Classification of PDE

MSO-203B

Indian Institute of Technology, Kanpur

kaushik@iitk.ac.in

October 18, 2016

Overview

Classification of PDE

- Introduction.

Overview

Classification of PDE

- Introduction.
- Examples.

Overview

Classification of PDE

- Introduction.
- Examples.
- Invariance Property.

Introduction

The Equation

Consider the Second order linear PDE:

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + cu_y + fu = g \ \text{ in } \ \Omega \tag{1}$$

and Ω is a open subset of $\mathbb{R}^2$ and a, b, c are C^1 function satisfying $a^2 + b^2 + c^2 \neq 0$.

Introduction

The Equation

Consider the Second order linear PDE:

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + cu_y + fu = g \text{ in } \Omega \qquad (1)$$

and Ω is a open subset of $\mathbb{R}^2$ and a, b, c are C^1 function satisfying $a^2 + b^2 + c^2 \neq 0$.

Question

- We want to Classify the equation in term of the sign of the discriminant.
- Does the equation remain invariant under co-ordinate transformation.

Basics

- The operator $L_0(u) = au_{xx} + 2bu_{xy} + cu_{yy}$ consisting of the second order terms of L is called the principal part of L.
- The discriminant $\Delta(L)(x, y)$ is defined as follows:

$$\Delta(L)(x,y) = \det \begin{bmatrix} b(x,y) & a(x,y) \\ c(x,y) & b(x,y) \end{bmatrix} = \begin{vmatrix} b & a \\ c & b \end{vmatrix}$$

Introduction

Definitions

- The PDE (1) is called hyperbolic if $\Delta(L)(x, y) > 0$.

Introduction

Definitions

- The PDE (1) is called hyperbolic if $\Delta(L)(x, y) > 0$.
- The PDE (1) is called parabolic if $\Delta(L)(x, y) = 0$.

Introduction

Definitions

- The PDE (1) is called hyperbolic if $\Delta(L)(x, y) > 0$.
- The PDE (1) is called parabolic if $\Delta(L)(x, y) = 0$.
- The PDE (1) is called Elliptic if $\Delta(L)(x, y) < 0$.

Examples

- The Wave equation $u_{tt} - u_{xx} = 0$ is a hyperbolic equation.

Examples

- The Wave equation $u_{tt} - u_{xx} = 0$ is a hyperbolic equation.
- The Heat equation $u_t - u_{xx} = 0$ is an parabolic equation.

Introduction

Examples

- The Wave equation $u_{tt} - u_{xx} = 0$ is a hyperbolic equation.
- The Heat equation $u_t - u_{xx} = 0$ is an parabolic equation.
- The Laplace equation $u_{xx} + u_{tt} = 0$ is an elliptic equation.

Invariance Property

Invariance under a change of variable

Consider a $F \in C^1$ given by $F(x,y) = (\theta(x,y), \eta(x,y))$ whose Jacobian satisfies

$$J(x,y) = \begin{vmatrix} \theta_x & \theta_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

at each point $(x,y) \in \Omega$.

Define $w(\theta,\eta) = u(x(\theta,\eta), y(\theta,\eta))$

Transforming the old equation

Using Chain rule one has,

$$
\begin{aligned}
u_x &= w_\theta \theta_x + w_\eta \eta_x \\
u_y &= w_\theta \theta_y + w_\eta \eta_y \\
u_{xx} &= w_{\theta\theta}(\theta_x)^2 + 2w_{\theta\eta}\theta_x\eta_x + w_{\eta\eta}(\eta_x)^2 + w_\theta\theta_{xx} + w_\eta\eta_{xx} \\
u_{yy} &= w_{\theta\theta}(\theta_y)^2 + 2w_{\theta\eta}\theta_y\eta_y + w_{\eta\eta}(\eta_y)^2 + w_\theta\theta_{yy} + w_\eta\eta_{yy} \\
u_{xy} = u_{yx} &= w_{\theta\theta}\theta_x\theta_y + w_{\theta\eta}(\theta_x\eta_y + \eta_x\theta_y) + w_{\eta\eta}\theta_x\eta_y + w_\theta\theta_{xy} + w_\eta\eta_{xy}
\end{aligned}
$$

Invariance Property

The Transformed Equation

Substituting the partial derivatives of u onto (1) we get

$$\bar{L}(w) = Aw_{\theta\theta} + 2Bw_{\theta\eta} + Cw_{\eta\eta} + Dw_{\theta} + Ew_{\eta} + Fw = G$$

where,

$$\begin{aligned} A(\theta, \eta) &= a{\theta_x}^2 + 2b\theta_x\theta_y + c{\theta_y}^2 \\ B(\theta, \eta) &= a\theta_x\eta_x + b(\theta_x\eta_y + \eta_x\theta_y) + c\eta_y\theta_y \\ C(\theta, \eta) &= a{\eta_x}^2 + 2b\eta_x\eta_y + c{\eta_y}^2 \end{aligned}$$

Invariance Property

Observation

Note that the coefficients A,B and C satisfies the following:

$$\begin{bmatrix} B & A \\ C & B \end{bmatrix} = \begin{bmatrix} \theta_x & \theta_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} b & a \\ c & b \end{bmatrix} \begin{bmatrix} \theta_x & \theta_y \\ \eta_x & \eta_y \end{bmatrix}^t$$

Invariance Property

Observation

Note that the coefficients A,B and C satisfies the following:

$$\begin{bmatrix} B & A \\ C & B \end{bmatrix} = \begin{bmatrix} \theta_x & \theta_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} b & a \\ c & b \end{bmatrix} \begin{bmatrix} \theta_x & \theta_y \\ \eta_x & \eta_y \end{bmatrix}^t$$

Conclusion

Since we have that $J(x, y) \neq 0$ hence we have the equation is invariant under transformation.

The End