ASSIGNMENT V MSO 202 A

RESIDUE FORMULA, REMOVABLE SINGULARITIES, LAURENT SERIES

Exercise 0.1 : Show that $z = \pi/2$ is a simple pole of $\frac{\cos(z)}{(z-\pi/2)^2}$.

Exercise 0.2 : Locate poles a of f and find residue $\mathrm{res}_a f$ of $f(z) = \frac{1}{1+z^4}$ at a.

Exercise 0.3 : The aim of this exercise is to prove that $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \frac{\pi}{\sin \pi a}$ $(0 < a < 1)$ as an application of Residue Formula. For $0 < a < 1$, consider the function $f(z) = \frac{e^{az}}{1+e^z}$ and let γ_R denote the rectangular curve with parametrization

$$\gamma_1(t) = t \text{ for } -R \leqslant t \leqslant R, \quad \gamma_2(t) = R + it \text{ for } 0 \leqslant t \leqslant 2\pi,$$

$$\gamma_3(t) = -t + 2\pi i \text{ for } -R \leqslant t \leqslant R, \quad \gamma_4(t) = -R - it \text{ for } -2\pi \leqslant t \leqslant 0.$$

Verify the following:

(1) The only simple pole of f inside γ_R is at $a = \pi i$.
(2) The residue $\mathrm{res}_{\pi i} f$ of f at πi is equal to $-e^{a\pi i}$.
(3) $\sum_{j=1}^{4} \int_{\gamma_j} f(z)dz = -2\pi i e^{a\pi i}$.
(4) $\int_{\gamma_1} f(z)dz \to \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx$ as $R \to \infty$.
(5) $\int_{\gamma_3} f(z)dz \to -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx$ as $R \to \infty$.
(6) $\int_{\gamma_2} f(z)dz \to 0$ as $R \to \infty$ and $\int_{\gamma_4} f(z)dz \to 0$ as $R \to \infty$.

Exercise 0.4 : The aim of this exercise is to evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$ as an application of Residue Formula. Let $f(z) = \frac{1}{1+z^4}$. Let γ_R be union of $[-R, R]$ and semicircle C_R:

$$z_1(t) = t \ (-R \leqslant t \leqslant R), \ z_2(t) = Re^{it} \ (0 \leqslant t \leqslant \pi).$$

Verify the following:

(1) $e^{i\pi/4}, e^{3i\pi/4}$ are the only poles inside γ_R if $R > 1$.
(2) Compute $\mathrm{res}_{e^{i\pi/4}} f$ and $\mathrm{res}_{e^{i3\pi/4}} f$.
(3) $\int_{-R}^{R} \frac{1}{1+x^4}dx + \int_{C_R} f(z)dz = 2\pi i(\mathrm{res}_{e^{i\pi/4}} f + \mathrm{res}_{e^{i3\pi/4}} f)$.
(4) $\lim_{R\to\infty} \int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

Exercise 0.5 : Show that any polynomial $p(z)$, the function $p(z)\sin(1/z)$ has essential singularity at 0.

Exercise 0.6 : Write Laurent series of f around a and determine the type of the singularity at a (removable/pole/essential):

(1) $\frac{e^z}{(z-1)^3}$, $a = 1$;
(2) $(z-1)\cos(1/(z+2))$, $a = -2$;
(3) $(z - \sin z)/z^3$, $a = 0$.