* loop: $\Phi = MI$. The field of *one* long wire is $B = \frac{\mu_0 I}{2\pi s} \Rightarrow \Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^{2a} \frac{1}{s} a\,ds = \frac{\mu_0 I a}{2\pi}\ln 2$, so the *total* flux is

$$\Phi = 2\Phi_1 = \frac{\mu_0 I a \ln 2}{\pi} \Rightarrow M = \frac{\mu_0 a \ln 2}{\pi} \Rightarrow \boxed{\mathcal{E} = \frac{\mu_0 k a \ln 2}{\pi},} \text{ in magnitude.}$$

Direction: The net flux (through the big loop), due to I in the little loop, is *into the page.* (Why? Field lines point *in*, for the inside of the little loop, and *out* everywhere outside the little loop. The big loop encloses *all* of the former, and only *part* of the latter, so *net* flux is *inward.*) This flux is *increasing*, so the induced current in the big loop is such that *its* field points *out* of the page: it flows $\boxed{\text{counterclockwise.}}$

Problem 7.22

$B = \mu_0 n I \Rightarrow \Phi_1 = \mu_0 n I \pi R^2$ (flux through a single turn). In a length l there are nl such turns, so the total flux is $\Phi = \mu_0 n^2 \pi R^2 I l$. The self-inductance is given by $\Phi = LI$, so the self-inductance per unit length is $\boxed{\mathcal{L} = \mu_0 n^2 \pi R^2.}$

Problem 7.23

The field of one wire is $B_1 = \frac{\mu_0}{2\pi}\frac{I}{s}$, so $\Phi = 2 \cdot \frac{\mu_0 I}{2\pi} \cdot l \int_{\epsilon}^{d-\epsilon} \frac{ds}{s} = \frac{\mu_0 I l}{\pi}\ln\left(\frac{d-\epsilon}{\epsilon}\right)$. The ϵ in the numerator is negligible (compared to d), but in the denominator we *cannot* let $\epsilon \to 0$, else the flux is *infinite.*

$\boxed{L = \frac{\mu_0 l}{\pi}\ln(d/\epsilon)}$. Evidently the size of the wire itself is critical in determining L.

Problem 7.24

(a) In the quasistatic approximation $\mathbf{B} = \frac{\mu_0}{2\pi s}\hat{\boldsymbol{\phi}}$. So $\Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{s} h\,ds = \frac{\mu_0 I h}{2\pi}\ln(b/a)$.

This is the flux through *one* turn; the *total* flux is N times Φ_1: $\Phi = \frac{\mu_0 N h}{2\pi}\ln(b/a) I_0 \cos(\omega t)$. So

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\mu_0 N h}{2\pi}\ln(b/a) I_0 \omega \sin(\omega t) = \frac{(4\pi\times 10^{-7})(10^3)(10^{-2})}{2\pi}\ln(2)(0.5)(2\pi\, 60)\sin(\omega t)$$

$= \boxed{2.61\times 10^{-4}\sin(\omega t)}$ (in volts), where $\omega = 2\pi\, 60 = 377/\text{s}$. $I_r = \frac{\mathcal{E}}{R} = \frac{2.61\times 10^{-4}}{500}\sin(\omega t)$

$= \boxed{5.22\times 10^{-7}\sin(\omega t)}$ (amperes).

(b) $\mathcal{E}_b = -L\frac{dI_r}{dt}$; where (Eq. 7.27) $L = \frac{\mu_0 N^2 h}{2\pi}\ln(b/a) = \frac{(4\pi\times10^{-7})(10^6)(10^{-2})}{2\pi}\ln(2) = 1.39\times 10^{-3}$ (henries).

Therefore $\mathcal{E}_b = -(1.39\times 10^{-3})(5.22\times 10^{-7}\,\omega)\cos(\omega t) = \boxed{-2.74\times 10^{-7}\cos(\omega t)}$ (volts).