Heat Equation: Existence and Uniqueness

MSO-203B

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Introduction

Heat Equation

We will be interested in the question of existence and uniqueness of the equation:

$$u_t = c^2 u_{xx}; \ \text{ in } \ (0, l) \times (0, \infty)$$
$$u(x, 0) = f(x); \ \text{ for } \ x \in (0, l)(\text{Initial Condition})$$
$$u(0, t) = u(l, t) = 0; \ \text{ for } \ t \in (0, \infty)(\text{Boundary Condition})$$

We want our solution $u(x, t) \to 0$ as $x \to \infty$.

Solution via Separation of Variable

Solution of the Heat Equation

We look for solution $u(x,t) = X(x)T(t)$ of the Heat equation satifying the Initial and Boundary conditions. Hence one has,

$$u_{xx} = X''(x)T(t) \text{ and } u_t = X(x)T'(t)$$

and hence we have,

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

and the BC becomes

$$u(0,t) = X(0)T(t) = 0$$

and

$$u(1,t) = X(1)T(t) = 0$$

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Solution of the Heat Equation

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Non-trivial Solution

If $f(x)$ is not zero for all $0 < x < 1$ then $T(t)$ cannot be zero and the equations are satisfied only if

$$X(0) = X(1) = 0$$

and so the problem boils down to solving

$$X''(x) + \lambda X(x) = 0 \text{ for } 0 < x < 1$$

with the condition that $X(0) = X(1) = 0$

Solution of Heat Equation

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- When $\lambda = 0$ there are still no eigenfunctions.
- When $\lambda > 0$ we have, $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ which gives $\lambda_n = n^2\pi^2$ for $n \in \mathbb{N}$ and the eigenfunctions are $X_n(x) = B_n \sin(n\pi x)$.

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Solving for $T(t)$

Since the nontrivial solutions arose for $\lambda_n = n^2\pi^2$ we have

$$T'(t) = -n^2\pi^2 T(t)$$

whose solutions are

$$T_n = c_n \exp(-n^2\pi^2 t) \quad \text{for} \quad n \in \mathbb{N}$$

Solution of Heat Equation

Incorporating the boundary conditions

Using the Principle of Superposition we have,

$$u(x,t) = \sum_{0}^{\infty} u_n(x,t) = \sum_{0}^{\infty} B_n \sin(n\pi x) \exp(n^2\pi^2 t)$$

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The Final Solution

Using Fourier Sine series we have,

$$u(x,t) = \sum_{0}^{\infty} B_n \sin(n\pi x) \exp(n^2\pi^2 t)$$

where $B_n = 2\int_0^1 \sin(m\pi x) f(x) dx$.

Uniqueness of Solution to the Heat Equation

Uniqueness Theorem

The Heat equation given by

$$u_t = c^2 u_{xx}; \ \text{ in } \ (0,1) \times (0,\infty)$$
$$u(x,0) = f(x); \ \text{ for } \ x \in (0,1)$$
$$u(0,t) = u(1,t) = 0; \ \text{ for } \ t \in (0,\infty)$$

has atmost one solution in $u \in C^2_1(\overline{(0,1) \times (0,\infty)})$.

Uniqueness of Solution to the Heat Equation

Proof of uniqueness Theorem

Consider the two solution u_1 and u_2 in $C_1^2(\overline{(0,1) \times (0,\infty)})$ to the Heat equation.
Define, $v = u_1 - u_2$ then one had that v satisfies

$$v_t = c^2 v_{xx}; \text{ in } (0,1) \times (0,\infty)$$
$$v(x,0) = 0; \text{ for } x \in (0,1)$$
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Auxiliary Function

Define $V : (0,\infty) \to \mathbb{R}$ as follows:

$$V(t) = \int_0^1 v^2(x,t)dt$$

Uniqueness of Solution to the Heat Equation

Proof of Uniqueness Theorem

We have, $V'(t) = 2\int_0^1 vv_t dt = 2\int_0^1 vv_{xx} dt$ which implies that

$$V'(t) = -2\int_0^1 v_x^2 dt \leq 0$$

which would imply $V(0) = 0$ by the initial conditions hence $V = 0$.