Maximum Principle- Elliptic Equation

MSO-203B

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Introduction

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Can one comment on the uniqueness of the solution of the Laplace equation with Dirichlet boundary condition with or without the prior knowledge of existence

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Answer

There are many ways to get information about the uniqueness of the solution. The most versatile way is to use The Maximum Principle, provided it exists.

Maximum Principle for the Laplace Equation

Let Ω be bounded connected domain in $\mathbb{R}^2$ and $u \in C^2(\bar{\Omega})$ be the solution of the equation $\Delta u = 0$. Then the maximum and the minimum value of u are attained on $\partial\Omega$.

Theorems

Maximum Principle for the Laplace Equation

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Uniqueness under Dirichlet boundary Condition

If Ω is a bounded connected domain in $\mathbb{R}^2$ then there exists a unique solution $u \in C^2(\bar{\Omega})$ of the problem

$$\Delta u = f \text{ in } \Omega; \quad u = g \text{ on } \partial\Omega \tag{1}$$

provided $f \in C(\Omega)$ and $g \in (\partial\Omega)$

Proof of the Uniqueness theorem

Let u and v be two solution of the equation (1).
Define $w = u - v$.
Clearly $w \in C^2(\Omega)$ and w satisfies the following equation:

$$\Delta w = 0 \text{ in } \Omega; \;\; w = 0 \text{ on } \partial\Omega$$

Uniqueness Theorem

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Applying the Maximum Principle

$$0 = \min_{\partial\Omega} w(x) \leq \min_{\Omega} w(x) \leq \max_{\Omega} w(x) \leq \max_{\partial\Omega} w(x) = 0$$

Maximum Principle

Proof of Maximum Principle

If u attains its maximum at (x_0, y_0) then one has,

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) \leq 0$$

Since $\Delta u = u_{xx} + u_{yy} = 0$ we doesn't reach a contradiction.

Maximum Principle

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Modifying u

Define, $v(x, y) = u(x, y) + \epsilon((x - x_0)^2 + (y - y_0)^2)$ for $(x, y) \in \Omega$.

Maximum Principle

Remark

Note that $\Delta v > 0$ which is a contradiction to the fact that v attains maximum in the interior.

Observation 2

If u attains a maximum at (x_0, y_0) then so does v and moreover we have

$$\max_{\Omega} v \leq v(x_0, y_0) \leq \max_{\partial\Omega} v$$

.

Proof of Maximum Principle

By the construction of v we have that

$$u(x,y) \leq v(x,y) \leq v(x_0,y_0) = u(x_0,y_0) + \epsilon(x_0^2 + y_0^2) \leq \max_{\partial\Omega} u + \epsilon l$$

where $l = \max_{z \in \partial\Omega} |z|^2$.
Letting $l \to 0$ we have

$$u(x,y) \leq \max_{\partial\Omega} u$$

and hence

$$\max_{\Omega} u \leq \max_{\partial\Omega} u$$