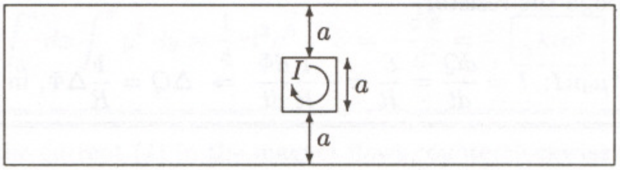
where $r = \sqrt{b^2 + z^2}$ and $\sin\bar{\theta} = b/r$. Evidently $\Phi = \frac{\mu_0 I \pi a^2}{r} \left. \frac{\sin^2\theta}{2} \right|_0^{\bar{\theta}} = \boxed{\dfrac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{3/2}}}$, the same as in (a)!!

(c) Dividing off I ($\Phi_1 = M_{12} I_2$, $\Phi_2 = M_{21} I_1$): $\boxed{M_{12} = M_{21} = \dfrac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

Problem 7.21

$$\mathcal{E} = -\frac{d\Phi}{dt} = -M\frac{dI}{dt} = -Mk.$$



It's hard to calculate M using a current in the little loop, so, exploiting the equality of the mutual inductances, I'll find the flux through the *little* loop when a current I flows in the *big* loop: $\Phi = MI$. The field of *one* long wire is $B = \frac{\mu_0 I}{2\pi s} \Rightarrow \Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^{2a} \frac{1}{s} a\, ds = \frac{\mu_0 I a}{2\pi}\ln 2$, so the *total* flux is

$$\Phi = 2\Phi_1 = \frac{\mu_0 I a \ln 2}{\pi} \Rightarrow M = \frac{\mu_0 a \ln 2}{\pi} \Rightarrow \boxed{\mathcal{E} = \frac{\mu_0 k a \ln 2}{\pi},} \text{ in magnitude.}$$

Direction: The net flux (through the big loop), due to I in the little loop, is *into the page.* (Why? Field lines point *in*, for the inside of the little loop, and *out* everywhere outside the little loop. The big loop encloses *all* of the former, and only *part* of the latter, so *net* flux is *inward.*) This flux is *increasing*, so the induced current in the big loop is such that *its* field points *out* of the page: it flows $\boxed{\text{counterclockwise.}}$

Problem 7.22

$B = \mu_0 n I \Rightarrow \Phi_1 = \mu_0 n I \pi R^2$ (flux through a single turn). In a length l there are nl such turns, so the total flux is $\Phi = \mu_0 n^2 \pi R^2 I l$. The self-inductance is given by $\Phi = LI$, so the self-inductance per unit length is $\boxed{\mathcal{L} = \mu_0 n^2 \pi R^2.}$

Problem 7.23

The field of one wire is $B_1 = \frac{\mu_0}{2\pi}\frac{I}{s}$, so $\Phi = 2 \cdot \frac{\mu_0 I}{2\pi} \cdot l \int\limits_{\epsilon}^{d-\epsilon} \frac{ds}{s} = \frac{\mu_0 I l}{\pi} \ln\left(\frac{d-\epsilon}{\epsilon}\right)$. The ϵ in the numerator is negligible (compared to d), but in the denominator we *cannot* let $\epsilon \to 0$, else the flux is *infinite*.

$\boxed{L = \dfrac{\mu_0 l}{\pi}\ln(d/\epsilon)}$. Evidently the size of the wire itself is critical in determining L.

Problem 7.24

(a) In the quasistatic approximation $\mathbf{B} = \dfrac{\mu_0}{2\pi s}\hat{\boldsymbol{\phi}}$. So $\Phi_1 = \dfrac{\mu_0 I}{2\pi}\displaystyle\int_a^b \frac{1}{s} h\, ds = \frac{\mu_0 I h}{2\pi}\ln(b/a)$.

This is the flux through *one* turn; the *total* flux is N times Φ_1: $\Phi = \dfrac{\mu_0 N h}{2\pi}\ln(b/a) I_0 \cos(\omega t)$. So

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\mu_0 N h}{2\pi}\ln(b/a) I_0 \omega \sin(\omega t) = \frac{(4\pi\times10^{-7})(10^3)(10^{-2})}{2\pi}\ln(2)(0.5)(2\pi\, 60)\sin(\omega t)$$

$= \boxed{2.61\times10^{-4}\sin(\omega t)}$ (in volts), where $\omega = 2\pi\, 60 = 377/\text{s}$. $I_r = \dfrac{\mathcal{E}}{R} = \dfrac{2.61\times10^{-4}}{500}\sin(\omega t)$

$= \boxed{5.22\times10^{-7}\sin(\omega t)}$ (amperes).

(b) $\mathcal{E}_b = -L\dfrac{dI_r}{dt}$; where (Eq. 7.27) $L = \frac{\mu_0 N^2 h}{2\pi}\ln(b/a) = \frac{(4\pi\times10^{-7})(10^6)(10^{-2})}{2\pi}\ln(2) = 1.39\times10^{-3}$ (henries).

Therefore $\mathcal{E}_b = -(1.39\times10^{-3})(5.22\times10^{-7}\,\omega)\cos(\omega t) = \boxed{-2.74\times10^{-7}\cos(\omega t)}$ (volts).

Ratio of amplitudes: $\dfrac{2.74 \times 10^{-7}}{2.61 \times 10^{-4}} = \boxed{1.05 \times 10^{-3}} = \dfrac{\mu_0 N^2 h\omega}{2\pi R} \ln(b/a).$

Problem 7.25

With I positive clockwise, $\mathcal{E} = -L\frac{dI}{dt} = Q/C$, where Q is the charge on the capacitor; $I = \frac{dQ}{dt}$, so $\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q = -\omega^2 Q$, where $\omega = \frac{1}{\sqrt{LC}}$. The general solution is $Q(t) = A\cos\omega t + B\sin\omega t$. At $t = 0$, $Q = CV$, so $A = CV$; $I(t) = \frac{dQ}{dt} = -A\omega\sin\omega t + B\omega\sin\omega t$. At $t = 0$, $I = 0$, so $B = 0$, and

$I(t) = -CV\omega\sin\omega t = \boxed{-V\sqrt{\dfrac{C}{L}}\sin\left(\dfrac{t}{\sqrt{LC}}\right)}.$

If you put in a resistor, the oscillation is "damped". This time $-L\frac{dI}{dt} = \frac{Q}{C} + IR$, so $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$. For an analysis of this case, see Purcell's *Electricity and Magnetism* (Ch. 8) or any book on oscillations and waves.

Problem 7.26

(a) $W = \frac{1}{2}LI^2$. $L = \mu_0 n^2 \pi R^2 l$ (Prob. 7.22) $\boxed{W = \dfrac{1}{2}\mu_0 n^2 \pi R^2 l I^2}$.

(b) $W = \frac{1}{2}\oint (\mathbf{A}\cdot\mathbf{I})dl$. $\mathbf{A} = (\mu_0 n I/2)R\,\hat{\boldsymbol{\phi}}$, at the surface (Eq. 5.70 or 5.71). So $W_1 = \frac{1}{2}\frac{\mu_0 n I}{2}RI \cdot 2\pi R$, for one turn. There are nl such turns in length l, so $W = \frac{1}{2}\mu_0 n^2\pi R^2 l I^2$. ✓

(c) $W = \frac{1}{2\mu_0}\int B^2\,d\tau$. $B = \mu_0 n I$, inside, and zero outside; $\int d\tau = \pi R^2 l$, so $W = \frac{1}{2\mu_0}\mu_0^2 n^2 I^2 \pi R^2 l = \frac{1}{2}\mu_0 n^2 \pi R^2 l I^2$. ✓

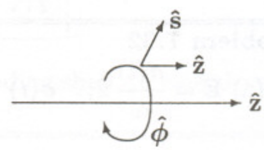
(d) $W = \frac{1}{2\mu_0}\left[\int B^2\,d\tau - \oint (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right]$. This time $\int B^2\,d\tau = \mu_0^2 n^2 I^2 \pi (R^2 - a^2) l$. Meanwhile, $\mathbf{A}\times\mathbf{B} = 0$ outside (at $s = b$). Inside, $\mathbf{A} = \frac{\mu_0 n I}{2} a\,\hat{\boldsymbol{\phi}}$ (at $s = a$), while $\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}$.

$\mathbf{A}\times\mathbf{B} = \frac{1}{2}\mu_0^2 n^2 I^2 a \underbrace{(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}})}_{\hat{\mathbf{s}}}$

points *inward* ("out" of the volume)

$\oint (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a} = \int (\frac{1}{2}\mu_0^2 n^2 I^2 a\hat{\mathbf{s}}) \cdot [a\,d\phi\,dz(-\hat{\mathbf{s}})] = -\frac{1}{2}\mu_0^2 n^2 I^2 a^2 2\pi l.$

$W = \frac{1}{2\mu_0}\left[\mu_0^2 n^2 I^2 \pi (R^2 - a^2) l + \mu_0^2 n^2 I^2 \pi a^2 l\right] = \frac{1}{2}\mu_0 n^2 I^2 R^2 \pi l$. ✓



Problem 7.27

$$B = \frac{\mu_0 n I}{2\pi s};\ W = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2\mu_0}\frac{\mu_0^2 n^2 I^2}{4\pi^2}\int \frac{1}{s^2} hr\,d\phi\,ds = \frac{\mu_0 n^2 I^2}{8\pi^2} h 2\pi \ln\left(\frac{b}{a}\right) = \boxed{\frac{1}{4\pi}\mu_0 n^2 I^2 h \ln(b/a)}.$$

$\boxed{L = \dfrac{\mu_0}{2\pi} n^2 h \ln(b/a)}$ (same as Eq. 7.27).

Problem 7.28

$$\oint \mathbf{B}\cdot d\mathbf{l} = B(2\pi s) = \mu_0 I_{\text{enc}} = \mu_0 I (s^2/R^2) \Rightarrow B = \frac{\mu_0 I s}{2\pi R^2}.$$

$$W = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2\mu_0}\frac{\mu_0^2 I^2}{4\pi^2 R^4}\int_0^R s^2 (2\pi s) l\,ds = \frac{\mu_0 I^2 l}{4\pi R^4}\left(\frac{s^4}{4}\right)\Big|_0^R = \frac{\mu_0 l}{16\pi} I^2 = \frac{1}{2}LI^2.$$

So $L = \dfrac{\mu_0}{8\pi} l$, and $\mathcal{L} = L/l = \boxed{\mu_0/8\pi,}$ independent of R!

Problem 7.29

(a) Initial current: $I_0 = \mathcal{E}_0/R$. So $-L\dfrac{dI}{dt} = IR \Rightarrow \dfrac{dI}{dt} = -\dfrac{R}{L}I \Rightarrow I = I_0 e^{-Rt/L}$, or $\boxed{I(t) = \dfrac{\mathcal{E}_0}{R}e^{-Rt/L}}$.

(b) $P = I^2 R = (\mathcal{E}_0/R)^2\,e^{-2Rt/L}R = \dfrac{\mathcal{E}_0^2}{R}e^{-2Rt/L} = \dfrac{dW}{dt}$.

$$W = \frac{\mathcal{E}_0^2}{R}\int_0^\infty e^{-2Rt/L}\,dt = \frac{\mathcal{E}_0^2}{R}\left(-\frac{L}{2R}e^{-2Rt/L}\right)\Big|_0^\infty = \frac{\mathcal{E}_0^2}{R}(0 + L/2R) = \boxed{\frac{1}{2}L\left(\mathcal{E}_0/R\right)^2}$$

(c) $W_0 = \frac{1}{2}LI_0^2 = \frac{1}{2}(\mathcal{E}_0/R)^2$. ✓

Problem 7.30

(a) $\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{\imath^3}I_1[3(\mathbf{a}_1\cdot\hat{\boldsymbol{\imath}})\hat{\boldsymbol{\imath}} - \mathbf{a}_1]$, since $\mathbf{m}_1 = I_1\mathbf{a}_1$. The flux through loop 2 is then

$$\Phi_2 = \mathbf{B}_1\cdot\mathbf{a}_2 = \frac{\mu_0}{4\pi}\frac{1}{\imath^3}I_1[3(\mathbf{a}_1\cdot\hat{\boldsymbol{\imath}})(\mathbf{a}_2\cdot\hat{\boldsymbol{\imath}}) - \mathbf{a}_1\cdot\mathbf{a}_2] = MI_1. \quad \boxed{M = \frac{\mu_0}{4\pi\imath^3}[3(\mathbf{a}_1\cdot\hat{\boldsymbol{\imath}})(\mathbf{a}_2\cdot\hat{\boldsymbol{\imath}}) - \mathbf{a}_1\cdot\mathbf{a}_2].}$$

(b) $\mathcal{E}_1 = -M\frac{dI_2}{dt}$, $\left.\frac{dW}{dt}\right|_1 = -\mathcal{E}_1 I_1 = MI_1\frac{dI_2}{dt}$. (This is the work done per unit time *against* the mutual emf in loop 1—hence the minus sign.) So (since I_1 is constant) $W_1 = MI_1I_2$, where I_2 is the final current in loop 2:

$$\boxed{W = \frac{\mu_0}{4\pi\imath^3}[3(\mathbf{m}_1\cdot\hat{\boldsymbol{\imath}})(\mathbf{m}_2\cdot\hat{\boldsymbol{\imath}}) - \mathbf{m}_1\cdot\mathbf{m}_2].}$$

Notice that this is *opposite in sign* to Eq. 6.35. In Prob. 6.21 we assumed that the magnitudes of the dipole moments were *fixed*, and we did not worry about the energy necessary to sustain the currents themselves—only the energy required to move them into position and rotate them into their final orientations. But in *this* problem we are including it *all*, and it is a curious fact that this merely changes the sign of the answer. For commentary on this subtle issue see R. H. Young, *Am. J. Phys.* **66**, 1043 (1998), and the references cited there.

Problem 7.31

The displacement current density (Sect. 7.3.2) is $\mathbf{J}_d = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2}\hat{\mathbf{z}}$. Drawing an "amperian loop" at radius s,

$$\oint\mathbf{B}\cdot d\mathbf{l} = B\cdot 2\pi s = \mu_0 I_{d_{\text{enc}}} = \mu_0\frac{I}{\pi a^2}\cdot\pi s^2 = \mu_0 I\frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 I s^2}{2\pi s a^2}; \quad \boxed{\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\boldsymbol{\phi}}.}$$

Problem 7.32

(a) $\mathbf{E} = \dfrac{\sigma(t)}{\epsilon_0}\hat{\mathbf{z}}$; $\sigma(t) = \dfrac{Q(t)}{\pi a^2} = \dfrac{It}{\pi a^2}$; $\boxed{\dfrac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}.}$

(b) $I_{d_{\text{enc}}} = J_d\pi s^2 = \epsilon_0\dfrac{dE}{dt}\pi s^2 = \boxed{I\dfrac{s^2}{a^2}.}$ $\oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{d_{\text{enc}}} \Rightarrow B\,2\pi s = \mu_0 I\dfrac{s^2}{a^2} \Rightarrow \mathbf{B} = \boxed{\dfrac{\mu_0 I}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.}$

(c) A surface current flows radially outward over the left plate; let $I(s)$ be the total current crossing a circle of radius s. The charge density (at time t) is

$$\sigma(t) = \frac{[I - I(s)]t}{\pi s^2}.$$

Since we are told this is independent of s, it must be that $I - I(s) = \beta s^2$, for some constant β. But $I(a) = 0$, so $\beta a^2 = I$, or $\beta = I/a^2$. Therefore $I(s) = I(1 - s^2/a^2)$.

$$B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0[I - I(s)] = \mu_0\frac{s^2}{a^2} \Rightarrow \boxed{\mathbf{B} = \frac{\mu_0}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.} \checkmark$$

Problem 7.33

(a) $\mathbf{J}_d = \epsilon_0\dfrac{\mu_0 I_0\omega^2}{2\pi}\cos(\omega t)\ln(a/s)\,\hat{\mathbf{z}}$. But $I_0\cos(\omega t) = I$. So $\boxed{\mathbf{J}_d = \dfrac{\mu_0\epsilon_0}{2\pi}\omega^2 I\ln(a/s)\,\hat{\mathbf{z}}.}$

(b) $I_d = \displaystyle\int\mathbf{J}_d\cdot d\mathbf{a} = \frac{\mu_0\epsilon_0\omega^2 I}{2\pi}\int_0^a\ln(a/s)(2\pi s\,ds) = \mu_0\epsilon_0\omega^2 I\int_0^a(s\ln a - s\ln s)ds$

$$= \mu_0\epsilon_0\omega^2 I\left[(\ln a)\tfrac{s^2}{2} - \tfrac{s^2}{2}\ln s + \tfrac{s^2}{4}\right]\Big|_0^a = \mu_0\epsilon_0\omega^2 I\left[\tfrac{a^2}{\cancel{2}}\cancel{\ln a} - \tfrac{a^2}{\cancel{2}}\cancel{\ln a} + \tfrac{a^2}{4}\right] = \boxed{\frac{\mu_0\epsilon_0\omega^2 I a^2}{4}.}$$

(c) $\boxed{\dfrac{I_d}{I} = \dfrac{\mu_0\epsilon_0\omega^2 a^2}{4}.}$ Since $\mu_0\epsilon_0 = 1/c^2$, $I_d/I = (\omega a/2c)^2$. If $a = 10^{-3}$ m, and $\frac{I_d}{I} = \frac{1}{100}$, so that $\frac{\omega a}{2c} = \frac{1}{10}$, $\omega = \frac{2c}{10a} = \frac{3\times10^8\ \mathrm{m/s}}{5\times10^{-3}\ \mathrm{m}}$, or $\omega = 0.6\times10^{11}/\mathrm{s} = \boxed{6\times10^{10}/\mathrm{s};}$ $\nu = \frac{\omega}{2\pi} \approx 10^{10}$ Hz, or 10^4 megahertz. (This is the *microwave* region, *way* above radio frequencies.)

Problem 7.34

Physically, this is the field of a point charge $-q$ at the origin, out to an expanding spherical shell of radius vt; outside this shell the field is zero. Evidently the shell carries the opposite charge, $+q$. *Mathematically*, using product rule #5 and Eq. 1.99:

$$\nabla\cdot\mathbf{E} = \theta(vt-r)\nabla\cdot\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}\right) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}\cdot\nabla[\theta(vt-r)] = -\frac{q}{\epsilon_0}\delta^3(\mathbf{r})\theta(vt-r) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\,\frac{\partial}{\partial r}\theta(vt-r).$$

But $\delta^3(\mathbf{r})\theta(vt-r) = \delta^3(\mathbf{r})\theta(t)$, and $\frac{\partial}{\partial r}\theta(vt-r) = -\delta(vt-r)$ (Prob. 1.45), so

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} = \boxed{-q\delta^3(\mathbf{r})\theta(t) + \frac{q}{4\pi r^2}\delta(vt-r).}$$

(For $t<0$ the field and the charge density are zero everywhere.)

Clearly $\nabla\cdot\mathbf{B} = 0$, and $\nabla\times\mathbf{E} = 0$ (since $\mathbf{E}$ has only an r component, and it is independent of θ and ϕ). There remains only the Ampére/Maxwell law, $\nabla\times\mathbf{B} = 0 = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial\mathbf{E}/\partial t$. Evidently

$$\mathbf{J} = -\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = -\epsilon_0\left\{-\frac{q}{4\pi\epsilon_0 r^2}\frac{\partial}{\partial t}\left[\theta(vt-r)\right]\right\}\hat{\mathbf{r}} = \boxed{\frac{q}{4\pi r^2}v\delta(vt-r)\,\hat{\mathbf{r}}.}$$

(The stationary charge at the origin does not contribute to $\mathbf{J}$, of course; for the expanding shell we have $\mathbf{J} = \rho\mathbf{v}$, as expected—Eq. 5.26.)

Problem 7.35

From $\nabla\cdot\mathbf{B} = \mu_0\rho_m$ it follows that the field of a point monopole is $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{\mathscr{r}^2}\hat{\mathscr{r}}$. The force law has the form $\mathbf{F} \propto q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right)$ (see Prob. 5.21—the c^2 is needed on dimensional grounds). The proportionality constant must be 1 to reproduce "Coulomb's law" for point charges at rest. So $\boxed{\mathbf{F} = q_m\left(\mathbf{B} - \dfrac{1}{c^2}\mathbf{v}\times\mathbf{E}\right).}$

Problem 7.36

Integrate the "generalized Faraday law" (Eq. 7.43iii), $\nabla\times\mathbf{E} = -\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}$, over the surface of the loop:

$$\int(\nabla\times\mathbf{E})\cdot d\mathbf{a} = \oint\mathbf{E}\cdot d\mathbf{l} = \mathcal{E} = -\mu_0\int\mathbf{J}_m\cdot d\mathbf{a} - \frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{a} = -\mu_0 I_{m_{\mathrm{enc}}} - \frac{d\Phi}{dt}.$$

But $\mathcal{E} = -L\dfrac{dI}{dt}$, so $\dfrac{dI}{dt} = \dfrac{\mu_0}{L}I_{m_{\mathrm{enc}}} + \dfrac{1}{L}\dfrac{d\Phi}{dt}$, or $I = \dfrac{\mu_0}{L}\Delta Q_m + \dfrac{1}{L}\Delta\Phi$, where ΔQ_m is the total magnetic charge passing through the surface, and $\Delta\Phi$ is the change in flux through the surface. If we use the *flat* surface, then $\Delta Q_m = q_m$ and $\Delta\Phi = 0$ (when the monopole is far away, $\Phi = 0$; the flux builds up to $\mu_0 q_m/2$ just before it passes through the loop; then it abruptly drops to $-\mu_0 q_m/2$, and rises back up to zero as the monopole disappears into the distance). If we use a huge balloon-shaped surface, so that q_m remains *inside* it on the far side, then $\Delta Q_m = 0$, but Φ rises monotonically from 0 to $\mu_0 q_m$. In either case,

$$\boxed{I = \frac{\mu_0 q_m}{L}.}$$

Problem 7.37

$$E = \frac{V}{d} \Rightarrow J_c = \sigma E = \frac{1}{\rho}E = \frac{V}{\rho d}.\ J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = \epsilon\frac{\partial}{\partial t}\left[\frac{V_0\cos(2\pi\nu t)}{d}\right] = \frac{\epsilon V_0}{d}\left[-2\pi\nu\sin(2\pi\nu t)\right].$$

The ratio of the amplitudes is therefore:

$$\frac{J_c}{J_d} = \frac{V_0}{\rho d}\frac{d}{2\pi\nu\epsilon V_0} = \frac{1}{2\pi\nu\epsilon\rho} = \left[2\pi(4\times10^8)(81)(8.85\times10^{-12})(0.23)\right]^{-1} = \boxed{2.41.}$$

Problem 7.38

The potential and field in this configuration are identical to those in the upper half of Ex. 3.8. Therefore:

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a}$$

where the integral is over the hemispherical surface just outside the sphere.

But I can with impunity *close* this surface: (because $E = 0$ down there anyway—inside a conductor).

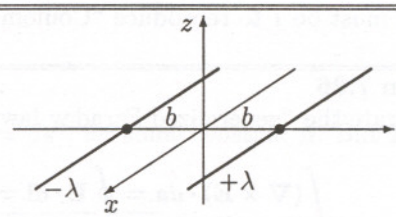
So $I = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}Q_{\text{enc}} = \frac{\sigma}{\epsilon_0}\int \sigma_e\, da$, where σ_e is the electric charge density on the surface of the hemisphere—to wit (Eq. 3.77) $\sigma_e = 3\epsilon_0 E_0\cos\theta$.

$$I = \frac{\sigma}{\epsilon_0}3\epsilon_0 E_0\int\cos\theta\, a^2\sin\theta\, d\theta\, d\phi = 3\sigma E_0 a^2 2\pi \underbrace{\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta}_{\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2} = \frac{1}{2}} = 3\sigma E_0\pi a^2.$$

But in this case $E_0 = V_0/d$, so $\boxed{I = \frac{3\sigma\pi V_0 a^2}{d}.}$

Problem 7.39

Begin with a different problem: two parallel wires carrying charges $+\lambda$ and $-\lambda$ as shown.



Field of one wire: $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$; potential: $V = -\frac{\lambda}{2\pi\epsilon_0}\ln(s/a)$.

Potential of combination: $V = \frac{\lambda}{2\pi\epsilon_0}\ln(s_-/s_+)$,

or $V(y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(y+b)^2+z^2}{(y-b)^2+z^2}\right\}$.

Find the locus of points of fixed V (i.e. equipotential surfaces):

$$e^{4\pi\epsilon_0 V/\lambda} \equiv \mu = \frac{(y+b)^2+z^2}{(y-b)^2+z^2} \Longrightarrow \mu(y^2-2yb+b^2+z^2) = y^2+2yb+b^2+z^2;$$

$$y^2(\mu-1)+b^2(\mu-1)+z^2(\mu-1)-2yb(\mu+1) = 0 \Longrightarrow y^2+z^2+b^2-2yb\beta = 0 \quad \left(\beta\equiv\frac{\mu+1}{\mu-1}\right);$$

$$(y-b\beta)^2+z^2+b^2-b^2\beta^2 = 0 \Longrightarrow (y-b\beta^2)+z^2 = b^2(\beta^2-1).$$

This is a *circle*, with center at $y_0 = b\beta = b\left(\frac{\mu+1}{\mu-1}\right)$ and radius $= b\sqrt{\beta^2-1} = b\sqrt{\frac{(\mu^2+2\mu+1)-(\mu^2-2\mu+1)}{(\mu-1)^2}} = \frac{2b\sqrt{\mu}}{\mu-1}$.

This suggests an image solution to the problem at hand. We want $y_0 = d$, radius $= a$, and $V = V_0$. These determine the parameters b, μ, and λ of the image solution:

$$\frac{d}{a} = \frac{y_0}{\text{radius}} = \frac{b\left(\frac{\mu+1}{\mu-1}\right)}{\frac{2b\sqrt{\mu}}{\mu-1}} = \frac{\mu+1}{2\sqrt{\mu}}. \quad \text{Call } \frac{d}{a} \equiv \alpha.$$

$$4\alpha^2\mu = (\mu+1)^2 = \mu^2 + 2\mu + 1 \Longrightarrow \mu^2 + (2-4\alpha^2)\mu + 1 = 0;$$

$$\mu = \frac{4\alpha^2 - 2 \pm \sqrt{4(1-2\alpha^2)^2 - 4}}{2} = 2\alpha^2 - 1 \pm \sqrt{1 - 4\alpha^2 + 4\alpha^4 - 1} = 2\alpha^2 - 1 \pm 2\alpha\sqrt{\alpha^2 - 1};$$

$$\frac{4\pi\epsilon_0 V_0}{\lambda} = \ln\mu \Longrightarrow \lambda = \frac{4\pi\epsilon_0 V_0}{\ln\left(2\alpha^2 - 1 \pm 2\alpha\sqrt{\alpha^2-1}\right)}. \quad \text{That's the line charge in the image problem.}$$

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{\sigma}{\epsilon_0}\lambda l.$$

The current per unit length is $i = \dfrac{I}{l} = \dfrac{\sigma\lambda}{\epsilon_0} = \dfrac{4\pi\sigma V_0}{\ln\left(2\alpha^2 - 1 \pm 2\alpha\sqrt{\alpha^2-1}\right)}$. Which sign do we want? Suppose the cylinders are far apart, $d \gg a$, so that $\alpha \gg 1$.

$$(\;) = 2\alpha^2 - 1 \pm 2\alpha^2\sqrt{1 - 1/\alpha^2} = 2\alpha^2 - 1 \pm 2\alpha^2\left[1 - \frac{1}{2\alpha^2} - \frac{1}{8\alpha^4} + \cdots\right]$$

$$= 2\alpha^2(1\pm1) - (1\pm1) \mp \frac{1}{4\alpha^2} \pm \cdots = \begin{cases} 4\alpha^2 - 2 - 1/2\alpha^2 + \cdots \approx 4\alpha^2 & (+\text{ sign}), \\ -1/4\alpha^2 & (-\text{ sign}). \end{cases}$$

The current must surely *decrease* with increasing α, so evidently the + sign is correct:

$$\boxed{i = \frac{4\pi\sigma V_0}{\ln\left(2\alpha^2 - 1 + 2\alpha\sqrt{\alpha^2-1}\right)}, \text{ where } \alpha = \frac{d}{a}.}$$

Problem 7.40

(a) The resistance of *one* disk (Ex. 7.1) is $dR = \frac{dz}{\sigma A} = \frac{\rho}{\pi r^2}\,dz$, where $r = \left(\frac{b-a}{L}\right)z + a$ is the radius of the disk. The total resistance is

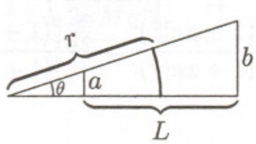
$$R = \frac{\rho}{\pi}\int_0^L \frac{1}{\left[\left(\frac{b-a}{L}\right)z + a\right]^2}\,dz = \frac{\rho}{\pi}\left(\frac{L}{b-a}\right)\left\{\frac{-1}{\left[\left(\frac{b-a}{L}\right)z + a\right]}\right\}\Bigg|_0^L = \frac{\rho L}{\pi(b-a)}\left[-\frac{1}{(b-a+a)} + \frac{1}{a}\right]$$

$$= \frac{\rho L}{\pi(b-a)}\left(\frac{b-a}{ab}\right) = \boxed{\frac{\rho L}{\pi ab}}.$$

(b) In Ex. 7.1 the current was parallel to the axis; here it certainly is *not*. (Nor is it radial with respect to the apex of the cone, since the ends are *flat*. This is not an easy configuration to solve exactly.)

(c) This time the flow *is* radial, and we can add the resistances of nested spherical shells: $dR = \dfrac{\rho}{A}\,dr$, where

$A = \displaystyle\int_0^\theta r^2\sin\theta\,d\theta\,d\phi = 2\pi r^2(-\cos\theta)\big|_0^\theta = 2\pi r^2(1-\cos\theta).$



$$R = \frac{\rho}{2\pi(1-\cos\theta)}\int_{r_a}^{r_b}\frac{1}{r^2}\,dr = \frac{\rho}{2\pi(1-\cos\theta)}\left(\frac{r_b-r_a}{r_ar_b}\right). \text{ Now } \frac{a}{r_a} = \frac{b}{r_b} = \sin\theta.$$

$$= \frac{\rho(b-a)}{2\pi ab}\frac{\sin\theta}{(1-\cos\theta)}. \text{ But } \sin\theta = \frac{b-a}{\sqrt{L^2+(b-a)^2}} \text{ and } \cos\theta = \frac{L}{\sqrt{L^2+(b-a)^2}}.$$

$$= \boxed{\frac{\rho(b-a)^2}{2\pi ab}\frac{1}{\left[\sqrt{L^2+(b-a)^2}-L\right]}.}$$

[Note that if $b-a \ll L$, then $\sqrt{L^2+(b-a)^2} \cong L\left[1+\frac{1}{2}\frac{(b-a)^2}{L^2}\right]$, and $R \cong \frac{\rho(b-a)^2}{2\pi ab}\frac{1}{(b-a)^2/2L} = \frac{\rho L}{\pi ab}$, as in (a).]

Problem 7.41

From Prob. 3.23, $\begin{cases} V_{\text{in}}(s,\phi) = \sum_{k=1}^{\infty} s^k b_k \sin(k\phi), & (s<a); \\ V_{\text{out}}(s,\phi) = \sum_{k=1}^{\infty} s^{-k} d_k \sin(k\phi), & (s>a). \end{cases}$

(We don't need the cosine terms, because V is clearly an *odd* function of ϕ.) At $s=a$, $V_{\text{in}} = V_{\text{out}} = V_0\phi/2\pi$. Let's start with V_{in}, and use Fourier's trick to determine b_k:

$\sum_{k=1}^{\infty} a^k b_k \sin(k\phi) = \frac{V_0\phi}{2\pi} \Rightarrow \sum_{k=1}^{\infty} a^k b_k \int_{-\pi}^{\pi}\sin(k\phi)\sin(k'\phi)\,d\phi = \frac{V_0}{2\pi}\int_{-\pi}^{\pi}\phi\sin(k'\phi)\,d\phi$. But

$\int_{-\pi}^{\pi}\sin(k\phi)\sin(k'\phi)\,d\phi = \pi\delta_{kk'}$, and

$\int_{-\pi}^{\pi}\phi\sin(k'\phi)\,d\phi = \left[\frac{1}{(k')^2}\sin(k'\phi) - \frac{\phi}{k'}\cos(k'\phi)\right]\Big|_{-\pi}^{\pi} = -\frac{2\pi}{k'}\cos(k'\phi) = -\frac{2\pi}{k'}(-1)^{k'}$. So

$\pi a^k b_k = \frac{V_0}{2\pi}\left[-\frac{2\pi}{k}(-1)^k\right]$, or $b_k = -\frac{V_0}{\pi k}\left(-\frac{1}{a}\right)^k$, and hence $V_{\text{in}}(s,\phi) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{s}{a}\right)^k\sin(k\phi)$.

Similarly, $V_{\text{out}}(s,\phi) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{a}{s}\right)^k\sin(k\phi)$. Both sums are of the form $S \equiv \sum_{k=1}^{\infty}\frac{1}{k}(-x)^k\sin(k\phi)$ (with $x = s/a$ for $r<a$ and $x = a/s$ for $r>a$). This series can be summed explicitly, using Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$): $S = \text{Im}\sum_{k=1}^{\infty}\frac{1}{k}(-x)^k e^{ik\phi} = \text{Im}\sum_{k=1}^{\infty}\frac{1}{k}\left(-xe^{i\phi}\right)^k$.

But $\ln(1+w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \frac{1}{4}w^4 \cdots = -\sum_{k=1}^{\infty}\frac{1}{k}(-w)^k$, so $S = -\text{Im}\left[\ln\left(1+xe^{i\phi}\right)\right]$.

Now $\ln\left(Re^{i\theta}\right) = \ln R + i\theta$, so $S = -\theta$, where

$$\tan\theta = \frac{\text{Im}\left(1+xe^{i\phi}\right)}{\text{Re}\left(1+xe^{i\phi}\right)} = \frac{\frac{1}{2i}\left[\left(1+xe^{i\phi}\right)-\left(1+xe^{-i\phi}\right)\right]}{\frac{1}{2}\left[\left(1+xe^{i\phi}\right)+\left(1+xe^{-i\phi}\right)\right]} = \frac{x\left(e^{i\phi}-e^{-i\phi}\right)}{i\left[2+x\left(e^{i\phi}+e^{-i\phi}\right)\right]} = \frac{x\sin\phi}{1+x\cos\phi}.$$

$$\boxed{\text{Conclusion:} \begin{cases} V_{\text{in}}(s,\phi) &= \dfrac{V_0}{\pi}\tan^{-1}\left(\dfrac{s\sin\phi}{a+s\cos\phi}\right), \quad (s<a); \\[2ex] V_{\text{out}}(s,\phi) &= \dfrac{V_0}{\pi}\tan^{-1}\left(\dfrac{a\sin\phi}{s+a\cos\phi}\right), \quad (s>a). \end{cases}}$$

(b) From Eq. 2.36, $\sigma(\phi) = -\epsilon_0\left\{\left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=a} - \left.\frac{\partial V_{\text{in}}}{\partial s}\right|_{s=a}\right\}$.

$$\frac{\partial V_{\text{out}}}{\partial s} = \frac{V_0}{\pi}\left\{\frac{1}{\left[1+\left(\frac{a\sin\phi}{s+a\cos\phi}\right)^2\right]}\frac{(-a\sin\phi)}{(s+a\cos\phi)^2}\right\} = -\frac{V_0}{\pi}\left[\frac{a\sin\phi}{(s+a\cos\phi)^2+(a\sin\phi)^2}\right]$$

$$= -\frac{V_0}{\pi}\left(\frac{a\sin\phi}{s^2+2as\cos\phi+a^2}\right);$$

$$\frac{\partial V_{\text{in}}}{\partial s} = \frac{V_0}{\pi}\left\{\frac{1}{\left[1+\left(\frac{s\sin\phi}{a+s\cos\phi}\right)^2\right]}\frac{[(a+s\cos\phi)\sin\phi - s\sin\phi\cos\phi]}{(a+s\cos\phi)^2}\right\} = \frac{V_0}{\pi}\left[\frac{a\sin\phi}{(a+s\cos\phi)^2+(s\sin\phi)^2}\right]$$

$$= \frac{V_0}{\pi}\left(\frac{a\sin\phi}{s^2+2as\cos\phi+a^2}\right).$$

$$\left.\frac{\partial V_{\text{in}}}{\partial s}\right|_{s=a} = -\left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=a} = \frac{V_0}{2\pi a}\left(\frac{\sin\phi}{1+\cos\phi}\right), \text{ so } \sigma(\phi) = \frac{\epsilon_0 V_0}{\pi a}\frac{\sin\phi}{(1+\cos\phi)} = \boxed{\frac{\epsilon_0 V_0}{\pi a}\tan(\phi/2).}$$

Problem 7.42

(a) Faraday's law says $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, so $\mathbf{E}=0 \Rightarrow \frac{\partial\mathbf{B}}{\partial t}=0 \Rightarrow \mathbf{B}(\mathbf{r})$ is independent of t.

(b) Faraday's law in integral form (Eq. 7.18) says $\oint\mathbf{E}\cdot d\mathbf{l} = -d\Phi/dt$. In the wire itself $\mathbf{E}=0$, so Φ through the loop is constant.

(c) Ampère-Maxwell$\Rightarrow$ $\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$, so $\mathbf{E}=0$, $\mathbf{B}=0 \Rightarrow \mathbf{J}=0$, and hence any current must be at the surface.

(d) From Eq. 5.68, a rotating shell produces a uniform magnetic field (inside): $\mathbf{B} = \frac{2}{3}\mu_0\sigma\omega a\hat{\mathbf{z}}$. So to *cancel* such a field, we need $\sigma\omega a = -\frac{3}{2}\frac{B_0}{\mu_0}$. Now $\mathbf{K} = \sigma\mathbf{v} = \sigma\omega a\sin\theta\,\hat{\boldsymbol{\phi}}$, so $\boxed{\mathbf{K} = -\frac{3B_0}{2\mu_0}\sin\theta\,\hat{\boldsymbol{\phi}}.}$

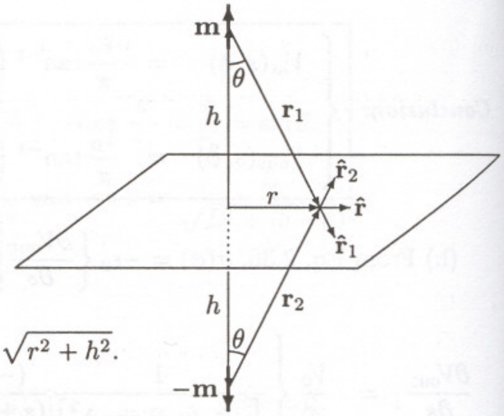
Problem 7.43

(a) To make the field *parallel* to the plane, we need image monopoles of the *same* sign (compare Figs. 2.13 and 2.14), so the image dipole points $\boxed{\text{down } (-z).}$

(b) From Prob. 6.3 (with $r\to 2z$):

$$F = \boxed{\frac{3\mu_0}{2\pi}\frac{m^2}{(2z)^4}}. \quad \frac{3\mu_0}{2\pi}\frac{m^2}{(2h)^4} = Mg \Rightarrow h = \boxed{\frac{1}{2}\left(\frac{3\mu_0 m^2}{2\pi Mg}\right)^{1/4}}.$$

(c) Using Eq. 5.87, and referring to the figure:



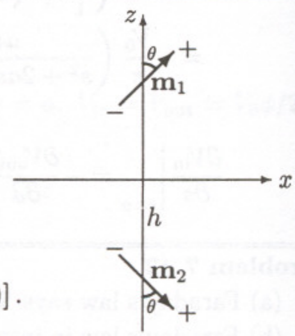
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{(r_1)^3}\left\{[3(m\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1)\,\hat{\mathbf{r}}_1 - m\hat{\mathbf{z}}] + [3(-m\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2)\,\hat{\mathbf{r}}_2 + m\hat{\mathbf{z}}]\right\}$$
$$= \frac{3\mu_0 m}{4\pi(r_1)^3}\left[(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1)\,\hat{\mathbf{r}}_1 - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2)\,\hat{\mathbf{r}}_2\right]. \quad \text{But } \hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1 = -\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2 = \cos\theta.$$
$$= -\frac{3\mu_0 m}{4\pi(r_1)^3}\cos\theta(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2). \quad \text{But } \hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2 = 2\sin\theta\,\hat{\mathbf{r}}.$$
$$= -\frac{3\mu_0 m}{2\pi(r_1)^3}\sin\theta\cos\theta\,\hat{\mathbf{r}}. \quad \text{But } \sin\theta = \frac{r}{r_1},\ \cos\theta = \frac{h}{r_1},\ \text{and } r_1 = \sqrt{r^2+h^2}.$$
$$= -\frac{3\mu_0 m h}{2\pi}\frac{r}{(r^2+h^2)^{5/2}}\,\hat{\mathbf{r}}.$$

Now $\mathbf{B} = \mu_0(\mathbf{K}\times\hat{\mathbf{z}}) \Rightarrow \hat{\mathbf{z}}\times\mathbf{B} = \mu_0\,\hat{\mathbf{z}}\times(\mathbf{K}\times\hat{\mathbf{z}}) = \mu_0\left[\mathbf{K} - \hat{\mathbf{z}}(\mathbf{K}\cdot\hat{\mathbf{z}})\right] = \mu_0\mathbf{K}$. (I used the BAC-CAB rule, and noted that $\mathbf{K}\cdot\hat{\mathbf{z}} = 0$, because the surface current is in the xy plane.)

$$\mathbf{K} = \frac{1}{\mu_0}(\hat{\mathbf{z}}\times\mathbf{B}) = -\frac{3mh}{2\pi}\frac{r}{(r^2+h^2)^{5/2}}(\hat{\mathbf{z}}\times\hat{\mathbf{r}}) = -\frac{3mh}{2\pi}\frac{r}{(r^2+h^2)^{5/2}}\,\hat{\boldsymbol{\phi}}. \quad \text{qed}$$

Problem 7.44

Say the angle between the dipole ($\mathbf{m}_1$) and the z axis is θ (see diagram).



The field of the image dipole ($\mathbf{m}_2$) is

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi}\frac{1}{(h+z)^3}\left[3(\mathbf{m}_2\cdot\hat{\mathbf{z}})\,\hat{\mathbf{z}} - \mathbf{m}_2\right]$$

for points on the z axis (Eq. 5.87). The torque on $\mathbf{m}_1$ is (Eq. 6.1)

$$\mathbf{N} = \mathbf{m}_1\times\mathbf{B} = \frac{\mu_0}{4\pi(2h)^3}\left[3(\mathbf{m}_2\cdot\hat{\mathbf{z}})(\mathbf{m}_1\times\hat{\mathbf{z}}) - (\mathbf{m}_1\times\mathbf{m}_2)\right].$$

But $\mathbf{m}_1 = m(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}})$, $\mathbf{m}_2 = m(\sin\theta\,\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{z}})$, so $\mathbf{m}_2\cdot\hat{\mathbf{z}} = -m\cos\theta$, $\mathbf{m}_1\times\hat{\mathbf{z}} = -m\sin\theta\,\hat{\mathbf{y}}$, and $\mathbf{m}_1\times\mathbf{m}_2 = 2m^2\sin\theta\cos\theta\hat{\mathbf{y}}$.

$$\mathbf{N} = \frac{\mu_0}{4\pi(2h)^3}\left[3m^2\sin\theta\cos\theta\,\hat{\mathbf{y}} - 2m^2\sin\theta\cos\theta\,\hat{\mathbf{y}})\right] = \frac{\mu_0 m^2}{4\pi(2h)^3}\sin\theta\cos\theta\,\hat{\mathbf{y}}.$$

Evidently the torque is zero for $\theta = 0$, $\pi/2$, or π. But 0 and π are clearly unstable, since the nearby ends of the dipoles (minus, in the figure) dominate, and they repel. The stable configuration is $\theta = \pi/2$: $\boxed{\text{parallel to the surface}}$ (contrast Prob. 4.6).

In this orientation, $\mathbf{B}(z) = -\frac{\mu_0 m}{4\pi(h+z)^3}\,\hat{\mathbf{x}}$, and the force on $\mathbf{m}_1$ is (Eq. 6.3):

$$\mathbf{F} = \boldsymbol{\nabla}\left[-\frac{\mu_0 m^2}{4\pi(h+z)^3}\right]\Bigg|_{z=h} = \frac{3\mu_0 m^2}{4\pi(h+z)^4}\,\hat{\mathbf{z}}\Bigg|_{z=h} = \frac{3\mu_0 m^2}{4\pi(2h)^4}\,\hat{\mathbf{z}}.$$

At equilibrium this force upward balances the weight Mg:

$$\frac{3\mu_0 m^2}{4\pi(2h)^4} = Mg \Rightarrow h = \boxed{\frac{1}{2}\left(\frac{3\mu_0 m^2}{4\pi M g}\right)^{1/4}}.$$

Incidentally, this is $(1/2)^{1/4} = 0.84$ times the height it would adopt in the orientation *perpendicular* to the plane (Prob. 7.43b).

Problem 7.45

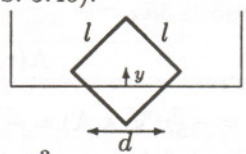
$\mathbf{f} = \mathbf{v} \times \mathbf{B}$; $\mathbf{v} = \omega a \sin\theta\,\hat{\boldsymbol{\phi}}$; $\mathbf{f} = \omega a B_0 \sin\theta(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}})$. $\mathcal{E} = \int \mathbf{f}\cdot d\mathbf{l}$, and $d\mathbf{l} = a\,d\theta\,\hat{\boldsymbol{\theta}}$.

So $\mathcal{E} = \omega a^2 B_0 \int_0^{\pi/2} \sin\theta(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}})\cdot\hat{\boldsymbol{\theta}}\,d\theta$. But $\hat{\boldsymbol{\theta}}\cdot(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = \hat{\mathbf{z}}\cdot(\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}}) = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$.

$\mathcal{E} = \omega a^2 B_0 \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \omega a^2 B_0 [\frac{\sin^2\theta}{2}]\Big|_0^{\pi/2} = \boxed{\frac{1}{2}\omega a^2 B_0}$ (same as the rotating disk in Ex. 7.4).

Problem 7.46

(a) In the "square" orientation (□), it falls at terminal velocity $\boxed{v_{\text{square}} = \frac{mgR}{B^2 l^2}}$ (Prob. 7.11). In the "diamond" orientation (◇), the magnetic force upward is $F = IBd$ (Prob. 5.40).



The flux is $\Phi = B\left[l^2 - (d/2)^2\right]$, and $d/2 = l/\sqrt{2} - y$,
so $\Phi = B\left[l^2 - (l/\sqrt{2} - y)^2\right]$.
$\mathcal{E} = -\frac{d\Phi}{dt} = -2B\left(l/\sqrt{2} - y\right)\frac{dy}{dt}$. But $\frac{dy}{dt} = -v$.
So $\mathcal{E} = 2Bv\left(l/\sqrt{2} - y\right) = IR \Rightarrow I = \frac{2Bv}{R}\left(l/\sqrt{2} - y\right)$; $F = 2\cdot\frac{2B^2v}{R}\left(l/\sqrt{2} - y\right)^2 = mg$ (at terminal velocity).

$\boxed{v_{\text{diamond}} = \frac{mgR}{4B^2\left(l/\sqrt{2} - y\right)^2}}$. (This works for negative y as well as positive, if you replace y by $|y|$.)

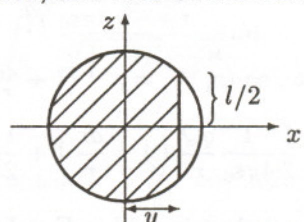
Thus $\frac{v_{\text{square}}}{v_{\text{diamond}}} = \left(\frac{mgR}{B^2l^2}\right)\frac{4B^2\left(l/\sqrt{2} - y\right)^2}{mgR} = \boxed{\left(\sqrt{2} - 2y/l\right)^2}$. At first $(y \sim l/\sqrt{2})$ the "diamond" falls faster; toward the halfway mark $(y \sim 0)$, the "square" falls twice as fast; then the diamond again takes over. The *total* time it takes for the square to fall is:

$$t_{\text{square}} = \frac{l}{v_{\text{square}}} = \boxed{\frac{B^2l^3}{mgR}}$$

(assuming it always goes at the terminal velocity, which—as we found in Prob. 7.11—is close to the truth, if the field is strong). For the diamond, t is

$$-\int \frac{dy}{v_{\text{diamond}}} = -\frac{8B^2}{mgR}\int_{l/\sqrt{2}}^{0} -\left(l/\sqrt{2} - y\right)^2 dy = \frac{8B^2}{mgR}\left[\frac{1}{3}\left(l/\sqrt{2} - y\right)^3\right]\Bigg|_{l/\sqrt{2}}^{0} = \frac{8B^2}{mgR}\frac{1}{3}\frac{l^3}{2\sqrt{2}} = \boxed{\frac{2\sqrt{2}}{3}\frac{B^2l^3}{mgR}}.$$

So $t_{\text{square}}/t_{\text{diamond}} = 3/2\sqrt{2} = 1.06$. The *"square" falls faster, overall.* If free to rotate, it would start out in the "diamond" orientation, switch to "square" for the middle portion, and then switch back to diamond, always trying to present the minimum *chord* at the field's edge.



(b) $F = IBl$; $\Phi = 2B\int_{-a}^{y}\sqrt{a^2 - x^2}\,dx$ (a = radius of circle).
$\mathcal{E} = -\frac{d\Phi}{dt} = -2B\sqrt{a^2 - y^2}\frac{dy}{dt} = 2Bv\sqrt{a^2 - y^2} = IR$.
$I = \frac{2Bv}{R}\sqrt{a^2 - y^2}$; $l/2 = \sqrt{a^2 - y^2}$. So $F = \frac{4B^2v}{R}(a^2 - y^2) = mg$.
$v_{\text{circle}} = \frac{mgR}{4B^2(a^2 - y^2)}$;

$$t_{\text{circle}} = \int_{+a}^{-a} -\frac{dy}{v} = \frac{4B^2}{mgR}\int_{-a}^{a}(a^2 - y^2)dy = \frac{4B^2}{mgR}\left(a^2y - \frac{1}{3}y^3\right)\Big|_{-a}^{a} = \frac{4B^2}{mgR}\left(\frac{4}{3}a^3\right) = \boxed{\frac{16}{3}\frac{B^2a^3}{mgR}}.$$

Problem 7.47

(a) In magnetostatics

$$\nabla\cdot\mathbf{B}=0,\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}\ \Rightarrow\ \mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'.$$

For Faraday electric fields (with $\rho=0$), therefore,

$$\nabla\cdot\mathbf{E}=0,\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\ \Rightarrow\ \mathbf{E}(\mathbf{r},t)=-\frac{1}{4\pi}\frac{\partial}{\partial t}\int\frac{\mathbf{B}(\mathbf{r}',t)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

(with the substitution $\mathbf{J}\to-\frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}$.)

(b) From Prob. 5.50a,

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{4\pi}\int\frac{\mathbf{B}(\mathbf{r}',t)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau',\ \text{so}\ \mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}.$$

[*Check:* $\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}(\nabla\times\mathbf{A})=-\frac{\partial\mathbf{B}}{\partial t}$, and we recover Faraday's law.]

(c) The Coulomb field is zero inside and $\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}=\frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi R^2}{r^2}\,\hat{\mathbf{r}}=\frac{\sigma R^2}{\epsilon_0 r^2}\,\hat{\mathbf{r}}$ outside. The Faraday field is $-\frac{\partial\mathbf{A}}{\partial t}$, where $\mathbf{A}$ is given (in the quasistatic approximation) by Eq. 5.67, with ω a function of time. Letting $\dot\omega\equiv d\omega/dt$,

$$\boxed{\mathbf{E}(r,\theta,\phi,t)=\begin{cases}\dfrac{\mu_0 R\dot\omega\sigma}{3}r\sin\theta\,\hat{\boldsymbol{\phi}} & (r<R),\\[2ex] \dfrac{\sigma R^2}{\epsilon_0 r^2}\,\hat{\mathbf{r}}+\dfrac{\mu_0 R^4\dot\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}} & (r>R).\end{cases}}$$

Problem 7.48

$qBR=mv$ (Eq. 5.3). If R is to stay fixed, then $qR\dfrac{dB}{dt}=m\dfrac{dv}{dt}=ma=F=qE$, or $E=R\dfrac{dB}{dt}$. But $\oint\mathbf{E}\cdot d\mathbf{l}=-\dfrac{d\Phi}{dt}$, so $E\,2\pi R=-\dfrac{d\Phi}{dt}$, so $-\dfrac{1}{2\pi R}\dfrac{d\Phi}{dt}=R\dfrac{dB}{dt}$, or $B=-\dfrac{1}{2}\left(\dfrac{1}{\pi R^2}\Phi\right)+$ constant. If at time $t=0$ the field is off, then the constant is zero, and $B(R)=\dfrac{1}{2}\left(\dfrac{1}{\pi R^2}\Phi\right)$ (in magnitude). Evidently the field at R must be *half* the *average* field over the cross-section of the orbit. qed

Problem 7.49

Initially, $\frac{mv^2}{r}=\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\Rightarrow T=\frac{1}{2}mv^2=\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}$. After the magnetic field is on, the electron circles in a new orbit, of radius r_1 and velocity v_1:

$$\frac{mv_1^2}{r_1}=\frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1^2}+qv_1B\Rightarrow T_1=\frac{1}{2}mv_1^2=\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1}+\frac{1}{2}qv_1r_1B.$$

But $r_1=r+dr$, so $(r_1)^{-1}=r^{-1}\left(1+\frac{dr}{r}\right)^{-1}\cong r^{-1}\left(1-\frac{dr}{r}\right)$, while $v_1=v+dv$, $B=dB$. To first order, then,

$$T_1=\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}\left(1-\frac{dr}{r}\right)+\frac{1}{2}q(vr)\,dB,\ \text{and hence}\ dT=T_1-T=\frac{qvr}{2}\,dB-\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\,dr.$$

Now, the induced electric field is $E=\frac{r}{2}\frac{dB}{dt}$ (Ex. 7.7), so $m\frac{dv}{dt}=qE=\frac{qr}{2}\frac{dB}{dt}$, or $m\,dv=\frac{qr}{2}dB$. The increase in kinetic energy is therefore $dT=d(\frac{1}{2}mv^2)=mv\,dv=\frac{qvr}{2}dB$. Comparing the two expressions, I conclude that $dr=0$. qed

Problem 7.50

$\mathcal{E} = -\dfrac{d\Phi}{dt} = -\alpha$. So the current in R_1 and R_2 is $I = \dfrac{\alpha}{R_1 + R_2}$; by Lenz's law, it flows counterclockwise. Now the voltage across R_1 (which voltmeter #1 measures) is $V_1 = IR_1 = \boxed{\dfrac{\alpha R_1}{R_1 + R_2}}$ (V_b is the *higher* potential), and $V_2 = -IR_2 = \boxed{\dfrac{-\alpha R^2}{R_1 + R_2}}$ (V_b is *lower*).

Problem 7.51

$$\mathcal{E} = vBh = -L\frac{dI}{dt};\; F = IhB = m\frac{dv}{dt};\; \frac{d^2v}{dt^2} = \frac{hB}{m}\frac{dI}{dt} = -\frac{hB}{m}\left(\frac{hB}{L}\right)v,\; \boxed{\frac{d^2v}{dt^2} = -\omega^2 v,} \text{ with } \boxed{\omega = \frac{hB}{\sqrt{mL}}}.$$

Problem 7.52

A point on the upper loop: $\mathbf{r}_2 = (a\cos\phi_2, a\sin\phi_2, z)$; a point on the lower loop: $\mathbf{r}_1 = (b\cos\phi_1, b\sin\phi_1, 0)$.

$$\begin{aligned}
\mathscr{r}^2 &= (\mathbf{r}_2 - \mathbf{r}_1)^2 = (a\cos\phi_2 - b\cos\phi_1)^2 + (a\sin\phi_2 - b\sin\phi_1)^2 + z^2 \\
&= a^2\cos^2\phi_2 - 2ab\cos\phi_2\cos\phi_1 + b^2\cos^2\phi_1 + a^2\sin^2\phi_2 - 2ab\sin\phi_1\sin\phi_2 + b^2\sin^2\phi_1 + z^2 \\
&= a^2 + b^2 + z^2 - 2ab(\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1) = a^2 + b^2 + z^2 - 2ab\cos(\phi_2 - \phi_1) \\
&= (a^2 + b^2 + z^2)[1 - 2\beta\cos(\phi_2 - \phi_1)] = \frac{ab}{\beta}[1 - 2\beta\cos(\phi_2 - \phi_1)].
\end{aligned}$$

$d\mathbf{l}_1 = b\,d\phi_1\,\hat{\boldsymbol{\phi}}_1 = b\,d\phi_1[-\sin\phi_1\,\hat{\mathbf{x}} + \cos\phi_1\,\hat{\mathbf{y}}]$; $d\mathbf{l}_2 = a\,d\phi_2\,\hat{\boldsymbol{\phi}}_2 = a\,d\phi_2[-\sin\phi_2\,\hat{\mathbf{x}} + \cos\phi_2\,\hat{\mathbf{y}}]$, so
$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = ab\,d\phi_1\,d\phi_2[\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2] = ab\cos(\phi_2 - \phi_1)\,d\phi_1\,d\phi_2$.

$$M = \frac{\mu_0}{4\pi}\oint\oint\frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}} = \frac{\mu_0}{4\pi}\frac{ab}{\sqrt{ab/\beta}}\iint\frac{\cos(\phi_2 - \phi_1)}{\sqrt{1 - 2\beta\cos(\phi_2 - \phi_1)}}\,d\phi_2\,d\phi_1.$$

Both integrals run from 0 to 2π. Do the ϕ_2 integral first, letting $u \equiv \phi_2 - \phi_1$:

$$\int_{-\phi_1}^{2\pi - \phi_1}\frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du = \int_0^{2\pi}\frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du$$

(since the integral runs over a complete cycle of $\cos u$, we may as well change the limits to $0 \to 2\pi$). Then the ϕ_1 integral is just 2π, and

$$M = \frac{\mu_0}{4\pi}\sqrt{ab\beta}\,2\pi\int_0^{2\pi}\frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du = \frac{\mu_0}{2}\sqrt{ab\beta}\int_0^{2\pi}\frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du.$$

(a) If a is small, then $\beta \ll 1$, so (using the binomial theorem)

$$\frac{1}{\sqrt{1 - 2\beta\cos u}} \cong 1 + \beta\cos u, \text{ and } \int_0^{2\pi}\frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du \cong \int_0^{2\pi}\cos u\,du + \beta\int_0^{2\pi}\cos^2 u\,du = 0 + \beta\pi,$$

and hence $M = (\mu_0\pi/2)\sqrt{ab\beta^3}$. Moreover, $\beta \cong ab/(b^2 + z^2)$, so $M \cong \boxed{\dfrac{\mu_0\pi a^2 b^2}{2(b^2 + z^2)^{3/2}}}$ (same as in Prob. 7.20).

(b) More generally,

$$(1+\epsilon)^{-1/2} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots \Longrightarrow \frac{1}{\sqrt{1-2\beta\cos u}} = 1 + \beta\cos u + \frac{3}{2}\beta^2\cos^2 u + \frac{5}{2}\beta^3\cos^3 u + \cdots,$$

so

$$M = \frac{\mu_0}{2}\sqrt{ab\beta}\left\{\int_0^{2\pi}\cos u\,du + \beta\int_0^{2\pi}\cos^2 u\,du + \frac{3}{2}\beta^2\int_0^{2\pi}\cos^3 u\,du + \frac{5}{2}\beta^3\int_0^{2\pi}\cos^4 u\,du + \cdots\right\}$$

$$= \frac{\mu_0}{2}\sqrt{ab\beta}\left[0 + \beta(\pi) + \frac{3}{2}\beta^2(0) + \frac{5}{2}\beta^3\left(\frac{3}{4}\pi\right) + \cdots\right] = \boxed{\frac{\mu_0\pi}{2}\sqrt{ab\beta^3}\left(1 + \frac{15}{8}\beta^2 + (\ )\beta^4 + \cdots\right).} \quad \text{qed}$$

Problem 7.53

Let Φ be the flux of $\mathbf{B}$ through a *single* loop of either coil, so that $\Phi_1 = N_1\Phi$ and $\Phi_2 = N_2\Phi$. Then

$$\mathcal{E}_1 = -N_1\frac{d\Phi}{dt}, \quad \mathcal{E}_2 = -N_2\frac{d\Phi}{dt}, \text{ so } \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}. \quad \text{qed}$$

Problem 7.54

(a) Suppose current I_1 flows in coil 1, and I_2 in coil 2. Then (if Φ is the flux through *one* turn):

$$\Phi_1 = I_1L_1 + MI_2 = N_1\Phi; \quad \Phi_2 = I_2L_2 + MI_1 = N_2\Phi, \quad \text{or } \Phi = I_1\frac{L_1}{N_1} + I_2\frac{M}{N_1} = I_2\frac{L_2}{N_2} + I_1\frac{M}{N_2}.$$

In case $I_1 = 0$, we have $\frac{M}{N_1} = \frac{L_2}{N_2}$; if $I_2 = 0$, we have $\frac{L_1}{N_1} = \frac{M}{N_2}$. Dividing: $\frac{M}{L_1} = \frac{L_2}{M}$, or $L_1L_2 = M^2$. qed

(b) $-\mathcal{E}_1 = \frac{d\Phi_1}{dt} = L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1\cos(\omega t)$; $-\mathcal{E}_2 = \frac{d\Phi_2}{dt} = L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2R$. qed

(c) Multiply the first equation by L_2: $L_1L_2\frac{dI_1}{dt} + L_2\frac{dI_2}{dt}M = L_2V_1\cos\omega t$. Plug in $L_2\frac{dI_2}{dt} = -I_2R - M\frac{dI_1}{dt}$.
$M^2\frac{dI_1}{dt} - MRI_2 - M^2\frac{dI_1}{dt} = L_2V_1\cos\omega t \Rightarrow \boxed{I_2(t) = -\frac{L_2V_1}{MR}\cos\omega t.}$ $L_1\frac{dI_1}{dt} + M\left(\frac{L_2V_1}{MR}\omega\sin\omega t\right) = V_1\cos\omega t$.

$$\frac{dI_1}{dt} = \frac{V_1}{L_1}\left(\cos\omega t - \frac{L_2}{R}\omega\sin\omega t\right) \Rightarrow \boxed{I_1(t) = \frac{V_1}{L_1}\left(\frac{1}{\omega}\sin\omega t + \frac{L_2}{R}\cos\omega t\right).}$$

(d) $\dfrac{V_{\text{out}}}{V_{\text{in}}} = \dfrac{I_2R}{V_1\cos\omega t} = \dfrac{-\frac{L_2V_1}{MR}\cos\omega t\,R}{V_1\cos\omega t} = -\dfrac{L_2}{M} = -\dfrac{N_2}{N_1}$. The ratio of the amplitudes is $\dfrac{N_2}{N_1}$. qed

(e) $P_{\text{in}} = V_{\text{in}}I_1 = (V_1\cos\omega t)\left(\frac{V_1}{L_1}\right)\left(\frac{1}{\omega}\sin\omega t + \frac{L_2}{R}\cos\omega t\right) = \boxed{\frac{(V_1)^2}{L_1}\left(\frac{1}{\omega}\sin\omega t\cos\omega t + \frac{L_2}{R}\cos^2\omega t\right).}$

$P_{\text{out}} = V_{\text{out}}I_2 = (I_2)^2R = \boxed{\frac{(L_2V_1)^2}{M^2R}\cos^2\omega t.}$ Average of $\cos^2\omega t$ is $1/2$; average of $\sin\omega t\cos\omega t$ is zero.

So $\langle P_{\text{in}}\rangle = \frac{1}{2}(V_1)^2\left(\frac{L_2}{L_1R}\right)$; $\langle P_{\text{out}}\rangle = \frac{1}{2}(V_1)^2\left[\frac{(L_2)^2}{M^2R}\right] = \frac{1}{2}(V_1)^2\left[\frac{(L_2)^2}{L_1L_2R}\right]$; $\boxed{\langle P_{\text{in}}\rangle = \langle P_{\text{out}}\rangle = \frac{(V_1)^2L_2}{2L_1R}.}$

Problem 7.55

(a) The continuity equation says $\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}$. Here the right side is independent of t, so we can integrate: $\rho(t) = (-\nabla\cdot\mathbf{J})t +$ constant. The "constant" may be a function of $\mathbf{r}$—it's only constant with respect to t. So, putting in the $\mathbf{r}$ dependence explicitly, and noting that $\nabla\cdot\mathbf{J} = -\dot\rho(\mathbf{r},0)$, $\rho(\mathbf{r},t) = \dot\rho(\mathbf{r},0)t + \rho(\mathbf{r},0)$. qed

(b) Suppose $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau$ and $\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau$. We want to show that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$; $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$, and $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, provided that $\mathbf{J}$ is independent of t.

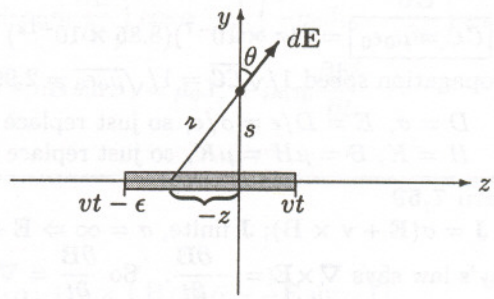
We know from Ch. 2 that Coulomb's law $\left(\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau\right)$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$ and $\boldsymbol{\nabla}\times\mathbf{E} = 0$. Since $\mathbf{B}$ is constant (in time), the $\boldsymbol{\nabla}\cdot\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{E}$ equations are satisfied. From Chapter 5 (specifically, Eqs. 5.45-5.48) we know that the Biot-Savart law satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. It remains only to check $\boldsymbol{\nabla}\times\mathbf{B}$. The argument in Sect. 5.3.2 carries through until the equation following Eq. 5.52, where I invoked $\boldsymbol{\nabla}'\cdot\mathbf{J} = 0$. In its place we now put $\boldsymbol{\nabla}'\cdot\mathbf{J} = -\dot{\rho}$:

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} - \frac{\mu_0}{4\pi}\int \underbrace{(\mathbf{J}\cdot\boldsymbol{\nabla})\frac{\hat{\boldsymbol{\imath}}}{\imath^2}}_{(-\mathbf{J}tp\boldsymbol{\nabla}')\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\ \ \text{(Eq. 5.52)}}\,d\tau \quad \text{(Eqs. 5.49-5.51)}$$

Integration by parts yields two terms, one of which becomes a surface integral, and goes to zero. The other is $\frac{\boldsymbol{\imath}}{\imath^3}\boldsymbol{\nabla}'\cdot\mathbf{J} = \frac{\hat{\boldsymbol{\imath}}}{\imath^2}(-\dot{\rho})$. So:

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} - \frac{\mu_0}{4\pi}\int\frac{\hat{\boldsymbol{\imath}}}{\imath^2}(-\dot{\rho})d\tau = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left\{\frac{1}{4\pi\epsilon_0}\int\frac{\rho\hat{\boldsymbol{\imath}}}{\imath^3}\,d\tau\right\} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}. \quad \text{qed}$$

Problem 7.56



(a) $dE_z = \dfrac{1}{4\pi\epsilon_0}\dfrac{(-\lambda)dz}{\imath^2}\sin\theta$

$\sin\theta = \dfrac{-z}{\imath}$; $\imath = \sqrt{z^2+s^2}$

$$E_z = \frac{\lambda}{4\pi\epsilon_0}\int\frac{z\,dz}{(z^2+s^2)^{3/2}} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-1}{\sqrt{z^2+s^2}}\right]\Bigg|_{vt-\epsilon}^{vt}$$

$$\boxed{E_z = \frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(vt-\epsilon)^2+s^2}} - \frac{1}{\sqrt{(vt)^2+s^2}}\right\}.}$$

(b)

$$\Phi_E = \frac{\lambda}{4\pi\epsilon_0}\int_0^a\left\{\frac{1}{\sqrt{(vt-\epsilon)^2+s^2}} - \frac{1}{\sqrt{(vt)^2+s^2}}\right\}2\pi s\,ds = \frac{\lambda}{2\epsilon_0}\left[\sqrt{(vt-\epsilon)^2+s^2} - \sqrt{(vt)^2+s^2}\right]\Bigg|_0^a$$

$$= \boxed{\frac{\lambda}{2\epsilon_0}\left[\sqrt{(vt-\epsilon)^2+a^2} - \sqrt{(vt)^2+a^2} - (\epsilon - vt) + (vt)\right].}$$

(c) $$I_d = \epsilon_0\frac{d\Phi_E}{dt} = \boxed{\frac{\lambda}{2}\left\{\frac{v(vt-\epsilon)}{\sqrt{(vt-\epsilon)^2+a^2}} - \frac{v(vt)}{\sqrt{(vt)^2+a^2}} + 2v\right\}.}$$

As $\epsilon \to 0$, $vt < \epsilon$ also $\to 0$, so $I_d \to \frac{\lambda}{2}(2v) = \lambda v = I$. With an infinitesimal gap we attribute the magnetic field to *displacement* current, instead of real current, but we get the same answer. qed

Problem 7.57

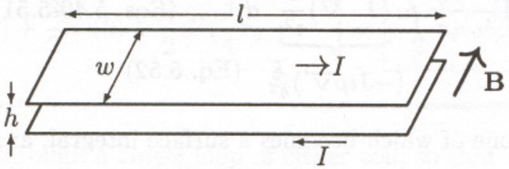
(a) $\nabla^2 V = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial(zf)}{\partial s}\right) + \dfrac{\partial^2(zf)}{\partial z^2} = \dfrac{z}{s}\dfrac{d}{ds}\left(s\dfrac{df}{ds}\right) = 0 \Rightarrow \dfrac{d}{ds}\left(s\dfrac{df}{ds}\right) = 0 \Rightarrow s\dfrac{df}{ds} = A$ (a constant) $\Rightarrow$ $A\dfrac{ds}{s} = df \Rightarrow f = A\ln(s/s_0)$ (s_0 another constant). But (ii) $\Rightarrow f(b) = 0$, so $\ln(b/s_0) = 0$, so $s_0 = b$, and

$V(s,z) = Az\ln(s/b)$. But (i) $\Rightarrow Az\ln(a/b) = -(I\rho z)/(\pi a^2)$, so $A = -\dfrac{I\rho}{\pi a^2}\dfrac{1}{\ln(a/b)}$; $\boxed{V(s,z) = -\dfrac{I\rho z}{\pi a^2}\dfrac{\ln(s/b)}{\ln(a/b)}.}$

(b) $\mathbf{E} = -\nabla V = -\dfrac{\partial V}{\partial s}\hat{\mathbf{s}} - \dfrac{\partial V}{\partial z}\hat{\mathbf{z}} = \dfrac{I\rho z}{\pi a^2}\dfrac{1}{s\ln(a/b)}\hat{\mathbf{s}} + \dfrac{I\rho}{\pi a^2}\dfrac{\ln(s/b)}{\ln(a/b)}\hat{\mathbf{z}} = \boxed{\dfrac{I\rho}{\pi a^2\ln(a/b)}\left(\dfrac{z}{s}\hat{\mathbf{s}} + \ln\left(\dfrac{s}{b}\right)\hat{\mathbf{z}}\right).}$

(c) $\sigma(z) = \epsilon_0\left[E_s(a^+) - E_s(a^-)\right] = \epsilon_0\left[\dfrac{I\rho}{\pi a^2\ln(a/b)}\left(\dfrac{z}{a}\right) - 0\right] = \boxed{\dfrac{\epsilon_0 I\rho z}{\pi a^3\ln(a/b)}.}$

Problem 7.58



(a) Parallel-plate capacitor: $E = \dfrac{1}{\epsilon_0}\sigma$; $V = Eh = \dfrac{1}{\epsilon_0}\dfrac{Q}{wl}h \Rightarrow C = \dfrac{Q}{V} = \dfrac{\epsilon_0 wl}{h} \Rightarrow \boxed{\mathcal{C} = \dfrac{\epsilon_0 w}{h}.}$

(b) $B = \mu_0 K = \mu_0\dfrac{I}{w}$; $\Phi = Bhl = \dfrac{\mu_0 I}{w}hl = LI \Rightarrow L = \dfrac{\mu_0 h}{w}l \Rightarrow \boxed{\mathcal{L} = \dfrac{\mu_0 h}{w}.}$

(c) $\boxed{\mathcal{CL} = \mu_0\epsilon_0} = (4\pi\times10^{-7})(8.85\times10^{-12}) = \boxed{1.112\times10^{-17}\,\mathrm{s^2/m^2}.}$

(Propagation speed $1/\sqrt{\mathcal{LC}} = 1/\sqrt{\mu_0\epsilon_0} = 2.999\times10^8\,\mathrm{m/s} = c$.)

(d) $\left.\begin{array}{l} D = \sigma,\ E = D/\epsilon = \sigma/\epsilon, \text{ so just replace } \epsilon_0 \text{ by } \epsilon; \\ H = K,\ B = \mu H = \mu K, \text{ so just replace } \mu_0 \text{ by } \mu. \end{array}\right\}$ $\boxed{\mathcal{LC} = \epsilon\mu;}$ $\boxed{v = 1/\sqrt{\epsilon\mu}.}$

Problem 7.59

(a) $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B})$; $\mathbf{J}$ finite, $\sigma = \infty \Rightarrow \mathbf{E} + (\mathbf{v}\times\mathbf{B}) = 0$. Take the curl: $\nabla\times\mathbf{E} + \nabla\times(\mathbf{v}\times\mathbf{B}) = 0$. But Faraday's law says $\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$. So $\dfrac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})$. qed

(b) $\nabla\cdot\mathbf{B} = 0 \Rightarrow \oint\mathbf{B}\cdot d\mathbf{a} = 0$ for any closed surface. Apply this at time $(t+dt)$ to the surface consisting of $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{R}$:

$$\int_{\mathcal{S}'}\mathbf{B}(t+dt)\cdot d\mathbf{a} + \int_{\mathcal{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a} - \int_{\mathcal{S}}\mathbf{B}(t+dt)\cdot d\mathbf{a} = 0$$

(the sign change in the third term comes from switching *outward* $d\mathbf{a}$ to *inward* $d\mathbf{a}$).

$$d\Phi = \int_{\mathcal{S}'}\mathbf{B}(t+dt)\cdot d\mathbf{a} - \int_{\mathcal{S}}\mathbf{B}(t)\cdot d\mathbf{a} = \int_{\mathcal{S}}\underbrace{[\mathbf{B}(t+dt) - \mathbf{B}(t)]}_{\frac{\partial\mathbf{B}}{\partial t}dt\ \text{(for infinitesimal } dt)}\cdot d\mathbf{a} - \int_{\mathcal{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a}$$

$$d\Phi = \left\{\int_{\mathcal{S}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a}\right\}dt - \int_{\mathcal{R}}\mathbf{B}(t+dt)\cdot[(d\mathbf{l}\times\mathbf{v})\,dt] \quad \text{(Figure 7.13).}$$

Since the second term is already first order in dt, we can replace $\mathbf{B}(t+dt)$ by $\mathbf{B}(t)$ (the distinction would be second order):

$$d\Phi = dt\int_{\mathcal{S}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} - dt\oint_{\mathcal{C}}\underbrace{\mathbf{B}\cdot(d\mathbf{l}\times\mathbf{v})}_{(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}} = dt\left\{\int_{\mathcal{S}}\left(\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{a} - \int_{\mathcal{S}}\nabla\times(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{a}\right\}.$$

$$\frac{d\Phi}{dt} = \int_S \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B})\right]\cdot d\mathbf{a} = 0. \quad \text{qed}$$

Problem 7.60

(a)

$$\begin{aligned}
\nabla\cdot\mathbf{E}' &= (\nabla\cdot\mathbf{E})\cos\alpha + c(\nabla\cdot\mathbf{B})\sin\alpha = \frac{1}{\epsilon_0}\rho_e\cos\alpha + c\mu_0\rho_m\sin\alpha \\
&= \frac{1}{\epsilon_0}(\rho_e\cos\alpha + c\mu_0\epsilon_0\rho_m\sin\alpha) = \frac{1}{\epsilon_0}(\rho_e\cos\alpha + \frac{1}{c}\rho_m\sin\alpha) = \frac{1}{\epsilon_0}\rho_e'. \checkmark \\
\nabla\cdot\mathbf{B}' &= (\nabla\cdot\mathbf{B})\cos\alpha - \frac{1}{c}(\nabla\cdot\mathbf{E})\sin\alpha = \mu_0\rho_m\cos\alpha - \frac{1}{c\epsilon_0}\rho_e\sin\alpha \\
&= \mu_0(\rho_m\cos\alpha - \frac{1}{c\mu_0\epsilon_0}\rho_e\sin\alpha) = \mu_0(\rho_m\cos\alpha - c\rho_e\sin\alpha) = \mu_0\rho_m'. \checkmark \\
\nabla\times\mathbf{E}' &= (\nabla\times\mathbf{E})\cos\alpha + c(\nabla\times\mathbf{B})\sin\alpha = \left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\cos\alpha + c\left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\sin\alpha \\
&= -\mu_0(\mathbf{J}_m\cos\alpha - c\mathbf{J}_e\sin\alpha) - \frac{\partial}{\partial t}\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) = -\mu_0\mathbf{J}_m' - \frac{\partial\mathbf{B}'}{\partial t}. \checkmark \\
\nabla\times\mathbf{B}' &= (\nabla\times\mathbf{B})\cos\alpha - \frac{1}{c}(\nabla\times\mathbf{E})\sin\alpha = \left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cos\alpha - \frac{1}{c}\left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\sin\alpha \\
&= \mu_0(\mathbf{J}_e\cos\alpha + \frac{1}{c}\mathbf{J}_m\sin\alpha) + \mu_0\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) = \mu_0\mathbf{J}_e' + \mu_0\epsilon_0\frac{\partial\mathbf{E}'}{\partial t}. \checkmark
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{F}' &= q_e'(\mathbf{E}' + \mathbf{v}\times\mathbf{B}') + q_m'(\mathbf{B}' - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}') \\
&= \left(q_e\cos\alpha + \frac{1}{c}q_m\sin\alpha\right)\left[(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) + \mathbf{v}\times\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right)\right] \\
&\quad + (q_m\cos\alpha - cq_e\sin\alpha)\left[\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) - \frac{1}{c^2}\mathbf{v}\times(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha)\right] \\
&= q_e\Big[(\mathbf{E}\cos^2\alpha + c\mathbf{B}\sin\alpha\cos\alpha - c\mathbf{B}\sin\alpha\cos\alpha + \mathbf{E}\sin^2\alpha) \\
&\quad + \mathbf{v}\times\left(\mathbf{B}\cos^2\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin^2\alpha\right)\Big] \\
&\quad + q_m\Big[\left(\frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin^2\alpha + \mathbf{B}\cos^2\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha\right) \\
&\quad + \mathbf{v}\times\left(\frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha - \frac{1}{c^2}\mathbf{E}\sin^2\alpha - \frac{1}{c^2}\mathbf{E}\cos^2\alpha - \frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha\right)\Big] \\
&= q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) = \mathbf{F}. \quad \text{qed}
\end{aligned}$$

Chapter 8

Conservation Laws

Problem 8.1

Example 7.13.

$$\left.\begin{array}{l}\mathbf{E}=\dfrac{\lambda}{2\pi\epsilon_0}\dfrac{1}{s}\,\hat{\mathbf{s}}\\[2ex] \mathbf{B}=\dfrac{\mu_0 I}{2\pi}\dfrac{1}{s}\,\hat{\boldsymbol{\phi}}\end{array}\right\}\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{\lambda I}{4\pi^2\epsilon_0}\frac{1}{s^2}\,\hat{\mathbf{z}};$$

$$P=\int\mathbf{S}\cdot d\mathbf{a}=\int_a^b S\,2\pi s\,ds=\frac{\lambda I}{2\pi\epsilon_0}\int_a^b\frac{1}{s}\,ds=\frac{\lambda I}{2\pi\epsilon_0}\ln(b/a).$$

$$\text{But } V=\int_a^b\mathbf{E}\cdot d\mathbf{l}=\frac{\lambda}{2\pi\epsilon_0}\int_a^b\frac{1}{s}\,ds=\frac{\lambda}{2\pi\epsilon_0}\ln(b/a),\ \text{so } \boxed{P=IV.}$$

Problem 7.58.

$$\left.\begin{array}{l}\mathbf{E}=\dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}}\\[2ex] \mathbf{B}=\mu_0 K\,\hat{\mathbf{x}}=\dfrac{\mu_0 I}{w}\,\hat{\mathbf{x}}\end{array}\right\}\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{\sigma I}{\epsilon_0 w}\,\hat{\mathbf{y}};$$

$$P=\int\mathbf{S}\cdot d\mathbf{a}=Swh=\frac{\sigma I h}{\epsilon_0},\ \text{but } V=\int\mathbf{E}\cdot d\mathbf{l}=\frac{\sigma}{\epsilon_0}h,\ \text{so } \boxed{P=IV.}$$

Problem 8.2

(a) $\mathbf{E}=\dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}};\ \sigma=\dfrac{Q}{\pi a^2};\ Q(t)=It\ \Rightarrow\ \mathbf{E}(t)=\boxed{\dfrac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$

$$B\,2\pi s=\mu_0\epsilon_0\frac{\partial E}{\partial t}\pi s^2=\mu_0\epsilon_0\frac{I\pi s^2}{\pi\epsilon_0 a^2}\ \Rightarrow\ \mathbf{B}(s,t)=\boxed{\frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

(b) $u_{\text{em}}=\dfrac{1}{2}\left(\epsilon_0 E^2+\dfrac{1}{\mu_0}B^2\right)=\dfrac{1}{2}\left[\epsilon_0\left(\dfrac{It}{\pi\epsilon_0 a^2}\right)^2+\dfrac{1}{\mu_0}\left(\dfrac{\mu_0 I s}{2\pi a^2}\right)^2\right]=\boxed{\dfrac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2+(s/2)^2\right].}$

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}})=\boxed{-\frac{I^2 t}{2\pi^2\epsilon_0 a^4}\,s\,\hat{\mathbf{s}}.}$$

$$\frac{\partial u_{\text{em}}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4} 2c^2 t = \frac{I^2 t}{\pi^2 \epsilon_0 a^4}; \quad -\nabla\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\nabla\cdot(s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2\epsilon_0 a^2} = \frac{\partial u_{\text{em}}}{\partial t}. \checkmark$$

(c) $U_{\text{em}} = \int u_{\text{em}} w 2\pi s\, ds = 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b [(ct)^2 + (s/2)^2]s\, ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]\Bigg|_0^b$

$= \boxed{\frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{16}\right]}.$ Over a surface at radius b: $P_{\text{in}} = -\int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\left[b\,\hat{\mathbf{s}}\cdot(2\pi b w\,\hat{\mathbf{s}})\right] = \boxed{\frac{I^2 w t b^2}{\pi\epsilon_0 a^4}}.$

$\frac{dU_{\text{em}}}{dt} = \frac{\mu_0 w I^2 b^2}{2\pi a^4}2c^2 t = \frac{I^2 w t b^2}{\pi\epsilon_0 a^4} = P_{\text{in}}.$ ✓ (Set $b = a$ for *total*.)

Problem 8.3

$$\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \mu_0\epsilon_0\frac{d}{dt}\int \mathbf{S}\, d\tau.$$

The fields are constant, so the second term is zero. The force is clearly in the z direction, so we need

$$\begin{aligned}
(\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a})_z &= T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z = \frac{1}{\mu_0}\left(B_zB_x\,da_x + B_zB_y\,da_y + B_zB_z\,da_z - \frac{1}{2}B^2\,da_z\right)\\
&= \frac{1}{\mu_0}\left[B_z(\mathbf{B}\cdot d\mathbf{a}) - \frac{1}{2}B^2\,da_z\right].
\end{aligned}$$

Now $\mathbf{B} = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}}$ (inside) and $\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$ (outside), where $m = \frac{4}{3}\pi R^3(\sigma\omega R)$. (From Eq. 5.68, Prob. 5.36, and Eq. 5.86.) We want a surface that encloses the entire upper hemisphere—say a hemispherical cap just outside $r = R$ plus the equatorial circular disk.

Hemisphere:

$$\begin{aligned}
B_z &= \frac{\mu_0 m}{4\pi R^3}\left[2\cos\theta\,(\hat{\mathbf{r}})_z + \sin\theta\,(\hat{\boldsymbol{\theta}})_z\right] = \frac{\mu_0 m}{4\pi R^3}\left[2\cos^2\theta - \sin^2\theta\right] = \frac{\mu_0 m}{4\pi R^3}\left(3\cos^2\theta - 1\right).\\
d\mathbf{a} &= R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}};\ \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0 m}{4\pi R^3}(2\cos\theta)R^2\sin\theta\,d\theta\,d\phi;\ da_z = R^2\sin\theta\,d\theta\,d\phi\,\cos\theta;\\
B^2 &= \left(\frac{\mu_0 m}{4\pi R^3}\right)^2\left(4\cos^2\theta + \sin^2\theta\right) = \left(\frac{\mu_0 m}{4\pi R^3}\right)^2\left(3\cos^2\theta + 1\right).\\
(\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a})_z &= \frac{1}{\mu_0}\left(\frac{\mu_0 m}{4\pi R^3}\right)^2\left[\left(3\cos^2\theta - 1\right)2\cos\theta R^2\sin\theta\,d\theta\,d\phi - \frac{1}{2}\left(3\cos^2\theta + 1\right)R^2\sin\theta\cos\theta\,d\theta\,d\phi\right]\\
&= \mu_0\left(\frac{\sigma\omega R}{3}\right)^2\left[\frac{1}{2}R^2\sin\theta\cos\theta\,d\theta\,d\phi\right]\left(12\cos^2\theta - 4 - 3\cos^2\theta - 1\right)\\
&= \frac{\mu_0}{2}\left(\frac{\sigma\omega R^2}{3}\right)^2\left(9\cos^2\theta - 5\right)\sin\theta\cos\theta\,d\theta\,d\phi.
\end{aligned}$$

$$\begin{aligned}
(F_{\text{hemi}})_z &= \frac{\mu_0}{2}\left(\frac{\sigma\omega R^2}{3}\right)^2 2\pi\int_0^{\pi/2}\left(9\cos^3\theta - 5\cos\theta\right)\sin\theta\,d\theta = \mu_0\pi\left(\frac{\sigma\omega R^2}{3}\right)^2\left[-\frac{9}{4}\cos^4\theta + \frac{5}{2}\cos^2\theta\right]\Bigg|_0^{\pi/2}\\
&= \mu_0\pi\left(\frac{\sigma\omega R^2}{3}\right)^2\left(0 + \frac{9}{4} - \frac{5}{2}\right) = -\frac{\mu_0\pi}{4}\left(\frac{\sigma w R^2}{3}\right)^2.
\end{aligned}$$

Disk:

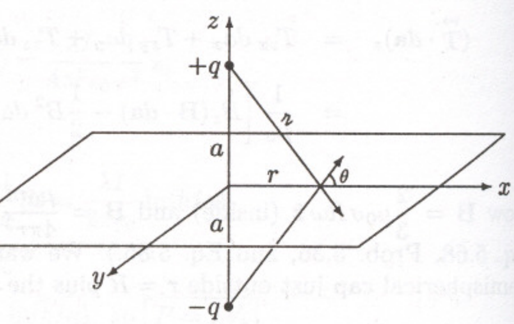
$$\begin{aligned}
B_z &= \frac{2}{3}\mu_0\sigma R\omega; \quad \mathbf{da} = r\,dr\,d\phi\,\hat{\boldsymbol{\phi}} = -r\,dr\,d\phi\,\hat{\mathbf{z}};\\
\mathbf{B}\cdot da &= -\frac{2}{3}\mu_0\sigma R\omega r\,dr\,d\phi; \quad B^2 = \left(\frac{2}{3}\mu_0\sigma R\omega\right)^2; \quad da_z = -r\,dr\,d\phi.\\
(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{da})_z &= \frac{1}{\mu_0}\left(\frac{2}{3}\mu_0\sigma R\omega\right)^2\left[-r\,dr\,d\phi + \frac{1}{2}r\,dr\,d\phi\right] = -\frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0\sigma R\omega\right)^2 r\,dr\,d\phi.\\
(F_{\text{disk}})_z &= -2\mu_0\left(\frac{\sigma\omega R}{3}\right)^2 2\pi\int_0^R r\,dr = -2\pi\mu_0\left(\frac{\sigma\omega R^2}{3}\right)^2.
\end{aligned}$$

Total:

$$\mathbf{F} = -\pi\mu_0\left(\frac{\sigma\omega R^2}{3}\right)^2\left(2+\frac{1}{4}\right)\hat{\mathbf{z}} = \boxed{-\pi\mu_0\left(\frac{\sigma\omega R^2}{2}\right)^2\hat{\mathbf{z}}} \text{ (agrees with Prob. 5.42).}$$

Problem 8.4

(a) $(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{da})_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z.$



But for the xy plane $da_x = da_y = 0$, and $da_z = -r\,dr\,d\phi$ (I'll calculate the force on the *upper* charge).

$$(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{da})_z = \epsilon_0\left(E_zE_z - \frac{1}{2}E^2\right)(-r\,dr\,d\phi).$$

Now $\mathbf{E} = \frac{1}{4\pi\epsilon_0}2\frac{q}{\mathscr{r}^2}\cos\theta\,\hat{\mathbf{r}}$, and $\cos\theta = \frac{r}{\mathscr{r}}$, so $E_z = 0$, $E^2 = \left(\frac{q}{2\pi\epsilon_0}\right)^2\frac{r^2}{(r^2+a^2)^3}$. Therefore

$$\begin{aligned}
F_z &= \frac{1}{2}\epsilon_0\left(\frac{q}{2\pi\epsilon_0}\right)^2 2\pi\int_0^{\in fty}\frac{r^3\,dr}{(r^2+a^2)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\int_0^\infty\frac{u\,du}{(u+a^2)^3} \quad \text{(letting } u\equiv r^2\text{)}\\
&= \frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\left[-\frac{1}{(u+a^2)} + \frac{a^2}{2\,(u+a^2)^2}\right]\Bigg|_0^\infty = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\left[0 + \frac{1}{a^2} - \frac{a^2}{2a^4}\right] = \boxed{\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}}. \checkmark
\end{aligned}$$

(b) In this case $\mathbf{E} = -\frac{1}{4\pi\epsilon_0}2\frac{q}{\mathscr{r}^2}\sin\theta\,\hat{\mathbf{z}}$, and $\sin\theta = \frac{a}{\mathscr{r}}$, so

$$E^2 = E_z^2 = \left(\frac{qa}{2\pi\epsilon_0}\right)^2\frac{1}{(r^2+a^2)^3}, \text{ and hence } (\overleftrightarrow{\mathbf{T}}\cdot\mathbf{da})_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2\frac{r\,dr\,d\phi}{(r^2+a^2)^3}. \text{ Therefore}$$

$$F_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2 2\pi\int_0^\infty\frac{r\,dr}{(r^2+a^2)^3} = -\frac{q^2a^2}{4\pi\epsilon_0}\left[-\frac{1}{4}\frac{1}{(r^2+a^2)^2}\right]_0^\infty = -\frac{q^2a^2}{4\pi\epsilon_0}\left[0+\frac{1}{4a^4}\right] = \boxed{-\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}}. \checkmark$$

Problem 8.5

(a) $E_x = E_y = 0,\ E_z = -\sigma/\epsilon_0$. Therefore

$$T_{xy} = T_{xz} = T_{yz} = \cdots = 0; \quad T_{xx} = T_{yy} = -\frac{\epsilon_0}{2}E^2 = -\frac{\sigma^2}{2\epsilon_0}; \quad T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E^2 = \frac{\sigma^2}{2\epsilon_0}.$$

$$\boxed{\overleftrightarrow{\mathbf{T}} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}.}$$

(b) $\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$ ($\mathbf{S} = 0$, since $\mathbf{B} = 0$); integrate over the xy plane: $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}}$ (negative because *outward* with respect to a surface enclosing the upper plate). Therefore

$$F_z = \int T_{zz}\,da_z = -\frac{\sigma^2}{2\epsilon_0}A, \text{ and the force per unit area is } \mathbf{f} = \frac{\mathbf{F}}{A} = \boxed{-\frac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{z}}.}$$

(c) $-T_{zz} = \boxed{\sigma^2/2\epsilon_0}$ is the momentum in the z direction crossing a surface perpendicular to z, per unit area, per unit time (Eq. 8.31).

(d) The recoil force is the momentum delivered per unit time, so the force per unit area on the top plate is

$$\boxed{\mathbf{f} = -\frac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{z}}} \quad \text{(same as (b)).}$$

Problem 8.6

(a) $\wp_{\text{em}} = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \epsilon_0 EB\,\hat{\mathbf{y}}$; $\quad \mathbf{p}_{\text{em}} = \boxed{\epsilon_0 EBAd\,\hat{\mathbf{y}}.}$

(b) $\mathbf{I} = \int_0^\infty \mathbf{F}\,dt = \int_0^\infty I(\mathbf{l}\times\mathbf{B})\,dt = \int_0^\infty IBd(\hat{\mathbf{z}}\times\hat{\mathbf{x}})\,dt = (Bd\,\hat{\mathbf{y}})\int_0^\infty\left(-\frac{dQ}{dt}\right)dt$ $= -(Bd\hat{\mathbf{y}})[Q(\infty) - Q(0)] = BQd\,\hat{\mathbf{y}}$. But the original field was $E = \sigma/\epsilon_0 = Q/\epsilon_0 A$, so $Q = \epsilon_0 EA$, and hence $\mathbf{I} = \boxed{\epsilon_0 EBAd\,\hat{\mathbf{y}};}$ as expected, the momentum originally stored in the fields (a) is delivered as a kick to the capacitor.

(c) $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{dB}{dt}ld$ (for a length l in the y direction). $-lE(d) + lE(0) = -ld\frac{dB}{dt} \Rightarrow$ $E(d) - E(0) = d\frac{dB}{dt}$. $\mathbf{F} = -\sigma AE(d)\,\hat{\mathbf{y}} + \sigma AE(0)\,\hat{\mathbf{y}} = -\sigma A[E(d) - E(0)]\,\hat{\mathbf{y}} = -\sigma Ad\frac{dB}{dt}\,\hat{\mathbf{y}}$. $\mathbf{I} = \int_0^\infty \mathbf{F}\,dt =$ $-(\sigma Ad\,\hat{\mathbf{y}})\int_0^\infty \frac{dB}{dt}\,dt = -(\sigma Ad\,\hat{\mathbf{y}})[B(\infty) - B(0)] = \sigma AdB\,\hat{\mathbf{y}}$. But $E = \frac{\sigma}{\epsilon_0}$, so $\mathbf{I} = \boxed{\epsilon_0 EBAd\,\hat{\mathbf{y}},}$ as before.

Problem 8.7

$\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}$ (for $a < r < R$; outside the solenoid $B = 0$). The force on a segment dr of spoke is

$$d\mathbf{F} = I'd\mathbf{l}\times\mathbf{B} = I'\mu_0 nI\,dr(\hat{\mathbf{r}}\times\hat{\mathbf{z}}) = -I'\mu_0 nI\,dr\,\hat{\boldsymbol{\phi}}.$$

The torque on the spoke is

$$\mathbf{N} = \int \mathbf{r}\times d\mathbf{F} = I'\mu_0 nI\int_a^R r\,dr(-\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = I'\mu_0 nI\frac{1}{2}\left(R^2 - a^2\right)(-\hat{\mathbf{z}}).$$

Therefore the angular momentum of the cylinders is $\mathbf{L} = \int \mathbf{N}\,dt = -\frac{1}{2}\mu_0 n I(R^2 - a^2)\,\hat{\mathbf{z}} \int I' dt$. But $\int I' dt = Q$, so

$$\boxed{\mathbf{L} = -\frac{1}{2}\mu_0 n I Q(R^2 - a^2)\,\hat{\mathbf{z}}} \quad \text{(in agreement with Eq. 8.35).}$$

Problem 8.8

(a)

$$\mathbf{E} = \left\{ \begin{array}{ll} 0, & (r < R) \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\,\hat{\mathbf{r}}, & (r > R) \end{array} \right\}; \quad \mathbf{B} = \left\{ \begin{array}{ll} \frac{2}{3}\mu_0 M\,\hat{\mathbf{z}}, & (r < R) \\ \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right], & (r > R) \end{array} \right\} \text{(Ex. 6.1)}$$

(where $m = \frac{4}{3}\pi R^3 M$); $\wp = \epsilon_0(\mathbf{E} \times \mathbf{B}) = \dfrac{\mu_0}{(4\pi)^2}\dfrac{Qm}{r^5}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}})\sin\theta$, and $(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\phi}}$, so

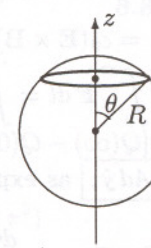
$$\boldsymbol{\ell} = \mathbf{r} \times \wp = \frac{\mu_0}{(4\pi)^2}\frac{mQ}{r^4}\sin\theta(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}).$$

But $(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) = -\hat{\boldsymbol{\theta}}$, and only the z component will survive integration, so (since $(\hat{\boldsymbol{\theta}})_z = -\sin\theta$):

$$\mathbf{L} = \frac{\mu_0 m Q}{(4\pi)^2}\,\hat{\mathbf{z}} \int \frac{\sin^2\theta}{r^4}\left(r^2 \sin\theta\,dr\,d\theta\,d\phi\right). \quad \int_0^{2\pi} d\phi = 2\pi; \quad \int_0^{\pi} \sin^3\theta\,d\theta = \frac{4}{3}; \quad \int_R^{\infty} \frac{1}{r^2}\,dr = \left(-\frac{1}{r}\right)\Big|_R^{\infty} = \frac{1}{R}.$$

$$\mathbf{L} = \frac{\mu_0 m Q}{(4\pi)^2}\,\hat{\mathbf{z}}\,(2\pi)\left(\frac{4}{3}\right)\left(\frac{1}{R}\right) = \boxed{\frac{2}{9}\mu_0 M Q R^2\,\hat{\mathbf{z}}.}$$

(b) Apply Faraday's law to the ring shown:



$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi r \sin\theta) = -\frac{d\Phi}{dt} = -\pi(r\sin\theta)^2\left(\frac{2}{3}\mu_0\frac{dM}{dt}\right)$$

$$\Rightarrow \boxed{\mathbf{E} = -\frac{\mu_0}{3}\frac{dM}{dt}(r\sin\theta)\,\hat{\boldsymbol{\phi}}.}$$

The force on a patch of surface (da) is $d\mathbf{F} = \sigma\mathbf{E}\,da = -\dfrac{\mu_0\sigma}{3}\dfrac{dM}{dt}(r\sin\theta)\,da\,\hat{\boldsymbol{\phi}}\ \left(\sigma = \dfrac{Q}{4\pi R^2}\right)$.

The torque on the patch is $d\mathbf{N} = \mathbf{r} \times d\mathbf{F} = -\dfrac{\mu_0\sigma}{3}\dfrac{dM}{dt}\left(r^2\sin\theta\right)da\,(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}})$. But $(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) = -\hat{\boldsymbol{\theta}}$, and we want only the z component ($\hat{\boldsymbol{\theta}}_z = -\sin\theta$):

$$\mathbf{N} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\,\hat{\mathbf{z}} \int r^2\sin^2\theta\left(r^2\sin\theta\,d\theta\,d\phi\right).$$

Here $r = R$; $\displaystyle\int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}$; $\displaystyle\int_0^{2\pi} d\phi = 2\pi$, so $\mathbf{N} = -\dfrac{\mu_0\sigma}{3}\dfrac{dM}{dt}\,\hat{\mathbf{z}}\,R^4\left(\dfrac{4}{3}\right)(2\pi) = \boxed{-\dfrac{2\mu_0}{9}QR^2\dfrac{dM}{dt}\,\hat{\mathbf{z}}.}$

$$\mathbf{L} = \int \mathbf{N}\,dt = -\frac{2\mu_0}{9}QR^2\,\hat{\mathbf{z}}\int_M^0 dM = \boxed{\frac{2\mu_0}{9}MQR^2\,\hat{\mathbf{z}}} \quad \text{(same as (a)).}$$

(c) Let the charge on the sphere at time t be $q(t)$; the charge *density* is $\sigma = \frac{q(t)}{4\pi R^2}$. The charge below ("south of") the ring in the figure is

$$q_s = \sigma\,(2\pi R^2)\int_0^\pi \sin\theta'\,d\theta' = \frac{q}{2}\,(-\cos\theta')\Big|_\theta^\pi = \frac{q}{2}(1+\cos\theta).$$

So the total current crossing the ring (flowing "north") is $I(t) = -\frac{1}{2}\frac{dq}{dt}(1+\cos\theta)$, and hence $\mathbf{K}(t) = \frac{I}{2\pi R\sin\theta}(-\hat{\boldsymbol\theta}) = \frac{1}{4\pi R}\frac{dq}{dt}\frac{(1+\cos\theta)}{\sin\theta}\,\hat{\boldsymbol\theta}$. The force on a patch of area da is $d\mathbf{F} = (\mathbf{K}\times\mathbf{B})\,da$.

$$\mathbf{B}_{\text{ave}} = \left[\frac{2}{3}\mu_0 M\,\hat{\mathbf{z}} + \frac{\mu_0}{4\pi}\frac{\frac{4}{3}\pi R^3 M}{R^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol\theta})\right]\frac{1}{2} = \frac{\mu_0 M}{6}[2\,\hat{\mathbf{z}} + 2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol\theta}];$$

$$\mathbf{K}\times\mathbf{B} = \frac{1}{4\pi R}\frac{dq}{dt}\frac{\mu_0 M}{6}\frac{(1+\cos\theta)}{\sin\theta}[2(\hat{\boldsymbol\theta}\times\hat{\mathbf{z}}) + 2\cos\theta\,\underbrace{(\hat{\boldsymbol\theta}\times\hat{\mathbf{r}})}_{-\hat{\phi}}].$$

$$d\mathbf{N} = R\,\hat{\mathbf{r}}\times d\mathbf{F} = \frac{\mu_0 M}{24\pi}\left(\frac{dq}{dt}\right)\frac{(1+\cos\theta)}{\sin\theta}2[\underbrace{\hat{\mathbf{r}}\times(\hat{\boldsymbol\theta}\times\hat{\mathbf{z}})}_{\hat{\boldsymbol\theta}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}) - \hat{\mathbf{z}}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol\theta})} - \cos\theta\,\underbrace{(\hat{\mathbf{r}}\times\hat{\phi})}_{-\hat{\theta}}]R^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{\mu_0 M}{12\pi}\left(\frac{dq}{dt}\right)(1+\cos\theta)R^2[\cos\theta\,\hat{\boldsymbol\theta} + \cos\theta\,\hat{\boldsymbol\theta}]\,d\theta\,d\phi = \frac{\mu_0 M R^2}{6\pi}\left(\frac{dq}{dt}\right)(1+\cos\theta)\cos\theta\,d\theta\,d\phi\,\hat{\boldsymbol\theta}.$$

The x and y components integrate to zero; $(\hat{\boldsymbol\theta})_z = -\sin\theta$, so (using $\int_0^{2\pi} d\phi = 2\pi$):

$$N_z = -\frac{\mu_0 M R^2}{6\pi}\left(\frac{dq}{dt}\right)(2\pi)\int_0^\pi(1+\cos\theta)\cos\theta\sin\theta\,d\theta = -\frac{\mu_0 M R^2}{3}\left(\frac{dq}{dt}\right)\left(\frac{\sin^2\theta}{2} - \frac{\cos^3\theta}{3}\right)\Big|_0^\pi$$

$$= -\frac{\mu_0 M R^2}{3}\left(\frac{dq}{dt}\right)\left(\frac{2}{3}\right) = -\frac{2\mu_0}{9}MR^2\frac{dq}{dt}. \quad \boxed{\mathbf{N} = -\frac{2\mu_0}{9}MR^2\frac{dq}{dt}\,\hat{\mathbf{z}}.}$$

Therefore

$$\mathbf{L} = \int\mathbf{N}\,dt = -\frac{2\mu_0}{9}MR^2\,\hat{\mathbf{z}}\int_Q^0 dq = \boxed{\frac{2\mu_0}{9}MR^2Q\,\hat{\mathbf{z}}}\ \text{(same as (a))}.$$

(I used the *average* field at the discontinuity—which is the correct thing to do—but in this case you'd get the same answer using either the inside field or the outside field.)

Problem 8.9

(a) $\mathcal{E} = -\frac{d\Phi}{dt}$; $\Phi = \pi a^2 B$; $B = \mu_0 n I_s$; $\mathcal{E} = I_r R$. So $\boxed{I_r = -\frac{1}{R}(\mu_0\pi a^2 n)\frac{dI_s}{dt}.}$

(b) $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} \Rightarrow E(2\pi a) = -\mu_0\pi a^2 n\frac{dI_s}{dt} \Rightarrow \mathbf{E} = -\frac{1}{2}\mu_0 an\frac{dI_s}{dt}\,\hat{\phi}$. $\mathbf{B} = \frac{\mu_0 I_r}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\,\hat{\mathbf{z}}$ (Eq. 5.38).

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(-\frac{\mu_0 an}{2}\frac{dI_s}{dt}\right)\left(\frac{\mu_0 I_r}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\right)(\hat{\phi}\times\hat{\mathbf{z}}) = \boxed{-\frac{1}{4}\mu_0 I_r\frac{dI_s}{dt}\frac{ab^2n}{(b^2+z^2)^{3/2}}\,\hat{\mathbf{r}}.}$$

Power:

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \int_{-\infty}^{\infty}(S)(2\pi a)\,dz = -\frac{1}{2}\pi\mu_0 a^2 b^2 n I_n \frac{dI_s}{dt}\int_{-\infty}^{\infty}\frac{1}{(b^2+z^2)^{3/2}}\,dx$$

The integral is $\left.\frac{z}{b^2\sqrt{z^2+b^2}}\right|_{-\infty}^{\infty} = \frac{1}{b^2} - \left(-\frac{1}{b^2}\right) = \frac{2}{b^2}$.

$$= -\left(\pi\mu_0 a^2 n\frac{dI_s}{dt}\right) I_r = (RI_r)I_r = I_r{}^2 R. \quad \text{qed}$$

Problem 8.10

According to Eqs. 3.104, 4.14, 5.87, and 6.16, the fields are

$$\mathbf{E} = \left\{\begin{array}{ll} -\dfrac{1}{3\epsilon_0}\mathbf{P}, & (r<R), \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p}], & (r>R), \end{array}\right\} \quad \mathbf{B} = \left\{\begin{array}{ll} \dfrac{2}{3}\mu_0\mathbf{M}, & (r<R), \\ \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{m}], & (r>R), \end{array}\right\}$$

where $\mathbf{p} = (4/3)\pi R^3\mathbf{P}$, and $\mathbf{m} = (4/3)\pi R^3\mathbf{M}$. Now $\mathbf{p} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$, and there are two contributions, one from inside the sphere and one from outside.

Inside:

$$\mathbf{p}_{\text{in}} = \epsilon_0\int\left(-\frac{1}{3\epsilon_0}\mathbf{P}\right)\times\left(\frac{2}{3}\mu_0\mathbf{M}\right)d\tau = -\frac{2}{9}\mu_0(\mathbf{P}\times\mathbf{M})\int d\tau = -\frac{2}{9}\mu_0(\mathbf{P}\times\mathbf{M})\frac{4}{3}\pi R^3 = \frac{8}{27}\mu_0\pi R^3(\mathbf{M}\times\mathbf{P}).$$

Outside:

$$\mathbf{p}_{\text{out}} = \epsilon_0\frac{1}{4\pi\epsilon_0}\frac{\mu_0}{4\pi}\int\frac{1}{r^6}\{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p}]\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{m}]\}\,d\tau.$$

Now $\hat{\mathbf{r}}\times(\mathbf{p}\times\mathbf{m}) = \mathbf{p}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}(\hat{\mathbf{r}}\cdot\mathbf{p})$, so $\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times(\mathbf{p}\times\mathbf{m})] = (\hat{\mathbf{r}}\cdot\mathbf{m})(\hat{\mathbf{r}}\times\mathbf{p})-(\hat{\mathbf{r}}\cdot\mathbf{p})(\hat{\mathbf{r}}\times\mathbf{m})$, whereas using the BAC-CAB rule directly gives $\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times(\mathbf{p}\times\mathbf{m})] = \hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})]-(\mathbf{p}\times\mathbf{m})(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})$. So $\{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p}]\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{m}]\} = -3(\mathbf{p}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\times\mathbf{m})+3(\mathbf{m}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\times\mathbf{p})+(\mathbf{p}\times\mathbf{m}) = 3\{\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})]-(\mathbf{p}\times\mathbf{m})\}+(\mathbf{p}\times\mathbf{m}) = -2(\mathbf{p}\times\mathbf{m})+3\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})]$.

$$\mathbf{p}_{\text{out}} = \frac{\mu_0}{16\pi^2}\int\frac{1}{r^6}\{-2(\mathbf{p}\times\mathbf{m})+3\,\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})]\}\,r^2\sin\theta\,dr\,d\theta\,d\phi.$$

To evaluate the integral, set the z axis along $(\mathbf{p}\times\mathbf{m})$; then $\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m}) = |\mathbf{p}\times\mathbf{m}|\cos\theta$. Meanwhile, $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}}+\sin\theta\sin\phi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}}$. But $\sin\phi$ and $\cos\phi$ integrate to zero, so the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms drop out, leaving

$$\begin{aligned}
\mathbf{p}_{\text{out}} &= \frac{\mu_0}{16\pi^2}\left(\int_0^\infty\frac{1}{r^4}\,dr\right)\left\{-2(\mathbf{p}\times\mathbf{m})\int\sin\theta\,d\theta\,d\phi+3|\mathbf{p}\times\mathbf{m}|\,\hat{\mathbf{z}}\int\cos^2\theta\sin\theta\,d\theta\,d\phi\right\} \\
&= \frac{\mu_0}{16\pi^2}\left.\left(-\frac{1}{3r^3}\right)\right|_R^\infty\left[-2(\mathbf{p}\times\mathbf{m})4\pi+3(\mathbf{p}\times\mathbf{m})\frac{4\pi}{3}\right] = -\frac{\mu_0}{12\pi R^3}(\mathbf{p}\times\mathbf{m}) \\
&= -\frac{\mu_0}{12\pi R^3}\left(\frac{4}{3}\pi R^3\mathbf{P}\right)\times\left(\frac{4}{3}\pi R^3\mathbf{M}\right) = \frac{4\mu_0}{27}R^3(\mathbf{M}\times\mathbf{P}).
\end{aligned}$$

$$\mathbf{p}_{\text{tot}} = \left(\frac{8}{27}+\frac{4}{27}\right)\mu_0 R^3(\mathbf{M}\times\mathbf{P}) = \boxed{\frac{4}{9}\mu_0 R^3(\mathbf{M}\times\mathbf{P}).}$$

Problem 8.11

(a) From Eq. 5.68 and Prob. 5.36,

$$\begin{cases} r < R: \ \mathbf{E} = 0, \ \mathbf{B} = \frac{2}{3}\mu_0 \sigma R \omega \, \hat{\mathbf{z}}, \ \text{ with } \sigma = \frac{e}{4\pi R^2}; \\ r > R: \ \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{e}{r^2}\,\hat{\mathbf{r}}, \ \mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), \ \text{with } m = \frac{4}{3}\pi\sigma\omega R^4. \end{cases}$$

The energy stored in the electric field is (Ex. 2.8):

$$W_E = \frac{1}{8\pi\epsilon_0}\frac{e^2}{R}.$$

The energy density of the internal magnetic field is:

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0 R\omega \frac{e}{4\pi R^2}\right)^2 = \frac{\mu_0\omega^2 e^2}{72\pi^2 R^2}, \text{ so } W_{B_{\text{in}}} = \frac{\mu_0 \omega^2 e^2}{72\pi^2 R^2}\frac{4}{3}\pi R^3 = \frac{\mu_0 e^2\omega^2 R}{54\pi}.$$

The energy density in the external magnetic field is:

$$u_B = \frac{1}{2\mu_0}\frac{\mu_0^2}{16\pi^2}\frac{m^2}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) = \frac{e^2\omega^2 R^4 \mu_0}{18(16\pi^2)}\frac{1}{r^6}\left(3\cos^2\theta + 1\right), \text{ so}$$

$$W_{B_{\text{out}}} = \frac{\mu_0 e^2\omega^2 R^4}{(18)(16)\pi^2}\int_R^\infty \frac{1}{r^6}r^2\,dr \int_0^\pi \left(3\cos^2\theta + 1\right)\sin\theta\,d\theta \int_0^{2\pi} d\phi = \frac{\mu_0 e^2\omega^2 R^4}{(18)(16)\pi^2}\left(\frac{1}{3R^3}\right)(4)(2\pi) = \frac{\mu_0 e^2 \omega^2 R}{108\pi}.$$

$$W_B = W_{B_{\text{in}}} + W_{b_{\text{out}}} = \frac{\mu_0 e^2\omega^2 R}{108\pi}(2+1) = \frac{\mu_0 e^2 \omega^2 R}{36\pi}; \ W = W_E + W_B = \boxed{\frac{1}{8\pi\epsilon_0}\frac{e^2}{R} + \frac{\mu_0 e^2\omega^2 R}{36\pi}}.$$

(b) Same as Prob. 8.8(a), with $Q \to e$ and $m \to \frac{1}{3}e\omega R^2$: $\boxed{\mathbf{L} = \frac{\mu_0 e^2 \omega R}{18\pi}\,\hat{\mathbf{z}}.}$

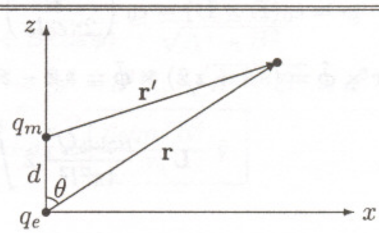
(c) $\frac{\mu_0 e^2}{18\pi}\omega R = \frac{\hbar}{2} \Rightarrow \omega R = \frac{9\pi\hbar}{\mu_0 e^2} = \frac{(9)(\pi)(1.05\times 10^{-34})}{(4\pi\times 10^{-7})(1.60\times 10^{-19})^2} = \boxed{9.23\times 10^{10}\,\text{m/s.}}$

$$\frac{1}{8\pi\epsilon_0}\frac{e^2}{R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = mc^2; \ \left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = 1 + \frac{2}{9}\left(\frac{9.23\times 10^{10}}{3\times 10^8}\right)^2 = 2.10\times 10^4;$$

$$R = \frac{(2.01\times 10^4)(1.6\times 10^{-19})^2}{8\pi(8.85\times 10^{-12})(9.11\times 10^{-31})(3\times 10^8)^2} = \boxed{2.95\times 10^{-11}\,\text{m};} \quad \omega = \frac{9.23\times 10^{-10}}{2.95\times 10^{-11}} = \boxed{3.13\times 10^{21}\,\text{rad/s.}}$$

Since ωR, the speed of a point on the equator, is 300 times the speed of light, this "classical" model is clearly unrealistic.

Problem 8.12



$$\mathbf{E} = \frac{q_e}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3};$$

$$\mathbf{B} = \frac{\mu_0 q_m}{4\pi}\frac{\mathbf{r}'}{r'^3} = \frac{\mu_0 q_m}{4\pi}\frac{(\mathbf{r} - d\,\hat{\mathbf{z}})}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}}.$$

Momentum density (Eq. 8.33):

$$\boldsymbol{\wp} = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0 q_e q_m}{(4\pi)^2}\frac{(-d)(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}.$$

Angular momentum density (Eq. 8.34):

$$\boldsymbol{\ell} = (\mathbf{r}\times\boldsymbol{\wp}) = -\frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{\mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}.\quad \text{But } \mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}}) = \mathbf{r}(\mathbf{r}\cdot\hat{\mathbf{z}}) - r^2\,\hat{\mathbf{z}} = r^2\cos\theta\,\hat{\mathbf{r}} - r^2\,\hat{\mathbf{z}}.$$

The x and y components will integrate to zero; using $(\hat{\mathbf{r}})_z = \cos\theta$, we have:

$$\begin{aligned}\mathbf{L} &= -\frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\hat{\mathbf{z}}\int\frac{r^2(\cos^2\theta-1)}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}r^2\sin\theta\,dr\,d\theta\,d\phi. \quad \text{Let } u\equiv\cos\theta: \\ &= \frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\hat{\mathbf{z}}\,(2\pi)\int_{-1}^{1}\int_{0}^{\infty}\frac{r\,(1-u^2)}{\left(r^2+d^2-2rdu\right)^{3/2}}\,du\,dr.\end{aligned}$$

Do the r integral first:

$$\int_0^\infty\frac{r\,dr}{\left(r^2+d^2-2rdu\right)^{3/2}} = \left.\frac{(ru-d)}{d(1-u^2)\sqrt{r^2+d^2-2rdu}}\right|_0^\infty = \frac{u}{d\,(1-u^2)}+\frac{d}{d\,(1-u^2)\,d} = \frac{u+1}{d\,(1-u^2)} = \frac{1}{d(1-u)}.$$

Then

$$\mathbf{L} = \frac{\mu_0 q_e q_m d}{8\pi}\,\hat{\mathbf{z}}\,\frac{1}{d}\int_{-1}^{1}\frac{(1-u^2)}{(1-u)}\,du = \frac{\mu_0 q_e q_m}{8\pi}\,\hat{\mathbf{z}}\int_{-1}^{1}(1+u)du = \frac{\mu_0 q_e q_m}{8\pi}\,\hat{\mathbf{z}}\left.\left(u+\frac{u^2}{2}\right)\right|_{-1}^{1} = \boxed{\frac{\mu_0 q_e q_m}{4\pi}\,\hat{\mathbf{z}}.}$$

Problem 8.13

(a) The rotating shell at radius b produces a solenoidal magnetic field:

$$\mathbf{B} = \mu_0 K\,\hat{\mathbf{z}},\ \text{where } K = \sigma_b\omega_b b, \text{ and } \sigma_b = -\frac{Q}{2\pi b l}.\ \text{So } \mathbf{B} = -\frac{\mu_0\omega_b Q}{2\pi l}\,\hat{\mathbf{z}}\ (a<s<b).$$

The shell at a also produces a magnetic field $(\mu_0\omega_a Q/2\pi l)\,\hat{\mathbf{z}}$, in the region $s<a$, so the total field inside the inner shell is

$$\mathbf{B} = \frac{\mu_0 Q}{2\pi l}\,(\omega_a-\omega_b)\,\hat{\mathbf{z}},\ (s<a).$$

Meanwhile, the electric field is

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}} = \frac{Q}{2\pi\epsilon_0 l s}\,\hat{\mathbf{s}},\quad (a<s<b).$$

$$\boldsymbol{\wp} = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \epsilon_0\left(\frac{Q}{2\pi\epsilon_0 l s}\right)\left(-\frac{\mu_0\omega_b Q}{2\pi l}\right)(\hat{\mathbf{s}}\times\hat{\mathbf{z}}) = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2 s}\,\hat{\boldsymbol{\phi}};\quad \boldsymbol{\ell} = \mathbf{r}\times\boldsymbol{\wp} = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2 s}(\mathbf{r}\times\hat{\boldsymbol{\phi}}).$$

Now $\mathbf{r}\times\hat{\boldsymbol{\phi}} = (s\,\hat{\mathbf{s}}+z\,\hat{\mathbf{z}})\times\hat{\boldsymbol{\phi}} = s\,\hat{\mathbf{z}} - z\,\hat{\mathbf{s}}$, and the $\hat{\mathbf{s}}$ term integrates to zero, so

$$\mathbf{L} = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2}\,\hat{\mathbf{z}}\int d\tau = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2}\pi(b^2-a^2)l\,\hat{\mathbf{z}} = \boxed{\frac{\mu_0\omega_b Q^2(b^2-a^2)}{4\pi l}\,\hat{\mathbf{z}}.}$$

(b) The extra electric field induced by the changing magnetic field due to the rotating shells is given by $E\,2\pi s = -\frac{d\Phi}{dt} \Rightarrow \mathbf{E} = -\frac{1}{2\pi s}\frac{d\Phi}{dt}\,\hat{\boldsymbol{\phi}}$, and in the region $a < s < b$

$$\Phi = \frac{\mu_0 Q}{2\pi l}(\omega_a - \omega_b)\,\pi a^2 - \frac{\mu_0 Q\omega_b}{2\pi l}\pi\left(s^2 - a^2\right) = \frac{\mu_0 Q}{2l}\left(\omega_a a^2 - \omega_b s^2\right);\ \mathbf{E}(s) = -\frac{1}{2\pi s}\frac{\mu_0 Q}{2l}\left(a^2\frac{d\omega_a}{dt} - s^2\frac{d\omega_b}{dt}\right)\hat{\boldsymbol{\phi}}.$$

In particular,

$$\mathbf{E}(a) = -\frac{\mu_0 Q a}{4\pi l}\left(\frac{d\omega_a}{dt} - \frac{d\omega_b}{dt}\right)\hat{\boldsymbol{\phi}},\ \text{and}\ \mathbf{E}(b) = -\frac{\mu_0 Q}{4\pi l b}\left(a^2\frac{d\omega_a}{dt} - b^2\frac{d\omega_b}{dt}\right)\hat{\boldsymbol{\phi}}.$$

The torque on a shell is $\mathbf{N} = \mathbf{r}\times q\mathbf{E} = qsE\,\hat{\mathbf{z}}$, so

$$\begin{aligned}
\mathbf{N}_a &= Qa\left(-\frac{\mu_0 Q a}{4\pi l}\right)\left(\frac{d\omega_a}{dt} - \frac{d\omega_b}{dt}\right)\hat{\mathbf{z}};\quad \mathbf{L}_a = \int_0^\infty \mathbf{N}_a\,dt = -\frac{\mu_0 Q^2 a^2}{4\pi l}(\omega_a - \omega_b)\,\hat{\mathbf{z}}.\\
\mathbf{N}_b &= -Qb\left(-\frac{\mu_0 Q}{4\pi l b}\right)\left(a^2\frac{d\omega_a}{dt} - b^2\frac{d\omega_b}{dt}\right)\hat{\mathbf{z}};\quad \mathbf{L}_b = \int_0^\infty \mathbf{N}_b\,dt = \frac{\mu_0 Q^2}{4\pi l}\left(a^2\omega_a - b^2\omega_b\right)\hat{\mathbf{z}}.\\
\mathbf{L}_{\text{tot}} &= \mathbf{L}_a + \mathbf{L}_b = \frac{\mu_0 Q^2}{4\pi l}\left(a^2\omega_a - b^2\omega_b - a^2\omega_a + a^2\omega_b\right)\hat{\mathbf{z}} = \boxed{-\frac{\mu_0 Q^2\omega_b}{4\pi l}(b^2 - a^2)\,\hat{\mathbf{z}}.}
\end{aligned}$$

Thus the reduction in the final mechanical angular momentum (b) is equal to the residual angular momentum in the fields (a). ✓

Problem 8.14

$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}$, $(s < R)$; $\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}$, where $\boldsymbol{\mathscr{r}} = (x - a, y, z)$.

$$\boldsymbol{\wp} = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \epsilon_0(\mu_0 n I)\left(\frac{q}{4\pi\epsilon_0}\right)\frac{1}{\mathscr{r}^3}(\boldsymbol{\mathscr{r}}\times\hat{\mathbf{z}}) = \frac{\mu_0 q n I}{4\pi\mathscr{r}^3}\left[y\,\hat{\mathbf{x}} - (x - a)\,\hat{\mathbf{y}}\right].$$

Linear Momentum.

$$\begin{aligned}
\mathbf{p} &= \int\boldsymbol{\wp}\,d\tau = \frac{\mu_0 q n I}{4\pi}\int\frac{y\,\hat{\mathbf{x}} - (x - a)\,\hat{\mathbf{y}}}{[(x - a)^2 + y^2 + z^2]^{3/2}}\,dx\,dy\,dz. \text{ The } \hat{\mathbf{x}} \text{ term is odd in } y\text{; it integrates to zero.}\\
&= -\frac{\mu_0 q n I}{4\pi}\,\hat{\mathbf{y}}\int\frac{(x - a)}{[(x - a)^2 + y^2 + z^2]^{3/2}}\,dx\,dy\,dz. \quad \text{Do the } z \text{ integral first :}
\end{aligned}$$

$$\left.\frac{z}{[(x - a)^2 + y^2]\sqrt{(x - a)^2 + y^2 + z^2}}\right|_{-\infty}^{\infty} = \frac{2}{[(x - a)^2 + y^2]}.$$

$$= -\frac{\mu_0 q n I}{2\pi}\,\hat{\mathbf{y}}\int\frac{(x - a)}{[(x - a)^2 + y^2]}\,dx\,dy. \quad \text{Switch to polar coordinates :}$$

$x = s\cos\phi$, $y = s\sin\phi$, $dx\,dy \Rightarrow s\,ds\,d\phi$; $[(x - a)^2 + y^2] = s^2 + a^2 - 2sa\cos\phi$.

$$= -\frac{\mu_0 q n I}{2\pi}\,\hat{\mathbf{y}}\int\frac{(s\cos\phi - a)}{(s^2 + a^2 - 2sa\cos\phi)}\,s\,ds\,d\phi$$

Now $\displaystyle\int_0^{2\pi}\frac{\cos\phi\,d\phi}{(A + B\cos\phi)} = \frac{2\pi}{B}\left(1 - \frac{A}{\sqrt{A^2 - B^2}}\right)$; $\displaystyle\int_0^{2\pi}\frac{d\phi}{(A + B\cos\phi)} = \frac{2\pi}{\sqrt{A^2 - B^2}}$.

Here $A^2 - B^2 = (s^2 + a^2)^2 - 4s^2a^2 = s^4 + 2s^2a^2 + a^4 - 4s^2a^2 = (s^2 - a^2)^2$; $\sqrt{A^2 - B^2} = a^2 - s^2$.

$$= \frac{\mu_0 q n I}{2a}\,\hat{\mathbf{y}}\int\left[1 - \left(\frac{a^2 + s^2}{a^2 - s^2}\right) + \frac{2a^2}{(a^2 - s^2)}\right]s\,ds = \frac{\mu_0 q n I}{a}\,\hat{\mathbf{y}}\int_0^R s\,ds = \boxed{\frac{\mu_0 q n I R^2}{2a}\,\hat{\mathbf{y}}.}$$

Angular Momentum.

$$\boldsymbol{\ell} = \mathbf{r}\times\boldsymbol{\wp} = \frac{\mu_0 q n I}{4\pi \mathscr{r}^3}\,\mathbf{r}\times[y\,\hat{\mathbf{x}} - (x-a)\,\hat{\mathbf{y}}] = \frac{\mu_0 q n I}{4\pi \mathscr{r}^3}\left\{z(x-a)\,\hat{\mathbf{x}} + zy\,\hat{\mathbf{y}} - [x(x-a)+y^2]\,\hat{\mathbf{z}}\right\}.$$

The $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms are odd in z, and integrate to zero, so

$$\mathbf{L} = -\frac{\mu_0 q n I}{4\pi}\,\hat{\mathbf{z}}\int \frac{x^2+y^2-xa}{[(x-a)^2+y^2+z^2]^{3/2}}\,dx\,dy\,dz.$$ The z integral is the same as before.

$$= -\frac{\mu_0 q n I}{2\pi}\,\hat{\mathbf{z}}\int \frac{x^2+y^2-xa}{[(x-a)^2+y^2]}\,dx\,dy = -\frac{\mu_0 q n I}{2\pi}\,\hat{\mathbf{z}}\int \frac{s-a\cos\phi}{(s^2+a^2-2sa\cos\phi)}\,s^2\,ds\,d\phi$$

$$= -\mu_0 q n I\,\hat{\mathbf{z}}\int\left[\frac{s^2}{a^2-s^2} + \left(1-\frac{a^2+s^2}{a^2-s^2}\right)\right] s\,ds = -\mu_0 q n I\,\hat{\mathbf{z}}\int_0^R \frac{s^2-s^2}{a^2-s^2}\,s\,ds = \boxed{\text{zero.}}$$

Problem 8.15

(a) If we're only interested in the work done on *free* charges and currents, Eq. 8.6 becomes $\dfrac{dW}{dt} = \displaystyle\int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J}_f)\,d\tau$. But $\mathbf{J}_f = \nabla\times\mathbf{H} - \dfrac{\partial\mathbf{D}}{\partial t}$ (Eq. 7.55), so $\mathbf{E}\cdot\mathbf{J}_f = \mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\dfrac{\partial\mathbf{D}}{\partial t}$. From product rule #6, $\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H})$, while $\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$, so $\mathbf{E}\cdot(\nabla\times\mathbf{H}) = -\mathbf{H}\cdot\dfrac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$. Therefore $\mathbf{E}\cdot\mathbf{J}_f = -\mathbf{H}\cdot\dfrac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\dfrac{\partial\mathbf{D}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$, and hence

$$\frac{dW}{dt} = -\int_{\mathcal{V}}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)d\tau - \oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}.$$

This is Poynting's theorem for the fields in matter. Evidently the Poynting vector, representing the power per unit area transported by the fields, is $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, and the rate of change of the electromagnetic energy density is $\dfrac{\partial u_{\text{em}}}{\partial t} = \mathbf{E}\cdot\dfrac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\dfrac{\partial\mathbf{B}}{\partial t}$.

For *linear* media, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \dfrac{1}{\mu}\mathbf{B}$, with ϵ and μ constant (in time); then

$$\frac{\partial u_{\text{em}}}{\partial t} = \epsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) + \frac{1}{2\mu}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}),$$

so $u_{\text{em}} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$. qed

(b) If we're only interested in the force on *free* charges and currents, Eq. 8.15 becomes $\mathbf{f} = \rho_f\mathbf{E} + \mathbf{J}_f\times\mathbf{B}$. But $\rho_f = \nabla\cdot\mathbf{D}$, and $\mathbf{J}_f = \nabla\times\mathbf{H} - \dfrac{\partial\mathbf{D}}{\partial t}$, so $\mathbf{f} = \mathbf{E}(\nabla\cdot\mathbf{D}) + (\nabla\times\mathbf{H})\times\mathbf{B} - \left(\dfrac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B}$. Now $\dfrac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) = \dfrac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} + \mathbf{D}\times\left(\dfrac{\partial\mathbf{B}}{\partial t}\right)$, and $\dfrac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$, so $\dfrac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} = \dfrac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \mathbf{D}\times(\nabla\times\mathbf{E})$, and hence $\mathbf{f} = \mathbf{E}(\nabla\cdot\mathbf{D}) - \mathbf{D}\times(\nabla\times\mathbf{E}) - \mathbf{B}\times(\nabla\times\mathbf{H}) - \dfrac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})$. As before, we can with impunity add the term $\mathbf{H}(\nabla\cdot\mathbf{B})$, so

$$\mathbf{f} = \{[\mathbf{E}(\nabla\cdot\mathbf{D}) - \mathbf{D}\times(\nabla\times\mathbf{E})] + [\mathbf{H}(\nabla\cdot\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{H})]\} - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}).$$

The term in curly brackets can be written as the divergence of a stress tensor (as in Eq. 8.21), and the last term is (minus) the rate of change of the momentum density, $\boldsymbol{\wp} = \mathbf{D}\times\mathbf{B}$.

Chapter 9

Electromagnetic Waves

Problem 9.1

$$\frac{\partial f_1}{\partial z} = -2Ab(z-vt)e^{-b(z-vt)^2};\ \frac{\partial^2 f_1}{\partial z^2} = -2Ab\left[e^{-b(z-vt)^2} - 2b(z-vt)^2e^{-b(z-vt)^2}\right];$$

$$\frac{\partial f_1}{\partial t} = 2Abv(z-vt)e^{-b(z-vt)^2};\ \frac{\partial^2 f_1}{\partial t^2} = 2Abv\left[-ve^{-b(z-vt)^2} + 2bv(z-vt)^2e^{-b(z-vt)^2}\right] = v^2\frac{\partial^2 f_1}{\partial z^2}. \checkmark$$

$$\frac{\partial f_2}{\partial z} = Ab\cos[b(z-vt)];\ \frac{\partial^2 f_2}{\partial z^2} = -Ab^2\sin[b(z-vt)];$$

$$\frac{\partial f_2}{\partial t} = -Abv\cos[b(z-vt)];\ \frac{\partial^2 f_2}{\partial t^2} = -Ab^2v^2\sin[b(z-vt)] = v^2\frac{\partial^2 f_2}{\partial z^2}. \checkmark$$

$$\frac{\partial f_3}{\partial z} = \frac{-2Ab(z-vt)}{[b(z-vt)^2+1]^2};\ \frac{\partial^2 f_3}{\partial z^2} = \frac{-2Ab}{[b(z-vt)^2+1]^2} + \frac{8Ab^2(z-vt)^2}{[b(z-vt)^2+1]^3};$$

$$\frac{\partial f_3}{\partial t} = \frac{2Abv(z-vt)}{[b(z-vt)^2+1]^2};\ \frac{\partial^2 f_3}{\partial t^2} = \frac{-2Abv^2}{[b(z-vt)^2+1]^2} + \frac{8Ab^2v^2(z-vt)^2}{[b(z-vt)^2+1]^3} = v^2\frac{\partial^2 f_3}{\partial z^2}. \checkmark$$

$$\frac{\partial f_4}{\partial z} = -2Ab^2ze^{-b(bz^2+vt)};\ \frac{\partial^2 f_4}{\partial z^2} = -2Ab^2\left[e^{-b(bz^2+vt)} - 2b^2z^2e^{-b(bz^2+vt)}\right];$$

$$\frac{\partial f_4}{\partial t} = -Abve^{-b(bz^2+vt)};\ \frac{\partial^2 f_4}{\partial t^2} = Ab^2v^2e^{-b(bz^2+vt)} \neq v^2\frac{\partial^2 f_4}{\partial z^2}.$$

$$\frac{\partial f_5}{\partial z} = Ab\cos(bz)\cos(bvt)^3;\ \frac{\partial^2 f_5}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3;\ \frac{\partial f_5}{\partial t} = -3Ab^3v^3t^2\sin(bz)\sin(bvt)^3;$$

$$\frac{\partial^2 f_5}{\partial t^2} = -6Ab^3v^3t\sin(bz)\sin(bvt)^3 - 9Ab^6v^6t^4\sin(bz)\cos(bvt)^3 \neq v^2\frac{\partial^2 f_5}{\partial z^2}.$$

Problem 9.2

$$\frac{\partial f}{\partial z} = Ak\cos(kz)\cos(kvt);\ \frac{\partial^2 f}{\partial z^2} = -Ak^2\sin(kz)\cos(kvt);$$

$$\frac{\partial f}{\partial t} = -Akv\sin(kz)\sin(kvt);\ \frac{\partial^2 f}{\partial t^2} = -Ak^2v^2\sin(kz)\cos(kvt) = v^2\frac{\partial^2 f}{\partial z^2}. \checkmark$$

Use the trig identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ to write

$$f = \boxed{\frac{A}{2}\{\sin[k(z+vt)] + \sin[k(z-vt)]\}},$$

which is of the form 9.6, with $g = (A/2)\sin[k(z - vt)]$ and $h = (A/2)\sin[k(z + vt)]$.

Problem 9.3

$$\begin{aligned}
(A_3)^2 &= \left(A_3e^{i\delta_3}\right)\left(A_3e^{-i\delta_3}\right) = \left(A_1e^{i\delta_1} + A_2e^{i\delta_2}\right)\left(A_1e^{-i\delta_1} + A_2e^{-i\delta_2}\right) \\
&= (A_1)^2 + (A_2)^2 + A_1A_2\left(e^{i\delta_1}e^{-i\delta_2} + e^{-i\delta_1}e^{i\delta_2}\right) = (A_1)^2 + (A_2)^2 + A_1A_2 2\cos(\delta_1 - \delta_2); \\
A_3 &= \boxed{\sqrt{(A_1)^2 + (A_2)^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}.} \\
A_3e^{i\delta_3} &= A_3(\cos\delta_3 + i\sin\delta_3) = A_1(\cos\delta_1 + i\sin\delta_1) + A_2(\cos\delta_2 + i\sin\delta_2) \\
&= (A_1\cos\delta_1 + A_2\cos\delta_2) + i(A_1\sin\delta_1 + A_2\sin\delta_2). \quad \tan\delta_3 = \frac{A_3\sin\delta_3}{A_3\cos\delta_3} = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}; \\
\delta_3 &= \boxed{\tan^{-1}\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right).}
\end{aligned}$$

Problem 9.4

The wave equation (Eq. 9.2) says $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$. Look for solutions of the form $f(z,t) = Z(z)T(t)$. Plug this in: $T\frac{d^2Z}{dz^2} = \frac{1}{v^2}Z\frac{d^2T}{dt^2}$. Divide by ZT : $\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{v^2T}\frac{d^2T}{dt^2}$. The left side depends only on z, and the right side only on t, so both must be constant. Call the constant $-k^2$.

$$\left\{\begin{array}{lcll}
\dfrac{d^2Z}{dz^2} &=& -k^2Z & \Rightarrow\ Z(z) = Ae^{ikz} + Be^{-ikz}, \\[2ex]
\dfrac{d^2T}{dt^2} &=& -(kv)^2T & \Rightarrow\ T(t) = Ce^{ikvt} + De^{-ikvt}.
\end{array}\right\}$$

(Note that k must be *real*, else Z and T blow up; with no loss of generality we can assume k is *positive.*)
$f(z,t) = \left(Ae^{ikz} + Be^{-ikz}\right)\left(Ce^{ikvt} + De^{-ikvt}\right) = A_1e^{i(kz+kvt)} + A_2e^{i(kz-kvt)} + A_3e^{i(-kz+kvt)} + A_4e^{i(-kz-kvt)}$.
The general linear combination of separable solutions is therefore

$$f(z,t) = \int_0^\infty \left[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)} + A_3(k)e^{i(-kz+\omega t)} + A_4(k)e^{i(-kz-\omega t)}\right] dk,$$

where $\omega \equiv kv$. But we can combine the third term with the first, by allowing k to run *negative* ($\omega = |k|v$ remains positive); likewise the second and the fourth:

$$f(z,t) = \int_{-\infty}^\infty \left[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)}\right] dk.$$

Because (in the end) we shall only want the the *real part* of f, it suffices to keep only *one* of these terms (since k goes negative, both terms include waves traveling in both directions); the second is traditional (though either would do). Specifically,

$$\mathrm{Re}(f) = \int_{-\infty}^\infty [\mathrm{Re}(A_1)\cos(kz+\omega t) - \mathrm{Im}(A_1)\sin(kz+\omega t) + \mathrm{Re}(A_2)\cos(kz-\omega t) - \mathrm{Im}(A_2)\sin(kz-\omega t)]\, dk.$$

The first term, $\cos(kz+\omega t) = \cos(-kz-\omega t)$, combines with the third, $\cos(kz-\omega t)$, since the negative k is picked up in the other half of the range of integration, and the second, $\sin(kz+\omega t) = -\sin(-kz-\omega t)$, combines with the fourth for the same reason. So the general solution, for our purposes, can be written in the form

$$\tilde{f}(z,t) = \int_{-\infty}^\infty \tilde{A}(k)e^{i(kz-\omega t)}\, dk \quad \text{qed (the tildes remind us that we want the real part).}$$

Problem 9.5

Equation 9.26 $\Rightarrow g_I(-v_1t)+h_R(v_1t)=g_T(-v_2t)$. Now $\frac{\partial g_I}{\partial z}=-\frac{1}{v_1}\frac{\partial g_I}{\partial t}$; $\frac{\partial h_R}{\partial z}=\frac{1}{v_1}\frac{\partial h_R}{\partial t}$; $\frac{\partial g_T}{\partial z}=-\frac{1}{v_2}\frac{\partial g_T}{\partial t}$.

Equation 9.27 $\Rightarrow -\frac{1}{v_1}\frac{\partial g_I(-v_1t)}{\partial t}+\frac{1}{v_1}\frac{\partial h_R(v_1t)}{\partial t}=-\frac{1}{v_2}\frac{\partial g_T(-v_2t)}{\partial t}\Rightarrow g_I(-v_1t)-h_R(v_1t)=\frac{v_1}{v_2}g_T(-v_2t)+\kappa$ (where κ is a constant).

Adding these equations, we get $2g_I(-v_1t)=\left(1+\frac{v_1}{v_2}\right)g_T(-v_2t)+\kappa$, or $g_T(-v_2t)=\left(\frac{2v_2}{v_1+v_2}\right)g_I(-v_1t)+\kappa'$ (where $\kappa'\equiv-\kappa\frac{v_2}{v_1+v_2}$). Now $g_I(z,t)$, $g_T(z,t)$, and $h_R(z,t)$ are each functions of a single variable u (in the first case $u=z-v_1t$, in the second $u=z-v_2t$, and in the third $u=z+v_1t$). Thus

$$\boxed{g_T(u)=\left(\frac{2v_2}{v_1+v_2}\right)g_I(v_1u/v_2)+\kappa'.}$$

Multiplying the first equation by v_1/v_2 and subtracting, $\left(1-\frac{v_1}{v_2}\right)g_I(-v_1t)-\left(1+\frac{v_1}{v_2}\right)h_R(v_1t)=\kappa\Rightarrow$

$$h_R(v_1t)=\left(\frac{v_2-v_1}{v_1+v_2}\right)g_I(-v_1t)-\kappa\left(\frac{v_2}{v_1+v_2}\right),\text{ or }\boxed{h_R(u)=\left(\frac{v_2-v_1}{v_1+v_2}\right)g_I(-u)+\kappa'.}$$

[The notation is tricky, so here's an example: for a sinusoidal wave,

$$\begin{cases} g_I = A_I\cos(k_1z-\omega t) = A_I\cos[k_1(z-v_1t)] \Rightarrow g_I(u)=A_I\cos(k_1u). \\ g_T = A_T\cos(k_2z-\omega t) = A_T\cos[k_2(z-v_2t)] \Rightarrow g_T(u)=A_T\cos(k_2u). \\ h_R = A_R\cos(-k_1z-\omega t) = A_R\cos[-k_1(z+v_1t)] \Rightarrow h_R(u)=A_R\cos(-k_1u). \end{cases}$$

Here $\kappa'=0$, and the boundary conditions say $\frac{A_T}{A_I}=\frac{2v_2}{v_1+v_2}$, $\frac{A_R}{A_I}=\frac{v_2-v_1}{v_1+v_2}$ (same as Eq. 9.32), and $\frac{v_1}{v^2}k_1=k_2$ (consistent with Eq. 9.24).]

Problem 9.6

(a) $T\sin\theta_+-T\sin\theta_-=ma\Rightarrow\boxed{T\left(\left.\frac{\partial f}{\partial z}\right|_{0+}-\left.\frac{\partial f}{\partial z}\right|_{0-}\right)=m\left.\frac{\partial^2 f}{\partial t^2}\right|_0.}$

(b) $\tilde{A}_I+\tilde{A}_R=\tilde{A}_T$; $T[ik_2\tilde{A}_T-ik_1(\tilde{A}_I-\tilde{A}_R)]=m(-\omega^2\tilde{A}_T)$, or $k_1(\tilde{A}_I-\tilde{A}_R)=\left(k_2-\frac{im\omega^2}{T}\right)\tilde{A}_T$.

Multiply first equation by k_1 and add: $2k_1\tilde{A}_I=\left(k_1+k_2-i\frac{m\omega^2}{T}\right)\tilde{A}_T$, or $\tilde{A}_T=\left(\frac{2k_1}{k_1+k_2-im\omega^2/T}\right)\tilde{A}_I$.

$$\tilde{A}_R=\tilde{A}_T-\tilde{A}_I=\frac{2k_1-(k_1+k_2-im\omega^2/T)}{k_1+k_2-im\omega^2/T}\tilde{A}_I=\left(\frac{k_1-k_2+im\omega^2/T}{k_1+k_2-im\omega^2/T}\right)\tilde{A}_I.$$

If the second string is massless, so $v_2=\sqrt{T/\mu_2}=\infty$, then $k_2/k_1=0$, and we have $\tilde{A}_T=\left(\frac{2}{1-i\beta}\right)\tilde{A}_I$, $\tilde{A}_R=\left(\frac{1+i\beta}{1-i\beta}\right)\tilde{A}_I$, where $\beta\equiv\frac{m\omega^2}{k_1T}=\frac{m(k_1v_1)^2}{k_1T}=\frac{mk_1}{T}\frac{T}{\mu_1}$, or $\boxed{\beta=m\frac{k_1}{\mu_1}.}$ Now $\left(\frac{1+i\beta}{1-i\beta}\right)=Ae^{i\phi}$, with $A^2=\left(\frac{1+i\beta}{1-i\beta}\right)\left(\frac{1-i\beta}{1+i\beta}\right)=1\Rightarrow A=1$, and $e^{i\phi}=\frac{(1+i\beta)^2}{(1-i\beta)(1+i\beta)}=\frac{1+2i\beta-\beta^2}{1+\beta^2}\Rightarrow$

$\tan\phi=\frac{2\beta}{1-\beta^2}$. Thus $A_Re^{i\delta_R}=e^{i\phi}A_Ie^{i\delta_I}\Rightarrow\boxed{A_R=A_I,}\ \boxed{\delta_R=\delta_I+\tan^{-1}\left(\frac{2\beta}{1-\beta^2}\right).}$

Similarly, $\left(\frac{2}{1-i\beta}\right)=Ae^{i\phi}\Rightarrow A^2=\left(\frac{2}{1-i\beta}\right)\left(\frac{2}{1+i\beta}\right)=\frac{4}{1+\beta^2}\Rightarrow A=\frac{2}{\sqrt{1+\beta^2}}.$

$Ae^{i\phi} = \dfrac{2(1+i\beta)}{(1-i\beta)(1+i\beta)} = \dfrac{2(1+i\beta)}{(1+\beta^2)} \Rightarrow \tan\phi = \beta$. So $A_T e^{i\delta_T} = \dfrac{2}{\sqrt{1+\beta^2}} e^{i\phi} A_I e^{i\delta_I}$;

$\boxed{A_T = \dfrac{2}{\sqrt{1+\beta^2}} A_I;}$ $\boxed{\delta_T = \delta_I + \tan^{-1}\beta.}$

Problem 9.7

(a) $F = T\dfrac{\partial^2 f}{\partial z^2}\Delta z - \gamma\dfrac{\partial f}{\partial t}\Delta z = \mu\Delta z\dfrac{\partial^2 f}{\partial t^2}$, or $\boxed{T\dfrac{\partial^2 f}{\partial z^2} = \mu\dfrac{\partial^2 f}{\partial t^2} + \gamma\dfrac{\partial f}{\partial t}.}$

(b) Let $\tilde{f}(z,t) = \tilde{F}(z)e^{-i\omega t}$; then $Te^{-i\omega t}\dfrac{d^2\tilde{F}}{dz^2} = \mu(-\omega^2)\tilde{F}e^{-i\omega t} + \gamma(-i\omega)\tilde{F}e^{-i\omega t} \Rightarrow$

$T\dfrac{d^2\tilde{F}}{dz^2} = -\omega(\mu\omega + i\gamma)\tilde{F}$, $\dfrac{d^2\tilde{F}}{dz^2} = -\tilde{k}^2\tilde{F}$, where $\tilde{k}^2 \equiv \dfrac{\omega}{T}(\mu\omega + i\gamma)$. Solution : $\tilde{F}(z) = \tilde{A}e^{i\tilde{k}z} + \tilde{B}e^{-i\tilde{k}z}$.

Resolve $\tilde{k}$ into its real and imaginary parts: $\tilde{k} = k + i\kappa \Rightarrow \tilde{k}^2 = k^2 - \kappa^2 + 2ik\kappa = \dfrac{\omega}{T}(\mu\omega + i\gamma)$.

$2k\kappa = \dfrac{\omega\gamma}{T} \Rightarrow \kappa = \dfrac{\omega\gamma}{2kT}$; $k^2 - \kappa^2 = k^2 - \left(\dfrac{\omega\gamma}{2T}\right)^2\dfrac{1}{k^2} = \dfrac{\mu\omega^2}{T}$; or $k^4 - k^2(\mu\omega^2/T) - (\omega\gamma/2T)^2 = 0 \Rightarrow$

$k^2 = \dfrac{1}{2}\left[(\mu\omega^2/T) \pm \sqrt{(\mu\omega^2/T)^2 + 4(\omega\gamma/2T)^2}\right] = \dfrac{\mu\omega^2}{2T}\left[1 \pm \sqrt{1+(\gamma/\mu\omega)^2}\right]$. But k is real, so k^2 is positive, so we need the plus sign: $k = \omega\sqrt{\dfrac{\mu}{2T}}\sqrt{1+\sqrt{1+(\gamma/\mu\omega)^2}}$. $\kappa = \dfrac{\omega\gamma}{2kT} = \dfrac{\gamma}{\sqrt{2T\mu}}\left[1+\sqrt{1+(\gamma/\mu\omega)^2}\right]^{-1/2}$.

Plugging this in, $\tilde{F} = Ae^{i(k+i\kappa)z} + Be^{-i(k+i\kappa)z} = Ae^{-\kappa z}e^{ikz} + Be^{\kappa z}e^{-ikz}$. But the B term gives an exponentially *increasing* function, which we don't want (I assume the waves are propagating in the $+z$ direction), so $B = 0$, and the solution is $\boxed{\tilde{f}(z,t) = \tilde{A}e^{-\kappa z}e^{i(kz-\omega t)}.}$ (The actual displacement of the string is the real part of this, of course.)

(c) The wave is attenuated by the factor $e^{-\kappa z}$, which becomes $1/e$ when

$z = \dfrac{1}{\kappa} = \boxed{\dfrac{\sqrt{2T\mu}}{\gamma}\sqrt{1+\sqrt{1+(\gamma/\mu\omega)^2}};}$ this is the characteristic penetration depth.

(d) This is the same as before, except that $k_2 \to k + i\kappa$. From Eq. 9.29, $\tilde{A}_R = \left(\dfrac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\tilde{A}_I$;

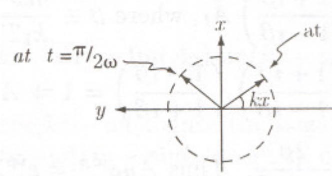
$\left(\dfrac{A_R}{A_I}\right)^2 = \left(\dfrac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\left(\dfrac{k_1 - k + i\kappa}{k_1 + k - i\kappa}\right) = \dfrac{(k_1-k)^2 + \kappa^2}{(k_1+k)^2 + \kappa^2}$. $\boxed{A_R = \sqrt{\dfrac{(k_1-k)^2+\kappa^2}{(k_1+k)^2+\kappa^2}}A_I}$

(where $k_1 = \omega/v_1 = \omega\sqrt{\mu_1/T}$, while k and κ are defined in part b). Meanwhile

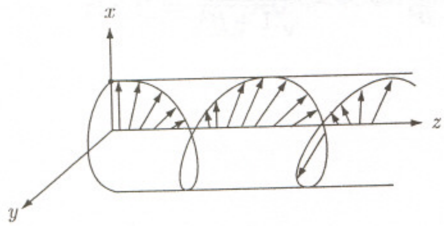
$\left(\dfrac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right) = \dfrac{(k_1 - k - i\kappa)(k_1 + k + i\kappa)}{(k_1+k)^2 + \kappa^2} = \dfrac{(k_1)^2 - k^2 - \kappa^2 - 2i\kappa k_1}{(k_1+k)^2+\kappa^2} \Rightarrow \boxed{\delta_R = \tan^{-1}\left(\dfrac{-2k_1\kappa}{(k_1)^2 - k^2 - \kappa^2}\right).}$

Problem 9.8

(a) $\mathbf{f}_v(z,t) = A\cos(kz - \omega t)\,\hat{\mathbf{x}}$; $\mathbf{f}_h(z,t) = A\cos(kz - \omega t + 90°)\,\hat{\mathbf{y}} = -A\sin(kz - \omega t)\,\hat{\mathbf{y}}$. Since $f_v^2 + f_h^2 = A^2$, the vector sum $\mathbf{f} = \mathbf{f}_v + \mathbf{f}_h$ lies on a circle of radius A. At time $t = 0$, $\mathbf{f} = A\cos(kz)\,\hat{\mathbf{x}} - A\sin(kz)\,\hat{\mathbf{y}}$. At time $t = \pi/2\omega$, $\mathbf{f} = A\cos(kz - 90°)\,\hat{\mathbf{x}} - A\sin(kz - 90°)\,\hat{\mathbf{y}} = A\sin(kz)\,\hat{\mathbf{x}} + A\cos(kz)\,\hat{\mathbf{y}}$. Evidently it circles $\boxed{\text{counterclockwise}}$. To make a wave circling the other way, use $\delta_h = -90°$.



(b)

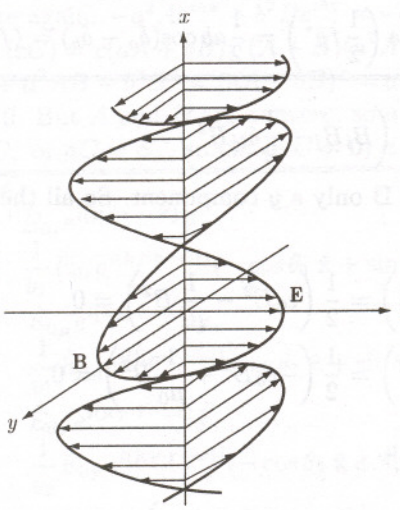


(c) Shake it around in a circle, instead of up and down.

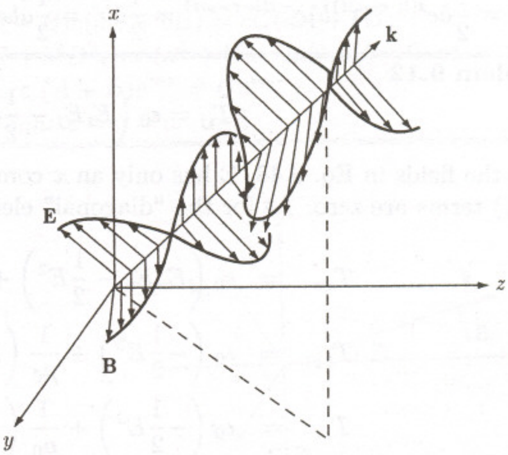
Problem 9.9

(a) $\boxed{\mathbf{k} = -\frac{\omega}{c}\hat{\mathbf{x}};\ \hat{\mathbf{n}} = \hat{\mathbf{z}}.}$ $\mathbf{k}\cdot\mathbf{r} = \left(-\frac{\omega}{c}\hat{\mathbf{x}}\right)\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = -\frac{\omega}{c}x;$ $\mathbf{k}\times\hat{\mathbf{n}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}.$

$$\boxed{\mathbf{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{z}};\quad \mathbf{B}(x,t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{y}}.}$$



(a)



(b)

(b) $\boxed{\mathbf{k} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}}{\sqrt{3}}\right);\ \hat{\mathbf{n}} = \frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}}{\sqrt{2}}.}$ (Since $\hat{\mathbf{n}}$ is parallel to the $x\,z$ plane, it must have the form $\alpha\,\hat{\mathbf{x}} + \beta\,\hat{\mathbf{z}}$; since $\hat{\mathbf{n}}\cdot\mathbf{k} = 0, \beta = -\alpha$; and since it is a unit vector, $\alpha = 1/\sqrt{2}$.)

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{\sqrt{3}c}(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}) = \frac{\omega}{\sqrt{3}c}(x+y+z);\ \hat{\mathbf{k}}\times\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 1 & 1 & 1\\ 1 & 0 & -1\end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{\mathbf{x}}+2\,\hat{\mathbf{y}}-\hat{\mathbf{z}}).$$

$$\boxed{\begin{aligned}\mathbf{E}(x,y,z,t) &= E_0\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z)-\omega t\right]\left(\frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}}{\sqrt{2}}\right);\\ \mathbf{B}(x,y,z,t) &= \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z)-\omega t\right]\left(\frac{-\hat{\mathbf{x}}+2\hat{\mathbf{y}}-\hat{\mathbf{z}}}{\sqrt{6}}\right).\end{aligned}}$$

Problem 9.10

$P = \frac{I}{c} = \frac{1.3\times 10^3}{3.0\times 10^8} = \boxed{4.3\times 10^{-6}\ \mathrm{N/m^2}.}$ For a perfect reflector the pressure is twice as great: $\boxed{8.6\times 10^{-6}\ \mathrm{N/m^2}.}$ Atmospheric pressure is $1.03\times 10^5\ \mathrm{N/m^2}$, so the pressure of light on a reflector is $(8.6\times 10^{-6})/(1.03\times 10^5) = \boxed{8.3\times 10^{-11}\ \text{atmospheres.}}$

Problem 9.11

$$\begin{aligned}\langle fg\rangle &= \frac{1}{T}\int_0^T a\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_a)\,b\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_b)\,dt\\ &= \frac{ab}{2T}\int_0^T[\cos(2\mathbf{k}\cdot\mathbf{r}-2\omega t+\delta_a+\delta_b)+\cos(\delta_a-\delta_b)]\,dt = \frac{ab}{2T}\cos(\delta_a-\delta_b)T = \frac{1}{2}ab\cos(\delta_a-\delta_b).\end{aligned}$$

Meanwhile, in the complex notation: $\tilde f = \tilde a e^{i\mathbf{k}\cdot\mathbf{r}-\omega t)}$, $\tilde g = \tilde b e^{i\mathbf{k}\cdot\mathbf{r}-\omega t)}$, where $\tilde a = a e^{i\delta_a}$, $\tilde b = b e^{i\delta_b}$. So $\frac{1}{2}\tilde f\tilde g^* = \frac{1}{2}\tilde a e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\tilde b^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \frac{1}{2}\tilde a\tilde b^* = \frac{1}{2}abe^{i(\delta_a-\delta_b)}$, $\quad \mathrm{Re}\left(\frac{1}{2}\tilde f\tilde g^*\right) = \frac{1}{2}ab\cos(\delta_a-\delta_b) = \langle fg\rangle$. qed

Problem 9.12

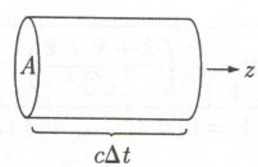
$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right).$$

With the fields in Eq. 9.48, **E** has only an x component, and **B** only a y component. So all the "off-diagonal" ($i \neq j$) terms are zero. As for the "diagonal" elements:

$$\begin{aligned}T_{xx} &= \epsilon_0\left(E_xE_x - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = \frac{1}{2}\left(\epsilon_0E^2 - \frac{1}{\mu_0}B^2\right) = 0.\\ T_{yy} &= \epsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_yB_y - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\epsilon_0E^2 + \frac{1}{\mu_0}B^2\right) = 0.\\ T_{zz} &= \epsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = -u.\end{aligned}$$

So $\boxed{T_{zz} = -\epsilon_0E_0^2\cos^2(kz-\omega t+\delta)}$ (all other elements zero).

The momentum of these fields is in the z direction, and it is being *transported* in the z direction, so *yes*, it does make sense that T_{zz} should be the only nonzero element in T_{ij}. According to Sect. 8.2.3, $-\overleftrightarrow{T}\cdot d\mathbf{a}$ is the rate at which momentum crosses an area $d\mathbf{a}$. Here we have *no* momentum crossing areas oriented in the x or y direction; the momentum per unit time per unit area flowing across a surface oriented in the z direction is $-T_{zz} = u = \wp c$ (Eq. 9.59), so $\Delta p = \wp cA\Delta t$, and hence $\Delta p/\Delta t = \wp cA =$ momentum per unit time crossing area A. Evidently $\boxed{\text{momentum flux density = energy density.}}$ ✓



Problem 9.13

$$R = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 \text{ (Eq. 9.86)} \Rightarrow \boxed{R = \left(\frac{1-\beta}{1+\beta}\right)^2} \text{ (Eq. 9.82), where } \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}.\quad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \text{ (Eq. 9.87)}$$

$$\Rightarrow \boxed{T = \beta\left(\frac{2}{1+\beta}\right)^2} \text{ (Eq. 9.82). [Note that } \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\mu_1}{\mu_2}\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}\frac{v_2}{v_1} = \frac{\mu_1}{\mu_2}\left(\frac{v_1}{v_2}\right)^2\frac{v_2}{v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} = \beta.]$$

$$T + R = \frac{1}{(1+\beta)^2}\left[4\beta + (1-\beta)^2\right] = \frac{1}{(1+\beta)^2}(4\beta + 1 - 2\beta + \beta^2) = \frac{1}{(1+\beta)^2}(1 + 2\beta + \beta^2) = 1. \checkmark$$

Problem 9.14

Equation 9.78 is replaced by $\tilde{E}_{0_I}\hat{\mathbf{x}} + \tilde{E}_{0_R}\hat{\mathbf{n}}_R = \tilde{E}_{0_T}\hat{\mathbf{n}}_T$, and Eq. 9.80 becomes $\tilde{E}_{0_I}\hat{\mathbf{y}} - \tilde{E}_{0_R}(\hat{\mathbf{z}}\times\hat{\mathbf{n}}_R) = \beta\tilde{E}_{0_T}(\hat{\mathbf{z}}\times\hat{\mathbf{n}}_T)$. The y component of the first equation is $\tilde{E}_{0_R}\sin\theta_R = \tilde{E}_{0_T}\sin\theta_T$; the x component of the second is $\tilde{E}_{0_R}\sin\theta_R = -\beta\tilde{E}_{0_T}\sin\theta_T$. Comparing these two, we conclude that $\sin\theta_R = \sin\theta_T = 0$, and hence $\theta_R = \theta_T = 0$. qed

Problem 9.15

$Ae^{iax} + Be^{ibx} = Ce^{icx}$ for all x, so (using $x = 0$), $A + B = C$.

Differentiate: $iaAe^{iax} + ibBe^{ibx} = icCe^{icx}$, so (using $x = 0$), $aA + bB = cC$.

Differentiate again: $-a^2Ae^{iax} - b^2Be^{ibx} = -c^2Ce^{icx}$, so (using $x = 0$), $a^2A + b^2B = c^2C$.

$a^2A + b^2B = c(cC) = c(aA + bB)$; $(A + B)(a^2A + b^2B) = (A + B)c(aA + bB) = cC(aA + bB)$;

$a^2A^2 + b^2AB + a^2AB + b^2B^2 = (aA + bB)^2 = a^2A^2 + 2abAB + b^2B^2$, or $(a^2 + b^2 - 2ab)AB = 0$, or

$(a - b)^2AB = 0$. But A and B are nonzero, so $a = b$. Therefore $(A + B)e^{iax} = Ce^{icx}$.

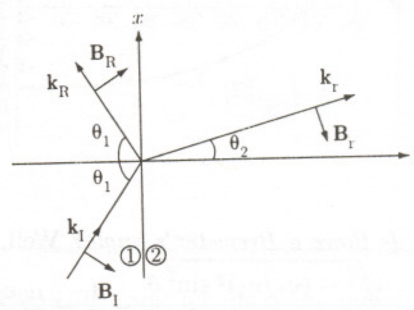
$a(A + B) = cC$, or $aC = cC$, so (since $C \neq 0$) $a = c$. *Conclusion:* $a = b = c$. qed

Problem 9.16

$$\left\{\begin{array}{lcl}\tilde{\mathbf{E}}_I &=& \tilde{E}_{0_I}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}},\\ \tilde{\mathbf{B}}_I &=& \dfrac{1}{v_1}\tilde{E}_{0_I}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}(-\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}});\end{array}\right\}$$

$$\left\{\begin{array}{lcl}\tilde{\mathbf{E}}_R &=& \tilde{E}_{0_R}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}},\\ \tilde{\mathbf{B}}_R &=& \dfrac{1}{v_1}\tilde{E}_{0_R}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}(\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}});\end{array}\right\}$$

$$\left\{\begin{array}{lcl}\tilde{\mathbf{E}}_T &=& \tilde{E}_{0_T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}},\\ \tilde{\mathbf{B}}_T &=& \dfrac{1}{v_2}\tilde{E}_{0_T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_2\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}});\end{array}\right\}$$



Boundary conditions: $\left\{\begin{array}{ll}\text{(i) } \epsilon_1E_1^\perp = \epsilon_2E_2^\perp, & \text{(iii) } \mathbf{E}_1^\parallel = \mathbf{E}_2^\parallel,\\ \text{(ii) } B_1^\perp = B_2^\perp, & \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_1^\parallel = \frac{1}{\mu_2}\mathbf{B}_2^\parallel.\end{array}\right.$

Law of refraction: $\dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{v_2}{v_1}$. [Note: $\mathbf{k}_I\cdot\mathbf{r} - \omega t = \mathbf{k}_R\cdot\mathbf{r} - \omega t = \mathbf{k}_T\cdot\mathbf{r} - \omega t$, at $z = 0$, so we can drop all exponential factors in applying the boundary conditions.]

Boundary condition (i): $0 = 0$ (trivial). Boundary condition (iii): $\boxed{\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}.}$

Boundary condition (ii): $\dfrac{1}{v_1}\tilde{E}_{0_I}\sin\theta_1 + \dfrac{1}{v_1}\tilde{E}_{0_R}\sin\theta_1 = \dfrac{1}{v_2}\tilde{E}_{0_T}\sin\theta_2 \Rightarrow \tilde{E}_{0_I} + \tilde{E}_{0_R} = \left(\dfrac{v_1\sin\theta_2}{v_2\sin\theta_1}\right)\tilde{E}_{0_T}$.

But the term in parentheses is 1, by the law of refraction, so this is the same as (ii).

Boundary condition (iv): $\dfrac{1}{\mu_1}\left[\dfrac{1}{v_1}\tilde{E}_{0_I}(-\cos\theta_1) + \dfrac{1}{v_1}\tilde{E}_{0_R}\cos\theta_1\right] = \dfrac{1}{\mu_2v_2}\tilde{E}_{0_T}(-\cos\theta_2) \Rightarrow$

$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \left(\dfrac{\mu_1v_1\cos\theta_2}{\mu_2v_2\cos\theta_1}\right)\tilde{E}_{0_T}$. Let $\boxed{\alpha \equiv \dfrac{\cos\theta_2}{\cos\theta_1};\ \beta \equiv \dfrac{\mu_1v_1}{\mu_2v_2}.}$ Then $\boxed{\tilde{E}_{0_I} - \tilde{E}_{0_R} = \alpha\beta\tilde{E}_{0_T}.}$

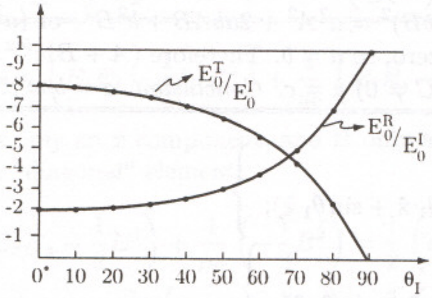
Solving for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$: $2\tilde{E}_{0_I} = (1 + \alpha\beta)\tilde{E}_{0_T} \Rightarrow \tilde{E}_{0_T} = \left(\dfrac{2}{1+\alpha\beta}\right)\tilde{E}_{0_I}$;

$\tilde{E}_{0_R} = \tilde{E}_{0_T} - \tilde{E}_{0_I} = \left(\dfrac{2}{1+\alpha\beta} - \dfrac{1+\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0_I} \Rightarrow \tilde{E}_{0_R} = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0_I}$.

Since α and β are positive, it follows that $2/(1+\alpha\beta)$ is positive, and hence the *transmitted* wave is *in phase* with the incident wave, and the (real) amplitudes are related by $\boxed{E_{0_T} = \left(\dfrac{2}{1+\alpha\beta}\right)E_{0_I}.}$ The *reflected* wave is

in phase if $\alpha\beta < 1$ and 180° out of phase if $\alpha\beta < 1$; the (real) amplitudes are related by $\boxed{E_{0_R} = \left|\dfrac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}.}$
These are the **Fresnel equations** for polarization perpendicular to the plane of incidence.

To construct the graphs, note that $\alpha\beta = \beta\dfrac{\sqrt{1-\sin^2\theta/\beta^2}}{\cos\theta} = \dfrac{\sqrt{\beta^2-\sin^2\theta}}{\cos\theta}$, where θ is the angle of incidence, so, for $\beta = 1.5$, $\alpha\beta = \dfrac{\sqrt{2.25-\sin^2\theta}}{\cos\theta}$.



Is there a Brewster's angle? Well, $E_{0_R} = 0$ would mean that $\alpha\beta = 1$, and hence that $\alpha = \dfrac{\sqrt{1-(v_2/v_1)^2\sin^2\theta}}{\cos\theta} = \dfrac{1}{\beta} = \dfrac{\mu_2 v_2}{\mu_1 v_1}$, or $1 - \left(\dfrac{v_2}{v_1}\right)^2 \sin^2\theta = \left(\dfrac{\mu_2 v_2}{\mu_1 v_1}\right)^2 \cos^2\theta$, so $1 = \left(\dfrac{v_2}{v_1}\right)^2 \left[\sin^2\theta + (\mu_2/\mu_1)^2\cos^2\theta\right]$. Since $\mu_1 \approx \mu_2$, this means $1 \approx (v_2/v_1)^2$, which is only true for optically indistinguishable media, in which case there is of *course* no reflection—but that would be true at *any* angle, not just at a special "Brewster's angle". [If μ_2 were substantially different from μ_1, and the relative velocities were just right, it *would* be possible to get a Brewster's angle for this case, at

$$\left(\frac{v_1}{v_2}\right)^2 = 1 - \cos^2\theta + \left(\frac{\mu_2}{\mu_1}\right)^2 \cos^2\theta \Rightarrow \cos^2\theta = \frac{(v_1/v_2)^2 - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\mu_2\epsilon_2/\mu_1\epsilon_1) - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\epsilon_2/\epsilon_1) - (\mu_1/\mu_2)}{(\mu_2/\mu_1) - (\mu_1/\mu_2)}.$$

But the media would be very peculiar.]

By the same token, δ_R is either always 0, or always π, for a given interface—it does not switch over as you change θ, the way it does for polarization in the plane of incidence. In particular, if $\beta = 3/2$, then $\alpha\beta > 1$, for

$$\alpha\beta = \frac{\sqrt{2.25 - \sin^2\theta}}{\cos\theta} > 1 \text{ if } 2.25 - \sin^2\theta > \cos^2\theta, \text{ or } 2.25 > \sin^2\theta + \cos^2\theta = 1. \checkmark$$

In general, for $\beta > 1$, $\alpha\beta > 1$, and hence $\delta_R = \pi$. For $\beta < 1$, $\alpha\beta < 1$, and $\delta_R = 0$.

At normal incidence, $\alpha = 1$, so Fresnel's equations reduce to $E_{0_T} = \left(\dfrac{2}{1+\beta}\right) E_{0_I}$; $E_{0_R} = \left|\dfrac{1-\beta}{1+\beta}\right| E_{0_I}$, consistent with Eq. 9.82.

Reflection and Transmission coefficients: $\boxed{R = \left(\dfrac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$ Referring to Eq. 9.116,

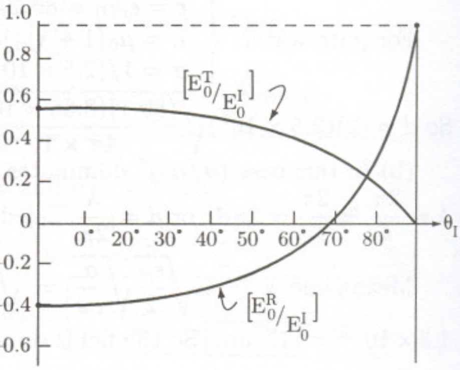
$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \boxed{\alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2.}$$

$$R + T = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1 - 2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1. \checkmark$$

Problem 9.17

Equation 9.106 $\Rightarrow \beta = 2.42$; Eq. 9.110 $\Rightarrow$

$\alpha = \dfrac{\sqrt{1-(\sin\theta/2.42)^2}}{\cos\theta}$.



(a) $\theta = 0 \Rightarrow \alpha = 1$. Eq. 9.109 $\Rightarrow \left(\dfrac{E_{0_R}}{E_{0_I}}\right) = \dfrac{\alpha-\beta}{\alpha+\beta} = \dfrac{1-2.42}{1+2.42} = -\dfrac{1.42}{3.42} = \boxed{-0.415;}$

$\left(\dfrac{E_{0_T}}{E_{0_I}}\right) = \dfrac{2}{\alpha+\beta} = \dfrac{2}{3.42} = \boxed{0.585.}$

(b) Equation 9.112 $\Rightarrow \theta_B = \tan^{-1}(2.42) = \boxed{67.5°.}$

(c) $E_{0_R} = E_{0_T} \Rightarrow \alpha - \beta = 2; \alpha = \beta + 2 = 4.42;$

$(4.42)^2\cos^2\theta = 1 - \sin^2\theta/(2.42)^2;$

$(4.42)^2(1-\sin^2\theta) = (4.42)^2 - (4.42)^2\sin^2\theta$

$= 1 - 0.171\sin^2\theta;\ 19.5 - 1 = (19.5 - 0.17)\sin^2\theta;$

$18.5 = 19.3\sin^2\theta;\ \sin^2\theta = 18.5/19.3 = 0.959;$

$\sin\theta = 0.979;\ \boxed{\theta = 78.3°.}$

Problem 9.18

(a) Equation 9.120 $\Rightarrow \tau = \epsilon/\sigma$. Now $\epsilon = \epsilon_0\epsilon_r$ (Eq. 4.34), $\epsilon_r \cong n^2$ (Eq. 9.70), and for glass the index of refraction is typically around 1.5, so $\epsilon \approx (1.5)^2 \times 8.85\times10^{-12} = 2\times10^{-11}\ \mathrm{C^2/N\,m^2}$, while $\sigma = 1/\rho \approx 10^{-12}\ \Omega\,\mathrm{m}$ (Table 7.1). Then $\tau = (2\times10^{-11})/10^{-12} = \boxed{20\ \mathrm{s}.}$ (But the resistivity of glass varies enormously from one type to another, so this answer could be off by a factor of 100 in either direction.)

(b) For silver, $\rho = 1.59\times10^{-8}$ (Table 7.1), and $\epsilon \approx \epsilon_0$, so $\omega\epsilon = 2\pi\times10^{10}\times8.85\times10^{-12} = 0.56$. Since $\sigma = 1/\rho = 6.25\times10^7 \gg \omega\epsilon$, the skin depth (Eq. 9.128) is

$$d = \frac{1}{\kappa} \cong \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2}{2\pi\times10^{10}\times6.25\times10^7\times4\pi\times10^{-7}}} = 6.4\times10^{-7}\ \mathrm{m} = 6.4\times10^{-4}\ \mathrm{mm}.$$

I'd plate silver to a depth of about $\boxed{0.001\ \mathrm{mm};}$ there's no point in making it any thicker, since the fields don't penetrate much beyond this anyway.

(c) For copper, Table 7.1 gives $\sigma = 1/(1.68\times10^{-8}) = 6\times10^7$, $\omega\epsilon_0 = (2\pi\times10^6)\times(8.85\times10^{-12}) = 6\times10^{-5}$. Since $\sigma \gg \omega\epsilon$, Eq. 9.126 $\Rightarrow k \approx \sqrt{\dfrac{\omega\sigma\mu}{2}}$, so (Eq. 9.129)

$$\lambda = 2\pi\sqrt{\frac{2}{\omega\sigma\mu_0}} = 2\pi\sqrt{\frac{2}{2\pi\times10^6\times6\times10^7\times4\pi\times10^{-7}}} = 4\times10^{-4}\ \mathrm{m} = \boxed{0.4\ \mathrm{mm}.}$$

¿From Eq. 9.129, the propagation speed is $v = \dfrac{\omega}{k} = \dfrac{\omega}{2\pi}\lambda = \lambda\nu = (4\times10^{-4})\times10^6 = \boxed{400\ \mathrm{m/s}.}$ In vacuum, $\lambda = \dfrac{c}{\nu} = \dfrac{3\times10^8}{10^6} = \boxed{300\ \mathrm{m};}\ v = c = \boxed{3\times10^8\ \mathrm{m/s}.}$ (But really, in a good conductor the skin depth is so small, compared to the wavelength, that the notions of "wavelength" and "propagation speed" lose their meaning.)

Problem 9.19

(a) Use the binomial expansion for the square root in Eq. 9.126:

$$\kappa \cong \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2-1\right]^{1/2} = \omega\sqrt{\frac{\epsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\epsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}.$$

So (Eq. 9.128) $d = \dfrac{1}{\kappa} \cong \dfrac{2}{\sigma}\sqrt{\dfrac{\epsilon}{\mu}}$. qed

For pure water, $\begin{cases} \epsilon = \epsilon_r\epsilon_0 = 80.1\,\epsilon_0 \quad \text{(Table 4.2)}, \\ \mu = \mu_0(1+\chi_m) = \mu_0(1-9.0\times10^{-6}) \cong \mu_0 \quad \text{(Table 6.1)}, \\ \sigma = 1/(2.5\times10^5) \quad \text{(Table 7.1)}. \end{cases}$

So $d = (2)(2.5\times10^5)\sqrt{\dfrac{(80.1)(8.85\times10^{-12})}{4\pi\times10^{-7}}} = \boxed{1.19\times10^4\text{ m.}}$

(b) In this case $(\sigma/\epsilon\omega)^2$ dominates, so (Eq. 9.126) $k \cong \kappa$, and hence (Eqs. 9.128 and 9.129) $\lambda = \dfrac{2\pi}{k} \cong \dfrac{2\pi}{\kappa} = 2\pi d$, or $d = \dfrac{\lambda}{2\pi}$. qed

Meanwhile $\kappa \cong \omega\sqrt{\dfrac{\epsilon\mu}{2}}\sqrt{\dfrac{\sigma}{\epsilon\omega}} = \sqrt{\dfrac{\omega\mu\sigma}{2}} = \sqrt{\dfrac{(10^{15})(4\pi\times10^{-7})(10^7)}{2}} = 8\times10^7$; $d = \dfrac{1}{\kappa} = \dfrac{1}{8\times10^7} = 1.3\times10^{-8} = \boxed{13\,\text{nm.}}$ So the fields do not penetrate far into a metal—which is what accounts for their opacity.

(c) Since $k \cong \kappa$, as we found in (b), Eq. 9.134 says $\phi = \tan^{-1}(1) = 45°$. qed

Meanwhile, Eq. 9.137 says $\dfrac{B_0}{E_0} \cong \sqrt{\epsilon\mu\dfrac{\sigma}{\epsilon\omega}} = \boxed{\sqrt{\dfrac{\sigma\mu}{\omega}}}$. For a typical metal, then, $\dfrac{B_0}{E_0} = \sqrt{\dfrac{(10^7)(4\pi\times10^{-7})}{10^{15}}} = \boxed{10^{-7}\,\text{s/m.}}$ (In vacuum, the ratio is $1/c = 1/(3\times10^8) = 3\times10^{-9}$ s/m, so the magnetic field is comparatively about 100 times larger in a metal.)

Problem 9.20

(a) $u = \dfrac{1}{2}\left(\epsilon E^2 + \dfrac{1}{\mu}B^2\right) = \dfrac{1}{2}e^{-2\kappa z}\left[\epsilon E_0^2\cos^2(kz-\omega t+\delta_E) + \dfrac{1}{\mu}B_0^2\cos^2(kz-\omega t+\delta_E+\phi)\right]$. Averaging over a full cycle, using $\langle\cos^2\rangle = \frac{1}{2}$ and Eq. 9.137:

$$\langle u\rangle = \frac{1}{2}e^{-2\kappa z}\left[\frac{\epsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right] = \frac{1}{4}e^{-2\kappa z}\left[\epsilon E_0^2 + \frac{1}{\mu}E_0^2\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] = \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left[1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right].$$

But Eq. 9.126 $\Rightarrow 1+\sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2} = \dfrac{2}{\epsilon\mu}\dfrac{k^2}{\omega^2}$, so $\langle u\rangle = \dfrac{1}{4}e^{-2\kappa z}\epsilon E_0^2\dfrac{2}{\epsilon\mu}\dfrac{k^2}{\omega^2} = \boxed{\dfrac{k^2}{2\mu\omega^2}E_0^2e^{-2\kappa z}}$. So the ratio of the magnetic contribution to the electric contribution is

$$\frac{\langle u_{\text{mag}}\rangle}{\langle u_{\text{elec}}\rangle} = \frac{B_0^2/\mu}{E_0^2\epsilon} = \frac{1}{\mu\epsilon}\mu\epsilon\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} = \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} > 1. \quad \text{qed}$$

(b) $\mathbf{S} = \dfrac{1}{\mu}(\mathbf{E}\times\mathbf{B}) = \dfrac{1}{\mu}E_0B_0e^{-2\kappa z}\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{z}}$; $\langle\mathbf{S}\rangle = \dfrac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi\,\hat{\mathbf{z}}$. [The average of the product of the cosines is $(1/2\pi)\int_0^{2\pi}\cos\theta\cos(\theta+\phi)\,d\theta = (1/2)\cos\phi$.] So $I = \dfrac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi = \dfrac{1}{2\mu}E_0^2e^{-2\kappa z}\left(\dfrac{K}{\omega}\cos\phi\right)$, while, from Eqs. 9.133 and 9.134, $K\cos\phi = k$, so $\boxed{I = \dfrac{k}{2\mu\omega}E_0^2e^{-2\kappa z}}$. qed

Problem 9.21

According to Eq. 9.147, $R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right)$, where $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$ $= \frac{\mu_1 v_1}{\mu_2 \omega}(k_2 + i\kappa_2)$ (Eqs. 9.125 and 9.146). Since silver is a good conductor ($\sigma \gg \epsilon\omega$), Eq. 9.126 reduces to $\kappa_2 \cong k_2 \cong \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\sqrt{\frac{\sigma}{\epsilon_2\omega}} = \sqrt{\frac{\sigma\omega\mu_2}{2}}$, so $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\sqrt{\frac{\sigma\omega\mu_2}{2}}(1+i) = \mu_1 v_1\sqrt{\frac{\sigma}{2\mu_2\omega}}(1+i)$.

Let $\gamma \equiv \mu_1 v_1\sqrt{\frac{\sigma}{2\mu_2\omega}} = \mu_0 c\sqrt{\frac{\sigma}{2\mu_0\omega}} = c\sqrt{\frac{\sigma\mu_0}{2\omega}} = (3\times10^8)\sqrt{\frac{(6\times10^7)(4\pi\times10^{-7})}{(2)(4\times10^{15})}} = 29.$ Then

$R = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2} = \boxed{0.93.}$ Evidently 93% of the light is reflected.

Problem 9.22

(a) We are told that $v = \alpha\sqrt{\lambda}$, where α is a constant. But $\lambda = 2\pi/k$ and $v = \omega/k$, so $\omega = \alpha k\sqrt{2\pi/k} = \alpha\sqrt{2\pi k}$. From Eq. 9.150, $v_g = \frac{d\omega}{dk} = \alpha\sqrt{2\pi}\frac{1}{2\sqrt{k}} = \frac{1}{2}\alpha\sqrt{\frac{2\pi}{k}} = \frac{1}{2}\alpha\sqrt{\lambda} = \frac{1}{2}v$, or $\boxed{v = 2v_g.}$

(b) $\frac{i(px-Et)}{\hbar} = i(kx-\omega t) \Rightarrow k = \frac{p}{\hbar},\ \omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar} = \frac{\hbar k^2}{2m}$. Therefore $\boxed{v = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m} = \frac{\hbar k}{2m}};$

$v_g = \frac{d\omega}{dk} = \frac{2\hbar k}{2m} = \frac{\hbar k}{m} = \boxed{\frac{p}{m}.}$ So $\boxed{v = \frac{1}{2}v_g.}$ Since $p = mv_c$ (where v_c is the classical speed of the particle), it follows that $\boxed{v_g \text{ (not } v\text{) corresponds to the classical veloctity.}}$

Problem 9.23

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} \Rightarrow F = -qE = -\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{a^3}\right)x = -k_{\text{spring}}x = -m\omega_0^2 x \text{ (Eq. 9.151). So } \boxed{\omega_0 = \sqrt{\frac{q^2}{4\pi\epsilon_0 m a^3}}.}$$

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(0.5\times10^{-10})^3}} = \boxed{7.16\times10^{15}\text{ Hz.}}\ \text{This is}\ \boxed{\text{ultraviolet.}}$$

¿From Eqs. 9.173 and 9.174,

$$A = \frac{nq^2}{2m\epsilon_0}\frac{f}{\omega_0^2},\left\{\begin{array}{lcl} N &=& \text{\# of molecules per unit volume} = \frac{\text{Avogadro's \#}}{22.4\text{ liters}} = \frac{6.02\times10^{23}}{22.4\times10^{-3}} = 2.69\times10^{25}, \\ f &=& \text{\# of electrons per molecule} = 2 \text{ (for H}_2\text{)}. \end{array}\right.$$

$$= \frac{(2.69\times10^{25})(1.6\times10^{-19})^2}{(9.11\times10^{-31})(8.85\times10^{-12})(4.5\times10^{16})^2} = \boxed{4.2\times10^{-5}} \text{ (which is about 1/3 the actual value);}$$

$$B = \left(\frac{2\pi c}{\omega_0}\right)^2 = \left(\frac{2\pi\times3\times10^8}{4.5\times10^{16}}\right)^2 = \boxed{1.8\times10^{-15}\text{ m}^2} \text{ (which is about 1/4 the actual value).}$$

So even this extremely crude model is in the right ball park.

Problem 9.24

Equation 9.170 $\Rightarrow n = 1 + \frac{Nq^2}{2m\epsilon_0}\frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}$. Let the denominator $\equiv D$. Then

$\frac{dn}{d\omega} = \frac{Nq^2}{2m\epsilon_0}\left\{\frac{-2\omega}{D} - \frac{(\omega_0^2-\omega^2)}{D^2}\left[2(\omega_0^2-\omega^2)(-2\omega)+\gamma^2 2\omega\right]\right\} = 0 \Rightarrow 2\omega D = (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)-\gamma^2\right]2\omega;$
$(\omega_0^2-\omega^2)^2+\gamma^2\omega^2 = 2(\omega_0^2-\omega^2)^2-\gamma^2(\omega_0^2-\omega^2)$, or $(\omega_0^2-\omega^2)^2 = \gamma^2(\omega^2+\omega_0^2-\omega^2) = \gamma^2\omega_0^2 \Rightarrow (\omega_0^2-\omega^2) = \pm\omega_0\gamma;$

$\omega^2 = \omega_0^2 \mp \omega_0\gamma$, $\omega = \omega_0\sqrt{1 \mp \gamma/\omega_0} \cong \omega_0(1 \mp \gamma/2\omega_0) = \omega_0 \mp \gamma/2$. So $\omega_2 = \omega_0 + \gamma/2$, $\omega_1 = \omega_0 - \gamma/2$, and the width of the anomalous region is $\boxed{\Delta\omega = \omega_2 - \omega_1 = \gamma.}$

From Eq. 9.171, $\alpha = \dfrac{Nq^2\omega^2}{m\epsilon_0 c}\dfrac{\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$, so at the maximum ($\omega = \omega_0$), $\alpha_{\max} = \dfrac{Nq^2}{m\epsilon_0 c\gamma}$.

At ω_1 and ω_2, $\omega^2 = \omega_0^2 \mp \omega_0\gamma$, so $\alpha = \dfrac{Nq^2\omega^2}{m\epsilon_0 c}\dfrac{\gamma}{\gamma^2\omega_0^2+\gamma^2\omega^2} = \alpha_{\max}\left(\dfrac{\omega^2}{\omega^2+\omega_0^2}\right)$. But

$$\frac{\omega^2}{\omega^2+\omega_0^2} = \frac{\omega_0^2 \mp \omega_0\gamma}{2\omega_0^2 \mp \omega_0\gamma} = \frac{1}{2}\frac{(1 \mp \gamma/\omega_0)}{(1 \mp \gamma/2\omega_0)} \cong \frac{1}{2}\left(1 \mp \frac{\gamma}{\omega_0}\right)\left(1 \pm \frac{\gamma}{2\omega_0}\right) \cong \frac{1}{2}\left(1 \mp \frac{\gamma}{2\omega_0}\right) \cong \frac{1}{2}.$$

So $\alpha \cong \frac{1}{2}\alpha_{\max}$ at ω_1 and ω_2. qed

Problem 9.25

$$k = \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2-\omega^2)}\right]. \quad v_g = \frac{d\omega}{dk} = \frac{1}{(dk/d\omega)}.$$

$$\frac{dk}{d\omega} = \frac{1}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2-\omega^2)} + \omega\sum f_j\frac{-(-2\omega)}{(\omega_j^2-\omega^2)^2}\right] = \frac{1}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum f_j\frac{(\omega_j^2+\omega^2)}{(\omega_j^2-\omega^2)^2}\right].$$

$\boxed{v_g = c\left[1 + \dfrac{Nq^2}{2m\epsilon_0}\sum f_j\dfrac{(\omega_j^2+\omega^2)}{(\omega_j^2-\omega^2)^2}\right]^{-1}}$. Since the second term in square brackets is ***positive***, it follows that

$\boxed{v_g < c,}$ whereas $v = \dfrac{\omega}{k} = c\left[1 + \dfrac{Nq^2}{2m\epsilon_0}\sum\dfrac{f_j}{(\omega_j^2-\omega^2)}\right]^{-1}$ is greater than c or less than c, depending on ω.

Problem 9.26

(a) From Eqs. 9.176 and 9.177, $\boldsymbol{\nabla}\times\tilde{\mathbf{E}} = -\dfrac{\partial\tilde{\mathbf{B}}}{\partial t} = i\omega\tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$; $\boldsymbol{\nabla}\times\tilde{\mathbf{B}} = \dfrac{1}{c^2}\dfrac{\partial\tilde{\mathbf{E}}}{\partial t} = -\dfrac{i\omega}{c^2}\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$.

In the terminology of Eq. 9.178:

$(\boldsymbol{\nabla}\times\tilde{\mathbf{E}})_x = \dfrac{\partial\tilde{E}_z}{\partial y} - \dfrac{\partial\tilde{E}_y}{\partial z} = \left(\dfrac{\partial\tilde{E}_{0_z}}{\partial y} - ik\tilde{E}_{0_y}\right)e^{i(kz-\omega t)}$. So (ii) $\dfrac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$.

$(\boldsymbol{\nabla}\times\tilde{\mathbf{E}})_y = \dfrac{\partial\tilde{E}_x}{\partial z} - \dfrac{\partial\tilde{E}_z}{\partial x} = \left(ik\tilde{E}_{0_x} - \dfrac{\partial\tilde{E}_{0_z}}{\partial x}\right)e^{i(kz-\omega t)}$. So (iii) $ikE_x - \dfrac{\partial E_z}{\partial x} = i\omega B_y$.

$(\boldsymbol{\nabla}\times\tilde{\mathbf{E}})_z = \dfrac{\partial\tilde{E}_y}{\partial x} - \dfrac{\partial\tilde{E}_x}{\partial y} = \left(\dfrac{\partial\tilde{E}_{0_y}}{\partial x} - \dfrac{\partial\tilde{E}_{0_x}}{\partial y}\right)e^{i(kz-\omega t)}$. So (i) $\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z$.

$(\boldsymbol{\nabla}\times\tilde{\mathbf{B}})_x = \dfrac{\partial\tilde{B}_z}{\partial y} - \dfrac{\partial\tilde{B}_y}{\partial z} = \left(\dfrac{\partial\tilde{B}_{0_z}}{\partial y} - ik\tilde{B}_{0_y}\right)e^{i(kz-\omega t)}$. So (v) $\dfrac{\partial B_z}{\partial y} - ikB_y = -\dfrac{i\omega}{c^2}E_x$.

$(\boldsymbol{\nabla}\times\tilde{\mathbf{B}})_y = \dfrac{\partial\tilde{B}_x}{\partial z} - \dfrac{\partial\tilde{B}_z}{\partial x} = \left(ik\tilde{B}_{0_x} - \dfrac{\partial\tilde{B}_{0_z}}{\partial x}\right)e^{i(kz-\omega t)}$. So (vi) $ikB_x - \dfrac{\partial B_z}{\partial x} = -\dfrac{i\omega}{c^2}E_y$.

$(\boldsymbol{\nabla}\times\tilde{\mathbf{B}})_z = \dfrac{\partial\tilde{B}_y}{\partial x} - \dfrac{\partial\tilde{B}_x}{\partial y} = \left(\dfrac{\partial\tilde{B}_{0_y}}{\partial x} - \dfrac{\partial\tilde{B}_{0_x}}{\partial y}\right)e^{i(kz-\omega t)}$. So (iv) $\dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2}E_z$.

This confirms Eq. 9.179. Now multiply (iii) by k, (v) by ω, and subtract: $ik^2E_x - k\dfrac{\partial E_z}{\partial x} - \omega\dfrac{\partial B_z}{\partial y} + i\omega kB_y = ik\omega B_y + \dfrac{i\omega^2}{c^2}E_x \Rightarrow i\left(k^2 - \dfrac{\omega^2}{c^2}\right)E_x = k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}$, or (i) $E_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}\right)$.

Multiply (ii) by k, (vi) by ω, and add: $k\dfrac{\partial E_z}{\partial y} - ik^2E_y + i\omega kB_x - \omega\dfrac{\partial B_z}{\partial x} = i\omega kB_x - \dfrac{i\omega^2}{c^2}E_y \Rightarrow i\left(\dfrac{\omega^2}{c^2} - k^2\right)E_y =$

$-k\frac{\partial E_z}{\partial y} + \omega\frac{\partial B_z}{\partial x}$, or (ii) $E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$.

Multiply (ii) by ω/c^2, (vi) by k, and add: $\frac{\omega}{c^2}\frac{\partial E_z}{\partial y} - i\frac{\omega k}{c^2}E_y + ik^2 B_x - k\frac{\partial B_z}{\partial x} = i\frac{\omega^2}{c^2}B_x - i\frac{\omega k}{c^2}E_y \Rightarrow$ $i\left(k^2 - \frac{\omega^2}{c^2}\right)B_x = k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}$, or (iii) $B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$.

Multiply (iii) by ω/c^2, (v) by k, and subtract: $i\frac{\omega k}{c^2}E_x - \frac{\omega}{c^2}\frac{\partial E_z}{\partial x} - k\frac{\partial B_z}{\partial y} + ik^2 B_y = i\frac{\omega^2}{c^2}B_y + \frac{i\omega k}{c^2}E_x \Rightarrow$ $i\left(k^2 - \frac{\omega^2}{c^2}\right)B_y = \frac{\omega}{c^2}\frac{\partial E_z}{\partial x} + k\frac{\partial B_z}{\partial y}$, or (iv) $B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$.

This completes the confirmation of Eq. 9.180.

(b) $\boldsymbol{\nabla}\cdot\tilde{\mathbf{E}} = \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} + \frac{\partial \tilde{E}_z}{\partial z} = \left(\frac{\partial \tilde{E}_{0_x}}{\partial x} + \frac{\partial \tilde{E}_{0_y}}{\partial y} + ik\tilde{E}_{0_z}\right)e^{i(kz-\omega t)} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0.$

Using Eq. 9.180, $\frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 E_z}{\partial x^2} + \omega\frac{\partial^2 B_z}{\partial x\partial y}\right) + \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 E_z}{\partial^2 y} - \omega\frac{\partial^2 B_z}{\partial x\partial y}\right) + ikE_z = 0,$

or $\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial^2 y} + \left[(\omega/c)^2 - k^2\right]E_z = 0.$

Likewise, $\boldsymbol{\nabla}\cdot\tilde{\mathbf{B}} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0 \Rightarrow$

$\frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 B_z}{\partial x^2} - \frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x\partial y}\right) + \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 B_z}{\partial y^2} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x\partial y}\right) + ikB_z = 0 \Rightarrow$

$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial^2 y} + \left[(\omega/c)^2 - k^2\right]B_z = 0.$

This confirms Eqs. 9.181. [You can also do it by putting Eq. 9.180 into Eq. 9.179 (i) and (iv).]

Problem 9.27

Here $E_z = 0$ (TE) and $\omega/c = k$ ($n = m = 0$), so Eq. 9.179(ii) $\Rightarrow E_y = -cB_x$, Eq. 9.179(iii) $\Rightarrow E_x = cB_y$, Eq. 9.179(v) $\Rightarrow \frac{\partial B_z}{\partial y} = i\left(kB_y - \frac{\omega}{c^2}E_x\right) = i\left(kB_y - \frac{\omega}{c}B_y\right) = 0$, Eq. 9.179(vi) $\Rightarrow \frac{\partial B_z}{\partial x} = i\left(kB_x + \frac{\omega}{c^2}E_y\right) = i\left(kB_x - \frac{\omega}{c}B_x\right) = 0$. So $\frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial y} = 0$, and since B_z is a function only of x and y, this says B_z is in fact a *constant* (as Eq. 9.186 also suggests). Now Faraday's law (in integral form) says $\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a}$, and Eq. 9.176 $\Rightarrow \frac{\partial \mathbf{B}}{\partial t} = -i\omega\mathbf{B}$, so $\oint \mathbf{E}\cdot d\mathbf{l} = i\omega\int \mathbf{B}\cdot d\mathbf{a}$. Applied to a cross-section of the waveguide this gives $\oint \mathbf{E}\cdot d\mathbf{l} = i\omega e^{i(kz-\omega t)}\int B_z\,da = i\omega B_z e^{i(kz-\omega t)}(ab)$ (since B_z is constant, it comes outside the integral). But if the boundary is just inside the metal, where $\mathbf{E} = 0$, it follows that $\boxed{B_z = 0.}$ So this would be a TEM mode, which we already know cannot exist for this guide.

Problem 9.28

Here $a = 2.28$ cm and $b = 1.01$ cm, so $\nu_{10} = \frac{1}{2\pi}\omega_{10} = \frac{c}{2a} = 0.66\times 10^{10}$ Hz; $\nu_{20} = 2\frac{c}{2a} = 1.32\times 10^{10}$ Hz; $\nu_{30} = 3\frac{c}{2a} = 1.97\times 10^{10}$ Hz; $\nu_{01} = \frac{c}{2b} = 1.49\times 10^{10}$ Hz; $\nu_{02} = 2\frac{c}{2b} = 2.97\times 10^{10}$ Hz; $\nu_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 1.62\times 10^{10}$ Hz. Evidently just four modes occur: $\boxed{\text{10, 20, 01, and 11.}}$

To get only *one* mode you must drive the waveguide at a frequency between ν_{10} and ν_{20}: $\boxed{0.66\times 10^{10} < \nu < 1.32\times 10^{10}\text{ Hz.}}$ $\lambda = \frac{c}{\nu}$, so $\lambda_{10} = 2a$; $\lambda_{20} = a$. $\boxed{2.28\text{ cm} < \lambda < 4.56\text{ cm.}}$

Problem 9.29

From Prob. 9.11, $\langle \mathbf{S}\rangle = \frac{1}{2\mu_0}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)$. Here (Eq. 9.176) $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$, $\tilde{\mathbf{B}}^* = \tilde{\mathbf{B}}_0^* e^{-i(kz-\omega t)}$, and, for the TE_{mn} mode (Eqs. 9.180 and 9.186)

$$\begin{aligned}
B_x^* &= \frac{-ik}{(\omega/c)^2-k^2}\left(\frac{-m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
B_y^* &= \frac{-ik}{(\omega/c)^2-k^2}\left(\frac{-n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
B_z^* &= B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_x &= \frac{i\omega}{(\omega/c)^2-k^2}\left(\frac{-n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
E_y &= \frac{-i\omega}{(\omega/c)^2-k^2}\left(\frac{-m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_z &= 0.
\end{aligned}$$

So

$$\begin{aligned}
\langle\mathbf{S}\rangle &= \frac{1}{2\mu_0}\left\{\frac{i\pi\omega B_0^2}{(\omega/c)^2-k^2}\left(\frac{m}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\hat{\mathbf{x}}\right.\\
&\quad+\frac{i\pi\omega B_0^2}{(\omega/c)^2-k^2}\left(\frac{n}{b}\right)\cos^2\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{y}}\\
&\quad\left.+\frac{\omega k\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{\mathbf{z}}\right\}.\\
\int\langle\mathbf{S}\rangle\cdot d\mathbf{a} &= \boxed{\frac{1}{8\mu_0}\frac{\omega k\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}ab\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]}.
\end{aligned}$$

[In the last step I used $\int_0^a\sin^2(m\pi x/a)\,dx = \int_0^a\cos^2(m\pi x/a)\,dx = a/2$; $\int_0^b\sin^2(n\pi y/b)\,dy = \int_0^b\cos^2(n\pi y/b)\,dy = b/2$.]

Similarly,

$$\begin{aligned}
\langle u\rangle &= \frac{1}{4}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^*+\frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)\\
&= \frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\
&\quad+\frac{1}{4\mu_0}\left\{B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right.\\
&\quad\left.+\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\right\}.
\end{aligned}$$

$$\int\langle u\rangle\,da = \boxed{\frac{ab}{4}\left\{\frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]+\frac{B_0^2}{4\mu_0}+\frac{1}{4\mu_0}\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]\right\}}.$$

These results can be simplified, using Eq. 9.190 to write $\left[(\omega/c)^2 - k^2\right] = (\omega_{mn}/c)^2$, $\epsilon_0\mu_0 = 1/c^2$ to eliminate ϵ_0, and Eq. 9.188 to write $\left[(m/a)^2 + (n/b)^2\right] = (\omega_{mn}/\pi c)^2$:

$$\int \langle \mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\omega k a b c^2}{8\mu_0\omega_{mn}^2} B_0^2; \quad \int \langle u\rangle\, da = \frac{\omega^2 ab}{8\mu_0\omega_{mn}^2} B_0^2.$$

Evidently

$$\frac{\text{energy per unit time}}{\text{energy per unit length}} = \frac{\int \langle \mathbf{S}\rangle \cdot d\mathbf{a}}{\int \langle u\rangle\, da} = \frac{kc^2}{\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2} = v_g \text{ (Eq. 9.192).} \quad \text{qed}$$

Problem 9.30

Following Sect. 9.5.2, the problem is to solve Eq. 9.181 with $E_z \neq 0, B_z = 0$, subject to the boundary conditions 9.175. Let $E_z(x, y) = X(x)Y(y)$; as before, we obtain $X(x) = A\sin(k_x x) + B\cos(k_x x)$. But the boundary condition requires $E_z = 0$ (and hence $X = 0$) when $x = 0$ and $x = a$, so $B = 0$ and $k_x = m\pi/a$. But this time $m = 1, 2, 3, \ldots$, but *not* zero, since $m = 0$ would kill X entirely. The same goes for $Y(y)$. Thus $\boxed{E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$ with $n, m = 1, 2, 3, \ldots$.

The rest is the same as for TE waves: $\boxed{\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}}$ is the cutoff frequency, the wave velocity is $v = c/\sqrt{1 - (\omega_{mn}/\omega)^2}$, and the group velocity is $v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$. The lowest TM mode is 11, with cutoff frequency $\omega_{11} = c\pi\sqrt{(1/a)^2 + (1/b)^2}$. So the ratio of the lowest TM frequency to the lowest TE frequency is $\dfrac{c\pi\sqrt{(1/a)^2 + (1/b)^2}}{(c\pi/a)} = \boxed{\sqrt{1 + (a/b)^2}.}$

Problem 9.31

(a) $\nabla\cdot\mathbf{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) = 0\ \checkmark$; $\nabla\cdot\mathbf{B} = \frac{1}{s}\frac{\partial}{\partial\phi}(B_\phi) = 0\ \checkmark$; $\nabla\times\mathbf{E} = \frac{\partial E_s}{\partial z}\,\hat{\boldsymbol{\phi}} - \frac{1}{s}\frac{\partial E_s}{\partial\phi}\,\hat{\mathbf{z}} = -\frac{E_0 k\sin(kz - \omega t)}{s}\,\hat{\boldsymbol{\phi}} \stackrel{?}{=} -\frac{\partial\mathbf{B}}{\partial t} = -\frac{E_0\omega}{c}\frac{\sin(kz - \omega t)}{s}\,\hat{\boldsymbol{\phi}}\ \checkmark$ (since $k = \omega/c$); $\nabla\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\,\hat{\mathbf{z}} = \frac{E_0 k}{c}\frac{\sin(kz - \omega t)}{s}\,\hat{\mathbf{s}} \stackrel{?}{=} \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \frac{E_0\omega}{c^2}\frac{\sin(kz - \omega t)}{s}\,\hat{\mathbf{s}}\ \checkmark$. Boundary conditions: $E^{\parallel} = E_z = 0\ \checkmark$; $B^{\perp} = B_s = 0\ \checkmark$.

(b) To determine λ, use Gauss's law for a cylinder of radius s and length dz:

$$\oint \mathbf{E}\cdot d\mathbf{a} = E_0\frac{\cos(kz - \omega t)}{s}(2\pi s)\,dz = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\lambda\,dz \Rightarrow \boxed{\lambda = 2\pi\epsilon_0 E_0\cos(kz - \omega t).}$$

To determine I, use Ampére's law for a circle of radius s (note that the displacement current through this loop is zero, since $\mathbf{E}$ is in the $\hat{\mathbf{s}}$ direction): $\oint \mathbf{B}\cdot d\mathbf{l} = \frac{E_0}{c}\frac{\cos(kz - \omega t)}{s}(2\pi s) = \mu_0 I_{\text{enc}} \Rightarrow \boxed{I = \frac{2\pi E_0}{\mu_0 c}\cos(kz - \omega t).}$

The charge and current on the outer conductor are precisely the $\boxed{\text{opposite}}$ of these, since $\mathbf{E} = \mathbf{B} = 0$ *inside* the metal, and hence the ***total*** enclosed charge and current must be zero.

Problem 9.32

$\tilde{f}(z, 0) = \int_{-\infty}^{\infty}\tilde{A}(k)e^{ikz}\,dk \Rightarrow \tilde{f}(z, 0)^* = \int_{-\infty}^{\infty}\tilde{A}(k)^* e^{-ikz}\,dk$. Let $l \equiv -k$; then $\tilde{f}(z, 0)^* = \int_{\infty}^{-\infty}\tilde{A}(-l)^* e^{ilz}(-dl) = \int_{-\infty}^{\infty}\tilde{A}(-l)^* e^{ilz}\,dl = \int_{-\infty}^{\infty}\tilde{A}(-k)^* e^{ikz}\,dk$ (renaming the dummy variable $l \to k$).

$f(z, 0) = \text{Re}\left[\tilde{f}(z, 0)\right] = \frac{1}{2}\left[\tilde{f}(z, 0) + \tilde{f}(z, 0)^*\right] = \int_{-\infty}^{\infty}\frac{1}{2}\left[\tilde{A}(k) + \tilde{A}(-k)^*\right]e^{ikz}\,dk$. Therefore

$\frac{1}{2}\left[\tilde{A}(k)+\tilde{A}(-k)^*\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(z,0)e^{-ikz}\,dz.$

Meanwhile, $\dot{f}(z,t)=\int_{-\infty}^{\infty}\tilde{A}(k)(-i\omega)e^{i(kz-\omega t)}\,dk \Rightarrow \dot{f}(z,0)=\int_{-\infty}^{\infty}[-i\omega\tilde{A}(k)]e^{ikz}\,dk.$

(Note that $\omega=|k|v$, here, so it does *not* come outside the integral.)

$$\begin{aligned}\dot{f}(z,0)^* &= \int_{-\infty}^{\infty}[i\omega\tilde{A}(k)^*]e^{-ikz}\,dk=\int_{-\infty}^{\infty}[i|k|v\tilde{A}(k)^*]e^{-ikz}\,dk=\int_{\infty}^{-\infty}[i|l|v\tilde{A}(-l)^*]e^{ilz}(-dl)\\ &= \int_{-\infty}^{\infty}[i|k|v\tilde{A}(-k)^*]e^{ikz}\,dk=\int_{-\infty}^{\infty}[i\omega\tilde{A}(-k)^*]e^{ikz}\,dk.\end{aligned}$$

$$\dot{f}(z,0) = \mathrm{Re}\left[\dot{f}(z,0)\right]=\frac{1}{2}\left[\dot{f}(z,0)+\dot{f}(z,0)^*\right]=\int_{-\infty}^{\infty}\frac{1}{2}[-i\omega\tilde{A}(k)+i\omega\tilde{A}(-k)^*]e^{ikz}\,dk.$$

$$\frac{-i\omega}{2}\left[\tilde{A}(k)-\tilde{A}(-k)^*\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\dot{f}(z,0)e^{-ikz}\,dz,\text{ or }\frac{1}{2}\left[\tilde{A}(k)-\tilde{A}(-k)^*\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{i}{\omega}\dot{f}(z,0)\right]e^{-ikz}\,dz.$$

Adding these two results, we get $\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(z,0)+\frac{i}{\omega}\dot{f}(z,0)\right]e^{-ikz}\,dz.}$ qed

---

Problem 9.33

(a) (i) *Gauss's law:* $\nabla\cdot\mathbf{E}=\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}=0.$ ✓

(ii) *Faraday's law:*

$$\begin{aligned}-\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{E}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\,\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\,\hat{\boldsymbol{\theta}}\\ &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[E_0\frac{\sin^2\theta}{r}\left(\cos u-\frac{1}{kr}\sin u\right)\right]\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}\left[E_0\sin\theta\left(\cos u-\frac{1}{kr}\sin u\right)\right]\hat{\boldsymbol{\theta}}.\end{aligned}$$

But $\frac{\partial}{\partial r}\cos u=-k\sin u;\ \frac{\partial}{\partial r}\sin u=k\cos u.$

$$= \frac{1}{r\sin\theta}\frac{E_0}{r}2\sin\theta\cos\theta\left(\cos u-\frac{1}{kr}\sin u\right)\hat{\mathbf{r}}-\frac{1}{r}E_0\sin\theta\left(-k\sin u+\frac{1}{kr^2}\sin u-\frac{1}{r}\cos u\right)\hat{\boldsymbol{\theta}}.$$

Integrating with respect to t, and noting that $\int\cos u\,dt=-\frac{1}{\omega}\sin u$ and $\int\sin u\,dt=\frac{1}{\omega}\cos u$, we obtain

$$\boxed{\mathbf{B}=\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u+\frac{1}{kr}\cos u\right)\hat{\mathbf{r}}+\frac{E_0\sin\theta}{\omega r}\left(-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right)\hat{\boldsymbol{\theta}}.}$$

(iii) *Divergence of* $\mathbf{B}$:

$$\begin{aligned}\nabla\cdot\mathbf{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2B_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{2E_0\cos\theta}{\omega}\left(\sin u+\frac{1}{kr}\cos u\right)\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{E_0\sin^2\theta}{\omega r}\left(-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right)\right]\\ &= \frac{1}{r^2}\frac{2E_0\cos\theta}{\omega}\left(k\cos u-\frac{1}{kr^2}\cos u-\frac{1}{r}\sin u\right)\\ &\quad+\frac{1}{r\sin\theta}\frac{2E_0\sin\theta\cos\theta}{\omega r}\left(-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right)\end{aligned}$$

$$= \frac{2E_0\cos\theta}{\omega r^2}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u - k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right) = 0. \checkmark$$

(iv) *Ampére/Maxwell:*

$$\begin{aligned}
\nabla\times\mathbf{B} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi} \\
&= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\frac{E_0\sin\theta}{\omega}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] - \frac{\partial}{\partial\theta}\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\right]\right\}\hat{\phi} \\
&= \frac{E_0\sin\theta}{\omega r}\left(k^2\sin u - \frac{2}{kr^3}\cos u - \frac{1}{r^2}\sin u - \frac{1}{r^2}\sin u + \frac{k}{r}\cos u + \frac{2}{r^2}\sin u + \frac{2}{kr^3}\cos u\right)\hat{\phi} \\
&= \frac{k}{\omega}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi}. \\
\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \frac{1}{c^2}\frac{E_0\sin\theta}{r}\left(\omega\sin u + \frac{\omega}{kr}\cos u\right)\hat{\phi} = \frac{1}{c^2}\frac{\omega}{k}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} \\
&= \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \nabla\times\mathbf{B}. \checkmark
\end{aligned}$$

(b) *Poynting Vector:*

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{E_0\sin\theta}{\mu_0 r}\left(\cos u - \frac{1}{kr}\sin u\right)\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\theta}\right. \\
&\quad\left. + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)(-\hat{\mathbf{r}})\right] \\
&= \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u) - \frac{1}{k^2r^2}\sin u\cos u\right]\hat{\theta}\right. \\
&\quad\left. - \sin\theta\left(-k\cos^2 u + \frac{1}{kr^2}\cos^2 u + \frac{1}{r}\sin u\cos u + \frac{1}{r}\sin u\cos u - \frac{1}{k^2r^3}\sin u\cos u - \frac{1}{kr^2}\sin^2 u\right)\hat{\mathbf{r}}\right\} \\
&= \boxed{\begin{aligned}&\frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u)\right]\hat{\theta}\right. \\ &\quad\left. + \sin\theta\left[\left(-\frac{2}{r} + \frac{1}{k^2r^3}\right)\sin u\cos u + k\cos^2 u + \frac{1}{kr^2}(\sin^2 u - \cos^2 u)\right]\hat{\mathbf{r}}\right\}.\end{aligned}}
\end{aligned}$$

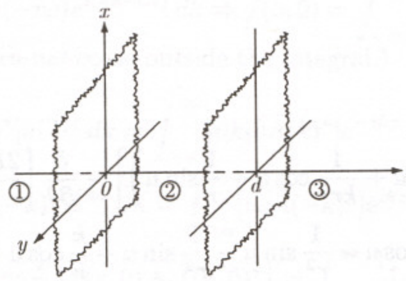
Averaging over a full cycle, using $\langle\sin u\cos u\rangle = 0$, $\langle\sin^2 u\rangle = \langle\cos^2 u\rangle = \frac{1}{2}$, we get the intensity:

$$\mathbf{I} = \langle\mathbf{S}\rangle = \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left(\frac{k}{2}\sin\theta\right)\hat{\mathbf{r}} = \boxed{\frac{E_0^2\sin^2\theta}{2\mu_0 c r^2}\hat{\mathbf{r}}.}$$

It points in the $\hat{\mathbf{r}}$ direction, and falls off as $1/r^2$, as we would expect for a spherical wave.

(c) $P = \int \mathbf{I}\cdot d\mathbf{a} = \frac{E_0^2}{2\mu_0 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{E_0^2}{2\mu_0 c}2\pi\int_0^\pi \sin^3\theta\,d\theta = \boxed{\frac{4\pi}{3}\frac{E_0^2}{\mu_0 c}}.$

Problem 9.34



$$z<0: \quad \begin{cases} \tilde{\mathbf{E}}_I(z,t) = \tilde{E}_I e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_I e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_R(z,t) = \tilde{E}_R e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_R e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$0<z<d: \quad \begin{cases} \tilde{\mathbf{E}}_r(z,t) = \tilde{E}_r e^{i(k_2 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_r(z,t) = \frac{1}{v_2}\tilde{E}_r e^{i(k_2 z - \omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_l(z,t) = \tilde{E}_l e^{i(-k_2 z - \omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_l(z,t) = -\frac{1}{v_2}\tilde{E}_l e^{i(-k_2 z - \omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$z>d: \quad \left\{ \tilde{\mathbf{E}}_T(z,t) = \tilde{E}_T e^{i(k_3 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{1}{v_3}\tilde{E}_T e^{i(k_3 z - \omega t)}\,\hat{\mathbf{y}}. \right.$$

Boundary conditions: $\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$, $\mathbf{B}_1^{\parallel} = \mathbf{B}_2^{\parallel}$, at each boundary (assuming $\mu_1 = \mu_2 = \mu_3 = \mu_0$):

$$z=0: \quad \begin{cases} \tilde{E}_I + \tilde{E}_R = \tilde{E}_r + \tilde{E}_l; \\ \\ \frac{1}{v_1}\tilde{E}_I - \frac{1}{v_1}\tilde{E}_R = \frac{1}{v_2}\tilde{E}_r - \frac{1}{v_2}\tilde{E}_l \Rightarrow \tilde{E}_I - \tilde{E}_R = \beta(\tilde{E}_r - \tilde{E}_l), \text{ where } \beta \equiv v_1/v_2. \end{cases}$$

$$z=d: \quad \begin{cases} \tilde{E}_r e^{ik_2 d} + \tilde{E}_l e^{-ik_2 d} = \tilde{E}_T e^{ik_3 d}; \\ \\ \frac{1}{v_2}\tilde{E}_r e^{ik_2 d} - \frac{1}{v_2}\tilde{E}_l e^{-ik_2 d} = \frac{1}{v_3}\tilde{E}_T e^{ik_3 d} \Rightarrow \tilde{E}_r e^{ik_2 d} - \tilde{E}_l e^{-ik_2 d} = \alpha\tilde{E}_T e^{ik_3 d}, \text{ where } \alpha \equiv v_2/v_3. \end{cases}$$

We have here four equations; the problem is to eliminate $\tilde{E}_R$, $\tilde{E}_r$, and $\tilde{E}_l$, to obtain a single equation for $\tilde{E}_T$ in terms of $\tilde{E}_I$.

Add the first two to eliminate $\tilde{E}_R$: $\quad 2\tilde{E}_I = (1+\beta)\tilde{E}_r + (1-\beta)\tilde{E}_l$;
Add the last two to eliminate $\tilde{E}_l$: $\quad 2\tilde{E}_r e^{ik_2 d} = (1+\alpha)\tilde{E}_T e^{ik_3 d}$;
Subtract the last two to eliminate $\tilde{E}_r$: $\quad 2\tilde{E}_l e^{-ik_2 d} = (1-\alpha)\tilde{E}^T e^{ik_3 d}$.

Plug the last two of these into the first:

$$\begin{aligned} 2\tilde{E}_I &= (1+\beta)\frac{1}{2}e^{-ik_2 d}(1+\alpha)\tilde{E}_T e^{ik_3 d} + (1-\beta)\frac{1}{2}e^{ik_2 d}(1-\alpha)\tilde{E}_T e^{ik_3 d} \\ 4\tilde{E}_I &= \left[(1+\alpha)(1+\beta)e^{-ik_2 d} + (1-\alpha)(1-\beta)e^{ik_2 d}\right]\tilde{E}_T e^{ik_3 d} \\ &= \left[(1+\alpha\beta)\left(e^{-ik_2 d} + e^{ik_2 d}\right) + (\alpha+\beta)\left(e^{-ik_2 d} - e^{ik_2 d}\right)\right]\tilde{E}_T e^{ik_3 d} \\ &= 2\left[(1+\alpha\beta)\cos(k_2 d) - i(\alpha+\beta)\sin(k_2 d)\right]\tilde{E}_T e^{ik_3 d}. \end{aligned}$$

Now the transmission coefficient is $T = \frac{v_3\epsilon_3 E_{T_0}^2}{v_1\epsilon_1 E_{I_0}^2} = \frac{v_3}{v_1}\left(\frac{\mu_0\epsilon_3}{\mu_0\epsilon_1}\right)\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \frac{v_1}{v_3}\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \alpha\beta\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2}$, so

$$\begin{aligned}
T^{-1} &= \frac{1}{\alpha\beta}\frac{|\tilde{E}_I|^2}{|\tilde{E}_T|^2} = \frac{1}{\alpha\beta}\left|\frac{1}{2}\left[(1+\alpha\beta)\cos(k_2d) - i(\alpha+\beta)\sin(k_2d)\right]e^{ik_3d}\right|^2 \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2\cos^2(k_2d) + (\alpha+\beta)^2\sin^2(k_2d)\right]. \quad \text{But } \cos^2(k_2d) = 1-\sin^2(k_2d). \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2 + (\alpha^2 + 2\alpha\beta + \beta^2 - 1 - 2\alpha\beta - \alpha^2\beta^2)\sin^2(k_2d)\right] \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2d)\right].
\end{aligned}$$

But $n_1 = \frac{c}{v_1}$, $n_2 = \frac{c}{v_2}$, $n_3 = \frac{c}{v_3}$, so $\alpha = \frac{n_3}{n_3}$, $\beta = \frac{n_2}{n_1}$.

$$= \boxed{\frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2(k_2d)\right].}$$

Problem 9.35

$T = 1 \Rightarrow \sin kd = 0 \Rightarrow kd = 0, \pi, 2\pi \ldots$. The *minimum* (nonzero) thickness is $d = \pi/k$. But $k = \omega/v = 2\pi\nu/v = 2\pi\nu n/c$, and $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ (Eq. 9.69), where (presumably) $\mu \approx \mu_0$. So $n = \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r}$, and hence $d = \frac{\pi c}{2\pi\nu\sqrt{\epsilon_r}} = \frac{c}{2\nu\sqrt{\epsilon_r}} = \frac{3\times 10^8}{2(10\times 10^9)\sqrt{2.5}} = 9.49\times 10^{-3}\,\text{m}$, or $\boxed{9.5\,\text{mm.}}$

Problem 9.36

From Eq. 9.199,

$$\begin{aligned}
T^{-1} &= \frac{1}{4(4/3)(1)}\left\{[(4/3)+1]^2 + \frac{[(16/9)-(9/4)][1-(9/4)]}{(9/4)}\sin^2(3\omega d/2c)\right\} \\
&= \frac{3}{16}\left[\frac{49}{9} + \frac{(-17/36)(-5/4)}{(9/4)}\sin^2(3\omega d/2c)\right] = \frac{49}{48} + \frac{85}{(48)(36)}\sin^2(3\omega d/2c). \\
T &= \frac{48}{49 + (85/36)\sin^2(3\omega d/2c)}.
\end{aligned}$$

Since $\sin^2(3\omega d/2c)$ ranges from 0 to 1, $T_{\min} = \frac{48}{49+(85/36)} = \boxed{0.935;}$ $T_{\max} = \frac{48}{49} = \boxed{0.980.}$ Not much variation, and the transmission is good (over 90%) for *all* frequencies. Since Eq. 9.199 is unchanged when you switch 1 and 3, the transmission is the same either direction, and the $\boxed{\text{fish sees you just as well as you see it.}}$

Problem 9.37

(a) Equation 9.91 $\Rightarrow \tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$; $\mathbf{k}_T\cdot\mathbf{r} = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = k_T(x\sin\theta_T + z\cos\theta_T) = xk_T\sin\theta_T + izk_T\sqrt{\sin^2\theta_T - 1} = kx + i\kappa z$, where

$$\begin{aligned}
k &\equiv k_T\sin\theta_T = \left(\frac{\omega n_2}{c}\right)\frac{n_1}{n_2}\sin\theta_I = \frac{\omega n_1}{c}\sin\theta_I, \\
\kappa &\equiv k_T\sqrt{\sin^2\theta_T - 1} = \frac{\omega n_2}{c}\sqrt{(n_1/n_2)^2\sin^2\theta_I - 1} = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}. \quad \text{So} \\
\tilde{\mathbf{E}}_T(\mathbf{r},t) &= \tilde{\mathbf{E}}_{0_T}e^{-\kappa z}e^{i(kx-\omega t)}. \quad \text{qed}
\end{aligned}$$

(b) $R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$. Here β is real (Eq. 9.106) and α is purely imaginary (Eq. 9.108); write $\alpha = ia$, with a real: $R = \left(\frac{ia-\beta}{ia+\beta}\right)\left(\frac{-ia-\beta}{-ia+\beta}\right) = \frac{a^2+\beta^2}{a^2+\beta^2} = \boxed{1.}$

(c) From Prob. 9.16, $E_{0_R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}$, so $R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = \left|\frac{1-ia\beta}{1+ia\beta}\right|^2 = \frac{(1-ia\beta)(1+ia\beta)}{(1+ia\beta)(1-ia\beta)} = \boxed{1.}$

(d) From the solution to Prob. 9.16, the transmitted wave is

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}}, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{z}}).$$

Using the results in (a): $\mathbf{k}_T\cdot\mathbf{r} = kx + i\kappa z - \omega t$, $\sin\theta_T = \frac{ck}{\omega n_2}$, $\cos\theta_T = i\frac{c\kappa}{\omega n_2}$:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\,\hat{\mathbf{y}}, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\left(-i\frac{c\kappa}{\omega n_2}\,\hat{\mathbf{x}} + \frac{ck}{\omega n_2}\,\hat{\mathbf{z}}\right).$$

We may as well choose the phase constant so that $\tilde{E}_{0_T}$ is *real.* Then

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{y}};\\
\mathbf{B}(\mathbf{r},t) &= \frac{1}{v_2}E_0 e^{-\kappa z}\frac{c}{\omega n_2}\mathrm{Re}\left\{[\cos(kx-\omega t) + i\sin(kx-\omega t)]\,[-i\kappa\,\hat{\mathbf{x}} + k\,\hat{\mathbf{z}}]\right\}\\
&= \frac{1}{\omega}E_0 e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\,\hat{\mathbf{x}} + k\cos(kx-\omega t)\,\hat{\mathbf{z}}\right]. \quad \text{qed}
\end{aligned}$$

(I used $v_2 = c/n_2$ to simplfy $\mathbf{B}$.)

(e) (i) $\nabla\cdot\mathbf{E} = \frac{\partial}{\partial y}\left[E_0 e^{-\kappa z}\cos(kx-\omega t)\right] = 0.$ ✓

$$\begin{aligned}
\text{(ii) } \nabla\cdot\mathbf{B} &= \frac{\partial}{\partial x}\left[\frac{E_0}{\omega}e^{-\kappa z}\kappa\sin(kx-\omega t)\right] + \frac{\partial}{\partial z}\left[\frac{E_0}{\omega}e^{-\kappa z}k\cos(kx-\omega t)\right]\\
&= \frac{E_0}{\omega}\left[e^{-\kappa z}\kappa k\cos(kx-\omega t) - \kappa e^{-\kappa z}k\cos(kx-\omega t)\right] = 0. \checkmark
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } \nabla\times\mathbf{E} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z}\,\hat{\mathbf{x}} + \frac{\partial E_y}{\partial x}\,\hat{\mathbf{z}}\\
&= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - E_0 e^{-\kappa z}k\sin(kx-\omega t)\,\hat{\mathbf{z}}.\\
-\frac{\partial\mathbf{B}}{\partial t} &= -\frac{E_0}{\omega}e^{-\kappa z}\left[-\kappa\omega\cos(kx-\omega t)\,\hat{\mathbf{x}} + k\omega\sin(kx-\omega t)\,\hat{\mathbf{z}}\right]\\
&= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - kE_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{z}} = \nabla\times\mathbf{E}. \checkmark
\end{aligned}$$

$$\begin{aligned}
\text{(iv) } \nabla\times\mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ B_x & 0 & B_z \end{vmatrix} = \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{\mathbf{y}}\\
&= \left[-\frac{E_0}{\omega}\kappa^2 e^{-\kappa z}\sin(kx-\omega t) + \frac{E_0}{\omega}e^{-\kappa z}k^2\sin(kx-\omega t)\right]\hat{\mathbf{y}} = (k^2-\kappa^2)\frac{E_0}{\omega}e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}.
\end{aligned}$$

$$\text{Eq. 9.202} \Rightarrow k^2-\kappa^2 = \left(\frac{\omega}{c}\right)^2\left[n_1^2\sin^2\theta_I - (n_1\sin\theta_I)^2 + (n_2)^2\right] = \left(\frac{n_2\omega}{c}\right)^2 = \omega^2\epsilon_2\mu_2.$$

$$\begin{aligned} &= \epsilon_2\mu_2\omega E_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}. \\ \mu_2\epsilon_2\frac{\partial \mathbf{E}}{\partial t} &= \mu_2\epsilon_2 E_0 e^{-\kappa z}\omega\sin(kx-\omega t)\,\hat{\mathbf{y}} = \boldsymbol{\nabla}\times\mathbf{B}\ \checkmark. \end{aligned}$$

(f)
$$\mathbf{S} = \frac{1}{\mu_2}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_2}\frac{E_0^2}{\omega}e^{-2\kappa z}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \cos(kx-\omega t) & 0 \\ \kappa\sin(kx-\omega t) & 0 & k\cos(kx-\omega t)\end{vmatrix}$$

$$= \boxed{\frac{E_0^2}{\mu_2\omega}e^{-2\kappa z}\left[k\cos^2(kx-\omega t)\,\hat{\mathbf{x}} - \kappa\sin(kx-\omega t)\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].}$$

Averaging over a complete cycle, using $\langle\cos^2\rangle = 1/2$ and $\langle\sin\cos\rangle = 0$, $\langle\mathbf{S}\rangle = \dfrac{E_0^2 k}{2\mu_2\omega}e^{-2\kappa z}\,\hat{\mathbf{x}}$. On average, then, no energy is transmitted in the z direction, only in the x direction (parallel to the interface). qed

Problem 9.38

Look for solutions of the form $\mathbf{E} = \mathbf{E}_0(x,y,z)e^{-i\omega t}$, $\mathbf{B} = \mathbf{B}_0(x,y,z)e^{-i\omega t}$, subject to the boundary conditions $\mathbf{E}^{\parallel} = 0$, $B^{\perp} = 0$ at all surfaces. Maxwell's equations, in the form of Eq. 9.177, give

$$\left\{\begin{array}{llll} \boldsymbol{\nabla}\cdot\mathbf{E} = 0 & \Rightarrow \boldsymbol{\nabla}\cdot\mathbf{E}_0 = 0; & \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & \Rightarrow \boldsymbol{\nabla}\times\mathbf{E}_0 = i\omega\mathbf{B}_0; \\ \boldsymbol{\nabla}\cdot\mathbf{B} = 0 & \Rightarrow \boldsymbol{\nabla}\cdot\mathbf{B}_0 = 0; & \boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} & \Rightarrow \boldsymbol{\nabla}\times\mathbf{B}_0 = -\frac{i\omega}{c^2}\mathbf{E}_0. \end{array}\right\}$$

From now on I'll leave off the subscript (0). The problem is to solve the (time independent) equations

$$\left\{\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{E} = 0; & \boldsymbol{\nabla}\times\mathbf{E} = i\omega\mathbf{B}; \\ \boldsymbol{\nabla}\cdot\mathbf{B} = 0; & \boldsymbol{\nabla}\times\mathbf{B} = -\frac{i\omega}{c^2}\mathbf{E}. \end{array}\right\}$$

From $\boldsymbol{\nabla}\times\mathbf{E} = i\omega\mathbf{B}$ it follows that I can get $\mathbf{B}$ once I know $\mathbf{E}$, so I'll concentrate on the latter for the moment.

$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = \boldsymbol{\nabla}\times(i\omega\mathbf{B}) = i\omega\left(-\dfrac{i\omega}{c^2}\mathbf{E}\right) = \dfrac{\omega^2}{c^2}\mathbf{E}$. So $\nabla^2 E_x = -\left(\dfrac{\omega}{c}\right)^2 E_x$; $\nabla^2 E_y = -\left(\dfrac{\omega}{c}\right)^2 E_y$; $\nabla^2 E_z = -\left(\dfrac{\omega}{c}\right)^2 E_z$. Solve each of these by separation of variables: $E_x(x,y,z) = X(x)Y(y)Z(z) \Rightarrow YZ\dfrac{d^2X}{dx^2} + ZX\dfrac{d^2Y}{dy^2} + XY\dfrac{d^2Z}{dz^2} = -\left(\dfrac{\omega}{c}\right)^2 XYZ$, or $\dfrac{1}{X}\dfrac{d^2X}{dx^2} + \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} + \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = -(\omega/c)^2$. Each term must be a constant, so $\dfrac{d^2X}{dx^2} = -k_x^2 X$, $\dfrac{d^2Y}{dy^2} = -k_y^2 Y$, $\dfrac{d^2Z}{dz^2} = -k_z^2 Z$, with $k_x^2 + k_y^2 + k_z^2 = -(\omega/c)^2$. The solution is

$$E_x(x,y,z) = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)][E\sin(k_z z) + F\cos(k_z z)].$$

But $\mathbf{E}^{\parallel} = 0$ at the boundaries $\Rightarrow E_x = 0$ at $y = 0$ and $z = 0$, so $D = F = 0$, and $E_x = 0$ at $y = b$ and $z = d$, so $k_y = n\pi/b$ and $k_z = l\pi/d$, where n and l are integers. A similar argument applies to E_y and E_z. *Conclusion:*

$$\begin{aligned} E_x(x,y,z) &= [A\sin(k_x x) + B\cos(k_x x)]\sin(k_y y)\sin(k_z z), \\ E_y(x,y,z) &= \sin(k_x x)[C\sin(k_y y) + D\cos(k_y y)]\sin(k_z z), \\ E_z(x,y,z) &= \sin(k_x x)\sin(k_y y)[E\sin(k_z z) + F\cos(k_z z)], \end{aligned}$$

where $k_x = m\pi/a$. (Actually, there is no reason at this stage to assume that k_x, k_y, and k_z are the *same* for all three components, and I should really affix a second subscript (x for E_x, y for E_y, and z for E_z), but in a moment we shall see that *in fact* they *do* have to be the same, so to avoid cumbersome notation I'll *assume* they are from the start.)

Now $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 \Rightarrow k_x[A\cos(k_x x) - B\sin(k_x x)]\sin(k_y y)\sin(k_z z) + k_y\sin(k_x x)[C\cos(k_y y) - D\sin(k_y y)]\sin(k_z z) + k_z\sin(k_x x)\sin(k_y y)[E\cos(k_z z) - F\sin(k_z z)] = 0$. In particular, putting in $x = 0$, $k_x A\sin(k_y y)\sin(k_z z) = 0$, and hence $A = 0$. Likewise $y = 0 \Rightarrow C = 0$ and $z = 0 \Rightarrow E = 0$. (Moreover, if the k's were *not* equal for different components, then by Fourier analysis this equation could not be satisfied (for all x, y, and z) unless the other three constants were *also* zero, and we'd be left with no field at all.) It follows that $-(Bk_x + Dk_y + Fk_z) = 0$ (in order that $\nabla \cdot \mathbf{E} = 0$), and we are left with

$$\boxed{\begin{array}{l}\mathbf{E} = B\cos(k_x x)\sin(k_y y)\sin(k_z z)\,\hat{\mathbf{x}} + D\sin(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{\mathbf{y}} + F\sin(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{\mathbf{z}},\\ \text{with } k_x = (m\pi/a),\ k_y = (n\pi/b),\ k_z = (l\pi/d)\ (l,\ m,\ n \text{ all integers}), \text{ and } Bk_x + Dk_y + Fk_z = 0.\end{array}}$$

The corresponding magnetic field is given by $\mathbf{B} = -(i/\omega)\nabla \times \mathbf{E}$:

$$\begin{aligned}
B_x &= -\frac{i}{\omega}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -\frac{i}{\omega}\left[Fk_y \sin(k_x x)\cos(k_y y)\cos(k_z z) - Dk_z \sin(k_x x)\cos(k_y y)\cos(k_z z)\right],\\
B_y &= -\frac{i}{\omega}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -\frac{i}{\omega}\left[Bk_z \cos(k_x x)\sin(k_y y)\cos(k_z z) - Fk_x \cos(k_x x)\sin(k_y y)\cos(k_z z)\right],\\
B_z &= -\frac{i}{\omega}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{i}{\omega}\left[Dk_x \cos(k_x x)\cos(k_y y)\sin(k_z z) - Bk_y \cos(k_x x)\cos(k_y y)\sin(k_z z)\right].
\end{aligned}$$

Or:

$$\boxed{\begin{aligned}\mathbf{B} \;=\; &-\frac{i}{\omega}(Fk_y - Dk_z)\sin(k_x x)\cos(k_y y)\cos(k_z z)\,\hat{\mathbf{x}} - \frac{i}{\omega}(Bk_z - Fk_x)\cos(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{\mathbf{y}}\\ &-\frac{i}{\omega}(Dk_x - Bk_y)\cos(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{\mathbf{z}}.\end{aligned}}$$

These *automatically* satisfy the boundary condition $B^{\perp} = 0$ ($B_x = 0$ at $x = 0$ and $x = a$, $B_y = 0$ at $y = 0$ and $y = b$, and $B_z = 0$ at $z = 0$ and $z = d$).

As a check, let's see if $\nabla \cdot \mathbf{B} = 0$:

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= -\frac{i}{\omega}(Fk_y - Dk_z)k_x \cos(k_x x)\cos(k_y y)\cos(k_z z) - \frac{i}{\omega}(Bk_z - Fk_x)k_y \cos(k_x x)\cos(k_y y)\cos(k_z z)\\
&\quad - \frac{i}{\omega}(Dk_x - Bk_y)k_z \cos(k_x x)\cos(k_y y)\cos(k_z z)\\
&= -\frac{i}{\omega}(Fk_x k_y - Dk_x k_z + Bk_z k_y - Fk_x k_y + Dk_x k_z - Bk_y k_z)\cos(k_x x)\cos(k_y y)\cos(k_z z) = 0. \checkmark
\end{aligned}$$

The boxed equations satisfy all of Maxwell's equations, and they meet the boundary conditions. For TE modes, we pick $E_z = 0$, so $F = 0$ (and hence $Bk_x + Dk_y = 0$, leaving only the overall amplitude undetermined, for given l, m, and n); for TM modes we want $B_z = 0$ (so $Dk_x - Bk_y = 0$, again leaving only one amplitude undetermined, since $Bk_x + Dk_y + Fk_z = 0$). In either case (TE_{lmn} or TM_{lmn}), the frequency is given by $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) = c^2\left[(m\pi/a)^2 + (n\pi/b)^2 + (l\pi/d)^2\right]$, or $\boxed{\omega = c\pi\sqrt{(m/a)^2 + (n/b)^2 + (l/d)^2}.}$

Chapter 10

Potentials and Fields

Problem 10.1

$$\begin{aligned}
\Box^2 V + \frac{\partial L}{\partial t} &= \nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = \nabla^2 V + \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) = -\frac{1}{\epsilon_0}\rho. \checkmark \\
\Box^2\mathbf{A} - \boldsymbol{\nabla}L &= \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \checkmark
\end{aligned}$$

Problem 10.2

(a) $W = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$. At $t_1 = d/c$, $x \geq d = ct_1$, so $\mathbf{E} = 0$, $\mathbf{B} = 0$, and hence $\boxed{W(t_1) = 0.}$

At $T_2 = (d+h)/c$, $ct_2 = d+h$:

$$\mathbf{E} = -\frac{\mu_0\alpha}{2}(d+h-x)\,\hat{\mathbf{z}}, \quad \mathbf{B} = \frac{1}{c}\frac{\mu_0\alpha}{2}(d+h-x)\,\hat{\mathbf{y}},$$

so $B^2 = \frac{1}{c^2}E^2$, and

$$\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \epsilon_0\left(E^2 + \frac{1}{\mu_0\epsilon_0}\frac{1}{c^2}E^2\right) = 2\epsilon_0 E^2.$$

Therefore

$$W(t_2) = \frac{1}{2}(2\epsilon_0)\frac{\mu_0^2\alpha^2}{4}\int_d^{(d+h)}(d+h-x)^2\,dx\,(lw) = \frac{\epsilon_0\mu_0^2\alpha^2 lw}{4}\left[-\frac{(d+h-x)^3}{3}\right]_d^{d+h} = \boxed{\frac{\epsilon_0\mu_0^2\alpha^2 lwh^3}{12}}.$$

(b) $\mathbf{S}(x) = \frac{1}{\mu_0}(\mathbf{B}\times\mathbf{E}) = \frac{1}{\mu_0 c}E^2[-\hat{\mathbf{z}}\times(\pm\hat{\mathbf{y}})] = \pm\frac{1}{\mu_0 c}E^2\,\hat{\mathbf{x}} = \boxed{\pm\frac{\mu_0\alpha^2}{4c}(ct-|x|)^2\,\hat{\mathbf{x}}}$

(plus sign for $x > 0$, as here). For $|x| > ct$, $\mathbf{S} = 0$.

So the energy per unit time entering the box in this time interval is

$$\frac{dW}{dt} = P = \int\mathbf{S}(d)\cdot d\mathbf{a} = \boxed{\frac{\mu_0\alpha^2 lw}{4c}(ct-d)^2.}$$

Note that no energy flows out the top, since $\mathbf{S}(d+h) = 0$.

(c) $W = \int_{t_1}^{t_2} P\,dt = \frac{\mu_0\alpha^2 lw}{4c}\int_{d/x}^{(d+h)/c}(ct-d)^2\,dt = \frac{\mu_0\alpha^2 lw}{4c}\left[\frac{(ct-d)^3}{3c}\right]_{d/c}^{(d+h)/c} = \boxed{\frac{\mu_0\alpha^2 lwh^3}{12c^2}}.$

Since $1/c^2 = \mu_0\epsilon_0$, this agrees with the answer to (a).

Problem 10.3

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}.}\quad \mathbf{B} = \nabla\times\mathbf{A} = \boxed{0.}$$

This is a funny set of potentials for a $\boxed{\text{stationary point charge}}$ q at the origin. ($V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, $\mathbf{A}=0$ would, of course, be the customary choice.) Evidently $\boxed{\rho = q\delta^3(\mathbf{r});\ \mathbf{J}=0.}$

Problem 10.4

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -A_0\cos(kx-\omega t)\,\hat{\mathbf{y}}(-\omega) = \boxed{A_0\omega\cos(kx-\omega t)\,\hat{\mathbf{y}},}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{z}}\frac{\partial}{\partial x}[A_0\sin(kx-\omega t)] = \boxed{A_0 k\cos(kx-\omega t)\,\hat{\mathbf{z}}.}$$

Hence $\nabla\cdot\mathbf{E}=0$ ✓, $\nabla\cdot\mathbf{B}=0$ ✓.

$$\nabla\times\mathbf{E} = \hat{\mathbf{z}}\frac{\partial}{\partial x}[A_0\omega\cos(kx-\omega t)] = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}},\qquad -\frac{\partial\mathbf{B}}{\partial t} = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}},$$

so $\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ ✓.

$$\nabla\times\mathbf{B} = -\hat{\mathbf{y}}\frac{\partial}{\partial x}[A_0 k\cos(kx-\omega t)] = A_0k^2\sin(kx-\omega t)\,\hat{\mathbf{y}},\qquad \frac{\partial\mathbf{E}}{\partial t} = A_0\omega^2\sin(kx-\omega t)\,\hat{\mathbf{y}}.$$

So $\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$ *provided* $\boxed{k^2 = \mu_0\epsilon_0\omega^2,}$ or, since $c^2 = 1/\mu_0\epsilon_0$, $\boxed{\omega = ck.}$

Problem 10.5

$$V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r};}\ \mathbf{A}' = \mathbf{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}} + \left(-\frac{1}{4\pi\epsilon_0}qt\right)\left(-\frac{1}{r^2}\,\hat{\mathbf{r}}\right) = \boxed{0.}$$

This gauge function transforms the "funny" potentials of Prob. 10.3 into the "ordinary" potentials of a stationary point charge.

Problem 10.6

Ex. 10.1: $\nabla\cdot\mathbf{A} = 0;\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Both Coulomb and Lorentz.}}$

Prob. 10.3: $\nabla\cdot\mathbf{A} = -\frac{qt}{4\pi\epsilon_0}\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = -\frac{qt}{\epsilon_0}\delta^3(\mathbf{r});\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Neither.}}$

Prob. 10.4: $\nabla\cdot\mathbf{A} = 0;\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Both.}}$

Problem 10.7

Suppose $\nabla\cdot\mathbf{A} \neq -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. (Let $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = \Phi$—some known function.) We want to pick λ such that $\mathbf{A}'$ and V' (Eq. 10.7) *do* obey $\nabla\cdot\mathbf{A}' = -\mu_0\epsilon_0\frac{\partial V'}{\partial t}$.

$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \nabla\cdot\mathbf{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \Phi + \Box^2\lambda.$$

This will be zero provided we pick for λ the solution to $\Box^2\lambda = -\Phi$, which by hypothesis (and in fact) we know how to solve.

We *could* always find a gauge in which $V' = 0$, simply by picking $\lambda = \int_0^t V\,dt'$. We *cannot* in general pick $\mathbf{A} = 0$—this would make $\mathbf{B} = 0$. [Finding such a gauge function would amount to expressing $\mathbf{A}$ as $-\nabla\lambda$, and we know that vector functions *cannot* in general be written as gradients—only if they happen to have curl zero, which $\mathbf{A}$ (ordinarily) does *not*.]

Problem 10.8

¿From the product rule:

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla\frac{1}{\mathscr{r}}\right), \quad \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right).$$

But $\nabla\frac{1}{\mathscr{r}} = -\nabla'\frac{1}{\mathscr{r}}$, since $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$. So

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) - \mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) + \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J}) - \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right).$$

But

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x} + \frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y} + \frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z},$$

and

$$\frac{\partial t_r}{\partial x} = -\frac{1}{c}\frac{\partial \mathscr{r}}{\partial x}, \quad \frac{\partial t_r}{\partial y} = -\frac{1}{c}\frac{\partial \mathscr{r}}{\partial y}, \quad \frac{\partial t_r}{\partial z} = -\frac{1}{c}\frac{\partial \mathscr{r}}{\partial z},$$

so

$$\nabla\cdot\mathbf{J} = -\frac{1}{c}\left[\frac{\partial J_x}{\partial t_r}\frac{\partial \mathscr{r}}{\partial x} + \frac{\partial J_y}{\partial t_r}\frac{\partial \mathscr{r}}{\partial y} + \frac{\partial J_z}{\partial t_r}\frac{\partial \mathscr{r}}{\partial z}\right] = -\frac{1}{c}\frac{\partial \mathbf{J}}{\partial t_r}\cdot(\nabla\mathscr{r}).$$

Similarly,

$$\nabla'\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r}).$$

[The first term arises when we differentiate with respect to the *explicit* $\mathbf{r}'$, and use the continuity equation.] thus

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\left[-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right] + \frac{1}{\mathscr{r}}\left[-\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right] - \nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial t} - \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$$

(the other two terms cancel, since $\nabla\mathscr{r} = -\nabla'\mathscr{r}$). Therefore:

$$\nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\left[-\frac{\partial}{\partial t}\int\frac{\rho}{\mathscr{r}}\,d\tau - \int\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau\right] = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}\,d\tau\right] - \frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}.$$

The last term is over the suface at "infinity", where $\mathbf{J} = 0$, so it's zero. Therefore $\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. ✓

Problem 10.9

(a) As in Ex. 10.2, for $t < r/c$, $\mathbf{A} = 0$; for $t > r/c$,

$$\mathbf{A}(r,t) = \left(\frac{\mu_0}{4\pi}\hat{\mathbf{z}}\right)2\int_0^{\sqrt{(ct)^2-r^2}} \frac{k(t-\sqrt{r^2+z^2}/c)}{\sqrt{r^2+z^2}}\,dz = \frac{\mu_0 k}{2\pi}\hat{\mathbf{z}}\left\{t\int_0^{\sqrt{(ct)^2-r^2}}\frac{dz}{\sqrt{r^2+z^2}} - \frac{1}{c}\int_0^{\sqrt{(ct)^2-r^2}} dz\right\}$$

$$= \left(\frac{\mu_0 k}{2\pi}\hat{\mathbf{z}}\right)\left[t\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) - \frac{1}{c}\sqrt{(ct)^2-r^2}\right].\quad \text{Accordingly,}$$

$$\mathbf{E}(r,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2\pi}\hat{\mathbf{z}}\left\{\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right)\left(\frac{1}{r}\right)\left(c+\frac{1}{2}\frac{2c^2t}{\sqrt{(ct)^2-r^2}}\right) - \frac{1}{2c}\frac{2c^2t}{\sqrt{(ct)^2-r^2}}\right\}$$

$$= -\frac{\mu_0 k}{2\pi}\hat{\mathbf{z}}\left\{\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \frac{ct}{\sqrt{(ct)^2-r^2}} - \frac{ct}{\sqrt{(ct)^2-r^2}}\right\}$$

$$= \boxed{-\frac{\mu_0 k}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)\hat{\mathbf{z}}}\quad \text{(or zero, for } t < r/c\text{).}$$

$$\mathbf{B}(r,t) = -\frac{\partial A_z}{\partial r}\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 k}{2\pi}\left\{t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right)\frac{\left[r\frac{1}{2}\frac{(-2r)}{\sqrt{(ct)^2-r^2}} - ct - \sqrt{(ct)^2-r^2}\right]}{r^2} - \frac{1}{2c}\frac{(-2r)}{\sqrt{(ct)^2-r^2}}\right\}\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 k}{2\pi}\left\{\frac{-ct^2}{r\sqrt{(ct)^2-r^2}} + \frac{r}{c\sqrt{(ct)^2-r^2}}\right\}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 k}{2\pi}\frac{(-c^2t^2+r^2)}{rc\sqrt{(ct)^2-r^2}}\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0 k}{2\pi rc}\sqrt{(ct)^2-r^2}\,\hat{\boldsymbol{\phi}}.}$$

(b) $\mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{q_0\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz$. But $\mathscr{r} = \sqrt{r^2+z^2}$, so the integrand is even in z:

$$\mathbf{A}(r,t) = \left(\frac{\mu_0 q_0}{4\pi}\hat{\mathbf{z}}\right)2\int_0^{\infty}\frac{\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz.$$

Now $z = \sqrt{\mathscr{r}^2-r^2} \Rightarrow dz = \frac{1}{2}\frac{2\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}} = \frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}}$, and $z=0 \Rightarrow \mathscr{r} = r$, $z = \infty \Rightarrow \mathscr{r} = \infty$. So:

$$\mathbf{A}(r,t) = \frac{\mu_0 q_0}{2\pi}\hat{\mathbf{z}}\int_r^{\infty}\frac{1}{\mathscr{r}}\delta\left(t-\frac{\mathscr{r}}{c}\right)\frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}}.$$

Now $\delta(t - \mathscr{r}/c) = c\delta(\mathscr{r} - ct)$ (Ex. 1.15); therefore $\mathbf{A} = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}}\,c\int_r^\infty \frac{\delta(\mathscr{r} - ct)}{\sqrt{\mathscr{r}^2 - r^2}}\,d\mathscr{r}$, so

$$\mathbf{A}(r,t) = \frac{\mu_0 q_0 c}{2\pi}\frac{1}{\sqrt{(ct)^2 - r^2}}\,\hat{\mathbf{z}} \quad \text{(or zero, if } ct < r\text{);}$$

$$\mathbf{E}(r,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{2c^2 t}{[(ct)^2 - r^2]^{3/2}}\,\hat{\mathbf{z}} = \boxed{\frac{\mu_0 q_0 c^3 t}{2\pi[(ct)^2 - r^2]^{3/2}}\,\hat{\mathbf{z}}} \quad \text{(or zero, for } t < r/c\text{);}$$

$$\mathbf{B}(r,t) = -\frac{\partial \mathbf{A}_z}{\partial t}\,\hat{\boldsymbol{\phi}} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{-2r}{[(ct)^2 - r^2]^{3/2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{-\mu_0 q_0 c r}{2\pi[(ct)^2 - r^2]^{3/2}}\,\hat{\boldsymbol{\phi}}} \quad \text{(or zero, for } t < r/c\text{).}$$

Problem 10.10

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}(t_r)}{\mathscr{r}}\,dl = \frac{\mu_0 k}{4\pi}\int \frac{(t - \mathscr{r}/c)}{\mathscr{r}}\,d\mathbf{l} = \frac{\mu_0 k}{4\pi}\left\{t\int\frac{d\mathbf{l}}{\mathscr{r}} - \frac{1}{c}\int d\mathbf{l}\right\}.$$

But for the complete loop, $\int d\mathbf{l} = 0$, so $\mathbf{A} = \frac{\mu_0 k t}{4\pi}\left\{\frac{1}{a}\int_1 d\mathbf{l} + \frac{1}{b}\int_2 d\mathbf{l} + 2\,\hat{\mathbf{x}}\int_a^b \frac{dx}{x}\right\}$. Here $\int_1 d\mathbf{l} = 2a\,\hat{\mathbf{x}}$ (inner circle), $\int_2 d\mathbf{l} = -2b\,\hat{\mathbf{x}}$ (outer circle), so

$$\mathbf{A} = \frac{\mu_0 k t}{4\pi}\left[\frac{1}{a}(2a) + \frac{1}{b}(-2b) + 2\ln(b/a)\right]\hat{\mathbf{x}} \Rightarrow \boxed{\mathbf{A} = \frac{\mu_0 k t}{2\pi}\ln(b/a)\,\hat{\mathbf{x}},} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \boxed{-\frac{\mu_0 k}{2\pi}\ln(b/a)\,\hat{\mathbf{x}}.}$$

The changing magnetic field induces the electric field. Since we only know $\mathbf{A}$ at *one point* (the center), we can't compute $\nabla \times \mathbf{A}$ to get $\mathbf{B}$.

Problem 10.11

In this case $\dot{\rho}(\mathbf{r},t) = \dot{\rho}(\mathbf{r},0)$ and $\dot{\mathbf{J}}(\mathbf{r},t) = 0$, so Eq. 10.29 $\Rightarrow$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',0) + \dot{\rho}(\mathbf{r}',0)t_r}{\mathscr{r}^2} + \frac{\dot{\rho}(\mathbf{r}',0)}{c\mathscr{r}}\right]\hat{\boldsymbol{\mathscr{r}}}\,d\tau', \text{ but } t_r = t - \frac{\mathscr{r}}{c} \text{ (Eq. 10.18), so}$$

$$= \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',0) + \dot{\rho}(\mathbf{r}',0)t}{\mathscr{r}^2} - \frac{\dot{\rho}(\mathbf{r}',0)(\mathscr{r}/c)}{\mathscr{r}^2} + \frac{\dot{\rho}(\mathbf{r}',0)}{c\mathscr{r}}\right]\hat{\boldsymbol{\mathscr{r}}}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t)}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'. \quad \text{qed}$$

Problem 10.12

In this approximation we're dropping the higher derivatives of $\mathbf{J}$, so $\dot{\mathbf{J}}(t_r) = \dot{\mathbf{J}}(t)$, and Eq. 10.31 $\Rightarrow$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{1}{\mathscr{r}^2}\left[\mathbf{J}(\mathbf{r}',t) + (t_r - t)\dot{\mathbf{J}}(\mathbf{r}',t) + \frac{\mathscr{r}}{c}\dot{\mathbf{J}}(\mathbf{r}',t)\right]\times\hat{\boldsymbol{\mathscr{r}}}\,d\tau', \text{ but } t_r - t = -\frac{\mathscr{r}}{c} \text{ (Eq. 10.18), so}$$

$$= \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'. \quad \text{qed}$$

Problem 10.13

At time t the charge is at $\mathbf{r}(t) = a[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$, so $\mathbf{v}(t) = \omega a[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$. Therefore $\boldsymbol{\mathscr{r}} = z\,\hat{\mathbf{z}} - a[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}]$, and hence $\mathscr{r}^2 = z^2 + a^2$ (of course), and $\mathscr{r} = \sqrt{z^2 + a^2}$.

$$\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v} = \frac{1}{\mathscr{r}}(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = \frac{1}{\mathscr{r}}\left\{-\omega a^2[-\sin(\omega t_r)\cos(\omega t_r) + \sin(\omega t_r)\cos(\omega t_r)]\right\} = 0, \text{ so } \left(1 - \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v}}{c}\right) = 1.$$

Therefore

$$V(z,t) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2+a^2}}};\ \mathbf{A}(z,t) = \boxed{\frac{q\omega a}{4\pi\epsilon_0 c^2\sqrt{z^2+a^2}}[-\sin(\omega t_r)\,\hat{\mathbf{x}} + \cos(\omega t_r)\,\hat{\mathbf{y}}),}\ \text{where } \boxed{t_r = t - \frac{\sqrt{z^2+a^2}}{c}}.$$

Problem 10.14

Term under square root in (Eq. 9.98) is:

$$\begin{aligned} I &= c^4t^2 - 2c^2t(\mathbf{r}\cdot\mathbf{v}) + (\mathbf{r}\cdot\mathbf{v})^2 + c^2r^2 - c^4t^2 - v^2r^2 + v^2c^2t^2 \\ &= (\mathbf{r}\cdot\mathbf{v})^2 + (c^2-v^2)r^2 + c^2(vt)^2 - 2c^2(\mathbf{r}\cdot\mathbf{v}t). \quad \text{put in } \mathbf{v}t = \mathbf{r}-\mathbf{R}^2. \\ &= (\mathbf{r}\cdot\mathbf{v})^2 + (c^2-v^2)r^2 + c^2(r^2+R^2-2\mathbf{r}\cdot\mathbf{R}) - 2c^2(r^2-\mathbf{r}\cdot\mathbf{R}) = (\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2 + c^2R^2. \end{aligned}$$

but

$$\begin{aligned} (\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2 &= ((\mathbf{R}+\mathbf{v}t)\cdot\mathbf{v})^2 - (\mathbf{R}+\mathbf{v}t)^2v^2 \\ &= (\mathbf{R}\cdot\mathbf{v})^2 + v^4t^2 + 2(\mathbf{R}\cdot\mathbf{v})v^2t - R^2v^2 - 2(\mathbf{R}\cdot\mathbf{v})tv^2 - v^2t^2v^2 \\ &= (\mathbf{R}\cdot\mathbf{v})^2 - R^2v^2 = R^2v^2\cos^2\theta - R^2v^2 = -R^2v^2(1-\cos^2\theta) \\ &= -R^2v^2\sin^2\theta. \end{aligned}$$

Therefore

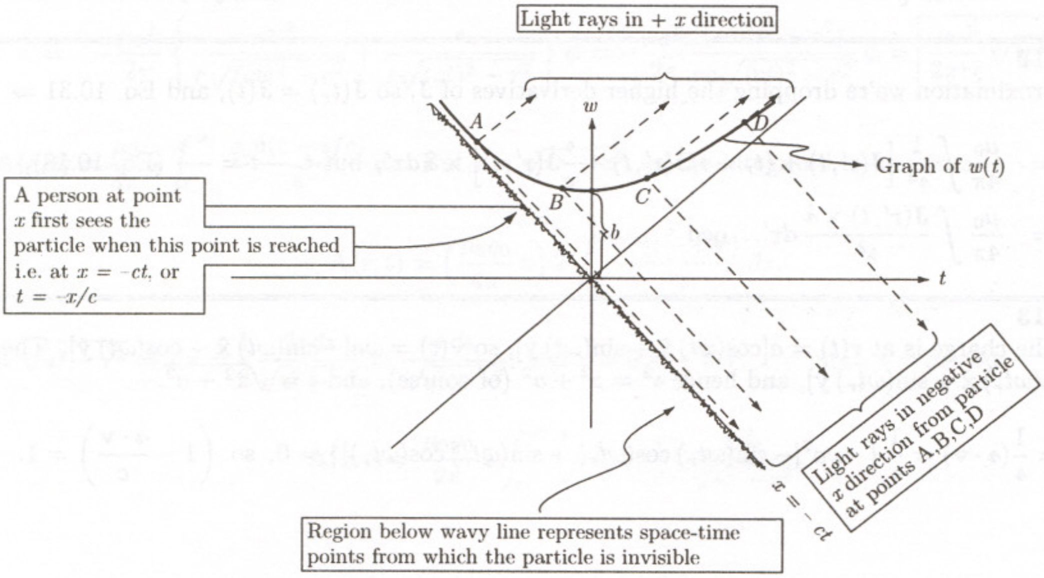
$$I = -R^2v^2\sin^2\theta + c^2R^2 = c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right).$$

Hence

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}. \quad \text{qed}$$

Problem 10.15

Once seen, from a given point x, the particle will forever remain in view—to disappear it would have to travel faster than light.



Problem 10.16

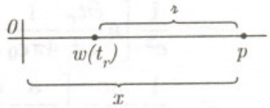
First calculate t_r: $t_r = t - |\mathbf{r} - \mathbf{w}(t_r)|/c \Rightarrow$

$-c(t_r - t) = x - \sqrt{b^2 + c^2 t_r^2} \Rightarrow c(t_r - t) + x = \sqrt{b^2 + c^2 t_r^2}$;

$c^2 t_r^2 - 2c^2 t_r t + c^2 t^2 + 2xct_r - 2xct + x^2 = b^2 + c^2 t_r^2$;

$2ct_r(x - ct) + (x^2 - 2xct + c^2t^2) = b^2$;

$2ct_r(x - ct) = b^2 - (x - ct)^2$, or $t_r = \dfrac{b^2 - (x - ct)^2}{2c(x - ct)}$.



Now $V(x,t) = \dfrac{1}{4\pi\epsilon_0}\dfrac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}$, and $\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \mathscr{r}(c - v)$; $\mathscr{r} = c(t - t_r)$.

$$v = \frac{1}{2}\frac{1}{\sqrt{b^2 + c^2t_r^2}}2c^2 t_r = \frac{c^2 t_r}{c(t_r - t) + x} = \frac{c^2 t_r}{ct_r + (x - ct)};\ (c - v) = \frac{c^2 t_r + c(x - ct) - c^2 t_r}{ct_r + (x - ct)} = \frac{c(x - ct)}{ct_r + (x - ct)};$$

$$\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \frac{c(t - t_r)c(x - ct)}{ct_r + (x - ct)} = \frac{c^2(t - t_r)(x - ct)}{ct_r + (x - ct)};\ ct_r + (x - ct) = \frac{b^2 - (x - ct)^2}{2(x - ct)} + (x - ct) = \frac{b^2 + (x - ct)^2}{2(x - ct)};$$

$$t - t_r = \frac{2ct(x - ct) - b^2 + (x - ct)^2}{2c(x - ct)} = \frac{(x - ct)(x + ct) - b^2}{2c(x - ct)} = \frac{(x^2 - c^2t^2 - b^2)}{2c(x - ct)}.\ \text{Therefore}$$

$$\frac{1}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}} = \left[\frac{b^2 + (x - ct)^2}{2(x - ct)}\right]\frac{1}{c^2(x - ct)}\frac{2c(x - ct)}{[2ct(x - ct) - b^2 + (x - ct)^2]} = \frac{b^2 + (x - ct)^2}{c(x - ct)\,[2ct(x - ct) - b^2 + (x - ct)^2]}.$$

The term in square brackets simplifies to $(2ct + x - ct)(x - ct) - b^2 = (x + ct)(x - ct) - b^2 = x^2 - c^2t^2 - b^2$.

So $\boxed{V(x,t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{b^2 + (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}}$.

Meanwhile

$$\begin{aligned}\mathbf{A} &= \left(\frac{V}{c^2}\right)\mathbf{v} = \frac{c^2 t_r}{ct_r + (x - ct)}\frac{V}{c^2}\,\hat{\mathbf{x}} = \left[\frac{b^2 - (x - ct)^2}{2c(x - ct)}\right]\frac{2(x - ct)}{b^2 + (x - ct)^2}\frac{q}{4\pi\epsilon_0}\frac{b^2 + (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}\,\hat{\mathbf{x}} \\ &= \boxed{\frac{q}{4\pi\epsilon_0 c}\frac{b^2 - (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}\,\hat{\mathbf{x}}.}\end{aligned}$$

Problem 10.17

From Eq. 10.33, $c(t - t_r) = \mathscr{r} \Rightarrow c^2(t - t_r)^2 = \mathscr{r}^2 = \boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}$. Differentiate with respect to t:

$2c^2(t - t_r)\left(1 - \dfrac{\partial t_r}{\partial t}\right) = 2\boldsymbol{\mathscr{r}}\cdot\dfrac{\partial\boldsymbol{\mathscr{r}}}{\partial t}$, or $c\mathscr{r}\left(1 - \dfrac{\partial t_r}{\partial t}\right) = \boldsymbol{\mathscr{r}}\cdot\dfrac{\partial\boldsymbol{\mathscr{r}}}{\partial t}$. Now $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$, so

$\dfrac{\partial\boldsymbol{\mathscr{r}}}{\partial t} = -\dfrac{\partial\mathbf{w}}{\partial t} = -\dfrac{\partial\mathbf{w}}{\partial t_r}\dfrac{\partial t_r}{\partial t} = -\mathbf{v}\dfrac{\partial t_r}{\partial t}$; $c\mathscr{r}\left(1 - \dfrac{\partial t_r}{\partial t}\right) = -\boldsymbol{\mathscr{r}}\cdot\mathbf{v}\dfrac{\partial t_r}{\partial t}$; $c\mathscr{r} = \dfrac{\partial t_r}{\partial t}(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = \dfrac{\partial t_r}{\partial t}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})$ (Eq. 10.64),

and hence $\dfrac{\partial t_r}{\partial t} = \dfrac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}$. qed

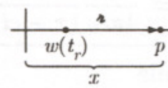
Now Eq. 10.40 says $\mathbf{A}(\mathbf{r},t) = \dfrac{\mathbf{v}}{c^2}V(\mathbf{r},t)$, so

$$\begin{aligned}
\frac{\partial \mathbf{A}}{\partial t} &= \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial t}V + \mathbf{v}\frac{\partial V}{\partial t}\right) = \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}V + \mathbf{v}\frac{\partial V}{\partial t}\right) \\
&= \frac{1}{c^2}\left[\mathbf{a}\frac{\partial t_r}{\partial t}\frac{1}{4\pi\epsilon_0}\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}} + \mathbf{v}\frac{1}{4\pi\epsilon_0}\frac{-qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\frac{\partial}{\partial t}(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})\right] \\
&= \frac{1}{c^2}\frac{qc}{4\pi\epsilon_0}\left[\frac{\mathbf{a}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\frac{\partial t_r}{\partial t} - \frac{\mathbf{v}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left(c\frac{\partial \mathscr{r}}{\partial t} - \frac{\partial \boldsymbol{\mathscr{r}}}{\partial t}\cdot\mathbf{v} - \boldsymbol{\mathscr{r}}\cdot\frac{\partial \mathbf{v}}{\partial t}\right)\right].
\end{aligned}$$

But $\mathscr{r} = c(t-t_r) \Rightarrow \dfrac{\partial \mathscr{r}}{\partial t} = c\left(1 - \dfrac{\partial t_r}{\partial t}\right)$, $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \Rightarrow \dfrac{\partial \boldsymbol{\mathscr{r}}}{\partial t} = -\mathbf{v}\dfrac{\partial t_r}{\partial t}$ (as above), and $\dfrac{\partial \mathbf{v}}{\partial t} = \dfrac{\partial \mathbf{v}}{\partial t_r}\dfrac{\partial t_r}{\partial t} = \mathbf{a}\dfrac{\partial t_r}{\partial t}$.

$$\begin{aligned}
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{\mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\frac{\partial t_r}{\partial t} - \mathbf{v}\left[c^2\left(1 - \frac{\partial t_r}{\partial t}\right) + v^2\frac{\partial t_r}{\partial t} - \boldsymbol{\mathscr{r}}\cdot\mathbf{a}\frac{\partial t_r}{\partial t}\right]\right\} \\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v} + \left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{\partial t_r}{\partial t}\right\} \\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v} + \left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right\} \\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[-c^2\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}) + c\mathscr{r}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + c\mathscr{r}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right] \\
&= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})\left(-\mathbf{v} + \frac{\mathscr{r}}{c}\mathbf{a}\right) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]. \quad \text{qed}
\end{aligned}$$

Problem 10.18



$\mathbf{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$. Here $\mathbf{v} = v\,\hat{\mathbf{x}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, and, for points to the *right*, $\hat{\boldsymbol{\mathscr{r}}} = \hat{\mathbf{x}}$. So $\mathbf{u} = (c-v)\,\hat{\mathbf{x}}$, $\mathbf{u}\times\mathbf{a} = 0$, and $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c-v)$.

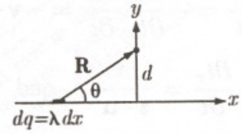
$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c-v)^3}(c^2 - v^2)(c-v)\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)^2}{(c-v)^3}\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}}; \\
\mathbf{B} &= \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E} = 0. \quad \text{qed}
\end{aligned}$$

For field points to the *left*, $\hat{\boldsymbol{\mathscr{r}}} = -\hat{\mathbf{x}}$ and $\mathbf{u} = -(c+v)\,\hat{\mathbf{x}}$, so $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c+v)$, and

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2 - v^2)(c+v)\,\hat{\mathbf{x}} = \boxed{\frac{-q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}};\ \mathbf{B} = 0.}$$

Problem 10.19

(a) $\mathbf{E} = \dfrac{\lambda}{4\pi\epsilon_0}(1 - v^2/c^2)\displaystyle\int \frac{\hat{\mathbf{R}}}{R^2}\frac{dx}{\left[1 - (v/c)^2\sin^2\theta\right]^{3/2}}$.



The horizontal components cancel; the vertical component of $\hat{\mathbf{R}}$ is $\sin\theta$ (see diagram). Here $d = R\sin\theta$, so $\dfrac{1}{R^2} = \dfrac{\sin^2\theta}{d^2}$; $-\dfrac{x}{d} = \cot\theta$, so $dx = -d(-\csc^2\theta)\,d\theta = \dfrac{d}{\sin^2\theta}\,d\theta$;

$\frac{1}{R^2}\,dx = \frac{d}{\sin^2\theta}\frac{\sin^2\theta}{d^2}\,d\theta = \frac{d\theta}{d}$. Thus

$$\begin{aligned}
\mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0}(1-v^2/c^2)\left(\frac{\hat{\mathbf{y}}}{d}\right)\int_0^\pi \frac{\sin\theta}{\left[1-(v/c)^2\sin^2\theta\right]^{3/2}}\,d\theta. \quad \text{Let } z \equiv \cos\theta, \text{ so } \sin^2\theta = 1-z^2. \\
&= \frac{\lambda(1-v^2/c^2)\,\hat{\mathbf{y}}}{4\pi\epsilon_0 d}\int_{-1}^{1}\frac{1}{\left[1-(v/c)^2+(v/c)^2z^2\right]^{3/2}}\,dz \\
&= \frac{\lambda(1-v^2/c^2)\,\hat{\mathbf{y}}}{4\pi\epsilon_0 d}\left[\frac{1}{(v/c)^3}\frac{z}{(c^2/v^2-1)\sqrt{(c/v)^2-1+z^2}}\right]\Bigg|_{-1}^{+1} \\
&= \frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0 d}\frac{c}{v}\frac{1}{(1-c^2/v^2)}\frac{2}{\sqrt{(c/v)^2-1+1}}\,\hat{\mathbf{y}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}\,\hat{\mathbf{y}}} \text{ (same as for a line charge at rest).}
\end{aligned}$$

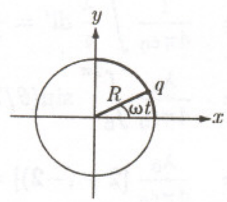
(b) $\mathbf{B} = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$ for each segment $dq = \lambda\,dx$. Since $\mathbf{v}$ is constant, it comes outside the integral, and the same formula holds for the *total* field:

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}) = \frac{1}{c^2}v\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) = \mu_0\epsilon_0 v\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}\,\hat{\mathbf{z}} = \frac{\mu_0}{4\pi}\frac{2\lambda v}{d}\,\hat{\mathbf{z}}.$$

But $\lambda v = I$, so $\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\frac{2I}{d}\,\hat{\boldsymbol{\phi}}}$ (the same as we got in magnetostatics, Eq. 5.36 and Ex. 5.7).

Problem 10.20

$\mathbf{w}(t) = R[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$;
$\mathbf{v}(t) = R\omega[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$;
$\mathbf{a}(t) = -R\omega^2[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] = -\omega^2\mathbf{w}(t)$;
$\boldsymbol{\mathscr{r}} = -\mathbf{w}(t_r)$;
$\mathscr{r} = R$;
$t_r = t - R/c$;
$\hat{\boldsymbol{\mathscr{r}}} = -[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}]$;



$$\begin{aligned}
\mathbf{u} &= c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}(t_r) = -c[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}] - \omega R[-\sin(\omega t_r)\,\hat{\mathbf{x}} + \cos(\omega t_r)\,\hat{\mathbf{y}}] \\
&= -\left\{[c\cos(\omega t_r) - \omega R\sin(\omega t_r)]\,\hat{\mathbf{x}} + [c\sin(\omega t_r) + \omega R\cos(\omega t_r)]\,\hat{\mathbf{y}}\right\}; \\
\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a}) &= (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a};\quad \boldsymbol{\mathscr{r}}\cdot\mathbf{a} = -\mathbf{w}\cdot(-\omega^2\mathbf{w}) = \omega^2R^2; \\
\boldsymbol{\mathscr{r}}\cdot\mathbf{u} &= R\left[c\cos^2(\omega t_r) - \omega R\sin(\omega t_r)\cos(\omega t_r) + c\sin^2(\omega t_r) + \omega R\sin(\omega t_r)\cos(\omega t_r)\right] = Rc;
\end{aligned}$$

$v^2 = (\omega R)^2$. So (Eq. 10.65):

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{R}{(Rc)^3}\left[\mathbf{u}\,(c^2-\omega^2R^2) + \mathbf{u}(\omega R)^2 - \mathbf{a}(Rc)\right] = \frac{q}{4\pi\epsilon_0}\frac{c\mathbf{u} - R\mathbf{a}}{(Rc)^2} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\left\{-[c^2\cos(\omega t_r) - \omega Rc\sin(\omega t_r)]\,\hat{\mathbf{x}} - [c^2\sin(\omega t_r) + \omega Rc\cos(\omega t_r)]\,\hat{\mathbf{y}}\right. \\
&\quad \left. + R^2\omega^2\cos(\omega t_r)\,\hat{\mathbf{x}} + R^2\omega^2\sin(\omega t_r)\,\hat{\mathbf{y}}\right\} \\
&= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\left\{\left[(\omega^2R^2 - c^2)\cos(\omega t_r) + \omega Rc\sin(\omega t_r)\right]\hat{\mathbf{x}} + \left[(\omega^2R^2 - c^2)\sin(\omega t_r) - \omega Rc\cos(\omega t_r)\right]\hat{\mathbf{y}}\right\}.}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{B} &= \frac{1}{c}\hat{\boldsymbol{\imath}} \times \mathbf{E} = \frac{1}{c}\left(\hat{\imath}_x E_y - \hat{\imath}_y E_x\right)\hat{\mathbf{z}} \\
&= -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\left\{\cos(\omega t_r)\left[(\omega^2R^2 - c^2)\sin(\omega t_r) - \omega Rc\cos(\omega t_r)\right]\right. \\
&\quad \left. - \sin(\omega t_r)\left[(\omega^2R^2 - c^2)\cos(\omega t_r) + \omega Rc\sin(\omega t_r)\right]\right\}\hat{\mathbf{z}} \\
&= -\frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\left[-\omega Rc\cos^2(\omega t_r) - \omega Rc\sin^2(\omega t_r)\right]\hat{\mathbf{z}} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\omega Rc\,\hat{\mathbf{z}} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{\omega}{Rc^2}\,\hat{\mathbf{z}}.}
\end{aligned}
$$

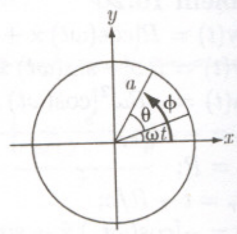
Notice that $\mathbf{B}$ is constant in time.

To obtain the field at the center of a circular *ring* of charge, let $q \to \lambda(2\pi R)$; for this ring to carry current I, we need $I = \lambda v = \lambda\omega R$, so $\lambda = I/\omega R$, and hence $q \to (I/\omega R)(2\pi R) = 2\pi I/\omega$. Thus $\mathbf{B} = \dfrac{2\pi I}{4\pi\epsilon_0}\dfrac{1}{Rc^2}\,\hat{\mathbf{z}}$, or, since $1/c^2 = \epsilon_0\mu_0$, $\boxed{\mathbf{B} = \dfrac{\mu_0 I}{2R}\,\hat{\mathbf{z}},}$ the same as Eq. 5.38, in the case $z = 0$.

Problem 10.21

$\lambda(\phi, t) = \lambda_0|\sin(\theta/2)|$, where $\theta = \phi - \omega t$. So the (retarded) scalar potential at the center is (Eq. 10.19)

$$
\begin{aligned}
V(t) &= \frac{1}{4\pi\epsilon_0}\int\frac{\lambda}{\imath}\,dl' = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda_0\left|\sin[(\phi-\omega t_r)/2]\right|}{a}a\,d\phi \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\int_0^{2\pi}\sin(\theta/2)\,d\theta = \frac{\lambda_0}{4\pi\epsilon_0}\left[-2\cos(\theta/2)\right]\Big|_0^{2\pi} \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\left[2-(-2)\right] = \boxed{\frac{\lambda_0}{\pi\epsilon_0}.}
\end{aligned}
$$



(Note: at fixed t_r, $d\phi = d\theta$, and it goes through one full cycle of ϕ *or* θ.)

Meanwhile $\mathbf{I}(\phi, t) = \lambda\mathbf{v} = \lambda_0\omega a\left|\sin[(\phi-\omega t)/2]\right|\hat{\boldsymbol{\phi}}$. From Eq. 10.19 (again)

$$
\mathbf{A}(t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}}{\imath}\,dl' = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{\lambda_0\omega a\left|\sin[(\phi-\omega t_r)/2]\right|\hat{\boldsymbol{\phi}}}{a}a\,d\phi.
$$

But $t_r = t - a/c$ is again constant, for the ϕ integration, and $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$.

$$
= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}\left|\sin[(\phi-\omega t_r)/2]\right|(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\,d\phi.
$$

Again, switch variables to $\theta = \phi - \omega t_r$, and integrate from $\theta = 0$ to $\theta = 2\pi$ (so we don't have to worry about the absolute value).

$$
= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}\sin(\theta/2)\left[-\sin(\theta+\omega t_r)\,\hat{\mathbf{x}} + \cos(\theta+\omega t_r)\,\hat{\mathbf{y}}\right]d\theta.
$$

Now

$$\int_0^{2\pi}\sin(\theta/2)\sin(\theta+\omega t_r)\,d\theta = \frac{1}{2}\int_0^{2\pi}[\cos(\theta/2+\omega t_r)-\cos(3\theta/2+\omega t_r)]\,d\theta$$

$$= \frac{1}{2}\left[2\sin(\theta/2+\omega t_r)-\frac{2}{3}\sin(3\theta/2+\omega t_r)\right]\Big|_0^{2\pi}$$

$$= \sin(\pi+\omega t_r)-\sin(\omega t_r)-\frac{1}{3}\sin(3\pi+\omega t_r)+\frac{1}{3}\sin(\omega t_r)$$

$$= -2\sin(\omega t_r)+\frac{2}{3}\sin(\omega t_r) = -\frac{4}{3}\sin(\omega t_r).$$

$$\int_0^{2\pi}\sin(\theta/2)\cos(\theta+\omega t_r)\,d\theta = \frac{1}{2}\int_0^{2\pi}[-\sin(\theta/2+\omega t_r)+\sin(3\theta/2+\omega t_r)]\,d\theta$$

$$= \frac{1}{2}\left[2\cos(\theta/2+\omega t_r)-\frac{2}{3}\cos(3\theta/2+\omega t_r)\right]\Big|_0^{2\pi}$$

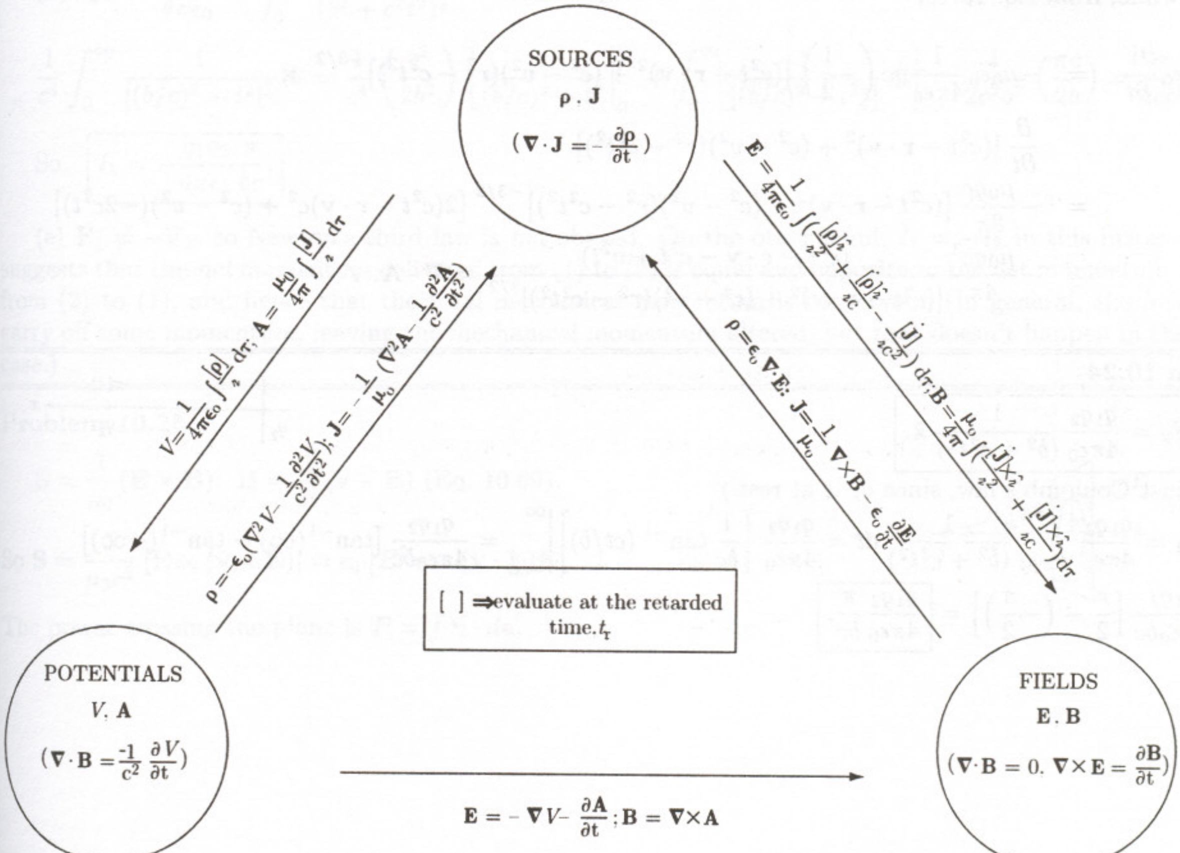
$$= \cos(\pi+\omega t_r)-\cos(\omega t_r)-\frac{1}{3}\cos(3\pi+\omega t_r)+\frac{1}{3}\cos(\omega t_r)$$

$$= -2\cos(\omega t_r)+\frac{2}{3}\cos(\omega t_r) = -\frac{4}{3}\cos(\omega t_r).$$

So

$$\mathbf{A}(t) = \frac{\mu_0\lambda_0\omega a}{4\pi}\left(\frac{4}{3}\right)[\sin(\omega t_r)\,\hat{\mathbf{x}}-\cos(\omega t_r)\,\hat{\mathbf{y}}] = \boxed{\frac{\mu_0\lambda_0\omega a}{3\pi}\{\sin[\omega(t-a/c)]\,\hat{\mathbf{x}}-\cos[\omega(t-a/c)]\,\hat{\mathbf{y}}\}.}$$

Problem 10.22



Problem 10.23

Using Product Rule #5, Eq. 10.43 ⇒

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{\mu_0}{4\pi}qc\mathbf{v}\cdot\boldsymbol{\nabla}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-1/2} \\
&= \frac{\mu_0 qc}{4\pi}\mathbf{v}\cdot\left\{-\frac{1}{2}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\boldsymbol{\nabla}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\right\} \\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left\{-2(c^2t-\mathbf{r}\cdot\mathbf{v})\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{v})+(c^2-v^2)\boldsymbol{\nabla}(r^2)\right\}.
\end{aligned}
$$

Product Rule #4 ⇒

$$
\begin{aligned}
\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{v}) &= \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{r})+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{r},\ \text{but}\ \boldsymbol{\nabla}\times\mathbf{r}=0, \\
(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{r} &= \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}) = v_x\,\hat{\mathbf{x}}+v_y\,\hat{\mathbf{y}}+v_z\,\hat{\mathbf{z}}=\mathbf{v},\ \text{and} \\
\boldsymbol{\nabla}(r^2) &= \boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{r}) = 2\mathbf{r}\times(\boldsymbol{\nabla}\times\mathbf{r})+2(\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{r}=2\mathbf{r}.\ \text{So}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{A} &= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left[-2(c^2t-\mathbf{r}\cdot\mathbf{v})\mathbf{v}+(c^2-v^2)2\mathbf{r}\right] \\
&= \frac{\mu_0 qc}{4\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left\{(c^2t-\mathbf{r}\cdot\mathbf{v})v^2-(c^2-v^2)(\mathbf{r}\cdot\mathbf{v})\right\}.
\end{aligned}
$$

But the term in curly brackets is : $c^2tv^2-v^2(\mathbf{r}\cdot\mathbf{v})-c^2(\mathbf{r}\cdot\mathbf{v})+v^2(\mathbf{r}\cdot\mathbf{v})=c^2(v^2t-\mathbf{r}\cdot\mathbf{v})$.

$$
= \frac{\mu_0 qc^3}{4\pi}\frac{(v^2t-\mathbf{r}\cdot\mathbf{v})}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}.
$$

Meanwhile, from Eq. 10.42,

$$
\begin{aligned}
-\mu_0\epsilon_0\frac{\partial V}{\partial t} &= -\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}qc\left(-\frac{1}{2}\right)\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\times \\
&\quad \frac{\partial}{\partial t}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right] \\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left[2(c^2t-\mathbf{r}\cdot\mathbf{v})c^2+(c^2-v^2)(-2c^2t)\right] \\
&= -\frac{\mu_0 qc^3}{4\pi}\frac{(c^2t-\mathbf{r}\cdot\mathbf{v}-c^2t+v^2t)}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}=\boldsymbol{\nabla}\cdot\mathbf{A}.\ \checkmark
\end{aligned}
$$

Problem 10.24

(a) $\boxed{\mathbf{F}_2=\frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{(b^2+c^2t^2)}\,\hat{\mathbf{x}}.}$

(This is just Coulomb's law, since q_1 is at rest.)

(b) $I_2=\frac{q_1q_2}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{1}{(b^2+c^2t^2)}\,dt=\frac{q_1q_2}{4\pi\epsilon_0}\left[\frac{1}{bc}\tan^{-1}(ct/b)\right]\Big|_{-\infty}^{\infty}=\frac{q_1q_2}{4\pi\epsilon_0 bc}\left[\tan^{-1}(\infty)-\tan^{-1}(-\infty)\right]$

$=\frac{q_1q_2}{4\pi\epsilon_0 bc}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=\boxed{\frac{q_1q_2}{4\pi\epsilon_0}\frac{\pi}{bc}.}$