# NOTES ON DERIVATION OF WEAK FORM

For an eq of the form

$$\mathbf{A}\mathbf{u} = \mathbf{b},$$

"self-adjoincy" or "symmetry" of $\mathbf{A}$ implies that,

$$(\mathbf{A}\mathbf{u}, \mathbf{v})_H = (\mathbf{A}\mathbf{v}, \mathbf{u})_H + \text{boundary terms}$$

If the operator $\mathbf{A}$ is self-adjoint, then the variational principle corresponding to $\mathbf{A}\mathbf{u} = \mathbf{b}$ is

$$\Pi = \frac{1}{2}(\mathbf{A}\mathbf{u}, \mathbf{u})_H - (\mathbf{b}, \mathbf{u})_H + \text{boundary terms}.$$

I skipped the proof in the class, but I will include it here: with $\Pi$ as above:

$$\delta\Pi = \delta\left\{\frac{1}{2}\int_V \mathbf{u}\cdot\mathbf{A}\mathbf{u}\,dV - \int_V \mathbf{b}\cdot\mathbf{u}\,dV + b.t.\right\}.$$

or

$$= \int_V \left\{\frac{1}{2}\left(\delta\mathbf{u}\cdot\mathbf{A}\mathbf{u} + \mathbf{u}\cdot\mathbf{A}\delta\mathbf{u}\right) + \mathbf{b}\cdot\delta\mathbf{u}\right\} dV + b.t.$$

$$= \int_V \delta\mathbf{u}\cdot\left[\mathbf{A}\mathbf{u} - \mathbf{b}\right] dV + b.t.$$

eg: Consider, as we did in the class:

$$EI\frac{d^4 w}{dx^4} = q(x) \qquad 0 \leqslant x \leqslant L.$$

$$EI = \text{constant}.$$

$$A = EI\frac{d^4}{dx^4} \qquad b = q(x).$$

To check for self-adjoincy,

$$EI\int_0^L v\frac{d^4 w}{dx^4}\,dx = EI\left\{ v\frac{d^3 w}{dx^3}\Big|_0^L - \int_0^L \frac{d^3 w}{dx^3}\cdot\frac{dv}{dx}\,dx \right\}.$$

$$= EI\left\{ v\big|_L Q_2 - v\big|_0 Q_1 - \int_0^L \frac{d^3 w}{dx^3}\frac{dv}{dx}\,dx \right\}$$

$$= \cancel{EI}\left\{ v\big|_L Q_2 - v\big|_0 Q_1 - EI\frac{d^2 w}{dx^2}\frac{dv}{dx}\Big|_0^L + \int_0^L EI\frac{d^2 w}{dx^2}\frac{d^2 v}{dx^2}\,dx \right\}$$

$$= \cancel{E}\; v\big|_L Q_2 - v\big|_0 Q_1 - \frac{dv}{dx}\Big|_L Q_4 + \frac{dv}{dx}\Big|_0 Q_3.$$

$$+ EI\underbrace{\int_0^L \frac{d^2 w}{dx^2}\frac{d^2 v}{dx^2}\,dx}.$$

This term is symmetric.
That proves self-adjoincy.

To get the weak form take the special case $v = \delta w$. Then:

$$EI\int_0^L \frac{d^2w}{dx^2}\,\delta\left(\frac{d^2w}{dx^2}\right)dx = \frac{1}{2}EI\int_0^L \delta\left(\left(\frac{d^2w}{dx^2}\right)^2\right)dx$$

Also, letting

$$\delta\Pi = \delta w\Big|_L Q_2 - \delta w\Big|_0 Q_1 - \delta\frac{dw}{dx}\Big|_L Q_4 + \delta\frac{dw}{dx}\Big|_0 Q_3 + \frac{1}{2}\int_0^L \delta\left\{EI\left(\frac{d^2w}{dx^2}\right)^2\right\}dx.$$

we have:

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx + w\Big|_L Q_2 - w\Big|_0 Q_1 - \frac{dw}{dx}\Big|_L Q_4 + \frac{dw}{dx}\Big|_0 Q_3.$$

Note, we have defined

$$EI\frac{d^2w}{dx^3}\Big|_0 = Q_1 \qquad EI\frac{d^3w}{dx^3}\Big|_L = Q_2.$$

$$EI\frac{d^2w}{dx^2}\Big|_0 = Q_3 \qquad EI\frac{d^2w}{dx^2}\Big|_L = Q_4.$$

In conventional strength of materials notation, negatives of $Q_1, Q_2$ are shear forces & negatives of $Q_3, Q_4$ are bending moments at $0, L$ respectively.