Solutions Manual

Elasticity: Theory, Applications and Numerics
Second Edition

By

Martin H. Sadd
Professor
Department of Mechanical Engineering & Applied Mechanics
University of Rhode Island
Kingston, Rhode Island

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Foreword

Exercises found at the end of each chapter are an important ingredient of the text as they provide homework for student engagement, problems for examinations, and can be used in class to illustrate other features of the subject matter. This solutions manual is intended to aid the instructors in their own particular use of the exercises. Review of the solutions should help determine which problems would best serve the goals of homework, exams or be used in class.

The author is committed to continual improvement of engineering education and welcomes feedback from users of the text and solutions manual. Please feel free to send comments concerning suggested improvements or corrections to sadd@egr.uri.edu. Such feedback will be shared with the text user community via the publisher's web site.

Martin H. Sadd
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1-1.

(a) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 4 + 1 = 6$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1 = 25$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ (vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1 + 0 + 2 + 0 + 0 + 0 + 0 + 0 + 4 = 7$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1 + 0 + 4 = 5$ (scalar)

(b) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 2 + 2 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 4 + 0 + 0 + 4 + 1 + 0 + 16 + 4 = 30$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \text{ (vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 4 + 4 + 0 + 0 + 2 + 1 + 0 + 4 + 2 = 17$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4 + 1 + 1 = 6$ (scalar)

(c) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 0 + 4 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 1 + 1 + 1 + 0 + 4 + 0 + 1 + 16 = 25$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ (vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0 = 3$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1 + 1 + 0 = 2$ (scalar)

1-2.

(a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

(b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

(c) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

1-3.

$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij} \Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0$

From Exercise 1-2(a): $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2(b): $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2(c): $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

1-4.

$$\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = \begin{bmatrix} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

1-5.

$$\det(a_{ij}) = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{312}a_{13}a_{21}a_{32}$$
$$+ \varepsilon_{321}a_{13}a_{22}a_{31} + \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{213}a_{12}a_{21}a_{33}$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$
$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

1-6.

$$45^{\circ} \text{ rotation about } x_1\text{-axis} \Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$\text{From Exercise 1-1(a)}: \ b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\text{From Exercise 1-1(b)}: \ b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

$$\text{From Exercise 1-1(c)}: \ b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

1-7.

$$Q_{ij}=\begin{bmatrix}\cos(x_1',x_1) & \cos(x_1',x_2)\\ \cos(x_2',x_1) & \cos(x_2',x_2)\end{bmatrix}=\begin{bmatrix}\cos\theta & \cos(90^o-\theta)\\ \cos(90^o+\theta) & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$b_i'=Q_{ij}b_j=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}=\begin{bmatrix}b_1\cos\theta+b_2\sin\theta\\ -b_1\sin\theta+b_2\cos\theta\end{bmatrix}$$

$$a_{ij}'=Q_{ip}Q_{jq}a_{pq}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}^T$$

$$=\begin{bmatrix}a_{11}\cos^2\theta+(a_{12}+a_{21})\sin\theta\cos\theta+a_{22}\sin^2\theta & a_{12}\cos^2\theta-(a_{11}-a_{22})\sin\theta\cos\theta-a_{21}\sin^2\theta\\ a_{21}\cos^2\theta-(a_{11}-a_{22})\sin\theta\cos\theta-a_{12}\sin^2\theta & a_{11}\sin^2\theta-(a_{12}+a_{21})\sin\theta\cos\theta+a_{22}\cos^2\theta\end{bmatrix}$$

1-8.

$$a'\delta_{ij}'=Q_{ip}Q_{jq}a\delta_{pq}=aQ_{ip}Q_{jp}=a\delta_{ij}$$

1-9.

$$\alpha'\delta_{ij}'\delta_{kl}'+\beta'\delta_{ik}'\delta_{jl}'+\gamma'\delta_{il}'\delta_{jk}'=Q_{im}Q_{jn}Q_{kp}Q_{lq}(\alpha\delta_{mn}\delta_{pq}+\beta\delta_{mp}\delta_{nq}+\gamma\delta_{mq}\delta_{np})$$
$$=\alpha Q_{im}Q_{jm}Q_{kp}Q_{lp}+\beta Q_{im}Q_{jn}Q_{km}Q_{\ln}+\gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}=\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}$$

1-10.

$$C_{ijkl}=\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}=\alpha\delta_{ij}\delta_{kl}+\beta(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$
$$=\alpha\delta_{kl}\delta_{ij}+\beta(\delta_{ki}\delta_{lj}+\delta_{kj}\delta_{li})=C_{klij}$$

1-11.

If $a=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$

$$I_a=a_{ii}=\lambda_1+\lambda_2+\lambda_3$$

$$II_a=\begin{vmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{vmatrix}+\begin{vmatrix}\lambda_2 & 0\\ 0 & \lambda_3\end{vmatrix}+\begin{vmatrix}\lambda_1 & 0\\ 0 & \lambda_3\end{vmatrix}=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3$$

$$III_a=\begin{vmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{vmatrix}=\lambda_1\lambda_2\lambda_3$$

1-12.

(a) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1,\ II_a = -2,\ III_a = 0$

$\therefore$ Characteristic Eqn is $-\lambda^3 - \lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$

Roots $\Rightarrow \lambda_1 = -2\ ,\ \lambda_2 = 0,\ \lambda_3 = 1$

$\lambda_1 = -2$ Case :

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ {n_1^{(1)}}^2 + {n_2^{(1)}}^2 + {n_3^{(1)}}^2 = 1 \end{matrix} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ \boldsymbol{n}^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2 = 0$ Case :

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ {n_1^{(2)}}^2 + {n_2^{(2)}}^2 + {n_3^{(2)}}^2 = 1 \end{matrix} \Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow \boldsymbol{n}^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3 = 1$ Case :

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ {n_1^{(3)}}^2 + {n_2^{(3)}}^2 + {n_3^{(3)}}^2 = 1 \end{matrix} \Rightarrow n_1 = n_2 = 0,\ n_3^{(3)} = 1 \Rightarrow \boldsymbol{n}^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1-12.

(b) $a_{ij}=\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\Rightarrow I_a=-4,\ II_a=3,\ III_a=0$

$\therefore$ Characteristic Eqn is $-\lambda^3-4\lambda^2-3\lambda=0\Rightarrow\lambda(\lambda^2+4\lambda+3)=0\Rightarrow\lambda(\lambda+3)(\lambda+1)=0$

Roots $\Rightarrow\lambda_1=-3$, $\lambda_2=-1$, $\lambda_3=0$

$\lambda_1=-3$ Case :

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}n_1^{(1)}\\ n_2^{(1)}\\ n_3^{(1)}\end{bmatrix}=0\Rightarrow\begin{array}{c}n_1^{(1)}+n_2^{(1)}=0\\ n_3^{(1)}=0\\ n_1^{(1)2}+n_2^{(1)2}+n_3^{(1)2}=1\end{array}\Rightarrow n_1^{(1)}=-n_2^{(1)}=\pm\sqrt{2}/2,\ \boldsymbol{n}^{(1)}=\pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2=-1$ Case :

$$\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=0\Rightarrow\begin{array}{c}-n_1^{(2)}+n_2^{(2)}=0\\ n_3^{(2)}=0\\ n_1^{(2)2}+n_2^{(2)2}+n_3^{(2)2}=1\end{array}\Rightarrow n_1=n_2=\pm\sqrt{2}/2\Rightarrow\boldsymbol{n}^{(2)}=\pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3=0$ Case :

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=0\Rightarrow\begin{array}{c}-2n_1^{(3)}+n_2^{(3)}=0\\ n_1^{(3)}-2n_2^{(3)}=0\\ n_1^{(3)2}+n_2^{(3)2}+n_3^{(3)2}=1\end{array}\Rightarrow n_1=n_2=0,\ n_3^{(3)}=1\Rightarrow\boldsymbol{n}^{(3)}=\pm(0,0,1)$$

The rotation matrix is given by $Q_{ij}=\sqrt{2}/2\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}$ and

$$a'_{ij}=Q_{ip}Q_{jp}a_{pq}=\frac{1}{2}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}^T=\begin{bmatrix}-3 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

1-12.

(c) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -2,\ II_a = 0,\ III_a = 0$

$\therefore$ Characteristic Eqn is $-\lambda^3 - 2\lambda^2 = 0$ or $\lambda^2(\lambda+2) = 0$

Roots $\Rightarrow \lambda_1 = -2$, $\lambda_2 = \lambda_3 = 0$

$\lambda_1 = -2$ Case :

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{matrix} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ \boldsymbol{n}^{(1)} = \pm\sqrt{2}/2(-1,1,0)$$

$\lambda_2 = \lambda_3 = 0$ Case :

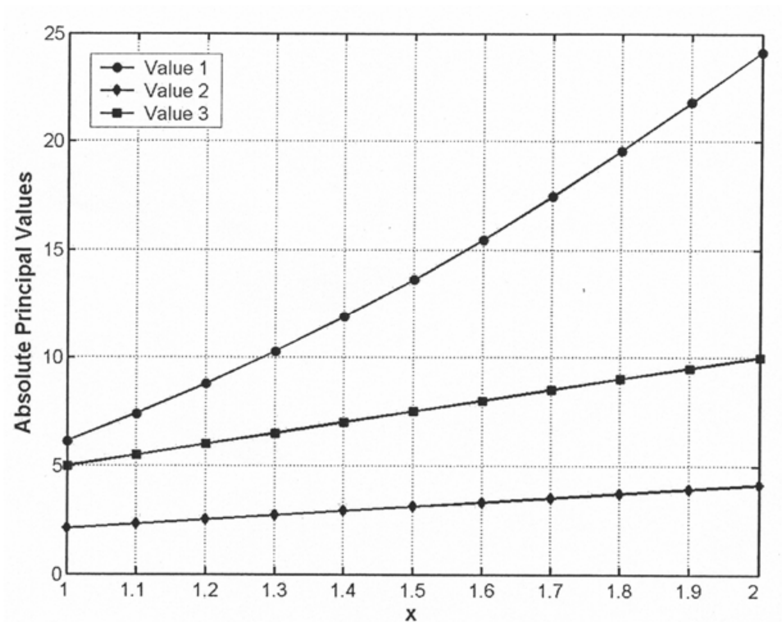
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1 + n_2 = 0 \\ n_1^{\ 2} + n_2^{\ 2} + n_3^{\ 2} = 1 \end{matrix} \Rightarrow n_1 = n_2\ ,\ n_3^{\ 2} = 1 - 2n_1^{\ 2} \Rightarrow \boldsymbol{n} = \pm(k,k,\sqrt{1\text{-}2k^2})$$

for arbitrary k, and thus directions are not uniquely determined. For convenience we may choose $k = \sqrt{2}/2$ and 0 to get $\boldsymbol{n}^{(2)} = \pm\sqrt{2}/2(1,1,0)$ and $\boldsymbol{n}^{(3)} = \pm(0,0,1)$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1-13*.



1-14.

(a) $\boldsymbol{u} = x_1\boldsymbol{e_1} + x_1x_2\boldsymbol{e_2} + 2x_1x_2x_3\boldsymbol{e_3}$

$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1x_2$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1x_2 & 2x_1x_2x_3 \end{vmatrix} = 2x_1x_3\boldsymbol{e_1} - 2x_2x_3\boldsymbol{e_2} + x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = 0\boldsymbol{e_1} + 0\boldsymbol{e_2} + 0\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 1 + x_1 + 2x_1x_2$$

(b) $\boldsymbol{u} = x_1^2\boldsymbol{e_1} + 2x_1x_2\boldsymbol{e_2} + x_3^3\boldsymbol{e_3}$

$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0\boldsymbol{e_1} - 0\boldsymbol{e_2} + 2x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 6x_3\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 4x_1 + 3x_3^2$$

(c) $\boldsymbol{u} = x_2^2\boldsymbol{e_1} + 2x_2x_3\boldsymbol{e_2} + 4x_1^2\boldsymbol{e_3}$

$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2\boldsymbol{e_1} - 8x_1\boldsymbol{e_2} - 2x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 8\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 3x_3$$

1-15.

$$a_i = -\frac{1}{2}\varepsilon_{ijk}a_{jk}$$

$$\varepsilon_{imn}a_i = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn}a_{jk} = -\frac{1}{2}\begin{vmatrix}\delta_{ii} & \delta_{im} & \delta_{in}\\ \delta_{ji} & \delta_{jm} & \delta_{jn}\\ \delta_{ki} & \delta_{km} & \delta_{kn}\end{vmatrix}a_{jk} = -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})a_{jk}$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) = -\frac{1}{2}(a_{mn} + a_{mn}) = -a_{mn}$$

$$\therefore\ a_{jk} = -\varepsilon_{ijk}a_i$$

1-16.

(a)

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi = \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi\boldsymbol{u}) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot\boldsymbol{u} + \phi(\nabla\cdot\boldsymbol{u})$$

(b)

$$\nabla\times(\phi\boldsymbol{u}) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_k = \nabla\phi\times\boldsymbol{u} + \phi(\nabla\times\boldsymbol{u})$$

$$\nabla\cdot(\boldsymbol{u}\times\boldsymbol{v}) = (\varepsilon_{ijk}u_jv_k)_{,i} = \varepsilon_{ijk}(u_jv_{k,i} + u_{j,i}v_k) = v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i} = \boldsymbol{v}\cdot(\nabla\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\nabla\times\boldsymbol{v})$$

$$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0 \text{ because of symmetry and antisymmetry in } jk$$

$$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

(c)

$$\nabla\cdot(\nabla\times\boldsymbol{u}) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0, \text{ because of symmetry and antisymmetry in } ij$$

$$\nabla\times(\nabla\times\boldsymbol{u}) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn} = (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$$

$$= \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$$

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_ju_{n,m} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_ju_{n,m} = u_nu_{n,i} - u_mu_{i,m}$$

$$= \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

1-17.

Cylindrical coordinates: $\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z$

$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$

$\hat{\boldsymbol{e}}_r = \cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2,\ \hat{\boldsymbol{e}}_\theta = -\sin\theta \boldsymbol{e}_1 + \cos\theta \boldsymbol{e}_2,\ \hat{\boldsymbol{e}}_z = \boldsymbol{e}_3$

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial\theta} = \hat{\boldsymbol{e}}_\theta\ ,\ \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial\theta} = -\hat{\boldsymbol{e}}_r\ ,\ \frac{\partial \hat{\boldsymbol{e}}_r}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_z}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_z}{\partial \theta} = \frac{\partial \hat{\boldsymbol{e}}_z}{\partial z} = 0$$

$$\nabla = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{\boldsymbol{e}}_r \frac{\partial f}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\boldsymbol{e}}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z$$

1-18.

Spherical coordinates : $\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\ x^2 = \xi^1 \sin\xi^2 \sin\xi^3\ ,\ x^3 = \xi^1 \cos\xi^2$

Scale factors :

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit vectors :

$$\hat{\boldsymbol{e}}_R = \cos\theta\sin\phi\boldsymbol{e}_1 + \sin\theta\sin\phi\boldsymbol{e}_2 + \cos\phi\boldsymbol{e}_3$$

$$\hat{\boldsymbol{e}}_\phi = \cos\theta\cos\phi\boldsymbol{e}_1 + \sin\theta\cos\phi\boldsymbol{e}_2 - \sin\phi\boldsymbol{e}_3$$

$$\hat{\boldsymbol{e}}_\theta = -\sin\theta\boldsymbol{e}_1 + \cos\theta\boldsymbol{e}_2$$

$$\frac{\partial\hat{\boldsymbol{e}}_R}{\partial R} = 0,\ \frac{\partial\hat{\boldsymbol{e}}_R}{\partial\phi} = \hat{\boldsymbol{e}}_\phi\ ,\ \frac{\partial\hat{\boldsymbol{e}}_R}{\partial\theta} = \sin\phi\hat{\boldsymbol{e}}_\theta$$

$$\frac{\partial\hat{\boldsymbol{e}}_\phi}{\partial R} = 0,\ \frac{\partial\hat{\boldsymbol{e}}_\phi}{\partial\phi} = -\hat{\boldsymbol{e}}_r\ ,\ \frac{\partial\hat{\boldsymbol{e}}_\phi}{\partial\theta} = \cos\phi\hat{\boldsymbol{e}}_\theta$$

$$\frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial R} = 0,\ \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\phi} = 0\ ,\ \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\theta} = -\cos\phi\hat{\boldsymbol{e}}_\phi$$

Using (1.9.12) - (1.9.16) $\Rightarrow$

$$\nabla = \hat{\boldsymbol{e}}_R\frac{\partial}{\partial R} + \hat{\boldsymbol{e}}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{e}}_\theta\frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}$$

$$\nabla f = \hat{\boldsymbol{e}}_R\frac{\partial f}{\partial R} + \hat{\boldsymbol{e}}_\phi\frac{1}{R}\frac{\partial f}{\partial\phi} + \hat{\boldsymbol{e}}_z\frac{1}{R\sin\phi}\frac{\partial f}{\partial\theta}$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(R\sin\phi u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}(Ru_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial f}{\partial\phi}) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta})$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial f}{\partial\phi}) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

1-18. Continued

$$\nabla \times \boldsymbol{u} = \left(\frac{1}{R^2 \sin\phi} [\frac{\partial}{\partial\phi}(R\sin\phi u_\theta) - \frac{\partial}{\partial\theta}(Ru_\phi)] \right) \hat{\boldsymbol{e}}_R + \left(\frac{1}{R\sin\phi} [\frac{\partial}{\partial\theta}(u_R) - \frac{\partial}{\partial R}(R\sin\phi u_\theta)] \right) \hat{\boldsymbol{e}}_\phi$$

$$+ \left(\frac{1}{R}\frac{\partial}{\partial R}[(Ru_\phi) - \frac{\partial}{\partial\phi}(u_R) \right) \hat{\boldsymbol{e}}_\theta$$

$$= \left[\frac{1}{R\sin\phi} \left(\frac{\partial}{\partial\phi}(\sin\phi u_\theta) - \frac{\partial u_\phi}{\partial\theta} \right) \right] \hat{\boldsymbol{e}}_{\boldsymbol{R}} + \left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(Ru_\theta) \right] \hat{\boldsymbol{e}}_{\boldsymbol{\varphi}}$$

$$+ \left[\frac{1}{R} \left(\frac{\partial}{\partial R}(Ru_\phi) - \frac{\partial u_R}{\partial\phi} \right) \right] \hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$$

2-1.

(a) $u = Axy,\ v = Bxz^2,\ w = C(x^2 + y^2)$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} Ay & \frac{1}{2}(Ax + Bz^2) & Cx \\ \cdot & 0 & Bxz + Cy \\ \cdot & \cdot & 0 \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \begin{bmatrix} 0 & \frac{1}{2}(Ax - Bz^2) & -Cx \\ -\frac{1}{2}(Ax - Bz^2) & 0 & Bxz - Cy \\ Cx & -Bxz + Cy & 0 \end{bmatrix}$$

(b) $u = Ax^2,\ v = Bxy,\ w = Cxyz$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} 2Ax & \frac{1}{2}By & \frac{1}{2}Cyz \\ \cdot & Bx & \frac{1}{2}Cxz \\ \cdot & \cdot & Cxy \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \begin{bmatrix} 0 & -\frac{1}{2}By & -\frac{1}{2}Cyz \\ \frac{1}{2}By & 0 & -\frac{1}{2}Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & 0 \end{bmatrix}$$

(c) $u = Ayz^3,\ v = Bxy^2,\ w = C(x^2 + z^2)$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} 0 & \frac{1}{2}(Az^3 + By^2) & \frac{1}{2}(3Ayz^2 + 2Cx) \\ \cdot & 2Bxy & 0 \\ \cdot & \cdot & 2Cz \end{bmatrix}$$

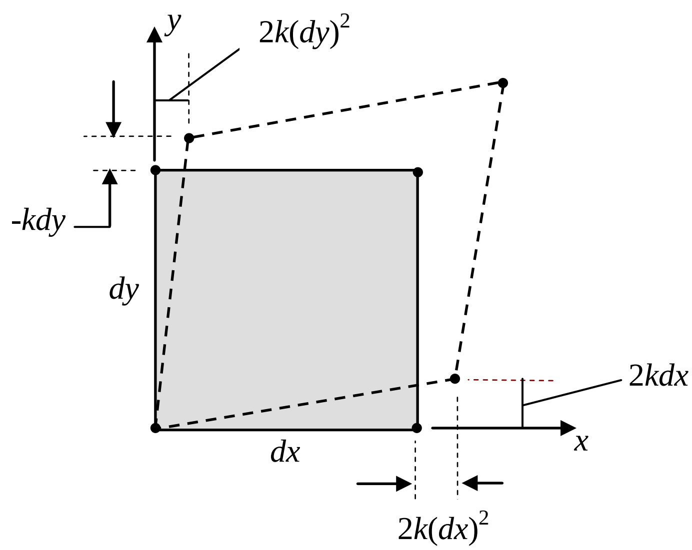
$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \begin{bmatrix} 0 & \frac{1}{2}(Az^3 - By^2) & \frac{1}{2}(3Ayz^2 - 2Cx) \\ -\frac{1}{2}(Az^3 - By^2)) & 0 & 0 \\ -\frac{1}{2}(3Ayz^2 - 2Cx) & 0 & 0 \end{bmatrix}$$

2-2.

$$u = k(x^2 + y^2),\ v = k(2x - y),\ w = 0$$

$$e_x = \frac{\partial u}{\partial x} = 2kx,\ e_y = \frac{\partial v}{\partial y} = -k,\ \gamma_{xy} = 2e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2ky + 2k = 2k(1+y)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(2k - 2ky) = k(1-y)$$



2-3.

Assume two - dimensional behavior in x, y - plane

$$e_x = \frac{\partial u}{\partial x} = C_1 \Rightarrow u = C_1 x + f(y)$$

$$e_y = \frac{\partial v}{\partial y} = -C_2 \Rightarrow v = -C_2 y + g(x)$$

$$e_z = \frac{\partial w}{\partial z} = 0 \Rightarrow w = h(x, y)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{df}{dy} + \frac{dg}{dx} = 0 \Rightarrow \frac{df}{dy} = -\frac{dg}{dx} = \text{constant} = a$$

$$\therefore\ f = ay + d_1 \text{ and } g = -ax + d_2$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h = f_1(x)$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \Rightarrow \frac{\partial h}{\partial x} = 0 \Rightarrow h = g_1(y)$$

$$\therefore\ h = \text{constant} = d_3$$

Combining the individual results and redefining the constants of integration gives

$$u = C_1 x - \omega_z y + u_o$$

$$v = -C_2 y + \omega_z x + v_o$$

$$w = w_o$$

Note the rigid body motion terms defined by relations (2.2.9)

2-4.

$$e_x = \frac{\partial u}{\partial x} = Az \Rightarrow u = Azx + f(y,z)$$

$$e_y = \frac{\partial v}{\partial y} = Az \Rightarrow v = Azy + g(x,z)$$

$$e_z = \frac{\partial w}{\partial z} = Bz \Rightarrow w = \frac{1}{2}Bz^2 + h(x,y)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x} = L(z)$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \Rightarrow Ay + \frac{\partial g}{\partial z} = -\frac{\partial h}{\partial y} \Rightarrow \frac{\partial g}{\partial z} = -Ay - \frac{\partial h}{\partial y} = M(x)$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \Rightarrow \frac{\partial h}{\partial x} = -Ax - \frac{\partial f}{\partial z} \Rightarrow \frac{\partial h}{\partial x} = -Ax - \frac{\partial f}{\partial z} = N(y)$$

From the previous three relations, $L'(z) = -N'(y)$, $L'(z) = -M'(x)$, $M'(x) = -N'(y)$

$\Rightarrow L'(z) = M'(x) = N'(y) = 0 \Rightarrow L(z) = a$, $M(x) = b$, $N(y) = c$

$\therefore \dfrac{\partial f}{\partial y} = L(z) = a \Rightarrow f = ay + f_1(x,z)$ and

$\dfrac{\partial f}{\partial z} = -Ax - N(y) = -Ax - c \Rightarrow f = -Axz - cz + f_2(x,y)$

To satisfy each form $f = -Axz + ay - cz + d_1$. Forms for g and h following in similar fashion giving

$g = Azy - ax + bz + d_2$ and $h = -\dfrac{1}{2}Ay^2 - by + cx + d_3$.

Combining the individual results and redefining the constants of integration gives

$u = Azx - \omega_z y + \omega_y z + u_o$

$v = Azy - \omega_x z + \omega_z x + v_o$

$w = \dfrac{1}{2}(Bz^2 - Ay^2) - \omega_y x + \omega_x y + w_o$

2-5.

$u^* = u_o - \omega_z y + \omega_y z$, $v^* = v_o - \omega_x z + \omega_z x$, $w^* = w_o - \omega_y x + \omega_x y$

$$e_x = \frac{\partial u^*}{\partial x} = 0,\; e_y = \frac{\partial v^*}{\partial y} = 0,\; e_z = \frac{\partial w^*}{\partial z} = 0$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x}\right) = \frac{1}{2}(-\omega_z + \omega_z) = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v^*}{\partial z} + \frac{\partial w^*}{\partial y}\right) = \frac{1}{2}(-\omega_x + \omega_x) = 0$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial w^*}{\partial x} + \frac{\partial u^*}{\partial z}\right) = \frac{1}{2}(-\omega_y + \omega_y) = 0$$

$$e_x = \frac{\partial u}{\partial x} = 0 \Rightarrow u = f(y,z)$$

$$e_y = \frac{\partial v}{\partial y} = 0 \Rightarrow v = g(z,x)$$

$$e_z = \frac{\partial w}{\partial z} = 0 \Rightarrow w = h(x,y)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x} = L(z)$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \Rightarrow \frac{\partial g}{\partial z} + \frac{\partial h}{\partial y} = 0 \Rightarrow \frac{\partial g}{\partial z} = -\frac{\partial h}{\partial y} = M(x)$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \Rightarrow \frac{\partial h}{\partial x} + \frac{\partial f}{\partial z} = 0 \Rightarrow \frac{\partial h}{\partial x} = -\frac{\partial f}{\partial z} = N(y)$$

From the previous three relations, $L'(z) = -N'(y)$, $L'(z) = -M'(x)$, $M'(x) = -N'(y)$

$\Rightarrow L'(z) = M'(x) = N'(y) = 0 \Rightarrow L(z) = a$, $M(x) = b$, $N(y) = c$

$\therefore \dfrac{\partial f}{\partial y} = L(z) = a \Rightarrow f = ay + f_1(x,z)$ and $\dfrac{\partial f}{\partial z} = -N(y) = -c \Rightarrow f = -cz + f_2(x,y)$

To satisfy each form $f = ay - cz + d_1$. Forms for g and h following in similar fashion giving $g = -ax + bz + d_2$ and $h = -by + cx + d_3$. Combining the individual results and redefining the constants of integration gives the rigid body motion form

$u = u_o - \omega_z y + \omega_y z$

$v = v_o - \omega_x z + \omega_z x$

$w = w_o - \omega_y x + \omega_x y$

2-6.

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Use two - dimensional transformation theory

$$e'_{ij} = \begin{bmatrix} e_r & e_{r\theta} \\ e_{r\theta} & e_\theta \end{bmatrix} = Q_{ip}Q_{jq}e_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e_x & e_{xy} \\ e_{xy} & e_y \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta & \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta \\ \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta & \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \end{bmatrix}$$

2-7.

$$e_a = e_x\cos^2 30^o + e_y\sin^2 30^o + 2e_{xy}\sin 30^o\cos 30^o \Rightarrow$$

$$0.001 = e_x\frac{3}{4} + e_y\frac{1}{4} + e_{xy}\frac{\sqrt{3}}{2}$$

$$e_b = e_x\cos^2 90^o + e_y\sin^2 90^o + 2e_{xy}\sin 90^o\cos 90^o \Rightarrow$$

$$0.002 = e_y$$

$$e_c = e_x\cos^2 150^o + e_y\sin^2 150^o + 2e_{xy}\sin 150^o\cos 150^o \Rightarrow$$

$$0.004 = e_x\frac{3}{4} + e_y\frac{1}{4} - e_{xy}\frac{\sqrt{3}}{2}$$

Solving for the strains : $e_x = 0.0027, e_y = 0.002,\ e_{xy} = -0.0017$

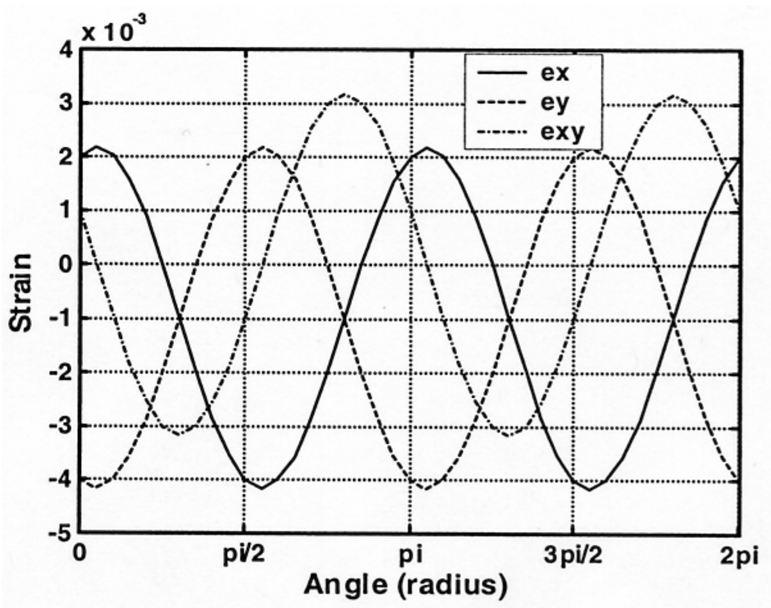
2-8*.

$$e_x' = -0.001 + 0.003\cos 2\theta + 0.001\sin 2\theta$$
$$e_y' = -0.001 - 0.003\cos 2\theta - 0.001\sin 2\theta$$
$$e_{xy}' = -0.003\sin 2\theta + 0.001\cos 2\theta$$

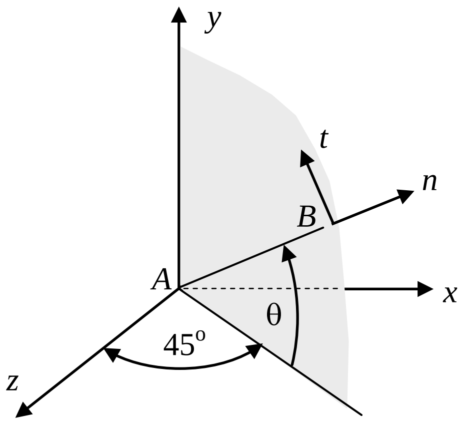
The maximum values are found to be

$$(e_x')_{\max} = 0.0022 \ @ \ \theta = 9.2^o, 189.2^o$$
$$(e_y')_{\max} = 0.0022 \ @ \ \theta = 99.2^o, 279.2^o$$
$$(e_{xy}')_{\max} = 0.0032 \ @ \ \theta = 144.2^o, 324.2^o$$

The MATLAB Plot is given by



2-9*.



Define directions n and t in the shaded plane as shown.

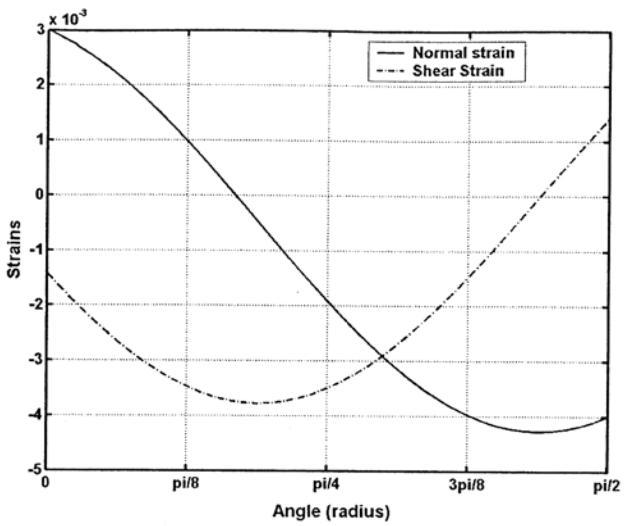
For direction $n: l_1 = \dfrac{1}{\sqrt{2}}\cos\theta,\ m_1 = \sin\theta,\ n_1 = \dfrac{1}{\sqrt{2}}\cos\theta$

For direction $t: l_2 = -\dfrac{1}{\sqrt{2}}\sin\theta,\ m_2 = \cos\theta,\ n_2 = -\dfrac{1}{\sqrt{2}}\sin\theta$

Relations (2.3.3) $\Rightarrow$

$$e_{nn} = e_x l_1^2 + e_y m_1^2 + e_z n_1^2 + 2(e_{xy} l_1 m_1 + e_{yz} m_1 n_1 + e_{zx} n_1 l_1)$$
$$= (3\cos^2\theta - 4\sin^2\theta - \sqrt{2}\sin 2\theta)\times 10^{-3}$$
$$e_{nt} = e_x l_1 l_2 + e_y m_1 m_2 + e_z n_1 n_2 + e_{xy}(l_1 m_2 + m_1 l_2) + e_{yz}(m_1 n_2 + n_1 m_2) + e_{zx}(n_1 l_2 + l_1 n_2)$$
$$= (-3.5\sin 2\theta - \sqrt{2}\cos 2\theta)\times 10^{-3}$$

The MATLAB plot of these relations are



2-10*.

MATLAB CODE

```
% Principal Value Problem
% Enter strain matrix
e=[2,-2,0;-2,-4,1;0,1,6]*0.001;
% Calculate principal values L and directions N
[N,L]=eig(e);
fprintf('Principal Values')
disp(diag(L)')
fprintf('Principal Directions')
N1=N(:,1)'
N2=N(:,2)'
N3=N(:,3)'
```

SCREEN OUTPUT

```
>> Principal Values   -0.0047    0.0026    0.0061
Principal Directions
N1 = -0.2852   -0.9543    0.0893
N2 = 0.9570   -0.2784    0.0815
N3 = -0.0529    0.1087    0.9927
```

2-11.

$$\text{Volume of Undeformed Element} = V_o = dxdydz$$

$$\text{Volume of Deformed Element} = V_f = (1+e_1)dx(1+e_2)dy(1+e_3)dz$$

$$= dxdydz + (e_1 + e_2 + e_3)dxdydz + \text{terms of } O(e^2, e^3) \approx (1+\vartheta)dxdydz = (1+\vartheta)V_o$$

$$\therefore \Delta V = V_f - V_o = (1+\vartheta)V_o - V_o = \vartheta V_o \Rightarrow \vartheta = \frac{\Delta V}{V_o}$$

2-12.

$$e_{ij} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & -4 & 1 \\ 0 & 1 & 6 \end{bmatrix} \times 10^{-3}$$

$$\text{Spherical Strain : } \tilde{e}_{ij} = \frac{1}{3}e_{kk}\delta_{ij} = \begin{bmatrix} 4/3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \times 10^{-3}$$

$$\text{Deviatoric Strain : } \hat{e}_{ij} = e_{ij} - \tilde{e}_{ij} = \begin{bmatrix} 2/3 & -2 & 0 \\ -2 & -16/3 & 1 \\ 0 & 1 & 14/3 \end{bmatrix} \times 10^{-3}$$

$$\hat{e}_{kk} = 2/3 - 16/3 + 14/3 = 0 \Rightarrow \text{No volumetric changes associated with } \hat{e}_{ij}$$

2-13.

$$e_x = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2 e_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}$$

$$e_y = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \Rightarrow 2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial y \partial x \partial y} + \frac{\partial^3 v}{\partial x \partial x \partial y}$$

If the displacements are single - valued and continuous, we can interchange the order of differentiation

$$\Rightarrow \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

2-14.

$$\eta_{ij} = \varepsilon_{ikl}\varepsilon_{jmp}e_{lp,km} = \begin{vmatrix} \delta_{ij} & \delta_{im} & \delta_{ip} \\ \delta_{kj} & \delta_{km} & \delta_{kp} \\ \delta_{lj} & \delta_{lm} & \delta_{lp} \end{vmatrix} e_{lp,km}$$

$$= \left[\delta_{ij}(\delta_{km}\delta_{lp} - \delta_{kp}\delta_{lm}) - \delta_{im}(\delta_{kj}\delta_{lp} - \delta_{kp}\delta_{lj}) + \delta_{ip}\left(\delta_{kj}\delta_{lm} - \delta_{km}\delta_{lj}\right)\right] e_{lp,km}$$

$$= \delta_{ij}(e_{ll,kk} - e_{kl,kl}) - e_{ll,ij} + e_{jk,ki} + e_{li,jl} - e_{ji,kk}$$

If indices k and l are taken to be the same, the first term will drop, and we get

$\eta_{ij} = -(e_{ij,kk} + e_{kk,ij} - e_{ik,jk} - e_{jk,ik}) = 0$, which is the same as (2.6.1) with $k = l$

$$\nabla \times \boldsymbol{e} \times \nabla = -\nabla \times \nabla \times \boldsymbol{e}$$

$$\nabla \times \boldsymbol{e} = \varepsilon_{imn}e_{jn,m} \Rightarrow \nabla \times \nabla \times \boldsymbol{e} = \varepsilon_{pqj}\varepsilon_{imn}e_{jn,mq} = \varepsilon_{imn}\varepsilon_{pqj}e_{nj,mq} = \eta_{ip}$$

2-15.

$$\eta_{ij}=\varepsilon_{ikl}\varepsilon_{jmp}e_{lp,km}$$
$$\eta_{11}=\varepsilon_{1kl}\varepsilon_{1mp}e_{lp,km}=e_{22,33}+e_{33,22}-2e_{23,23}$$
$$\eta_{22}=\varepsilon_{2kl}\varepsilon_{2mp}e_{lp,km}=e_{33,11}+e_{11,33}-2e_{31,31}$$
$$\eta_{33}=\varepsilon_{3kl}\varepsilon_{3mp}e_{lp,km}=e_{11,22}+e_{22,11}-2e_{12,12}$$
$$\eta_{12}=\varepsilon_{1kl}\varepsilon_{2mp}e_{lp,km}=-e_{33,12}+(-e_{12,3}+e_{23,1}+e_{31,2})_{,3}$$
$$\eta_{23}=\varepsilon_{2kl}\varepsilon_{3mp}e_{lp,km}=-e_{11,23}+(-e_{23,1}+e_{31,2}+e_{12,3})_{,1}$$
$$\eta_{31}=\varepsilon_{3kl}\varepsilon_{1mp}e_{lp,km}=-e_{22,31}+(-e_{31,2}+e_{12,3}+e_{23,1})_{,2}$$
$$\eta_{ij,j}=\eta_{i1,1}+\eta_{i2,2}+\eta_{i3,3}\Rightarrow$$
$$i=1:\ \eta_{1j,j}=\eta_{11,1}+\eta_{12,2}+\eta_{13,3}=\left(e_{22,331}+e_{33,221}-2e_{23,231}\right)+\left(-e_{33,122}+(-e_{12,3}+e_{23,1}+e_{31,2})_{,32}\right)$$
$$=\left(-e_{22,313}+(-e_{31,2}+e_{12,3}+e_{23,1})_{,23}\right)=0$$

Likewise for the $i=2$ and $i=3$ cases

2-16.

Starting with the first three compatibility equations,

$$\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}\Rightarrow\frac{\partial^4 e_x}{\partial y^2\partial z^2}+\frac{\partial^4 e_y}{\partial x^2\partial z^2}=2\frac{\partial^4 e_{xy}}{\partial x\partial y\partial z^2}\quad(a)$$
$$\frac{\partial^2 e_y}{\partial z^2}+\frac{\partial^2 e_z}{\partial y^2}=2\frac{\partial^2 e_{yz}}{\partial y\partial z}\Rightarrow\frac{\partial^4 e_y}{\partial z^2\partial x^2}+\frac{\partial^4 e_z}{\partial y^2\partial x^2}=2\frac{\partial^4 e_{yz}}{\partial y\partial z\partial x^2}\quad(b)$$
$$\frac{\partial^2 e_z}{\partial x^2}+\frac{\partial^2 e_x}{\partial z^2}=2\frac{\partial^2 e_{zx}}{\partial z\partial x}\Rightarrow\frac{\partial^4 e_z}{\partial x^2\partial y^2}+\frac{\partial^4 e_x}{\partial z^2\partial y^2}=2\frac{\partial^4 e_{zx}}{\partial z\partial x\partial y^2}\quad(c)$$
$$(a)+(c)-(b)\Rightarrow\frac{\partial^4 e_x}{\partial y^2\partial z^2}=\frac{\partial^3}{\partial x\partial y\partial z}\left(-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right)$$
$$(b)+(a)-(c)\Rightarrow\frac{\partial^4 e_y}{\partial z^2\partial x^2}=\frac{\partial^3}{\partial x\partial y\partial z}\left(-\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}\right)$$
$$(c)+(b)-(a)\Rightarrow\frac{\partial^4 e_z}{\partial x^2\partial y^2}=\frac{\partial^3}{\partial x\partial y\partial z}\left(-\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}\right)$$

Using the last three compatibility equations $(2.6.2)_{4,5,6}$,

$\dfrac{\partial^2}{\partial y\partial z}(2.6.2)_4$ gives the first fourth order equation

$\dfrac{\partial^2}{\partial z\partial x}(2.6.2)_5$ gives the second fourth order equation

$\dfrac{\partial^2}{\partial x\partial y}(2.6.2)_6$ gives the third fourth order equation

2-17.

The given strains must satisfy the compatibility equations (2.6.2)

$$(2.6.2)_1 \Rightarrow \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow 6Ay + 6Ax = 2B(2x+2y) \Rightarrow 6A = 4B \Rightarrow A = \frac{2}{3}B$$

$$(2.6.2)_{2,3,4,5,6} \Rightarrow 0 = 0$$

2-18.

$$u = v = 0,\ w = \frac{b}{2\pi}\tan^{-1}\frac{y}{x}$$

$$e_x = \frac{\partial u}{\partial x} = 0\,,\ e_y = \frac{\partial v}{\partial y} = 0,\ e_z = \frac{\partial w}{\partial z} = 0,\ e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{b}{4\pi}\frac{\partial}{\partial y}(\tan^{-1}\frac{y}{x}) = \frac{b}{4\pi}\frac{x}{x^2+y^2}$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{b}{4\pi}\frac{\partial}{\partial x}(\tan^{-1}\frac{y}{x}) = -\frac{b}{4\pi}\frac{y}{x^2+y^2}$$

The six compatibility equations (2.6.2) yield

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

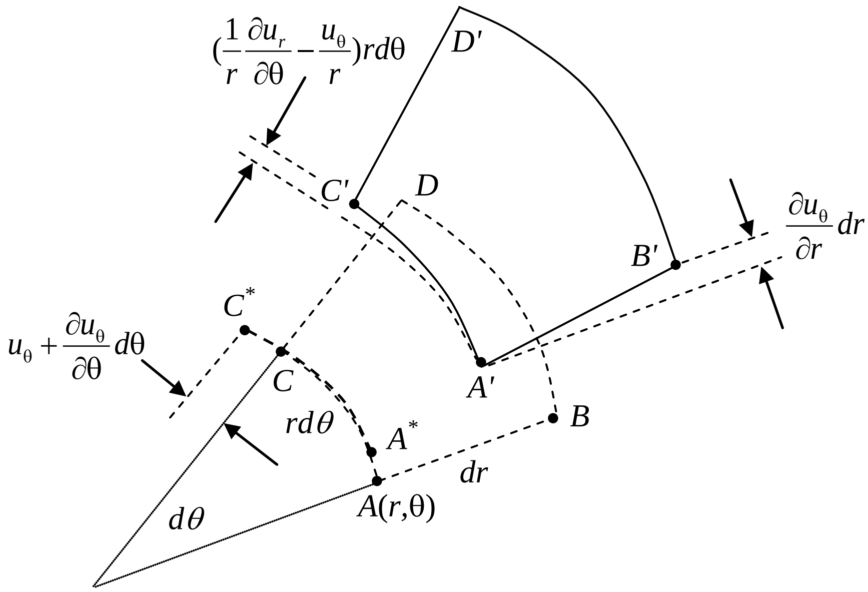
$$0 = -\frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right) \Rightarrow -\frac{\partial}{\partial x}(0) = 0 \Rightarrow 0 = 0$$

$$0 = -\frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{yz}}{\partial x}\right) \Rightarrow -\frac{\partial}{\partial y}(0) = 0 \Rightarrow 0 = 0$$

$$0 = 0$$

Thus all compatibility equations are satisfied even though the displacements are multi-valued.

2-19.



From the figure geometry,

$$e_r = \frac{A'B' - AB}{AB} = \frac{[(dr + \frac{\partial u_r}{\partial r} dr)^2 + (\frac{\partial u_\theta}{\partial r} dr)^2]^{1/2}}{dr} = \frac{\partial u_r}{\partial r}$$

$$e_\theta = \frac{A^*C^* - AC}{AC} + \frac{A'C' - A^*C^*}{A^*C^*}$$

$$= \frac{(u_\theta + \frac{\partial u_\theta}{\partial \theta} + rd\theta - u_\theta) - rd\theta}{rd\theta} + \frac{(r + u_r)d\theta - rd\theta}{rd\theta}$$

$$= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$e_{r\theta} = \frac{1}{2}(\angle CAB - \angle C'A'B') = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)$$

2-20.

$$e_r = \frac{\partial u_r}{\partial r}\,,\ e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right),\ e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

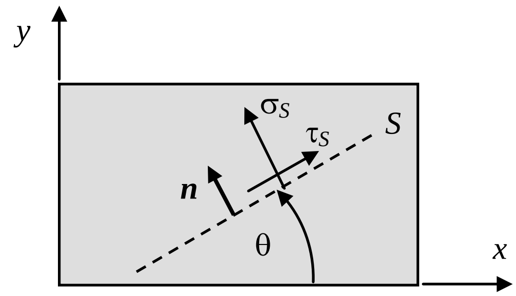
(a) $e_r = -\frac{A}{r^2}\,,\ e_\theta = \frac{1}{r}\left(\frac{A}{r} - B\sin\theta\right),\ e_{r\theta} = -\frac{B}{2r}\cos\theta$

(b) $e_r = 2Ar\,,\ e_\theta = \frac{1}{r}\left(Ar^2 + Br\cos\theta\right),\ e_{r\theta} = \frac{1}{2}\left(B\sin\theta - B\sin\theta\right) = 0$

(c) $e_r = 0\,,\ e_\theta = \frac{1}{r}\left(A\sin\theta + B\cos\theta - A\sin\theta - B\cos\theta\right) = 0$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}(A\cos\theta - B\sin\theta) + C - \frac{1}{r}(A\cos\theta - B\sin\theta + Cr)\right) = 0$$

3-1.



$$\sigma_{ij}=\begin{bmatrix} X & 0 & 0\\ 0 & Y & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$T_i^n=\sigma_{ij}n_j\ ,\ \boldsymbol{n}=(-\sin\theta,\cos\theta)$$

$$T_x^n=\sigma_x n_x=-X\sin\theta$$

$$T_y^n=\sigma_y n_y=Y\cos\theta$$

Stresses on oblique plane S :

$$\sigma_S=\sigma_y'=\sigma_x\sin^2\theta+\sigma_y\cos^2\theta-2\tau_{xy}\sin\theta\cos\theta=X\sin^2\theta+Y\cos^2\theta$$

$$\tau_S=\tau_{xy}'=-\sigma_x\sin\theta\cos\theta+\sigma_y\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta)$$

$$=-X\sin\theta\cos\theta+Y\sin\theta\cos\theta=(Y-X)\sin\theta\cos\theta$$

3-2*.

(a) $\sigma_{ij}=\begin{bmatrix} 2 & 1 & -4\\ 1 & 4 & 0\\ -4 & 0 & 1\end{bmatrix}$, $\boldsymbol{n}=(\cos\theta,\sin\theta,0)$

$$T_x^n=\sigma_x n_x+\tau_{xy}n_y+\tau_{xz}n_z=2\cos\theta+\sin\theta$$

$$T_y^n=\tau_{yx}n_x+\sigma_y n_y+\tau_{yz}n_z=\cos\theta+4\sin\theta$$

$$T_z^n=\tau_{zx}n_x+\tau_{zy}n_y+\sigma_z n_z=-4\cos\theta$$

$$\left|\boldsymbol{T}^n\right|=\sqrt{T_x^{n\,2}+T_y^{n\,2}+T_z^{n\,2}}=\sqrt{19+2\cos2\theta+6\sin2\theta}$$

(b) $\sigma_{ij}=\begin{bmatrix} 4 & 1 & 0\\ 1 & -6 & 2\\ 0 & 2 & 1\end{bmatrix}$, $\boldsymbol{n}=(\cos\theta,\sin\theta,0)$

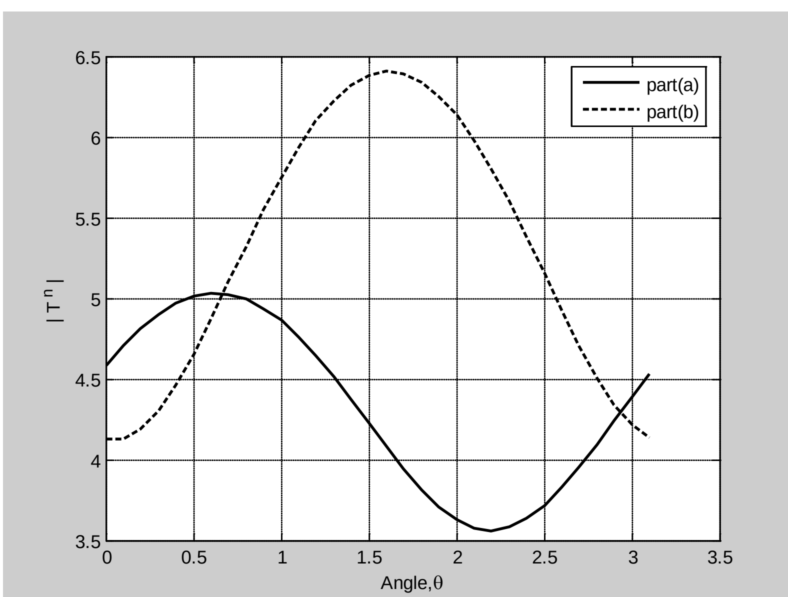
$$T_x^n=\sigma_x n_x+\tau_{xy}n_y+\tau_{xz}n_z=4\cos\theta+\sin\theta$$

$$T_y^n=\tau_{yx}n_x+\sigma_y n_y+\tau_{yz}n_z=\cos\theta-6\sin\theta$$

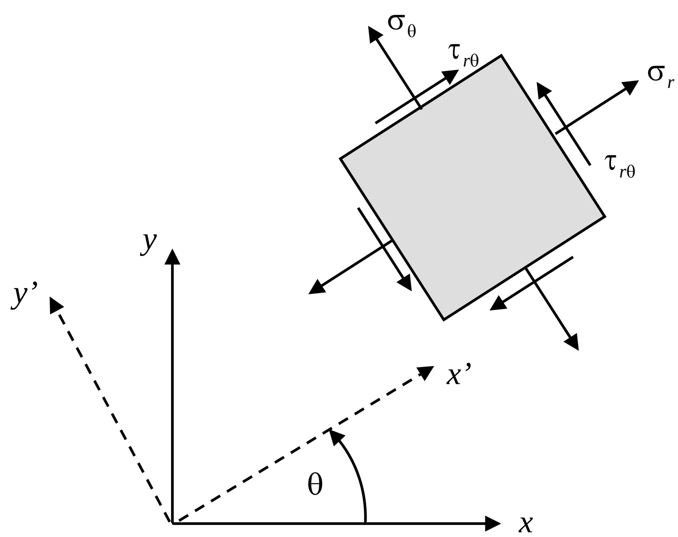
$$T_z^n=\tau_{zx}n_x+\tau_{zy}n_y+\sigma_z n_z=2\sin\theta$$

$$\left|\boldsymbol{T}^n\right|=\sqrt{T_x^{n\,2}+T_y^{n\,2}+T_z^{n\,2}}=\sqrt{29-12\cos2\theta-\sin2\theta}$$

MATLAB Plots :



3-3.



$$\sigma_r = \sigma'_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_\theta = \sigma'_y = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{r\theta} = \tau'_{xy} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy} \cos 2\theta$$

3-4.

Plane stress $\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = 0 \Rightarrow \begin{vmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{vmatrix} = 0$$

$$\Rightarrow \sigma^2 - (\sigma_x + \sigma_y)\sigma + \sigma_x\sigma_y - \tau_{xy}^2 = 0$$

$$\sigma_{1,2} = \frac{1}{2}[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x + \sigma_y)^2 - 4(\sigma_x\sigma_y - \tau_{xy}^2)}$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2)}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2)}$$

3-5.

Using principal coordinates with $\boldsymbol{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\sigma_{oct} = \sigma_{ij}n_in_j = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}\sigma_{kk} = \frac{1}{3}I_1$$

$$\tau_{oct}^2 = |\boldsymbol{T}^n|^2 - N^2 = \sigma_{ij}n_j\sigma_{ik}n_k - (\sigma_{ij}n_in_j)^2$$

$$= \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$= \frac{1}{9}\left[2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$= \frac{1}{9}\left[2(I_1^2 - 2I_2) - 2I_2\right] = \frac{1}{9}\left[2I_1^2 - 6I_2\right]$$

$$= \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

Next using given x,y,z axes

$$\sigma_{oct} = \frac{1}{3}\sigma_{kk} = \frac{1}{3}I_1 = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$\tau_{oct}^2 = \sigma_{ij}n_j\sigma_{ik}n_k - (\sigma_{ij}n_in_j)^2 = \frac{1}{9}\left[2I_1^2 - 6I_2\right]$$

$$= \frac{1}{9}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 + 6\tau_{yz}^2 + 6\tau_{zx}^2\right]$$

3-6.

$$\sigma_{1,2} = \frac{1}{2}[(\sigma_x+\sigma_y) \pm \sqrt{(\sigma_x+\sigma_y)^2 - 4(\sigma_x\sigma_y - \tau_{xy}^2)}$$

$$= \frac{1}{2}I_1 \pm \sqrt{\frac{1}{4}\left(I_1^2 - 4I_2\right)} = \frac{1}{2}I_1 \pm \sqrt{\frac{1}{4}I_1^2 - I_2}$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\frac{1}{4}I_1^2 - I_2}$$

$\therefore$ the principal and max shear stresses are functions of the invariants and thus must be independent of the coordinate system, so

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_r+\sigma_\theta}{2} \pm \sqrt{\left(\frac{\sigma_r-\sigma_\theta}{2}\right)^2 + \tau_{r\theta}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma_r-\sigma_\theta}{2}\right)^2 + \tau_{r\theta}^2}$$

3-7.

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \frac{3P}{4c^3}\begin{bmatrix} 2xy & c^2-y^2 \\ c^2-y^2 & 0 \end{bmatrix}$$

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{3P}{4c^3}\left[xy \pm \sqrt{(c^2-y^2)^2 + x^2y^2}\right]$$

Principal Directions:

$$\frac{3P}{4c^3}\begin{bmatrix} 2xy - xy \mp \sqrt{(c^2-y^2)^2+x^2y^2} & c^2-y^2 \\ c^2-y^2 & 0 - xy \mp \sqrt{(c^2-y^2)^2+x^2y^2} \end{bmatrix}\begin{bmatrix} n_1^{(1,2)} \\ n_2^{(1,2)} \end{bmatrix} = 0 \Rightarrow$$

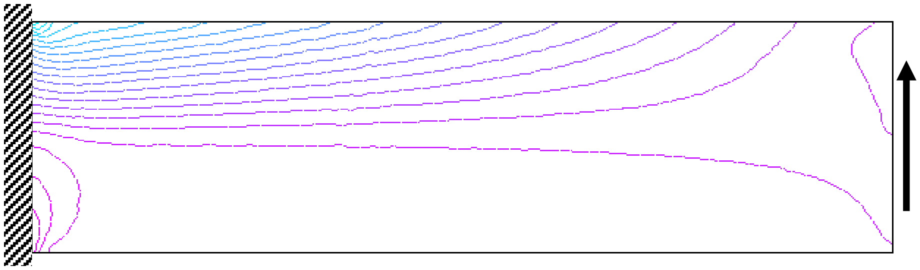
$$n_1^{(1,2)} = \frac{xy \pm \sqrt{(c^2-y^2)^2+x^2y^2}}{c^2-y^2}\, n_2^{(1,2)}$$

Thus the principal directions can be expressed by

$$\boldsymbol{n}^{(1,2)} = \left[\left(xy \pm \sqrt{(c^2-y^2)^2+x^2y^2}\right)\boldsymbol{e}_1 + \left(c^2-y^2\right)\boldsymbol{e}_2\right] \times \text{constant}$$

3-8*.

Using MATLAB the first principal stress contours are:



3-9.

Setting up the principal value problem with $\sigma_{ij} = Pf_{ij}(x_k) \Rightarrow$

$$(\sigma_{ij} - \sigma\delta_{ij})n_j = 0 \Rightarrow \left(Pf_{ij}(x_k) - \sigma\delta_{ij}\right)n_j = 0 \Rightarrow \left(f_{ij}(x_k) - \frac{\sigma}{P}\delta_{ij}\right)n_j = 0$$

and thus the principal values of this problem would be of the form

$$\frac{\sigma_{1,2,3}}{P} = g_{1,2,3}(x_k) \Rightarrow \sigma_{1,2,3} = Pg_{1,2,3}(x_k)$$

The principal directions follow from

$$\left(Pf_{ij}(x_k) - \sigma_{1,2,3}\delta_{ij}\right)n_j^{(1,2,3)} = 0 \Rightarrow \left(Pf_{ij}(x_k) - Pg_{1,2,3}\delta_{ij}\right)n_j^{(1,2,3)} = 0 \Rightarrow \left(f_{ij}(x_k) - g_{1,2,3}\delta_{ij}\right)n_j^{(1,2,3)} = 0$$

and thus the principal directions will not depend on P

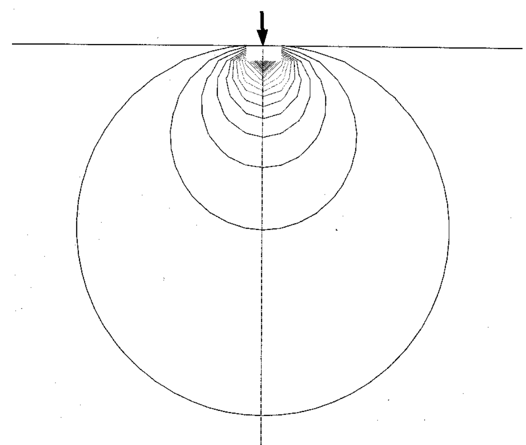
3-10*.

$$\sigma_x = -\frac{2Px^2y}{\pi(x^2+y^2)^2}\ ,\ \sigma_y = -\frac{2Py^3}{\pi(x^2+y^2)^2}\ ,\ \tau_{xy} = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2)} = \sqrt{\left(\frac{-Py(x^2-y^2)}{\pi(x^2+y^2)^2}\right)^2 + \left(\frac{-2Pxy^2)}{\pi(x^2+y^2)^2}\right)^2} = \frac{Py}{\pi(x^2+y^2)}$$

MATLAB Code and Plot :

```
% Exercise 3-6
% Maximum Shear Stress Contours for Flamant Problem
clear;
% Generation of variable space
[x,y]=meshgrid(-3:0.05:3,0:0.05:3);
% Calculation of Nondimensional Maximum Shear Stress
sxy=y./(x.^2+y.^2);
hold on
axis equal
axis off
% Plot contours with reversed y-axis
contour(x,-y,sxy,20);
```



3-11.

$$|\mathrm{T}^n|^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

$$N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

$$S^2 = \left|\boldsymbol{T}^n\right|^2 - N^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2$$

$$= n_1^2 n_2^2(\sigma_1-\sigma_2)^2 + n_2^2 n_3^2(\sigma_2-\sigma_3)^2 + n_3^2 n_1^2(\sigma_3-\sigma_1)^2$$

3-12.

Using $S_{\max}=\frac{1}{2}(\sigma_1-\sigma_3)$ in the equation of circle: $S^2+(N-\sigma_3)(N-\sigma_1)=0 \Rightarrow$

$\frac{1}{4}(\sigma_1-\sigma_3)^2+(N-\sigma_3)(N-\sigma_1)=0 \Rightarrow N=\frac{1}{2}(\sigma_1+\sigma_3)$

Using (3.4.9) $\Rightarrow$

$$n_1^2=\frac{S^2+(N-\sigma_2)(N-\sigma_3)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}=\frac{\frac{1}{4}(\sigma_1-\sigma_3)^2+\left(\frac{1}{2}(\sigma_1+\sigma_3)-\sigma_2\right)\left(\frac{1}{2}(\sigma_1+\sigma_3)-\sigma_3\right)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}=\frac{1}{2}$$

$$n_2^2=\frac{S^2+(N-\sigma_3)(N-\sigma_1)}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)}=\frac{0}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)}=0$$

$$n_3^2=\frac{S^2+(N-\sigma_1)(N-\sigma_2)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)}=\frac{\frac{1}{4}(\sigma_1-\sigma_3)^2+\left(\frac{1}{2}(\sigma_1+\sigma_3)-\sigma_1\right)\left(\frac{1}{2}(\sigma_1+\sigma_3)-\sigma_2\right)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)}=\frac{1}{2}$$

3-13.

$\sigma_{ij}=\begin{bmatrix}3&1&1\\1&0&2\\1&2&0\end{bmatrix}$ is the given stress matrix and using the previously determined principal directions,

$Q_{ij}=\begin{bmatrix}2/\sqrt{6}&1/\sqrt{6}&1/\sqrt{6}\\-1/\sqrt{3}&1/\sqrt{3}&1/\sqrt{3}\\0&-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$ would be the proper transformation matrix

Using standard tensor transformation theory

$$\sigma'_{ij}=Q_{im}Q_{jn}\sigma_{mn}=\begin{bmatrix}2/\sqrt{6}&1/\sqrt{6}&1/\sqrt{6}\\-1/\sqrt{3}&1/\sqrt{3}&1/\sqrt{3}\\0&-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}\begin{bmatrix}3&1&1\\1&0&2\\1&2&0\end{bmatrix}\begin{bmatrix}2/\sqrt{6}&1/\sqrt{6}&1/\sqrt{6}\\-1/\sqrt{3}&1/\sqrt{3}&1/\sqrt{3}\\0&-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}^T$$

$$=\begin{bmatrix}4&0&0\\0&1&0\\0&0&-2\end{bmatrix}$$

3-14.

The characteristic equation for the deviatoric stress tensor is given by $\det[\hat{\sigma}_{ij} - \sigma_d \delta_{ij}] = 0$.

Since $\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$, this equation can be written as

$\det[\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} - \sigma_d \delta_{ij}] = \det[\sigma_{ij} - (\frac{1}{3}\sigma_{kk} + \sigma_d)\delta_{ij}] = 0$.

However, the characteristic equation for the stress tensor σ_{ij} is $\det[\sigma_{ij} - \sigma\delta_{ij}] = 0$, and thus it follows that $\sigma = \frac{1}{3}\sigma_{kk} + \sigma_d$ or $\sigma_d = \sigma - \frac{1}{3}\sigma_{kk}$.

If $\boldsymbol{n}$ is the principal direction of the stress tensor σ_{ij}, then $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, and this can be written as $\left(\sigma_{ij} - (\frac{1}{3}\sigma_{kk} + \sigma_d)\delta_{ij}\right)n_j = 0$ or $(\hat{\sigma}_{ij} - \sigma_d\delta_{ij})n_j = 0$. Therefore $\boldsymbol{n}$ also the principal direction of the deviatoric stress $\hat{\sigma}_{ij}$.

3-15.

(a) $\sigma_{ij} = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 4 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} = \begin{bmatrix} 7/3 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 7/3 \end{bmatrix}$

$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \begin{bmatrix} -1/3 & 1 & -4 \\ 1 & 5/3 & 0 \\ -4 & 0 & -4/3 \end{bmatrix}$

(b) $\sigma_{ij} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & -6 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} = \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$

$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \begin{bmatrix} 13/3 & 1 & 0 \\ 1 & -17/3 & 2 \\ 0 & 2 & 4/3 \end{bmatrix}$

3-16.

With $\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow \sigma_1 = 4,\ \sigma_2 = 1,\ \sigma_3 = -2$

$$\sigma_{vonMises} = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} = \frac{1}{\sqrt{2}}[(3)^2 + (3)^2 + (6)^2]^{1/2} = \sqrt{27} = 5.196$$

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = 1$$

$$\tau_{oct} = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} = \frac{1}{3}[(3)^2 + (3)^2 + (6)^2]^{1/2} = \frac{1}{3}\sqrt{54} = \sqrt{6} = 2.449$$

3-17.

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det[\sigma_{ij} - \sigma\delta_{ij}] = \begin{vmatrix} -\sigma & \tau & 0 \\ \tau & -\sigma & 0 \\ 0 & 0 & -\sigma \end{vmatrix} = \sigma(\tau^2 - \sigma^2) = 0$$

$$\Rightarrow \sigma_1 = \tau\,,\ \sigma_2 = 0, \sigma_3 = -\tau$$

$\sigma_1 = \tau$ Case:

$$(\sigma_{ij} - \sigma_1\delta_{ij})n_j^{(1)} = 0 \Rightarrow \boldsymbol{n}^{(1)} = \pm 1/\sqrt{2}(1,1,0)$$

$\sigma_2 = 0$ Case:

$$(\sigma_{ij} - \sigma_2\delta_{ij})n_j^{(2)} = 0 \Rightarrow \boldsymbol{n}^{(2)} = \pm(0,0,1)$$

$\sigma_3 = -\tau$ Case:

$$(\sigma_{ij} - \sigma_3\delta_{ij})n_j^{(3)} = 0 \Rightarrow \boldsymbol{n}^{(3)} = \pm 1/\sqrt{2}(1,-1,0)$$

For the octahedral plane, $n_1 = n_2 = n_3 = 1/\sqrt{3}$

$$\left|\boldsymbol{T}^n\right|^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = \frac{1}{3}(\tau^2 + \tau^2) = \frac{2}{3}\tau^2$$

$$N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{\tau}{3} - \frac{\tau}{3} = 0$$

$$S = \sqrt{\left|\boldsymbol{T}^n\right|^2 - N^2} = \sqrt{\frac{2}{3}}\tau = \sqrt{6}\tau/3$$

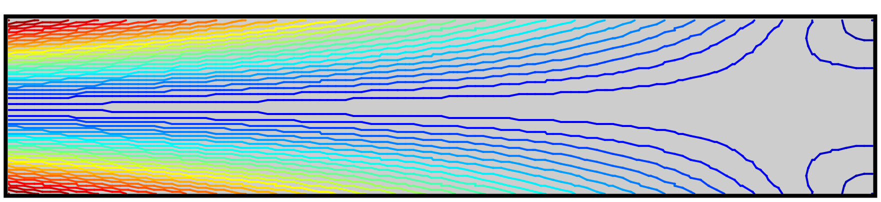
3-18*.

From Exercise 3-7, $\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \dfrac{3P}{4c^3}\begin{bmatrix} 2xy & c^2 - y^2 \\ c^2 - y^2 & 0 \end{bmatrix}$

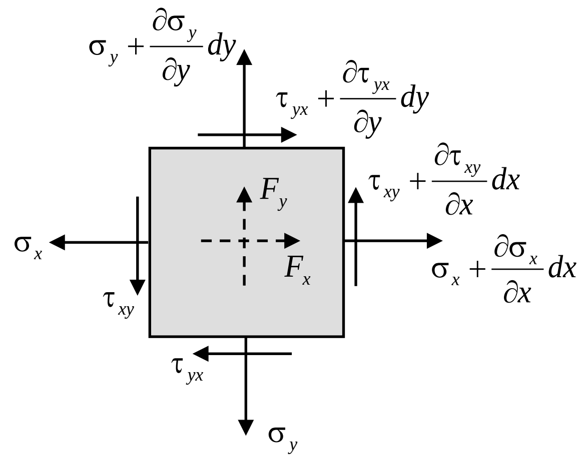
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{3P}{4c^3}\left[xy \pm \sqrt{(c^2 - y^2)^2 + x^2y^2}\right],\ \sigma_3 = 0$$

$$\sigma_{vonMises} = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} = [\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2]^{1/2}$$

MATLAB stress contours:



3-19.



$$\sum F_x = 0 \Rightarrow$$

$$(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx)dy - \sigma_x dy$$

$$+ (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy)dx - \tau_{yx} dx + F_x dxdy = 0$$

Divide through by $dxdy$ and let $dx, dy \to 0$

$$\Rightarrow \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0$$

$$\sum F_y = 0 \Rightarrow$$

$$(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy)dx - \sigma_y dx$$

$$+ (\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx)dy - \tau_{xy} dy + F_y dxdy = 0$$

Divide through by $dxdy$ and let $dx, dy \to 0$

$$\Rightarrow \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0$$

$$\sum M = 0 \ (@\, center) \Rightarrow$$

$$(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy)dx \frac{dy}{2} + \tau_{yx} dx \frac{dy}{2}$$

$$- (\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx)dy \frac{dx}{2} - \tau_{xy} dy \frac{dx}{2} = 0$$

Divide through by $dxdy$ and let $dx, dy \to 0$

$$\Rightarrow \tau_{xy} = \tau_{yx}$$

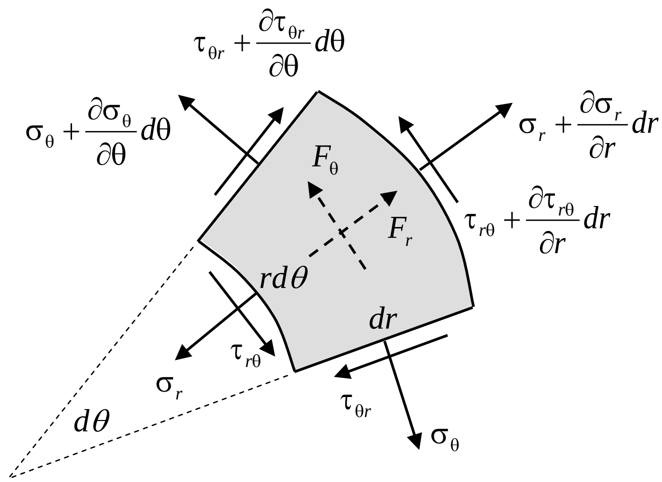
3-20.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) \equiv 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2}\right) \equiv 0$$

$\therefore$ both equilibrium equations are identically satisfied

3-21.



$$\sum F_r = 0 \Rightarrow$$

$$(\sigma_r + \frac{\partial\sigma_r}{\partial r}dr)(r+dr)d\theta - \sigma_r r d\theta + (\tau_{\theta r} + \frac{\partial\tau_{\theta r}}{\partial\theta}d\theta)dr\cos(d\theta/2)$$

$$-\tau_{\theta r}dr\cos(d\theta/2) - (\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta}d\theta)dr\sin(d\theta/2) - \sigma_\theta dr\sin(d\theta/2) + F_r r dr d\theta = 0$$

Divide through by $rdrd\theta$ and let $dr, d\theta \to 0$

$$\Rightarrow \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

$$\sum F_\theta = 0 \Rightarrow$$

$$(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial r}dr)(r+dr)d\theta - \tau_{r\theta} r d\theta + (\tau_{\theta r} + \frac{\partial\tau_{\theta r}}{\partial\theta}d\theta)dr\sin(d\theta/2) + \tau_{\theta r}dr\sin(d\theta/2)$$

$$(\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta}d\theta)dr\cos(d\theta/2) - \sigma_\theta dr\cos(d\theta/2) + F_\theta r dr d\theta = 0$$

Divide through by $rdrd\theta$ and let $dr, d\theta \to 0$

$$\Rightarrow \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + 2\frac{\tau_{\theta r}}{r} + F_\theta = 0$$

$$\sum M = 0 \ (@\, center) \Rightarrow$$

$$(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial r}dr)(r+dr)d\theta\frac{dr}{2} + \tau_{r\theta} r d\theta\frac{dr}{2} - (\tau_{\theta r} + \frac{\partial\tau_{\theta r}}{\partial\theta}d\theta)dr\frac{rd\theta}{2} - \tau_{\theta r}dr\frac{rd\theta}{2} = 0$$

Divide through by $rdrd\theta$ and let $dr, d\theta \to 0$

$$\Rightarrow \tau_{r\theta} = \tau_{\theta r}$$

3-22.

$$\sigma_x = -\frac{My}{I},\ \tau_{xy} = \frac{V(R^2 - y^2)}{3I},\ \sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + F_x = 0 \Rightarrow -\frac{y}{I}\frac{dM}{dx} - \frac{2y}{3I}V \neq 0$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + F_y = 0 \Rightarrow \frac{(R^2 - y^2)}{3I}\frac{dV}{dx} \neq 0$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + F_z = 0 \Rightarrow 0 = 0$$

$\therefore$ The approximate mechanics of materials stresses
do not satisfy the equilibrium equations.

3-23.

$$\sigma_x = \sigma_x(x),\ \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0,\ F_x = \rho g,\ F_y = F_z = 0$$

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + F_x = 0 \Rightarrow \frac{\partial\sigma_x}{\partial x} + \rho g = 0 \Rightarrow \sigma_x = -\int_l^x \rho g dx = \rho g(l - x)$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + F_y = 0 \Rightarrow 0 = 0$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + F_z = 0 \Rightarrow 0 = 0$$

3-24.

$$\sigma_{ij} = -p\delta_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

$$\sigma_{ij,j} + F_i = 0 \Rightarrow -p_{,j}\delta_{ij} + F_i = 0 \Rightarrow p_{,i} = F_i \ \ or\ \ \nabla p = F$$

Using a scalar approach

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + F_x = 0 \Rightarrow \frac{\partial p}{\partial x} = F_x$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + F_y = 0 \Rightarrow \frac{\partial p}{\partial y} = F_y$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + F_z = 0 \Rightarrow \frac{\partial p}{\partial z} = F_z$$

$\therefore \nabla p = F$

3-25.

$$\boldsymbol{\sigma} = \boldsymbol{e_r T_r} + \boldsymbol{e_\theta T_\theta} + \boldsymbol{e_z T_z}$$
$$\boldsymbol{T_r} = \sigma_r \boldsymbol{e_r} + \tau_{r\theta}\boldsymbol{e_\theta} + \tau_{rz}\boldsymbol{e_z}$$
$$\boldsymbol{T_\theta} = \tau_{r\theta}\boldsymbol{e_r} + \sigma_\theta \boldsymbol{e_\theta} + \tau_{\theta z}\boldsymbol{e_z}$$
$$\boldsymbol{T_z} = \tau_{rz}\boldsymbol{e_r} + \tau_{\theta z}\boldsymbol{e_\theta} + \sigma_z \boldsymbol{e_z}$$

From (1.9.14) or exercise 1-16, $\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$

$$\Rightarrow \nabla \cdot \boldsymbol{\sigma} = \frac{\partial \boldsymbol{T_r}}{\partial r} + \frac{1}{r}\boldsymbol{T}_r + \frac{1}{r}\frac{\partial \boldsymbol{T_\theta}}{\partial \theta} + \frac{\partial \boldsymbol{T_z}}{\partial z}$$
$$= \frac{\partial}{\partial r}(\sigma_r \boldsymbol{e_r} + \tau_{r\theta}\boldsymbol{e_\theta} + \tau_{rz}\boldsymbol{e_z}) + \frac{1}{r}(\sigma_r \boldsymbol{e_r} + \tau_{r\theta}\boldsymbol{e_\theta} + \tau_{rz}\boldsymbol{e_z})$$
$$+ \frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{r\theta}\boldsymbol{e_r} + \sigma_\theta \boldsymbol{e_\theta} + \tau_{\theta z}\boldsymbol{e_z}) + \frac{\partial}{\partial z}\left(\tau_{rz}\boldsymbol{e_r} + \tau_{\theta z}\boldsymbol{e_\theta} + \sigma_z \boldsymbol{e_z}\right)$$
$$= \frac{\partial \sigma_r}{\partial r}\boldsymbol{e_r} + \frac{\partial \tau_{r\theta}}{\partial r}\boldsymbol{e_\theta} + \frac{\partial \tau_{rz}}{\partial r}\boldsymbol{e}_z + \frac{1}{r}(\sigma_r \boldsymbol{e_r} + \tau_{r\theta}\boldsymbol{e_\theta} + \tau_{rz}\boldsymbol{e_z})$$
$$+ \frac{1}{r}\left(\frac{\partial \tau_{r\theta}}{\partial \theta}\boldsymbol{e_r} + \tau_{r\theta}\boldsymbol{e_\theta} + \frac{\partial \sigma_\theta}{\partial \theta}\boldsymbol{e_\theta} - \sigma_\theta \boldsymbol{e_r} + \frac{\partial \tau_{\theta z}}{\partial \theta}\boldsymbol{e_z}\right)$$
$$+ \frac{\partial \tau_{rz}}{\partial z}\boldsymbol{e_r} + \frac{\partial \tau_{\theta z}}{\partial z}\boldsymbol{e_\theta} + \frac{\partial \sigma_z}{\partial z}\boldsymbol{e_z}$$

Collecting terms in each coordinate direction and adding
the appropriated body force gives

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r = 0$$
$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0$$
$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + F_z = 0$$

4-1.

From relation (4.2.1)

$$\sigma_x = C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx}$$
$$\sigma_y = C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx}$$
$$\sigma_z = C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx}$$
$$\tau_{xy} = C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx}$$
$$\tau_{yz} = C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx}$$
$$\tau_{zx} = C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx}$$

While from (4.2.3)

$$\sigma_x = C_{1111}e_x + C_{1122}e_y + C_{1133}e_z + 2C_{1112}e_{xy} + 2C_{1123}e_{yz} + 2C_{1131}e_{zx}$$
$$\sigma_y = C_{2211}e_x + C_{2222}e_y + C_{2233}e_z + 2C_{2212}e_{xy} + 2C_{2223}e_{yz} + 2C_{2231}e_{zx}$$
$$\sigma_z = C_{3311}e_x + C_{3322}e_y + C_{3333}e_z + 2C_{3312}e_{xy} + 2C_{3323}e_{yz} + 2C_{3331}e_{zx}$$
$$\tau_{xy} = C_{1211}e_x + C_{1222}e_y + C_{1233}e_z + 2C_{1212}e_{xy} + 2C_{1223}e_{yz} + 2C_{1231}e_{zx}$$
$$\tau_{yz} = C_{2311}e_x + C_{2322}e_y + C_{2333}e_z + 2C_{2312}e_{xy} + 2C_{2323}e_{yz} + 2C_{2331}e_{zx}$$
$$\tau_{zx} = C_{3111}e_x + C_{3122}e_y + C_{3133}e_z + 2C_{3112}e_{xy} + 2C_{3123}e_{yz} + 2C_{3131}e_{zx}$$

Comparing these two relations implies the result

$$C_{ij} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} \end{bmatrix}$$

4-2.

$$\sigma_{ij} = C_{ijkl}e_{kl} \ , \ \sigma_{ji} = C_{jikl}e_{kl}$$

$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl}e_{kl} = C_{jikl}e_{kl} \Rightarrow C_{ijkl} = C_{jikl}$ which is $(4.2.4)_1$

$$\sigma_{ij} = C_{ijkl}e_{kl} = \frac{1}{2}C_{ijkl}e_{kl} + \frac{1}{2}C_{ijkl}e_{kl}$$
$$= \frac{1}{2}C_{ijkl}e_{kl} + \frac{1}{2}C_{ijlk}e_{lk} = \frac{1}{2}(C_{ijkl} + C_{ijlk})e_{kl} = C^*_{ijkl}e_{kl}$$

but from the definition of C^*_{ijkl} it is symmetric in k and l, and thus $C_{ijkl} = C_{ijlk}$ which is $(4.2.4)_2$

4-3.

$$\sigma_{ij} = C_{ijkl}e_{kl} = (\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk})e_{kl}$$

$$= \alpha e_{kk}\delta_{ij} + \beta e_{ij} + \gamma e_{ji} = \alpha e_{kk}\delta_{ij} + (\beta + \gamma)e_{ij}$$

letting $\alpha = \lambda$ and $(\beta + \gamma) = 2\mu$,

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

4-4.

From (4.2.6) $\Rightarrow$

$$\sigma_{ij} = C_{ijkl}e_{kl} = (\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk})e_{kl}$$

$$= \alpha e_{kk}\delta_{ij} + \beta e_{ij} + \gamma e_{ji} = \alpha e_{kk}\delta_{ij} + \beta e_{ij} + \gamma e_{ij}$$

Comparing this with relation (4.2.7) $\Rightarrow$

$\alpha = \lambda$ and $\beta = \mu$, $\gamma = \mu \Rightarrow C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$

Since $\lambda = k - \frac{2}{3}\mu \Rightarrow C_{ijkl} = \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + (k - \frac{2}{3}\mu)\delta_{ij}\delta_{kl}$

Since $\mu = \dfrac{E}{2(1+\nu)}$ and $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \Rightarrow$

$$C_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

4-5.

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \Rightarrow \sigma_{kk} = (3\lambda + 2\mu)e_{kk} \Rightarrow e_{kk} = \frac{\sigma_{kk}}{(3\lambda + 2\mu)}$$

$$(4.2.7) \Rightarrow e_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu}e_{kk}\delta_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{kk}\delta_{ij}$$

This will equal form (4.2.10) providing $\dfrac{1}{2\mu} = \dfrac{1+\nu}{E}$ and $\dfrac{\lambda}{2\mu(3\lambda + 2\mu)} = \dfrac{\nu}{E}$

and this will be true if $E = \dfrac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}$ and $\nu = \dfrac{\lambda}{2(\lambda + \mu)}$

4-6.

From exercise 4 - 3,

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)} = \mu\left(2 + \frac{\lambda}{\lambda + \mu}\right) = \mu(2 + 2\nu) \Rightarrow \mu = \frac{E}{2(1+\nu)}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \Rightarrow \frac{1}{2\nu} = \frac{(\lambda + \mu)}{\lambda} = 1 + \frac{\mu}{\lambda} = 1 + \frac{E}{2(1+\nu)\lambda}$$

$$\therefore \frac{E}{2(1+\nu)\lambda} = \frac{1}{2\nu} - 1 = \frac{1-2\nu}{2\nu} \Rightarrow \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

4-7.

If σ_i and n_i^σ are the principal values and directions for the stress, then $(\sigma_{\text{ij}} - \sigma_{\underline{i}}\delta_{ij})n_i^\sigma = 0$

For isotropic materials, $\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$ and thus

$$(\sigma_{\text{ij}} - \sigma_{\underline{i}}\delta_{ij})n_i^\sigma = (\lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \sigma_{\underline{i}}\delta_{ij})n_i^\sigma = 0 \Rightarrow \left(e_{ij} - \frac{1}{2\mu}(\sigma_{\underline{i}} - \lambda e_{kk})\delta_{ij}\right)n_i^\sigma = 0$$

which can be written as $(e_{ij} - e_{\underline{i}}\delta_{ij})n_i^\sigma = 0$ where $e_i = \dfrac{1}{2\mu}(\sigma_i - \lambda e_{kk})$

However, the principal value problem for the strain can be expressed as

$(e_{ij} - e_{\underline{i}}\delta_{ij})n_i^e = 0$ where e_i and n_i^e are the principal values and directions for the strain.

These results for isotropic materials then imply that $n_i^\sigma = n_i^e$ and $\sigma_i = 2\mu e_i + \lambda e_{kk}$

4-8.

$$e_a = e_x \cos^2 60^o + e_y \sin^2 60^o + 2e_{xy} \sin 60^o \cos 60^o \Rightarrow$$

$$300 \times 10^{-6} = e_x \frac{1}{4} + e_y \frac{3}{4} + e_{xy} \frac{\sqrt{3}}{2}$$

$$e_b = e_x \cos^2 90^o + e_y \sin^2 90^o + 2e_{xy} \sin 90^o \cos 90^o \Rightarrow$$

$$400 \times 10^{-6} = e_y$$

$$e_a = e_x \cos^2 120^o + e_y \sin^2 120^o + 2e_{xy} \sin 120^o \cos 120^o \Rightarrow$$

$$100 \times 10^{-6} = e_x \frac{1}{4} + e_y \frac{3}{4} - e_{xy} \frac{\sqrt{3}}{2}$$

Solving for the strains : $e_x = -400 \times 10^{-6}, e_y = 400 \times 10^{-6}, \ e_{xy} = \dfrac{200}{\sqrt{3}} \times 10^{-6}$

All other strain components are assumed to vanish.

Using Hooke's law with elastic moduli for steel ($\lambda = 111\text{GPa}, \mu = 80.2\text{GPa}$)

$\sigma_x = \lambda(e_x + e_y) + 2\mu e_x = -64.16\text{MPa}$

$\sigma_y = \lambda(e_x + e_y) + 2\mu e_y = 64.16\text{MPa}$

$\tau_{xy} = 2\mu e_{xy} = 18.52\text{MPa}$

4-9.

$$e_x = \frac{\partial u}{\partial x} = -\frac{M(1-\nu^2)}{EI} y \ , \ e_y = \frac{\partial v}{\partial y} = \frac{M(1+\nu)\nu}{EI} y$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{M(1-\nu^2)}{EI} x + \frac{M(1-\nu^2)}{EI} x = 0 \ , \ e_z = e_{xz} = e_{yz} = 0$$

Hooke's law gives the following in - plane stress components

$$\sigma_x = \lambda(e_x + e_y) + 2\mu e_x = -\frac{My}{I}$$

$$\sigma_y = \lambda(e_x + e_y) + 2\mu e_y = 0$$

$$\tau_{xy} = 2\mu e_{xy} = 0$$

Thus this state of stress corresponds to pure bending of a rectangular beam in the $x,y-$plane

4-10.

From Table 4 - 1,

$E = 3k(1-2\nu)$ and so if $E > 0$ and $k > 0$, then $\nu < \frac{1}{2}$

$E = 2\mu(1+\nu)$ and so if $E > 0$ and $\mu > 0$, then $\nu > -1$

$$\therefore -1 < \nu < \frac{1}{2}$$

Noting that $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, and if we choose the more restrictive inequality

$$0 < \nu < \frac{1}{2} \Rightarrow \lambda > 0$$

4-11.

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \begin{cases} 0 \ , \ \nu = 0 \\ 2E/5 \ , \nu = 1/4 \\ \infty \ , \ \nu = 1/2 \end{cases}$$

$$\mu = \frac{E}{2(1+\nu)} = \begin{cases} E/2 \ , \ \nu = 0 \\ 2E/5 \ , \nu = 1/4 \\ E/3 \ , \ \nu = 1/2 \end{cases}$$

$$k = \frac{E}{3(1-2\nu)} = \begin{cases} E/3 \ , \ \nu = 0 \\ 2E/3 \ , \nu = 1/4 \\ \infty \ , \ \nu = 1/2 \end{cases}$$

For $\nu = 1/2$, λ and k become unbounded.

4-12.

(a) Aluminum : $E = 68.9GPa$, $\nu = 0.34$, $\mu = 25.7GPa$, $k = 71.8GPa$

$$\text{Simple Tension : } e_{ij} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\frac{\nu}{E}\sigma & 0 \\ 0 & 0 & -\frac{\nu}{E}\sigma \end{bmatrix} = \begin{bmatrix} 2.17 & 0 & 0 \\ 0 & -0.74 & 0 \\ 0 & 0 & -0.74 \end{bmatrix} \times 10^{-3}$$

$$\text{Pure Shear : } e_{ij} = \begin{bmatrix} 0 & \tau/2\mu & 0 \\ \tau/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1.46 & 0 \\ 1.46 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

$$\text{Hydrostatic Compression : } e_{ij} = \begin{bmatrix} -\frac{p}{3k} & 0 & 0 \\ 0 & -\frac{p}{3k} & 0 \\ 0 & 0 & -\frac{p}{3k} \end{bmatrix} = \begin{bmatrix} -2.32 & 0 & 0 \\ 0 & -2.32 & 0 \\ 0 & 0 & -2.32 \end{bmatrix} \times 10^{-3}$$

(b) Steel : $E = 207GPa$, $\nu = 0.29$, $\mu = 80.2GPa$, $k = 164GPa$

$$\text{Simple Tension : } e_{ij} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\frac{\nu}{E}\sigma & 0 \\ 0 & 0 & -\frac{\nu}{E}\sigma \end{bmatrix} = \begin{bmatrix} 1.45 & 0 & 0 \\ 0 & -0.42 & 0 \\ 0 & 0 & -0.42 \end{bmatrix} \times 10^{-3}$$

$$\text{Pure Shear : } e_{ij} = \begin{bmatrix} 0 & \tau/2\mu & 0 \\ \tau/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.935 & 0 \\ 0.935 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

$$\text{Hydrostatic Compression : } e_{ij} = \begin{bmatrix} -\frac{p}{3k} & 0 & 0 \\ 0 & -\frac{p}{3k} & 0 \\ 0 & 0 & -\frac{p}{3k} \end{bmatrix} = \begin{bmatrix} -1.02 & 0 & 0 \\ 0 & -1.02 & 0 \\ 0 & 0 & -1.02 \end{bmatrix} \times 10^{-3}$$

(c) Rubber : $E = 0.0019GPa$, $\nu = 0.499$, $\mu = 0.000654GPa$, $k = 0.326GPa$

$$\text{Simple Tension : } e_{ij} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\frac{\nu}{E}\sigma & 0 \\ 0 & 0 & -\frac{\nu}{E}\sigma \end{bmatrix} = \begin{bmatrix} 7894 & 0 & 0 \\ 0 & -3939 & 0 \\ 0 & 0 & -3939 \end{bmatrix} \times 10^{-3}$$

$$\text{Pure Shear : } e_{ij} = \begin{bmatrix} 0 & \tau/2\mu & 0 \\ \tau/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5351 & 0 \\ 5351 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

$$\text{Hydrostatic Compression : } e_{ij} = \begin{bmatrix} -\frac{p}{3k} & 0 & 0 \\ 0 & -\frac{p}{3k} & 0 \\ 0 & 0 & -\frac{p}{3k} \end{bmatrix} = \begin{bmatrix} -511 & 0 & 0 \\ 0 & -511 & 0 \\ 0 & 0 & -511 \end{bmatrix} \times 10^{-3}$$

4-13.

$$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij}\ ,\ \hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\ ,\ \tilde{e}_{ij} = \frac{1}{3}e_{kk}\delta_{ij}\ ,\ \hat{e}_{ij} = e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \Rightarrow \sigma_{kk} = (3\lambda + 2\mu)e_{kk} \Rightarrow \tilde{\sigma}_{ij} = 3k\tilde{e}_{ij}$$

$$\hat{\sigma}_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \frac{1}{3}(3\lambda + 2\mu)e_{kk}\delta_{ij} = 2\mu(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}) = 2\mu\hat{e}_{ij}$$

4-14.

With $\sigma_x = 2\sigma_y\ ,\ \sigma_z = \tau_{xz} = \tau_{yz} = 0,$ Hooke's law gives

$$e_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{1}{E}\left[\sigma_x - \frac{1}{2}\nu\sigma_x\right] = \frac{2-\nu}{2E}\sigma_x$$

$$\Rightarrow \sigma_x = \frac{2E}{2-\nu}e_x \text{ and the slope of the uniaxial stress - strain curve is } \frac{2E}{2-\nu}$$

4-15.

For steel, $E = 207GPa\ ,\ \nu = 0.29$

$$e_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{1}{207}\left[20 - 0.29(30)\right]\times 10^{-3} = 5.4\times 10^{-5}$$

$$e_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] = \frac{1}{207}\left[30 - 0.29(20)\right]\times 10^{-3} = 11.7\times 10^{-5}$$

$$e_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = \frac{1}{207}\left[-0.29(50)\right]\times 10^{-3} = -7.0\times 10^{-5}$$

$$\Delta x = l_x e_x = (300)(5.4\times 10^{-5}) = 0.0164mm$$

$$\Delta y = l_y e_y = (200)(11.7\times 10^{-5}) = 0.0234mm$$

$$\Delta z = l_z e_z = (4)(-7.0\times 10^{-5}) = -0.00028mm$$

4-16.

For steel, $E = 207GPa\ ,\ \nu = 0.29$

$$e_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{1}{207}\left[-50 - 0.29(50)\right]\times 10^{-3} = -3.1\times 10^{-4}$$

$$e_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] = \frac{1}{207}\left[50 - 0.29(-50)\right]\times 10^{-3} = 3.1\times 10^{-4}$$

$$e_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = \frac{1}{207}\left[-0.29(-50+50)\right]\times 10^{-3} = 0$$

$$\Delta x = l_x e_x = (300)(-3.1\times 10^{-4}) = -0.093mm$$

$$\Delta y = l_y e_y = (200)(3.1\times 10^{-4}) = 0.062mm$$

$$\Delta z = l_z e_z = (4)(0) = 0$$

4-17.

No axial strain $\Rightarrow e_x = 0$ and no transverse stresses $\Rightarrow \sigma_{\mathrm{y}} = \sigma_{\mathrm{z}} = 0$

Hooke's law (4.4.4) then gives

$$e_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha T = 0 \Rightarrow \sigma_{\mathrm{x}} = -E\alpha T$$

$$e_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] + \alpha T = \nu E\alpha T + \alpha T = (1+\nu)\alpha T$$

$$e_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] + \alpha T = (1+\nu)\alpha T$$

4-18.

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - (3\lambda + 2\mu)\alpha(T - T_o)\delta_{ij} \;\Rightarrow$$

$$\sigma_x = \lambda(e_x + e_y + e_z) + 2\mu e_x - (3\lambda + 2\mu)\alpha(T - T_o)$$

$$= (\lambda + 2\mu)e_x + \lambda(e_y + e_z) - (3\lambda + 2\mu)\alpha(T - T_o)$$

Using Table 4-1 $\Rightarrow$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_x + \nu(e_y + e_z)] - \frac{E}{(1-2\nu)}\alpha(T - T_o)$$

Similarly

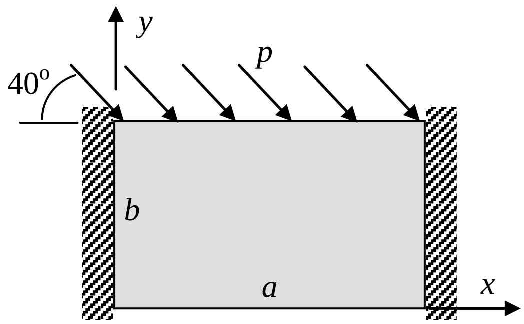
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_y + \nu(e_z + e_x)] - \frac{E}{(1-2\nu)}\alpha(T - T_o)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_z + \nu(e_x + e_y)] - \frac{E}{(1-2\nu)}\alpha(T - T_o)$$

Shear stresses follow from the original equation

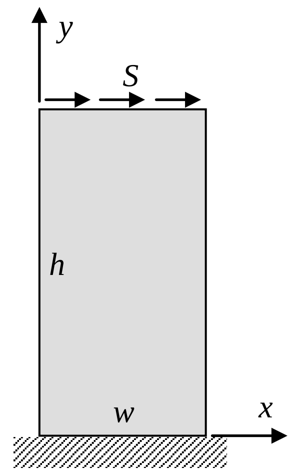
$$\tau_{xy} = 2\mu e_{xy} = \frac{E}{1+\nu}e_{xy}\ ,\ \tau_{yz} = \frac{E}{1+\nu}e_{yz}\ ,\ \tau_{zx} = \frac{E}{1+\nu}e_{zx}$$

5-1.



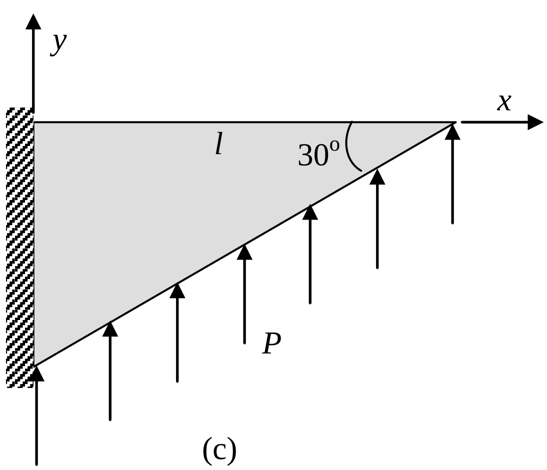
(a)

$$T_x(x,0) = T_y(x,0) = 0$$
$$T_x(x,b) = p\cos 40^o \ , \ T_y(x,b) = -p\sin 40^o$$
$$u(0,y) = v(0,y) = 0$$
$$u(a,y) = v(a,y) = 0$$



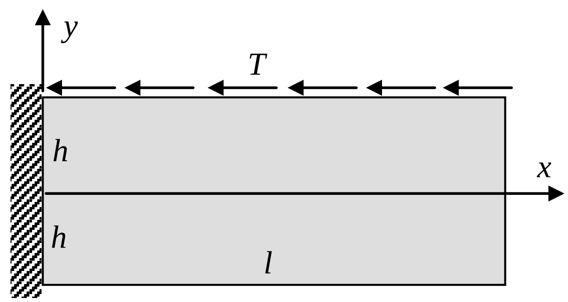
(b)

$$T_x(0,y) = T_y(0,y) = 0$$
$$T_x(w,y) = T_y(w,y) = 0$$
$$T_x(x,h) = S \ , \ T_y(x,h) = 0$$
$$u(x,0) = v(x,0) = 0$$



(c)

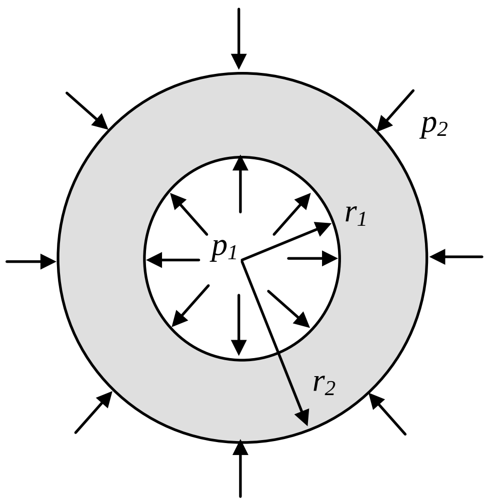
$$T_x(x,0) = \tau_{xy}(x,0) = 0$$
$$T_y(x,0) = \sigma_y(x,0) = 0$$
$$T_x(x,-x\tan 30^o) = 0 \ , \ T_y(x,-x\tan 30^o) = P$$
$$u(0,y) = v(0,y) = 0$$



(d)

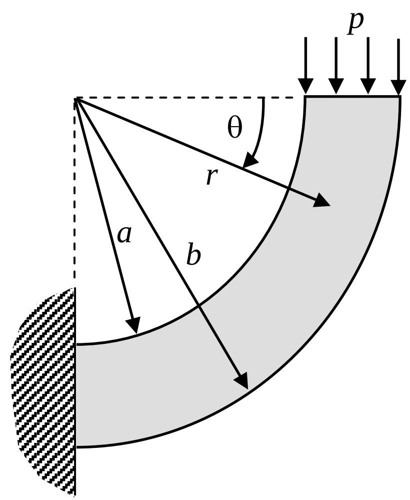
$$T_x(l,y) = \sigma_x(l,y) = 0$$
$$T_y(l,y) = \tau_{xy}(l,y) = 0$$
$$T_x(x,h) = \tau_{xy}(x,h) = -T$$
$$T_y(x,h) = \sigma_y(x,h) = 0$$
$$T_x(x,-h) = -\tau_{xy}(x,-h) = 0$$
$$T_y(x,-h) = -\sigma_y(x,-h) = 0$$
$$u(0,y) = v(0,y) = 0$$

5-1. Continued



(e)

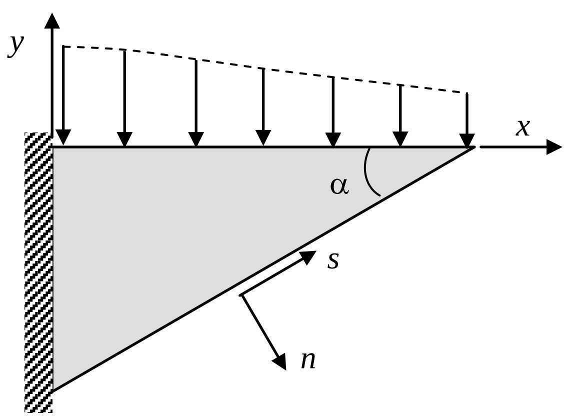
$$T_r(r_1,\theta) = p_1 \ , \ T_\theta(r_1,\theta) = 0$$
$$T_r(r_2,\theta) = -p_2 \ , \ T_\theta(r_2,\theta) = 0$$



(f)

$$T_r(a,\theta) = T_\theta(a,\theta) = 0$$
$$T_r(b,\theta) = T_\theta(b,\theta) = 0$$
$$T_r(r,0) = 0, \ T_\theta(r,0) = -p$$
$$u_r(r,\pi/2) = u_\theta(r,\pi/2) = 0$$

5-2.



$$\text{Bottom Surface}: n_x = \sin\alpha \ , \ n_y = -\cos\alpha$$
$$T_x = \sigma_x n_x + \tau_{xy} n_y = 0 \ , \ T_y = \tau_{xy} n_x + \sigma_y n_y = 0$$
$$T_n = T_x \sin\alpha - T_y \cos\alpha$$
$$T_s = T_x \cos\alpha + T_y \sin\alpha$$
$$\therefore \text{ If } T_x = T_y = 0 \Rightarrow T_n = T_s = 0$$

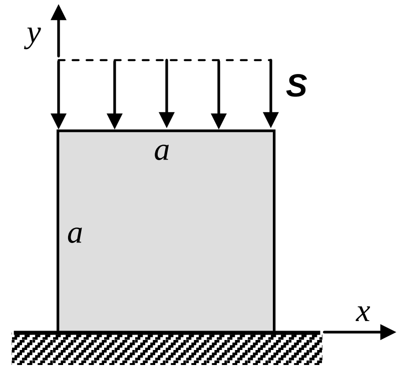
5-3.

(a) $u(x,0) = v(x,0) = 0$

$$\sigma_x(0,y) = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} = 0\ ,\ \tau_{xy}(0,y) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\sigma_x(a,y) = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} = 0\ ,\ \tau_{xy}(a,y) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\sigma_y(x,a) = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} = -S\ ,\ \tau_{xy}(x,a) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$



(a)

(b) $u(0,y) = v(0,y) = 0$

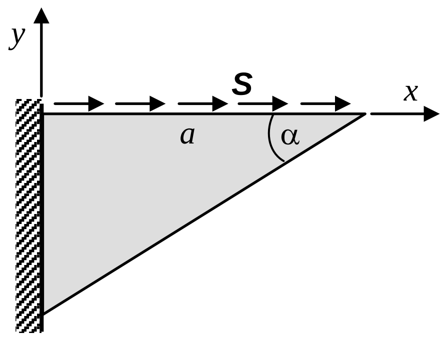
$$\sigma_y(x,0) = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} = 0\ ,\ \tau_{xy}(x,0) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = S$$

$$T_x(x,(x-a)\tan\alpha) = \sigma_x n_x + \tau_{xy} n_y = 0 \Rightarrow$$

$$\left(\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}\right)\sin\alpha + \left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)(-\cos\alpha) = 0$$

$$T_y(x,(x-a)\tan\alpha) = \tau_{xy} n_x + \sigma_y n_y = 0 \Rightarrow$$

$$\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)\sin\alpha + \left(\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}\right)(-\cos\alpha) = 0$$



(b)

5-4.

(a) Material (1):

$u^{(1)}(x,0) = v^{(1)}(x,0) = 0$

$\sigma_x^{(1)}(0,y) = 0\ ,\ \tau_{xy}^{(1)}(0,y) = 0$

$\sigma_x^{(1)}(a,y) = 0\ ,\ \tau_{xy}^{(1)}(a,y) = 0$

Material (2):

$\sigma_y^{(2)}(x,h_1+h_2) = -S\ ,\ \tau_{xy}^{(2)}(x,h_1+h_2) = 0$

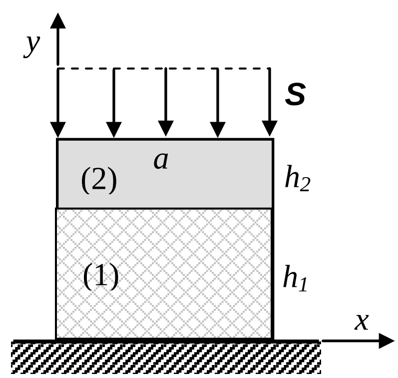
$\sigma_x^{(2)}(0,y) = 0\ ,\ \tau_{xy}^{(2)}(0,y) = 0$

$\sigma_x^{(2)}(a,y) = 0\ ,\ \tau_{xy}^{(2)}(a,y) = 0$

Interface Conditions:

$u^{(1)}(x,h_1) = u^{(2)}(x,h_1)\ ,\ v^{(1)}(x,h_1) = v^{(2)}(x,h_1)$

$\sigma_y^{(1)}(x,h_1) = \sigma_y^{(2)}(x,h_1)\ ,\ \tau_{xy}^{(1)}(x,h_1) = \tau_{xy}^{(2)}(x,h_1)$



(a)

5-4. Continued

(a) Material (1):

Bounded stresses & displacements at $r = 0$

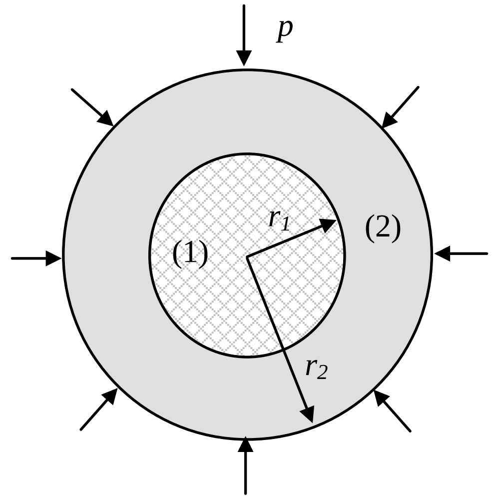
Material (2):

$\sigma_r^{(2)}(r_2,\theta) = -p$, $\tau_{r\theta}^{(2)}(r_2,\theta) = 0$

Interface Conditions:

$u_r^{(1)}(r_2,\theta) = u_r^{(2)}(r_2,\theta)$, $u_\theta^{(1)}(r_2,\theta) = u_\theta^{(1)}(r_2,\theta)$

$\sigma_r^{(1)}(r_2,\theta) = \sigma_r^{(2)}(r_2,\theta)$, $\tau_{r\theta}^{(1)}(r_2,\theta) = \tau_{r\theta}^{(2)}(r_2,\theta)$



(b)

5-5.

(a) Material (1):

$u^{(1)}(x,0) = v^{(1)}(x,0) = 0$

$\sigma_x^{(1)}(0,y) = 0$, $\tau_{xy}^{(1)}(0,y) = 0$

$\sigma_x^{(1)}(a,y) = 0$, $\tau_{xy}^{(1)}(a,y) = 0$

Material (2):

$\sigma_y^{(2)}(x,h_1+h_2) = -S$, $\tau_{xy}^{(2)}(x,h_1+h_2) = 0$

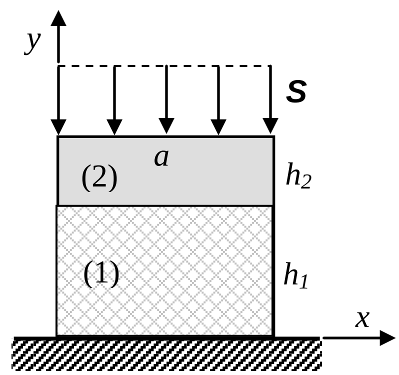
$\sigma_x^{(2)}(0,y) = 0$, $\tau_{xy}^{(2)}(0,y) = 0$

$\sigma_x^{(2)}(a,y) = 0$, $\tau_{xy}^{(2)}(a,y) = 0$

Interface Conditions:

$v^{(1)}(x,h_1) = v^{(2)}(x,h_1)$

$\sigma_y^{(1)}(x,h_1) = \sigma_y^{(2)}(x,h_1)$, $\tau_{xy}^{(1)}(x,h_1) = \tau_{xy}^{(2)}(x,h_1) = 0$



(a)

(a) Material (1):

Bounded stresses & displacements at $r = 0$

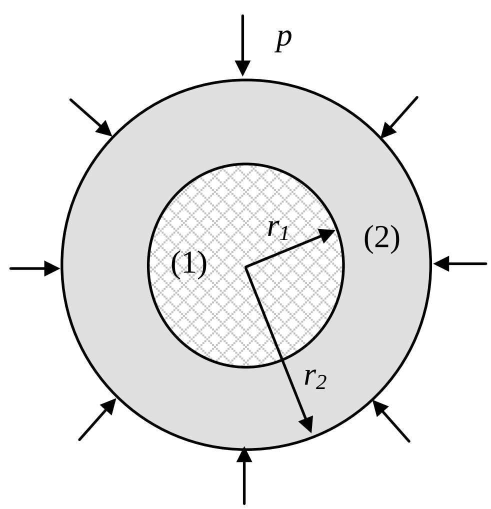
Material (2):

$\sigma_r^{(2)}(r_2,\theta) = -p$, $\tau_{r\theta}^{(2)}(r_2,\theta) = 0$

Interface Conditions:

$u_r^{(1)}(r_2,\theta) = u_r^{(2)}(r_2,\theta)$

$\sigma_r^{(1)}(r_2,\theta) = \sigma_r^{(2)}(r_2,\theta)$, $\tau_{r\theta}^{(1)}(r_2,\theta) = \tau_{r\theta}^{(2)}(r_2,\theta) = 0$



(b)

5-6.

(b) $\int_0^w T_x(x,0)dx = -Sw$, $\int_0^w T_y(x,0)dx = 0$, $\int_0^w T_y(x,0)xdx = Swh$

(c) $\int_{-l\tan 30^o}^0 T_x(0,y)dy = 0$, $\int_{-l\tan 30^o}^0 T_y(0,y)dy = -Pl/\cos 30^o$, $\int_{-l\tan 30^o}^0 T_x(0,y)ydy = Pl^2/2\cos 30^o$

(d) $\int_{-h}^h T_x(0,y)dy = Tl$, $\int_{-h}^h T_y(0,y)dy = 0$, $\int_{-h}^h T_x(0,y)ydy = Tlh$

5-7.

The compatibility relations (5.1.2) with $k = l$ are given by $e_{ij,kk} + e_{kk,ij} - e_{ik,jk} - e_{jk,ik} = 0$

Using Hooke's law $(5.1.4)_2$ into these relations $\Rightarrow$

$$\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1+\nu}(\sigma_{mm,kk}\delta_{ij} + \sigma_{mm,ij}\delta_{kk} - \sigma_{mm,jk}\delta_{ik} - \sigma_{mm,ik}\delta_{jk})$$

$$= \frac{\nu}{1+\nu}(\sigma_{mm,kk}\delta_{ij} + 3\sigma_{mm,ij} - \sigma_{mm,ji} - \sigma_{mm,ij}) = \frac{\nu}{1+\nu}(\sigma_{mm,kk}\delta_{ij} + \sigma_{mm,ij})$$

From the equilibrium equations $\sigma_{ij,j} = -F_i$ and using this result in the above relation gives

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1+\nu}\sigma_{mm,kk}\delta_{ij} - F_{i,j} - F_{j,i}$$

For the case $i = j$, the above relation gives $\sigma_{ii,kk} = -\frac{1+\nu}{1-\nu}F_{i,i}$, and using this result back into the compatibility statement yields the desired result

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i}$$

5-8.

Equaton (5.4.1) : $\sigma_{ij} = (\lambda+\mu)u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$

Using in equilibrium equations : $\sigma_{ij,j} + F_i = 0 \Rightarrow$

$(\lambda+\mu)u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) + F_i = 0$

$(\lambda+\mu)u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) + F_i = 0$

$\mu u_{i,kk} + (\lambda+\mu)u_{k,ki} + F_i = 0$

5-9.

Relation (5.4.3) gives

$$\mu u_{i,kk} + (\lambda+\mu)u_{k,ki} + F_i = 0 \Rightarrow$$

$$u_{i,kk} + \frac{\lambda+\mu}{\mu}u_{k,ki} + F_i = 0$$

$$u_{i,kk} + \frac{\dfrac{E\nu}{(1+\nu)(1-2\nu)} + \dfrac{E}{2(1+\nu)}}{\dfrac{E}{2(1+\nu)}}u_{k,ki} + F_i = 0$$

$$u_{i,kk} + \left(\frac{2\nu}{(1-2\nu)} + 1\right)u_{k,ki} + F_i = 0$$

$$u_{i,kk} + \frac{1}{1-2\nu}u_{k,ki} + F_i = 0$$

5-10.

$$\text{Strain field}: \; e_x = e_y = -\frac{\nu\rho gz}{E},\; e_z = \frac{\rho gz}{E},\; e_{xy} = e_{yz} = e_{xz} = 0$$

$$e_x = \frac{\partial u}{\partial x} = -\frac{\nu\rho gz}{E} \Rightarrow u = -\frac{\nu\rho gzx}{E} + f(y,z)$$

$$e_y = \frac{\partial v}{\partial y} = -\frac{\nu\rho gz}{E} \Rightarrow v = -\frac{\nu\rho gzy}{E} + g(x,z)$$

$$e_z = \frac{\partial w}{\partial z} = \frac{\rho gz}{E} \Rightarrow w = \frac{\rho gz^2}{2E} + h(x,y)$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial f(y,z)}{\partial y} = -\frac{\partial g(x,z)}{\partial x} = F(z) \Rightarrow$$

$$f(y,z) = F(z)y + G(z)\;,\;\; g(x,z) = -F(z)x + H(z)$$

$$e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \Rightarrow -\frac{\nu\rho gy}{E} + \frac{\partial g(x,z)}{\partial z} + \frac{\partial h(x,y)}{\partial y} = 0 \Rightarrow$$

$$-\frac{\nu\rho gy}{E} - F'(z)x + \frac{\partial h(x,y)}{\partial y} = -H'(z) \Rightarrow F'(z) = \frac{\partial^2 h(x,y)}{\partial x \partial y} \Rightarrow F(z) = C_1 z + C_2$$

and thus $H'(z)$ also must equal a constant $\Rightarrow H(z) = C_3 z + C_4$

$$e_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \Rightarrow \frac{\partial h(x,y)}{\partial x} - \frac{\nu\rho gx}{E} + \frac{\partial f(y,z)}{\partial z} = 0 \Rightarrow$$

$$\frac{\partial h(x,y)}{\partial x} - \frac{\nu\rho gx}{E} + F'(z)y = -G'(z) \Rightarrow G(z) = C_5 z + C_6$$

Also from the above relations it can be shown that

$$\frac{\partial^2 f(y,z)}{\partial y \partial z} = \frac{\partial^2 g(x,z)}{\partial x \partial z} \Rightarrow F'(z) = 0 \Rightarrow C_1 = \frac{\partial^2 h(x,y)}{\partial x \partial y} = 0 \Rightarrow$$

$$h(x,y) = \frac{\nu\rho g}{2E}(x^2 + y^2) - C_5 x - C_3 y + C_7$$

Applying the boundary conditons of no displacement and rotation at $(0,0,l)$ gives

$$u = -\frac{\nu\rho gxz}{E},\; v = -\frac{\nu\rho gyz}{E}\;,\; w = \frac{\rho g}{2E}[z^2 + \nu(x^2 + y^2) - l^2]$$

5-11.

$\sigma_x = Axy$, $\sigma_y = 0$, $\tau_{xy} = B + Cy^2$

Equilibrium Equations :

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0 \Rightarrow Ay + 2Cy = 0 \Rightarrow C = -A/2$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \Rightarrow 0 = 0$$

Compatibility Equations (7.2.7) with no body forces :

$$\nabla^2(\sigma_x + \sigma_y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(Axy + 0) = 0 \;\therefore\; \text{satisfied}$$

If we consider the rectangular domain $0 \geq x \geq l$, $-h \geq y \geq h$ with $l >> h$, the stress field gives a linearly varying bending stress and a quadratic shear stress distribution in the beam domain, while satisfying the following boundary conditions :

$\sigma_x(0, y) = 0$, $\tau_{xy}(0, y) = B + Cy^2$

$\sigma_x(l, y) = Aly$, $\tau_{xy}(l, y) = B + Cy^2$

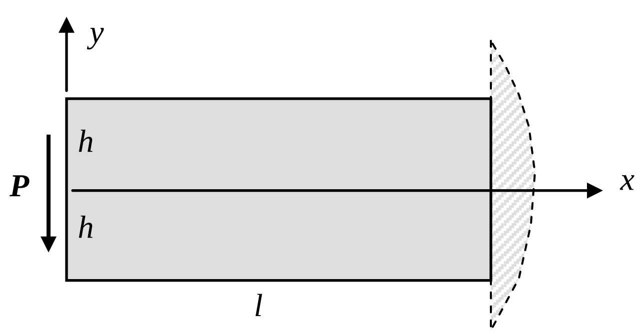
$\sigma_y(x, \pm h) = 0$, $\tau_{xy}(x, \pm h) = B + Ch^2$

Note that if $B = -Ch^2$, then the shear stress will vanish on top and bottom of beam

This suggests that the field could be used to represent a cantilever beam bending problem supported at the right and under end loading as shown in the figure. For this case, strength of materials theory $\Rightarrow$

$$\sigma = \sigma_x - \frac{My}{I} = \frac{Pxy}{I} \;,\; \tau = \tau_{\mathrm{xy}} = \frac{VQ}{It} = \frac{P}{2I}(h^2 - y^2)$$

With constants $A = \frac{P}{I}$, $B = \frac{Ph^2}{2I}$, $C = -\frac{P}{2I}$, the elasticity results match exactly with SOM theory.



5-12.

$$\sigma_x = c[y^2 + \nu(x^2 - y^2)] \ , \ \sigma_y = c[x^2 + \nu(y^2 - x^2)] \ , \sigma_z = c\nu(x^2 + y^2)$$

$$\tau_{xy} = -2c\nu xy \, , \ \tau_{yz} = \tau_{zx} = 0 \ , \ c \neq 0$$

To be a solution to elasticity problem, stress field must satisfy both equilibrium and compatibility.

Equilibrium equation check (no body forces) :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \Rightarrow 2c\nu x - 2c\nu x + 0 = 0 \ \ \text{(checks)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \Rightarrow -2c\nu y + 2c\nu y + 0 = 0 \ \text{(checks)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \Rightarrow 0 + 0 + 0 = 0 \ \text{(checks)}$$

Beltrami - Michell compatibility equation check :

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}(\sigma_x + \sigma_y + \sigma_z) = 0 \Rightarrow$$

$$c(1+\nu)\nabla^2[y^2 + \nu(x^2 - y^2)] + \frac{\partial^2}{\partial x^2}[c(1+\nu)(x^2 + y^2)] = 0 \Rightarrow 4c(1+\nu) = 0 \ \ \text{(does not check)}$$

$\therefore$ stresses satisfy equilibrium but not compatibility, and thus are not a proper elasticity solution.

5-13*.

Stress field for problem (a) :

$$\sigma_x^{(a)} = -\frac{2Px^2y}{\pi(x^2+y^2)^2}, \sigma_y^{(a)} = -\frac{2Py^3}{\pi(x^2+y^2)^2}, \tau_{xy}^{(a)} = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}$$

Stress field for problem (b) :

$$\sigma_x^{(b)} = -\left(\frac{Px^2y}{\pi(x^2+y^2)^2} + \frac{P(x+a)^2y}{\pi[(x+a)^2+y^2]^2}\right)$$

$$\sigma_y^{(b)} = -\left(\frac{Py^3}{\pi(x^2+y^2)^2} + \frac{Py^3}{\pi[(x+a)^2+y^2]^2}\right)$$

$$\tau_{xy}^{(b)} = -\left(\frac{Pxy^2}{\pi(x^2+y^2)^2} + \frac{P(x+a)y^2}{\pi[(x+a)^2+y^2]^2}\right)$$

At points far away from the loadings, $(|x|,|y|) \to \infty \Rightarrow (x+a) \approx x$ and $(y+a) \approx y$

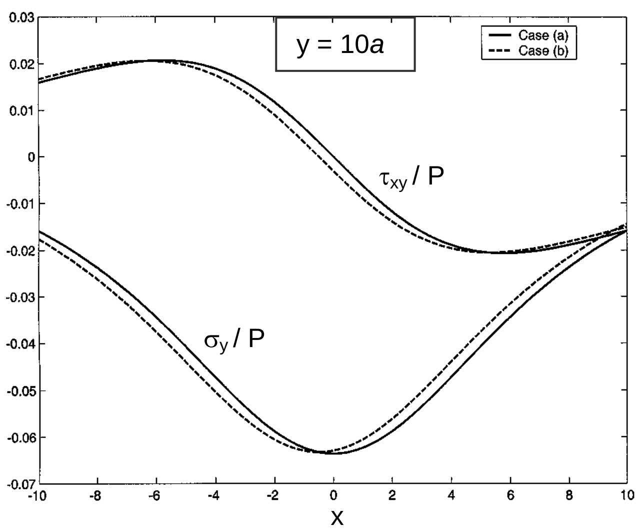
Investigating the horizontal normal stress :

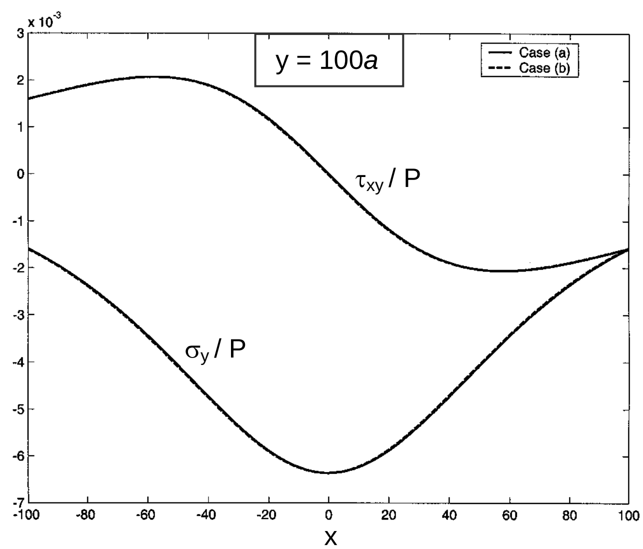
$$\sigma_x^{(b)} = -\left(\frac{Px^2y}{\pi(x^2+y^2)^2} + \frac{P(x+a)^2y}{\pi[(x+a)^2+y^2]^2}\right) \approx -\left(\frac{Px^2y}{\pi(x^2+y^2)^2} + \frac{P(x)^2y}{\pi[(x)^2+y^2]^2}\right)$$

$\approx -\dfrac{2Px^2y}{\pi(x^2+y^2)^2} \approx \sigma_x^{(a)}$, and similar arguements can be made for the other stress components.

Thus at large distances from the loading, the stresses from problems (a) and (b) are identical.

The following MATALB results for the specific cases of $y = 10a$ and $y = 100a$ are shown below





6-1.

$$dU=\int_0^{\sigma_x}(\sigma+\frac{\partial\sigma}{\partial x}dx)d(u+\frac{\partial u}{\partial x}dx)dydz-\int_0^{\sigma_x}\sigma dudydz+\int_0^{\sigma_x}F_x dudxdydz$$

$$=\int_0^{\sigma_x}\left(\sigma dudydz+\sigma d\left(\frac{\partial u}{\partial x}\right)dxdydz+\frac{\partial\sigma}{\partial x}dudxdydz+\frac{\partial\sigma}{\partial x}d\left(\frac{\partial u}{\partial x}\right)(dx)^2dydz\right)$$

$$-\int_0^{\sigma_x}\sigma dudydz+\int_0^{\sigma_x}F_x dudxdydz$$

$$=\int_0^{\sigma_x}\left(\sigma d\left(\frac{\partial u}{\partial x}\right)dxdydz+\left(\frac{\partial\sigma}{\partial x}+F_x\right)dudxdydz\right)=\int_0^{\sigma_x}\sigma d\left(\frac{\partial u}{\partial x}\right)dxdydz$$

$$=\int_0^{\sigma_x}\sigma d\left(\frac{\sigma}{E}\right)dxdydz=\frac{\sigma_x^2}{2E}dxdydz$$

Thus the strain energy density is given by

$$U=\frac{dU}{dxdydz}=\frac{\sigma_x^2}{2E}=\frac{Ee_x^2}{2}=\frac{1}{2}\sigma_x e_x$$

6-2.

$$U(\boldsymbol{e})=\frac{1}{2}\lambda e_{jj}e_{kk}+\mu e_{ij}e_{ij}$$

Now $I_e=e_{jj}=e_{kk}$, $II_e=\frac{1}{2}(e_{ii}e_{jj}-e_{ij}e_{ij})\Rightarrow e_{ij}e_{ij}=I_e^2-2II_e$

$$\therefore U(\boldsymbol{e})=\frac{1}{2}\lambda I_e^2+\mu(I_e^2-2II_e)=(\frac{1}{2}\lambda+\mu)I_e^2-2\mu II_e$$

Also $U(\boldsymbol{\sigma})=\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij}-\frac{\nu}{2E}\sigma_{jj}\sigma_{kk}$

$\sigma_{kk}=I_1$ and $\sigma_{ij}\sigma_{ij}=I_1^2-2I_2$

$$\therefore U(\boldsymbol{\sigma})=\frac{1+\nu}{2E}(I_1^2-2I_2)-\frac{\nu}{2E}I_1^2=\frac{1}{2E}\left(I_1^2-2(1+\nu)I_2\right)$$

6-3.

$$U=\frac{1}{2}\sigma_{ij}e_{ij}=\frac{1}{2}(\lambda e_{kk}\delta_{ij}+2\mu e_{ij})e_{ij}=\frac{1}{2}\lambda e_{jj}e_{kk}+\mu e_{ij}e_{ij}$$

$$=\frac{1}{2}\lambda(e_x+e_y+e_z)^2+\mu(e_x^2+e_y^2+e_z^2+2e_{xy}^2+2e_{yz}^2+2e_{zx}^2)$$

$$=\frac{1}{2}\lambda(e_x+e_y+e_z)^2+\mu(e_x^2+e_y^2+e_z^2+\frac{1}{2}\gamma_{xy}^2+\frac{1}{2}\gamma_{yz}^2+\frac{1}{2}\gamma_{zx}^2)$$

$$U=\frac{1}{2}\sigma_{ij}e_{ij}=U=\frac{1}{2}\sigma_{ij}\left(\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)=\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij}-\frac{\nu}{2E}\sigma_{jj}\sigma_{kk}$$

$$=\frac{1+\nu}{2E}(\sigma_x^2+\sigma_y^2+\sigma_z^2+2\tau_{xy}^2+2\tau_{yz}^2+2\tau_{zx}^2)-\frac{\nu}{2E}(\sigma_x+\sigma_y+\sigma_z)^2$$

6-4.

$$U(\boldsymbol{e})=\frac{1}{2}\lambda e_{jj}e_{kk}+\mu e_{ij}e_{ij}$$

$$\frac{\partial U}{\partial e_{mn}}=\frac{1}{2}\lambda\left(e_{jj}\frac{\partial e_{kk}}{\partial e_{mn}}+\frac{\partial e_{jj}}{\partial e_{mn}}e_{kk}\right)+\mu\left(e_{ij}\frac{\partial e_{ij}}{\partial e_{mn}}+\frac{\partial e_{ij}}{\partial e_{mn}}e_{ij}\right)$$

Now since $\dfrac{\partial e_{ij}}{\partial e_{mn}}=\delta_{im}\delta_{jn}$

$$\frac{\partial U}{\partial e_{mn}}=\frac{1}{2}\lambda\left(e_{jj}\delta_{km}\delta_{kn}+\delta_{jm}\delta_{jn}e_{kk}\right)+\mu\left(e_{ij}\delta_{im}\delta_{jn}+\delta_{im}\delta_{jn}e_{ij}\right)$$

$$=\frac{1}{2}\lambda(e_{jj}\delta_{mn}+e_{kk}\delta_{mn})+2\mu e_{mn}=\lambda e_{kk}\delta_{mn}+2\mu e_{mn}=\sigma_{mn}$$

6-5.

$$\sigma_{ij}=\frac{\partial U}{\partial e_{ij}}\Rightarrow\frac{\partial\sigma_{ij}}{\partial e_{kl}}=\frac{\partial^2 U}{\partial e_{kl}\partial e_{ij}}\Rightarrow\frac{\partial\sigma_{kl}}{\partial e_{ij}}=\frac{\partial^2 U}{\partial e_{ij}\partial e_{kl}}$$

Since U is a continuous function, the order of differentiation can be interchanged giving

$$\frac{\partial\sigma_{ij}}{\partial e_{kl}}=\frac{\partial\sigma_{kl}}{\partial e_{ij}}$$

Likewise starting with $e_{ij}=\dfrac{\partial U}{\partial\sigma_{ij}}$ and following similar steps would give $\dfrac{\partial e_{ij}}{\partial\sigma_{kl}}=\dfrac{\partial e_{kl}}{\partial\sigma_{ij}}$

Now since $\dfrac{\partial\sigma_{ij}}{\partial e_{kl}}=C_{ijkl}$ and $\dfrac{\partial\sigma_{kl}}{\partial e_{ij}}=C_{klij}\Rightarrow C_{ijkl}=C_{klij}$

6-6.

$$U_v=\frac{1}{2}\tilde{\sigma}_{ij}\tilde{e}_{ij}=\frac{1}{2}\left(\frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\frac{1}{3}e_{ll}\delta_{ij}\right)=\frac{1}{6}\sigma_{kk}e_{jj}$$

$$=\frac{1}{6}\sigma_{kk}\left(\frac{1-2\nu}{E}\sigma_{jj}\right)=\frac{1-2\nu}{6E}\sigma_{jj}\sigma_{kk}=\frac{1-2\nu}{6E}(\sigma_x+\sigma_y+\sigma_z)^2$$

$$U_d=U-U_v=\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij}-\frac{\nu}{2E}\sigma_{jj}\sigma_{kk}-\frac{1-2\nu}{6E}\sigma_{jj}\sigma_{kk}=\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij}-\frac{1+\nu}{6E}\sigma_{jj}\sigma_{kk}$$

$$=\frac{1}{4\mu}\sigma_{ij}\sigma_{ij}-\frac{1}{12\mu}\sigma_{jj}\sigma_{kk}=\frac{1}{12\mu}[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)]$$

6-7.

$$\sigma_{kk} = I_1 \text{ and } \sigma_{ij}\sigma_{ij} = I_1^2 - 2I_2$$

$$U_v = \frac{1-2\nu}{6E}\sigma_{jj}\sigma_{kk} = \frac{1-2\nu}{6E}I_1^2$$

$$U_d = U - U_v = \frac{1}{4\mu}\sigma_{ij}\sigma_{ij} - \frac{1}{12\mu}\sigma_{jj}\sigma_{kk} = \frac{1}{4\mu}(I_1^2 - 2I_2) - \frac{1}{12\mu}I_1^2$$

$$= \frac{1}{12\mu}\left(3I_1^2 - 6I_2 - I_1^2\right) = \frac{1}{6\mu}\left(I_1^2 - 3I_2\right)$$

6-8.

$$U_d = \frac{1}{12\mu}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]$$

$$\text{From Exercise 3-5} \Rightarrow \tau_{oct}^2 = \frac{1}{9}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 + 6\tau_{yz}^2 + 6\tau_{zx}^2\right]$$

$$\therefore U_d = \frac{1}{12\mu}9\tau_{oct}^2 = \frac{3}{4\mu}\tau_{oct}^2 = \frac{3}{2}\frac{1+\nu}{E}\tau_{oct}^2$$

6-9.

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U = \frac{1+\nu}{2E}(\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2) - \frac{\nu}{2E}(\sigma_x + \sigma_y)^2$$

$$= \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{1+\nu}{E}\tau_{xy}^2$$

6-10.

$$\text{Given stress field: } \sigma_x = -\frac{3M}{2c^3}y, \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

$$U = \frac{\sigma_x^2}{2E} = \frac{1}{2E}\frac{9M^2}{4c^6}y^2 = \frac{9M^2}{8Ec^6}y^2$$

$$U_T = \iiint U dV = \int_0^1\int_0^{2l}\int_{-c}^{c}\frac{9M^2}{8Ec^6}y^2 dV = \frac{9M^2}{8Ec^6}(2l)\int_{-c}^{c}y^2 dy = \frac{3M^2 l}{2Ec^3} = \frac{M^2 l}{EI}$$

6-11.

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0\ ,\ \tau_{xz} = -\mu\alpha y\ ,\ \tau_{yz} = -\mu\alpha x$$

$$U = \frac{1+\nu}{2E}(2\mu^2\alpha^2 x^2 + 2\mu^2\alpha^2 y^2) = \frac{1+\nu}{E}\mu^2\alpha^2(x^2+y^2) = \frac{E\alpha^2 r^2}{4(1+\nu)}$$

Total Strain Energy :

$$U_T = \iiint_V U dV = \frac{1+\nu}{E}\mu^2\alpha^2\int_0^L\iint_A (x^2+y^2)dAdx$$

$$= \frac{1+\nu}{E}\mu^2\alpha^2 L\int_0^{2\pi}\int_0^R r^2 rdrd\theta = \frac{1+\nu}{E}\mu^2\alpha^2 L\frac{\pi R^4}{2}$$

$$= \frac{1+\nu}{2E}\pi\mu^2\alpha^2 LR^4 = \frac{E\pi\alpha^2 LR^4}{8(1+\nu)}$$

6-12.

Reciprocal Theorem : $\int_S T_i^{(1)}u_i^{(2)}dS + \int_V F_i^{(1)}u_i^{(2)}dV = \int_S T_i^{(2)}u_i^{(1)}dS + \int_V F_i^{(2)}u_i^{(1)}dV$

Choose first state as : $u_i^{(1)} = Ax_i$, $F_i^{(1)} = 0$, $T_i^{(1)} = 3KAn_i$, and second state as $u_i, F_i, T_i \Rightarrow$

$$\int_S 3KAn_i u_i dS = \int_S T_i Ax_i dS + \int_V F_i Ax_i dV \Rightarrow \int_S n_i u_i dS = \frac{1}{3K}\left(\int_S T_i x_i dS + \int_V F_i x_i dV\right)$$

From the Divergence Theorem : $\int_S n_i u_i dS = \int_V u_{i,i}\, dV = \int_V e_{ii}\, dV$

$$\therefore \Delta V = \int_V e_{ii}\, dV = \frac{1}{3K}\left(\int_S T_i x_i dS + \int_V F_i x_i dV\right)$$

6-13.

$$w_j = \sin\frac{j\pi x}{l} \Rightarrow w = \sum_{j=1}^{N} c_j \sin\frac{j\pi x}{l}$$

$$\Pi = \int_0^l \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - q_o w\right]dx$$

$$= \int_0^l \left[\frac{EI}{2}\left(\sum_{j=1}^{N} c_j\left(\frac{j\pi}{l}\right)^2\left(-\sin\frac{j\pi x}{l}\right)\right)^2 - q_o\sum_{j=1}^{N} c_j \sin\frac{j\pi x}{l}\right]dx$$

$N = 2$ Case:

$$\Pi = \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l \left(c_1\sin\frac{\pi x}{l} + 4c_2\sin\frac{2\pi x}{l}\right)^2 dx - q_o\int_0^l \left(c_1\sin\frac{\pi x}{l} + c_2\sin\frac{2\pi x}{l}\right)dx$$

$$= \frac{EI\pi^4}{2l^4}\left(\frac{1}{2}c_1^2 l + 8c_2^2 l\right) - q_o\frac{2c_1 l}{\pi}$$

$$\frac{\partial\Pi}{\partial c_1} = 0 \Rightarrow \frac{EI\pi^4}{2l^4}c_1 l - q_o\frac{2l}{\pi} = 0$$

$$\frac{\partial\Pi}{\partial c_2} = 0 \Rightarrow \frac{EI\pi^4}{2l^4}16c_2 l = 0$$

Solving for the coefficients: $c_1 = \dfrac{4q_o l^4}{EI\pi^5}$, $c_2 = 0$, and thus the approximate solution is

$w = \dfrac{4q_o l^4}{EI\pi^5}\sin\dfrac{\pi x}{l}$, and at mid-span $w(l/2) = \dfrac{4q_o l^4}{EI\pi^5} = 0.0131\dfrac{q_o l^4}{EI}$

Results from Example 6.2 are $w = \dfrac{q_o l^2}{24EI}x(l-x)$, and at mid-span $w(l/2) = \dfrac{q_o l^4}{96EI} = 0.0104\dfrac{q_o l^4}{EI}$

The exact solution is $w = \dfrac{q_o x}{24EI}(l^3 + x^3 - 2lx^2)$, and thus $w(l/2) = \dfrac{5q_o l^4}{384EI} = 0.015\dfrac{q_o l^4}{EI}$

6-14.

$$\sigma_x = -\frac{My}{I},\ M = EI\frac{d^2w}{dx^2} \Rightarrow \sigma_x = -E\frac{d^2w}{dx^2}y$$

Example 6-2: $\sigma_x = -E\dfrac{d^2w}{dx^2}y = \dfrac{q_o l^2}{12I}y$, $(\sigma_x)_{\max} = \sigma_x(l/2) = 0.0833\dfrac{q_o l^2}{I}y$

Exercise 6-10: $\sigma_x = -E\dfrac{d^2w}{dx^2}y = \dfrac{4q_o l^2}{I\pi^3}\sin\dfrac{\pi x}{l}y$, $(\sigma_x)_{\max} = \sigma_x(l/2) = 0.129\dfrac{q_o l^2}{I}y$

Exact Solution: $\sigma_x = -E\dfrac{d^2w}{dx^2}y = \dfrac{q_o}{2I}(lx - x^2)y$, $(\sigma_x)_{\max} = \sigma_x(l/2) = 0.125\dfrac{q_o l^2}{I}y$

7-1.

$$\begin{aligned}\sigma_x &= \lambda(e_x+e_y)+2\mu e_x \\ \sigma_y &= \lambda(e_x+e_y)+2\mu e_y\end{aligned} \Rightarrow \begin{aligned}\sigma_x+\sigma_y &= 2(\lambda+\mu)(e_x+e_y) \\ \sigma_x-\sigma_y &= 2\mu(e_x-e_y)\end{aligned} \Rightarrow$$

$$2e_x = \frac{1}{2(\lambda+\mu)}(\sigma_x+\sigma_y)+\frac{1}{2\mu}(\sigma_x-\sigma_y) = \frac{(1+\nu)(1-2\nu)}{E}(\sigma_x+\sigma_y)+\frac{1+\nu}{E}(\sigma_x-\sigma_y)$$

$$2e_y = \frac{1}{2(\lambda+\mu)}(\sigma_x+\sigma_y)-\frac{1}{2\mu}(\sigma_x-\sigma_y) = \frac{(1+\nu)(1-2\nu)}{E}(\sigma_x+\sigma_y)-\frac{1+\nu}{E}(\sigma_x-\sigma_y)$$

$$e_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x-\nu\sigma_y]$$

$$e_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y-\nu\sigma_x]$$

$$\tau_{xy} = 2\mu e_{xy} \Rightarrow e_{xy} = \frac{1}{2\mu}\tau_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

7-2.

Navier's Equations:

$$\sigma_x = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial u}{\partial x},\ \sigma_y = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial v}{\partial y},\ \tau_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \Rightarrow \frac{\partial}{\partial x}\left(\lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right) + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \Rightarrow \frac{\partial}{\partial x}\left(\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right) + \frac{\partial}{\partial y}\left(\lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial v}{\partial y}\right) + F_y = 0$$

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2})\right) + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

$$\mu\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2})\right) + (\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

Beltrami - Michell Compatibility Equation:

$$e_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y],\ e_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x],\ e_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y]\right) + \frac{\partial^2}{\partial x^2}\left(\frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]\right) = 2\frac{\partial^2}{\partial x \partial y}\left(\frac{1+\nu}{E}\tau_{xy}\right) \Rightarrow$$

$$(1-\nu)\nabla^2(\sigma_x + \sigma_y) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}$$

But from equilibrium equations,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \Rightarrow -\frac{\partial F_x}{\partial x} = \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \Rightarrow -\frac{\partial F_y}{\partial y} = \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial y \partial x}$$

$\therefore\ 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$, and using this result into compatibility relation gives

$$(1-\nu)\nabla^2(\sigma_x + \sigma_y) = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

7-3.

$$\text{Navier equations}: \mu\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x=0\ ,\ \mu\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y=0$$

$$\frac{\partial}{\partial y}\left\{\mu\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x\right\}-\frac{\partial}{\partial x}\left\{\mu\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y\right\}=0\Rightarrow$$

$$\frac{\partial}{\partial y}\nabla^2 u=\frac{\partial}{\partial x}\nabla^2 v\ \ \cdots\text{(a)}$$

$$\frac{\partial}{\partial x}\left\{\mu\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x\right\}+\frac{\partial}{\partial y}\left\{\mu\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y\right\}=0\Rightarrow$$

$$\frac{\partial}{\partial x}\nabla^2 u=-\frac{\partial}{\partial y}\nabla^2 v\ \ \cdots\text{(b)}$$

$$\frac{\partial}{\partial x}\text{(b)}+\frac{\partial}{\partial y}\text{(a)}=0\Rightarrow\ \ \nabla^4 u=0$$

$$\frac{\partial}{\partial x}\text{(a)}-\frac{\partial}{\partial y}\text{(b)}=0\Rightarrow\ \ \nabla^4 v=0$$

7-4.

Choose total solution as sum, $\sigma_z^{(T)}=\sigma_z+\sigma_z^{(c)}$

$\sigma_z=\nu(\sigma_x+\sigma_y)$, $\sigma_z^{(c)}=Ax+By+C$

For zero resultant force at ends with cross-section $\hat{A}$:

$$R_z=\int_{\hat{A}}\sigma_z^{(T)}\,dA=0\Rightarrow\int_{\hat{A}}(\sigma_z+\sigma_z^{(c)})\,dA=0\Rightarrow$$

$$\int_{\hat{A}}\sigma_z\,dA=-\int_{\hat{A}}(Ax+By+C)\,dA=-C\hat{A}\ \text{(for principal axes)}\Rightarrow C=-\frac{1}{\hat{A}}\int_{\hat{A}}\sigma_z\,dA$$

For zero resultant moments at ends:

$$M_x=\int_{\hat{A}}\sigma_z^{(T)}\,ydA=0\Rightarrow\int_{\hat{A}}(\sigma_z+\sigma_z^{(c)})\,ydA=0\Rightarrow$$

$$\int_{\hat{A}}\sigma_z\,ydA=-\int_{\hat{A}}(Axy+By^2+Cy)\,dA=-BI_x\ \text{(for principal axes) , where } I_x=\int_{\hat{A}}y^2\,dA$$

$$\Rightarrow B=-\frac{1}{I_x}\int_{\hat{A}}\sigma_z\,ydA$$

$$M_y=\int_{\hat{A}}\sigma_z^{(T)}\,xdA=0\Rightarrow\int_{\hat{A}}(\sigma_z+\sigma_z^{(c)})\,xdA=0\Rightarrow$$

$$\int_{\hat{A}}\sigma_z\,xdA=-\int_{\hat{A}}(Ax^2+Bxy+Cx)\,dA=-AI_y\ \text{(for principal axes) , where } I_y=\int_{\hat{A}}x^2\,dA$$

$$\Rightarrow A=-\frac{1}{I_y}\int_{\hat{A}}\sigma_z\,xdA$$

7-5.

Stresses must satisfy equilbrium and compatibility equations:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x=0\Rightarrow\frac{\partial}{\partial x}(kxy)+\frac{\partial}{\partial y}\left(-\frac{1}{2}ky^2\right)=0\Rightarrow 0=0$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_y=0\Rightarrow\frac{\partial}{\partial x}\left(-\frac{1}{2}ky^2\right)+\frac{\partial}{\partial y}(kx)=0\Rightarrow 0=0$$

$$\nabla^2(\sigma_x+\sigma_y)=-\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)\Rightarrow\nabla^2(kxy+kx)=0\Rightarrow 0=0$$

The out-of-plane normal stress must also satisfy $\sigma_z=\nu(\sigma_x+\sigma_y)$

$\nu(\sigma_x+\sigma_y)=\nu(kxy+kx)=\nu kx(1+y)$, which also checks

7-6.

$$\begin{aligned}e_x&=\frac{1}{E}(\sigma_x-\nu\sigma_y)\\ e_y&=\frac{1}{E}(\sigma_y-\nu\sigma_x)\end{aligned}\Rightarrow\begin{aligned}\sigma_x+\sigma_y&=\frac{E}{1-\nu}(e_x+e_y)\\ \sigma_x-\sigma_y&=\frac{E}{1+\nu}(e_x-e_y)\end{aligned}\Rightarrow$$

$$2\sigma_x=\frac{E}{1-\nu}(e_x+e_y)+\frac{E}{1+\nu}(e_x-e_y)\Rightarrow\sigma_x=\frac{E}{1-\nu^2}(e_x+\nu e_y)$$

$$2\sigma_y=\frac{E}{1-\nu}(e_x+e_y)-\frac{E}{1+\nu}(e_x-e_y)\Rightarrow\sigma_y=\frac{E}{1-\nu^2}(e_y+\nu e_x)$$

$$e_{xy}=\frac{1+\nu}{E}\tau_{xy}\Rightarrow\tau_{xy}=\frac{E}{1+\nu}e_{xy}$$

7-7.

$$\sigma_x = \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right),\ \sigma_y = \frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right),\ \tau_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

Substitute into equilibrium equations :

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x = 0 \Rightarrow \frac{\partial}{\partial x}\left[\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+F_x = 0 \Rightarrow$$

$$\frac{E}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^2 v}{\partial x\partial y}\right)+\frac{E}{2(1+\nu)}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y\partial x}\right)+F_x = 0 \Rightarrow$$

$$\mu\nabla^2 u+\frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x = 0$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_y = 0 \Rightarrow \frac{\partial}{\partial x}\left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)\right]+F_y = 0 \Rightarrow$$

$$\frac{E}{2(1+\nu)}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+\frac{E}{1-\nu^2}\left(\frac{\partial^2 v}{\partial y^2}+\nu\frac{\partial^2 u}{\partial y\partial x}\right)+F_y = 0 \Rightarrow$$

$$\mu\nabla^2 v+\frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y = 0$$

Beltrami - Michell Compatibility Equation :

$$e_x = \frac{1}{E}(\sigma_x-\nu\sigma_y)\ ,\ e_y = \frac{1}{E}(\sigma_y-\nu\sigma_x)\ ,\ e_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

$$\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x\partial y} \Rightarrow \frac{\partial^2}{\partial y^2}\left(\frac{1}{E}(\sigma_x-\nu\sigma_y)\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}(\sigma_y-\nu\sigma_x)\right) = 2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\tau_{xy}\right) \Rightarrow$$

$$\frac{1}{(1+\nu)}\nabla^2(\sigma_x+\sigma_y) = 2\frac{\partial^2\tau_{xy}}{\partial x\partial y}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}$$

But from equilibrium equations,

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x = 0 \Rightarrow -\frac{\partial F_x}{\partial x} = \frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\tau_{xy}}{\partial x\partial y}$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_y = 0 \Rightarrow -\frac{\partial F_y}{\partial y} = \frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial x}$$

$\therefore\ 2\dfrac{\partial^2\tau_{xy}}{\partial x\partial y}+\dfrac{\partial^2\sigma_x}{\partial x^2}+\dfrac{\partial^2\sigma_y}{\partial y^2} = -\left(\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}\right)$, and using this result into compatibility relation gives

$$\nabla^2(\sigma_x+\sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

7-8.

$e_z = \frac{\partial w}{\partial z} \Rightarrow w = e_z z + f(x,y)$, where $f(x,y)$ is an arbitrary function

Note that $e_z = -\frac{\nu}{1-\nu}(e_x + e_y) = \text{function of } x, y$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \Rightarrow \frac{\partial v}{\partial z} = -\frac{\partial e_z}{\partial y} z - \frac{\partial f}{\partial y} \Rightarrow v = -\frac{1}{2}\frac{\partial e_z}{\partial y} z^2 - \frac{\partial f}{\partial y} z + g(x,y)$$

$$e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial e_z}{\partial x} z - \frac{\partial f}{\partial x} \Rightarrow v = -\frac{1}{2}\frac{\partial e_z}{\partial x} z^2 - \frac{\partial f}{\partial x} z + h(x,y)$$

where g and h are arbitrary functions

$\therefore$ These results imply that the displacements will depend on the out - of - plane coordinate z, and thus the field will be three - dimensional.

7-9.

The three non - vanishing compatibility relations which were not included in the plane stress formulation are :

$$\frac{\partial^2 e_y}{\partial z^2}+\frac{\partial^2 e_z}{\partial y^2}=2\frac{\partial^2 e_{yz}}{\partial y\partial z}\Rightarrow\frac{\partial^2 e_z}{\partial y^2}=0$$

$$\frac{\partial^2 e_z}{\partial x^2}+\frac{\partial^2 e_x}{\partial z^2}=2\frac{\partial^2 e_{zx}}{\partial z\partial x}\Rightarrow\frac{\partial^2 e_z}{\partial x^2}=0$$

$$\frac{\partial^2 e_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}\right)\Rightarrow\frac{\partial^2 e_z}{\partial x\partial y}=0$$

Intergrating these three results gives :

$$\frac{\partial^2 e_z}{\partial x^2}=0\Rightarrow\frac{\partial e_z}{\partial x}=f(y)$$

$$\frac{\partial^2 e_z}{\partial y^2}=0\Rightarrow\frac{\partial e_z}{\partial y}=g(x)$$

$$\frac{\partial^2 e_z}{\partial x\partial y}=0\Rightarrow f'(y)=g'(x)=0\Rightarrow f(y)=\text{constant}=a\ \text{ and }\ g(x)=b$$

$$\frac{\partial e_z}{\partial x}=f(y)=a\Rightarrow e_z=ax+F(y)$$

$$\frac{\partial e_z}{\partial y}=g(x)=b\Rightarrow F'(y)=b\Rightarrow F(y)=by+c$$, where c is an arbitrary constant

$$\therefore e_z=ax+by+c$$

Relation $(7.2.2)_3\Rightarrow e_z=-\dfrac{\nu}{E}(\sigma_x+\sigma_y)=-\dfrac{\nu}{1-\nu}(e_x+e_y)$

In general the in - plane stress or strain field will not be linear, and so the result from integrating the compatibility relations will not be satisfied for a general problem.

7-10.

Plane stress to plane strain :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{\frac{E}{1-\nu^2}\frac{\nu}{1-\nu}}{(1+\frac{\nu}{1-\nu})(1-2\frac{\nu}{1-\nu})} = \frac{E}{1-\nu^2}\frac{\nu(1-\nu)}{(1-\nu+\nu)(1-\nu-2\nu)} = \frac{E\nu}{(1+\nu)(1-3\nu)}$$

$$\mu = \frac{E}{2(1+\nu)} = \frac{\frac{E}{1-\nu^2}}{2(1+\frac{\nu}{1-\nu})} = \frac{E}{1-\nu^2}\frac{1-\nu}{2(1-\nu+\nu)} = \frac{E}{2(1+\nu)} = \mu$$

Plane strain to plane stress :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{\frac{E(1+2\nu)}{(1+\nu)^2}\frac{\nu}{1+\nu}}{(1+\frac{\nu}{1+\nu})(1-2\frac{\nu}{1+\nu})} = \frac{E(1+2\nu)}{(1+\nu)^2}\frac{\nu(1+\nu)}{(1+\nu+\nu)(1+\nu-2\nu)} = \frac{E\nu}{(1+\nu)(1-\nu)}$$

$$\mu = \frac{E}{2(1+\nu)} = \frac{\frac{E(1+2\nu)}{(1+\nu)^2}}{2(1+\frac{\nu}{1+\nu})} = \frac{E(1+2\nu)}{(1+\nu)^2}\frac{1+\nu}{2(1+\nu+\nu)} = \frac{E}{2(1+\nu)} = \mu$$

Notice that the shear modulus does not change for either case!

7-11.

(a) Equation $(7.1.5)_1$: $\mu\nabla^2 u + (\lambda+\mu)\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)+F_x = 0$

Converting to plane stress using the results from Exercise 7-10 $\Rightarrow$

$$\mu\nabla^2 u + \left(\frac{E\nu}{(1+\nu)(1-\nu)}+\frac{E}{2(1+\nu)}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x = 0$$

which reduces to: $\mu\nabla^2 u + \dfrac{E}{2(1-\nu)})\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)+F_x = 0$

Likewise for relation $(7.1.5)_2$

Equation (7.1.7): $\nabla^2(\sigma_x+\sigma_y) = -\dfrac{1}{1-\nu}\left(\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}\right)$

$$\Rightarrow \nabla^2(\sigma_x+\sigma_y) = -\frac{1}{1-\dfrac{\nu}{1+\nu}}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \Rightarrow \nabla^2(\sigma_x+\sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

(b) Equation $(7.2.5)_1$: $\mu\nabla^2 u + \dfrac{E}{2(1-\nu)})\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)+F_x = 0$

Converting to plane strain using the results from Table 7-1 $\Rightarrow$

$$\mu\nabla^2 u + \frac{\dfrac{E}{1-\nu^2}}{2(1-\dfrac{\nu}{1-\nu})})\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x = 0 \Rightarrow \mu\nabla^2 u + \frac{E}{2(1+\nu)(1-\nu)})\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x = 0$$

which reduces to: $\mu\nabla^2 u + (\lambda+\mu)\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)+F_x = 0$

Likewise for relation $(7.2.5)_2$

Equation (7.2.7): $\nabla^2(\sigma_x+\sigma_y) = -(1+\nu)\left(\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}\right)$

$$\Rightarrow \nabla^2(\sigma_x+\sigma_y) = -(1+\frac{\nu}{1-\nu})\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \Rightarrow \nabla^2(\sigma_x+\sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

7-12.

(a) Plane Strain $\rightarrow$ Plane Stress : $E \rightarrow \frac{E(1+2\nu)}{(1+\nu)^2}$, $\nu \rightarrow \frac{\nu}{1+\nu}$, $\mu \rightarrow \mu$, $\lambda \rightarrow \frac{E\nu}{(1+\nu)(1-\nu)}$

$$\sigma_x = \lambda(e_x + e_y) + 2\mu e_x \rightarrow \frac{E\nu}{(1+\nu)(1-\nu)}(e_x + e_y) + \frac{E}{1+\nu}e_x = \frac{E}{1-\nu^2}(e_x + \nu e_y)$$

Likewise : $\sigma_y = \lambda(e_x + e_y) + 2\mu e_y \rightarrow \frac{E\nu}{(1+\nu)(1-\nu)}(e_x + e_y) + \frac{E}{1+\nu}e_y = \frac{E}{1-\nu^2}(e_y + \nu e_x)$

$$\tau_{xy} = 2\mu e_{xy} \rightarrow 2\mu e_{xy} = \frac{E}{1+\nu}\tau_{xy}$$

Results properly match with expressions given in Exercise 7 - 6.

(b) Plane Stress $\rightarrow$ Plane Strain : $E \rightarrow \frac{E}{1-\nu^2}$, $\nu \rightarrow \frac{\nu}{1-\nu}$, $\mu \rightarrow \mu$, $\lambda \rightarrow \frac{E\nu}{(1+\nu)(1-3\nu)}$

$$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \rightarrow \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y]$$

$$e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \rightarrow \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

$$e_{xy} = \frac{1+\nu}{E}\tau_{xy} \rightarrow \frac{1+\frac{\nu}{1-\nu}}{\frac{E}{1-\nu^2}}\tau_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

Results properly match with expressions given in Exercise 7 - 1.

7-13*.

$$\text{Plane Stress Results}: \ u=-\frac{Mxy}{EI}\ ,\ v=\frac{M}{2EI}[\nu y^2+x^2-l^2]\ ,\ -l\le x\le l$$

$$\text{Plane Stress} \ \rightarrow \ \text{Plane Strain}: \ E\rightarrow\frac{E}{1-\nu^2},\ \nu\rightarrow\frac{\nu}{1-\nu}$$

$$\therefore\ \text{Plane Strain Results}: \ u=-\frac{Mxy}{EI}(1-\nu^2)\ ,\ v=\frac{M(1-\nu^2)}{2EI}\left[\frac{\nu}{1-\nu}y^2+x^2-l^2\right]$$

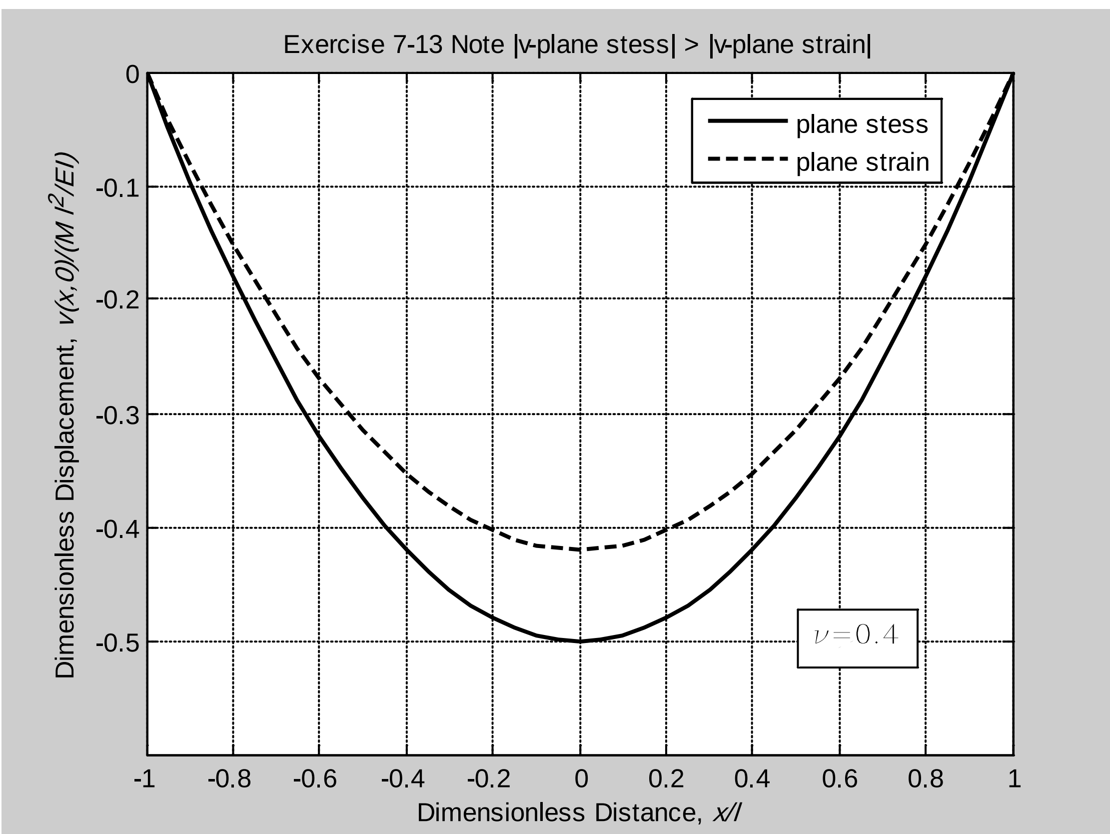
Along x - aixs $(y=0)$:

$$v_{P.Stress}=\frac{M}{2EI}[x^2-l^2]\Rightarrow\frac{v_{P.Stress}}{Ml^2/EI}=\frac{1}{2}\left[\left(\frac{x}{l}\right)^2-1\right]$$

$$v_{P.Strain}=\frac{M(1-\nu^2)}{2EI}[x^2-l^2]\Rightarrow\frac{v_{P.Strain}}{Ml^2/EI}=\frac{(1-\nu^2)}{2}\left[\left(\frac{x}{l}\right)^2-1\right]$$

When Poisson's ratio $\rightarrow 0$, the two displacements become identical.

Using MATLAB, plots of the plane stress and plane strain displacements are given by



7-14*.

$$\text{Plane Strain Result}: \ u_r = \frac{T(1+\nu)}{E}\left[(1-2\nu)r + \frac{r_1^2}{r}\right]$$

$$\text{Plane Strain} \ \rightarrow \ \text{Plane Stress}: \ E \rightarrow \frac{E(1+2\nu)}{(1+\nu)^2}, \ \nu \rightarrow \frac{\nu}{1+\nu}$$

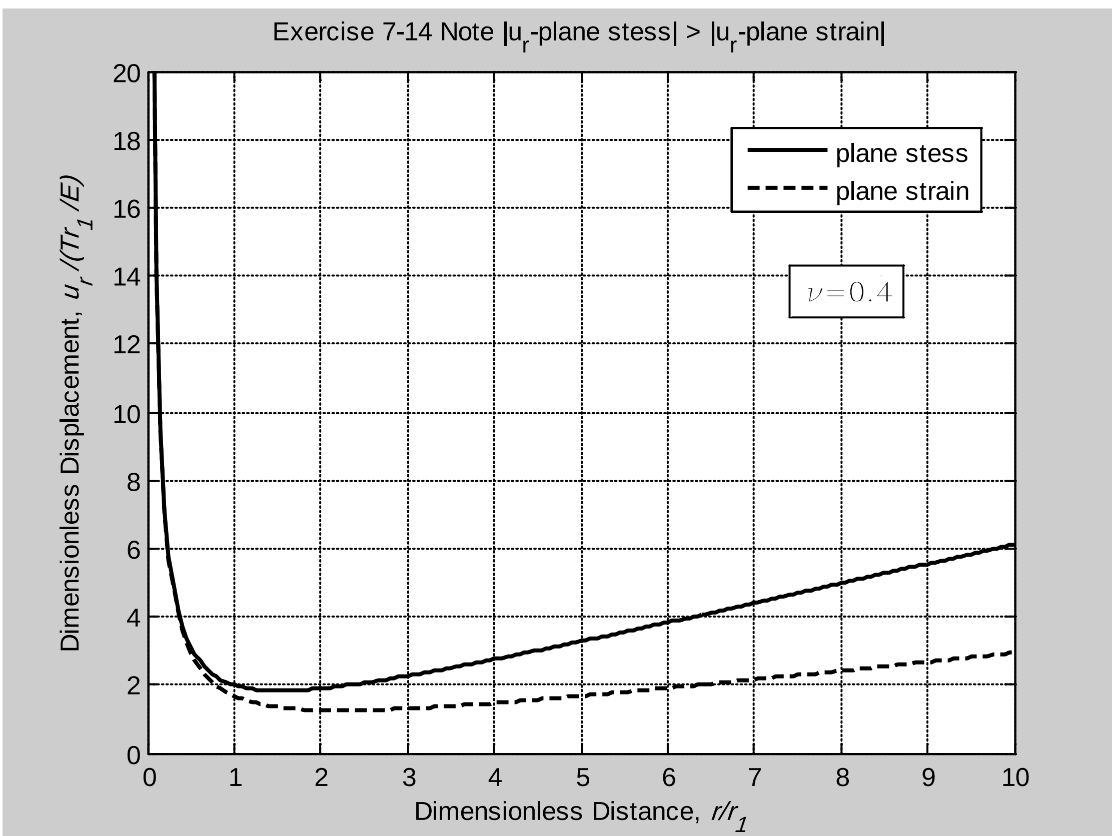
$$u_r = \frac{T(1+\frac{\nu}{1+\nu})}{\frac{E(1+2\nu)}{(1+\nu)^2}}\left[\left(1-2\frac{\nu}{1+\nu}\right)r + \frac{r_1^2}{r}\right] = T\frac{1+\nu}{E}\left[\frac{1-\nu}{1+\nu}r + \frac{r_1^2}{r}\right]$$

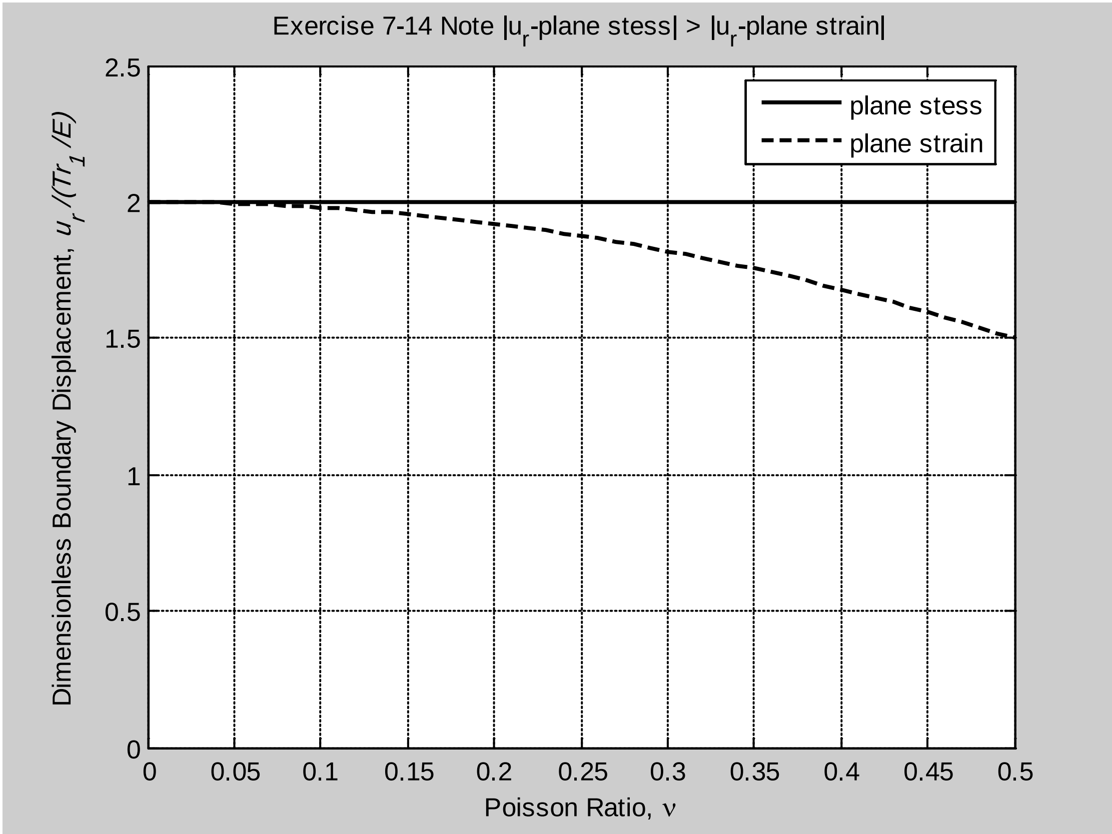
Non - dimensionalizing :

$$\frac{(u_r)_{P.Strain}}{Tr_1/E} = (1+\nu)\left[(1-2\nu)\frac{r}{r_1} + \frac{1}{r/r_1}\right], \ \frac{(u_r)_{P.Stress}}{Tr_1/E} = (1+\nu)\left[\frac{1-\nu}{1+\nu}\frac{r}{r_1} + \frac{1}{r/r_1}\right]$$

When Poisson's ratio $\rightarrow 0$, the two displacements become identical.

Using MATLAB, plots of the plane stress and plane strain displacements are given by





7-15.

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + V \ , \ \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V \ , \ \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} \ , \ F_x = -\frac{\partial V}{\partial x} \ , \ F_y = -\frac{\partial V}{\partial y}$$

$$\text{Plane Strain}: \ \nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \Rightarrow$$

$$\nabla^2\left(\frac{\partial^2\phi}{\partial y^2} + V + \frac{\partial^2\phi}{\partial x^2} + V\right) = \frac{1}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \Rightarrow \nabla^4\phi = -\frac{1-2\nu}{1-\nu}\nabla^2 V$$

$$\text{Plane Stress}: \ \nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \Rightarrow$$

$$\nabla^2\left(\frac{\partial^2\phi}{\partial y^2} + V + \frac{\partial^2\phi}{\partial x^2} + V\right) = (1+\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \Rightarrow \nabla^4\phi = -(1-\nu)\nabla^2 V$$

7-16.

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(u_r \cos\theta - u_\theta \sin\theta) = \frac{\partial u_r}{\partial x}\cos\theta + u_r \frac{\partial}{\partial x}\cos\theta - \frac{\partial u_\theta}{\partial x}\sin\theta - u_\theta \frac{\partial}{\partial x}\sin\theta$$

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(u_r \sin\theta + u_\theta \cos\theta) = \frac{\partial u_r}{\partial y}\sin\theta + u_r \frac{\partial}{\partial y}\sin\theta + \frac{\partial u_\theta}{\partial y}\cos\theta + u_\theta \frac{\partial}{\partial y}\cos\theta$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial}{\partial y}(u_r \cos\theta - u_\theta \sin\theta) + \frac{\partial}{\partial x}(u_r \sin\theta + u_\theta \cos\theta)\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_r}{\partial y}\cos\theta + u_r \frac{\partial}{\partial y}\cos\theta - \frac{\partial u_\theta}{\partial y}\sin\theta - u_\theta \frac{\partial}{\partial y}\sin\theta + \frac{\partial u_r}{\partial x}\sin\theta + u_r \frac{\partial}{\partial x}\sin\theta + \frac{\partial u_\theta}{\partial x}\cos\theta + u_\theta \frac{\partial}{\partial x}\cos\theta\right)$$

$$\frac{\partial}{\partial x}\cos\theta = \frac{\sin^2\theta}{r}, \quad \frac{\partial}{\partial y}\sin\theta = \frac{\cos^2\theta}{r}, \quad \frac{\partial}{\partial x}\sin\theta = \frac{\partial}{\partial y}\cos\theta = -\frac{\sin\theta\cos\theta}{r}$$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} - \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$e_r = e_x \cos^2\theta + e_y \sin^2\theta + 2e_{xy}\sin\theta\cos\theta$$

$$= \left(\frac{\partial u_r}{\partial x}\cos\theta + u_r \frac{\partial}{\partial x}\cos\theta - \frac{\partial u_\theta}{\partial x}\sin\theta - u_\theta \frac{\partial}{\partial x}\sin\theta\right)\cos^2\theta$$

$$+ \left(\frac{\partial u_r}{\partial y}\sin\theta + u_r \frac{\partial}{\partial y}\sin\theta + \frac{\partial u_\theta}{\partial y}\cos\theta + u_\theta \frac{\partial}{\partial y}\cos\theta\right)\sin^2\theta$$

$$+ \begin{pmatrix} \dfrac{\partial u_r}{\partial y}\cos\theta + u_r \dfrac{\partial}{\partial y}\cos\theta - \dfrac{\partial u_\theta}{\partial y}\sin\theta - u_\theta \dfrac{\partial}{\partial y}\sin\theta \\ + \dfrac{\partial u_r}{\partial x}\sin\theta + u_r \dfrac{\partial}{\partial x}\sin\theta + \dfrac{\partial u_\theta}{\partial x}\cos\theta + u_\theta \dfrac{\partial}{\partial x}\cos\theta \end{pmatrix}\sin\theta\cos\theta$$

$$= \frac{\partial u_r}{\partial r}$$

Likewise,

$$e_\theta = e_x \sin^2\theta + e_y \cos^2\theta - 2e_{xy}\sin\theta\cos\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$e_{r\theta} = -e_x \sin\theta\cos\theta + e_y \sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta) = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

7-17.

Strain - Displacement Relations :

$$e_r = \frac{\partial u_r}{\partial r}\ ,\ e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\ ,\ e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$2r\frac{\partial e_{r\theta}}{\partial \theta} = \frac{\partial^2 u_r}{\partial \theta^2} + r\frac{\partial^2 u_\theta}{\partial r \partial \theta} - \frac{\partial u_\theta}{\partial \theta}$$

$$r^2\frac{\partial e_\theta}{\partial r} = r^2\left(-\frac{1}{r^2}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + \frac{1}{r}\left(\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_\theta}{\partial r \partial \theta}\right)\right)$$

$$= -u_r - \frac{\partial u_\theta}{\partial \theta} + r\left(\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_\theta}{\partial r \partial \theta}\right)$$

$$\frac{\partial}{\partial r}\left(2r\frac{\partial e_{r\theta}}{\partial \theta} - r^2\frac{\partial e_\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial^2 u_r}{\partial \theta^2} + u_r - r\frac{\partial u_r}{\partial r}\right) = \frac{\partial^3 u_r}{\partial r \partial \theta^2} - r\frac{\partial^2 u_r}{\partial r^2} = \frac{\partial^2 e_r}{\partial \theta^2} - r\frac{\partial e_r}{\partial r} \Rightarrow$$

$$\frac{\partial}{\partial r}\left(2r\frac{\partial e_{r\theta}}{\partial \theta} - r^2\frac{\partial e_\theta}{\partial r}\right) + r\frac{\partial e_r}{\partial r} - \frac{\partial^2 e_r}{\partial \theta^2} = 0$$

7-18.

$$\sigma_r = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right) + 2\mu\frac{\partial u_r}{\partial r}$$

$$\sigma_\theta = \lambda\left(\frac{\partial u_r}{\partial r}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right) + 2\mu\frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

Equilibrium equations :

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{(\sigma_r - \sigma_\theta)}{r} + F_r = 0 \Rightarrow$$

$$\lambda\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right) + 2\mu\frac{\partial^2 u_r}{\partial r^2} + \frac{\mu}{r}\frac{\partial}{\partial \theta}\left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right] + \frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r} - \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right) + F_r = 0$$

$$\mu\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right) + (\lambda + \mu)\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0 \Rightarrow$$

$$\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{\lambda}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right) + \frac{2\mu}{r^2}\frac{\partial}{\partial \theta}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + \frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + F_\theta = 0$$

$$\mu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + (\lambda + \mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + F_\theta = 0$$

Beltrami - Michell equation : (Note $\sigma_x + \sigma_y = \sigma_r + \sigma_\theta$)

$$\nabla^2(\sigma_r + \sigma_\theta) = -\frac{1}{1-\nu}(\nabla \cdot \boldsymbol{F}) = -\frac{1}{1-\nu}\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right)$$

7-19.

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\partial u_r}{\partial r} + \frac{\nu}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + \nu\frac{\partial u_r}{\partial r}\right)$$

$$\tau_{r\theta} = \frac{E}{2(1+\nu)}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

Equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{(\sigma_r - \sigma_\theta)}{r} + F_r = 0 \Rightarrow$$

$$\frac{E}{1-\nu^2}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{\nu}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\right) + \frac{1}{r}\frac{E}{2(1+\nu)}\frac{\partial}{\partial \theta}\left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right]$$

$$+\frac{E}{1-\nu^2}\frac{1}{r}\left(\frac{\partial u_r}{\partial r} + \frac{\nu}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) - \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) - \nu\frac{\partial u_r}{\partial r}\right) + F_r = 0 \Rightarrow$$

$$\mu\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right) + \frac{E}{2(1-\nu)}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0 \Rightarrow$$

$$\frac{E}{2(1+\nu)}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{E}{1-\nu^2}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + \nu\frac{\partial u_r}{\partial r}\right) + \frac{E}{1+\nu}\frac{1}{r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + F_\theta = 0$$

$$\mu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + \frac{E}{2(1-\nu)}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + F_\theta = 0$$

Beltrami - Michell equation: (Note $\sigma_x + \sigma_y = \sigma_r + \sigma_\theta$)

$$\nabla^2(\sigma_r + \sigma_\theta) = -(1+\nu)(\nabla \cdot \boldsymbol{F}) = -(1+\nu)\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right)$$

7-20.

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2}=\cos^2\theta\frac{\partial^2}{\partial r^2}+\sin^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$

$$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}-\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial y^2}=\sin^2\theta\frac{\partial^2}{\partial r^2}+\cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)-2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$

$$\frac{\partial^2}{\partial x\partial y}=\sin\theta\cos\theta\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)-(\cos^2\theta-\sin^2\theta)\left(\frac{1}{r^2}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2}=\sin^2\theta\frac{\partial^2\phi}{\partial r^2}+\cos^2\theta\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)-2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)$$

$$\sigma_y=\frac{\partial^2\phi}{\partial x^2}=\cos^2\theta\frac{\partial^2\phi}{\partial r^2}+\sin^2\theta\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)+2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)$$

$$\tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}=-\sin\theta\cos\theta\left(\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)+(\cos^2\theta-\sin^2\theta)\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)$$

Use stress transformation rules:

$$\sigma'_r=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+2\tau_{xy}\sin\theta\cos\theta\Rightarrow$$

$$=\left[\sin^2\theta\frac{\partial^2\phi}{\partial r^2}+\cos^2\theta\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)-2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)\right]\cos^2\theta$$

$$+\left[\cos^2\theta\frac{\partial^2\phi}{\partial r^2}+\sin^2\theta\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)+2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)\right]\sin^2\theta$$

$$+2\left[-\sin\theta\cos\theta\left(\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)+(\cos^2\theta-\sin^2\theta)\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)\right]\sin\theta\cos\theta$$

$$=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma'_\theta=\sigma_x\sin^2\theta+\sigma_y\cos^2\theta-2\tau_{xy}\sin\theta\cos\theta=\frac{\partial^2\phi}{\partial r^2}$$

$$\tau'_{r\theta}=-\sigma_x\sin\theta\cos\theta+\sigma_y\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta)=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

Equilibrium check follows through direct substitution.

7-21.

$$e_r = \frac{\partial u_r}{\partial r} = 0 \Rightarrow u_r = f(\theta)$$

$$e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) = 0 \Rightarrow \frac{\partial u_\theta}{\partial \theta} = -u_r = -f(\theta) \Rightarrow u_\theta = -\int f(\theta)d\theta + g(r)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = 0 \Rightarrow \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0 \Rightarrow$$

$$\frac{1}{r}f'(\theta) + g'(r) + \frac{1}{r}\int f(\theta)d\theta - \frac{1}{r}g(r) = 0 \Rightarrow f'(\theta) + \int f(\theta)d\theta = g(r) - rg'(r) = \text{constant} = K$$

$g(r) - rg'(r) = K \Rightarrow g(r) = Cr + K$, but the constant K must be zero to satisfy $e_{r\theta} = 0$

$$f'(\theta) + \int f(\theta)d\theta = 0 \Rightarrow f''(\theta) + f(\theta) = 0 \Rightarrow f(\theta) = A\sin\theta + B\cos\theta$$

$\therefore u_r = A\sin\theta + B\cos\theta$, $u_\theta = A\cos\theta - B\sin\theta + Cr$

Check Cartesian Form :

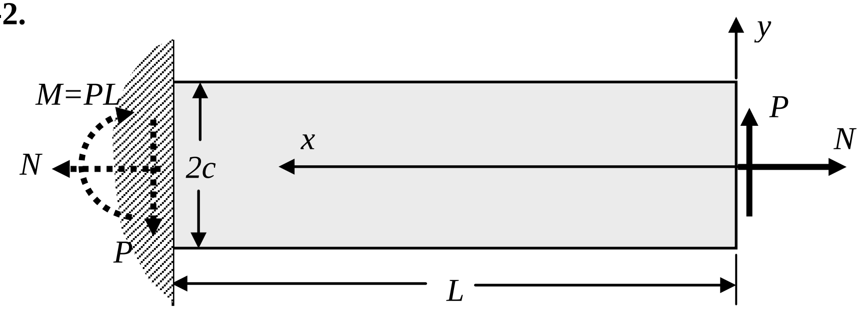
$$u = u_r\cos\theta - u_\theta\sin\theta$$
$$= (A\sin\theta + B\cos\theta)\cos\theta - (A\cos\theta - B\sin\theta + Cr)\sin\theta = B - Cr\sin\theta = B - Cy$$
$$v = u_r\sin\theta + u_\theta\cos\theta$$
$$= (A\sin\theta + B\cos\theta)\sin\theta + (A\cos\theta - B\sin\theta + Cr)\cos\theta = A + Cx$$

which is the same as (2.2.9)

8-1.

$$\phi = A_{40}x^4 + A_{22}x^2y^2 + A_{04}y^4$$
$$\nabla^4\phi = 24A_{40} + 8A_{22} + 24A_{04} = 0 \Rightarrow$$
$$3A_{40} + A_{22} + 3A_{04} = 0$$

8-2.



$$\phi=\frac{3P}{4c}\left(xy-\frac{xy^3}{3c^2}\right)+\frac{N}{4c}y^2$$

By inspeciton $\nabla^4\phi=0$

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2}=-\frac{3Pxy}{2c^3}+\frac{N}{2c}\ ,\ \sigma_y=\frac{\partial^2\phi}{\partial x^2}=0\ ,\ \tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}=-\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right)$$

Boundary Conditions:

$\sigma_y(x,\pm c)=0$, satisfied identically

$\tau_{xy}(x,\pm c)=0$, satisfied by inspection

$\int_{-c}^{c}\sigma_x(0,y)dy=N\Rightarrow\int_{-c}^{c}\frac{N}{2c}dy=N$, satisfied

$\int_{-c}^{c}\tau_{xy}(0,y)dy=-P\Rightarrow-\frac{3P}{4c}\int_{-c}^{c}\left(1-\frac{y^2}{c^2}\right)dy=-P$, satisfied

$\int_{-c}^{c}\sigma_x(0,y)ydy=0\Rightarrow\int_{-c}^{c}\frac{N}{2c}ydy=0$, satisfied

$\int_{-c}^{c}\sigma_x(L,y)dy=N\Rightarrow\int_{-c}^{c}\left(-\frac{3PLy}{2c^3}+\frac{N}{2c}\right)dy=N$, satisfied

$\int_{-c}^{c}\tau_{xy}(L,y)dy=-P\Rightarrow-\frac{3P}{4c}\int_{-c}^{c}\left(1-\frac{y^2}{c^2}\right)dy=-P$, satisfied

$\int_{-c}^{c}\sigma_x(L,y)ydy=-PL\Rightarrow\int_{-c}^{c}\left(-\frac{3PLy^2}{2c^3}+\frac{Ny}{2c}\right)dy=-PL$, satisfied

$N=0$ Case:

$$\sigma_x=-\frac{3Pxy}{2c^3}\ ,\ \sigma_y=0\ ,\ \tau_{xy}=-\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right)$$

Strength of Materials Solution: $M(x)=Px,\ V(x)=-P$

$$\sigma_x=-\frac{My}{I}=-\frac{3Pxy}{2c^3}\ ,\ \sigma_y=0\ ,\ \tau_{xy}=\frac{VQ}{It}=-\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right)$$

$\therefore$ Elasticity and Strength of Materials give identical results.

8-3.

$$\sigma_x = -\frac{3Pxy}{2c^3} + \frac{N}{2c}\ ,\ \sigma_y = 0\ ,\ \tau_{xy} = -\frac{3P}{4c}(1-\frac{y^2}{c^2})$$

$$\frac{\partial u}{\partial x} = e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}(-\frac{3Pxy}{2c^3} + \frac{N}{2c}) \Rightarrow u = \frac{1}{E}(-\frac{3Px^2y}{4c^3} + \frac{N}{2c}x) + f(y)$$

$$\frac{\partial v}{\partial y} = e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -\frac{\nu}{E}(-\frac{3Pxy}{2c^3} + \frac{N}{2c}) \Rightarrow v = -\frac{\nu}{E}(-\frac{3Pxy^2}{4c^3} + \frac{N}{2c}y) + g(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2e_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{3(1+\nu)P}{2Ec}(1-\frac{y^2}{c^2})$$

$$\Rightarrow -\frac{3Px^2}{4Ec^3} + f'(y) + \frac{3P\nu y^2}{4Ec^3} + g'(x) = -\frac{3(1+\nu)P}{2Ec}(1-\frac{y^2}{c^2})$$

Rearranging and separating the variables $\Rightarrow$

$$-\frac{3Px^2}{4Ec^3} + g'(x) = -\frac{3(1+\nu)P}{2Ec}(1-\frac{y^2}{c^2}) - f'(y) - \frac{3P\nu y^2}{4Ec^3} = \text{constant} = \omega_o$$

$$\therefore\ g(x) = \frac{Px^3}{4Ec^3} + \omega_o x + v_o\ ,\ f(y) = -\frac{P\nu y^3}{4Ec^3} - \frac{3(1+\nu)P}{2Ec}(y - \frac{y^3}{3c^2}) - \omega_o y + u_o$$

$$u = -\frac{3Px^2y}{4Ec^3} + \frac{N}{2Ec}x - \frac{P\nu y^3}{4Ec^3} - \frac{3(1+\nu)P}{2cE}(y - \frac{y^3}{3c^2}) - \omega_o y + u_o$$

$$v = \frac{3P\nu xy^2}{4Ec^3} - \frac{N\nu}{2Ec}y + \frac{Px^3}{4Ec^3} + \omega_o x + v_o$$

In order to complete the solution, we must choose additional boundary conditions to properly constrain the cantilever beam; thus choose

$$\frac{\partial v(L,0)}{\partial x} = 0 \Rightarrow \frac{3PL^2}{4Ec^3} + \omega_o = 0 \Rightarrow \omega_o = -\frac{3PL^2}{4Ec^3}$$

$$u(L,0) = 0 \Rightarrow u_o = \frac{N}{2Ec}L$$

$$v(L,0) = 0 \Rightarrow v_o = -\frac{PL^3}{4Ec^3} - \omega_o L = \frac{PL^3}{2Ec^3}$$

Note that we cannot ensure pointwise conditions such as

$u(L,y) = 0$ and $v(L,y) = 0$ with our approximate St. Venant type solution

For the case $N = 0$

$$v(x,0) = \frac{Px^3}{4Ec^3} - \frac{3PL^2}{4Ec^3}x + \frac{PL^3}{2Ec^3} = \frac{P}{4Ec^3}(x^3 - 3L^2x + 2L^3)$$

From Strength of Materials $v(x) = \frac{P}{6EI}(x^3 - 3L^2x + 2L^3) = \frac{P}{4Ec^3}(x^3 - 3L^2x + 2L^3)$

Therefore the two displacement solutions are the same!

8-4.

$$\phi = C_1x^2 + C_2x^2y + C_3y^3 + C_4y^5 + C_5x^2y^3$$

$$\nabla^4\phi = 0 \Rightarrow 120C_4y + 24C_5y = 0 \Rightarrow C_5 = -5C_4$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 6C_3y + C_4(20y^3 - 30x^2y),\ \sigma_y = \frac{\partial^2\phi}{\partial x^2} = 2C_1 + 2C_2y - 10C_4y^3$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -2C_2x + 30C_4xy^2$$

Boundary Conditions :

$$\tau_{xy}(x,c) = 0 \Rightarrow C_2 = 15C_4c^2$$

$$\sigma_y(x,c) = -q \Rightarrow C_1 + C_2c - 5C_4c^3 = -q/2$$

$$\tau_{xy}(x,-c) = 0 \Rightarrow C_2 = 15C_4c^2 \text{ (same)}$$

$$\sigma_y(x,-c) = 0 \Rightarrow C_1 - C_2c + 5C_4c^3 = 0$$

Conditions at free end $x = 0$:

$$\int_{-c}^{c}\sigma_x(0,y)dy = 0 \Rightarrow \int_{-c}^{c}(6C_3y + 20C_4y^3)dy = 0 \text{ , satisfied}$$

$$\int_{-c}^{c}\tau_{xy}(0,y)dy = 0 \Rightarrow 0 = 0 \text{ ,satisfied}$$

$$\int_{-c}^{c}\sigma_x(0,y)ydy = 0 \Rightarrow \int_{-c}^{c}(6C_3y^2 + 20C_4y^4)dy = 4C_3c^3 + 8C_4c^5 = 0$$

Solving for the four constants $\Rightarrow$ $C_1 = -q/4,\ C_2 = -3q/8c, C_3 = q/20c, C_4 = -q/40c^3$

$$\therefore\ \sigma_x = \frac{3}{10}qy - \frac{q}{4c^3}(y^3 - 3x^2y),\ \sigma_y = -\frac{q}{2} - \frac{qy}{4c}\left(3 - \frac{y^2}{c^2}\right),\ \tau_{xy} = \frac{3qx}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

Check remaining conditions at fixed end $x = L$:

$$\int_{-c}^{c}\sigma_x(L,y)dy = 0 \Rightarrow \int_{-c}^{c}\left(\frac{3}{10}qy - \frac{q}{4c^3}(y^3 - 3L^2y)\right)dy = 0\text{, satisfied}$$

$$\int_{-c}^{c}\tau_{xy}(L,y)dy = qL \Rightarrow \frac{3qL}{4c}\int_{-c}^{c}\left(1 - \frac{y^2}{c^2}\right)dy = qL\text{, satisfied}$$

$$\int_{-c}^{c}\sigma_x(L,y)ydy = qL^2/2 \Rightarrow \int_{-c}^{c}\left(\frac{3}{10}qy - \frac{q}{4c^3}(y^3 - 3L^2y)\right)ydy = qL^2/2\text{, satisfied}$$

8-5.

$$\phi = \frac{s}{4}\left(xy + \frac{ly^2}{c} + \frac{ly^3}{c^2} - \frac{xy^2}{c} - \frac{xy^3}{c^2}\right)$$

$$\phi_{,x} = \frac{s}{4}\left(y - \frac{y^2}{c} - \frac{y^3}{c^2}\right) \Rightarrow \phi_{,xx} = \phi_{,xxx} = \phi_{,xxxx} = 0$$

$$\phi_{,xy} = \frac{s}{4}\left(1 - \frac{2y}{c} - \frac{3y^2}{c^2}\right) \Rightarrow \phi_{,xxyy} = 0$$

$$\phi_{,yy} = \frac{s}{4}\left(\frac{2l}{c} + \frac{6ly}{c^2} - \frac{2x}{c} - \frac{6xy}{c^2}\right) \Rightarrow \phi_{,yyyy} = 0$$

$$\therefore \nabla^4\phi = 0$$

$$\sigma_x = \phi_{,yy} = \frac{s}{4}\left(\frac{2l}{c} + \frac{6ly}{c^2} - \frac{2x}{c} - \frac{6xy}{c^2}\right) = \frac{s}{2c}\left(1 + \frac{3y}{c}\right)(l - x)$$

$$\sigma_y = \phi_{,xx} = 0 \text{ , } \tau_{xy} = -\phi_{,xy} = -\frac{s}{4}\left(1 - \frac{2y}{c} - \frac{3y^2}{c^2}\right) = -\frac{s}{4}\left(1 + \frac{y}{c}\right)\left(1 - \frac{3y}{c}\right)$$

Boundary Conditions:

$\sigma_y(x, \pm c) = 0$, satisfied

$\tau_{xy}(x, c) = s$, satisfied

$\tau_{xy}(x, -c) = 0$, satisfied

Conditions at fixed end $x = 0$:

$$\int_{-c}^{c} \sigma_x(0, y)dy = \frac{sl}{2c}\int_{-c}^{c}\left(1 + \frac{3y}{c}\right)dy = sl \text{ , satisfied}$$

$$\int_{-c}^{c} \tau_{xy}(0, y)dy = -\frac{s}{4}\int_{-c}^{c}\left(1 - \frac{2y}{c} - \frac{3y^2}{c^2}\right)dy = 0 \text{ , satisfied}$$

$$\int_{-c}^{c} \sigma_x(0, y)ydy = \frac{sl}{2c}\int_{-c}^{c}\left(1 + \frac{3y}{c}\right)ydy = slc \text{ , satisfied}$$

Conditions at free end $x = l$: Note that $\sigma_x(l, y) = 0$

$$\therefore \int_{-c}^{c} \sigma_x(l, y)dy = 0 \ \ \& \ \int_{-c}^{c} \sigma_x(l, y)ydy = 0 \text{ , satisfied}$$

$$\int_{-c}^{c} \tau_{xy}(l, y)dy = -\frac{s}{4}\int_{-c}^{c}\left(1 - \frac{2y}{c} - \frac{3y^2}{c^2}\right)dy = 0 \text{ , satisfied}$$

8-6.

$$\phi = C_1 xy + C_2 \frac{x^3}{6} + C_3 \frac{x^3 y}{6} + C_4 \frac{xy^3}{6} + C_5 \frac{x^3 y^3}{9} + C_6 \frac{xy^5}{20}$$

$$\nabla^4 \phi = 0 \Rightarrow 8C_5 xy + 6C_6 xy = 0 \Rightarrow C_6 = -\frac{4}{3} C_5$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = C_4 xy + \frac{2}{3} C_5 x^3 y + C_6 xy^3 \text{ , } \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = C_2 x + C_3 xy + \frac{2}{3} C_5 xy^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\left(C_1 + \frac{1}{2} C_3 x^2 + \frac{1}{2} C_4 y^2 + C_5 x^2 y^2 + \frac{1}{4} C_6 y^4 \right)$$

Boundary Conditions:

$$\sigma_y(x, c) = -px/L \Rightarrow C_2 + C_3 c + \frac{2}{3} C_5 c^3 = -p/L$$

$$\sigma_y(x, -c) = 0 \Rightarrow C_2 - C_3 c - \frac{2}{3} C_5 c^3 = 0$$

$$\tau_{xy}(x, \pm c) = 0 \Rightarrow C_1 + \frac{1}{2} C_4 c^2 + \frac{1}{4} C_6 c^4 = 0 \text{ and } \frac{1}{2} C_3 + C_5 c^2 = 0$$

$\int_{-c}^{c} \sigma_x(0, y) dy = 0$, indentically satisfied

$$\int_{-c}^{c} \tau_{xy}(0, y) dy = 0 \Rightarrow C_1 + \frac{1}{6} C_4 c^2 + \frac{1}{20} C_6 c^4 = 0$$

$\int_{-c}^{c} \sigma_x(0, y) y dy = 0$, indentically satisfied

$\int_{-c}^{c} \sigma_x(L, y) dy = 0$, indentically satisfied

$\int_{-c}^{c} \tau_{xy}(L, y) dy = \frac{1}{2} pL$, satisfied using previous results

$\int_{-c}^{c} \sigma_x(L, y) y dy = \frac{1}{6} pL^2$, satisfied using previous results

Using the six conditions from differential equation and boundary conditons on top, bottom and free end determines the six constants

$$C_1 = -\frac{pc}{40L}, C_2 = -\frac{p}{2L}, C_3 = -\frac{3p}{4Lc}, C_4 = \frac{3p}{10Lc}, C_5 = \frac{3p}{8Lc^3}, C_6 = -\frac{p}{2Lc^3}$$

$$\sigma_x = \frac{pxy}{20Lc^3}(5x^2 - 10y^2 + 6c^2) \text{ , } \sigma_y = -\frac{px}{4Lc^3}(2c^3 + 3c^2 y - y^3)$$

$$\tau_{xy} = \frac{p}{8Lc^3} \left(\frac{8}{40} c^4 + 3c^2 x^2 - \frac{6}{5} c^2 y^2 - 3x^2 y^2 + y^4 \right)$$

8-7.

$$\phi = c_1 y^2 + c_2 y^3 + c_3 y^4 + c_4 y^5 + c_5 x^2 + c_6 x^2 y + c_7 x^2 y^2 + c_8 x^2 y^3$$

$$\nabla^4 \phi = 0 \Rightarrow 24c_3 + 120c_4 y + 8c_7 + 24c_8 y = 0 \Rightarrow 3c_3 + c_7 = 0 \text{ and } 5c_4 + c_8 = 0$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2c_1 + 6c_2 y + 12c_3 y^2 + 20c_4 y^3 + 2c_7 x^2 + 6c_8 x^2 y$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2c_5 + 2c_6 y + 2c_7 y^2 + 2c_8 y^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -2c_6 x - 4c_7 xy - 6c_8 xy^2$$

Boundary Conditions:

$$\sigma_y(x, \pm c) = 0 \Rightarrow 2c_5 \pm 2c_6 c + 2c_7 c^2 \pm 2c_8 c^3 = 0$$

$$\tau_{xy}(x, -c) = 0 \Rightarrow 2c_6 - 4c_7 c + 6c_8 c^2 = 0$$

$$\tau_{xy}(x, c) = -\tau_o x / l \Rightarrow 2c_6 + 4c_7 c + 6c_8 c^2 = \tau_o / l$$

$$\int_{-c}^{c} \sigma_x(0, y) dy = 0 \Rightarrow c_1 + 2c_3 c^2 = 0$$

$$\int_{-c}^{c} \tau_{xy}(0, y) dy = 0 \text{, indentically satisfied}$$

$$\int_{-c}^{c} \sigma_x(0, y) y dy = 0 \Rightarrow c_2 + 2c_4 c^2 = 0$$

$$\int_{-c}^{c} \sigma_x(l, y) dy = \tau_o l / 2 \text{ , } \int_{-c}^{c} \tau_{xy}(l, y) dy = 0 \text{ , } \int_{-c}^{c} \sigma_x(l, y) y dy = \tau_o lc / 2$$

Using the eight conditions from differential equation and boundary conditons on top, bottom and free end determines the eight constants

$$c_1 = \frac{\tau_o c}{12l},\ c_2 = \frac{\tau_o}{20l},\ c_3 = -\frac{\tau_o}{24cl},\ c_4 = -\frac{\tau_o}{40c^2 l},\ c_5 = -\frac{\tau_o c}{8l},\ c_6 = -\frac{\tau_o}{8l},\ c_7 = \frac{\tau_o}{8cl},\ c_8 = \frac{\tau_o}{8c^2 l}$$

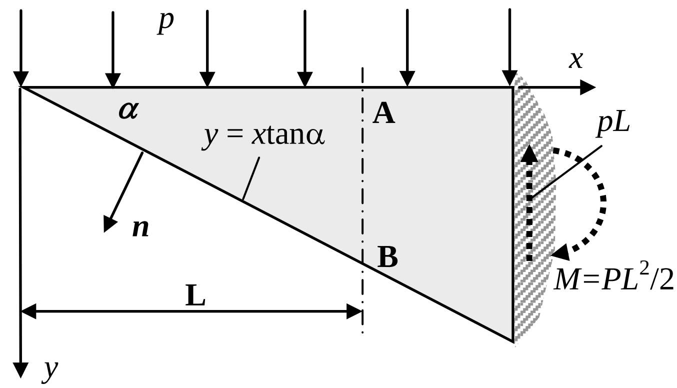
Using these values, the boundary conditions at $x = l$ are found to be satisfied

$$\sigma_x = \frac{\tau_o c}{6l} + \frac{3\tau_o}{10l} y - \frac{\tau_o}{2cl} y^2 - \frac{\tau_o}{2c^2 l} y^3 + \frac{\tau_o}{4cl} x^2 + \frac{3\tau_o}{4c^2 l} x^2 y$$

$$\sigma_y = -\frac{\tau_o c}{4l} - \frac{\tau_o}{4l} y + \frac{\tau_o}{4cl} y^2 + \frac{\tau_o}{4c^2 l} y^3$$

$$\tau_{xy} = \frac{\tau_o}{4l} x - \frac{\tau_o}{2cl} xy - \frac{3\tau_o}{4c^2 l} xy^2$$

8-8.



$$\phi = K\left[-x^2\tan\alpha + xy + (x^2+y^2)(\alpha - \tan^{-1}\frac{y}{x})\right],$$

$$K = \frac{p\cot\alpha}{2(1-\alpha\cot\alpha)}$$

$$\phi_{,xxxx} = K\left[\frac{-16xy^3}{(x^2+y^2)^3}\right],\ \phi_{,yyyy} = K\left[\frac{16x^3y}{(x^2+y^2)^3}\right],$$

$$\phi_{,xxyy} = K\left[\frac{8xy^3-8x^3y}{(x^2+y^2)^3}\right]\ \therefore \nabla^4\phi = 0$$

$$\sigma_x = \phi_{,yy} = 2K\left[\alpha - \tan^{-1}\frac{y}{x} - \frac{xy}{x^2+y^2}\right], \sigma_y = \phi_{,xx} = 2K\left[\alpha - \tan\alpha - \tan^{-1}\frac{y}{x} + \frac{xy}{x^2+y^2}\right]$$

$$\tau_{xy} = -\phi_{,xy} = -2K\frac{y^2}{x^2+y^2}$$

Boundary Conditions:

$\sigma_y(x,0) = -p$, $\tau_{xy}(x,0) = 0$

$T_x(x, x\tan\alpha) = \sigma_x n_x + \tau_{xy} n_y = 0$, $T_y(x, x\tan\alpha) = \tau_{xy} n_x + \sigma_y n_y = 0$, where $n_x = -\sin\alpha$, $n_y = \cos\alpha$

Choosing $x = L$ to be the location of the built-in end, the resultant forces and moments must satisfy

$$\int_0^{L\tan\alpha}\sigma_x(L,y)dy = 0,\ \int_0^{L\tan\alpha}\tau_{xy}(L,y)dy = -pL,\ \int_0^{L\tan\alpha}\sigma_x(L,y)ydy = -\frac{1}{2}pL^2$$

$\sigma_y(x,0) = 2K[\alpha - \tan\alpha] = -p$, $\tau_{xy}(x,0) = 0$ (satisfied)

$$\sigma_x n_x + \tau_{xy} n_y\Big|_{y=x\tan\alpha} = \frac{p\cos^2\alpha\sin\alpha}{1-\alpha\cot\alpha} - \frac{p\cos^2\alpha\sin\alpha}{1-\alpha\cot\alpha} = 0 \ \text{(satisfied)}$$

$$\tau_{xy} n_x + \sigma_y n_y\Big|_{y=x\tan\alpha} = \frac{p\cos\alpha\sin^2\alpha}{1-\alpha\cot\alpha} - \frac{p\cos\alpha\sin^2\alpha}{1-\alpha\cot\alpha} = 0 \ \text{(satisfied)}$$

$$\int_0^{L\tan\alpha}\sigma_x(L,y)dy = 2K\int_0^{L\tan\alpha}(\alpha - \tan^{-1}\frac{y}{L} - \frac{Ly}{L^2+y^2})dy$$

$$= 2K\left[\alpha y - [y\tan^{-1}\frac{y}{L} - \frac{L}{2}\log(L^2+y^2)] - \frac{L}{2}\log(L^2+y^2)\right]_0^{L\tan\alpha}$$

$$= 2K\left[\alpha y - y\tan^{-1}\frac{y}{L}\right]_0^{L\tan\alpha} = [\alpha L\tan\alpha - \alpha L\tan\alpha] = 0 \ \text{(satisfied)}$$

$$\int_0^{L\tan\alpha}\tau_{xy}(L,y)dy = -2K\int_0^{L\tan\alpha}\frac{y^2}{L^2+y^2}dy = -2KL(\tan\alpha - \alpha) = -pL \ \text{(satisfied)}$$

$$\int_0^{L\tan\alpha}\sigma_x(L,y)ydy = 2K\int_0^{L\tan\alpha}(\alpha - \tan^{-1}\frac{y}{L} - \frac{Ly}{L^2+y^2})ydy$$

$$= K\left[\alpha y^2 + (L^2 - y^2)\tan^{-1}\frac{y}{L} - Ly\right]_0^{L\tan\alpha} = -\frac{1}{2}pL^2 \ \text{(satisfied)}$$

8-8. Continued

Evaluation of stresses at $x = L = 1$ for $\alpha = 30^o = \pi/6 : \Rightarrow h = L\tan 30^o = 1/\sqrt{3}$

$$2K = \frac{p\cot\alpha}{(1-\alpha\cot\alpha)} = \frac{p}{(\tan\alpha - \alpha)} = \frac{p}{\dfrac{1}{\sqrt{3}} - \dfrac{\pi}{6}}$$

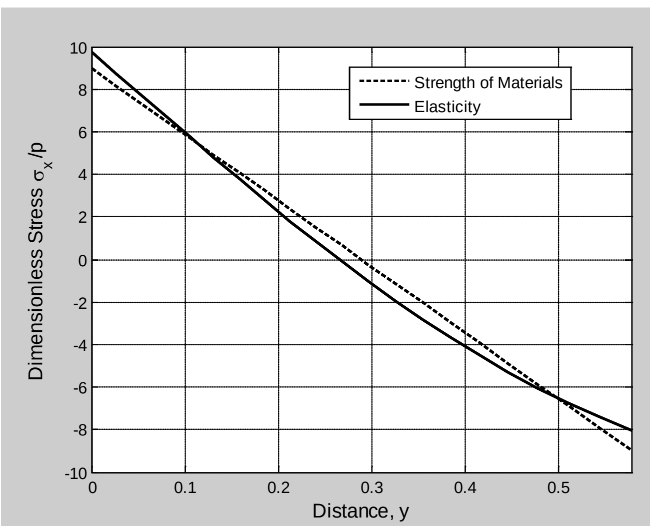
$$\sigma_x = 2K\left[\alpha - \tan^{-1} y - \frac{y}{1+y^2}\right],\ \tau_{xy} = -2K\frac{y^2}{1+y^2}$$

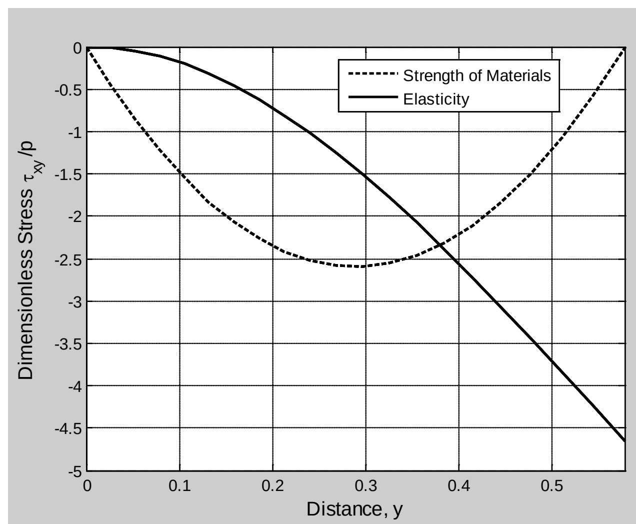
Strength of Materials stresses :

$$\sigma_x = \frac{M\overline{y}}{I} = \frac{\frac{1}{2}pL^2(\frac{h}{2} - y)}{h^3/12} = 9p\left(1 - 2\sqrt{3}\frac{y}{L}\right) \Rightarrow \frac{\sigma_x}{p} = 18\left(\frac{1}{2} - \frac{y}{h}\right)$$

$$\tau_{xy} = \frac{VQ}{It} = \frac{-pL(\frac{h^2}{4} - \overline{y}^2)/2}{h^3/12} = -\frac{18p}{L^2}y(L - y\sqrt{3}) \Rightarrow \frac{\tau_{xy}}{p} = -\frac{18}{\sqrt{3}}\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

MATLAB Plots :





8-9*.

Converting from Cartesian to polar form gives :

$$\phi = K[r^2(\alpha-\theta) + r^2\sin\theta\cos\theta - r^2\cos^2\theta\tan\alpha] = K\left[r^2(\alpha-\theta) + \frac{r^2}{2}\sin 2\theta - \frac{r^2}{2}(1+\cos 2\theta)\tan\alpha\right]$$

$$\phi_{,r} = 2K[r(\alpha-\theta) + r\sin\theta\cos\theta - r\cos^2\theta\tan\alpha]$$

$$\phi_{,rr} = 2K[(\alpha-\theta) + \sin\theta\cos\theta - \cos^2\theta\tan\alpha] = \sigma_\theta$$

$$\phi_{,\theta} = K[-r^2 + r^2\cos 2\theta + r^2\sin 2\theta\tan\alpha] \ \ , \ \ \phi_{,r\theta} = 2Kr[-1+\cos 2\theta + \sin 2\theta\tan\alpha] = -\tau_{r\theta}$$

$$\phi_{,\theta\theta} = 2K[-r^2\sin 2\theta + r^2\cos 2\theta\tan\alpha]$$

$$\nabla^2\phi = \phi_{,rr} + \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} = 2K[-\sin 2\theta + \cos 2\theta\tan\alpha] \Rightarrow \nabla^4\phi = 0$$

Boundary Conditions :

$$\sigma_\theta(r,0) = 2K[\alpha - \tan\alpha] = 2\frac{p\cot\alpha}{2(1-\alpha\cot\alpha)}[\alpha-\tan\alpha] = \frac{p(\alpha\cot\alpha - 1)}{(1-\alpha\cot\alpha)} = -p \text{ , satisfies}$$

$$\sigma_\theta(r,\alpha) = 2K[\sin\alpha\cos\alpha - \cos^2\alpha\tan\alpha] = 0 \text{ , satisfies}$$

$$\tau_{r\theta}(r,0) = -\phi_{,r\theta}(r,0) = 0 \text{ , satisfies}$$

$$\tau_{r\theta}(r,\alpha) = -\phi_{,r\theta}(r,\alpha) = 2Kr[-1+\cos 2\alpha + \sin 2\alpha\tan\alpha] = 2Kr[-1+1-2\sin^2\alpha + 2\sin^2\alpha] = 0 \text{ , satisfies}$$

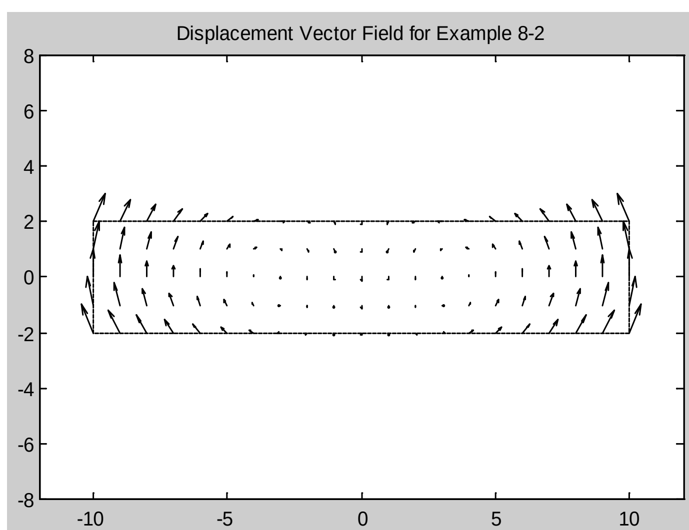
Other conditions and plots would be the same as Exercise 8 - 8.

8-10*.

Example 8 - 2, Displacement Field Solution : $u = -\dfrac{Mxy}{EI}$, $v = \dfrac{M}{2EI}[\nu y^2 + x^2 - l^2]$

MATLAB Solution Code and Plot :

```
% Elasticity 2e - Prof. M. Sadd
% Displacement Vector Plots
% Beam Problem Example 8-2
clc;clear all;clf
nu=0.3;L=10;c=2;
[x,y]=meshgrid(-L:L/10:L,-c:c/2:c);
ux=-x.*y;
uy=(nu*y.^2+x.^2-L)/2;
quiver(x,y,ux,uy,'color','k')
hold on
axis equal
axis([-1.2*L,1.2*L,-4*c,4*c])
rectangle('Position',[-L,-c,2*L,2*c],'linestyle','--')
% X=[-L,-L,L,L,-L];Y=[-c,c,c,-c,-c];
% line(X,Y,'color','k','linestyle','--')
title('Displacement Vector Field for Example 8-2')
```



8-11*.

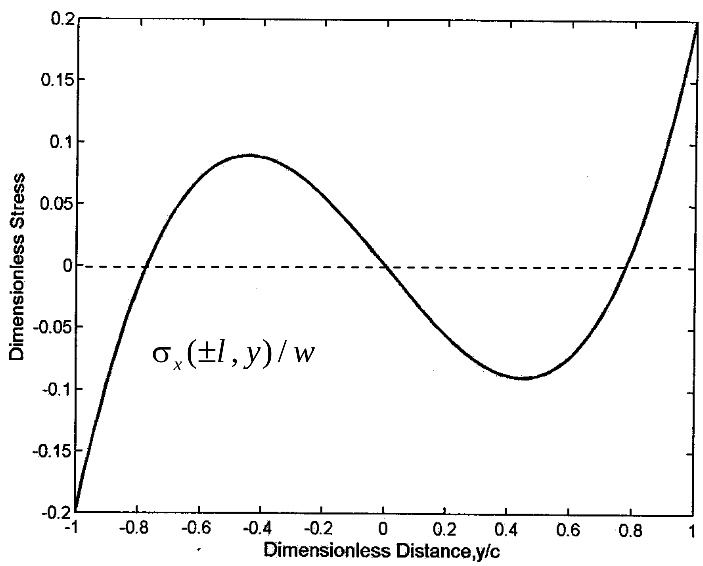
From Example 8 - 3, the normal stress was given by

$$\sigma_x = \frac{w}{2I}(l^2 - x^2)y + \frac{w}{I}\left(\frac{y^3}{3} - \frac{c^2 y}{5}\right)$$

At each end $x = \pm l$, $\sigma_x(\pm l, y) = \dfrac{w}{I}\left(\dfrac{y^3}{3} - \dfrac{c^2 y}{5}\right) = \dfrac{3w}{2}\left(\dfrac{1}{3}\dfrac{y^3}{c^3} - \dfrac{1}{5}\dfrac{y}{c}\right)$

This stress gives zero resultant force but clearly does not vanish itself.

A MATALB plot of this distribution is shown.



8-12*.

Results from Example 8 - 4 :

$$\sigma_x = \beta^2 \sin\beta x[(A\sinh\beta y + C(\beta y\sinh\beta y + 2\cosh\beta y) + B\cosh\beta y + D(\beta y\cosh\beta y + 2\sinh\beta y)]$$

$$C = \frac{-q_o \sinh\dfrac{\pi c}{l}}{2\dfrac{\pi^2}{l^2}\left[\dfrac{\pi c}{l} + \sinh\dfrac{\pi c}{l}\cosh\dfrac{\pi c}{l}\right]} \;,\; D = \frac{q_o \cosh\dfrac{\pi c}{l}}{2\dfrac{\pi^2}{l^2}\left[\dfrac{\pi c}{l} - \sinh\dfrac{\pi c}{l}\cosh\dfrac{\pi c}{l}\right]}$$

$$A = -D(\beta c\tanh\beta c + 1) \;,\; B = -C(\beta c\coth\beta c + 1) \;,\; \beta = \frac{\pi}{l}$$

$$\sigma_x = -\frac{q_o}{2}\sinh\frac{\pi c}{l}\sin\frac{\pi x}{l}\left[\frac{\pi y\cosh\dfrac{\pi y}{l} + 2l\sinh\dfrac{\pi y}{l} - \left(\pi c\tanh\dfrac{\pi c}{l} + l\right)\sinh\dfrac{\pi y}{l}}{\pi c + l\sinh\dfrac{\pi c}{l}\cosh\dfrac{\pi c}{l}}\right.$$

$$\left. + \frac{\pi y\sinh\dfrac{\pi y}{l} + 2l\cosh\dfrac{\pi y}{l} - \left(\pi c\coth\dfrac{\pi c}{l} + l\right)\cosh\dfrac{\pi y}{l}}{\pi c + l\sinh\dfrac{\pi c}{l}\cosh\dfrac{\pi c}{l}}\right]$$

For the case $l >> c$:

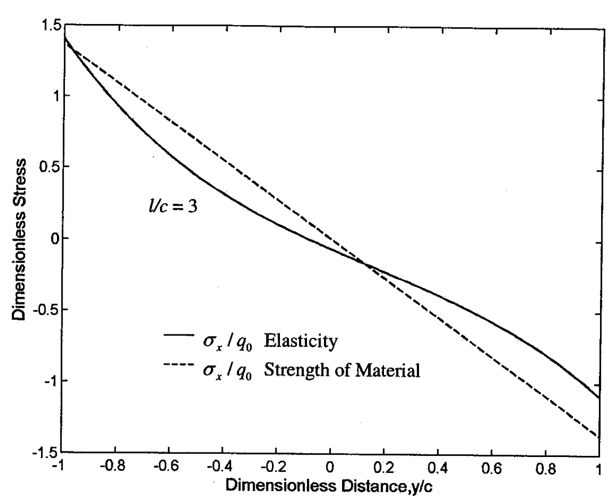
$$D \approx -\frac{3q_o l^5}{4c^3\pi^5} \;,\; C \approx 0 \;,\; A \approx -D \;,\; B \approx 0$$

$$\sigma_x \approx -\frac{3q_o l^3}{4c^3\pi^3}\left(\frac{\pi y}{l}\cosh\frac{\pi y}{l} + \sinh\frac{\pi y}{l}\right)\sin\frac{\pi x}{l} \approx -\frac{3q_o l^2}{2c^3\pi^2}\,y\sin\frac{\pi x}{l}$$

Strength of Materials Theory : $\sigma_x = -\dfrac{My}{I} = -\dfrac{\dfrac{q_o l^2}{\pi^2}\sin\dfrac{\pi x}{l}\;y}{2c^3/3} = -\dfrac{3q_o l^2}{2c^3\pi^2}\,y\sin\dfrac{\pi x}{l}$

and thus strength of materials and elasticity predictions are the same

For the case $l/c = 3$, the MATALB plot is shown



8-13.

For the axisymmetric case, the stresses were given by

$$\sigma_r = 2a_3 \log r + \frac{a_1}{r^2} + a_3 + 2a_2 \ , \ \sigma_\theta = 2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2 \ , \ \tau_{r\theta} = 0$$

Using strain - displacement and Hooke's law for plane stress

$$e_r = \frac{\partial u_r}{\partial r} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{1}{E}\left[2a_3 \log r + \frac{a_1}{r^2} + a_3 + 2a_2 - \nu\left(2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2\right)\right] \Rightarrow$$

$$u_r = \frac{1}{E}\left[2a_3(r \log r - r) - \frac{a_1}{r} + a_3 r + 2a_2 r - \nu\left(2a_3(r \log r - r) + \frac{a_1}{r} + 3a_3 r + 2a_2 r\right)\right] + f(\theta)$$

$$= \frac{1}{E}\left[-\frac{(1+\nu)}{r}a_1 + 2(1-\nu)a_3 r \log r - (1+\nu)a_3 r + 2a_2(1-\nu)r\right] + f(\theta)$$

$$e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$= \frac{1}{E}\left[2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2 - \nu\left(2a_3 \log r + \frac{a_1}{r^2} + a_3 + 2a_2\right)\right] \Rightarrow$$

$$\frac{\partial u_\theta}{\partial \theta} = \frac{r}{E}\left[2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2 - \nu\left(2a_3 \log r + \frac{a_1}{r^2} + a_3 + 2a_2\right)\right] - u_r = \frac{4r}{E}a_3 - f(\theta) \Rightarrow$$

$$u_\theta = \frac{4r\theta}{E}a_3 - \int f(\theta)d\theta + g(r)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \frac{1+\nu}{E}\tau_{r\theta} = 0 \Rightarrow f'(\theta) + rg'(r) + \int f(\theta)d\theta - g(r) = 0 \Rightarrow$$

$$f'(\theta) + \int f(\theta)d\theta = g(r) - rg'(r) = \text{constant} = K \Rightarrow$$

$$f(\theta) = A\sin\theta + B\cos\theta \text{ and } g(r) = Cr + D$$

Note that D must be dropped for consistent tangential displacement, and thus

$$u_r = \frac{1}{E}\left[-\frac{(1+\nu)}{r}a_1 + 2(1-\nu)a_3 r \log r - (1+\nu)a_3 r + 2a_2(1-\nu)r\right] + A\sin\theta + B\cos\theta$$

$$u_\theta = \frac{4r\theta}{E}a_3 + A\cos\theta - B\sin\theta + Cr$$

8-14.

$$\phi = a_4\theta \Rightarrow \sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0\ ,\ \sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = 0\ ,\ \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{a_4}{r^2}$$

Boundary Condition : $\tau_{r\theta}(r_2) = \tau \Rightarrow a_4 = \tau r_2^2 \quad \therefore\ \tau_{r\theta} = \tau\dfrac{r_2^2}{r^2}$

Displacements : $e_r = \dfrac{du_r}{dr} = 0 \Rightarrow u_r = \text{constant}$, but $u_r(r_1) = 0, \therefore u_r = 0$

$$e_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta} = \frac{1+\nu}{E}\tau\frac{r_2^2}{r^2} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \Rightarrow$$

$$\frac{du_\theta}{dr} - \frac{u_\theta}{r} = \frac{2(1+\nu)\tau r_2^2}{E}\frac{1}{r^2} \Rightarrow \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = \frac{2(1+\nu)\tau r_2^2}{E}\frac{1}{r^3}$$

Solution : $u_\theta = -\dfrac{(1+\nu)\tau r_2^2}{E}\dfrac{1}{r} + Cr$, but $u_\theta(r_1) = 0 \Rightarrow C = \dfrac{(1+\nu)\tau r_2^2}{E}\dfrac{1}{r_1^2}$

$$\therefore u_\theta = \frac{(1+\nu)\tau r_2^2}{E}\left(\frac{r}{r_1^2} - \frac{1}{r}\right)$$

8-15.

Axisymmetric Case : $\boldsymbol{u} = u_r\boldsymbol{e_r}$, and from Example 1 - 3,

$$\nabla = \boldsymbol{e_r}\frac{d}{dr}\ ,\ \nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{d}{dr}(ru_r)\ ,\ \nabla(\nabla\cdot\boldsymbol{u}) = \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_r)\right)\boldsymbol{e_r}$$

$$\nabla^2\boldsymbol{u} = \nabla^2(u_r\boldsymbol{e_r}) = \left(\nabla^2 u_r - \frac{u_r}{r^2}\right)\boldsymbol{e_r} = \left(\frac{1}{r}\frac{d}{dr}(r\frac{du_r}{dr}) - \frac{u_r}{r^2}\right)\boldsymbol{e_r}$$

Navier's Equations with no body force : $\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = 0 \Rightarrow$

$$\mu\left(\frac{1}{r}\frac{d}{dr}(r\frac{du_r}{dr}) - \frac{u_r}{r^2}\right)\boldsymbol{e_r} + (\lambda+\mu)\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_r)\right)\boldsymbol{e_r} = 0 \Rightarrow$$

$$(\lambda+2\mu)\left(\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right) = 0 \Rightarrow$$

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{C}{r^2} = 0 \quad \text{or} \quad \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_r)\right) = 0$$

Integrating the differential equation $\Rightarrow\ u_r = C_1 r + \dfrac{C_2}{r}$

Using the strain - displacement relations,

$$e_r = \frac{du_r}{dr} = C_1 - \frac{C_2}{r^2}\ ,\ e_\theta = \frac{u_r}{r} = C_1 + \frac{C_2}{r^2}$$

The stresses follow from Hooke's law (plane strain case)

$$\sigma_r = \lambda(e_r + e_\theta) + 2\mu e_r = \lambda(2C_1) + 2\mu\left(C_1 - \frac{C_2}{r^2}\right)$$

$$\sigma_\theta = \lambda(e_r + e_\theta) + 2\mu e_\theta = \lambda(2C_1) + 2\mu\left(C_1 + \frac{C_2}{r^2}\right)$$

$$\tau_{r\theta} = 0$$

Note these stresses do not contain the logarithmic terms given in the general solution (8.3.8).

8-16.

Using relations (7.6.1) and (7.6.2) and Exercise 7-1

$$e_r = \frac{du_r}{dr} = \frac{1+\nu}{E}[(1-\nu)\sigma_r - \nu\sigma_\theta] = \frac{1+\nu}{E}\left[(1-\nu)\left(\frac{A}{r^2}+B\right) - \nu\left(-\frac{A}{r^2}+B\right)\right]$$

$$= \frac{1+\nu}{E}\left[\frac{A}{r^2}+(1-2\nu)B\right] \Rightarrow u_r = \frac{1+\nu}{E}r[(1-2\nu)B - \frac{A}{r^2}]$$

Using the values determined for A and B then gives

$$u_r = \frac{1+\nu}{E}\left[-\frac{r_1^2 r_2^2(p_2-p_1)}{r_2^2-r_1^2}\frac{1}{r} + (1-2\nu)\frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2-r_1^2}r\right]$$

8-17.

General Axisymmetric Solution : $\sigma_r = \dfrac{A}{r^2}+B$, $\sigma_\theta = -\dfrac{A}{r^2}+B$

Plane Strain Hooke's Law :

$$e_r = \frac{1+\nu}{E}[(1-\nu)\sigma_r - \nu\sigma_\theta] = \frac{1+\nu}{E}\left[\frac{A}{r^2}+(1-2\nu)B\right]$$

$$e_\theta = \frac{1+\nu}{E}[(1-\nu)\sigma_\theta - \nu\sigma_r] = \frac{1+\nu}{E}\left[-\frac{A}{r^2}+(1-2\nu)B\right]$$

Strain - displacement relation : $e_\theta = \dfrac{u_r}{r} \Rightarrow$

$$u_r = \frac{1+\nu}{E}\left[-\frac{A}{r}+(1-2\nu)Br\right]$$

Boundary Conditions :

$$\sigma_r(r_2) = 0 \Rightarrow \frac{A}{r_2^2}+B = 0 \Rightarrow B = -\frac{A}{r_2^2}$$

$$u_r(r_1) = \delta \Rightarrow \frac{1+\nu}{E}\left[-\frac{A}{r_1}+(1-2\nu)Br_1\right] = \delta$$

Solving for the constants A and B gives

$$A = -\frac{\delta E}{1+\nu}\left[\frac{r_1 r_2^2}{r_2^2+(1-2\nu)r_1^2}\right],\ B = -A/r_2^2 = \frac{\delta E}{1+\nu}\left[\frac{r_1}{r_2^2+(1-2\nu)r_1^2}\right]$$

8-18.

General plane strain axisymmetric solution was given by relations (8.4.1) and (8.4.5) $\Rightarrow$

$$\sigma_r = \frac{A}{r^2} + B \ , \ \sigma_\theta = -\frac{A}{r^2} + B \ , \ u_r = \frac{1+\nu}{E}\left[-\frac{A}{r} + (1-2\nu)Br\right]$$

Material 1:

$$\sigma_r^{(1)} = \frac{A^{(1)}}{r^2} + B^{(1)} \ , \ \sigma_\theta^{(1)} = -\frac{A^{(1)}}{r^2} + B^{(1)} \ ,$$

but for finite stresses at $r = 0 \Rightarrow A^{(1)} = 0$

$$\therefore \sigma_r^{(1)} = \sigma_\theta^{(1)} = B^{(1)} \ , \ u_r^{(1)} = \frac{(1+\nu_1)(1-2\nu_1)}{E_1} B^{(1)} r$$

Material 2:

$$\sigma_r^{(2)} = \frac{A^{(2)}}{r^2} + B^{(2)} \ , \ \sigma_\theta^{(2)} = -\frac{A^{(2)}}{r^2} + B^{(2)} \ ,$$

Boundary conditon $\sigma_r^{(2)}(r_2) = -p \Rightarrow \dfrac{A^{(2)}}{r_2^2} + B^{(2)} = -p \Rightarrow B^{(2)} = -p - \dfrac{A^{(2)}}{r_2^2}$

$$\therefore \sigma_r^{(2)} = \frac{A^{(2)}}{r^2} - \frac{A^{(2)}}{r_2^2} - p \ , \ u_r^{(2)} = \frac{1+\nu_2}{E_2}\left[-\frac{A^{(2)}}{r^2} - (1-2\nu_2)\left(p + \frac{A^{(2)}}{r_2^2}\right)\right] r$$

Matching conditions @ $r = r_1$:

$$u_r^{(1)}(r_1) = u_r^{(2)}(r_1) \Rightarrow \frac{(1+\nu_1)(1-2\nu_1)}{E_1} B^{(1)} = \frac{1+\nu_2}{E_2}\left[-\frac{A^{(2)}}{r_1^2} - (1-2\nu_2)\left(p + \frac{A^{(2)}}{r_2^2}\right)\right]$$

$$\sigma_r^{(1)}(r_1) = \sigma_r^{(2)}(r_1) \Rightarrow B^{(1)} = \frac{A^{(2)}}{r_1^2} - \frac{A^{(2)}}{r_2^2} - p$$

Solve matching condition relations for $B^{(1)}$ and $A^{(2)}$ to complete solution.

8-19.

Solutions from Exercise 8-18:

$$\text{Material 1}: \sigma_r^{(1)} = \sigma_\theta^{(1)} = B^{(1)} \ , \ u_r^{(1)} = \frac{(1+\nu_1)(1-2\nu_1)}{E_1} B^{(1)} r$$

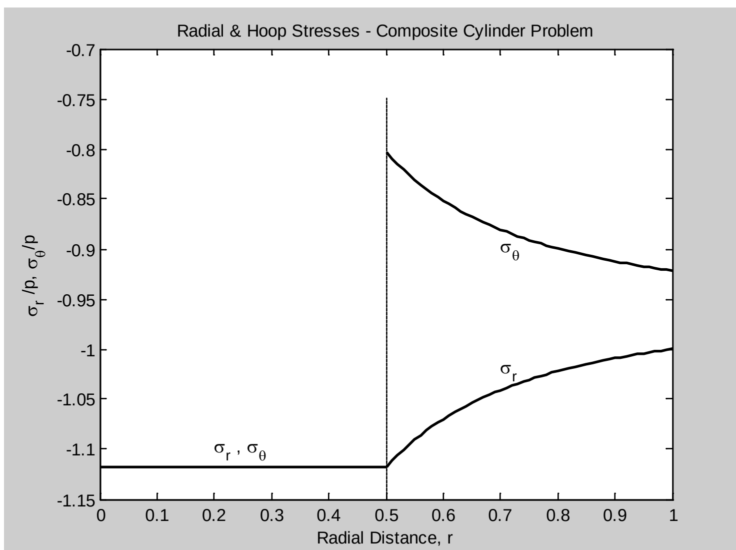
$$\text{Material 2}: \ \sigma_r^{(2)} = \frac{A^{(2)}}{r^2} - \frac{A^{(2)}}{r_2^2} - p \ , \ \sigma_\theta^{(2)} = -\frac{A^{(2)}}{r^2} - \frac{A^{(2)}}{r_2^2} - p$$

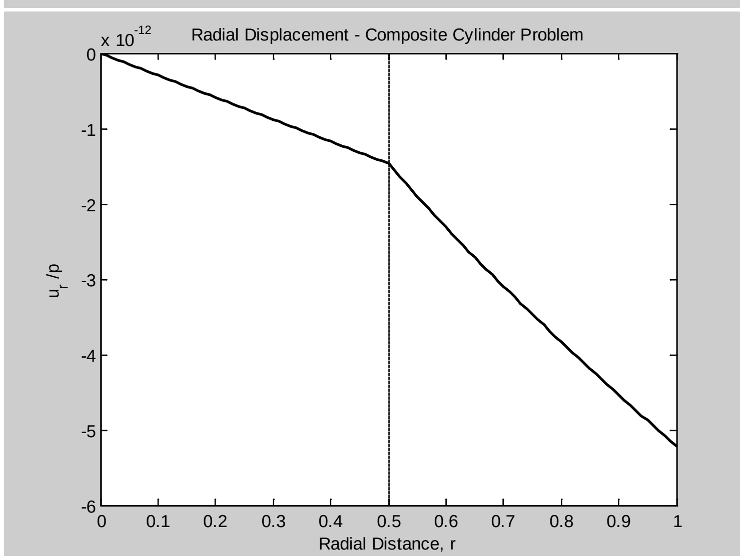
$$u_r^{(2)} = \frac{1+\nu_2}{E_2}\left[-\frac{A^{(2)}}{r^2} - (1-2\nu_2)\left(p + \frac{A^{(2)}}{r_2^2}\right)\right] r$$

Two Equations From Matching Conditions @ $r = r_1$:

$$\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) A^{(2)} - B^{(1)} = p \ , \ \left(\frac{1}{r_1^2} + \frac{(1-2\nu_2)}{r_2^2}\right) A^{(2)} - \left(\frac{E_2(1+\nu_1)(1-2\nu_1)}{E_1(1+\nu_2)}\right) B^{(1)} = -(1-2\nu_2)p$$

Using MATLAB to Solve Matching Relations and Then Calculate and Plot Stresses and Displacements for the Case $r_1 = 0.5, \ r_2 = 1.0$ Note that σ_θ is not continuous across the interface.





8-20.

General solution from $(8.4.3)_2$

$$\sigma_\theta = -\frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

For the case with $p_1 = p$ and $p_2 = 0$

$$\sigma_\theta = \frac{r_1^2 p}{r_2^2 - r_1^2}\left(1 + \frac{r_2^2}{r^2}\right)$$

From the definitions : $t = r_2 - r_1$ and $r_o = \frac{1}{2}(r_1 + r_2) \Rightarrow$

$$r_2^2 - r_1^2 = (r_2 - r_1)(r_2 + r_1) = 2tr_o$$

Thin - walled cylinder $\Rightarrow r_1 \approx r_2 \approx r_o \approx r$

$$\therefore \sigma_\theta = \frac{r_1^2 p}{2tr_o}\left(1 + \frac{r_2^2}{r^2}\right) \approx \frac{r_o^2 p}{2tr_o}(1+1) \approx \frac{pr_o}{t}$$

8-21.

Solution $(8.3.9)_2$ without Rigid - Body Motion : $u_\theta = \frac{4r\theta}{E}a_3$

Cyclic Jump Condition : $u_\theta(r,2\pi) - u_\theta(r,0) = \alpha r \Rightarrow \frac{8\pi r}{E}a_3 = \alpha r \Rightarrow a_3 = \frac{\alpha E}{8\pi}$

General Stress Field Solution :

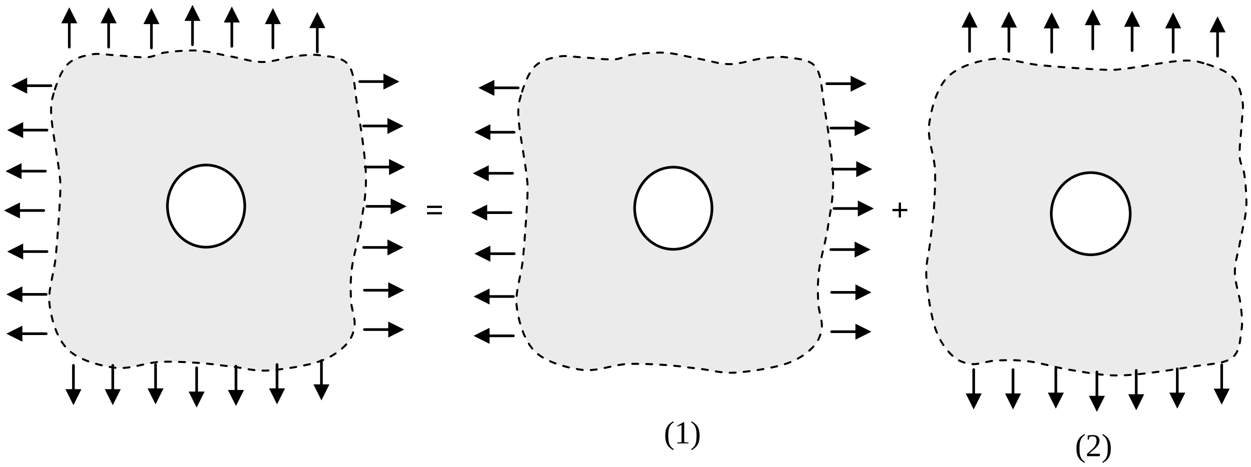
$$\sigma_r = a_3(1+\log r) + \frac{a_1}{r^2} + 2a_2 \ , \ \sigma_\theta = a_3(3 + 2\log r) - \frac{a_1}{r^2} + 2a_2$$

Boundary Conditions : $\sigma_r(r_i) = a_3(1+\log r_i) + \frac{a_1}{r_i^2} + 2a_2 = 0$, $\sigma_r(r_o) = a_3(1+\log r_o) + \frac{a_1}{r_o^2} + 2a_2 = 0$

Solving for Constants : $a_1 = \frac{2r_i^2 r_o^2 \log(r_o / r_i)}{r_o^2 - r_i^2}a_3$, $a_2 = \frac{r_i^2(1+2\log r_i) - r_o^2(1+2\log r_o)}{2(r_o^2 - r_i^2)}a_3$

$\therefore$ All Constants Are Determined and the Stress Field Is Found

8-22.



Using superposition as shown, the solution to the biaxial loading problem is given by

$$\sigma_r = \sigma_r^{(1)} + \sigma_r^{(2)} \ , \ \sigma_\theta = \sigma_\theta^{(1)} + \sigma_\theta^{(2)} \ , \ \tau_{r\theta} = \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)}$$

The solution to problem (1) follows directly from Example 8 - 7, (8.4.15)

$$\sigma_r^{(1)} = \frac{T}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta^{(1)} = \frac{T}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta}^{(1)} = -\frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

The solution to problem (2) is given by the same expressions with $\theta \to \theta + \frac{\pi}{2}$

$$\sigma_r^{(2)} = \frac{T}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos(2\theta+\pi)$$

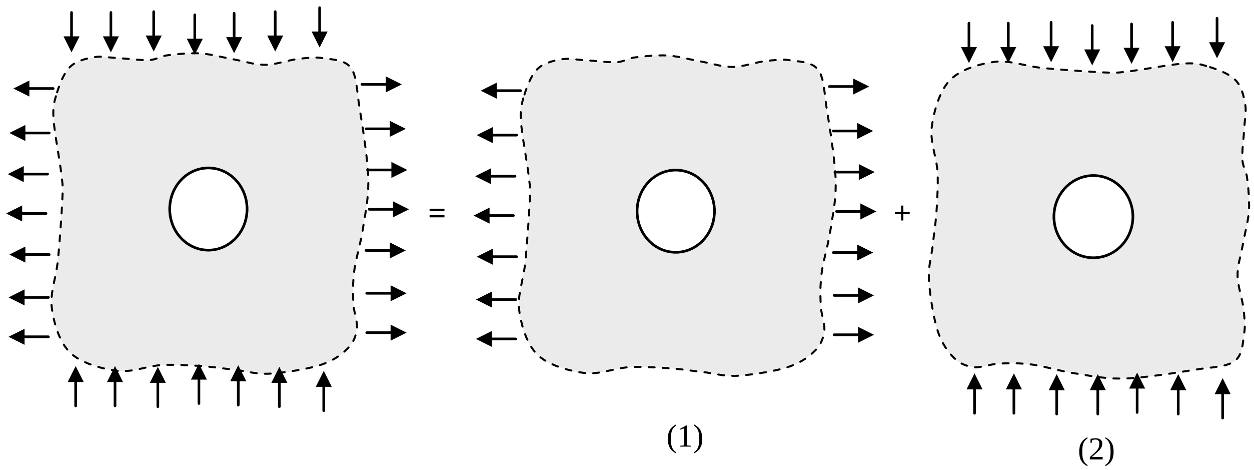
$$\sigma_\theta^{(2)} = \frac{T}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos(2\theta+\pi)$$

$$\tau_{r\theta}^{(2)} = -\frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin(2\theta+\pi)$$

Since $\cos(2\theta+\pi) = -\cos 2\theta$, $\sin(2\theta+\pi) = -\sin 2\theta$

$$\therefore \sigma_r = \sigma_r^{(1)} + \sigma_r^{(2)} = T\left(1-\frac{a^2}{r^2}\right), \ \sigma_\theta = \sigma_\theta^{(1)} + \sigma_\theta^{(2)} = T\left(1+\frac{a^2}{r^2}\right), \ \tau_{r\theta} = \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)} = 0$$

8-23*.



Using superposition as shown, the solution to the biaxial loading problem is given by

$$\sigma_r = \sigma_r^{(1)} + \sigma_r^{(2)} \ , \ \sigma_\theta = \sigma_\theta^{(1)} + \sigma_\theta^{(2)} \ , \ \tau_{r\theta} = \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)}$$

The solution to problem (1) follows directly from Example 8 - 7, (8.4.15)

$$\sigma_r^{(1)} = \frac{T}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta^{(1)} = \frac{T}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta}^{(1)} = -\frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

The solution to problem (2) is given by the same expressions with $T \to -T$ and $\theta \to \theta + \dfrac{\pi}{2}$

$$\sigma_r^{(2)} = -\frac{T}{2}\left(1-\frac{a^2}{r^2}\right) - \frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos(2\theta+\pi)$$

$$\sigma_\theta^{(2)} = -\frac{T}{2}\left(1+\frac{a^2}{r^2}\right) + \frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos(2\theta+\pi)$$

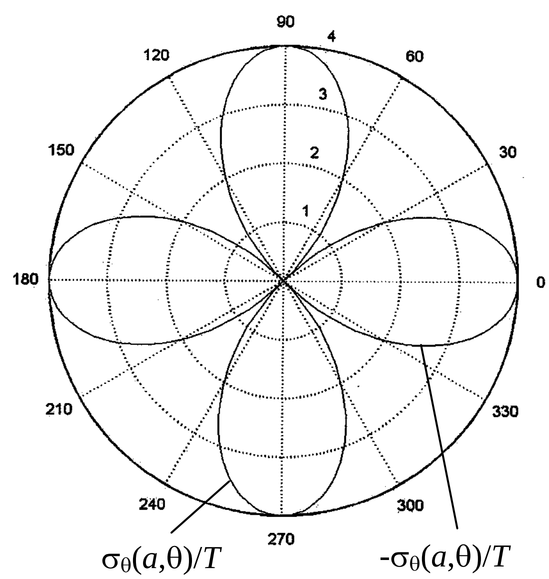
$$\tau_{r\theta}^{(2)} = \frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin(2\theta+\pi)$$

$$\therefore \sigma_r = \sigma_r^{(1)} + \sigma_r^{(2)} = T\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \sigma_\theta^{(1)} + \sigma_\theta^{(2)} = -T\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)} = -T\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

Note as $r \to \infty$ with $\theta = \pm 45^\circ$, $\sigma_r = \sigma_\theta = 0$, $\tau_{r\theta} = \pm T$



8-24.

$$\phi = \frac{\tau_o r^2}{\pi}\left[\sin^2\theta\log r + \theta\sin\theta\cos\theta - \sin^2\theta\right] = \frac{\tau_o r^2}{\pi}\left[\sin^2\theta\log r + \frac{1}{2}\theta\sin 2\theta - \sin^2\theta\right]$$

$$\phi_{,r} = \frac{\tau_o r}{\pi}\left[2\sin^2\theta\log r + \theta\sin 2\theta - \sin^2\theta\right],\ \phi_{,\theta} = \frac{\tau_o r^2}{\pi}\left[\sin 2\theta\log r + \theta\cos 2\theta - \frac{1}{2}\sin 2\theta\right]$$

$$\phi_{,rr} = \frac{\tau_o}{\pi}\left[2\sin^2\theta\log r + \theta\sin 2\theta + \sin^2\theta\right],\ \phi_{,\theta\theta} = \frac{2\tau_o r^2}{\pi}\left[\cos 2\theta\log r - \theta\sin 2\theta\right]$$

Do Laplacian First and Simplify :

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = \frac{2\tau_o\log r}{\pi} \Rightarrow \nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{2\tau_o\log r}{\pi}\right) = 0$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\frac{\tau_o}{\pi}(\theta\sin 2\theta + \sin^2\theta - 2\cos^2\theta\log r)$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \frac{\tau_o}{\pi}\sin\theta(2\sin\theta\log r + 2\theta\cos\theta + \sin\theta)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -\frac{\tau_o}{\pi}(\sin 2\theta\log r + \frac{1}{2}\sin 2\theta + \theta\cos 2\theta)$$

Boundary Conditions :

$\sigma_\theta(r,0) = 0$, satisfied

$\tau_{r\theta}(r,0) = 0$, satisfied

$\sigma_\theta(r,\pi) = 0$, satisfied

$\tau_{r\theta}(r,\pi) = -\tau_o \Rightarrow -\dfrac{\tau_o}{\pi}(\pi\cos 2\pi) = -\tau_o$, satisfied

Stresses are singular at the origin where the abrupt change in loading occurs.

8-25.

General Flamant Solution:

$$\phi = (a_{12} r \log r + a_{15} r\theta)\cos\theta + (b_{12} r \log r + b_{15} r\theta)\sin\theta$$

$$\sigma_r = \frac{1}{r}[(a_{12} + 2b_{15})\cos\theta + (b_{12} - 2a_{15})\sin\theta]$$

$$\sigma_\theta = \frac{1}{r}[a_{12}\cos\theta + b_{12}\sin\theta]$$

$$\tau_{r\theta} = \frac{1}{r}[a_{12}\sin\theta - b_{12}\cos\theta]$$

Boundary Conditions:

$$\sigma_\theta(r,\alpha) = \tau_{r\theta}(r,\alpha) = \sigma_\theta(r,\beta) = \tau_{r\theta}(r,\beta) = 0 \Rightarrow a_{12} = b_{12} = 0$$

$$\therefore \sigma_r = \frac{2}{r}[b_{15}\cos\theta - a_{15}\sin\theta]\ ,\ \sigma_\theta = \tau_{r\theta} = 0$$

$$X = -\int_\alpha^\beta \sigma_r(a,\theta) a\cos\theta d\theta = b_{15}[(\sin 2\alpha - \sin 2\beta) + 2(\alpha - \beta)] + a_{15}(\cos 2\alpha - \cos 2\beta)$$

$$Y = -\int_\alpha^\beta \sigma_r(a,\theta) a\sin\theta d\theta = a_{15}[-(\sin 2\beta - \sin 2\alpha) + 2(\alpha - \beta)] + b_{15}(\cos 2\alpha - \cos 2\beta)$$

Solving the previous two relations for a_{15} and b_{15} gives

$$a_{15} = \frac{X(\cos 2\alpha - \cos 2\beta) + Y[(\sin 2\alpha - \sin 2\beta) + 2(\alpha - \beta)]}{(\cos 2\alpha - \cos 2\beta)^2 + (\sin 2\alpha - \sin 2\beta)^2 - 4(\alpha - \beta)^2}$$

$$b_{15} = \frac{X[(\sin 2\alpha - \sin 2\beta) + 2(\alpha - \beta)] - Y(\cos 2\alpha - \cos 2\beta)}{(\cos 2\alpha - \cos 2\beta)^2 + (\sin 2\alpha - \sin 2\beta)^2 - 4(\alpha - \beta)^2}$$

and thus the problem is solved.

8-26.

The solution for the single vertical downward force acting at the origin is given by

$$\phi = \frac{\mathrm{P}}{\pi} r\theta\cos\theta = \frac{\mathrm{P}}{\pi} x\tan^{-1}\left(\frac{y}{x}\right) = PF(x,y)\ ,\ \text{where } F(x,y) = \frac{x}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

Using superposition and taking the limit as $d \to 0$, the stress function for the moment problem is

$$\begin{aligned}
\phi_M &= \lim_{d\to\infty}\left[\phi(x,y) - \phi(x+d,y)\right] = -\lim_{d\to\infty}\left[Pd\,\frac{F(x+d,y)-F(x,y)}{d}\right] \\
&= -M\lim_{d\to\infty}\left[\frac{F(x+d,y)-F(x,y)}{d}\right] = -M\,\frac{\partial F}{\partial x} = -d\,\frac{\partial\phi}{\partial x} \\
&= -d\,\frac{\partial}{\partial x}\left(\frac{\mathrm{P}}{\pi}x\tan^{-1}\left(\frac{y}{x}\right)\right) = -\frac{\mathrm{M}}{\pi}\left(-\frac{xy}{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right)\right) = -\frac{\mathrm{M}}{\pi}(\theta - \sin\theta\cos\theta)
\end{aligned}$$

and the stresses follow from the standard differentiation relations

$$\sigma_r = -\frac{4M}{\pi r^2}\sin\theta\cos\theta$$

$$\sigma_\theta = 0$$

$$\tau_{r\theta} = -\frac{2M}{\pi r^2}\sin^2\theta$$

8-27*.

$$\sigma_x = -\frac{p}{2\pi}[2(\theta_2 - \theta_1) + (\sin 2\theta_2 - \sin 2\theta_1)]$$

$$\sigma_y = -\frac{p}{2\pi}[2(\theta_2 - \theta_1) - (\sin 2\theta_2 - \sin 2\theta_1)]$$

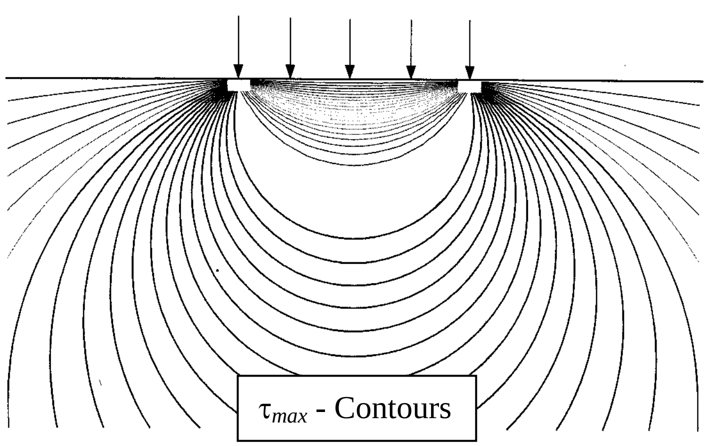
$$\tau_{xy} = \frac{p}{2\pi}[\cos 2\theta_2 - \cos 2\theta_1]$$

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \pm\sqrt{\left(-\frac{p}{2\pi}(\sin 2\theta_2 - \sin 2\theta_1)\right)^2 + \left(\frac{p}{2\pi}[\cos 2\theta_2 - \cos 2\theta_1]\right)^2}$$

$$= \pm\frac{p}{2\pi}\sqrt{2 - 2(\sin 2\theta_2 \sin 2\theta_1 + \cos 2\theta_2 \cos 2\theta_1)}$$

$$= \pm\frac{p\sqrt{2}}{2\pi}\sqrt{1 - \cos 2(\theta_1 - \theta_2)} = \pm\frac{p}{\pi}\sin(\theta_1 - \theta_2) = \mp\frac{p}{\pi}\sin(\theta_2 - \theta_1)$$



8-28.

Since: $\sigma_r = -\frac{2X}{\pi r}\cos\theta$, $\sigma_\theta = \tau_{r\theta} = 0 \Rightarrow$

$$\sigma_x = \sigma_r \cos^2\theta = -\frac{2X}{\pi r}\cos^3\theta,\ \sigma_y = \sigma_r \sin^2\theta = -\frac{2X}{\pi r}\cos\theta\sin^2\theta,\ \tau_{xy} = \sigma_r \sin\theta\cos\theta = -\frac{2X}{\pi r}\cos^2\theta\sin\theta$$

But, $dX = trd\theta/\sin\theta \Rightarrow$

$$d\sigma_x = -\frac{2t}{\pi}\frac{\cos^3\theta}{\sin\theta} \Rightarrow$$

$$\sigma_x = -\frac{2t}{\pi}\int_{\theta_1}^{\theta_2}\frac{\cos^3\theta}{\sin\theta}d\theta = -\frac{2t}{\pi}\int_{\theta_1}^{\theta_2}\left[\frac{\cos\theta}{\sin\theta} - \sin\theta\cos\theta\right]d\theta = \frac{t}{2\pi}\left[4\log\left(\frac{\sin\theta_1}{\sin\theta_2}\right) - \cos 2\theta_2 + \cos 2\theta_1\right]$$

$$d\sigma_y = -\frac{2t}{\pi}\sin\theta\cos\theta d\theta \Rightarrow$$

$$\sigma_y = -\frac{2t}{\pi}\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta d\theta = \frac{t}{2\pi}\left[\cos 2\theta_2 - \cos 2\theta_1\right]$$

$$d\tau_{xy} = -\frac{2t}{\pi}\cos^2\theta d\theta \Rightarrow$$

$$\tau_{xy} = -\frac{2t}{\pi}\int_{\theta_1}^{\theta_2}\cos^2\theta d\theta = -\frac{t}{2\pi}\left[2(\theta_2 - \theta_1) + \sin 2\theta_2 - \sin 2\theta_1\right]$$

8-29.

From $(8.4.36)_1$ and Exercise 8 - 28, Using $Y = pds$, $X = tds \Rightarrow$

$$\sigma_x = -\frac{2Yx^2y}{\pi(x^2+y^2)^2} - \frac{2Xx^3}{\pi(x^2+y^2)^2} \Rightarrow d\sigma_x = -\frac{2p(s)}{\pi}\frac{(x-s)^2y}{[(x-s)^2+y^2]^2}ds - \frac{2t(s)}{\pi}\frac{(x-s)^3}{[(x-s)^2+y^2]^2}ds$$

$$\therefore \sigma_x = -\frac{2y}{\pi}\int_{-a}^{a}\frac{p(s)(x-s)^2}{[(x-s)^2+y^2]^2}ds - \frac{2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)^3}{[(x-s)^2+y^2]^2}ds$$

$$\sigma_y = -\frac{2Yy^3}{\pi(x^2+y^2)^2} - \frac{2Xxy^2}{\pi(x^2+y^2)^2} \Rightarrow d\sigma_y = -\frac{2p(s)}{\pi}\frac{y^3}{[(x-s)^2+y^2]^2}ds - \frac{2t(s)}{\pi}\frac{(x-s)y^2}{[(x-s)^2+y^2]^2}ds$$

$$\therefore \sigma_y = -\frac{2y^3}{\pi}\int_{-a}^{a}\frac{p(s)}{[(x-s)^2+y^2]^2}ds - \frac{2y^2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)}{[(x-s)^2+y^2]^2}ds$$

$$\tau_{xy} = -\frac{2Yxy^2}{\pi(x^2+y^2)^2} - \frac{2Xx^2y}{\pi(x^2+y^2)^2} \Rightarrow d\tau_{xy} = -\frac{2p(s)}{\pi}\frac{(x-s)y^2}{[(x-s)^2+y^2]^2}ds - \frac{2t(s)}{\pi}\frac{(x-s)^2y}{[(x-s)^2+y^2]^2}ds$$

$$\therefore \tau_{xy} = -\frac{2y^2}{\pi}\int_{-a}^{a}\frac{p(s)(x-s)}{[(x-s)^2+y^2]^2}ds - \frac{2y}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)^2}{[(x-s)^2+y^2]^2}ds$$

8-30.

$$\phi = r^{\lambda}[A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta]$$

$$\sigma_r = -r^{\lambda-2}[A\lambda(\lambda-1)\sin\lambda\theta + B\lambda(\lambda-1)\cos\lambda\theta$$
$$+ C(\lambda^2-5\lambda+4)\sin(\lambda-2)\theta + D(\lambda^2-5\lambda+4)\cos(\lambda-2)\theta]$$

$$\sigma_\theta = \lambda(\lambda-1)r^{\lambda-2}[A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta]$$

$$\tau_{r\theta} = -(\lambda-1)r^{\lambda-2}[A\lambda\cos\lambda\theta - B\lambda\sin\lambda\theta + C(\lambda-2)\cos(\lambda-2)\theta - D(\lambda-2)\sin(\lambda-2)\theta]$$

Boundary condtions on the problem gave $\lambda = \frac{3}{2}$, $D=-B$, $C = -\frac{\lambda A}{(\lambda-2)} = 3A$

Substituting these results into the general stress relations gives

$$\sigma_r = -\frac{3}{4}\frac{1}{\sqrt{r}}\left[A(\sin\frac{3}{2}\theta + 5\sin\frac{\theta}{2}) + B(\cos\frac{3}{2}\theta + \frac{5}{3}\cos\frac{\theta}{2})\right]$$

$$\sigma_\theta = \frac{3}{4}\frac{1}{\sqrt{r}}\left[A(\sin\frac{3}{2}\theta - 3\sin\frac{\theta}{2}) + B(\cos\frac{3}{2}\theta - \cos\frac{\theta}{2})\right]$$

$$\tau_{r\theta} = -\frac{3}{4}\frac{1}{\sqrt{r}}\left[A(\cos\frac{3}{2}\theta - \cos\frac{\theta}{2}) - B(\sin\frac{3}{2}\theta - \frac{1}{3}\sin\frac{\theta}{2})\right]$$

Changing the angular coordinate : $\vartheta = \pi - \theta \Rightarrow \theta = \pi - \vartheta$

$$\sigma_r = -\frac{3}{4}\frac{1}{\sqrt{r}}\left[A(-\cos\frac{3}{2}\vartheta + 5\cos\frac{\vartheta}{2}) + B(-\sin\frac{3}{2}\vartheta + \frac{5}{3}\sin\frac{\vartheta}{2})\right]$$
$$= -\frac{3}{2}\frac{A}{\sqrt{r}}\cos\frac{\vartheta}{2}(3-\cos\vartheta) - \frac{B}{2\sqrt{r}}\sin\frac{\vartheta}{2}(1-3\cos\vartheta)$$

$$\sigma_\theta = \frac{3}{4}\frac{1}{\sqrt{r}}\left[A(-\cos\frac{3}{2}\vartheta - 3\cos\frac{\vartheta}{2}) + B(-\sin\frac{3}{2}\vartheta - \sin\frac{\vartheta}{2})\right]$$
$$= -\frac{3}{2}\frac{A}{\sqrt{r}}\cos\frac{\vartheta}{2}(1+\cos\vartheta) - \frac{3B}{2\sqrt{r}}\sin\frac{\vartheta}{2}(1+\cos\vartheta)$$

$$\tau_{r\theta} = -\frac{3}{4}\frac{1}{\sqrt{r}}\left[A(-\sin\frac{3}{2}\vartheta - \sin\frac{\vartheta}{2}) - B(-\cos\frac{3}{2}\vartheta - \frac{1}{3}\cos\frac{\vartheta}{2})\right]$$
$$= \frac{3}{2}\frac{A}{\sqrt{r}}\sin\frac{\vartheta}{2}(1+\cos\vartheta) + \frac{B}{2\sqrt{r}}\cos\frac{\vartheta}{2}(1-3\cos\vartheta)$$

8-31*.

Mode I:

$$\sigma_r = -\frac{3}{2}\frac{A}{\sqrt{r}}\cos\frac{\vartheta}{2}(3-\cos\vartheta)$$

$$\sigma_\theta = -\frac{3}{2}\frac{A}{\sqrt{r}}\cos\frac{\vartheta}{2}(1+\cos\vartheta)$$

$$\tau_{r\theta} = \frac{3}{2}\frac{A}{\sqrt{r}}\sin\frac{\vartheta}{2}(1+\cos\vartheta)$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_r-\sigma_\theta}{2}\right)^2+\tau_{r\theta}^2}$$

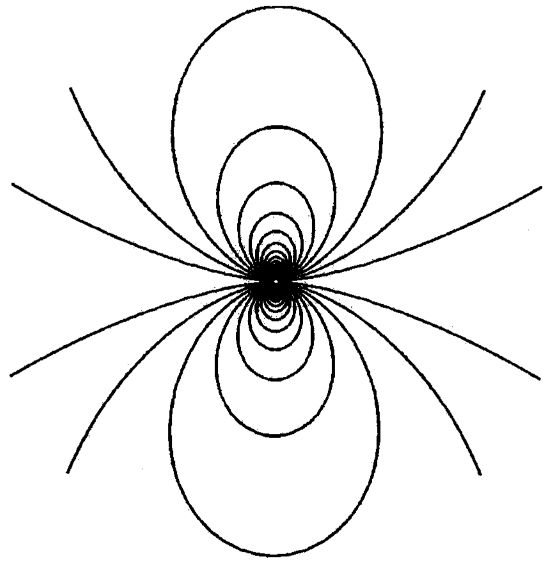
$$= \frac{3A}{2\sqrt{r}}\sqrt{(1-\cos\vartheta)^2\cos^2\frac{\vartheta}{2}+\sin^2\frac{\vartheta}{2}(1+\cos\vartheta)^2}$$

$$= \frac{3A}{2\sqrt{r}}\sqrt{4\sin^4\frac{\vartheta}{2}\cos^2\frac{\vartheta}{2}+\sin^2\frac{\vartheta}{2}4\cos^4\frac{\vartheta}{2}}$$

$$= \frac{3A}{2\sqrt{r}}\sqrt{4\sin^2\frac{\vartheta}{2}\cos^2\frac{\vartheta}{2}\left(\sin^2\frac{\vartheta}{2}+\cos^2\frac{\vartheta}{2}\right)}$$

$$= \frac{3A}{2\sqrt{r}}2\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2} = \frac{3A}{2\sqrt{r}}\sin\vartheta$$

We wish to plot contours of τ_{max}/A



τ_{max} - Contours – Mode I

Mode II:

$$\sigma_r = -\frac{B}{2\sqrt{r}}\sin\frac{\vartheta}{2}(1-3\cos\vartheta)$$

$$\sigma_\theta = -\frac{3B}{2\sqrt{r}}\sin\frac{\vartheta}{2}(1+\cos\vartheta)$$

$$\tau_{r\theta} = \frac{B}{2\sqrt{r}}\cos\frac{\vartheta}{2}(1-3\cos\vartheta)$$

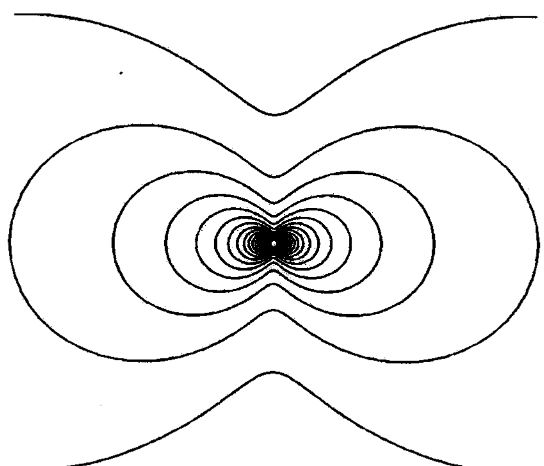
$$\tau_{max} = \sqrt{\left(\frac{\sigma_r-\sigma_\theta}{2}\right)^2+\tau_{r\theta}^2}$$

$$= \frac{B}{2\sqrt{r}}\sqrt{\sin^2\frac{\vartheta}{2}(1+3\cos\vartheta)^2+\cos^2\frac{\vartheta}{2}(1-3\cos\vartheta)^2}$$

$$= \frac{B}{2\sqrt{r}}\sqrt{\left(\sin^2\frac{\vartheta}{2}+\cos^2\frac{\vartheta}{2}\right)+9\cos^2\vartheta\left(\sin^2\frac{\vartheta}{2}+\cos^2\frac{\vartheta}{2}\right)-6\cos\theta\left(\cos^2\frac{\vartheta}{2}-\sin^2\frac{\vartheta}{2}\right)}$$

$$= \frac{B}{2\sqrt{r}}\sqrt{(1+3\cos^2\vartheta)}$$

We wish to plot contours of τ_{max}/B



τ_{max} - Contours – Mode II

8-32.

Using $w = Ar^{\lambda} f(\theta)$ in $\nabla^2 w = 0 \Rightarrow$

$$\lambda(\lambda-1)r^{\lambda-2} f + \frac{1}{r}\lambda r^{\lambda-1} f + \frac{1}{r^2}\lambda r^{\lambda} f'' = 0 \Rightarrow f'' + \lambda^2 f = 0$$

Solution : $f = A\sin\lambda\theta + B\cos\lambda\theta$, Choosing only the odd part gives $f = A\sin\lambda\theta$

The Stresses from This Displacement Solution Are :

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = 0 \ , \ \tau_{rz} = \mu\frac{\partial w}{\partial r} = \mu A r^{\lambda-1}\sin\lambda\theta \ , \ \tau_{\theta z} = \frac{\mu}{r}\frac{\partial w}{\partial\theta} = \mu A r^{\lambda-1}\cos\lambda\theta$$

Zero Stress Boundary Conditions on Crack Surfaces : $\Rightarrow$

$$\tau_{\theta z}(r, \pm\pi) = 0 \Rightarrow \cos\lambda\pi = 0 \Rightarrow \lambda\pi = n\pi/2 \ , \ n = 1,3,5,\cdots \ \therefore \lambda = n/2 \ , \ n = 1,3,5,\cdots$$

Finite Displacements at $r \to 0 \Rightarrow \lambda > 0$

Singular Stresses at $r \to 0 \Rightarrow \lambda < 1$

$\therefore 0 < \lambda < 1 \Rightarrow \lambda = 1/2$ and So the Displacements and Stresses Become

$$w = A\sqrt{r}\sin\frac{\theta}{2} \ , \ \tau_{\theta z} = \frac{\mu A}{2\sqrt{r}}\cos\frac{\theta}{2} \ , \ \tau_{rz} = \frac{\mu A}{2\sqrt{r}}\sin\frac{\theta}{2}$$

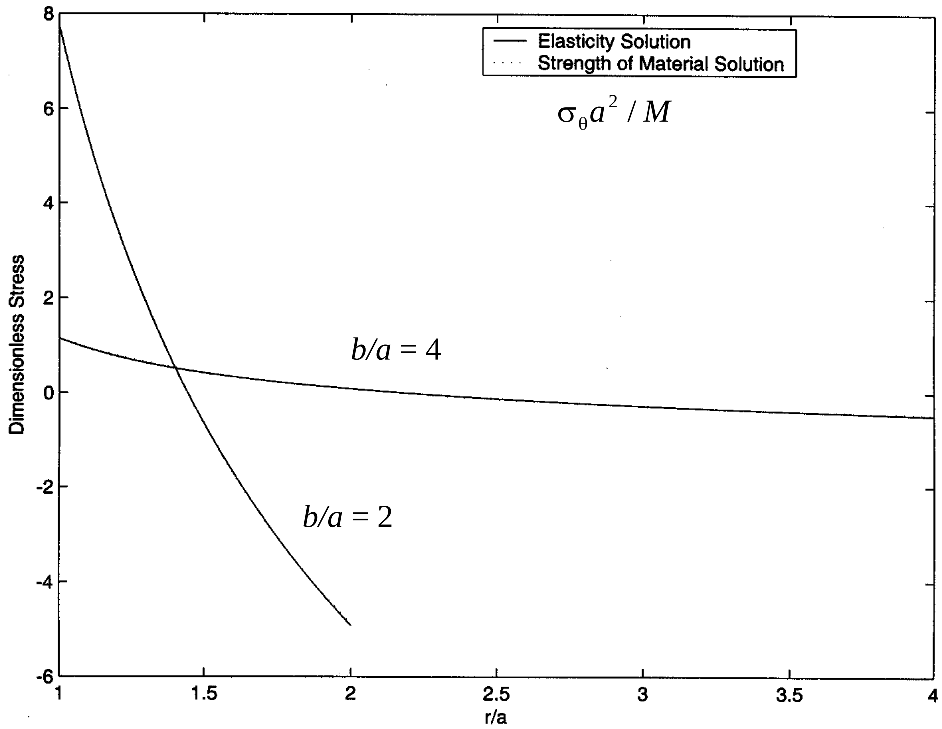
8-33*.

$$\sigma_\theta = -\frac{4M}{N}\left[-\frac{a^2b^2}{r^2}\log(\frac{b}{a}) + b^2\log(\frac{r}{b}) + a^2\log(\frac{a}{r}) + b^2 - a^2\right]$$

with $N = (b^2 - a^2)^2 - 4a^2b^2[\log(\frac{b}{a})]^2$

Strength of Materials Solution:

$$\sigma_\theta = -\frac{M(r-B)}{rA(R-B)}\ ,\ A = b - a\ ,\ B = \frac{b-a}{\log(b/a)}\ ,\ R = (a+b)/2$$



8-34.

$$\phi = (Ar^3 + \frac{B}{r} + Cr + Dr\log r)\cos\theta$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$$

Boundary Conditions:

$$\sigma_r(a,\theta) = \tau_{r\theta}(a,\theta) = 0 \Rightarrow 2Aa - \frac{2B}{a^3} + \frac{D}{a} = 0$$

$$\sigma_r(b,\theta) = \tau_{r\theta}(b,\theta) = 0 \Rightarrow 2Ab - \frac{2B}{b^3} + \frac{D}{b} = 0$$

$$\int_a^b \sigma_\theta(r,0)dr = T \Rightarrow 3A(b^2 - a^2) + \frac{B}{a^2b^2}(b^2 - a^2) + D\log\left(\frac{b}{a}\right) = T$$

$$\int_a^b \sigma_\theta(r,0)rdr = M + \frac{T(a+b)}{2} = M_R \Rightarrow$$

$$2A(b^3 - a^3) + \frac{2B}{ab}(b-a) + D(b-a) = M_R$$

$$\int_a^b \tau_{r\theta}(r,0)dr = 0 \text{ , identically satisfied}$$

$$\int_a^b \sigma_\theta(r,\pi/2)dr = 0 \text{ , identically satisfied}$$

$$\int_a^b \sigma_\theta(r,\pi/2)rdr = -M_R \;\cdots\; \text{appears not to be satisfied}$$

$$\int_a^b \tau_{r\theta}(r,\pi/2)dr = T \Rightarrow A(b^2 - a^2) + \frac{B}{a^2b^2}(b^2 - a^2) + D\log\left(\frac{b}{a}\right) = T$$

These boundary condition relations can be solved giving the results

$$A = \frac{T}{2N} \text{ , } B = -\frac{Ta^2b^2}{2N} \text{ , } D = -\frac{T(a^2+b^2)}{N} \text{ but also gives } M_R = 0 \Rightarrow M = -\frac{T(a+b)}{2}$$

So the problem is solved for the special case with $M = -\dfrac{T(a+b)}{2}$. Note however, that by adding the pure bending case given by solution (8.2.61), the solution for an arbitrary end moment may be generated.

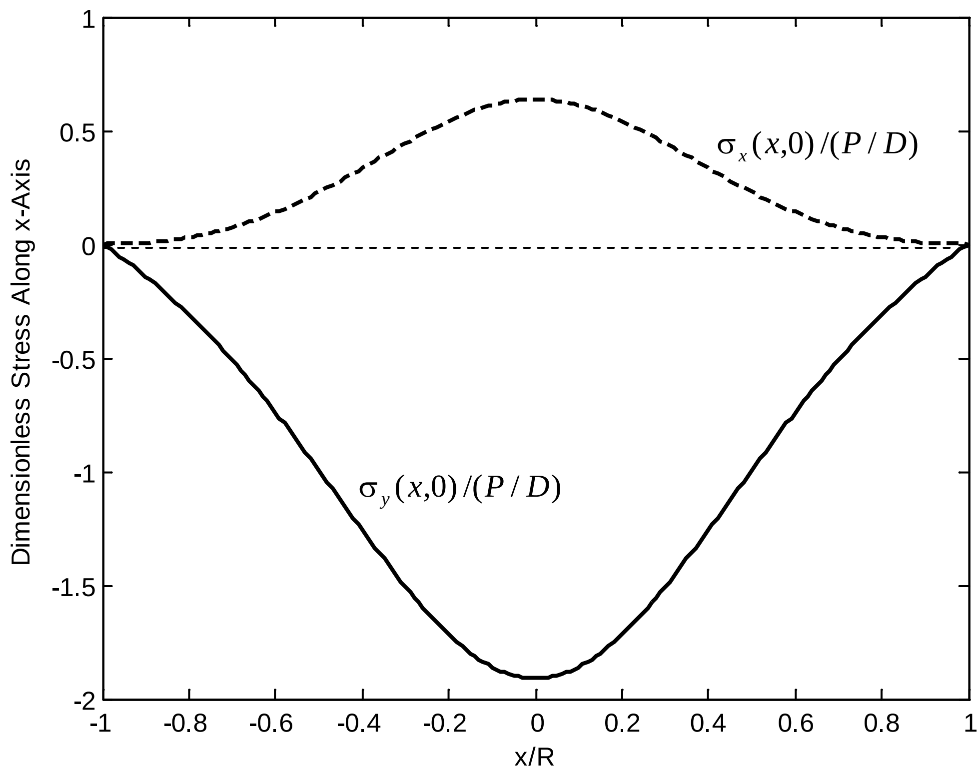
8-35*.

For disk problem shown in Figure 8 - 35, the two normal stresses on $y = 0$ are

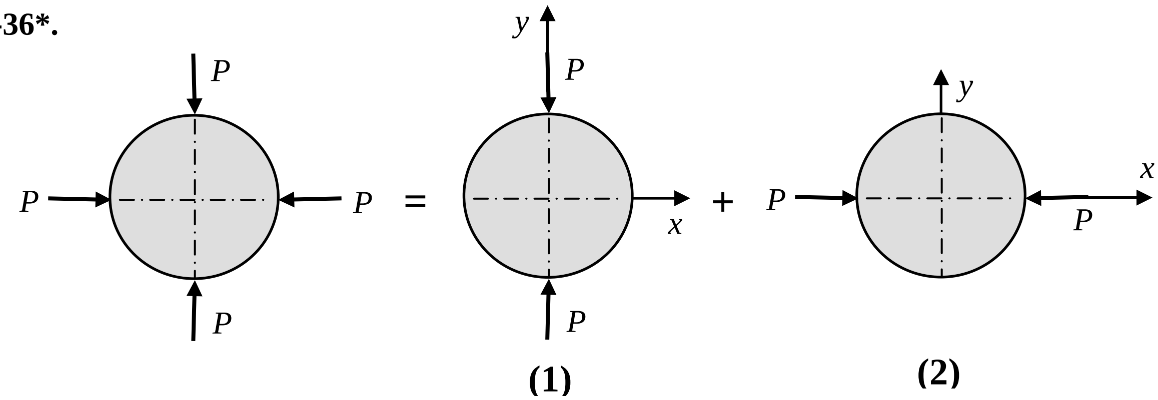
$$\sigma_x(x,0) = \frac{2P}{\pi D}\left[\frac{D^2 - 4x^2}{D^2 + 4x^2}\right]^2 \Rightarrow \sigma_x(x,0)/(P/D) = \frac{2}{\pi}\left[\frac{1-(x/R)^2}{1+(x/R)^2}\right]^2$$

$$\sigma_y(x,0) = -\frac{2P}{\pi D}\left[\frac{4D^4}{(D^2 + 4x^2)^2} - 1\right] \Rightarrow \sigma_y(x,0)/(P/D) = -\frac{2}{\pi}\left[\frac{4}{(1+(x/R)^2)^2} - 1\right]$$

MATLAB Plots :



8-36*.



Use superposition of the two problems as shown using simple interchange of x and y

$$\sigma_x^{(1)} = -\frac{2P}{\pi}\left[\frac{(R-y)x^2}{r_1^4} + \frac{(R+y)x^2}{r_2^4} - \frac{1}{D}\right]$$

$$\sigma_y^{(1)} = -\frac{2P}{\pi}\left[\frac{(R-y)^3}{r_1^4} + \frac{(R+y)^3}{r_2^4} - \frac{1}{D}\right]$$

$$\tau_{xy}^{(1)} = \frac{2P}{\pi}\left[\frac{(R-y)^2 x}{r_1^4} - \frac{(R+y)^2 x}{r_2^4}\right],\ r_{1,2} = \sqrt{x^2 + (R \mp y)^2}$$

$$\sigma_x^{(2)} = -\frac{2P}{\pi}\left[\frac{(R-x)^3}{r_1'^4} + \frac{(R+x)^3}{r_2'^4} - \frac{1}{D}\right]$$

$$\sigma_y^{(2)} = -\frac{2P}{\pi}\left[\frac{(R-x)y^2}{r_1'^4} + \frac{(R+x)y^2}{r_2'^4} - \frac{1}{D}\right]$$

$$\tau_{xy}^{(2)} = \frac{2P}{\pi}\left[\frac{(R-x)^2 y}{r_1'^4} - \frac{(R+x)^2 y}{r_2'^4}\right],\ r'_{1,2} = \sqrt{y^2 + (R \mp x)^2}$$

$$\sigma_x = \sigma_x^{(1)} + \sigma_x^{(2)} = -\frac{2P}{\pi}\left[\frac{(R-y)x^2}{r_1^4} + \frac{(R+y)x^2}{r_2^4} + \frac{(R-x)^3}{r_1'^4} + \frac{(R+x)^3}{r_2'^4} - \frac{2}{D}\right]$$

$$\sigma_y = \sigma_y^{(1)} + \sigma_y^{(2)} = -\frac{2P}{\pi}\left[\frac{(R-y)^3}{r_1^4} + \frac{(R+y)^3}{r_2^4} + \frac{(R-x)y^2}{r_1'^4} + \frac{(R+x)y^2}{r_2'^4} - \frac{2}{D}\right]$$

$$\tau_{xy} = \tau_{xy}^{(1)} + \tau_{xy}^{(2)} = \frac{2P}{\pi}\left[\frac{(R-y)^2 x}{r_1^4} - \frac{(R+y)^2 x}{r_2^4} + \frac{(R-x)^2 y}{r_1'^4} - \frac{(R+x)^2 y}{r_2'^4}\right]$$

On y-axis $(x = 0)$

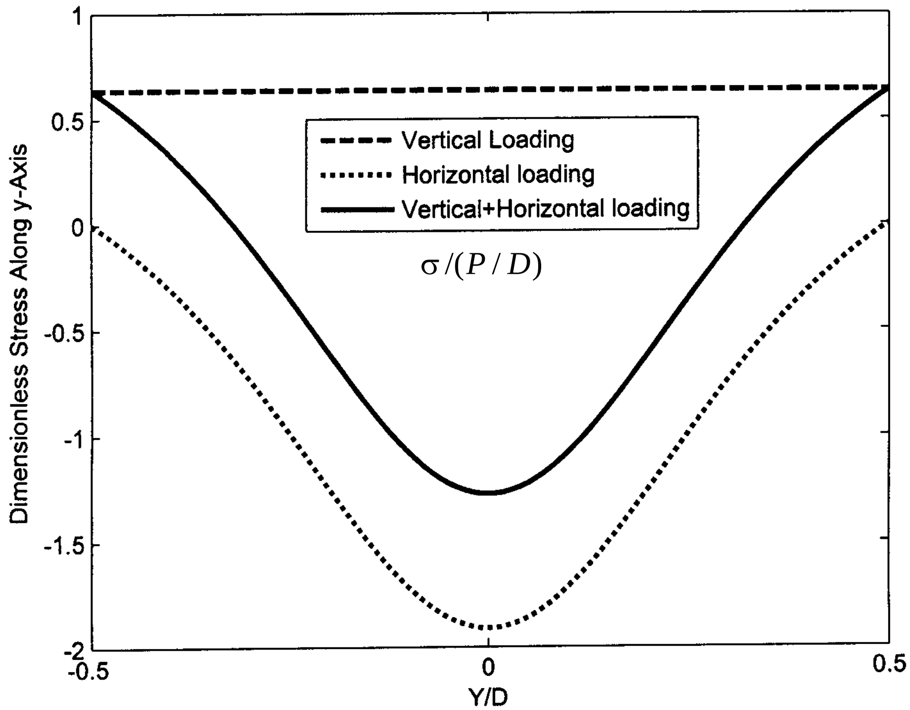
$$\sigma_x = -\frac{4P}{\pi}\left[\frac{R^3}{(y^2+R^2)^2} - \frac{1}{D}\right],\ \sigma_y = -\frac{4P}{\pi}\left[\frac{Ry^2}{(y^2+R^2)^2} + \frac{R}{R^2+y^2} - \frac{1}{D}\right],\ \tau_{xy} = 0$$

Corresponding results from Example 8-10

$$\sigma_x(0,y) = \frac{2P}{\pi D},\ \sigma_y(0,y) = -\frac{2P}{\pi}\left[\frac{2}{D-2y} + \frac{2}{D+2y} - \frac{1}{D}\right],\ \tau_{xy}(0,y) = 0$$

8-36*. Continued

MATLAB Comparison Plot:



8-37.

$$\sigma_r = -\frac{(3+\nu)}{8}\rho\omega^2 r^2 + \frac{C_1}{2} + \frac{C_2}{r^2}$$

$$\sigma_\theta = -\frac{1+3\nu}{8}\rho\omega^2 r^2 + \frac{C_1}{2} - \frac{C_2}{r^2}$$

$$\sigma_r(a) = 0 \Rightarrow -\frac{(3+\nu)}{8}\rho\omega^2 a^2 + \frac{C_1}{2} + \frac{C_2}{a^2} = 0$$

$$\sigma_r(b) = 0 \Rightarrow -\frac{(3+\nu)}{8}\rho\omega^2 b^2 + \frac{C_1}{2} + \frac{C_2}{b^2} = 0$$

Solving the boundary condition relations for C_1 and C_2 gives

$$C_1 = \frac{(3+\nu)}{4}\rho\omega^2(a^2+b^2)\ ,\ C_2 = -\frac{(3+\nu)}{8}\rho\omega^2 a^2 b^2 \Rightarrow$$

$$\sigma_\theta = -\frac{1+3\nu}{8}\rho\omega^2 r^2 + \frac{(3+\nu)}{8}\rho\omega^2(a^2+b^2) + \frac{(3+\nu)}{8r^2}\rho\omega^2 a^2 b^2$$

$$(\sigma_\theta)_{\max} = \sigma_\theta(a) = -\frac{1+3\nu}{8}\rho\omega^2 a^2 + \frac{(3+\nu)}{8}\rho\omega^2(a^2+b^2) + \frac{(3+\nu)}{8}\rho\omega^2 b^2$$

when $b >> a$, $(\sigma_\theta)_{\max} \approx \frac{(3+\nu)}{8}\rho\omega^2 b^2$

From Example 8 - 11

$$\sigma_{\max} = \sigma_r(0) = \sigma_\theta(0) = \frac{3+\nu}{8}\rho\omega^2 b^2\ ,\ (b >> a)$$

$\therefore$ the maximum stress in the annular disk is twice that of the solid disk

9-1.

$\sigma_x = \sigma_y = \tau_{xy} = 0$

Equilibrium Equations $(3.6.5)_{1,2}$ (no body forces) $\Rightarrow$

$$\frac{\partial \tau_{xz}}{\partial z} = 0\ ,\ \frac{\partial \tau_{yz}}{\partial z} = 0$$

Beltrami - Michell Compatibility Relations $(5.3.4)_{1,2,3,4}$ $\Rightarrow$

$$\frac{\partial^2 \sigma_z}{\partial x^2} = \frac{\partial^2 \sigma_z}{\partial y^2} = \frac{\partial^2 \sigma_z}{\partial z^2} = \frac{\partial^2 \sigma_z}{\partial x \partial y} = 0$$

Integrating the compatibility results,

$$\frac{\partial^2 \sigma_z}{\partial x^2} = 0 \Rightarrow \frac{\partial \sigma_z}{\partial x} = f(y,z)\ ,\ \frac{\partial^2 \sigma_z}{\partial x \partial y} = 0 \Rightarrow \frac{\partial \sigma_z}{\partial x} = g(x,z)$$

$$\therefore\ f(y,z) = g(x,z) = F(z) \Rightarrow \frac{\partial \sigma_z}{\partial x} = F(z) \Rightarrow \sigma_z = xF(z) + \hat{f}(y,z)$$

In similar fashion, it can be shown that $\dfrac{\partial \sigma_z}{\partial y} = G(z) \Rightarrow \sigma_z = yG(z) + \hat{g}(x,z)$

$$\frac{\partial^2 \sigma_z}{\partial z^2} = 0 \Rightarrow \frac{\partial^3 \sigma_z}{\partial x \partial z^2} = 0 \Rightarrow F''(z) = 0 \Rightarrow F(z) = A_1 z + B_1$$

Likewise we can show that $G''(z) = 0 \Rightarrow G(z) = A_2 z + B_2$

$$\frac{\partial \sigma_z}{\partial x} = F(z) = A_1 z + B_1 \Rightarrow \sigma_z = A_1 zx + B_1 x + \hat{f}(y,z)$$

$$\frac{\partial \sigma_z}{\partial y} = G(z) = A_2 z + B_2 \Rightarrow \frac{\partial \hat{f}(y,z)}{\partial y} = A_2 z + B_2 \Rightarrow \hat{f}(y,z) = A_2 zy + B_2 y + \hat{F}(z)$$

$$\frac{\partial^2 \sigma_z}{\partial z^2} = 0 \Rightarrow \hat{F}''(z) = 0 \Rightarrow \hat{F}(z) = Cz + D$$

Combining these results yields the general form

$$\sigma_z = C_1 x + C_2 y + C_3 z + C_4 xz + C_5 yz + C_6$$

9-2.

With no warping displacement, $u = -\alpha yz$, $v = \alpha xz$, $w = 0$

$$e_x = e_y = e_z = e_{xy} = 0 \text{ , } e_{xz} = -\frac{\alpha y}{2} \text{ , } e_{yz} = \frac{\alpha x}{2}$$

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \text{ , } \tau_{xz} = -\mu\alpha y \text{ , } \tau_{yz} = \mu\alpha x$$

By inspection the equilibrium and compatibility relations are all satisfied

Boundary conditions on lateral surface :

$$T_x^n = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z = 0 \Rightarrow 0 = 0$$

$$T_y^n = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z = 0 \Rightarrow 0 = 0$$

$$T_z^n = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = 0 \Rightarrow -\mu\alpha y n_x + \mu\alpha x n_y = -\mu\alpha y \frac{dy}{ds} - \mu\alpha x \frac{dx}{ds}$$

$$= -\frac{\mu\alpha}{2}\frac{d}{ds}(x^2 + y^2) \neq 0 \text{ in general (vanishes only for circular case)}$$

Thus assumed displacement field does not satisfy all boundary conditions on lateral side

Finally check boundary conditions on ends; from relations (9.3.14)

$$P_x = \iint_R T_x^n dxdy = -\mu\alpha \iint_R ydxdy = 0 \text{ (using centroidal axes)}$$

$$P_y = \iint_R T_y^n dxdy = \mu\alpha \iint_R xdxdy = 0 \text{ (using centroidal axes)}$$

$$P_z = \iint_R T_z^n dxdy = 0 \text{ (since } T_z^n = 0)$$

$$M_x = \iint_R yT_z^n dxdy = 0 \text{ (since } T_z^n = 0)$$

$$M_y = \iint_R xT_z^n dxdy = 0 \text{ (since } T_z^n = 0)$$

$$M_z = \iint_R (xT_y^n - yT_x^n) dxdy = \mu\alpha \iint_R (x^2 + y^2) dxdy = T \Rightarrow$$

$$T = \mu\alpha J_p \text{ , where } J_p = \iint_R r^2 dA = \text{polar moment of inertia of cross-section}$$

9-3.

Displacement Field : $u = -\alpha z(y-b)$, $v = \alpha z(x-a)$, $w = w(x,y)$

Strain Field : $e_x = e_y = e_z = e_{xy} = 0$, $e_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \alpha(y-b)\right)$, $e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \alpha(x-a)\right)$

Stress Field : $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$, $\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} - \alpha(y-b)\right)$, $\tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \alpha(x-a)\right)$

Equilibrium Equations :

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

Compatibility Equation with usual Prandtl Stress Function :

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -2\mu\alpha \Rightarrow \nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\mu\alpha$$

Boundary Conditions on lateral surface :

$$T_x^n \equiv 0 \ , \ T_y^n \equiv 0 , \ T_z^n = \tau_{xz} n_x + \tau_{yz} n_y = 0 \Rightarrow \frac{d\phi}{ds} = 0 \Rightarrow \phi = \text{constant on boundary}$$

For boundary conditions on ends of cylinder, introduce new coordinates :

$$x' = x - a \ , \ y' = y - b \Rightarrow dx = dx' \ , \ dy = dy'$$

$$P_x = \iint_R T_x^n dxdy = \iint_{R'} T_x^{n'} dx'dy' = 0, \text{ and likewise for } P_y \text{ and } P_z$$

$$M_x = \iint_R (y-b) T_z^n dxdy = \iint_{R'} y' T_z^{n'} dx'dy' = 0 \ , \text{ likewise for } M_y$$

$$M_z = \iint_R \left((x-a)T_y^n - (y-b)T_x^n\right) dxdy = \iint_{R'} \left(x' T_y^{n'} - y' T_x^{n'}\right) dx'dy' = 2\iint_{R'} \phi' \, dx'dy' = T$$

Boundary conditions for displacement formulation

$$\tau_{xz} n_x + \tau_{yz} n_y = \left(\frac{\partial w}{\partial x} - (y-b)\alpha\right) n_x + \left(\frac{\partial w}{\partial y} + (x-a)\alpha\right) n_y = 0 \Rightarrow$$

$$\frac{dw}{dn} = \frac{\alpha}{2} \frac{d}{ds}\left((x-a)^2 + (y-b)^2\right)$$

$$T = \mu \iint_R \left(\alpha[(x-a)^2 + (y-b)^2] + (x-a)\frac{\partial w}{\partial y} - (y-b)\frac{\partial w}{\partial x}\right) dxdy$$

$$= \mu \iint_{R'} \left(\alpha(x'^2 + y'^2) + x'\frac{\partial w}{\partial y'} - y'\frac{\partial w}{\partial x'}\right) dx'dy'$$

So the new coordinate system will not change the general formulation.

9-4.

$$\frac{\partial\psi}{\partial x} = -\frac{1}{\alpha}\frac{\partial w}{\partial y}\ ,\ \frac{\partial\psi}{\partial y} = \frac{1}{\alpha}\frac{\partial w}{\partial x}$$

$$\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} - \alpha y\right) \Rightarrow \tau_{xz} = \mu\alpha\left(\frac{\partial\psi}{\partial y} - y\right)$$

$$\tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \alpha x\right) \Rightarrow \tau_{yz} = \mu\alpha\left(-\frac{\partial\psi}{\partial x} + x\right)$$

Equilibrium equations are satisfied identically, and compatibility equations give

$$\mu\alpha\left(\frac{\partial^2\psi}{\partial y^2} - 1\right) - \mu\alpha\left(\frac{\partial^2\psi}{\partial x^2} + 1\right) = -2\mu\alpha \Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \text{ in } R$$

Boundary condtions :

$$\tau_{xz}n_x + \tau_{yz}n_y = 0 \Rightarrow \mu\alpha\left(\frac{\partial\psi}{\partial y} - y\right)\frac{dy}{ds} - \mu\alpha\left(-\frac{\partial\psi}{\partial x} + x\right)\frac{dx}{ds} = 0 \Rightarrow$$

$$\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} = x\frac{dx}{ds} + y\frac{dy}{ds} = \frac{d}{ds}\left(\frac{1}{2}(x^2 + y^2)\right) \Rightarrow \psi = \frac{1}{2}(x^2 + y^2) \text{ on } S$$

9-5.

Resultant shear stress : $\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$

$$\tau^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = (\nabla\phi)^2 = \phi_{,i}\phi_{,i}\ ,\ i = 1,2$$

$$\nabla^2\tau^2 = \nabla^2\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = (\phi_{,i}\phi_{,i})_{,jj} = (2\phi_{,i}\phi_{,ij})_{,j} = 2(\phi_{,i}\phi_{,ijj} + \phi_{,ij}\phi_{,ij})$$

but $\phi_{,ijj} = (\phi_{,jj})_{,i} = (-2\mu\alpha)_{,i} = 0 \Rightarrow$

$$\nabla^2\tau^2 = 2\phi_{,ij}\phi_{,ij} = 2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right] \geq 0$$

$\therefore\ \tau^2$ is subharmonic $\Rightarrow$ maximum shear stress will always occur on section boundary.

9-6.

Polar Coordinate Formulation : $u_r = u_z = 0$, $u_\theta = \alpha rz \Rightarrow$

$e_r = e_\theta = e_z = e_{rz} = e_{r\theta} = 0$, $e_{\theta z} = \frac{1}{2}\alpha r \Rightarrow$

$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = \tau_{r\theta} = 0$, $\tau_{\theta z} = \mu\alpha r \Rightarrow$ Equilibrium Equations are Satisfied

Using Stress Function Approach, Governing Equation : $\nabla^2\phi(r) = -2\mu\alpha \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -2\mu\alpha$

Integrating $\Rightarrow$ $\phi = -\frac{1}{2}\mu\alpha r^2 + C_1 \log r + C_2$

For Bounded Solution at $r = 0 \Rightarrow C_1 = 0$, With $\phi(a) = 0 \Rightarrow C_2 = \frac{1}{2}\mu\alpha a^2$

$\therefore \phi = -\frac{\mu\alpha}{2}(r^2 - a^2)$

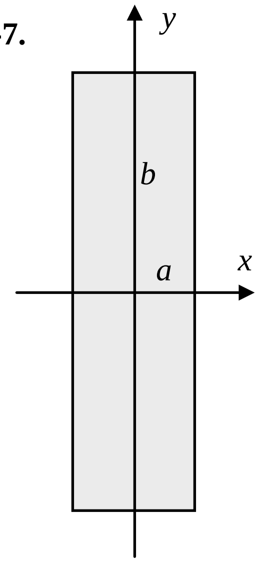
From Membrane Analogy : $\tau_{\theta z} = -\frac{d\phi}{dr} = \mu\alpha r$

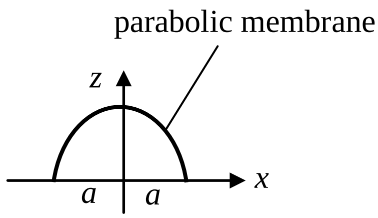
Check with Previous Solution of Ellipitical Section Case with $a = b$

Equation (9.4.3) $\Rightarrow$ $K = -\mu\alpha\frac{a^4}{2a^2} = -\frac{\mu\alpha}{2}a^2$

Equation (9.4.2) $\Rightarrow$ $\phi = -\frac{\mu\alpha}{2}a^2\left(\frac{r^2}{a^2} - 1\right) = -\frac{\mu\alpha}{2}(r^2 - a^2)$, $\therefore$ Checks

9-7.





$$\nabla^2 z = -2\mu\alpha \Rightarrow \frac{d^2 z}{dx^2} = -2\mu\alpha \Rightarrow z = -\mu\alpha x^2 + C_1 x + C_2$$

Boundary conditions:

$$z(a) = 0 \Rightarrow -\mu\alpha a^2 + C_1 a + C_2 = 0 \ , \ z(-a) = 0 \Rightarrow -\mu\alpha a^2 - C_1 a + C_2 = 0$$

Solving for the constants $\Rightarrow C_1 = 0 \ , \ C_2 = \mu\alpha a^2$

$$\therefore \ z = \mu\alpha(a^2 - x^2)$$

Stress: $\tau_{yz} = -\dfrac{dz}{dx} = -2\mu\alpha x \Rightarrow \tau_{\max} = 2\mu\alpha a$

Load carrying capacity: $T = 2V = 2\iint_R z\,dx\,dy = 4\mu\alpha b\int_{-a}^{a}(a^2 - x^2)dx = \dfrac{16}{3}\mu\alpha a^3 b$

9-8.

Stress field for torsion of ellipse section :

$$\tau_{xz} = -\frac{2a^2\mu\alpha}{a^2+b^2}y = -\frac{2Ty}{\pi ab^3}\ ,\ \tau_{yz} = \frac{2b^2\mu\alpha}{a^2+b^2}x = \frac{2Tx}{\pi ba^3}$$

Resulting strain field :

$$\gamma_{xz} = -\frac{2Ty}{\pi\mu ab^3}\ ,\ \gamma_{yz} = \frac{2Tx}{\pi\mu ba^3}$$

Strain - Displacement relations :

$$\frac{\partial w}{\partial x} = \gamma_{xz} + \alpha y = -\frac{2Ty}{\pi\mu ab^3} + \alpha y = -\frac{2Ty}{\pi\mu ab^3} + \frac{T(a^2+b^2)}{\pi a^3 b^3 \mu}y = \frac{Ty}{\pi\mu a^3 b^3}(b^2-a^2) \Rightarrow$$

$$w = \frac{Txy}{\pi\mu a^3 b^3}(b^2-a^2) + f(y)$$

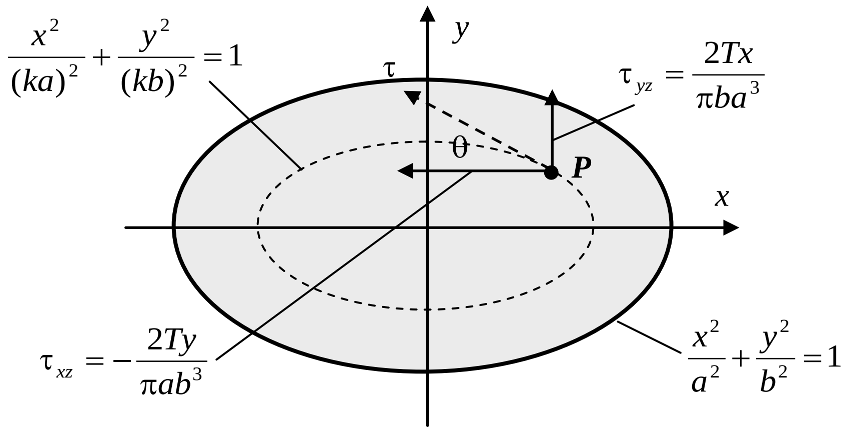
$$\frac{\partial w}{\partial y} = \gamma_{yz} - \alpha x = \frac{2Tx}{\pi\mu ba^3} - \alpha x = \frac{Tx}{\pi\mu a^3 b^3}(b^2-a^2) \Rightarrow$$

$$\frac{Tx}{\pi\mu a^3 b^3}(b^2-a^2) = \frac{Tx}{\pi\mu a^3 b^3}(b^2-a^2) + f'(y) \Rightarrow f'(y) = 0 \Rightarrow f = \text{constant} = C$$

$$w = \frac{Txy}{\pi\mu a^3 b^3}(b^2-a^2) + C\ ,\ \text{but}\ w(0,0) = 0 \Rightarrow C = 0$$

$$\therefore w = \frac{T(b^2-a^2)}{\pi\mu a^3 b^3}xy$$

9-9.



At a generic point P in the cross - section : $\tan\theta = \frac{\tau_{yz}}{\tau_{xz}} = -\frac{\frac{2Tx}{\pi ba^3}}{\frac{2Ty}{\pi ab^3}} = -\frac{b^2}{a^2}\frac{x}{y}$

Ellipse with same ratio of major to minor axes : $\frac{x^2}{(ka)^2} + \frac{y^2}{(kb)^2} = 1 \Rightarrow y = kb\sqrt{1 - \frac{x^2}{(ka)^2}}$

Tangent to ellipse : $\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$

$\therefore$ Tangent slope $= \tan\theta$

9-10.

For equilateral triangular section,

$\phi = K(x - \sqrt{3}y + 2a)(x + \sqrt{3}y + 2a)(x - a)\ ,\ K = -\frac{\mu\alpha}{6a}$

$T = 2\iint_R \phi dxdy = T = 2\int_{-2a}^{a}\int_{-\sqrt{3}(x+2a)/3}^{\sqrt{3}(x+2a)/3} \phi dydx$

$= 2\int_{-2a}^{a}\left(-\frac{2\sqrt{3}\mu\alpha(x-a)(x+2a)^3}{27a}\right)dx = \frac{9\sqrt{3}}{5}\mu\alpha a^4 = \frac{27}{5\sqrt{3}}\mu\alpha a^4$

$= \frac{3}{5}\mu\alpha I_p$, where $I_p = 3\sqrt{3}a^4$

9-11.

For torsion of elliptical section : $T = \frac{\pi a^3 b^3 \mu\alpha}{a^2 + b^2}$

Polar moment of inertia of elliptical section :

$$I_p = I_x + I_y = \frac{\pi}{4} ab(a^2 + b^2)$$

$$T = \frac{\pi a^3 b^3 \mu\alpha}{a^2 + b^2} = \frac{\pi}{4} ab(a^2 + b^2) \frac{4a^2 b^2 \mu\alpha}{(a^2 + b^2)^2} = \frac{4a^2 b^2 \mu\alpha}{(a^2 + b^2)^2} I_p$$

For the circular section case $(a = b)$: $T = \mu\alpha I_p$

From Exercise 9 - 9, the torsion of equilateral triangular section : $T = \frac{3}{5}\mu\alpha I_p$

9-12*.

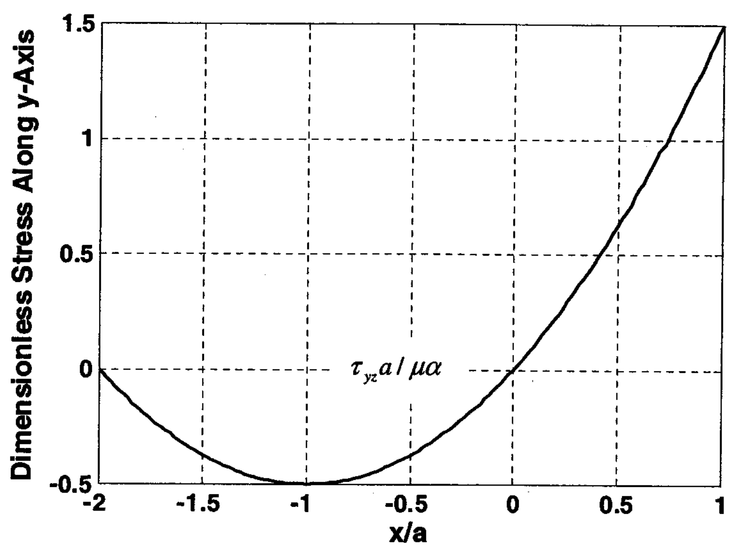
For torsion of equilateral triangular section :

$$\tau_{xz} = \frac{\mu\alpha}{a}(x - a)y \ , \ \tau_{yz} = \frac{\mu\alpha}{2a}(x^2 + 2ax - y^2)$$

Along $y = 0$: $\tau_{xz} = 0$, $\tau_{yz} = \frac{\mu\alpha}{2a} x(x + 2a) = \tau$

$$\tau_{\max} = \tau(x = a) = \frac{3\mu\alpha a}{2} \ , \ \tau_{\min} = \tau(x = -a) = -\frac{\mu\alpha a}{2} \ , \ \tau = 0 \text{ at } x = 0 \text{ and } -2a$$

MATLAB Plot :

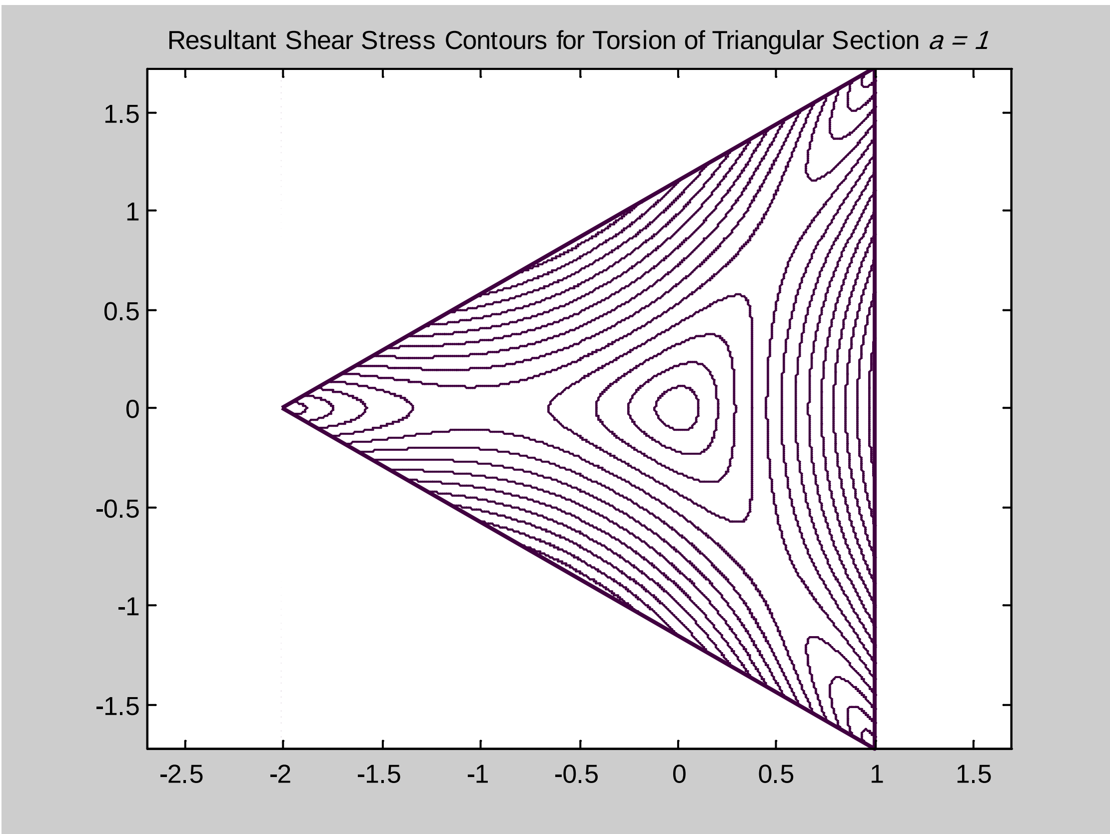


9-13*.

$$\tau_{xz} = \frac{\mu\alpha}{a}(x-a)y \ , \ \tau_{yz} = \frac{\mu\alpha}{2a}(x^2 + 2ax - y^2) \ , \ \tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$$

Use MATLAB to Calculate and Plot Contours of Resultant Shear Stress τ:

Note the Highest Stresses Occur on Boundary Where Contours Are Closest Together



9-14.

Try $\phi = K(x-a)(y - m_1 x)(y + m_2 x)$

Clearly this stress function satisfies the zero boundary conditions on each side

$\nabla^2\phi = K[(2 - 6m_1 m_2)x - 2(m_1 - m_2)y + 2(m_1 m_2 - 1)a]$

For $\nabla^2\phi = -2\mu\alpha \Rightarrow m_1 = m_2 \ , \ m_1 m_2 = \dfrac{1}{3} \Rightarrow m_1 = m_2 = \dfrac{1}{\sqrt{3}}$ and $K = \dfrac{3\mu\alpha}{2a}$

Therefore in order to satisfy the differential equation, $m_1 = m_2 = \dfrac{1}{\sqrt{3}} \Rightarrow$

that the section boundary must be an equilateral triangle.

9-15.

From Example 9 - 3, $\phi = K(a^2 - x^2 + cy^2)(a^2 + cx^2 - y^2)$

Differential Equation : $\nabla^2\phi = -2\mu\alpha \Rightarrow$

$K(-4a^2 + 4ca^2 - 12cx^2 + 2c^2x^2 + 2x^2 + 2y^2 + 2c^2y^2 - 12cy^2) = -2\mu\alpha \Rightarrow$

$K\left(-4a^2 + 4ca^2 + (-12c + 2c^2 + 2)x^2 + (2 + 2c^2 - 12c)y^2\right) = -2\mu\alpha$

Relation must be true for all x and y : $\Rightarrow (2 + 2c^2 - 12c) = 0 \Rightarrow c = 3 - \sqrt{8} = 3 - 2\sqrt{2}$

and thus $K = \dfrac{-2\mu\alpha}{4a^2(c-1)} = -\dfrac{\mu\alpha}{4a^2(1-\sqrt{2})}$

Shear Stresses :

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -\frac{(1-\sqrt{2})\mu\alpha y}{2a^2}\left(a^2 + (3-2\sqrt{2})x^2 - y^2\right) - \frac{(1+\sqrt{2})\mu\alpha y}{2a^2}\left(a^2 + (3-2\sqrt{2})y^2 - x^2\right)$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = \frac{(1+\sqrt{2})\mu\alpha x}{2a^2}\left(a^2 + (3-2\sqrt{2})x^2 - y^2\right) + \frac{(1-\sqrt{2})\mu\alpha x}{2a^2}\left(a^2 + (3-2\sqrt{2})y^2 - x^2\right)$$

On boundaries $x = \pm\sqrt{(a^2 + cy^2)}$:

$$\tau_{xz} = -\frac{(1-\sqrt{2})\mu\alpha y}{2a^2}\left(a^2 + (3-2\sqrt{2})x^2 - y^2\right),\ \tau_{yz} = \frac{(1+\sqrt{2})\mu\alpha x}{2a^2}\left(a^2 + (3-2\sqrt{2})x^2 - y^2\right)$$

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \frac{(2-\sqrt{2})a^2 + (8-6\sqrt{2})y^2}{a^2}\mu\alpha\sqrt{(3+2\sqrt{2})a^2 + (4-2\sqrt{2})y^2}$$

Maximum stress occurs at $y = 0$: $\tau_{\max} = \sqrt{2}\mu\alpha a$

A similar analysis on boundaries $y = \pm\sqrt{(a^2 + cx^2)} \Rightarrow \tau_{\max} = \sqrt{2}\mu\alpha a$ at $x = 0$

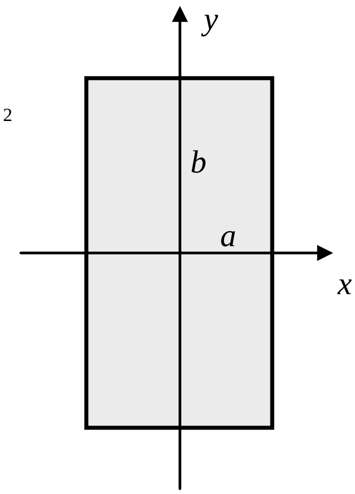
9-16.

Try boundary product form : $\phi = (x-a)(x+a)(y-b)(y+b)$

$\nabla^2\phi = \nabla^2(x^2y^2 - x^2b^2 - y^2a^2 + a^2b^2) = 2y^2 - 2b^2 + 2x^2 - 2a^2$

Clearly $\nabla^2\phi$ cannot be made to equal $-2\mu\alpha$ for all x and y

$\therefore$ the boundary equation scheme will not work for this shape.



9-17.

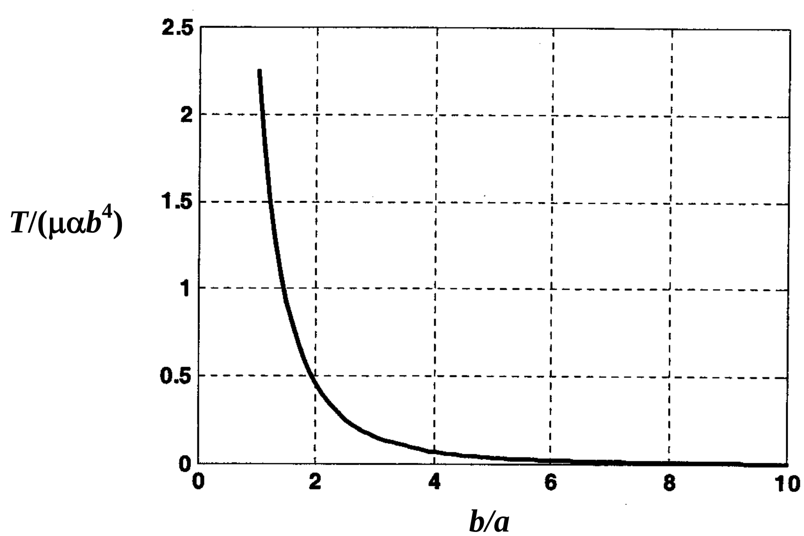
From relation (9.5.12) : $T = \dfrac{16\mu\alpha a^3 b}{3} - \dfrac{1024\mu\alpha a^4}{\pi^5}\displaystyle\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}$

Non - dimensional form : $\dfrac{T}{\mu\alpha b^4} = \dfrac{16}{3}\left(\dfrac{a}{b}\right)^3 - \dfrac{1024}{\pi^5}\left(\dfrac{a}{b}\right)^4\displaystyle\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}$

For $\dfrac{\mathrm{b}}{\mathrm{a}} \to 10 \Rightarrow \left(\dfrac{a}{b}\right)^3 \to 10^{-3}$, $\left(\dfrac{a}{b}\right)^4 \to 10^{-4}$, $\tanh\dfrac{n\pi b}{2a} \to 1 \Rightarrow$

$\dfrac{T}{\mu\alpha b^4} \approx \dfrac{16}{3}\left(\dfrac{a}{b}\right)^3$

MATLAB Plot :



9-18.

$$
\begin{aligned}
T &= \frac{16}{3}\mu\alpha\sum_{i=1}^{N} a_i^3 b_i \\
&= \frac{16}{3}\mu\alpha\left(\left(\frac{t}{2}\right)^3\left(\frac{a}{2}\right)+2\left(\frac{t}{2}\right)^3\left(\frac{b}{2}\right)\right) \\
&= \frac{1}{3}(a+2b)t^3\mu\alpha
\end{aligned}
$$

9-19.

$$\phi = K(b^2-r^2)(1-\frac{2a\cos\theta}{r}) = K\left(b^2-x^2-y^2+2ax-\frac{2b^2ax}{x^2+y^2}\right)$$

$$\nabla^2\phi = \phi_{,xx}+\phi_{,xx} = 2K\left[\left(-1+2b^2ax\frac{3y^2-x^2}{(x^2+y^2)^3}\right)+\left(-1+2b^2ax\frac{x^2-3y^2}{(x^2+y^2)^3}\right)\right] = -4K$$

$\therefore\ \nabla^2\phi = -2\mu\alpha \Rightarrow K = \dfrac{\mu\alpha}{2}$ and the governing differential equation is satisfied

Clearly on boundary $r = b$ and $r = 2a\cos\theta$, the stress function will vanish as required

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = \mu\alpha\left(-1+\frac{2b^2ax}{(x^2+y^2)^2}\right)y = \mu\alpha\left(-1+\frac{2b^2a\cos\theta}{r^3}\right)r\sin\theta$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = -\mu\alpha\left(a-x-b^2a\frac{y^2-x^2}{(x^2+y^2)^2}\right) = -\mu\alpha\left(a-r\cos\theta-b^2a\frac{\sin^2\theta-\cos^2\theta}{r^2}\right)$$

9-20*.

From Exercise 9 - 17, the stress were found to be

$$\tau_{xz} = \mu\alpha\left(-1+\frac{2b^2ax}{(x^2+y^2)^2}\right)y = \mu\alpha\left(-1+\frac{2b^2a\cos\theta}{r^3}\right)r\sin\theta$$

$$\tau_{yz} = -\mu\alpha\left(a-x-b^2a\frac{y^2-x^2}{(x^2+y^2)^2}\right) = -\mu\alpha\left(a-r\cos\theta-b^2a\frac{\sin^2\theta-\cos^2\theta}{r^2}\right)$$

On keyway boundary $r = b$:

$$\tau_{xz} = \mu\alpha(-b+2a\cos\theta)\sin\theta\ ,\ \tau_{yz} = -\mu\alpha(-b+2a\cos\theta)\cos\theta \ \Rightarrow$$

$$\tau_{keyway} = \sqrt{\tau_{xz}^2+\tau_{yz}^2} = \mu\alpha(2a\cos\theta - b)$$

Maximum value occurs at $\theta = 0 \ \Rightarrow (\tau_{\max})_{keyway} = \mu\alpha(2a-b)$

Maximum value for the case $\frac{b}{a}\to 0 \ \Rightarrow (\tau_{\max})_{keyway} = \mu\alpha a\left(2-\frac{b}{a}\right) \approx 2\mu\alpha a$

On shaft boundary $r = 2a\cos\theta$:

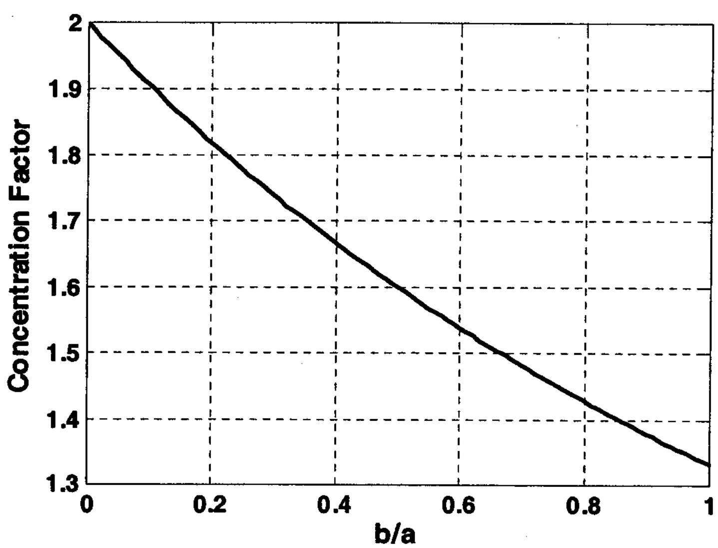
$$\tau_{xz} = \frac{\mu\alpha}{4a}(b^2-4a^2\cos^2\theta)\frac{\sin 2\theta}{\cos^2\theta}\ ,\ \tau_{yz} = -\frac{\mu\alpha}{4a}(b^2-4a^2\cos^2\theta)\frac{\cos 2\theta}{\cos^2\theta} \ \Rightarrow$$

$$\tau_{shaft} = \sqrt{\tau_{xz}^2+\tau_{yz}^2} = \frac{\mu\alpha}{4a\cos^2\theta}(b^2-4a^2\cos^2\theta) = \frac{\mu\alpha}{4a}(b^2\sec^2\theta - 4a^2)$$

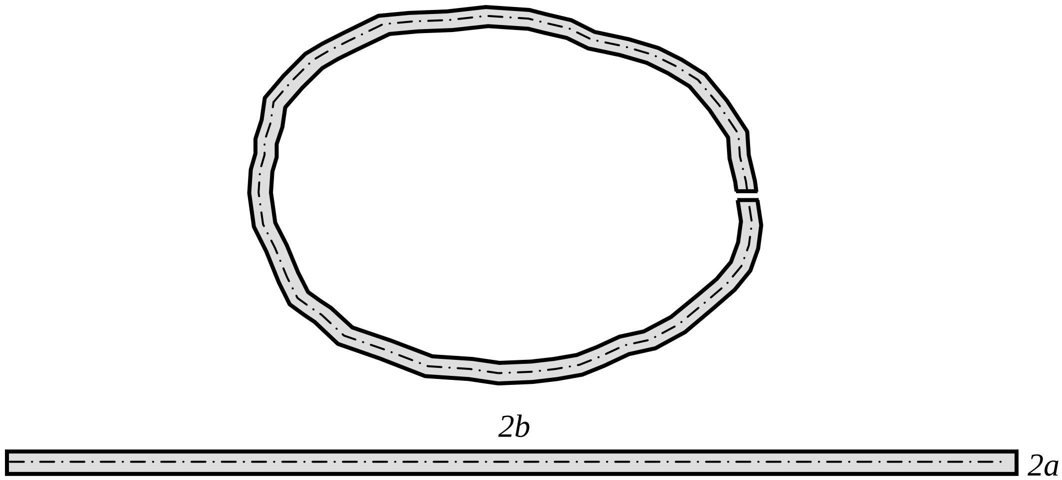
Maximum value for the case $\frac{b}{a}\to 0 \ \Rightarrow (\tau_{\max})_{shaft} \approx \mu\alpha a$

$\therefore$ For the small notch case the concentration factor is given by $\dfrac{(\tau_{\max})_{keyway}}{(\tau_{\max})_{shaft}} \approx \dfrac{2\mu\alpha a}{\mu\alpha a} \approx 2$

MATLAB Plot :



9-21.



(Unfold Open Tube Into a Rectangular Strip)

From relation (9.6.8), for the closed tube

$$\tau_{CT} = \frac{T}{2A_c t}\text{ , where } A_c = \text{area enclosed by tube centerline}$$

For the open tube, simply open the section to form a strip as shown, and then relations $(9.5.15)_{2,3}$ can be used to develop approximate equation for the stress $\Rightarrow$

$$\tau_{OT} \approx \frac{3}{8}\frac{T}{a^2 b} \approx \frac{3}{2}\frac{T}{aA_s}$$

For the same torque T

$$\frac{\tau_{OT}}{\tau_{CT}} \approx \frac{\frac{3}{2}\frac{T}{aA_s}}{\frac{T}{2A_c t}} \approx 6\frac{A_c}{A_s}\text{ , but since } A_c >> A_s \Rightarrow \frac{\tau_{OT}}{\tau_{CT}} >> 1 \Rightarrow \tau_{OT} >> \tau_{CT}$$

$\therefore$ Stresses are higher in open tube and thus closed tube is stronger

9-22.

From Example 9 - 7, the stress $\tau_{\theta z}$ was given by

$$\tau_{\theta z} = -\frac{C\mu rz}{(r^2+z^2)^{5/2}} \text{ , where } C = -\frac{T}{2\pi\mu(\frac{2}{3} - \cos\varphi + \frac{1}{3}\cos^3\varphi)}$$

The maximum stress occurs on boundary where $z = \cos\varphi\sqrt{r^2+z^2}$, $r = \sin\varphi\sqrt{r^2+z^2}$

$$\therefore \ (\tau_{\theta z})_{\max} = -C\mu\frac{\sin\varphi\cos^4\varphi}{z^3}$$

For the case $z = l$, $\varphi = 20^o \Rightarrow \dfrac{(\tau_{\theta z})_{\max}}{(T/l^3)} = 11.9$

From strength of materials theory

$$\tau_{\theta z} = \frac{Tr}{(\pi r^4/2)} = \frac{2T}{\pi r^3}$$

At $z = l \Rightarrow r = l\tan\varphi = l\tan 20^o \Rightarrow$

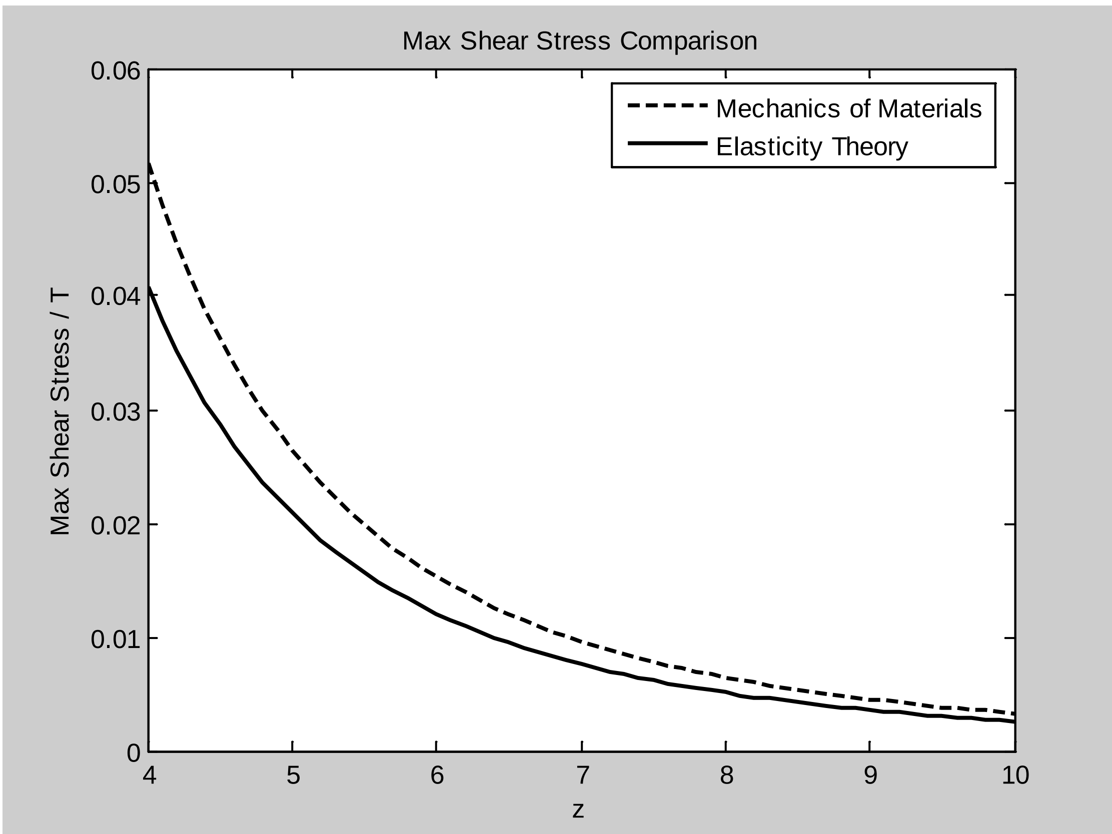
$$\tau_{\theta z} = \frac{2T}{\pi l^3}\cot^3\varphi \Rightarrow \frac{(\tau_{\theta z})_{\max}}{(T/l^3)} = 13.2$$

9-23*.

From Exercise 9 - 22: $(\tau_{\theta z})_{max} = \dfrac{T\sin\varphi\cos^4\varphi}{2\pi z^3\left(\dfrac{2}{3} - \cos\varphi + \dfrac{1}{3}\cos^3\varphi\right)}$

From Mechanics of Materials Theory: $(\tau_{\theta z})_{max} = \dfrac{Tr}{(\pi r^4/2)} = \dfrac{2T}{\pi r^3} = \dfrac{2T}{\pi z^3\tan^3\varphi}$

Using MATLAB to Calculate and Plot the Two Predicitons :



9-24.

Stress field solution from Example 9 - 8 :

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

$$\tau_{xz} = -\frac{P}{4I_x}\frac{1+2\nu}{1+\nu}xy \ , \ \tau_{yz} = \frac{P}{I_x}\frac{3+2\nu}{8(1+\nu)}\left(a^2 - y^2 - \frac{1-2\nu}{3+2\nu}x^2\right) , \ \sigma_z = -\frac{P}{I_x}y(l-z)$$

Using Hooke's law and Strain - Displacement relations

$$e_x = \frac{\partial u}{\partial x} = -\frac{\nu}{E}\sigma_z = \frac{P\nu}{EI_x}y(l-z) \Rightarrow u = \frac{P\nu}{EI_x}y(l-z)x + f(y,z)$$

$$e_y = \frac{\partial v}{\partial y} = -\frac{\nu}{E}\sigma_z = \frac{P\nu}{EI_x}y(l-z) \Rightarrow v = \frac{P\nu}{2EI_x}y^2(l-z) + g(x,z)$$

$$e_z = \frac{\partial w}{\partial z} = \frac{1}{E}\sigma_z = -\frac{P}{EI_x}y(l-z) \Rightarrow w = -\frac{P}{EI_x}y\left(lz - \frac{z^2}{2}\right) + h(x,y)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2\mu}\tau_{xy} = 0 \Rightarrow f_{,y} + g_{,x} = -\frac{P\nu}{EI_x}(l-z)x$$

$$e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2\mu}\tau_{xz} \Rightarrow f_{,z} + h_{,x} = -\frac{Pxy}{2EI_x}$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2\mu}\tau_{yz} \Rightarrow g_{,z} + h_{,y} = \frac{P(3+2\nu)}{4EI_x}\left(a^2 - y^2 - \frac{1-2\nu}{3+2\nu}x^2\right) + \frac{P\nu}{2EI_x}y^2 + \frac{P}{EI_x}\left(lz - \frac{z^2}{2}\right)$$

Differentiating the above relations $\Rightarrow$

$$f_{,yy} = 0 \ , \ f_{,zz} = 0$$

$$f_{,yz} + g_{,xz} = \frac{P\nu}{EI_x}x \ , \ f_{,zy} + h_{,xy} = -\frac{Px}{2EI_x} \ , \ g_{,zx} + h_{,yx} = -\frac{P(1-2\nu)}{2EI_x}x$$

Solving the above relations $\Rightarrow$ $f_{,yz} = 0 \ , \ g_{,xz} = \frac{P\nu}{EI_x}x \ , \ h_{,xy} = -\frac{Px}{2EI_x}$

$$f_{,yy} = f_{,zz} = f_{,yz} = 0 \Rightarrow \text{(see exercise 7 - 9)} \ \ f = c_1 y + c_2 z + c_3$$

$$\therefore f_{,y} + g_{,x} = -\frac{P\nu}{EI_x}(l-z)x \Rightarrow g_{,x} = -\frac{P\nu}{EI_x}(l-z)x - c_1 \Rightarrow g = -\frac{P\nu}{2EI_x}(l-z)x^2 - c_1 x + g_1(z)$$

$$f_{,z} + h_{,x} = -\frac{Pxy}{2EI_x} \Rightarrow h_{,x} = -\frac{Pxy}{2EI_x} - c_2 \Rightarrow h = -\frac{Px^2y}{4EI_x} - c_2 x + h_1(y)$$

$$g_{,z} + h_{,y} = \frac{P\nu x^2}{2EI_x} + g_1'(z) - \frac{Px^2}{4EI_x} + h_1'(y) = \frac{P(3+2\nu)}{4EI_x}\left(a^2 - y^2 - \frac{1-2\nu}{3+2\nu}x^2\right) + \frac{P\nu y^2}{2EI_x} + \frac{P}{EI_x}\left(lz - \frac{z^2}{2}\right) \Rightarrow$$

$$g_1'(z) + h_1'(y) = \frac{P(3+2\nu)}{4EI_x}\left(a^2 - y^2\right) + \frac{P\nu y^2}{2EI_x} + \frac{P}{EI_x}\left(lz - \frac{z^2}{2}\right) \Rightarrow$$

$$g_1'(z) - \frac{P}{EI_x}\left(lz - \frac{z^2}{2}\right) = -h_1'(y) + \frac{P(3+2\nu)}{4EI_x}\left(a^2 - y^2\right) + \frac{P\nu y^2}{2EI_x} = \text{constant} = c_4 \Rightarrow$$

9-24. Continued

$$g_1'(z)=\frac{P}{EI_x}\left(lz-\frac{z^2}{2}\right)+c_4 \Rightarrow g_1(z)=\frac{P}{EI_x}\left(\frac{l}{2}z^2-\frac{z^3}{6}\right)+c_4z+c_5$$

$$h_1'(y)=\frac{P(3+2\nu)}{4EI_x}\left(a^2-y^2\right)+\frac{P\nu}{2EI_x}y^2-c_4 \Rightarrow h_1(y)=\frac{P(3+2\nu)}{4EI_x}\left(a^2y-\frac{y^3}{3}\right)+\frac{P\nu}{6EI_x}y^3-c_4y+c_6$$

Collecting these results determines the form of the displacement field

$$u=\frac{P\nu}{EI_x}y(l-z)x+c_1y+c_2z+c_3$$

$$v=\frac{P\nu}{2EI_x}y^2(l-z)+-\frac{P\nu}{2EI_x}(l-z)x^2-c_1x+\frac{P}{EI_x}\left(\frac{l}{2}z^2-\frac{z^3}{6}\right)+c_4z+c_5$$

$$w=-\frac{P}{EI_x}y\left(lz-\frac{z^2}{2}\right)-\frac{Px^2y}{4EI_x}-c_2x+\frac{P(3+2\nu)}{4EI_x}\left(a^2y-\frac{y^3}{3}\right)+\frac{P\nu}{6EI_x}y^3-c_4y+c_6$$

Finally applying the fixity conditions:

$$u(0,0,0)=0 \Rightarrow c_3=0\ ,\ v(0,0,0)=0 \Rightarrow c_5=0\ ,\ w(0,0,0)=0 \Rightarrow c_6=0$$

$$\omega_z(0,0,0)=0 \Rightarrow \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \Rightarrow c_1=-c_1 \Rightarrow c_1=0$$

$$\omega_y(0,0,0)=0 \Rightarrow \frac{\partial w}{\partial x}=\frac{\partial u}{\partial z} \Rightarrow -c_2=c_2 \Rightarrow c_2=0$$

$$\omega_x(0,0,0)=0 \Rightarrow \frac{\partial v}{\partial z}=\frac{\partial w}{\partial y} \Rightarrow -c_4=\frac{P(3+2\nu)}{4EI_x}a^2-c_4 \Rightarrow c_4=\frac{P(3+2\nu)}{8EI_x}a^2$$

Thus the final form of the displacement field is given by

$$u=\frac{P\nu}{EI_x}y(l-z)x$$

$$v=\frac{P\nu}{2EI_x}(y^2-x^2)(l-z)+\frac{P}{EI_x}\left(\frac{l}{2}z^2-\frac{z^3}{6}\right)+\frac{P(3+2\nu)a^2}{8EI_x}z$$

$$w=-\frac{P}{EI_x}y\left(lz-\frac{z^2}{2}\right)-\frac{Px^2y}{4EI_x}+\frac{P(3+2\nu)}{4EI_x}\left(a^2y-\frac{y^3}{3}\right)+\frac{P\nu}{6EI_x}y^3-\frac{P(3+2\nu)}{8EI_x}a^2y$$

Beam Deflection Comparison with Strength of Materials Solution:

$$v_{S.O.M.}=\frac{Pz^2}{6EI}(3l-z)$$

$$v(0,0,z)=\frac{P}{EI_x}\left(\frac{l}{2}z^2-\frac{z^3}{6}\right)+\frac{P(3+2\nu)a^2}{8EI_x}z=\frac{Pz^2}{6EI}(3l-z)+\frac{P(3+2\nu)a^2}{8EI_x}z$$

For $z=$ constant, the u - displacement solution predicts a bilinear form 'xy', thus indicating that plane sections do not remain plane

9-25*.

Elasticity solution from relation (9.9.18)

$$\tau_{yz}(0,y)=\frac{P}{2I_x}(b^2-y^2)+\frac{\nu P}{6(1+\nu)I_x}\left[-a^2-\frac{12a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cosh\frac{n\pi y}{a}}{\cosh\frac{n\pi b}{a}}\right],\; I_x=\frac{1}{12}(2a)(2b)^3=\frac{4}{3}ab^3$$

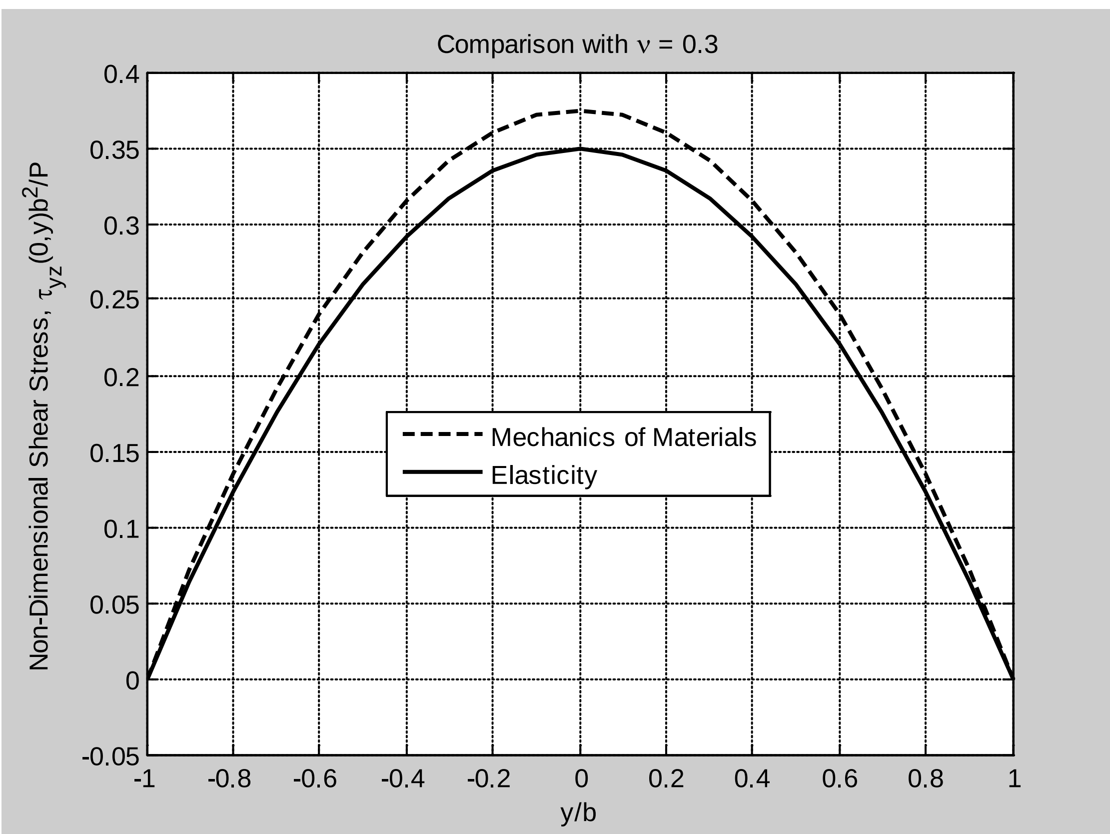
$$=\frac{3P}{8ab}\left(1-\frac{y^2}{b^2}\right)+\frac{\nu P}{8(1+\nu)ab^3}\left[-a^2-\frac{12a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cosh\frac{n\pi y}{a}}{\cosh\frac{n\pi b}{a}}\right]$$

$$\frac{\tau_{yz}(0,y)b^2}{P}=\frac{3}{8}\left(\frac{b}{a}\right)\left(1-\frac{y^2}{b^2}\right)-\frac{\nu}{8(1+\nu)}\left(\frac{a}{b}\right)\left[1+\frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cosh\frac{n\pi y}{a}}{\cosh\frac{n\pi b}{a}}\right]$$

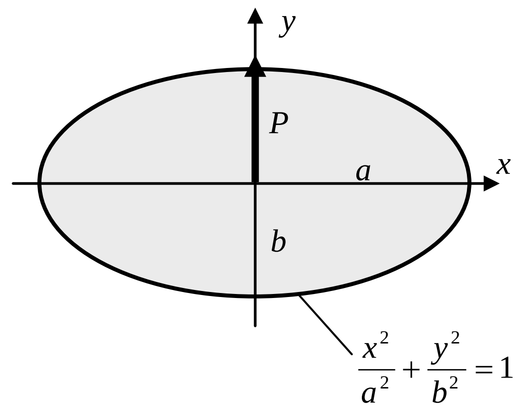
Strength of Materials Solution

$$\tau_{yz}=\frac{P}{2I_x}(b^2-y^2)=\frac{3P}{8ab}\left(1-\frac{y^2}{b^2}\right)\Rightarrow\frac{\tau_{yz}b^2}{P}=\frac{3}{8}\left(\frac{b}{a}\right)\left(1-\frac{y^2}{b^2}\right)$$

MATLAB Plot $(a=b)$ Case :



9-26.



For the elliptical section with $P_y = P$ and no torsion

$$\nabla^2\psi = -\frac{\nu}{1+\nu}\frac{P}{I_x}x \ \text{ in } R\ , \quad \frac{d\psi}{ds} = -\frac{1}{2}\frac{P}{I_x}y^2\frac{dx}{ds} \ \text{ on } S$$

Use boundary shape to formulate a simplified solution, so choose $\psi(x,y) = f(x,y) + h(x)$

$$\nabla^2\psi = -\frac{\nu}{1+\nu}\frac{P}{I_x}x \Rightarrow \nabla^2 f = -\frac{\nu}{1+\nu}\frac{P}{I_x}x - \frac{d^2h}{dx^2} \ \text{ in } R$$

$$\frac{d\psi}{ds} = -\frac{1}{2}\frac{P}{I_x}y^2\frac{dx}{ds} \Rightarrow \frac{df}{ds} = -\left(\frac{dh}{dx} + \frac{P}{2I_x}y^2\right)\frac{dx}{ds} \ \text{ on } S$$

On the boundary, $y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$, and so if we choose $\frac{dh}{dx} = -\frac{Pb^2}{2I_x}\left(1-\frac{x^2}{a^2}\right) \Rightarrow$

$\frac{df}{ds} = 0 \Rightarrow f = \text{constant} = 0$ on S, and the governing differential equation becomes

$$\nabla^2 f = -\frac{\nu}{1+\nu}\frac{P}{I_x}x - \frac{Pb^2}{I_x a^2}x = -\frac{Px}{I_x}\left(\frac{b^2}{a^2} + \frac{\nu}{1+\nu}\right) \ \text{ in } R$$

The boundary condition $f = 0$ can be satisfied if we choose f of the form

$f = Kx\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$, where K is a constant to be determined. Substituting this form into the differential equation $\Rightarrow K = -\frac{Pa^2b^2}{2I_x(3b^2+a^2)}\left(\frac{b^2}{a^2} + \frac{\nu}{1+\nu}\right) = -\frac{Pb^2}{2I_x}\frac{(1+\nu)b^2 + \nu a^2}{(3b^2+a^2)(1+\nu)}$

Thus the form for f now satisfies both the governing equation and boundary condition

$$\tau_{xz} = \frac{\partial\psi}{\partial y} = \frac{\partial f}{\partial y} = 2K\frac{xy}{b^2} = -\frac{Pa^2}{I_x(3b^2+a^2)}\left(\frac{b^2}{a^2} + \frac{\nu}{1+\nu}\right)xy = -\frac{Pxy}{I_x}\frac{(1+\nu)b^2 + \nu a^2}{(3b^2+a^2)(1+\nu)}$$

$$\tau_{yz} = -\frac{\partial\psi}{\partial x} - \frac{P}{2I_x}y^2 = -\left(\frac{\partial f}{\partial x} + \frac{dh}{dx}\right) - \frac{P}{2I_x}y^2 = -\left(K\left(\frac{3x^2}{a^2} + \frac{y^2}{b^2} - 1\right) - \frac{Pb^2}{2I_x}\left(1 - \frac{x^2}{a^2}\right)\right) - \frac{P}{2I_x}y^2$$

$$= -K\left(\frac{3x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \frac{Pb^2}{2I_x}\left(1 - \frac{x^2}{a^2}\right) - \frac{P}{2I_x}y^2 = -K\left(\frac{3x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \frac{Pb^2}{2I_x}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$= \frac{Pb^2}{2I_x}\frac{(1+\nu)b^2 + \nu a^2}{(3b^2+a^2)(1+\nu)}\left(\frac{3x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \frac{Pb^2}{2I_x}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

For the case circular case, $a = b \Rightarrow$

$$\tau_{xz} = -\frac{P}{4I_x}\frac{1+2\nu}{1+\nu}xy\ , \quad \tau_{yz} = \frac{P}{I_x}\frac{3+2\nu}{8(1+\nu)}\left[a^2 - y^2 - \frac{1-2\nu}{3+2\nu}x^2\right]$$

10-1.

$$z = x + iy\,,\ \bar{z} = z - iy \Rightarrow x = \frac{1}{2}(z + \bar{z})\,,\ y = \frac{1}{2}(z - \bar{z})$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

Solving previous lines $\Rightarrow \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)\,,\quad \dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) - \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

$$\nabla^4 = \nabla^2\nabla^2 = 16\frac{\partial^4}{\partial z^2 \partial \bar{z}^2}$$

10-2.

Integrating relation (10.2.6) $\frac{\partial^4\phi}{\partial z^2\partial\bar{z}^2}=0\Rightarrow$

$$\frac{\partial^4\phi}{\partial z^2\partial\bar{z}}=a(z)$$

$$\frac{\partial^4\phi}{\partial z\partial\bar{z}}=\int a(z)dz+b(\bar{z})=c(z)+b(\bar{z})$$

$$\frac{\partial^4\phi}{\partial z}=\bar{z}c(z)+\int b(\bar{z})d\bar{z}+d(z)=\bar{z}c(z)+e(\bar{z})+d(z)$$

$$\phi=\bar{z}\int c(z)dz+ze(\bar{z})+\int d(z)dz+f(\bar{z})$$

$$=\bar{z}g(z)+ze(\bar{z})+h(z)+f(\bar{z})$$

Now since ϕ must be a real function,

$$\phi=Re[\bar{z}g(z)+ze(\bar{z})+h(z)+f(\bar{z})]$$

$$Re[\bar{z}g(z)]=\frac{1}{2}[\bar{z}g(z)+zg(\bar{z})]$$

$$Re[ze(\bar{z})]=\frac{1}{2}[ze(\bar{z})+\bar{z}e(z)]$$

$$Re[h(z)+f(\bar{z})]=\frac{1}{2}[h(z)+h(\bar{z})+f(\bar{z})+f(z)]$$

$$\therefore\ \phi=\frac{1}{2}[\bar{z}g(z)+zg(\bar{z})+ze(\bar{z})+\bar{z}e(z)+h(z)+h(\bar{z})+f(\bar{z})+f(z)]$$

$$=\frac{1}{2}\left(z\left(g(\bar{z})+e(\bar{z})\right)+\bar{z}\left(g(z)+e(z)\right)+\left(h(z)+f(z)\right)+\left(h(\bar{z})+f(\bar{z})\right)\right)$$

Letting $\gamma(\mathrm{z})=g(z)+e(z)$, $\psi(z)=h(z)+f(z)\Rightarrow$

$$\phi=\frac{1}{2}\left(z\,\overline{\gamma(\mathrm{z})}+\bar{\mathrm{z}}\gamma(\mathrm{z})+\psi(z)+\overline{\psi(z)}\right)=Re\left(\bar{\mathrm{z}}\gamma(\mathrm{z})+\psi(z)\right)$$

10-3.

Use Navier Equations: $\mu\nabla^2\mathrm{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathrm{u}) = 0$ to construct the complex form

$$\left[\mu\nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] + i\left[\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] = 0$$

Using the complex displacement $U = u + iv$, $\overline{U} = u - iv$, and relations from Exercise 10-1,

it can be shown that $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = \frac{\partial U}{\partial z}+\overline{\frac{\partial U}{\partial z}}$, and the complex Navier equation can be written as

$$2\mu\frac{\partial^2 U}{\partial z\partial\bar{z}} + (\lambda+\mu)\frac{\partial}{\partial\bar{z}}\left(\frac{\partial U}{\partial z}+\overline{\frac{\partial U}{\partial z}}\right) = 0$$

Integrating this result $\Rightarrow$ $2\mu\frac{\partial U}{\partial z} + (\lambda+\mu)\left(\frac{\partial U}{\partial z}+\overline{\frac{\partial U}{\partial z}}\right) = f'(z)$

Next take complex conjugate $2\mu\overline{\frac{\partial U}{\partial z}} + (\lambda+\mu)\left(\frac{\partial U}{\partial z}+\overline{\frac{\partial U}{\partial z}}\right) = \overline{f'(z)}$

Eliminating the $\overline{\frac{\partial U}{\partial z}}$ term from the previous two relations $\Rightarrow$

$$4\mu(\lambda+2\mu)\frac{\partial U}{\partial z} = (\lambda+3\mu)f'(z) - (\lambda+\mu)\overline{f'(z)}$$

Integrate again $\Rightarrow 4\mu(\lambda+2\mu)U = (\lambda+3\mu)f(z) - (\lambda+\mu)z\overline{f'(z)} + g(\bar{z})$

Rearranging and redefining arbitrary functions of intergration f and g gives

$2\mu U = \kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}$, where κ is given by (10.2.10).

10-4.

$$2\mu(u+iv)=\kappa\gamma(z)-z\overline{\gamma'(z)}-\overline{\psi(z)}$$

$$2\mu\frac{\partial}{\partial\bar{z}}(u+iv)=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\left(\kappa\gamma(z)-z\overline{\gamma'(z)}-\overline{\psi(z)}\right)$$

$$=-z\gamma''(\bar{z})-\psi'(\bar{z})=-[z\overline{\gamma''(z)}-\overline{\psi'(z)}]$$

Now $\sigma_y-\sigma_x+2i\tau_{xy}=2[\bar{z}\gamma''(z)+\psi'(z)]\Rightarrow$

$$-[z\overline{\gamma''(z)}-\overline{\psi'(z)}]=-\frac{1}{2}(\sigma_y-\sigma_x-2i\tau_{xy})$$

Using Hooke's law, $\sigma_x-\sigma_y+2i\tau_{xy}=2\mu(e_x-e_y+2ie_{xy})$

$$\therefore\ 2\mu\frac{\partial}{\partial\bar{z}}(u+iv)=-[z\overline{\gamma''(z)}-\overline{\psi'(z)}]=\mu(e_x-e_y+2ie_{xy})$$

10-5.

$$\phi(z,\bar{z})=\frac{1}{2}\left(z\,\overline{\gamma(z)}+\bar{z}\gamma(z)+\chi(z)+\overline{\chi(z)}\right)=\mathrm{Re}\left(\bar{z}\gamma(z)+\chi(z)\right)$$

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2}=-\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\phi=-\left(\frac{\partial^2\phi}{\partial z^2}-2\frac{\partial^2\phi}{\partial z\partial\bar{z}}+\frac{\partial^2\phi}{\partial \bar{z}^2}\right)$$

$$\sigma_y=\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)\phi=\left(\frac{\partial^2\phi}{\partial z^2}+2\frac{\partial^2\phi}{\partial z\partial\bar{z}}+\frac{\partial^2\phi}{\partial \bar{z}^2}\right)$$

$$\therefore\ \sigma_x+\sigma_y=4\frac{\partial^2\phi}{\partial z\partial\bar{z}}=2[\gamma'(z)+\overline{\gamma'(z)}]$$

$$\sigma_y-\sigma_x=2\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial \bar{z}^2}\right)$$

$$\tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}=-i\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\phi=-i\left(\frac{\partial^2\phi}{\partial z^2}-\frac{\partial^2\phi}{\partial \bar{z}^2}\right)$$

$$\therefore\ \sigma_y-\sigma_x+2i\tau_{xy}=4\frac{\partial^2\phi}{\partial z^2}=2[\,\bar{z}\gamma''(z)+\psi'(z)]$$

$$2\sigma_x=2Re[\gamma'(z)+\overline{\gamma'(z)}]-2Re[\,\bar{z}\gamma''(z)+\psi'(z)]\Rightarrow$$

$$\sigma_x=2Re[\gamma'(z)-\frac{1}{2}\bar{z}\gamma''(z)-\frac{1}{2}\psi'(z)]$$

$$\sigma_y=(\sigma_x+\sigma_y)-\sigma_x=2[\gamma'(z)+\overline{\gamma'(z)}]-2Re[\gamma'(z)-\frac{1}{2}\bar{z}\gamma''(z)-\frac{1}{2}\psi'(z)]$$

$$=2Re[\gamma'(z)+\frac{1}{2}\bar{z}\gamma''(z)+\frac{1}{2}\psi'(z)]$$

$$\tau_{xy}=Im[\,\bar{z}\gamma''(z)+\psi'(z)]$$

10-6.

From the stress transformation laws in Chapter 3

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\tau_{r\theta} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$\sigma_r + \sigma_\theta = \sigma_x + \sigma_y$$
$$\sigma_\theta - \sigma_r = (\sigma_y - \sigma_x)\cos 2\theta - 2\tau_{xy}\sin 2\theta$$
$$2\tau_{r\theta} = (\sigma_y - \sigma_x)\sin 2\theta + 2\tau_{xy}\cos 2\theta$$
$$\therefore\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} = (\sigma_y - \sigma_x)(\cos 2\theta + i\sin 2\theta) - 2\tau_{xy}(\sin 2\theta - i\cos 2\theta)$$
$$= (\sigma_y - \sigma_x + 2i\tau_{xy})(\cos 2\theta + i\sin 2\theta) = (\sigma_y - \sigma_x + 2i\tau_{xy})e^{2i\theta}$$

From the displacement transformation laws in Appendix B

$$u_r = u\cos\theta + v\sin\theta\ ,\ u_\theta = -u\sin\theta + v\cos\theta \Rightarrow$$
$$u_r + iu_\theta = u\cos\theta + v\sin\theta + i(-u\sin\theta + v\cos\theta) = (u + iv)(\cos\theta - i\sin\theta) = (u + iv)e^{-i\theta}$$

10-7.

$\gamma(z) = Aiz^2$, $\psi(z) = -Aiz^2$, $A = A_R + iA_I$

$$\sigma_x + \sigma_y = 2[\gamma'(z) + \overline{\gamma'(z)}] = 2[2Aiz + 2\overline{Aiz}] = 4[Aiz - A i\bar{z}] = -8(A_R y + A_I x)$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\gamma''(z) + \psi'(z)] = 2[2\bar{z}Ai - 2Aiz] = 8(A_R y + iA_I y) \Rightarrow$$
$$\sigma_y - \sigma_x = 8A_R y \; , \; \tau_{xy} = 4A_I y$$

Solving for the normal stresses $\Rightarrow \sigma_x = -4A_I x - 8A_R y$, $\sigma_y = -4A_I x$

Displacements :

$$2\mu(u + iv) = \kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)} = \kappa Aiz^2 - 2z\,\overline{Aiz} + \overline{Aiz^2}$$
$$= \kappa(iA_R - A_I)(x^2 - y^2 + 2ixy) + 2(x^2 + y^2)(iA_R + A_I)$$
$$-\left([(x^2 - y^2)A_I + 2xyA_R] + i[(x^2 - y^2)A_R - 2xyA_I]\right) \Rightarrow$$
$$u = -\frac{1}{2\mu}\left[2(\kappa+1)xyA_R + [(\kappa-1)x^2 - (\kappa+3)y^2]A_I\right]$$
$$v = \frac{1}{2\mu}\left[2(1-\kappa)xyA_I + [(\kappa+1)x^2 + (3-\kappa)y^2]A_R\right]$$

Considering a pure bending problem of a rectangular beam ($-l \le x \le l, -c \le y \le c$), the boundary conditions would be

$$\sigma_y(x, \pm c) = 0 \Rightarrow A_I = 0$$
$$\tau_{xy}(x, \pm c) = 0 \Rightarrow A_I = 0$$
$$\tau_{xy}(\pm l, y) = 0 \Rightarrow A_I = 0$$
$$\int_{-c}^{c} \sigma_x(\pm l, y)dy = 0 \Rightarrow 0 = 0, \therefore \text{ satisfied}$$
$$\int_{-c}^{c} \sigma_x(\pm l, y)ydy = \pm M \Rightarrow -\frac{16}{3}A_R c^3 = \pm M \Rightarrow A_R = \mp\frac{3M}{16c^3}$$

$\therefore$ with the indicated values of A_I and A_R, stress field will solve the pure bending problem

10-8.

$\gamma(z) = Az$, $\psi(z) = B/z$ where $A = A_R + iA_I$, $B = B_R + iB_I$

$$\sigma_r + \sigma_\theta = 2[\gamma'(z) + \overline{\gamma'(z)}] = 4A_R$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2[\bar{z}\gamma''(z) + \psi'(z)]e^{2i\theta} = 2\psi'(z)e^{2i\theta} - 2\frac{B}{r^2 e^{2i\theta}}e^{2i\theta} = -\frac{2}{r^2}(B_R + iB_I)$$

$$\therefore \sigma_\theta - \sigma_r = -\frac{2}{r^2}B_R \ , \ \tau_{r\theta} = -\frac{1}{r^2}B_I$$

Solving for the normal stresses $\Rightarrow$

$$\sigma_r = 2A_R + \frac{1}{r^2}B_R \ , \ \sigma_\theta = 2A_R - \frac{1}{r^2}B_R$$

Consider the axisymmetric cylinder problem shown in Figure 8 - 8 with boundary conditions

$$\tau_{r\theta} = 0 \Rightarrow B_I = 0$$

$$\sigma_r(r_1) = p_1 \Rightarrow 2A_R + \frac{1}{r_1^2}B_R = p_1$$

$$\sigma_r(r_2) = p_2 \Rightarrow 2A_R + \frac{1}{r_2^2}B_R = p_2$$

Solving the previous two expressions for A_R and B_R and backsubstituting gives

$$\sigma_r = \frac{r_1^2 r_2^2(p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2} \ , \ \sigma_\theta = -\frac{r_1^2 r_2^2(p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

which are identical to previous solution given by relations (8.4.3).

10-9.

$$\gamma(z)=0\Rightarrow\gamma'(z)=\gamma''(z)=0$$
$$\psi(z)=A/z\Rightarrow\psi'(z)=-A/z^2$$
$$\sigma_r+\sigma_\theta=2[\gamma'(z)+\overline{\gamma'(z)}]=0$$
$$\sigma_\theta-\sigma_r+2i\tau_{r\theta}=2[\bar{z}\gamma''(z)+\psi'(z)]e^{2i\theta}=2\psi'(z)e^{2i\theta}=-2\frac{A}{r^2e^{2i\theta}}e^{2i\theta}=-\frac{2A}{r^2}$$

Solving for the individual stresses $\Rightarrow\sigma_r=-\sigma_\theta=\dfrac{A}{r^2}\,,\ \tau_{r\theta}=0$

Boundary Conditons : $\sigma_r(a)=-p\Rightarrow A=-pa^2$

$$\therefore\ \sigma_r=-\frac{pa^2}{r^2}\,,\ \sigma_\theta=\frac{pa^2}{r^2}\,,\ \tau_{r\theta}=0$$

Displacements :

$$2\mu(u_r+iu_\theta)=e^{-i\theta}[\kappa\gamma(z)-\overline{z\gamma'(z)}-\overline{\psi(z)}]=-e^{-i\theta}\frac{A}{re^{-i\theta}}=-\frac{A}{r}$$

Solving for the individual displacement components :

$$u_r=\frac{pa^2}{2\mu}\frac{1}{r}\,,\ u_\theta=0$$

10-10.

From Exercise 10 - 9 : $2\mu(u_r+iu_\theta)=-\dfrac{A}{r}\Rightarrow 2\mu u_r=-\dfrac{A}{r}$

Boundary Condition : $u_r(a)=\delta\Rightarrow A=-2\mu\delta a$

$$\therefore u_r=\frac{\delta a}{r}\,,\ u_\theta=0$$

From Exercise 10 - 9 : $\sigma_r=-\sigma_\theta=\dfrac{A}{r^2}\,,\ \tau_{r\theta}=0\Rightarrow$

$$\sigma_r=-\sigma_\theta=-\frac{2\mu\delta a}{r^2}\,,\ \tau_{r\theta}=0$$

10-11.

$$\gamma(z) = -\frac{X+iY}{2\pi(1+\kappa)}\log z\,,\ \psi(z) = \frac{\kappa(X-iY)}{2\pi(1+\kappa)}\log z\ ,\ \text{let}\ C = \frac{X+iY}{2\pi(1+\kappa)}$$

$$2\mu U = \kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)} = -\kappa C\log z + \overline{C}e^{2i\theta} - \kappa C\log\bar{z}$$
$$= -2\kappa C\log r + \overline{C}e^{2i\theta}$$

when $|z| \to 0 \Rightarrow r \to 0 \Rightarrow 2\mu U \to \infty$

when $|z| \to \infty \Rightarrow r \to \infty \Rightarrow 2\mu U \to \infty$

$$F_k = X_k + iY_k = \oint_{C_k}(T_x^n + iT_y^n)\,ds = i[\gamma(z) + z\overline{\gamma'(z)} + \overline{\psi(z)}]_{C_k}\ ,\ \text{for clockwise interior circuit}$$

$$= i\left[-\frac{X+iY}{2\pi(1+\kappa)}(\log r + i\theta) - \frac{X-iY}{2\pi(1+\kappa)}e^{2i\theta} + \frac{\kappa(X+iY)}{2\pi(1+\kappa)}(\log r - i\theta)\right]_{C_k}$$

$$= i\left[-\frac{X+iY}{2\pi(1+\kappa)}2\pi i - \frac{\kappa(X+iY)}{2\pi(1+\kappa)}2\pi i\right] = X+iY$$

Stresses on $r = a$:

$$\sigma_r + \sigma_\theta = 2[\gamma'(z) + \overline{\gamma'(z)}] = -2\left[\frac{X+iY}{2\pi a(1+\kappa)}(\cos\theta - i\sin\theta) + \frac{X-iY}{2\pi a(1+\kappa)}(\cos\theta + i\sin\theta)\right]$$
$$= -\frac{2(X\cos\theta + Y\sin\theta)}{\pi a(1+\kappa)}$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2[\bar{z}\gamma''(z) + \psi'(z)]e^{2i\theta} = 2\left[\frac{X+iY}{2\pi a(1+\kappa)}e^{-i\theta} + \frac{X-iY}{2\pi a(1+\kappa)}e^{i\theta}\right]$$
$$= \frac{1}{\pi a(1+\kappa)}\left[(1+\kappa)(X\cos\theta + Y\sin\theta) + i(1-\kappa)(Y\cos\theta - X\sin\theta)\right]$$

Solving for the individual components

$$\sigma_r = -\frac{(\kappa+3)}{2\pi a(1+\kappa)}(X\cos\theta + Y\sin\theta)$$

$$\sigma_\theta = \frac{(\kappa-1)}{2\pi a(1+\kappa)}(X\cos\theta + Y\sin\theta)$$

$$\tau_{r\theta} = \frac{(1-\kappa)}{2\pi a(1+\kappa)}(Y\cos\theta - X\sin\theta)$$

10-12.

$$M=\int_A^B(xT_y^n-yT_x^n)\,ds=-\int_A^B\left[x\,d\left(\frac{\partial\phi}{\partial x}\right)+y\,d\left(\frac{\partial\phi}{\partial y}\right)\right]ds$$

$$=-\left[x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right]_A^B+\phi\Big|_A^B$$

$$=-\frac{1}{2}\left[z\gamma(\bar z)+z\bar z\gamma'(z)+\bar z z\gamma'(\bar z)+\bar z\gamma(z)+z\psi(z)+\bar z\psi(\bar z)\right]_A^B$$

$$+\frac{1}{2}\left(z\,\overline{\gamma(z)}+\bar z\gamma(z)+\chi(z)+\overline{\chi(z)}\right)_A^B$$

$$=Re[\chi(z)-z\psi(z)-z\bar z\gamma'(z)]_A^B$$

10-13.

$$\gamma(z) = -\frac{X+iY}{2\pi(1+\kappa)}\log z\ ,\ \ \psi(z) = \frac{X-iY}{2\pi(1+\kappa)}\kappa\log z + \frac{X+iY}{2\pi(1+\kappa)}\frac{a^2}{z^2} + \frac{iM}{2\pi z}$$

$$\sigma_r + \sigma_\theta = 2[\gamma'(z) + \overline{\gamma'(z)}] = -2\left[\frac{X+iY}{2\pi(1+\kappa)}\frac{1}{r}e^{-i\theta} + \frac{X-iY}{2\pi(1+\kappa)}\frac{1}{r}e^{i\theta}\right] = -\frac{2(X\cos\theta + Y\sin\theta)}{\pi r(1+\kappa)}$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2[\,\bar{z}\gamma''(z) + \psi'(z)]e^{2i\theta}$$

$$= 2\left[re^{-i\theta}\frac{X+iY}{2\pi(1+\kappa)}\frac{1}{r^2}e^{-2i\theta} + \frac{X-iY}{2\pi(1+\kappa)}\frac{\kappa}{r}e^{-i\theta} - \frac{X+iY}{2\pi(1+\kappa)}\frac{2a^2}{r^3}e^{-3i\theta} - \frac{iM}{2\pi r^2}e^{-2i\theta}\right]e^{2i\theta}$$

$$= \frac{X+iY}{\pi r(1+\kappa)}e^{-i\theta} + \frac{X-iY}{\pi r(1+\kappa)}\kappa e^{i\theta} - \frac{2a^2(X+iY)}{\pi r^3(1+\kappa)}e^{-3i\theta} - \frac{iM}{\pi r^2} \Rightarrow$$

$$\sigma_\theta - \sigma_r = \left[\frac{1}{\pi r} - \frac{2a^2}{\pi r^3(1+\kappa)}\right](X\cos\theta + Y\sin\theta)$$

$$\tau_{r\theta} = \frac{(1-\kappa)r^2 - 2a^2}{2\pi r^3(1+\kappa)}(Y\cos\theta - X\sin\theta) - \frac{M}{2\pi r^2}$$

$$\sigma_r = \frac{2a^2 - (3+\kappa)r^2}{2\pi r^3(1+\kappa)}(X\cos\theta + Y\sin\theta)\ ,\ \sigma_\theta = \frac{(\kappa-1)r^2 - 2a^2}{2\pi r^3(1+\kappa)}(X\cos\theta + Y\sin\theta)$$

Note that all stresses vanish at ∞

Net Force on C Enclosing Origin : $\oint_C (T_x^n + iT_y^n)\,ds = -i[\gamma(z) + z\overline{\gamma'(z)} + \overline{\psi(z)}\,]_C$

$$= -i\left[-i\frac{X+iY}{(1+\kappa)} + 0 - i\frac{X+iY}{(1+\kappa)}\kappa\right] = -(X+iY)$$

Moment Over C $= Re[\chi(z) - z\psi(z) - z\bar{z}\gamma'(z)]_C = 0 - M + 0 = -M$

$\therefore$ Resultant Force and Moments match

$$2\mu(u+iv) = \kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}$$

$$= -\kappa\frac{X+iY}{2\pi(1+\kappa)}\log z + z\frac{X-iY}{2\pi(1+\kappa)}\frac{1}{z} - \kappa\frac{X+iY}{2\pi(1+\kappa)}\log\bar{z} - \frac{X-iY}{2\pi(1+\kappa)}\frac{a^2}{\bar{z}^2} + \frac{iM}{2\pi\bar{z}}$$

$$= -\kappa\frac{X+iY}{\pi(1+\kappa)}\log r + \frac{X-iY}{2\pi(1+\kappa)}\left(1 - \frac{a^2}{r^2}\right)e^{2i\theta} + \frac{iM}{2\pi r}e^{i\theta}$$

$$2\mu(u+iv)\Big|_{r=a} = -\kappa\frac{X+iY}{\pi(1+\kappa)}\log a + \frac{iM}{2\pi a}e^{i\theta}$$

Now the rigid-body motion of the inclusion must be of the form

$(u+iv) = (u_o + iv_o) + (-\theta_o y + i\theta_o x) \Rightarrow$

$$u_o = \frac{-\kappa X\log a}{2\pi\mu(1+\kappa)}\ ,\ \ v_o = \frac{-\kappa Y\log a}{2\pi\mu(1+\kappa)}\ ,\ \theta_o = \frac{M}{4\pi\mu a^2}$$

10-14.

$$\gamma(z)=\frac{T}{4}\left(z-\frac{2a^2}{\kappa z}\right)\Rightarrow\gamma'(z)=\frac{T}{4}\left(1+\frac{2a^2}{\kappa z^2}\right)\Rightarrow\gamma''(z)=-\frac{Ta^2}{\kappa z^3}$$

$$\psi(z)=-\frac{T}{2}\left(z-\frac{\kappa-1}{2z}a^2+\frac{a^4}{\kappa z^3}\right)\Rightarrow\psi'(z)=-\frac{T}{2}\left(1+\frac{\kappa-1}{2z^2}a^2-\frac{3a^4}{\kappa z^4}\right)$$

The given forms are appropriate since they match with relations (10.4.7) and (10.4.8) with no resultant force on the hole and with $\sigma_x^\infty=T,\ \sigma_y^\infty=\tau_{xy}^\infty=0$

$$\sigma_x+\sigma_y=2[\gamma'(z)+\overline{\gamma'(z)}]=\frac{T}{2}\left(2+\frac{2a^2}{\kappa z^2}+\frac{2a^2}{\kappa\bar{z}^2}\right)=T\left(1+\frac{a^2}{\kappa}\left(\frac{1}{z^2}+\frac{1}{\bar{z}^2}\right)\right)=T\left(1+\frac{2a}{\kappa r^2}\cos 2\theta\right)$$

$$\sigma_y-\sigma_x+2i\tau_{xy}=2[\bar{z}\gamma''(z)+\psi'(z)]=2\left(-\bar{z}\frac{Ta^2}{\kappa z^3}-\frac{T}{2}\left(1+\frac{\kappa-1}{2z^2}a^2-\frac{3a^4}{\kappa z^4}\right)\right)$$

$$=-2T\left(\frac{a^2\bar{z}}{\kappa z^3}+\frac{1}{2}+\frac{\kappa-1}{4z^2}a^2-\frac{3a^4}{2\kappa z^4}\right)$$

As $r\to\infty$: $\sigma_x^\infty+\sigma_y^\infty=T$, $\sigma_y^\infty-\sigma_x^\infty+2i\tau_{xy}^\infty=-T\Rightarrow\sigma_x^\infty=T$, $\sigma_y^\infty=\tau_{xy}^\infty=0$

Check Conditions on Plug Boundary $\Rightarrow$

$$2\mu(u+iv)\big|_{z=ae^{i\theta}}=\left[\kappa\gamma(z)-z\overline{\gamma'(z)}-\overline{\psi(z)}\right]_{z=ae^{i\theta}}$$

$$=\left[\kappa\frac{T}{4}\left(z-\frac{2a^2}{\kappa z}\right)-z\frac{T}{4}\left(1+\frac{2a^2}{\kappa\bar{z}^2}\right)+\frac{T}{2}\left(\bar{z}-\frac{\kappa-1}{2\bar{z}}a^2+\frac{a^4}{\kappa\bar{z}^3}\right)\right]_{z=ae^{i\theta}}$$

$$=\frac{T}{4}\left[\kappa ae^{i\theta}-2ae^{-i\theta}-ae^{i\theta}-\frac{2a}{\kappa}e^{3i\theta}+2ae^{-i\theta}-(\kappa-1)ae^{i\theta}+\frac{2a}{\kappa}e^{3i\theta}\right]=0$$

10-15.

$$u_r + iu_\theta = (u+iv)e^{-i\theta} = \frac{1}{2\mu}\left[\kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}\right]e^{-i\theta}$$

$\therefore$ Boundary condition on C :

$$\frac{1}{2\mu}\left[\kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}\right]e^{-i\theta}\bigg|_{z=\zeta} = h(\zeta) \text{ , with } \zeta = z\big|_{r=1} = e^{i\theta} \text{ , } \bar{\zeta} = e^{-i\theta} = \frac{1}{\zeta}$$

Multiplying by $\dfrac{1}{2\pi i(\zeta - z)}$ and integrating around the boundary contour C

$$\frac{e^{-i\theta}}{2\mu}\left[\kappa\frac{1}{2\pi i}\oint_C \frac{\gamma(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_C \frac{\zeta\overline{\gamma'(\zeta)}}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_C \frac{\overline{\psi(\zeta)}}{\zeta - z}d\zeta\right] = \frac{1}{2\pi i}\oint_C \frac{h(\zeta)}{\zeta - z}d\zeta$$

Using series form for complex potentials : $\gamma(z) = \sum a_n z^n$, $\psi(z) = \sum b_n z^n$ in boundary relation $\Rightarrow$

$$\frac{e^{-i\theta}}{2\mu}\left[\kappa\gamma(z) - a_1 z - 2a_2 - \psi(0)\right] = \frac{1}{2\pi i}\oint_C \frac{h(\zeta)}{\zeta - z}d\zeta \Rightarrow$$

$$\gamma(z) = \frac{1}{\kappa}\left[\frac{2\mu e^{i\theta}}{2\pi i}\oint_C \frac{h(\zeta)}{\zeta - z}d\zeta + \bar{a}_1 z - 2\bar{a}_2\right]$$, where the constant term $\psi(0)$ has been dropped

Next starting with the conjugate of the boundary relation $\Rightarrow$

$$\frac{1}{2\mu}\left[\kappa\,\overline{\gamma(z)} - \bar{z}\gamma'(z) - \psi(z)\right]e^{i\theta}\bigg|_{z=\zeta} = \overline{h(\zeta)} \Rightarrow$$

$$\frac{e^{i\theta}}{2\mu}\left[\kappa\frac{1}{2\pi i}\oint_C \frac{\overline{\gamma(\zeta)}}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_C \frac{\bar{\zeta}\gamma'(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_C \frac{\psi(\zeta)}{\zeta - z}d\zeta\right] = \frac{1}{2\pi i}\oint_C \frac{\overline{h(\zeta)}}{\zeta - z}d\zeta \Rightarrow$$

$$\psi(z) = \frac{-2\mu e^{-i\theta}}{2\pi i}\oint_C \frac{\overline{h(\zeta)}}{\zeta - z}d\zeta + \kappa\,\overline{\gamma(0)} + \frac{a_1}{z} - \frac{\gamma'(z)}{z}$$, dropping the constant term $\Rightarrow$

$$\psi(z) = \frac{-2\mu e^{-i\theta}}{2\pi i}\oint_C \frac{\overline{h(\zeta)}}{\zeta - z}d\zeta + \frac{a_1}{z} - \frac{\gamma'(z)}{z}$$

10-16.

From (10.5.18), $\sigma_r + \sigma_\theta = \sigma_x + \sigma_y = -\frac{2Fa}{\pi}\left[\frac{1}{(z^2 - a^2)} + \frac{1}{(\bar{z}^2 - a^2)} + \frac{1}{2a^2}\right]$

$$= -\frac{2F}{\pi}\left[\frac{a-x}{(x-a)^2 + y^2} + \frac{a+x}{(x+a)^2 + y^2} - \frac{1}{a^2}\right]$$

From (8.4.69) for loading along the x - axis

$$\sigma_x = -\frac{2P}{\pi}\left[\frac{(R-x)y^2}{(y^2 + (R-x)^2)^2} + \frac{(R+x)y^2}{(y^2 + (R+x)^2)^2} - \frac{1}{D}\right]$$

$$\sigma_y = -\frac{2P}{\pi}\left[\frac{(R-x)^3}{(y^2 + (R-x)^2)^2} + \frac{(R+x)^3}{(y^2 + (R+x)^2)^2} - \frac{1}{D}\right]$$

Adding these stresses and noting that $R = a$, $D = 2a \Rightarrow$

$$\sigma_x + \sigma_y = -\frac{2P}{\pi}\left[\frac{a-x}{(x-a)^2 + y^2} + \frac{a+x}{(x+a)^2 + y^2} - \frac{1}{a^2}\right]$$

$\therefore$ the results from Chapters 8 and 10 are identical

10-17.

From general solution (10.6.2)

$$\sigma_x + \sigma_y = -\frac{1}{\pi(1+\kappa)}\left(\frac{X+iY}{z}+\frac{X-iY}{\bar{z}}\right) = -\frac{1}{\pi(1+\kappa)}\left(\frac{P}{(x+iy)}+\frac{P}{(x-iy)}\right) = -\frac{2Px}{\pi(1+\kappa)r^2}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = \frac{P}{\pi(1+\kappa)}\frac{x-iy}{(x+iy)^2}+\frac{\kappa P}{\pi(1+\kappa)}\frac{1}{(x+iy)}$$

$$= \frac{P}{\pi(1+\kappa)r^4}\left[(x^3-3xy^2)+i(y^3-3x^2y)\right]+\frac{\kappa P}{\pi(1+\kappa)r^2}(x-iy) \Rightarrow$$

$$\sigma_y - \sigma_x = \frac{Px}{\pi(1+\kappa)r^4}(x^2-3y^2+\kappa r^2)$$

$$\tau_{xy} = \frac{Py}{2\pi(1+\kappa)r^4}(y^2-3x^2-\kappa r^2) = \frac{Py}{2\pi(1+\kappa)r^4}\left[4y^2-(3+\kappa)r^2\right]$$

$$\sigma_x = -\frac{Px}{2\pi(1+\kappa)r^4}\left[4x^2+(\kappa-1)r^2\right],\ \sigma_y = \frac{Px}{2\pi(1+\kappa)r^4}\left[4x^2+(\kappa-5)r^2\right]$$

In polar coordinates

$$\sigma_x = -\frac{P\cos\theta}{2\pi(1+\kappa)r}\left[4\cos^2\theta+\kappa-1\right]$$

$$\sigma_y = -\frac{P\cos\theta}{2\pi(1+\kappa)r}\left[4\cos^2\theta+\kappa-5\right]$$

$$\tau_{xy} = -\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-\kappa-3\right]$$

$$\sigma_r = \frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos 2\theta+\tau_{xy}\sin 2\theta$$

$$= -\frac{P\cos\theta}{\pi(1+\kappa)r}-\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-3-\kappa\right]\cos 2\theta-\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-\kappa-3\right]\sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x+\sigma_y}{2}-\frac{\sigma_x-\sigma_y}{2}\cos 2\theta-\tau_{xy}\sin 2\theta$$

$$= -\frac{P\cos\theta}{\pi(1+\kappa)r}+\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-3-\kappa\right]\cos 2\theta+\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-\kappa-3\right]\sin 2\theta$$

$$\tau_{r\theta} = \frac{\sigma_y-\sigma_x}{2}\sin 2\theta+\tau_{xy}\cos 2\theta$$

$$= -\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-3-\kappa\right]\sin 2\theta-\frac{P\sin\theta}{2\pi(1+\kappa)r}\left[4\sin^2\theta-\kappa-3\right]\cos 2\theta$$

10-18.

With far-field stresses $\sigma_x^\infty = \sigma_y^\infty = S, \tau_{xy}^\infty = 0$, the general solution form (10.4.7) yields

$$\gamma(z) = \frac{S}{2}z + \sum_{n=1}^{\infty} a_n z^{-n} \ , \ \psi(z) = \sum_{n=1}^{\infty} b_n z^{-n}$$

From previous work in Example 10-6, relations (10.6.12)

$$a_1 = 0 \ , \ a_n = 0 \ (n \geq 2) \ , \ b_1 = -Sa^2 \ , \ b_2 = 0 \ , \ b_3 = a^2 a_1 = 0 \ , \ b_n = 0 \ (n \geq 4) \Rightarrow$$

$$\gamma(z) = \frac{S}{2}z \ , \ \psi(z) = \frac{-Sa^2}{z}$$

$$\sigma_r + \sigma_\theta = 2[\gamma'(z) + \overline{\gamma'(z)}] = 2\left[\frac{S}{2} + \frac{S}{2}\right] = 2S$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2[\bar{z}\gamma''(z) + \psi'(z)]e^{2i\theta} = 2\frac{Sa^2}{z^2}e^{2i\theta} = 2\frac{Sa^2}{r^2}$$

Solving for the stresses

$$\sigma_r = S\left(1 - \frac{a^2}{r^2}\right), \ \sigma_\theta = S\left(1 + \frac{a^2}{r^2}\right), \ \tau_{r\theta} = 0 \ , \text{ which matches with previous solution (8.4.9)}$$

10-19.

Circular Case : $z = w(\zeta) = \dfrac{R}{\zeta}$

On boundary in ζ-plane : $\zeta = e^{i\theta} \Rightarrow z = R\,e^{-i\theta}$

$\therefore$ boundaries match appropriately

Point at infinity : $z \to \infty \Rightarrow \zeta \to 0$

Elliptical Case : $z = w(\zeta) = R\left(\dfrac{1}{\zeta} + m\zeta\right)$

On boundary in ζ-plane : $\zeta = e^{i\theta} \Rightarrow$

$$z = R\left(e^{-i\theta} + me^{i\theta}\right) = R[(1+m)\cos\theta + i(m-1)\sin\theta]$$

$$= x + iy \ , \text{ where } x = R(1+m)\cos\theta \text{ and } y = R(m-1)\sin\theta$$

Note that $\dfrac{x^2}{R^2(1+m)^2} + \dfrac{y^2}{R^2(1-m)^2} = 1 \Rightarrow$ elliptical boundary in z-plane

Point at infinity : $z \to \infty \Rightarrow \zeta \to 0$

10-20.

From equation (10.7.14) : $\sigma_\varphi(\varphi) = S\left(\dfrac{2m+1-2\cos 2\varphi - m^2}{m^2 - 2m\cos 2\varphi + 1}\right)$

To find maximum value set $\dfrac{d\sigma_\varphi}{d\varphi} = 0 \Rightarrow$

$4\sin 2\varphi(m^2 - 2m\cos 2\varphi + 1) - 4m\sin 2\varphi(2m+1-2\cos 2\varphi - m^2) = 0 \Rightarrow$

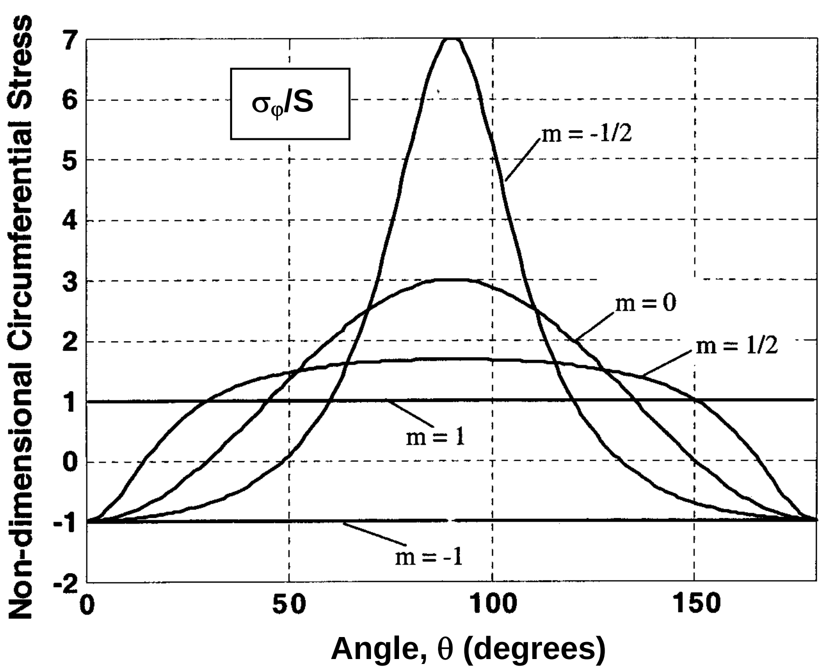
$4\sin 2\varphi(m+1)(m-1)^2 = 0$, considering general case with $m \neq \pm 1 \Rightarrow$

$\sin 2\varphi = 0 \Rightarrow \varphi = 0, \dfrac{\pi}{2}, \pi, \cdots$ (note that $\varphi = 0$ corresponds to a minimum value)

For the case $\varphi = \dfrac{\pi}{2}$, $\cos 2\varphi = -1 \Rightarrow$

$\sigma_\varphi(\dfrac{\pi}{2}) = S\left(\dfrac{2m+3-m^2}{m^2+2m+1}\right) = -S\dfrac{(m-3)(m+1)}{(m+1)^2} = -S\dfrac{m-3}{m+1}$

MATLAB Plot :



10-21.

Elliptical hole problem with far - field stress : $\sigma_x^\infty = \sigma_y^\infty = 0\,,\tau_{xy}^\infty = S$

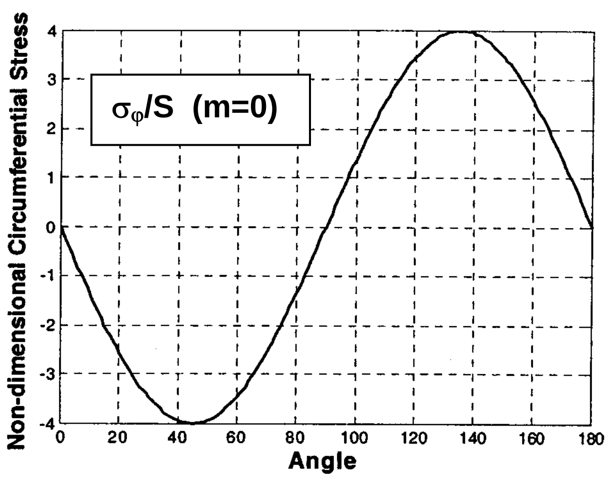
$$\frac{d\gamma(\zeta)}{dz} = \frac{iS}{m-\zeta^2} = \frac{\gamma_1'(\zeta)}{w'(\zeta)}$$

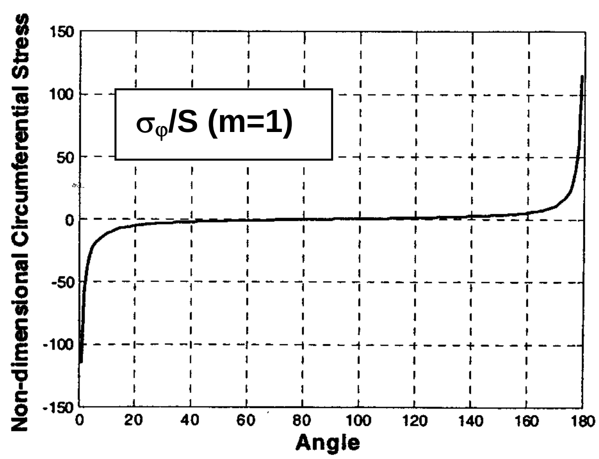
$$\sigma_\rho + \sigma_\varphi = 2\left(\frac{\gamma_1'(\zeta)}{w'(\zeta)} + \frac{\overline{\gamma_1'(\zeta)}}{\overline{w'(\zeta)}}\right) = 2\left(\frac{iS}{m-\zeta^2} + \frac{-iS}{m-\overline{\zeta}^2}\right)$$

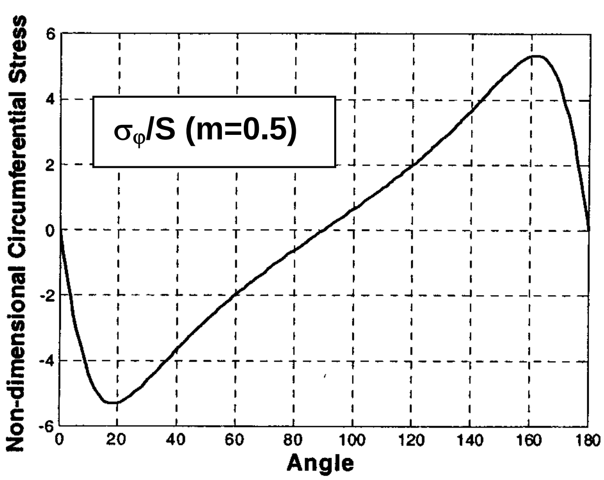
$$= 2\left(\frac{iS(\zeta^2-\overline{\zeta}^2)}{m^2 - m(\zeta^2+\overline{\zeta}^2) + \zeta^2\overline{\zeta}^2}\right) = 2\left(\frac{-2S\sin 2\varphi}{m^2 - 2m\cos 2\varphi + 1}\right)$$

On hole boundary, $\sigma_\rho = 0 \Rightarrow \sigma_\varphi = -\dfrac{4S\sin 2\varphi}{m^2 - 2m\cos 2\varphi + 1}$

MATLAB Plot :







10-22.

From Example 10 - 8 :

$$\sigma_x + \sigma_y = SRe\left(\frac{r_\beta e^{i\beta}}{\sqrt{rr_\alpha}\, e^{i(\theta+\alpha)/2}} + \frac{r_\beta e^{-i\beta}}{\sqrt{rr_\alpha}\, e^{-i(\theta+\alpha)/2}} - 1\right) = SRe\left(\frac{r_\beta}{\sqrt{rr_\alpha}}\left[e^{i(\beta-\frac{\theta+\alpha}{2})} + e^{-i(\beta-\frac{\theta+\alpha}{2})}\right] - 1\right)$$

$$= S\left(\frac{2r_\beta}{\sqrt{rr_\alpha}}\cos\left(\beta - \frac{\theta+\alpha}{2}\right) - 1\right)$$

Near crack tip : $r_\alpha \approx 2a$, $r_\beta \approx a$, $\cos\left(\beta - \frac{\theta+\alpha}{2}\right) = \cos\left(\frac{\theta}{2} + \left(\frac{\alpha}{2} - \beta\right)\right) \approx \cos\frac{\theta}{2} \Rightarrow$

$$\sigma_x + \sigma_y \approx \frac{2Sa}{\sqrt{2ar}}\cos\frac{\theta}{2}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = Sa^2\left(\frac{r_\beta(e^{i\beta} - e^{-i\beta})}{(rr_\alpha)^{3/2}}e^{-3i(\theta+\alpha)/2} + \frac{1}{a^2}\right)$$

$$= \frac{2Sa^2 i r_\beta \sin\beta}{(rr_\alpha)^{3/2}}\left(\cos\left(\frac{3(\theta+\alpha)}{2}\right) - i\sin\left(\frac{3(\theta+\alpha)}{2}\right)\right) + S$$

Near crack tip : $r_\alpha \approx 2a$, $r_\beta \approx a$, and $r_\beta \sin\beta = r\sin\theta = 2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\sigma_y - \sigma_x + 2i\tau_{xy} \approx \frac{2Sa^2 i r_\beta \sin\beta}{(rr_\alpha)^{3/2}}\left(\cos\left(\frac{3(\theta+\alpha)}{2}\right) - i\sin\left(\frac{3(\theta+\alpha)}{2}\right)\right)$$

$$\approx \frac{2Sa}{\sqrt{2ar}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\sin\frac{3\theta}{2} + i\cos\frac{3\theta}{2}\right) \Rightarrow$$

$$\sigma_y - \sigma_x = \frac{2Sa}{\sqrt{2ar}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}\ ,\ \tau_{xy} = \frac{Sa}{\sqrt{2ar}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Solving for the individual stresses and defining the stress intensity factor as $K_I = S\sqrt{\pi a} \Rightarrow$

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

10-23.

$$2\mu(u+iv)=\kappa\gamma(z)-z\overline{\gamma'(z)}-\overline{\psi(z)}$$
$$=\kappa\frac{S}{4}\left(z\sqrt{z^2-a^2}-z\right)-z\frac{S}{4}\left(\frac{2\bar{z}}{\sqrt{\bar{z}^2-a^2}}-1\right)-\frac{S}{2}\left(\bar{z}-\frac{a^2}{\sqrt{\bar{z}^2-a^2}}\right)$$
$$=\kappa\frac{S}{4}\left(z\sqrt{z^2-a^2}-z\right)+\frac{S}{2}\left(\frac{a^2-z\bar{z}}{\sqrt{\bar{z}^2-a^2}}+\frac{z}{2}-\bar{z}\right)$$

Near the crack tip :

$$2\mu(u+iv)\approx\kappa\frac{S}{2}\sqrt{rr_\alpha}e^{i(\theta+\alpha)/2}+\frac{S}{2}(a^2-r_\beta^2)\frac{1}{\sqrt{rr_\alpha}}e^{i(\theta+\alpha)/2}$$
$$\approx\kappa\frac{S}{2}\sqrt{rr_\alpha}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)-\frac{S}{2}\frac{2ar\cos\theta}{\sqrt{rr_\alpha}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$
$$\approx\kappa K_I\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)-K_I\sqrt{\frac{r}{2\pi}}\cos\theta\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$
$$\approx K_I\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2}(\kappa-\cos\theta)+i\sin\frac{\theta}{2}(\kappa-\cos\theta)\right)$$
$$\approx K_I\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left((\kappa-1)+2\sin^2\frac{\theta}{2}\right)+iK_I\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left((\kappa+1)-2\cos^2\frac{\theta}{2}\right)\Rightarrow$$

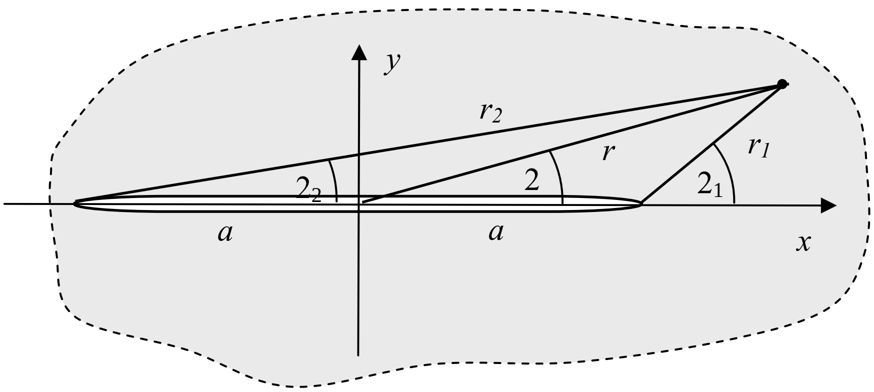
$$u=\frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\frac{\kappa-1}{2}+\sin^2\frac{\theta}{2}\right),\quad v=\frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\frac{\kappa+1}{2}-\cos^2\frac{\theta}{2}\right)$$

10-24.

$$\text{Stress Function}: Z(z) = \frac{Sz}{\sqrt{z^2 - a^2}} - \frac{S}{2}\ , \ A = \frac{S}{2}$$

Use polar coordinate s to explore stress field (see figure) :

$$z = re^{i\theta}\ ,\ z - a = r_1 e^{i\theta_1}\ ,\ z + a = r_2 e^{i\theta_2}$$



$$\sigma_x = \operatorname{Re} Z(z) - y \operatorname{Im} Z'(z) - A$$

$$= \frac{Sr}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) - \frac{S}{2} - r\sin\theta \frac{Sa^2}{(r_1 r_2)^{3/2}} \sin\left(\frac{3(\theta_1 + \theta_2)}{2}\right) - \frac{S}{2}$$

$$\sigma_y = \operatorname{Re} Z(z) + y \operatorname{Im} Z'(z) + A$$

$$= \frac{Sr}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) - \frac{S}{2} + r\sin\theta \frac{Sa^2}{(r_1 r_2)^{3/2}} \sin\left(\frac{3(\theta_1 + \theta_2)}{2}\right) + \frac{S}{2}$$

$$\tau_{xy} = -y \operatorname{Re} Z'(z) = r\sin\theta \frac{Sa^2}{(r_1 r_2)^{3/2}} \cos\left(\frac{3(\theta_1 + \theta_2)}{2}\right)$$

Check boundary conditions :

On crack surfaces :

$$\sigma_y = \frac{Sr}{\sqrt{r_1 r_2}} \cos\left(\frac{\pi}{2}\right) - \frac{S}{2} + r\sin(0) \frac{Sa^2}{(r_1 r_2)^{3/2}} \sin\left(\frac{3\pi}{2}\right) + \frac{S}{2} = 0$$

$$\tau_{xy} = r\sin(0) \frac{Sa^2}{(r_1 r_2)^{3/2}} \cos\left(\frac{3\pi}{2}\right) = 0\ ,\ \therefore \text{checks}$$

Far field :

$$r, r_1, r_2 \to \infty \text{ with } \theta = \theta_1 = \theta_2 = \pi/2 \Rightarrow$$

$$\sigma_x \to S - \frac{S}{2} - 0 - \frac{S}{2} = 0\ ,\ \sigma_y \to S - \frac{S}{2} + 0 + \frac{S}{2} = S\ ,\ \tau_{xy} \to 0$$

$\therefore$ stresses check

10-25.

For the skew - symmetric problem, along the x - axis ($y = 0$), the normal stress σ_y vanishes $\Rightarrow$

$$\sigma_y = Re[2\gamma'(z) + \bar{z}\gamma''(z) + \psi'(z)] = 0 \Rightarrow 2\gamma'(z) + z\gamma''(z) + \psi'(z) = -iB \Rightarrow$$

$$\psi'(z) = -iB - 2\gamma'(z) - z\gamma''(z)$$

$$\therefore \sigma_x = Re[2\gamma'(z) - \bar{z}\gamma''(z) - \psi'(z)] = 4Re(\gamma'(z)) - 2yIm(\gamma''(z))$$

$$\sigma_y = Re[2\gamma'(z) + \bar{z}\gamma''(z) + \psi'(z)] = 2yIm(\gamma''(z))$$

$$\tau_{xy} = Im[\bar{z}\gamma''(z) + \psi'(z)] = -2Im(\gamma'(z)) - 2yRe(\gamma''(z)) - B$$

Defining the Westergaard function $Z(z) = 2\gamma'(z) \Rightarrow$

$$\sigma_x = 2\,\mathrm{Re}\, Z(z) - y\,\mathrm{Im}\, Z'(z)$$

$$\sigma_y = y\,\mathrm{Im}\, Z'(z)$$

$$\tau_{xy} = -\,\mathrm{Im}\, Z(z) - y\,\mathrm{Re}\, Z'(z) - B$$

11-1.

From Hooke's, $\dfrac{\partial \sigma_{ij}}{\partial e_{kl}} = C_{ijkl}$

$$\therefore \frac{\partial \sigma_{ij}}{\partial e_{kl}} = \frac{\partial \sigma_{kl}}{\partial e_{ij}} \Rightarrow C_{ijkl} = C_{klij} \Rightarrow C_{ij} = C_{ji}$$

and thus the general 6×6 anisotropic stiffness matrix is symmetric and thus implies that only 21 independent elastic moduli exist

11-2.

It is equivalent to either do three rotations about the coordinate axes or
three reflections about each of the coordinate planes. Choose the three reflections.

Reflection about the x_1, x_2 – plane: $Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Transformation law for the elasticity tensor, $C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq} \Rightarrow$

$C_{14} = C_{1123} = Q_{1m}Q_{1n}Q_{2p}Q_{3q}C_{mnpq} = -C_{1123} \Rightarrow C_{14} = 0$

$C_{15} = C_{1131} = Q_{1m}Q_{1n}Q_{3p}Q_{1q}C_{mnpq} = -C_{1131} \Rightarrow C_{15} = 0$

$C_{24} = C_{2223} = Q_{2m}Q_{2n}Q_{2p}Q_{3q}C_{mnpq} = -C_{2223} \Rightarrow C_{24} = 0$

$C_{25} = C_{2231} = Q_{2m}Q_{2n}Q_{3p}Q_{1q}C_{mnpq} = -C_{2231} \Rightarrow C_{25} = 0$

$C_{34} = C_{3323} = Q_{3m}Q_{3n}Q_{2p}Q_{3q}C_{mnpq} = -C_{3323} \Rightarrow C_{34} = 0$

$C_{46} = C_{2312} = Q_{2m}Q_{3n}Q_{1p}Q_{2q}C_{mnpq} = -C_{2312} \Rightarrow C_{46} = 0$

$C_{56} = C_{3112} = Q_{3m}Q_{1n}Q_{1p}Q_{2q}C_{mnpq} = -C_{3112} \Rightarrow C_{56} = 0$

$C_{35} = C_{3331} = Q_{3m}Q_{3n}Q_{3p}Q_{1q}C_{mnpq} = -C_{3331} \Rightarrow C_{35} = 0$

Reflection about the x_2, x_3 – plane: $Q_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Transformation law for the elasticity tensor, $C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq} \Rightarrow$

$C_{16} = C_{1112} = Q_{1m}Q_{1n}Q_{1p}Q_{2q}C_{mnpq} = -C_{1112} \Rightarrow C_{16} = 0$

$C_{26} = C_{2212} = Q_{2m}Q_{2n}Q_{1p}Q_{2q}C_{mnpq} = -C_{2212} \Rightarrow C_{26} = 0$

$C_{36} = C_{3312} = Q_{3m}Q_{3n}Q_{1p}Q_{2q}C_{mnpq} = -C_{3312} \Rightarrow C_{36} = 0$

$C_{45} = C_{2231} = Q_{2m}Q_{3n}Q_{3p}Q_{1q}C_{mnpq} = -C_{2331} \Rightarrow C_{45} = 0$

These two transformations $\Rightarrow C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ \cdot & C_{22} & C_{23} & 0 & 0 & 0 \\ \cdot & \cdot & C_{33} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & C_{44} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & C_{55} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix}$

giving the desired form for orthotropic materials with nine independent elastic constants.
Employing the final (third) reflection about the x_3, x_1 – plane would not produce any new relations.

11-3.

Material Symmetry: $C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq}$

For this case, the following moduli are zero:

$C_{14} = C_{1123} = C_{15} = C_{1131} = C_{16} = C_{1112} = C_{24} = C_{2223} = 0$

$C_{25} = C_{2231} = C_{26} = C_{2212} = C_{34} = C_{3323} = C_{35} = C_{3331} = 0$

$C_{36} = C_{3312} = C_{45} = C_{2331} = C_{46} = C_{2312} = C_{56} = C_{3112} = 0$

and: $C_{44} = C_{55}$, $C_{66} = (C_{11} - C_{22})/2$

The particular rotation is given by:

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C_{55} Check:

$$\begin{aligned} C'_{55} &= C'_{3131} = Q_{3m}Q_{1n}Q_{3p}Q_{1q}C_{mnpq} = Q_{33}Q_{1n}Q_{33}Q_{1q}C_{3n3q} = Q_{1n}Q_{1q}C_{3n3q} \\ &= Q_{11}Q_{11}C_{3131} + Q_{11}Q_{12}C_{3132} + Q_{12}Q_{11}C_{3231} + Q_{12}Q_{12}C_{3232} \\ &= \cos^2\theta C_{55} + \sin^2\theta C_{44} = (\cos^2\theta + \sin^2\theta)C_{55} = C_{55} \end{aligned}$$

C_{22} Check:

$$\begin{aligned} C'_{22} &= C'_{2222} = Q_{2m}Q_{2n}Q_{2p}Q_{2q}C_{mnpq} \\ &= Q_{21}^4C_{1111} + Q_{21}^2Q_{22}^2C_{1122} + Q_{21}^2Q_{22}^2C_{1212} + Q_{21}^2Q_{22}^2C_{1221} \\ &\quad + Q_{21}^2Q_{22}^2C_{2112} + Q_{21}^2Q_{22}^2C_{2121} + Q_{22}^2Q_{21}^2C_{2211} + Q_{22}^4C_{2222} \\ &= \sin^4\theta C_{11} + 2\sin^2\theta\cos^2\theta C_{12} + 4\sin^2\theta\cos^2\theta C_{66} + \cos^4\theta C_{22} \\ &= C_{11}\sin^4\theta + 2C_{12}\sin^2\theta\cos^2\theta + 2(C_{11} - C_{12})\sin^2\theta\cos^2\theta + C_{22}\cos^4\theta \\ &= C_{22}(\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta) = C_{22}(\sin^2\theta + \cos^2\theta)^2 = C_{22} \end{aligned}$$

11-4.

C_{ij} must be positive definite and so all principal minors $p_i > 0$
and all diagonal elements are > 0

Orthotropic Case :

$$p_1 > 0 \Rightarrow \begin{vmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{vmatrix} > 0 \Rightarrow C_{22}C_{33} > C_{23}^2$$

$$p_2 > 0 \Rightarrow \begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} > 0 \Rightarrow C_{11}C_{33} > C_{13}^2$$

$$p_3 > 0 \Rightarrow \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} > 0 \Rightarrow C_{11}C_{22} > C_{12}^2$$

$$p_4 > 0 \Rightarrow C_{11}\begin{vmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{vmatrix} - C_{12}\begin{vmatrix} C_{12} & C_{23} \\ C_{13} & C_{33} \end{vmatrix} + C_{13}\begin{vmatrix} C_{12} & C_{22} \\ C_{13} & C_{23} \end{vmatrix} > 0 \Rightarrow$$

$$C_{11}C_{22}C_{33} + 2C_{12}C_{23}C_{31} > C_{11}C_{23}^2 + C_{22}C_{13}^2 + C_{33}C_{12}^2$$

Transversely Isotropic Case :

$$p_1 = p_2 > 0 \Rightarrow \begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} > 0 \Rightarrow C_{11}C_{33} > C_{13}^2$$

$$p_3 > 0 \Rightarrow \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{11} \end{vmatrix} > 0 \Rightarrow C_{11}^2 > C_{12}^2$$

$$p_4 > 0 \Rightarrow C_{11}\begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} - C_{12}\begin{vmatrix} C_{12} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} + C_{13}\begin{vmatrix} C_{12} & C_{11} \\ C_{13} & C_{13} \end{vmatrix} > 0 \Rightarrow C_{33}(C_{11} + C_{12}) > 2C_{13}^2$$

Isotropic Case : $\mu > 0$

$$p_1 = p_2 = p_3 > 0 \Rightarrow \begin{vmatrix} \lambda + 2\mu & \lambda \\ \lambda & \lambda + 2\mu \end{vmatrix} > 0 \Rightarrow 4\mu(\lambda + 2\mu) > 0 \Rightarrow \lambda + 2\mu > 0$$

$$p_4 > 0 \Rightarrow \begin{vmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{vmatrix} > 0 \Rightarrow 4\mu^2(3\lambda + 2\mu) > 0 \Rightarrow \lambda + \frac{2}{3}\mu > 0$$

11-5.

For orthotropic materials, $S_{ij}=\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ \cdot & S_{22} & S_{23} & 0 & 0 & 0 \\ \cdot & \cdot & S_{33} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & S_{44} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & S_{55} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & S_{66} \end{bmatrix}$

Positive definite strain energy function $\Rightarrow$ that each principal minor of S_{ij} be positive $\Rightarrow$

$$\begin{vmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{vmatrix} > 0 \Rightarrow S_{11}S_{22} > S_{12}^2 \Rightarrow \frac{1}{E_1}\frac{1}{E_2} > \left(-\frac{\nu_{21}}{E_2}\right)^2 \Rightarrow \nu_{21}^2 < \frac{E_2}{E_1}$$

$$\begin{vmatrix} S_{22} & S_{23} \\ S_{23} & S_{33} \end{vmatrix} > 0 \Rightarrow S_{22}S_{33} > S_{23}^2 \Rightarrow \frac{1}{E_2}\frac{1}{E_3} > \left(-\frac{\nu_{32}}{E_3}\right)^2 \Rightarrow \nu_{32}^2 < \frac{E_3}{E_2}$$

$$\begin{vmatrix} S_{33} & S_{13} \\ S_{13} & S_{11} \end{vmatrix} > 0 \Rightarrow S_{33}S_{11} > S_{13}^2 \Rightarrow \frac{1}{E_3}\frac{1}{E_1} > \left(-\frac{\nu_{31}}{E_3}\right)^2 \Rightarrow \nu_{31}^2 < \frac{E_3}{E_1}$$

$$\therefore \nu_{ij}^2 < \frac{E_i}{E_j}$$

Using moduli E_i and E_j from Table 11-1 $\Rightarrow$

S-Glass/Epoxy: $\nu_{12}^2 < \dfrac{E_1}{E_2} = \dfrac{50}{17} = 2.94$

Boron/Epoxy: $\nu_{12}^2 < \dfrac{E_1}{E_2} = \dfrac{205}{20} = 10.25$

Carbon/Epoxy: $\nu_{12}^2 < \dfrac{E_1}{E_2} = \dfrac{205}{10} = 20.5$

Kevlar49/Epoxy: $\nu_{12}^2 < \dfrac{E_1}{E_2} = \dfrac{76}{5.5} = 13.82$

Thus the inequalities $\Rightarrow$ ν_{12} could be greater than one.

11-6.

For material with a plane of symmetry and with $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \Rightarrow$

$$\begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{yz} \\ 2e_{zx} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & C_{13} & 0 & 0 & S_{16} \\ \cdot & S_{22} & C_{23} & 0 & 0 & S_{26} \\ \cdot & \cdot & S_{33} & 0 & 0 & S_{36} \\ \cdot & \cdot & \cdot & S_{44} & S_{45} & 0 \\ \cdot & \cdot & \cdot & \cdot & S_{55} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & S_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau_{yz} \\ \tau_{zx} \\ 0 \end{bmatrix} \Rightarrow$$

$e_x = e_y = e_z = e_{xy} = 0$, $2e_{yz} = S_{44}\tau_{yz} + S_{45}\tau_{zx}$, $2e_{zx} = S_{45}\tau_{yz} + S_{55}\tau_{zx}$

Compatibility equations :

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow 0 = 0$$

$$\frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = 2\frac{\partial^2 e_{yz}}{\partial y \partial z} \Rightarrow 0 = 0$$

$$\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = 2\frac{\partial^2 e_{zx}}{\partial z \partial x} \Rightarrow 0 = 0$$

$$\frac{\partial^2 e_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right) \Rightarrow 0 = 0$$

$$\frac{\partial^2 e_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right) \Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right) = 0 \Rightarrow$$

$$\frac{\partial}{\partial x}\left[-\frac{\partial}{\partial x}(S_{44}\tau_{yz} + S_{45}\tau_{xz}) + \frac{\partial}{\partial y}(S_{54}\tau_{yz} + S_{55}\tau_{xz})\right] = 0 \Rightarrow$$

$$-\frac{\partial}{\partial x}(S_{44}\tau_{yz} + S_{45}\tau_{xz}) + \frac{\partial}{\partial y}(S_{54}\tau_{yz} + S_{55}\tau_{xz}) = f(y)$$

$$\frac{\partial^2 e_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right) \Rightarrow \frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{yz}}{\partial x}\right) = 0 \Rightarrow$$

$$\frac{\partial}{\partial y}\left[-\frac{\partial}{\partial y}(S_{54}\tau_{yz} + S_{55}\tau_{xz}) + \frac{\partial}{\partial x}(S_{44}\tau_{yz} + S_{45}\tau_{xz})\right] = 0 \Rightarrow$$

$$-\frac{\partial}{\partial y}(S_{54}\tau_{yz} + S_{55}\tau_{xz}) + \frac{\partial}{\partial x}(S_{44}\tau_{yz} + S_{45}\tau_{xz}) = g(x)$$

Adding the previous results $\Rightarrow f(y) + g(x) = 0 \Rightarrow f(y) = -g(x) = \text{constant} = C$

$$-\frac{\partial}{\partial x}(S_{44}\tau_{yz} + S_{45}\tau_{xz}) + \frac{\partial}{\partial y}(S_{54}\tau_{yz} + S_{55}\tau_{xz}) = C$$

11-7.

Homogeneous equation :

$S_{44}\psi_{xx} - 2S_{45}\psi_{xy} + S_{55}\psi_{yy} = 0$

Look for solutions of the form $\psi(x, y) = f(x + \mu y)$, where μ is a parameter $\Rightarrow$

$(S_{44} - 2S_{45}\mu + S_{55}\mu^2) f'' = 0 \Rightarrow$

$S_{55}\mu^2 - 2S_{45}\mu + S_{44} = 0 \cdots$ characteristic equation

Solving the quadratic for the roots $\mu_i \Rightarrow \mu_{1,2} = \dfrac{S_{45} \pm \sqrt{S_{45}^2 - S_{44}S_{55}}}{S_{55}}$

$\mu_1 + \mu_2 = 2\dfrac{S_{45}}{S_{55}}$, $\mu_1\mu_2 = \dfrac{S_{44}}{S_{55}}$

The original differential equation can be written as

$$\frac{S_{44}}{S_{55}}\frac{\partial^2\psi}{\partial x^2} - 2\frac{S_{45}}{S_{55}}\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} = 0 \Rightarrow$$

$$\mu_1\mu_2\frac{\partial^2\psi}{\partial x^2} - (\mu_1 + \mu_2)\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} = 0 \Rightarrow$$

$$\left(\frac{\partial}{\partial y} - \mu_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - \mu_2\frac{\partial}{\partial x}\right)\psi = 0$$

11-8.

Relations (11.5.3)

$$B_{11}=\frac{S_{11}S_{33}-S_{13}^2}{S_{33}}\ ,\ B_{12}=\frac{S_{12}S_{33}-S_{13}S_{23}}{S_{33}}\ ,\ B_{22}=\frac{S_{22}S_{33}-S_{23}^2}{S_{33}}$$

$$B_{16}=\frac{S_{16}S_{33}-S_{13}S_{36}}{S_{33}}\ ,\ B_{66}=\frac{S_{66}S_{33}-S_{36}^2}{S_{33}}\ ,\ B_{26}=\frac{S_{26}S_{33}-S_{23}S_{36}}{S_{33}}$$

Using these relations into Hooke's law for plane strain $\Rightarrow$

$$e_x=B_{11}\sigma_x+B_{12}\sigma_y+B_{16}\tau_{xy}=\frac{S_{11}S_{33}-S_{13}^2}{S_{33}}\sigma_x+\frac{S_{12}S_{33}-S_{13}S_{23}}{S_{33}}\sigma_y+\frac{S_{16}S_{33}-S_{13}S_{36}}{S_{33}}\tau_{xy}$$

Now we wish to develop plane stress relations with $\sigma_z=0\Rightarrow S_{13}=S_{23}=0$

$$\therefore\ e_x=S_{11}\sigma_x+S_{12}\sigma_y+S_{16}\tau_{xy}$$

$$e_y=B_{12}\sigma_x+B_{22}\sigma_y+B_{26}\tau_{xy}=\frac{S_{12}S_{33}-S_{13}S_{23}}{S_{33}}\sigma_x+\frac{S_{22}S_{33}-S_{23}^2}{S_{33}}\sigma_y+\frac{S_{26}S_{33}-S_{23}S_{36}}{S_{33}}\tau_{xy}$$

$$=S_{12}\sigma_x+S_{22}\sigma_y+S_{26}\tau_{xy}$$

$$2e_{xy}=B_{16}\sigma_x+B_{26}\sigma_y+B_{66}\tau_{xy}=\frac{S_{16}S_{33}-S_{13}S_{36}}{S_{33}}\sigma_x+\frac{S_{26}S_{33}-S_{23}S_{36}}{S_{33}}\sigma_y+\frac{S_{66}S_{33}-S_{36}^2}{S_{33}}\tau_{xy}$$

$$=S_{16}\sigma_x+S_{26}\sigma_y+S_{66}\tau_{xy}$$

11-9.

Case 2 : $\mu_1=\mu_2=\mu=\alpha+i\beta\ ,\ \mu_3=\mu_4=\bar{\mu}$

Governing Equation : $D_1D_2D_3D_4\,\phi=0$

with $D_1=D_2=\dfrac{\partial}{\partial y}-\mu\dfrac{\partial}{\partial x}\ ,\ D_3=D_4=\dfrac{\partial}{\partial y}-\bar{\mu}\dfrac{\partial}{\partial x}$

Integrating as before yields the form

$\phi=2Re[\bar{z}_1F_1(z_1)+F_2(z_1)]$, where $z_1=x+\mu y$

In terms of the variable z_1, this formulation is similar to the isotropic case given by (10.2.7)

Another scheme of demonstrating this result is to expand the governing equation as

$$\left(\frac{\partial}{\partial y}-\mu\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}-\bar{\mu}\frac{\partial}{\partial x}\right)^2\phi=0\Rightarrow$$

$$\mu^2\bar{\mu}^2\phi_{,xxxx}-2\mu\bar{\mu}(\mu+\bar{\mu})\phi_{,xxxy}+(\mu^2+4\mu\bar{\mu}+\bar{\mu}^2)\phi_{,xxyy}-2(\mu+\bar{\mu})\phi_{,xyyy}+\phi_{,yyyy}=0$$

$$\mu\bar{\mu}=\alpha^2+\beta^2\ ,\ \mu+\bar{\mu}=2\alpha\ ,\ \mu^2+4\mu\bar{\mu}+\bar{\mu}^2=(\mu+\bar{\mu})^2+2\mu\bar{\mu}=4\alpha^2+2(\alpha^2+\beta^2)$$

Through a change in variables, this equation can be transformed into the standard biharmonic form for isotropic theory.

11-10.

$$S_{11}\mu^4 - 2S_{16}\,\mu^3 + (2S_{12} + S_{66})\mu^2 - 2S_{26}\mu + S_{22} = 0$$

For orthotropic materials $S_{16} = S_{26} = 0 \Rightarrow$

$$S_{11}\mu^4 + (2S_{12} + S_{66})\mu^2 + S_{22} = 0$$

Solving as a quadratic equation for μ^2 :

$$\mu_{1,2}^2 = \frac{1}{2}\left[-\left(\frac{2S_{12} + S_{66}}{S_{11}}\right) \pm \sqrt{\left(\frac{2S_{12} + S_{66}}{S_{11}}\right)^2 - 4\frac{S_{22}}{S_{11}}}\right]$$

For S - Glass/Epoxy material :

$$\frac{S_{22}}{S_{11}} = \frac{E_1}{E_2} = \frac{50}{17} = 2.94\ ,\ \frac{2S_{12} + S_{66}}{S_{11}} = -2\nu_{12} + \frac{E_1}{\mu_{12}} = -2(0.27) + \frac{50}{7} = 6.6$$

$$\therefore\ \mu_{1,2}^2 = \frac{1}{2}\left[-6.6 \pm \sqrt{(6.6)^2 - 4(2.94)}\right] = -0.48, -6.12 \Rightarrow$$

$$\mu_{1,2} = \pm 0.69i, \pm 2.47i$$

11-11.

$$\phi = F_1(z_1) + \overline{F_1(z_1)} + F_2(z_2) + \overline{F_2(z_2)}$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = \mu_1^2 F_1''(z_1) + \bar{\mu}_1^2 F_1''(\bar{z}_1) + \mu_2^2 F_2''(z_2) + \bar{\mu}_2^2\, F_2''(\bar{z}_2)$$

$$= \mu_1^2 F_1''(z_1) + \overline{\mu_1^2 F_1''(z_1)} + \mu_2^2 F_2''(z_2) + \overline{\mu_2^2 F_2''(z_2)}$$

$$= 2Re[\mu_1^2 F_1''(z_1) + \mu_2^2 F_2''(z_2)]$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = F_1''(z_1) + F_1''(\bar{z}_1) + F_2''(z_2) + F_2''(\bar{z}_2)$$

$$= F_1''(z_1) + \overline{F_1''(z_1)} + F_2''(z_2) + \overline{F_2''(z_2)}$$

$$= 2Re[F_1''(z_1) + F_2''(z_2)]$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\mu_1 F_1''(z_1) - \bar{\mu}_1 F_1''(\bar{z}_1) - \mu_2 F_2''(z_2) - \bar{\mu}_2\, F_2''(\bar{z}_2)$$

$$= -\mu_1 F_1''(z_1) - \overline{\mu_1 F_1''(z_1)} - \mu_2 F_2''(z_2) - \overline{\mu_2 F_2''(z_2)}$$

$$= -2Re[\mu_1 F_1''(z_1) + \mu_2 F_2''(z_2)]$$

11-12.

$\sigma_x = 2Re[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)]$
$\sigma_y = 2Re[\Phi_1'(z_1) + \Phi_2'(z_2)]$
$\tau_{xy} = -2Re[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)]$

For plane stress,

$$e_x = S_{11}\sigma_x + S_{12}\sigma_y + S_{16}\tau_{xy}$$
$$= S_{11}Re[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)] + 2S_{12}Re[\Phi_1'(z_1) + \Phi_2'(z_2)] - 2S_{16}Re[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)]$$
$$= 2Re[p_1\Phi_1'(z_1) + p_2\Phi_2'(z_2)] = \frac{\partial u}{\partial x}$$

$$e_y = S_{12}\sigma_x + S_{22}\sigma_y + S_{26}\tau_{xy} = 2Re[q_1\mu_1\Phi_1'(z_1) + q_2\mu_2\Phi_2'(z_2)] = \frac{\partial v}{\partial y}$$

$$2e_{xy} = S_{16}\sigma_x + S_{26}\sigma_y + S_{66}\tau_{xy} = 2Re[(p_1\mu_1 + q_1)\Phi_1'(z_1) + (p_2\mu_2 + q_2)\Phi_2'(z_2)] = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

where $p_i = S_{11}\mu_i^2 - S_{16}\mu_i + S_{12}$ and $q_i = S_{12}\mu_i - S_{26} + (S_{22})/\mu_i$

Integrating the stain - displacement relations $\Rightarrow$

$u = 2Re[p_1\Phi_1'(z_1) + p_2\Phi_2'(z_2)] + f(y)$
$v = 2Re[q_1\Phi_1'(z_1) + q_2\Phi_2'(z_2)] + g(x)$

Using these results in the expression for the shear strain $\Rightarrow$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2Re[p_1\mu_1\Phi_1'(z_1) + p_2\mu_2\Phi_2'(z_2)] + f'(y) + 2Re[q_1\mu_1\Phi_1'(z_1) + q_2\mu_2\Phi_2'(z_2)] + g'(x)$$
$$= 2Re[(p_1\mu_1 + q_1)\Phi_1'(z_1) + (p_2\mu_2 + q_2)\Phi_2'(z_2)] \Rightarrow$$

$f'(y) + g'(x) = 0 \Rightarrow f'(y) = -g'(x) = \text{constant} = C \Rightarrow f(y) = Cy + C_1,\ g(x) = Cx + C_2$

Note that f and g are rigid - body motion terms and may be dropped

11-13.

Stresses :

$$\sigma_x = 2Re[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)]$$
$$\sigma_y = 2Re[\Phi_1'(z_1) + \Phi_2'(z_2)]$$
$$\tau_{xy} = -2Re[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)]$$

From stress transformation theory,

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= 2Re[(\mu_1^2\cos^2\theta + \sin^2\theta - 2\mu_1\sin\theta\cos\theta)\Phi_1'(z_1) + (\mu_2^2\cos^2\theta + \sin^2\theta - 2\mu_2\sin\theta\cos\theta)\Phi_2'(z_2)]$$
$$= 2Re[(\sin\theta - \mu_1\cos\theta)^2\Phi_1'(z_1) + (\sin\theta - \mu_2\cos\theta)^2\Phi_2'(z_2)]$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$
$$= 2Re[(\mu_1^2\sin^2\theta + \cos^2\theta - 2\mu_1\sin\theta\cos\theta)\Phi_1'(z_1) + (\mu_2^2\sin^2\theta + \cos^2\theta + 2\mu_2\sin\theta\cos\theta)\Phi_2'(z_2)]$$
$$= 2Re[(\cos\theta + \mu_1\sin\theta)^2\Phi_1'(z_1) + (\cos\theta + \mu_2\sin\theta)^2\Phi_2'(z_2)]$$
$$\tau_{r\theta} = -\sigma_x\sin\theta\cos\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= 2Re[(-\mu_1^2\sin\theta\cos\theta + \sin\theta\cos\theta - \mu_1\cos^2\theta + \mu_1\sin^2\theta)\Phi_1'(z_1)$$
$$+ (-\mu_2^2\sin\theta\cos\theta + \sin\theta\cos\theta - \mu_2\cos^2\theta + \mu_2\sin^2\theta)\Phi_2'(z_2)]$$
$$= 2Re[(\sin\theta - \mu_1\cos\theta)(\cos\theta + \mu_1\sin\theta)\Phi_1'(z_1) + (\sin\theta - \mu_2\cos\theta)(\cos\theta + \mu_2\sin\theta)\Phi_2'(z_2)]$$

Displacements :

$$u = 2\mathrm{Re}[p_1\Phi_1(z_1) + p_2\Phi_2(z_2)]$$
$$v = 2\mathrm{Re}[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)]$$

From displacement transformation theory,

$$u_r = u\cos\theta - v\sin\theta$$
$$= 2Re[(p_1\cos\theta + q_1\sin\theta)\Phi_1(z_1) + (p_2\cos\theta + q_2\sin\theta)\Phi_2(z_2)]$$
$$u_\theta = -u\sin\theta + v\cos\theta$$
$$= 2Re[(q_1\cos\theta - p_1\sin\theta)\Phi_1(z_1) + (q_2\cos\theta - p_2\sin\theta)\Phi_2(z_2)]$$

11-14*.

General characteristic equation :

$$S_{11}\mu^4 - 2S_{16}\,\mu^3 + (2S_{12} + S_{66})\mu^2 - 2S_{26}\mu + S_{22} = 0$$

For orthotropic materials $S_{16} = S_{26} = 0 \Rightarrow$

$$S_{11}\mu^4 + (2S_{12} + S_{66})\mu^2 + S_{22} = 0$$

Solving as a quadratic equation for μ^2 :

$$\mu_{1,2}^2 = \frac{1}{2}\left[-\left(\frac{2S_{12} + S_{66}}{S_{11}}\right) \pm \sqrt{\left(\frac{2S_{12} + S_{66}}{S_{11}}\right)^2 - 4\frac{S_{22}}{S_{11}}}\right]$$

Defining $\mu_{1,2} = i\beta_{1,2} \Rightarrow \beta_{1,2}^2 = -\mu_{1,2}^2$

For the isotropic case, $\dfrac{S_{22}}{S_{11}} = 1$, $\dfrac{2S_{12} + S_{66}}{S_{11}} = 2 \Rightarrow \beta_{1,2}^2 = -\dfrac{1}{2}\left[-2 \pm \sqrt{2^2 - 4}\right] = 1$

Using MATLAB :

S - Glass/Epoxy : $\beta_{1,2} = 0.693, 2.474$

Boron/Epoxy : $\beta_{1,2} = 0.577, 5.545$

Carbon/Epoxy : $\beta_{1,2} = 0.788, 5.747$

Kevlar49/Epoxy : $\beta_{1,2} = 0.643, 5.784$

11-15.

With $u = v = 0$, $w = w(x,y) \Rightarrow e_x = e_y = e_z = e_{xy} = 0$,

and therefore Hooke's law for monclinic materials becomes

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ \cdot & C_{22} & C_{23} & 0 & 0 & C_{26} \\ \cdot & \cdot & C_{33} & 0 & 0 & C_{36} \\ \cdot & \cdot & \cdot & C_{44} & C_{45} & 0 \\ \cdot & \cdot & \cdot & \cdot & C_{55} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2e_{yz} \\ 2e_{zx} \\ 0 \end{bmatrix} \Rightarrow$$

$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$

$$\tau_{xz} = 2C_{45}e_{yz} + 2C_{55}e_{xz} = C_{45}\frac{\partial w}{\partial y} + C_{55}\frac{\partial w}{\partial x}$$

$$\tau_{yz} = 2C_{44}e_{yz} + 2C_{45}e_{xz} = C_{44}\frac{\partial w}{\partial y} + C_{45}\frac{\partial w}{\partial x}$$

Equilbrium Eqns. $\Rightarrow \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = 0 \Rightarrow C_{55}\dfrac{\partial^2 w}{\partial x^2} + 2C_{45}\dfrac{\partial^2 w}{\partial x \partial y} + C_{44}\dfrac{\partial^2 w}{\partial y^2} = 0$

Looking for solutions $w = F(x + \mu y)$ gives $(C_{55} + 2C_{45}\mu + C_{44}\mu^2)F'' = 0$

Cancelling the common F'' term produces the quadratic characteristic equation

$C_{44}\mu^2 + 2C_{45}\mu + C_{55} = 0$

Solving for the two roots gives : $\mu = \dfrac{-C_{45} \pm \sqrt{C_{45}^2 - C_{44}C_{55}}}{C_{44}}$

As previously shown $C_{44}C_{55} > C_{45}^2 \Rightarrow$ roots are complex conjugate pairs $(\mu, \bar{\mu})$

and so general solution to governing equation can be written as

$w = F_1(x + \mu y) + F_2(x + \bar{\mu} y) = F_1(z^*) + F_2(\bar{z}^*)$, where $z^* = x + \mu y$

Since w must be real $\Rightarrow w = F_1(z^*) + \overline{F}_1(\bar{z}^*) = 2Re\{F_1(z^*)\} = 2Re\{F(z^*)\}$

and the stresses then can be written as

$\tau_{xz} = 2Re\{(\mu C_{45} + C_{55})F'(z^*)\}$

$\tau_{yz} = 2Re\{(\mu C_{44} + C_{45})F'(z^*)\}$

11-16*.

For the case : $X = 0$, $\mu_i = i\beta_i$

$$A_1 = \frac{(X+\mu_2 Y)}{2i\pi(\mu_2-\mu_1)} = \frac{\beta_2 Y}{2i\pi(\beta_2-\beta_1)}\ ,\ A_2 = \frac{(X+\mu_1 Y)}{2i\pi(\mu_1-\mu_2)} = \frac{\beta_1 Y}{2i\pi(\beta_1-\beta_2)}$$

$$\Phi_1'(z_1) = \frac{A_1}{z_1} = \frac{-\beta_1\beta_2 Yy - i\beta_2 Yx}{2\pi(\beta_2-\beta_1)(x^2+\beta_1^2 y^2)}\ ,\ \Phi_2'(z_2) = \frac{A_2}{z_2} = \frac{-\beta_1\beta_2 Yy - i\beta_2 Yx}{2\pi(\beta_1-\beta_2)(x^2+\beta_2^2 y^2)}$$

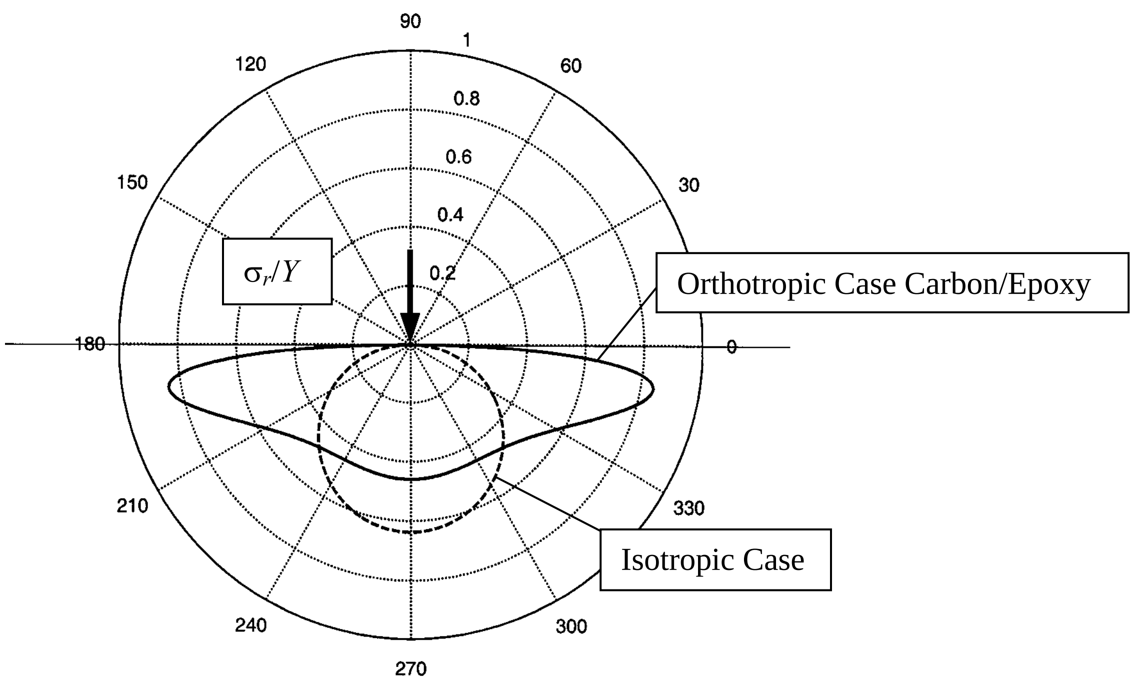
$$\begin{aligned}\sigma_r &= 2\,\mathrm{Re}[(\sin\theta-\mu_1\cos\theta)^2\Phi_1'(z_1)+(\sin\theta-\mu_2\cos\theta)^2\Phi_2'(z_2)]\\ &= -\frac{Y\beta_1\beta_2(\beta_1+\beta_2)\sin\theta}{\pi r(\cos^2\theta+\beta_1^2\sin^2\theta)(\cos^2\theta+\beta_2^2\sin^2\theta)}\\ &= \frac{\beta_1\beta_2 Y\sin\theta}{\pi r(\beta_1-\beta_2)}\left[\frac{1+\cos^2\theta-\beta_1^2}{\cos^2\theta+\beta_1^2\sin^2\theta} - \frac{1+\cos^2\theta-\beta_2^2}{\cos^2\theta+\beta_2^2\sin^2\theta}\right]\end{aligned}$$

$$\begin{aligned}\sigma_\theta &= 2\,\mathrm{Re}[(\cos\theta+\mu_1\sin\theta)^2\Phi_1'(z_1)+(\cos\theta+\mu_2\sin\theta)^2\Phi_2'(z_2)]\\ &= -\frac{\beta_1\beta_2 Y\sin\theta}{\pi r(\beta_1-\beta_2)}\left[\frac{\cos^2\theta+\beta_1^2\sin^2\theta}{\cos^2\theta+\beta_1^2\sin^2\theta} - \frac{\cos^2\theta+\beta_2^2\sin^2\theta}{\cos^2\theta+\beta_2^2\sin^2\theta}\right] = 0\end{aligned}$$

$$\begin{aligned}\tau_{r\theta} &= 2\,\mathrm{Re}[(\sin\theta-\mu_1\cos\theta)(\cos\theta+\mu_1\sin\theta)\Phi_1'(z_1)\\ &\quad +(\sin\theta-\mu_2\cos\theta)(\cos\theta+\mu_2\sin\theta)\Phi_2'(z_2)]\\ &= -\frac{\beta_1\beta_2 Y\cos\theta}{\pi r(\beta_1-\beta_2)} + \frac{\beta_1\beta_2 Y\cos\theta}{\pi r(\beta_1-\beta_2)} = 0\end{aligned}$$

For the isotropic case, $\beta_1 = \beta_2 = 1 \Rightarrow \sigma_r = -\dfrac{2Y\sin\theta}{\pi r}$, $\sigma_\theta = \tau_{r\theta} = 0$

MATLAB Plots for orthotropic and isotropic cases :



11-17*.

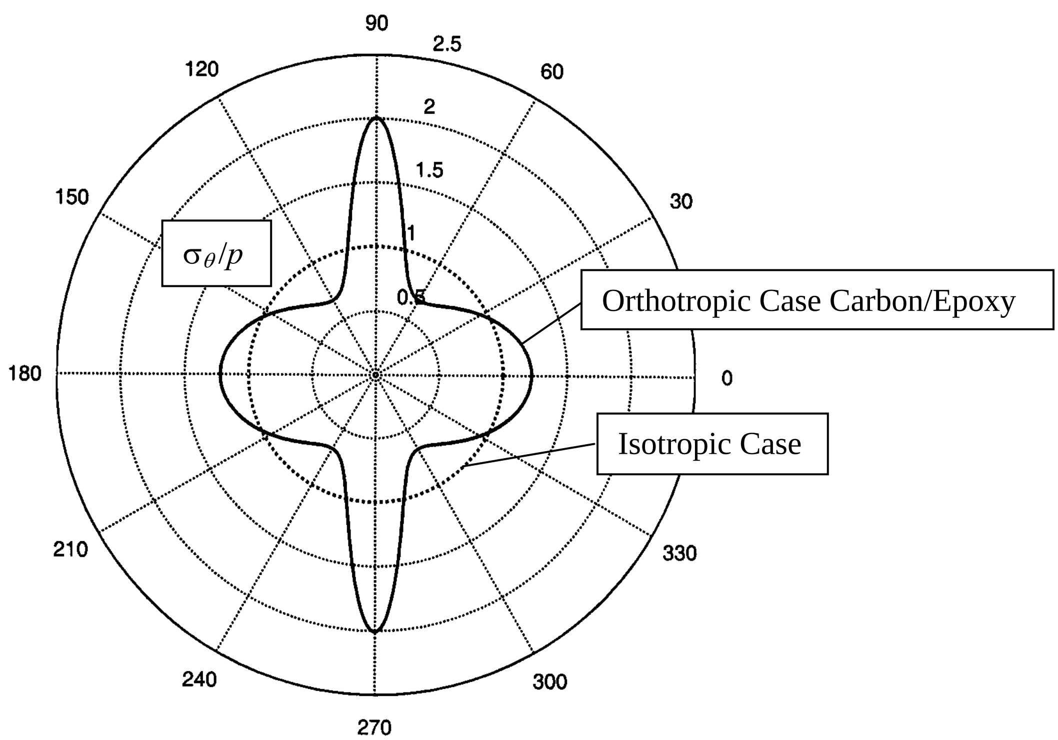
The hoop stress solution for the pressurized circular hole problem was given by (11.5.47)

$$\sigma_\theta = p\,\mathrm{Re}\left\{\frac{ie^{-i\theta}}{(\sin\theta-\mu_1\cos\theta)(\sin\theta-\mu_2\cos\theta)}\right.$$
$$\cdot[(\mu_1\mu_2 - i\mu_1 - i\mu_2)\sin^3\theta + i(\mu_1\mu_2-2)\sin^2\theta\cos\theta$$
$$\left. + (2\mu_1\mu_2-1)\sin\theta\cos^2\theta + (\mu_1+\mu_2-i)\cos^3\theta]\right\}$$

For the orthotropic case with $\mu_{1,2} = i\beta_{1,2} \Rightarrow$

$$\sigma_\theta = p\,\mathrm{Re}\left\{\frac{ie^{-i\theta}}{(\sin\theta-i\beta_1\cos\theta)(\sin\theta-i\beta_2\cos\theta)}\right.$$
$$\cdot[(-\beta_1\beta_2 - \beta_1 + \beta_2)\sin^3\theta + i(\beta_1\beta_2-2)\sin^2\theta\cos\theta$$
$$\left. + (-2\beta_1\beta_2-1)\sin\theta\cos^2\theta + i(\beta_1+\beta_2-1)\cos^3\theta]\right\}$$

MATLAB Plot : (Note higher stress regions for orthotropic materials)



11-18*.

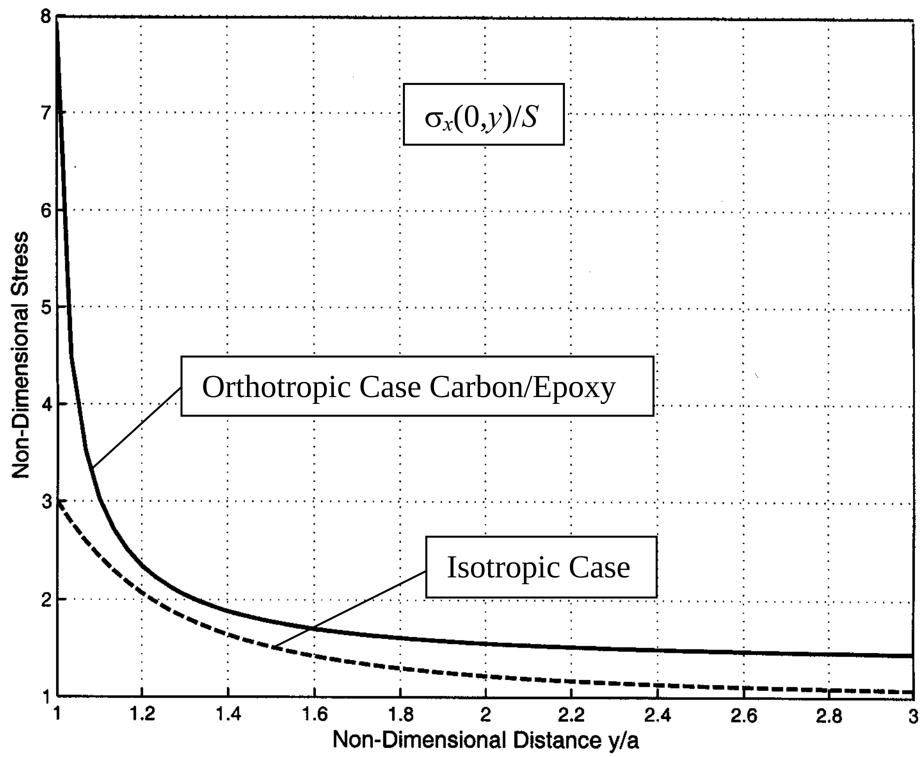
From solution (11.5.51) for the circular hole case

$$\sigma_x(0,y) = S + \frac{S}{(\beta_1-\beta_2)}\left[-\frac{\beta_1^2}{(1-\beta_1)}\left(\frac{\beta_1 y}{\sqrt{a^2+\beta_1^2(y^2-a^2)}}-1\right)+\frac{\beta_2^2}{(1-\beta_2)}\left(\frac{\beta_2 y}{\sqrt{a^2+\beta_2^2(y^2-a^2)}}-1\right)\right]$$

Solution for the isotropic case comes from (8.4.15)

$$\sigma_x(0,y) = \sigma_\theta(\pi/2,y) = \frac{S}{2}\left(1+\frac{a^2}{y^2}\right)+\frac{S}{2}\left(1+\frac{3a^4}{y^4}\right) = S+\frac{S}{2}\left(\frac{a^2}{y^2}+\frac{3a^4}{y^4}\right)$$

MATLAB Plots :



11-19.

From solution (11.5.50) with $a \to 0$,

$$\sigma_x = S + Re\left[-\frac{iSb\mu_1^2}{(\mu_1-\mu_2)(i\mu_1 b)}\left(\frac{z_1}{\sqrt{z_1^2-\mu_1^2 b^2}}-1\right)+\frac{iSb\mu_2^2}{(\mu_1-\mu_2)(i\mu_2 b)}\left(\frac{z_2}{\sqrt{z_2^2-\mu_2^2 b^2}}-1\right)\right]$$

$$= S + Re\left[-\frac{S\mu_1}{(\mu_1-\mu_2)}\left(\frac{z_1}{\sqrt{z_1^2-\mu_1^2 b^2}}-1\right)+\frac{S\mu_2}{(\mu_1-\mu_2)}\left(\frac{z_2}{\sqrt{z_2^2-\mu_2^2 b^2}}-1\right)\right]$$

Now for $x = 0$, and $y \ge b$, $z_1 = \mu_1 y$ and $z_2 = \mu_2 y \Rightarrow$

$$\sigma_x = S + Re\left[-\frac{S\mu_1}{(\mu_1-\mu_2)}\left(\frac{y}{\sqrt{y^2-b^2}}-1\right)+\frac{S\mu_2}{(\mu_1-\mu_2)}\left(\frac{y}{\sqrt{y^2-b^2}}-1\right)\right] = S + Re\left[S - S\left(\frac{y}{\sqrt{y^2-b^2}}\right)\right]$$

$$= 2S - S\left(\frac{y}{\sqrt{y^2-b^2}}\right)$$

and likewise for the other stress components,

$$\sigma_y = Re\left[-\frac{S}{(\mu_1-\mu_2)\mu_1}\left(\frac{z_1}{\sqrt{z_1^2-\mu_1^2 b^2}}-1\right)+\frac{S}{(\mu_1-\mu_2)\mu_2}\left(\frac{z_2}{\sqrt{z_2^2-\mu_2^2 b^2}}-1\right)\right] = \frac{S}{Re(\mu_1\mu_2)}\left(\frac{y}{\sqrt{y^2-b^2}}-1\right)$$

$$\tau_{xy} = -Re\left[-\frac{S}{(\mu_1-\mu_2)}\left(\frac{z_1}{\sqrt{z_1^2-\mu_1^2 b^2}}-1\right)+\frac{S}{(\mu_1-\mu_2)}\left(\frac{z_2}{\sqrt{z_2^2-\mu_2^2 b^2}}-1\right)\right] = 0$$

Note the singular nature of the stresses at the crack tip of order $O(1/\sqrt{r})$

11-20.

$$\Phi_1(z_1) = A_1 z_1 + \frac{Sa^2\mu_2}{2(\mu_1-\mu_2)}\left(z_1+\sqrt{z_1^2-a^2}\right)^{-1} \Rightarrow \Phi_1'(z_1) = A_1 - \frac{Sa^2\mu_2}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_1^2-a^2}\left(z_1+\sqrt{z_1^2-a^2}\right)}$$

$$\Phi_2(z_2) = A_2 z_2 - \frac{Sa^2\mu_1}{2(\mu_1-\mu_2)}\left(z_2+\sqrt{z_2^2-a^2}\right)^{-1} \Rightarrow \Phi_2'(z_1) = A_2 - \frac{Sa^2\mu_1}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_2^2-a^2}\left(z_2+\sqrt{z_2^2-a^2}\right)}$$

$$\sigma_x = 2Re[\mu_1^2\Phi_1'(z_1)+\mu_2^2\Phi_2'(z_2)]$$

$$= 2Re\left[A_1\mu_1^2 - \frac{Sa^2\mu_2\mu_1^2}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_1^2-a^2}\left(z_1+\sqrt{z_1^2-a^2}\right)} + A_2\mu_2^2 - \frac{Sa^2\mu_1\mu_2^2}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_2^2-a^2}\left(z_2+\sqrt{z_2^2-a^2}\right)}\right]$$

$$\sigma_y = 2Re[\Phi_1'(z_1)+\Phi_2'(z_2)]$$

$$= 2Re\left[A_1 - \frac{Sa^2\mu_2}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_1^2-a^2}\left(z_1+\sqrt{z_1^2-a^2}\right)} + A_2 - \frac{Sa^2\mu_1}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_2^2-a^2}\left(z_2+\sqrt{z_2^2-a^2}\right)}\right]$$

$$\tau_{xy} = -2Re[\mu_1\Phi_1'(z_1)+\mu_2\Phi_2'(z_2)]$$

$$= -2Re\left[A_1\mu_1 - \frac{Sa^2\mu_1\mu_2}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_1^2-a^2}\left(z_1+\sqrt{z_1^2-a^2}\right)} + A_2\mu_2 - \frac{Sa^2\mu_1\mu_2}{2(\mu_1-\mu_2)}\frac{1}{\sqrt{z_2^2-a^2}\left(z_2+\sqrt{z_2^2-a^2}\right)}\right]$$

Far - Field Behavior : $|z_1| \to \infty$, $|z_2| \to \infty$

$$\sigma_x = 2Re[A_1\mu_1^2 + A_2\mu_2^2]$$

$$= 2\left[\frac{(\alpha_2^2+\beta_2^2)(\alpha_1^2-\beta_1^2)S}{2[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]} + \frac{(\alpha_1^2-\beta_1^2-2\alpha_1\alpha_2)(\alpha_2^2-\beta_2^2)S}{2[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]} - \frac{2\alpha_2\beta_2[\alpha_2(\alpha_1^2-\beta_1^2)-\alpha_1(\alpha_2^2-\beta_2^2)]S}{2\beta_2[(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)]}\right]$$

$$= 0$$

$$\sigma_y = 2Re[A_1 + A_2] = 2\left[\frac{[(\alpha_2^2+\beta_2^2)+(\alpha_1^2-\beta_1^2-2\alpha_1\alpha_2)]S}{2[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]}\right] = \frac{[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]}{[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]}S = S$$

$$\tau_{xy} = -2Re[A_1\mu_1 + A_2\mu_2]$$

$$= -2\left[\frac{\alpha_1(\alpha_2^2+\beta_2^2)S}{2[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]} + \frac{\alpha_2(\alpha_1^2-\beta_1^2-2\alpha_1\alpha_2)S}{2[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2]} - \frac{\beta_2[\alpha_2(\alpha_1^2-\beta_1^2)-\alpha_1(\alpha_2^2-\beta_2^2)]S}{2\beta_2[(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)]}\right]$$

$$= 0$$

11-20. Continued

At Crack Tip :

$$z_1 - a = \hat{z}_1 = r\cos\theta + \mu_1 r\sin\theta \ , \ z_1 \approx a \ , \ z_1 + a \approx 2a \Rightarrow$$

$$\frac{1}{\sqrt{z_1^2 - a^2}\left(z_1 + \sqrt{z_1^2 - a^2}\right)} \approx \frac{1}{\sqrt{2a\hat{z}_1}\left(a + \sqrt{2a\hat{z}_1}\right)} \approx \frac{1}{a\sqrt{2a\hat{z}_1}}$$

$$z_2 - a = \hat{z}_2 = r\cos\theta + \mu_2 r\sin\theta \ , \ z_2 \approx a \ , \ z_2 + a \approx 2a \Rightarrow$$

$$\frac{1}{\sqrt{z_2^2 - a^2}\left(z_2 + \sqrt{z_2^2 - a^2}\right)} \approx \frac{1}{a\sqrt{2a\hat{z}_2}}$$

$$\sigma_x = Re\left[\frac{S\sqrt{a}\mu_1\mu_2}{\sqrt{2}(\mu_1 - \mu_2)}\left(\frac{\mu_2}{\sqrt{\hat{z}_2}} - \frac{\mu_1}{\sqrt{\hat{z}_1}}\right)\right] = Re\left[\frac{S\sqrt{a}\mu_1\mu_2}{\sqrt{2r}(\mu_1 - \mu_2)}\left(\frac{\mu_2}{\sqrt{\cos\theta + \mu_2\sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta + \mu_1\sin\theta}}\right)\right]$$

$$= \frac{S\sqrt{a}}{\sqrt{2r}} Re\left[\frac{\mu_1\mu_2}{\mu_1 - \mu_2}\left(\frac{\mu_2}{\sqrt{\cos\theta - \mu_2\sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta - \mu_1\sin\theta}}\right)\right]$$

Likewise at the crack tip, the other stress components reduce to

$$\sigma_y = \frac{S\sqrt{a}}{\sqrt{2r}} Re\left[\frac{1}{\mu_1 - \mu_2}\left(\frac{\mu_1}{\sqrt{\cos\theta - \mu_2\sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta - \mu_1\sin\theta}}\right)\right]$$

$$\tau_{xy} = \frac{S\sqrt{a}}{\sqrt{2r}} Re\left[\frac{\mu_1\mu_2}{\mu_1 - \mu_2}\left(\frac{1}{\sqrt{\cos\theta - \mu_1\sin\theta}} - \frac{1}{\sqrt{\cos\theta - \mu_2\sin\theta}}\right)\right]$$

11-21.

From Exercise 11-15:

$$w = 2Re\{F(z^*)\}\ ,\ \tau_{xz} = 2Re\{(\mu C_{45} + C_{55})F'(z^*)\}\ ,\ \tau_{yz} = 2Re\{(\mu C_{44} + C_{45})F'(z^*)\}$$

From characteristic equation: $\mu C_{45} + C_{55} = -\mu(C_{45} + \mu C_{44})$

Choosing $F(z^*) = A\sqrt{z^*}$, where $A = -\sqrt{2}K_3\mu/(C_{55} + \mu C_{45}) \Rightarrow F'(z^*) = A/2\sqrt{z^*}$

$$w = 2Re\{F(z^*)\} = 2Re\{A\sqrt{r}\sqrt{\cos\theta + \mu\sin\theta}\}$$

$$= -K_3\sqrt{2r}Re\left\{\frac{\mu\sqrt{\cos\theta + \mu\sin\theta}}{C_{55} + \mu C_{45}}\right\} = K_3\sqrt{2r}Re\left\{\frac{\sqrt{\cos\theta + \mu\sin\theta}}{C_{45} + \mu C_{44}}\right\}$$

$$\tau_{xz} = 2Re\{(\mu C_{45} + C_{55})F'(z^*)\} = \frac{1}{2\sqrt{r}}Re\left\{A\frac{\mu C_{45} + C_{55}}{\sqrt{\cos\theta + \mu\sin\theta}}\right\} = -\frac{K_3}{\sqrt{2r}}Re\left\{\frac{\mu}{\sqrt{\cos\theta + \mu\sin\theta}}\right\}$$

$$\tau_{yz} = 2Re\{(\mu C_{44} + C_{45})F'(z^*)\} = \frac{1}{2\sqrt{r}}Re\left\{A\frac{\mu C_{44} + C_{45}}{\sqrt{\cos\theta + \mu\sin\theta}}\right\} = \frac{K_3}{\sqrt{2r}}Re\left\{\frac{1}{\sqrt{\cos\theta + \mu\sin\theta}}\right\}$$

Note $\tau_{yz}(r, \pm\pi) = \frac{K_3}{\sqrt{2r}}Re\left\{\frac{1}{\sqrt{-1}}\right\} = \frac{K_3}{\sqrt{2r}}Re\{-i\} = 0$

11-22.

$$\sigma_r = \frac{E_r}{1-\nu_{\theta r}\nu_{r\theta}}\left(\frac{du}{dr} - \nu_{\theta r}\frac{u}{r}\right),\ \sigma_\theta = \frac{E_\theta}{1-\nu_{\theta r}\nu_{r\theta}}\left(\frac{u}{r} - \nu_{r\theta}\frac{du}{dr}\right),\ \frac{\nu_{\theta r}}{E_\theta} = \frac{\nu_{r\theta}}{E_r}$$

Equilibrium Eqn : $\dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 \Rightarrow$

$$\frac{E_r}{1-\nu_{\theta r}\nu_{r\theta}}\left(\frac{d^2u}{dr^2} + \nu_{\theta r}\frac{1}{r}\frac{du}{dr} - \nu_{\theta r}\frac{u}{r^2}\right) + \frac{1}{1-\nu_{\theta r}\nu_{r\theta}}\left[(E_r - E_\theta\nu_{r\theta})\frac{du}{dr} + (E_r\nu_{\theta r} - E_\theta)\frac{u}{r^2}\right] = 0$$

$$E_r\frac{d^2u}{dr^2} + E_r\frac{1}{r}\frac{du}{dr} - E_\theta\frac{u}{r^2} = 0 \Rightarrow \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - n^2\frac{u}{r^2} = 0\text{, where } n^2 = E_\theta / E_r$$

Looking for solutions of the form : $u = Cr^m \Rightarrow$

$m(m-1) + m - n^2 = 0 \Rightarrow m^2 = n^2 \Rightarrow m = \pm n$, $\therefore$ Solution is $u = Ar^n + Br^{-n}$

$$\sigma_r = \frac{E_r}{1-\nu_{\theta r}\nu_{r\theta}}\left(\frac{du}{dr} - \nu_{\theta r}\frac{u}{r}\right) = A\frac{E_r}{1-\nu_{\theta r}\nu_{r\theta}}(n-\nu_{\theta r})r^{n-1} - B\frac{E_r}{1-\nu_{\theta r}\nu_{r\theta}}(n+\nu_{\theta r})r^{-n-1}$$

$$\sigma_\theta = \frac{E_\theta}{1-\nu_{\theta r}\nu_{r\theta}}\left(\frac{u}{r} - \nu_{r\theta}\frac{du}{dr}\right) = A\frac{E_r n}{1-\nu_{\theta r}\nu_{r\theta}}(n-\nu_{\theta r})r^{n-1} + B\frac{E_r n}{1-\nu_{\theta r}\nu_{r\theta}}(n+\nu_{\theta r})r^{-n-1}$$

Rewriting the radial stress form as : $\sigma_r = C_1 r^{n-1} + C_2 r^{-n-1}$ and applying the Boundary Conditions :

$$\sigma_r(a) = 0,\ \sigma_r(b) = -p \Rightarrow C_1 = -\frac{pb^{n+1}}{b^{2n} - a^{2n}},\ C_2 = \frac{pb^{n+1}a^{2n}}{b^{2n} - a^{2n}}$$

$$\therefore \sigma_r = -\frac{pb^{n+1}}{b^{2n} - a^{2n}}\left(r^{n-1} - a^{2n}r^{-n-1}\right),\ \sigma_\theta = -\frac{pb^{n+1}n}{b^{2n} - a^{2n}}\left(r^{n-1} + a^{2n}r^{-n-1}\right)$$

12-1.

$$\text{Thermal Strains: } e_{ij}^{(T)} = \alpha(T - T_o)\delta_{ij}$$

$$\text{Mechanical Strains: } e_{ij}^{(M)} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$e_{ij} = e_{ij}^{(M)} + e_{ij}^{(T)} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_o)\delta_{ij} \Rightarrow$$

$$e_{kk} = \frac{1-2\nu}{E}\sigma_{kk} + 3\alpha(T - T_o) \Rightarrow \sigma_{kk} = \frac{E}{1-2\nu}\left(e_{kk} - 3\alpha(T - T_o)\right)$$

$$\therefore \sigma_{ij} = \frac{E}{1+\nu}\left[e_{ij} + \frac{\nu}{1-2\nu}\left(e_{kk} - 3\alpha(T - T_o)\right)\delta_{ij} - \alpha(T - T_o)\delta_{ij}\right]$$

$$= \lambda e_{kk} + 2\mu e_{ij} - (3\lambda + 2\mu)\alpha(T - T_o)\delta_{ij}$$

12-2.

$$\text{Strain Compatibility: } e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

$$\text{Hooke's Law: } e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_o)\delta_{ij}$$

Using Hooke's law in the strain compatibility relations and setting $k = l \Rightarrow$

$$\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} =$$

$$\frac{\nu}{1+\nu}(\sigma_{mm,kk}\delta_{ij} + \sigma_{mm,ij}\delta_{kk} - \sigma_{mm,jk}\delta_{ik} - \sigma_{mm,ik}\delta_{jk}) - \frac{E\alpha}{1+\nu}(T_{,kk}\delta_{ij} + T_{,ij}\delta_{kk} - T_{,jk}\delta_{ik} - T_{,ik}\delta_{jk})$$

Note that with zero body forces, $\sigma_{\mathrm{ij,j}} = 0 \Rightarrow$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1+\nu}\sigma_{mm,kk}\delta_{ij} - \frac{E\alpha}{1+\nu}(T_{,kk}\delta_{ij} + T_{,ij})$$

Setting $i = j \Rightarrow \sigma_{mm,kk} = -\dfrac{2E\alpha}{1-\nu}T_{,kk}$

Combining these results gives

$$\sigma_{ij,kk} + \frac{1}{(1+\nu)}\sigma_{kk,,ij} = -\frac{E\alpha}{1+\nu}(T_{,ij} + \frac{1+\nu}{1-\nu}\delta_{ij}T_{,kk})$$

12-3.

Unrestricted thermal expansion : $e_{ij} = \alpha T(x,y,z)\delta_{ij} \Rightarrow$

$e_x = e_y = e_z = \alpha T$, $e_{xy} = e_{yz} = e_{zx} = 0$

Using strain compatibility relations $\Rightarrow$

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = 2\frac{\partial^2 e_{yz}}{\partial y \partial z} \Rightarrow \frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = 2\frac{\partial^2 e_{zx}}{\partial z \partial x} \Rightarrow \frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

The above three relations $\Rightarrow \dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial^2 T}{\partial z^2} = 0$

$$\frac{\partial^2 e_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} \right) \Rightarrow \frac{\partial^2 e_x}{\partial y \partial z} = 0 \Rightarrow \frac{\partial^2 T}{\partial y \partial z} = 0$$

$$\frac{\partial^2 e_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} \right) \Rightarrow \frac{\partial^2 e_y}{\partial z \partial x} = 0 \Rightarrow \frac{\partial^2 T}{\partial z \partial x} = 0$$

$$\frac{\partial^2 e_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} \right) \Rightarrow \frac{\partial^2 e_z}{\partial x \partial y} = 0 \Rightarrow \frac{\partial^2 T}{\partial x \partial y} = 0$$

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \frac{\partial T}{\partial x} = f(y,z) \text{ , } \frac{\partial^2 T}{\partial x \partial y} = 0 \Rightarrow \frac{\partial T}{\partial x} = g(x,z) \text{ , } \frac{\partial^2 T}{\partial z \partial x} = 0 \Rightarrow \frac{\partial T}{\partial x} = h(x,y)$$

$\therefore \dfrac{\partial T}{\partial x} = \text{constant} = a$, and following similar steps it can be shown that $\dfrac{\partial T}{\partial y} = b$, $\dfrac{\partial T}{\partial z} = c$

$$\frac{\partial T}{\partial x} = a \Rightarrow T = ax + F(y,z)$$

$$\frac{\partial T}{\partial y} = b \Rightarrow \frac{\partial F}{\partial y} = b \Rightarrow F = by + G(z)$$

$$\frac{\partial T}{\partial z} = c \Rightarrow \frac{dG}{dz} = c \Rightarrow G = cz + d$$

$\therefore T = ax + by + cz + d$

12-4.

For Plane Stress :

$$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha(T - T_o)\ ,\ e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha(T - T_o)\ ,\ e_{xy} = \frac{1+\nu}{E}\tau_{xy} \Rightarrow$$

$$\sigma_x = \frac{E}{1-\nu^2}(e_x + \nu e_y) - \frac{E\alpha}{1-\nu}(T - T_o)$$

$$\sigma_y = \frac{E}{1-\nu^2}(e_y + \nu e_x) - \frac{E\alpha}{1-\nu}(T - T_o)$$

$$\tau_{xy} = \frac{E}{1+\nu}e_{xy}$$

$$T_x^n = \sigma_x n_x + \tau_{xy} n_y = (T_x^n)_s \Rightarrow$$

$$\left[\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) - \frac{E\alpha}{1-\nu}(T - T_o)\right]n_x + \left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]n_y = (T_x^n)_s$$

$$T_y^n = \tau_{xy} n_x + \sigma_y n_y = (T_y^n)_s \Rightarrow$$

$$\left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]n_x + \left[\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) - \frac{E\alpha}{1-\nu}(T - T_o)\right]n_y = (T_y^n)_s$$

12-5.

Plane Strain :

$\sigma_x = \lambda(e_x + e_y) + 2\mu e_x - \alpha(3\lambda + 2\mu)(T - T_o)$, $\sigma_y = \lambda(e_x + e_y) + 2\mu e_y - \alpha(3\lambda + 2\mu)(T - T_o)$, $\tau_{xy} = 2\mu e_{xy}$

Solving for the strains $\Rightarrow$

$$e_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y + E\alpha(T - T_o)] \, , \; e_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x + E\alpha(T - T_o)] \, , \; e_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

Using these strains in the compatibility relation $\dfrac{\partial^2 e_x}{\partial y^2} + \dfrac{\partial^2 e_y}{\partial x^2} = 2\dfrac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow$

$$\frac{\partial^2}{\partial y^2}\big([(1-\nu)\sigma_x - \nu\sigma_y + E\alpha(T - T_o)]\big) + \frac{\partial^2}{\partial x^2}\big([(1-\nu)\sigma_y - \nu\sigma_x + E\alpha(T - T_o)]\big) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \Rightarrow$$

$$(1-\nu)\nabla^2(\sigma_x + \sigma_y) + E\alpha\nabla^2 T = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}$$

But from equilibrium equations with zero body forces,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \Rightarrow \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \, , \; \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \Rightarrow \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial y \partial x} = 0$$

$\therefore \; 2\dfrac{\partial^2 \tau_{xy}}{\partial x \partial y} + \dfrac{\partial^2 \sigma_x}{\partial x^2} + \dfrac{\partial^2 \sigma_y}{\partial y^2} = 0$ and thus the previous compatibility statement reduces to

$$\nabla^2(\sigma_x + \sigma_y) + \frac{E\alpha}{1-\nu}\nabla^2 T = 0$$

To determine the corresponding plane stress result, simply use the transformation Table 12-1 $\Rightarrow$

$$E \to \frac{E(1+2\nu)}{(1+\nu)^2} \, , \; \nu \to \frac{\nu}{1+\nu} \, , \; \alpha \to \frac{1+\nu}{1+2\nu}\alpha \Rightarrow \frac{E\alpha}{1-\nu} \to E\alpha \, ,$$

and thus the compatibility relation for plane stress becomes $\nabla^2(\sigma_x + \sigma_y) + E\alpha\nabla^2 T = 0$

12-6.

From Example 12 - 1, the stresses were given by

$$\sigma_x = -\beta^2[C_2 \cosh\beta x + C_3 x \sinh\beta x]\sin\beta y - E\alpha T_o \sin\beta y$$

$$\sigma_y = \beta^2\left[C_2 \cosh\beta x + C_3(x\sinh\beta x + \frac{2}{\beta}\cosh\beta x)\right]\sin\beta y$$

$$\tau_{xy} = -\beta^2\left[C_2 \sinh\beta x + C_3(x\cosh\beta x + \frac{1}{\beta}\sinh\beta x)\right]\cos\beta y$$

Stress free boundary conditions : $\sigma_x(\pm a, y) = \tau_{xy}(\pm a, y) = 0 \Rightarrow$

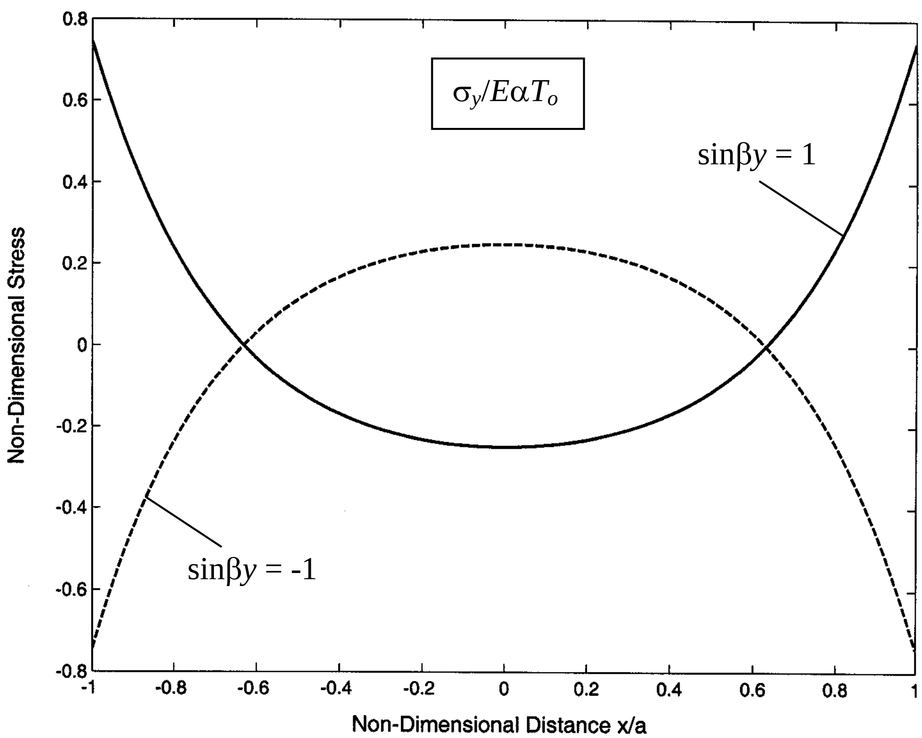
$$C_2 \cosh\beta a + C_3 a \sinh\beta a = -E\alpha T_o / \beta^2$$

$$C_2 \sinh\beta a + C_3(a\cosh\beta a + \frac{1}{\beta}\sinh\beta a) = 0$$

Solving for the two constants $\Rightarrow$

$$C_2 = -\frac{E\alpha T_o(a\beta\cosh\beta a + \sinh\beta a)}{\beta^2 a(\beta + \sinh\beta a\cosh\beta a)}\ ,\ C_3 = \frac{E\alpha T_o \sinh\beta a}{\beta a(\beta + \sinh\beta a\cosh\beta a)}$$

MATLAB Plots $(a = 1, \beta = 2)$:



12-7.

For the radially symmetric case,

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right),\ \nabla^4 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right)\right)$$

$\therefore$ the plane stress governing equation $\nabla^4\phi + E\alpha\nabla^2 T = 0 \Rightarrow$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right)\right) + E\alpha\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0\ ,\ \text{and since } \sigma_r = \frac{1}{r}\frac{d\phi}{dr} \Rightarrow$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right)\right)\right) = -E\alpha\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

Integrating this result three times $\Rightarrow$

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right)\right) = -E\alpha r\frac{dT}{dr} + C_1 \Rightarrow$$

$$\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right) = -E\alpha T + C_1\log r + C_2 \Rightarrow$$

$$r^2\sigma_r = -E\alpha\int Tdr + C_1\int r\log r dr + C_2 r^2 + C_3 \Rightarrow$$

$$\sigma_r = -\frac{E\alpha}{r^2}\int Tdr + \frac{C_1}{4}(2\log r - 1) + C_2 + \frac{C_3}{r^2}$$

12-8.

Using axisymmetric stress forms :

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\frac{u}{r} - (1+\nu)\alpha T\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr} - (1+\nu)\alpha T\right]$$

Into Equilibrium Eqn. : $\dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 \Rightarrow$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1+\nu)\alpha\frac{dT}{dr}\ ,\ \text{which can be written as}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = (1+\nu)\alpha\frac{dT}{dr}$$

Integrating the governing equation $\Rightarrow$

$$\frac{1}{r}\frac{d}{dr}(ru) = (1+\nu)\alpha T + \text{constant}\ ,\ \text{intergrating again} \Rightarrow$$

$$u = A_1 r + \frac{A_2}{r} + \frac{(1+\nu)\alpha}{r}\int Trdr$$

12-9.

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+h_o=0 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-h_o$$

Integrating $\Rightarrow T=C_1\log r+C_2-\frac{h_o}{4}r^2$

Boundary Conditions :

$T(0)$ must remain bounded $\Rightarrow C_1=0$

$$T(a)=T_o \Rightarrow C_2=T_o+\frac{h_o}{4}a^2$$

$$\therefore T=T_o+\frac{h_o}{4}(a^2-r^2)$$

Governing equation for plane strain, $\nabla^4\phi+\frac{E\alpha}{1-\nu}\nabla^2 T=0 \Rightarrow \nabla^4\phi=\frac{E\alpha h_o}{1-\nu}\Rightarrow$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right)\right)\right)=\frac{E\alpha h_o}{1-\nu}$$

Integrating the governing equation three times $\Rightarrow$

$$\sigma_r=\frac{C_1}{4}(2\log r-1)+C_2+\frac{C_3}{r^2}+\frac{E\alpha h_o}{16(1-\nu)}r^2$$

$$\sigma_\theta=\frac{d^2\phi}{dr^2}=\frac{d}{dr}(r\sigma_r)=\frac{C_1}{4}(2\log r+1)+C_2-\frac{C_3}{r^2}+\frac{3E\alpha h_o}{16(1-\nu)}r^2\ ,\ \tau_{r\theta}=0$$

$$\sigma_z=\nu(\sigma_r+\sigma_\theta)-E\alpha(T-T_o)=\nu(\sigma_r+\sigma_\theta)-E\alpha\frac{h_o}{4}(a^2-r^2)$$

Since the cylinder is solid, the stresses must be bounded at $r=0 \Rightarrow C_1=C_3=0$

Boundary condition $\sigma_r(a)=0 \Rightarrow C_2=-\frac{E\alpha h_o}{16(1-\nu)}a^2$

$$\therefore \sigma_r=\frac{E\alpha h_o}{16(1-\nu)}\left(r^2-a^2\right),\ \sigma_\theta=\frac{E\alpha h_o}{16(1-\nu)}\left(3r^2-a^2\right),\ \tau_{r\theta}=0$$

$$\sigma_z=\nu\left(\frac{E\alpha h_o}{4(1-\nu)}r^2-\frac{E\alpha h_o}{8(1-\nu)}a^2\right)-E\alpha\frac{h_o}{4}(a^2-r^2)$$

12-10.

From the general solution given by (12.7.6),

$$u = A_1 r + \frac{A_2}{r} + \frac{(1+\nu)\alpha}{r}\int^r T\xi d\xi$$

Bounded displacements at $r = 0 \Rightarrow A_2 = 0$

$$u(a) = 0 \Rightarrow A_1 = -\frac{(1+\nu)\alpha}{a^2}\int^a T\xi d\xi$$

$$\therefore u = \frac{(1+\nu)\alpha}{r}\int^r T\xi d\xi - \frac{(1+\nu)\alpha r}{a^2}\int^a T\xi d\xi$$

For the case where $T = T_o$:

$$u = \frac{(1+\nu)\alpha T_o}{r}\int^r \xi d\xi - \frac{(1+\nu)\alpha T_o r}{a^2}\int^a \xi d\xi = (1+\nu)\alpha T_o\left(\frac{1}{r}\frac{r^2}{2} - \frac{r}{a^2}\frac{a^2}{2}\right) = 0$$

$$\sigma_r = \frac{E}{1-\nu^2}[e_r + \nu e_\theta - (1+\nu)\alpha(T - T_o)] = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial r} + \nu\frac{u}{r} - (1+\nu)\alpha(T - T_o)\right] = 0$$

$$\sigma_\theta = \frac{d^2\phi}{dr^2} = \frac{d}{dr}(r\sigma_r) = 0$$

12-11.

The plane stress governing equation $\nabla^4\phi + E\alpha\nabla^2 T = 0$ with axisymmetry reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right)\right) + E\alpha\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

This result can be integrated to give

$$\phi = A_1 \log\frac{r}{a} + A_2\left(\frac{r}{a}\right)^2 \log\frac{r}{a} + A_3\left(\frac{r}{a}\right)^2 + A_4 - E\alpha\int_a^r\left(\frac{1}{\xi}\int_a^\xi T\eta d\eta\right)d\xi$$

and the resulting stress field then becomes

$$\sigma_r = \frac{1}{r}\frac{d\phi}{dr} = \frac{A_1}{r^2} + \frac{A_2}{a^2}(2\log\frac{r}{a}+1) + \frac{2A_3}{a^2} - \frac{E\alpha}{r^2}\int_a^r Trdr$$

$$\sigma_\theta = \frac{d^2\phi}{dr^2} = \frac{d}{dr}(r\sigma_r) = -\frac{A_1}{r^2} + \frac{A_2}{a^2}(2\log\frac{r}{a}+3) + \frac{2A_3}{a^2} - E\alpha\left(T - \frac{1}{r^2}\int_a^r Trdr\right)$$

With zero tractions on $r = a$ and $r = b$, general boundary conditions (12.5.9) $\Rightarrow$

$$\phi(a) = 0 \Rightarrow A_3 + A_4 = 0$$

$$\phi(b) = 0 \Rightarrow A_1 \log\frac{b}{a} + A_2\left(\frac{b}{a}\right)^2 \log\frac{r}{a} + A_3\left(\frac{b}{a}\right)^2 + A_4 - E\alpha\int_a^b\left(\frac{1}{\xi}\int_a^\xi T\eta d\eta\right)d\xi$$

$$\frac{d\phi(a)}{dr} = 0 \Rightarrow A_1 + A_2 + 2A_3 = 0$$

$$\frac{d\phi(b)}{dr} = 0 \Rightarrow \frac{A_1}{b^2} + \frac{A_2}{a^2}(2\log\frac{b}{a}+1) + \frac{2A_3}{a^2} - \frac{E\alpha}{b^2}\int_a^b Trdr = 0$$

Solving for the constants gives

$$A_1 = \frac{E\alpha}{N}\left\{\left[2\left(\frac{b}{a}\right)^2 \log\frac{b}{a}(2\log\frac{b}{a}+1) + \left(\frac{b}{a}\right)^2 - 1\right]\int_a^b Trdr - 4\left(\frac{b}{a}\right)^2 \log\frac{b}{a}\int_a^b Tr\log\frac{r}{a}dr\right\}$$

$$A_2 = \frac{E\alpha}{N}\left\{\left[2\left(\frac{b}{a}\right)^2 \log\frac{b}{a} - \left(\frac{b}{a}\right)^2 + 1\right]\int_a^b Trdr - 2\left[\left(\frac{b}{a}\right)^2 - 1\right]\int_a^b Tr\log\frac{r}{a}dr\right\}$$

$$A_3 = -A_4 = \frac{E\alpha}{N}\left\{-2\left(\frac{b}{a}\right)^2\left(\log\frac{b}{a}\right)^2\int_a^b Trdr + \left[2\left(\frac{b}{a}\right)^2 \log\frac{b}{a} + \left(\frac{b}{a}\right)^2 - 1\right]\int_a^b Tr\log\frac{r}{a}dr\right\}$$

where $N = 4\left(\frac{b}{a}\log\frac{b}{a}\right)^2 - \left[\left(\frac{b}{a}\right)^2 - 1\right]^2$

12-12.

Spherical Coordinates with Spherical Symmetry :

$$e_R = \frac{\partial u_R}{\partial R}\ ,\ \ e_\phi = e_\theta = \frac{u_R}{R}\ ,\ e_{R\phi} = e_{\phi\theta} = e_{\theta R} = 0$$

$$\sigma_R = \lambda(e_R + e_\phi + e_\theta) + 2\mu e_R - (3\lambda + 2\mu)\alpha(T - T_o)$$

$$= (\lambda + 2\mu)\frac{\partial u_R}{\partial R} + 2\lambda\frac{u_R}{R} - (3\lambda + 2\mu)\alpha(T - T_o)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u_R}{\partial R} + 2\nu\frac{u_R}{R}\right] - \frac{E}{(1-2\nu)}\alpha(T - T_o)$$

Likewise,

$$\sigma_\phi = \sigma_\theta = \lambda(e_R + e_\phi + e_\theta) + 2\mu e_\phi - (3\lambda + 2\mu)\alpha(T - T_o)$$

$$= (\lambda + 2\mu)\frac{u_R}{R} + 2\lambda\frac{\partial u_R}{\partial R} - (3\lambda + 2\mu)\alpha(T - T_o)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left[\frac{u_R}{R} + \nu\frac{\partial u_R}{\partial R}\right] - \frac{E}{(1-2\nu)}\alpha(T - T_o)$$

$$\tau_{R\phi} = \tau_{\phi\theta} = \tau_{\theta R} = 0$$

Using these results into the equilibrium equations $\Rightarrow$

$$\frac{d\sigma_\mathrm{R}}{dR} + \frac{1}{R}(2\sigma_R - \sigma_\phi - \sigma_\theta) = 0 \Rightarrow \frac{d}{dR}\left(\frac{1}{R^2}\frac{d}{dR}(R^2 u_R)\right) = \frac{1+\nu}{1-\nu}\alpha\frac{dT}{dR}$$

Integrating this differential equation $\Rightarrow$

$$u_R = \frac{1+\nu}{1-\nu}\alpha\frac{1}{R^2}\int^R T\xi^2 d\xi + C_1 R + \frac{C_2}{R^2}$$

Note : $\displaystyle \frac{\partial u_R}{\partial R} = \frac{1+\nu}{1-\nu}\alpha\left(T - \frac{2}{R^3}\int^R T\xi^2 d\xi\right) + C_1 - \frac{2C_2}{R^3}$

Using these results back into Hooke's Law gives the stresses

$$\sigma_R = -\frac{2\alpha E}{1-\nu}\frac{1}{R^3}\int^R T\xi^2 d\xi + \frac{EC_1}{1-2\nu} - \frac{2EC_2}{1+\nu}\frac{1}{R^3}$$

$$\sigma_\phi = \sigma_\theta = \frac{\alpha E}{1-\nu}\frac{1}{R^3}\int^R T\xi^2 d\xi + \frac{EC_1}{1-2\nu} + \frac{EC_2}{1+\nu}\frac{1}{R^3} - \frac{\alpha ET}{1-\nu}$$

12-13.

Spherical Symmetry Problem :

$$\nabla^2 T = 0 \Rightarrow \frac{d}{dR}\left(R^2 \frac{dT}{dR}\right) = 0 \Rightarrow T = -\frac{C_1}{R} + C_2$$

Temperature Boundary Conditions : $T(a) = T_i,\ T(b) = 0 \Rightarrow C_1 = -\frac{T_i ab}{b-a}\ ,\ C_2 = -\frac{T_i a}{b-a}$

$$\therefore T = \frac{T_i a}{b-a}\left(\frac{b}{R} - 1\right) \Rightarrow \int^R T\xi^2 d\xi = \frac{T_i a}{b-a}\int_0^R (b\xi - \xi^2)d\xi = \frac{T_i a}{b-a}\left(\frac{1}{2}bR^2 - \frac{1}{3}R^3\right)$$

$$\sigma_R(b) = 0 \Rightarrow -\frac{2\alpha E}{1-\nu}\frac{T_i a}{b-a}\frac{1}{6} + \frac{EC_1}{1-2\nu} - \frac{2EC_2}{1+\nu}\frac{1}{b^3} = 0$$

$$\sigma_R(a) = 0 \Rightarrow -\frac{2\alpha E}{1-\nu}\frac{T_i a}{b-a}\left(\frac{1}{2}\frac{b}{a} - \frac{1}{3}\right) + \frac{EC_1}{1-2\nu} - \frac{2EC_2}{1+\nu}\frac{1}{a^3} = 0$$

Solving for the constants, $C_1 = \frac{(1-2\nu)\alpha T_i a}{3(1-\nu)(b^3 - a^3)}(b^2 + ab - 2a^2)\ ,\ C_2 = -\frac{(1+\nu)\alpha T_i a^3 b^3}{2(1-\nu)(b^3 - a^3)}$

Inserting these expressions into the general solution from Exercise 12 - 11 gives the stress field

$$\sigma_R = \frac{E\alpha T_i}{1-\nu}\frac{ab}{b^3 - a^3}\left[a + b - \frac{1}{R}(b^2 + ab + a^2) + \frac{a^2 b^2}{R^3}\right]$$

$$\sigma_\phi = \sigma_\theta = \frac{E\alpha T_i}{1-\nu}\frac{ab}{b^3 - a^3}\left[a + b - \frac{1}{2R}(b^2 + ab + a^2) - \frac{a^2 b^2}{R^3}\right]$$

On the inner surface $R = a$ with $b = a(1+\varepsilon)$

$$\sigma_\phi = \sigma_\theta = \frac{E\alpha T_i}{1-\nu}\frac{b}{b^3 - a^3}(a^2 + ab - 2b^2) = \frac{E\alpha T_i}{2(1-\nu)}\frac{1+\varepsilon}{[(1+\varepsilon)^3 - 1]}[1 + (1+\varepsilon) - 2(1+\varepsilon)^2]$$

$$\approx \frac{E\alpha T_i}{2(1-\nu)}(-1 - \frac{2}{3}\varepsilon)$$

On the outer surface $R = b$ with $b = a(1+\varepsilon)$

$$\sigma_\phi = \sigma_\theta = \frac{E\alpha T_i}{2(1-\nu)}\frac{a}{b^3 - a^3}(b^2 + ab - 2a^2) = \frac{E\alpha T_i}{2(1-\nu)}\frac{1+\varepsilon}{[(1+\varepsilon)^3 - 1]}[(1+\varepsilon)^2 + (1+\varepsilon) - 2]$$

$$\approx \frac{E\alpha T_i}{2(1-\nu)}(1 - \frac{2}{3}\varepsilon)$$

12-14.

Using definitions of the complex and integrated temperature function it was determined that $T = \frac{\partial t_R}{\partial x} = \frac{\partial t_I}{\partial y}$, and this suggested the decomposition of the displacement field $u = u' + \beta t_R$, $v = v' + \beta t_I$, where β is a constant to be determined

Using this decomposition definition in Hooke's Law for plane strain

$$\begin{aligned}
\sigma_x &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} - \alpha(3\lambda + 2\mu)T \\
&= \lambda\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) + 2\mu\frac{\partial u'}{\partial x} - \alpha(3\lambda + 2\mu)T + \beta\left[\lambda\left(\frac{\partial t_R}{\partial x} + \frac{\partial t_I}{\partial y}\right) + 2\mu\frac{\partial t_R}{\partial x}\right] \\
&= \lambda\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) + 2\mu\frac{\partial u'}{\partial x} - \alpha(3\lambda + 2\mu)T + \beta\left[2\lambda\frac{\partial t_R}{\partial x} + 2\mu\frac{\partial t_R}{\partial x}\right] \\
&= \lambda\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) + 2\mu\frac{\partial u'}{\partial x} - \alpha(3\lambda + 2\mu)T + 2\beta(\lambda + \mu)\frac{\partial t_R}{\partial x} \\
&= \lambda\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) + 2\mu\frac{\partial u'}{\partial x} + [-\alpha(3\lambda + 2\mu) + 2\beta(\lambda + \mu)]T
\end{aligned}$$

Now if the factor $-\alpha(3\lambda + 2\mu) + 2\beta(\lambda + \mu) = 0$, the temperature terms will vanish.

This will occur if $\beta = \frac{\alpha(3\lambda + 2\mu)}{2(\lambda + \mu)} = \alpha(1 + \nu)$, and thus for this case $\sigma_x = \sigma_x'$, $\sigma_y = \sigma_y'$ and $\tau_{xy} = \tau_{xy}'$

For the plane stress case, simply use the interchange of elastic constants in Table 12-1 to show that

$$\beta = \frac{1 + \nu}{1 + 2\nu}\alpha\left(1 + \frac{\nu}{1 + \nu}\right) = \alpha$$

12-15.

From relations (12.8.15)

$$\gamma(z) = A\left(\log z - \frac{z^2}{r_i^2 + r_o^2}\right),\ \psi(z) = A\left(\log z - \frac{r_i^2 r_o^2}{z^2(r_i^2 + r_o^2)}\right)$$

Stress free boundary conditions :

$$(\sigma_r - i\tau_{r\theta})_{r=r_i,r_o} = \left(\gamma'(z) + \overline{\gamma'(z)} - e^{2i\theta}[\bar{z}\gamma''(z) + \psi'(z)]\right)_{r=r_i,r_o} = 0$$

$$\gamma'(z) + \overline{\gamma'(z)} - e^{2i\theta}[\bar{z}\gamma''(z) + \psi'(z)] = \left[\frac{1}{z} - \frac{2z}{r_i^2 + r_o^2} + \frac{1}{\bar{z}} - \frac{2\bar{z}}{r_i^2 + r_o^2} + e^{2i\theta}\left(\frac{\bar{z}}{z^2} + \frac{2\bar{z}}{r_i^2 + r_o^2} - \frac{1}{z} - \frac{2r_i^2 r_o^2}{(r_i^2 + r_o^2)z^3}\right)\right]$$

At $z = r_i e^{i\theta}$

$$(\sigma_r - i\tau_{r\theta})_{r=r_i} = \left[\frac{e^{-i\theta}}{r_i} - \frac{2r_i e^{i\theta}}{r_i^2 + r_o^2} + \frac{e^{i\theta}}{r_i} - \frac{2r_i e^{-i\theta}}{r_i^2 + r_o^2} + e^{2i\theta}\left(\frac{e^{-3i\theta}}{r_i} + \frac{2r_i e^{i\theta}}{r_i^2 + r_o^2} - \frac{e^{-i\theta}}{r_i} - \frac{2r_i^2 r_o^2 e^{-3i\theta}}{(r_i^2 + r_o^2)r_i^3}\right)\right]$$

$$= \left(\frac{2}{r_i} - \frac{2r_i}{r_i^2 + r_o^2} - \frac{2r_o^2}{r_i(r_i^2 + r_o^2)}\right)e^{-i\theta} = 0$$

Likewise at $z = r_o e^{i\theta}$

$$(\sigma_r - i\tau_{r\theta})_{r=r_o} = \left[\frac{e^{-i\theta}}{r_o} - \frac{2r_o e^{i\theta}}{r_i^2 + r_o^2} + \frac{e^{i\theta}}{r_o} - \frac{2r_o e^{-i\theta}}{r_i^2 + r_o^2} + e^{2i\theta}\left(\frac{e^{-3i\theta}}{r_o} + \frac{2r_o e^{i\theta}}{r_i^2 + r_o^2} - \frac{e^{-i\theta}}{r_o} - \frac{2r_i^2 r_o^2 e^{-3i\theta}}{(r_i^2 + r_o^2)r_o^3}\right)\right]$$

$$= \left(\frac{2}{r_o} - \frac{2r_o}{r_i^2 + r_o^2} - \frac{2r_i^2}{r_o(r_i^2 + r_o^2)}\right)e^{-i\theta} = 0$$

$\therefore$ stress free boundary conditions are satisfied

12-16.

Conduction Equation :

$$\nabla^2 T = 0 \Rightarrow \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

Using separation of variables, let $T(r,\theta) = f(r)g(\theta) \Rightarrow$

$$f''g + \frac{1}{r}f'g + \frac{1}{r^2}fg'' = 0 \Rightarrow \frac{r^2\left[f''(r) + \frac{1}{r}f'(r)\right]}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = \text{constant} = c^2 \Rightarrow$$

$$f''(r) + \frac{1}{r}f'(r) + \frac{c^2}{r^2}f(r) = 0 \text{ and } g''(\theta) + c^2 g(\theta) = 0$$

Solution to θ - equation is $g(\theta) = A\sin c\theta + B\cos c\theta$,

but temperature field should be odd function of θ with $c = 1 \Rightarrow g(\theta) = A\sin\theta$

Solution to r - equation is $f(r) = C_1 r + \dfrac{C_2}{r}$

$\therefore$ Temperature solution is $T = \left(C_1 r + \dfrac{C_2}{r}\right)\sin\theta$, where we have absorbed the constant A in C_1 and C_2

Insulated boundary condition on hole : $\dfrac{\partial T(a,\theta)}{\partial r} = 0 \Rightarrow C_1 - \dfrac{C_2}{a^2} = 0 \Rightarrow C_2 = a^2 C_1$

Conditions at Infinity : $T(\infty,\theta) = \dfrac{q}{k}y \Rightarrow C_1 = \dfrac{q}{k}$

$$\therefore T = \frac{q}{k}\left(r + \frac{a^2}{r}\right)\sin\theta$$

12-17.

The potentials from Example 12 - 4 where given by

$$\gamma(z) = A\log z\ ,\ \psi(z) = -A\left(\frac{a^2}{z^2} + \log z + 1\right),\ \text{where } A = -\frac{2i\mu q a^2\alpha}{(1+\kappa)k}$$

$$\sigma_r + \sigma_\theta = 2\left(\gamma'(z) + \overline{\gamma'(z)}\right) = 4Re\left(\gamma'(z)\right) = 4Re\left(\frac{A}{z}\right) = -\frac{8\mu q a^2\alpha}{(1+\kappa)kr}\sin\theta\ ,$$

For plane stress, $\kappa = \dfrac{3-\nu}{1+\nu} \Rightarrow 1+\kappa = \dfrac{4}{1+\nu} \Rightarrow \sigma_r + \sigma_\theta = -\dfrac{E\alpha q a^2}{kr}\sin\theta$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2e^{2i\theta}\left(\bar{z}\gamma''(z) + \psi'(z)\right) = 2Ae^{2i\theta}\left(-\frac{\bar{z}}{z^2} + 2\frac{a^2}{z^3} - \frac{1}{z}\right)$$

$$= -2Ae^{2i\theta}\left(\frac{1}{r}e^{-3i\theta} - \frac{2a^2}{r^3}e^{-3i\theta} + \frac{1}{r}e^{-i\theta}\right)$$

$$= -\frac{8\mu q a^2\alpha i}{(1+\kappa)kr}\left(\cos\theta - \frac{a^2}{r^2}\cos\theta + i\frac{a^2}{r^2}\sin\theta\right)$$

Separating real and imaginary parts $\Rightarrow$

$$\sigma_\theta - \sigma_r = -\frac{E\alpha q a^3}{kr^3}\sin\theta \text{ and } \tau_{r\theta} = \frac{1}{2}\frac{E\alpha q a}{k}\left(\frac{a}{r} - \frac{a^3}{r^3}\right)\cos\theta$$

Solving for the individual normal stresses $\Rightarrow$

$$\sigma_r = -\frac{1}{2}\frac{E\alpha q a}{k}\left(\frac{a}{r} - \frac{a^3}{r^3}\right)\sin\theta\ ,\ \sigma_\theta = -\frac{1}{2}\frac{E\alpha q a}{k}\left(\frac{a}{r} + \frac{a^3}{r^3}\right)\sin\theta$$

12-18*.

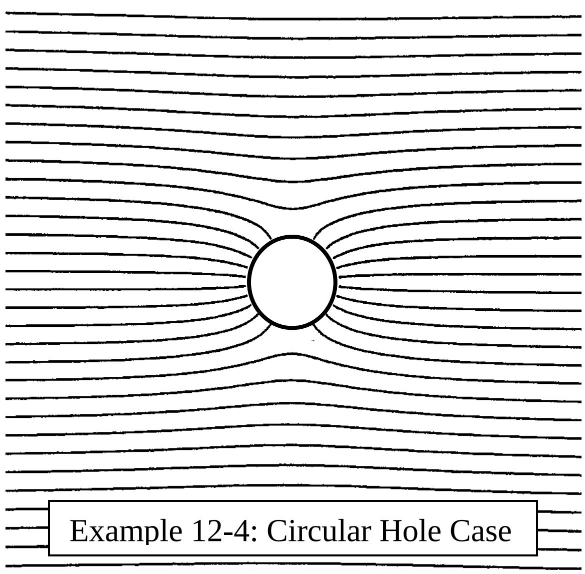
Temperature Field for Example 12 - 4 : $T = \frac{q}{k}\left(r + \frac{a^2}{r}\right)\sin\theta$

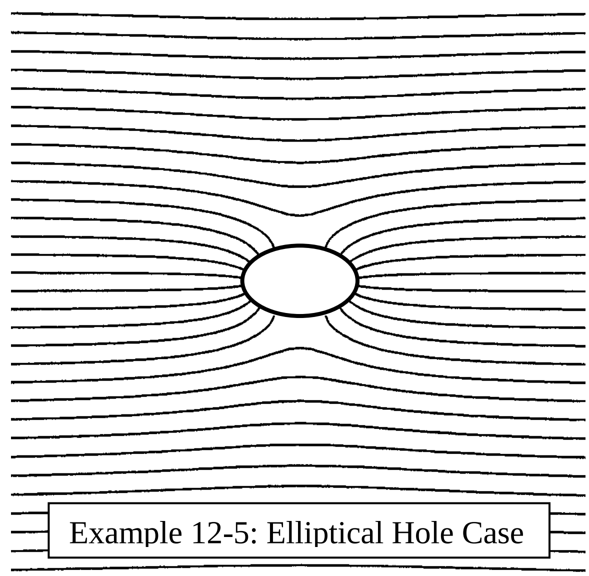
Temperature Field for Example 12 - 5 : $T = \frac{q}{k}\left(\rho + \frac{a^2}{\rho}\right)\sin\theta$,

where ρ, θ are the coordinates in the mapped region with

$$z = R\left(\rho e^{i\theta} + \frac{m}{\rho}e^{-i\theta}\right) \Rightarrow x = R\left(\rho + \frac{m}{\rho}\right)\cos\theta \text{ and } y = R\left(\rho + \frac{m}{\rho}\right)\sin\theta$$

MATLAB Plots of isotherms for each problem :





12-19.

To extract the circular hole case from Example 12 - 5, let $a = b \Rightarrow m = 0$ and $R = a \Rightarrow$

$$z = a\zeta \Rightarrow re^{i\theta} = a\rho e^{i\theta} \Rightarrow \rho = \frac{r}{a}$$

$$\sigma_\rho = -\frac{E\alpha qa}{2kh(\theta)}\rho(\rho^2 + m)[\rho^4 - \rho^2(1+m^2) + m^2]\sin\theta$$

$$= -\frac{E\alpha qa}{2k\rho^8}\rho\rho^2(\rho^4 - \rho^2)\sin\theta$$

$$= -\frac{E\alpha qa}{2k}\left(\frac{a}{r} - \frac{a^3}{r^3}\right)\sin\theta$$

$$\sigma_\theta = -\frac{E\alpha qa}{2kh(\theta)}\rho(\rho^2 + m)\{[\rho^4 + \rho^2(1+m)^2 + m^2]\sin\theta - 2\rho^2 m\sin 3\theta\}$$

$$= -\frac{E\alpha qa}{2k\rho^8}\rho\rho^2(\rho^4 + \rho^2)\sin\theta$$

$$= -\frac{E\alpha qa}{2k}\left(\frac{a}{r} + \frac{a^3}{r^3}\right)\sin\theta$$

$$\tau_{\rho\theta} = \frac{E\alpha qa}{2kh(\theta)}\rho(\rho^2 - m)[\rho^4 - \rho^2(1+m^2) + m^2]\cos\theta$$

$$= \frac{E\alpha qa}{2k\rho^8}\rho\rho^2(\rho^4 - \rho^2)\cos\theta$$

$$= \frac{E\alpha qa}{2k}\left(\frac{a}{r} - \frac{a^3}{r^3}\right)\cos\theta$$

12-20*.

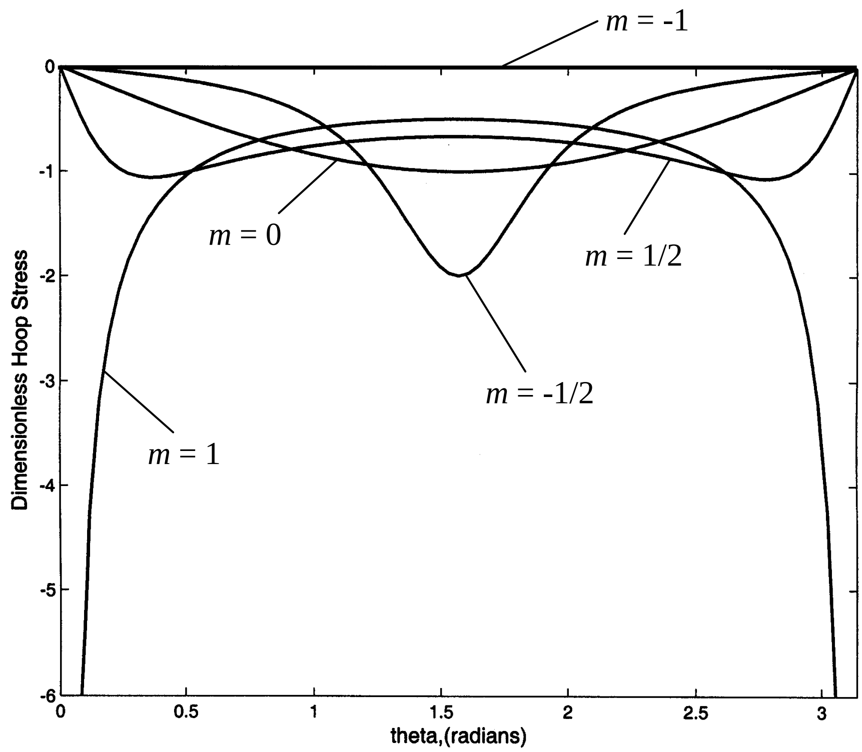
From Example 12 - 5, the non - dimensional hoop stress around hole boundary is gvien by equation $(12.8.32)_2$ with $\rho = 1$

$$\bar{\sigma}_\theta = \frac{\sigma_\theta}{E\alpha qa/k} = -\frac{(1+m)\{[(1+(1+m)^2+m^2]\sin\theta - 2m\sin 3\theta\}}{2(1-2m\cos 2\theta+m^2)^2}$$

$$= -\frac{(1+m)[(1+m+m^2)\sin\theta - m\sin 3\theta]}{(1-2m\cos 2\theta+m^2)^2}$$

Note : $\bar{\sigma}_\theta(\pi/2) = -1/(1+m)$

MATLAB Plots for $m = 0, \pm 1/2, \pm 1$:



12-21.

Uncoupled energy equation for steady state case : $q_{i,i} = 0$

Anisotropic heat conduction law : $q_i = -k_{ij}T_{,j}$

Combining these two relations $\Rightarrow k_{ij}T_{,ij} = 0$, which for two - dimensions gives

$$k_{xx}\frac{\partial^2 T}{\partial x^2} + 2k_{xy}\frac{\partial^2 T}{\partial x \partial y} + k_{yy}\frac{\partial^2 T}{\partial y^2} = 0$$

Looking for solutions : $T = T(x+\lambda y) \Rightarrow (k_{xx} + 2k_{xy}\lambda + k_{yy}\lambda^2)T'' = 0$

$\therefore$ we get the characteristic equation : $k_{xx} + 2k_{xy}\lambda + k_{yy}\lambda^2 = 0$

with solution : $\lambda = \dfrac{1}{2k_{yy}}\left[-2k_{xy} \pm \sqrt{4k_{xy}^2 - 4k_{xx}k_{yy}}\right]$

with $k_{xx}k_{yy} > k_{xy}^2$ the roots will be complex conjugate pairs :

$$\lambda = -\frac{k_{xy}}{k_{yy}} \pm i\sqrt{\frac{k_{xx}k_{yy} - k_{xy}^2}{k_{yy}}} \Rightarrow \lambda, \bar{\lambda}$$

So the general solution becomes : $T(x,y) = F_1(x+\lambda y) + F_2(x+\bar{\lambda}y)$

but since the temperature must be real $\Rightarrow$

$T(x,y) = F(x+\lambda y) + \overline{F(x+\lambda y)} = 2Re\{F(z^*)\}$, where $z^* = x+\lambda y$

13-1.

$$\phi = x^2 + 4y^2 \ , \ \varphi = R^2\boldsymbol{e}_3 = (x^2+y^2+z^2)\boldsymbol{e}_3$$
$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\varphi_z}{\partial y} = 2x + 2y = 2(x+y)$$
$$v = \frac{\partial\phi}{\partial y} - \frac{\partial\varphi_z}{\partial x} = 8y + 2x = 2(x+4y)$$
$$w = \frac{\partial\phi}{\partial z} = 0$$

Since $\nabla^2\boldsymbol{u} = 0$ and $\nabla(\nabla\cdot\boldsymbol{u}) = 0 \Rightarrow$ Navier's equations will be satisfied.

13-2.

$$\boldsymbol{u} = \nabla\phi + \nabla\times\varphi$$
$$e_{kk} = \nabla\cdot\boldsymbol{u} = \nabla\cdot(\nabla\phi + \nabla\times\varphi) = \nabla^2\phi = \phi_{,kk}$$
$$\omega = \frac{1}{2}\nabla\times\boldsymbol{u} = \frac{1}{2}\nabla\times(\nabla\phi+\nabla\times\varphi) = \frac{1}{2}\nabla\times\nabla\times\varphi = \frac{1}{2}\left(\nabla(\nabla\cdot\varphi) - \nabla^2\varphi\right) = -\frac{1}{2}\nabla^2\varphi$$

or $\omega_i = -\dfrac{1}{2}\varphi_{i,kk}$

13-3.

Navier's Equation: $\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = 0$

Using the vector identity $(1.8.5)_9$ $\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u} \Rightarrow$

$\nabla(\nabla\cdot\boldsymbol{u}) = \nabla\times(\nabla\times\boldsymbol{u}) + \nabla^2\boldsymbol{u}$ and using this result in Navier's equation $\Rightarrow$

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\left(\nabla\times(\nabla\times\boldsymbol{u}) + \nabla^2\boldsymbol{u}\right) = 0 \Rightarrow$$
$$(\lambda+2\mu)\nabla^2\boldsymbol{u} + (\lambda+\mu)\left(\nabla\times(\nabla\times\boldsymbol{u})\right) = 0$$

Taking the divergence of this result gives

$$\nabla\cdot\left[(\lambda+2\mu)\nabla^2\boldsymbol{u} + (\lambda+\mu)\left(\nabla\times(\nabla\times\boldsymbol{u})\right)\right] = 0 \Rightarrow (\lambda+2\mu)\nabla^2(\nabla\cdot\boldsymbol{u}) = 0$$

Taking the Laplacian of the original form of Navier's equations $\Rightarrow$

$$\nabla^2\left(\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})\right) = 0 \Rightarrow \nabla^4\boldsymbol{u} = -\frac{\lambda+\mu}{\mu}\nabla\nabla^2(\nabla\cdot\boldsymbol{u}) = 0$$

Thus the displacement vector is biharmonic, and the stress and strain will also be biharmonic.

13-4.

$$\text{Lame's Potential}: \nabla^2\phi = 0\ ,\ \psi = 0 \Rightarrow 2\mu u_i = \phi_{,i}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{4\mu}(\phi_{,ij} + \phi_{,ji}) = \frac{1}{2\mu}\phi_{,ij}$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \frac{\lambda}{2\mu}\phi_{,kk}\delta_{ij} + \phi_{,ij}\ ,\ \text{but } \phi_{,kk} = 0$$

$$\therefore \sigma_{ij} = \phi_{,ij}$$

13-5.

$$\text{Galerkin Vector Representation}:\ 2\mu\boldsymbol{u} = 2(1-\nu)\nabla^2\boldsymbol{V} - \nabla(\nabla\cdot\boldsymbol{V}) \Rightarrow \boldsymbol{u} = \frac{1-\nu}{\mu}\nabla^2\boldsymbol{V} - \frac{1}{2\mu}\nabla(\nabla\cdot\boldsymbol{V})$$

$$\nabla^2\boldsymbol{u} = \frac{1-\nu}{\mu}\nabla^4\boldsymbol{V} - \frac{1}{2\mu}\nabla^2[\nabla(\nabla\cdot\boldsymbol{V})]$$

$$\nabla(\nabla\cdot\boldsymbol{u}) = \nabla\left[\frac{1-\nu}{\mu}\nabla^2(\nabla\cdot\boldsymbol{V}) - \frac{1}{2\mu}\nabla^2(\nabla\cdot\boldsymbol{V})\right] = \frac{1-2\nu}{2\mu}\nabla[\nabla^2(\nabla\cdot\boldsymbol{V})] = \frac{1-2\nu}{2\mu}\nabla^2[\nabla(\nabla\cdot\boldsymbol{V})]$$

$$\text{Navier's Equation}:\ \mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \boldsymbol{F} = 0 \Rightarrow$$

$$\mu\left[\frac{1-\nu}{\mu}\nabla^4\boldsymbol{V} - \frac{1}{2\mu}\nabla^2[\nabla(\nabla\cdot\boldsymbol{V})]\right] + (\lambda+\mu)\left[\frac{1-2\nu}{2\mu}\nabla^2[\nabla(\nabla\cdot\boldsymbol{V})]\right] + \boldsymbol{F} = 0 \Rightarrow$$

$$(1-\nu)\nabla^4\boldsymbol{V} + \left(-\frac{1}{2} + (\lambda+\mu)\frac{1-2\nu}{2\mu}\right)\nabla^2[\nabla(\nabla\cdot\boldsymbol{V})] + \boldsymbol{F} = 0 \Rightarrow$$

$$(1-\nu)\nabla^4\boldsymbol{V} + \boldsymbol{F} = 0 \text{ or}$$

$$\nabla^4\boldsymbol{V} = -\frac{\boldsymbol{F}}{1-\nu}$$

13-6.

Helmholtz Representation : $\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{\varphi}$

From (13.1.3) $\Rightarrow \nabla\cdot\boldsymbol{u} = \nabla^2\phi$ and $\nabla\times\boldsymbol{u} = -\nabla^2\boldsymbol{\varphi}$

Galerkin Vector Representation :

$$2\mu\boldsymbol{u} = 2(1-\nu)\nabla^2\boldsymbol{V} - \nabla(\nabla\cdot\boldsymbol{V}) \Rightarrow \boldsymbol{u} = \frac{1-\nu}{\mu}\nabla^2\boldsymbol{V} - \frac{1}{2\mu}\nabla(\nabla\cdot\boldsymbol{V})$$

$$\nabla\cdot\boldsymbol{u} = \frac{1-\nu}{\mu}\nabla\cdot\nabla^2\boldsymbol{V} - \frac{1}{2\mu}\nabla^2(\nabla\cdot\boldsymbol{V}) = \nabla^2\left[\frac{1-2\nu}{2\mu}(\nabla\cdot\boldsymbol{V})\right]$$

$$\nabla\times\boldsymbol{u} = \frac{1-\nu}{\mu}\nabla\times\nabla^2\boldsymbol{V} - \frac{1}{2\mu}\nabla\times\nabla(\nabla\cdot\boldsymbol{V}) = \nabla^2\left[\frac{1-\nu}{\mu}(\nabla\times\boldsymbol{V})\right]$$

Comparing results for $\nabla\cdot\boldsymbol{u}$ and $\nabla\times\boldsymbol{u} \Rightarrow \phi = \frac{1-2\nu}{2\mu}(\nabla\cdot\boldsymbol{V})$, $\boldsymbol{\varphi} = -\frac{1-\nu}{\mu}(\nabla\times\boldsymbol{V})$

Note that $\nabla\times\boldsymbol{\varphi} = \frac{1-\nu}{\mu}\left[\nabla^2\boldsymbol{V} - \nabla(\nabla\cdot\boldsymbol{V})\right]$ which provides proper match for Galerkin form.

13-7.

Galerkin Vector Representation :

$$\boldsymbol{u} = \frac{1}{2\mu}\left(2(1-\nu)\nabla^2\boldsymbol{V} - \nabla(\nabla\cdot\boldsymbol{V})\right) \Rightarrow u_i = \frac{1}{2\mu}\left(2(1-\nu)V_{i,kk} - V_{k,ki}\right) \Rightarrow$$

$$u_{i,j} = \frac{1}{2\mu}\left(2(1-\nu)V_{i,kkj} - V_{k,kij}\right), \; u_{j,i} = \frac{1}{2\mu}\left(2(1-\nu)V_{j,kki} - V_{k,kji}\right),$$

$$u_{k,k} = \frac{1}{2\mu}\left(2(1-\nu)V_{k,mmk} - V_{m,mkk}\right) = \frac{1}{2\mu}\left(2(1-\nu)V_{m,mkk} - V_{m,mkk}\right) = \frac{1-2\nu}{2\mu}V_{m,mkk}$$

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

$$= \lambda\frac{1-2\nu}{2\mu}V_{m,mkk}\delta_{ij} + \frac{1}{2}\left(2(1-\nu)V_{i,kkj} + 2(1-\nu)V_{j,kki} - 2V_{k,kij}\right)$$

$$= \nu V_{m,mkk}\delta_{ij} + (1-\nu)V_{i,kkj} + (1-\nu)V_{j,kki} - V_{k,kij}$$

Specific components :

$$\sigma_{11} = \sigma_x = \nu V_{m,mkk} + (1-\nu)V_{1,kk1} + (1-\nu)V_{1,kk1} - V_{k,k11}$$

$$= \nu\nabla^2(\nabla\cdot\boldsymbol{V}) + 2(1-\nu)\frac{\partial}{\partial x}\nabla^2 V_x - \frac{\partial^2}{\partial x^2}(\nabla\cdot\boldsymbol{V}) = 2(1-\nu)\frac{\partial}{\partial x}\nabla^2 V_x + \left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)(\nabla\cdot\boldsymbol{V})$$

The other normal stress expressions follow in similar fashion

$$\sigma_{12} = \tau_{xy} = (1-\nu)V_{1,kk2} + (1-\nu)V_{2,kk1} - V_{k,k12}$$

$$= (1-\nu)\frac{\partial}{\partial y}\nabla^2 V_x + (1-\nu)\frac{\partial}{\partial x}\nabla^2 V_y - \frac{\partial^2}{\partial x\partial y}(\nabla\cdot\boldsymbol{V})$$

$$= (1-\nu)\left(\frac{\partial}{\partial y}\nabla^2 V_x + \frac{\partial}{\partial x}\nabla^2 V_y\right) - \frac{\partial^2}{\partial x\partial y}(\nabla\cdot\boldsymbol{V})$$

The other shear stress expressions follow in similar fashion

13-8.

Cylindrical Coordinate Case : $\boldsymbol{V} = V_r\boldsymbol{e}_r + V_\theta\boldsymbol{e}_\theta + V_z\boldsymbol{e}_z$

From Section 1-9 or Exercise 1-16 : $\nabla = \boldsymbol{e_r}\frac{\partial}{\partial r} + \boldsymbol{e_\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}$

$$\nabla^2(V_r\boldsymbol{e}_r) = \nabla\cdot\nabla(V_r\boldsymbol{e}_r) = \nabla\cdot\left[\boldsymbol{e_r}\frac{\partial}{\partial r}(V_r\boldsymbol{e}_r) + \boldsymbol{e_\theta}\frac{1}{r}\frac{\partial}{\partial\theta}(V_r\boldsymbol{e}_r) + \boldsymbol{e}_z\frac{\partial}{\partial z}(V_r\boldsymbol{e}_r)\right]$$

$$= \nabla\cdot\left[\boldsymbol{e_r}\frac{\partial V_r}{\partial r}\boldsymbol{e_r} + \boldsymbol{e_\theta}\frac{1}{r}\left(V_r\boldsymbol{e_\theta} + \frac{\partial V_r}{\partial\theta}\boldsymbol{e_r}\right) + \boldsymbol{e}_z\frac{\partial V_r}{\partial z}\boldsymbol{e_r}\right]$$

$$= \left(\boldsymbol{e_r}\frac{\partial}{\partial r} + \boldsymbol{e_\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right)\cdot\left[\boldsymbol{e_r}\frac{\partial V_r}{\partial r}\boldsymbol{e_r} + \boldsymbol{e_\theta}\frac{1}{r}\left(V_r\boldsymbol{e_\theta} + \frac{\partial V_r}{\partial\theta}\boldsymbol{e_r}\right) + \boldsymbol{e}_z\frac{\partial V_r}{\partial z}\boldsymbol{e_r}\right]$$

$$= \left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2} + \frac{\partial^2 V_r}{\partial z^2} - \frac{1}{r^2}V_r\right)\boldsymbol{e_r} + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\boldsymbol{e_\theta} = \left(\nabla^2 V_r - \frac{1}{r^2}V_r\right)\boldsymbol{e_r} + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\boldsymbol{e_\theta}$$

In similar fashion

$$\nabla^2(V_\theta\boldsymbol{e_\theta}) = \left(\nabla^2 V_\theta - \frac{V_\theta}{r^2}\right)\boldsymbol{e_\theta} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta}\boldsymbol{e_r}\ ,\ \nabla^2(V_z\boldsymbol{e}_z) = \nabla^2 V_z\boldsymbol{e}_z$$

To determine biharmonic forms, reapply the previous Laplacian operators

$$\nabla^4(V_r\boldsymbol{e}_r) = \nabla^2\nabla^2(V_r\boldsymbol{e}_r) = \nabla^2\left[\left(\nabla^2 V_r - \frac{V_r}{r^2}\right)\boldsymbol{e_r} + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\boldsymbol{e_\theta}\right] = \nabla^2\left[\left(\nabla^2 V_r - \frac{V_r}{r^2}\right)\boldsymbol{e_r}\right] + \nabla^2\left[\frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\boldsymbol{e_\theta}\right]$$

$$= \left[\nabla^2\left(\nabla^2 V_r - \frac{V_r}{r^2}\right) - \frac{1}{r^2}\left(\nabla^2 V_r - \frac{V_r}{r^2}\right)\right]\boldsymbol{e_r} + \frac{2}{r^2}\frac{\partial}{\partial\theta}\left(\nabla^2 V_r - \frac{V_r}{r^2}\right)\boldsymbol{e_\theta}$$

$$+ \left[\nabla^2\left(\frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\right) - \frac{1}{r^2}\left(\frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\right)\right]\boldsymbol{e_\theta} - \frac{2}{r^2}\frac{\partial}{\partial\theta}\left(\frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\right)\boldsymbol{e_r}$$

$$= \left[\left(\nabla^2 - \frac{1}{r^2}\right)^2 V_r - \frac{4}{r^4}\frac{\partial^2 V_r}{\partial\theta^2}\right]\boldsymbol{e_r} + \left[\frac{4}{r^2}\left(\nabla^2 - \frac{1}{r^2}\right)\frac{\partial V_r}{\partial\theta}\right]\boldsymbol{e_\theta}$$

The other two biharmonic forms follow in similar fashion

13-9.

Proposed solution to Boussinesq's Problem is combination of Galerkin vector and Lame potential

$V_x = V_y = 0$, $V_z = AR$, $\phi = B\log(R+z)$ with $A = \dfrac{P}{2\pi}$, $B = -\dfrac{(1-2\nu)P}{2\pi}$

These forms satisfy the governing equations: $\nabla^2\phi = 0$ and $\nabla^4 V_z = 0$

Displacements:

$$u = \frac{1}{2\mu}\frac{\partial\phi}{\partial x} - \frac{1}{2\mu}\frac{\partial^2 V_z}{\partial x\partial z} = \frac{Px}{4\pi\mu R}\left(\frac{z}{R^2} - \frac{1-2\nu}{R+z}\right)$$

$$v = \frac{1}{2\mu}\frac{\partial\phi}{\partial y} - \frac{1}{2\mu}\frac{\partial^2 V_z}{\partial y\partial z} = \frac{Py}{4\pi\mu R}\left(\frac{z}{R^2} - \frac{1-2\nu}{R+z}\right)$$

$$w = \frac{1}{2\mu}\frac{\partial\phi}{\partial z} - \frac{1}{2\mu}\left(2(1-\nu)\nabla^2 V_z - \frac{\partial^2 V_z}{\partial z^2}\right) = \frac{P}{4\pi\mu R}\left(2(1-\nu) + \frac{z^2}{R^2}\right)$$

Stresses:

$$\sigma_x = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)V_z$$

$$= \frac{1}{2}\frac{P(1-2\nu)x^2}{\pi R^3(R+z)} - \frac{1}{2}\frac{P(1-2\nu)}{\pi R(R+z)} + \frac{1}{2}\frac{P(1-2\nu)x^2}{R^2(R+z)^2} - \frac{\nu Pz}{\pi R^3} - \frac{3}{2}\frac{Px^2 z}{\pi R^5} + \frac{1}{2}\frac{Pz}{\pi R^3}$$

$$= -\frac{P}{\pi R^2}\left[\frac{3x^2 z}{R^3} - (1-2\nu)\left(\frac{z}{R} - \frac{R}{R+z} + \frac{x^2(2R+z)}{R(R+z)^2}\right)\right]$$

$$\sigma_y = \frac{\partial^2\phi}{\partial y^2} + \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)V_z = -\frac{P}{2\pi R^2}\left[\frac{3y^2 z}{R^3} - (1-2\nu)\left(\frac{z}{R} - \frac{R}{R+z} + \frac{y^2(2R+z)}{R(R+z)^2}\right)\right]$$

$$\sigma_z = \frac{\partial^2\phi}{\partial z^2} + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z = -\frac{3Pz^3}{2\pi R^5}$$

$$\tau_{xy} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^3 V_z}{\partial x\partial y\partial z} = -\frac{P}{2\pi R^2}\left[\frac{3xyz}{R^3} - \frac{(1-2\nu)(2R+z)xy}{R(R+z)^2}\right]$$

$$\tau_{yz} = \frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial}{\partial y}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z = -\frac{3Pyz^2}{2\pi R^5}$$

$$\tau_{zx} = \frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial}{\partial x}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z = -\frac{3Pxz^2}{2\pi R^5}$$

Boundary Conditions Check:

$\tau_{yz}(x,y,0) = \tau_{zx}(x,y,0) = \sigma_z(x,y,0) = 0$ (except at $R = 0$, stresses are singular at origin)

Finally it can be shown that on any plane $z =$ constant, $\int_0^\infty \sigma_z(r,z)2\pi r dr = -P$

13-10.

Proposed solution to Cerruti's Problem is combination of Galerkin vector and Lame potential

$$V_x = AR,\ V_y = 0,\ V_z = Bx\log(R+z)\ ,\ \phi = \frac{Cx}{R+z}\ ,\ \text{where } A = \frac{P}{4\pi(1-\nu)}\ ,\ B = \frac{(1-2\nu)P}{4\pi(1-\nu)},\ C = \frac{(1-2\nu)P}{2\pi}$$

These forms satisfy the governing equations: $\nabla^2\phi = 0$ and $\nabla^4 V_z = 0$

Displacements:

$$u = \frac{1}{2\mu}\frac{\partial\phi}{\partial x} + \frac{1}{2\mu}\left(2(1-\nu)\nabla^2 V_x - \frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right)\right)$$

$$= \frac{1}{2\mu}\left[\frac{1}{2\pi}\frac{P(1-2\nu)}{R+z} - \frac{1}{2\pi}\frac{P(1-2\nu)x^2}{(R+z)^2 R} + \frac{P}{2\pi R} + \frac{Px^2}{2\pi R^3}\right] = \frac{P}{4\pi\mu R}\left[1 + \frac{x^2}{R^2} + (1-2\nu)\left(\frac{R}{R+z} - \frac{x^2}{(R+z)^2}\right)\right]$$

$$v = \frac{1}{2\mu}\frac{\partial\phi}{\partial y} - \frac{1}{2\mu}\left(\frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right)\right) = \frac{Pxy}{4\pi\mu R}\left(\frac{1}{R^2} - \frac{1-2\nu}{(R+z)^2}\right)$$

$$w = \frac{1}{2\mu}\frac{\partial\phi}{\partial z} + \frac{1}{2\mu}\left(2(1-\nu)\nabla^2 V_z - \frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right)\right) = \frac{Px}{4\pi\mu R}\left(\frac{z}{R^2} + \frac{1-2\nu}{R+z}\right)$$

Stresses:

$$\sigma_x = \frac{\partial^2\phi}{\partial x^2} + 2(1-\nu)\frac{\partial}{\partial x}\nabla^2 V_x + \left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right)$$

$$= \frac{3P(1-2\nu)x}{2\pi(R+z)^2 R} + \frac{P(1-2\nu)x^3}{\pi(R+z)^3 R^2} + \frac{P(1-2\nu)x^3}{2\pi(R+z)^2 R^2} - \frac{Px}{\pi R^3} - \frac{\nu Px}{\pi R^3} + \frac{3Px}{2\pi R^3} - \frac{3Px^3}{2\pi R^5}$$

$$= \frac{Px}{2\pi R^3}\left[-\frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left(R^2 - y^2 - \frac{2Ry^2}{R+z}\right)\right]$$

$$\sigma_y = \frac{\partial^2\phi}{\partial y^2} + \left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right) = \frac{Px}{2\pi R^3}\left[-\frac{3y^2}{R^2} - \frac{(1-2\nu)}{(R+z)^2}\left(3R^2 - x^2 - \frac{2Rx^2}{R+z}\right)\right]$$

$$\sigma_z = \frac{\partial^2\phi}{\partial z^2} + 2(1-\nu)\frac{\partial}{\partial z}\nabla^2 V_z + \left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right) = -\frac{3Pxz^2}{2\pi R^5}$$

$$\tau_{xy} = \frac{\partial^2\phi}{\partial x\partial y} + (1-\nu)\frac{\partial}{\partial y}\nabla^2 V_x - \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right) = \frac{Py}{2\pi R^3}\left[-\frac{3x^2}{R^2} - \frac{(1-2\nu)}{(R+z)^2}\left(R^2 - x^2 + \frac{2Rx^2}{R+z}\right)\right]$$

$$\tau_{yz} = \frac{\partial^2\phi}{\partial y\partial z} + (1-\nu)\frac{\partial}{\partial y}\nabla^2 V_z - \frac{\partial^2}{\partial y\partial z}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right) = -\frac{3Pxyz}{2\pi R^5}$$

$$\tau_{zx} = \frac{\partial^2\phi}{\partial z\partial x} + (1-\nu)\left(\frac{\partial}{\partial z}\nabla^2 V_x + \frac{\partial}{\partial x}\nabla^2 V_z\right) - \frac{\partial^2}{\partial z\partial x}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right) = -\frac{3Px^2 z}{2\pi R^5}$$

Boundary Conditions Check:

$\tau_{yz}(x,y,0) = \tau_{zx}(x,y,0) = \sigma_z(x,y,0) = 0$ (except at $R = 0$, stresses are singular at origin)

Finally it can be shown that on any plane $x =$ constant, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma_x(x,y,z)\,dydz = -P$

13-11.

Papkovich Representation : $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}\right]$

From formulation result (13.4.3), $2\mu\boldsymbol{u} = 2\mu\left[\boldsymbol{h} - \frac{1}{1-2\nu}\nabla\phi\right] \Rightarrow$

$2\mu\boldsymbol{h} = \boldsymbol{A}\ ,\ \frac{2\mu}{1-2\nu}\phi = B + \frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}$

Now since $\nabla^2\boldsymbol{h} = -\frac{\boldsymbol{F}}{\mu} \Rightarrow \nabla^2\boldsymbol{A} = -2\boldsymbol{F}$

$$B = \frac{2\mu}{1-2\nu}\phi - \frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)} \Rightarrow$$

$$\nabla^2 B = \nabla^2\left(\frac{2\mu}{1-2\nu}\phi - \frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}\right) = \frac{2\mu}{1-2\nu}\nabla^2\phi - \frac{1}{4(1-\nu)}\left(2(\nabla\cdot\boldsymbol{A}) + (\nabla^2\boldsymbol{A})\cdot\boldsymbol{R}\right)$$

Using formulation result (13.4.6) $\Rightarrow$

$$\nabla^2\phi = \frac{1-2\nu}{2(1-\nu)}\left[\frac{\boldsymbol{R}\cdot\boldsymbol{F}}{2\mu} + \frac{1}{2}\nabla^2(\boldsymbol{R}\cdot\boldsymbol{h})\right] = \frac{1-2\nu}{2(1-\nu)}\left[\frac{\boldsymbol{R}\cdot\boldsymbol{F}}{2\mu} + \frac{1}{2}\left(2(\nabla\cdot\boldsymbol{h}) + \boldsymbol{R}\cdot(\nabla^2\boldsymbol{h})\right)\right]$$

$$= \frac{1-2\nu}{2(1-\nu)}\left[\frac{\boldsymbol{R}\cdot\boldsymbol{F}}{2\mu} + \frac{(\nabla\cdot\boldsymbol{A})}{2\mu} - \frac{\boldsymbol{R}\cdot\boldsymbol{F}}{2\mu}\right] = \frac{1-2\nu}{4\mu(1-\nu)}[\nabla\cdot\boldsymbol{A}]$$

Combining these results $\Rightarrow$

$$\nabla^2 B = \frac{2\mu}{1-2\nu}\frac{1-2\nu}{4\mu(1-\nu)}(\nabla\cdot\boldsymbol{A}) - \frac{1}{4(1-\nu)}\left(2(\nabla\cdot\boldsymbol{A}) - 2(\boldsymbol{R}\cdot\boldsymbol{F})\right)$$

$$= \frac{1}{2(1-\nu)}[\nabla\cdot\boldsymbol{A} - \nabla\cdot\boldsymbol{A} + (\boldsymbol{R}\cdot\boldsymbol{F})] = \frac{\boldsymbol{R}\cdot\boldsymbol{F}}{2(1-\nu)}$$

13-12.

General Papkovich Representation : $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \dfrac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}\right]$

Axisymmetric Form - Boussinesq Potentials :

$A_r = A_\theta = 0$, $A_z = A_z(r,z)$, $B = B(r,z)$, with $\nabla^2 B = 0$ and $\nabla^2 A_z = 0 \Rightarrow$

$$u_r = -\frac{1}{2\mu}\frac{\partial}{\partial r}\left(B + \frac{A_z z}{4(1-\nu)}\right) \text{ , } u_\theta = 0 \text{ , } u_z = \frac{1}{2\mu}\left[A_z - \frac{\partial}{\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right)\right]$$

$$e_r = -\frac{1}{2\mu}\frac{\partial^2}{\partial r^2}\left(B + \frac{A_z z}{4(1-\nu)}\right) \text{ , } e_\theta = -\frac{1}{2\mu r}\frac{\partial}{\partial r}\left(B + \frac{A_z z}{4(1-\nu)}\right) \text{ , } e_z = \frac{1}{2\mu}\frac{\partial}{\partial z}\left[A_z - \frac{\partial}{\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right)\right]$$

$$e_{rz} = -\frac{1}{4\mu}\frac{\partial^2}{\partial r\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right) + \frac{1}{4\mu}\frac{\partial}{\partial r}\left[A_z - \frac{\partial}{\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right)\right]$$

$$= \frac{1}{4\mu}\frac{\partial A_z}{\partial r} - \frac{1}{2\mu}\frac{\partial^2}{\partial r\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$\sigma_r = \lambda(e_r + e_\theta + e_z) + 2\mu e_r$$

$$= -\frac{\lambda}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(B + \frac{A_z z}{4(1-\nu)}\right) + \frac{\lambda}{2\mu}\frac{\partial A_z}{\partial z} - \frac{\partial^2}{\partial r^2}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$= -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{A_z z}{4(1-\nu)}\right) + \frac{\nu}{1-2\nu}\frac{\partial A_z}{\partial z} - \frac{\partial^2}{\partial r^2}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

Likewise,

$$\sigma_\theta = -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{A_z z}{4(1-\nu)}\right) + \frac{\nu}{1-2\nu}\frac{\partial A_z}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$\sigma_z = -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{A_z z}{4(1-\nu)}\right) + \frac{\nu}{1-2\nu}\frac{\partial A_z}{\partial z} + \frac{\partial A_z}{\partial z} - \frac{\partial^2}{\partial z^2}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$\tau_{rz} = 2\mu e_{rz} = \frac{1}{2}\frac{\partial A_z}{\partial r} - \frac{\partial^2}{\partial r\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

13-13.

Boussinesq Problem - Papkovich Solution

$$A_z = \frac{2(1-\nu)P}{\pi R}\ ,\quad B = \frac{(1-2\nu)P}{2\pi}\log(R+z)$$

Using results from Exercise 13-12,

$$u_r = -\frac{1}{2\mu}\frac{\partial}{\partial r}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$= -\frac{1}{2\mu}\frac{\partial}{\partial r}\left(\frac{(1-2\nu)P}{2\pi}\log(R+z) + \frac{Pz}{2\pi R}\right) = \frac{P}{4\pi\mu R}\left[\frac{rz}{R^2} - \frac{(1-2\nu)r}{R+z}\right]$$

$$u_z = \frac{1}{2\mu}\left[A_z - \frac{\partial}{\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right)\right] = \frac{P}{4\pi\mu R}\left[2(1-\nu) + \frac{z^2}{R^2}\right],\ u_\theta = 0$$

$$\sigma_r = -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{A_z z}{4(1-\nu)}\right) + \frac{\nu}{1-2\nu}\frac{\partial A_z}{\partial z} - \frac{\partial^2}{\partial r^2}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$= -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{Pz}{2\pi R}\right) + \frac{\nu}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{2(1-\nu)P}{\pi R}\right) - \frac{\partial^2}{\partial r^2}\left(\frac{(1-2\nu)P}{2\pi}\log(R+z) + \frac{Pz}{2\pi R}\right)$$

$$= \frac{P}{2\pi R^2}\left[-\frac{3r^2 z}{R^3} + \frac{(1-2\nu)R}{R+z}\right]$$

$$\sigma_\theta = -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{Pz}{2\pi R}\right) + \frac{\nu}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{2(1-\nu)P}{\pi R}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{(1-2\nu)P}{2\pi}\log(R+z) + \frac{Pz}{2\pi R}\right)$$

$$= \frac{(1-2\nu)P}{2\pi R^2}\left[\frac{z}{R} - \frac{R}{R+z}\right]$$

$$\sigma_z = -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{Pz}{2\pi R}\right) + \frac{\nu}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{2(1-\nu)P}{\pi R}\right) - \frac{\partial}{\partial z}\left(\frac{2(1-\nu)P}{\pi R}\right) - \frac{\partial^2}{\partial z^2}\left(\frac{(1-2\nu)P}{2\pi}\log(R+z) + \frac{Pz}{2\pi R}\right)$$

$$= -\frac{3Pz^3}{2\pi R^5}$$

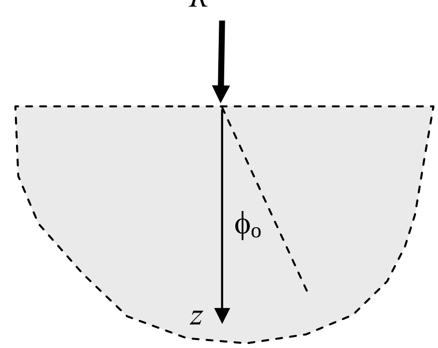
$$\tau_{rz} = 2\mu e_{rz} = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{2(1-\nu)P}{\pi R}\right) - \frac{\partial^2}{\partial r\partial z}\left(\frac{(1-2\nu)P}{2\pi}\log(R+z) + \frac{Pz}{2\pi R}\right) = -\frac{3P\,rz^2}{2\pi R^5}$$

Note $u_r > 0 \Rightarrow \frac{rz}{R^2} > \frac{(1-2\nu)r}{R+z} \Rightarrow \frac{z}{R} > \frac{(1-2\nu)R}{R+z}$, and defining $\cos\phi = \frac{z}{R} \Rightarrow$

$$\cos\phi > \frac{(1-2\nu)}{1+\cos\phi} \Rightarrow \cos^2\phi + \cos\phi > (1-2\nu)$$

$\therefore u_r > 0$ for $\phi \le \phi_o$ where ϕ_o is determined from

$$\cos^2\phi_o + \cos\phi_o - (1-2\nu) = 0$$



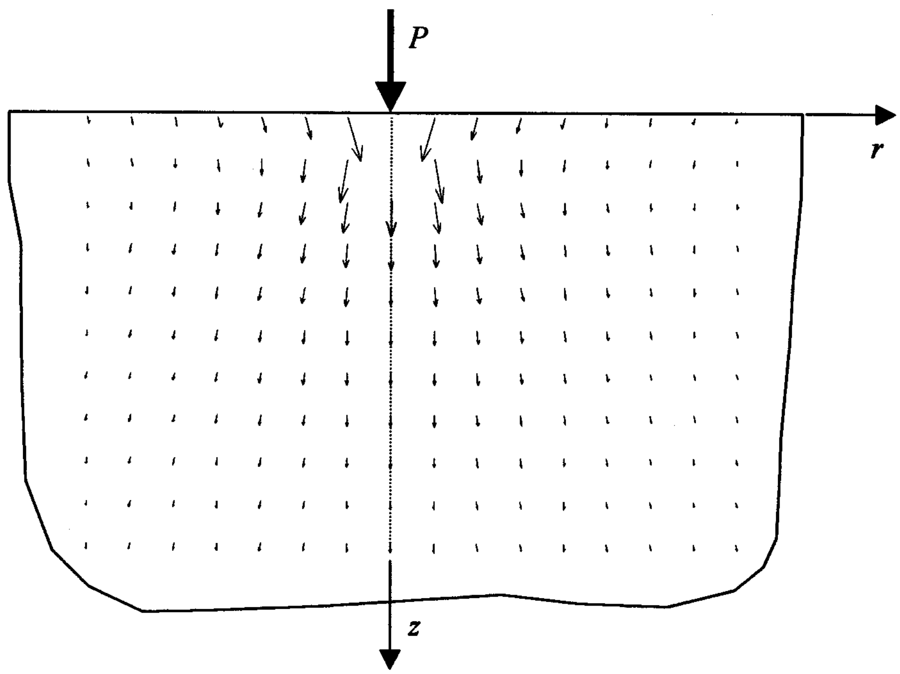
13-14*.

Displacement Field - Boussinesq Problem :

$$u_r = \frac{P}{4\pi\mu R}\left[\frac{rz}{R^2} - \frac{(1-2\nu)r}{R+z}\right],\ u_z = \frac{P}{4\pi\mu R}\left[2(1-\nu) + \frac{z^2}{R^2}\right],\ u_\theta = 0$$

Note surface displacements : $u_r(r,0) = -\dfrac{P(1-2\nu)}{4\pi\mu r} < 0$, $u_z(r,0) = \dfrac{P(1-\nu)}{2\pi\mu r} > 0$

MATLAB Vector Distribution Plot :



13-15.

General Papkovich Representation : $2\mu \boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}\right]$

Axisymmetric Form - Boussinesq Potentials :

$A_r = A_\theta = 0$, $A_z = A_z(r,z)$, $B = B(r,z)$, with $\nabla^2 B = 0$ and $\nabla^2 A_z = 0 \Rightarrow$

$$2\mu u_r = -\frac{\partial}{\partial r}\left(B + \frac{A_z z}{4(1-\nu)}\right) = -\frac{\partial B}{\partial r} - \frac{z}{4(1-\nu)}\frac{\partial A_z}{\partial r}$$

$$2\mu u_z = A_z - \frac{\partial}{\partial z}\left(B + \frac{A_z z}{4(1-\nu)}\right) = A_z - \frac{\partial B}{\partial z} - \frac{A_z}{4(1-\nu)} - \frac{z}{4(1-\nu)}\frac{\partial A_z}{\partial z}$$

$$= \frac{3-4\nu}{4(1-\nu)}A_z - \frac{\partial B}{\partial z} - \frac{z}{4(1-\nu)}\frac{\partial A_z}{\partial z}$$

From Exercise 13 - 12

$$\sigma_z = \lambda(e_r + e_\theta + e_z) + 2\mu e_z$$

$$= -\frac{\nu}{1-2\nu}\nabla^2\left(\frac{A_z z}{4(1-\nu)}\right) + \frac{\nu}{1-2\nu}\frac{\partial A_z}{\partial z} + \frac{\partial A_z}{\partial z} - \frac{\partial^2}{\partial z^2}\left(B + \frac{A_z z}{4(1-\nu)}\right)$$

$$\sigma_z(r,0) = 0 \Rightarrow -\frac{\nu}{4(1-\nu)(1-2\nu)}\frac{\partial A_z}{\partial z} + \frac{\nu}{1-2\nu}\frac{\partial A_z}{\partial z} + \frac{\partial A_z}{\partial z} - \frac{\partial^2 B}{\partial z^2} - \frac{1}{2(1-\nu)}\frac{\partial A_z}{\partial z} = 0$$

$$\Rightarrow \frac{1}{2}\frac{\partial A_z}{\partial z} = \frac{\partial^2 B}{\partial z^2}$$

Integrating $\Rightarrow$ $A_z = 2\frac{\partial B}{\partial z}$, where we have dropped the arbitrary function of integration

13-16.

Two - Dimensional Papkovich Representation : $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}\right]$

where $\boldsymbol{A} = A_1(x,y)\boldsymbol{e_1} + A_2(x,y)\boldsymbol{e_2}$ and $B = B(x,y)$

$$2\mu u = A_1 - \frac{\partial}{\partial x}\left[B + \frac{(xA_1 + yA_2)}{4(1-\nu)}\right] = A_1 - \frac{\partial B}{\partial x} - \frac{1}{4(1-\nu)}\left[x\frac{\partial A_1}{\partial x} + A_1 + y\frac{\partial A_2}{\partial x}\right]$$

$$= \frac{1}{4(1-\nu)}\left[(3-4\nu)A_1 - x\frac{\partial A_1}{\partial x} - y\frac{\partial A_2}{\partial x}\right] - \frac{\partial B}{\partial x}$$

Likewise

$$2\mu v = \frac{1}{4(1-\nu)}\left[(3-4\nu)A_2 - x\frac{\partial A_1}{\partial y} - y\frac{\partial A_2}{\partial y}\right] - \frac{\partial B}{\partial y}$$

Forming the complex displacement

$$2\mu(u+iv) = \frac{1}{4(1-\nu)}\left\{(3-4\nu)(A_1 + iA_2) - \left[\left(x\frac{\partial A_1}{\partial x} + y\frac{\partial A_2}{\partial x}\right) + i\left(x\frac{\partial A_1}{\partial y} + y\frac{\partial A_2}{\partial y}\right)\right] - \left(\frac{\partial B}{\partial x} + i\frac{\partial B}{\partial y}\right)\right\}$$

Define $\gamma(z) = \frac{A_1 + iA_2}{4(1-\nu)}$ and $\psi(z) = \frac{\partial B}{\partial x} - i\frac{\partial B}{\partial y}$ with $\kappa = 3-4\nu \Rightarrow$

$$2\mu(u+iv) = \kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}$$

13-17.

Trial Papkovich Functions for Kelvin Problem: $B=0\,,\; A_z=\dfrac{P}{2\pi R}$

From Exercise 13-12

$$2\mu u_r=-\frac{\partial}{\partial r}\left(\frac{A_z z}{4(1-\nu)}\right)=-\frac{\partial}{\partial r}\left(\frac{Pz}{8\pi R(1-\nu)}\right)=\frac{Prz}{8\pi(1-v)R^3}$$

$$2\mu u_z=\left[A_z-\frac{\partial}{\partial z}\left(\frac{A_z z}{4(1-\nu)}\right)\right]=\left[\frac{P}{2\pi R}-\frac{\partial}{\partial z}\left(\frac{Pz}{8\pi R(1-\nu)}\right)\right]=\left[\frac{P}{2\pi R}-\frac{Pr^2}{8\pi R^3(1-\nu)}\right]$$

$$=\frac{P}{8\pi(1-v)}\left[\frac{4(1-\nu)}{R}-\frac{r^2}{R^3}\right]=\frac{P}{8\pi(1-v)}\left[\frac{2(1-2\nu)}{R}+\frac{1}{R}+\frac{z^2}{R^3}\right]$$

$$2\mu u_\theta=0$$

This displacement field matches that given in relation $(13.3.10)_1$ for the Kelvin Problem

13-18.

For the Doublet Force Problem, use superposition of two Kelvin Problems and take limit $d\to 0$

$$\sigma_r^D=\lim_{d\to 0}[\sigma_r(r,z)-\sigma_r(r,z+d)]=-d\lim_{d\to 0}\left[\frac{\sigma_r(r,z+d)-\sigma_r(r,z)}{d}\right]=-d\,\frac{\partial\sigma_r}{\partial z}$$

$$=-\frac{D}{8\pi(1-\nu)}\frac{\partial}{\partial z}\left[(1-2\nu)z(r^2+z^2)^{-3/2}-3r^2z(r^2+z^2)^{-5/2}\right]$$

Likewise for the other nonzero stress components

$$\sigma_z^D=\frac{D}{8\pi(1-\nu)}\frac{\partial}{\partial z}\left[(1-2\nu)z(r^2+z^2)^{-3/2}+3z^3(r^2+z^2)^{-5/2}\right]$$

$$\sigma_\theta^D=-\frac{D}{8\pi(1-\nu)}\frac{\partial}{\partial z}\left[(1-2\nu)z(r^2+z^2)^{-3/2}\right]$$

$$\tau_{rz}^D=\frac{D}{8\pi(1-\nu)}\frac{\partial}{\partial z}\left[(1-2\nu)r(r^2+z^2)^{-3/2}+3rz^2(r^2+z^2)^{-5/2}\right]$$

Using the transformation relations (B.6) to go from cylindrical to spherical coordinates

$$\sigma_R=\sigma_r\sin^2\phi+\sigma_z\cos^2\phi+2\tau_{rz}\sin\phi\cos\phi\,,\text{ with }\sin\phi=r(r^2+z^2)^{-1/2}=\frac{r}{R}\,,\cos\phi=z(r^2+z^2)^{-1/2}=\frac{z}{R}\Rightarrow$$

$$\sigma_R=-\frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\left[-\sin^2\phi+\frac{2(2-\nu)}{1+\nu}\cos^2\phi\right]$$

$$\tau_{R\phi}=(\sigma_r-\sigma_z)\sin\phi\cos\phi-\tau_{rz}(\sin^2\phi-\cos^2\phi)\Rightarrow$$

$$\tau_{R\phi}=-\frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\sin\phi\cos\phi$$

13-19.

Since it is expected that the Center of Dilatation will produce a spherically symmetric stress state, we consider a special direction in the x, z - plane as shown to apply the superposition principle. For a Doublet acting along the z - axis, the solution was given in Exercise 13 - 18 as

$$\sigma_R^z = -\frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\left[-\sin^2\phi + \frac{2(2-\nu)}{1+\nu}\cos^2\phi\right],\ \tau_{R\phi}^z = -\frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\sin\phi\cos\phi$$

For a Doublet acting along the x - axis, replace ϕ by $\frac{\pi}{2}-\phi$ in the previous expressions $\Rightarrow$

$$\sigma_R^x = -\frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\left[-\cos^2\phi + \frac{2(2-\nu)}{1+\nu}\sin^2\phi\right],\ \tau_{R\phi}^x = \frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\sin\phi\cos\phi$$

For a Doublet acting along the y - axis, replace ϕ by $\frac{\pi}{2}$ in the previous z - axis expressions $\Rightarrow$

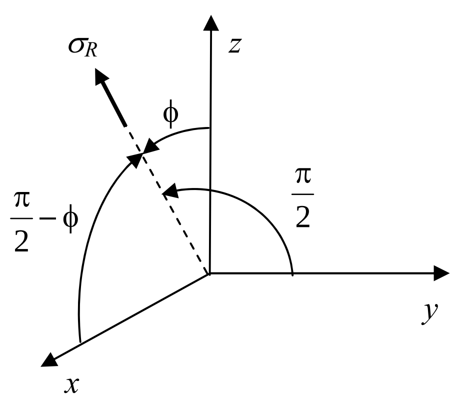
$$\sigma_R^y = \frac{2(1+\nu)D}{8\pi(1-\nu)R^3},\ \tau_{R\phi}^y = 0$$

Applying superposition $\Rightarrow$

$$\sigma_R = \sigma_R^x + \sigma_R^y + \sigma_R^z = -\frac{2(1+\nu)D}{8\pi(1-\nu)R^3}\left[-1 + \frac{2(2-\nu)}{1+\nu} - 1\right] = -\frac{(1-2\nu)D}{2\pi(1-\nu)R^3}$$

$$\tau_{R\phi} = \tau_{R\phi}^x + \tau_{R\phi}^y + \tau_{R\phi}^z = 0$$

These results should then be valid for all directions



13-20.

For the spherically symmetric case : $u_R = u(R), u_\phi = u_\theta = 0$, and relation $(A.12)_1$ reduces to

$$\nabla^2 u - \frac{2u}{R^2} = 0 \Rightarrow \frac{d^2u}{dR^2} + \frac{2}{R}\frac{du}{dR} - \frac{2}{R^2}u = 0$$

Look for solutions of the form : $u = AR^m$ and substitue into governing equ. $\Rightarrow$

$m(m-1) + 2m - 2 = 0 \Rightarrow m^2 + m - 2 = 0 \Rightarrow (m-1)(m+2) = 0 \Rightarrow$ roots $m = 1, -2$

$$\therefore u = C_1 R + \frac{C_2}{R^2}$$

Using relations (A.2) into (A.9) gives the stress relations

$$\sigma_R = \lambda\left(\frac{du}{dR} + 2\frac{u}{R}\right) + 2\mu\frac{du}{dR} = (\lambda + 2\mu)\frac{du}{dR} + 2\lambda\frac{u}{R}$$

$$\sigma_\phi = \sigma_\theta = \lambda\left(\frac{du}{dR} + 2\frac{u}{R}\right) + 2\mu\frac{u}{R} = 2(\lambda + \mu)\frac{u}{R} + \lambda\frac{du}{dR}$$

which allowes the stresses to be written as : $\sigma_R = K_1 - \dfrac{2K_2}{R^3}$, $\sigma_\phi = \sigma_\theta = K_1 + \dfrac{K_2}{R^3}$

Note that $C_1 = \dfrac{1-2\nu}{2\mu(1+\nu)}K_1$, $C_2 = \dfrac{K_2}{2\mu}$

13-21.

From Exercise 13 - 20 $\Rightarrow$ $\sigma_R = K_1 - \dfrac{2K_2}{R^3}$, $\sigma_\phi = \sigma_\theta = K_1 + \dfrac{K_2}{R^3}$

Boundary Conditions on problem (raduis of cavity taken as a) :

$$\sigma_R(a) = 0 \Rightarrow K_1 - \frac{2K_2}{a^3} = 0 \Rightarrow K_1 = \frac{2K_2}{a^3}$$

$$\sigma_R(\infty) = \sigma_\phi(\infty) = \sigma_\theta(\infty) = S \Rightarrow K_1 = S \Rightarrow K_2 = Sa^3/2$$

$\therefore$ the stresses are given by : $\sigma_R = S\left[1 - \dfrac{a^3}{R^3}\right]$, $\sigma_\phi = \sigma_\theta = S\left[1 + \dfrac{a^3}{2R^3}\right]$

$\sigma_{max} = \sigma_\phi(a) = \sigma_\theta(a) = \dfrac{3}{2}S \Rightarrow$ Stress Concentration Factor $= K = \dfrac{3}{2}$

From section 8.4.2, the corresponding two - dimensional result was $K_{2-D} = 2$

So $K_{2-D} > K_{3-D}$ which is to be expected since a three - dimensional domain has an additional dimension to relieve or reduce the stress concentration.

13-22.

From Exercise 13 - 20 $\Rightarrow \sigma_R = K_1 - \frac{2K_2}{R^3}$, $\sigma_\phi = \sigma_\theta = K_1 + \frac{K_2}{R^3}$

Boundary Conditions on spherical shell problem :

$\sigma_R(R_1) = -p_1$, $\sigma_R(R_2) = -p_2 \Rightarrow$

$K_1 = \frac{p_1 R_1^3 - p_2 R_2^3}{R_2^3 - R_1^3}$, $K_2 = \frac{(p_1 - p_2) R_1^3 R_2^3}{2(R_2^3 - R_1^3)}$ and the stresses become

$$\sigma_R = \frac{p_1 R_1^3 - p_2 R_2^3}{R_2^3 - R_1^3} - \frac{(p_1 - p_2) R_1^3 R_2^3}{(R_2^3 - R_1^3)} \frac{1}{R^3}$$

$$\sigma_\phi = \sigma_\theta = \frac{p_1 R_1^3 - p_2 R_2^3}{R_2^3 - R_1^3} + \frac{(p_1 - p_2) R_1^3 R_2^3}{2(R_2^3 - R_1^3)} \frac{1}{R^3}$$

For the special case : $p_1 = p$, $p_2 = 0$

$$\sigma_R = \frac{pR_1^3}{R_2^3 - R_1^3}\left[1 - \frac{R_2^3}{R^3}\right], \ \sigma_\phi = \sigma_\theta = \frac{pR_1^3}{R_2^3 - R_1^3}\left[1 + \frac{R_2^3}{2R^3}\right]$$

13-23*.

From Exercise 13 - 20 :

$$u = C_1 R + \frac{C_2}{R^2},\ \sigma_R = K_1 - \frac{2K_2}{R^3},\ \sigma_\phi = \sigma_\theta = K_1 + \frac{K_2}{R^3}$$

with $C_1 = \frac{1-2\nu}{2\mu(1+\nu)} K_1$, $C_2 = \frac{K_2}{2\mu}$

Inclusion Problem Boundary Conditions :

$$\sigma_R(\infty) = S \Rightarrow K_1 = S \Rightarrow C_1 = \frac{1-2\nu}{2\mu(1+\nu)} S$$

$$u(a) = 0 \Rightarrow C_1 a + \frac{C_2}{a^2} = 0 \Rightarrow C_2 = -C_1 a^3 \Rightarrow K_2 = -\frac{1-2\nu}{1+\nu} S a^3$$

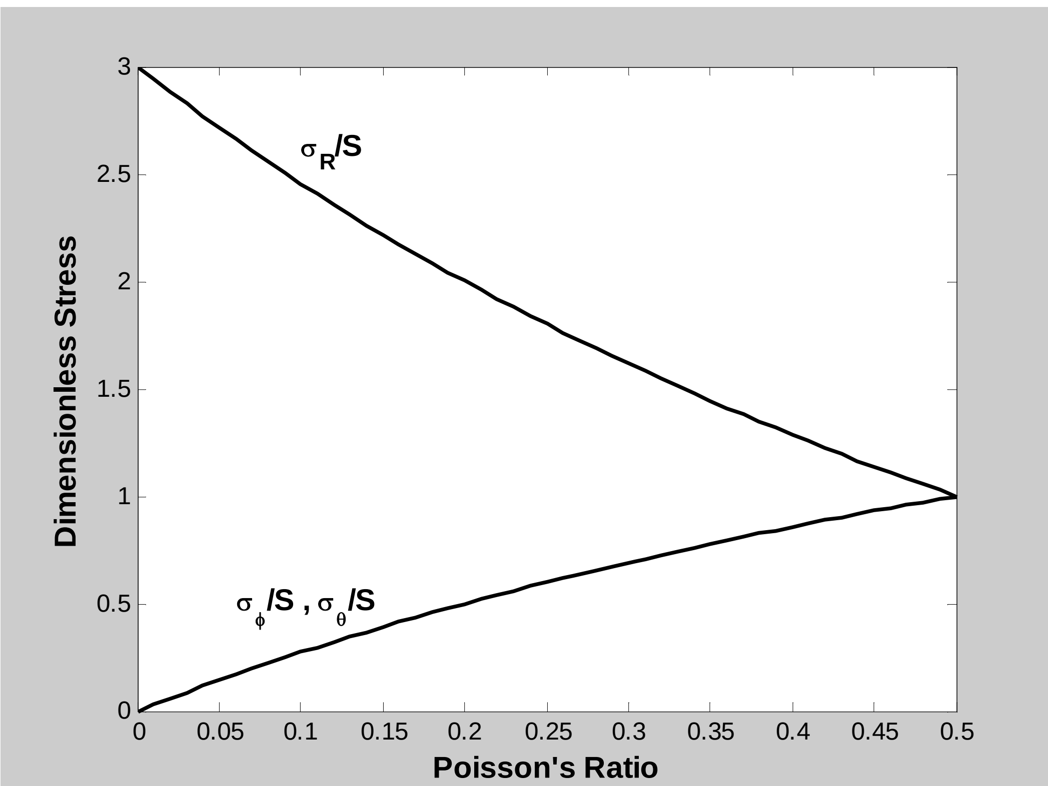
$$\therefore \sigma_R = S\left[1 + 2\frac{1-2\nu}{1+\nu}\frac{a^3}{R^3}\right],\ \sigma_\phi = \sigma_\theta = S\left[1 - \frac{1-2\nu}{1+\nu}\frac{a^3}{R^3}\right]$$

For the case with $\nu = 1/2$: $\sigma_R = \sigma_\phi = \sigma_\theta = S$

On the boundary of the inclusion $R = a$:

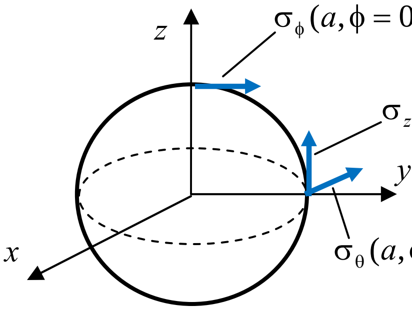
$$\sigma_R = S\left[1 + 2\frac{1-2\nu}{1+\nu}\right] = \frac{3(1-\nu)}{1+\nu} S\ ,\ \sigma_\phi = \sigma_\theta = S\left[1 - \frac{1-2\nu}{1+\nu}\right] = \frac{3\nu}{1+\nu} S$$

MATLAB Plots of boundary stresses :



13-24.

Given stress relations for the spherical cavity problem with tension S along z direction :



$$\sigma_\phi(a,\phi=0) = -\frac{3+15\nu}{2(7-5\nu)}S$$

$$\sigma_z(a,z=0) = \frac{27-15\nu}{2(7-5\nu)}S$$

$$\sigma_\theta(a,\phi=\pi/2) = \frac{15\nu-3}{2(7-5\nu)}S$$

(a) Tension S along x and z directions :

$$\sigma_{\max} = \sigma_z(a,z=0)+\sigma_\phi(a,\phi=0)$$

$$= \frac{27-15\nu}{2(7-5\nu)}S - \frac{3+15\nu}{2(7-5\nu)}S = \frac{24-30\nu}{2(7-5\nu)}S$$

(b) Tension S along z and compression x directions :

$$\sigma_{\max} = \sigma_z(a,z=0)-\sigma_\phi(a,\phi=0)$$

$$= \frac{27-15\nu}{2(7-5\nu)}S + \frac{3+15\nu}{2(7-5\nu)}S = \frac{15}{7-5\nu}S$$

which also correspondes to a pure shear far - field loading

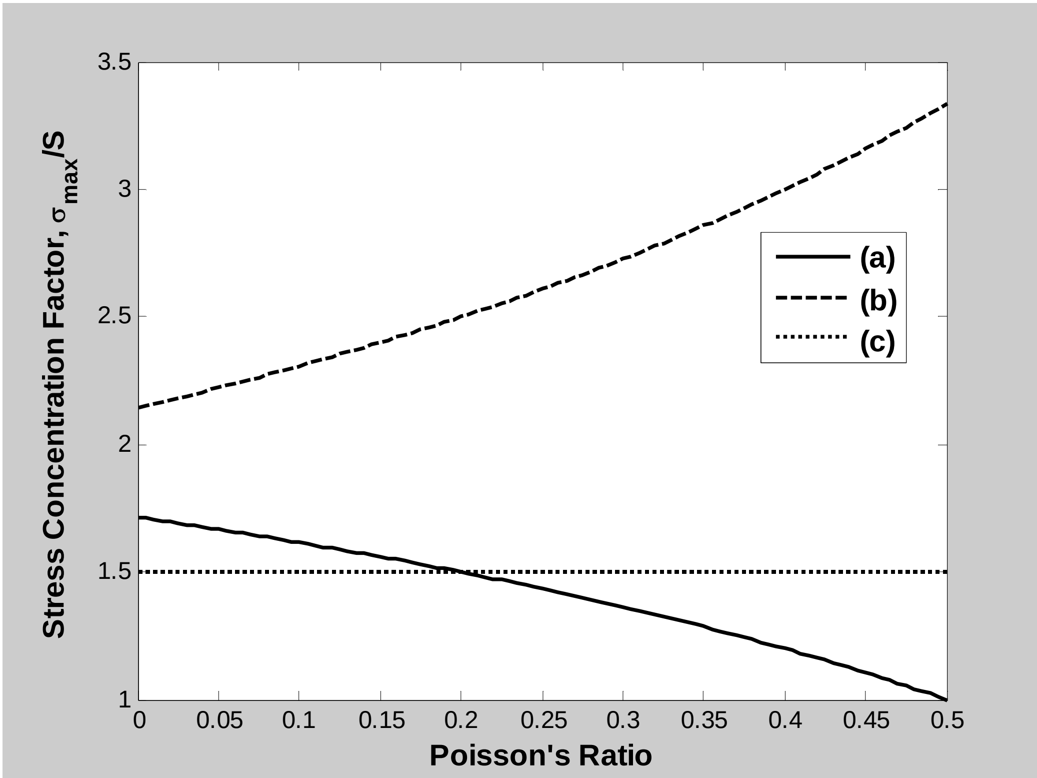
(c) Tension S along x, y and z directions :

$$\sigma_{\max} = \sigma_z(a,z=0)+\sigma_\phi(a,\phi=0)+\sigma_\theta(a,\phi=\pi/2)$$

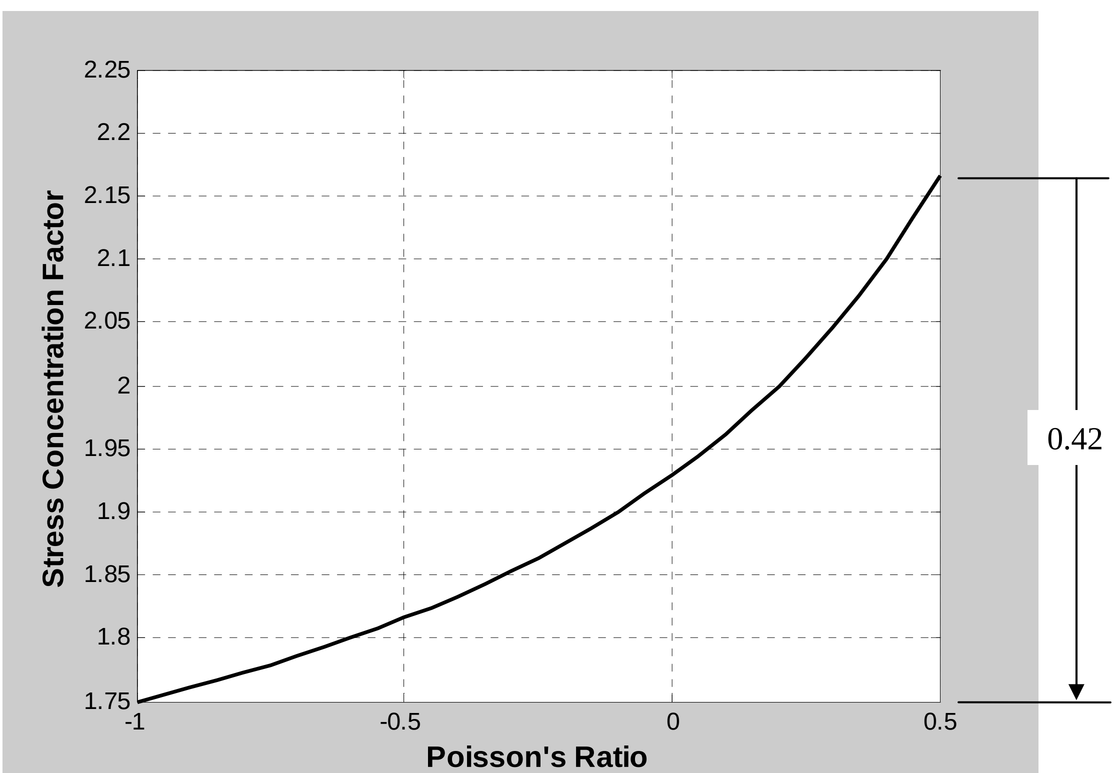
$$= \frac{27-15\nu}{2(7-5\nu)}S - \frac{3+15\nu}{2(7-5\nu)}S + \frac{15\nu-3}{2(7-5\nu)}S = \frac{21-15\nu}{2(7-5\nu)}S = \frac{3}{2}S$$

which matches with results of Exercise 13 - 21.

13-25*.


Stress Concentration Factor, σ_{max}/S
3.5
3
2.5
2
1.5
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Poisson's Ratio
(a)
(b)
(c)

13-26*.


Stress Concentration Factor
2.25
2.2
2.15
2.1
2.05
2
1.95
1.9
1.85
1.8
1.75
-1
-0.5
0
0.5
Poisson's Ratio
0.42

13-27.

Use the Morera stress function with $\Phi_{13} = -\frac{1}{2} z\phi_{,1}$, $\Phi_{23} = -\frac{1}{2} z\phi_{,2}$, $\Phi_{12,12} = -\frac{\nu}{2}\nabla^2\phi$, $\phi = \phi(x, y)$

$\sigma_{11} = -2\,\Phi_{23,23} = \left(z\phi_{,2}\right)_{,23} = \phi_{,22}$

$\sigma_{22} = -2\,\Phi_{31,31} = \left(z\phi_{,1}\right)_{,31} = \phi_{,11}$

$\sigma_{33} = -2\,\Phi_{12,12} = \nu\nabla^2\phi = \nu(\sigma_{11} + \sigma_{22})$

$\sigma_{12} = -\Phi_{12,33} + \Phi_{23,13} + \Phi_{13,23} = -0 + \left(-\frac{1}{2} z\phi_{,2}\right)_{,13} + \left(-\frac{1}{2} z\phi_{,1}\right)_{,23} = -\frac{1}{2}\phi_{,21} - -\frac{1}{2}\phi_{,12} = -\phi_{,12}$

$\sigma_{23} = -\Phi_{23,11} + \Phi_{13,21} + \Phi_{12,31} = -\left(-\frac{1}{2} z\phi_{,2}\right)_{,11} + \left(-\frac{1}{2} z\phi_{,1}\right)_{,21} + 0 = 0$

$\sigma_{31} = -\Phi_{31,22} + \Phi_{12,32} + \Phi_{23,12} = -\left(-\frac{1}{2} z\phi_{,1}\right)_{,22} + 0 + \left(-\frac{1}{2} z\phi_{,2}\right)_{,12} = 0$

$\therefore$ stresses are given by the usual form of the Airy stress function $\phi(x, y)$

14-1.

The governing compatibility equation is $\frac{\partial^2 e_y}{\partial x^2}+\frac{\partial^2 e_x}{\partial y^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}$

and for plane strain using relation (7.2.2) this would give

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\sigma_y-\frac{\nu}{E}\sigma_x\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\sigma_x-\frac{\nu}{E}\sigma_y\right)-2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\tau_{xy}\right)=0$$

Introducing the usual Airy stress function gives

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial x^2}-\frac{\nu}{E}\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial y^2}-\frac{\nu}{E}\frac{\partial^2\phi}{\partial x^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\frac{\partial^2\phi}{\partial x\partial y}\right)=0$$

We could also simply start with plane strain relation (14.1.5) and use conversion Table 7 -1.

For the special case of $E=E(x)$ and $\nu=\text{constant}$:

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial x^2}-\frac{\nu}{E}\frac{\partial^2\phi}{\partial y^2}\right)+\frac{1}{E}\frac{\partial^4\phi}{\partial y^4}-\frac{\nu}{E}\frac{\partial^4\phi}{\partial x^2\partial y^2}+2(1+\nu)\left(\frac{1}{E}\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial^3\phi}{\partial x\partial y^2}\right)=0\Rightarrow$$

$$\frac{1}{E}\left(\frac{\partial^4\phi}{\partial x^4}+\frac{\partial^4\phi}{\partial y^4}\right)+2\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\frac{\partial^2\phi}{\partial x^2}-2\frac{\nu}{E}\frac{\partial^4\phi}{\partial x^2\partial y^2}-2\nu\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial^3\phi}{\partial x\partial y^2}$$

$$-\nu\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\frac{\partial^2\phi}{\partial y^2}+2(1+\nu)\left(\frac{1}{E}\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial^3\phi}{\partial x\partial y^2}\right)=0\Rightarrow$$

$$\frac{1}{E}\left(\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right)+2\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\frac{\partial^2\phi}{\partial y^2}+2\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial^3\phi}{\partial x\partial y^2}=0\Rightarrow$$

$$\frac{1}{E}\left(\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right)+2\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)=0$$

14-2.

$$\sigma_x = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} = (k+2)\mu\frac{\partial u}{\partial x} + k\mu\frac{\partial v}{\partial y}$$
$$\sigma_y = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\frac{\partial u}{\partial x} = (k+2)\mu\frac{\partial v}{\partial y} + k\mu\frac{\partial u}{\partial x}$$
$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \mu(x) = \mu_o(1+ax),\ \lambda(x) = k\mu(x)$$

Using the new form of Hooke's in equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \Rightarrow$$
$$\frac{\partial}{\partial x}\left((k+2)\mu\frac{\partial u}{\partial x} + k\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) = 0$$
$$\mu_o(1+ax)\left(k\frac{\partial \vartheta}{\partial x} + 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \mu_o a\left(k\vartheta + 2\frac{\partial u}{\partial x}\right) = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \Rightarrow$$
$$\mu_o(1+ax)\left(k\frac{\partial \vartheta}{\partial y} + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) + \mu_o a\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

14-3.

$$\nu = \text{constant},\ E(x) = E_o e^{ax} \Rightarrow \frac{\partial E}{\partial x} = aE_o e^{ax} = aE \Rightarrow \frac{1}{E}\frac{\partial E}{\partial x} = a \Rightarrow \frac{1}{E^2}\frac{\partial E}{\partial x} = \frac{a}{E}$$

$$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}(\phi_{,yy} - \nu\phi_{,xx})$$

$$e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{E}(\phi_{,xx} - \nu\phi_{,yy})$$

$$e_{xy} = \frac{1+\nu}{E}\tau_{xy} = -\frac{1+\nu}{E}\phi_{,xy}$$

$$\text{Compatibility Relation}: \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}(\phi_{,yy} - \nu\phi_{,xx})\right) + \frac{\partial^2}{\partial x^2}\left(\frac{1}{E}(\phi_{,xx} - \nu\phi_{,yy})\right) = 2\frac{\partial^2}{\partial x \partial y}\left(-\frac{1+\nu}{E}\phi_{,xy}\right)$$

$$\frac{1}{E}(\phi_{,yyyy} - \nu\phi_{,xxyy}) + \frac{\partial}{\partial x}\left(\frac{1}{E}(\phi_{,xxx} - \nu\phi_{,yyx}) - \frac{1}{E^2}\frac{\partial E}{\partial x}(\phi_{,xx} - \nu\phi_{,yy})\right) = -2(1+\nu)\left(\frac{1}{E}\phi_{,xxyy} - \frac{1}{E^2}\frac{\partial E}{\partial x}\phi_{,xyy}\right)$$

$$\frac{1}{E}(\phi_{,yyyy} - \nu\phi_{,xxyy}) + \frac{1}{E}\left(\phi_{,xxxx} - \nu\phi_{,yyxx} - a(\phi_{,xxx} - \nu\phi_{,yyx})\right) - \frac{a}{E}\left(\phi_{,xxx} - \nu\phi_{,yyx} - a(\phi_{,xx} - \nu\phi_{,yy}\right)$$

$$= -\frac{2(1+\nu)}{E}(\phi_{,xxyy} - a\phi_{,xyy}) \Rightarrow$$

$$\nabla^4\phi - 2a\frac{\partial}{\partial x}(\nabla^2\phi) + a^2\nabla^2\phi - a^2(1+\nu)\frac{\partial^2\phi}{\partial y^2} = 0$$

14-4.

For the hollow cylinder problem the displacements were given by: $u = Ar^{-(n+k)/2} + Br^{(-n+k)/2}$

Using the relations for the stresses gives

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\frac{u}{r}\right] = \frac{E}{1-\nu^2}\left[A\left(-\frac{(n+k)}{2} + \nu\right)r^{-(n+k+2)/2} + B\left(\frac{(-n+k)}{2} + \nu\right)r^{(-n+k-2)/2}\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr}\right] = \frac{E}{1-\nu^2}\left[A\left(1 - \frac{(n+k)}{2}\nu\right)r^{-(n+k+2)/2} + B\left(1 - \frac{(-n+k)}{2}\nu\right)r^{(-n+k-2)/2}\right]$$

Boundary Conditions:

$$\sigma_r(a) = p_i \Rightarrow A\left(-\frac{n+k-2\nu}{2}\right)a^{-(n+k+2)/2} + B\left(\frac{-n+k+2\nu}{2}\right)a^{(-n+k-2)/2} = \frac{1-\nu^2}{E}p_i$$

$$\sigma_r(b) = p_o \Rightarrow A\left(-\frac{n+k-2\nu}{2}\right)b^{-(n+k+2)/2} + B\left(\frac{-n+k+2\nu}{2}\right)b^{(-n+k-2)/2} = \frac{1-\nu^2}{E}p_o$$

Solving the two equations for A and $B \Rightarrow$

$$A = \frac{1-\nu^2}{E}\frac{\left(\frac{-n+k+2\nu}{2}\right)\left(p_i b^{(-n+k-2)/2} - p_o a^{(-n+k-2)/2}\right)}{\left(-\frac{n+k-2\nu}{2}\right)\left(\frac{-n+k+2\nu}{2}\right)\left(a^{-(n+k+2)/2}b^{(-n+k-2)/2} - a^{(-n+k-2)/2}b^{-(n+k+2)/2}\right)}$$

$$= \frac{1-\nu^2}{E}\frac{2\left(p_i b^{(-n+k-2)/2} - p_o a^{(-n+k-2)/2}\right)}{(-n-k+2\nu)\left(a^{-(n+k+2)/2}b^{(-n+k-2)/2} - a^{(-n+k-2)/2}b^{-(n+k+2)/2}\right)}$$

$$B = \frac{1-\nu^2}{E}\frac{2\left(p_o a^{-(n+k+2)/2} - p_i b^{-(n+k+2)/2}\right)}{(-n+k+2\nu)\left(a^{-(n+k+2)/2}b^{(-n+k-2)/2} - a^{(-n+k-2)/2}b^{-(n+k+2)/2}\right)}$$

Back substituting into the stress realtions gives

$$\sigma_r = -\frac{a^{-n/2}b^{-n/2}r^{(-2-k+n)/2}}{b^k - a^k}\left[-a^{k+n/2}b^{(2+k)/2}p_o + a^{n/2}b^{(2+k)/2}p_o r^k + a^{(2+k)/2}b^{n/2}p_i(b^k - r^k)\right]$$

$$\sigma_\theta = \frac{a^{-n/2}b^{-n/2}r^{(-2-k+n)/2}}{b^k - a^k}\left[\frac{(a^{(2+k)/2}b^{n/2}p_i - a^{n/2}b^{(2+k)/2}p_o)r^k(2+k\nu-n\nu)}{k-n+2\nu}\right.$$
$$\left. + \frac{a^{k/2}b^{k/2}(-ab^{(k+n)/2}p_i + a^{(k+n)/2}bp_o)(-2+k\nu+n\nu)}{k+n-2\nu}\right]$$

14-5.

From the general solution :

$$\sigma_r = -\frac{a^{-n/2}b^{-n/2}r^{(-2-k+n)/2}}{b^k - a^k}\left[-a^{k+n/2}b^{(2+k)/2}p_o + a^{n/2}b^{(2+k)/2}p_o r^k + a^{(2+k)/2}b^{n/2}p_i(b^k - r^k)\right]$$

$$\sigma_\theta = \frac{a^{-n/2}b^{-n/2}r^{(-2-k+n)/2}}{b^k - a^k}\left[\frac{(a^{(2+k)/2}b^{n/2}p_i - a^{n/2}b^{(2+k)/2}p_o)r^k(2+k\nu-n\nu)}{k-n+2\nu} + \frac{a^{k/2}b^{k/2}(-ab^{(k+n)/2}p_i + a^{(k+n)/2}bp_o)(-2+k\nu+n\nu)}{k+n-2\nu}\right]$$

For the special case of internal pressure only : $p_o = 0 \Rightarrow$

$$\sigma_r = -\frac{a^{-n/2}b^{-n/2}r^{(-2-k+n)/2}}{b^k - a^k}\left[a^{(2+k)/2}b^{n/2}p_i(b^k - r^k)\right] = -\frac{p_i a^{(2+k-n)/2}}{b^k - a^k}\left[(r^{(-2+k+n)/2} - b^k r^{(-2-k+n)/2}\right]$$

$$\sigma_\theta = \frac{a^{-n/2}b^{-n/2}r^{(-2-k+n)/2}}{b^k - a^k}\left[\frac{(a^{(2+k)/2}b^{n/2}p_i)r^k(2+k\nu-n\nu)}{k-n+2\nu} + \frac{a^{k/2}b^{k/2}(-ab^{(k+n)/2}p_i)(-2+k\nu+n\nu)}{k+n-2\nu}\right]$$

$$= \frac{p_i a^{(2+k-n)/2}}{b^k - a^k}\left[\frac{2+k\nu-n\nu}{k-n+2\nu}r^{(-2+k+n)/2} + \frac{2-k\nu-n\nu}{k+n-2\nu}b^k r^{(-2-k+n)/2}\right]$$

14-6.

With $k = \sqrt{n^2+4-4n\nu} \Rightarrow k^2 = n^2 + 4(1-n\nu)$

For the case with $0 \le \nu \le 1/2, n \ge 0 \Rightarrow \sqrt{n^2+4} \ge k \ge \sqrt{n^2+4-2n}$

$\sqrt{n^2+4} - 2 + n \ge -2 + k + n \ge \sqrt{n^2 - 2n + 4} - 2 + n \Rightarrow$

$-2 + k + n \ge 0$ (equality with $n = 0$)

$-\sqrt{n^2+4} - 2 + n \le -2 - k + n \le -\sqrt{n^2 - 2n + 4} - 2 + n \Rightarrow$

$-2 - k + n \le 0$ (equality with $n = 0$)

$\frac{2}{k-n} \le \frac{2+k\nu-n\nu}{k-n+2\nu} \le \frac{2+(k-n)/2}{k-n+1} \Rightarrow \frac{2+k\nu-n\nu}{k-n+2\nu} \ge 1$ (equality with $n = 0$)

$\frac{2}{k+n} \le \frac{2-k\nu-n\nu}{k+n-2\nu} \le \frac{2-(k+n)/2}{k+n-1} \Rightarrow \frac{2-k\nu-n\nu}{k+n-2\nu} \le 1$ (equality with $n = 0$)

Using this results in relations (14.2.7) it follows that $\sigma_r < 0$, $\sigma_\theta > 0$

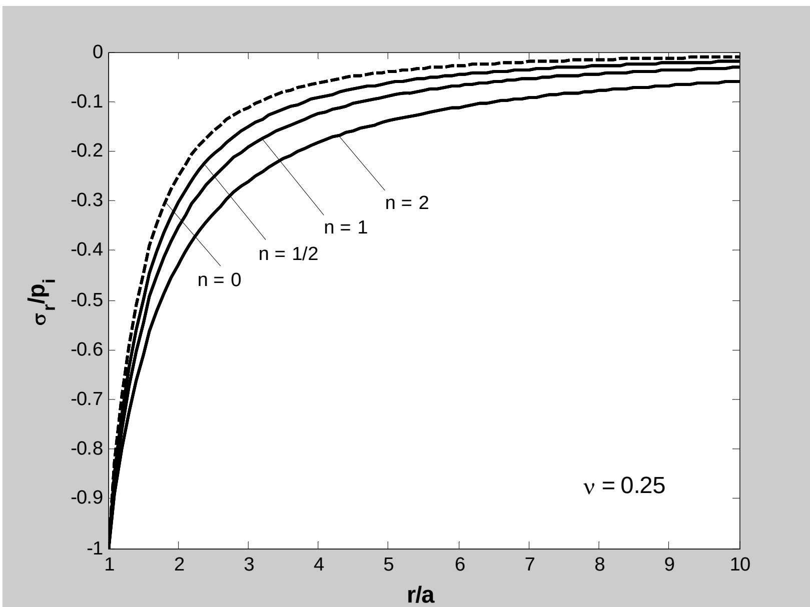
14-7.

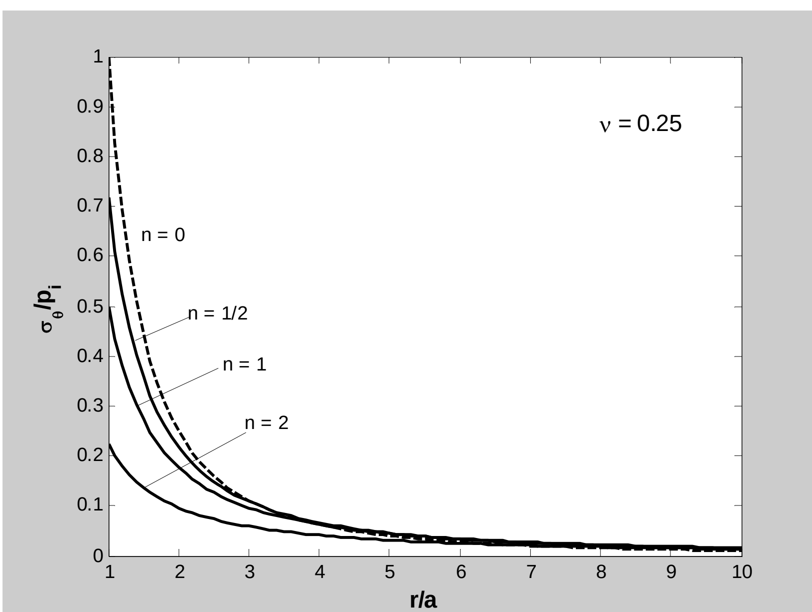
Use the special solution for the case of internal pressure only and let $b \to \infty$

$$\sigma_r = -\frac{p_i a^{(2+k-n)/2}}{b^k - a^k}\left[(r^{(-2+k+n)/2} - b^k r^{(-2-k+n)/2}\right] \to -p_i a^{(2+k-n)/2} r^{(-2-k+n)/2} = -p_i\left(\frac{a}{r}\right)^{(2+k-n)/2}$$

$$\sigma_\theta = \frac{p_i a^{(2+k-n)/2}}{b^k - a^k}\left[\frac{2+k\nu-n\nu}{k-n+2\nu} r^{(-2+k+n)/2} + \frac{2-k\nu-n\nu}{k+n-2\nu} b^k r^{(-2-k+n)/2}\right]$$

$$\to p_i a^{(2+k-n)/2}\frac{2-k\nu-n\nu}{k+n-2\nu} r^{(-2-k+n)/2} = p_i \frac{2-k\nu-n\nu}{k+n-2\nu}\left(\frac{a}{r}\right)^{(2+k-n)/2}$$





14-8.

For the case $\nu = 0$ and $n = 3 \Rightarrow k = \sqrt{13} > n$

Solution is given by relations (14.3.11)

$\sigma_r = 0,\ \sigma_\theta = \rho\omega^2 r^2$

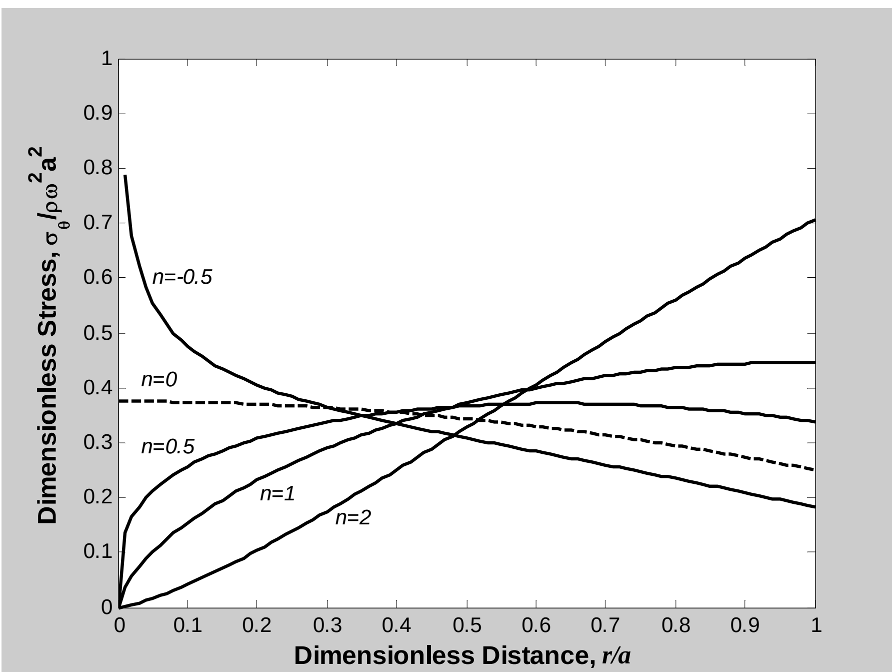
Displacement solution follows from (14.3.6):

$$u(r) = \cancel{A}r^{-(n+k)/2} + Br^{(\sqrt{13}-3)/2} + \frac{\rho\omega^2 a^3}{E_o} \Rightarrow$$

$$u(0) = \frac{\rho\omega^2 a^3}{E_o} > 0$$

14-9*.

MATLAB plot of hoop stress in rotating disk problem



Similar to the radial stress behavior, the hoop stress becomes unbounded at the origin for the case with $n < 0$, while for $n > 0$, the hoop stress goes to zero at the origin.

14-10.

For the case with $\sigma_r = \sigma_\theta$, relation (14.3.2) $\Rightarrow \frac{\mathrm{d}u}{\mathrm{dr}} = \frac{u}{\mathrm{r}}$

Intergrating gives the displacement soluton $u = C_1 r$

Using Hooke's law $\Rightarrow \sigma_r = \frac{E(r)}{1-\nu^2(r)}\left[\frac{du}{dr} + \nu(r)\frac{u}{r}\right] = \frac{C_1 E(r)}{1+\nu(r)}$

Using equilibrium equations : $\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \Rightarrow \frac{d}{dr}\left(\frac{C_1 E(r)}{1+\nu(r)}\right) = -\rho\omega^2 r$

Intergrating gives the soluton : $\sigma_r = \frac{C_1 E(r)}{1+\nu(r)} = -\rho\omega^2 r^2 + C_2$

Using the boundary conditon $\sigma_r(a) = 0 \Rightarrow C_2 = \frac{\rho\omega^2 a^2}{2}$

and $\therefore \frac{E(r)}{1+\nu(r)} = \frac{1}{2C_1}\rho\omega^2(a^2 - r^2)$

14-11.

$$\sigma_x = \lambda(e_x + e_z) + 2\mu e_x = 2\mu\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 2\mu\frac{\partial u}{\partial x} + \lambda\vartheta \ , \ \vartheta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

$$= 2\mu\left(\frac{\partial u}{\partial x} + \frac{\lambda}{2\mu}\vartheta\right) = 2\mu\left(\frac{\partial u}{\partial x} + \frac{\frac{2\mu\nu}{1-2\nu}}{2\mu}\vartheta\right) = 2\mu\left(\frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\vartheta\right) = 2\mu(z)\left(\frac{\partial u}{\partial x} + \nu^*\vartheta\right)$$

where $\nu^* = \dfrac{\nu}{1-2\nu}$

Likewise for z-component: $\sigma_z = \lambda(e_z + e_x) + 2\mu e_z = 2\mu(z)\left(\dfrac{\partial w}{\partial z} + \nu^*\vartheta\right)$

$$\tau_{xz} = 2\mu e_{xz} = \mu(z)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \mu(z)\vartheta$$

Equilibrium equation: $\dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xz}}{\partial z} = 0 \Rightarrow \dfrac{\partial}{\partial x}\left[2\mu(z)\left(\dfrac{\partial u}{\partial x} + \nu^*\vartheta\right)\right] + \dfrac{\partial}{\partial z}\left[\mu(z)\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right)\right] = 0 \Rightarrow$

$$2\mu(z)\left(\frac{\partial^2 u}{\partial x^2} + \nu^*\frac{\partial\vartheta}{\partial x}\right) + \mu(z)\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x\partial z}\right) + \frac{d\mu(z)}{dz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \Rightarrow$$

$$\nabla^2 u + (1+2\nu^*)\frac{\partial\vartheta}{\partial x} + \frac{1}{\mu(z)}\frac{d\mu(z)}{dz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

Likewise: $\dfrac{\partial\tau_{xz}}{\partial x} + \dfrac{\partial\sigma_z}{\partial z} = 0 \Rightarrow \dfrac{\partial}{\partial x}\left[\mu(z)\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right)\right] + \dfrac{\partial}{\partial z}\left[2\mu(z)\left(\dfrac{\partial w}{\partial z} + \nu^*\vartheta\right)\right] = 0 \Rightarrow$

$$\nabla^2 w + (1+2\nu^*)\frac{\partial\vartheta}{\partial z} + \frac{2}{\mu(z)}\frac{d\mu(z)}{dz}\left(\frac{\partial w}{\partial z} + \nu^*\vartheta\right) = 0$$

Special case: $\mu(z) = \mu_o e^{\alpha z} \Rightarrow \dfrac{d\mu(z)}{dz} = \alpha\mu_o e^{\alpha z} \Rightarrow \dfrac{1}{\mu(z)}\dfrac{d\mu(z)}{dz} = \alpha$

Thus the equilibrium equations reduce to:

$$\nabla^2 u + (1+2\nu^*)\frac{\partial\vartheta}{\partial x} + \alpha\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\nabla^2 w + (1+2\nu^*)\frac{\partial\vartheta}{\partial z} + 2\alpha\left(\frac{\partial w}{\partial z} + \nu^*\vartheta\right) = 0$$

14-12.

$$\sigma_r = \lambda(e_r + e_\theta + e_z) + 2\mu e_r = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_r}{\partial r} = 2\mu\frac{\partial u_r}{\partial r} + \lambda\vartheta \text{ , } \vartheta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

$$= 2\mu\left(\frac{\partial u_r}{\partial r} + \frac{\lambda}{2\mu}\vartheta\right) = 2\mu\left(\frac{\partial u_r}{\partial r} + \frac{\frac{2\mu\nu(z)}{1-2\nu(z)}}{2\mu}\vartheta\right) = 2\mu\left(\frac{\partial u_r}{\partial r} + \frac{\nu(z)}{1-2\nu(z)}\vartheta\right) = 2\mu\left(\frac{\partial u_r}{\partial r} + \nu^*(z)\vartheta\right)$$

$$\sigma_z = \lambda(e_r + e_\theta + e_z) + 2\mu e_z = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z} = 2\mu\frac{\partial u_z}{\partial z} + \lambda\vartheta$$

$$= 2\mu\left(\frac{\partial u_z}{\partial z} + \frac{\lambda}{2\mu}\vartheta\right) = 2\mu\left(\frac{\partial u_z}{\partial z} + \nu^*(z)\vartheta\right)$$

$$\sigma_\theta = \lambda(e_r + e_\theta + e_z) + 2\mu e_\theta = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{u_r}{r} = 2\mu\frac{u_r}{r} + \lambda\vartheta$$

$$= 2\mu\left(\frac{u_r}{r} + \frac{\lambda}{2\mu}\vartheta\right) = 2\mu\left(\frac{u_r}{r} + \nu^*(z)\vartheta\right)$$

$$\tau_{zr} = 2\mu e_{zr} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

14-13.

Using transformation (14.5.5) into governing equation (14.5.4) $\Rightarrow$

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)=0 \Rightarrow \frac{\partial}{\partial x}\left(\mu\frac{\partial}{\partial x}\left(\frac{W}{\sqrt{\mu}}\right)\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial}{\partial y}\left(\frac{W}{\sqrt{\mu}}\right)\right)=0 \Rightarrow$$

$$\frac{\partial}{\partial x}\left(\mu\left(\frac{1}{\sqrt{\mu}}\frac{\partial W}{\partial x}-\frac{W}{2\mu\sqrt{\mu}}\frac{\partial\mu}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\mu\left(\frac{1}{\sqrt{\mu}}\frac{\partial W}{\partial y}-\frac{W}{2\mu\sqrt{\mu}}\frac{\partial\mu}{\partial y}\right)\right)=0$$

$$\frac{\partial}{\partial x}\left(\sqrt{\mu}\frac{\partial W}{\partial x}-\frac{W}{2\sqrt{\mu}}\frac{\partial\mu}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sqrt{\mu}\frac{\partial W}{\partial y}-\frac{W}{2\sqrt{\mu}}\frac{\partial\mu}{\partial y}\right)=0$$

$$\sqrt{\mu}\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+\frac{1}{2}\left(\frac{1}{2\mu}\frac{1}{\sqrt{\mu}}(\mu_x^2+\mu_y^2)-\frac{1}{\sqrt{\mu}}(\mu_{xx}+\mu_{yy})\right)=0 \Rightarrow$$

$$\mu\nabla^2 W+\frac{1}{2}\left(\frac{1}{2\mu}(\mu_x^2+\mu_y^2)-\mu_{xx}-\mu_{yy}\right)W=0$$

Separation of Variables: $W = X(x)Y(y)\ ,\ \mu=\mu_o p(x)q(y) \Rightarrow$

$$\mu_o pq\left(X''Y+XY''\right)+\frac{1}{2}\left(\frac{1}{2\mu_o pq}(\mu_o^2 p'^2 q^2+\mu_o^2 p^2 q'^2)-\mu_o p''q-\mu_o pq''\right)W=0$$

Divide through by $\mu_o pqXY \Rightarrow$

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{1}{4}\frac{p'^2}{p^2}+\frac{1}{4}\frac{q'^2}{q^2}-\frac{1}{2}\frac{p''}{p}-\frac{1}{2}\frac{q''}{q}=0 \Rightarrow$$

$$\frac{X''}{X}+\frac{1}{4}\frac{p'^2}{p^2}-\frac{1}{2}\frac{p''}{p}=-\left(\frac{Y''}{Y}+\frac{1}{4}\frac{q'^2}{q^2}-\frac{1}{2}\frac{q''}{q}\right)=\text{constant}=-n^2 \Rightarrow$$

$$X_{xx}+\left[n^2+\frac{1}{4}\left(\frac{p_x}{p}\right)^2-\frac{1}{2}\left(\frac{p_{xx}}{p}\right)\right]X=0\ ,\ Y_{xx}+\left[-n^2+\frac{1}{4}\left(\frac{q_x}{q}\right)^2-\frac{1}{2}\left(\frac{q_{yy}}{q}\right)\right]Y=0$$

14-14.

Using the functions, $p(x)=e^{\alpha|x|}$, $q(y)=e^{\beta|y|}$

$$\frac{1}{2p}\frac{d^2p}{dx^2}-\frac{1}{4}\left(\frac{1}{p}\frac{dp}{dx}\right)^2=a_o\Rightarrow\frac{\alpha^2}{2}-\frac{1}{4}(\alpha)^2=\frac{\alpha^2}{4}=a_o$$

$$\frac{1}{2q}\frac{d^2q}{dy^2}-\frac{1}{4}\left(\frac{1}{q}\frac{dq}{dy}\right)^2=b_o\Rightarrow\frac{\beta^2}{2}-\frac{1}{4}(\beta)^2=\frac{\beta^2}{4}=b_o$$

14-15.

For the circular section case, we have axisymmetry, no warping displacement and the section rotates as a rigid body $\Rightarrow u_r=u_z=0$, $u_\theta=\alpha rz$

The strains then follow from relation (A.2) as $e_r=e_\theta=e_z=e_{rz}=e_{r\theta}=0$, $e_{\theta z}=\dfrac{\alpha r}{2}$

The stress can then be calculated using (A.8) $\sigma_r=\sigma_\theta=\sigma_z=\tau_{rz}=\tau_{r\theta}=0$, $\tau_{\theta z}=\alpha\mu r$

For the axisymmetric case:

$$\frac{\partial}{\partial x}=\frac{d}{dr}\frac{\partial r}{\partial x}=\frac{x}{r}\frac{d}{dr}\ ,\ \frac{\partial}{\partial y}=\frac{d}{dr}\frac{\partial r}{\partial y}=\frac{y}{r}\frac{d}{dr}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\mu}\frac{\partial\phi}{\partial y}\right)=-2\alpha\Rightarrow\frac{x}{r}\frac{d}{dr}\left(\frac{1}{\mu}\frac{x}{r}\frac{d\phi}{dr}\right)+\frac{y}{r}\frac{d}{dr}\left(\frac{1}{\mu}\frac{y}{r}\frac{d\phi}{dr}\right)=-2\alpha\Rightarrow$$

$$\frac{x^2+y^2}{r}\frac{d}{dr}\left(\frac{1}{\mu}\frac{1}{r}\frac{d\phi}{dr}\right)+\frac{1}{\mu}\frac{x}{r}\frac{d\phi}{dr}\frac{dx}{dr}+\frac{1}{\mu}\frac{y}{r}\frac{d\phi}{dr}\frac{dy}{dr}=-2\alpha\Rightarrow$$

$$\frac{d}{dr}\left(\frac{1}{\mu}\right)\frac{d\phi}{dr}+\frac{1}{\mu}\frac{1}{r}\frac{d\phi}{dr}+\frac{1}{\mu}\frac{d^2\phi}{dr^2}=-2\alpha\Rightarrow\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\mu}\frac{d\phi}{dr}\right)=-2\alpha$$

$$\tau_{\theta z}=-\sqrt{\tau_{xz}^2+\tau_{yz}^2}=-\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial x}\right)^2}=-\sqrt{\left(\frac{x}{r}\frac{d\phi}{dr}\right)^2+\left(\frac{y}{r}\frac{d\phi}{dr}\right)^2}=-\frac{d\phi}{dr}$$

14-16.

From relation (14.6.9) $\Rightarrow \dfrac{d}{dr}\left(\dfrac{r}{\mu}\dfrac{d\phi}{dr}\right) = -2\alpha r$

Integrating $\Rightarrow \dfrac{r}{\mu}\dfrac{d\phi}{dr} = -\alpha r^{2} + C_1 \Rightarrow \dfrac{d\phi}{dr} = -\alpha\mu r + \dfrac{C_1\mu}{r} \Rightarrow$

$\phi = -\alpha\int \mu(r) r dr + C_1\int \dfrac{\mu(r)}{r} dr + C_2$

As discussed in text, $C_1 = 0$, and boundary condition $\phi(a) = 0 \Rightarrow C_2 = \alpha\int^{a} \mu(r) r dr$

$\therefore$ solution is $\phi(r) = \alpha\int_r^a \xi\mu(\xi)d\xi$

$\tau_{\theta z} = -\dfrac{d\phi}{dr} \Rightarrow \tau_{\theta z} = \alpha r\mu(r)$

$J = T/\alpha = \dfrac{2\pi}{\alpha}\int_0^a \tau_{\theta z} r^2 dr = 2\pi\int_0^a r^3\mu(r)dr$

14-17.

Using the gradation model: $\mu(r) = \mu_o\left(1+\dfrac{n}{a}r\right)^m$, solutuon (14.6.11) gives

$\phi(r) = \alpha\int_r^a \xi\mu(\xi)d\xi = \alpha\mu_o\int_r^a \xi\left(1+\dfrac{n}{a}\xi\right)^m d\xi$

For the $m = 1$ case, this easily integrates out to

$\phi(r) = \dfrac{\mu_o\alpha}{2}(a^2 - r^2) + \dfrac{\mu_o\alpha n}{3}(a^2 - \dfrac{r^3}{a})$

while for the $m = -1$ case, gives

$$\phi(r) = \mu_o\alpha\int_r^a \frac{\xi}{1+\dfrac{n}{a}\xi} d\xi = \mu_o\alpha\left[\frac{a}{n}\xi - \left(\frac{a}{n}\right)^2 \log\left(1+\frac{n}{a}\xi\right)\right]_r^a$$

$$= -\mu_o\alpha\left[\frac{a}{n}r - \left(\frac{a}{n}\right)^2 \log\left|1+\frac{n}{a}r\right|\right] + \mu_o\alpha\left[\frac{a^2}{n} - \left(\frac{a}{n}\right)^2 \log|1+n|\right]$$

Using relation $(14.6.12)_1 \Rightarrow \tau_{\theta z} = \alpha r\mu(r) = \mu_o\alpha r\left(1+\dfrac{n}{a}r\right)^m$

14-18.

Using the gradation model: $\mu(r)=\mu_o e^{-\frac{n}{a}r}$, solutuon (14.6.11) gives

$$\phi(r)=\alpha\int_r^a \xi\mu(\xi)d\xi=\alpha\mu_o\int_r^a \xi e^{-\frac{n}{a}\xi}d\xi=\alpha\mu_o\left[\left(-\frac{n}{a}-1\right)\frac{e^{-\frac{n}{a}\xi}}{(n^2/a^2)}\right]_r^a$$

$$=\mu_o\alpha e^{-\frac{n}{a}r}\left(\frac{ar}{n}+\left(\frac{a}{n}\right)^2\right)-\mu_o\alpha a^2\left(\frac{1}{n}+\frac{1}{n^2}\right)e^{-n}$$

Using relation $(14.6.12)_1 \Rightarrow \tau_{\theta z}=\alpha r\mu(r)=\mu_o\alpha r e^{-\frac{n}{a}r}$

14-19.

Relation $(14.6.12)_1$: $\tau_{\theta z}=\alpha r\mu(r)\Rightarrow$

$$\frac{d\tau_{\theta z}}{dr}=\alpha r\mu'(r)+\alpha\mu(r)=0\Rightarrow r_o=-\frac{\mu(r_o)}{\mu'(r_o)}$$, where r_o is location of max stress point

Since $r<a\Rightarrow \mu'(r)<-\dfrac{\mu(r)}{a}$

Using the gradation model: $\mu(r)=\mu_o\left(1+\frac{n}{a}r\right)^m\Rightarrow \mu'(r)=\mu_o m\left(\frac{n}{a}\right)\left(1+\frac{n}{a}r\right)^{m-1}\Rightarrow$

$$r_o=-\frac{\mu(r_o)}{\mu'(r_o)}=-m\left(1+\frac{n}{a}r_o\right)\left(\frac{a}{n}\right)\Rightarrow r_o=-\frac{a}{n(1+m)}$$

Using the gradation model: $\mu(r)=\mu_o e^{-\frac{n}{a}r}\Rightarrow \mu'(r)=\mu_o\left(\frac{n}{a}\right)e^{-\frac{n}{a}r}\Rightarrow$

$$r_o=-\frac{\mu(r_o)}{\mu'(r_o)}=\frac{a}{n}$$

14-20.

Using the gradation model $(14.6.13)_2$: $\mu(r) = \mu_o e^{-\frac{n}{a}r}$, the torsional rigidity is

$$J = T/\alpha = \frac{2\pi}{\alpha}\int_0^a \tau_{\theta z} r^2 dr = 2\pi\int_0^a r^3\mu(r)dr = 2\pi\mu_o\int_0^a r^3 e^{-\frac{n}{a}r}dr$$

$$= 2\pi\mu_o\left[\left(\frac{r^3}{(-n/a)} - \frac{3r^2}{(n/a)^2} + \frac{6r}{(-n/a)^3} - \frac{6}{(n/a)^4}\right)e^{-\frac{n}{a}r}\right]_0^a$$

$$= 2\pi\mu_o a^4\left[\frac{6}{n^4} - \left(\frac{1}{n} + \frac{3}{n^2} + \frac{6}{n^3} + \frac{6}{n^4}\right)e^{-n}\right]$$

For small n : $e^{-n} = 1 - n + \frac{n^2}{2!} - \cdots \Rightarrow 1 - e^{-n} = n - \frac{n^2}{2!} + \frac{n^3}{3!}\cdots$

$$J \approx 2\pi\mu_o a^4\left[\left(n - \frac{n^2}{2!} + \frac{n^3}{3!} - \cdots\right)\frac{6}{n^4} - \left(\frac{1}{n} + \frac{3}{n^2} + \frac{6}{n^3}\right)\left(1 - n + \frac{n^2}{2!} - \cdots\right)\right]$$

$$\approx 2\pi\mu_o a^4\left[\frac{1}{4} - \frac{1}{5}n + O(n^2)\right] \approx \frac{\pi a^4\mu_o}{2}\left[1 - \frac{4n}{5} + O(n^2)\right]$$

For the homogeneous case, $n = 0$, and thus $J = \frac{\pi a^4\mu_o}{2}$ which checks with previous study

15-1.

Using complex variable methods, choose the potentials

$$\gamma(z)=\frac{i\mu b}{4\pi(1-\nu)}\log z\ ,\ \ \psi(z)=-\frac{i\mu\bar{b}}{4\pi(1-\nu)}\log z$$

$$2\mu(u+iv)=\kappa\gamma(z)-z\overline{\gamma'(z)}-\overline{\psi(z)}\Rightarrow \text{(for plane strain case)}$$

$$u+iv=(3-4\nu)\frac{ib}{8\pi(1-\nu)}\log z+z\frac{i\bar{b}}{8\pi(1-\nu)}\frac{1}{\bar{z}}-\frac{ib}{8\pi(1-\nu)}\log\bar{z}$$

$$=(3-4\nu)\frac{i(b_x+ib_y)}{8\pi(1-\nu)}(\log r+i\theta)+\frac{i(b_x-ib_y)}{8\pi(1-\nu)}(\cos 2\theta+i\sin 2\theta)-\frac{i(b_x+ib_y)}{8\pi(1-\nu)}(\log r-i\theta)$$

Check cyclic property:

$$[u+iv]_C=(3-4\nu)\frac{ib}{8\pi(1-\nu)}(2\pi i)-\frac{ib}{8\pi(1-\nu)}(-2\pi i)=-\frac{b}{4(1-\nu)}[(3-4\nu)+1]=-b$$

Separating real and imaginary parts $\Rightarrow$

$$u=-\frac{b_x}{2\pi}\left[\tan^{-1}\frac{y}{x}+\frac{1}{2(1-\nu)}\frac{xy}{x^2+y^2}\right]+\frac{b_y}{8\pi(1-\nu)}\left[(2\nu-1)\log(x^2+y^2)+\frac{x^2-y^2}{x^2+y^2}\right]$$

$$v=\frac{b_y}{2\pi}\left[\tan^{-1}\frac{y}{x}+\frac{1}{2(1-\nu)}\frac{xy}{x^2+y^2}\right]+\frac{b_x}{8\pi(1-\nu)}\left[(1-2\nu)\log(x^2+y^2)+\frac{x^2-y^2}{x^2+y^2}\right]$$

Stress Field:

$$\sigma_x+\sigma_y=2\left(\gamma'(z)+\overline{\gamma'(z)}\right)=4Re\left[\frac{i\mu b}{4\pi(1-\nu)}\frac{1}{z}\right]=\frac{\mu}{\pi r(1-\nu)}(b_x\sin\theta-b_y\cos\theta)$$

$$\sigma_y-\sigma_x+2i\tau_{xy}=2\left(\bar{z}\gamma''(z)+\psi'(z)\right)=-\frac{i\mu}{2\pi r(1-\nu)}\left[be^{-3i\theta}+\bar{b}e^{-i\theta}\right]$$

$$=-\frac{\mu i}{2\pi r(1-\nu)}\left[b_x\cos 3\theta+b_y\sin 3\theta+b_x\cos\theta-b_y\sin\theta\right]$$

$$-\frac{\mu}{2\pi r(1-\nu)}\left[b_y\cos\theta+b_x\sin\theta-b_y\cos 3\theta+b_x\sin 3\theta\right]$$

$$\sigma_y-\sigma_x=-\frac{\mu}{2\pi r(1-\nu)}\left[b_y\cos\theta+b_x\sin\theta-b_y\cos 3\theta+b_x\sin 3\theta\right]$$

$$\tau_{xy}=-\frac{\mu}{4\pi r(1-\nu)}\left[b_x\cos 3\theta+b_y\sin 3\theta+b_x\cos\theta-b_y\sin\theta\right]$$

$$\sigma_x=\frac{\mu}{4\pi r(1-\nu)}\left[3b_x\sin\theta-b_y\cos\theta-b_y\cos 3\theta+b_x\sin 3\theta\right]$$

$$\sigma_y=\frac{\mu}{4\pi r(1-\nu)}\left[b_x\sin\theta-3b_y\cos\theta+b_y\cos 3\theta-b_x\sin 3\theta\right]$$

15-1. Continued

Expressing the stresses in terms of Cartesian coordinates $\Rightarrow$

$$\sigma_x = \frac{\mu}{2\pi(1-\nu)}\left(b_x \frac{y(3x^2+y^2)}{(x^2+y^2)^2} - b_y \frac{x(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$$\sigma_y = \frac{\mu}{2\pi(1-\nu)}\left(b_x \frac{y(y^2-x^2)}{(x^2+y^2)^2} - b_y \frac{x(x^2+3y^2)}{(x^2+y^2)^2}\right)$$

$$\tau_{xy} = -\frac{\mu}{2\pi(1-\nu)}\left(b_x \frac{x(x^2-y^2)}{(x^2+y^2)^2} + b_y \frac{y(x^2-y^2)}{(x^2+y^2)^2}\right)$$

15-2.

Edge Dislocation :

$$u = \frac{b}{2\pi}\left[\tan^{-1}\frac{y}{x} + \frac{1}{2(1-\nu)}\frac{xy}{x^2+y^2}\right],\ v = -\frac{b}{2\pi}\left[\frac{1-2\nu}{4(1-\nu)}\log(x^2+y^2) - \frac{1}{2(1-\nu)}\frac{y^2}{x^2+y^2}\right]$$

Cyclic Behavior : $[u]_C = \frac{b}{2\pi}\left[\tan^{-1}\frac{y}{x}\right]_C = b\ ,\ [v]_C = 0$

Strain Field :

$$e_x = \frac{\partial u}{\partial x} = \frac{b}{2\pi}\left[-\frac{y}{x^2+y^2} + \frac{1}{2(1-\nu)}\frac{(x^2+y^2)y - 2x^2y}{(x^2+y^2)^2}\right] = \frac{b}{2\pi}\left[-\frac{y}{x^2+y^2} + \frac{1}{2(1-\nu)}\frac{y(y^2-x^2)}{(x^2+y^2)^2}\right]$$

$$= \frac{b}{2\pi r^4}\left[-y(x^2+y^2) + \frac{y(y^2-x^2)}{2(1-\nu)}\right] = \frac{b}{4\pi r^4(1-\nu)}[y(y^2-x^2) - 2(1-\nu)y(x^2+y^2)]$$

$$e_y = \frac{\partial v}{\partial y} = -\frac{b}{2\pi}\left[\frac{1-2\nu}{4(1-\nu)}\frac{2y}{x^2+y^2} - \frac{1}{2(1-\nu)}\frac{(x^2+y^2)2y - 2y^3}{(x^2+y^2)^2}\right]$$

$$= -\frac{b}{2\pi r^4}\left[\frac{1-2\nu}{2(1-\nu)}y(x^2+y^2) - \frac{yx^2}{(1-\nu)}\right] = -\frac{b}{4\pi r^4(1-\nu)}[(1-2\nu)y(x^2+y^2) - 2yx^2]$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{b}{4\pi}\left[\frac{x}{x^2+y^2} + \frac{1}{2(1-\nu)}\frac{x(x^2-y^2)}{(x^2+y^2)^2}\right] - \frac{b}{4\pi}\left[\frac{1-2\nu}{4(1-\nu)}\frac{2x}{x^2+y^2} + \frac{1}{2(1-\nu)}\frac{2xy^2}{(x^2+y^2)^2}\right]$$

$$= \frac{b}{4\pi r^4}\left[x(x^2+y^2) + \frac{x(x^2-y^2)}{2(1-\nu)}\right] - \frac{b}{4\pi r^4}\left[\frac{1-2\nu}{2(1-\nu)}x(x^2+y^2) + \frac{xy^2}{(1-\nu)}\right]$$

$$= \frac{b}{4\pi r^4(1-\nu)}x(x^2-y^2)$$

Plane Strain Stress Field :

$$\sigma_x = \frac{2\mu\nu}{1-2\nu}(e_x+e_y) + 2\mu e_x = -\frac{b\mu}{2\pi r^4(1-\nu)}y(y^2+3x^2) = -\frac{b\mu}{2\pi(1-\nu)}\frac{y(3x^2+y^2)}{(x^2+y^2)^2}$$

$$\sigma_y = \frac{2\mu\nu}{1-2\nu}(e_x+e_y) + 2\mu e_y = \frac{b\mu}{2\pi r^4(1-\nu)}y(x^2-y^2) = \frac{b\mu}{2\pi(1-\nu)}\frac{y(x^2-y^2)}{(x^2+y^2)^2}$$

$$\tau_{xy} = 2\mu e_{xy} = \frac{b\mu}{2\pi r^4(1-\nu)}x(x^2-y^2) = \frac{b\mu}{2\pi(1-\nu)}\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

In Cylindrical Coordinates :

$$\sigma_r = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta = -\frac{b\mu}{2\pi(1-\nu)r}\sin\theta$$

$$\sigma_\theta = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta = -\frac{b\mu}{2\pi(1-\nu)r}\sin\theta$$

$$\tau_{r\theta} = -\sigma_x\sin\theta\cos\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) = \frac{b\mu}{2\pi(1-\nu)r}\cos\theta$$

15-3.

Screw Dislocation :

$$u = v = 0 \ , \ w = \frac{b}{2\pi}\tan^{-1}\frac{y}{x}$$

Cyclic Behavior : $[w]_C = \dfrac{b}{2\pi}\left[\tan^{-1}\dfrac{y}{x}\right]_C = b$

Strain Field :

$$e_x = e_y = e_z = e_{xy} = 0 \ , \ e_{yz} = \frac{1}{2}\frac{\partial w}{\partial y} = \frac{b}{4\pi}\frac{x}{x^2+y^2} \ , \ e_{xz} = \frac{1}{2}\frac{\partial w}{\partial x} = -\frac{b}{4\pi}\frac{y}{x^2+y^2}$$

Stress Field :

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{yz} = 2\mu e_{yz} = \frac{b\mu}{2\pi}\frac{x}{x^2+y^2} \ , \ \tau_{xz} = 2\mu e_{xz} = -\frac{b\mu}{2\pi}\frac{y}{x^2+y^2}$$

In Cylindrical Coordinates :

$$\sigma_r = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta = 0$$

$$\sigma_\theta = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta = 0$$

$$\sigma_z = 0$$

$$\tau_{r\theta} = -\sigma_x\sin\theta\cos\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) = 0$$

$$\tau_{\theta z} = \tau_{yz}\cos\theta - \tau_{zx}\sin\theta = \frac{b\mu}{2\pi}\left[\frac{r\cos\theta}{r^2}\cos\theta + \frac{r\sin\theta}{r^2}\sin\theta\right] = \frac{b\mu}{2\pi r}$$

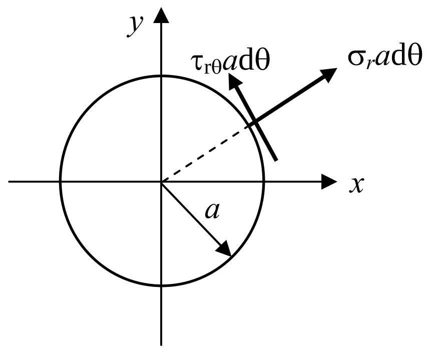
$$\tau_{zr} = \tau_{yz}\sin\theta + \tau_{zx}\cos\theta = \frac{b\mu}{2\pi}\left[\frac{r\cos\theta}{r^2}\sin\theta - \frac{r\sin\theta}{r^2}\cos\theta\right] = 0$$

15-4.

Stress Solution for Edge Dislocation :

$$\sigma_r = \sigma_\theta = -\frac{bB}{r}\sin\theta \ , \ \tau_{r\theta} = \frac{bB}{r}\cos\theta$$

Sum Forces on Cylinder of radius a ,



$$\sum F_x = \int_0^{2\pi}(a\sigma_r\cos\theta - a\tau_{r\theta}\sin\theta)d\theta$$

$$= -2bB\int_0^{2\pi}\sin\theta\cos\theta\, d\theta$$

$$= -bB\int_0^{2\pi}\sin 2\theta\, d\theta = 0$$

$$\sum F_y = \int_0^{2\pi}(a\sigma_r\sin\theta + a\tau_{r\theta}\cos\theta)d\theta$$

$$= bB\int_0^{2\pi}(-\sin^2\theta + \cos^2\theta)d\theta$$

$$= bB\int_0^{2\pi}\cos 2\theta\, d\theta = 0$$

15-5.

Stress Field for Screw Dislocation :

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0 \ , \ \tau_{\theta z} = \frac{\mu b}{2\pi r}$$

On the ends of a cylinder of finite length, $\tau_{\theta z}$ will not vanish, and the resultant twisting moment is given by

$$T = \int_0^a\int_0^{2\pi}\tau_{\theta z} rrd\theta dr = \int_0^a\int_0^{2\pi}\frac{\mu b}{2\pi}rd\theta dr = \mu b\int_0^a rdr = \frac{\mu ba^2}{2}$$

15-6.

Strain Energy Density for Screw Dislocation :

$$U = \frac{1}{2}\tau_{\theta z}\gamma_{\theta z} = \frac{1}{2}\frac{\mu b^2}{4\pi^2 r^2}$$

Strain Energy for Screw Dislocation in Region $R_o \le r \le R_c$:

$$W_{screw} = \int_0^{2\pi}\int_{R_c}^{R_o}U\, rdrd\theta = \frac{1}{2}2\pi\int_{R_c}^{R_o}\frac{\mu b^2}{4\pi^2 r}dr$$

$$= \frac{\mu b^2}{4\pi}\log\frac{R_o}{R_c}$$

15-7.

Stress Field for Edge Dislocation : $\sigma_r = \sigma_\theta = -\frac{bB}{r}\sin\theta\ ,\ \tau_{r\theta} = \frac{bB}{r}\cos\theta$

Strain Field for Edge Dislocation (Plane Strain) :

$$e_r = \frac{1+\nu}{E}[(1-\nu)\sigma_r - \nu\sigma_\theta] = \frac{1+\nu}{E}(1-2\nu)\sigma_r = \frac{1-2\nu}{2\mu}\sigma_r = -\frac{1-2\nu}{2\mu}\frac{bB}{r}\sin\theta$$

$$e_\theta = \frac{1+\nu}{E}[(1-\nu)\sigma_\theta - \nu\sigma_r] = \frac{1+\nu}{E}(1-2\nu)\sigma_\theta = \frac{1-2\nu}{2\mu}\sigma_\theta = -\frac{1-2\nu}{2\mu}\frac{bB}{r}\sin\theta$$

$$\gamma_{r\theta} = \frac{1}{\mu}\tau_{r\theta} = \frac{bB}{\mu r}\cos\theta$$

Strain Energy Density for Edge Dislocation :

$$U = \frac{1}{2}(\sigma_r e_r + \sigma_\theta e_\theta + \tau_{r\theta}\gamma_{r\theta}) = \frac{1-2\nu}{2\mu}\frac{b^2B^2}{r^2}\sin^2\theta + \frac{b^2B^2}{2\mu r^2}\cos^2\theta$$

$$= \frac{b^2B^2}{2\mu r^2}(\sin^2\theta + \cos^2\theta) - \frac{\nu}{\mu}\frac{b^2B^2}{r^2}\sin^2\theta = \frac{b^2B^2}{2\mu r^2}(1-2\nu\sin^2\theta)$$

Strain Energy for Edge Dislocation in Region $R_o \le r \le R_c$:

$$W_{screw} = \int_0^{2\pi}\int_{R_c}^{R_o} U\, r dr d\theta = \frac{b^2B^2}{2\mu}\int_0^{2\pi}\int_{R_c}^{R_o}\frac{1}{r}(1-2\nu\sin^2\theta) dr d\theta$$

$$= \frac{b^2B^2}{2\mu}2\pi(1-\nu)\log\frac{R_o}{R_c} = \frac{\mu b^2}{4\pi(1-\nu)}\log\frac{R_o}{R_c}$$

15-8.

Kelvin State with Unit Loads : $\phi^\alpha = 0$, $\psi_i^\alpha = -C\,\dfrac{\delta_{i\alpha}}{R}$, where $C = \dfrac{1}{8\pi(1-\nu)}$

$$2\mu u_i^\alpha = (\phi^\alpha + x_k\,\psi_k^\alpha)_{,i} - 4(1-\nu)\,\psi_i^\alpha = -C(\frac{x_\alpha}{R})_{,i} + 4C(1-\nu)\frac{\delta_{i\alpha}}{R} = C\left(\frac{x_\alpha x_i}{R^3} + (3-4\nu)\frac{\delta_{i\alpha}}{R}\right) \Rightarrow$$

$$u_i^\alpha = \frac{C}{2\mu R}\left(\frac{x_\alpha x_i}{R^2} + (3-4\nu)\,\delta_{i\alpha}\right)$$

$$\sigma_{ij}^\alpha = \phi_{,ij} - 2\nu\,\delta_{ij}\,\psi_{k,k}^\alpha - (1-2\nu)(\psi_{i,j}^\alpha + \psi_{j,i}^\alpha) + x_k\psi_{k,ij}^\alpha$$

$$\psi_{k,k}^\alpha = C\,\frac{\delta_{k\alpha}\,x_k}{R^3} = C\,\frac{x_\alpha}{R^3}\ ,\ \psi_{i,j}^\alpha = C\,\frac{\delta_{i\alpha}\,x_j}{R^3}\ ,\ \psi_{j,i}^\alpha = C\,\frac{\delta_{j\alpha}\,x_i}{R^3}\ ,\ \psi_{k,ij}^\alpha = -\frac{3C}{R^5}\delta_{k\alpha}\,x_i x_j + \frac{C}{R^3}\delta_{k\alpha}\,\delta_{ij} \Rightarrow$$

$$\sigma_{ij}^\alpha = -\frac{C}{R^3}\left(\frac{3x_\alpha x_i x_j}{R^2} + (1-2\nu)(\delta_{i\alpha}\,x_j + \delta_{j\alpha}\,x_i - \delta_{ij}\,x_\alpha)\right)$$

For the specail case $a_\alpha = [0,0,1]$: $\phi^z = 0$, $\psi_i^z = \left[0\,,0,-\dfrac{C}{R}\right] \Rightarrow$

$$u_i^z = \frac{C}{2\mu R}\left(\frac{zx_i}{R^2} + (3-4\nu)\,\delta_{i3}\right) \Rightarrow$$

$$u_x^z = \frac{C}{2\mu R}\frac{zx}{R^2} = \frac{Czx}{2\mu R^3}\ ,\ u_y^z = \frac{Czy}{2\mu R^3}\, ,\ u_z^z = \frac{C}{2\mu R}\left(\frac{z^2}{R^2} + (3-4\nu)\right)$$

$$\sigma_{ij}^z = -\frac{C}{R^3}\left(\frac{3zx_i x_j}{R^2} + (1-2\nu)(\delta_{i3}\,x_j + \delta_{j3}\,x_i - \delta_{ij}\,z)\right) \Rightarrow$$

$$\sigma_x^z = -\frac{C}{R^3}\left(\frac{3zx^2}{R^2} - (1-2\nu)z\right),\ \sigma_y^z = -\frac{C}{R^3}\left(\frac{3zy^2}{R^2} - (1-2\nu)z\right),\ \sigma_z^z = -\frac{C}{R^3}\left(\frac{3z^3}{R^2} - (1-2\nu)z\right)$$

$$\tau_{xy}^z = -\frac{C}{R^3}\frac{3zxy}{R^2} = -\frac{3Czxy}{R^5}\ ,\ \tau_{yz}^z = -\frac{C}{R^3}\left(\frac{3z^2 y}{R^2} + (1-2\nu)\,y\right),\ \tau_{zx}^z = -\frac{C}{R^3}\left(\frac{3z^2 x}{R^2} + (1-2\nu)x\right)$$

Using the displacement vector transformation relations (B.8),

$$u_R = u_x^z \sin\phi\cos\theta + u_y^z \sin\phi\sin\theta + u_z^z \cos\phi = \frac{2C(1-\nu)}{\mu}\,\frac{\cos\phi}{R}$$

$$u_\phi = u_x^z \cos\phi\cos\theta + u_y^z \cos\phi\sin\theta - u_z^z \sin\phi = -\frac{C(3-4\nu)}{2\mu}\,\frac{\sin\phi}{R}$$

$$u_\theta = -u_x^z \sin\theta + u_y^z \cos\theta = 0$$

15-8. Continued

Using the stress transformation relations (B.9),

$$\sigma_R = \sigma_x^z \sin^2\phi\cos^2\theta + \sigma_y^z \sin^2\phi\sin^2\theta + \sigma_z^z \cos^2\phi$$
$$+ 2\tau_{xy}^z \sin^2\phi\sin\theta\cos\theta + 2\tau_{yz}^z \sin\phi\cos\phi\sin\theta + 2\tau_{zx}^z \sin\phi\cos\phi\cos\theta$$
$$= -2C(2-\nu)\frac{\cos\phi}{R^2}$$

$$\sigma_\phi = \sigma_x^z \cos^2\phi\cos^2\theta + \sigma_y^z \cos^2\phi\sin^2\theta + \sigma_z^z \sin^2\phi$$
$$+ 2\tau_{xy}^z \cos^2\phi\sin\theta\cos\theta - 2\tau_{yz}^z \sin\phi\cos\phi\sin\theta - 2\tau_{zx}^z \sin\phi\cos\phi\cos\theta$$
$$= C(1-2\nu)\frac{\cos\phi}{R^2}$$

$$\sigma_\theta = \sigma_x^z \sin^2\theta + \sigma_y^z \cos^2\theta - 2\tau_{xy}^z \sin\theta\cos\theta = C(1-2\nu)\frac{\cos\phi}{R^2}$$

$$\tau_{R\phi} = \sigma_x^z \sin\phi\cos\phi\cos^2\theta + \sigma_y^z \sin\phi\cos\phi\sin^2\theta - \sigma_z^z \sin\phi\cos\phi$$
$$+ 2\tau_{xy}^z \sin\phi\cos\phi\sin\theta\cos\theta - \tau_{yz}^z(\sin^2\phi - \cos^2\phi)\sin\theta$$
$$- \tau_{zx}^z(\sin^2\phi - \cos^2\phi)\cos\theta$$
$$= C(1-2\nu)\frac{\sin\phi}{R^2}$$

$$\tau_{\phi\theta} = -\sigma_x^z \cos\phi\sin\theta\cos\theta + \sigma_y^z \cos\phi\sin\theta\cos\theta + \tau_{xy}^z \cos\phi(\cos^2\theta - \sin^2\theta)$$
$$- \tau_{yz}^z \sin\phi\cos\theta + \tau_{zx}^z \sin\phi\sin\theta$$
$$= 0$$

$$\tau_{\theta R} = -\sigma_x^z \sin\phi\sin\theta\cos\theta + \sigma_y^z \sin\phi\sin\theta\cos\theta + \tau_{xy}^z \sin\phi(\cos^2\theta - \sin^2\theta)$$
$$+ \tau_{yz}^z \cos\phi\cos\theta - \tau_{zx}^z \cos\phi\sin\theta$$
$$= 0$$

15-9.

Center of Compression : $\phi^o = -\frac{3}{2(1-2\nu)}\frac{1}{R}$, $\psi_i^o = \frac{x_i}{2(1-2\nu)}\frac{1}{R^3}$

Noting that $R^2 = x_k x_k$, $R_{,i} = \frac{x_i}{R}$, $\left(\frac{1}{R}\right)_{,i} = -\frac{x_i}{R^3} \Rightarrow$

$$2\mu u_i^o = (\phi^o + x_k\,\psi_{\mathrm{k}}^o)_{,i} - 4(1-\nu)\,\psi_{\mathrm{i}}^o = \left(-\frac{3}{2(1-2\nu)}\frac{1}{R} + \frac{1}{2(1-2\nu)}\frac{1}{R}\right)_{,i} - \frac{2(1-\nu)}{(1-2\nu)}\frac{x_i}{R^3}$$

$$= \frac{1}{(1-2\nu)}\frac{x_i}{R^3} - \frac{2(1-\nu)}{(1-2\nu)}\frac{x_i}{R^3} = -\frac{x_i}{R^3} \Rightarrow$$

$$u_i^o = -\frac{x_i}{2\mu R^3}$$

$$e_{kk}^o = -\frac{1}{2\mu}\left(\frac{x_k}{R^3}\right)_{,k} = 0 \ , \ e_{ij}^o = \frac{1}{2}\left(u_{i,j}^o + u_{j,i}^o\right) = -\frac{1}{2\mu}\left(-\frac{3x_i x_j}{R^5} + \frac{\delta_{ij}}{R^3}\right)$$

$$\sigma_{ij}^o = \lambda e_{kk}^o \delta_{ij} + 2\mu e_{ij}^o = \frac{3x_i x_j}{R^5} - \frac{\delta_{ij}}{R^3} = \frac{1}{R^3}\left(\frac{3x_i x_j}{R^2} - \delta_{ij}\right)$$

For spherical displacement components, use transformation relations (B.8)

$$u_R^o = u_x^o \sin\phi\cos\theta + u_y^o \sin\phi\sin\theta + u_z^o \cos\phi = -\frac{1}{2\mu R^2}$$

$$u_\phi^o = u_x^o \cos\phi\cos\theta + u_y^o \cos\phi\sin\theta - u_z^o \sin\phi = 0$$

$$u_\theta^o = -u_x^o \sin\theta + u_y^o \cos\theta = 0$$

For spherical stress components, use transformation relations (B.9)

$$\sigma_R^o = \sigma_x^o \sin^2\phi\cos^2\theta + \sigma_y^o \sin^2\phi\sin^2\theta + \sigma_z^o \cos^2\phi$$
$$+ 2\tau_{xy}^o \sin^2\phi\sin\theta\cos\theta + 2\tau_{yz}^o \sin\phi\cos\phi\sin\theta + 2\tau_{zx}^o \sin\phi\cos\phi\cos\theta$$
$$= \frac{2}{R^3}$$

$$\sigma_\phi^o = \sigma_x^o \cos^2\phi\cos^2\theta + \sigma_y^o \cos^2\phi\sin^2\theta + \sigma_z^o \sin^2\phi$$
$$+ 2\tau_{xy}^o \cos^2\phi\sin\theta\cos\theta - 2\tau_{yz}^o \sin\phi\cos\phi\sin\theta - 2\tau_{zx}^o \sin\phi\cos\phi\cos\theta$$
$$= -\frac{1}{R^3}$$

$$\sigma_\theta^o = \sigma_x^o \sin^2\theta + \sigma_y^o \cos^2\theta - 2\tau_{xy}^o \sin\theta\cos\theta = -\frac{1}{R^3}$$

15-9. Continued

$$\tau_{R\phi}^o = \sigma_x^o \sin\phi\cos\phi\cos^2\theta + \sigma_y^o \sin\phi\cos\phi\sin^2\theta - \sigma_z^o \sin\phi\cos\phi$$
$$+ 2\tau_{xy}^o \sin\phi\cos\phi\sin\theta\cos\theta - \tau_{yz}^o(\sin^2\phi - \cos^2\phi)\sin\theta$$
$$- \tau_{zx}^o(\sin^2\phi - \cos^2\phi)\cos\theta$$
$$= 0$$

$$\tau_{\theta\phi}^o = -\sigma_x^o \cos\phi\sin\theta\cos\theta + \sigma_y^o \cos\phi\sin\theta\cos\theta + \tau_{xy}^o \cos\phi(\cos^2\theta - \sin^2\theta)$$
$$- \tau_{yz}^o \sin\phi\cos\theta + \tau_{zx}^o \sin\phi\sin\theta$$
$$= 0$$

$$\tau_{\theta R}^o = -\sigma_x^o \sin\phi\sin\theta\cos\theta + \sigma_y^o \sin\phi\sin\theta\cos\theta + \tau_{xy}^o \sin\phi(\cos^2\theta - \sin^2\theta)$$
$$+ \tau_{yz}^o \cos\phi\cos\theta - \tau_{zx}^o \cos\phi\sin\theta$$
$$= 0$$

15-10.

The state for a line of Centers of Dilatation along x_1 - axis from 0 to a is given by,

$$ {}^{x}\boldsymbol{\mathcal{S}}^{o}(\boldsymbol{x}) = -\int_0^a \boldsymbol{\mathcal{S}}^{o}(\boldsymbol{x};\lambda)d\lambda \text{ , where } \boldsymbol{\mathcal{S}}^{o}(\boldsymbol{x};\lambda) = \boldsymbol{\mathcal{S}}^{o}(x_1 - a, x_2, x_3) \Rightarrow $$

$$ {}^{x}u_1^o = \frac{1}{2\mu}\int_0^a \frac{(x_1-\lambda)}{[(x_1-\lambda)^2 + x_2^2 + x_3^3]^{3/2}} d\lambda = \frac{x_1}{2\mu}\int_0^a \frac{d(x_1-\lambda)^2}{[(x_1-\lambda)^2 + x_2^2 + x_3^3]^{3/2}} $$

$$ = \frac{1}{2\mu}\left(\frac{1}{\hat{R}} - \frac{1}{R}\right) \text{, where } \hat{R} = \sqrt{(x_1-a)^2 + x_2^2 + x_3^3} \text{ and } R = \sqrt{x_1^2 + x_2^2 + x_3^3} $$

$$ {}^{x}u_2^o = \frac{x_2}{2\mu}\int_0^a \frac{d\lambda}{[(x_1-\lambda)^2 + x_2^2 + x_3^3]^{3/2}} = -\frac{x_2}{2\mu}\int_0^a \frac{d(x_1-\lambda)}{[(x_1-\lambda)^2 + x_2^2 + x_3^3]^{3/2}} $$

$$ = \frac{1}{2\mu}\left(\frac{1}{R}\frac{x_1x_2}{x_2^2 + x_3^3} - \frac{1}{\hat{R}}\frac{(x_1-a)x_2}{x_2^2 + x_3^3}\right) = \frac{1}{2\mu}\frac{x_2}{x_2^2 + x_3^3}\left(\frac{x_1}{R} - \frac{(x_1-a)}{\hat{R}}\right) $$

$$ {}^{x}u_3^o = \frac{x_3}{2\mu}\int_0^a \frac{d\lambda}{[(x_1-\lambda)^2 + x_2^2 + x_3^3]^{3/2}} = -\frac{x_3}{2\mu}\int_0^a \frac{d(x_1-\lambda)}{[(x_1-\lambda)^2 + x_2^2 + x_3^3]^{3/2}} $$

$$ = \frac{1}{2\mu}\frac{x_3}{x_2^2 + x_3^3}\left(\frac{x_1}{R} - \frac{(x_1-a)}{\hat{R}}\right) $$

15-11.

Isotropic Self - Consistent Crack Distribution Case with $\nu = 0.5$:

$$\varepsilon = \frac{45(\nu - \bar{\nu})(2-\bar{\nu})}{16(1-\bar{\nu}^2)(10\nu - 3\nu\bar{\nu} - \bar{\nu})} = \frac{45(0.5-\bar{\nu})(2-\bar{\nu})}{16(1-\bar{\nu}^2)(5-1.5\bar{\nu}-\bar{\nu})}$$

$$= \frac{45(0.5)(1-2\bar{\nu})(2-\bar{\nu})}{16(1-\bar{\nu}^2)(2.5)(2-\bar{\nu})} = \frac{9}{16}\frac{(1-2\bar{\nu})}{(1-\bar{\nu}^2)} \Rightarrow$$

$16\varepsilon\bar{\nu}^2 - 18\bar{\nu} + (9-16\varepsilon) = 0$, solving by quadratic formula $\Rightarrow \bar{\nu} = \dfrac{9 \pm \sqrt{9^2 - 16\varepsilon(9-16\varepsilon)}}{16\varepsilon}$

When $\varepsilon = 9/16 \Rightarrow$

$\bar{\nu} = 0$

$$\frac{\bar{E}}{E} = 1 - \frac{16(1-\bar{\nu}^2)(10-3\bar{\nu})\varepsilon}{45(2-\bar{\nu})} = 1 - \frac{16(10)9}{45(2)16} = 1 - 1 = 0$$

$$\frac{\bar{\mu}}{\mu} = 1 - \frac{32(1-\bar{\nu})(5-\bar{\nu})\varepsilon}{45(2-\bar{\nu})} = 1 - \frac{32(5)9}{45(2)16} = 1 - 1 = 0$$

Dilute Case with $\nu = 0.5$:

$$\varepsilon = \frac{45(\nu - \bar{\nu})(2-\nu)}{16(1-\nu^2)(10\bar{\nu} - 3\nu\bar{\nu} - \nu)} = \frac{45(0.5-\bar{\nu})(2-0.5)}{16(1-0.25)(10\bar{\nu}-1.5\bar{\nu}-0.5)} = \frac{45(1-2\bar{\nu})}{8(17\bar{\nu}-1)} \Rightarrow$$

$$\bar{\nu} = \frac{45+8\varepsilon}{90+136\varepsilon}$$

When $\varepsilon = 9/16 \Rightarrow$

$$\bar{\nu} = \frac{11}{37} = 0.297$$

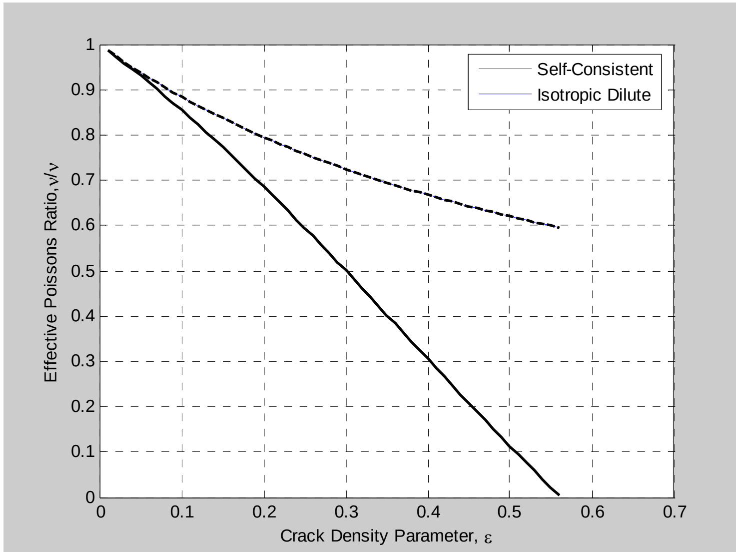
$$\frac{\bar{E}}{E} = \frac{45(2-\nu)}{45(2-\nu)+16(1-\nu^2)(10-3\nu)\varepsilon} = \frac{45(1.5)}{45(1.5)+16(0.75)(8.5)\varepsilon}$$

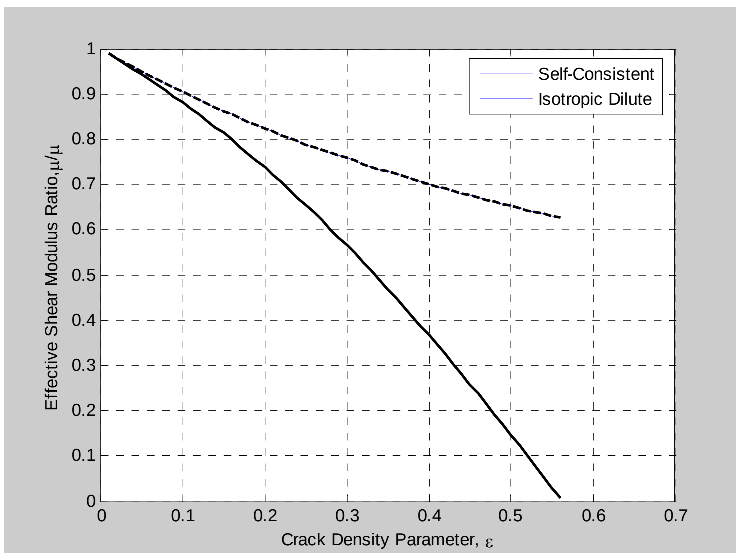
$$= \frac{1}{1+\frac{68}{45}\varepsilon} = \frac{1}{1+\frac{68}{45}\frac{9}{16}} = \frac{1}{1.85} = 0.54$$

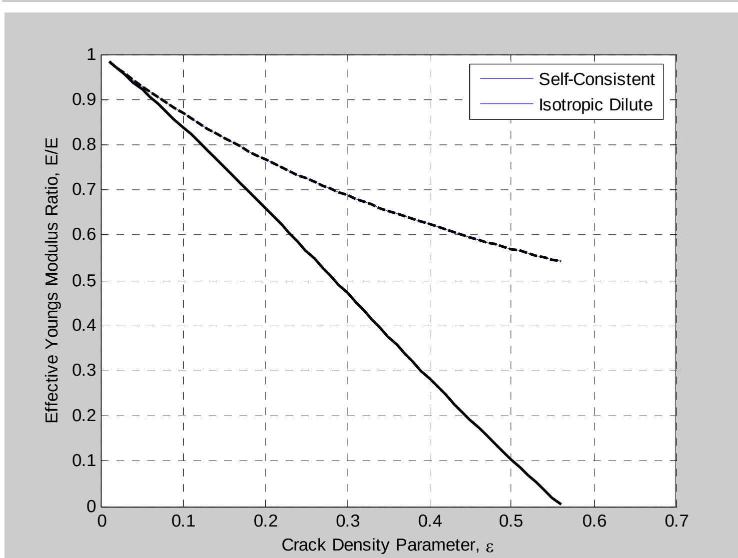
$$\frac{\bar{\mu}}{\mu} = \frac{45(2-\nu)}{45(2-\nu)+32(1-\nu)(5-\nu)\varepsilon} = \frac{45(1.5)}{45(1.5)+32(0.5)(4.5)\varepsilon}$$

$$= \frac{1}{1+\frac{48}{45}\varepsilon} = \frac{1}{1+\frac{48}{45}\frac{9}{16}} = \frac{1}{1.6} = 0.625$$

15-11. Continued
MATLAB Plots of effective moduli ratios and comparison with isotropic dilute case :







15-12.

For Two - Dimensional Couple Stress Theory :

$$e_x = \frac{\partial u}{\partial x},\ e_y = \frac{\partial v}{\partial y},\ e_{xy} = \frac{\partial v}{\partial x} - \phi_z,\ e_{yx} = \frac{\partial u}{\partial y} + \phi_z \text{ , with } \phi_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 e_x}{\partial y^2} = \frac{\partial^3 u}{\partial y^2 \partial x},\ \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y},$$

$$\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x} - \phi_z\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x} - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) = \frac{1}{2}\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial y^2 \partial x}\right) \Rightarrow$$

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

Standard principles of calculus $\Rightarrow \dfrac{\partial^2 \phi_z}{\partial x \partial y} = \dfrac{\partial^2 \phi_z}{\partial y \partial x}$

$$\frac{\partial \phi_z}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}\right),\ \frac{\partial e_{xy}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \phi_z\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) \Rightarrow$$

$$\frac{\partial \phi_z}{\partial x} = \frac{\partial e_{xy}}{\partial x} - \frac{\partial e_x}{\partial y}$$

$$\frac{\partial \phi_z}{\partial y} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 u}{\partial y^2}\right),\ \frac{\partial e_{xy}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \phi_z\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) = \frac{1}{2}\left(\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2}\right) \Rightarrow$$

$$\frac{\partial \phi_z}{\partial y} = \frac{\partial e_y}{\partial x} - \frac{\partial e_{xy}}{\partial y}$$

Stresses : $\sigma_x = \lambda(e_x + e_y) + (2\mu + \kappa)e_x$, $\sigma_y = \lambda(e_x + e_y) + (2\mu + \kappa)e_y$, $\tau_{xy} = (2\mu + \kappa)e_{xy} \Rightarrow$

$$e_x = \frac{1}{(2\mu + \kappa)}[\sigma_x - \nu(\sigma_x + \sigma_y)],\ e_y = \frac{1}{(2\mu + \kappa)}[\sigma_y - \nu(\sigma_x + \sigma_y)],\ e_{xy} = \frac{1}{(2\mu + \kappa)}\tau_{xy}$$

where $\nu = \lambda/(2\lambda + 2\mu + \kappa)$

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \Rightarrow \frac{\partial^2}{\partial y^2}[\sigma_x - \nu(\sigma_x + \sigma_y)] + \frac{\partial^2}{\partial x^2}[\sigma_y - \nu(\sigma_x + \sigma_y)] = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \Rightarrow$$

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\nabla^2(\sigma_x + \sigma_y) = \frac{\partial^2}{\partial x \partial y}(\tau_{xy} + \tau_{yx})$$

$$m_{xz} = \gamma\frac{\partial \phi_z}{\partial x},\ m_{yz} = \gamma\frac{\partial \phi_z}{\partial y} \Rightarrow \frac{\partial m_{xz}}{\partial y} = \frac{\partial m_{yz}}{\partial x}$$

Now $m_{xz} = \gamma\dfrac{\partial \phi_z}{\partial x} = \gamma\left(\dfrac{\partial e_{xy}}{\partial x} - \dfrac{\partial e_x}{\partial y}\right)$ and $e_{xy} = \dfrac{1}{2(2\mu + \kappa)}(\tau_{xy} + \tau_{yx})$, $e_x = \dfrac{1}{(2\mu + \kappa)}[\sigma_x - \nu(\sigma_x + \sigma_y)] \Rightarrow$

$$m_{xz} = \frac{\gamma}{2(2\mu + \kappa)}\frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - \frac{2\gamma}{2(2\mu + \kappa)}\frac{\partial}{\partial y}[\sigma_x - \nu(\sigma_x + \sigma_y)] = l^2\frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - 2l^2\frac{\partial}{\partial y}[\sigma_x - \nu(\sigma_x + \sigma_y)]$$

where $l^2 = \dfrac{\gamma}{2(2\mu + \kappa)}$, Likewise for the relation $m_{yz} = 2l^2\dfrac{\partial}{\partial x}[\sigma_y - \nu(\sigma_x + \sigma_y)] - l^2\dfrac{\partial}{\partial y}(\tau_{xy} + \tau_{yx})$

15-13.

Couple Stress Function Formulation :

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\partial y} \ , \ \sigma_y = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Psi}{\partial x\partial y} \ , \ \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial y^2} \ , \ \tau_{yx} = -\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial x^2}$$

$$m_{xz} = \frac{\partial\Psi}{\partial x} \ , \ m_{yz} = \frac{\partial\Psi}{\partial y}$$

Using these forms in the equlibrium relations (14.4.7) $\Rightarrow$

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} = 0 \Rightarrow \frac{\partial^3\Phi}{\partial x\partial y^2} - \frac{\partial^3\Psi}{\partial x^2\partial y} - \frac{\partial^3\Phi}{\partial x\partial y^2} + \frac{\partial^3\Psi}{\partial y\partial x^2} = 0, \text{ checks}$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \Rightarrow -\frac{\partial^3\Phi}{\partial x^2\partial y} - \frac{\partial^3\Psi}{\partial x\partial y^2} + \frac{\partial^3\Phi}{\partial y\partial x^2} + \frac{\partial^3\Psi}{\partial x\partial y^2} = 0, \text{ checks}$$

$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \tau_{xy} - \tau_{yx} = 0 \Rightarrow \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial x^2} = 0, \text{ checks}$$

14-14.

Couple Stress Function Formulation :

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\partial y}\ ,\ \sigma_y = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Psi}{\partial x\partial y}\ ,\ \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial y^2}\ ,\ \tau_{yx} = -\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial x^2}$$

$$m_{xz} = \frac{\partial\Psi}{\partial x}\ ,\ m_{yz} = \frac{\partial\Psi}{\partial y}$$

Using these forms in the compatibility relations (14.4.13) :

$$\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} - \nu\nabla^2(\sigma_x + \sigma_y) = \frac{\partial^2}{\partial x\partial y}(\tau_{xy} + \tau_{yx}) \Rightarrow$$

$$\frac{\partial^4\Phi}{\partial y^4} - \frac{\partial^4\Psi}{\partial x\partial y^3} + \frac{\partial^4\Phi}{\partial x^4} + \frac{\partial^4\Psi}{\partial x^3\partial y} - \nu\nabla^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = -\frac{\partial^4\Phi}{\partial x^2\partial y^2} - \frac{\partial^4\Psi}{\partial x\partial y^3} - \frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Psi}{\partial y\partial x^3} \Rightarrow$$

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} - \nu\nabla^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = 0 \Rightarrow (1-\nu)\nabla^4\Phi = 0 \Rightarrow \nabla^4\Phi = 0$$

$$\frac{\partial m_{xz}}{\partial y} = \frac{\partial m_{yz}}{\partial x} \Rightarrow \frac{\partial^2\Psi}{\partial y\partial x} = \frac{\partial^2\Psi}{\partial x\partial y}\ ,\ \text{satisfied identically}$$

$$m_{xz} = l^2\frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - 2l^2\frac{\partial}{\partial y}[\sigma_x - \nu(\sigma_x + \sigma_y)] \Rightarrow$$

$$\frac{\partial\Psi}{\partial x} = l^2\frac{\partial}{\partial x}\left(\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} - 2\frac{\partial^2\Phi}{\partial x\partial y}\right) - 2l^2\frac{\partial}{\partial y}\left[\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\partial y} - \nu\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right)\right] \Rightarrow$$

$$\frac{\partial}{\partial x}\left(\Psi - l^2\nabla^2\Psi\right) = -2(1-\nu)l^2\frac{\partial}{\partial y}\left(\nabla^2\Phi\right)$$

Likewise using $m_{yz} = 2l^2\frac{\partial}{\partial x}[\sigma_y - \nu(\sigma_x + \sigma_y)] - l^2\frac{\partial}{\partial y}(\tau_{xy} + \tau_{yx}) \Rightarrow$

$$\frac{\partial}{\partial y}\left(\Psi - l^2\nabla^2\Psi\right) = 2(1-\nu)l^2\frac{\partial}{\partial x}\left(\nabla^2\Phi\right)$$

Following the suggested differentiations on the top of page 393 in text $\Rightarrow$

$$\nabla^2\Psi - l^2\nabla^4\Psi = 0$$

15-15.

Stress Concentration Around Circular Hole - Micropolar Case

Hoop Stress Solution :

$$\sigma_\theta = \frac{T}{2}(1-\cos 2\theta) - \frac{A_1}{r^2} + \left(\frac{6A_2}{r^4} - \frac{6A_4}{r^4}\right)\cos 2\theta - \frac{2A_5}{l_1 r}\left[\frac{3l_1}{r}K_o(r/l_1) + \left(1+\frac{6l_1^2}{r^2}\right)K_1(r/l_1)\right]\cos 2\theta$$

where $A_1 = -\frac{T}{2}a^2$, $A_2 = -\frac{Ta^4(1-F)}{4(1+F)}$, $A_3 = \frac{Ta^2}{2(1+F)}$, $A_4 = \frac{4T(1-\nu)a^2 l_2^2}{1+F}$, $A_5 = -\frac{Tal_1F}{(1+F)K_1(a/l_1)}$

$$F = 8(1-\nu)\frac{l_2^2}{l_1^2}\left[4 + \frac{a^2}{l_1^2} + \frac{2a}{l_1}\frac{K_o(a/l_1)}{K_1(a/l_1)}\right]^{-1}$$

On Hole Boundary : $r = a$

$$\sigma_\theta(a,\theta) = \frac{T}{2}(1-\cos 2\theta) + \frac{T}{2} + \left(-\frac{6T(1-F)}{4(1+F)} - \frac{24T(1-\nu)l_2^2}{a^2(1+F)}\right)\cos 2\theta$$

$$+\frac{2TF}{(1+F)K_1(a/l_1)}\left[\frac{3l_1}{a}K_o(a/l_1) + \left(1+\frac{6l_1^2}{a^2}\right)K_1(a/l_1)\right]\cos 2\theta$$

$$= T + T\left[-\frac{1}{2} - \frac{6(1-F)}{4(1+F)} - \frac{24(1-\nu)l_2^2}{a^2(1+F)} + \frac{6Fl_1}{(1+F)a}\frac{K_o(a/l_1)}{K_1(a/l_1)} + \frac{2F}{(1+F)} + \frac{12Fl_1^2}{(1+F)a^2}\right]\cos 2\theta$$

$$= T + \frac{T}{1+F}\left[-\frac{1}{2}(1+F) - \frac{3(1-F)}{2} - \frac{24(1-\nu)l_2^2}{a^2} + \frac{6Fl_1}{a}\frac{K_o(a/l_1)}{K_1(a/l_1)} + 2F + 12\frac{Fl_1^2}{a^2}\right]\cos 2\theta$$

But from definition of $F \Rightarrow \frac{K_o(a/l_1)}{K_1(a/l_1)} = \frac{l_1}{2a}\left[\frac{8(1-\nu)}{F}\frac{l_2^2}{l_1^2} - 4 - \frac{a^2}{l_1^2}\right]$ and thus

$$\sigma_\theta(a,\theta) = T + \frac{T}{1+F}\left[-\frac{1}{2}(1+F) - \frac{3(1-F)}{2} - \frac{24(1-\nu)l_2^2}{a^2} + \frac{3Fl_1^2}{a^2}\left(\frac{8(1-\nu)}{F}\frac{l_2^2}{l_1^2} - 4 - \frac{a^2}{l_1^2}\right) + 2F + 12\frac{Fl_1^2}{a^2}\right]\cos 2\theta$$

$$= T + \frac{T}{1+F}\left[-\frac{1}{2}(1+F) - \frac{3(1-F)}{2} - 3F + 2F\right] = T\left(1 - \frac{2\cos 2\theta}{1+F}\right)$$

Clearly this expression takes on a maximum value when $\cos 2\theta = -1 \Rightarrow \theta = \pm\pi/2$

$$\therefore (\sigma_\theta)_{\max} = \sigma_\theta(a,\pi/2) = T\left(1+\frac{2}{1+F}\right) = T\left(\frac{3+F}{1+F}\right)$$

When $l_1 = l_2 = l = 0 \Rightarrow$

$$F = 8(1-\nu)\frac{l_2^2}{l_1^2}\left[4 + \frac{a^2}{l_1^2} + \frac{2a}{l_1}\frac{K_o(a/l_1)}{K_1(a/l_1)}\right]^{-1} = 8(1-\nu)\frac{l_2^2}{4l_1^2 + a^2 + 2al_1\frac{K_o(a/l_1)}{K_1(a/l_1)}} = 0 \Rightarrow$$

$$(\sigma_\theta)_{\max} = 3T$$

15-16.

Constitutive Equations for Linear Isotropic Elastic Materials with Voids

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij} \; , \; h_i = \alpha\phi_{,i} \; , \; g = -\omega\dot{\phi} - \xi\phi - \beta e_{kk}$$

Under Plane Stress Condtions:

$$\sigma_x = \lambda(e_x + e_y + e_z) + 2\mu e_x + \beta\phi$$

$$\sigma_y = \lambda(e_x + e_y + e_z) + 2\mu e_y + \beta\phi$$

$$\sigma_z = \lambda(e_x + e_y + e_z) + 2\mu e_z + \beta\phi = 0 \Rightarrow (\lambda + 2\mu)e_z = -\beta\phi - \lambda(e_x + e_y)$$

$$\tau_{xy} = 2\mu e_{xy} \; , \; \tau_{xz} = \tau_{yz} = 0$$

Using the relation for $e_z \Rightarrow$

$$\sigma_x = \lambda\left(e_x + e_y + \frac{-\beta\phi - \lambda(e_x + e_y)}{\lambda + 2\mu} \right) + 2\mu e_x + \beta\phi = \lambda\frac{2\mu}{\lambda + 2\mu}(e_x + e_y) + \frac{2\mu}{\lambda + 2\mu}\beta\phi + 2\mu e_x$$

$$\sigma_y = \lambda\frac{2\mu}{\lambda + 2\mu}(e_x + e_y) + \frac{2\mu}{\lambda + 2\mu}\beta\phi + 2\mu e_y$$

Combining these results gives: $\sigma_{ij} = \dfrac{2\mu}{\lambda + 2\mu}(\lambda e_{kk} + \beta\phi)\delta_{ij} + 2\mu e_{ij}$, where $e_{kk} = e_x + e_y$

$$g = -\omega\dot{\phi} - \xi\phi - \beta e_{kk} = -\omega\dot{\phi} - \xi\phi - \beta(e_x + e_y + e_z) = -\omega\dot{\phi} - \xi\phi - \beta\left(e_x + e_y + \frac{-\beta\phi - \lambda(e_x + e_y)}{\lambda + 2\mu} \right)$$

$$= -\omega\dot{\phi} - \xi\phi - \frac{\beta^2\phi}{\lambda + 2\mu} - \beta\left(e_x + e_y - \frac{\lambda(e_x + e_y)}{\lambda + 2\mu} \right) = -\omega\dot{\phi} - \left(\xi - \frac{\beta^2}{\lambda + 2\mu} \right)\phi - \frac{2\mu\beta}{\lambda + 2\mu}(e_x + e_y)$$

15-17.

For elasticity with voids, the plane stress compatibility relation is the same as that originally developed in Chapter 7, given by (7.2.6)

$$\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}$$

From the stress - strain relations developed in Exercise 14 - 16

$$\sigma_x=\lambda\frac{2\mu}{\lambda+2\mu}(e_x+e_y)+\frac{2\mu}{\lambda+2\mu}\beta\phi+2\mu e_x=\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}e_x+\frac{2\lambda\mu}{\lambda+2\mu}e_y+\frac{2\mu\beta\phi}{\lambda+2\mu}$$

$$=\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(e_x+\frac{\lambda}{2(\lambda+\mu)}e_y\right)+\frac{2\mu\beta\phi}{\lambda+2\mu}$$

$$\sigma_y=\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(e_y+\frac{\lambda}{2(\lambda+\mu)}e_x\right)+\frac{2\mu\beta\phi}{\lambda+2\mu}$$

These relations can be cast in the identical form as the original classical elasticity results given in Exercise 7 - 6,

$$\hat\sigma_x=\frac{\hat E}{1-\hat\nu^2}(e_x+\hat\nu e_y)\ ,\ \hat\sigma_y=\frac{\hat E}{1-\hat\nu^2}(e_y+\hat\nu e_x)\ ,\ \text{with}$$

$$\hat\sigma_x=\sigma_x-\frac{2\mu\beta\phi}{\lambda+2\mu}\ ,\ \hat\sigma_y=\sigma_y-\frac{2\mu\beta\phi}{\lambda+2\mu}\ ,\ \hat E=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\ ,\ \hat\nu=\frac{\lambda}{2(\lambda+\mu)}$$

Thus the solution for the strains follows as

$$e_x=\frac{1}{\hat E}(\hat\sigma_x-\hat\nu\hat\sigma_y)\ ,\ e_y=\frac{1}{\hat E}(\hat\sigma_y-\hat\nu\hat\sigma_x)\ \text{and also}\ \ e_{xy}=\frac{1}{2\mu}\tau_{xy}=\frac{1+\hat\nu}{\hat E}\tau_{xy}$$

Substituting these into the strain compatibility relation and following similar steps as in Exercise 7 - 7 $\Rightarrow$

$$\frac{1}{(1+\hat\nu)}\nabla^2(\hat\sigma_x+\hat\sigma_y)=2\frac{\partial^2\tau_{xy}}{\partial x\partial y}+\frac{\partial^2\hat\sigma_x}{\partial x^2}+\frac{\partial^2\hat\sigma_y}{\partial y^2}=-\frac{2\mu\beta}{\lambda+2\mu}\nabla^2\phi\Rightarrow$$

$$\nabla^2(\hat\sigma_x+\hat\sigma_y)+\frac{3\lambda+2\mu}{2(\lambda+\mu)}\frac{2\mu\beta}{\lambda+2\mu}\nabla^2\phi=0\Rightarrow$$

$$\nabla^2(\sigma_x+\sigma_y)-\frac{4\mu\beta}{\lambda+2\mu}\nabla^2\phi+\frac{3\lambda+2\mu}{(\lambda+\mu)}\frac{\mu\beta}{\lambda+2\mu}\nabla^2\phi=0\Rightarrow$$

$$\nabla^2(\sigma_x+\sigma_y)-\frac{\mu\beta}{\lambda+\mu}\nabla^2\phi=0$$

Introducing the usual Airy stress function $\sigma_x=\dfrac{\partial^2\psi}{\partial x^2}\ ,\ \sigma_y=\dfrac{\partial^2\psi}{\partial y^2}\Rightarrow$

$$\nabla^4\psi-\frac{\mu\beta}{\lambda+\mu}\nabla^2\phi=0$$

15-18.

Hoop stress solution for elasticity with voids

$$\sigma_\theta = \frac{T}{2}\left\{\left(1+\frac{a^2}{r^2}\right)+\cos 2\theta\left[a^2\frac{F''(r)}{F(a)}-\left(1+3\frac{a^4}{r^4}\right)\right]\right\}$$

$$F(r)=1+\frac{h^2}{N}\left[\frac{1}{r^2}+\frac{2hK_2(r/h)}{a^3K_2'(a/h)}\right] \Rightarrow F'(r)=\frac{h^2}{N}\left[-\frac{2}{r^3}+\frac{2K_2'(r/h)}{a^3K_2'(a/h)}\right] \Rightarrow F''(r)=\frac{h^2}{N}\left[\frac{6}{r^4}+\frac{2K_2''(r/h)}{a^3hK_2'(a/h)}\right]$$

When $N=\dfrac{1}{2},\ L=\dfrac{a}{h}=2 \Rightarrow h=\dfrac{a}{2} \Rightarrow$

$$F(a)=1+\frac{a^2}{2}\left[\frac{1}{a^2}+\frac{K_2(2)}{a^2K_2'(2)}\right],\ F''(r)=\frac{a^2}{2}\left[\frac{6}{r^4}+\frac{4K_2''(2r/a)}{a^4K_2'(2)}\right]$$

From the properties of Bessel Functions:

$$K_2'(x)=-K_1(x)-\frac{2}{x}K_2(x)$$

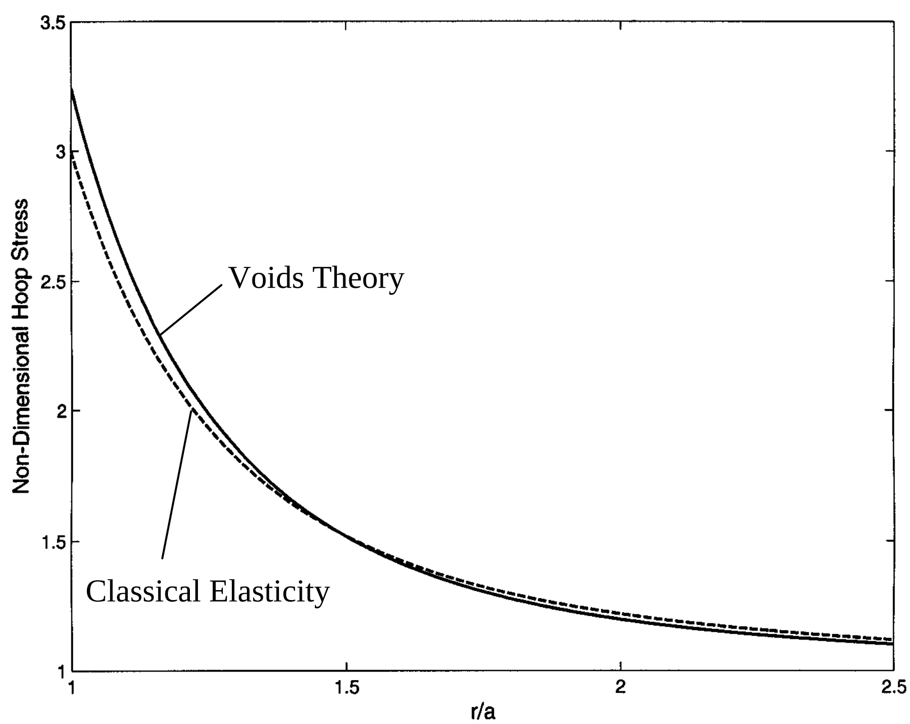
$$K_2''(x)=K_0(x)+\frac{1}{x}K_1(x)+\frac{2}{x}\left(K_1(x)+\frac{2}{x}K_2(x)\right)+\frac{2}{x^2}K_2(x)$$

$$\therefore\ \sigma_\theta(r,\pi/2)/T=\frac{1}{2}\left\{\left(1+\frac{a^2}{r^2}\right)-a^2\frac{F''(r)}{F(a)}+\left(1+3\frac{a^4}{r^4}\right)\right\}$$

From classical elasticity, the corresponding result is

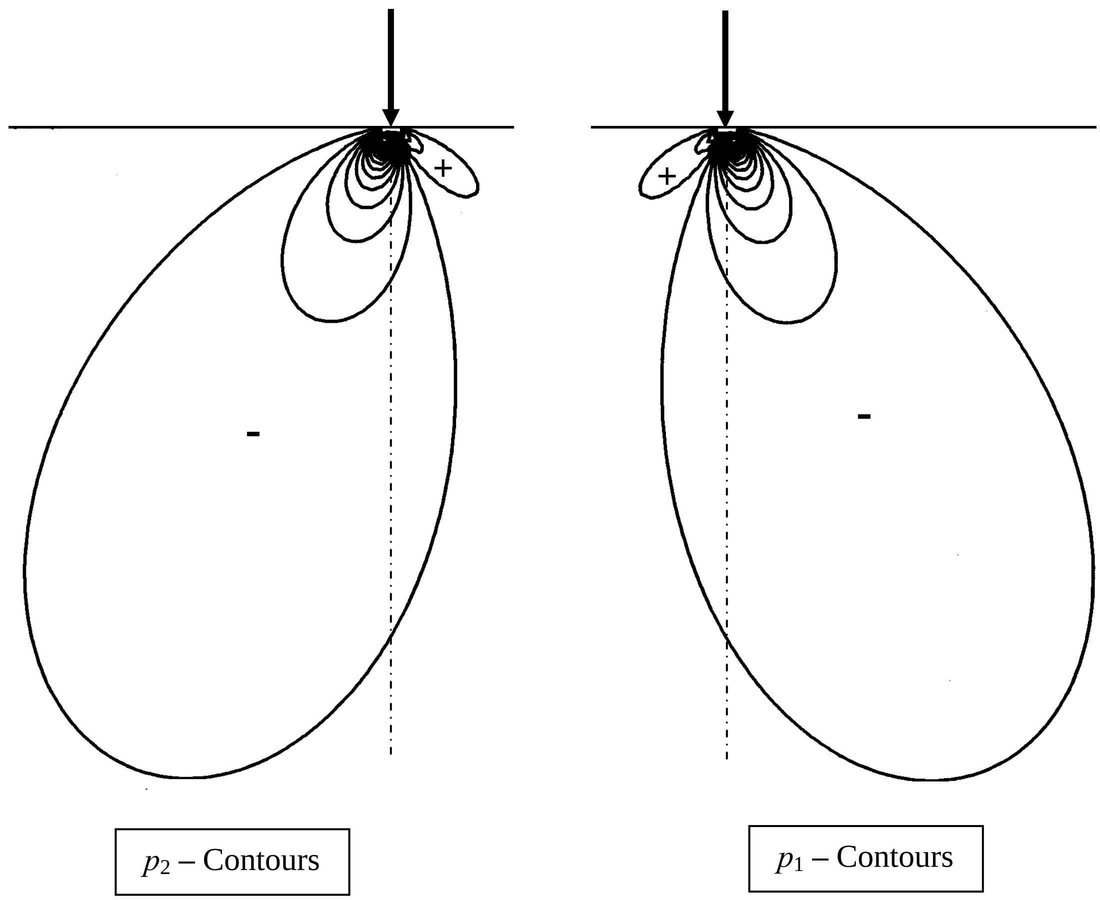
$$\sigma_\theta(r,\pi/2)/T=\frac{1}{2}\left(2+\frac{a^2}{r^2}+\frac{3a^4}{r^4}\right)$$

MATLAB Plots:



15-19.

MATLAB Contour Plots of Doublet Mechanics Solution –
p_1 and p_2 Microstresses for Flamant Problem



16-1.

Linear Approximation : $u(x, y) = c_1 + c_2 x + c_3 y$

Evaluating at each node $\Rightarrow$

$u(x_1, y_1) = u_1 = c_1 + c_2 x_1 + c_3 y_1$

$u(x_2, y_2) = u_2 = c_1 + c_2 x_2 + c_3 y_2$

$u(x_3, y_3) = u_3 = c_1 + c_2 x_3 + c_3 y_3$

Solving the three equations for the constants $\Rightarrow$

$$c_1 = \frac{1}{2A_e}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)\ ,\ c_2 = \frac{1}{2A_e}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)\ ,\ c_3 = \frac{1}{2A_e}(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)$$

where where A_e is the area of the element, and $\alpha_i = x_j y_k - x_k y_j$, $\beta_i = y_j - y_k$, $\gamma_i = x_k - x_j$

Back substituting these results into the original form for $u(x, y)$ $\Rightarrow$

$$\begin{aligned} u(x, y) &= \frac{1}{2A_e}[(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) + (\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)x + (\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)y] \\ &= \frac{1}{2A_e}(\alpha_1 + \beta_1 x + \gamma_1 y)u_1 + \frac{1}{2A_e}(\alpha_2 + \beta_2 x + \gamma_2 y)u_2 + \frac{1}{2A_e}(\alpha_3 + \beta_3 x + \gamma_3 y)u_3 \\ &= \sum_{i=1}^{3} u_i \psi_i(x, y) \end{aligned}$$

where $\psi_i(x, y) == \frac{1}{2A_e}(\alpha_i + \beta_i x + \gamma_i y)$ are the interpolation or approximation functions

16-2.

For the constant strain triangular element,

$[K]=h_e A_e [B]^T[C][B]$, with

$$[B]=\frac{1}{2A_e}\begin{bmatrix}\beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0\\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3\\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3\end{bmatrix},\ [C]=\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\Rightarrow$$

$$[K]=\frac{h_e}{4A_e}\begin{bmatrix}\beta_1 & 0 & \gamma_1\\ 0 & \gamma_1 & \beta_1\\ \beta_2 & 0 & \gamma_2\\ 0 & \gamma_2 & \beta_2\\ \beta_3 & 0 & \gamma_3\\ 0 & \gamma_3 & \beta_3\end{bmatrix}\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0\\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3\\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3\end{bmatrix}$$

$$=\frac{h_e}{4A_e}\begin{bmatrix}\beta_1 & 0 & \gamma_1\\ 0 & \gamma_1 & \beta_1\\ \beta_2 & 0 & \gamma_2\\ 0 & \gamma_2 & \beta_2\\ \beta_3 & 0 & \gamma_3\\ 0 & \gamma_3 & \beta_3\end{bmatrix}\begin{bmatrix}\beta_1 C_{11} & \gamma_1 C_{12} & \beta_2 C_{11} & \gamma_2 C_{12} & \beta_3 C_{11} & \gamma_3 C_{12}\\ \beta_1 C_{12} & \gamma_1 C_{22} & \beta_2 C_{12} & \gamma_2 C_{22} & \beta_3 C_{12} & \gamma_3 C_{22}\\ \gamma_1 C_{66} & \beta_1 C_{66} & \gamma_2 C_{66} & \beta_2 C_{66} & \gamma_3 C_{66} & \beta_3 C_{66}\end{bmatrix}$$

$$=\frac{h_e}{4A_e}\begin{bmatrix}\beta_1^2 C_{11}+\gamma_1^2 C_{66} & \beta_1\gamma_1 C_{12}+\beta_1\gamma_1 C_{66} & \beta_1\beta_2 C_{11}+\gamma_1\gamma_2 C_{66} & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\end{bmatrix}$$

16-3.

In general, the body force vector is given by $\{F\} = h_e \int_{\Omega_e} [\psi]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} dxdy$

$$[\psi]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{bmatrix} \psi_1 & 0 & \psi_2 & 0 & \psi_3 & 0 \\ 0 & \psi_1 & 0 & \psi_2 & 0 & \psi_3 \end{bmatrix}^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

$$= \begin{bmatrix} F_x\psi_1 \\ F_y\psi_1 \\ F_x\psi_2 \\ F_y\psi_2 \\ F_x\psi_3 \\ F_y\psi_3 \end{bmatrix}$$

For the case of element - wise constant body forces,

$\int_{\Omega_e} F_x\psi_1 dxdy = F_x \int_{\Omega_e} \psi_1 dxdy = \frac{1}{3} F_x A_e$, and likewise for all other components

$$\therefore \{F\} = \frac{h_e A_e}{3} \{F_x \quad F_y \quad F_x \quad F_y \quad F_x \quad F_y\}^T$$

16-4.

For the linear triangular element, with constant boundary tractions

$$h_e \int_{\Gamma_{12}} [\psi]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds = h_e \int_{\Gamma_{12}} \begin{Bmatrix} \psi_1 T_x^n \\ \psi_1 T_y^n \\ \psi_2 T_x^n \\ \psi_2 T_y^n \\ \psi_3 T_x^n \\ \psi_3 T_y^n \end{Bmatrix} ds = \frac{h_e L_{12}}{2} \begin{Bmatrix} T_x^n \\ T_y^n \\ T_x^n \\ T_y^n \\ 0 \\ 0 \end{Bmatrix}_{12}$$

Now $h_e \int_{\Gamma_{12}} \psi_1 T_x^n ds = h_e T_x^n \int_{\Gamma_{12}} \psi_1 ds = h_e T_x^n \int_{\Gamma_{12}} \frac{1}{2A_e} (\alpha_1 + \beta_1 x + \gamma_1 y) ds = \frac{h_e L_{12}}{2}$

Integrals of the other components involving ψ_1 and ψ_2 follow identical patterns, while integrals containing the ψ_3 terms will vanish since $\psi_3 = 0$ on Γ_{12}.

Collecting all of the integral evaluations $\Rightarrow$

$$h_e \int_{\Gamma_{12}} [\psi]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds = \frac{h_e L_{12}}{2} \begin{Bmatrix} T_x^n \\ T_y^n \\ T_x^n \\ T_y^n \\ 0 \\ 0 \end{Bmatrix}_{12}$$

16-5.

For the isotropic plane stress case :

$$C_{11} = C_{22} = \frac{E}{1-\nu^2}\ ,\ C_{12} = \frac{E\nu}{1-\nu^2}\ ,\ C_{66} = \mu = \frac{E}{2(1+\nu)}$$

For element 1 in Example 15-1, $\beta_1 = -1, \beta_2 = 1, \beta_3 = 0, \gamma_1 = 0, \gamma_2 = -1, \gamma_3 = 1, A_1 = 1/2\ , h = 1$

$$\Rightarrow \beta_1^2 C_{11} + \gamma_1^2 C_{66} = -\frac{E}{1-\nu^2}\ ,\ \beta_1\gamma_1 C_{12} + \beta_1\gamma_1 C_{66} = 0,\ \beta_1\beta_2 C_{11} + \gamma_1\gamma_2 C_{66} = -\frac{E}{1-\nu^2}\ , \ldots$$

and similarly for all other components in the stiffness matrix $\Rightarrow$

$$[\boldsymbol{K}^{(1)}] = \frac{E}{2(1-\nu^2)}\begin{bmatrix} 1 & 0 & -1 & \nu & 0 & -\nu \\ \cdot & \frac{1-\nu}{2} & \frac{1-\nu}{2} & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 \\ \cdot & \cdot & \frac{3-\nu}{2} & -\frac{1+\nu}{2} & -\frac{1-\nu}{2} & \nu \\ \cdot & \cdot & \cdot & \frac{3-\nu}{2} & \frac{1-\nu}{2} & -1 \\ \cdot & \cdot & \cdot & \cdot & \frac{1-\nu}{2} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

For element 2, $\beta_1 = 0, \beta_2 = 1, \beta_3 = -1, \gamma_1 = -1, \gamma_2 = 0, \gamma_3 = 1, A_1 = 1/2\ , h = 1$

$$\Rightarrow \beta_1^2 C_{11} + \gamma_1^2 C_{66} = \frac{E}{2(1+\nu)}\ ,\ \beta_1\gamma_1 C_{12} + \beta_1\gamma_1 C_{66} = 0,\ \beta_1\beta_2 C_{11} + \gamma_1\gamma_2 C_{66} = 0\ , \ldots$$

and similarly for all other components in the stiffness matrix $\Rightarrow$

$$[\boldsymbol{K}^{(2)}] = \frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{1-\nu}{2} & 0 & 0 & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & \frac{1-\nu}{2} \\ \cdot & 1 & -\nu & 0 & \nu & -1 \\ \cdot & \cdot & 1 & 0 & -1 & \nu \\ \cdot & \cdot & \cdot & \frac{1-\nu}{2} & \frac{1-\nu}{2} & -\frac{1-\nu}{2} \\ \cdot & \cdot & \cdot & \cdot & \frac{3-\nu}{2} & -\frac{1-\nu}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{3-\nu}{2} \end{bmatrix}$$

16-6.

From Example 15 - 1, the reduced global system matrix was given by

$$\begin{bmatrix} K_{33}^{(1)} & K_{34}^{(1)} & K_{35}^{(1)} & K_{36}^{(1)} \\ \cdot & K_{44}^{(1)} & K_{45}^{(1)} & K_{46}^{(1)} \\ \cdot & \cdot & K_{55}^{(1)} + K_{33}^{(2)} & K_{56}^{(1)} + K_{34}^{(2)} \\ \cdot & \cdot & \cdot & K_{66}^{(1)} + K_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} T/2 \\ 0 \\ T/2 \\ 0 \end{Bmatrix} \Rightarrow$$

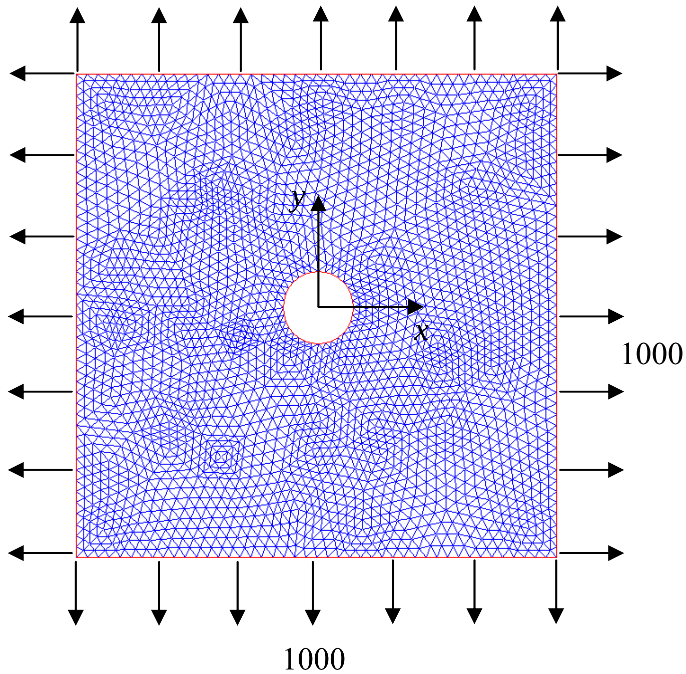
$$\frac{E}{2(1-\nu)} \begin{bmatrix} \frac{3-\nu}{2} & -\frac{1+\nu}{2} & -\frac{1-\nu}{2} & \nu \\ \cdot & \frac{3-\nu}{2} & \frac{1-\nu}{2} & -1 \\ \cdot & \cdot & \frac{3-\nu}{2} & 0 \\ \cdot & \cdot & \cdot & \frac{3-\nu}{2} \end{bmatrix} \begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} T/2 \\ 0 \\ T/2 \\ 0 \end{Bmatrix} \Rightarrow$$

$$10^9 \begin{bmatrix} 151.8 & -69 & -41.4 & 27.6 \\ \cdot & 151.8 & 41.4 & -110.4 \\ \cdot & \cdot & 151.8 & 0 \\ \cdot & \cdot & \cdot & 151.8 \end{bmatrix} \begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} T/2 \\ 0 \\ T/2 \\ 0 \end{Bmatrix}$$

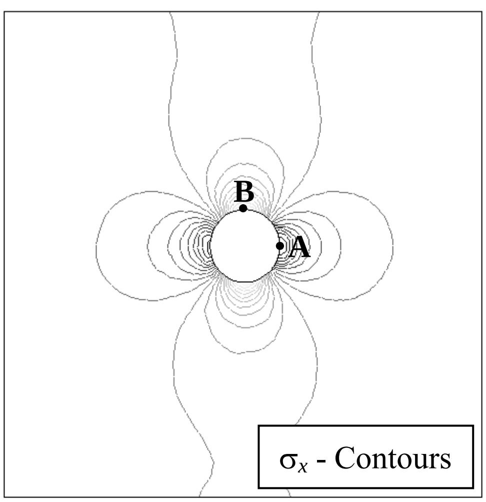
Solving the system $\Rightarrow \begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0.492 \\ 0.081 \\ 0.441 \\ -0.030 \end{Bmatrix} T \times 10^{-11}\, m$

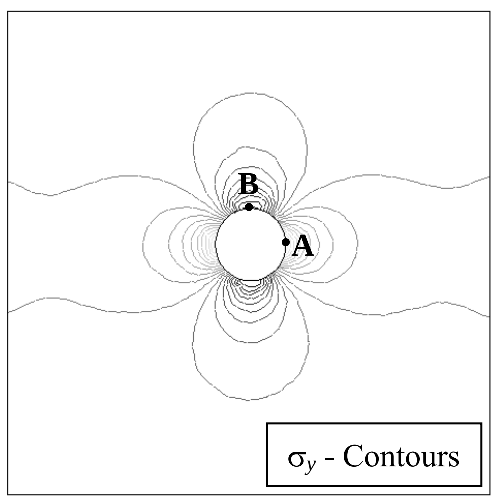
16-7.

MATLAB PDE Toolbox: FEA Solution to Biaxial Loading Problem in Exercise 8-14



FEA Model:
($w/d \approx 7$, 2792 Nodes, 5376 Elements)



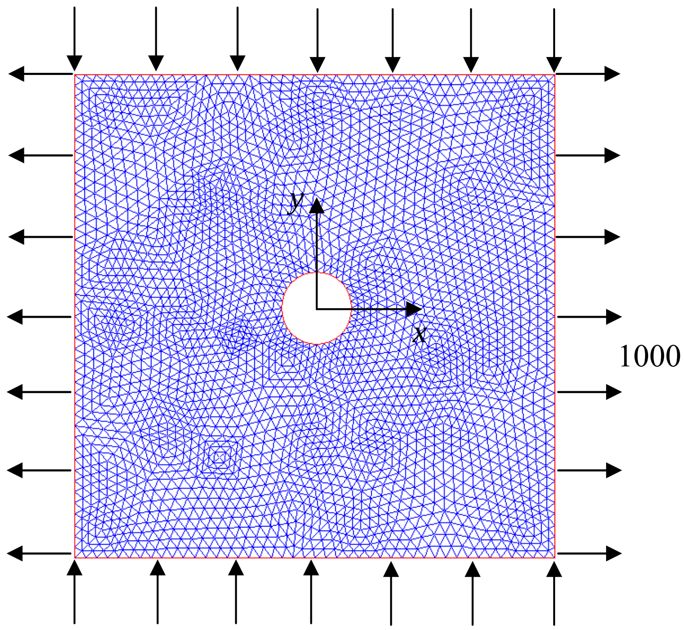


Comparison of σ_x Results:
FEA Solution: $(\sigma_x)_A \cong 200$, $(\sigma_x)_B \cong 1900$
Analytical Solution: $(\sigma_x)_A \cong 0$, $(\sigma_x)_B \cong 2000$

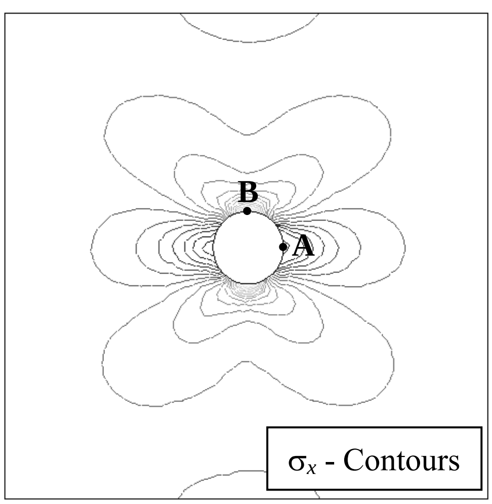
Comparison of σ_y Results:
FEA Solution: $(\sigma_y)_A \cong 1900$, $(\sigma_y)_B \cong 300$
Analytical Solution: $(\sigma_y)_A \cong 2000$, $(\sigma_y)_B \cong 0$

16-8.

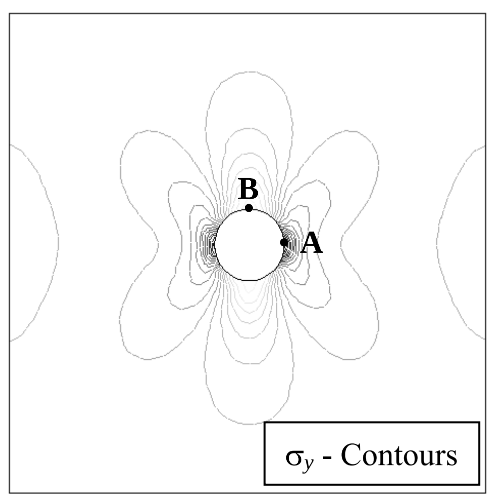
MATLAB PDE Toolbox: FEA Solution to Biaxial-Shear Loading Problem in Exercise 8-15 (Figure 8-15)



FEA Model:
($w/d \approx 7$, 2792 Nodes, 5376 Elements)



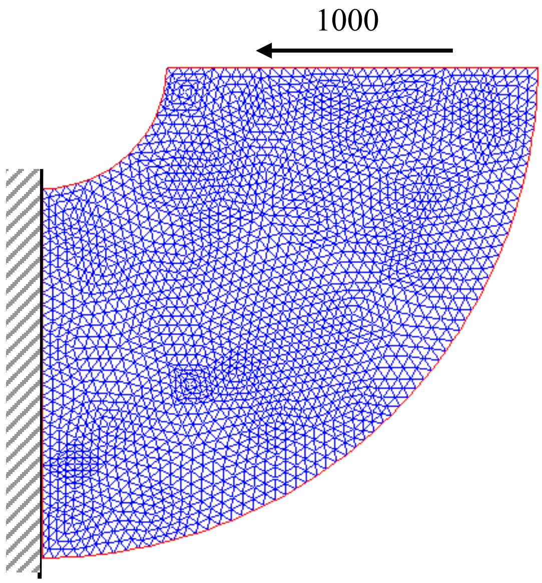
Comparison of σ_x Results:
FEA Solution: $(\sigma_x)_A \cong -270$, $(\sigma_x)_B \cong 3300$
Analytical Solution: $(\sigma_x)_A \cong 0$, $(\sigma_x)_B \cong 4000$



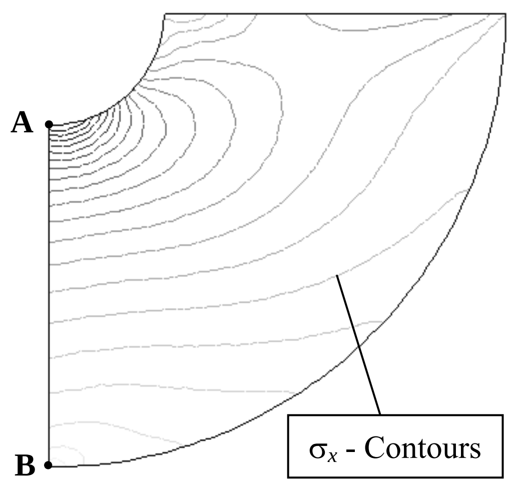
Comparison of σ_y Results:
FEA Solution: $(\sigma_y)_A \cong -3500$, $(\sigma_y)_B \cong 280$
Analytical Solution: $(\sigma_y)_A \cong 4000$, $(\sigma_y)_B \cong 0$

16-9.

MATLAB PDE Toolbox: FEA Solution to Curved Beam Problem shown in Figure 8-32. Choose case *b/a* = 4, to provide simple match with analytical results given in Figure 8-33



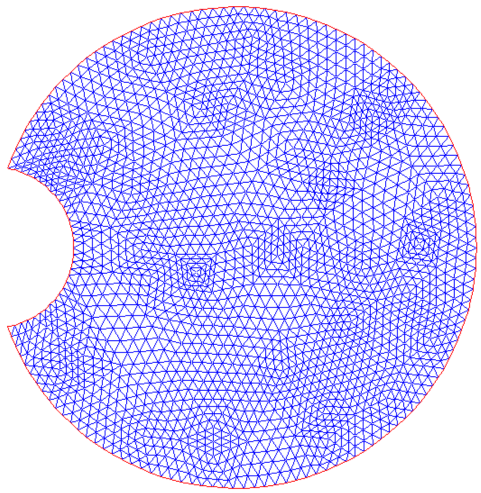
FEA Model:
b/a = 4 , 2009 Nodes , 3872 Elements



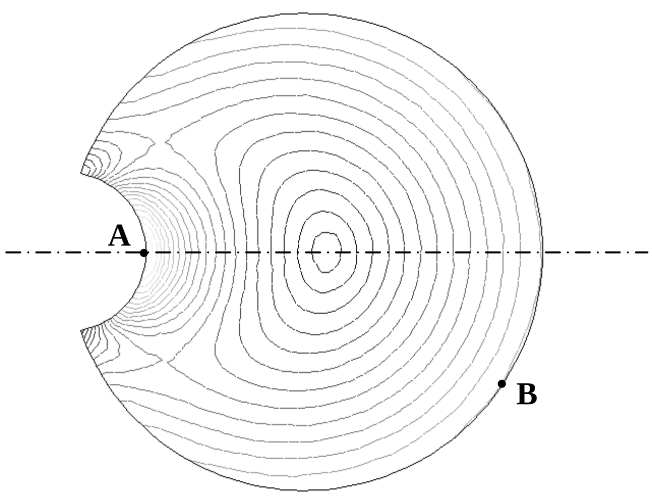
Comparison of σ_x Results:
FEA Solution: $(\sigma_x)_A \cong -8973$, $(\sigma_x)_B \cong 3875$
Analytical Solution: $(\sigma_x)_A \cong -10{,}000$, $(\sigma_x)_B \cong 2571$

16-10.

MATLAB PDE Toolbox: FEA Solution to Torsion Problem in Exercise 9-17/18



FEA Model:
(b = 0.267, a = 0.8, 2129 Nodes, 4096 Elements)



FEA | ∇ϕ | - Contours

$$\text{Resultant shear stress: } \tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2} = \sqrt{(\nabla\phi)^2} = |\nabla\phi|$$

$$\text{Analytical Results: } \tau_A \approx 2\tau_B\text{ ; FEA Results: } \quad |\nabla\phi|_A \approx 1.2\ ,\ |\nabla\phi|_B \approx 0.75$$

16-11.

For the two - dimensional, plane strain case :

$$G_{ij} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{ij} + r_{,i}r_{,j}\right]$$

Using the fact that $r_{,i} = \dfrac{x_i}{r} \Rightarrow$

$$G_{ij,k} = \frac{1}{8\pi\mu(1-\nu)}\left[-(3-4\nu)\frac{r_{,k}}{r}\delta_{ij} + (r_{,i}r_{,j})_{,k}\right]$$

$$= \frac{1}{8\pi\mu(1-\nu)r}\left[-(3-4\nu)r_{,k}\delta_{ij} + r_{,i}\delta_{jk} + r_{,j}\delta_{ik} - 2r_{,i}r_{,j}r_{,k}\right]$$

The stress form is given by

$$T_{ikj} = \lambda G_{lj,l}\delta_{ik} + \mu(G_{ij,k} + G_{kj,i})$$

$$= \frac{\lambda}{8\pi\mu(1-\nu)r}\left[-(3-4\nu)r_{,l}\delta_{lj} + r_{,l}\delta_{jl} + r_{,j}\delta_{ll} - 2r_{,l}r_{,j}r_{,l}\right]$$

$$+ \frac{1}{8\pi(1-\nu)r}\left[-(3-4\nu)r_{,k}\delta_{ij} + r_{,i}\delta_{jk} + r_{,j}\delta_{ik} - 2r_{,i}r_{,j}r_{,k}\right]$$

$$+ \frac{1}{8\pi(1-\nu)r}\left[-(3-4\nu)r_{,i}\delta_{kj} + r_{,k}\delta_{ji} + r_{,j}\delta_{ik} - 2r_{,i}r_{,j}r_{,k}\right]$$

$$= \frac{2\nu}{8\pi(1-2\nu)(1-\nu)r}\left[-2(1-2\nu)r_{,j}\delta_{ik}\right]$$

$$+ \frac{1}{8\pi(1-\nu)r}\left[-2(1-2\nu)(r_{,k}\delta_{ij} + r_{,i}\delta_{jk}) + 2r_{,j}\delta_{ik} - 4r_{,i}r_{,j}r_{,k}\right]$$

$$= \frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)r_{,j}\delta_{ik} - (1-2\nu)(r_{,k}\delta_{ij} + r_{,i}\delta_{jk}) - 2r_{,i}r_{,j}r_{,k}\right]$$

$$p_{ij} = T_{ikj}n_k = \frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)r_{,j}n_i - (1-2\nu)\left(\frac{\partial r}{\partial n}\delta_{ij} + r_{,i}n_j\right) - 2r_{,i}r_{,j}\frac{\partial r}{\partial n}\right]$$

$$= -\frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)\left(\frac{\partial r}{\partial n}\delta_{ij} + r_{,i}n_j - r_{,j}n_i\right) + 2r_{,i}r_{,j}\frac{\partial r}{\partial n}\right]$$

where we have used $r_{,k}n_k = \dfrac{\partial r}{\partial n}$