State Space Control

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Solution of State-Space Equation

State space equations could be solved by following simple ODE solving procedure. Thus, at any time t_f the response of the state space system in standard form could be expressed as:

$$x(t_f) = e^{At_f} x(0) + \int_0^{t_f} e^{A(t_f-\tau)} B u(\tau)\, d\tau$$

The expression e^{At} is also known as state transition matrix and could be solved by procedures like Eigen-vector representation, Cayley-Hamilton Method and Resolvent Matrix Method.

$$e^{At} = I + \sum_{k=1}^{\infty} (At)^k / k! = \sum_{i=1}^{n} t_i e^{\lambda_i t} q_i$$

t_i, q_i are left & right eigen - vectors

Invariance of Eigen Values

$$\dot{x} = Ax + Bu, \qquad Use \qquad x = Tz$$

$$T\dot{z} = ATz + Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\left|\lambda I - T^{-1}AT\right| = \left|\lambda T^{-1}T - T^{-1}AT\right|$$

$$= \left|T^{-1}(\lambda I - A)T\right|$$

$$= \left|T^{-1}\right|\left|\lambda I - A\right|\left|T\right|$$

$$= \left|\lambda I - A\right|$$

How do we check the System Stability?

In state space form, the stability of a system depends on the Eigen Values of A which may be obtained from the characteristic equation as follows. If the real parts of the roots of this equation are strictly negative then the system is considered to be asymptotically stable

$$|\lambda I - A| = 0$$

$$e.g.$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$\lambda = -1, -2, -3 : \quad Stable \quad System$$

Controllability & Observability of Dynamic Systems

Controllability of a System

- A state x_1 of a system is "controllable" if all initial conditions x_0 at any previous time t can be transferred to x_1 in a finite time by some control function $u(t,x_0)$.
- If all the states are controllable then the system is completely controllable
- If controllability is restricted to depend upon t_0, then the system is *controllable at time t_0*.
- If a particular output can be obtained from any arbitrary x_0 at t_0, then the system is *output controllable*.

How to test the Controllability of a system?

- A system is state controllable at $t=t_0$ if there exists a continuous input u(t) such that it will drive the initial states $x(t_0)$ to any final state $x(t_f)$ within a finite time interval (t_f-t_0)

- The Controllability matrix for a system (A,B) is defined as:

$$C=[B \quad AB \quad A^2B \quad .. \quad A^{n-1}B]$$

- The state matrix A is of size *nxn.*

- The system is fully controllable iff Rank(*C*) = n

The concept of Stabilizability

- In general, controllability is considered to be a very strong constraint for a multi-degrees of freedom system.
- Hence, in practice, there exists an weaker definition of controllability – this is known as stabilizability.
- Let us consider the following system which is represented in modal or block diagram form:

$$\dot{x} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

- This system has two roots one at 3 and hence unstable, the other at -2 and hence stable. However, the control effort exists only for the unstable mode at 3 and hence the system is partly controllable or stabilizable.

Observability of a System

A state $x_1(t)$ at some given time is 'observable' if knowledge of the input u(t) and output y(t) over a finite segment of time completely determines $x_1(t)$.

The Observability matrix for a system (A,C) is defined as:

$$O^T = [C \quad C\,A \;\ldots\ldots\; C\,A^{n-1}]$$

The state matrix A is of size *n x n.*

The system is fully Observable iff Rank(*O*) = n

Example 1:

Check whether the following system is controllable and observable

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Solution 1:

- The order of the Plant is 2 here. Let us obtain the Controllability and the Observability Matrices

- Following earlier definitions:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$O = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- Since both *C* and *O* are of rank 2 which equals to the order of the system – hence, this system is fully controllable and observable.

Assignment: Check Controllability and Observability of the given system

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

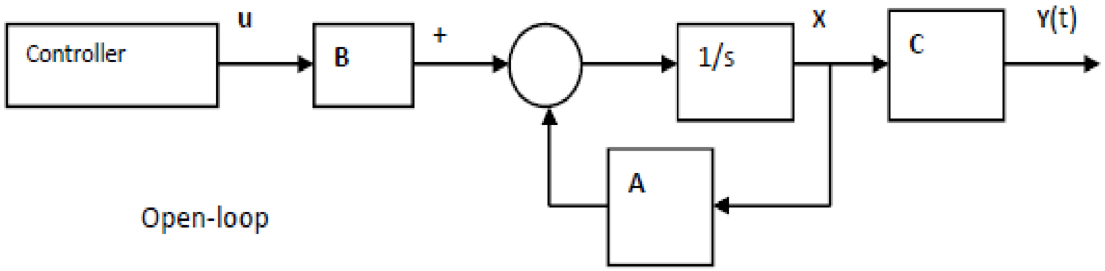
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Full State Feedback Control

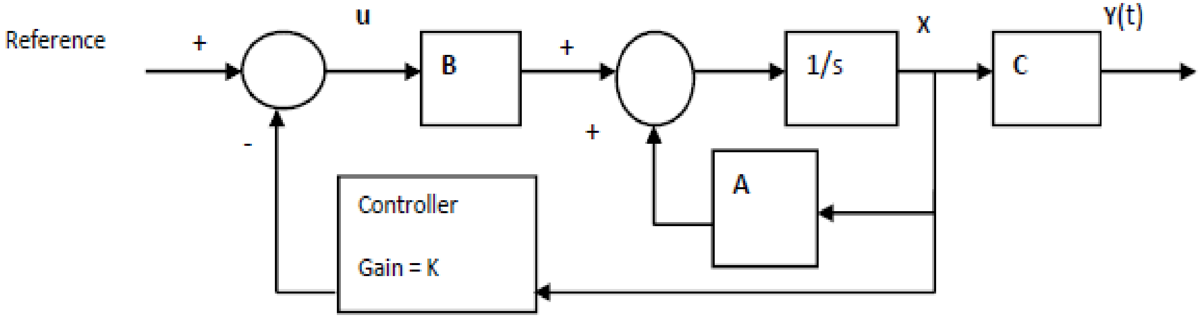
Introduction to Full-state feedback control

- Using the transfer function based technique; compensators are designed to predominantly control the response of second-order systems in frequency domain. By adjusting, the control gain, poles and zeroes of the compensator, the adverse effect of the system is compensated.
- The effect of higher-order poles are either neglected or compensated separately using notch filters.
- In case of full-state feed-back control, on the other hand, controllers could be designed to regulate the behavior of all the poles of the system.
- Although, such design is based on idealistic assumption of sensing all the states of the system, in reality, only some of the states are measured while the rest are estimated using numerical simulation.

Graphical Representation



Open-loop



Closed-loop

A System in Control Canonical Form

Let us consider the following system in control canonical form:

$$\left|s\mathbf{I}-\mathbf{A}\right| = s^{n} + a_{n-1}s^{n-1} + \dots\dots + a_1 s + a_0 = 0;$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

Controller Design

Let us define the control-law as

$$u = -K\,X$$

where, the control-gains ***K*** are represented in a matrix-form. For a single input system, ***u*** becomes scalar and consequently ***K*** will have a vector-form as

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$

The new state space equation could be written as

$$\dot{\mathbf{X}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{X}$$
$$\mathbf{Y} = \mathbf{C}\mathbf{X}$$

Controller Design – contd..

The characteristic equation corresponding to the closed-loop plant may be expanded as:

$$\left|sI-(A-BK)\right| = s^{n} + (a_{n-1}+k_{n})s^{n-1} + (a_{n-2}+k_{n-1})s^{n-2} + \cdots + (a_{0}+k_{1}) = 0$$

When the desired roots of the closed-loop system

$$\Lambda_{c} = \left[\lambda_{c_1} \quad \cdots \quad \lambda_{c_i} \quad \cdots \quad \lambda_{c_n}\right]$$

are known, the desired characteristic equation may be obtained as:

$$\prod_{i=1}^{n}(s-\lambda_{c_i}) = 0$$

$$\text{or, } s^{n} + d_{n-1}s^{n-1} + \cdots + d_{0} = 0$$

By comparing the coefficients of the polynomials of desired and initial characteristic polynomial one can get the elements of control gain vector **K** as

$$k_i = d_{i-1} - a_{i-1}, \quad \text{for } i = 1 \cdots n$$

Example: Controller Design for a Canonical System

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Characteristic Equation: $s^4 + 4s^3 + 3s^2 + 2s + 1 = 0$

The Controller Gain Structure

Let us consider the desired roots of the new system to be [-1,-2,-5,-10]. Then, the desired characteristic polynomial may be written as:

$$s^4 + 18s^3 + 97s^2 + 180s + 100 = 0$$

The initial characteristic equation was:

$$s^4 + 4s^3 + 3s^2 + 2s + 1 = 0$$

Hence, the controller gain may be obtained as:

$$K = [99, 178, 94, 14]$$

Assignment

Consider a third order system with the following governing equation:

$$\frac{d^3y}{dt^3}+7\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=u$$

Obtain, the state space representation of the system. Design a controller such that damping factor will be 0.6 and the settling time less than 1 second.

Full state feedback control for system in non-canonical form

- If the system is not in control canonical form, you have to find out the proper transformation matrix T to convert the system into canonical form.
- If x is the state vector corresponding to non-canonical form along with the corresponding state-space parameters A, B and C and z is the state vector in canonical form along with system parameters given by A_c, B_c and C_c, then, considering T to be the transformation matrix between the two linear systems such that :

$$\mathbf{x} = \mathbf{T}\,\mathbf{z}, \quad \text{then the state space equation in non-canonical form}$$

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}, \text{gets transformed to Canonical form as}$$

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\,\mathbf{A}\,\mathbf{T}\,\mathbf{z} + \mathbf{T}^{-1}\,\mathbf{B}\,\mathbf{u} = \mathbf{A}_c\,\mathbf{z} + \mathbf{B}_c\,\mathbf{u}$$

Full state feedback control in non-canonical form contd..

- The first task here is to find out the controllability matrix corresponding to the canonical form.
- How do we find it without knowing the transformation matrix?
- Well, we can find out the roots of the characteristic equation by evaluating the determinant of $[sI-A]^{-1}$
- Once we know the roots, we can write the new plant matrix in canonical form (see the standard form discussed before)
- In order to obtain the controllability matrix you also need to know the B matrix, for a single input system it is simply

$$B = \begin{bmatrix} 0 & 0 & . & . & 1 \end{bmatrix}^T$$

A System not in Control Canonical Form

After evaluating the controllability matrix related to the canonical form, you can find the controllability matrix corresponding to the non-canonical form as

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\,\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} = \mathbf{T}\hat{\mathbf{C}}_c$$

This controllability matrix can be used along with the controllability matrix corresponding to canonical form to obtain the transformation matrix between the two systems as:

$$\mathbf{T} = \hat{\mathbf{C}}\,\hat{\mathbf{C}}_c^{-1}$$

Now, you can represent the system to canonical form and obtain the corresponding gain as $\mathbf{K_c}$.
Then, the gain for non-canonical form **K** could be written as

$$\mathbf{K} = \mathbf{K}_C\,\mathbf{T}^{-1}$$

Controller Design using Ackermann's algorithm

For a single input system, one can use a direct relationship to find the controller gain **K** by using Ackermann's formulation as follows:

$$\mathbf{K} = \mathbf{R}\,\hat{\mathrm{C}}^{-1}\,\boldsymbol{\Psi}(\mathbf{A})$$

$$\text{with } \mathbf{R} = \begin{bmatrix} 0 & \cdots & \cdots & 1 \end{bmatrix},$$

$$\hat{\mathrm{C}} = [\mathbf{B}\ \ \mathbf{AB} \ldots\ldots\ldots\ \mathbf{A}^{\mathbf{n-1}}\mathbf{B}]$$

$$\text{and } \boldsymbol{\Psi}(\mathbf{A}) = \mathbf{A}^{n} + d_{n-1}\mathbf{A}^{n-1} + d_{n-2}\mathbf{A}^{n-2} + \ldots\ldots + d_0\mathbf{I}$$

where, d_i are the coefficients of the desired characteristic polynomial.

This is based on the fact that a matrix satisfies it's own characteristic equation, which is also known as Cayley-Hamilton's theorem

Where to place the Closed-loop poles?

- The placement of the pole often becomes one of the important prerogatives of the controller design. Given a freedom, you should design a system such that it is predominantly second order in nature. This implies that the higher order poles should be placed at least five times away from the real part of the second order poles.
- However, from the energy point of view, you should not place the closed loop poles quite far away from the open loop poles as the gain requirement would increase proportionately.
- The choice of B matrix also places an important role as the lesser controllable systems require higher gains.

Butterworth pole configurations

Following an optimization procedure, it is shown that the closed loop poles could be placed such that the characteristic equation is

$$\left(\frac{s}{\omega}\right)^{2k} = (-1)^{k+1}$$

Where, k is the number of poles required.

It can be shown that for k=1, you need to place a single pole on the –ve real axis at a distance ω from the origin. For, k=2, the radial distance remains unchanged, however, the poles will be complex and at angle 45^0 from the imaginary axis. These Configurations are known as Butterworth pole configuration.

Assignment:

A SDOF system has the following mass, stiffness and damping constant in appropriate units, m=0.1, c=0.01 and k=0.8; design a full-state feedback control, with an actuator influence matrix B^T = [0 1] and a forcing function 0.1u(t) (u(t) – unit step function), such that the desired eigen-values are at -1± 2j, respectively.

Special References for this lecture

- Feedback Control of Dynamic Systems, Frankline, Powell and Emami, Pearson

- Control System Design, Bernard Friedland, Dover Publications Inc