# ME361 – Manufacturing Science Technology

Measurements and Metrology

Dr. Mohit Law

mlaw@iitk.ac.in

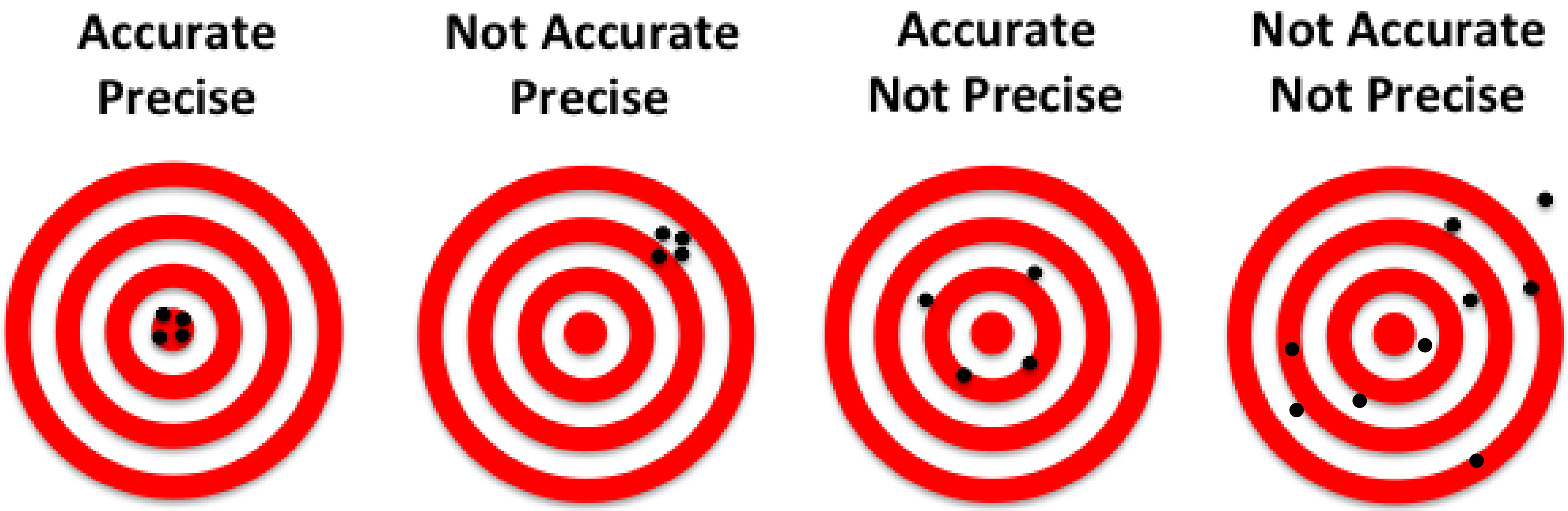
# Measurements and metrology

Metrology is concerned with the establishment, reproduction, conservation and transfer of units of measurements and their standards

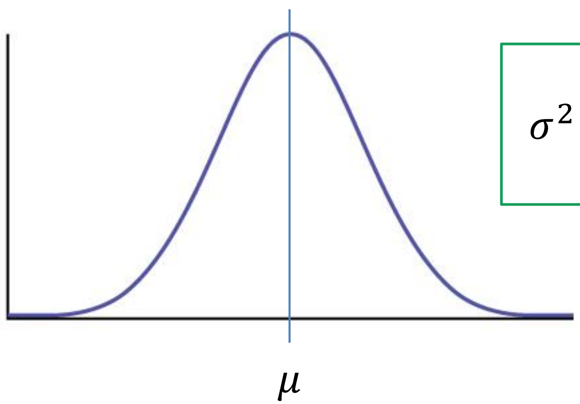
- Measurements and equipment for measurements
- Standards
- Limits, fits and gauges
- Testing and calibration
- Machine tool metrology
- Measuring machines (CMMs, etc.)
- Statistical quality control
- Regression analysis and response models
- Design of experiments

*Since we cannot know all that there is to be known about anything, we ought to know a little about everything* - Blaise Pascal

# Accuracy and precision



# Distribution of measured data




Mean

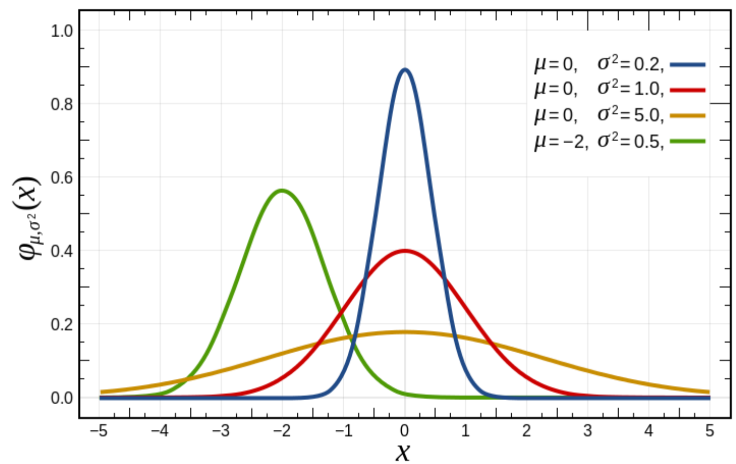
$$\mu = \frac{\sum_{i=1}^{n} y_i}{n}$$

Variance

$$\sigma^2 = \sum_{i=i}^{n} \frac{(y_i - \mu)^2}{n-1}$$

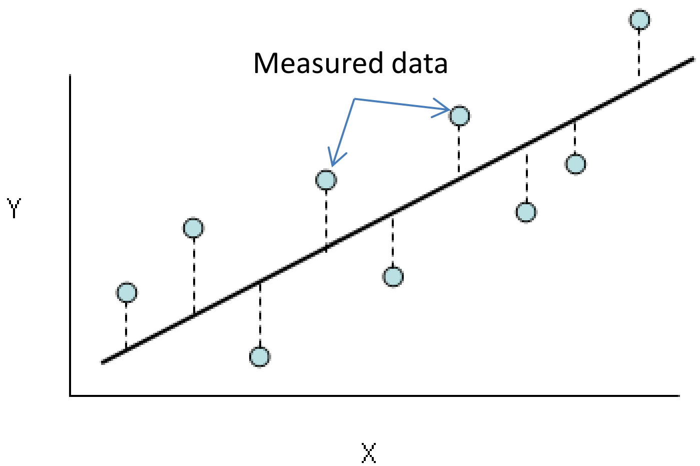
Probability density function

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$



# Regression analysis – method of least squares

Regression analysis is a statistical process for estimating the relationships among independent and dependent (response) variables. i.e. fit a model to measured data.




Suggested relationship between independent and dependent variables

$$y = b_0 + b_1 x + \epsilon$$

$\epsilon \rightarrow$ experimental error with $\mu = 0$ and variance, $\sigma^2$

Predicted response with fitted values, $\hat{b}_0$ and $\hat{b}_1$

$$\hat{y} = \hat{b}_0 + \hat{b}_1 x$$

Least square method to minimize the sum of squared deviations of the responses from their vertical projections onto the fitted line

$$SS(\hat{b}_0,\hat{b}_1) = \sum_n (y_i - \hat{y}_i)^2 = \sum_n \left(y_i - \hat{b}_0 - \hat{b}_1 x_i\right)^2$$

# Regression analysis – method of least squares

Least square method to minimize the sum of squared deviations of the responses from their vertical projections onto the fitted line

$$\boxed{SS(\hat{b}_0,\hat{b}_1) = \sum_n (y_i - \hat{y}_i)^2 = \sum_n \left(y_i - \hat{b}_0 - \hat{b}_1 x_i\right)^2}$$

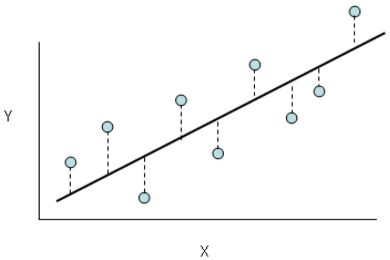
To find fitted values, $\hat{b}_0$ and $\hat{b}_1$, diff w.r.t $\hat{b}_0$ and $\hat{b}_1$

$$\boxed{\sum_n \hat{b}_0 + \hat{b}_1 \sum_n x_i = \sum_n \hat{y}_i} \quad \text{(a)}$$

$$\boxed{\hat{b}_0 \sum_n x_i + \hat{b}_1 \sum_n {x_i}^2 = \sum_n x_i y_i} \quad \text{(b)}$$

Solve (a) and (b) simultaneously to obtain the least-squares fit

$$\boxed{\hat{y} = \hat{b}_0 + \hat{b}_1 x}$$



# Response models from design of experiments

- Any statistical process control measure for metrology and quality improvement assumes response is known

- What is a response model?
- Can we design experiments to observe effects of changing parameters and variables on the response?
- Do parameters interact with each other to influence overall response? Can experiments be designed to study these interactions?
- Can experiments be designed to reveal statistical and random errors?

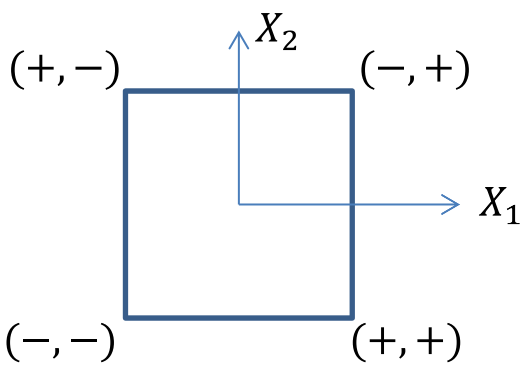
Design of experiments

# Design of experiments (DOE)

Full two – level factorial design

- Considers two levels of each variable
- Levels are commonly referred to as low (-) and high (+)
- When there are $k$ factors, there exist $2^k$ combinations of high and low levels, i.e. total number of tests will be $2^k$
- Low level is coded as $-1$ and high level as $+1$

Consider 2 factors (variables, $X_1$ and $X_2$) with 2 levels $k = 2$; Total tests = $2^k = 4$




| | Coded variables | | | Actual variables | | |
|---|---|---|---|---|---|---|
| | Main effects | | Interaction effects | Main effects | | Interaction effects |
| Test | $x_1$ | $x_2$ | $x_1x_2$ | $X_1$ | $X_2$ | $X_1X_2$ |
| 1 | $-1$ | $-1$ | $+1$ | $-1$ | $-1$ | $+1$ |
| 2 | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ | $-1$ |
| 3 | $-1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ |
| 4 | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ |

# DOE - $2^3$ factorial design

Problem statement:

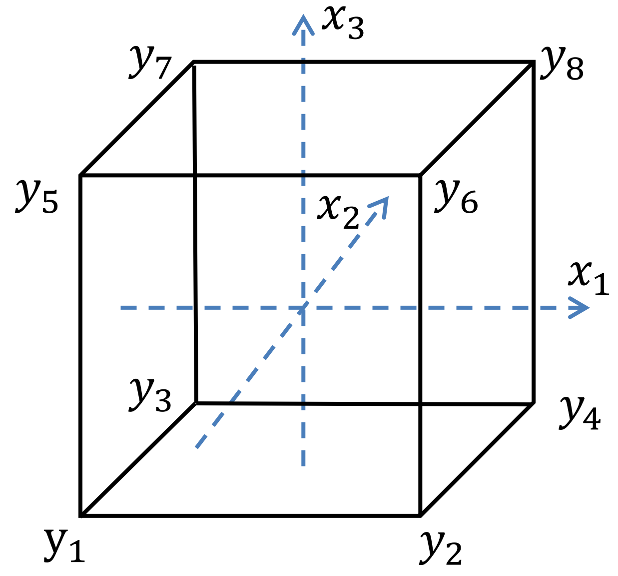
Desired to determine the effects of spindle speed, $\mathrm{X}_1 \rightarrow x_1$ (RPM), feed rate, $\mathrm{X}_2 \rightarrow x_2$(mm/rev) and corner radius, $\mathrm{X}_3 \rightarrow x_3$(mm) on the surface finish of a turned surface. Each has two levels, high (+1) and low (-1).

Visualize the design as a cube

→ each corner represents a test, and its response $y_i$

→ Each of the three dimensions represents one of the three coded variables that are centered about zero

→ each plane represents a $2^2$ experiment



# DOE - $2^3$ factorial design – main effects

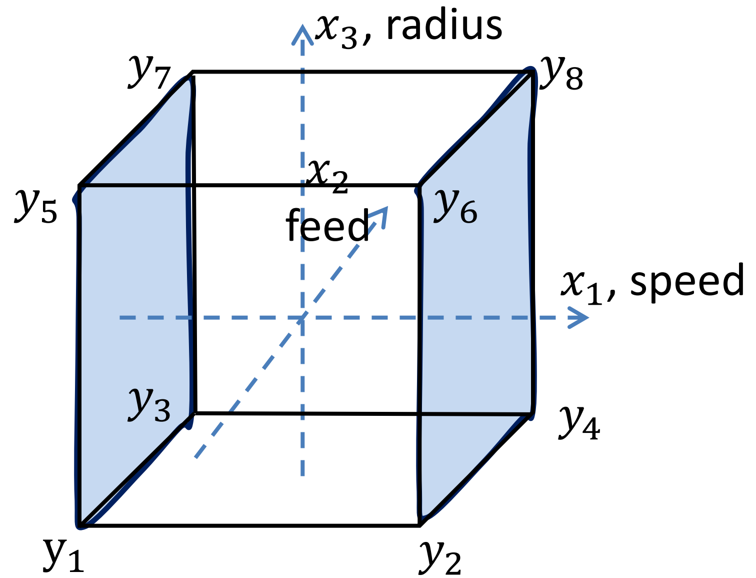
How to evaluate the individual and joint influences of speed, feed and corner radius on the surface finish?

You may ask:

How is the surface finish influenced by a change in any one of the three variables while the other two are held constant?

<u>Consider this question for the speed ($x_1$).</u>

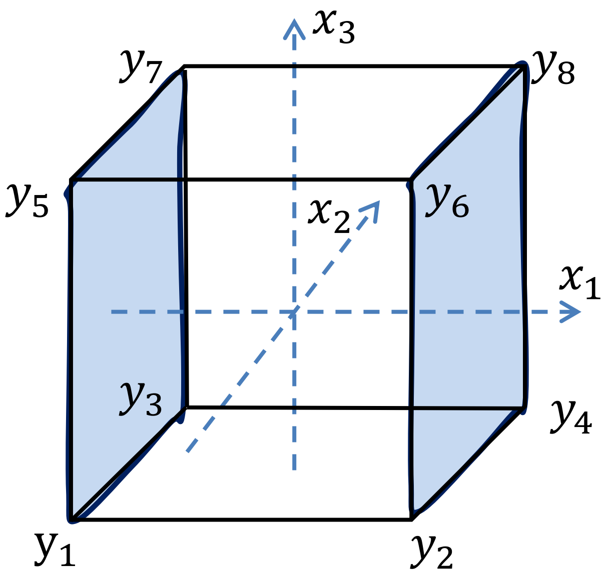
- Four contrasts of test results that indicate how surface finish changes when speed is changed from low to high, keeping feed and corner radius fixed.
    - $y_1$ and $y_2$
    - $y_3$ and $y_4$
    - $y_5$ and $y_6$
    - $y_7$ and $y_8$



# DOE - $2^3$ factorial design – main effects

<u>Speed $x_1$ is varied while feed and corner radius are held constant.</u>

Geometrical representation




Following pairs may be compared to see the effect of the speed

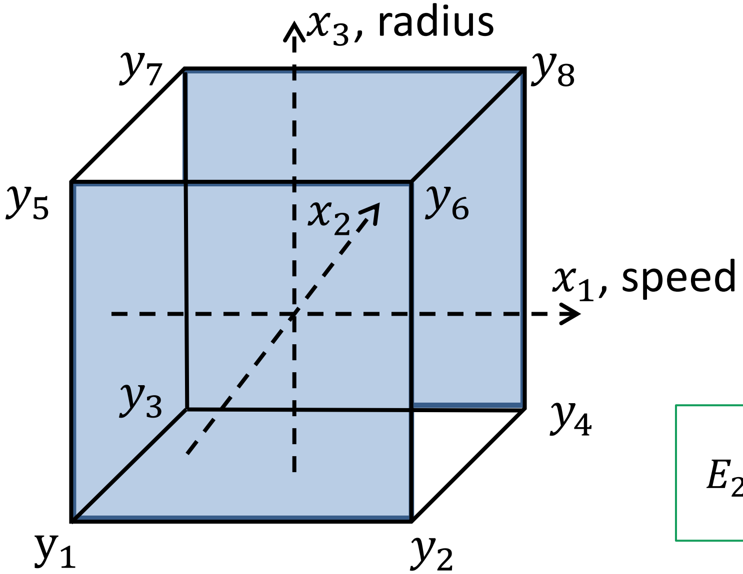
- $y_1$ and $y_2$; $y_3$ and $y_4$
- $y_5$ and $y_6$; $y_7$ and $y_8$

Average effect of the speed, $E_1$, i.e. main effect:

$$\boxed{E_1 = \left(\frac{1}{4}\right)[(y_2 - y_1) + (y_4 - y_3) + (y_6 - y_5) + (y_8 - y_7)]}$$

# DOE - $2^3$ factorial design – main effects

Feed, $x_2$ is varied while speed and corner radius are held constant.




Following pairs may be compared to see the effect of the feed

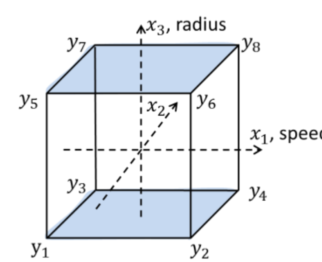
- $y_1$ and $y_3$; $y_2$ and $y_4$
- $y_5$ and $y_7$; $y_6$ and $y_8$

Average effect of the feed, $E_2$, i.e. main effect:

$$E_2 = \left(\frac{1}{4}\right)[(y_3 - y_1) + (y_4 - y_2) + (y_7 - y_5) + (y_8 - y_6)]$$

Corner radius, $x_3$ is varied while speed and feed are held constant.

$$E_3 = \left(\frac{1}{4}\right)[(y_5 - y_1) + (y_6 - y_2) + (y_7 - y_3) + (y_8 - y_4)]$$



# DOE - $2^3$ factorial design – interaction effects

You may ask:

How much different the speed ($X_1$) effect is when the feed ($X_2$) is at its low value as opposed to the feed ($X_2$) being at its high value?

Or, equivalently:

How much different the feed ($X_2$) effect is when the speed ($X_1$) is at its low value as opposed to the speed ($X_1$) being at its high value?

Consider the first case above:

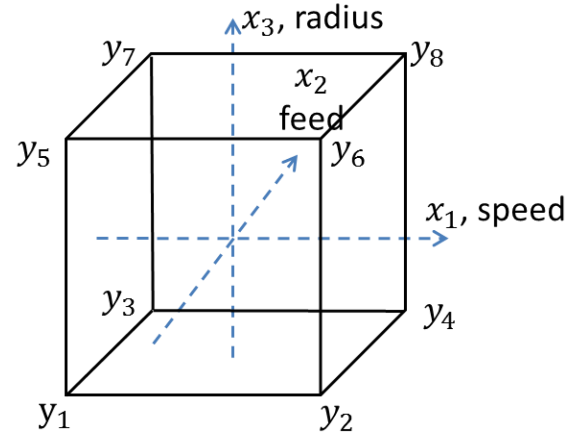
- When $X_2$ is low (${X_2}^-$), two situations exist:

  ${X_2}^-, {X_3}^-$ and ${X_2}^-, {X_3}^+$

  and

- When $X_2$ is high (${X_2}^+$), two situations exist:

  ${X_2}^+, {X_3}^-$ and ${X_2}^+, {X_3}^+$

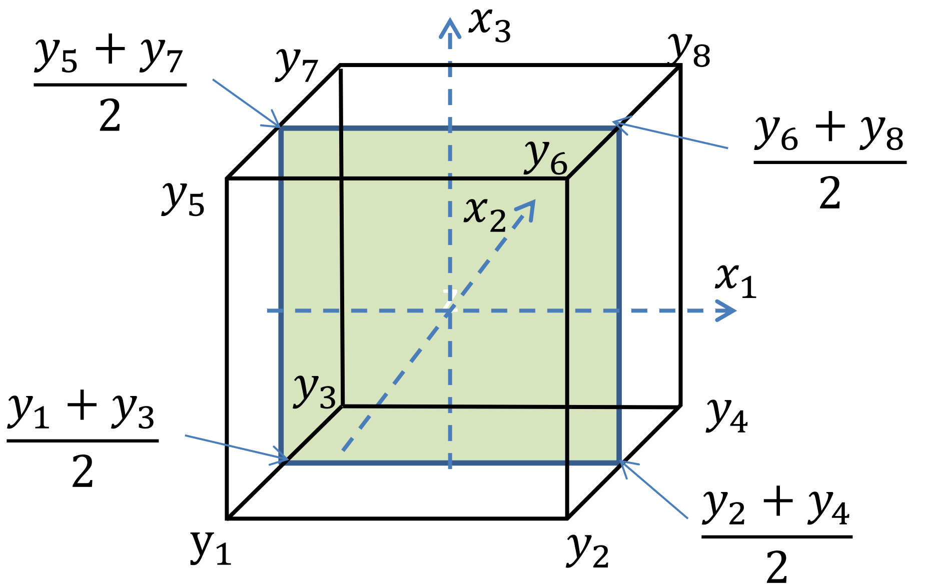



**This is the difference in the effect of $X_1$ between the average of the first $(X_2 = {X_2}^-)$ and the second pairs above $(X_2 = {X_2}^+)$**

# DOE - $2^3$ factorial design – interaction effects

<u>Interaction between speed and feed:</u>

Is difference in the effect of $X_1$ between the average of the first $(X_2 = X_2{}^{-})$ and the second pairs above $(X_2 = X_2{}^{+})$




$$\boxed{E_{12} = \frac{y_1 + y_4 + y_5 + y_8}{4} - \frac{y_2 + y_3 + y_6 + y_7}{4}}$$

# DOE - $2^3$ factorial design – interaction effects

Interaction between speed and feed:

$$E_{12} = \frac{y_1 + y_4 + y_5 + y_8}{4} - \frac{y_2 + y_3 + y_6 + y_7}{4}$$

Similarly, the other two-way, and three-way interactions are

Interaction between speed and corner radius:

$$E_{13} = \frac{y_1 + y_3 + y_6 + y_8}{4} - \frac{y_2 + y_4 + y_5 + y_7}{4}$$

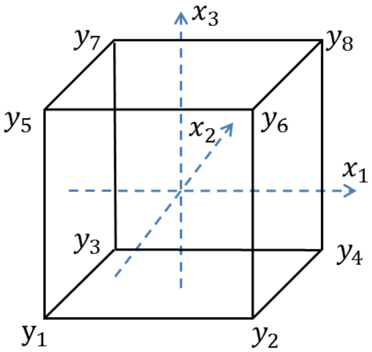
Interaction between feed and corner radius:

$$E_{23} = \frac{y_1 + y_4 + y_7 + y_8}{4} - \frac{y_3 + y_4 + y_5 + y_6}{4}$$

Interaction between speed, feed, and corner radius:

$$E_{123} = \frac{y_2 + y_3 + y_5 + y_8}{4} - \frac{y_1 + y_4 + y_6 + y_7}{4}$$

# DOE - $2^3$ factorial design – general method




| | Coded variables | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Main effects | | | Interaction effects | | | | Response |
| Test | $x_1$ | $x_2$ | $x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ | $x_1x_2x_3$ | $y$ |
| 1 | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $+1$ | $-1$ | $y_1$ |
| 2 | $+1$ | $-1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $y_2$ |
| 3 | $-1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ | $+1$ | $y_3$ |
| 4 | $+1$ | $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$ | $y_4$ |
| 5 | $-1$ | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $y_5$ |
| 6 | $+1$ | $-1$ | $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $y_6$ |
| 7 | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ | $y_7$ |
| 8 | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $y_8$ |

Use the calculation matrix, and add all elements in each column for the relevant effect, example:

$$E_1 = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{4}$$

# DOE - $2^3$ factorial design – response model

Model has the response of the form of:

$$y = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_{12}x_1x_2 + b_{13}x_1x_3 + b_{23}x_2x_3 + b_{123}x_1x_2x_3 + \epsilon$$

$\epsilon$ represents random noise or experimental error & has expected value of 0

Coefficients:

$$\hat{b}_0 = I = 1/8(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8)$$

$$\hat{b}_. = E_./2$$

$\hat{b}_. = \hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_{12}, \ldots$
$\hat{b}_{13}, \hat{b}_{23}, \hat{b}_{123}$

From the calculation matrix, sum the relevant column

$$E_1 = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{4}$$

$$E_2 = \frac{-y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 + y_8}{4}$$

$$\vdots$$

$$E_{123} = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{4}$$

| | Coded variables | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Main effects | | | Interaction effects | | | | Response |
| Test | $x_1$ | $x_2$ | $x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ | $x_1x_2x_3$ | $y$ |
| 1 | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $+1$ | $-1$ | $y_1$ |
| 2 | $+1$ | $-1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $y_2$ |
| 3 | $-1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ | $+1$ | $y_3$ |
| 4 | $+1$ | $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$ | $y_4$ |
| 5 | $-1$ | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $y_5$ |
| 6 | $+1$ | $-1$ | $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $y_6$ |
| 7 | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ | $y_7$ |
| 8 | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $y_8$ |

# DOE - $2^3$ factorial design – response model

Estimate the response as:

$$\boxed{\hat{y} = \hat{b}_0 + \hat{b}_1 x_1 + \hat{b}_2 x_2 + \hat{b}_3 x_3 + \hat{b}_{12} x_1 x_2 + \hat{b}_{13} x_1 x_3 + \hat{b}_{23} x_2 x_3 + \hat{b}_{123} x_1 x_2 x_3}$$

Transform coded space response to the un-coded space, i.e. actual variables

$$\boxed{x = \frac{2X - X^+ - X^-}{X^+ - X^-}} \rightarrow \boxed{x_1 = \frac{2X_1 - {X_1}^+ - {X_1}^-}{{X_1}^+ - {X_1}^-};} \quad \boxed{x_2 = \frac{2X_2 - {X_2}^+ - {X_2}^-}{{X_2}^+ - {X_2}^-}} \quad \ldots$$

$X^+ \rightarrow$ high level of variable, and $X^- \rightarrow$ low level of variable

Estimate of the response in terms of actual variables:

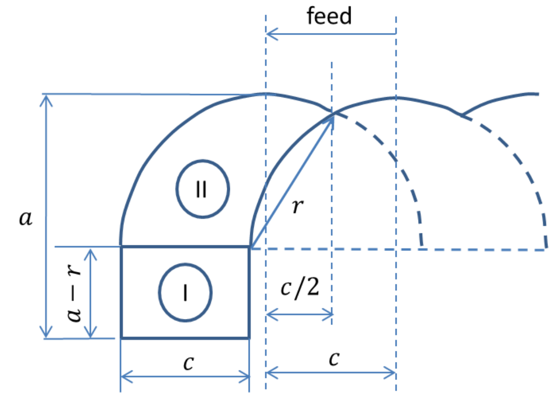
$$\boxed{\hat{y} = \hat{b}_0 + \hat{b}_1 X_1 + \hat{b}_2 X_2 + \hat{b}_3 X_3 + \hat{b}_{12} X_1 X_2 + \hat{b}_{13} X_1 X_3 + \hat{b}_{23} X_2 X_3 + \hat{b}_{123} X_1 X_2 X_3}$$

# DOE - $2^3$ factorial design – response model

<u>Recalling problem statement:</u>

Desired to determine the effects of spindle speed, $X_1 \rightarrow x_1$ (RPM), feed rate, $X_2 \rightarrow x_2$(mm/rev) and corner radius, $X_3 \rightarrow x_3$(mm) on the surface finish of a turned surface. Each has two levels, high (+1) and low (-1).

Recall from turning, how surface quality was thought to be a $f$(feed, nose radius)




Estimate of the response in terms of actual variables:

$$\hat{y} = \hat{b}_0 + \hat{b}_1 X_1 + \hat{b}_2 X_2 + \hat{b}_3 X_3 + \hat{b}_{12} X_1 X_2 + \hat{b}_{13} X_1 X_3 + \hat{b}_{23} X_2 X_3 + \hat{b}_{123} X_1 X_2 X_3$$

$$\hat{y} \approx \hat{b}_0 + \hat{b}_2 X_2 + \hat{b}_3 X_3 + \hat{b}_{23} X_2 X_3$$

# Consider a $2^{10}$ factorial design

A 10 variable experiment with two levels for each variable would need $2^{10}$ = 1024 tests

This test, will have the following variable effects:

| | |
|---|---|
| 1 | Mean response |
| 10 | Main effects |
| 45 | Two-factors interaction effects |
| 120 | Three-factor interaction effects |
| 210 | Four-factor interaction effects |
| 252 | Five-factor interaction effects |
| 210 | Six-factor interaction effects |
| 120 | Seven-factor interaction effects |
| 45 | Eight-factor interaction effects |
| 10 | Nine-factor interaction effects |
| 1 | Ten-factor interaction effect |
| 1024 | Mean + 1023 variable effects |

Generally, higher-order variable interactions are negligible and can be ignored

$\downarrow$

We then use fractional factorial designs:

$$2^k \rightarrow 2^{k-n}$$