ASSIGNMENT III MSO 202 A

CAUCHY'S THEOREM, CAUCHY INTEGRAL FORMULAS, AND LIOUVILLE'S THEOREM

Exercise 0.1 : The aim of this exercise is to derive the following formula using Cauchy's Theorem:

$$\int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Verify the following:

(1) For $R > 0$, consider the closed curve γ (boundary of the sector at 0 of angle $\pi/4$) with parametrization

$$\gamma_1(t) = t,\ 0 \leqslant t \leqslant R, \quad \gamma_2(t) = Re^{it},\ 0 \leqslant t \leqslant \frac{\pi}{4}, \quad \gamma_3(t) = -te^{i\frac{\pi}{4}},\ -R \leqslant t \leqslant 0.$$

Then the integral of e^{iz^2} over γ equals 0.

(2) $\int_{\gamma_1} e^{-z^2}dz$ converges to $\int_0^\infty \cos(t^2)dt + i\int_0^\infty \sin(t^2)dt$ as $R \to \infty$.

(3) $\int_{\gamma_2} e^{iz^2}dz \to 0$ as $R \to \infty$ (Hint. Use $\sin(2t) \geqslant \frac{4t}{\pi}$ $(0 \leqslant t \leqslant \frac{\pi}{4})$).

(4) $\int_{\gamma_3} e^{iz^2}dz \to e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2}$ as $R \to \infty$ (Hint. Use $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$).

Solution.

(1) Note that γ is a closed curve and e^{iz^2} is holomorphic inside γ. By Cauchy's Theorem, $\sum_{i=1}^4 \int_{\gamma_i} e^{iz^2}dz = \int_\gamma e^{iz^2}dz = 0$.

(2) $\int_{\gamma_1} e^{iz^2}dz = \int_0^R e^{it^2}dt \to \int_0^\infty e^{it^2}dt = \int_0^\infty \cos(t^2)dt + i\int_0^\infty \sin(t^2)dt$.

(3) $\int_{\gamma_2} e^{iz^2}dz = \int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2}Rie^{it}dt = \int_0^{\frac{\pi}{4}} e^{iR^2e^{2it}}Rie^{it}dt$
$= \int_0^{\frac{\pi}{4}} e^{iR^2(\cos(2t)+i\sin(2t))}Rie^{it}dt$. Since $\sin(2t) \geqslant \frac{4t}{\pi}$, we obtain

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \leqslant \int_0^{\frac{\pi}{4}} e^{-R^2\sin(2t)}Rdt \leqslant \int_0^{\frac{\pi}{4}} e^{-R^2\frac{4t}{\pi}}Rdt \to 0 \text{ as } R \to \infty.$$

(4) $\int_{\gamma_3} e^{iz^2}dz = \int_{-R}^0 e^{it^2e^{i\frac{\pi}{2}}}(-e^{i\frac{\pi}{4}})dt = -\int_0^R e^{-t^2}e^{i\frac{\pi}{4}}dt$
$\to -e^{i\frac{\pi}{4}}\int_0^\infty e^{-t^2}dt$. But $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$, and hence

$$\int_{\gamma_3} e^{iz^2}dz \to -e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{\pi}}{2\sqrt{2}}(1+i) \text{ as } R \to \infty.$$

It follows that $\int_0^\infty \cos(t^2)dt + i\int_0^\infty \sin(t^2)dt = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$. Now compare the imaginary parts.

Exercise 0.2 : The aim of this exercise is to derive the following formula using Cauchy's Theorem:

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

Verify the following:

(1) Consider the indented semicircle γ (with $0 < r < R$) given by

$$\gamma_1(t) = t \ (-R \leqslant t \leqslant -r), \ \gamma_2(t) = re^{-it} \ (-\pi \leqslant t \leqslant 0),$$

$$\gamma_3(t) = t \ (r \leqslant t \leqslant R), \ \gamma_4(t) = Re^{it} \ (0 \leqslant t \leqslant \pi).$$

Then the integral of $f(z) = \frac{e^{iz}-1}{z}$ over γ is 0.

(2) $\int_{\gamma_1} \frac{e^{iz}-1}{z}dz \to \int_{-\infty}^0 \frac{e^{it}-1}{t}dt$ as $R \to \infty$ and $r \to 0$.

(3) $\int_{\gamma_2} \frac{e^{iz}-1}{z}dz \to 0$ as $r \to 0$.

(4) $\int_{\gamma_3} \frac{e^{iz}-1}{z}dz \to \int_0^\infty \frac{e^{it}-1}{t}dt$ as $R \to \infty$ and $r \to 0$.

(5) $\int_{\gamma_4} \frac{e^{iz}-1}{z}dz \to -i\pi$ as $R \to \infty$.

Solution.

(1) By Cauchy's Theorem, $\sum_{i=1}^4 \int_{\gamma_i} e^{iz^2}dz = 0$.

(2) $\int_{\gamma_1} \frac{e^{iz}-1}{z}dz = \int_{-R}^{-r} \frac{e^{it}-1}{t}dt \to \int_{-\infty}^0 \frac{e^{it}-1}{t}dt$ as $R \to \infty$ and $r \to 0$.

(3) $\int_{\gamma_2} \frac{e^{iz}-1}{z}dz = -i\int_{-\pi}^0 (e^{ire^{-it}} - 1)dt$. On the other hand,

$$e^{ire^{-it}} - 1 = \sum_{n=1}^\infty \frac{(ire^{-it})^n}{n!} = r\sum_{n=1}^\infty r^{n-1}\frac{(ie^{-it})^n}{n!},$$

and hence $\int_{\gamma_2} \frac{e^{iz}-1}{z}dz \to 0$ as $r \to 0$.

(4) $\int_{\gamma_3} \frac{e^{iz}-1}{z}dz = \int_r^R \frac{e^{it}-1}{t}dt \to \int_0^\infty \frac{e^{it}-1}{t}dt$ as $R \to \infty$ and $r \to 0$.

(5) $\int_{\gamma_4} \frac{e^{iz}-1}{z}dz = \int_0^\pi \frac{e^{iRe^{it}}-1}{Re^{it}}(iRe^{it})dt = i\int_0^\pi (e^{iRe^{it}} - 1) = i\int_0^\pi e^{iRe^{it}} - i\pi$. On the other hand,

$$\left| i\int_0^\pi e^{R(-\sin(t)+i\cos(t))}dt \right| \leqslant \int_0^\pi e^{-R\sin(t)}dt \to 0 \text{ as } R \to \infty,$$

and hence $\int_{\gamma_4} \frac{e^{iz}-1}{z}dz \to -i\pi$ as $R \to \infty$.

It follows that $\int_{-\infty}^\infty \frac{e^{it}-1}{t}dt = i\pi$. The desired conclusion now follows from the fact that $\frac{\sin(x)}{x}$ is the real part of $\frac{1}{i}\frac{e^{ix}-1}{x}$.

Exercise 0.3 : For $a > 0$, let γ be the circle $|z - ia| = a$. Whether $\int_\gamma \frac{1}{z^2+a^2}dz$ depends on a ? Justify your answer.

Solution. Note that $f(z) = \frac{1}{z+ia}$ is holomorphic on $|z - ia| < a$. Hence, by Cauchy's Integral formula,

$$\int_\gamma \frac{1}{z^2 + a^2}dz = \int_\gamma \frac{\frac{1}{z+ia}}{z - ia}dz = \frac{1}{z + ia}|_{z=ia} = \frac{1}{2ia}.$$

Hence the answer is Yes.

Exercise 0.4 : Compute the Taylor series of $\log z$ in the disc $|z-i| = \frac{1}{2}$.

Solution. The Taylor series of f around a is given by

$$fz) = \sum_{n=0}^{\infty} a_n(z - a)^n, \text{ where } a_n = \frac{f^{(n)}(i)}{n!}.$$

Note that $a_0 = \log i$, $a_1 = \frac{1}{z}|_{z=i} = -i$, and more generally

$$a_n = \frac{f^{(n)}(i)}{n!} = (-1)^{n+1}\frac{1}{i^n}\frac{1}{n!}(n - 1)! = -i^n\frac{1}{n}.$$

Hence the Taylor series of $\log z$ is given by

$$\log i + \sum_{n=1}^{\infty} \frac{-i^n}{n}(z - i)^n \ (z \in \mathbb{D}_{\frac{1}{2}}(i)).$$

Exercise 0.5 : Let f be entire and k a positive integer. If

$$|f(z)| \leqslant C|z^k| \ (z \in \mathbb{C})$$

for some $C > 0$ then show that f is a polynomial of degree at most k.

Solution. Since f is entire, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \frac{f^{(n)}(0)}{n!}$. Since $|f(z)| \leqslant |z^n|$, $a_0 = a_1 = \cdots = a_{k-1} = 0$. Thus

$$\left|\sum_{n=k}^{\infty} a_n z^n\right| \leq C|z|^k,$$

and hence $g(z) := \left|\sum_{n=k}^{\infty} a_n z^{n-k}\right| \leq C$. Thus the entire function g is bounded and hence constant, by Liouville's Theorem. Hence $a_n = 0$ for $n > k$, that is, $f(z) = a_k z^k$.

Exercise 0.6 : Let f be an entire function such that $|f(a)| \leqslant |f(z)|$ $(z \in \mathbb{C})$ for some $a \in \mathbb{C}$. Show that either $f(a) = 0$ or f is constant.

Solution. We may assume that $f(a) \neq 0$. But then $g(z) = \frac{1}{f(z)}$ is an entire function which is bounded (since $|g(z)| \leqslant \frac{1}{|f(a)|}$ for all $z \in \mathbb{C}$). By Liouville's Theorem, g is a constant function, and hence so is f.

Exercise 0.7 : What are all entire functions f which satisfy $f(x) = e^{x^2}$ for all $x = 1, \frac{1}{2}, \frac{1}{3}, \cdots,$. Justify your answer.

Solution. Define the function $g(z) = f(z) - e^{z^2}$. The sequence $\{1/n\}$ converges to 0, and g vanishes at every point in $\{1/n\}$. Hence, by the Identity Theorem, g is identically 0. Hence e^{z^2} is the only entire function with the above property.

Remark. There are infinitely many real differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(x) = e^{x^2}$ for all $x = 1, \frac{1}{2}, \frac{1}{3}, \cdots,$.

Exercise 0.8 : Let f and g be two entire functions. Show that if $f(z)g(z) = 0$ for all $z \in \mathbb{C}$ then either $f(z) = 0$ for all $z \in \mathbb{C}$ or $g(z) = 0$ for all $z \in \mathbb{C}$.

Solution. Suppose $f(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. By continuity of f, $f(z) \neq 0$ for z in some disc centered at z_0. Since $f(z)g(z) = 0$ for all $z \in \mathbb{C}$, we must have $g(z) = 0$ in that disc. By Identity Theorem, g must be identically 0.