First Order PDE and Method of Characteristics

MSO-203B

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Overview

Topics to be covered

- Method of Characteristics.
- Solving First order linear PDE with boundary data.
- Non-Characteristics.
- Solving Quasilinear Equations.

Introduction

Question

We need to find a $u \in C^1(\Omega)$ where Ω is a open subset in $\mathbb{R}^2$ such that

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \ \text{ in } \ \Omega$$

with a, b and c being some continuous function in $\Omega \times \mathbb{R}$.

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Expectations

- One can only expect solution in a small neighbourhood of the inital data.
- It may happen that no solution $u \in C^1(\Omega)$ may exists in classical sense. Is there any way out of that.

Method of Characteristics

Linear First Order Equation

Consider the problem:

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c(x,y) \text{ in } \Omega \tag{1}$$

with a, b and c being some continuous function in Ω.

Method of Characteristics

Observation 1

- Note that equation (1) can be written as

$$(a(x,y), b(x,y), c(x,y)).(u_x, u_y, -1) = 0$$

- Define $G = \{(x, y, u(x,y)) : (x,y) \in \Omega\}$ be the graph of u.

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- Hence the vector $(a(x,y), b(x,y), c(x,y))$ must lie in the tangent plane of G at (x, y).
- Finding solution is equivalent to finding a surface such that at all points $(a(x,y), b(x,y), c(x,y))$ lies in the tangent plane.

Method of Characteristics

Observation 2

- To construct a required surface G we start by trying to find a curve $C(s)$ on G such that the derivative at each point $(x(s), y(s), z(s))$ is equal to the vector $(a(x, y), b(x, y), c(x, y))$.

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- One obtains the following

$$x'(s) = a((x(s), y(s))$$
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- Taking the union of all such curves provide us with the required surface G.

Illustration

Example 1

Consider the equation $u_t + au_x = 0$ with a being a constant.

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Solution

Note that

$$u_t + au_x = (u_t, u_x).(1, a) = 0$$

Hence the *Characteristics Curves* are given by

$$x'(s) = a \implies x(s) = as + c_1 \tag{2}$$

$$t'(s) = 1 \implies t(s) = s + c_2 \tag{3}$$

$$z'(s) = 0 \implies z(s) = c_3. \tag{4}$$

where c_i are constants for $i = 1, 2, 3$.

Solution

From (2) and (3) we get $x = at + c$ for some constant c depending only on c_1 and c_2.

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- The Characteristic Curves will be given by $x = at + c$ and $z = c_3$.
- Define $u(x, t) = z(x, t)$.
- So, $u(x, t) = f(x - at)$ for some function $f \in C^1(\mathbb{R})$.

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Verification

If $u(x, t) = f(x - at)$ then $u_x(x, t) = f'(x - at)$ and $u_t(x, t) = -af'(x - at)$ and hence $u_t + au_x = 0$.

Example 2

Handling the initial conditions

Consider the problem:

$$u_t + au_x = 0; \;\; u(x,0) = g(x)$$

where a is a constant and $g \in C^1(\mathbb{R})$ is given.

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Geometric View

We proceed as above to solve the problem with an extra assumption that the curve $(x, 0, g(x))$ is contained in the surface given by the graph of the solution.

- Let $\Gamma = \{(x, 0, g(x))\}$ be a curve in $\mathbb{R}^3$
- For a fixed $A = (x_0, 0, g(x_0))$ on Γ construct the Integral curves starting from A.

Solution

Analytic Solution

Parametrizing the curve Γ by r, we look for solutions of the Characteristic equation given by

$$\frac{dx}{ds}(r,s) = a$$
$$\frac{dt}{ds}(r,s) = 1$$
$$\frac{dz}{ds}(r,s) = 0$$

for a fixed r and with initial conditions

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for a fixed r and with initial conditions

$$x(r,0) = r$$
$$t(r,0) = 0$$
$$z(r,0) = g(r)$$

Solution

Characteristic Curves

Solving the Characteristic equation one has

$$(x(r,s), t(r,s), z(r,s)) = (as + r, s, g(r))$$

Hence one has

$$u(x,t) = z(r(x,t), s(x,t)) = g(x - at)$$

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Hence one has

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Remark

Note that $u(x,t)$ is constant along the line $x - at = c$ for any constant c.

Example 3

Consider the problem

$$xu_x + yu_y = u + 1; \quad u|_\Gamma = x^2$$

where Γ is the parabola $y = x^2$.

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Solution

Let Γ is parametrized by r as (r, r^2, r^2). For r fixed the characteristic equations are given as:

$$x'(r,s) = x; \quad x(r,0) = r$$
$$y'(r,s) = y; \quad y(r,0) = r^2$$
$$z'(r,s) = z + 1; \quad z(r,0) = r^2$$

Solution

Solving the characteristic equations we have,

$$x(r,s) = re^s$$
$$y(r,s) = r^2 e^s$$
$$z(r,s) = (r^2+1)e^s - 1$$

Hence the solution is given by $u(x,y) = \frac{x^2}{y} + y - 1$.

Non-Characteristic

Question

Will the problem

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \text{ in } \Omega$$

admit a solution for any boundary condition?

Non-Characteristic

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Will the problem

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \text{ in } \Omega$$

admit a solution for any boundary condition?

Answer

No, it may not. Consider the problem

$$u_t + au_x = 0; \ u|_\Gamma = g(x)$$

From the earlier examples we have seen that $u(x, t) = f(x - at)$ is a solution for some $f \in C^1(\mathbb{R})$.
Now if $\Gamma = \{x - at = c\}$ then g is constant along Γ, which may not be the case.

Non-Characteristic

The Way Out - Definition

Γ is called a Non-Characteristic for the Cauchy problem

$$a(x,y)u_x + b(x,y)u_y = c(x,y); \;\; u|_\Gamma = g$$

if

$$(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r)).(-\gamma_2'(r),\gamma_1'(r)) \neq 0 \tag{5}$$

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Example

If $\Gamma = \{x - at = c\}$ then $\Gamma(r) = (\gamma_1(r),\gamma_2(r)) = (c + a^2 r, ar)$

- $(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r)).(-\gamma_2'(r),\gamma_1'(r)) = (a,1).(-a,a^2) = 0$
- Hence Γ is a Non-Characteristic for the equation $u_t + au_x = 0$.

Quasilinear Equation

Example 3

Consider the Burger's Equation:

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Solution

Let $\Gamma = (r, 0)$ be the parametrization of the x-axis. Γ is a Non-Characteristic since $(0,1).(r^2,1) = 1$.
Hence the Characteristic Equation is given by

$$\begin{aligned} x'(s) &= z; \;\; x(r,0) = r, \\ t'(s) &= 1; \;\; t(r,0) = 0, \\ z'(s) &= 0; \;\; z(r,0) = r^2 \end{aligned}$$

Quasilinear Equation

Solution

Solving the characteristic equations we get

$$x(r,s) = r^2 s + r$$
$$t(r,s) = s$$
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Hence we have, $u(x,t) = (x - ut)^2$

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Hence we have, $u(x,t) = (x - ut)^2$

Remark

Note that the solution is not an explicit solution but an implicit one.

The End