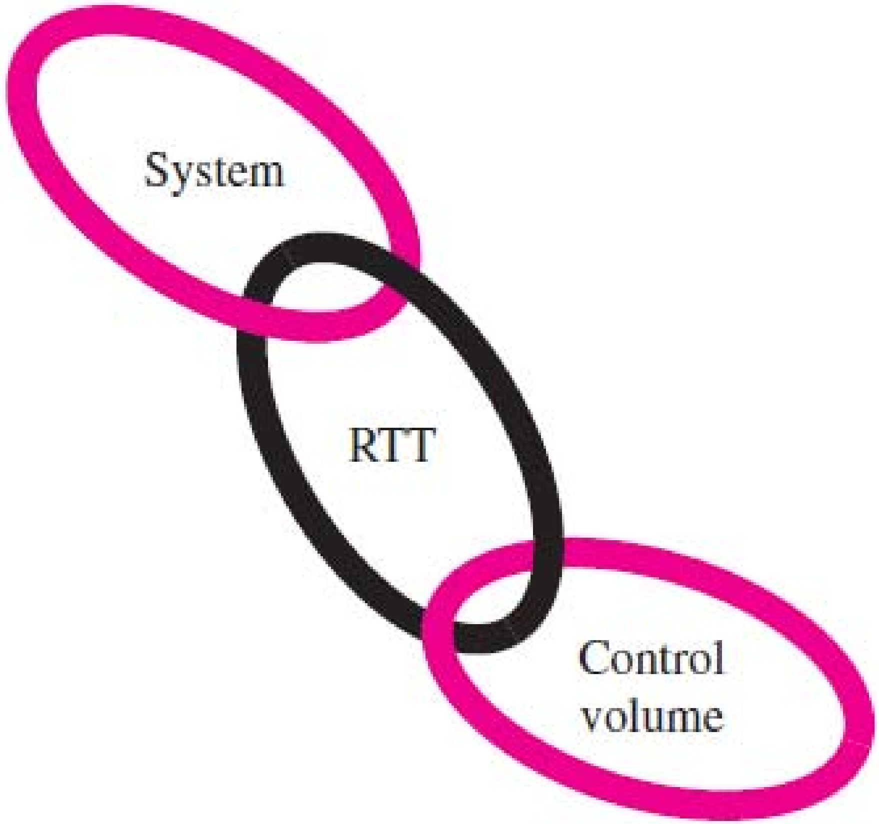
Reynolds Transport Theorem (RTT)

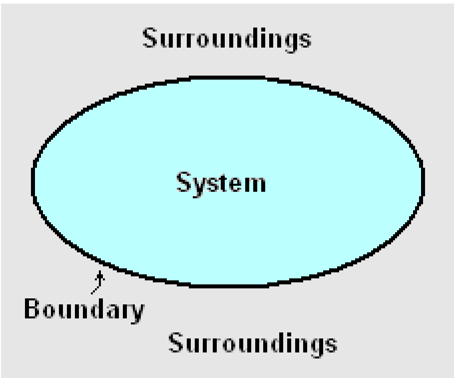
Connection between Eulerian and Lagrangian descriptions

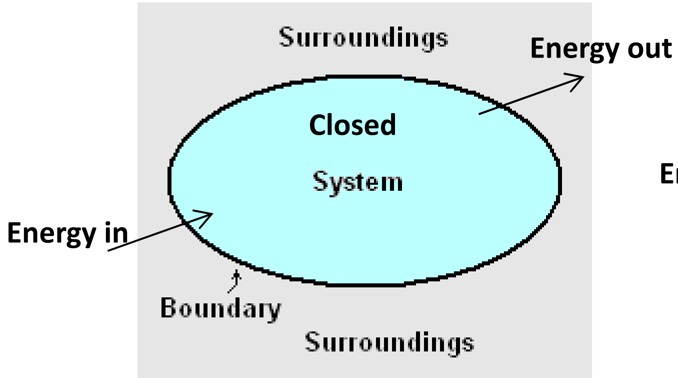


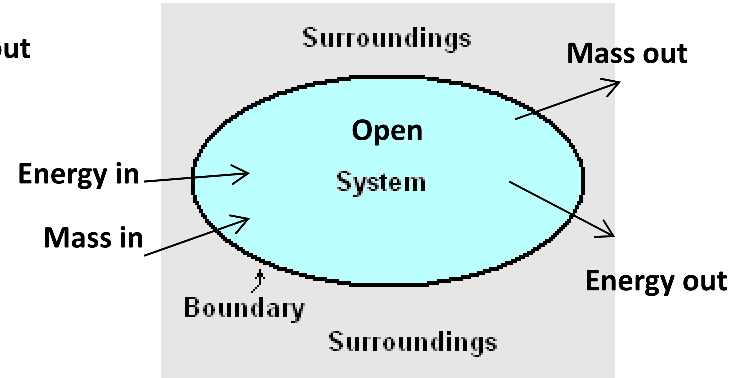
System and Control Volume

- A system is defined as a quantity of matter in space on which attention is paid in the analysis of a problem. It does not exchange mass with surroundings
- Everything external to the system is called the surroundings
- The system is separated from surroundings by the system boundaries

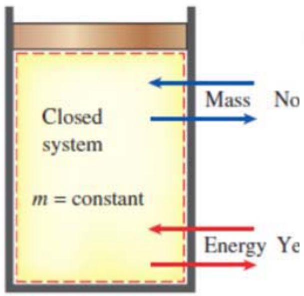
Boundaries may be (i) solid or (ii) imaginary. They may be fixed or moving.

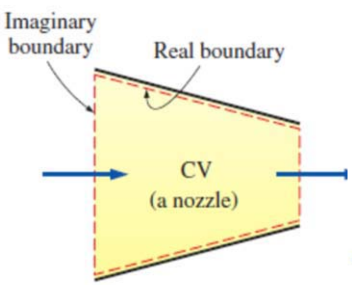


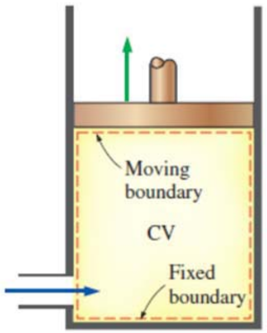


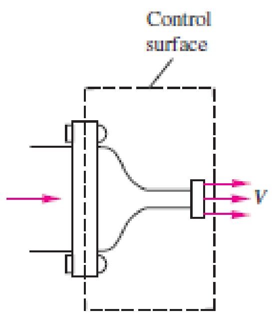


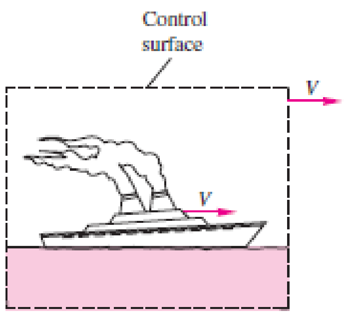
4th category: Isolated system

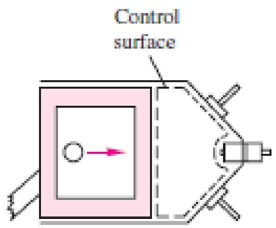




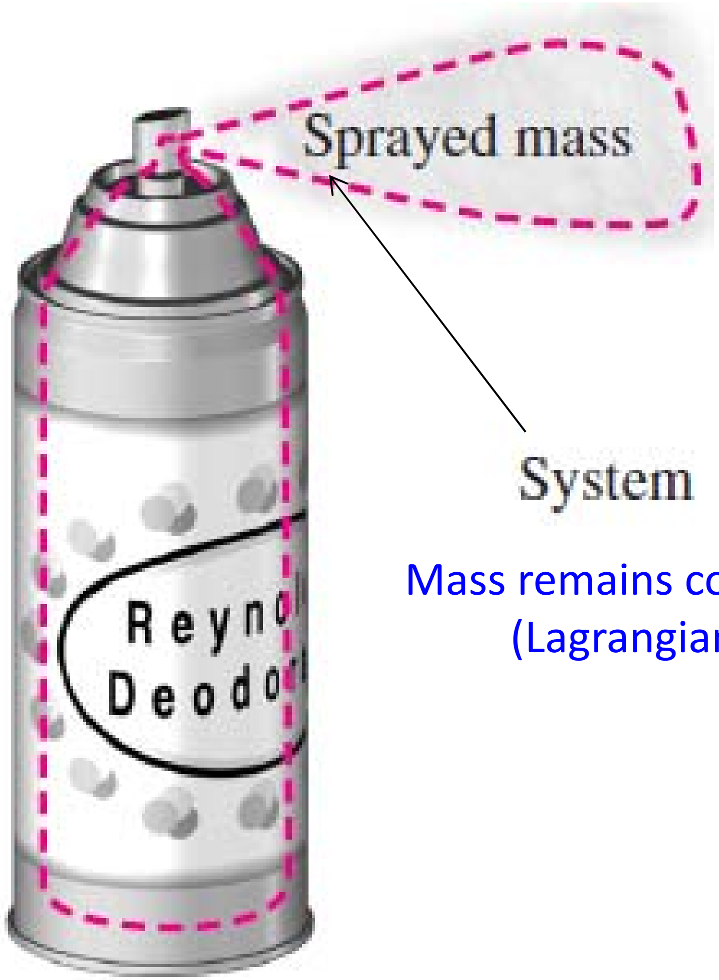








System and Control Volume



Mass remains constant
(Lagrangian)

Mass does not remain constant
(Eulerian)

Approaches to study fluid motion

System: It is collection of matter of fixed identity (Closed System).

The motion of an individual fluid particle or group of particles is studied as they move through space. The motion of a particle (an infinitesimal) is studied by considering an infinitesimal fluid element leading to differential equations thus the method is called ***differential*** approach.

Advantages: Direct application of physical laws (Mass, momentum energy etc.)
Disadvantages: Mathematics associated with this approach can become somewhat complicated, usually leading to a set of partial differential equations.

Control Volume: A geometric entity (fixed or moving, rigid or deformable) in space through which fluid flows (Open System)

The study of a region of space as fluid flows through it. This is called integral method as a finite region is studied.
Advantages: It is easier to apply without the knowledge each individual fluid particle.
Disadvantages: The physical laws apply to matter and not directly to regions of space, so we have to perform some mathematics to convert physical laws from their system formulation to a control volume formulation.

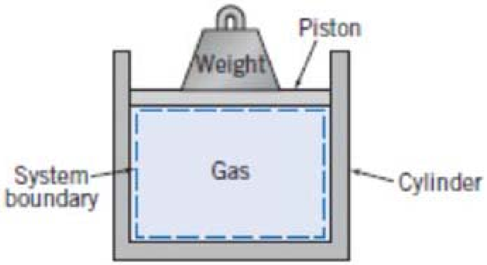
Basic Laws for a System:

$$\left.\frac{dm}{dt}\right|_{system} = 0 \quad \text{where } m_{system} = \int_{m(system)} dm = \int_{\forall(system)} \rho d\forall \longleftarrow$$ Mass Conservation

Newtons 2nd Law: $\longrightarrow$ $$\vec{F} = \left.\frac{d\vec{P}}{dt}\right|_{system} \quad \text{where } \vec{F}_{system} = \int_{m(system)} \vec{V} dm = \int_{\forall(system)} \vec{V} \rho d\forall$$

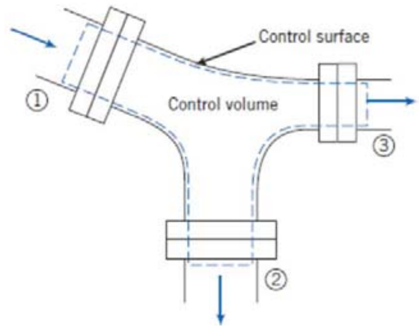
Conservation of mass (system)

$$\frac{dm_{sys}}{dt} = 0$$



Conservation of mass (CV)

$$\frac{\partial m_{CV}}{\partial t} = \text{mass inflow rate at inlet} - \text{mass outflow rate at outlet}$$



i. B of ***a system*** $\boldsymbol{B_{sys}}$ at a given instant,

$$B_{sys} = \lim_{\delta V \to 0} \sum_i b_i (\rho_i \delta V_i) = \int_{sys} \rho b \, dV$$

δm_i for i^{th} fluid particle in the system

where δV_i : Volume of i^{th} fluid particle

And Time rate of change of B_{sys},

$$\frac{dB_{sys}}{dt} = \frac{d\left(\int_{sys} \rho b \, dV\right)}{dt}$$

RTT connects $\dfrac{dm_{sys}}{dt}$ with $\dfrac{\partial m_{CV}}{\partial t}$

ii. B of fluid ***in a control volume*** $\boldsymbol{B_{cv}}$

$$B_{cv} = \lim_{\delta V \to 0} \sum_i b_i (\rho_i \delta V_i) = \int_{cv} \rho b \, dV$$

and

$$\frac{dB_{cv}}{dt} = \frac{d\left(\int_{cv} \rho b \, dV\right)}{dt}$$

Only difference from B of a system

Extensive and Intensive Properties

Let's set a fundamental equation of physical parameters

$$B = m\beta$$

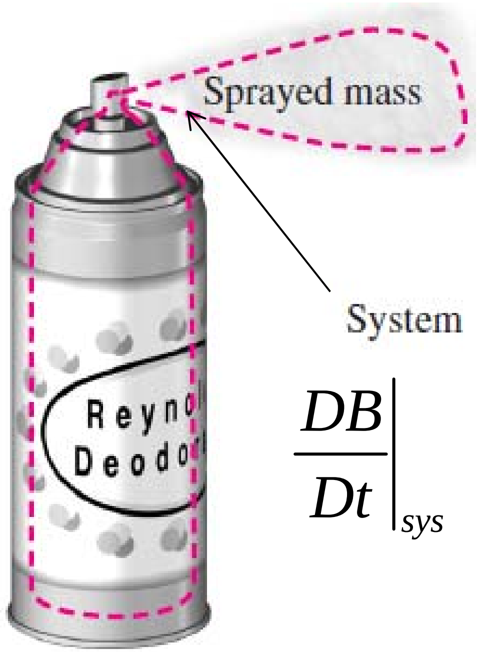
where B: Fluid property which is *proportional to amount of mass* (***Extensive*** property)
β: *B per unit mass* (Independent to the mass) (***Intensive*** property)

Examples a) If $\vec{B} = m\vec{V}$ (Linear momentum): Extensive property

Then $\vec{\beta} = \vec{V}$, (Velocity) : Intensive property

b) If $B = \frac{1}{2}mV^2$ (Kinetic energy) : Extensive property

then, $\beta = \frac{1}{2}V^2$: Intensive property



$$\left.\frac{DB}{Dt}\right|_{sys}$$

Control Volume

$$\frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall$$

outflow

$$\int_{CS} \beta\rho\left(\vec{V}.\vec{n}\right)(dA)$$

Reynolds Transport Theorem (RTT)

For an extensive property B and intensive property β, RTT can be written as

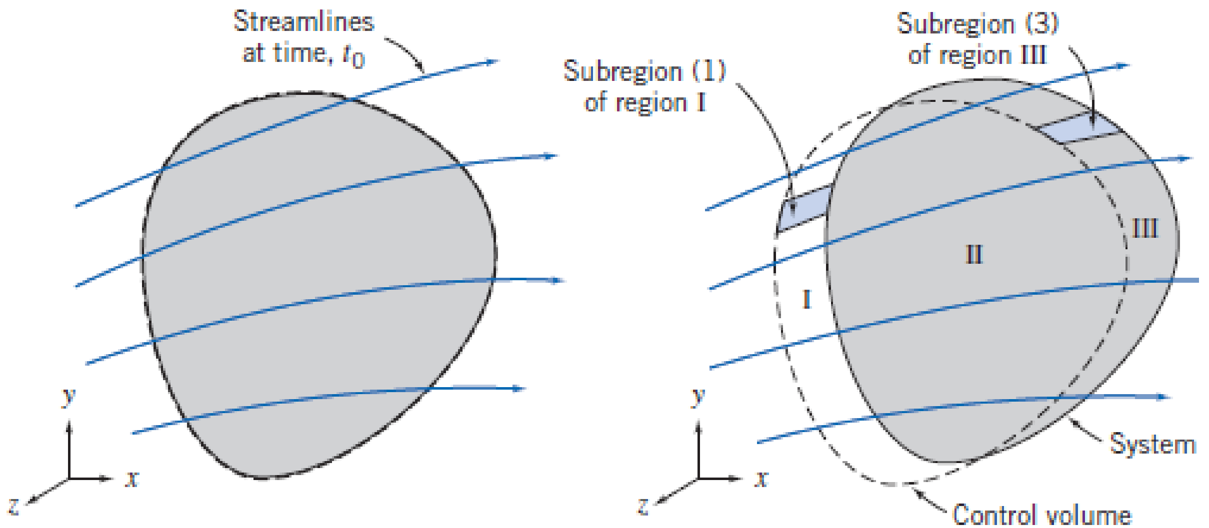
$$\left.\frac{DB}{Dt}\right|_{sys} \leftarrow \left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho\left(\vec{V}.\vec{n}\right)(dA)$$

Material derivative

$\vec{n}$ is the unit normal vector to area dA

Converts Lagrangian (system-based) description to Eulerian (CV-based) description for an arbitrary volume of fluid.

Proof for fixed CV



(a) Time, t_0 (b) Time, $t_0 + \Delta t$

At $t = t_0 + \Delta t$

Regions I and II together consist CV

Regions II and III together consist System

System and CV coincide

System is partially outside CV

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{B_S|_{t_0+\Delta t} - B_S|_{t_0}}{\Delta t}$$

$$B_S\big|_{t_0+\Delta t} = (B_{II} + B_{III})\big|_{t_0+\Delta t} = (B_{CV} - B_I + B_{III})\big|_{t_0+\Delta t}$$ and $$B_S\big|_{t_0} = B_{CV}\big|_{t_0}$$

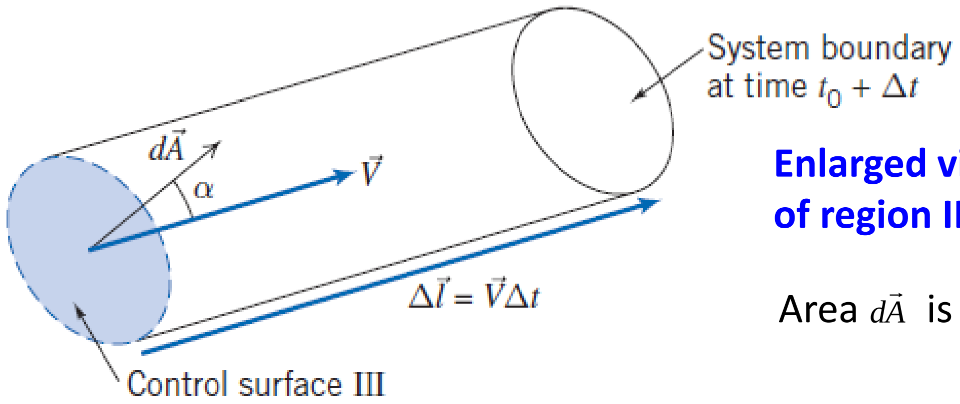
$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{(B_{CV} - B_I + B_{III})\big|_{t_0+\Delta t} - B_{CV}\big|_{t_0}}{\Delta t}$$

Since the limit of a sum is equal to the sum of the limits

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{B_{CV}|_{t_0+\Delta t} - B_{CV}|_{t_0}}{\Delta t} + \lim_{\Delta t \to 0} \frac{(B_{III})\big|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{(B_I)\big|_{t_0+\Delta t}}{\Delta t}$$

1 2 3

1 $$\lim_{\Delta t \to 0} \frac{B_{CV}|_{t_0+\Delta t} - B_{CV}|_{t_0}}{\Delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t}\int_{CV} \beta \rho \, d\mathcal{V}$$



Enlarged view of a typical subregion (subregion (3)) of region III

Area $d\vec{A}$ is at an angle α to its length $\Delta\vec{l}$

2 $\quad dB_{III}\big|_{t_0+\Delta t} = \left(\beta\rho d\forall\right)\big|_{t_0+\Delta t} \qquad \Delta\vec{l} = \vec{V}\Delta t \qquad d\forall = (\Delta l)(dA)\cos\alpha = \left(\Delta\vec{l}\right).\left(d\vec{A}\right) = \vec{V}.\left(d\vec{A}\right)(\Delta t)$

$$dB_{III}\big|_{t_0+\Delta t} = \beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)\Delta t$$

$$\lim_{\Delta t\to 0}\frac{(B_{III})\big|_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{CS_{III}} dB_{III}\big|_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{CS_{III}}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)\Delta t}{\Delta t} = \int_{CS_{III}}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$$

3 $\quad \lim_{\Delta t\to 0}\frac{(B_I)\big|_{t_0+\Delta t}}{\Delta t} = -\int_{CS_I}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV}\beta\rho d\forall + \int_{CS_I}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right) + \int_{CS_{III}}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$$

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV}\beta\rho d\forall + \int_{CS}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$$

Physical Interpretation of RTT

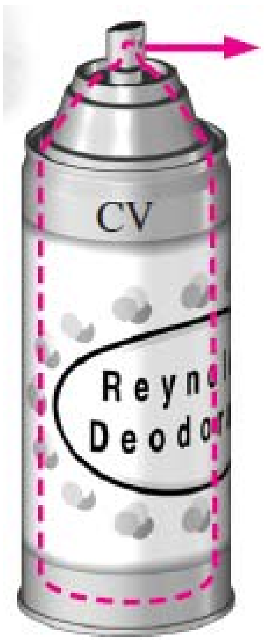
$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$$

Rate of change of *B* of the system

Rate of change of *B* in the CV

Rate of efflux of *B* through the CS

The theorem can also be proved for moving /deformable CVs



$$B = m \text{ and } \beta = 1 \qquad \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall < 0$$

$$\left.\frac{dB}{dt}\right|_{sys} = 0 \qquad \int_{CS} \beta\rho\left(\vec{V}\right).\left(d\vec{A}\right) > 0$$

Conservation of Mass for Fixed CV

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho\left(\vec{V}\right).\left(d\vec{A}\right) \qquad B = m \text{ and } \beta = 1$$

Principle of conservation of mass $\left.\frac{dB}{dt}\right|_{sys} = 0$

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho\left(\vec{V}\right).\left(d\vec{A}\right) = 0$$

Conservation of mass in integral form

Also for fixed CV: $\int_{CV} \frac{\partial}{\partial t}\left(\rho d\forall\right) + \int_{CS} \rho\left(\vec{V}\right).\left(d\vec{A}\right) = 0$ Since $d\forall$ is not changing

Steady Flow: $\cancel{\frac{\partial}{\partial t}\int_{CV} \rho d\forall} + \int_{CS} \rho\left(\vec{V}\right).\left(d\vec{A}\right) = 0 \Rightarrow \int_{CS} \rho\left(\vec{V}\right).\left(d\vec{A}\right) = 0$

Incompressible Flow: $\int_{CV} \frac{\partial}{\partial t}\left(\rho d\forall\right) + \int_{CS} \rho\left(\vec{V}\right).\left(d\vec{A}\right) \Rightarrow \rho \cancel{\frac{\partial \forall}{\partial t}} + \int_{CS} \rho\left(\vec{V}\right).\left(d\vec{A}\right) = 0$

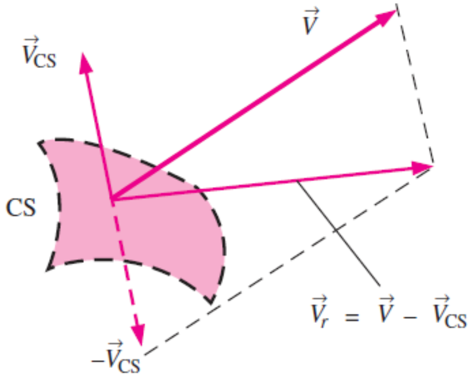
$$\int_{CS} \left(\vec{V}\right).\left(d\vec{A}\right) = 0$$

True for both steady and unsteady cases

Uniform Flow over areas: $\sum\left(\vec{V}\right).\left(d\vec{A}\right) = 0$

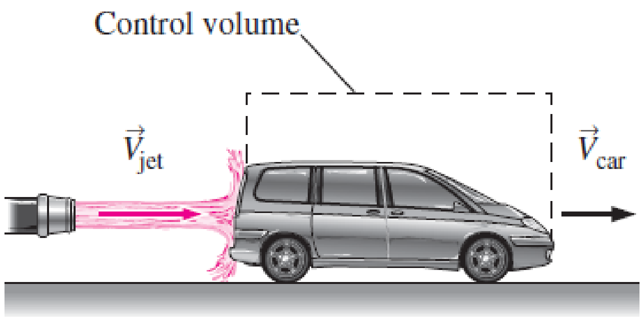
Moving and/or deforming CV

The absolute fluid velocity in the last term is replaced by the **relative velocity**



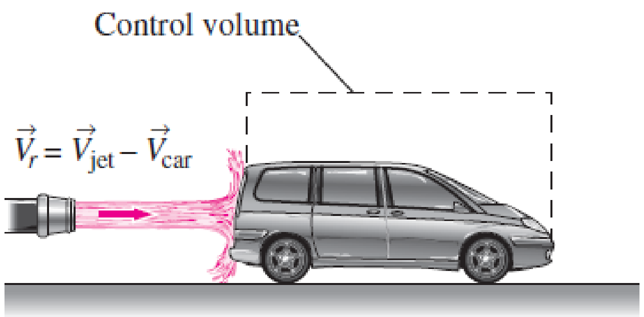
Relative Velocity: $\vec{V}_r = \vec{V} - \vec{V}_{CS}$

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho\left(\vec{V}_r\right).\left(d\vec{A}\right)$$



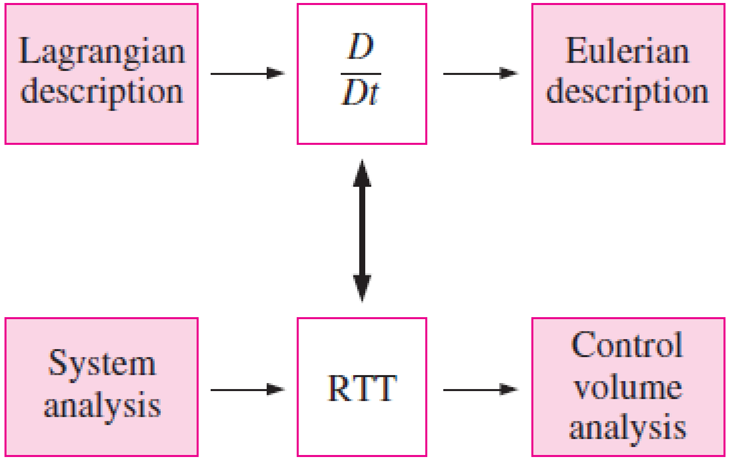
For a CV that moves and/or deforms with time, the time derivative must be applied *after* integration

Relative reference frame:



$\vec{V}_r$ is the fluid velocity expressed relative to a coordinate system moving *with* the control volume.

Relationship between Material Derivative and RTT



$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \left(\vec{V}.\vec{\nabla}\right)B$$ Material Derivative

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$$ **RTT**

The RTT for finite volumes (integral analysis) is analogous to the material derivative for infinitesimal volumes (differential analysis).

Reynolds transport theorem can be thought of as the integral counterpart of the material derivative.

In either case, the total rate of change of some property following an identified portion of fluid consists of two parts:
(a) There is a local or unsteady part that accounts for changes in the flow field with time.
(b) There is also an advective part that accounts for the movement of fluid from one region of the flow to another (compare the second term on the right-hand sides).

Just as the material derivative can be applied to any fluid property, scalar or vector, the Reynolds transport theorem can be applied to any scalar or vector property as well.

Conservation of Momentum for Fixed CV

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{CV}\beta\rho d\cancel{V} + \int_{CS}\beta\rho\left(\vec{V}\right).\left(d\vec{A}\right)$$

$$\vec{B} = m\vec{V} \text{ and } \beta = \vec{V}$$

Principle of conservation of mass

$$\left.\frac{d\left(m\vec{V}\right)}{dt}\right|_{sys} = \vec{F}$$

$$\left.\frac{d\left(m\vec{V}\right)}{dt}\right|_{sys} = \vec{F} = \frac{\partial}{\partial t}\int_{CV}\rho\vec{V}d\cancel{V} + \int_{CS}\rho\vec{V}\left(\vec{V}\right).\left(d\vec{A}\right)$$

Conservation of momentum in integral form

$$\vec{F} = \vec{F}_s + \vec{F}_B$$

$\vec{F}_s$: Surface force, all forces acting at the CS

$\vec{F}_B$: Body forces (Gravity, electromagnetic, buoyancy etc.)

Surface forces usually come from pressure, shear and interaction with solid objects/surfaces

Conservation of Momentum for Fixed CV

$$\vec{F} = \vec{F}_s + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \rho \vec{V} d\forall + \int_{CS} \rho \vec{V}\left(\vec{V}\right).\left(d\vec{A}\right)$$

Steady Flow:

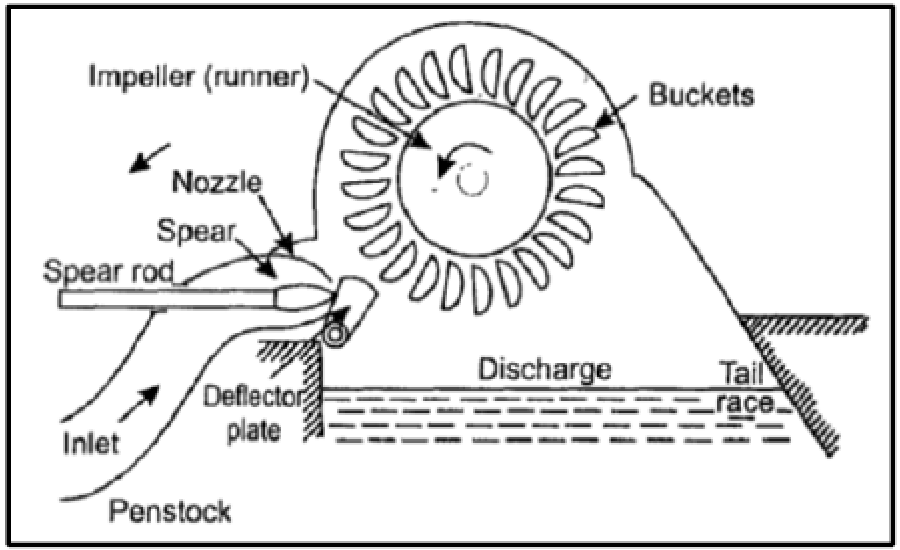
$$\vec{F} = \vec{F}_s + \vec{F}_B = \rho\int_{CS} \vec{V}\left(\vec{V}\right).\left(d\vec{A}\right)$$

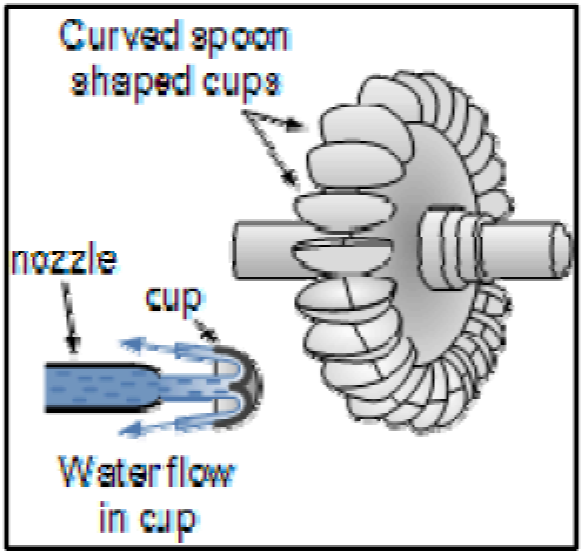
Incompressible Flow:

$$\vec{F} = \vec{F}_s + \vec{F}_B = \rho\frac{\partial}{\partial t}\int_{CV} \vec{V} d\forall + \rho\int_{CS} \vec{V}\left(\vec{V}\right).\left(d\vec{A}\right)$$

Steady Incompressible Flow:

$$\vec{F} = \vec{F}_s + \vec{F}_B = \rho\int_{CS} \vec{V}\left(\vec{V}\right).\left(d\vec{A}\right)$$





Application: Hydraulic turbine such as Pelton Wheel

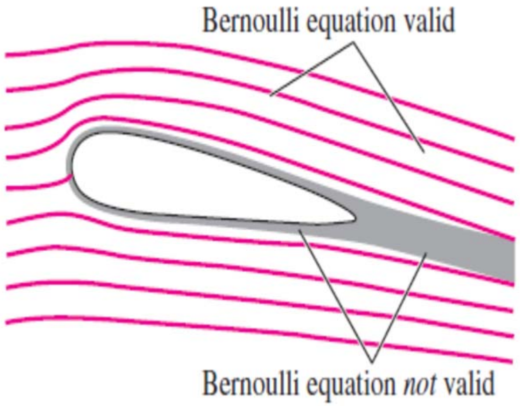
Momentum conservation along a streamline: (Bernoulli Equation)
Energy conservation: integral formulation

Very useful for steady, frictionless flows; essentially a ‘differential’, not ‘integral’ formulation

THE BERNOULLI EQUATION

The **Bernoulli equation** is *an approximate relation between pressure, velocity, and elevation,* and is valid in *regions of steady, incompressible flow where net frictional forces are negligible.*

Conservation of linear momentum principle



Key approximations:

Steady   *Frictionless*   *No shaft work*   *Incompressible flow*

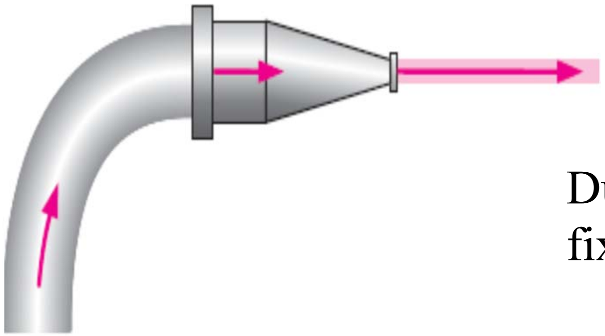
Valid along a streamline   *No heat transfer*

Viscous effects are negligibly small compared to inertial, gravitational, and pressure effects.

Inviscid flow approximation cannot be valid for an entire flow field of practical interest. (This approximation *is* reasonable in certain *regions* of many practical flows called *inviscid regions where the flow irrotational).*

They are *not* regions where the fluid itself is inviscid or frictionless, but rather they are regions where net viscous or frictional forces are negligibly small compared to other forces acting on fluid particles.

Acceleration of a Fluid Particle



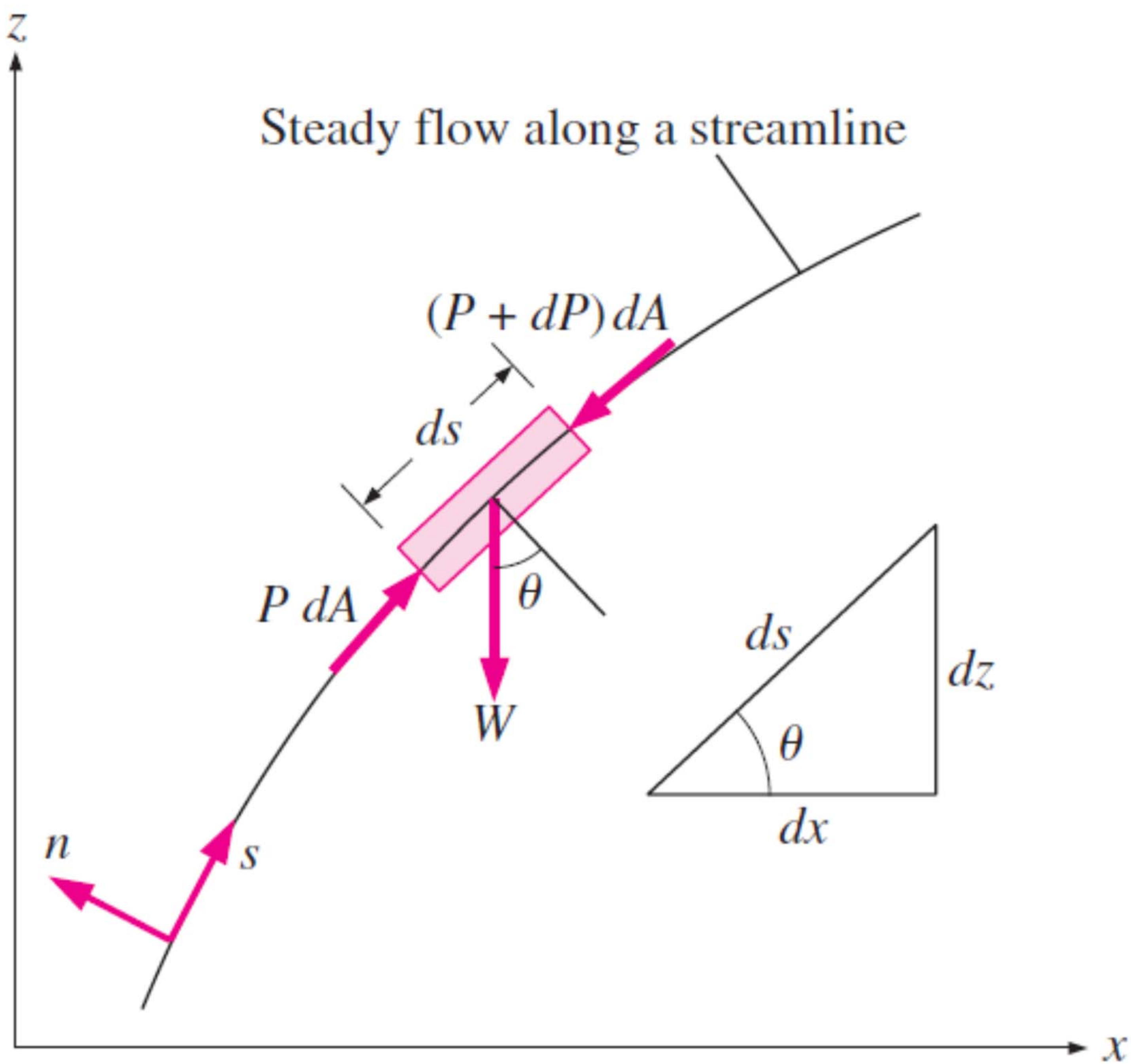
During steady flow, a fluid may not accelerate in time at a fixed point, but it may accelerate in space.

Steady simply means *no change with time at a specified location*, but the value of a quantity may change from one location to another.

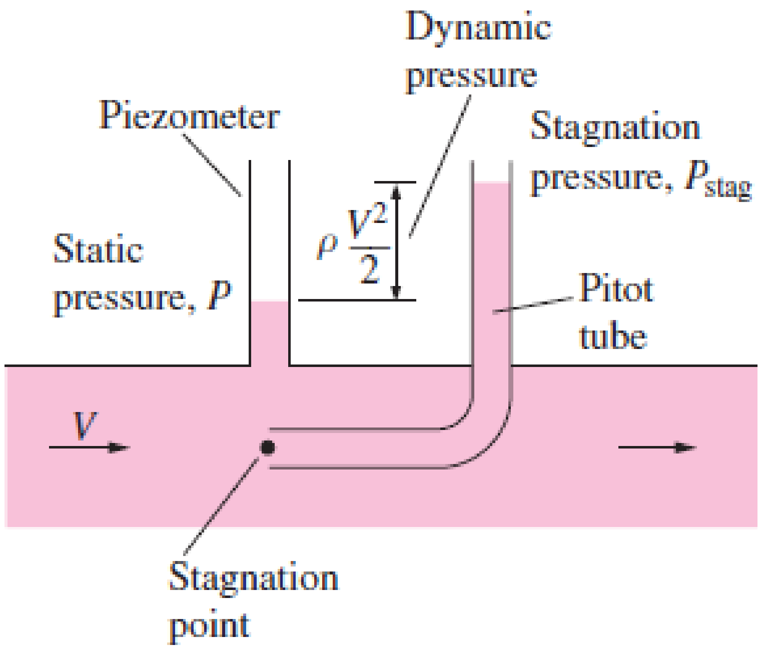
$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} dt \qquad \text{and} \qquad \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t}$$

In steady flow $\frac{\partial V}{\partial t} = 0$ and thus $V = V(s)$, and the acceleration in the s-direction becomes

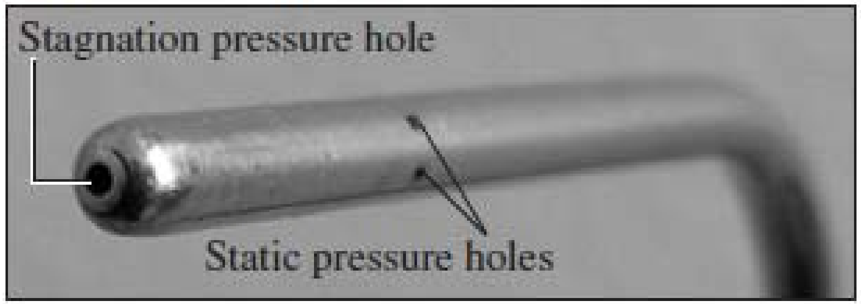
$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = V\frac{\partial V}{\partial s}$$



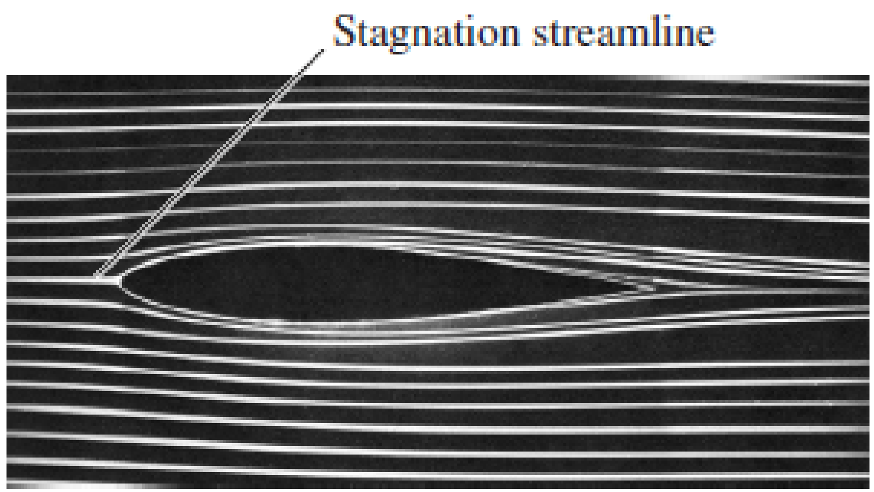
Steady flow along a streamline
(P + dP) dA
ds
P dA
θ
W
ds
dz
θ
dx
n
s
z
x

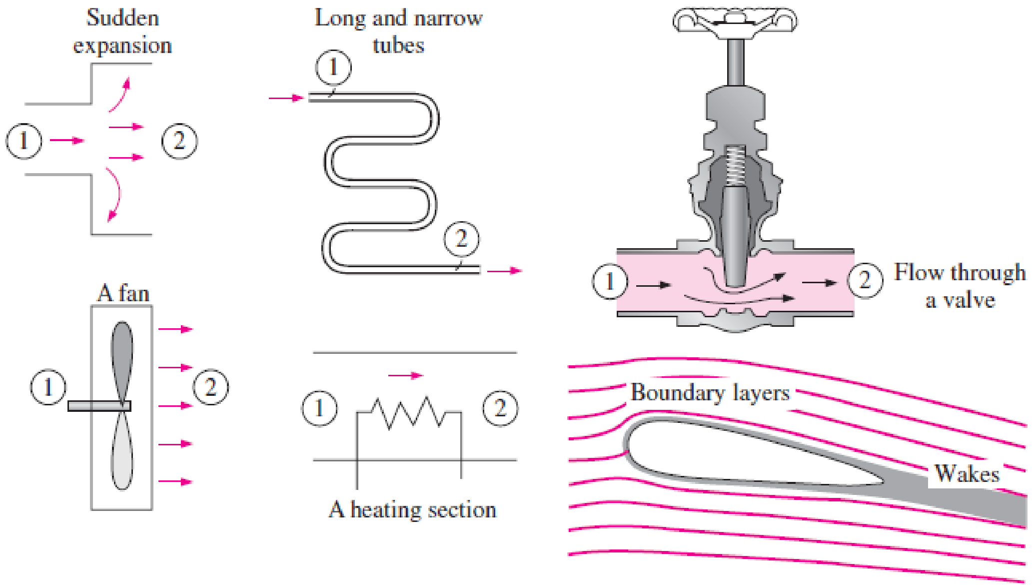


$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

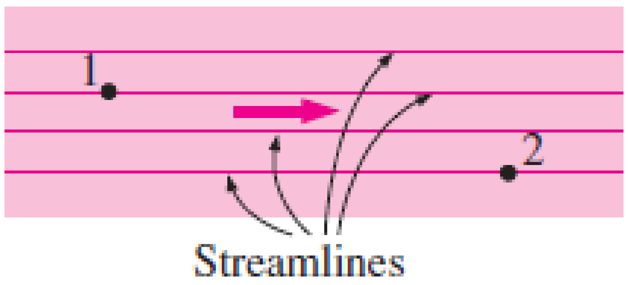


Pitot-static probe





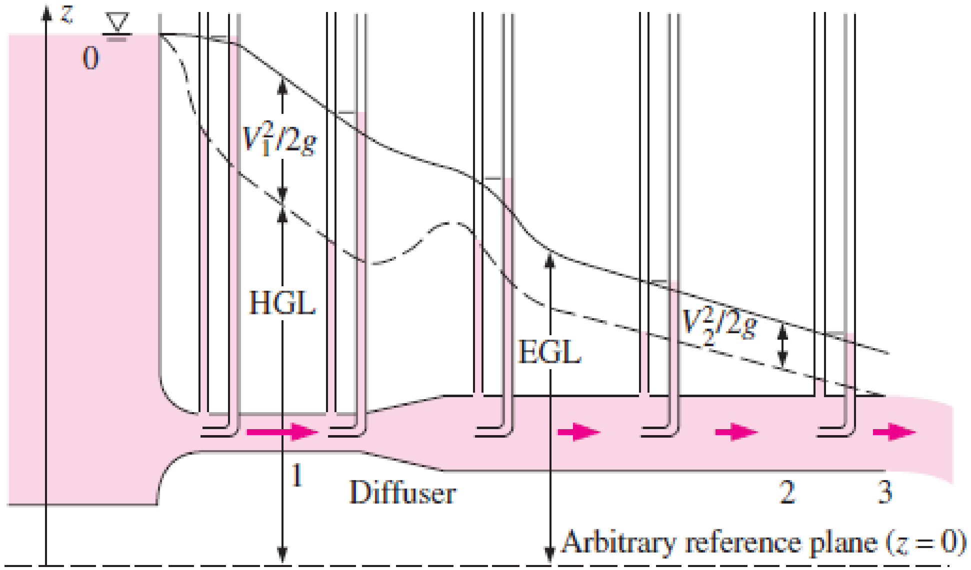
Frictional effects and components that disturb the streamlined structure of flow in a flow section



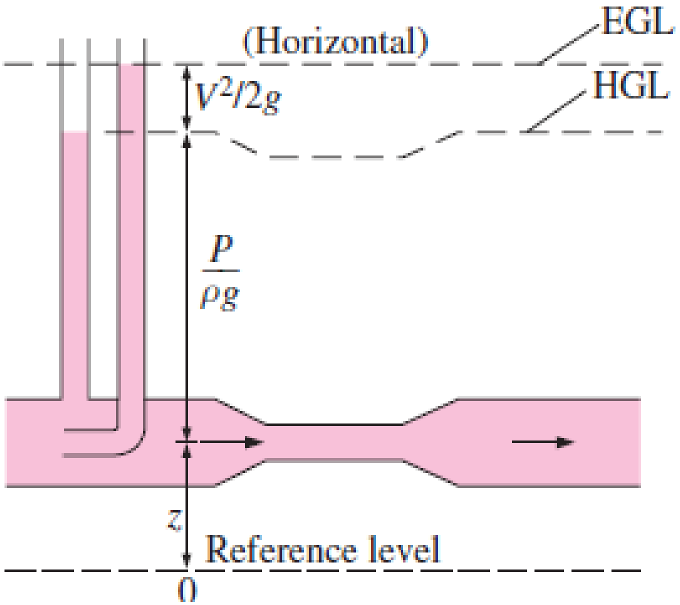
$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

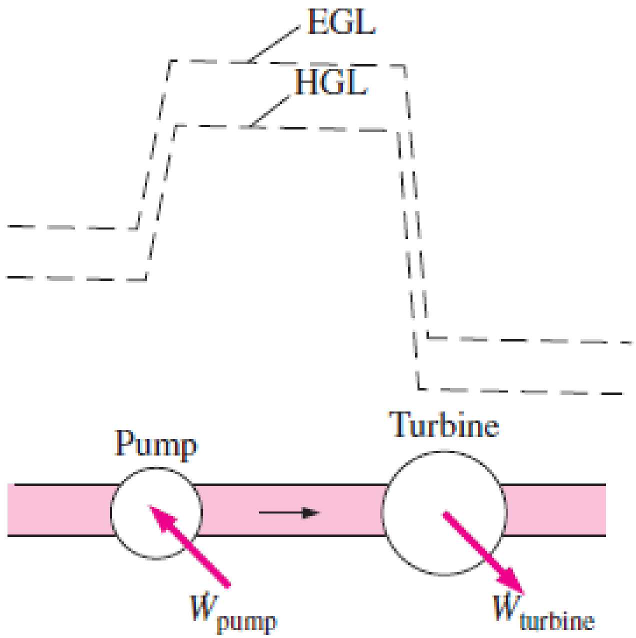
When the flow is irrotational, the Bernoulli equation becomes applicable between any two points along the flow
(not just on the same streamline).

The Bernoulli equation is valid for general *three-dimensional* flow as well, as long as it is applied along the same streamline.



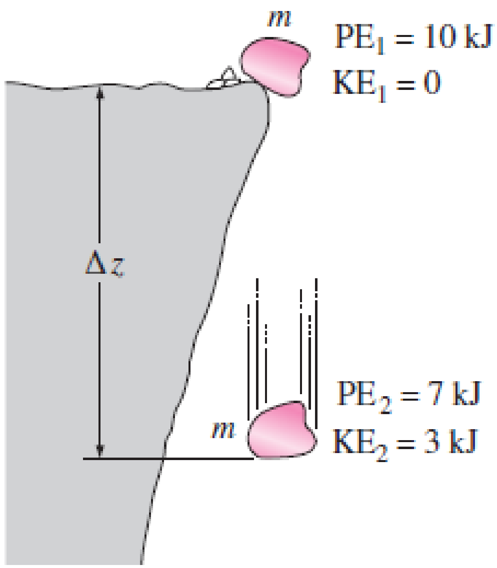
The *hydraulic grade line* (HGL) and the *energy grade line* (EGL) for free discharge from a reservoir through a horizontal pipe with a diffuser.

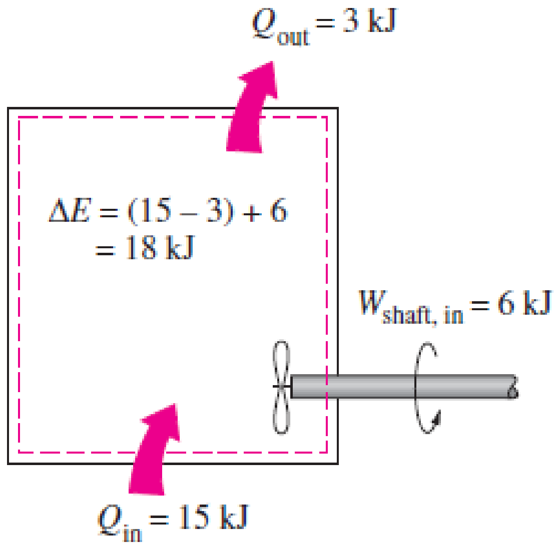




In an idealized Bernoulli-type flow, EGL is horizontal and its height remains constant.

Energy Equation





$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{net in}} = \frac{dE_{\text{sys}}}{dt} \quad \text{or} \quad \dot{Q}_{\text{net in}} + \dot{W}_{\text{net in}} = \frac{d}{dt}\int_{\text{sys}} \rho e \, dV$$

The change in the energy content of a system is equal to the difference between the energy input and the energy output, and the conservation of energy principle for any system can be expressed simply as $E_{in} - E_{out} = \Delta E$

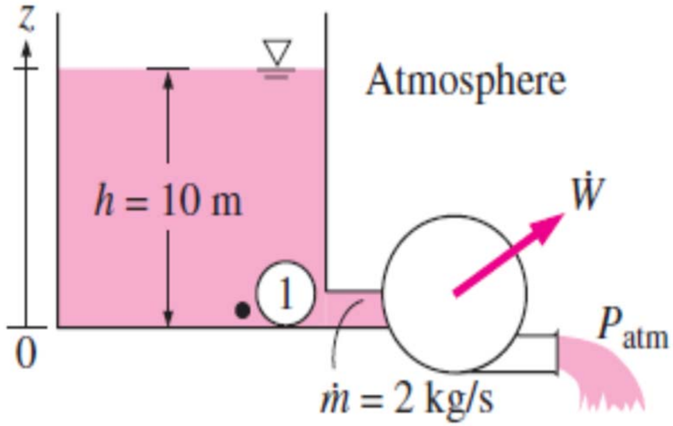
The **mechanical energy** can be defined as *the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine*.

Kinetic and potential energies are the familiar forms of mechanical energy.

Thermal energy is not mechanical energy, however, since it cannot be converted to work directly and completely (the second law of thermodynamics).

A pump transfers mechanical energy to a fluid by raising its pressure.

A turbine extracts mechanical energy from a fluid by dropping its pressure. Therefore, the pressure of a flowing fluid is also associated with its mechanical energy.



$$\dot{W}_{\max} = \dot{m}\frac{p_1 - p_{atm}}{\rho} = \dot{m}\frac{\rho g h}{\rho} = \dot{m} g h$$

$$e_{mech} = \frac{p}{\rho} + \frac{V^2}{2} + gz$$

where P/ρ is the *flow energy*, $\frac{V^2}{2}$ is the kinetic energy, gz *potential energy* of the fluid, all per unit mass, (kJ/kg)

$$e_{mech} = \frac{p_2 - p_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$