MTH204A

1. If $\sim$ is an equivalence relation on a non-empty set A, then the set of equivalence classes of $\sim$ forms a partition of A. On the other hand, if $\{A_i\}_{i\in I}$ is a partition of a non-empty set A, then there is an equivalence relation on A whose equivalence classes are precisely the sets A_i, $i \in I$.

2. Let G be a group. Show that
   (i) identity element in G is unique.
   (ii) For each $a \in G$, a^{-1} is unique.
   (iii) $(a^{-1})^{-1} = a$
   (iv) $(a * b)^{-1} = b^{-1} * a^{-1}$ in $(G, *)$.

3. Define $\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times} = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z}, (a,n) = 1\}$

   Show that $\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times} = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid \exists\, \bar{c} \in \mathbb{Z}/n\mathbb{Z}$ s.t. $\bar{a}\bar{c} = \bar{c}\bar{a} = \bar{1}\}$

   Define $\cdot$ on $\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times}$ by
   $\bar{a} \cdot \bar{b} = \overline{ab}$ for $\bar{a}, \bar{b} \in \left(\mathbb{Z}/n\mathbb{Z}\right)^{\times}$

   Show that $\left(\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times}, \cdot\right)$ is a group.

4. Show that $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \times)$

5. Show that any two cyclic groups of same order are isomorphic

6. Determine all the subgroups of $(\mathbb{Z}, +)$

7. Let a, n are integers with $(a,n) = 1$ & $n > 0$. Prove

that $a^{\phi(n)} \equiv 1 \pmod n$

~~Show that ... for many ...~~

8 (i) Show that $x^p = x \;\forall\, x \in \mathbb{Z}/p\mathbb{Z}$.

(ii) If G is a finite group of prime order, then show that G is cyclic

9. Classify groups of order ≤ 5.

10. Prove that $a\mathbb{Z} + b\mathbb{Z} = \gcd(a,b)\,\mathbb{Z}$

$$a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a,b)\,\mathbb{Z}$$

11. (i) Show that if every element in a group is its own inverse, then the group is abelian

(ii) If $(a\cdot b)^n = a^n b^n \;\forall\, a,b \in G$, then G is abelian.

12. Show that rational numbers with odd denominators forms a group with respect to addition.

13. Prove that S_n, D_n are non-abelian for $n \geq 3$.

14. If G is a finite group, show that $\exists\, n \in \mathbb{N}$ s.t. $a^n = 1 \;\forall\, a \in G$.

15. If G is a group of even order, prove that it has an element $a \neq 1$ s.t. $a^2 = 1$

16. (i) Recall the definition of $GL_n(\mathbb{Z}/p\mathbb{Z})$. Show that $GL_n(\mathbb{Z}/p\mathbb{Z})$ forms a group w.r.to matrix multiplication

(ii) Determine the order of the group $GL_n(\mathbb{Z}/p\mathbb{Z})$.

(Here p is a prime).

17: Let G be a group s.t. the intersection of all of its subgps different from $\{1\}$ is a subgroup different from $\{1\}$. Prove that every element in G has finite order.

18. (i) prove that every subgroup of a cyclic group

is cyclic.

(ii) Let G be a finite cyclic group. For every $d \mid n$, show that G has a unique subgroup of order d.

(iii) Show that G in (ii) has $\phi(n)$ many generators.

19: Let G be a cyclic group of order n. If $d \mid n$, show that no. of elements of order d in $G = \phi(d)$

If $d \nmid n$ then no. of elements of order $d = 0$.

20. If N & M are normal subgp of G, prove that NM is also a normal subgroup of G.

21. If N & M are normal subgroups of G & $N \cap M = \{1\}$ Then show that $\forall\, n \in N, \forall\, m \in M, nm = mn$.

22. If N is the only subgroup of order $|N|$ (which is finite) in G, Then show that N is a normal subgroup of G.

23. Show that any subgroup of index 2 is normal in G.

24. Show that $\langle -1 \rangle$ is a normal subgp in Q_8.

25. Show that A_3 is normal in S_3 & $S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z}$. Show that none of the two cycles are normal in S_3.

26. Determine all groups of order 6.

27. If a cyclic subgroup T of G is normal in G, show that every subgroup of T is normal in G.

28. If N is normal in G, prove that $o(\bar{a})$ in $\frac{G}{N}$ divides $o(a)$ in G.

29. If N is normal in a finite group G s.t $[G:N]$ & $|N|$ are relatively prime, then show that any element $x \in G$, satisfying $x^{|N|} = e$ must be in N.

30. Verify Isomorphism theorems I & III in D8, Q_8 & $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$ through various examples (4)

31. Let G be finite group. If $O(x) = n$, show that $O(x^a) = \frac{n}{\gcd(a,n)}$. If G is abelian & has elements x, y of order ~~(m,n)~~ m, n respectively then show that there exists an element $z \in G$ of order $= \mathrm{lcm}(m,n)$

32. Prove that (i) $\phi(p^r) = p^r - p^{r-1}$

(ii) $\phi(mn) = \phi(n)\,\phi(m)$ if $(m,n) = 1$.

33 Let G be a finite group & if $\exists$ an automorphism $f: G \to G$ ~~o~~ s.t. $f(x) = x^{-1}$ for strictly more that 75% elements x of G, then prove that $f(x) = x^{-1}\ \forall x \in G$ & G is an abelian group.

34. Given an example of a non-abelian group which sends each of its ~~ele~~ precisely 75% elements to their inverses.

35. Compute the Centre of D_n, Q_8, $GL_n(\mathbb{R})$.

36. If $G/Z(G)$ is cyclic, then prove that G is abelian.

37. Let G be a group of order 2n. ~~Sup~~ Let us assume that half of the elements of G are of order 2 & the other half forms a subgroup H of order n (& H has no element of order 2). Prove that H is of odd order & is an abelian subgroup of G.

38. Let G be the group of polynomials ($\mathbb{R}[X]$) in one variable with coefficient in $\mathbb{R}$. Define the map I by

$I(0) = 0$

& $I(p(x)) = \int p(x)\,dx$. Show that I is a group homomorphism & determine the kernel.

39. Prove that if a group contains exactly one element of order 2, then that element is in the centre of the group.

40 Let A, B are abelian groups. Show that there is a natural group structure on Hom(A, B), the set of groups homomorphism from A to B, which makes Hom(A, B) an abelian group.

41. Determine Hom$(\mathbb{Z},\mathbb{Z})$, Automorphism$(\mathbb{Z},\mathbb{Z})$ Hom$(\mathbb{Z}, A)$ where A is any abelian group.

42. Prove that any group of order p^2, with p - a prime is abelian.

43. Let G be a finite group of order n & r be an integer relatively prime to n, then show that $g \in G \Rightarrow g = x^r$ for some $x \in G$.

44. Show that A_4 has no subgroup of order 6.

45. Determine the kernel & image of the ~~map~~ natural map $\phi: \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$

46 Deduce from Isomorphism theorems that there is a 1-1 correspondence between the subgroups of G containing N & subgroup of G/N. More precisely, there is a bijection from the set of subgroups of G which contains N onto the subgroups of G/N. In particular, every subgroup of G/N is of the form A/N for some subgroup A of G containing N.

47 Explicitely identify $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ inside A_4.

48. If G is group of order n & p is the smallest prime dividing order of G, then any subgroup of index p is normal in G.

49. Prove that the size of a conjugacy class containing an element $\mid |G|$

where G is a finite group & for $x \in G$
Congugacy class of $x = \{gxg^{-1} \mid g \in G\}$

50. Show that any Group of p-power order has nontrivial centre, where p is a prime.

51. (i) Prove Cauchy's theorem I: If G is a finite abelian group & $p \mid |G|$ then G contains an element of order p.

(ii) Deduce that ~~general same~~ the same result holds for a general finite non-abelian group.

52. Show that if a group has no nontrivial subgroup, then G is finite of prime order.