Root Locus Method Part 2

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Routh Stability: Practice Problem

- Consider the TF: $\dfrac{s+8}{s^5-s^4+4s^3-4s^2+3s-2}$
- Find out the RHP Poles and Stability of the system

Routh Stability Practice

- Consider a TF: $\dfrac{s+8}{s^5 - s^4 + 4s^3 - 4s^2 + 3s - 2}$

s^5	1	4	3
s^4	-1	-4	-2
s^3	ξ	1	0
s^2	$\dfrac{1-4\xi}{\xi}$	-2	0
s^1	$\dfrac{2\xi^2+1-4\xi}{1-4\xi}$	0	0
s^0	-2	0	0

Steady State Error: Practice Problem

- Consider the TF $\frac{K\,(s+7)}{s(s+5)(s+8)(s+12)}$ with unity feedback.
- Find for what value of K it will yield a steady state error of 0.01 for an input of 0.1t.
- What is the minimum possible steady state error for the same input.

$$e(\infty) = \frac{1/10}{K_v} = 0.01; \text{ where } K_v = \frac{7K}{5x8x12} = 10. \text{ Thus, } K = 685.71.$$

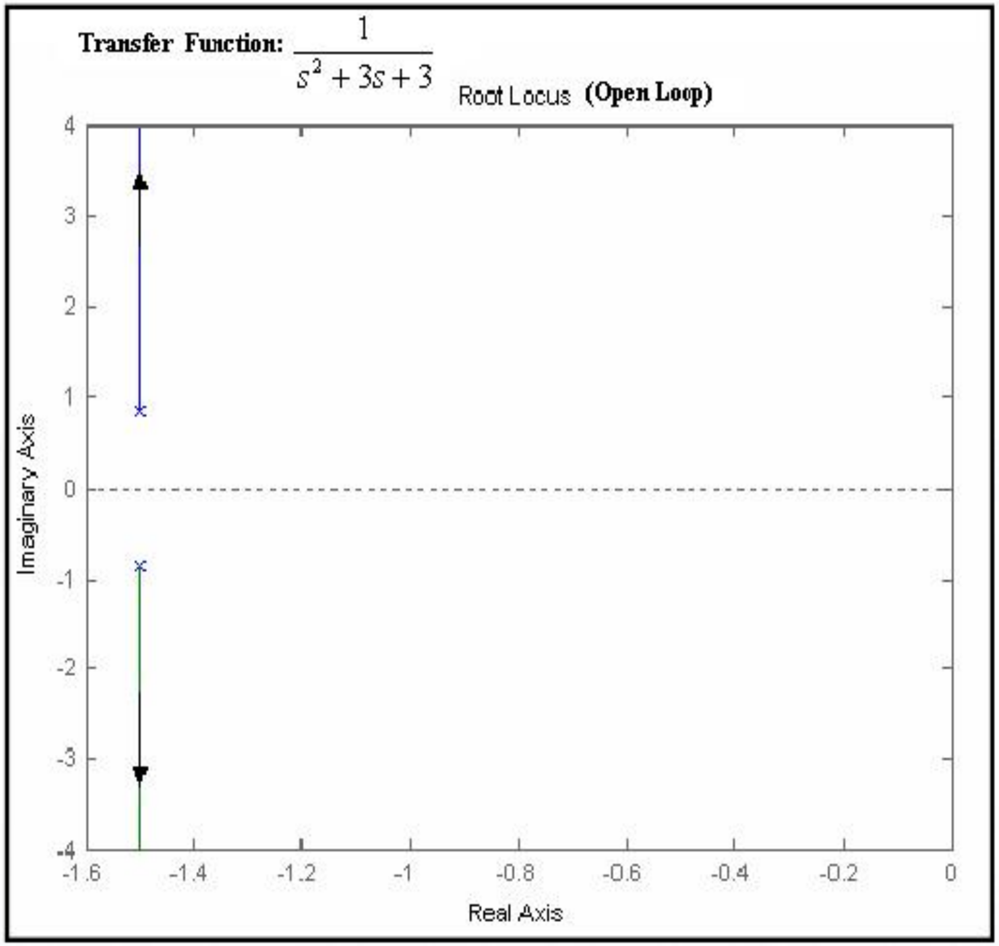
c. The minimum error will occur for the maximum gain before instability. Using the Routh-Hurwitz Criterion along with $T(s) = \dfrac{K(s+7)}{s^4 + 25s^3 + 196s^2 + (480+K)s + 7K}$:

s^4	1	196	$7K$	For Stability
s^3	25	$480+K$		
s^2	$4420-K$	$175K$		$K < 4420$
s^1	$-K^2 - 435K + 2121600$			$-1690.2 < K < 1255.2$
s^0	$175K$			$K > 0$

Thus, for stability and minimum error $K = 1255.2$. Thus, $K_v = \dfrac{7K}{5x8x12} = 18.3$ and

$$e(\infty) = \frac{1/10}{K_v} = \frac{1/10}{18.3} = 0.0055.$$

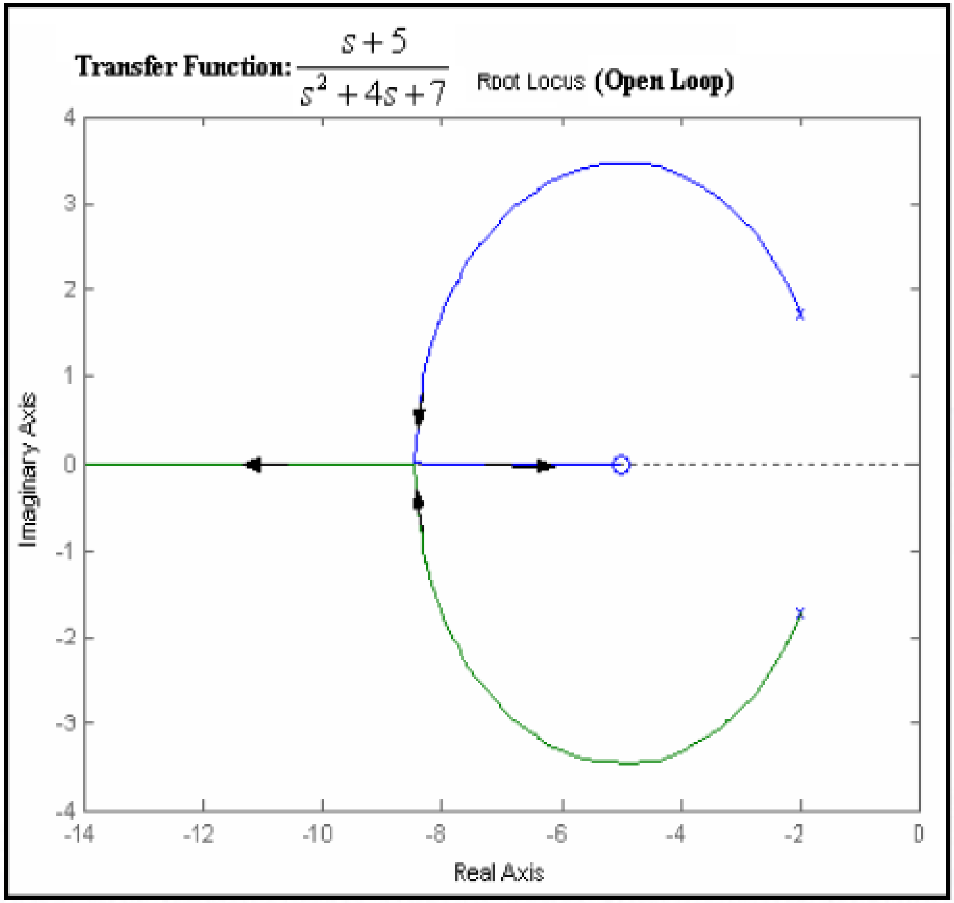
Back to Root Locus



✓ Number of root loci – as many as open loop poles

✓Origin at poles

Root Locus of the Open Loop Transfer Function

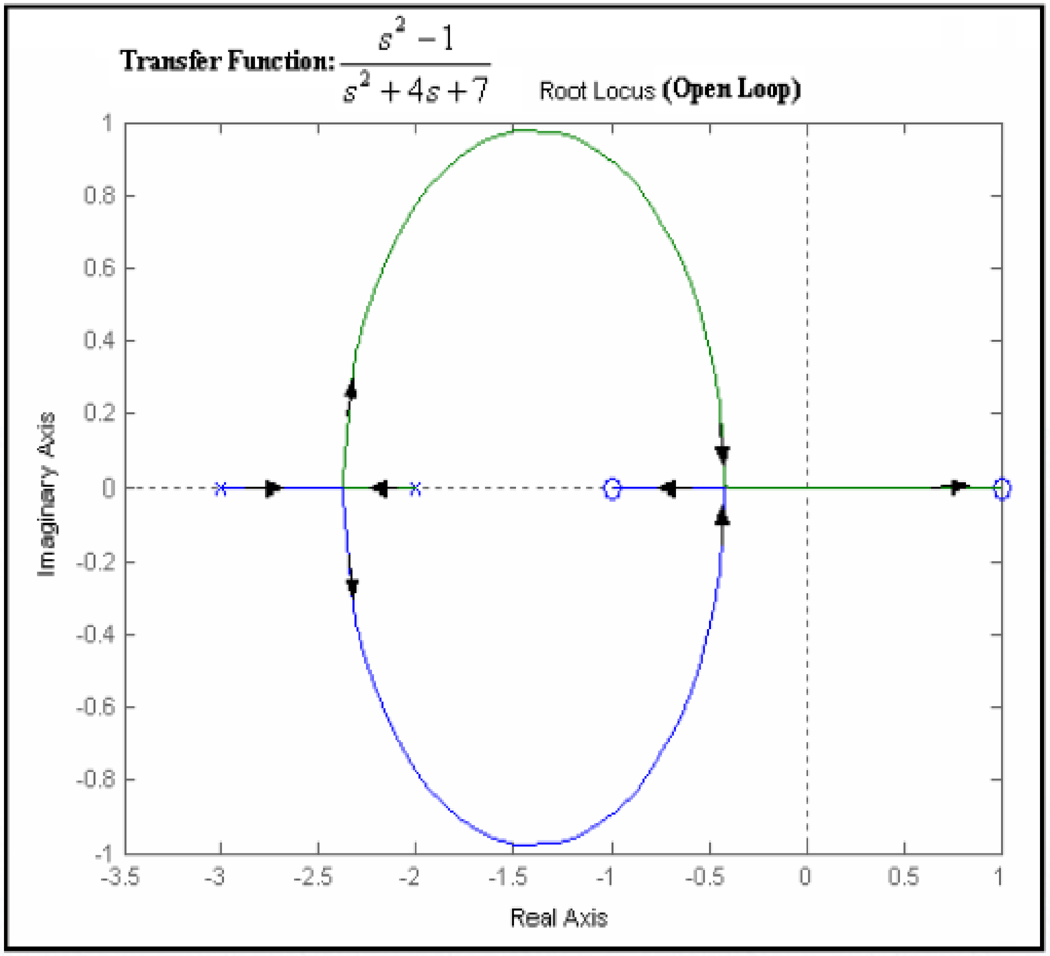


✓Symmetry always about the real axis.

✓Termination as $K \rightarrow \alpha$ m open loop poles to finite zeros

Of the open loop system, n-m number of poles approach zeroes at infinity.

Root Locus of the Open Loop Transfer Function



✓Real axis Segment: - Root locus exists on the left of an odd number of real axis finite open loop poles and zeroes

Root Locus of the Open Loop Transfer Function

Angle of the Asymptotes

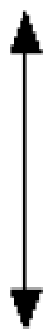
- The root loci approaches the zeros at infinity along asymptotes. The real axis intercept and angle of which are given by the following rules:

$$R_{\text{int}} = \frac{\sum Poles - \sum Zeros}{m}, \; m = n_p - n_z$$

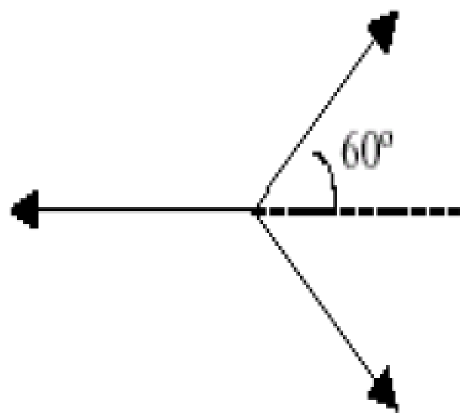
$$\Phi_{\text{int}} = \frac{(2i+1)}{m} 180^0, i = 0,1,2..(m-1)$$

m=1

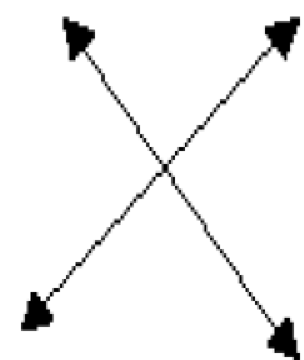
m=2



m=3



m=4



Jω intercept: Use $1+KG(j\omega)H(j\omega) = 0$

- Find ω and K and for any Complex Pole Find Angle

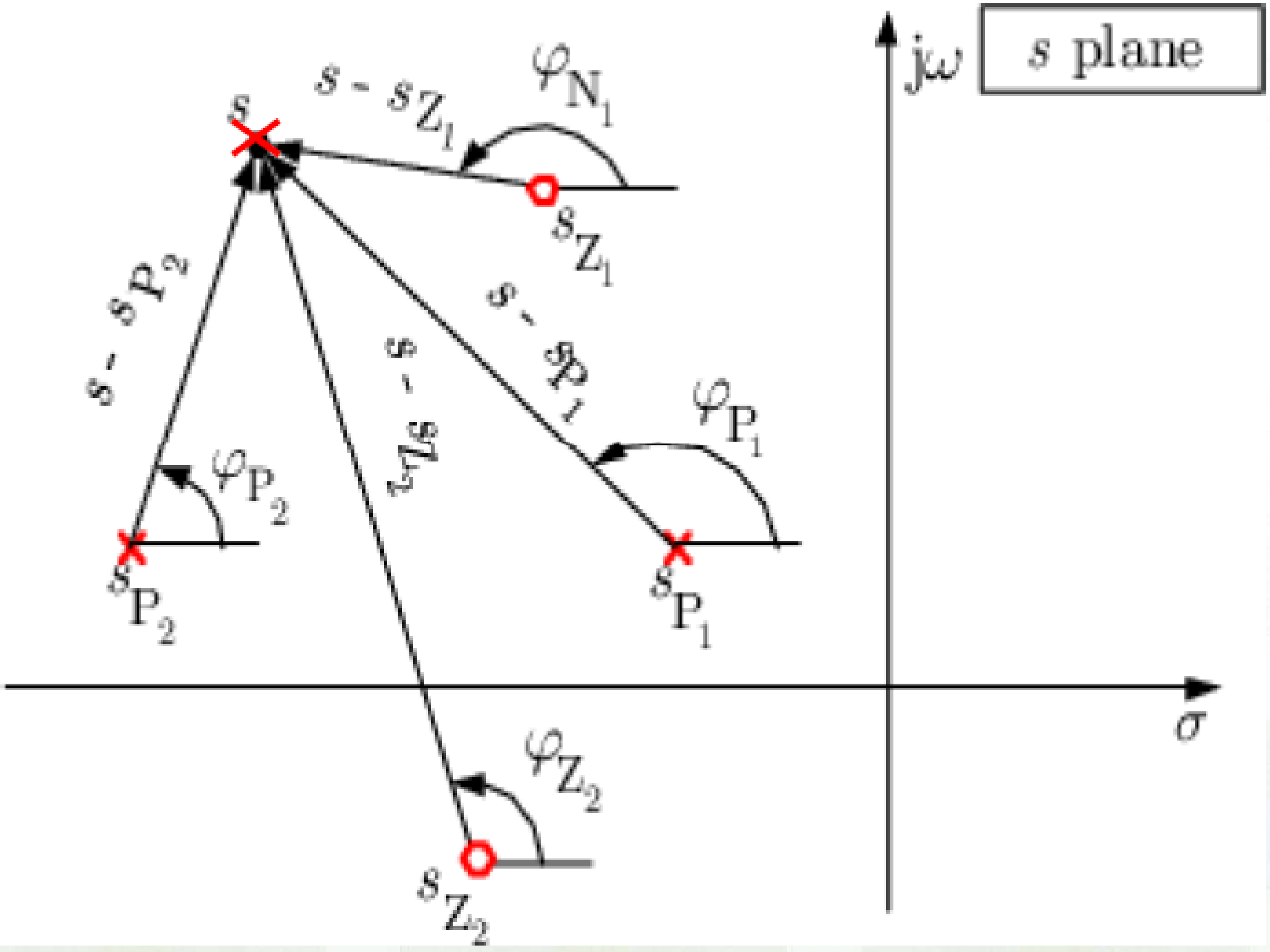
- Another way: Use Routh's Array, find stability condition for a complete zero row,

Obtain, K, go back to the upper row, obtain ω

- Take points along the imaginary axis and check the phase condition
- Try the problem $T=K(s+3)/(s(s+1)(s+2)(s+4))$ (OLTF with unity feedback)

$-K^2 - 65K + 720 = 0 \Rightarrow K = 9.65$

Angle Checking at a Test Point



Departure and Arrival Angle

- **Angle of Departure from a complex Pole with 'r' multiplicity:**

$$\boxed{r\varphi_{l,dep} = \sum \psi_i - \sum_{i \neq l} \varphi_i - (180^\circ + 360^\circ(l-1)), l = 1..r}$$

- **Angle of Arrival at a complex Zero with 'r' multiplicity:**

$$\boxed{r\psi_{l,arr} = \sum \varphi_i - \sum_{i \neq l} \psi_i + (180^\circ + 360^\circ(l-1)), l = 1..r}$$

❑ Example:

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)} = KF(s)$$

$$F(s) = \frac{1}{s(s+2)(s+3)}$$

(1) 3 Loci Starting From $P_1 = 0, P_2 = -2, \quad P_3 = -3$

(2) All Roots Loci Terminate at ∞

Open Loop Zeros $= \infty$

$$\angle F(s) = -\angle s - \angle s+2 - \angle s+3 = (2i+1)180^{o}, i = 0,1,2..$$

$$|F(s)| = \frac{1}{K}, Or, \quad \left|\frac{K}{s(s+2)(s+3)}\right| = 1$$

(a) Root Loci on The Real Axis

(i) Positive Real Axis → $\angle s = \angle s+2 = \angle s+3 = 0^{o}$

(ii) $s: 0 \leftrightarrow -2, \quad \angle = 180^{o}, \quad \angle s = 180^{\circ}, \quad \angle s+2 = \angle s+3 = 0$

(iii) $s: -2 \leftrightarrow -3, \quad \angle = 360^{o}, \quad \angle s = 180^{\circ}, \quad \angle s+2 = 180^{\circ}, \quad \angle s+3 = 0$

(iv) $s: \triangleright -3, \quad \angle = 540^{o}, \quad \angle s = 180^{\circ}, \quad \angle s+2 = 180^{\circ}, \quad \angle s+3 = 180^{0}$

Six basic rules of Root-Locus Construction

1. ‘n’ branches of root locus starts at the open loop poles and ‘m’ of them meet the zeroes of the same
2. Loci are on the real axis to the left of odd number of poles and zeroes
3. For large s and K, n-m of the root loci are asymptotic. Get Asymptote angle and Real intercept by using the earlier equations
4. Calculate the Angle of Departure from poles and arrival at zeroes by using the rule derived
5. Calculate the ‘jω’ crossing using Routh’s stability
6. Calculate the break away and break-in points using $dK/d\sigma = 0$

Consider another system with unity feedback.

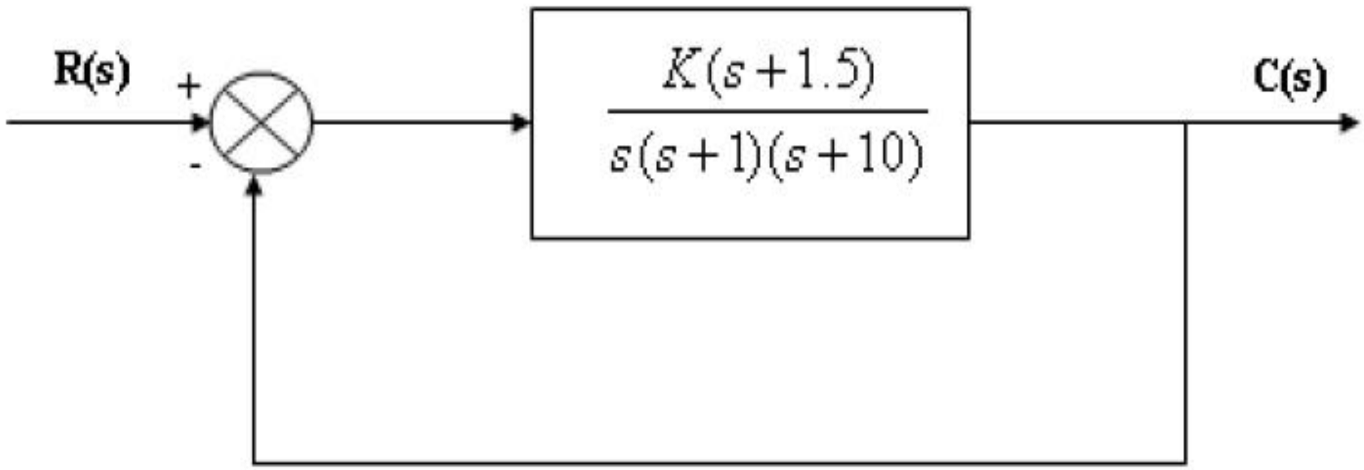
$$GH = \frac{10}{(s+2)(s+p_1)}, H = 1$$

$$\mathrm{KGH} = \frac{10}{(s+2)(s+p_1)}, \mathrm{CLT} = \frac{10}{s^2+(p_1+2)s+2p_1+10} = \frac{10}{(s^2+2s+10)+p_1(s+2)}$$

$$= \frac{10/(s^2+2s+10)}{1+p_1(s+2)/(s^2+2s+10)}$$

Find the root locus for gain p_1 and TF: $(s+2)/(s^2+2s+10)$

Assignment: Find the position of the leftmost pole by using root locus corresponding to the Control Gains K= 7 and 40



Effect of Different Parameters on Root Locus

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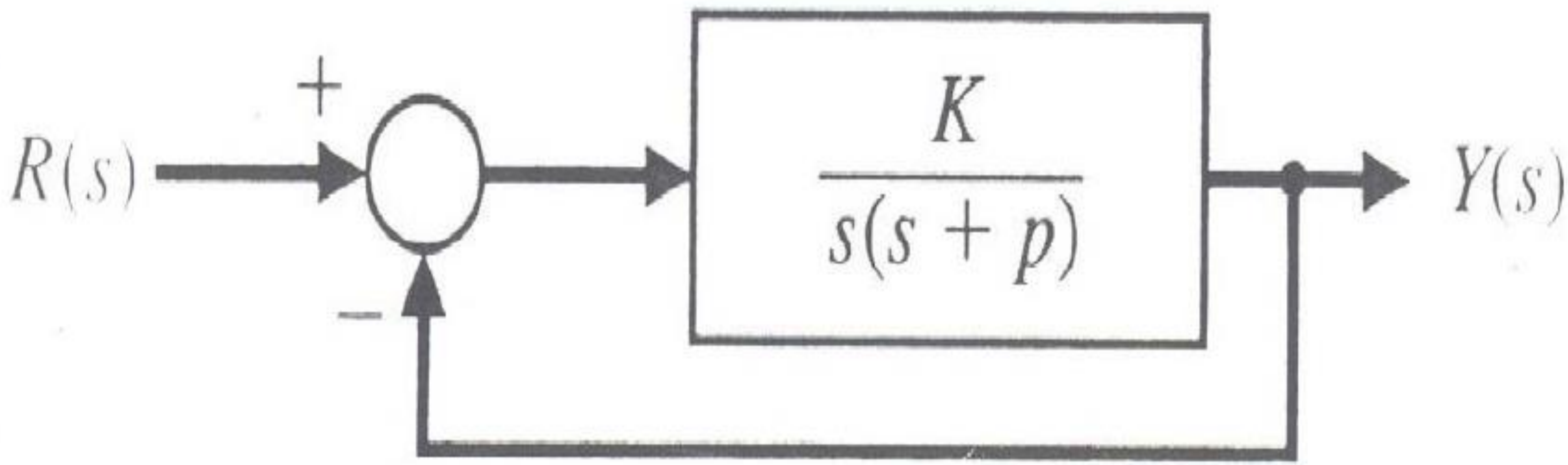
This Lecture Contains

- Feasible Design Space for a second order system
- Effect of Additional Zero
- Effect of Additional Pole and Zero

Example: Design of a second order system



Select the Gain K and pole p such that in a step response OS <5% and the settling time corresponding to 2% of final value will be less than 4 seconds

Feasible region of the Design

Step 1: Consider the closed loop transfer function for the system and compare it with a standard form

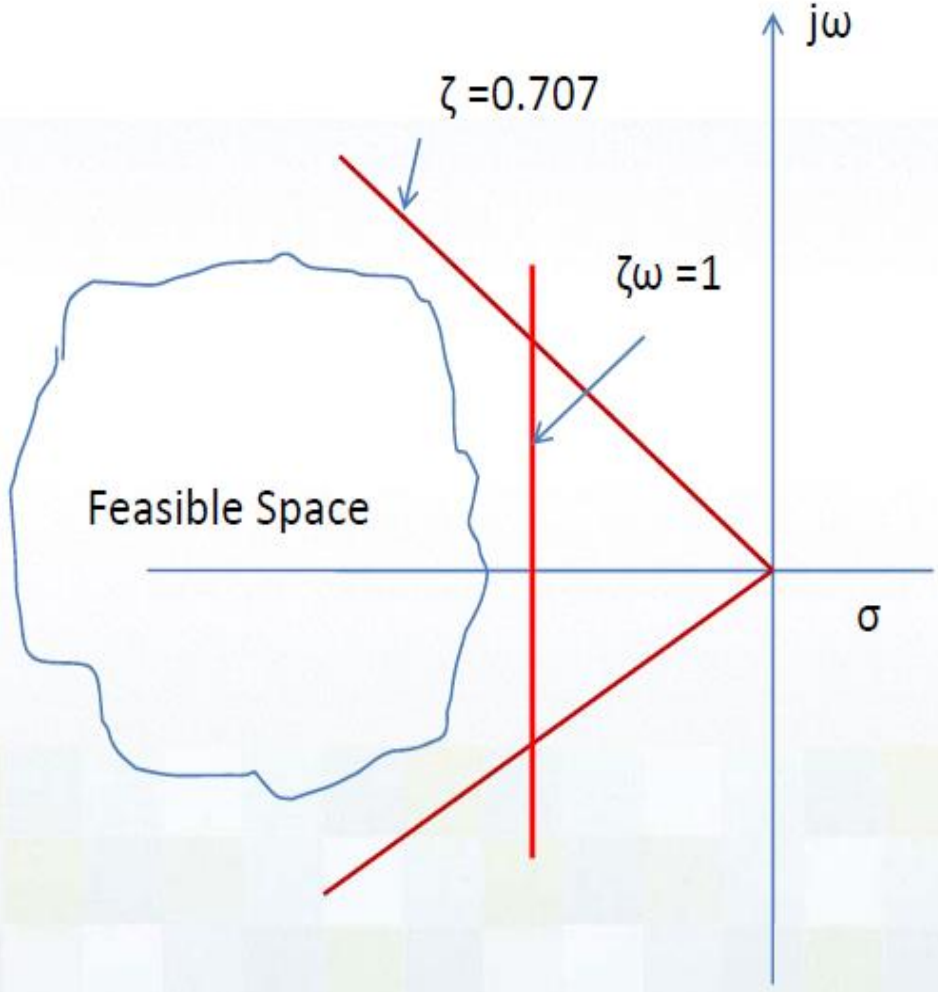
$$\boxed{\frac{k}{s^2+ps+k}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

Step 2: It is evident from comparison that in this case:

$$k=\omega_n^{\ 2},\quad p=2\xi\omega_n$$

The design specification tells us that: $\zeta\omega_n > 1$, Also , the Overshoot specification tells us that ζ should be greater than 0.707. The feasible design space is shown in the following figure.

Feasible Design Space



Choice of Poles and it's effect

- If we choose two extreme points from the design space, then the closed loop pole locations are -1 +/- j1 and the closed loop transfer function will be

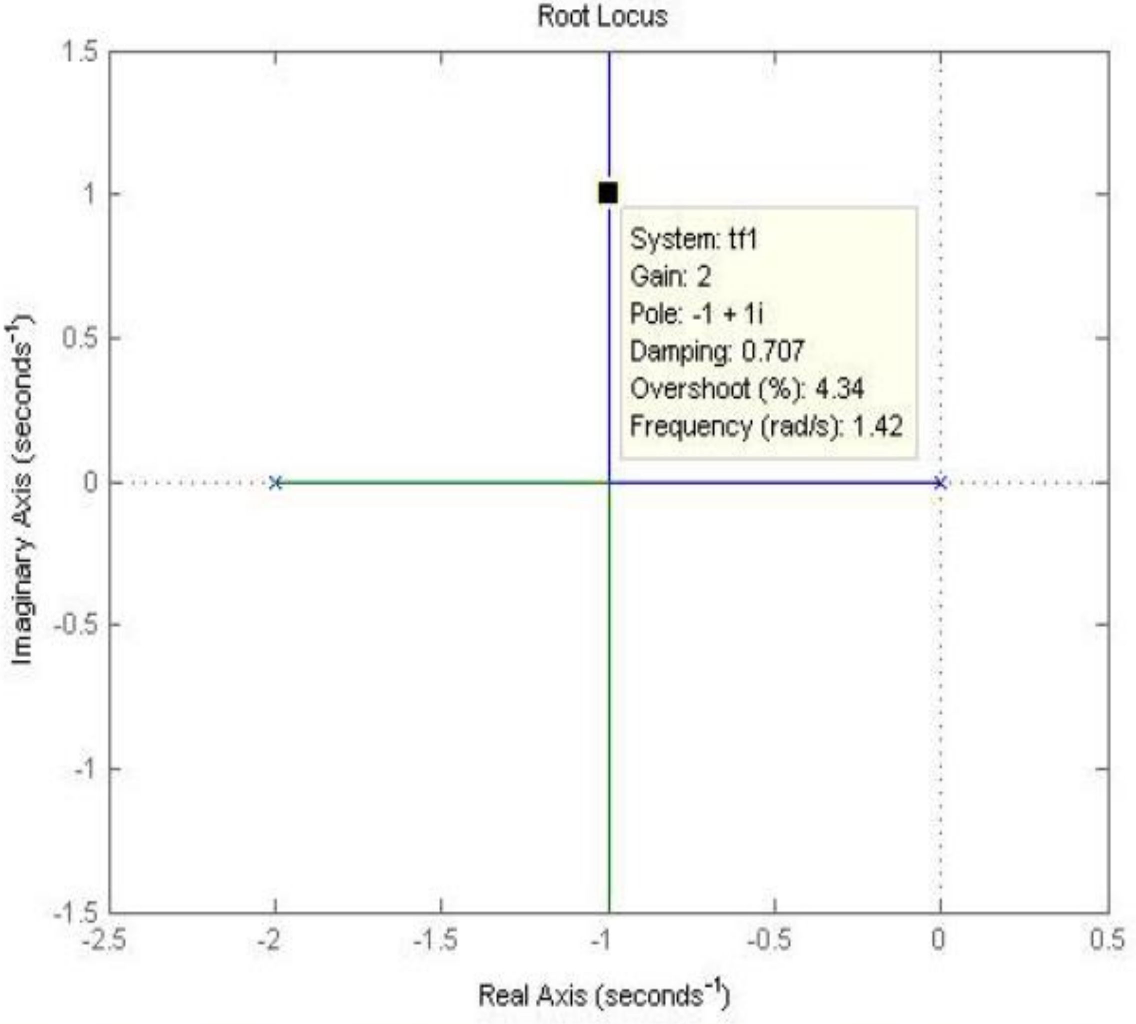
$$\frac{2}{s^2+2s+2}$$

- The open loop transfer function is:

$$\frac{1}{s(s+2)}$$

- The corresponding root locus is shown hereafter. The root locus may help in choosing other control gains.

Root Locus plot of the System



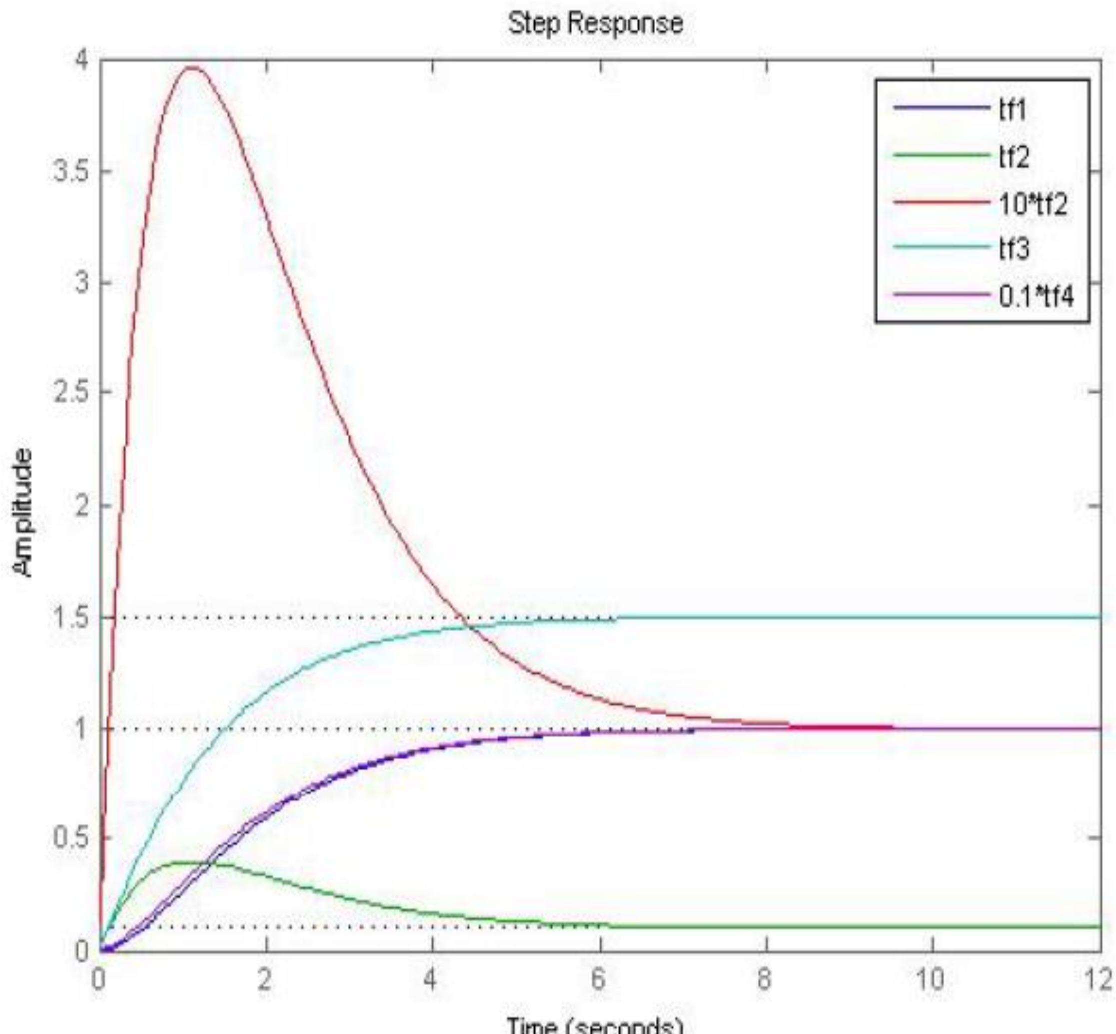
Effect of additional zero

- Consider a transfer function with two complex poles and one additional zero
- Normalised Transfer function

$$\boxed{H(s) = \frac{(s/\alpha\zeta\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}}$$

- If α is large, zero will have little effect. When α is about 1, the zero may increase the overshoot, without influencing the settling time

The effect of zero



$$tf_1 = \frac{1}{s^2 + 2s + 1}$$

$$tf_2 = \frac{(s + 0.1)}{s^2 + 2s + 1}$$

$$tf_3 = \frac{(s + 1.5)}{s^2 + 2s + 1}$$

$$tf_4 = \frac{(s + 10)}{s^2 + 2s + 1}$$

Effects of Additional pole and zero

- For a second order system with no finite zero, the transient parameters are given by: $t_r = 1.8/\omega_n$, O.S. = .05 for ζ =.7, $t_s = 4/\zeta\omega$
- A zero in the LHP will increase OS if it is within a factor of 4 of the real part of complex poles
- A non-minimum phase will depress the OS
- If the additional pole is within a factor of 4, then the rise time will increase significantly

System with Additional Pole and Zero

$$T(s) = \frac{(\omega_n^2 / a)(s + a)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(1 + \tau s)}$$

Find out the effect of 'a' and 'T' on the system response corresponding to a step input