Wave Equation: Existence and Uniqueness

MSO-203B

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Introduction

Question

We want to discuss the question of Existence and Uniqueness of the equation

$$u_{tt} - c^2 u_{xx} = 0 \text{ on } \mathbb{R} \times (0, \infty)$$

subject to the initial conditions

$$u(x, 0) = f(x)$$

and

$$u_t(x, 0) = g(x)$$

Existence Question

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We have using chain rule,

$$u_x = w_T T_x + w_S S_x = w_T + w_S$$

$$u_{xx} = w_{TT} + 2w_{TS} + w_{SS}$$

$$u_t = w_T T_t + w_S S_t = w_T(-c) + w_S(c) = c(w_S - w_T)$$

$$u_{tt} = c(cw_{SS} - cw_{ST} + cw_{TT} - cw_{ST}) = c^2(w_{SS} - w_{ST} + w_{TT})$$

Existence Result

Canonical Form

Hence the canonical form is given by $w_{ST} = 0$ whose solution is given by $w(S, T) = F(S) + G(T)$.
Hence the solution of our problem is

$$u(x, y) = F(x - ct) + G(x + ct)$$

for arbitrary smooth F and G respectively.

Existence result

Incorporating the Boundary Condition

Using the condition $u(x,0) = f(x)$ we have,

$$u(x,0) = F(x) + G(x) = f(x)$$

Again using the condition $u_t(x,0) = g(x)$ we have,

$$u_t(x,0) = -cF'(x) + cG'(x) = g(x)$$

Using the above two conditions one deduce that

$$G(x) = \frac{1}{2}[f(x) - f(0)] + \frac{1}{2c}\int_0^x g(s)ds$$

and

$$F(x) = \frac{1}{2}[f(x) + f(0)] - \frac{1}{2c}\int_0^x g(s)ds$$

Existence Result

d'Alembert's Formula

Therefore the solution to our initial value wave equation is given by

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

An example

Problem

Find the solution of the wave equation given by:

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = \sin x, \quad u_t(x,0) = x^2$$

Solution

Using d'Alembert's Formula one has,

$$u(x,t) = \frac{1}{2}[\sin(x-ct) + \sin(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} y^2 dy$$

which implies,

$$u(x,t) = \sin x \cos ct + x^2 t + \frac{1}{3}c^2 t^3$$

Uniqueness Result

Energy Methods

Define the energy functional

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + c^2 u_x^2)dx$$

where $u = u(x,t)$ is a smooth solution of the wave equation such that the ∇u is square summable for each $t \geq 0$.

Uniqueness Result

Energy Methods

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Question

What is the energy of the system given by $E(t)$?

Uniqueness Result

Conservation of Energy

Multiplying u_t with $u_{tt} - c^2 u_{xx} = 0$ and integrating by parts we have,

$$\int_{-\infty}^{\infty} u_t u_{tt} dx = \int_{-\infty}^{\infty} c^2 u_{xx} u_t dx$$
$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right) dx = c^2 u_x u_t|_{-\infty}^{\infty} - c^2 \int_{-\infty}^{\infty} u_x u_{xt} dx$$

hence one has,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right) dx = -c^2 \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{u_x^2}{2}\right) dx$$

that is

$$E'(t) = \frac{d}{dt} \int_{-\infty}^{\infty} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2\right) dx = 0$$

Uniqueness Result

Uniqueness Theorem

The problem:

$$u_{tt} - c^2 u_{xx} = h(x,t), \text{ on } \mathbb{R} \times \{t > 0\}$$
$$u(x,0) = f(x); \ \ u_t(x,0) = g(x), \text{ on } \mathbb{R}$$

has a unique solution.

Proof

Let $u(x,t) = u_1(x,t) - u_2(x,t)$ where u_1 and u_2 are two solutions of the problem. Then u solves the homogeneous problem with the initial data

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = 0, \ \ u_t(x,0) = 0$$

Uniqueness Result

Proof

Since $E(t) = E(0) = 0$ by the energy estimate we have

$$\frac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + c^2 u_x^2)dx = 0$$

which implies $u_t = 0$ and $u_x = 0$.
Thus u is constant in x and t, but $u(x,0) = 0$ so the constant is zero.
Hence there exists a unique solution to the initial data inhomogeneous wave equation.