ME623: Finite Element Methods in Engineering Mechanics

Instructor: Sumit Basu

Email: sbasu@iitk.ac.in

Phone office: (0512 259) 7506

Office: NL211A

Phone residence: (0512 259) 8511

Coursenotes, assignments and announcements will all be made in Backpack (http://www.usebackpack.com).
Coursecode is 063085

The Google Calendar for the course is available at https://calendar.google.com/calendar?cid=ZHluc3VtaXQuYmFzdUBnbWFpbC5jb20

Evaluation Policy

The endsem and midsem exams will be the only exams that we will conduct. Each of these will carry 50% weight. These will be ‘pen and paper’ tests based on theoretical aspects of the FE method.

Several assignments meant to help your understanding of the techniques will be given from time to time. If you want to learn the method seriously, I suggest that you try these out.

Coding workshops will be held in the Linux Lab in the New Core Building.

Books

Elementary books covering a wide range of problems

- Concepts and applications of Finite element analysis: Cook, Malkus and Plesha, John Wiley and Sons, 2003.
- T.R. Chandrupatla and A.D. Belegundu, Introduction to Finite Elements in Engineering, Second Edition, Prentice-Hall, 1997.
- Daryl Logan, A First Course in Finite Element Method, Thomson, India Edition

Somewhat more advanced texts dealing less with procedures and more with ideas behind the technique

- K-J Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall of India, 1990
- O C Zienkiewicz and R L Taylor, The Finite Element Method: Its Basics and Fundamentals, Butterworth Heinemann, 2007
- J N Reddy, An Introduction To Finite Element Method, Tata McGraw Hill, 2005

Why do we need to simulate things?

Seeing is believing.

Sophisticated experiments can tell everything. Why do we need the FE method?

- ❑Experimental results are subject to interpretation. Interpretations are as good as the competence of the experimenter.
- ❑Experiments, especially sophisticated ones, can be expensive
- ❑There are regimes of mechanical material behaviour that experiments cannot probe.
- ❑Generality of behaviour is often not apparent from experiments.

Experiments and simulations are like two legs of a human being. You need both to walk and it does not matter which you use first!

FEA has become extremely important in engineering design

- The FEM has attained a level of generality that makes it suitable for a very wide range of engineering and scientific problems that require solution of governing partial differential equations with well defined boundary conditions. These include, solid and fluid mechanics, structural mechanics, biomechanics, quantum mechanics, electromagnetics, geomechanics …

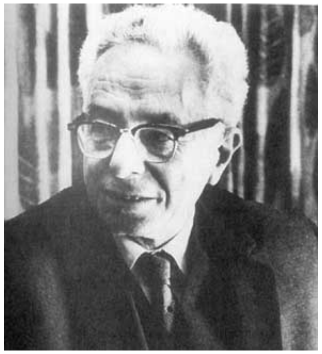
- The mathematical basis of the FEM is well understood and the discretisation equations for FEM arise out of well established principles.

- FEM simulations form important part of safety assessment procedures for Reactor Pressure Vessels in nuclear power plants, aircraft industry, crashworthiness of automobiles, earthquake resistance of buildings etc.

- Powerful commercial softwares that combine modelling algorithms with the power of FEM are now easily available. Extermely realistic problems may be solved given the enormous increase in computational power.

How did it evolve?

1943: Richard Courant, a mathematician described a piecewise polynomial solution for the torsion problem of a shaft of arbitrary cross section. Even holes.
The early ideas of FEA date back to a 1922 book
by Hurwitz and Courant.



His work was not noticed by engineers
and the procedure was impractical at the time due to the lack of digital computers.

1888-1972: b in Lublintz Germany

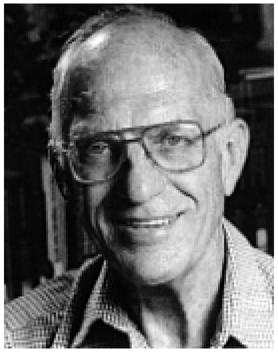
Student of Hilbert and Minkowski in Gottingen Germany

Ph.D in 1910 under Hilbert's supervision.

1934: moved to New York University, founded the Courant Institute

In the **1950**s: work in the aircraft industry introduced FE to practicing engineers. A classic paper described FE work that was prompted by a need to analyze delta wings, which are too short for beam theory to be reliable.

1960: The name "finite element" was coined by structural engineer Ray Clough of the University of California



Professor emeritus of Structural Engineering at UC Berkley

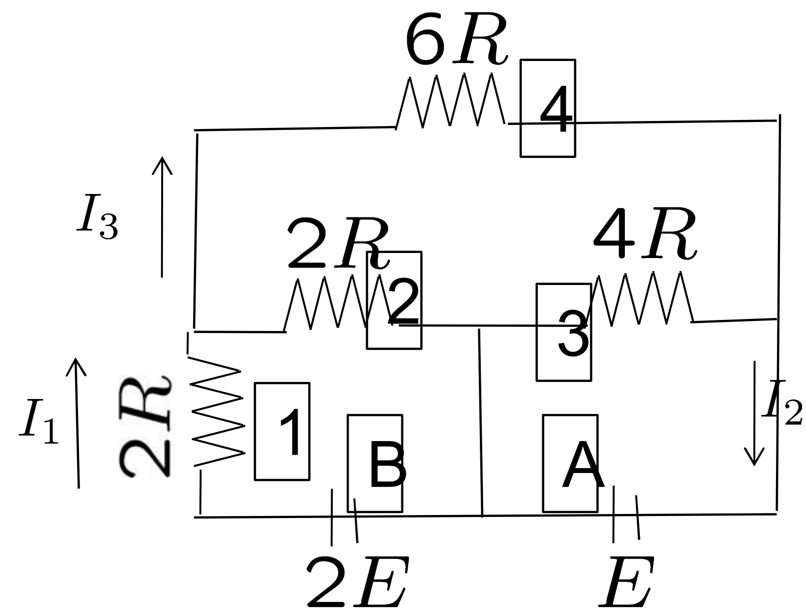
Ph.D from MIT

Well known earthquake engineer

By **1963** the mathematical validity of FE was recognized and the method was expanded from its structural beginnings to include heat transfer, groundwater flow, magnetic fields, and other areas.

Large general-purpose FE software began to appear in the 1970s.
By the late **1980**s the software was available on microcomputers, complete with color graphics and pre- and post-processors.
By the mid **1990**s roughly 40,000 papers and books about FE and its applications had been published.

Solving Engineering Problems



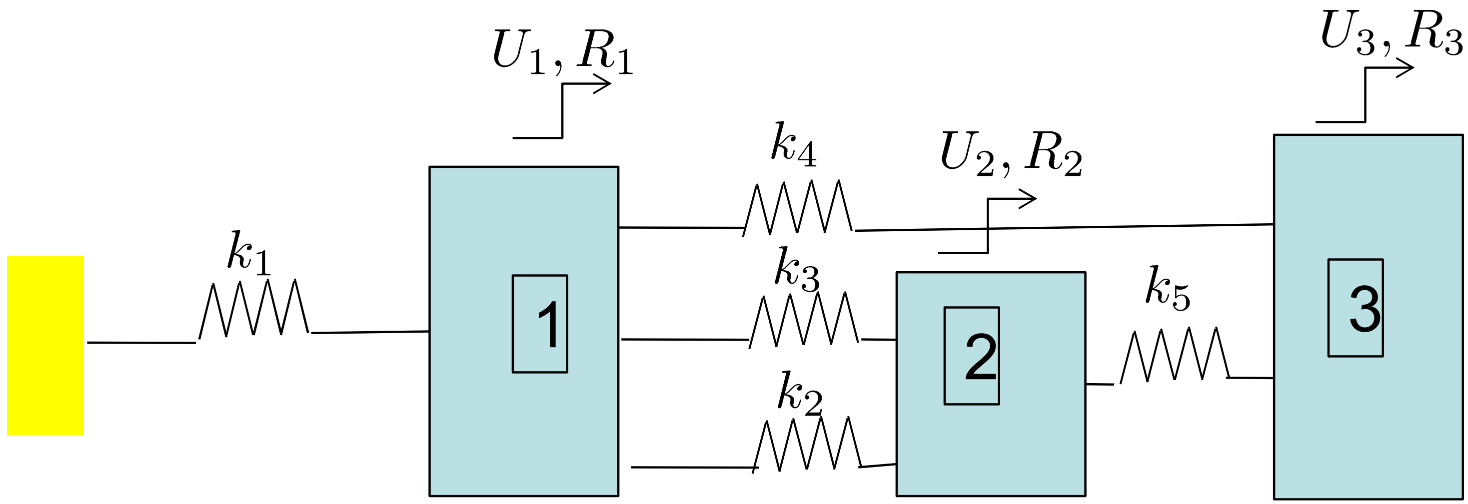
To solve engineeringing problems on the computer, one of the common goals is to cast the problem in a matrix form. Then we can use the numerical methods for solving linear system of equations. For example, for the simple circuit shown here,

$$
\begin{aligned}
2E &= 2RI_1 + 2R(I_1 - I_3) \\
E &= 4R(I_2 - I_1) \\
0 &= 6RI_3 + 4R(I_3 - I_2) + 2R(I_3 - I_1)
\end{aligned}
$$

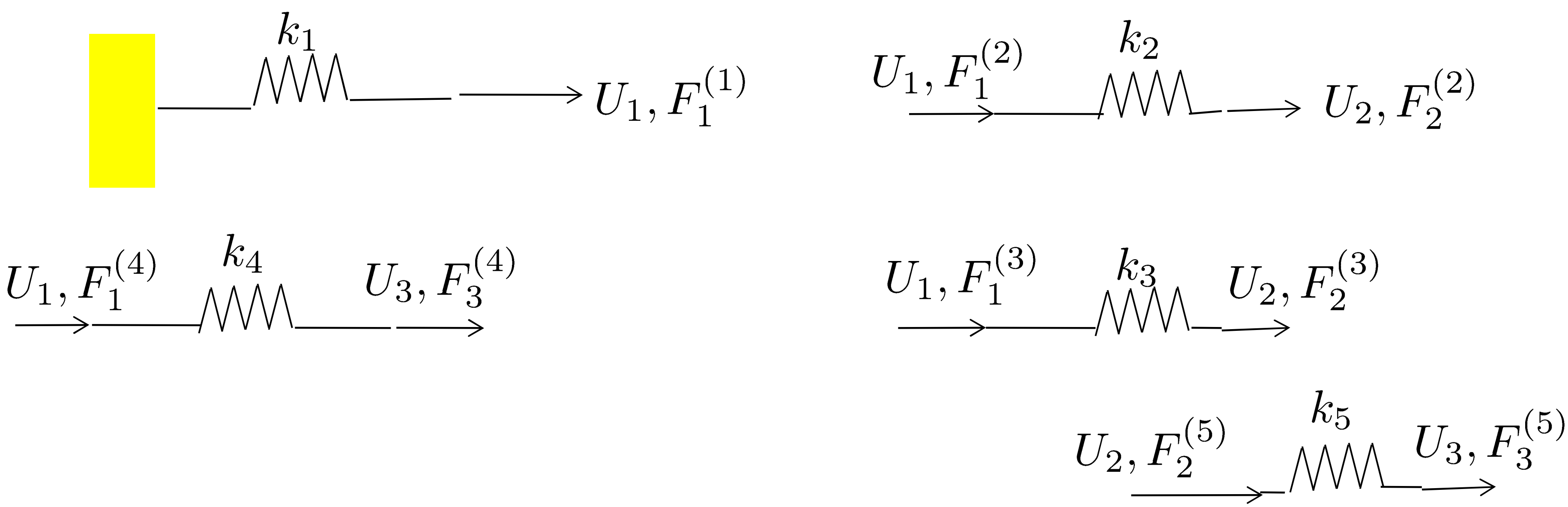
These equations can be cast in the form:

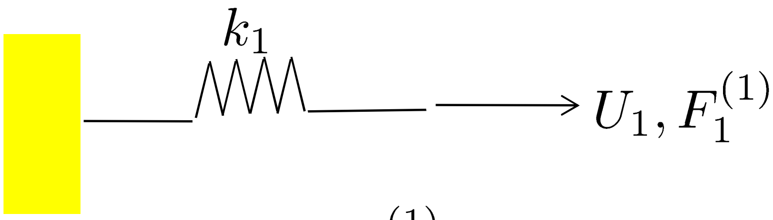
$$\begin{pmatrix} 4R & 0 & -2R \\ 0 & 4R & -4R \\ -2R & -4R & 12R \end{pmatrix} \left\{ \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} \right\} = \left\{ \begin{array}{c} 2E \\ E \\ 0 \end{array} \right\},$$

and hence can be easily solved for the unknowns I_1, I_2 and I_3. The procedure is similar for equivalent spring-mass problems.

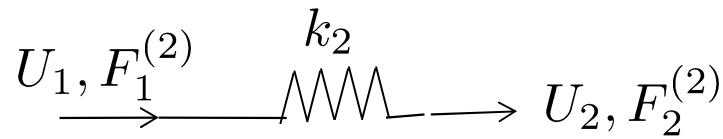


Define $F_i^{(j)}$ where $i = 1, 2, 3$ and $j = 1, \ldots 5$

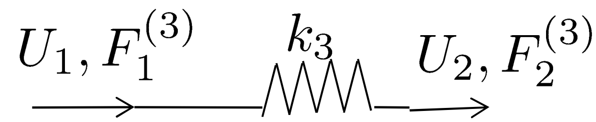




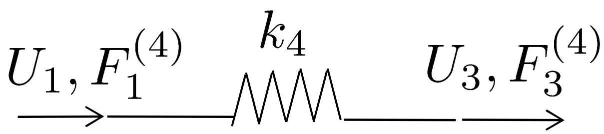
$$k_1 U_1 = F_1^{(1)}$$



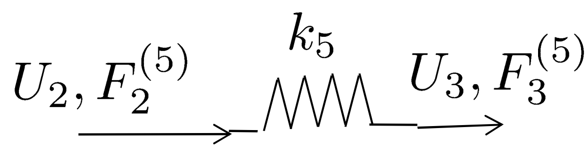
$$\begin{aligned} F_2^{(2)} &= k_2(U_2 - U_1) \\ F_1^{(2)} &= -k_2(U_2 - U_1) \end{aligned}$$



$$\begin{aligned} F_2^{(3)} &= k_3(U_2 - U_1) \\ F_1^{(3)} &= -k_3(U_2 - U_1) \end{aligned}$$



$$\begin{aligned} F_3^{(4)} &= k_4(U_3 - U_1) \\ F_1^{(4)} &= -k_4(U_3 - U_1) \end{aligned}$$



$$\begin{aligned} F_2^{(5)} &= k_5(U_3 - U_2) \\ F_2^{(5)} &= -k_5(U_3 - U_2) \end{aligned}$$

Finally, free body diagrams of each of the nodes $1, 2$ and 3 give

$$\begin{aligned} F_1^{(1)} + F_1^{(2)} + F_1^{(3)} + F_1^{(4)} &= R_1 \\ F_2^{(2)} + F_2^{(3)} + F_2^{(5)} &= R_2 \\ F_3^{(4)} + F_3^{(5)} &= R_3. \end{aligned}$$

Now we replace the forces on the left hand sides in terms of the stiffnesses and displacements. Also, U_1, U_2, U_3 are the unknowns in this problem. So, we get

$$\begin{pmatrix} (k_1 + k_2 + k_3 + k_4) & -(k_2 + k_3) & -k_4 \\ -(k_2 + k_3) & (k_2 + k_3 + k_5) & -k_5 \\ -k_4 & -k_5 & (k_4 + k_5) \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}.$$

The above set of equations govern the equilibium of the system of springs in this problem. They are expressed as (in analogy with a single linear spring)

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}.$$

The same problem can be solved using a energy based approach. It turns out that such an approach is more suited for numerical implementations. The energy principle (which we will deal with in detail later) states that the total potential energy of the system should be a minimum. The total potential energy of the system is

$$\Pi = \mathcal{U} - \mathcal{W},$$

where, $\mathcal{U}$ is the strain energy stored and $\mathcal{W}$ is the potential of the external forces.

The potential of the loads is

$$\mathcal{W} = U_1 R_1 + U_2 R_2 + U_3 R_3,$$

which can be expressed as

$$\mathcal{W} = \boldsymbol{U}^T \boldsymbol{F}.$$

Further, noting that the strain energy stored in a linear spring of stiffness k is $(1/2)ku^2$, the strain energy of the system is

$$\Pi = \frac{1}{2}\boldsymbol{U}^T \boldsymbol{K} \boldsymbol{U}. \text{ (verify this yourself.)}$$

Thus

$$\Pi = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} - \boldsymbol{U}^T\boldsymbol{F}.$$

Finding the extremum of Π with respect to U_i involves solving

$$\frac{\partial \Pi}{\partial U_i} = 0 \text{ for } i = 1, 2, 3.$$

The extremum is a minimum (can be checked formally) because the maximum is obviously infinity. Again, verify that, the above procedure applied to the Π derived above gives:

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}.$$

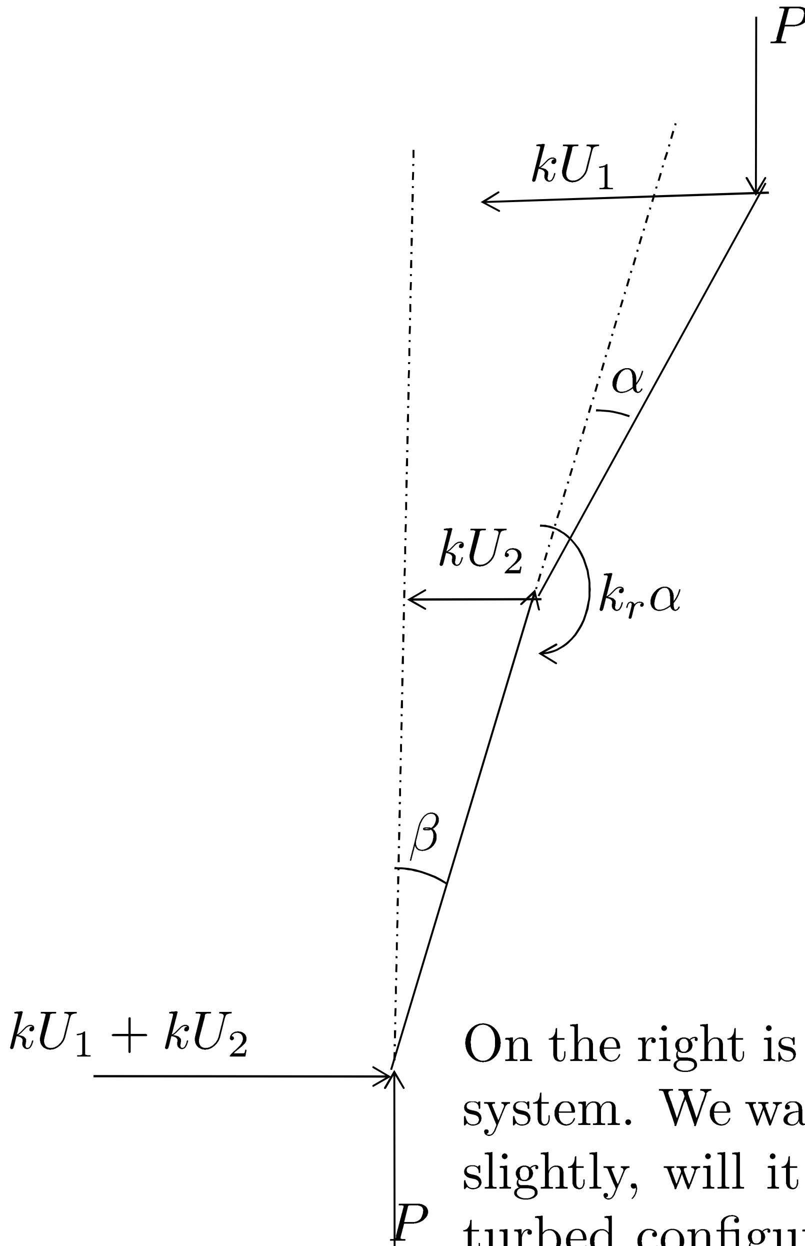
Minimising the potential energy of the system provides us with an alternate way to arrive at the equations of motion of a system.

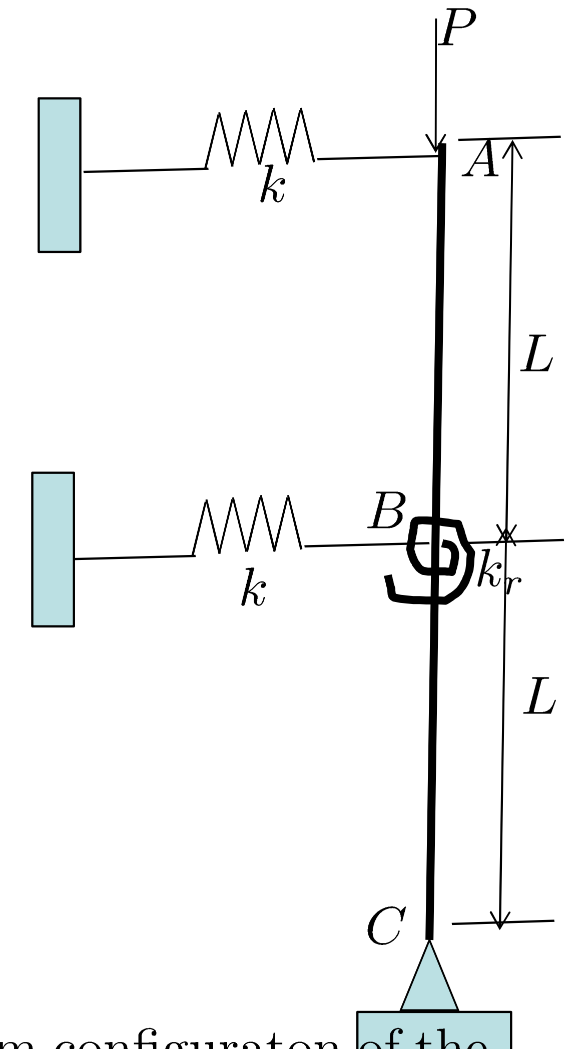
We can also investigate problems of stability where we ask if, given that a steady state solution to a problem is known, whether a small perturbation of the system will take it to another solution.

To investigate the stability of a system we will need to consider a generalised eigenvalue problem

$$\boldsymbol{K}\boldsymbol{U} = \lambda \boldsymbol{B}\boldsymbol{U}.$$

If a solution exists to this problem, λ_i and $\boldsymbol{U}_i$ that satisfy this are called *eigenvalues* and *eigenvectors* of $\boldsymbol{K}$. We will look at a simple problem to better understand this concept.





On the right is the equilibrium configuraton of the system. We want to know if we perturb the system slightly, will it be stable? On the left is the perturbed configuration and its free body diagram.

Strain energy stored in the system

$$\mathcal{U} = \frac{1}{2}\left(kU_1^2 + kU_2^2 + k_r\alpha^2\right).$$

Vertical movement of the external load is

$$\delta_v = L - L\cos(\alpha+\beta) + L - L\cos\beta = L\left[1 - \cos(\alpha+\beta) + 1 - \cos\beta\right].$$

Using a series approximation for the cosines assuming a small perturbation, we have

$$\delta_v = L\left[\frac{(\alpha+\beta)^2}{2} + \frac{\beta^2}{2}\right].$$

Further, for small perturbations

$$\alpha = \frac{U_1 - 2U_2}{L}, \beta = \frac{U_2}{L}, \alpha + \beta = \frac{U_1 - U_2}{L}.$$

Potential of the external loads is given as

$$\mathcal{W} = P\delta_v.$$

The potential energy thus can be written in terms of the displacements as

$$
\begin{aligned}
\Pi &= \frac{1}{2}kU_1^2 + \frac{1}{2}kU_2^2 + \frac{1}{2}k_r\left(\frac{U_1 - 2U_2}{L}\right)^2 \\
&- \frac{P}{2L}\left(U_1 - U_2\right)^2 - \frac{P}{2L}U_2^2.
\end{aligned}
$$

Now using

$$\frac{\partial \Pi}{\partial U_1} = \frac{\partial \Pi}{\partial U_2} = 0,$$

we get

$$\begin{pmatrix} kL + k_r/L & -2k_r/L \\ -2k_r/L & kL + 4k_r/L \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = P \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}.$$

Note that this is an eigenvalue problem and existence of non trivial solutions for the load P indicates that multiple solutions to this are possible.

Solving problems governed by differential equations: finite difference techniques

Mathematical model: an equation of motion

$$\frac{du}{dt} = f(t, u)$$

for $t > 0$ and $u = u_0$ at $t = 0$

Use

$$\left(\frac{du}{dt}\right)_i \simeq \frac{[u(t_{i+1}) - u(t_i)]}{t_{i+1} - t_i}$$

$$\Rightarrow u_{i+1} = u_i + \Delta t f(u_i, t_i)$$

Euler's explicit scheme or first order Runge Kutta scheme

Consider the motion of a pendulum:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$\theta(0) = \theta_0$ and $\dot{\theta}(0) = v_0$.
Solution to this system is

$$\theta(t) = \frac{v_0}{\lambda}\sin\lambda t + \theta_0\cos\lambda t$$

where $\lambda = \sqrt{g/l}$

Alternately,

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= -\lambda^2\theta\end{aligned}$$

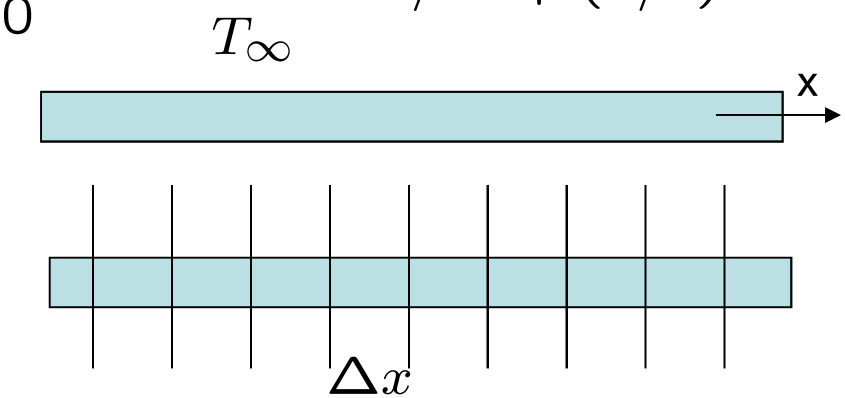
$\Rightarrow$

$$\begin{aligned}\theta_{i+1} &= \theta_i + \Delta t v_i \\ v_{i+1} &= v_i - \Delta t\lambda^2\theta_i\end{aligned}$$

As another example, consider the problem of heat conduction in a bar with specified boundary conditions.

At $x = L$
$dT/dx + (h/k)T = 0$

At $x = 0$
$T = T_0$



The governing deq is

$$kA\frac{d^2T}{dx^2} + hP(T_\infty - T) = 0$$

Using $T_\infty = 0$ arbitrarily

$$\left(\frac{d^2T}{dx^2}\right)_{x=x_i} \simeq \left(\frac{T_{i-1} - 2T_i + T_{i+1}}{(\Delta x)^2}\right)$$

$\Rightarrow$

$$-T_{i-1} + [2 + (m\Delta x)^2]T_i - T_{i+1} = 0$$

where $m = \sqrt{hP/kA}$

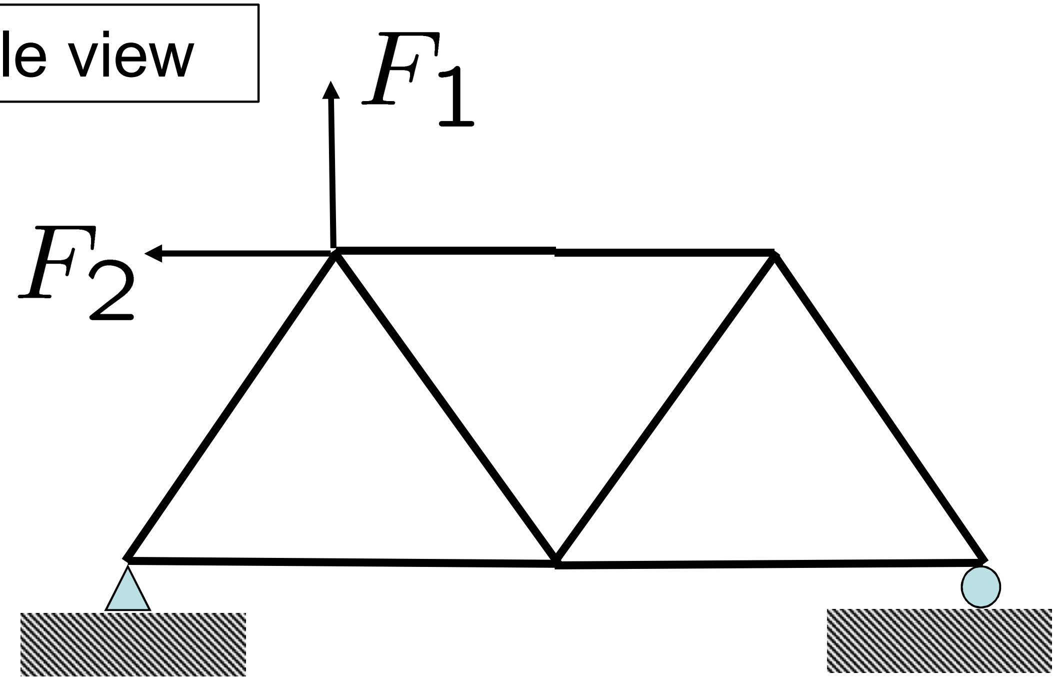
$$
\begin{aligned}
-T_0 + DT_1 - T_2 &= 0 \\
-T_1 + DT_2 - T_3 &= 0 \\
\cdots\cdots\cdots\cdot &\quad \cdots \\
-T_{N-1} + DT_N - T_{N+1} &= 0
\end{aligned}
$$

This is a *tridiagonal* system of equations, where, for a point i, the temperatures at $i-1$ and $i+1$ only appear in the ith equation. When put in the matrix form, only the diagonal and the sub-diagonals above and below it will be populated. Additionally,

$$
\begin{aligned}
\frac{T_{N+1} - T_N}{\Delta x} + \frac{h}{k} T_N &= 0 \\
\Rightarrow T_{N+1} &= \left(1 - \frac{h\Delta x}{k}\right) T_N
\end{aligned}
$$

The FEM procedure: a simple view

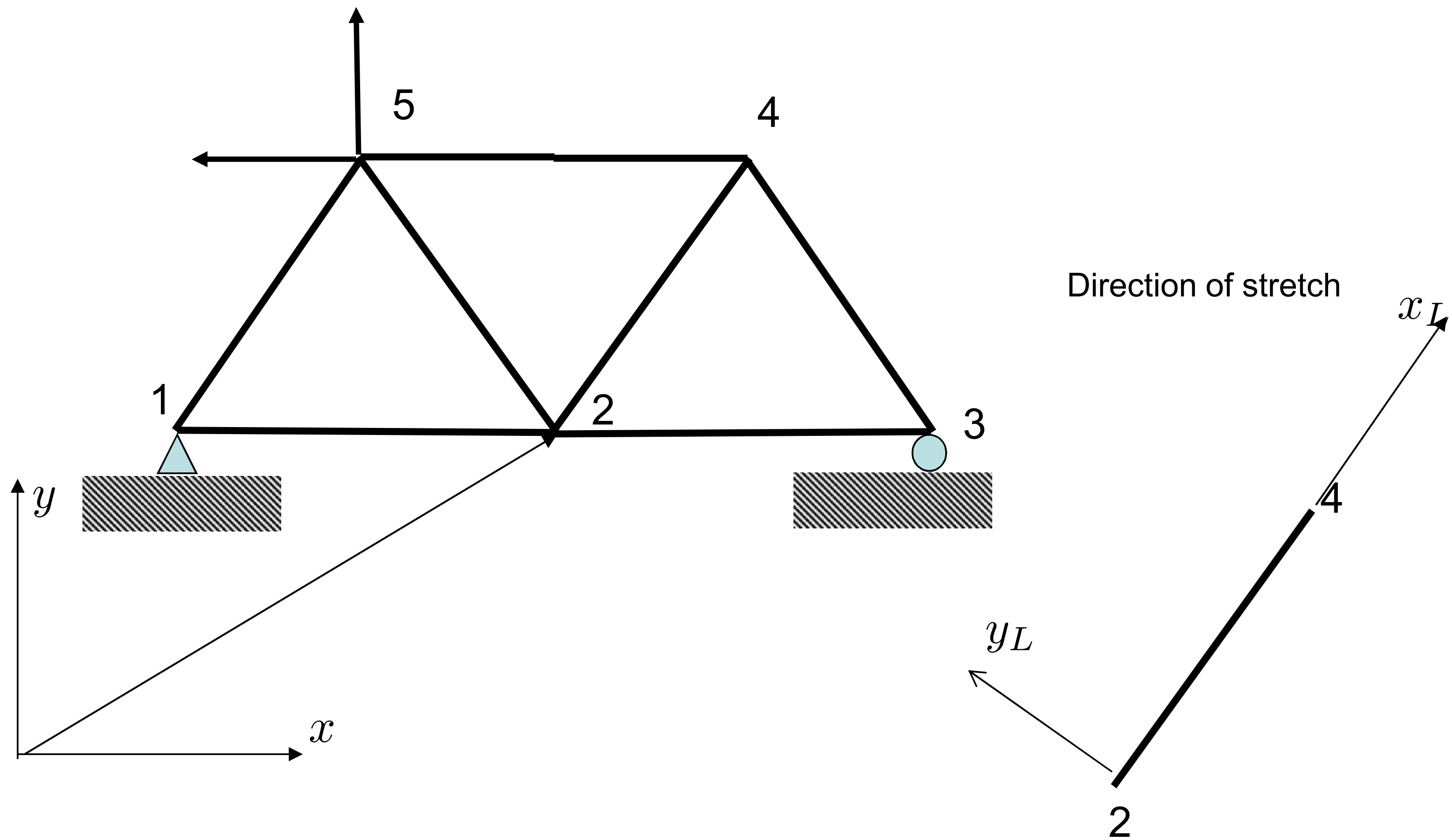
A 2-d truss with elements that can only withstand tension.



For any truss element, uniaxial elastic constitutive relations are valid.

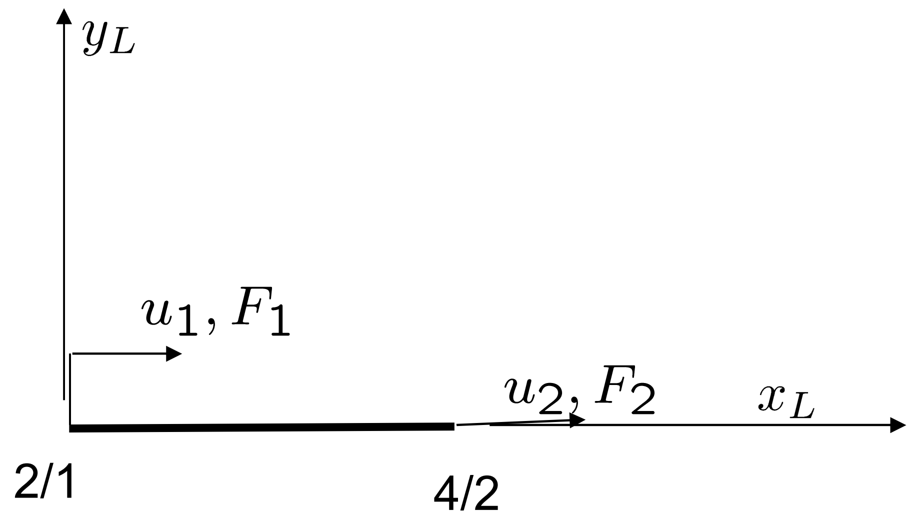
$$\begin{aligned} F &= \sigma A \\ &= EA\epsilon \\ &= EA\frac{l - l_0}{l_0} \\ &= k\delta \end{aligned}$$

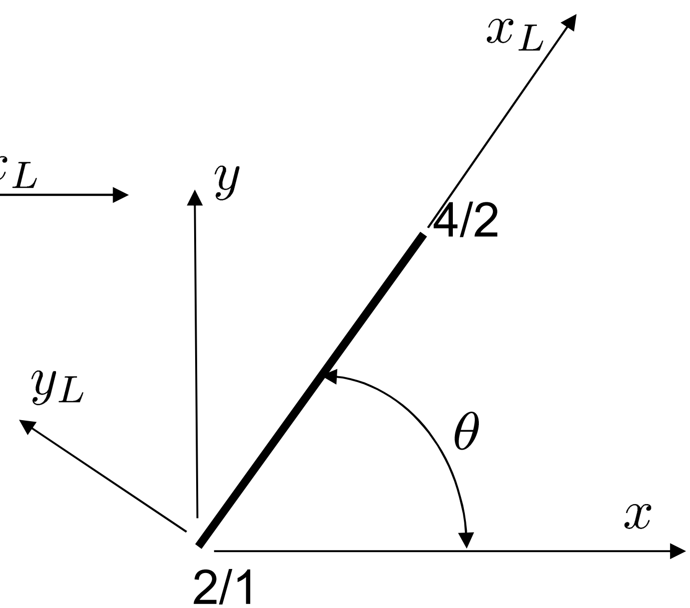
where, $k = \frac{AE}{l_0}$ and $\delta = l - l_0$.



Each node in a planar truss has 2 degrees of freedom (dof). Also, each node can be located in a global (x, y) coordinate system.

The equilibrium equation for every element in the truss can be written in a *local* coordinate system (x_L, y_L). In this system, the element has only one dof per node, it can only stretch along x_L. The orientation of a generic element can be defined by an angle θ wrt the global system.





$$
\begin{aligned}
F_2 &= k(u_2 - u_1) \\
F_1 &= k(-u_2 + u_1)
\end{aligned}
$$

$$
\left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\} = \left(\begin{array}{cc} k & -k \\ -k & k \end{array} \right) \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}
$$

The force balance for a generic element can be expressed in the form of a 4×4 matrix by considering the possibility rigid body movement along y_L.

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} = \begin{pmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Local force vector (ndof X nnod,1)

Displacement vector (ndof X nnod, 1)

$$\boldsymbol{f} = \boldsymbol{K}_L \boldsymbol{u}$$

Local stiffness matrix in local coordinate system (ndof X nnod, ndof X nnod)

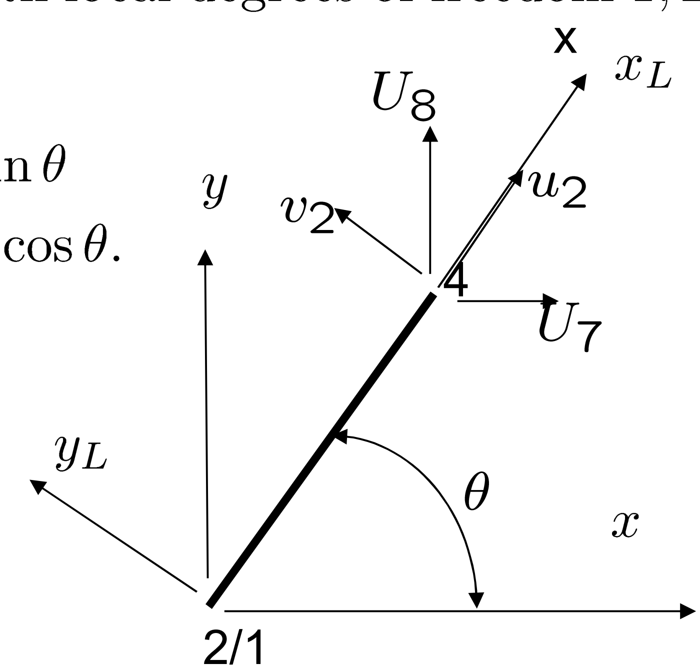
We will now express the local quantites in the global coordinates. The numbering for the global vectors is based on a continuous numbering of the dofs i.e. for node i the dof's are $(i-1)*ndof+1,\ldots,i*ndof$. Thus, the transformation for the displacement components at a node is given as:

$$
\begin{aligned}
u_2 &= U_7\cos\theta + U_8\sin\theta \\
v_2 &= -U_7\sin\theta + U_8\cos\theta.
\end{aligned}
$$

$$
\left\{ \begin{array}{c} u_2 \\ v_2 \end{array} \right\} = \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right) \left\{ \begin{array}{c} U_7 \\ U_8 \end{array} \right\}
$$

Also, in a similar manner, for the local node 1 with local degrees of freedom $1,2$ and global dof's $3,4$,

$$
\begin{aligned}
u_1 &= U_3\cos\theta + U_4\sin\theta \\
v_1 &= -U_3\sin\theta + U_4\cos\theta.
\end{aligned}
$$



The transformation from local to global quantities for both the nodes can be expressed in one equation as:

$$\left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \right\} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \left\{ \begin{array}{c} U_3 \\ U_4 \\ U_7 \\ U_8 \end{array} \right\}$$

Note that the transformation matrix is orthogonal.

$$\boldsymbol{u} = \boldsymbol{T}\boldsymbol{U}$$

Similarly the relation between the global and local components of the force vector can be written as

$$\boldsymbol{f} = \boldsymbol{T}\boldsymbol{F}$$

Now we can easily derive the relationship between the global and local stiffness matrices.

$$\begin{aligned} \boldsymbol{f} &= \boldsymbol{K}_L \boldsymbol{u} \\ \boldsymbol{T}\boldsymbol{F} &= \boldsymbol{K}_L \boldsymbol{T}\boldsymbol{U} \\ \boldsymbol{F} &= \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}\boldsymbol{U} \end{aligned}$$

Thus, the global equilibrium for the generic element can be written as:

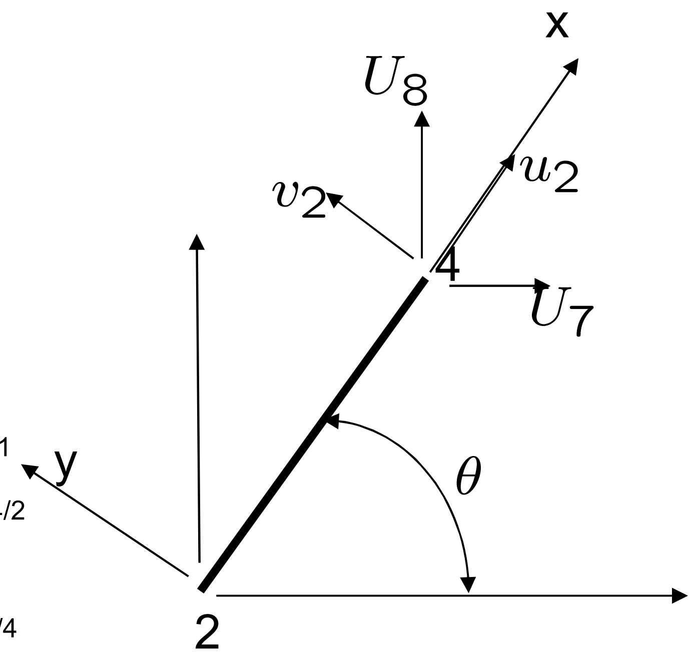
$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

where,

$$\boldsymbol{K} = \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}$$

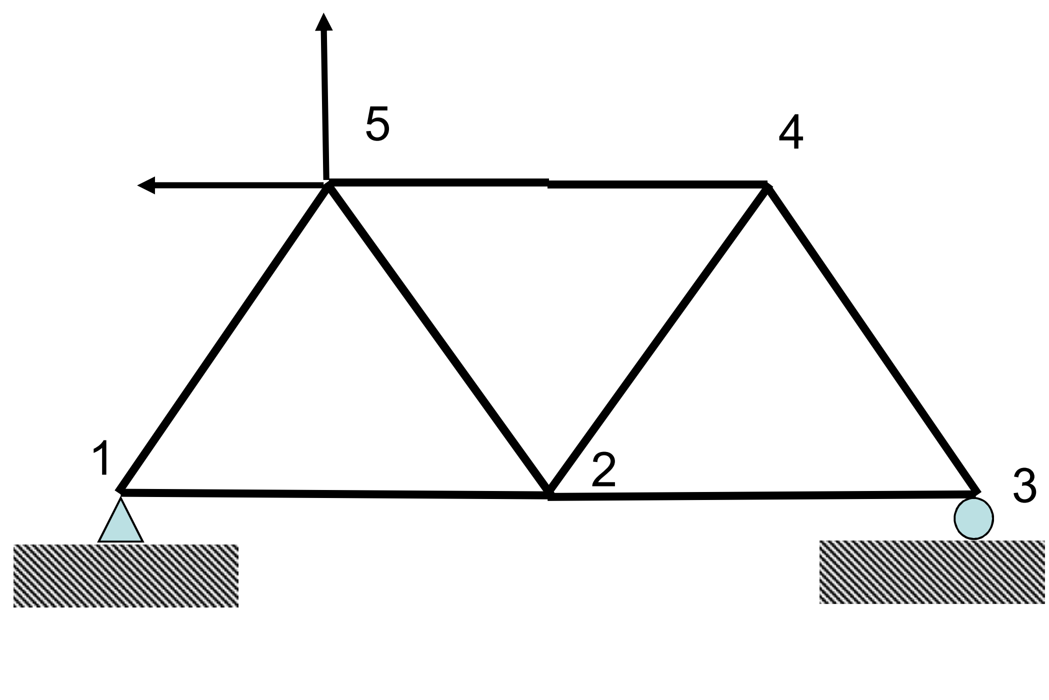
Assembly procedure

$$\begin{Bmatrix}F_{x1}\\F_{y1}\\F_{x2}\\F_{y2}\end{Bmatrix} = \boldsymbol{T}^T \begin{array}{c} \begin{array}{cccc}3/1 & 4/2 & 7/3 & 8/4\end{array}\\ \begin{pmatrix}k & 0 & -k & 0\\ 0 & 0 & 0 & 0\\ -k & 0 & k & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\end{array} \boldsymbol{T} \begin{array}{cc}\begin{Bmatrix}u_1\\v_1\\u_2\\v_2\end{Bmatrix} & \begin{array}{c}3/1\\4/2\\7/3\\8/4\end{array}\end{array}$$

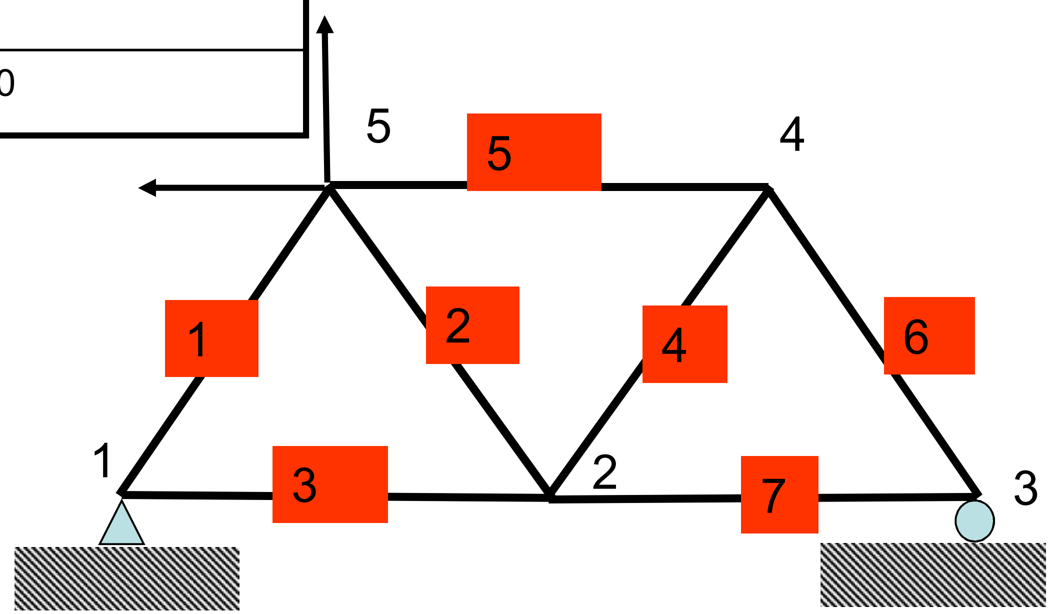


Eg: 2,2 → 4,4

3,1 → 7,3



Elem no	Local dof	Destination
1	1	1
	2	2
	3	9
	4	10
2	1	3
	2	4
	3	9
	4	10



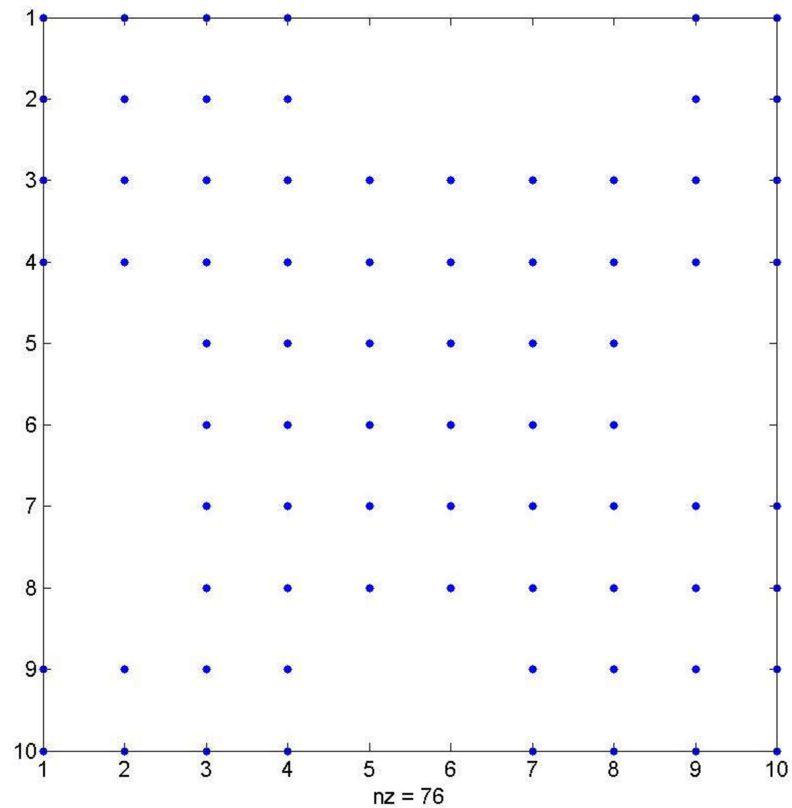
After assembling elements 1 and 2

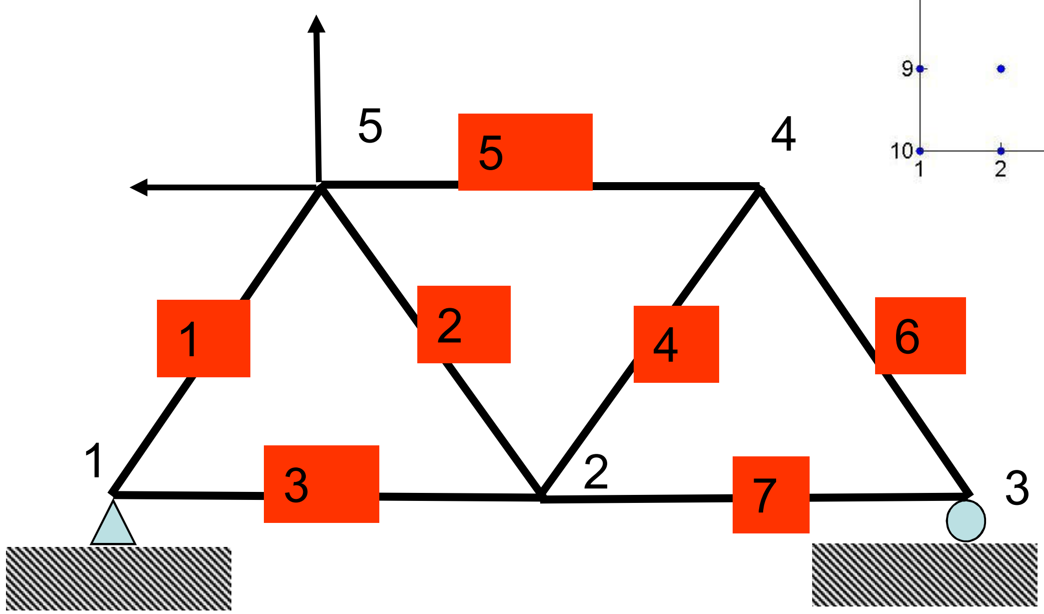
$$\begin{pmatrix}
K^1_{11} & K^1_{12} & 0 & 0 & 0 & 0 & 0 & 0 & K^1_{13} & K^1_{14} \\
K^1_{21} & K^1_{22} & 0 & 0 & 0 & 0 & 0 & 0 & K^1_{23} & K^1_{24} \\
0 & 0 & K^2_{11} & K^2_{12} & 0 & 0 & 0 & 0 & K^2_{13} & K^2_{14} \\
0 & 0 & K^2_{21} & K^2_{22} & 0 & 0 & 0 & 0 & K^2_{23} & K^2_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
K^1_{31} & K^1_{32} & K^2_{31} & K^2_{32} & 0 & 0 & 0 & 0 & K^1_{33} + K^2_{33} & K^1_{34} + K^2_{34} \\
K^1_{41} & K^1_{42} & K^2_{41} & K^2_{42} & 0 & 0 & 0 & 0 & K^1_{43} + K^2_{43} & K^1_{44} + K^2_{44}
\end{pmatrix}$$

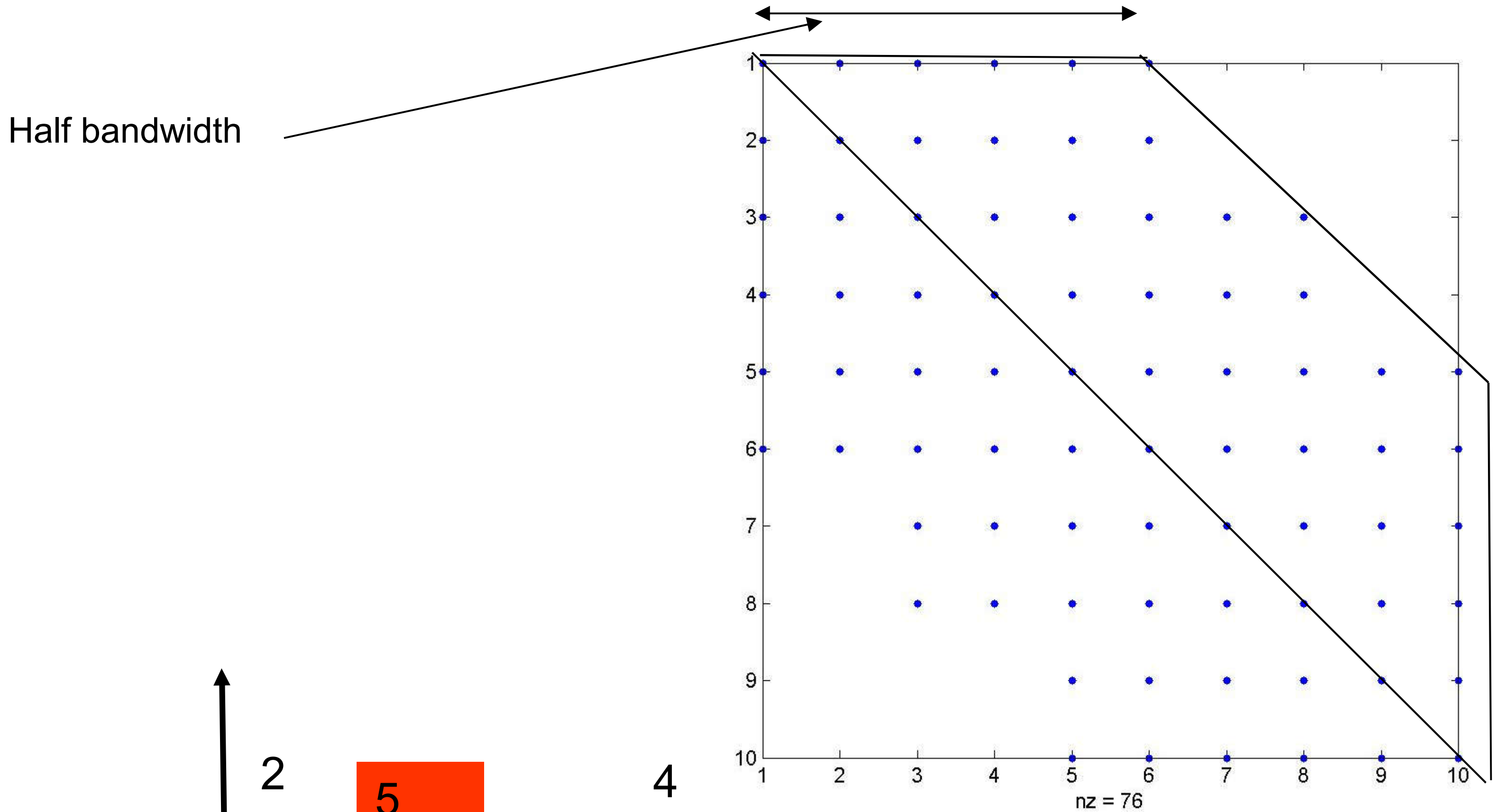
X+X	X+X	X	X					X	X
X+X	X+X	X	X					X	X
X	X	X+X+X	X+X+X			X	X	X	X
X	X	X+X+X	X+X+X			X	X	X	X
		X	X	X+X	X+X	X	X		
		X	X	X+X	X+X	X	X		
		X+X	X+X	X+X	X+X	X+X+X	X+X+X	X	X
		X+X	X+X	X+X	X+X	X+X+X	X+X+X	X	X
X	X	X	X			X	X	X+X+X	X+X+X
X	X	X	X			X	X	X+X+X	X+X+X

Stiffness matrix is symmetric, diagonally dominant, sparse and banded.

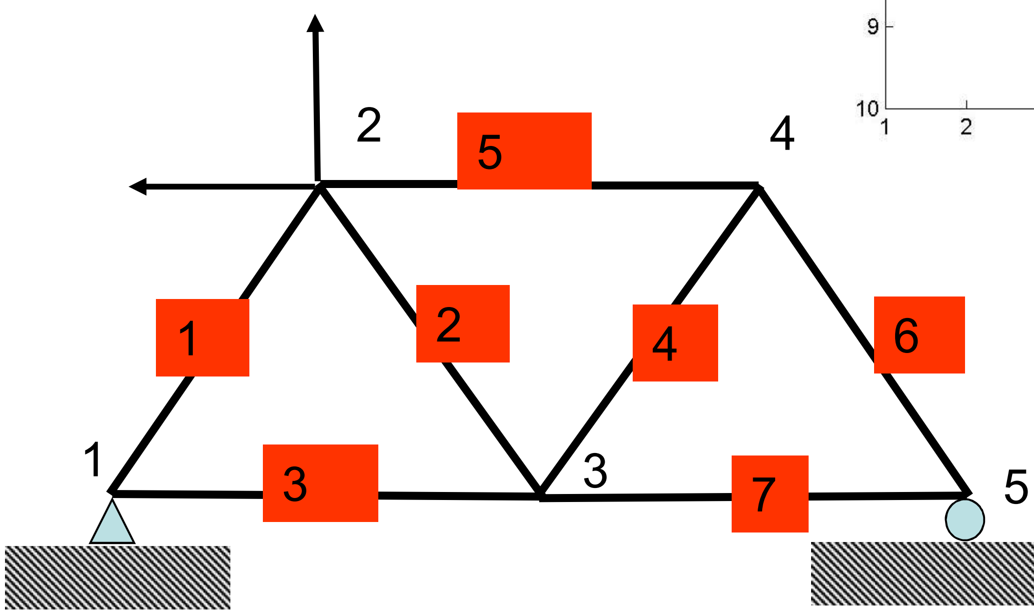
Sparsity pattern







Half bandwidth
1
2
3
4
5
6
7
8
9
10
nz = 76



1
2
3
4
5
1
2
3
4
5
6
7

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

Global displacement vector

Global force vector

Global stiffness matrix

Notes:

Global stiffness matrix is singular i.e. it has zero eigenvalues

Hence it cannot be inverted!

Boundary conditions

Force specified: eg. dof 9 and 10 in our example

Displacement specified: eg. dof 1,2,and 6 in our example

Both forces and displacements cannot be specified at the same dof.

$$\boldsymbol{F} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_1 \\ F_2 \end{array} \right\} \qquad \boldsymbol{U} = \left\{ \begin{array}{c} 0 \\ 0 \\ U_3 \\ U_4 \\ U_5 \\ 0 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \end{array} \right\}$$

Naïve approach for imposing displacement boundary conditions

$$\begin{pmatrix} L & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} & K_{19} & K_{1,10} \\ K_{21} & L & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} & K_{29} & K_{2,10} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} & K_{39} & K_{3,10} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & K_{49} & K_{4,10} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} & K_{5,10} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & L & K_{67} & K_{68} & K_{69} & K_{6,10} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} & K_{7,10} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & K_{89} & K_{8,10} \\ K_{91} & K_{92} & K_{93} & K_{94} & K_{95} & K_{96} & K_{97} & K_{98} & K_{99} & K_{9,10} \\ K_{10,1} & K_{10,2} & K_{10,3} & K_{10,4} & K_{10,5} & K_{10,6} & K_{10,7} & K_{10,8} & K_{10,9} & K_{10,10} \end{pmatrix}$$

L=a very large number. Also replace the corresponding dofs in the rhs vector by LXspecified displacement value

$$U_1 = L\delta_1 - \frac{K_{12}U_2 + \ldots K_{1,10}U_{10}}{L} \simeq \delta_1$$

The “proper” way of imposing displacement constraints

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= f_1 \\
a_2 x + b_2 y + c_2 z &= f_2 \\
a_3 x + b_3 y + c_3 z &= f_3
\end{aligned}
$$

Suppose $y = \delta$ (known).

$$a_1 x + c_1 z = f_1 - b_1 \delta$$

$$a_3 x + c_3 z = f_3 - b_3 \delta$$

$$
\begin{pmatrix} a_1 & 0 & c_1 \\ 0 & 1 & 0 \\ a_3 & 0 & c_3 \end{pmatrix}
\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}
=
\begin{Bmatrix} f_1 \\ \delta \\ f_3 \end{Bmatrix}
- \delta
\begin{Bmatrix} b_1 \\ 0 \\ b_3 \end{Bmatrix}
$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_6 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_6 \\ \vdots \end{Bmatrix} - \delta_1 \begin{Bmatrix} 0 \\ 0 \\ K_{15} \\ 0 \\ \vdots \end{Bmatrix} - \delta_2 \begin{Bmatrix} 0 \\ \vdots \\ K_{25} \\ 0 \\ \vdots \end{Bmatrix} - \delta_6 \begin{Bmatrix} 0 \\ \vdots \\ K_{65} \\ 0 \\ \vdots \end{Bmatrix}$$

Suppose dof k is specified

Traspose negative of the specified value X kth column to the right

Replace k th row and columns in the stiffness matrix to zero

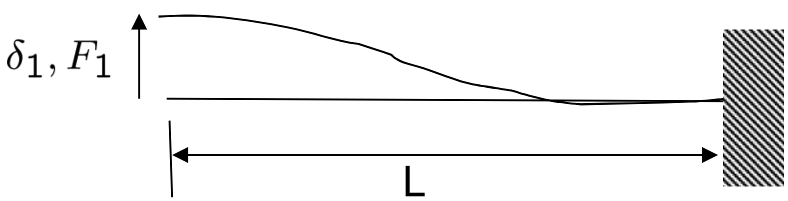
Replace K(k,k) by 1

Set F(k)=specified value

Repeat above steps for all specified dofs.

Stiffness matrix from basics: beam elements

$$\theta_1 = \theta_2 = \delta_2 = 0$$



$$EI\frac{d^4w}{dx^4} = p(x) = 0$$

$$w = \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

At $x = 0$ $\quad \frac{dw}{dx} = 0 \Rightarrow C_3 = 0$

At $x = L$ $\quad \frac{dw}{dx} = 0, w = 0 \Rightarrow$

$$C_1L = -2C_2, C_4 = -\frac{1}{6}C_2L^2$$

$$\begin{aligned}\frac{d^2w}{dx^2} &= \frac{M(x)}{EI}\\ \frac{dM}{dx} &= V\\ \Rightarrow \qquad & EI\frac{d^3w}{dx^3} = V. \end{aligned} \tag{1}$$

At $x = 0$

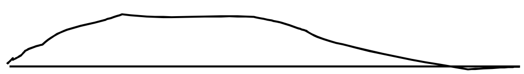
$$\frac{d^3w}{dx^3} = \frac{F_1}{EI} \Rightarrow C_1 = \frac{F_1}{EI}$$

Thus,

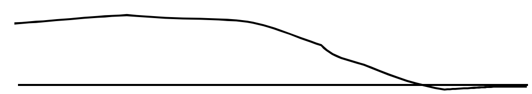
$$K_{11} = \frac{F_1}{\delta_1} = \frac{F_1}{w(x=0)} = \frac{12EI}{L^3}$$

Intuitive but not easy

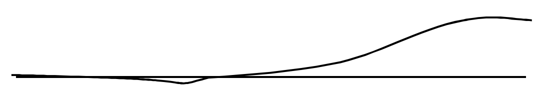
$$\boldsymbol{K}_L = \frac{EI}{L}\begin{pmatrix} 12/L^2 & -6/L & -12/L^2 & -6/L \\ & 4 & 6/L & 2 \\ & & 12/L^2 & 6/L \\ & & & 4 \end{pmatrix}$$



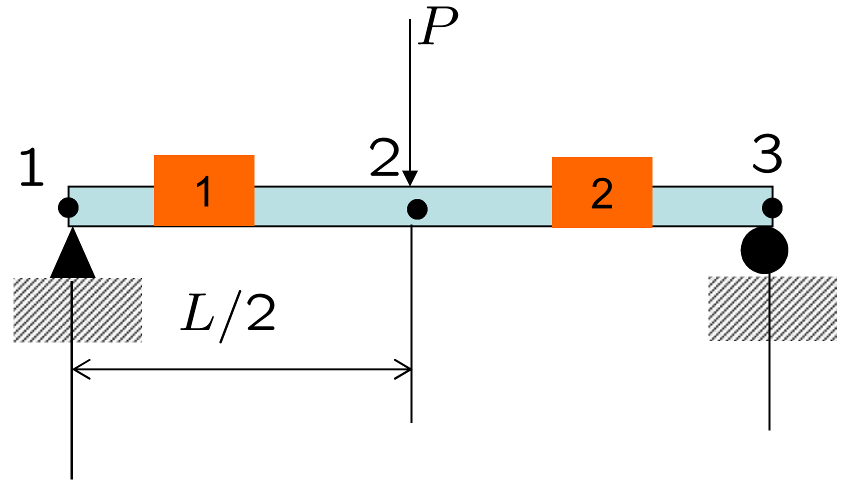
22



12



13



For each element

$$\boldsymbol{K}_L^{1/2} = \frac{8EI}{L^3} \begin{pmatrix} 12 & 6(L/2) & -12 & 6(L/2) \\ 6L/2 & 4(L/2)^2 & -6(L/2) & 2(L/2)^2 \\ -12 & -6(L/2) & 12 & -6(L/2) \\ 6(L/2) & 2(L/2)^2 & -6(L/2) & 4(L/2)^2 \end{pmatrix}$$

The assembled global stiffness

$$\boldsymbol{K}_G = \frac{8EI}{L^3}\begin{pmatrix} 12 & 6(L/2) & -12 & 6(L/2) & 0 & 0 \\ 6L/2 & 4(L/2)^2 & -6(L/2) & 2(L/2)^2 & 0 & 0 \\ -12 & -6(L/2) & 12+12 & -6(L/2)-6(L/2) & -12 & 6(L/2) \\ 6(L/2) & 2(L/2)^2 & -6(L/2)+6(L/2) & 4(L/2)^2+4(L/2)^2 & -6(L/2) & 2(L/2)^2 \\ 0 & 0 & -12 & -6(L/2) & 12 & -6(L/2) \\ 0 & 0 & 6(L/2) & 2(L/2)^2 & -6(L/2) & 4(L/2)^2 \end{pmatrix}$$

The global force vector

$$\boldsymbol{F} = \left\{ \begin{array}{c} 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \end{array} \right\}$$

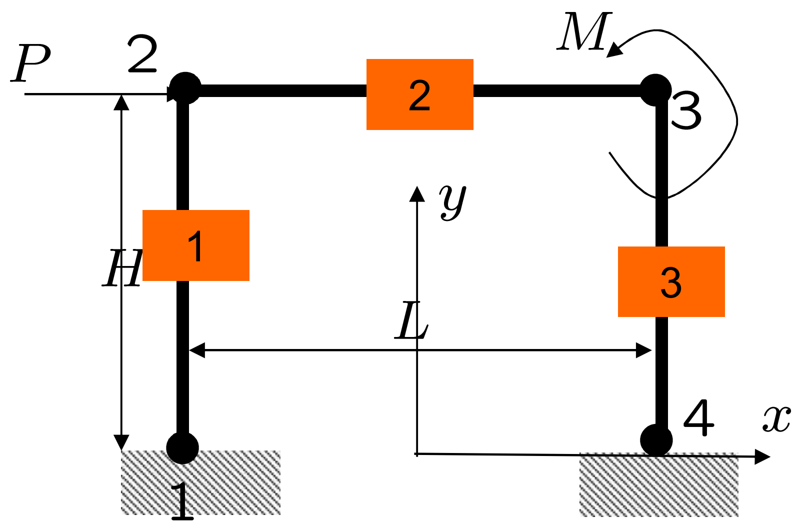
The global displacement vector

$$\boldsymbol{F} = \left\{ \begin{array}{c} w_1 = 0 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 = 0 \\ \theta_3 \end{array} \right\}$$

Finally we solve

$$\frac{8EI}{L^3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4(L/2)^2 & -6(L/2) & 2(L/2)^2 & 0 & 0 \\ 0 & -6(L/2) & 12+12 & -6(L/2)-6(L/2) & 0 & 6(L/2) \\ 0 & 2(L/2)^2 & -6(L/2)+6(L/2) & 4(L/2)^2+4(L/2)^2 & 0 & 2(L/2)^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6(L/2) & 2(L/2)^2 & 0 & 4(L/2)^2 \end{pmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

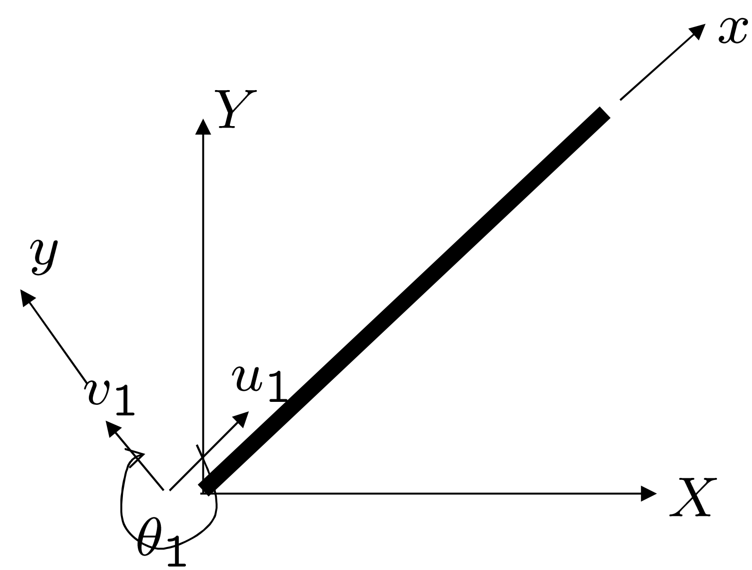
Beams in 2-d



The local stiffness for any element (say 2)

$$
\boldsymbol{K}_L^2 = \begin{pmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & -6EI/L^2 & -4EI/L & 0 & 6EI/L^2 & -2EI/L
\end{pmatrix}
$$

Transformation Matrix for a 2-d beam element



$$\boldsymbol{u}_1 = \left\{ \begin{array}{c} u_1 \\ v_1 \\ \theta_1 \end{array} \right\} = \left(\begin{array}{ccc} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{array} \right) \left\{ \begin{array}{c} U_1 \\ V_1 \\ \Theta_1 \end{array} \right\} = \boldsymbol{t}\boldsymbol{U}$$

$$\boldsymbol{u} = \left(\begin{array}{cc} \boldsymbol{t} & 0 \\ 0 & \boldsymbol{t} \end{array} \right) \boldsymbol{U} = \boldsymbol{T}\boldsymbol{U}$$

$$\boldsymbol{K}_G = \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}$$

Solving the equations

The stiffness matrix in the worst case is $N \times N$. We can take advantage of its sparsity, diagonal dominance and symmetry to design storage and solution methodologies. The basic problem is to start from

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}$$

and determine the unknown displacements $\boldsymbol{U}$ by

$$\boldsymbol{U} = \boldsymbol{K}^{-1}\boldsymbol{F}$$

We will look at a few numerical techniques for tackling this problem. In particular, we will look at direct methods. There are other iterative methods like conjugate gradient schemes that are suitable for FE calculations in large parallel computing environments.

Basic Gauss elimination:an example

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ -6 & 12 & 0 & -6 \\ -6 & 0 & 12 & -6 \\ 0 & -6 & -6 & 12 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 0 \\ 20 \\ 0 \end{Bmatrix}$$

Original problem

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & -2 & 10 & -6 \\ 0 & -6 & -6 & 12 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 20 \\ 40 \\ 0 \end{Bmatrix}$$

1st elimination:

Row2- Row 1*(-6/18)

Row3-Row1*(-6/18)

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & 0 & 9.6 & -7.2 \\ 0 & 0 & -7.2 & 8.4 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 20 \\ 44 \\ 12 \end{Bmatrix}$$

2nd elimination

Row3-Row2*(-2/10)

Row4-Row2*(-6/10)

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & 0 & 9.6 & -7.2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 20 \\ 44 \\ 45 \end{Bmatrix}$$

3rd elimination:

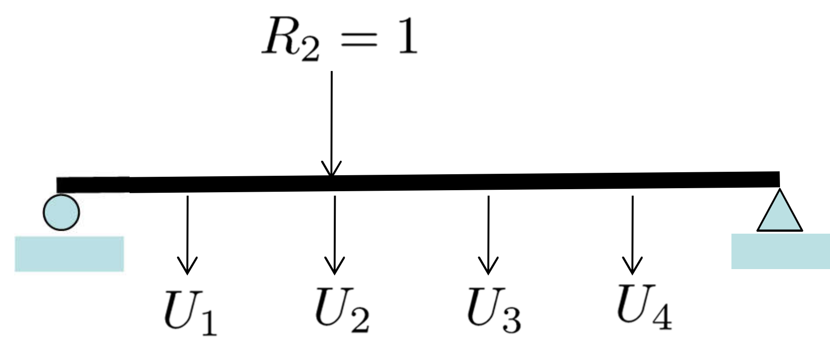
Row4-Row3*(-7.2/9.6)

To solve for $u_1, \dots u_4$, we now *back substitute*:

$$\begin{aligned}
u_4 &= 45/3 \\
u_3 &= (44 + 7.2U_4)/9.6 = 15.83 \\
u_2 &= (20 + 2U_3 + 6U_4)/10 = 14.17 \\
u_1 &= (60 + 6U_2 + 6U_3)/18 = 13.33
\end{aligned}$$

Note that, at any stage if we encounter a zero on the diagonal, the procedure will fail. It is advisable to resort to *pivoting* to take care of this issue.

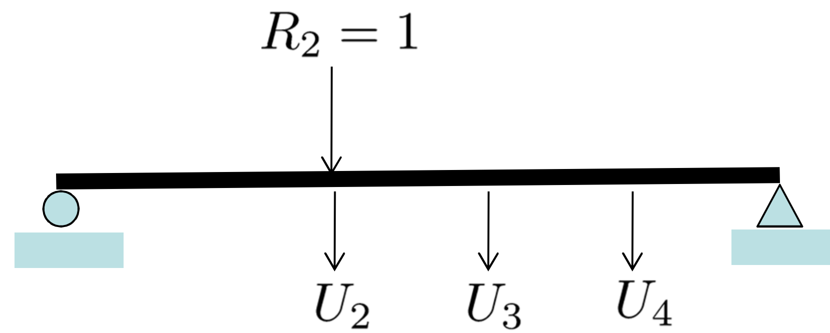
A physical picture of the Gauss elimination procedure



To illustrate the Gauss elimination procedure, consider a simply supported beam as shown in the figure. We wish to solve for dof's $U_1 \ldots U_4$. Assume that the final equation of equilibrium for this problem is

$$\begin{pmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

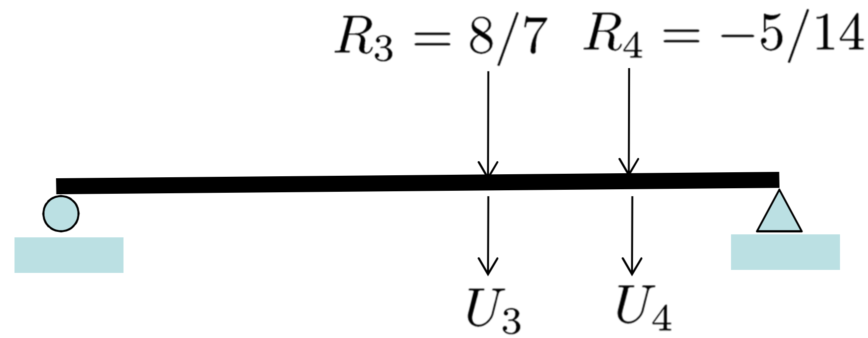
The first elimination will be equivalent to getting rid of the first dof U_1 by modifying the load accordingly.



Thus the physical problem being solved after the first elimination is

$$\begin{pmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{pmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

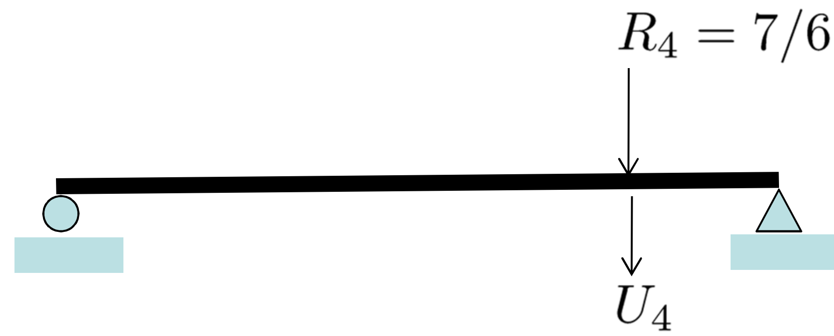
The physical problem now has 3 dof's.



The next elimination results in:

$$\begin{pmatrix} \frac{15}{7} & -\frac{20}{7} \\ -\frac{20}{7} & \frac{65}{14} \end{pmatrix} \left\{ \begin{array}{c} U_3 \\ U_4 \end{array} \right\} = \left\{ \begin{array}{c} \frac{8}{7} \\ -\frac{5}{14} \end{array} \right\}$$

Note now that a different loading has to be applied to maintain the equivalence with the original problem.



Finlly we get

$$\frac{5}{6}U_4 = \frac{7}{6}.$$

Thus, from a physical standpoint, we can arge that we should not encounter a zero diagonal element while performing a Gauss elimination on a FE stiffness matrix. This is because the i th diagonal at any stage of elimination represents the stiffness of the structure when $(i-1)$ dof's have been released. This stiffness should be positive if the structure is stable.

The numerics of Gauss elimination

In this chapter we will look at methods to solve the linear system of simultaneous equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where, the bold capitals denote matrices and bold small symbols denote vectors.

It is particularly easy to solve a $n \times n$ *upper triangular system* like

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \cdots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + a_{34}x_4 \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{nn}x_n &= b_n
\end{aligned}$$

Clearly,

$$x_n = \frac{b_n}{a_{nn}}.$$

Given that $a_{jj} \neq 0$, for $j = n-1, \ldots 1$, we can perform *backward substitution* to obtain:

$$x_j = \frac{1}{a_{jj}} \left(b_j - \sum_{k=j+1}^{n} a_{jk} x_k \right).$$

To count the number of operations, we assume that multiplication/division requires the longest time (though, division takes somewhat longer than multiplication in a computer) and addition/subtraction takes short times. In each step, we have $n-j$ multiplications and 1 division. Thus, for the entire back substitution process, we have

$$n^2 - \frac{n(n-1)}{2} + n = \frac{n^2}{2} + \frac{3n}{2},$$

multiplications/divisions. We also have $n-j-1$ additions and 1 subtraction per step, which leads to

$$\frac{n^2+n}{2}$$

addition/subtraction operations. The number of arithmetic operations in back substitution is:

$$\frac{n^2}{2} + O(n)$$

In the *Gauss elimination method*, a series of "elimination" steps transforms the system $\boldsymbol{Ax} = \boldsymbol{b}$ to an upper traingular system $\boldsymbol{Rx} = \boldsymbol{c}$, which is mathematically equivalent (may not be exactly equivalent due to round off errors) to the original system. Two operations *permutation of rows* and *scaling of rows* are performed in the elimination process. The process goes through the following steps:

- Form the augmented matrix

$$[\boldsymbol{A}|\boldsymbol{b}] \, .$$

- Set

$$\boldsymbol{A}^0 = \boldsymbol{A}, \boldsymbol{b}^0 = \boldsymbol{b}$$

- For $r \in 1, \ldots n$, find $a_{r1} \neq 0$. Such an element must exist as otherwise $\boldsymbol{A}$ will be singular. Permute 1st and r th rows. Let the new augmented matrix be

$$\left[\tilde{\boldsymbol{A}}^0 | \tilde{\boldsymbol{b}}^0\right] \, .$$

- For $j = 2, \ldots n$, multiply row 1 by q_{j1} and subtract the result from j th row, where,

$$q_{j1} = \frac{\tilde{a}^0_{j1}}{\tilde{a}^0_{11}}, a^1_{ji} = \tilde{a}^0_{ji} - q_{j1}\tilde{a}^0_{1i}, b^1_j = \tilde{b}^0_j - q_{j1}\tilde{b}^0_1 .$$

After the last step, the result is

$$[\boldsymbol{A}^1|\boldsymbol{b}^1] = \left(\begin{array}{cccc|c} \tilde{a}_{11}^0 & \tilde{a}_{12}^0 & \dots & \tilde{a}_{1n}^0 & b_1^0 \\ 0 & a_{21}^1 & \dots & a_{2n}^1 & b_2^1 \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1}^1 & \dots & a_{nn}^1 & b_n^1 \end{array}\right)$$

We can express the transformation $[\boldsymbol{A}|\boldsymbol{b}] \to [\tilde{\boldsymbol{A}}^0|\tilde{\boldsymbol{b}}^0]$ as $[\tilde{\boldsymbol{A}}^0|\tilde{\boldsymbol{b}}^0] = \boldsymbol{P}_1[\boldsymbol{A}|\boldsymbol{b}]$, where $\boldsymbol{P}$ is a *permutation matrix*. This matrix has the form

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & & & & & r & & \end{array} \\ \begin{array}{c} 1 \\ \\ \\ \\ \\ r \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots & \dots & \dots & 0 \\ & & & & \ddots & \vdots & \dots & 0 \\ 1 & & \dots & & & 0 & \dots & 0 \\ & & & \vdots & & & \ddots & \vdots \\ 0 & & & & & & & 1 \end{pmatrix} \end{array}$$

Similarly, we can represent $[\boldsymbol{A}^1|\boldsymbol{b}^1] = \boldsymbol{G}_1[\tilde{\boldsymbol{A}}^0|\tilde{\boldsymbol{b}}^0]$, where $\boldsymbol{G}_1$ is called a *Frobenius matrix* and is given by:

$$\begin{pmatrix} 1 & & & \\ -q_{21} & 1 & & \\ \vdots & & \ddots & \\ -q_{n1} & \dots & & 1 \end{pmatrix}$$

It can be shown that $\det \boldsymbol{P}_1 = \det \boldsymbol{G}_1 = 1$ and $\boldsymbol{P}_1^{-1} = \boldsymbol{P}_1$. The inverse of $\boldsymbol{G}_1$ is

$$\boldsymbol{G}_1^{-1} = \begin{pmatrix} 1 & & & \\ q_{21} & 1 & & \\ \vdots & & \ddots & \\ q_{n1} & \dots & & 1 \end{pmatrix}$$

Clearly, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has the same solution as $\boldsymbol{A}^1\boldsymbol{x} = \boldsymbol{b}^1$ as

$$\boldsymbol{G}_1\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{x} = \boldsymbol{G}_1\boldsymbol{P}_1\boldsymbol{b} = \boldsymbol{b}^1.$$

The series of steps

$$[\boldsymbol{A}|\boldsymbol{b}] \to [\boldsymbol{A}^1|\boldsymbol{b}^1] \to \cdots \to [\boldsymbol{A}^{n-1}|\boldsymbol{b}^{n-1}] \to [\boldsymbol{R}|\boldsymbol{c}],$$

leads to an upper triangular matrix $\boldsymbol{R}$, where

$$[\boldsymbol{R}|\boldsymbol{c}] = \boldsymbol{G}_{n-1}\boldsymbol{P}_{n-1}\ldots\boldsymbol{G}_1\boldsymbol{P}_1[\boldsymbol{A}|\boldsymbol{b}],$$

is equivalent to the original system. Here,

$$\boldsymbol{P}_i = \begin{array}{cc} & \begin{array}{ccccccccc} &&&i&&r&&& \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ r \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 &&&&&&&& \\ & \ddots &&&&&&& \\ && 1 &&&&&& \\ &&& 0 & \cdots & 1 & \cdots && \\ &&& \vdots & 1 &&&& \\ &&& \vdots &&&&& \\ &&& 1 & \cdots & 0 &&& \\ &&&&&& 1 && \\ &&&&&&& \ddots & \\ &&&&&&&& 1 \end{pmatrix} \end{array} \quad \boldsymbol{G}_i = \begin{array}{cc} & \begin{array}{ccccc} &&i&& \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ && 1 && \\ && -q_{i+1,i} & 1 & \\ && \vdots && \ddots \\ && -q_{ni} &&& 1 \end{pmatrix} \end{array}$$

The element $a_{r1} = \tilde{a}^0_{11}$ is called the *pivot element* and is chosen so that

$$|a_{r1}| = \max_{1 \le j \le n} |a_{j1}|.$$

Permutation of rows is called *column pivoting.* Though *total pivoting* where columns are also permuted is possible, it is expensive and rarely used.

We can perform an operation count for Gauss elimination. For simplicity, we will not count the steps required for pivoting. Thus,

- The kth elimination step involves

$$a^k_{ij} = a^{k-1}_{ij} - \frac{a^{k-1}_{ik}}{a^{k-1}_{kk}} a^{k-1}_{kj}, b^k_i = b^{k-1}_i - \frac{a^{k-1}_{ik}}{a^{k-1}_{kk}} b^{k-1}_k$$

  for $i, j = k \ldots n$.

  - This involves $n - k$ divisions to form $a^{k-1}_{ik}/a^{k-1}_{kk}$
  - followed by $(n-k)(n-k+1)$ multiplications.
  - Total number of multiplications/divisions over $k = 1, \ldots n-1$

$$\sum_{k=1}^{n-1} (n-k)(n-k+2) = \frac{2n^3 + 3n^2 - 5n}{6} = O(n^3)$$

Example

As an example, we solve the system

$$\begin{pmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 7 \\ 4 \end{Bmatrix}$$

Step 1: pivoting

$$\left(\begin{array}{ccc|c} \boxed{3} & 1 & 6 & 2 \\ 2 & 1 & 3 & 7 \\ 1 & 1 & 1 & 4 \end{array} \right)$$

Step 2: elimination

Instead of 0, a_{j1} can be stored.

$$\left(\begin{array}{ccc|c} 3 & 1 & 6 & 2 \\ 2/3 & 1/3 & -1 & 17/3 \\ 1/3 & \boxed{2/3} & -1 & 10/3 \end{array} \right)$$

Step 3: pivoting

$$\left(\begin{array}{ccc|c} 3 & 1 & 6 & 2 \\ 1/3 & 2/3 & -1 & 10/3 \\ 2/3 & 1/3 & -1 & 17/3 \end{array}\right)$$

Step 4: elimination

$$\left(\begin{array}{ccc|c} 3 & 1 & 6 & 2 \\ 1/3 & 2/3 & -1 & 10/3 \\ 2/3 & 1/2 & -1/2 & 4 \end{array}\right)$$

Step 5: back substitute

$$\begin{aligned} x_3 &= -8 \\ x_2 &= \frac{3}{2}\left(\frac{10}{3} - x_3\right) = -7 \\ x_1 &= \frac{1}{3}\left(2 - x_2 - 6x_3\right) = 19 \end{aligned}$$

Solution also using the demo code `gauss_elim.m`

After the elimination is complete, the transformed matrix can be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 2/3 & -1 \\ 0 & 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix},$$

which is same as the matrix we started with (with two rows permuted). This is called a LU decomposition.

Solution also using the demo code `LU.m`

Formally,

$$\boldsymbol{L} = \begin{pmatrix} l_{11} & & & 0 \\ l_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ l_{n1} & \dots & l_{n.n-1} & \end{pmatrix}, \boldsymbol{U} = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ & r_{22} & \dots & r_{2n} \\ & & \ddots & \\ 0 & & & r_{nn} \end{pmatrix}$$

are the factor in a LU decomposition such that

$$\boldsymbol{LU} = \boldsymbol{PA},$$

where,

$$\boldsymbol{P} = \boldsymbol{P}_{n-1} \dots \boldsymbol{P}_1.$$

In the i th elimination step, i.e. $[\boldsymbol{A}^{i-1}|\boldsymbol{b}^{i-1}] \rightarrow [\boldsymbol{A}^{i}|\boldsymbol{b}^{i}]$, we store the factors $q_{i+1,i} \ldots q_{n,i}$ of the Frobenius matrix $\boldsymbol{G}_i^{-1}$ in place of the zeros created. Thus, we work with the augmented matrix:

$$\left(\begin{array}{ccccccc|c}
r_{11} & r_{12} & \cdots & r_{1i} & r_{1,i+1} & \cdots & r_{1n} & c_1 \\
\lambda_{21} & r_{22} & \cdots & r_{2i} & r_{2,i+1} & \cdots & r_{2n} & c_2 \\
\lambda_{31} & \lambda_{32} & \cdots & r_{3i} & r_{3,i+1} & \cdots & r_{3n} & c_3 \\
\vdots & \vdots & \ddots & & \vdots & & \vdots & \vdots \\
\lambda_{i1} & \lambda_{i2} & & r_{ii} & r_{i,i+1} & & r_{in} & c_i \\
\lambda_{i+1,1} & \lambda_{i+1,2} & & \lambda_{i+1,i} & a^i_{i+1,i+1} & \cdots & a^i_{i+1,n} & b^i_{i+1} \\
\vdots & & & \vdots & \vdots & & \vdots & \vdots \\
\lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{ni} & a^i_{n,i+1} & \cdots & a^i_{nn} & b^i_n
\end{array}\right)$$

Here, the λ_{ki} are pertutations of q_{ki} and the matrix above finally leads to $\boldsymbol{L}$ and $\boldsymbol{U}$. Once the decomposition is done, two sets of equations need to be solved to obtain the solution $\boldsymbol{x}$:

$$\boldsymbol{Ly} = \boldsymbol{Pb}, \text{ and } \boldsymbol{Ux} = \boldsymbol{y}.$$

Exercise

The importance of the pivoting process can be understood from the following example:

$$\begin{pmatrix} 10^{-4} & 1 \\ 1 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

The exact solution to this is $x_1 = 1.00010001$ and $x_2 = 0.99989999$. Let us first do this without pivoting. The elimination step leads to

$$\left(\begin{array}{cc|c} 0.1 \cdot 10^{-3} & 0.1 \cdot 10^{1} & 0.1 \cdot 10^{1} \\ 0 & -0.1 \cdot 10^{5} & -0.1 \cdot 10^{5} \end{array} \right).$$

After backward substitution, the solution is $x_1 = 0, x_2 = 1$. With pivoting, the two rows are first swapped leading to:

$$\left(\begin{array}{cc|c} 0.1 \cdot 10^{1} & 0.1 \cdot 10^{1} & 0.2 \cdot 10^{1} \\ 0 & 0.1 \cdot 10^{1} & 0.1 \cdot 10^{1} \end{array} \right).$$

which gives the correct result $x_1 = x_2 = 1$.

Sometimes, even pivoting cannot lead us to the correct solution and a *scaling* is neccessary. For example, multiplying the first row of the matrix in this example by a large number (say 20000), we get the system

$$\begin{pmatrix} 2 & 20000 \\ 1 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \begin{Bmatrix} 20000 \\ 2 \end{Bmatrix}$$

With or without pivoting, this system gives the incorrect solution $x_1 = 0, x_2 = 1$. However, we can apply a *equilibriation step* to this system before solving it, by multiplying it with a diagonal matrix, i.e.

$$\boldsymbol{DAx} = \boldsymbol{Db},$$

where

$$D_{ii} = \sum_{j=1}^{n} \left(|a_{ij}|\right)^{-1}.$$

An even better method for stabilising matrices that have elements of very different size is to apply a *total pivoting* where both rows and columns are shifted in a equilibriation step.

Computation of inverse by LU decomposition

In principle, the inverse of a square matrix $\boldsymbol{A}$ can be computed by first LU decomposing $\boldsymbol{PA}$. Solution of the systems

$$\boldsymbol{L}\boldsymbol{y}^i = \boldsymbol{P}\boldsymbol{e}^i, \boldsymbol{U}\boldsymbol{x}^i = \boldsymbol{y}^i,$$

where, $\boldsymbol{e}^i$ are the Cartesian basis vectors, yields the inverse $\boldsymbol{A}^{-1} = [\boldsymbol{x}^1\ \boldsymbol{x}^2\ \ldots\ \boldsymbol{x}^n]$. The practical implementation is illustrated through an example.

Exercise

Consider again the matrix

$$\begin{pmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

Find its inverse. Step 1: pivoting

$$\left(\begin{array}{ccc|ccc} \boxed{3} & 1 & 6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step 2: elimination

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 6 & 1 & 0 & 0 \\ 0 & 1/3 & -1 & -2/3 & 0 & 1 \\ 0 & \boxed{2/3} & -1 & -1/3 & 1 & 0 \end{array}\right)$$

Step 3: Permute rows

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2/3 & -1 & -1/3 & 0 & 1 \\ 0 & 1/3 & -1 & -2/3 & 1 & 0 \end{array}\right)$$

Step 4: Next elimination

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2/3 & -1 & -1/3 & 0 & 1 \\ 0 & 0 & -1/2 & -1/2 & 1 & -1/2 \end{array}\right)$$

Step 4: Back substitution

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 0 & -5 & 12 & -6 \\ 0 & 2/3 & 0 & 2/3 & -2 & 2 \\ 0 & 0 & -1/2 & -1/2 & 1 & -1/2 \end{array}\right)$$

Back substitution leads to the inverse matrix:

$$A^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ 1 & -3 & 3 \\ 1 & -2 & 1 \end{pmatrix}$$

Matrix norms and conditioning

Assuming a basic exposure to linear algebra, we will quickly recapitulate some essential aspects. We have a vector space $\mathbb{K}^n$ of all n dimensional vectors. That is, we deal with vectors like $\boldsymbol{x} \in \mathbb{K}^n$ where, $\boldsymbol{x}^T = \langle x_1 \; x_2 \; \ldots \; x_n \rangle$, where x_i are components of the vector with respect to a fixed Cartesian basis $\boldsymbol{e}_i$,

$$\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i.$$

A mapping $||\cdot|| : \mathbb{K}^n \rightarrow \mathbb{R}$ is called a *vector norm* if it satisfies:

- *Definiteness:* $||\boldsymbol{x}|| \geq 0, ||\boldsymbol{x}|| = 0, \Rightarrow \boldsymbol{x} = 0, \forall \boldsymbol{x} \in \mathbb{K}^n$.
- *Homogenity:* $||\alpha \boldsymbol{x}|| = |\alpha| ||\boldsymbol{x}||, \alpha \in \mathbb{K}, \boldsymbol{x} \in \mathbb{K}^n$
- *Triangle inequality:* $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n$

The norm can be defined over any vector space V over $\mathbb{K}$. The resulting pair $\{V, ||\cdot||\}$ is called a *normed vector space*

Examples of vector norms are the L_2 norm

$$||\boldsymbol{x}||_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2},$$

or the more general L_p norm

$$||\boldsymbol{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

More useful examples include the L_∞ norm and the L_1 norm defined as

$$||\boldsymbol{x}||_\infty = \max_{i=1,\ldots n} |x_i|, ||\boldsymbol{x}||_1 = \sum_{i=1}^{n} |x_i|.$$

A mapping $(\boldsymbol{x}, \boldsymbol{y}) : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is called a *scalar product* if it satisfies

- *Symmetry:* $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x}) \;\; \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}$
- *Linearity:* $(\alpha \boldsymbol{x} + \beta \boldsymbol{y}, \boldsymbol{z}) = \alpha(\boldsymbol{x}, \boldsymbol{z}) + \beta(\boldsymbol{y}, \boldsymbol{z})$, $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{K}^n$, $\alpha, \beta \in \mathbb{K}$
- *Definiteness:* $(\boldsymbol{x}, \boldsymbol{x}) \in \mathbb{R}, (\boldsymbol{x}, \boldsymbol{x}) > 0 \; \boldsymbol{x} \in \mathbb{K}^n/\{0\}$.

We will mostly use the Euclidean scalar product

$$(\boldsymbol{x}, \boldsymbol{y})_2 = \sum_{j=1}^{n} x_j y_j.$$

Two vectors are orthogonal if

$$(\boldsymbol{x}, \boldsymbol{y})_2 = 0.$$

A set of vectors in $\{\boldsymbol{a}^1, \boldsymbol{a}^2 \ldots \boldsymbol{a}^n\}$, with $\boldsymbol{a}^i \neq 0$ is called an *orthogonal basis* if $(\boldsymbol{a}^i, \boldsymbol{a}^j)_2 = 0$ for $i \neq j$. Further, they are called an *orthonormal basis* if, additionally, $(\boldsymbol{a}^k, \boldsymbol{a}^k) = 1$ for $k = 1, \ldots n$.

With an orthonormal basis in place, we can easily see that

$$(\boldsymbol{x}, \boldsymbol{a}^i)_2 = \left(\sum \alpha_j \boldsymbol{a}^j, \boldsymbol{a}^i\right)_2 = \alpha_i.$$

This leads to the representation

$$\boldsymbol{x} = \sum_{i=1}^{n} \left(\boldsymbol{x}, \boldsymbol{a}^i\right)_2 \boldsymbol{a}^i.$$

Further, it can easily be seen that

$$||\boldsymbol{x}||_2^2 = (\boldsymbol{x}, \boldsymbol{x})_2 = \sum_{i=1}^{n} \left|(\boldsymbol{x}, \boldsymbol{a}^i)_2\right|^2.$$

Consider the vector space of all $n \times n$ matrices $\boldsymbol{A} \in \mathbb{K}^{n \times n}$. The norm induced by a vector norm $||\cdot||$ is

$$||\boldsymbol{A}|| = \max\left(\frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||} : ||\boldsymbol{x}|| \neq 0\right),$$

for $\boldsymbol{x} \in \mathbb{K}^n$. Through this norm, we are trying to determine the direction that is amplified most by $\boldsymbol{A}$.

A matrix norm satisfies

$$||\boldsymbol{A}\boldsymbol{x}|| \leq ||\boldsymbol{A}||||\boldsymbol{x}||, \boldsymbol{x} \in \mathbb{K}^n, \boldsymbol{A} \in \mathbb{K}^{n \times n}$$

and,

$$||\boldsymbol{A}\boldsymbol{B}|| \leq ||\boldsymbol{A}||||\boldsymbol{B}||, \boldsymbol{A}, \boldsymbol{B} \in \mathbb{K}^{n \times n}.$$

Further, a *natural matrix norm* satisfies

$$||\boldsymbol{I}|| = 1.$$

The natural matrix norm generated from the Euclidean norm is called the *spectral norm.* Note that it is not

$$||\boldsymbol{A}|| = \left(\sum_{j,k=1}^{n} |a_{jk}^2| \right)^{1/2},$$

as, for this norm, $||\boldsymbol{I}|| = \sqrt{n}$. If the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are λ, then the spectral norm is

$$||\boldsymbol{A}||_2 = \max |\lambda| \,.$$

Now, we can do an error analysis of the linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

This implies that if the matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ are faulty by $\delta\boldsymbol{A}$ and $\delta\boldsymbol{b}$ respectively, we wish to find the error $\delta\boldsymbol{x}$ in the solution of the system

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}},$$

with $\tilde{\boldsymbol{A}} = \boldsymbol{A} + \delta\boldsymbol{A}, \tilde{\boldsymbol{b}} = \boldsymbol{b} + \delta\boldsymbol{b}$ and $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \delta\boldsymbol{x}$.

We state the so-called *perturbation theorem* without proof. Let $\boldsymbol{A} \in \mathbb{K}^{n\times n}$ be an invertible matrix such that $||\delta\boldsymbol{A}|| \leq ||\boldsymbol{A}^{-1}||^{-1}$, then, the perturbed matrix $\tilde{\boldsymbol{A}} = \boldsymbol{A} + \delta\boldsymbol{A}$ is also invertible and

$$\frac{||\delta\boldsymbol{x}||}{||\boldsymbol{x}||} \leq \frac{\mathrm{cond}\boldsymbol{A}}{1 - \mathrm{cond}\boldsymbol{A}||\delta\boldsymbol{A}||||\boldsymbol{A}||}\left\{\frac{||\delta\boldsymbol{b}||}{||\boldsymbol{b}||} + \frac{||\delta\boldsymbol{A}||}{||\boldsymbol{A}||}\right\},$$

where $\mathrm{cond}\boldsymbol{A} = ||\boldsymbol{A}||||\boldsymbol{A}^{-1}||$.

Further, it can be shown that for the spectral norm

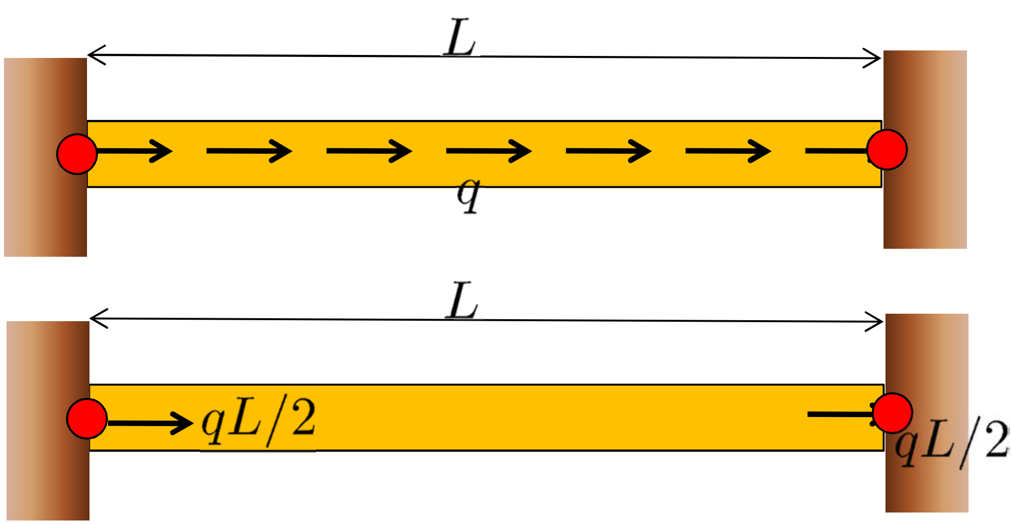
$$\mathrm{cond}_2\boldsymbol{A} = \frac{|\lambda_{max}|}{|\lambda_{min}|},$$

where λ_{max} and λ_{min} are the largest and smallest eigenvalues of $\boldsymbol{A}$.

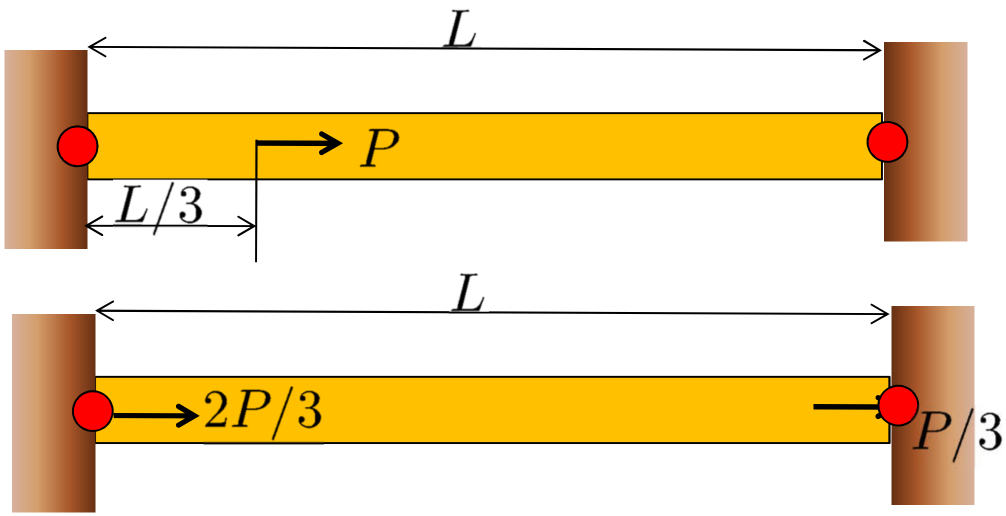
For cases where $\mathrm{cond}||\boldsymbol{A}||||\boldsymbol{A}^{-1}|| << 1$, for $\mathrm{cond}\boldsymbol{A} \sim 10^s$, and relative errors in $\boldsymbol{A}$ and $\boldsymbol{b} \sim 10^{-k}$, the relative error in $\boldsymbol{x}$ will be 10^{s-k}.

Loads and stresses

So far we have applied only *nodal* point loads to a structure. What happens when there are *distributed* loads? We need to convert the distributed load to nodal loads using an ad-hoc or a consistent scheme. Ways of calculating consistent nodal loads will be dealt with later. Now we look at some 'intuitive' ways of apportioning distributed loads to the nodes.



Both the systems have the same reaction forces at the nodes, qL/A at each end acting in the direction opposite to the externally applied load. This is an example of a consistent nodal load corresponding to a distributed load.

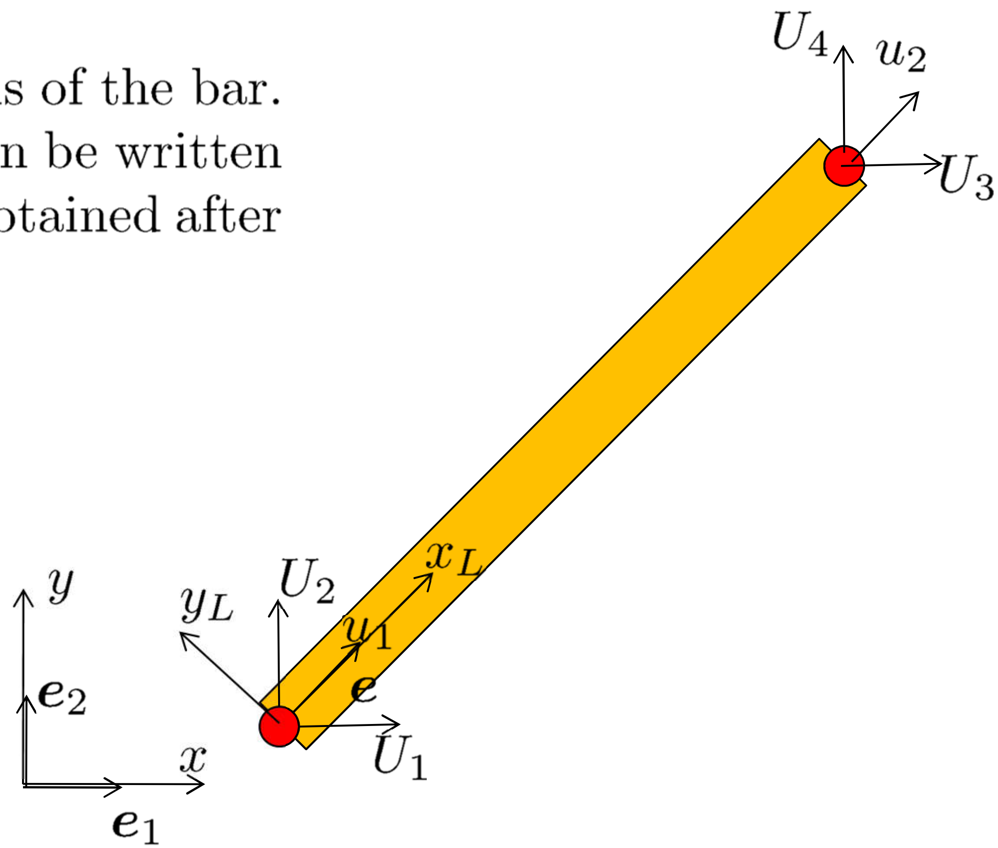

L
P
$L/3$
L
$2P/3$
$P/3$

Let $\boldsymbol{e}$ be the unit vector along the axis of the bar. The displacement at nodes 1 and 2 can be written in terms of the global displacements obtained after solving:

$$
\begin{aligned}
u_1 \boldsymbol{e} &= U_1 \boldsymbol{e}_1 + U_2 \boldsymbol{e}_2, \\
u_2 \boldsymbol{e} &= U_3 \boldsymbol{e}_1 + U_4 \boldsymbol{e}_2.
\end{aligned}
$$

The stress in the bar is:

$$\sigma = \frac{E}{L} \left(u_2 - u_1 \right)$$

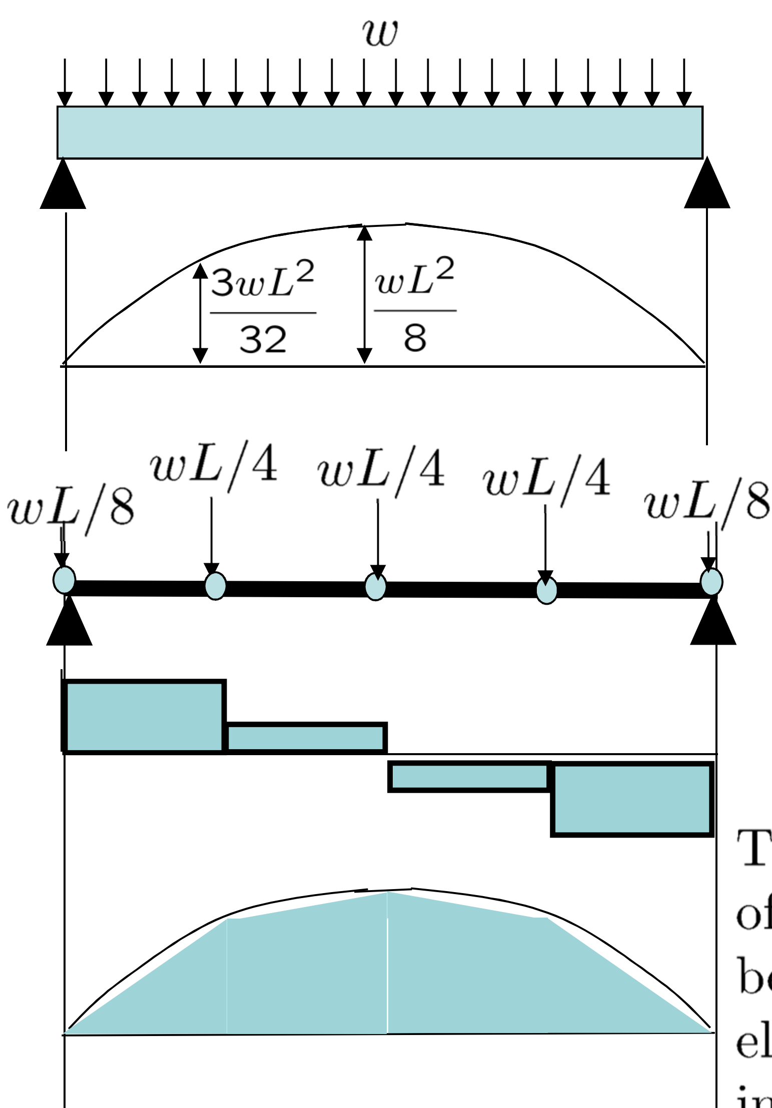


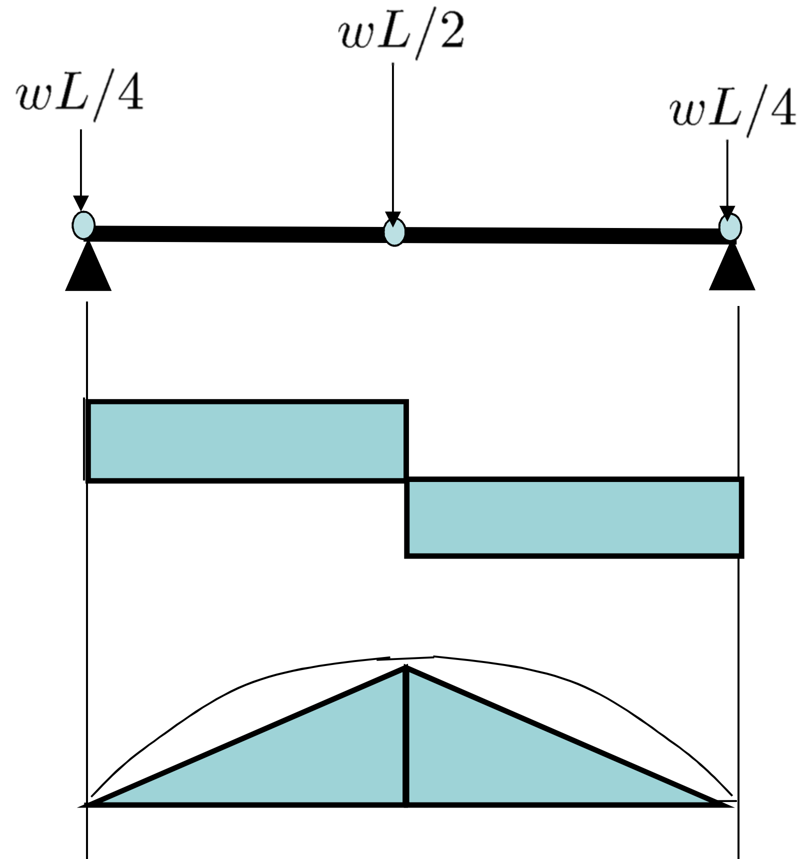
Thus the stress can be written in terms of the global displacements as:

$$\sigma = \frac{E}{L} \left[(U_3 - U_1) \boldsymbol{e}_1 \cdot \boldsymbol{e} + (U_4 - U_2) \boldsymbol{e}_2 \cdot \boldsymbol{e} \right]$$

Note here that $\boldsymbol{e} \cdot \boldsymbol{e}_1$ and $\boldsymbol{e} \cdot \boldsymbol{e}_2$ are the direction cosines of the axis of the bar with respect to the global axes.

For a beam with distributed loads, again the same strategy is adopted to determine the consistent nodal loads. In the following example, we use two beam elements to discretise the given structure on the right. Both systems yield the same reaction forces.



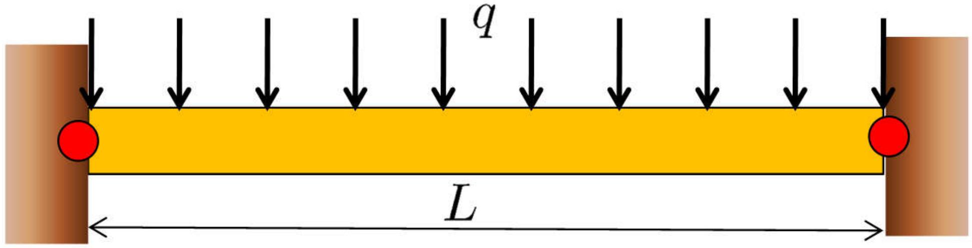


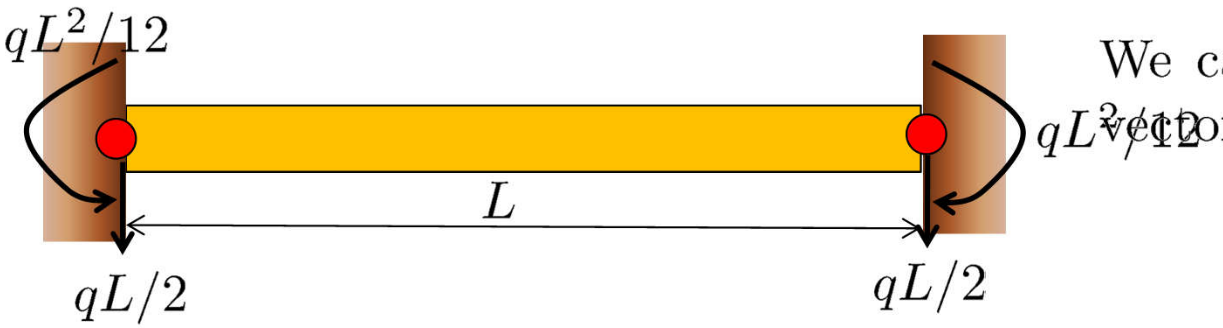
The bending moment and the shear force diagrams of the nodally loaded beams differ from the actual beam but become more accurate as the number of elements into which the structure is discretised is increased.

To find the bending moment at the ends, we can solve:

$$EI\frac{d^4w}{dx^4} = q,$$

to see that the bending moments at $x = 0, L$ are $qL^2/12, -qL^2/12$.





The consistent nodal load vector is

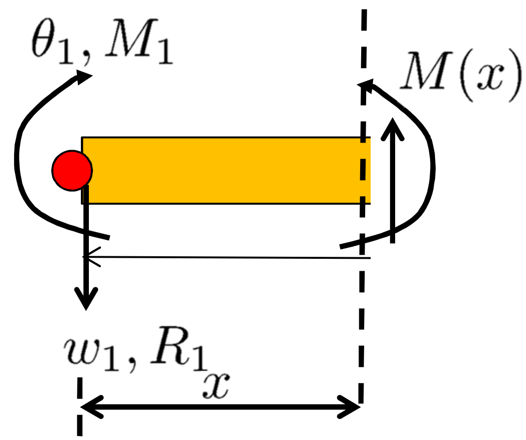
$$\boldsymbol{F} = \left\{ \begin{array}{c} -qL/2 \\ -qL^2/12 \\ -qL/2 \\ qL^2/12 \end{array} \right\}$$

We can also use a lumped force vector

$$\boldsymbol{F} = \left\{ \begin{array}{c} -qL/2 \\ 0 \\ -qL/2 \\ 0 \end{array} \right\}$$

where, the moments are not accounted for. This is actually not a bad choice at all.

After we have solved for all degrees of freedom, for any element e we can isolate its nodal degrees of freedom. Let these be $\boldsymbol{u}^T = \langle w_1 \; \theta_1 \; w_2 \; \theta_2 \rangle$.

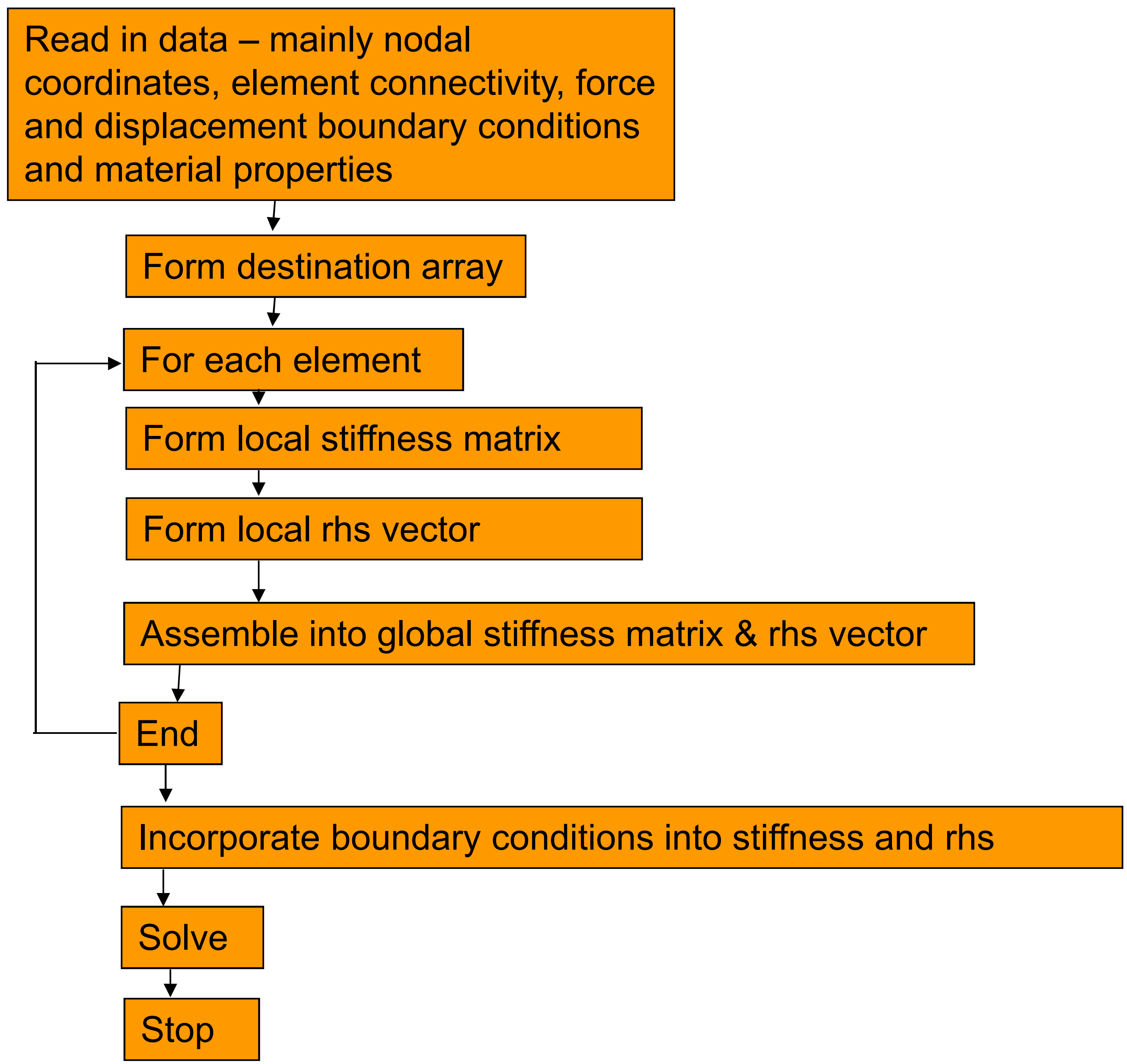


$$
\begin{aligned}
M(x) &= -\left(-R_1 x + M_1\right) \\
&= \left(\frac{12EI}{L^3}x + \frac{6EI}{L^2}\right) w_1 + \left(-\frac{6EI}{L^2}x - \frac{4EI}{L}\theta_1\right)\theta_1 + \\
&\quad \left(-\frac{12EI}{L^3}x - \frac{6EI}{L_2}\right) w_2 + \left(-\frac{6EI}{L^2}x - \frac{2EI}{L}\right)\theta_2
\end{aligned}
$$

Then, stress over the cross section at x is

$$\sigma(x) = \frac{My}{I}.$$

The FEM scheme of things



Assignment 1
Basic FE procedures

Pr.1: Use the Matlab ode solver `ode23` to solve the problem of the pendulum done in the class. Recall that the problem is governed by the ordinary differential equation

$$\ddot{\theta} + \frac{g}{l}\theta = 0,$$

with $\theta = \theta_0$ and $\dot{\theta} = v_0$ at $t = 0$. Take, $l = 1$, $g = 10$, and the initial position to be 10^0 and initial velocity to be 0. Plot the numerical solution $\theta(t)$ and the closed form solution together in a graph of θ versus t.

Pr.2: One dimensional heat conduction along a slender bar lying in $0 \leq x \leq L$, is governed by the equation

$$kA\frac{d^2T}{dx^2} + hP(T_\infty - T) = 0,$$

where A is the cross sectional area and P the perimeter of the cross section of the bar. At $x = 0$, the temperature is held at T_0 while at $x = L$, $dT/dx + (h/k)T = 0$.