MECHANICAL VIBRATIONS

THEORY AND APPLICATIONS

S. GRAHAM KELLY

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S. Graham Kelly

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About The Author

S. Graham Kelly received a B.S. in engineering science and mechanics, in 1975, a M.S in engineering mechanics, and a Ph.D. in engineering mechanics in 1979, all from Virginia Tech.

He served on the faculty of the University of Notre Dame from 1979 to 1982. Since 1982, Dr. Kelly has served on the faculty at The University of Akron where he has been active in teaching, research, and administration.

Besides vibrations, he has taught undergraduate courses in statics, dynamics, mechanics of solids, system dynamics, fluid mechanics, compressible fluid mechanics, engineering probability, numerical analysis, and freshman engineering. Dr. Kelly's graduate teaching includes courses in vibrations of discrete systems, vibrations of continuous systems, continuum mechanics, hydrodynamic stability, and advanced mathematics for engineers. Dr. Kelly is the recipient of the 1994 Chemstress award for Outstanding Teacher in the College of Engineering at the University of Akron.

Dr. Kelly is also known for his distinguished career in academic administration. His service includes stints as Associate Dean of Engineering, Associate Provost, and Dean of Engineering from 1998 to 2003. While serving in administration, Dr. Kelly continued teaching at least one course per semester.

Since returning to the faculty full-time in 2003, Dr. Kelly has enjoyed more time for teaching, research, and writing projects. He regularly advises graduate students in their research work on topics in vibrations and solid mechanics. Dr. Kelly is also the author of *System Dynamics and Response, Advanced Vibration Analysis, Advanced Engineering Mathematics with Modeling Applications, Fundamentals of Mechanical Vibrations* (*First* and *Second Editions*) and *Schaum's Outline in Theory and Problems in Mechanical Vibrations.*

Preface

Engineers apply mathematics and science to solve problems. In a traditional undergraduate engineering curriculum, students begin their academic career by taking courses in mathematics and basic sciences such as chemistry and physics. Students begin to develop basic problem-solving skills in engineering courses such as statics, dynamics, mechanics of solids, fluid mechanics, and thermodynamics. In such courses, students learn to apply basic laws of nature, constitutive equations, and equations of state to develop solutions to abstract engineering problems.

Vibrations is one of the first courses where students learn to apply the knowledge obtained from mathematics and basic engineering science courses to solve practical problems. While the knowledge about vibrations and vibrating systems is important, the problem-solving skills obtained while studying vibrations are just as important. The objectives of this book are twofold: to present the basic principles of engineering vibrations and to present them in a framework where the reader will advance his/her knowledge and skill in engineering problem solving.

This book is intended for use as a text in a junior- or senior-level course in vibrations. It could be used in a course populated by both undergraduate and graduate students. The latter chapters are appropriate for use as a stand-alone graduate course in vibrations. The prerequisites for such a course should include courses in statics, dynamics, mechanics of materials, and mathematics using differential equations. Some material covered in a course in fluid mechanics is included, but this material can be omitted without a loss in continuity.

Chapter 1 is introductory, reviewing concepts such as dynamics, so that all readers are familiar with the terminology and procedures. Chapter 2 focuses on the elements that comprise mechanical systems and the methods of mathematical modeling of mechanical systems. It presents two methods of the derivation of differential equations: the free-body diagram method and the energy method, which are used throughout the book. Chapters 3 through 5 focus on single degree-of-freedom (SDOF) systems. Chapter 6 is focused solely on two degree-of-freedom systems. Chapters 7 through 9 focus on general multiple degree-of-freedom systems. Chapter 10 provides a brief overview of continuous systems. The topic of Chapter 11 is the finite-element methods, which is a numerical method with its origin in energy methods, allowing continuous systems to be modeled as discrete systems. Chapter 12 introduces the reader to nonlinear vibrations, while Chapter 13 provides a brief introduction to random vibrations.

The references at the end of this text list many excellent vibrations books that address the topics of vibration and design for vibration suppression. There is a need for this book, as it has several unique features:

- Two benchmark problems are studied throughout the book. Statements defining the generic problems are presented in Chapter 1. Assumptions are made to render SDOF models of the systems in Chapter 2 and the free and forced vibrations of the systems studied in Chapters 3 through 5, including vibration isolation. Two degree-of-freedom system models are considered in Chapter 6, while MDOF models are studied in

Chapters 7 through 9. A continuous-systems model for one benchmark problem is considered in Chapter 10 and solved using the finite-element method in Chapter 11. A random-vibration model of the other benchmark problem is considered in Chapter 13. The models get more sophisticated as the book progresses.
- Most vibration problems (certainly ones encountered by undergraduates) involve the planar motion of rigid bodies. Thus, a free-body diagram method based upon D'Alembert's principle is developed and used for rigid bodies or systems of rigid bodies undergoing planar motion.
- An energy method called the equivalent systems method is developed for SDOF systems without introducing Lagrange's equations. Lagrange's equations are reserved for MDOF systems.
- Most chapters have a *Further Examples* section which presents problems using concepts presented in several sections or even several chapters of the book.
- MATLAB® is used in examples throughout the book as a computational and graphical aid. All programs used in the book are available at the specific book website accessible through www.cengage.com/engineering.
- The Laplace transform method and the concept of the transfer function (or the impulsive response) is used in MDOF problems. The sinusoidal transfer function is used to solve MDOF problems with harmonic excitation.
- The topic of design for vibration suppression is covered where appropriate. The design of vibration isolation for harmonic excitation is covered in Chapter 4, vibration isolation from pulses is covered in Chapter 5, design of vibration absorbers is considered in Chapter 6, and vibration isolation problems for general MDOF systems is considered in Chapter 9.

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The author acknowledges the support and encouragement of numerous people in the preparation of this book. Suggestions for improvement were taken from many students at The University of Akron. The author would like to especially thank former students Ken Kuhlmann for assistance with the problem involving the rotating manometer in Chapter 12, Mark Pixley for helping with the original concept of the prototype for the software package available at the website, and J.B. Suh for general support. The author also expresses gratitude to Chris Carson, Executive Director, Global Publishing; Chris Shortt, Publisher, Global Engineering; Randall Adams, Senior Acquisitions Editor; and Hilda Gowans, Senior Developmental Editor, for encouragement and guidance throughout the project. The author also thanks George G. Adams, Northeastern University; Cetin Cetinkaya, Clarkson University; Shanzhong (Shawn) Duan, South Dakota State University; Michael J. Leamy, Georgia Institute of Technology; Colin Novak, University of Windsor; Aldo Sestieri, University La Sapienza Roma; and Jean Zu, University of Toronto, for their valuable comments and suggestions for making this a better book. Finally, the author expresses appreciation to his wife, Seala Fletcher-Kelly, not only for her support and encouragement during the project but for her help with the figures as well.

S. Graham Kelly

Contents

Chapter 1

INTRODUCTION

1.1 THE STUDY OF VIBRATIONS

Vibrations are oscillations of a mechanical or structural system about an equilibrium position. Vibrations are initiated when an inertia element is displaced from its equilibrium position due to an energy imparted to the system through an external source. A restoring force, or a conservative force developed in a potential energy element, pulls the element back toward equilibrium. When work is done on the block of Figure 1.1(a) to displace it from its equilibrium position, potential energy is developed in the spring. When the block is released the spring force pulls the block toward equilibrium with the potential energy being converted to kinetic energy. In the absence of non-conservative forces, this transfer of energy is continual, causing the block to oscillate about its equilibrium position. When the pendulum of Figure 1.1(b) is released from a position above its equilibrium position the moment of the gravity force pulls the particle, the pendulum bob, back toward equilibrium with potential energy being converted to kinetic energy. In the absence of non-conservative forces, the pendulum will oscillate about the vertical equilibrium position.

Non-conservative forces can dissipate or add energy to the system. The block of Figure 1.2(a) slides on a surface with a friction force developed between the block and the surface. The friction force is non-conservative and dissipates energy. If the block is given a displacement from equilibrium and released, the energy dissipated by the friction force eventually causes the motion to cease. Motion is continued only if additional energy is added to the system as by the externally applied force in Figure 1.2(b).

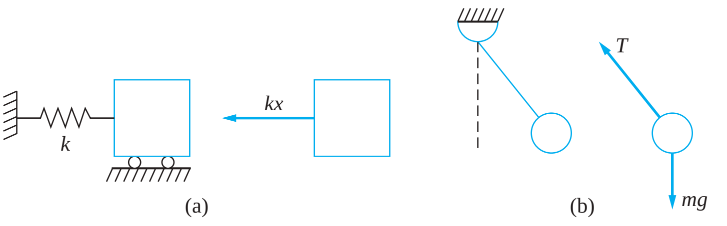

FIGURE 1.1
(a) When the block is displaced from equilibrium, the force developed in the spring (as a result of the stored potential energy) pulls the block back toward the equilibrium position. (b) When the pendulum is rotated away from the vertical equilibrium position, the moment of the gravity force about the support pulls the pendulum back toward the equilibrium position.

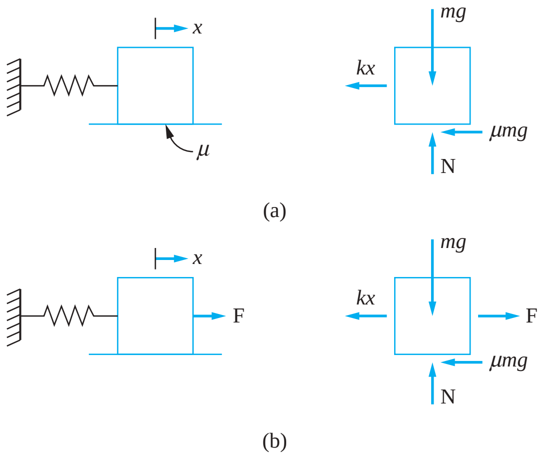

FIGURE 1.2
(a) Friction is a non-conservative force which dissipates the total energy of the system. (b) The external force is a non-conservative force which does work on the system

Vibrations occur in many mechanical and structural systems. If uncontrolled, vibration can lead to catastrophic situations. Vibrations of machine tools or machine tool chatter can lead to improper machining of parts. Structural failure can occur because of large dynamic stresses developed during earthquakes or even wind-induced vibration. Vibrations induced by an unbalanced helicopter blade while rotating at high speeds can lead to the blade's failure and catastrophe for the helicopter. Excessive vibrations of pumps, compressors, turbomachinery, and other industrial machines can induce vibrations of the surrounding structure, leading to inefficient operation of the machines while the noise produced can cause human discomfort.

Vibrations can be introduced, with beneficial effects, into systems in which they would not naturally occur. Vehicle suspension systems are designed to protect passengers from discomfort when traveling over rough terrain. Vibration isolators are used to protect structures from excessive forces developed in the operation of rotating machinery. Cushioning is used in packaging to protect fragile items from impulsive forces.

Energy harvesting takes unwanted vibrations and turns them into stored energy. An energy harvester is a device that is attached to an automobile, a machine, or any system that is undergoing vibrations. The energy harvester has a seismic mass which vibrates when excited, and that energy is captured electronically. The principle upon which energy harvesting works is discussed in Chapter 4.

Micro-electromechanical (MEMS) systems and nano-electromechanical (NEMS) systems use vibrations. MEMS sensors are designed using concepts of vibrations. The tip of an atomic force microscope uses vibrations of a nanotube to probe a specimen. Applications to MEMS and NEMS are sprinkled throughout this text.

Biomechanics is an area where vibrations are used. The human body is modeled using principles of vibration analysis. Chapter 7 introduces a three-degree-of-freedom model of a human hand and upper arm proposed by Dong, Dong, Wu, and Rakheja in the *Journal of Biomechanics.*

The study of vibrations begins with the mathematical modeling of vibrating systems. Solutions to the resulting mathematical problems are obtained and analyzed. The solutions are used to answer basic questions about the vibrations of a system as well as to determine how unwanted vibrations can be reduced or how vibrations can be introduced into a system with beneficial effects. Mathematical modeling leads to the development of principles governing the behavior of vibrating systems.

The purpose of this chapter is to provide an introduction to vibrations and a review of important concepts which are used in the analysis of vibrations. This chapter begins with the mathematical modeling of vibrating systems. This section reviews the intent of the modeling and outlines the procedure which should be followed in mathematical modeling of vibrating systems.

The coordinates in which the motion of a vibrating system is described are called the generalized coordinates. They are defined in Section 1.3, along with the definition of degrees of freedom. Section 1.4 presents the terms which are used to classify vibrations and describe further how this book is organized.

Section 1.5 is focused on dimensional analysis, including the Buckingham Pi theorem. This is a topic which is covered in fluid mechanics courses but is given little attention in solid mechanics and dynamics courses. It is important for the study of vibrations, as is steady-state amplitudes of vibrating systems are written in terms of non-dimensional variables for an easier understanding of dependence on parameters.

Simple harmonic motion represents the motion of many undamped systems and is presented in Section 1.6.

Section 1.7 provides a review of the dynamics of particles and rigid bodies used in this work. Kinematics of particles is presented and is followed by kinematics of rigid bodies undergoing planar motion. Kinetics of particles is based upon Newton's second law applied to a free-body diagram (FBD). A form of D'Almebert's principle is used to analyze problems involving rigid bodies undergoing planar motion. Pre-integrated forms of Newton's second law, the principle of work and energy, and the principle of impulse and momentum are presented.

Section 1.8 presents two benchmark problems which are used throughout the book to illustrate the concepts presented in each chapter. The benchmark problems will be reviewed at the end of each chapter. Section 1.9 presents further problems for additional study. This section will be present at the end of most chapters and will cover problems that use concepts from more than one section or even more than one chapter. Every chapter, including this one, ends with a summary of the important concepts covered and of the important equations introduced in that chapter.

Differential equations are used in Chapters 3, 4, and 5 to model single degree-of-freedom (SDOF) systems. Systems of differential equations are used in Chapters 6, 7, 8, and 9 to study multiple degree-of-freedom systems. Partial differential equations are used in Chapter 10 to study continuous systems. Chapter 11 introduces an approximate method for the solution of partial differential equations. Chapter 12 uses nonlinear differential equations to model nonlinear systems. Chapter 13 uses stochastic differential equations to study random vibrations. Differential equations are not the focus of this text, although methods of solution are presented. The reader is referred to a text on differential equations for a more thorough understanding of the mathematical methods employed.

1.2 MATHEMATICAL MODELING

Solution of an engineering problem often requires mathematical modeling of a physical system. The modeling procedure is the same for all engineering disciplines, although the details of the modeling vary between disciplines. The steps in the procedure are presented and the details are specialized for vibrations problems.

1.2.1 PROBLEM IDENTIFICATION

The system to be modeled is abstracted from its surroundings, and the effects of the surroundings are noted. Known constants are specified. Parameters which are to remain variable are identified.

The intent of the modeling is specified. Possible intents for modeling systems undergoing vibrations include analysis, design, and synthesis. Analysis occurs when all parameters are specified and the vibrations of the system are predicted. Design applications include parametric design, specifying the parameters of the system to achieve a certain design objective, or designing the system by identifying its components.

1.2.2 ASSUMPTIONS

Assumptions are made to simplify the modeling. If all effects are included in the modeling of a physical system, the resulting equations are usually so complex that a mathematical solution is impossible. When assumptions are used, an approximate physical system is modeled. An approximation should only be made if the solution to the resulting approximate problem is easier than the solution to the original problem and with the assumption that the results of the modeling are accurate enough for the use they are intended.

Certain implicit assumptions are used in the modeling of most physical systems. These assumptions are taken for granted and rarely mentioned explicitly. Implicit assumptions used throughout this book include:

1. Physical properties are continuous functions of spatial variables. This *continnum assumption* implies that a system can be treated as a continuous piece of matter. The continuum assumption breaks down when the length scale is of the order of the mean free path of a molecule. There is some debate as to whether the continuum assumption is valid in modeling new engineering materials, such as carbon nanotubes. Vibrations of nanotubes where the length-to-diameter ratio is large can be modeled reasonably using the continuum assumption, but small length-to-diameter ratio nanotubes must be modeled using molecular dynamics. That is, each molecule is treated as a separate particle.
2. The earth is an inertial reference frame, thus allowing application of Newton's laws in a reference frame fixed to the earth.

3. Relativistic effects are ignored. (Certaintly, velocities encountered in the modeling of vibrations problems are much less than the speed of light).

4. Gravity is the only external force field. The acceleration due to gravity is 9.81 m/s^2 (32.2 ft/s^2) on the surface of the earth.

5. The systems considered are not subject to nuclear reactions, chemical reactions, external heat transfer, or any other source of thermal energy.

6. All materials are linear, isotropic, and homogeneous.

7. The usual assumptions of mechanics of material apply. This includes plane sections remaining plane for beams in bending and circular sections under torsional loads do not warp.

Explicit assumptions are those specific to a particular problem. An explicit assumption is made to eliminate negligible effects from the analysis or to simplify the problem while retaining appropriate accuracy. An explicit assumption should be verified, if possible, on completion of the modeling.

All physical systems are inherently nonlinear. Exact mathematical modeling of any physical system leads to nonlinear differential equations, which often have no analytical solution. Since exact solutions of linear differential equations can usually be determined easily, assumptions are often made to *linearize* the problem. A linearizing assumption leads either to the removal of nonlinear terms in the governing equations or to the approximation of nonlinear terms by linear terms.

A *geometric nonlinearity* occurs as a result of the system's geometry. When the differential equation governing the motion of the pendulum bob of Figure 1.1(b) is derived, a term equal to $\sin\theta$ (where θ is the angular displacement from the equilibrium position) occurs. If θ is small, $\sin\theta \approx \theta$ and the differential equation is linearized. However, if aerodynamic drag is included in the modeling, the differential equation is still nonlinear.

If the spring in the system of Figure 1.1(a) is nonlinear, the force-displacement relation in the spring may be $F = k_1x + k_3x^3$. The resulting differential equation that governs the motion of the system is nonlinear. This is an example of a *material nonlinearity*. The assumption is often made that either the amplitude of vibration is small (such that $k_3x^3 \ll k_1x$ and the nonlinear term neglected).

Nonlinear systems behave differently than linear systems. If linearization of the differential equation occurs, it is important that the results are checked to ensure that the linearization assumption is valid.

When analyzing the results of mathematical modeling, one has to keep in mind that the mathematical model is only an approximation to the true physical system. The actual system behavior may be somewhat different than that predicted using the mathematical model. When aerodynamic drag and all other forms of friction are neglected in a mathematical model of the pendulum of Figure 1.1(b) then perpetual motion is predicted for the situation when the pendulum is given an initial displacement and released from rest. Such perpetual motion is impossible. Even though neglecting aerodynamic drag leads to an incorrect time history of motion, the model is still useful in predicting the period, frequency, and amplitude of motion.

Once results have been obtained by using a mathematical model, the validity of all assumptions should be checked.

1.2.3 BASIC LAWS OF NATURE

A basic law of nature is a physical law that applies to all physical systems regardless of the material from which the system is constructed. These laws are observable, but cannot be derived from any more fundamental law. They are empirical. There exist only a few basic laws of nature: conservation of mass, conservation of momentum, conservation of energy, and the second and third laws of thermodynamics.

Conservation of momentum, both linear and angular, is usually the only physical law that is of significance in application to vibrating systems. Application of the principle of conservation of mass to vibrations problems is trivial. Applications of the second and third laws of thermodynamics do not yield any useful information. In the absence of thermal energy, the principle of conservation of energy reduces to the mechanical work-energy principle, which is derived from Newton's laws.

1.2.4 CONSTITUTIVE EQUATIONS

Constitutive equations provide information about the materials of which a system is made. Different materials behave differently under different conditions. Steel and rubber behave differently because their constitutive equations have different forms. While the constitutive equations for steel and aluminum are of the same form, the constants involved in the equations are different. Constitutive equations are used to develop force-displacement relationships for mechanical components that are used in modeling vibrating systems.

1.2.5 GEOMETRIC CONSTRAINTS

Application of geometric constraints is often necessary to complete the mathematical modeling of an engineering system. Geometric constraints can be in the form of kinematic relationships between displacement, velocity, and acceleration. When application of basic laws of nature and constitutive equations lead to differential equations, the use of geometric constraints is often necessary to formulate the requisite boundary and initial conditions.

1.2.6 DIAGRAMS

Diagrams are often necessary to gain a better understanding of the problem. In vibrations, one is interested in forces and their effects on a system. Hence, a *free-body diagram* (*FBD*), which is a diagram of the body abstracted from its surrounding and showing the effect of those surroundings in the form of forces, is drawn for the system. Since one is interested in modeling the system for all time, a FBD is drawn at an arbitrary instant of time.

Two types of forces are illustrated on a FBD: body forces and surface forces. A *body force* is applied to a particle in the interior of the body and is a result of the body existence in an external force field. An implicit assumption is that gravity is the only external force field surrounding the body. The gravity force $-(mg)$ is applied to the center of mass and is directed toward the center of the earth, usually taken to be the downward direction, as shown in Figure 1.3.

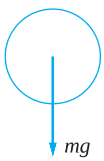

FIGURE 1.3
The gravity force is directed toward the center of the earth, usually taken as the vertical direction.

Surface forces are drawn at a particle on the body's boundary as a result of the interaction between the body and its surroundings. An external surface force is a reaction between the body and its external surface. Surface forces may be acting at a single point on the boundary of the body, as shown in Figure 1.4(a), or they may be distributed over the surface of the

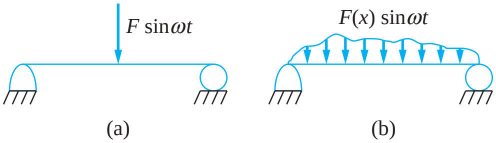

FIGURE 1.4
(a) A surface force applied to the beam may be concentrated at a single point. (b) A surface force also may be a distributed load, as shown on the beam.

body, as illustrated in Figure 1.4(b). Surface forces also may be the resultant of a stress distribution.

In analyzing vibrations, FBDs are generally drawn at an arbitrary instant in the motion of the body. Forces are labeled in terms of coordinates and system parameters. Constitutive laws and geometric constraints are taken into consideration. An FBD drawn and annotated as described, is ready for the basic laws of nature to be applied.

1.2.7 MATHEMATICAL SOLUTION

The mathematical modeling of a physical system results in the formulation of a mathematical problem. The modeling is not complete until the appropriate mathematics is applied and a solution obtained.

The type of mathematics required is different for different types of problems. Modeling of many statics, dynamics, and mechanics of solids problems leads only to algebraic equations. Mathematical modeling of vibrations problems leads to differential equations.

Exact analytical solutions, when they exist, are preferable to numerical or approximate solutions. Exact solutions are available for many linear problems, but for only a few nonlinear problems.

1.2.8 PHYSICAL INTERPRETATION OF MATHEMATICAL RESULTS

After the mathematical modeling is complete, there is still work to be done. Vibrations is an applied science—the results must mean something. The end result may be generic: to determine the frequency response of a system due to a harmonic force where a non-dimensional form of the frequency response would be a great help in understanding the behavior of the system. The reason for the mathematical modeling may be more specific: to analyze a specific system to determine the maximum displacement. It only remains to substitute given numbers. The objective of the mathematical modeling dictates the form of the physical interpretation of the results.

The mathematical modeling of a vibrations problem is analyzed from the beginning (where the conservation laws are applied to a FBD) to the end (where the results are used). A variety of different systems are analyzed, and the results of the modeling applied.

1.3 GENERALIZED COORDINATES

Mathematical modeling of a physical system requires the selection of a set of variables that describes the behavior of the system. *Dependent variables* are the variables that describe the physical behavior of the system. Examples of dependent variables are displacement of a particle in a dynamic system, the components of the velocity vector in a fluid flow problem, the temperature in a heat transfer problem, or the electric current in an AC circuit problem. *Independent variables* are the variables with which the dependent variables change. That is, the dependent variables are functions of the independent variables. An independent variable for most dynamic systems and electric circuit problems is time. The temperature distribution in a heat transfer problem may be a function of spatial position as well as time. The dependent variables in most vibrations problems are the displacements of specified particles from the system's equilibrium position while time is the independent variable.

Coordinates are kinematically independent if there is no geometric relationship between them. The coordinates in Figure 1.5(a) are kinematically dependent because

$$x = r_2\theta \tag{1.1}$$

and

$$y = r_1\theta = \frac{r_1}{r_2} \tag{1.2}$$

In Figure 1.5(b), the cables have some elasticity which is modeled by springs. The coordinates x, y, and θ are kinematically independent, because Equations (1.1) and (1.2) are not applicable due to the elasticity of the cables.

The number of *degrees of freedom* for a system is the number of kinematically independent variables necessary to completely describe the motion of every particle in the system. Any set of n kinematically independent coordinate for a system with n degrees of freedom is called a set of *generalized coordinates*. The number of degrees of freedom used in analyzing a system is unique, but the choice of generalized coordinates used to describe the motion of the system is not unique. The generalized coordinates are the dependent variables for a vibrations problem and are functions of the independent variable, time. If the time history of the generalized coordinates is known, the displacement, velocity, and acceleration of any particle in the system can be determined by using kinematics.

A single particle free to move in space has three degrees of freedom, and a suitable choice of generalized coordinates is the cartesian coordinates (x, y, z) of the particle with respect to a fixed reference frame. As the particle moves in space, its position is a function of time.

A unrestrained rigid body has six degrees of freedom, three coordinates for the displacement of its mass center, and angular rotation about three coordinate axes, as shown in Figure 1.6(a). However constraints may reduce that number. A rigid body undergoing planar motion has three possible degrees of freedom, the displacement of its mass center in

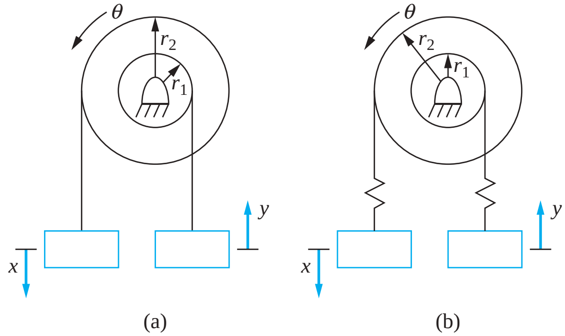

FIGURE 1.5
(a) The coordinates x, y, and θ are kinematically dependent, because there exists a kinematic relationship between them. (b) The coordinates x, y, and θ are kinematically independent, because there is no kinematic relation between them due to the elasticity of the cables modeled here as springs.

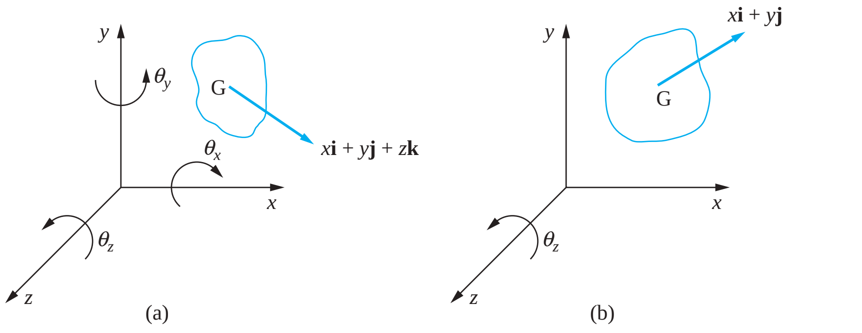

FIGURE 1.6
(a) The general three-dimensional motion of a rigid body has six degrees of freedom. Its mass center is free to move in three coordinate directions, and rotation may occur about three axes. (b) A rigid body undergoing planar motion has at most three degree of freedom. Its mass center can move in two directions, and rotation occurs only about an axis perpendicular to the plane of motion.

a plane, and angular rotation about one axis, as illustrated in Figure 1.6(b). Two rigid bodies undergoing planar motion have six degrees of freedom, but they may be connected in a manner which constrains them and reduces the number of degrees of freedom.

EXAMPLE 1.1

Each of the systems of Figure 1.7 is in equilibrium in the position shown and undergoes planar motion. All bodies are rigid. Specify, for each system, the number of degrees of freedom and recommend a set of generalized coordinates.

SOLUTION

(a) The system has one degree of freedom. If θ, the clockwise angular displacement of the bar from the system's equilibrium position, is chosen as the generalized coordinate, then a

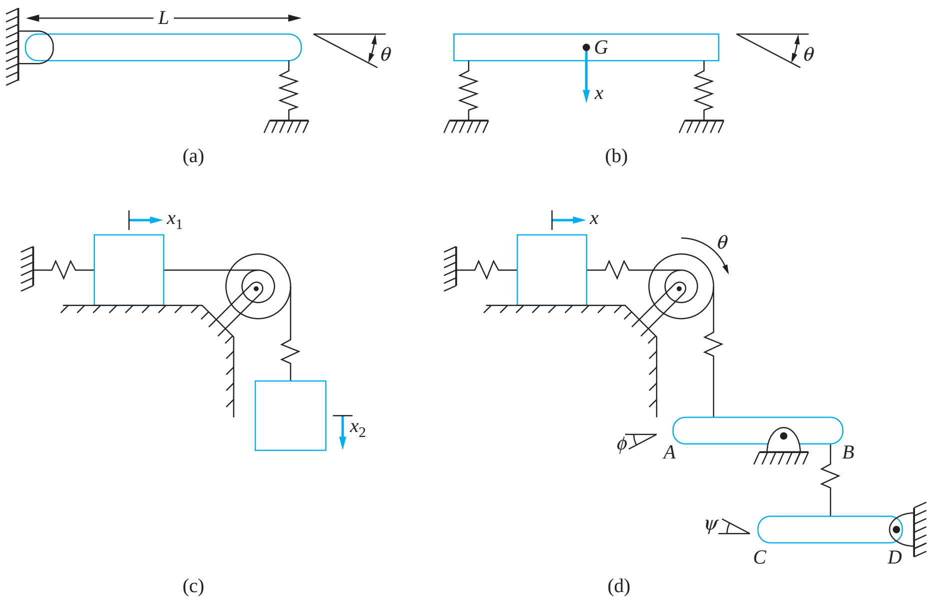

FIGURE 1.7
(a) through (d) Systems of Example 1.1. Possible generalized coordinates are indicated.

particle initially a distance a from the fixed support has a horizontal position $a\cos\theta$ and a vertical displacement $a\sin\theta$.

(b) The system has two degrees of freedom, assuming it is constrained from side-to-side motion. If θ, the clockwise angular displacement of the bar measured from its equilibrium position, and x, the displacement of the bar's mass center measured from equilibrium, are chosen as generalized coordinates, then the displacement of a particle a distance d to the right of the mass center is $x + d\sin\theta$. An alternate choice for the generalized coordinates is x_1, the displacement of the right end of the bar, and x_2, the displacement of the left end of the bar, both measured from equilibrium.

(c) The system has two degrees of freedom. The sliding block is rigidly connected to the pulley, but the pulley is connected by a spring to the hanging block. Two possible degrees of freedom are x_1 (the displacement of the sliding block from equilibrium) and x_2 (the displacement of the hanging mass from the system's equilibrium position). An alternate choice of generalized coordinates are θ (the clockwise angular rotation of the pulley from equilibrium) and x_2.

(d) The system has four degrees of freedom. The sliding block is connected by an elastic cable to the pulley. The pulley is connected by an elastic cable to bar AB, which is connected by a spring to bar CD. A possible set of generalized coordinates (all from equilibrium) is x, the displacement of the sliding block; θ, the clockwise angular rotation of the pulley; ϕ, the counterclockwise angular rotation of bar AB; and ψ, the clockwise angular rotation of bar CD.

The systems of Example 1.1 are assumed to be composed of rigid bodies. The relative displacement of two particles on a rigid body remains fixed as motion occurs. Particles in an elastic body may move relative to one another as motion occurs. Particles A and C lie along the neutral axis of the cantilever beam of Figure 1.8, while particle B is in the cross section obtained by passing a perpendicular plane through the neutral axis at A. Because of the assumption that plane sections remain plane during displacement, the displacements of particles A and B are related. However, the displacement of particle C relative to particle A depends on the loading of the beam. Thus, the displacements of A and C are kinematically independent. Since A and C represent arbitrary particles on the beam's neutral axis, it is inferred that there is no kinematic relationship between the displacements of any two particles along the neutral axis. Since there are an infinite number of particles along the neutral axis, the cantilever beam has an infinite number of degrees of freedom. In this case, an independent spatial variable x, which is the distance along the neutral axis to a particle when the beam is in equilibrium, is defined. The dependent variable, displacement, is a function of the independent variables x and time, $w(x, t)$.

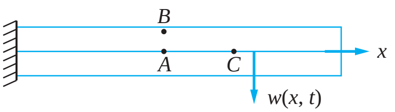

FIGURE 1.8
The transverse displacements of particles A and B are equal from elementary beam theory. However, no kinematic relationship exists between the displacements of particle A and B particle C. The beam has an infinite number of degrees of freedom and is a continuous system.

1.4 CLASSIFICATION OF VIBRATION

Vibrations are classified by the number of degrees of freedom necessary for their modeling, the type of forcing they are subject to, and the assumptions used in the modeling. Vibrations of systems that have a finite number of degrees of freedom are called *discrete systems*. A system with one degree of freedom is called a *single degree-of-freedom (SDOF) system*. A system with two or more degrees of freedom is called a *multiple degree-of-freedom (MDOF) system*. A system with an infinite number of degrees of freedom is called a *continuous system* or *distributed parameter* system.

If the vibrations are initiated by an initial energy present in the system and no other source is present, the resulting vibrations are called *free vibrations*. If the vibrations are caused by an external force or motion, the vibrations are called *farced vibrations*. If the external input is periodic, the vibrations are *harmonic*. Otherwise, the vibrations are said to be *transient*. If the input is stochastic, the vibrations are said to be *random*.

If the vibrations are assumed to have no source of energy dissipation, they are called *undamped*. If a dissipation source is present, the vibrations are called *damped* and are further characterized by the form of damping. For example, if viscous damping is present, they are called *viscously damped.*

If assumptions are made to render the differential equations governing the vibrations linear, the vibrations are called *linear*. If the governing equations are *nonlinear*, then so are the vibrations.

Mathematical modeling of SDOF systems is the topic of Chapter 2. Free vibrations of SDOF systems are covered in Chapter 3 (first undamped, then viscously damped, and finally with other forms of damping). Forced vibrations of SDOF systems are covered in Chapter 4 (harmonic) and Chapter 5 (transient). Chapter 6 discusses the special case of two degree-of-freedom systems from the derivation of the differential equations to forced vibrations. The more general MDOF systems are considered in Chapters 7 through 9. Chapter 7 focuses on the modeling of MDOF systems, Chapter 8 on the free vibration response of undamped and damped systems, and Chapter 9 on the forced response of MDOF systems. Chapters 10 and 11 consider continuous systems. The exact free and forced response of continuous systems is covered in Chapter 10, while Chapter 11 presents a numerical method called the finite-element method, which is used to approximate continuous systems with a discrete systems model. Chapter 12 covers nonlinear vibrations. Finally, Chapter 13 covers random vibrations.

1.5 DIMENSIONAL ANALYSIS

An engineer want to run tests to find the correlation between a single dependent variable and four independent variables,

$$y = f(x_1, x_2, x_3, x_4) \tag{1.3}$$

There are ten values of each independent variable. Changing one variable at a time requires 10,000 tests. The expense and time required to run these tests are prohibitive.

A better method to organize the tests is to use non-dimensional variables. The Buckingham Pi theorem states that you count the number of variables, including the dependent variable: call it n. Then count the number of basic dimensions involved in the variables; call it r. Then you need $n - r$ dimensionless variables or π groups. If $n = 6$ and $n = 3$ there are three π groups, and the relation has a non-dimensional form of

$$\pi_1 = f(\pi_2, \pi_3) \tag{1.4}$$

where π_1 is a dimensionless group of parameters involving the dependent variable and π_2 and π_3 are dimensionless groups that involve only the independent parameters.

Usually, the dimensionless parameters have physical meaning. For example, in fluid mechanics when it is desired to find the drag force acting on an airfoil, it is proposed that

$$D = f(v,\ L,\ \rho,\ \mu,\ c) \tag{1.5}$$

where D is the drag force, v is the velocity of the flow, L is the length of the airfoil, ρ is the mass density of the fluid, μ is the viscosity of the fluid, and c is the speed of sound in the fluid. There are six variables which involve three dimensions. Thus, the Buckingham Pi theorem yields a formulation involving three π groups. The result is

$$C_D = f(Re,\ M) \tag{1.6}$$

where the drag coefficient is

$$C_D = \frac{D}{\frac{1}{2}\rho v^2 L} \tag{1.7}$$

the Reynolds number is

$$Re = \frac{\rho v L}{\mu} \tag{1.8}$$

and the Mach number is

$$M = \frac{v}{c} \tag{1.9}$$

The drag coefficient is the ratio of the drag force to the inertia force, the Reynolds number is the ratio of the inertia force to the viscous force, and the Mach number is the ratio to the velocity of the flow to the speed of sound.

Dimensional analysis also can be used when a known relationship exists between a single dependent variable and a number of dimensional variables. The algebra leads to a relationship between a dimensionless variable involving the dependent parameter and non-dimensional variables involving the independent parameters.

EXAMPLE 1.2

A dynamic vibration absorber is added to a primary system to reduce its amplitude. The absorber is illustrated in Figure 1.9 and studied in Chapter 6. The steady-state amplitude of the primary system is dependent upon six parameters:

- m_1, the mass of the primary system
- m_2, the absorber mass
- k_1, the stiffness of the primary system
- k_2, the absorber stiffness
- F_0, the amplitude of excitation
- ω, the frequency of excitation

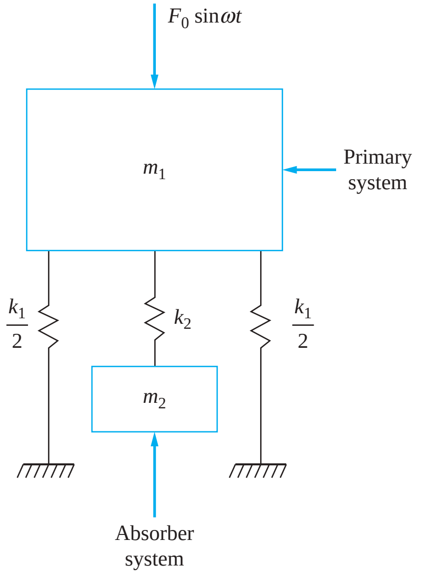

FIGURE 1.9
Example 1.2 is to determine the non-dimensional form of the steady-state amplitude of the primary system when an absorber system is added.

The equation for the dimensional amplitude is

$$X_1 = F_0 \left| \frac{k_2 - m_2\omega^2}{m_1 m_2 \omega^2 - (k_2 m_1 + k_1 m_2 + k_2 m_2)\omega^2 + k_1 k_2} \right| \tag{a}$$

Non-dimensionalize this relationship.

SOLUTION

The dimensional variables involve three independent basic dimensions: mass, length, and time. The Buckingham Pi theorem predicts that the non-dimensional relationship between X_1 and the parameters involve $7 - 3 = 4$ non-dimensional parameters. Factor k_2 out of the numerator and $k_1 k_2$ out of the denominator, resulting in

$$X_1 = \frac{F_0}{k_1} \left| \frac{1 - \dfrac{m_2\omega^2}{k_2}}{\dfrac{m_1 m_2 \omega^4}{k_1 k_2} - \left(\dfrac{m_1}{k_1} + \dfrac{m_2}{k_2} + \dfrac{m_2}{k_1}\right)\omega^2 + 1} \right| \tag{b}$$

Multiply both sides by $\dfrac{k_1}{F_0}$, making both sides dimensionless. Define $\pi_1 = \dfrac{k_1 x_1}{F_0}$ and $\pi_2 = \dfrac{m_2\omega^2}{k_2}$, leading to

$$\pi_1 = \left| \frac{1 - \pi_2}{\dfrac{m_1\omega^2}{k_1}\pi_2 - \pi_2 + \left(\dfrac{m_1}{k_1} + \dfrac{m_2}{k_1}\right)\omega^2 + 1} \right| \tag{c}$$

Define $\pi_3 = \dfrac{m_1\omega^2}{k_1}$. The final dimensional term in Equation (c) becomes

$$\left(\frac{m_1}{k_1} + \frac{m_2}{k_1}\right)\omega_2 = \pi_3\left(1 + \frac{m_2}{m_1}\right) = \pi_3(1 + \pi_4) \quad \text{(d)}$$

The non-dimensional form of Equation (a) is

$$\pi_1 = \left|\frac{1 - \pi_2}{\pi_3\pi_2 - \pi_2 + (1 + \pi_4)\,\pi_3 + 1}\right| \quad \text{(e)}$$

1.6 SIMPLE HARMONIC MOTION

Consider a motion represented by

$$x(t) = A\cos\omega t + B\sin\omega t \quad \text{(1.10)}$$

Such a motion is referred to as simple harmonic motion. Use of the trigonometric identity

$$\sin(\omega t + \phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi \quad \text{(1.11)}$$

in Equation (1.10) gives

$$x(t) = X\sin(\omega t + \phi) \quad \text{(1.12)}$$

where

$$X = \sqrt{A^2 + B^2} \quad \text{(1.13)}$$

and

$$\phi = \tan^{-1}\left(\frac{A}{B}\right) \quad \text{(1.14)}$$

Equation (1.12) is illustrated in Figure 1.10. The amplitude, X, is the maximum displacement from equilibrium. The response is cyclic. The period is the time required to execute one cycle, is determined by

$$T = \frac{2\pi}{\omega} \quad \text{(1.15)}$$

and is usually measured in seconds (s). The reciprocal of the period is the number of cycles executed in one second and is called the frequency

$$f = \frac{\omega}{2\pi} \quad \text{(1.16)}$$

The unit of cycles/second is designated as one hertz (Hz). As the system executes one cycle, the argument of the trigonometric function goes through 2π radians. Thus,

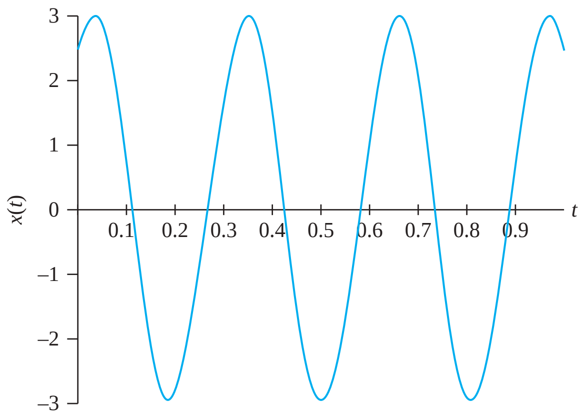

FIGURE 1.10
Illustration of simple harmonic motion in which $\phi > 0$ and the response lags a pure sinusoid.

1 cycle $= 2\pi$ radians and the frequency becomes

$$f = \left(\frac{\omega}{2\pi}\ \text{cycle/s}\right)(2\pi\ \text{rad/cycle}) = \omega\ \text{rad/s} \tag{1.17}$$

Thus, ω is the circular frequency measured in rad/s. The frequency also may be expressed in term of revolutions per minute (rpm) by noting that one revolution is the same as one cycle and there are 60 s in one minute,

$$\omega\ \text{rpm/s} = (\omega\ \text{rad/s})\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) \tag{1.18}$$

The phase angle ϕ represents the lead or lag between the response and a purely sinusoidal response. If $\phi > 0$, the response is said to "lag" a pure sinusoid, and if $\phi < 0$, the response is said to "lead" the sinusoid.

EXAMPLE 1.3

The response of a system is given by

$$x(t) = 0.003\ \cos(30t) + 0.004\ \sin(30t)\ \text{m} \tag{a}$$

Determine (a) the amplitude of motion, (b) the period of motion, (c) the frequency in Hz, (d) the frequency in rad/s, (e) the frequency in rpm, (f) the phase angle, and (g) the response in the form of Equation (1.12)

SOLUTION

(a) The amplitude is given by Equation (1.13) which results in

$$X = \sqrt{0.003^2 + 0.004^2}\ \text{m} = 0.005\ \text{m} \tag{b}$$

(b) The period of motion is

$$T = \frac{2\pi}{30}\ \text{s} = 0.209\ \text{s} \tag{c}$$

(c) The frequency in hertz is

$$f = \frac{1}{T} = \frac{1}{0.209\ \text{s}} = 4.77\ \text{Hz} \tag{d}$$

(d) The frequency in rad/s is

$$\omega = 2\pi f = 30 \text{ rad/s} \tag{e}$$

(e) The frequency in revolutions per minute is

$$\omega = \left(20\frac{\text{rad}}{\text{s}}\right)\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 191.0 \text{ rpm} \tag{f}$$

(f) The phase angle is

$$\phi = \tan^{-1}\left(\frac{0.003}{0.004}\right) = 0.643 \text{ rad} \tag{g}$$

(g) Written in the form of Equation (1.12), the response is

$$x(t) = 0.005 \sin(30t + 0.643) \text{ m} \tag{h}$$

1.7 REVIEW OF DYNAMICS

A brief review of dynamics is presented to familiarize the reader with the notation and methods used in this text. The review begins with kinematics of particles and progresses to kinematics of rigid bodies. Kinetics of particles is presented, followed by kinetics of rigid bodies undergoing planar motion.

1.7.1 KINEMATICS

The location of a particle on a rigid body at any instant of time can be referenced to a fixed cartesian reference frame, as shown in Figure 1.11. Let **i**, **j**, and **k** be unit vectors parallel to the x, y, and z axes, respectively. The particle's position vector is given by

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{1.19}$$

from which the particle's velocity and acceleration are determined

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k} \tag{1.20}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k} \tag{1.21}$$

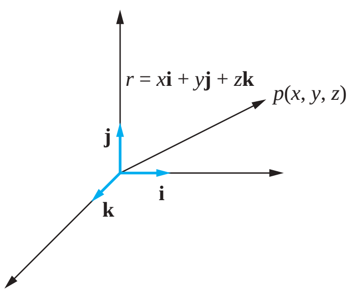

FIGURE 1.11
Illustration of the position vector for a particle in three-dimensional space.

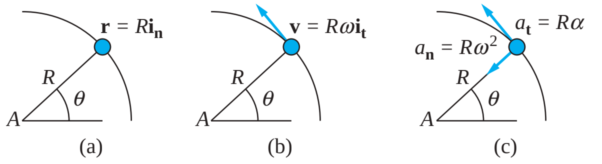

FIGURE 1.12
(a) The position vector for a particle moving in a circular path. (b) The velocity for such a particle is instantaneously tangent to the path of motion. (c) The particle has two components of acceleration. One component is instantaneously tangent to the path, while the other is directed from the particle to the center of rotation.

where a dot above a quantity represents differentiation of that quantity with respect to time.

The motion of a particle moving in a circular path centered at A is illustrated in Figure 1.12. The motion is characterized by an angular coordinate θ measured positive counterclockwise. The rate of rotation

$$\dot{\theta} = \omega \tag{1.22}$$

is called the angular speed and has units of rad/s, assuming the unit of time is in seconds. The angular acceleration is defined by

$$\alpha = \ddot{\theta} \tag{1.23}$$

and has units of rad/s^2.

The position vector of the particle is

$$r = R\mathbf{i_n} \tag{1.24}$$

where R is the radius of the circle and $\mathbf{i_n}$ is a unit vector instantaneously directed toward the particle from the center of rotation. Define $\mathbf{i_t}$ as the unit vector instantaneously tangent to the circle in the direction of increasing θ and instantaneously perpendicular to $\mathbf{i_n}$. Noting that $\frac{d\mathbf{i_t}}{dt} = -\omega\mathbf{i_n}$ and $\frac{d\mathbf{i_n}}{dt} = -\omega\mathbf{i_n}$, the velocity is

$$\mathbf{v} = \dot{\mathbf{r}} = R\frac{d\mathbf{i_n}}{dt} = R\omega\mathbf{i_t} \tag{1.25}$$

The particle's acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d(R\omega\mathbf{i_t})}{dt} = R\frac{d\omega}{dt}\mathbf{i_t} + R\omega\frac{d\mathbf{i_t}}{dt} = R\alpha\mathbf{i_t} - R\omega^2\mathbf{i_n} \tag{1.26}$$

Now consider a rigid body undergoing planar motion. That is (1) the mass center moves in a plane, say the x-y plane and (2) rotation occurs only about an axis perpendicular to the plane (the z axis), as illustrated in Figure 1.13. Consider two particles on the rigid body, A and B, and locate their position vectors $\mathbf{r_A}$ and $\mathbf{r_B}$. The relative position vector $\mathbf{r_{B/A}}$ lies in the x-y plane. The triangle rule for vector addition yields

$$\mathbf{r_B} = \mathbf{r_A} + \mathbf{r_{B/A}} \tag{1.27}$$

Differentiation of Equation (1.27) with respect to time yields

$$\mathbf{v_B} = \mathbf{v_A} + \mathbf{v_{B/A}} \tag{1.28}$$

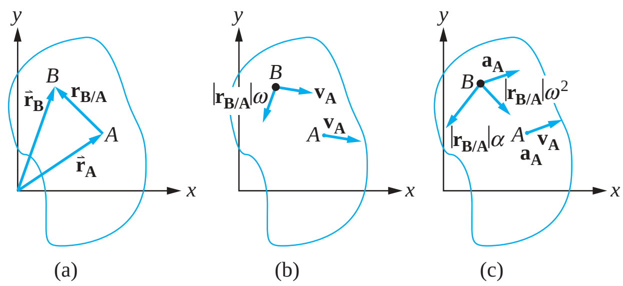

FIGURE 1.13
(a) The triangle rule for vector addition is used to define the relative position vector. (b) For a rigid body undergoing planar motion, the velocity of *B* viewed from *A* is that of a particle moving in a circular path centered at *A*. (c) The relative acceleration is that of a particle moving in a circular path centered at *A*.

Since rotation occurs only about the *z* axis, the motion of *B* (as viewed from *A*) is that of a particle moving in a circular path of radius $|\mathbf{r}_{B/A}|$ Thus, the magnitude of relative velocity is given by Equation (1.25) as

$$v_{B/A} = |\mathbf{r}_{B/A}|\omega \tag{1.29}$$

and its direction is tangent to the circle made by the motion of particle *B*, which is perpendicular to $\mathbf{r}_{B/A}$. The total velocity of particle *B* is given by Equation (1.28) and lies in the *x*-*y* plane.

Differentiating of Equation (1.28) with respect to time yields

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{1.30}$$

The acceleration of particle *B* viewed from particle *A* is the acceleration of a particle moving in a circular path centered at *A* as

$$\mathbf{a}_B = |\mathbf{r}_{B/A}|\alpha\mathbf{i}_t - r\omega^2\mathbf{i}_n \tag{1.31}$$

Equations (1.28) and (1.30) are known as the relative velocity and relative acceleration equations, respectively. They and Equations (1.29) and (1.31) are the only equations necessary for the study of rigid-body kinematics of bodies undergoing planar motion.

1.7.2 KINETICS

The basic law for kinetics of particles is Newton's second law of motion

$$\sum \mathbf{F} = m\mathbf{a} \tag{1.32}$$

where the sum of the forces is applied to a free-body diagram of the particle. A rigid body is a collection of particles. Writing an equation similar to Equation (1.32) for each particle in the rigid body and adding the equations together leads to

$$\sum \mathbf{F} = m\bar{\mathbf{a}} \tag{1.33}$$

where $\bar{\mathbf{a}}$ is the acceleration of the mass center of the body and the forces are summed on a free-body diagram of the rigid body. Equation (1.33) applies to all rigid bodies.

A moment equation is necessary in many problems. The moment equation for a rigid body undergoing planar motion is

$$\sum M_G = \bar{I}\alpha \tag{1.34}$$

where ***G*** is the mass center of the rigid body and $\bar{I}$ is the mass moment of inertia about an axis parallel to the *z* axis that passes through the mass center.

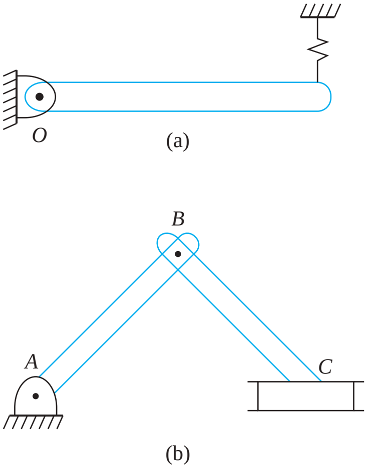

FIGURE 1.14
(a) Rotation about a fixed axis at O. (b) AB has a fixed axis of rotation at A, but BC does not have a fixed axis of rotation.

Equations (1.33) and (1.34) can be used to solve rigid-body problems for planar motion. In general, the force equation of Equation (1.33) yields two independent equations, and the moment equation of Equation (1.35) yields one. If the axis of rotation is fixed, Equation (1.33) may be replaced by

$$\sum M_O = I_O\alpha \tag{1.35}$$

where I_O is the moment of inertia about the axis of rotation. In Figure 1.14(a), O is a fixed axis of rotation, and Equation (1.35) is applicable. In Figure 1.14(b), link BC has does not have a fixed axis of rotation, and Equation (1.35) is not applicable.

Recall that a system of forces and moments acting on a rigid body can be replaced by a force equal to the resultant of the force system applied at any point on the body and a moment equal to the resultant moment of the system about the point where the resultant force is applied. The resultant force and moment act equivalently to the original system of forces and moments. Thus Equations (1.33) and (1.34) imply that the system of external forces and moments acting on a rigid body is equivalent to a force equal to $m\bar{\mathbf{a}}$ applied at the body's mass center and a resultant moment equal to $\bar{I}\alpha$. This latter resultant system is called the system of effective forces. The equivalence of the external forces and the effective forces is illustrated in Figure 1.15.

The previous discussion suggests a solution procedure for rigid-body kinetics problems. Two free-body diagrams are drawn for a rigid body. One free-body diagram shows all external forces and moments acting on the rigid body. The second free-body diagram shows the

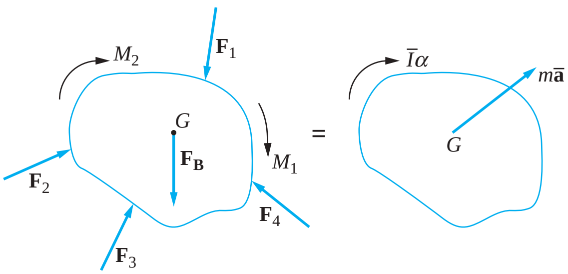

FIGURE 1.15
The system of external forces and moments acting on a rigid body undergoing planar motion is equivalent to the system of effective forces, a force equal to $m\,\bar{a}$ applied at the mass center, and a moment equal to $\bar{I}\,\alpha$.

effective forces. If the problem involves a system of rigid bodies, it may be possible to draw a single free-body diagram showing the external forces acting on the system of rigid bodies and one free-body diagram showing the effective forces of all of the rigid bodies. Equations (1.33) and (1.34) are equivalent to

$$\sum \mathbf{F}_{\text{ext}} = \sum \mathbf{F}_{\text{eff}} \tag{1.36}$$

and

$$\sum M_{O_{\text{ext}}} = \sum M_{O_{\text{eff}}} \tag{1.37}$$

taken about any point O on the rigid body. Equations (1.36) and (1.37) are statements of D' Alembert's principle applied to a rigid body undergoing planar motion.

EXAMPLE 1.4

The slender rod $(\bar{I} = \frac{1}{12}mL^2)$ AC of Figure 1.16(a) of mass m is pinned at B and held horizontally by a cable at C. Determine the angular acceleration of the bar immediately after the cable is cut.

SOLUTION

Immediately after the cable is cut, the angular velocity is zero. The bar has a fixed axis of rotation at B. Applying Equation (1.35)

$$\sum M_B = \sum I_B \alpha \tag{a}$$

to the FBD of Figure 1.16(b) and taking moments as positive clockwise, we have

$$mg\frac{L}{4} = I_B \alpha \tag{b}$$

The parallel-axis theorem is used to calculate I_B as

$$I_B = \bar{I} + md^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{4}\right)^2 = \frac{7}{48}mL^2 \tag{c}$$

Substituting into Equation (b) and solving for α yields

$$\alpha = \frac{12g}{7L} \tag{d}$$

ALTERNATIVE METHOD

Free-body diagrams showing effective and external forces are shown in Figure 1.16(c). The appropriate moment equation is

$$(\Sigma M_B)_{\text{ext}} = (\Sigma M_B)_{\text{eff}} \tag{e}$$

leading to

$$mg\frac{L}{4} = \frac{1}{12}mL^2 + \left(m\frac{L}{4}\alpha\right)\left(\frac{L}{4}\right) \tag{f}$$

and $\alpha = \dfrac{12g}{7L}$.

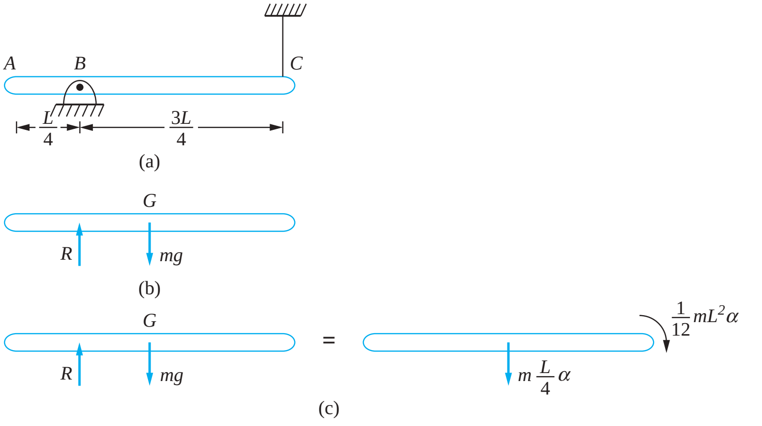

FIGURE 1.16
(a) System of Example 1.4 where the slender rod is pinned at B and held by the cable at C. (b) FBD of bar immediately after cable is cut. The problem involves rotation about a fixed axis at B, so $\sum M_B = I_B\alpha$. (c) FBD's showing external forces and effective forces immediately after cable is cut.

EXAMPLE 1.5

Determine the angular acceleration of the pulley of Figure 1.17.

SOLUTION

Consider the system of rigid bodies composed of the pulley and the two blocks. If α is the counterclockwise angular acceleration of the pulley, then, assuming no slip between the pulley and the cables, block A has a downward acceleration of $r_A\alpha$ and block B has an upward acceleration of $r_B\alpha$.

Summing moments about the center of the pulley, neglecting axle friction in the pulley, and using the free-body diagrams of Figure 1.17(b) assuming moments are positive counterclockwise yields

$$\sum M_{O_{\text{ext}}} = \sum M_{O_{\text{eff}}}$$

$$m_A g r_A - m_B g r_B = I_P\alpha + m_B r_A^2\alpha + m_B r_B^2\alpha$$

Substituting given values leads to $\alpha = 7.55$ rad/s^2.

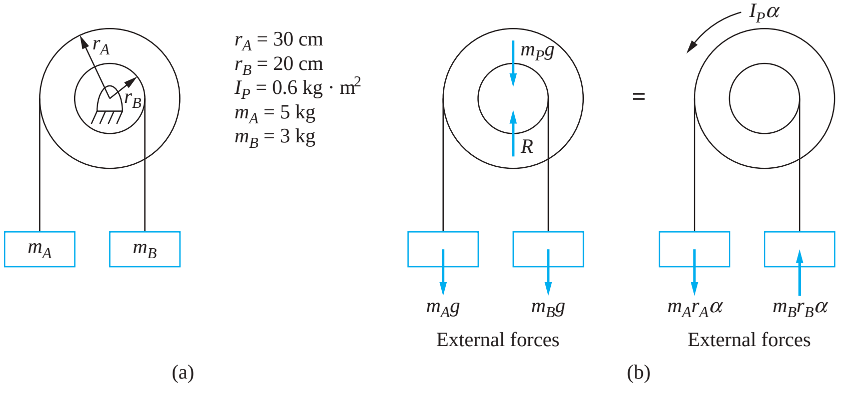

FIGURE 1.17
(a) System of Example 1.4. (b) FBDs showing external forces and effective forces.

1.7.3 PRINCIPLE OF WORK AND ENERGY

The kinetic energy of a rigid body undergoing planar motion is the sum of the translational kinetic energy and the rotational kinetic energy

$$T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \bar{I} \omega^2 \tag{1.38}$$

If the body has a fixed axis of rotation at O, the kinetic energy is

$$T = I_O \omega^2 \tag{1.39}$$

The work done by a force, F, acting on a rigid body as the point of application of the force travels between two points described by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ is

$$U_{A\to B} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F} \cdot d\mathbf{r} \tag{1.40}$$

where $d\mathbf{r}$ is a differential position vector in the direction of motion. The work done by a moment acting on a rigid body in planar motion is

$$U_{A\to B} = \int_{\theta_A}^{\theta_B} M \, d\theta \tag{1.41}$$

If the work of a force is independent of the path taken from A to B, the force is called *conservative*. Examples of conservative forces are spring forces, gravity forces, and normal forces. A potential energy function, $V(\mathbf{r})$, can be defined for conservative forces. The work done by a conservative force can be expressed as a difference in potential energies

$$U_{A\to B} = V_A - V_B \tag{1.42}$$

Since the system of external forces is equivalent to the system of effective forces, the total work done on a rigid body in planar motion is

$$U_{A\to B} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} m\bar{\mathbf{a}} \cdot d\mathbf{r} + \int_{\theta_A}^{\theta_B} \bar{I}\alpha \, d\theta \tag{1.43}$$

When integrated, the right-hand side of Equation (1.43) is equal to the difference in the kinetic energy of the rigid body between A and B. Thus Equation (1.43) yields the principle of work-energy,

$$T_B - T_A = U_{A\to B} \tag{1.44}$$

If all forces are conservative, Equation (1.42) is used in Equation (1.44) and the result is the principle of conservation of energy

$$T_A + V_A = T_B + V_B \tag{1.45}$$

If some external forces are conservative and some are non-conservative, then

$$U_{A\to B} = V_A - V_B + U_{A\to B_{NC}} \tag{1.46}$$

where $U_{A\to B_{NC}}$ is the work done by all non-conservative forces. Equation (1.44) becomes

$$T_A + V_A + U_{A\to B_{NC}} = T_B + V_B \tag{1.47}$$

Equation (1.47) is the most general form of the principle of work and energy.

EXAMPLE 1.6

Express the kinetic energy of each of the systems of Figure 1.18 in terms of the specified generalized coordinates at an arbitrary instant.

SOLUTION

(a) The system is a SDOF system. The angular velocity of the bar is $\dot{\theta}$. The velocity of the mass center of the bar is related to the angular velocity of the bar using the relative velocity equation $\bar{v} = \frac{L}{6}\dot{\theta}$. The kinetic energy of the system is calculated using Equation (1.38) as

$$T = \frac{1}{2}m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 = \frac{1}{18}mL^2\dot{\theta}^2 \quad \text{(a)}$$

(b) The system has two degrees of freedom. The kinetic energy is calculated using Equation (1.38) as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 \quad \text{(b)}$$

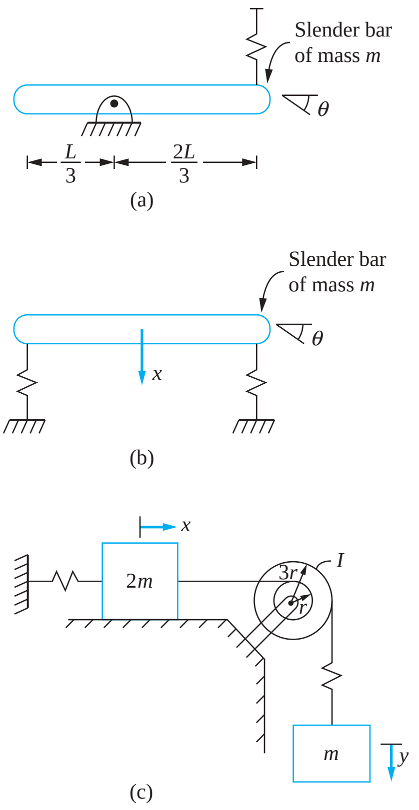

FIGURE 1.18
Systems of Example 1.6: (a) SDOF system; (b) two degree-of-freedom system with one rigid body; and (c) two degree-of-freedom system composed of three rigid bodies.

(c) The system has two degrees of freedom. The angular rotation of the pulley is related to the displacement of the sliding block by $\theta = \frac{x}{r}$. The displacement of the hanging mass is independent of x. The kinetic energy is the sum of the kinetic energies of the sliding mass, the pulley, and the hanging mass:

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}\left(2m + \frac{I}{r^2}\right)\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \tag{c}$$

1.7.4 PRINCIPLE OF IMPULSE AND MOMENTUM

The impulse of the force $\mathbf{F}$ between t_1 and t_2 is defined as

$$I_{1\to 2} = \int_{t_1}^{t_2} \mathbf{F}dt \tag{1.48}$$

The total angular impulse of a system of forces and moments about a point O is

$$J_{O_{1\to 2}} = \int_{t_1}^{t_2} \sum M_O\, dt \tag{1.49}$$

The system momenta at a given time are defined by the system's linear momentum

$$L = m\overline{\mathbf{v}} \tag{1.50}$$

and its angular momentum about its mass center for a rigid body undergoing planar motion

$$H_G = \bar{I}\omega \tag{1.51}$$

Integrating Equations (1.33) and (1.34) between arbitrary times t_1 and t_2 leads to

$$\mathbf{L}_1 + \mathbf{I}_{1\to 2} = \mathbf{L}_2 \tag{1.52}$$

and

$$H_{G_1} + J_{G_{1\to 2}} = H_{G_2} \tag{1.53}$$

Equations (1.52) and (1.53) summarize the principle of impulse and momentum for a system. For a particle application, Equation (1.52) is usually sufficient. For a rigid body undergoing planar motion, Equation (1.52) can be written (in general) in component form as two scalar equations. Equation (1.53) is not a vector equation and represents one equation.

Using an equivalent force system argument similar to that used to obtain Equations (1.36) and (1.37), it is deduced from Equations (1.52) and (1.53) that the system of applied impulses is equivalent to the difference between the system momenta at t_1 and the system momenta at t_2. This form of the principle of impulse and momentum, convenient for problem solution, is illustrated in Figure 1.19 for a rigid body undergoing planar motion.

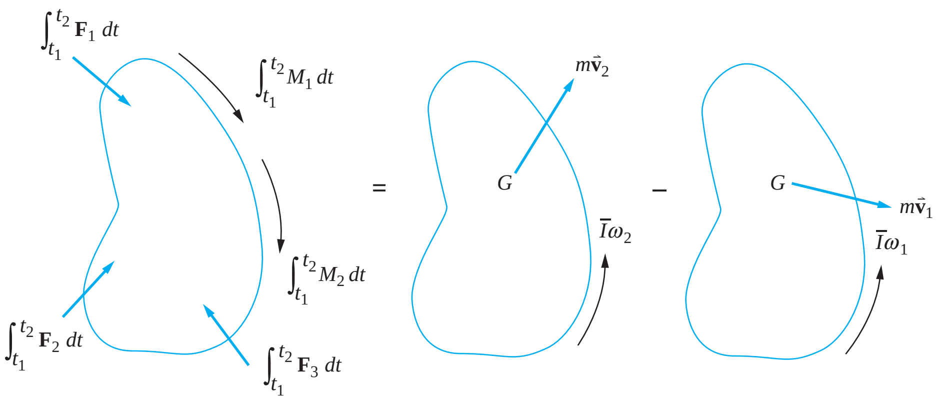

FIGURE 1.19
Illustration of the principle of impulse and momentum.

EXAMPLE 1.7

The slender rod of mass m of Figure 1.20 is swinging through a vertical position with an angular velocity ω_1 when it is struck at A by a particle of mass $m/4$ moving with a speed v_p. Upon impact the particle sticks to the bar. Determine (a) the angular velocity of the bar and particle immediately after impact, (b) the maximum angle through which the bar and particle will swing after impact, and (c) the angular acceleration of the bar and particle when they reach the maximum angle.

SOLUTION

(a) Let t_1 occur immediately before impact and t_2 occur immediately after impact. Consider the bar and the particle as a system. During the time of impact, the only external impulses are due to gravity and the reactions at the pin support. The principle of impulse and momentum is used in the following form:

$$\begin{pmatrix}\text{External angular}\\ \text{impulses about } O\\ \text{between } t_1 \text{and } t_2\end{pmatrix} = \begin{pmatrix}\text{Angular momentum}\\ \text{about } O\\ \text{at } t_2\end{pmatrix} - \begin{pmatrix}\text{Angular momentum}\\ \text{about } O\\ \text{at } t_1\end{pmatrix}$$

Using the momentum diagrams of Figure 1.20(b), this becomes

$$0 = \left(m\frac{L}{2}\omega_2\right)\left(\frac{L}{2}\right) + \left(\frac{m}{4}a\omega_2\right)(a) + \frac{1}{12}mL^2\omega_2 - \left[\left(m\frac{L}{2}\omega_1\right)\left(\frac{L}{2}\right) - \left(\frac{m}{4}v_p\right)(a) + \frac{1}{12}mL^2\omega_1\right] \qquad \text{(a)}$$

which is solved to yield

$$\omega_2 = \frac{4L^2\omega_1 - 3v_p a}{4L^2 + 3a^2} \qquad \text{(b)}$$

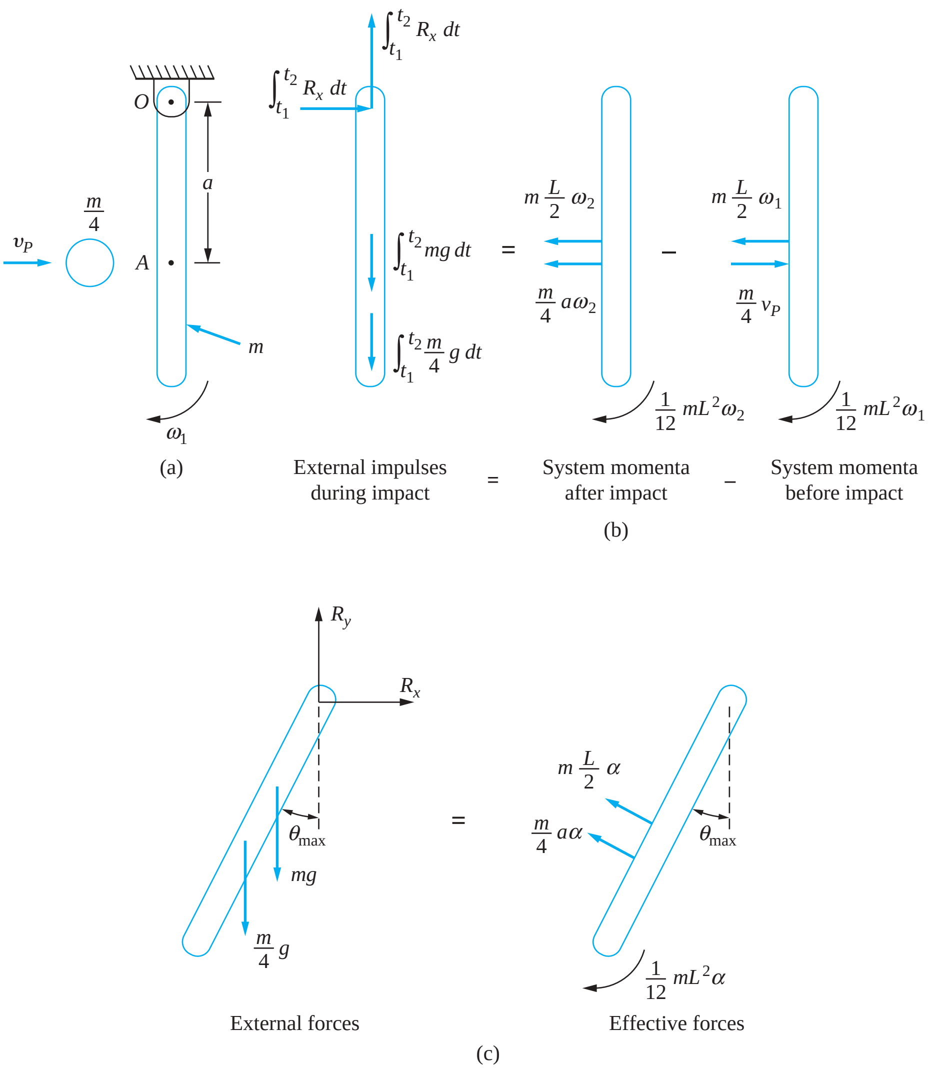

FIGURE 1.20
(a) Slender rod of Example 1.7 swinging through the vertical position with angular velocity ω_1 when it is struck by a particle moving with a velocity v_p a distance a from the pin support. (b) Impulse and momentum diagrams for the time immediately before impact and the time immediately after impact. (c) FBDs when the bar swings through its maximum angle.

(b) Let t_3 be the time when the bar and particle assembly attains its maximum angle. Gravity forces are the only external forces that do work; hence conservation of energy applies between t_2 and t_3. Thus, from Equation (1.45),

$$T_2 + V_2 = T_3 + V_3 \tag{c}$$

The potential energy of a gravity force is the magnitude of the force times the distance its point of application is above a horizontal datum plane. Choosing the datum as the horizontal plane through the support, using Equation (1.38) for the kinetic energy of a rigid body, and noting $T_3 = 0$ yields

$$\frac{1}{2}m\left(\frac{L}{2}\omega_2\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\omega_2^2 + \frac{1}{2}\frac{m}{4}(a\omega_2)^2 - mg\frac{L}{2} - \frac{mg}{4}a$$
$$= -mg\frac{L}{2}\cos\theta_{\max} - \frac{m}{4}ga\cos\theta_{\max} \quad \text{(d)}$$

which is solved to yield

$$\theta_{\max} = \cos^{-1}\left[1 - \frac{(4L^2 + 3a^2)\omega_2^2}{g(12L + 6a)}\right] \quad \text{(e)}$$

(c) The bar attains its maximum angle at t_3, $\omega_3 = 0$. Summing moments about O using the free-body diagrams of Figure 1.20(c) assuming moments and positive clockwise gives

$$\left(\sum M_O\right)_{\text{ext}} = \left(\sum M_O\right)_{\text{eff}} \quad \text{(f)}$$

$$-(mg)\left(\frac{L}{2}\sin\theta_{\max}\right) - \left(\frac{mg}{4}\right)(a\sin\theta_{\max})$$
$$= \left(m\frac{L}{2}\alpha\right)\left(\frac{L}{2}\right) + \left(\frac{m}{4}a\alpha\right)(a) + \frac{1}{12}mL^2\alpha \quad \text{(g)}$$

which is solved to yield

$$\alpha = -\frac{(6L + 3a)g\sin\theta_{\max}}{4L^2 + 3a^2} \quad \text{(h)}$$

1.8 TWO BENCHMARK EXAMPLES

Two benchmark examples will be followed throughout the text. The basic problems are introduced here. Their mathematical models, assuming a SDOF system, are constructed in Chapter 2 and analyzed under various forcing conditions in Chapters 3 through 5. Two degree-of-freedom models are introduced in Chapter 6, and more general MDOF system models are introduced in Chapter 7 and analyzed in Chapters 8 and 9. The first example continues into Chapters 10 and 11 using a continuous system analysis. The second example is continued into Chapter 13 using a random excitation.

1.8.1 MACHINE ON THE FLOOR OF AN INDUSTRIAL PLANT

A 2000 lb machine is placed on the floor of an industrial plant, as shown in Figure 1.21(a). The floor is supported by a W14 × 30 steel beam. The beam is 20 ft long, fixed at one end, and pinned at the other. The machine is placed twelve ft from the fixed end, as shown in Figure 1.21(b). The beam has a cross-sectional area of 8.85 in^2 and a cross-sectional

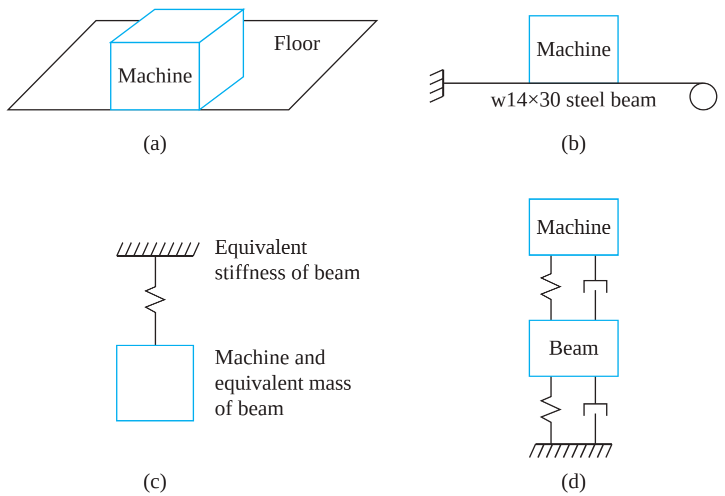

FIGURE 1.21
(a) The analysis of a machine placed on a floor in an industrial plant is one of the benchmark problems. (b) The problem has been idealized as a machine mounted on a fixed-pinned beam. (c) SODF model of mass on beam accounting for inertia effects of beam. (d) A two degree-of-freedom model of the machine when a vibration isolator is placed between the machine and the beam.

moment of inertia of 291 in^4. The beam's weight per unit foot is 30 lb. Steel has an elastic modulus of 30×10^6 psi. The basic model is that of a machine on an elastic beam.

Initially, the beam is modeled as a mass-less spring whose stiffness is calculated from static-beam deflection theory. The inertia of the spring is then taken into account by calculating an equivalent mass for the beam such that its kinetic energy is approximately that of the kinetic energy of a particle lumped at the location of the machine. This model is shown in Figure 1.21(c). In Chapter 3, the natural frequency of the system is calculated, and the free response of the system is examined when subject to an impulsive load.

First, the beam is modeled without damping. Then the hysteretic damping is modeled by an equivalent viscous damping model. The machine develops a harmonic force while operating and the steady-state vibrations of the beam are examined. Then the beam is assumed to be rigid, and a vibration isolator is designed to protect the beam from large forces generated during operation of the machine. The machine could be subject to a harmonic excitation (Chapter 4) or an impulsive loading (Chapter 5).

The inertia of the beam is lumped at the location of the mass and a two-degree-of-freedom system is assumed as shown in Figure 1.21(d). Natural frequencies of the two degree-of-freedom system are determined, and the forced response is calculated (Chapter 6). The same vibration isolator designed for the rigid beam is placed between the machine and the beam, a multiple degree-of-freedom model is assumed (Chapter 7), and the natural frequencies and mode shapes are calculated (Chapter 8). Then the performance of the vibration isolator is evaluated (Chapter 9).

A continuous system model is described in Chapter 10, when natural frequencies are approximated using the Rayleigh-Ritz method. The forced response is obtained by a finite-element method in Chapter 11.

1.8.2 SUSPENSION SYSTEM FOR A GOLF CART

The design of a suspension system for an automobile is complicated. Some models require up to eighteen degrees of freedom. The suspension system must be able to handle a wide variety of road contours. Suspension system performance is often analyzed using random vibration theory. Thus, a complete analysis is beyond the scope of this book. The focus is

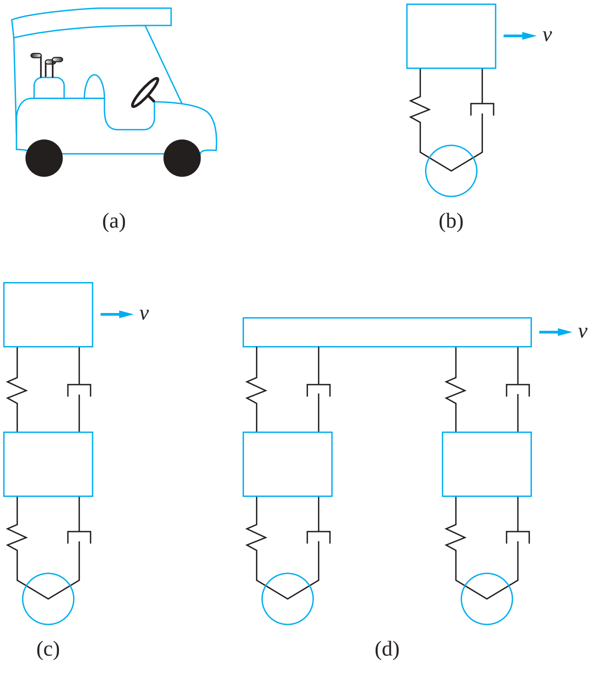

FIGURE 1.22
(a) A suspension system for a small vehicle such as a golf cart is the second benchmark problem. (b) In early chapters, the golf cart is modeled as a SDOF system. (c) The analysis grows in complexity as the chapters progress. In later chapters, the mass of the wheel is taken into account. (d) The distribution of mass on the body is considered.

instead on a simplified model of the suspension system, as shown in Figure 1.22, where this could serve as the model of a suspension system for a golf cart.

The mass of the empty golf cart is 300 kg. Two golfers and their clubs could add an addition 300 kg to the mass of the vehicle.

A simplified model for the suspension system is developed in Chapter 2. The analysis of the golf cart when it encounters a sudden change in terrain contour is analyzed in Chapter 3, while its performance under a sustained bumpy terrain contour is considered in Chapter 4. Its performance when it encounters a hole in the road considered in Chapter 5. A two degree-of-freedom model (which includes the mass of the axle and wheels) is used in Chapter 6. In Chapter 7, a multiple degree-of-freedom model is developed for the vehicle assuming the front wheels are independent of the rear wheels and the body has a distribution of mass, as shown in Figure 1.22(c). The natural frequencies of the MDOF model are calculated in Chapter 8, while the forced response is considered in Chapter 9. The effect of a random input is described in Chapter 13.

1.9 FURTHER EXAMPLES

EXAMPLE 1.8

The slender bar of Example 1.4 and Figure 1.16 is pinned at A and held in the horizontal position by a cable. The cable is cut at $t = 0$.

(a) What is the bar's angular velocity after it has rotated through 10°?
(b) What are the reactions at the pin support after it has rotated through 10°?

SOLUTION

(a) Let position 1 refer to the bar immediately after the cable is cut. Let position 2 refer to the bar after it has rotated through 10°. All external forces are conservative; thus, conservation of energy applies between positions 1 and 2 as

$$T_1 + V_1 = T_2 + V_2 \tag{a}$$

Take the datum for potential energy calculations for the gravity force to be position 1, then $V_1 = 0$, and $V_2 = -\frac{mgL}{3}\sin 10°$. The kinetic energy in position 1 is zero, and

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_2^2 \tag{b}$$

Kinematics (the relative velocity equation) is used to relate the velocity of the mass center to the angular velocity of the bar so that $\bar{v} = \frac{L}{3}\omega$. Substituting into Equation (a), we have

$$0 = \frac{1}{2}m\left(\frac{L}{3}\omega\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_2^2 - \frac{mgL}{3}\sin 10° \tag{c}$$

which is solved to yield

$$\omega = \sqrt{\frac{24g}{7L}\sin 10°} = 0.818\sqrt{\frac{g}{L}} \tag{d}$$

(b) Summing moments about the pin support on the free-body diagrams after the body has rotated through 10° are illustrated in Figure 1.23. Taking moments about the pin support yields $\alpha = \frac{12g}{7L}$, which is the same as the initial value. This is to be expected, as the external forces are constant, which implies uniformly accelerated motion. Summing forces using the free-body diagrams according to $(\Sigma\mathbf{F})_{\text{ext}} = (\Sigma\mathbf{F})_{\text{eff}}$ give

$$\begin{aligned} R_x\mathbf{i} + (R_y - mg)\mathbf{j} = {} & m\frac{L}{3}\left(\frac{12g}{7L}\right)(-\sin 10°\mathbf{i} - \cos 10°\mathbf{j}) \\ & + m\frac{L}{3}\left(\frac{24g}{7L}\sin 10°\right)(-\cos 10°\mathbf{i} + \sin 10°\mathbf{j}) \end{aligned} \tag{e}$$

By equating coefficients of the unit vectors, the reactions are determined as

$$R_x = -\frac{4mg}{7}\sin 10°(1 + 2\cos 10°) = -0.295mg \tag{f}$$

$$R_y = mg\left(1 - \frac{4}{7}\cos 10° + \frac{8}{7}\sin^2 10°\right) = 0.472mg \tag{g}$$

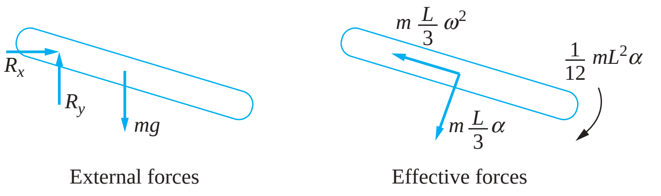

FIGURE 1.23
FBDs after bar of Example 1.8 has rotated through 10°.

EXAMPLE 1.9

Determine the acceleration of the block Figure 1.24(a).

SOLUTION

The acceleration of the block is assumed to be upward, which is consistent with the assumed direction of the angular acceleration of the disk. The point on the disk where the cable is in contact with it has the same acceleration ($r\alpha$) as the cable. Assuming the cable is inextensible, it has the same acceleration as the block. Summing moments about the mass center by applying $(\sum M_O)_{\text{ext}} = (\sum M_O)_{\text{eff}}$ to the FBDs shown in Figure 1.24(b) leads to

$$M - mgr = mr\alpha(r) + I\alpha \tag{a}$$

Solving for α gives

$$\alpha = \frac{M - mgr}{I + mr^2} = \frac{(18\ \text{N}\cdot\text{m}) - (1.3\ \text{kg})(9.81\ \text{m/s}^2)(0.3\ \text{m})}{0.09\ \text{kg}\cdot\text{m}^2 + (1.3\ \text{kg})(0.3\ \text{m})^2} = 68.5\ \text{rad/s}^2 \tag{b}$$

The acceleration of the block is

$$\alpha = r\alpha = (0.3\ \text{m})(68.5\ \text{rad/s}^2) = 20.5\ \text{m/s}^2 \tag{c}$$

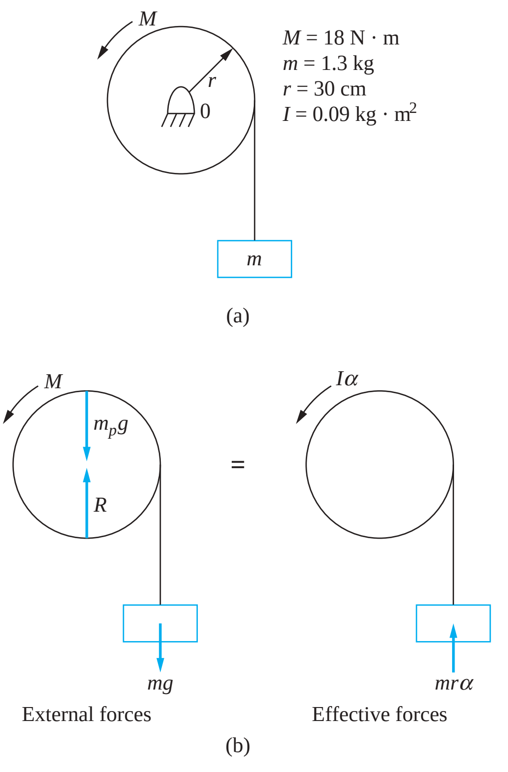

FIGURE 1.24
(a) System of Example 1.9. (b) FBDs drawn at an arbitrary instant showing the external forces and the effective forces.

EXAMPLE 1.10

A thin disk of mass 5 kg, radius 20 cm, and attached to a spring of stiffness 2000 N/m is in equilibrium when it is subject to an applied force $P = 10$ N. The coefficient of friction between the disk and the surface is 0.1.

(a) What is the maximum displacement of the disk from its equilibrium position, assuming no slipping between the disk and the surface?
(b) What is the angular acceleration of the disk immediately after it reaches its maximum displacement?
(c) Is the no-slip assumption correct?

SOLUTION

(a) Let position 1 refer to the position when the disk is in equilibrium, and let position 2 refer to the position when the disk reaches its maximum displacement. Application of the principle of work and energy between position 1 and position 2 for the disk gives

$$T_1 + V_1 + U_{1\to 2_{NC}} = T_2 + V_2 \tag{a}$$

The kinetic energy of the disk in position 1 is zero, because the disk is at rest. The kinetic energy of the disk in position 2 is zero, because the disk reaches its maximum displacement. The only source of potential energy is the spring force. The potential energy in the spring in position 1 is zero, as the spring is unstretched. Letting x be the maximum displacement, the potential energy in position 2 is

$$V_2 = \frac{1}{2}kx^2 \tag{b}$$

The friction force does no work, since the disk rolls without slipping. Thus, the velocity of the point where the friction force is applied is zero. The only non-conservative force is the applied force P. Its work is

$$U_{1\to 2_{NC}} = \int_0^x P dx = Px \tag{c}$$

Substituting into Equation (a),

$$Px = \frac{1}{2}kx^2 \tag{d}$$

or

$$x = \frac{2P}{k} = \frac{2(10\text{ N})}{2000\text{ N/m}} = 0.01\text{ m} \tag{e}$$

(b) Summing moments about the contact point as $(\sum M_O)_{\text{ext}} = (\sum M_O)_{\text{eff}}$ and using the free-body diagrams drawn immediately after the disk reaches its maximum displacement (illustrated in Figure 1.25) yields

$$-kxr + Pr = \frac{1}{2}mr^2\alpha + m\bar{a}r \tag{f}$$

If the disk rolls without slipping, the velocity of the point of contact is identically zero, and its acceleration only has an upward component of $r\omega^2$. Application of the horizontal

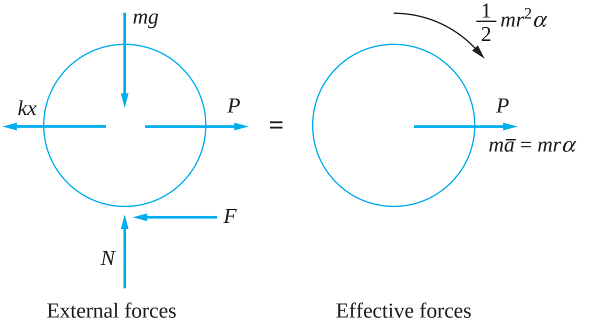

FIGURE 1.25
FBDs of system in Example 1.10. Summing moments about the point of contact helps to solve for the angular acceleration assuming no slipping. Summing moments about the mass center finds the friction force which is checked against the maximum value to determine if slipping occurs.

component of the relative acceleration equation between the point of contact and the mass center yields $\bar{a} = r\alpha$. Substituting this result into Equation (b) leads to

$$\alpha = \frac{2(P - kx)}{3mr} = \frac{2[10\text{ N} - (2000\text{ N/m})(0.01\text{ m})]}{3(5\text{ kg})(0.2\text{ m})} = 6.67\text{ rad/s}^2 \tag{g}$$

(c) Summing moments about the mass center as $(\sum M_C)_{\text{ext}} = (\sum M_C)_{\text{eff}}$ and using the free-body diagrams of Figure 1.25 yields

$$Fr = \frac{1}{2}mr^2\alpha \Rightarrow F = \frac{1}{2}mr\alpha \tag{h}$$

The maximum value of α from when the motion is initiated to when the disk reaches its maximum displacement should be used in the calculation. The maximum value occurs in position 1 when

$$\alpha = \frac{2P}{3mr} = \frac{2(10\text{ N})}{3(5\text{ kg})(0.2\text{ m})} = 6.67\text{ rad/s}^2 \tag{i}$$

and

$$F = \frac{1}{2}mr\alpha = \frac{1}{2}(5\text{ kg})(0.2\text{ m})(6.67\text{ rad/s}^2) = 3.33\text{ N} \tag{j}$$

The maximum available friction force is $\mu mg = 0.1(5\text{ kg})(9.81\text{ m/s}^2) = 4.91$ N. Since the friction force is less than the maximum allowable friction force, the disk rolls without slipping.

EXAMPLE 1.11

A baseball player holds a bat with a centroidal moment of inertia $\bar{I}$ a distance a from the bats mass center. His "bat speed" is the angular velocity with which he swings the bat. The pitched ball is a fastball which reaches the batter with a velocity v. Assuming his swing is a rigid-body rotation about an axis perpendicular to his hands, where should the batter hit the ball to minimize the impulse felt by his hands?

SOLUTION

When the better hits the ball, it exerts an impulse on the bat: call it B. Since the batter is holding the bat, he feels an impulse as he hits the ball: call it P. The effect of hitting the

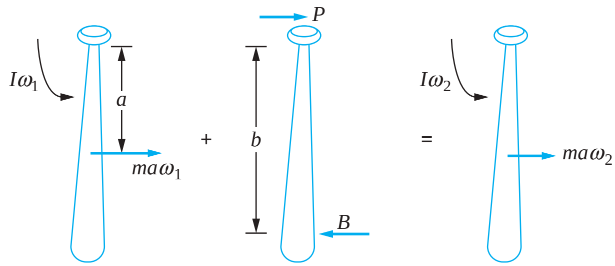

FIGURE 1.26
Impulse momentum diagrams for Example 1.11 as batter hits ball.

ball is to change the bat speed from ω_1 to ω_2. The impulse momentum diagrams of the bat during the time are shown in Figure 1.26.

Applying the principle of linear impulse and momentum to Figure 1.26 leads to

$$ma\omega_1 + P - B = ma\omega_2 \tag{a}$$

Application of the principle of angular impulse and angular momentum about an axis through the batter's hands yields

$$\bar{I}\omega_1 + ma\omega_1(a) - B(b) = \bar{I}\omega_2 + ma\omega_2(a) \tag{b}$$

Solving Equation (b) for B, we have

$$B = \frac{(\bar{I} + ma^2)}{b}(\omega_2 - \omega_1) \tag{c}$$

Substituting Equation (c) into Equation (a) and solving for P leads to

$$P = (\omega_1 - \omega_2)\left(\frac{\bar{I} + ma^2}{b} - ma\right) \tag{d}$$

Thus, $P = 0$ if

$$b = a + \frac{\bar{I}}{ma} \tag{e}$$

Thus, the angular impulse felt by the batter is zero if b satisfies Equation (e). The location of b is called the center of percussion.

1.10 SUMMARY

1.10.1 IMPORTANT CONCEPTS

- Vibrations are oscillations about an equilibrium position.
- Assumptions may be implicit (such as the continuum assumption) or explicit (such as neglecting all forms of friction).

- The number of degrees of freedom used in a system model is the number of kinematically independent coordinates necessary to describe the motion of every particle in the system.
- Vibrations are classified as free or forced, damped or undamped, linear or nonlinear, continuous or discrete, and deterministic or random.
- The Buckingham Pi theorem allows calculation of the number of dimensionless parameters which are involved in the non-dimensional formulation of an equation derived from a physical law.
- Kinematics of particles tracks the motion of particles through space through their position vector, velocity vector, and acceleration.
- A particle moving in a circular path has a velocity that is instantaneously tangent to the circle at the point where the particle is located.
- A particle moving in a circular path has two components of acceleration: a tangential component and a normal component.
- A rigid body undergoes planar motion in the x-y plane if the path of the mass center lies in x-y plane, and rotation occurs only about the z axis.
- The relative velocity and relative acceleration equations are used to analyze rigid body dynamics.
- A free-body diagram (FBD) is a diagram of the body, which has been abstracted from its surroundings, showing the effect of the surroundings in the form of forces.
- Body forces are forces that are applied within the body and are due to an external force field such as gravity.
- Surface forces are applied to the boundary of the body as a result of contact between the body and its surroundings.
- Newton's second law is a basic law of nature written for a particle.
- D'Alembert's principle applied to a rigid body undergoing planar motion reveals that the system of external forces is equivalent to the system of effective forces. The effective forces are a force equal to $m\bar{\mathbf{a}}$ applied at the mass center and a couple equal to $\bar{I}\alpha$.
- The principle of work and energy is a pre-integrated form of Newton's second law, The integration occurs over the path of motion.
- Conservative forces are forces whose work is independent of the path. A potential energy function, which is a function of position, is defined for conservative forces such that the work done by the force is the difference in potential energies.
- The principle of impulse and momentum is a pre-integrated form of Newton's second law, The integration occurs over time.

1.10.2 IMPORTANT EQUATIONS

Simple harmonic motion

$$x(t) = A \sin(\omega t + \phi) \tag{1.12}$$

Velocity and acceleration of a particle

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{1.20}$$

$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{1.21}$$

Velocity and acceleration of a particle moving in a circular path

$$\mathbf{v} = R\omega\mathbf{i}_t \tag{1.25}$$

$$\mathbf{a} = R\alpha\mathbf{i}_t - R\omega^2\mathbf{i}_n \tag{1.26}$$

Relative velocity equations

$$\mathbf{v_B} = \mathbf{v_A} + \mathbf{v_{B/A}} \tag{1.28}$$

$$v_{B/A} = |\mathbf{r_{B/A}}|\omega \tag{1.29}$$

Relative acceleration equations

$$\mathbf{a_B} = \mathbf{a_A} + \mathbf{a_{B/A}} \tag{1.30}$$

$$\mathbf{a_B} = |\mathbf{r_{B/A}}|\alpha\mathbf{i_t} - r\omega^2\mathbf{i_n} \tag{1.31}$$

Newton's second law as applied to a particle

$$\sum \mathbf{F} = m\mathbf{a} \tag{1.32}$$

Newton's second law for a rigid body

$$\sum \mathbf{F} = m\bar{\mathbf{a}} \tag{1.33}$$

Moment equation for a rigid body undergoing planar motion

$$\sum M_G = \bar{I}\alpha \tag{1.34}$$

D'Alembert's principle for rigid bodies undergoing planar motion

$$(\sum \mathbf{F})_{\text{ext}} = (\sum \mathbf{F})_{\text{eff}} \tag{1.36}$$

$$(\sum M_O)_{\text{ext}} = (\sum M_O)_{\text{eff}} \tag{1.37}$$

Work done by a force

$$U_{A\to B} = \int_{\mathbf{r_A}}^{\mathbf{r_B}} \mathbf{F}\cdot d\mathbf{r} \tag{1.40}$$

Principle of work and energy

$$T_A + V_A + U_{A\to B_{NC}} = T_B + V_B \tag{1.47}$$

Impulse due to a force

$$\mathbf{I}_{1\to 2} = \int_{t_1}^{t_2} \mathbf{F}dt \tag{1.48}$$

Principle of impulse and momentum

$$\mathbf{I}_1 + \mathbf{I}_{1\to 2} = \mathbf{I}_2 \tag{1.52}$$

Principle of angular impulse and angular momentum

$$H_{G_1} - J_{G_{1\to 2}} = H_{G_2} \tag{1.53}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For questions 1.1 through 1.10, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

1.1 The earth can be taken to be an inertial reference frame.
1.2 Systems undergoing mechanical vibrations are not subject to nuclear reactions is an example of an explicit assumption.
1.3 A basic law of nature is proven only empirically.
1.4 The point of application of surface forces is anywhere in the body.
1.5 The number of degrees of freedom necessary to model a mechanical system is not unique.
1.6 Distributed parameter systems are another name for discrete systems.
1.7 The Buckingham Pi theorem is used to predict how many non-dimensional variables are used in a dimensionless formulation of a dimensional relationship.
1.8 A rigid body undergoing planar motion has at most three degrees of freedom.
1.9 A particle traveling in a circular path has a velocity which is in the direction of the radius.
1.10 The principle of work and energy is derived from Newton's second law integrated over time.

Questions 1.11 through 1.25 require a short answer.

1.11 What is the continuum assumption, and what does it imply?
1.12 What is the difference between explicit and implicit assumptions?
1.13 How are constitutive equations used in vibrations modeling?
1.14 What is a free-body diagram (FBD)? How is it used in modeling mechanical systems?
1.15 What does the following equation represent

$$x(t) = X\sin(\omega t + \phi)$$

1.16 In the equation of Problem 1.15 define (a) X, (b) ω, and (c) ϕ.
1.17 The phase angle for a mechanical system is calculated as 26°. Does the response lead or lag a pure sinusoid?
1.18 What is the distinction between a particle and a rigid body?
1.19 What are the criteria for a rigid body to undergo planar motion?
1.20 The acceleration of a particle traveling in a circular path has two components. What are they?
1.21 Particle A and particle B are fixed particles on a rigid body undergoing planar motion. Describe the motion of particle B by an observer fixed at particle A.
1.22 How is the equation $\sum \mathbf{F} = m\mathbf{a}$ applied to a vibrating particle?
1.23 What are the effective forces for a rigid body undergoing planar motion?
1.24 The kinetic energy of a rigid body undergoing planar motion consists of two terms. What are they? What does each represent?
1.25 State the principle of impulse and momentum.

126–1.33 How many degrees of freedom are required to model the system of Figures SP 1.26 through 1.33? Identify a set of generalized coordinates which can be used to analyze the system's motion for each system.

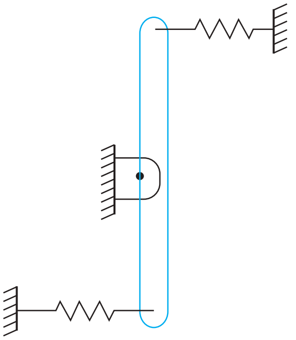

FIGURE SP 1.26

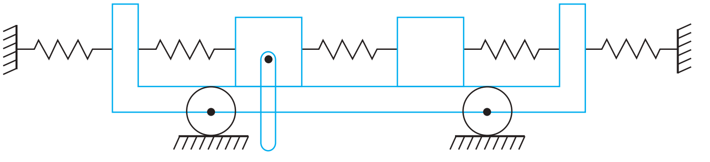

FIGURE SP 1.27

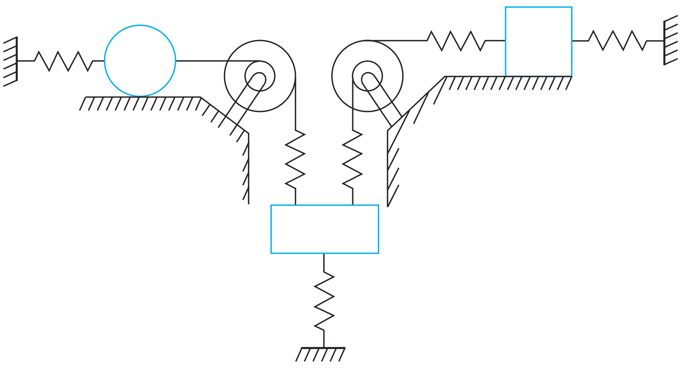

FIGURE SP 1.28

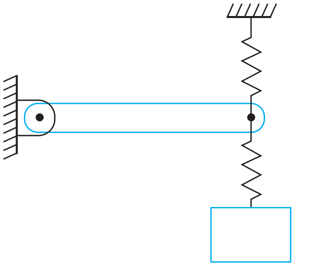

FIGURE SP 1.29

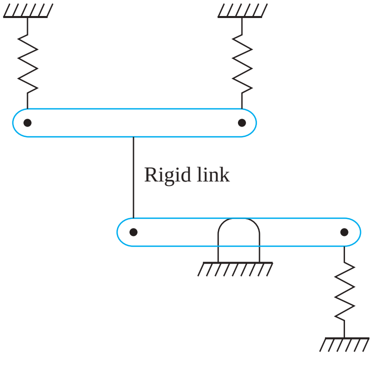

FIGURE SP 1.30

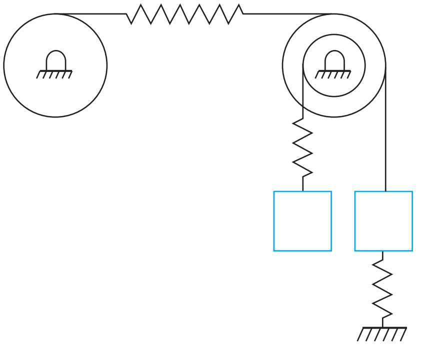

FIGURE SP 1.31

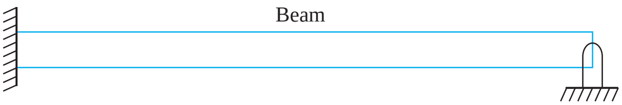

FIGURE SP 1.32

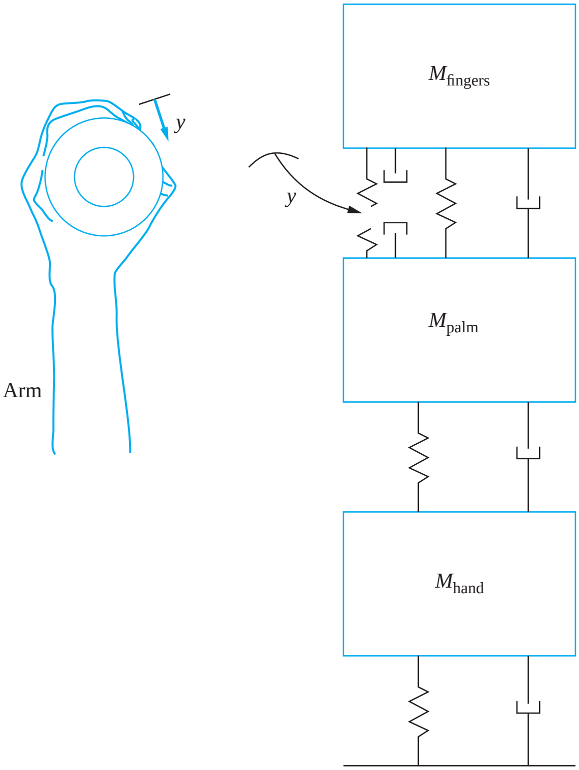

FIGURE SP 1.33

Questions 1.34 through 1.43 require short calculations.

1.34 A particle has a uniform acceleration of 2 m/s^2. If the particle starts from rest at $t = 0$.

(a) Determine the velocity of the particle at $t = 5$ s.
(b) Determine how far the particle travels in 5 s.

1.35 A particle starts at the origin of a Cartesian coordinate system and moves with a velocity vector $v = 2 \cos 2t\,\mathbf{i} + 3 \sin 2t\,\mathbf{j} + 0.4\,\mathbf{k}$ m/s.

(a) Determine the magnitude and direction of the particle's acceleration at $t = \pi$ s.
(b) Determine the particle's position at $t = \pi$ s.

1.36 A particle is traveling in a circular path of radius 3 m. The particle starts at $\theta = 0$ at $t = 0$ and has a constant speed of 2 m/s.

(a) Where is the particle at $t = 2$ s?
(b) What is the acceleration of the particle at $t = 2$ s?

1.37 A rigid body of mass 2 kg undergoes planar motion. At a given instant, the acceleration of its mass center is $(5\mathbf{i} + 3\mathbf{j})$ m/s^2, and it rotates about the z, axis with a clockwise angular acceleration of 10 rad/s^2. What are the effective forces at this instant? Where on the body are they applied?

1.38 The velocity of a particle of mass 0.1 kg is $(9\mathbf{i} + 11\mathbf{j})$ m/s. Calculate the kinetic energy of the particle.

1.39 The velocity of the mass center of a rigid body of mass 3 kg undergoing planar motion is $(3\mathbf{i} + 4\mathbf{j})$ m/s. The mass center is 20 cm from the fixed axis of rotation. Calculate the angular velocity of the body at this instant.

1.40 The kinetic energy of a body that rotates about its centroidal axis is 100 J. The centroidal mass moment of inertia is 0.03 kg • m^2. Calculate the angular velocity of the body.

1.41 The speed of the mass center of a rigid body undergoing planar motion of mass 5 kg is 4 m/s. It rotates about the z axis with a clockwise angular velocity of 20 rad/s. The mass moment of inertia of the body about its centroidal axis is 0.08 kg • m^2. Calculate the kinetic energy of the body.

1.42 An impulsive force of magnitude 12,000 N is applied to a particle for 0.03 s. What is the total impulse imparted by this force?

1.43 The force of Figure SP1.43 is applied to a particle of mass 3 kg at rest in equilibrium.

(a) What is the total impulse imparted to the particle?
(b) What is the velocity of the particle at $t = 2$ s?
(c) What is the velocity of the particle at 5 s?

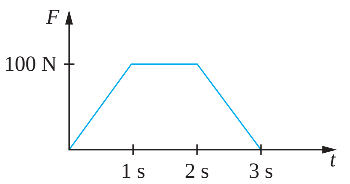

FIGURE SP 1.43

1.44 A particle of mass 2 kg is subject to a constant force of 6 N, as shown in Figure SP1.44. How far has the particle traveled after 10 s if the particle's velocity is 4 m/s initially?

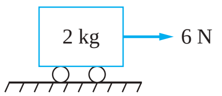

FIGURE SP 1.44

1.45 Match the quantity with the appropriate units (units may be used more than once, and some units may not be used).

(a) acceleration, a	(i) N • s
(b) velocity, v	(ii) m/s^2
(c) impulse, I	(iii) rad/s^2
(d) kinetic energy, T	(iv) m/s
(e) linear momentum, $\mathbf{L}$	(v) J

(f) work done by a force, $W_{1\rightarrow 2}$ (vi) rad/s
(g) angular velocity, ω (vii) m
(h) angular acceleration, α (viii) rad
(i) force, F (ix) N

CHAPTER PROBLEMS

1.1 The one-dimensional displacement of a particle is

$$x(t) = 0.5e^{-0.2t}\sin 5t \text{ m}$$

(a) What is the maximum displacement of the particle?
(b) What is the maximum velocity of the particle?
(c) What is the maximum acceleration of the particle?

1.2 The one-dimensional displacement of a particle is

$$x(t) = 0.5e^{-0.2t}\sin(5t + 0.24) \text{ m}$$

(a) What is the maximum displacement of the particle?
(b) What is the maximum velocity of the particle?
(c) What is the maximum acceleration of the particle?

1.3 At the instant shown in Figure P1.3, the slender rod has a clockwise angular velocity of 5 rad/s and a counterclockwise angular acceleration of 14 rad/s^2. At the instant shown, determine (a) the velocity of point P and (b) the acceleration of point P.

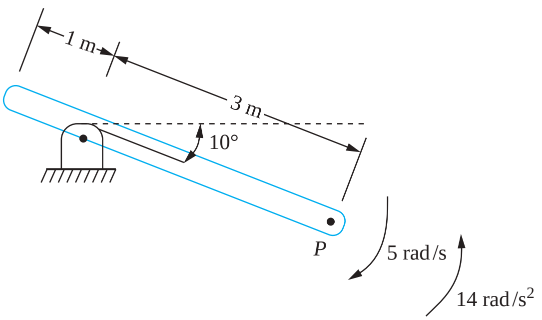

FIGURE P1.3

1.4 A $t = 0$, a particle of mass 1.2 kg is traveling with a speed of 10 m/s that is increasing at a rate of 0.5 m/s^2. The local radius of curvature at this instant is 50 m. After the particle travels 100 m, the radius of curvature of the particle's path is 50 m.

(a) What is the speed of the particle after it travels 100 m?
(b) What is the magnitude of the particle's acceleration after it travels 100 m?
(c) How long does it take the particle to travel 100 m?
(d) What is the external force acting on the particle after it travels 100 m?

1.5 The machine of Figure P1.5 has a vertical displacement $x(t)$. The machine has a component which rotates with a constant angular speed ω. The center of mass of the rotating component is a distance e from the axis of rotation. The center of mass of the rotating component is as shown at $t = 0$. Determine the vertical component of the acceleration of the rotating component.

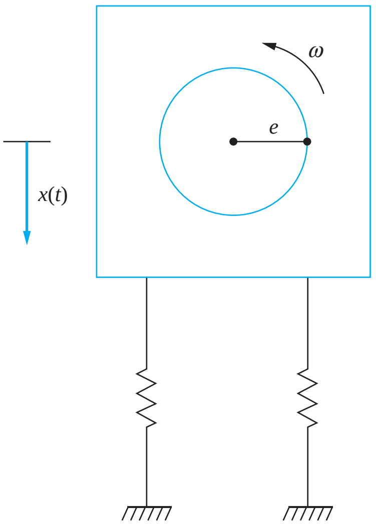

FIGURE P1.5

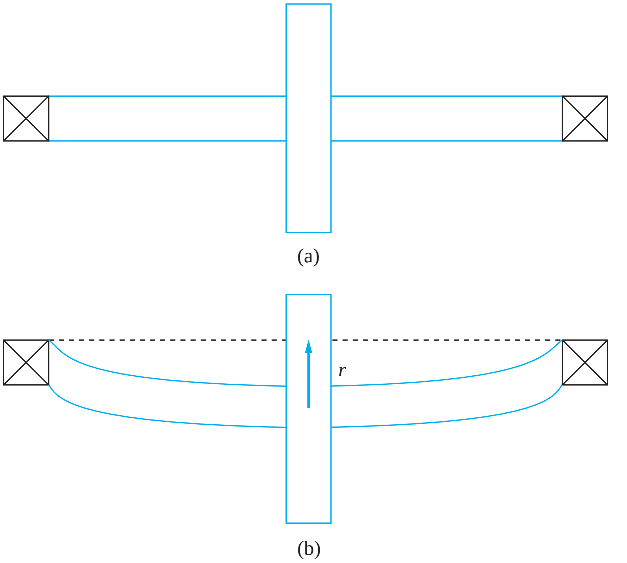

FIGURE P1.6

1.6 The rotor of Figure P1.6 consists of a disk mounted on a shaft. Unfortunately, the disk is unbalanced, and the center of mass is a distance e from the center of the shaft. As the disk rotates, this causes a phenomena called "whirl", where the disk bows. Let r be the instantaneous distance from the center of the shaft to the original axis of the shaft and θ be the angle made by a given radius with the horizontal. Determine the acceleration of the mass center of the disk.

1.7 A 2 ton truck is traveling down an icy, 10° hill at 50 mph when the driver sees a car stalled at the bottom of the hill 250 ft away. As soon as he sees the stalled car, the driver applies his brakes, but due to the icy conditions, a braking force of only 2000 N is generated. Does the truck stop before hitting the car?

1.8 The contour of a bumpy road is approximated by

$$y(x) = 0.03 \sin(0.125x) \text{ m}$$

What is the amplitude of the vertical acceleration of the wheels of an automobile as it travels over this road at a constant horizontal speed of 40 m/s?

1.9 The helicopter of Figure P1.9 has a horizontal speed of 110 ft/s and a horizontal acceleration of 3.1 ft/s^2. The main blades rotate at a constant speed of 135 rpm. At the instant shown, determine the velocity and acceleration of particle A.

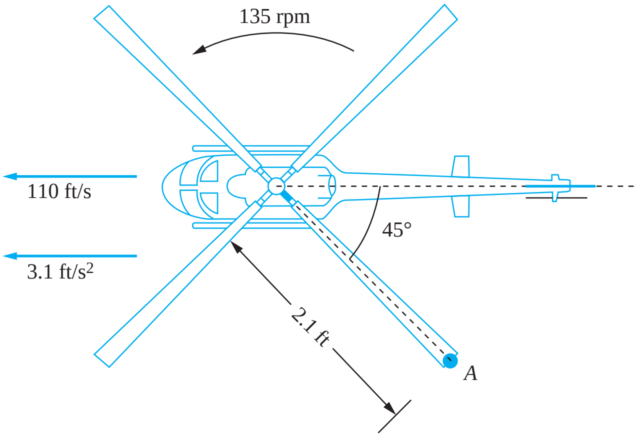

FIGURE P1.9

1.10 For the system shown in Figure P1.10, the angular displacement of the thin disk is given by $\theta(t) = 0.03 \sin\left(30t + \frac{\pi}{4}\right)$ rad. The disk rolls without slipping on the surface. Determine the following as functions of time.

(a) The acceleration of the center of the disk.
(b) The acceleration of the point of contact between the disk and the surface.
(c) The angular acceleration of the bar.
(d) The vertical displacement of the block.

(*Hint:* Assume small angular oscillations ϕ of the bar. Then $\sin\phi \approx \phi$.)

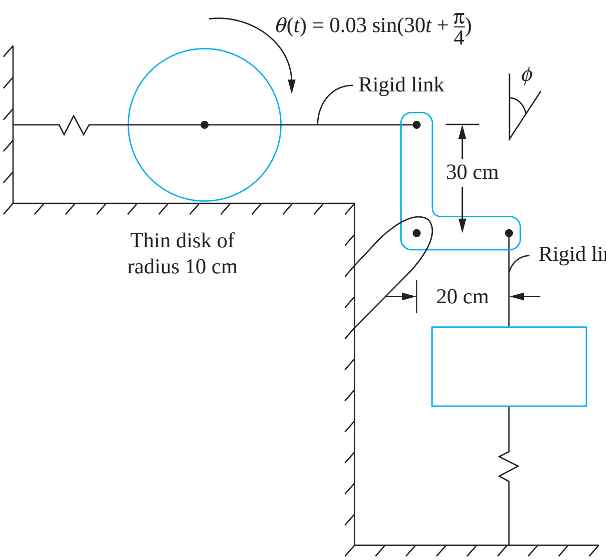

FIGURE P1.10

1.11 The velocity of the block of the system of Figure P1.11 is $\dot{y} = 0.02 \sin 20t$ m/s downward.

(a) What is the clockwise angular displacement of the pulley?
(b) What is the displacement of the cart?

1.12 A 60-lb block is connected by an inextensible cable through the pulley to the fixed surface, as shown in Figure P1.12. A 40 lb weight is attached to the pulley, which is free to move vertically. A force of magnitude $P = 100(1 + e^{-t})$ lb tows the block. The system is released from rest at $t = 0$.

(a) What is the acceleration of the 60 lb block as a function of time?
(b) How far does the block travel up the incline before it reaches a velocity of 2 ft/s?

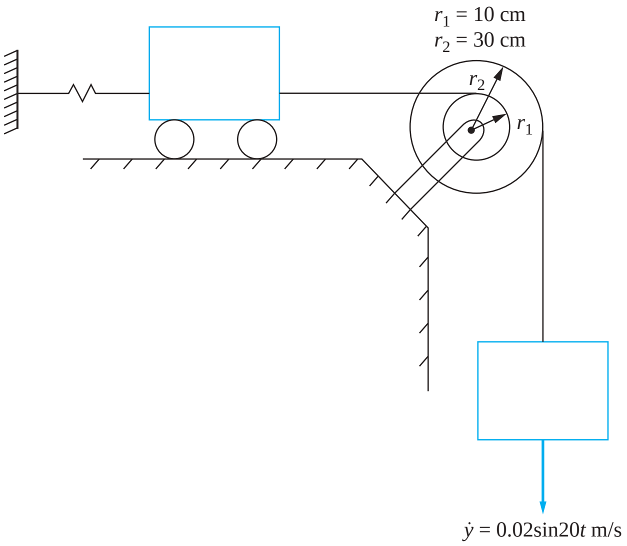

FIGURE P1.11

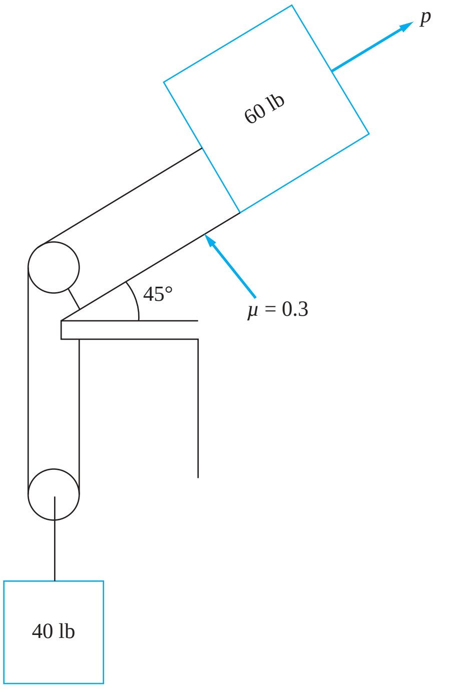

FIGURE P1.12

1.13 Repeat Problem 1.8 for a force of $P = 100t$ N.

1.14 Figure P1.14 shows a schematic diagram of a one-cylinder reciprocating one-cylinder engine. If at the instant of time shown the piston has a velocity v and an acceleration a, determine (a) the angular velocity of the crank and (b) the angular acceleration of the crank in terms of v, a, the crank radius r, the connecting rod length ℓ, and the crank angle θ.

1.15 Determine the reactions at A for the two-link mechanism of Figure P1.15. The roller at C rolls on a frictionless surface.

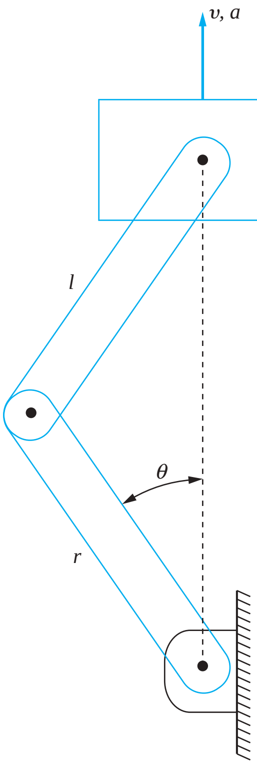

FIGURE P1.14

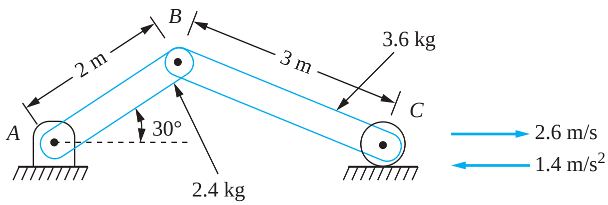

FIGURE P1.15

1.16 Determine the angular acceleration of each of the disks in Figure P1.16.

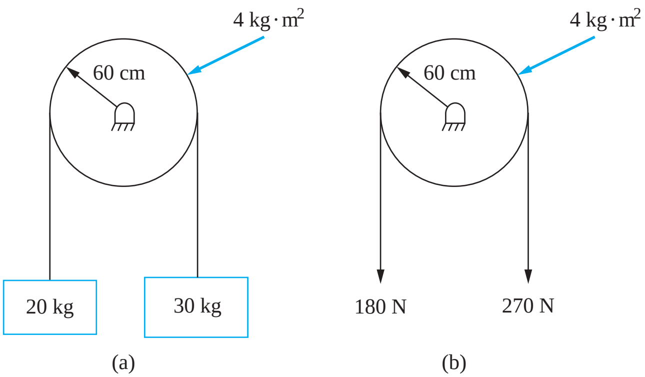

FIGURE P1.16

1.17 Determine the reactions at the pin support and the applied moment if the bar of Figure P1.17 has a mass of 50 g.

1.18 The disk of Figure P1.18 rolls without slipping. Assume if $P = 18$ N.

(a) Determine the acceleration of the mass center of the disk.
(b) Determine the angular acceleration of the disk.

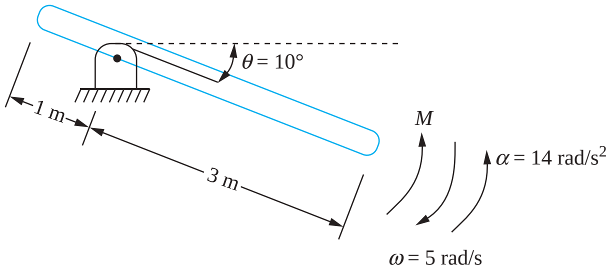

FIGURE P1.17

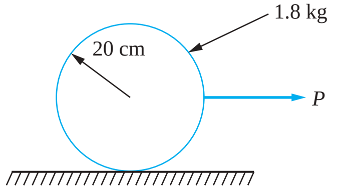

FIGURE P1.18

1.19 The coefficient of friction between the disk of Figure P1.18 and the surface is 0.12. What is the largest force that can be applied such that the disk rolls without slipping?

1.20 The coefficient of friction between the disk of Figure P1.18 and the surface is 0.12. If $P = 22$ N, what are the following?

(a) Acceleration of the mass center.
(b) Angular acceleration of the disk.

1.21 The 3 kg block of Figure P1.21 is displaced 10 mm downward and then released from rest.

(a) What is the maximum velocity attained by the 3-kg block?
(b) What is the maximum angular velocity attained by the disk?

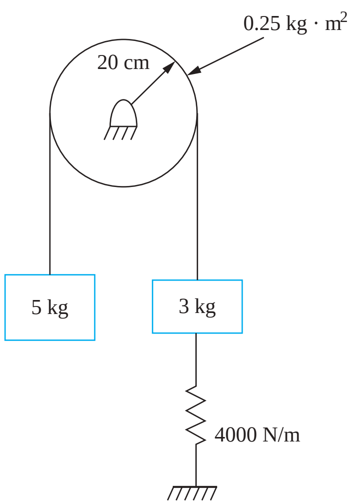

FIGURE P1.21

1.22 The center of the thin disk of Figure P1.22 is displaced 15 mm and released. What is the maximum velocity attained by the disk, assuming no slipping between the disk and the surface?

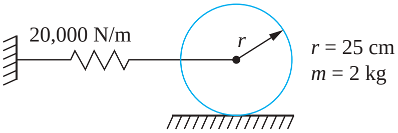

FIGURE P1.22

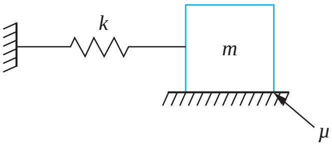

FIGURE P1.23

1.23 The block of Figure P1.23 is given a displacement δ and then released.

(a) What is the minimum value of δ such that motion ensues?

(b) What is the minimum value of δ such that the block returns to its equilibrium position without stopping?

1.24 The five-blade ceiling fan of Figure P1.24 operates at 60 rpm. The distance between the mass center of a blade and the axis of rotation is 0.35 m. What is its total kinetic energy?

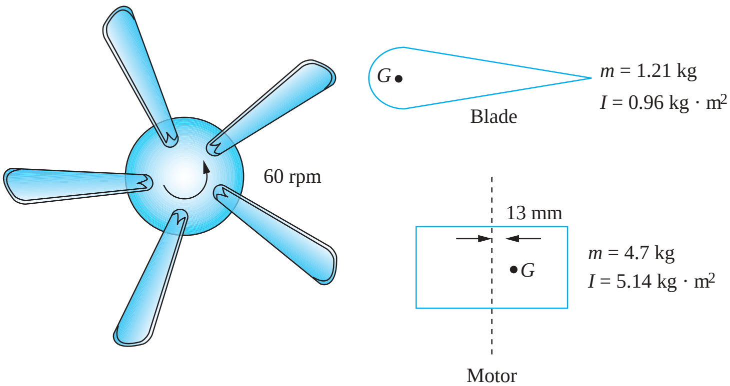

FIGURE P1.24

1.25 The U-tube manometer shown in Figure P1.25 rotates about axis *A*-*A* at a speed of 40 rad/s. At the instant shown, the column of liquid moves with a

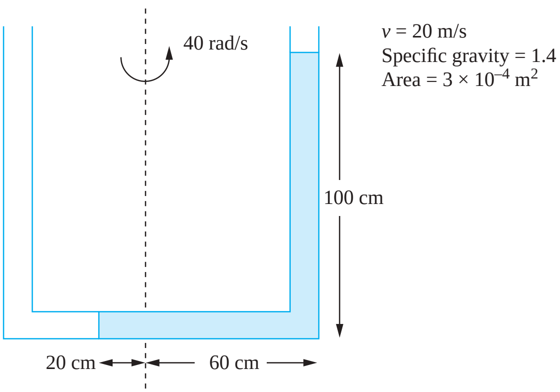

FIGURE P1.25

speed of 20 m/s relative to the manometer. Calculate the total kinetic energy of the column of liquid in the manometer.

1.26 The displacement function for the simply supported beam of Figure P1.26 is

$$y(x, t) = c \sin\left(\frac{\pi x}{L}\right)\cos\left(\pi^2\sqrt{\frac{EI}{\rho A L^4}}t\right)$$

where $c = 0.003$ m and t is in seconds. Determine the kinetic energy of the beam.

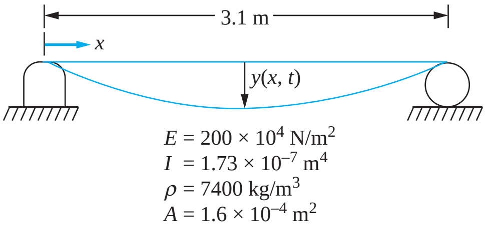

FIGURE P1.26

1.27 The block of Figure P1.27 is displaced 1.5 cm from equilibrium and released.

(a) What is the maximum velocity attained by the block?
(b) What is the acceleration of the block immediately after it is released?

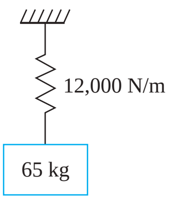

FIGURE P1.27

1.28 The slender rod of Figure P1.28 is released from the horizontal position when the spring attached at A is stretched 10 mm and the spring attached at B is unstretched.

(a) What is the acceleration of the bar immediately after it is released?
(b) What is the maximum angular velocity attained by the bar?

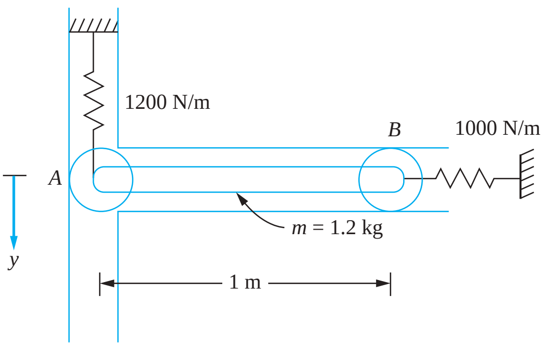

FIGURE P1.28

1.29 Let x be the displacement of the left end of the bar of the system in Figure P1.29. Let θ represent the clockwise angular rotation of the bar.

(a) Express the kinetic energy of the system at an arbitrary instant in terms of $\dot{x}$ and $\dot{\theta}$.
(b) Express the potential energy of an arbitrary instant in terms of x and θ.

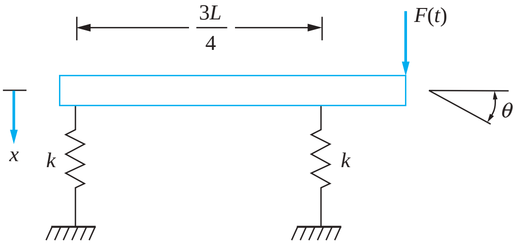

FIGURE P1.29

1.30 Repeat Problem 1.29 using as coordinates x_1, which is the displacement of the mass center, and x_2, which is the displacement of the point of attachment of the spring that is a distance $3L/4$ from the left end.

1.31 Let θ represent the clockwise angular displacement of the pulley of the system in Figure P1.31 from the system's equilibrium position.

(a) Express the potential energy of the system at an arbitrary instant in terms of θ.
(b) Express the kinetic energy of the system at an arbitrary instant in terms of $\dot{\theta}$.

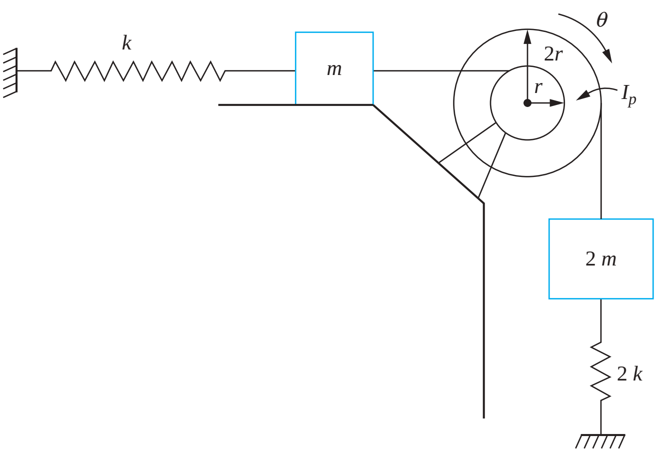

FIGURE P1.31

1.32 A 20 ton railroad car is coupled to a 15 ton car by moving the 20 ton car at 5 mph toward the stationary 15 ton car.

(a) What is the resulting speed of the two-car coupling?
(b) What would the resulting speed be if the 15 ton car is moving at 5 mph toward a stationary 20 ton car?

1.33 The 15 kg block of Figure P1.33 is moving with a velocity of 3 m/s at $t = 0$ when the force $F(t)$ is applied to the block.

(a) Determine the velocity of the block at $t = 2$ s.
(b) Determine the velocity of the block at $t = 4$ s.
(c) Determine the block's kinetic energy at $t = 4$ s.

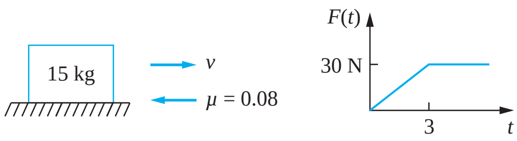

FIGURE P1.33

1.34 A 400 kg forging hammer is mounted on four identical springs, each of stiffness $k = 4200$ N/m. During the forging process, a 110 kg hammer, which is part of the machine, is dropped from a height of 1.4 m onto an anvil, as shown in Figure P1.34.

(a) What is the resulting velocity of the entire machine after the hammer is dropped?
(b) What is the maximum displacement of the machine?

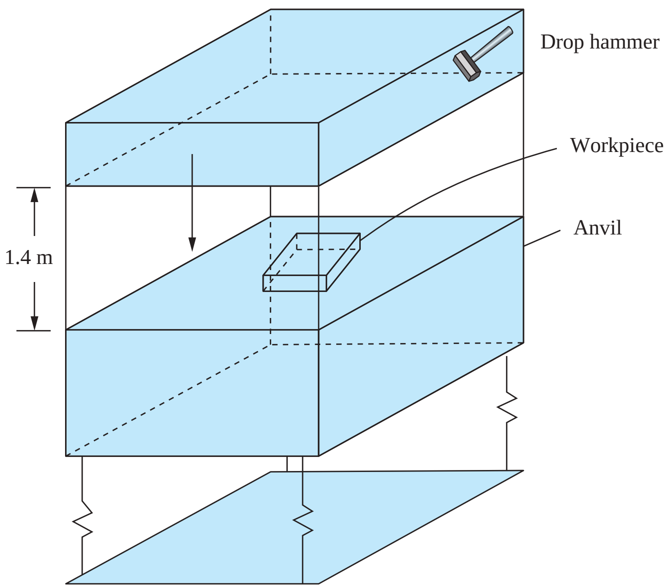

FIGURE P1.34

1.35 The motion of a baseball bat in a ballplayer's hands is approximated as a rigid-body motion about an axis through the player's hands, as shown in Figure P1.35. The bat has a centroidal moment of inertia I. The player's "bat speed" is ω, and the velocity of the pitched ball is v. Determine the distance from the player's hand along the bat where the batter should strike the ball to minimize the impulse felt by the his/her hands. Does the distance change if the player "chokes up" on the bat, reducing the distance from G to his/her hands.

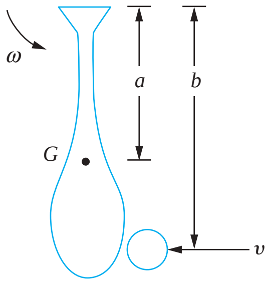

FIGURE P1.35

1.36 A playground ride has a centroidal moment of inertia of 17 slug · ft^2. Three children of weights 50 lb, 50 lb, and 55 lb are on the ride, which is rotating at 60 rpm. The children are 30 in. from the center of the ride. A father stops the ride by grabbing it with his hands. What is the impulse felt by the father?

Problems 1.37 through 1.39 present different problems that are to be formulated in non-dimensional form. For each problem answer the following.

(a) What are the dimensions involved in each of the parameters?
(b) How many dimensionless parameters does the Buckingham Pi theorem predict are in the non-dimensional formulation of the relation between the natural frequencies and the other parameters?
(c) Develop a set of dimensionless parameters.

1.37 The natural frequencies of a thermally loaded fixed-fixed beam (Figure P1.37) are a function of the material properties of the beam, including:

E, the elastic modulus of the beam
ρ, the mass density of the beam
α, the coefficient of thermal expansion

The geometric properties of the beam are

A, its cross-sectional area
I, its cross section moment of inertia
L, its length

Also,

ΔT, the temperature difference between the installation and loading

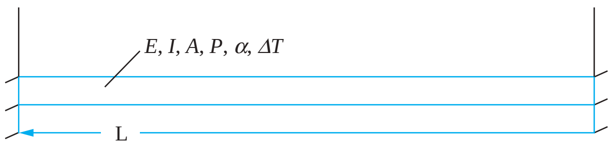

FIGURE P1.37

1.38 The drag force F on a circular cylinder due to vortex shedding is a function of

U, the velocity of the flow
μ, the dynamic viscosity of the fluid
ρ, the mass density of the fluid
L, the length of the cylinder
D, the diameter of the cylinder

1.39 The principal normal stress σ due to forcing of a beam with a concentrated harmonic excitation is a function of

F_0, the amplitude of loading
ω, the frequency of the loading
E, the elastic modulus of the beam
ρ, the mass density of the beam
A, the beam's cross-sectional area
I, the beam's cross-sectional moment of inertia
L, the beam's length
a, the location of the load along the axis of the beam

1.40 A MEMS system is undergoing simple harmonic motion according to

$$x(t) = [3.1 \sin(2 \times 10^5 t + 0.48) + 4.8 \cos(2 \times 10^5 t + 1.74)]\ \mu\text{m}$$

(a) What is the period of motion?
(b) What is the frequency of motion in Hz?
(c) What is the amplitude of motion?
(d) What is the phase and does it lead or lag?
(e) Plot the displacement.

1.41 The force that causes simple harmonic motion in the mass-spring system of Figure P1.41 is $F(t) = 35 \sin 30t$ N. The resulting displacement of the mass is $x(t) = 0.002 \sin(30t - \pi)$m.

(a) What is the period of the motion?
(b) The amplitude of displacement is $X = \frac{F_0}{k}M$ where F_0 is the amplitude of the force and M is a dimensionless factor called the magnification factor. Calculate M.
(c) M has the form

$$M = \frac{1}{\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|}$$

where ω_n is called the natural frequency. If $\omega_n < \omega$, then $\phi = \pi$; otherwise $\phi = 0$. Calculate ω_n.

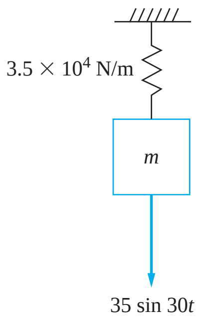

FIGURE P1.41

1.42 The displacement vector of a particle is

$$\mathbf{r}(t) = [2\sin 20t\,\mathbf{i} + 3\cos 20t\,\mathbf{j}]\ \text{mm}$$

(a) Describe the trajectory of the particle.
(b) How long does it take the particle to make one circuit around the path?

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Chapter 2

MODELING OF SDOF SYSTEMS

2.1 INTRODUCTION

The basic components of a mechanical system are inertia, stiffness, damping, and a source of work or energy. *Inertia components* store kinetic energy. *Stiffness components* store potential energy. *Damping components* dissipate energy. *Energy sources* provide energy to the system.

This chapter begins with a discussion of potential energy sources, mainly springs. Springs store potential energy, but they don't require motion to do so. The helical coil spring serves as the model for all linear springs. Structural components, such as bars undergoing longitudinal motion, shafts under rotational motion, and beams undergoing transverse vibrations, all store potential energy and can be modeled as springs. Combinations of springs may be replaced by a single spring of an equivalent stiffness. Hanging springs acting under gravity store potential energy when in static equilibrium. However, the potential energy stored in the spring due to deflection from its equilibrium position cancels with the potential energy due to gravity for a linear system, when modeling a linear system.

Viscous damping refers to any form of damping in which the friction force is proportional to the velocity. Viscous dampers are inserted into mechanical systems because they add a linear term in the differential equation. The energy dissipated due to the viscous damping force is considered and an equivalent viscous damping coefficient is calculated for a combination of viscous dampers.

An *inertia element* is anything that has mass or stores kinetic energy. The principles of dynamics reviewed in Chapter 1 govern the motion of inertia elements. An equivalent mass can be calculated for a SDOF system when it includes several inertia elements. The inertia effects of springs and entrained fluids are taken into account with an equivalent mass model.

The energy source could be an initial energy present in the system, or it could be an input to the system in terms of an external force or an imposed motion.

The derivation of differential equations governing the motion of a SDOF is considered. The free-body diagram method applies Newton's second law or D'Alembert's principle to free-body diagrams drawn at an arbitrary instant. Nonlinear differential equations are linearized through application of a small angle or small displacement assumption.

The equivalent systems method only applies for linear systems. It uses the model of a linear mass-spring and viscous-damper system for any linear SDOF system. The kinetic energy calculated at an arbitrary instant is used to determine an equivalent mass. The potential energy is used to determine an equivalent stiffness. The work done by viscous damping forces is used to calculate an equivalent viscous damping coefficient. The work done by external forces is used to calculate an equivalent force.

A second-order linear ordinary differential equation which governs the motion of a SDOF system results from either method. The equation may be homogeneous (in the case of free vibrations) or non-homogeneous (in the case of forced vibrations).

2.2 SPRINGS

2.2.1 INTRODUCTION

A *spring* is a flexible mechanical link between two particles in a mechanical system. In reality a spring itself is a continuous system. However, the inertia of the spring is usually small compared to other elements in the mechanical system and is neglected. Under this assumption the force applied to each end of the spring is the same.

The length of a spring when it is not subject to external forces is called its *unstretched length*. Since the spring is made of a flexible material, the force F that must be applied to the spring to change its length by x is some continuous function of x,

$$F = f(x) \tag{2.1}$$

The appropriate form of $f(x)$ is determined by using the constitutive equation for the spring's material. Since $f(x)$ is infinitely differentiable at $x = 0$, it can be expanded by a Taylor series about $x = 0$ (a MacLaurin expansion):

$$F = k_0 + k_1x + k_2x^2 + k_3x^3 + \cdots \tag{2.2}$$

Since x is the spring's change in length from its unstretched length, when $x = 0$, $F = 0$. Thus $k_0 = 0$. When x is positive, the spring is in tension. When x is negative, the spring is in compression. Many materials have the same properties in tension and compression. That is, if a tensile force F is required to lengthen the spring by δ, then a compressive force of the same magnitude F is required to shorten the spring by δ. For these materials, $f(-x) = -f(x)$, or f is an odd function of x. The Taylor series expansion of an odd function cannot contain even powers. Thus, Equation (2.2) becomes

$$F = k_1x + k_3x^3 + k_5x^5 + \cdots \tag{2.3}$$

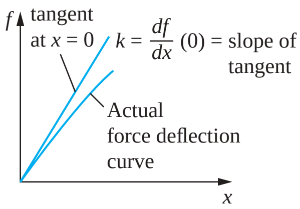


FIGURE 2.1
The spring stiffness is the derivative of the force displacement relation at $x = 0$.

All springs are inherently nonlinear. However in many situations x is small enough that the nonlinear terms of Equation (2.3) are small compared with $k_1 x$. A *linear spring* obeys a force-displacement law of

$$F = kx \tag{2.4}$$

where k is called the *spring stiffness* or *spring constant* and has dimensions of force per length. Thus, for a linear spring, $k = \frac{df}{dx}\big|_{x=0}$, which is illustrated in Figure 2.1.

The work done by a force is calculated according to Equation (1.40). For a linear system where the spring force is applied to a particle whose displacement is x, in the horizontal direction the force is represented by $-kx\mathbf{i}$, and the differential displacement vector is $dx\mathbf{i}$. The work done by the spring force as its point of application moves from a position described by x_1 to a position described by x_2 is

$$U_{1\to 2} = \int_{x_1}^{x_2} (-kx)\,dx = k\frac{x_1^2}{2} - k\frac{x_2^2}{2} \tag{2.5}$$

Since the work depends upon the initial and final position of the point of application of the spring force and not the path of the system, the spring force is conservative. A *potential energy function* can be defined for a spring as

$$V(x) = \frac{1}{2}kx^2 \tag{2.6}$$

where x is the change in the length of the spring from its unstretched length.

A *torsional spring* is a link in a mechanical system where application of a torque leads to an angular displacement between the ends of the torsional spring. A linear torsional spring has a relationship between an applied moment M and the angular displacement θ of

$$M = k_t\theta \tag{2.7}$$

where the *torsional stiffness* k_t has dimensions of force times length. The potential energy function for a torsional spring is

$$V = \frac{1}{2}k_t\theta^2 \tag{2.8}$$

2.2.2 HELICAL COIL SPRINGS

The helical coil spring is used in applications such as industrial machines and vehicle suspension systems. Consider a spring manufactured from a rod of circular cross section of diameter D. The shear modulus of the rod is G. The rod is formed into a coil of N turns of radius r. It is assumed that the coil radius is much larger than the radius of the rod and that the normal to the plane of one coil nearly coincides with the axis of the spring.

Consider a helical coil spring when subject to an axial load F. Imagine cutting the rod with a knife at an arbitrary location in a coil, slicing the spring in two sections. The cut exposes an internal shear force F and an internal resisting torque Fr, as illustrated in

FIGURE 2.2
A spring is subject to a force F along its axis. A section cut of the spring reveals its cross section has a shear force F and a torque Fr where r is the coil radius.

Figure 2.2. Assuming elastic behavior, the shear stress due to the resisting torque varies linearly with distance from the center of the rod to a maximum of

$$\tau_{\max} = \frac{FrD}{2J} = \frac{16Fr}{\pi D^3} \tag{2.9}$$

where $J = (\pi D^4)/32$ is the polar moment of inertia of the rod. The shear stress due to the shear force varies nonlinearly with distance from the neutral axis. For $r/D \gg 1$ the maximum shear stress due to the internal shear force is much less than the maximum shear stress due to the resisting torque, and its effect is neglected.

Principles of mechanics of materials can be used to show that the total change in length of the spring due to an applied force F is

$$x = \frac{64Fr^3N}{GD^4} \tag{2.10}$$

Comparing Equation (2.10) with Equation (2.4) leads to the conclusion that under the assumptions stated a helical coil spring can be modeled as a linear spring of stiffness

$$k = \frac{GD^4}{64Nr^3} \tag{2.11}$$

EXAMPLE 2.1

A tightly wound spring is made from a 20-mm-diameter bar of 0.2% C-hardened steel ($G = 80 \times 10^9$ N/m^2). The coil diameter is 20 cm. The spring has 30 coils. What is the largest force that can be applied such that the elastic strength in shear of 220×10^6 N/m^2 is not exceeded? What is the change in length of the spring when this force is applied?

SOLUTION

Assuming the shear stress due to the shear force is negligible, the maximum shear stress in the spring when a force F is applied is

$$\tau = \frac{FrD}{2J} = F\frac{(0.1\ \text{m})(0.02\ \text{m})}{\frac{2\pi}{32}(0.02\ \text{m})^4} = 6.37 \times 10^4 F$$

Thus the maximum allowable force is

$$F_{\max} = \frac{220 \times 10^6\ \text{N/m}^2}{6.37 \times 10^4} = 3.45 \times 10^3\ \text{N}$$

The stiffness of this spring is calculated by using Equation (2.11):

$$k = \frac{(80 \times 10^9\ \text{N/m}^2)(0.02\text{m})^4}{(64)(30)(0.1\text{m}^3)} = 6.67 \times 10^3\ \frac{\text{N}}{\text{m}}$$

The total changes in length of the spring due to application of the maximum allowable force is

$$\Delta = \frac{F}{k} = 0.518\ \text{m}$$

2.2.3 ELASTIC ELEMENTS AS SPRINGS

Application of a force F to the block of mass m of Figure 2.3 results in a displacement x. The block is attached to a uniform thin rod of elastic modulus E, unstretched length L, and cross-sectional area A. Application of the force results in a uniform normal strain in the rod of

$$\boldsymbol{\varepsilon} = \frac{F}{AE} = \frac{x}{L} \tag{2.12}$$

The strain energy per volume is the area under the stress–strain curve, which for an elastic bar:

$$s = \frac{1}{2}\boldsymbol{\sigma\varepsilon} = \frac{1}{2}E\boldsymbol{\varepsilon}^2 \tag{2.13}$$

The total strain energy is

$$S = sV = \frac{1}{2}E\boldsymbol{\varepsilon}^2 AL = \frac{1}{2}(EA/L)x^2 \tag{2.14}$$

If the force is suddenly removed, the block will oscillate about its equilibrium position. The initial strain energy is converted to kinetic energy and vice versa, a process which continues indefinitely. If the mass of the rod is small compared to the mass of the block, then inertia of the rod is negligible and the rod behaves as a discrete spring. From strength of materials, the force F required to change the length of the rod by x is

$$F = \frac{AE}{L}x \tag{2.15}$$

A comparison of Equation (2.15) with Equation (2.4) implies that the stiffness of the rod is

$$k = \frac{AE}{L} \tag{2.16}$$

The motion of a particle attached to an elastic element can be modeled as a particle attached to a linear spring, provided the mass of the element is small compared to the mass of the particle and a linear relationship between force and displacement exists for the element. In Figure 2.4, a particle of mass m is attached to the midspan of a simply supported beam of length L, elastic modulus E, and cross-sectional moment of inertia I. The transverse displacement of the midspan of the beam due to an applied static load F is

$$x = \frac{L^3}{48EI}F \tag{2.17}$$

Thus a linear relationship exists between transverse displacement and static load. Hence if the mass of the beam is small, the vibrations of the particle can be modeled as the vertical motion of a particle attached to a spring of stiffness

$$k = \frac{48EI}{L^3} \tag{2.18}$$

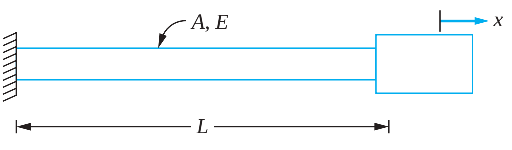


FIGURE 2.3
Longitudinal vibrations of a mass attached to the end of a uniform thin rod can be modeled as a linear mass-spring system with $k = AE/L$.

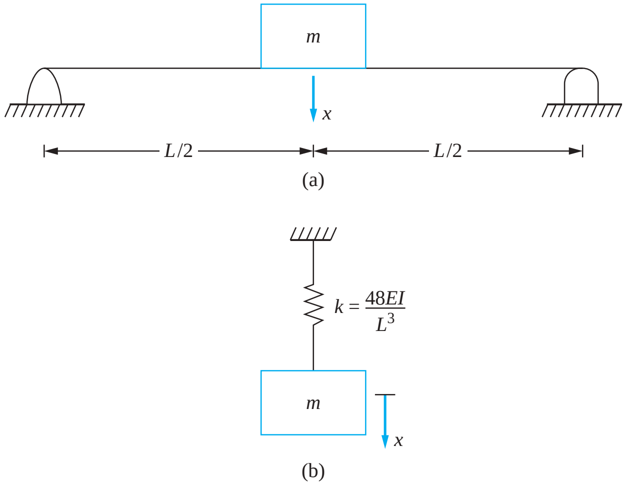


FIGURE 2.4
The transverse vibrations of a machine attached to the midspan of a simply supported beam (a) modeled by a mass-spring system and the stiffness of the spring is 48 EI/L^3. (b) provided the mass of the beam is small in comparison to the mass of the machine.

In general the transverse vibrations of a particle attached to a beam can be modeled as those of a particle attached to a linear spring. Let $w(z)$ represent the displacement function of the beam due to a concentrated unit load applied at $z = a$. Then the displacement at $z = a$ due to a load F applied at $z = a$ is

$$x = \omega(a)F \tag{2.19}$$

Then the spring stiffness for a particle placed at $z = a$ is

$$k = \frac{1}{\omega(a)} \tag{2.20}$$

EXAMPLE 2.2

A 200-kg machine is attached to the end of a cantilever beam of length $L = 2.5$ m, elastic modulus $E = 200 \times 10^9$ N/m^2, and cross-sectional moment of inertia 1.8×10^{-6} m^4. Assuming the mass of the beam is small compared to the mass of the machine, what is the stiffness of the beam?

SOLUTION

From Table D.2 the deflection equation for a cantilever beam with a concentrated unit load at $z = L$ is

$$\omega(z) = \frac{1}{EI}\left(-\frac{1}{6}z^3 + \frac{L}{2}z^2\right) \tag{a}$$

The deflection at the end of the beam is

$$\omega(L) = \frac{1}{EI}\left(-\frac{L^3}{6} + \frac{L}{2}L^2\right) = \frac{L^3}{3EI} \tag{b}$$

The stiffness of the cantilever beam at its end is

$$k = \frac{3\,EI}{L^3} = \frac{3\,(200 \times 10^9\ \text{N/m}^2)\,(1.8 \times 10^{-6}\ \text{m}^4)}{(2.5\ \text{m})^3} = 6.91 \times 10^4\ \text{N/m} \tag{c}$$

Equation (2.18) is used for the stiffness of a pinned-pinned beam at its midspan. The equation for the stiffness of a cantilever beam at its end is

$$k = \frac{3EI}{L^3} \tag{2.21}$$

The equivalent stiffness of a fixed-fixed beam at its midspan is

$$k = \frac{192EI}{L^3} \tag{2.22}$$

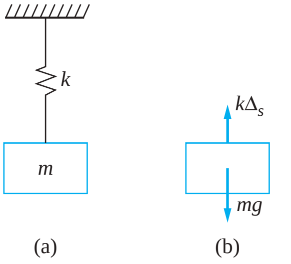


FIGURE 2.5
(a) The spring has a static spring force when the system is in static equilibrium.
(b) FBD of the mass when the system is in equilibrium.

2.2.4 STATIC DEFLECTION

When a spring is not in its unstreched length when a system is in equilibrium, the spring has a static deflection. When the system of Figure 2.5(b) is in equilibrium a static force in the spring is necessary to balance the gravity force. From the FBD of Figure 2.5(b) the force in the spring is $F_s = mg$. Since the force is the stiffness times the change in length from its unstretched length, the static deflection is calculated as

$$\Delta_s = \frac{mg}{k} \tag{2.23}$$

EXAMPLE 2.3

Determine the static deflection of the spring in the system of Figure 2.6(a).

SOLUTION

The FBDs of the system in its equilibrium position are shown in Figure 2.6(b). Summing forces to zero on the FBD of the left hand block $\Sigma F = 0$ leads to

$$T_1 = m_1 g - k\Delta_s \tag{a}$$

Summing moments about the center of the disk leads to $\Sigma M_O = 0$, as

$$m_2 g r_2 - (m_1 g - k\Delta_s) r_1 = 0 \tag{b}$$

from which the static deflection is determined as

$$\Delta_s = \frac{m_1 g r_1 - m_2 g r_2}{k r_1} \tag{c}$$

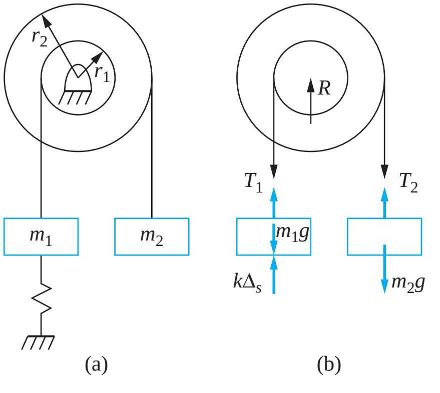


FIGURE 2.6
(a) System of Example 2.3. (b) FBDs of system when it is in equilibrium.

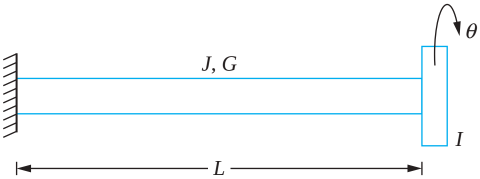


FIGURE 2.7
The rotational motion of the thin disk attached to the shaft are modeled by the torsional oscillations of a disk attached to a torsional spring of stiffness $k_t = \frac{JG}{L}$.

Torsional oscillations occur in the system of Figure 2.7. A thin disk of mass moment of inertia I is attached to a circular shaft of length L, shear modulus G, and polar moment of inertia J. When the disk is rotated through an angle θ from its equilibrium position, a moment

$$M = \frac{JG}{L}\theta \tag{2.24}$$

develops between the disk and the shaft. Thus, if the polar mass moment of inertia of the shaft is small compared with I, then the shaft acts as a torsional spring of stiffness

$$k_t = \frac{JG}{L} \tag{2.25}$$

2.3 SPRINGS IN COMBINATION

Often, in applications, springs are placed in combination. It is convenient, for purposes of modeling and analysis, to replace the combination of springs by a single spring of an equivalent stiffness, k_{eq}. The equivalent stiffness is determined such that the system with a combination of springs has the same displacement, x, as the equivalent system when both systems are subject to the same force, F. A model SDOF system consisting of a block attached to a spring of an equivalent stiffness is illustrated in Figure 2.8. The resultant force acting on the block is

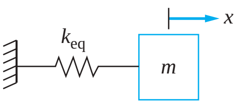


FIGURE 2.8
Combination of springs replaced by a single spring so that the system behaves identically to the original system.

$$F = k_{eq}x \tag{2.26}$$

2.3.1 PARALLEL COMBINATION

The springs in the system of Figure 2.9 are in *parallel*. The displacement of each spring in the system is the same, but the resultant force acting on the block is the sum of the forces developed in the parallel springs. If x is the displacement of the block, then the force developed in the ith spring is $k_i x$ and the resultant is

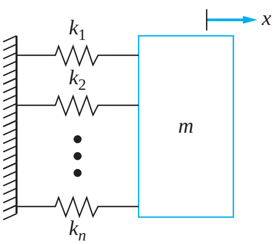


FIGURE 2.9
Each of the n springs in the parallel combination has the same displacement, but the resultant force acting on the FBD of the block is the sum of the individual spring forces.

$$F = k_1x + k_2x + \cdots + k_nx = \left(\sum_{i=1}^{n} k_i\right)x \tag{2.27}$$

Equating the forces from Equations (2.26) and (2.27) leads to

$$k_{eq} = \sum_{i=1}^{n} k_i \tag{2.28}$$

2.3.2 SERIES COMBINATION

The springs in Figure 2.10 are in *series*. The force developed in each spring is the same and equal to the force acting on the block. The displacement of the block is the sum of the

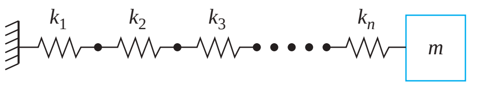


FIGURE 2.10
The springs in the series combination each develop the same force, but the total displacement of the combination is the sum of the individual changes in length.

changes in length of the springs in the series combination. If x_i is the change in length of the ith spring, then

$$x = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i \tag{2.29}$$

Since the force is the same in each spring, $x_i = F/k$ and Equation (2.29) becomes

$$x = \sum_{i=1}^{n} \frac{F}{k_i} \tag{2.30}$$

Since the series combination is to be replaced by a spring of an equivalent stiffness, Equation (2.26) is used in Equation (2.30), leading to

$$k_{\text{eq}} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{k_i}} \tag{2.31}$$

Electrical circuit components also can be placed in series and parallel and the effect of the combination replaced by a single component with an equivalent value. The equivalent capacitance of capacitors in parallel or series is calculated like that of springs in parallel or series. The equivalent resistance of resistors in series is the sum of the resistances, whereas the equivalent resistance of resistors in parallel is calculated by using an equation similar to Equation (2.31).

EXAMPLE 2.4

Model each of the systems of Figure 2.11 by a mass attached to a single spring of an equivalent stiffness. The system of Figure 2.11(c) is to be modeled by a disk attached to a torsional spring of an equivalent stiffness.

SOLUTION

(a) The steps involved in modeling the system of Figure 2.11(a) by the system of Figure 2.8 are shown in Figure 2.12. Equation (2.28) is used to replace the two parallel springs by an equivalent spring of stiffness $3k$. The three springs on the left of the mass are then in series, and Equation (2.31) is used to obtain an equivalent stiffness.

If the mass in Figure 2.11(a) is given a displacement x to the right, then the spring on the left of the mass will increase in length by x, while the spring on the right of the mass will decrease in length by x. Thus, each spring will exert a force to the left on the mass. The spring forces add; the springs behave as if they are in parallel. Hence Equation (2.28) is used to replace these springs by the equivalent spring shown in Figure 2.12(c).

(b) The deflection of the simply supported beam due to a unit load at $x = 2$ m is calculated using Table D.2

$$\omega(z = 2\text{ m}) = \omega\left(\frac{2L}{3}\right) = \frac{4L^3}{243EI} \tag{a}$$

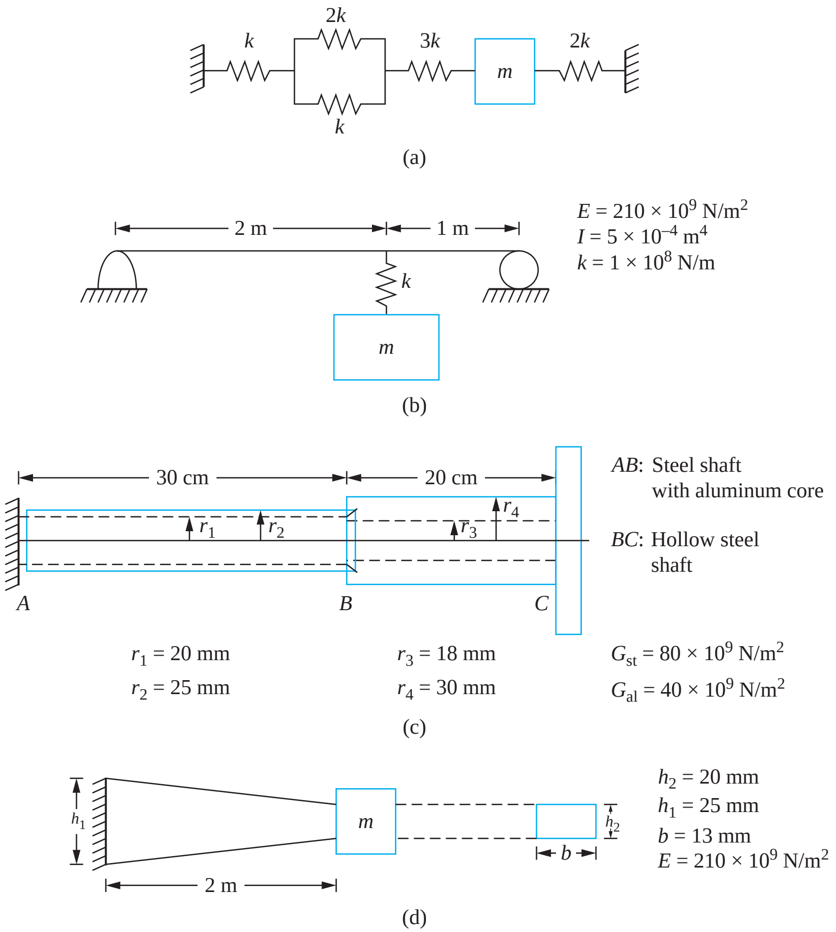


FIGURE 2.11
Systems for Example 2.4.

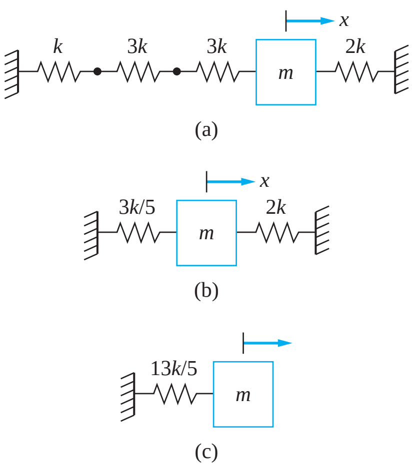


FIGURE 2.12
Steps in replacing the combination of springs in Figure 2.11 (a) using a single spring of an equivalent stiffness.

from which the equivalent stiffness is obtained

$$k_1 = \frac{243EI}{4L^3} = \frac{243(210 \times 10^9\ \text{N/m}^2)(5 \times 10^{-4}\ \text{m}^4)}{4(3\,\text{m})^3} = 2.36 \times 10^8\ \text{N/m} \qquad \textbf{(b)}$$

The displacement of the block of mass m equals the displacement of the beam at the location where the spring is attached plus the change in length of the spring. Hence the beam and spring act as a series combination. Equation (2.31) is used to calculate their equivalent stiffness

$$k_{\text{eq}} = \frac{1}{\dfrac{1}{2.36 \times 10^8\ \text{N/m}} + \dfrac{1}{1 \times 10^8\ \text{N/m}}} = 7.03 \times 10^7\ \text{N/m} \qquad \textbf{(c)}$$

(c) The aluminum core of shaft AB is rigidly bonded to the steel shell. Thus the angular rotation at B is the same for both materials. The total resisting torque transmitted to section BC is the sum of the torque developed in the aluminum core and the torque developed in the steel shell. Thus the aluminum core and steel shell of shaft AB behave as two torsional springs in parallel. The resisting torque in shaft AB is the same as the resisting torque in shaft BC. The angular displacement at C is the angular displacement of B plus the angular displacement of C relative to B. Thus shafts AB and BC behave as two torsional springs in series. In view of the preceding discussion and using Equations (2.28) and (2.31), the equivalent stiffness of shaft AC is

$$k_{t_{\text{eq}}} = \frac{1}{\dfrac{1}{k_{t_{AB_{\text{al}}}} + k_{t_{AB_{\text{st}}}}} + \dfrac{1}{k_{t_{BC}}}} \qquad \textbf{(d)}$$

where the torsional stiffness of a shaft is $k_t = JG/L$ and

$$k_{t_{AB_{\text{al}}}} = \frac{\dfrac{\pi}{32}(0.04\ \text{m})^4\left(40 \times 10^9\ \dfrac{\text{N}}{\text{m}^2}\right)}{0.3\,\text{m}} = 3.35 \times 10^4\ \frac{\text{N}\cdot\text{m}}{\text{rad}} \qquad \textbf{(e)}$$

$$k_{t_{AB_{\text{st}}}} = \frac{\dfrac{\pi}{32}[(0.05\ \text{m})^4 - (0.04\ \text{m})^4]\left(80 \times 10^9\dfrac{\text{N}}{\text{m}^2}\right)}{0.3\ \text{m}} = 9.66 \times 10^4\ \frac{\text{N}\cdot\text{m}}{\text{rad}} \qquad \textbf{(f)}$$

$$k_{t_{BC}} = = \frac{\dfrac{\pi}{32}[(0.06\,\text{m})^4 - (0.036\,\text{m})^4]\left(80 \times 10^9\dfrac{\text{N}}{\text{m}^2}\right)}{0.2\,\text{m}} = 4.43 \times 10^5\ \frac{\text{N}\cdot\text{m}}{\text{rad}} \qquad \textbf{(g)}$$

Substitution of these values into the equation for k_{eq} gives

$$k_{t,\text{eq}} = 1.01 \times 10^5\,\text{N}\cdot\text{m/rad} \qquad \textbf{(h)}$$

(d) Under the assumption that the rate of taper of the bar is small the following mechanics of materials equation is used to calculate the change in length of the bar due to a unit load applied at its end:

$$\Delta = \int_0^L \frac{dz}{AE} \qquad \textbf{(i)}$$

The area varies linearly over the length of the bar $A = \left(h_1 - \frac{h_1 - h_2}{L}z\right)b$. The change in length is

$$\Delta = \frac{1}{bE}\int_0^L \frac{dz}{h_1 - \frac{h_1 - h_2}{L}z} = \frac{1}{bE}\left(\frac{-L}{h_1 - h_2}\right)\ln\left(h_1 - \frac{h_1 - h_2}{L}z\right)\Bigg|_0^L = \frac{L}{bE(h_1 - h_2)}\ln\left(\frac{h_1}{h_2}\right)$$

$$= \frac{2\text{ m}}{(0.013\text{ m})\,(210 \times 10^9\text{ N/m}^2)(0.025\text{ m} - 0.02\text{ m})}\ln\frac{0.025\text{ m}}{0.02\text{ m}}$$

$$= 3.27 \times 10^{-8}\text{m/N} \tag{j}$$

Thus, the equivalent stiffness of the shaft is

$$k_{\text{eq}} = \frac{1}{\Delta} = \frac{1}{3.27 \times 10^{-8}\text{ m/N}} = 3.06 \times 10^7\text{ N/m} \tag{k}$$

2.3.3 General Combination of Springs

A single degree-of-freedom (SDOF) system is defined such that every particle is kinematically related to every other particle. Consider a system with n springs of stiffnesses $k_1, k_2, \ldots, kn$. Assume the jth spring is attached at a point where the relation between the displacement of the point of attachment and the generalized coordinate x is $x_j = \gamma jx$ for $j = 1, 2, \ldots, n$. The potential energy in a spring is $V = \frac{1}{2}kx^2$ where x is the change in length of the spring from its unstretched length. The total potential energy in the n springs is

$$V = \sum_{i=2}^{n}\left[\frac{1}{2}k_i(\gamma_i x)^2\right]$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n}k_i\alpha_i^2\right)x^2$$

$$= \frac{1}{2}k_{\text{eq}}x^2 \tag{2.32}$$

Equation (2.32) shows that (for analysis purpose) it is possible to replace a combination of springs in a linear SDOF system by a single spring of equivalent stiffness at the location described by the generalized coordinate x. The criterion for the equivalent stiffness is that the potential energy of the equivalent spring and the potential energy of the original system be equivalent at all times.

When using an angular coordinate as the generalized coordinate, the potential energy of a SDOF linear system is

$$V = \frac{1}{2}k_{t,\text{eq}}\theta^2 \tag{2.33}$$

where $k_{t,\text{eq}}$ is an equivalent, torsional viscous-damping coefficient.

EXAMPLE 2.5

The system of Figure 2.13 moves in a horizontal plane. Replace the system of springs by (a) a single spring of equivalent stiffness when x is the displacement of the block of mass 2 kg and is used as the generalized coordinate and (b) a spring of an equivalent torsional stiffness when the clockwise angular rotation of the disk θ is used as the generalized coordinate.

SOLUTION

(a) When the block of mass 2 kg moves through a displacement x, as shown in Figure 2.13, and assuming the cable connecting the block to the disk is inextensible, the point of contact between the disk and the cable have the same velocity. The velocity of the cable is $\dot{x}$, and the velocity of a point on the outer edge of the inner disk is $r\dot{\theta}$. Thus,

$$\dot{x} = r\dot{\theta} \tag{a}$$

Let y be the displacement of the cable attached to the 1 kg block. Its direction is opposite that of the other block. Assuming the cable is inextensible, the velocity of the cable $\dot{y}$ is the same as the velocity of the point on the disk in contact with the cable which is $\frac{3}{2}r\dot{\theta}$ leading to

$$\dot{y} = \frac{3}{2}r\dot{\theta} \tag{b}$$

Equations (a) and (b) are combined, leading to

$$\dot{y} = \frac{3}{2}\dot{x} \tag{c}$$

which is true for all time. Integrating and setting $y(0) = x(0) = 0$ leads to

$$y = \frac{3}{2}x \tag{d}$$

The total potential energy developed in the system at an arbitrary time in terms of x is the sum of the potential energies in the springs

$$V = \frac{1}{2}(3000\text{ N/m})x^2 + \frac{1}{2}(1000\text{ N/m})\left(\frac{3}{2}x\right)^2$$
$$= \frac{1}{2}(5250\text{N/m})x^2 \tag{e}$$

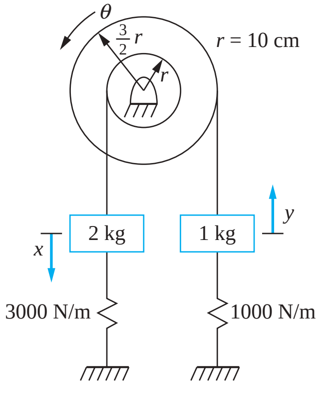


FIGURE 2.13
System of Example 2.5 is in a horizontal plane. The combination of springs are replaced by a single spring of an equivalent stiffness, so the potential energy of the original system is equal to the potential energy of the equivalent spring at any instant.

The equivalent stiffness of a spring placed on the 2 kg block to model the potential energy of the system is 5250 N/m.

(b) Using Equations (a) and (b) to give relations between x and θ and y and θ leads to the total potential energy in the system, which is written using θ as the generalized coordinate as

$$V = \frac{1}{2}(3000\,\mathrm{N/m})(r\theta)^2 + \frac{1}{2}(1000\,\mathrm{N/m})\left(\frac{3}{2}r\theta\right)^2 \tag{f}$$

Substituting $r = 0.1$ m gives

$$V = \frac{1}{2}\left(52.5\ \frac{\mathrm{N\cdot m}}{r}\right)\theta^2 \tag{g}$$

Thus, the equivalent torsional stiffness of the system when using θ as the generalized coordinate is 52.5 N·m/rad, which implies that the springs can be replaced by a single torsional spring of stiffness 52.5 N·m/rad attached to the pulley.

2.4 OTHER SOURCES OF POTENTIAL ENERGY

Any conservative force has an associated potential energy function. In addition to the spring force, this includes gravity, buoyancy, and a parallel-plate capacitor. Gravity and buoyancy are considered.

2.4.1 GRAVITY

The force due to the presence of a body of mass m in a gravitational field is mg directed toward the center of the earth applied at the mass center of the body. *Gravity* is a conservative force with a potential energy of

$$V = mgh \tag{2.34}$$

where h is the distance of the mass center above a reference position (the datum). The potential energy is a function of only the vertical position of the mass center.

EXAMPLE 2.6

A bar is hanging in equilibrium in the position shown in Figure 2.14(a). Determine the potential energy of the bar in terms of θ the counterclockwise angular position of the bar from its equilibrium position when (a) the datum is taken to be the horizontal plane at the bottom of the bar when in equilibrium, (b) the datum is taken as the horizontal plane through the mass center when the bar is in equilibrium, and (c) the datum is taken to be the horizontal plane through the pin support.

SOLUTION

(a) As the bar swings through an angle θ, as illustrated in Figure 2.14(b), the mass center is a distance

$$h = \frac{L}{2} + \frac{L}{2}(1 - \cos\theta) \tag{a}$$

and has a potential energy with respect to the datum of

$$V = mg\frac{L}{2}(2 - \cos\theta) \tag{b}$$

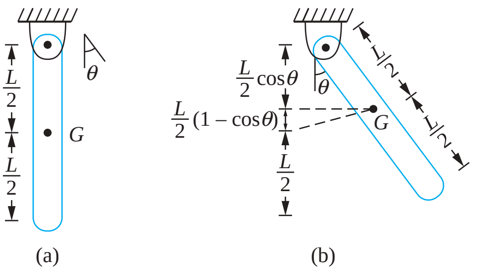


FIGURE 2.14
(a) The point of application of the gravity force mg is at the mass center of the bar.
(b) Diagram of a bar for an arbitrary value of θ, illustrating the geometry used in the calculation of the potential energy.

(b) Using a horizontal plane through G as a datum, we have

$$V = mg\frac{L}{2}(1 - \cos\theta) \qquad \text{(c)}$$

(c) Using a horizontal plane through O as a datum, we have

$$V = -mg\frac{L}{2}\cos\theta \qquad \text{(d)}$$

EXAMPLE 2.7

Calculate the total potential energy of the system of Figure 2.15 as the mass is displaced a distance x downward form the system's equilibrium position. Use a horizontal plane through the mass when the system is in equilibrium as the datum.

SOLUTION

When the system is in equilibrium, the spring has a static deflection, $\Delta = \frac{mg}{k}$. Thus, as the mass moves down a distance x from the equilibrium position, the potential energy in the spring is

$$V = \frac{1}{2}k(x + \Delta)^2 \qquad \text{(a)}$$

Adding to this, the potential energy due to gravity $Vg = -mgx$ yields

$$\begin{aligned} V &= \frac{1}{2}k(x + \Delta)^2 - mgx \\ &= \frac{1}{2}k\left(x + \frac{mg}{k}\right)^2 - mgx \\ &= \frac{1}{2}\left(kx^2 - 2mgx + \frac{m^2g^2}{k}\right) - mgx \\ &= \frac{1}{2}kx^2 + V_0 \qquad \text{(b)} \end{aligned}$$

where $V_0 = \frac{m^2g^2}{2k}$ is the potential energy in the spring when the system is in equilibrium. Thus, the total potential energy is expressed as the potential energy of the spring with respect to the equilibrium position plus the potential energy of the system when it is in equilibrium.

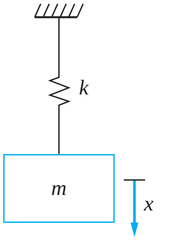


FIGURE 2.15
The potential energy due to gravity cancels with the potential energy of the static spring force as the mass moves from equilibrium.

2.4.2 BUOYANCY

When a solid body is submerged in a liquid or floating on the interface of a liquid and air, a force acts vertically upward on the body because of the variation of hydrostatic pressure. This force is called the *buoyant force*. Archimedes' principle states that the buoyant force acting on a floating or submerged body is equal to the weight of the liquid displaced by the body.

EXAMPLE 2.8

A sphere of mass 2.5 kg and radius 10 cm is hanging from a spring of stiffness 1000 N/m in a fluid of mass density 1200 kg/m^3. What is the static deflection of the spring?

SOLUTION

The spring force must balance with the gravity force and the buoyancy force as shown on the free-body diagram in Figure 2.16.

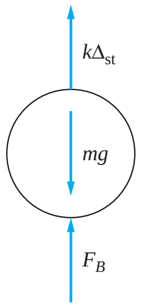


FIGURE 2.16 FBD of a sphere attached to a spring and submerged in a liquid.

$$k\Delta_{st} + F_B - mg = 0$$

Archimedes' principle is used to calculate the buoyant force as

$$F_B = \frac{4}{3}\rho g \pi r^3 = \frac{4}{3}(1200\ \text{kg/m}^3)\pi(9.81\ \text{m/s}^2)(0.1\ \text{m})^3 = 49.3\ \text{N}$$

The static deflection is calculated as

$$\Delta_{st} = \frac{mg - F_B}{k} = \frac{(2.5\ \text{kg})(9.81\ \text{m/s}^2) - 49.3\ \ \text{N}}{1000\ \text{N/m}} = -0.0185\ \text{m}$$

Consider a body floating stably on a liquid-air interface. The buoyant force balances with the gravity force. If the body is pushed farther into the liquid, the buoyant force increases. If the body is then released, it seeks to return to its equilibrium configuration. The buoyant force does work, which is converted into kinetic energy and oscillations about the equilibrium position ensue.

The circular cylinder of Figure 2.17 has a cross-sectional area A and floats stably on the surface of a fluid of density ρ. When the cylinder is in equilibrium, it is subject to a buoyant force mg and its center of gravity is a distance Δ from the surface. Let x be the vertical displacement of the center of gravity of the cylinder from this position. The additional

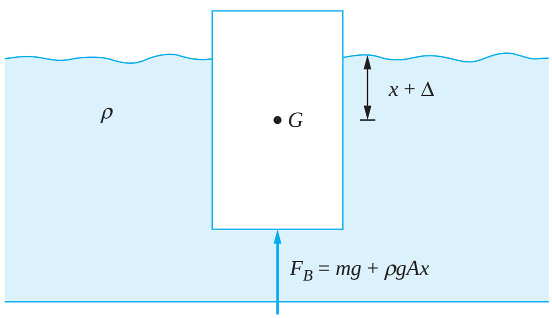


FIGURE 2.17 Oscillations of a cylinder on a free surface can be modeled by a SDOF system where the buoyant force is the source of potential energy.

volume displaced by the cylinder is xA. According to Archimedes' principle, the buoyant force is

$$F_B = mg + \rho gAx \tag{2.35}$$

Calculations show that the work done by the buoyant force as the cylinder's center of gravity moves between positions x_1 and x_2 is

$$U_{1\to2} = \frac{1}{2}\rho gAx_1^2 - \frac{1}{2}\rho gAx_2^2 \tag{2.36}$$

and is independent of path. Hence the buoyant force is conservative. Its effect on the cylinder is the same as that of a linear spring of stiffness ρgA. The oscillations of the cylinder on the liquid-gas interface can be modeled by a SDOF mass-spring system.

2.5 VISCOUS DAMPING

Viscous damping occurs in a mechanical system when a component of the system is in contact with a viscous liquid. The *damping force* is usually proportional to the velocity

$$F = cv \tag{2.37}$$

where c is called the *viscous damping coefficient* and has dimensions of (force)(time)/ (length).

Viscous damping is often added to mechanical systems as a means of vibration control. Viscous damping leads to an exponential decay in amplitude of free vibrations and a reduction in amplitude in forced vibrations caused by a harmonic excitation. In addition, the presence of viscous damping gives rise to a linear term in the governing differential equation, and thus does not significantly complicate the mathematical modeling of the system. A mechanical device called a *dashpot* is added to mechanical systems to provide viscous damping. A schematic of a dashpot in a one degree-of-freedom system is shown in Figure 2.18(a). The free-body diagram of the rigid body, Figure 2.18(b), shows the viscous force in the opposite direction of the positive velocity.

A simple dashpot configuration is shown in Figure 2.19(a). The upper plate of the dashpot is connected to a rigid body. As the body moves, the plate slides over a reservoir of viscous liquid of dynamic viscosity μ. The area of the plate in contact with the liquid is A. The shear stress developed between the fluid and the plate creates a resultant friction force acting on the plate. Assume the reservoir is stationary and the upper plate slides over the

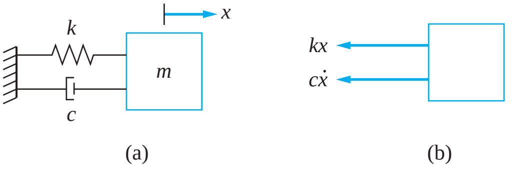


FIGURE 2.18
(a) Schematic of SDOF mass-spring-dashpot system. (b) Dashpot force is $c\dot{x}$ and opposes the direction of positive velocity.

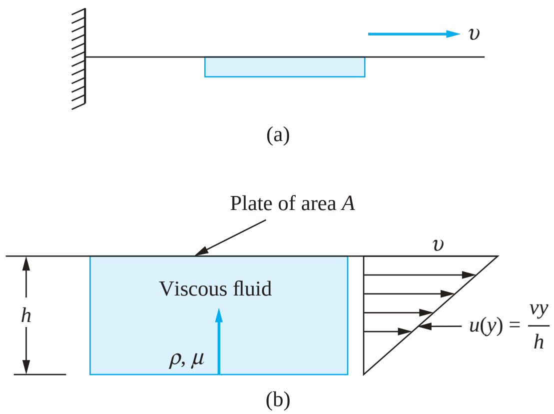


FIGURE 2.19
(a) Simple dashpot model where plate slides over a fixed reservoir of a viscous liquid. (b) Since h is small, a linear velocity profile is assumed in the liquid.

liquid with a velocity v. The reservoir depth h is small enough that the velocity profile in the liquid can be approximated as linear, as illustrated in Figure 2.19(b). If y is a coordinate measured upward from the bottom of the reservoir,

$$u(y) = v\frac{y}{h} \tag{2.38}$$

The shear stress developed on the plate is determined from Newton's viscosity law

$$\tau = \mu\frac{du}{dy} = \mu\frac{v}{h} \tag{2.39}$$

The viscous force acting on the plate is

$$F = \tau A = \frac{\mu A}{h}v \tag{2.40}$$

Comparison of Equation (2.40) with Equation (2.37) shows that the damping coefficient for this dashpot is

$$c = \frac{\mu A}{h} \tag{2.41}$$

Equation (2.41) shows that a large damping force is achieved with a very viscous fluid, a small h, and a large A. A dashpot design with these parameters is often impractical and thus the device of Figure 2.19(a) is rarely actually used as a dashpot.

This analysis assumes the plate moves with a constant velocity. During the motion of a mechanical system, the dashpot is connected to a particle which has a time-dependent velocity. The changing velocity of the plate leads to unsteady effects in the liquid. If the reservoir depth h is small, the unsteady effects are small and can be neglected.

A more practical dashpot is a piston-cylinder arrangement, as shown in Figure 2.20. The piston slides in a cylinder of viscous liquid. Because of the motion, a pressure difference

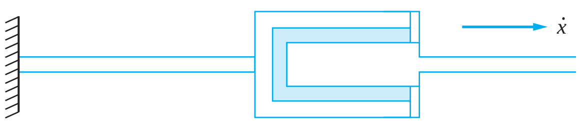


FIGURE 2.20
A piston and cylinder device that serves as a viscous damper.

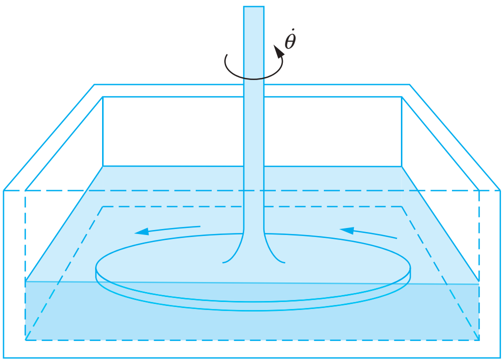


FIGURE 2.21
A disk rotates in a dish of a viscous liquid, producing a moment about the axis of the shaft and acting as a torsional viscous damper.

is formed across the head of the piston which is proportional to the velocity of the piston. The pressure times the area of the head is the damping force.

A torsional viscous damper is illustrated in Figure 2.21. The shaft is rigidly connected to a point on a body undergoing torsional oscillations. As the disk rotates in a dish of viscous liquid, a net moment due to the shear stresses developed on the face of the disk acts about the axis of rotation. The moment is proportional to the angular velocity of the shaft

$$M = c_t\dot{\theta} \tag{2.42}$$

where c_t is the torsional viscous damping coefficient and has dimensions of force-length-time.

Any form of damping where the damping force is proportional to the velocity is referred to as *viscous damping*. Viscous damping can be produced by a body moving through a magnetic field, a body oscillating on the surface of a lake, or by the oscillations of a column of liquid in a U-tube manometer.

The schematic representation for viscous damping when present in mechanical systems is shown in Figure 2.22. The force developed in the dashpot is equal to and opposite of the force from the damper on the body. The force resists the motion of the system and is drawn to show it acting in the opposite direction of the velocity. The direction of the force takes care of itself. If the velocity is negative, the actual damping force is acting in the direction of positive velocity. However, it is drawn on the FBD in the direction of negative velocity and has a negative value, thus being in the positive direction.

The viscous damping force is the damping coefficient times the velocity of the point where the dashpot is attached acting in the opposite direction of the positive velocity of that point.

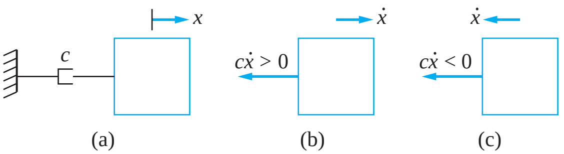


FIGURE 2.22
(a) Schematic of a viscous damper in a mechanical system. (b) The viscous damping force is always drawn as the opposite of the direction of positive velocity. (c) When velocity is negative, the viscous clamping force is still drawn to the left, but since it is negative, it goes toward the right.

EXAMPLE 2.9

Draw a FBD for the system of Figure 2.23(a) at an arbitrary instant using θ as the dependent variable and labeling the forces in terms of $\dot{\theta}$

SOLUTION

The FBD is shown in Figure 2.23(b). The velocity of particle A at an arbitrary instant is $\frac{L}{4}\dot{\theta}$ upward, while the velocity of particle B is $\frac{3L}{4}\dot{\theta}$ downward.

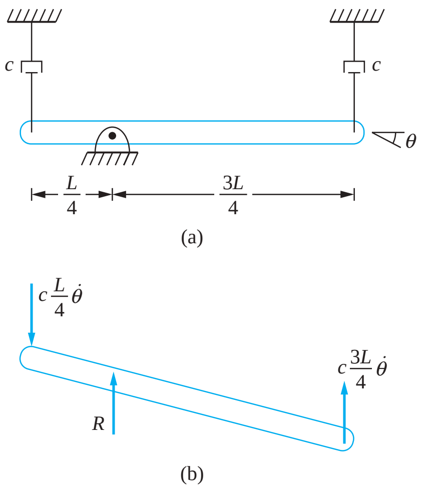


FIGURE 2.23
(a) System of Example 2.9. (b) FBD of system. The force from the viscous damper on the body is equal to and opposite the force from the body on the viscous damper. The force is always drawn opposite to the positive velocity of the point to which it is attached.

2.6 ENERGY DISSIPATED BY VISCOUS DAMPING

Rewriting the principle of work and energy, Equation (1.47) applied to a system is

$$U_{1\to 2_{NC}} = T_2 + V_2 - (T_1 + V_1) \tag{2.43}$$

and shows that work done by non-conservative forces is the difference in total energies.

Viscous damping is a non-conservative force. After application of viscous damping, $T_2 + V_2 < T_1 + V_1$, and the work done by viscous damping is negative. The viscous damping force always opposes the direction of motion. The work done by a viscous damper between the initial position is described by $x = 0$ and an arbitrary position

$$U_{1\to 2} = -\int_0^x c\dot{x}\,dx \tag{2.44}$$

The work done by discrete viscous dampers in a SDOF system is the sum of the work done by individual dampers. For a SDOF system, the displacement of all particles is kinematically related. In a system with n viscous dampers, the displacement of the ith viscous damper is related to the generalized coordinate by $x_i = \gamma_i x$. The total work done by the viscous dampers is

$$U_{1\to 2} = -\sum_{i=1}^{n}\int_0^{x_i} c_i\dot{x}_i\,dx_i \tag{2.45}$$

Equation (2.45) is rewritten by introducing the relationship between x_i and x as

$$U_{1\to 2} = -\sum_{i=1}^{n}\int_0^x c_i(\gamma_i\dot{x})\,d(\gamma_i x)$$

$$= -\sum_{i=1}^{n}\int_0^x c_i(\gamma_i^2\dot{x})\,dx \tag{2.46}$$

Now that the integrals all have the same variable of integration and limits, the order of summation and integration are interchanged to yield

$$U_{1\to 2} = -\int_0^x\left(\sum_{i=1}^{n} c_i\gamma_i^2\right)\dot{x}dx$$

$$= -\int_0^x c_{\mathrm{eq}}\,\dot{x}dx \tag{2.47}$$

Hence, an equivalent viscous-damping coefficient can be determined for any SDOF system.

If an angular coordinate θ is used as a generalized coordinate, Equation (2.47) is modified as

$$U_{1\to 2} = -\int_0^x c_{t,\mathrm{eq}}\,\dot{\theta}\,d\theta \tag{2.48}$$

where $c_{t,\mathrm{eq}}$ is an equivalent, torsional viscous-damping coefficient.

EXAMPLE 2.10

The system of Figure 2.24 moves in a horizontal plane.

(a) Determine the equivalent viscous-damping coefficient for the system if x is the displacement of the 2 kg block and is used as the generalized coordinate.

(b) Determine the equivalent, torsional viscous-damping coefficient θ if the clockwise angular displacement of the disk is used as the generalized coordinate.

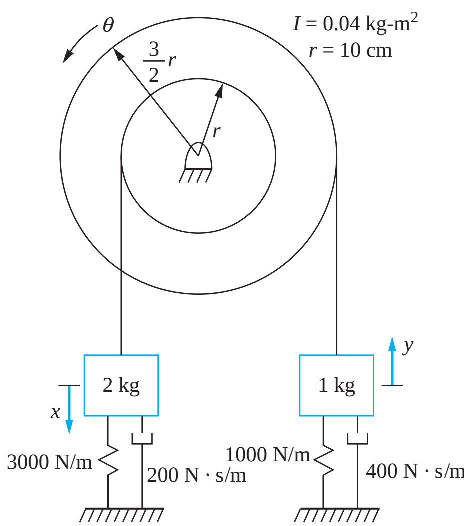


FIGURE 2.24
System for Examples 2.10 and 2.11.

SOLUTION

(a) Using kinematics, it is found that the relation between the downward displacement of the 2 kg block x and the upward displacement of the 1 kg block y is $y = \frac{3}{2}x$. Calculating the work done by the viscous dampers as the system moves between the initial position and an arbitrary position, we have

$$U_{1\to 2} = -\int_0^x (200\ \text{N}\cdot\text{s/m})\ \dot{x}\ dx - \int_0^x (400\,\text{N}\cdot\text{s/m})\left(\frac{3}{2}\dot{x}\right)d\left(\frac{3}{2}x\right)$$

$$= -\int_0^x (1100\ \text{N}\cdot\text{s/m})\ \dot{x}\ dx \qquad \text{(a)}$$

Thus, $c_{eq} = 1100\ \text{N}\cdot\text{s/n}$

(b) Kinematics is used to determine that $x = r\theta$ and $y = \frac{3}{2}r\theta$ where $r = 0.1$ m. Calculating the work done by the viscous dampers as the system moves from an initial position to an arbitrary position, we have

$$U_{1\to 2} = -\int_0^\theta (200\ \text{N}\cdot\text{s/m})[(0.1\text{m})\dot{\theta}]d[(0.1\ \text{m})\theta] - \int_0^\theta (400\,\text{N}\cdot\text{s/m})\left[\frac{3}{2}(0.1\ \text{m})\dot{\theta}\right]$$

$$\times\ d\left[\frac{3}{2}(0.1\,\text{m})\theta\right] = -\int_0^\theta \left(11\frac{\text{N}\cdot\text{m}\cdot\text{s}}{\text{rad}}\right)\dot{\theta}\,d\theta \qquad \text{(b)}$$

Thus, $c_{t,eq} = 11\ \text{N}\cdot\text{m}\cdot\text{s/rad}$

2.7 INERTIA ELEMENTS

A particle's mass is the only inertia property for the particle. The distribution of mass about the mass center is also important for a rigid body undergoing planar motion. It is described by a property of the rigid body called the *centroidal moment of inertia*, defined by

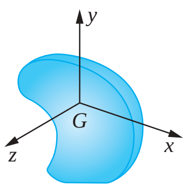
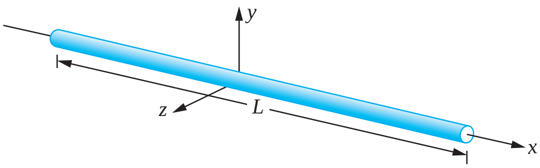
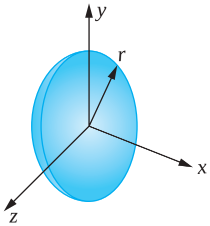
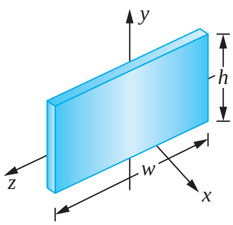
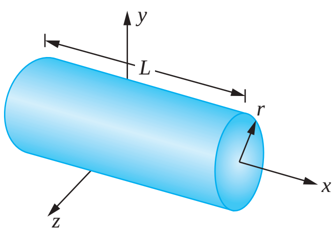
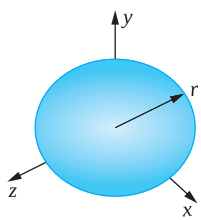
$$\bar{I} = \int_m [(x - \bar{x})^2 + (y - \bar{y})]^2\ dm \qquad (2.49)$$

when the coordinates of the rigid body's mass center are $(\bar{x}, \bar{y})$. The integration is carried out over the entire mass of the rigid body. The centroidal moment of inertia has been calculated for common shapes, and the results are tabulated in Table 2.1.

2.7.1 Equivalent Mass

The kinetic energy of a particle is $\frac{1}{2}mv^2$. The kinetic energy of a rigid body undergoing planar motion is $\frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$. For a linear SDOF system, the displacement of any particle in the system is kinematically dependent upon x. Consider a system composed of n bodies, particle, and rigid bodies undergoing planar motion. There exists a β_i such that the displacement of the mass center of the ith body is $\bar{x}_i = \beta_i x$, and there exists a ν_i such that the angular rotation of the ith body is $\theta_i = \nu_i x$. If the ith body is a particle, then

TABLE 2.1 Moments of inertia of three-dimensional bodies

Body	**General Shape**	**Centroidal Moments of Inertia**
General shape		$\bar{I}_x = \int (y^2 + z^2)\, dm$ $\bar{I}_y = \int (x^2 + z^2)\, dm$ $\bar{I}_z = \int (x^2 + y^2)\, dm$
Slender rod		$\bar{I}_x \approx 0$ $\bar{I}_y = \frac{1}{12}mL^2$ $\bar{I}_z = \frac{1}{12}mL^2$
Thin disk		$\bar{I}_x = \frac{1}{2}mr^2$ $\bar{I}_y = \frac{1}{4}mr^2$ $\bar{I}_z = \frac{1}{4}mr^2$
Thin plate		$\bar{I}_x = \frac{1}{12}m(w^2 + h^2)$ $\bar{I}_y = \frac{1}{12}mw^2$ $\bar{I}_z = \frac{1}{12}mh^2$
Circular cylinder		$\bar{I}_x = \frac{1}{12}mr^2$ $\bar{I}_y = \frac{1}{12}m(3r^2 + L^2)$ $\bar{I}_z = \frac{1}{12}m(3r^2 + L^2)$
Sphere		$\bar{I}_x = \frac{2}{5}mr^2$ $\bar{I}_y = \frac{2}{5}mr^2$ $\bar{I}_z = \frac{2}{5}mr^2$

$v_i = 0$. The total kinetic energy of the system is the sum of the kinetic energies of all bodies in the system:

$$T = \sum_{i=1}^{n}\left(\frac{1}{2}m_i v_i^2 + \frac{1}{2}\bar{I}_i\omega_i^2\right)$$

$$\sum_{i=1}^{n}\left[\frac{1}{2}m_i(\beta_i\dot{x})^2 + \frac{1}{2}\bar{I}_i(\nu_i\dot{x})^2\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n}(m_i\beta_i^2 + \bar{I}_i\nu_i^2)\right]\dot{x}^2$$

$$= \frac{1}{2}m_{\text{eq}}\dot{x}^2 \tag{2.50}$$

Thus, any single degree-of-freedom system has an equivalent mass defined by Equation (2.50).

If an angular coordinate is used as the generalized coordinate, the kinetic energy is written as

$$T = \frac{1}{2}I_{\text{eq}}\dot{\theta}^2 \tag{2.51}$$

where I_{eq} is an equivalent moment of inertia.

EXAMPLE 2.11

The system of Figure 2.24 moves in a horizontal plane.

(a) Determine the equivalent mass when x (the displacement of the 2 kg block) is used as the generalized coordinate.

(b) Determine the equivalent moment of inertia when θ (the clockwise angular rotation of the disk) is used as the generalized coordinate.

SOLUTION

During the solution of Example 2.10, it is determined that if y is the upward displacement of the 1 kg block, then $y = \frac{3}{2}x$ and $\theta = \frac{x}{r} = \frac{x}{0.1\text{ m}} = 10x$. The total kinetic energy is the kinetic energy of the blocks plus the kinetic energy of the disk:

$$T = \frac{1}{2}(2\text{ kg})\dot{x}^2 + \frac{1}{2}(1\text{ kg})\dot{y}^2 + \frac{1}{2}(0.0\ 4\text{ kg}\cdot\text{m}^2)\dot{\theta}^2$$

$$= \frac{1}{2}(2\text{ kg})\dot{x}^2 + \frac{1}{2}(1\text{ kg})\left(\frac{3}{2}\dot{x}\right)^2 + \frac{1}{2}(0.04\text{ kg}\cdot\text{m}^2)(10\dot{x}\text{ m}^{-1})^2$$

$$= \frac{1}{2}(8.25\text{ kg})\dot{x}^2 \tag{a}$$

Thus, the equivalent mass is 8.25 kg.

(b) During the solution of Example 2.10, it is shown that $y = \frac{3}{2}r\theta = \frac{3}{2}(0.1\text{ m})\theta$

$$T = \frac{1}{2}(2\text{ kg})\dot{x}^2 + \frac{1}{2}(1\text{ kg})\dot{y}^2 + \frac{1}{2}(0.04\text{ kg}\cdot\text{m}^2)\dot{\theta}^2$$

$$= \frac{1}{2}(2\text{ kg})[(0.1\text{m})\dot{\theta}]^2 + \frac{1}{2}(1\text{kg})\left[\frac{3}{2}(0.1\text{ m})\dot{\theta}\right]^2 + \frac{1}{2}(0.04\text{ kg}\cdot\text{m}^2)\dot{\theta}^2$$

$$= \frac{1}{2}(0.0825\text{ kg}\cdot\text{m}^2)\dot{\theta}^2 \quad \textbf{(b)}$$

Thus, if all of the inertia were concentrated on the disk, the disk would have a moment of inertia of 0.0825 kg · m².

2.7.2 INERTIA EFFECTS OF SPRINGS

When a force is applied to displace the block of Figure 2.25(a) from its equilibrium position, the work done by the force is converted into strain energy stored in the spring. If the block is held in this position and then released, the strain energy is converted to kinetic energy of both the block and the spring. If the mass of the spring is much smaller than the mass of the block, its kinetic energy is negligible. In this case the inertia of the spring has negligible effect on the motion of the block, and the system is modeled using one degree of freedom. The generalized coordinate is usually chosen as the displacement of the block.

If the mass of the spring is comparable to the mass of the block, the single degree-of-freedom assumption is not valid. The particles along the axis of the spring are kinematically independent from each other and from the block. The spring should be modeled as a continuous system.

If the mass of the spring is much smaller than the mass of the block, but not negligible, a reasonable one degree-of-freedom approximation can be made by approximating the spring's inertia effects. The actual system of Figure 2.25(a) is modeled by the ideal system of Figure 2.25(b) in which the spring is massless. The mass of the block in Figure 2.25(a) is greater than the mass of the actual block to account for inertia effects of the spring. The value of m_{eq} is calculated such that the kinetic energy of the system of Figure 2.25(b) is the same as the kinetic energy of the system of Figure 2.25(a) including the kinetic energy of the spring, when the velocities of both blocks are equal. Unfortunately, calculation of the exact kinetic energy of the spring requires a continuous system analysis. Thus, an approximation to the spring's kinetic energy is used.

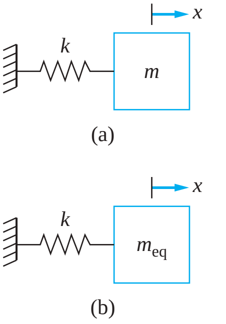


FIGURE 2.25
(a) Potential energy developed in the spring is converted into kinetic energy for both the block and the spring. (b) An equivalent mass is used to approximate inertia effects of the spring.

Let $x(t)$ be the generalized coordinate describing the motion of both the block of Figure 2.25(a) and the block of Figure 2.25(b). The kinetic energy of the system of Figure 2.25(a) is

$$T = T_s + \frac{1}{2}m\dot{x}^2 \tag{2.52}$$

where T_s is the kinetic energy of the spring. The kinetic energy of the system of Figure 2.25(b) is

$$T = \frac{1}{2}m_{eq}\dot{x}^2 \tag{2.53}$$

The spring in Figure 2.25(a) is uniform, has an unstretched length l and a total mass m_s. Define the coordinate z along the axis of the spring, measured from its fixed end, as defined in Figure 2.26. The coordinate z measures the distance of a particle from the fixed end in the spring's unstretched state. The displacement of a particle on the spring, $u(z)$, is assumed explicitly independent of time and a linear function of z such that $u(0) = 0$ and $u(l) = x$,

$$u(z) = \frac{x}{l}z \tag{2.54}$$

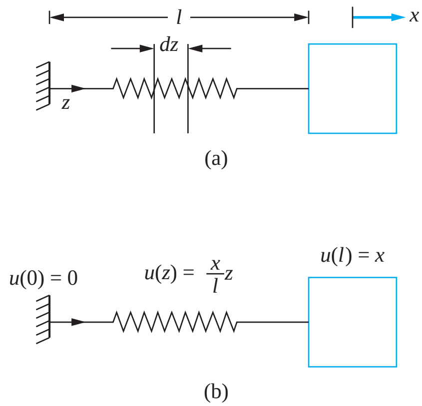


FIGURE 2.26
(a) The coordinate z is measured along the axis of the spring form its fixed end when the system is in equilibrium, $0 \leq z \leq \ell$. (b) The displacement of the spring is assumed as a linear function of z.

Equation (2.54) represents the displacement function of a uniform spring when it is statically stretched. Consider a differential element of length dz, located a distance z from the spring's fixed end. The kinetic energy of the differential element is

$$dT_s = \frac{1}{2}\dot{u}^2(z)\,dm = \frac{1}{2}\dot{u}^2(z)\frac{m_s}{l}dz \tag{2.55}$$

The total kinetic energy of the spring is

$$T_s = \int dT_s = \int_0^1 \frac{1}{2}\frac{m_s}{l}\left(\frac{\dot{x}z}{l}\right)^2 dz = \frac{1}{2}\frac{m_s}{l^3}\dot{x}^2\frac{z^3}{3}\bigg|_0^l = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2 \tag{2.56}$$

Equating T from Equations (2.52) and (2.53) and using T_s from Equation (2.56) gives

$$m_{\text{eq}} = m + \frac{m_s}{3} \tag{2.57}$$

Equation (2.57) can be interpreted as follows: The inertia effects of a linear spring with one end fixed and the other end connected to a moving body can be approximated by placing a particle whose mass is one-third of the mass of the spring at the point where the spring is connected to the body.

The preceding statement is true for all springs where use of a linear displacement function of the form of Equation (2.54) is justified. This is valid for helical coil springs, bars that are modeled as springs for longitudinal vibrations, and shafts acting as torsional springs.

EXAMPLE 2.12

The springs in the system of Figure 2.27(a) are all identical, with stiffness k and mass m_s. Calculate the kinetic energy of the system in terms of θ (t), including the inertia effects of the springs.

SOLUTION

Each spring is replaced by a massless spring and a particle of mass $m_s/3$ at the point on the bar where the spring is attached as shown in Figure 2.27(b). The total kinetic energy of the

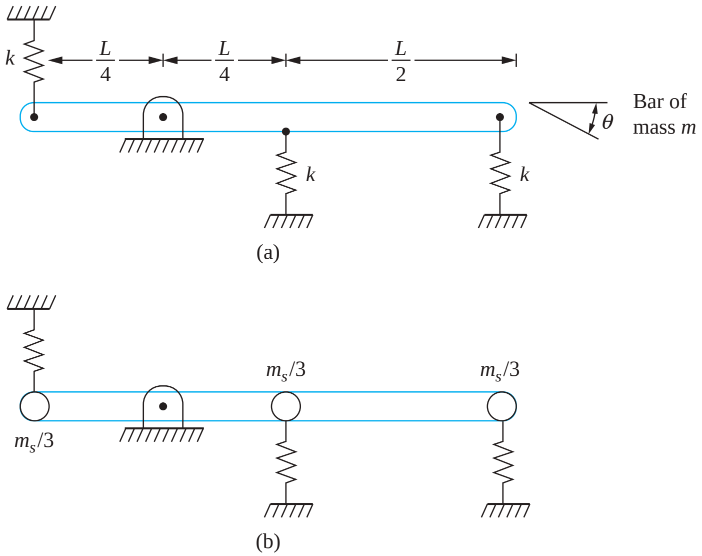


FIGURE 2.27
(a) System of Example 2.12. (b) Inertia effects of springs are approximated by placing a particle of mass $m_s/3$ at locations where springs are attached.

system of Figure 2.27(b) is the kinetic energy of the bar plus the kinetic energy of each of the particles

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2 + T_1 + T_2 + T_3$$

$$= \frac{1}{2}m\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}\frac{m_s}{3}\left(\frac{L}{4}\right)^2 + \frac{1}{2}\frac{m_s}{3}\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}\frac{m_s}{3}\left(\frac{3L}{4}\dot{\theta}\right)^2$$

$$= \frac{1}{2}\left(\frac{7m + 11m_s}{48}\right)L^2\dot{\theta}^2$$

EXAMPLE 2.13

The simply supported beam of Figure 2.28 is uniform and has a total mass of 100 kg. A machine of mass 350 kg is attached at B, as shown. What is the mass of a particle that should be placed at B to approximate the beam's inertia effects?

SOLUTION

Since the exact expression for the dynamic beam deflection is hard to obtain, an approximate displacement function is used in the calculation of the beam's kinetic energy. Let z be a coordinate along the beam's neutral axis. Assume that the time-dependent displacement of any particle a long the beam's neutral axis can be expressed as

$$y(z, t) = x(t)\omega(z) \qquad \text{(a)}$$

where $x(t)$ is the deflection of B. An appropriate approximation for $w(z)$ is the static deflection of the beam due to a concentrated load, P, applied at B, such that B has a unit deflection.

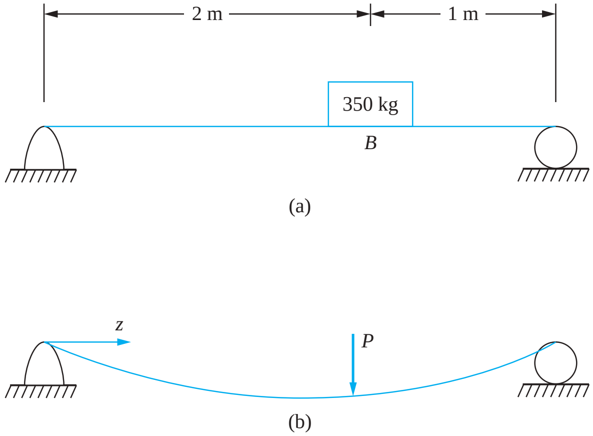


FIGURE 2.28
(a) System of Example 2.13.
(b) Static deflection of beam due to concentrated load at B.

By using the methods of Appendix D, the static deflection due to a concentrated load at B is found to be

$$[w(z) = \begin{cases} \dfrac{P}{18\,EI} z\left(\dfrac{8L^2}{9} - z^2\right) & 0 \le z \le \dfrac{2L}{3} \\ \dfrac{P}{18\,EI}\left(2z^3 - 6z^2L + \dfrac{44}{9}zL^2 - \dfrac{8}{9}L^3\right) & \dfrac{2L}{3} \le z \le L \end{cases} \tag{b}$$

The load required to cause a unit deflection at $z = 2L/3$

$$P = \frac{243EI}{4L^3} \tag{c}$$

Consider a differential element of length dz, located a distance z from the left support. The kinetic energy of the element is

$$dT = \frac{1}{2}\dot{y}^2(z, t)\, dm = \frac{1}{2}\dot{y}^2(z, t)\, \rho A\ \ dm \tag{d}$$

where ρ is the mass density of the beam and A is its cross-sectional area. The beam's total kinetic energy is calculated by integrating dT over the entire beam. Substituting the previous results for $w(x, t)$ in this integral leads to

$$T = \frac{1}{2}\rho A\left[\frac{1}{18EI}\left(\frac{243EI}{4L^3}\right)\right]^2 \dot{x}^2\left[\int_0^{2L/3} z^2\left(\frac{8L^2}{9} - z^2\right)^2 dz \right.$$
$$\left. + \int_{2L/3}^{L}\left(2z^3 - 6z^2L + \frac{44}{9}zL^2 - \frac{8}{9}L^3\right)^2\ dz\right] \tag{e}$$

The integral is evaluated yielding

$$T = \frac{1}{2}\,0.586\rho AL\ \dot{x}^2 \tag{f}$$

Noting that the total mass of the beam is ρAL, a particle of mass 58.6 kg should be added at B to approximate the inertia effects of the beam. The system of Figure 2.28(a) is modeled as a SDOF system with a particle of 408.6 kg located at B.

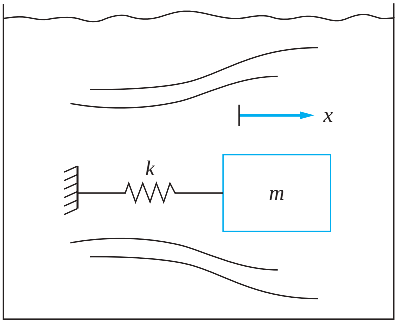


FIGURE 2.29
Oscillations of a submerged body create kinetic energy in a fluid. The inertia of the fluid can be approximated by a particle added to the mass of the body.

2.7.3 ADDED MASS

Consider a mass-spring system immersed in an inviscid fluid, as shown in Figure 2.29. The spring is stretched from its equilibrium configuration and the mass released. The ensuing motion of the mass causes motion in the surrounding fluid. The strain energy initially stored in the spring is converted to kinetic energy for both the mass and the fluid. Since the fluid is inviscid, energy is conserved

$$T_m + T_f + V = C \tag{2.58}$$

The inertia effects of the fluid can be included in an analysis by using a method similar to that used in Section 2.7.2 to account for the inertia effects of springs. An imagined particle is attached to the mass such that the kinetic energy of the particle is equal to the total kinetic energy of the fluid. If x is the displacement of the mass, the total kinetic energy of the system is $\frac{1}{2}m_{\text{eq}}\dot{x}^2$, where

$$m_{\text{eq}} = m + m_a \tag{2.59}$$

The mass of the particle is called the *added mass.*

The kinetic energy of the fluid is difficult to quantify. The motion of the body theoretically entrains fluid infinitely far away in all directions. The total kinetic energy of the fluid is calculated from

$$T_f = \frac{1}{2}\int\int\int \rho v^2\, dV \tag{2.60}$$

where v is the velocity of the fluid set in motion by the motion of the body. The integration is carried out from the body surface to infinity in all directions. If the integration of Equation (2.60) is carried out, the added mass is calculated from

$$m_a = \frac{T_f}{\frac{1}{2}\dot{x}^2} \tag{2.61}$$

Potential flow theory can be used to develop the velocity distribution in a fluid for a body moving through the fluid at a constant velocity. This velocity distribution is used in Equations (2.60) and (2.61) to calculate the added mass. Table 2.2 is adapted from Wendel (1956) and Patton (1965) and presents the added mass for common body shapes.

TABLE 2.2 Added mass for common two- and three-dimensional bodies (ρ is the mass density of the fluid)

Body	Added Mass
Sphere of diameter D	$\frac{1}{12}\pi\rho D^3$
Thin Circular disk of diameter D	$\frac{1}{3}\rho D^3$
Thin square plate of side h	$0.1195\pi\rho h^3$
Circular cylinder of length L, diameter D	$\frac{1}{4}\pi\rho D^2 L$
Thin flat plate of length L, width w	$\frac{1}{4}\pi\rho w^3 L$
Square cylinder of side h, length L	$0.3775\rho\pi h^2 L$
Cube of side h	$2.33\rho h^3$

TABLE 2.3 Added moments of inertia for common bodies (ρ is the mass density of the fluid)

Body	Added moment of inertia
Sphere	0
Circular cylinder	0
Any body rotating about axis of symmetry	0
Thin plate of length L, rotating about axis in the plane of the surface area of plate, perpendicular to direction for which L is defined	$0.0078125\pi\rho L^4$
Disk of diameter D rotating about a diameter	$\frac{1}{90}\rho D^5$

Rotational motion of a body in a fluid also imparts motion to the fluid resulting in rotational kinetic energy of the fluid. The inertia effects of the fluid are taken into account by adding a disk of an appropriate moment of inertia to the rotating body. If ω is the angular velocity of the body, the added mass moment of inertia is calculated from

$$I_a = \frac{T_f}{\frac{1}{2}\omega^2} \tag{2.62}$$

Note that the added mass moment of inertia is zero if the body is rotating about an axis of symmetry. Both the added mass and added moment of inertia terms are negligible for bodies moving in gases. Table 2.3 presents added moments of inertia for a few common bodies. It is adapted from Wendel (1956).

2.8 EXTERNAL SOURCES

A *non-conservative force* is one whose work depends upon the path traveled by the particle to which the force is attached. Viscous damping and externally applied forces are examples of non-conservative forces. The work done by an external force is

$$U_{1\to 2} = \int_{x_1}^{x_2} F(t)\,dx = \int_{t_1}^{t_2} F(t)\dot{x}\,dt \tag{2.63}$$

where $x(t_1) = x_1$ and $x(t_2) = x_2$.

Let x represent the generalized coordinate defined for a SDOF system. Suppose n external forces are applied to the system whose points of application are $x_i = \varepsilon_i x, i = 1, 2, \cdots$, n. The total work by the external forces are

$$U_{1\to 2} = \sum_{i=1}^{n}\int_{t_1}^{t_2} F_i\,(t)\dot{x}_i\;dt = \sum_{i=1}^{n}\int_{t_2}^{t_2} F_i(t)\varepsilon_i\;\dot{x}dt = \int_{t_1}^{t_2}\left(\sum_{i\,=\,1}^{n}\varepsilon_i F_i(t)\right)\dot{x}\;dt$$

$$= \int_{t_1}^{t_2} F_{\text{eq}}(t)\dot{x}dt \tag{2.64}$$

The power delivered by an external force $F(t)$ is

$$P = \frac{dU}{dt} = F(t)\dot{x} \tag{2.65}$$

Work is a cumulative effect, whereas power is instantaneous.

Sinusoidal forces are easy to generate by an actuator. Sometimes the dynamics of the system provides harmonic forces, such as reciprocating engines or any type of rotating machinery. *Impulsive forces* are large forces generated over a short period of time, such as the action of a hammer. *Transient forces* are generated over a period of time.

EXAMPLE 2.14

An applied force has the form $F(t) = 100\sin(50t)$ N.

(a) Determine the work done by the force between time 0 and an arbitrary time t if $x(t) = 0.002\sin(50t - 0.15)$ m.

(b) Determine the work done by the force between 0 s and 0.01 s.

(c) Determine the power delivered by the force at 0.01 s.

SOLUTION

(a) The work done by the force is

$$W(t) = \int_0^t (100\sin 50t\ \text{N})(0.002\ \text{m})(50\ \text{rad/s})\cos(50t - 0.15)dt$$

$$= 10\int_0^t \sin(50t)\cos(50t - 0.15)dt$$

$$= -\frac{1}{20}\cos(100t - 0.15) + \frac{1}{20}\cos(0.15) + 5\sin(0.15)t$$

$$= 0.049 + 0.747t - 0.05\cos(100t - 0.15)$$

(b) The work between 0 s and 0.01 s is $W(0.01)$

$$W(0.01) = -\frac{1}{20}\cos(0.85) + \frac{1}{20}\cos(0.15) + \frac{1}{20}\sin(0.15) = 0.0239\ \text{N}\cdot\text{m}$$

(c) The power delivered to the system at $t = 0.01$ s is

$$P = F(t)\dot{x} = [100\sin(0.5)\,\text{N}]\big[(0.002\,\text{m})(50\ \text{rad/s})\cos(0.5 - 0.15)\big]$$

$$= 4.50\ \text{N}\cdot\text{m/s}$$

Motion input is generated by kinematic mechanism, such as a cam and follower system or a Scotch yoke. Motion input also occurs through the wheels on a car following the road contour. The work done by the motion input depends upon the system. Consider a mass-spring

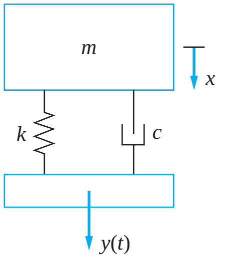


FIGURE 2.30
A mass-spring and viscous-damper system with the spring and viscous damper attached to a moveable support. The motion of the support induces both the spring force and viscous-damping force to do work on the system.

and viscous-damper system of Figure 2.30. The spring and viscous damper are connected to a moveable support which has a prescribed displacement $y(t)$. The motion causes work in the spring and viscous damper. If x is the chosen generalized coordinate and represents the displacement of the mass, the change in length of the spring is $y - x$ and the velocity developed in the viscous damper is $\dot{y} - \dot{x}$. The work done by the parallel combination of the spring and viscous damper on the body is

$$
\begin{aligned}
U_{1\to 2} &= \int_{x_1}^{x_2} [k(y - x) + c(\dot{y} - \dot{x})]\,dx \\
&= \int_{x_1}^{x_2} (-kx - c\dot{x})\,dx + \int_{x_1}^{x_2} (ky + c\dot{y})\,dx \\
&= V_1 - V_2 + U_{1\to 2_{NC,d}} + + \int_{x_1}^{x_2} (ky + c\dot{y})\,dx
\end{aligned}
\tag{2.66}
$$

where $U_{1\to 2_{NC,d}}$ is the work done by the non-conservative damping force. Hence, the equivalent force due to the motion input is

$$F_{eq} = ky + c\dot{y} \tag{2.67}$$

EXAMPLE 2.15

A car is traveling on a bumpy road that is approximated by

$$y(z) = 0.002\sin(2\pi z)\text{ m} \tag{a}$$

The car has a constant horizontal velocity of 60 m/s. The car is modeled using a simplified suspension system consisting of a mass attached to a spring in parallel with a viscous damper. The spring and viscous damper combination is attached to the wheels' axis which follow the road contour.

(a) What is the time dependent displacement imparted to the suspension system?
(b) What is the acceleration imparted to the suspension system?
(c) What is the equivalent force felt by the vehicle through a suspension system of stiffness 20,000 N/m and damping coefficient 1000 N · s/m?

SOLUTION

(a) The car is traveling at a constant speed of 60 m/s; thus, in time t, it travels $z = 60t$. The displacement imparted to the vehicle is

$$y(t) = 0.002\sin[2\pi(60t)] = 0.002\sin(120\pi t) \tag{b}$$

(b) The acceleration imparted to the suspension system is

$$\ddot{y} = -(0.002)(120\pi)^2\sin(120\pi t) = -2.84 \times 10^2\sin(120\pi t)\ \text{m/s}^2 \tag{c}$$

(c) The equivalent force is given by Equation (2.65) as

$$
\begin{aligned}
F_{eq} &= (20000\,\text{N/m})\,[0.002\sin(120t)\,\text{m}] + (1000\,\text{N}\cdot\text{s/m})\,(120)[0.002\cos(120t)\text{m/s}] \\
&= [40\sin(120t) + 240\cos(120t)]\,\text{N}
\end{aligned}
\tag{d}
$$

2.9 FREE-BODY DIAGRAM METHOD

Newton's laws, as formulated in Chapter 1, are applied to free-body diagrams of vibrating systems to derive the governing differential equation. The following steps are used in application to a SDOF system.

1. A generalized coordinate is chosen. This variable could represent the displacement of a particle in the system. If rotational motion is involved, the generalized coordinate could represent an angular displacement.
2. Free-body diagrams are drawn showing the system at an arbitrary instant of time. In line with the methods of Section 1.7, two free-body diagrams are drawn. One free-body diagram shows all external forces acting on the system. The second free-body diagram shows all effective forces acting on the system. Recall that the effective forces are a force equal to $m\bar{\mathbf{a}}$, applied at the mass center and a couple equal to $\bar{I}\alpha$.

   The forces drawn on each free-body diagram are annotated for an arbitrary instant. The direction of each force and moment are drawn consistent with the positive direction of the generalized coordinate. Geometry, kinematics, constitutive equations, and other laws valid for specific systems can be used to specify the external and effective forces.
3. The appropriate form of Newton's law is applied to the FBD. If the FBD is that of a particle, the appropriate conservation law is $\Sigma\mathbf{F} = m\mathbf{a}$. If the FBD is that of a rigid body undergoing planar motion, the conservation laws are $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$ and $\Sigma M_G = \bar{I}\alpha$. If the external and effective force method is used, the appropriate equations are $(\Sigma\mathbf{F})_{\text{ext}} = (\Sigma\mathbf{F})_{\text{eff}}$.
4. Applicable assumptions are used along with algebraic manipulation. The result is a governing differential equation.

Forces are drawn on the FBDs at an arbitrary instant. The force from the spring on the FBD (from Newton's third law) is equal and opposite to the force from the body on the spring. If the spring is stretched, it is in tension, and the force in the spring pulls on the spring, as shown in Figure 2.31(a). Equal and opposite to it is the spring force acting *away* from the body. If the spring is in compression, the force in the spring pushes against

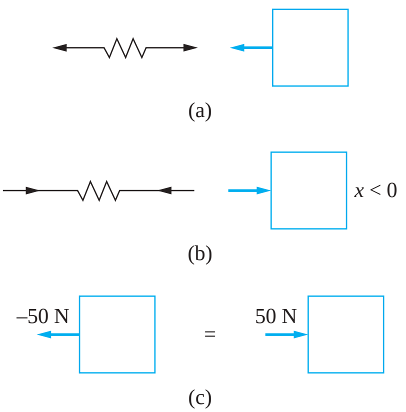


FIGURE 2.31
(a) Spring is in tension where force from a spring on a block is away from the block. (b) Spring is in compression were the force from a spring on a block pushes on the block. (c) A − 50 N force pulling on the block is equivalent to a 50 N force pushing on the block.

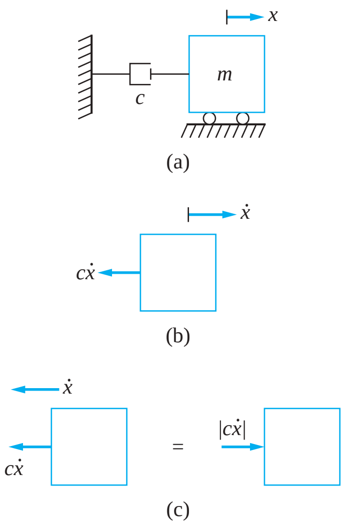


FIGURE 2.32
The sign of the viscous-damping force takes care of itself if it is drawn to the opposite of the positive motion of the point to which the viscous damper is attached.

the spring, as shown in Figure 2.31(b). Equal and opposite again, the spring force is acting *against* the body. Let x represent the displacement of the particle to which the spring is attached. If the spring force is drawn for a positive value of x, it is labeled kx and is drawn acting away from the body. Now if the spring is in compression, x takes on a negative value. If the spring force is drawn acting *away* from the body and x is negative, it is actually acting *against* the body as shown in Figure 2.31(c). Thus, the spring force is always drawn in the direction opposite to the that of positive displacement of the point to which it is attached. Then the direction of the spring force always takes care of itself.

The force from a viscous damper always opposes the direction of motion of the point to which it is attached on a FBD of a SDOF system. If x represents the displacement of the particle to which a viscous damper is attached, then its velocity is $\dot{x}$. The force from the viscous damper drawn on the FBD opposes the direction of positive $\dot{x}$. If the velocity of the particle is in the opposite direction and $\dot{x}$ is negative, it is the same situation shown Figure 2.32(c) where a negative force on a FBD is actually in the opposite direction. Thus, the force from a viscous damper always opposes the direction of positive motion of the particle to which it is attached. Like the spring force, the direction always takes care of itself.

When the effective force diagram is drawn, the effective forces are drawn to be consistent with the positive direction of the generalized coordinates.

EXAMPLE 2.16

The block of Figure 2.33(a) slides on a frictionless surface. Derive the differential equation governing the motion of the system using x as the displacement of the system from its equilibrium position and as the generalized coordinate.

SOLUTION

The free-body diagram of Figure 2.33(b) shows the forces acting on the block at an arbitrary instant. The spring force is kx and is drawn away from the block, indicating the spring is in tension for a positive x. The damping force is labeled $c\dot{x}$ and is drawn opposite the positive direction of motion.

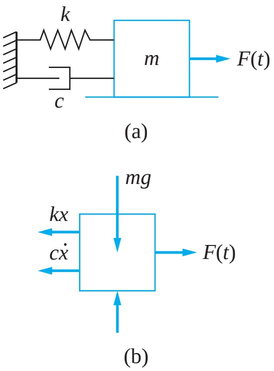


FIGURE 2.33
(a) System of Example 2.16. Mass-spring and viscous-damper system sliding on a frictionless surface with an external force.

Applying Newton's law to the free-body diagram in the x direction leads to

$$-kx - c\dot{x} + F(t) = m\ddot{x} \tag{a}$$

Rearranging the equation so that all terms involving the generalized coordinate are on one side yields

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{b}$$

Equation (b) is the governing differential equation. The values of $x(0)$ and $\dot{x}(0)$ must be specified before solving.

EXAMPLE 2.17

A thin disk of mass moment of inertia I is attached to a fixed shaft of length L. The polar moment of inertia of the shaft is J and it is made from a material of shear modulus G, as shown in Figure 2.34(a). A moment $M(t)$ is applied to the disk. Derive the differential equation governing the clockwise angular displacement of the disk θ.

SOLUTION

The effect of the shaft is to produce a resisting moment

$$M = \frac{JG}{L}\theta \tag{a}$$

on the disk. The disk undergoes pure rotational motion about the axis of the shaft. A FBD of the disk at an arbitrary time is shown in Figure 2.34(b). Applying $\Sigma M_G = \bar{I}\alpha$ to the disk and noting that $\alpha = \ddot{\theta}$ leads to

$$-\frac{JG}{L}\theta + M(t) = I\ddot{\theta} \tag{b}$$

$$I\ddot{\theta} + \frac{JG}{L}\theta = M(t) \tag{c}$$

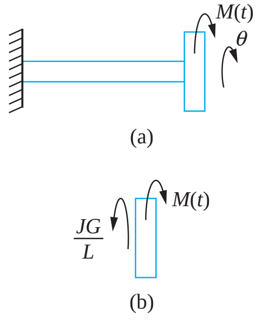


FIGURE 2.34
(a) System of Example 2.17. The angular displacement of the disk θ is the chosen generalized coordinate. (b) FBD of the system at an arbitrary instant.

EXAMPLE 2.18

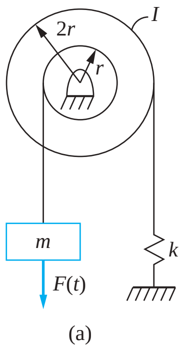
The system of Figure 2.35 lies in a horizontal plane on a frictionless surface. Derive the differential equation governing the displacement of the mass.

SOLUTION

Let x represent the displacement of the mass. The disks move together. Assuming the cable connecting the block to the disk is inextensible, the change in length of the cable is x, which must be the amount of cable taken up or let out by the disk. If θ represents the clockwise angular rotation of the disk, the amount of cable let out is equal to the arc length subtended by θ as

$$x = r\theta \tag{a}$$

Equation (a) is valid for all time. It can be differentiated leading to $\dot{x} = r\dot{\theta}$ and $\ddot{x} = r\ddot{\theta}$. This is consistent with use of the relative velocity and relative acceleration equations applied between the center of the disk and the point instantaneously releasing the cable. The acceleration of the point also has a component equal to $r\dot{\theta}^2$ directed toward the center of rotation. Using the same principle, the spring is stretched by $2x$.



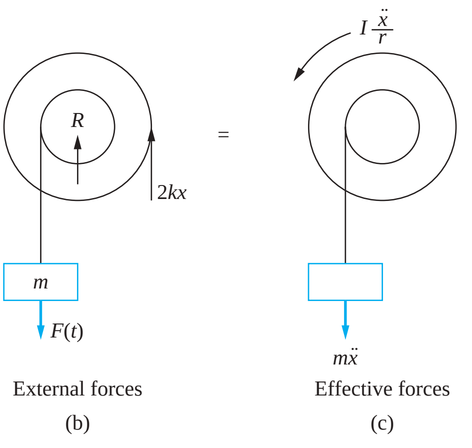


FIGURE 2.35
(a) System of Example 2.18 lies in a horizontal plane. (b) FBDs of the system at an arbitrary instant. The system consists of the disk and the block.

FBD's illustrating the external forces for the system and the effective forces are shown in Figure 2.35(b). Applying $(\Sigma M_O)_{ext} = (\Sigma M_O)_{eff}$ to these FBDs yields

$$-k(2x)(2r) + rF(t) = I\left(\frac{\ddot{x}}{r}\right) + m\ddot{x}(r) \tag{b}$$

which is rearranged to

$$\left(\frac{1}{r} + mr\right)\ddot{x} + 4krx = rF(t) \tag{c}$$

EXAMPLE 2.19

A thin disk of mass m and radius r, $\bar{I} = \frac{1}{2}mr^2$, has a spring of stiffness k, and has a viscous damper of damping coefficient c attached at its mass center, as shown in Figure 2.36(a). The disk rolls without slipping. Derive a differential equation governing the displacement of the mass center.

SOLUTION

Let x be the displacement of the disk's mass center. When the disk rolls without slipping the friction force is less than the maximum available friction force μN where N is the normal force. The point of contact between the disk and the surface has a velocity of zero. Use of the relative velocity equation between the point of contact and the center of mass yields

$$\bar{\mathbf{v}} = v_C + \mathbf{v}_{G/C} = r\omega\mathbf{i} \tag{a}$$

The mass center only has a velocity and an acceleration in the horizontal direction; thus, Equation (a) can be differentiated to yield

$$\bar{a} = r\alpha \tag{b}$$

When the disk rolls without slipping, the kinematic condition of Equation (b) exists between the disk's angular acceleration and the acceleration of the mass center. Noting that

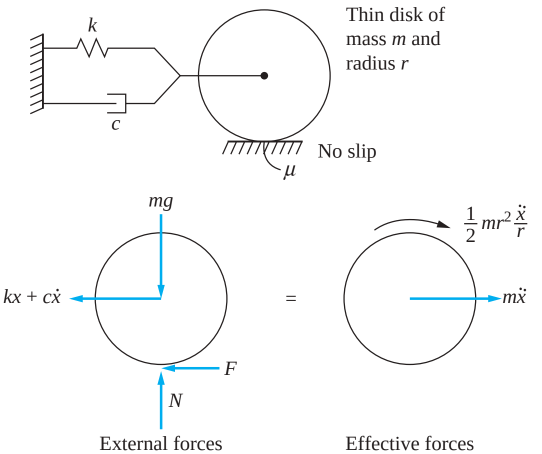


FIGURE 2.36
(a) System of Example 2.19. The disk rolls without slipping. (b) FBDs of the system at an arbitrary instant. The friction force is less than the maximum available friction, and a kinematic relationship exists between the angular acceleration and the acceleration of the mass center.

$\bar{a} = \ddot{x}$, FBDs of the disk at an arbitrary instant are shown in Figure 2.36(b). Summing moments on these FBDs according to $(\Sigma M_C)_{\text{ext}} = (\Sigma M_C)_{\text{eff}}$ leads to

$$-kx(r) - c\dot{x}(r) = \frac{1}{2}mr^2\left(\frac{\ddot{x}}{r}\right) + m\ddot{x}(r) \tag{c}$$

$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = 0 \tag{d}$$

EXAMPLE 2.20

An accelerometer used in micro-electromechanical (MEMS) applications is shown in Figure 2.37(a). The accelerometer consists of a rigid bar between two massless fixed-fixed beams that are acting like springs. The bar is free to vibrate in the surrounding medium, which provides viscous damping. Derive a differential equation for the free vibrations of the accelerometer using a one degree-of-freedom model.

SOLUTION

The system is modeled, as in Figure 2.37(b), as a rigid bar attached to two identical springs. The mass of the bar is

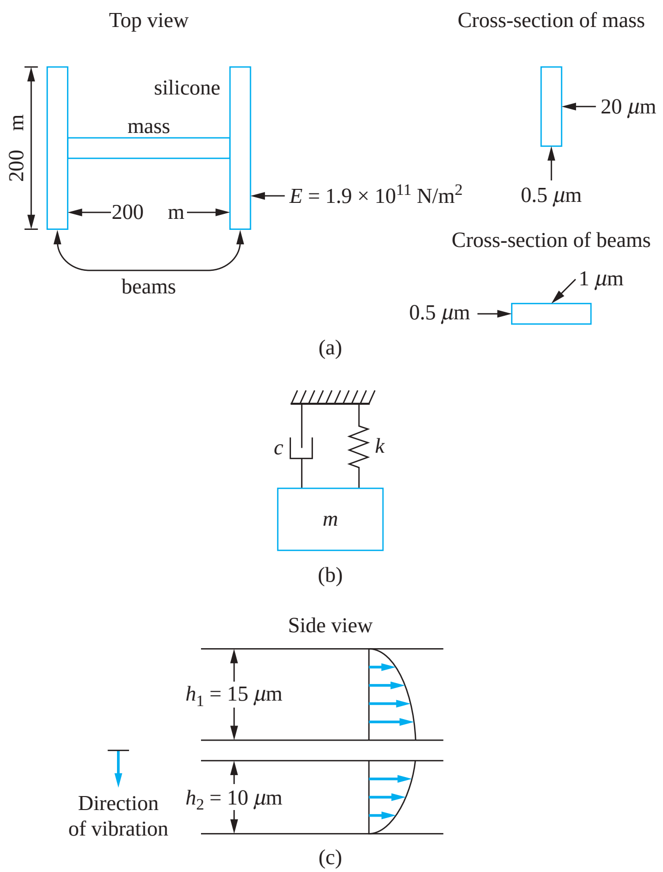


FIGURE 2.37
(a) MEMS accelerometer consists of a rigid bar between two fixed-fixed beams which vibrates in a viscous liquid. (b) SDOF model of system. (c) Calculation of viscous damping coefficient.

$$m_{eq} = \rho dtL$$
$$= \left(2.3\frac{\text{g}}{\text{cm}^3}\right)\left(\left(\frac{100\,\text{cm}}{\text{m}}\right)^3\left(\left(\frac{1\,\text{kg}}{1000\,\text{g}}\right)(20\times10^{-6}\,\text{m})(0.5\times10^{-6})\right.\right.$$
$$\times(200\times10^{-6}\text{m}) = 4.6\times10^{-12}\text{kg} \quad \text{(a)}$$

The moment of inertia of the cross section of one beam is

$$I = \frac{1}{12}th^3 = \frac{1}{12}(0.5\times10^{-6})(1.0\times10^{-6}\text{m})^3 = 4.17\times10^{-26}\ \text{m}^4 \quad \text{(b)}$$

The equivalent stiffness is twice the stiffness of a fixed-fixed beam at its midspan. From Appendix D, it is calculated as

$$k_{eq} = 2\left(\frac{192EI}{L^3}\right)$$
$$= 2\frac{192(1.9\times10^{11}\text{N/m}^2)(4.17\times10^{-26}\text{m}^4)}{(200\times10^{-6}\ \text{m})^3} = 0.380\,\text{N/m} \quad \text{(c)}$$

An equivalent viscous-damping coefficient is calculated using an approximate linear velocity profile in the surrounding fluid. The fluid on the top and bottom of the beam is in motion due to the vibrations of the beam as shown in Figure 2.37(c). The fluid above the beam has a velocity profile of

$$u(y) = \frac{v}{h_1}y \quad \text{(d)}$$

where y is a coordinate into the fluid from the fixed surface. The shear stress acting on the beam is calculated using Newton's viscosity law as

$$\tau = \mu\frac{du}{dy} = \mu\frac{v}{h_1} \quad \text{(e)}$$

and the resultant force on the surface of the beam is

$$F_1 = \tau Ld = \mu Ld\frac{v}{h_1} \quad \text{(f)}$$

Using a similar analysis, the force on the lower surface of the beam is

$$F_2 = \mu Ld\frac{v}{h_2} \quad \text{(g)}$$

The total damping force is expressed as

$$F = \mu Ld\left(\frac{1}{h_1} + \frac{1}{h_2}\right)v \quad \text{(h)}$$

from which the equivalent viscous damping coefficient is calculated as

$$c_{eq} = \mu Ld\left(\frac{1}{h_1} + \frac{1}{h_2}\right)$$
$$= (740\times10^{-6}\text{N}\cdot\text{s/m})(200\times10^{-6}\,\text{m})(20\times10^{-6}\,\text{m})$$
$$\left(\frac{1}{15\times10^{-6}\,\text{m}} + \frac{1}{10\times10^{-6}\,\text{m}}\right)$$

$$= 4.93 \times 10^{-7}\,\text{N} \cdot \text{s/m} \tag{i}$$

The mathematical model for the free response of the system is

$$4.6 \times 10^{-12}\ddot{x} + 4.93 \times 10^{-7}\dot{x} + 0.380x = 0 \tag{j}$$

2.10 STATIC DEFLECTIONS AND GRAVITY

Static deflections are present in springs due to an initial source of potential energy, usually gravity. The *static force* developed in the springs form an equilibrium condition with the gravity forces. The generalized coordinate is generally measured from the equilibrium position of the system. For a linear system, when the differential equation governing the motion is derived, the equilibrium condition appears in the differential equation. It is, of course, set equal to zero. The static spring forces cancel with the gravity forces that cause them in the differential equation. Thus, neither are drawn on the FBD showing the external forces.

EXAMPLE 2.21

A hanging mass-spring and viscous-damper system is illustrated inFigure 2.38(a). Derive the differential equation governing the motion of the system.

SOLUTION

Let x measure the displacement of the mass (positive downward) from the system's equilibrium position. When the system is in equilibrium, a static spring force is developed due to gravity. Summing forces to zero on the FBD (drawn when the system is in equilibrium, as shown in Figure 2.38(b)) leads to the equilibrium condition

$$mg - k\Delta_S = 0 \tag{a}$$

where Δ_s is the static deflection in the spring.

When the mass has deflected a distance x downward, the spring force is the spring force that is present in equilibrium $k\Delta_s$ plus the additional force developed from equilibrium kx. Applying $\Sigma\mathbf{F} = m\mathbf{a}$ in the downward direction to the FBD of the particle (drawn at an arbitrary instant, as shown in Figure 2.38(c)) leads to

$$mg - k(x + \Delta_s) - c\dot{x} + F(t) = m\ddot{x} \tag{b}$$

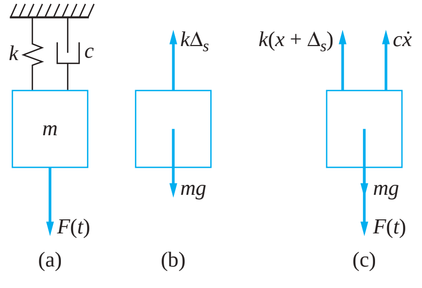


FIGURE 2.38
(a) System of Example 2.21. (b) FBD of the system drawn when the system is in equilibrium. (c) FBD drawn at an arbitrary instant. The differential equation governing the motion of the system is the same as the sliding mass-spring-viscous system without friction.

which rearranges to

$$m\ddot{x} + c\dot{x} + kx = F(t) + mg - k\Delta_S \quad \text{(c)}$$

Using the equilibrium condition, Equation (a) in Equation (c) gives

$$m\ddot{x} + c\dot{x} + kx = F(t) \quad \text{(d)}$$

The equation governing the displacement of the hanging mass-spring and viscous-damper system is the same as the sliding mass-spring and viscous-damper system.

The hanging mass-spring and viscous-damper system can be analyzed by considering it FBD, shown again in Figure 2.39. The FBD can be broken down by drawing a FBD showing the spring, viscous damper, and external forces plus a FBD showing the gravity and static spring force. The resultant of the gravity and static spring force is zero, so one only needs the first FBD. It is not necessary to show the static spring force or gravity on the FBD.

The above result, not needing to show the gravity force or the static spring force on the FBD, is valid only for deriving the differential equation of motion. If another goal (such as obtaining a reaction) is desired, the static spring forces and gravity must be included on the FBD.

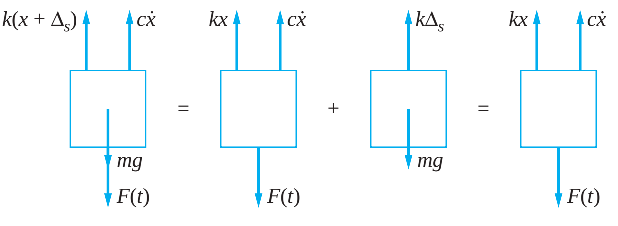


FIGURE 2.39
(a) FBD of hanging mass-spring and viscous-damper system can be drawn such that it is the same as the FBD of the sliding mass-spring and viscous-damper system.

EXAMPLE 2.22

Consider the system of Figure 2.40(a). Let x describe the downward displacement of m_1 from the system's equilibrium position.

(a) Derive the differential equation governing $x(t)$.

(b) Determine the reaction at the center of the disk at the pin support in terms of x, $\dot{x}$, and $\ddot{x}$.

SOLUTION

A FBD of the system in equilibrium is shown in Figure 2.40(b). Summing moments about the pin support to zero with positive moments counterclockwise leads to

$$m_1 g(2r) - k\Delta_{s1}(2r) - m_2 g(r) + k\Delta_{s2}(r) = 0 \quad \text{(a)}$$

FBDs illustrating the external forces and effective forces at an arbitrary instant are shown in Figure 2.40(c). Using and $(\Sigma M_O)_{\text{ext}}(\Sigma M_O)_{\text{eff}}$ on these FBDs lead to

$$-k(x+\Delta_{s1})(2r) + m_1 g(2r) - k\left(\frac{x}{2} - \Delta_{s2}\right)(r) - m_2 g(r)$$

$$= m_1\ddot{x}(2r) + m_2\frac{\ddot{x}}{2}(r) + I\frac{\ddot{x}}{2r} \quad \text{(b)}$$

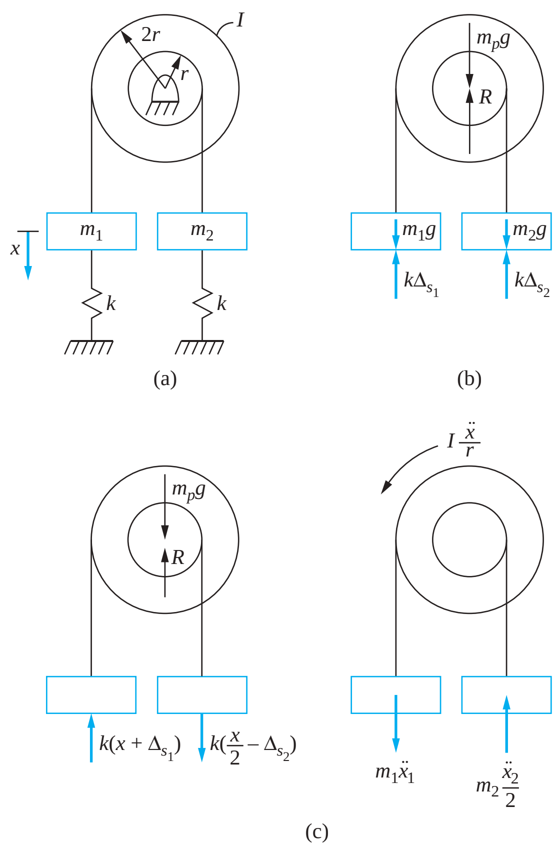


FIGURE 2.40
(a) System of Example 2.22. (b) FBD of static equilibrium position. (c) FBDs of system at an arbitrary instant.

which cleans up to

$$\left(\frac{1}{2r} + 2rm_1 + \frac{r}{2}m_2\right)\ddot{x} + \frac{5}{2}krx = m_1g(2r) - k\Delta_{s1}(2r) - m_2g(r) + k\Delta_Q(r) \quad \textbf{(c)}$$

Use of Equation (a) in Equation (c) gives

$$\left(\frac{1}{2r} + 2rm_1 + rm_2\right)\ddot{x} + \frac{5}{2}krx = 0 \quad \textbf{(d)}$$

(b) Applying $(\Sigma\mathbf{F})_{\text{ext}} = (\Sigma\mathbf{F})_{\text{eff}}$ in the vertical direction to the FBD of external forces, positive downward yields

$$m_pg + m_1g + m_2g - k(x + \Delta_{s1}) + k\left(\frac{x}{2} - \Delta_{s2}\right) - R = m_1\ddot{x} - m_2\frac{\ddot{x}}{2} \quad \textbf{(e)}$$

which is solved for R as

$$R = m_pg + m_1g + m_2g - \frac{1}{2}kx - k(\Delta_{s1} - \Delta_{s2}) + \left(\frac{1}{2}m_2 - m_1\right)\ddot{x} \quad \textbf{(f)}$$

From this point, it is assumed that for all linear systems the generalized coordinate will be measured from the system's equilibrium position, and the only goal is to derive the differential equation. Then the static spring force and the gravity force that causes it will not be drawn on a FBD showing external forces.

2.11 SMALL ANGLE OR DISPLACEMENT ASSUMPTION

Nonlinear differential equations occur when the generalized coordinate appears nonlinearly in the differential equation. Examples of nonlinear differential equations are

$$m\ddot{x} + c\dot{x} + k_1 x + k_3 x^3 = 0 \tag{2.68a}$$

$$m\ddot{x} + a\dot{x}^2 + k_1 x = 0 \tag{2.68b}$$

$$\ddot{\theta} + 3\ddot{\theta}\cos\theta + 200\cos\theta\sin\theta = 0 \tag{2.68c}$$

Equation (2.68a) occurs for a mass-spring and viscous-damper system when the spring has a cubic nonlinearity. Equation (2.68b) occurs for a system where air resistance is included in the modeling. An equation such as Equation (2.68c) could occur in the modeling of the vibrations of a bar about the equilibrium position.

The exact solution of few nonlinear equations are known. Methods to handle nonlinearities in differential equation (mostly approximate methods) are considered in Chapter 12. A linearization method is sought for the differential equations. It is clear that linearization of Equations (2.68a) or (2.68b) simply requires neglecting the nonlinear terms in comparison to the linear terms. The linearization of Equation (2.68c) is not quite as simple.

EXAMPLE 2.23

Derive the differential equation governing the motion of the simple pendulum of Figure 2.41(a) using θ as the counterclockwise angular displacement of the pendulum from the system's horizontal equilibrium position and as the generalized coordinate.

SOLUTION

The FBD of the system at an arbitrary time is illustrated in Figure 2.41(b). Summing moments about the fixed axis of rotation O using $\Sigma M_0 = I_O\alpha$ leads to

$$-mgL\sin\theta = mL^2\ddot{\theta} \tag{a}$$

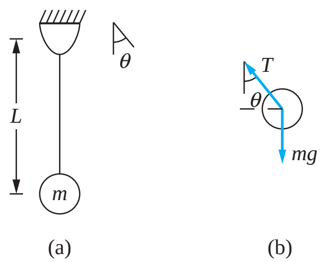


FIGURE 2.41
(a) System of Example 2.23. (b) FBD of particle at arbitrary instant.

Equation (a) is arranged to

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{b}$$

The differential equation derived in Example 2.23 is nonlinear because $\sin\theta$ is a transcendental, not linear, function of θ. Consider the Taylor series expansion for $\sin\theta$ about $\theta = 0$ as

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots \tag{2.69}$$

Suppose $\theta = 0.1$ rad. Thus,

$$\begin{aligned}\sin(0.1) &= 0.1 - \frac{(0.1)^3}{6} + \frac{(0.1)^5}{120} - \cdots \\ &= 0.1 - 1.67 \times 10^{-4} + 8.33 \times 10^{-8} - \cdots \\ &= 0.099833 + \cdots\end{aligned} \tag{2.70}$$

Thus, the approximation for a small θ of

$$\sin\theta \approx \theta \tag{2.71}$$

for $\theta = 0.1$ rad $= 5.1°($ has an error of 1.167 percent. This provides confidence in the *small angle approximation*. Using this approximation in the differential equation of Example 2.23 gives

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{2.72}$$

which is a linear differential equations.

Consistent with the small angle approximation, truncation of Taylor series expansions about $\theta = 0$ for other trigonometric functions yields

$$\cos\theta \approx 1 \tag{2.73}$$

$$\tan\theta \approx 0 \tag{2.74}$$

$$1 - \cos\theta \approx \frac{1}{2}\theta^2 \tag{2.75}$$

The small angle assumption may be made *a priori,* before the differential equation is derived. Consider the spring in the system Figure 2.42(a). It has an unstretched length ℓ. When the bar rotates through an angle θ, the spring moves to a new position, as shown in Figure 2.42(b). The change in length of the spring is

$$\delta = \sqrt{(\ell + L\sin\theta)^2 + (L - L\cos\theta)^2} - \ell \tag{2.76}$$

It is consistent with the small angle assumption to approximate the change in length of the spring by $L\theta$. The spring force would be at an angle θ to the vertical. However, it is also consistent with the small angle assumption to draw the spring force vertically and label it $kL\,\theta$, as shown in Figure 2.42(c). The distance for taking moments about the pin support is $L\cos\theta \approx L$.

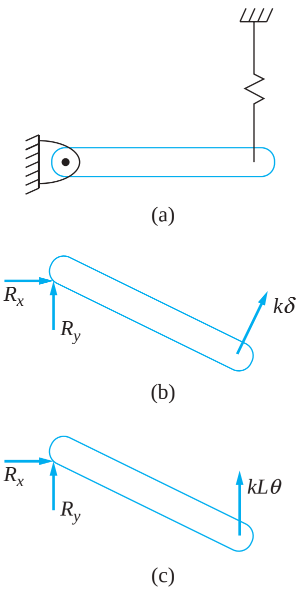


FIGURE 2.42
(a) The spring has an unstretched length ℓ. (b) When the system moves to a new position described by the generalized coordinate θ, the change in length of the spring is a nonlinear function of θ. (c) Consistent with the small angle assumption, the spring force is drawn vertically and labeled $kL\theta$.

EXAMPLE 2.24

Derive the differential equation governing the motion of the bar of Figure 2.43(a). Use θ as the clockwise angular displacement of the bar from the system's equilibrium position and as the chosen generalized coordinate. Assume a small θ.

SOLUTION

The small angle assumption will be used; thus, the differential equation will be linearized. Static deflections exist in the springs due to gravity. The static equilibrium position is defined by an angle θ_s, and θ is measured relative to this angle. It is assumed that θ_s is small and does not affect the lengths required for the moments. Indeed, under these conditions, θ_s is taken to be zero without loss of generality.

FBDs showing the external forces and the effective forces at an arbitrary instant are shown in Figure 2.42(b). The forces are drawn on the FBD with the small angle assumption already made. The spring forces are labeled assuming small displacements with sin $\theta \approx \theta$. They also remain vertical, which is consistent with the small angle assumption. The damping force is labeled as $c\frac{L}{6}\dot{\theta}$, which is derived from the relative velocity equation but is drawn vertical to be consistent with the small angle assumption.

This problem involves rotation about a fixed axis at O, so either $\Sigma M_O = I_O\alpha$ or $(\Sigma M_O)_{\text{ext}} = (\Sigma M_O)_{\text{eff}}$ is applicable. The latter is used here, applying $(\Sigma M_O)_{\text{ext}} = (\Sigma M_O)_{\text{eff}}$ to the FBDs of Figure 2.43(b) and leading to

$$-k\frac{L}{3}\theta\left(\frac{L}{3}\right) - k\frac{2}{3}L\theta\left(\frac{2L}{3}\right) - c\frac{L}{6}\dot{\theta}\left(\frac{L}{6}\right) = \frac{1}{12}mL^2\ddot{\theta} + m\frac{L}{6}\ddot{\theta}\left(\frac{L}{6}\right) \quad \text{(a)}$$

Rearranging Equation (a) gives

$$4m\ddot{\theta} + c\dot{\theta} + 20k\theta = 0 \quad \text{(b)}$$

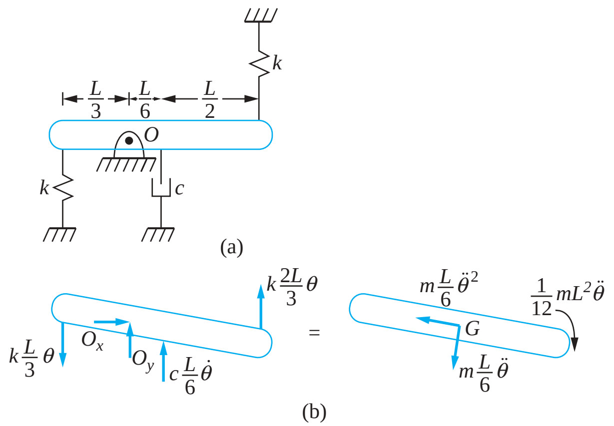


FIGURE 2.43
(a) System of Example 2.24. (b) FBDs drawn at an arbitrary instant using the small angle assumption, ignoring static spring forces and the gravity forces that cause them.

2.12 EQUIVALENT SYSTEMS METHOD

It has been shown that the potential energy for a linear SDOF system with chosen generalized coordinate x can be expressed as $V = \frac{1}{2}k_{eq}x^2 + V_0$ where V_0 is the potential energy in its equilibrium position, the kinetic energy is expressed as $T = \frac{1}{2}m_{eq}\dot{x}^2$, the work done by the viscous-damping forces as the generalized coordinate moves between x_1 and x_2 can be written as $U_{1\to 2} = -\int_{x_1}^{x_2} c_{eq}\dot{x}dx$, and the work done by all other external forces between times t_1 and t_2 is $\int_{t_1}^{t_2} F_{eq}\dot{x}dt$. Application of the principle of work and energy between position 1 and position 2 for the system where $x(t_1) = x_1$ and position 2 defines an arbitrary position of the system

$$T_1 + V_1 + U_{1\to 2} = T + V + V_0 \tag{2.77}$$

Substituting the given expression for both kinetic and potential energy and separating the work done by both viscous and external forces leads to

$$T_1 + V_1 - \int_{x_1}^{x} c_{eq}\dot{x}dx + \int_{t_1}^{t} F_{eq}\dot{x}dt = \frac{1}{2}m_{eq}\dot{x}^2 + \frac{1}{2}k_{eq}x^2 + V_0 \tag{2.78}$$

Noting that T_1, V_1, and V_0 represent kinetic and potential energy at a specific instant of time and therefore are constants, differentiation of Equation (2.78) with respect to time gives

$$-\frac{d}{dt}\left(\int_{x_1}^{x} c_{eq}\dot{x}dx\right) + \frac{d}{dt}\left(\int_{t_1}^{t} F_{eq}\dot{x}dt\right) = \frac{1}{2}m_{eq}\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}k_{eq}\frac{d}{dt}(x^2) \tag{2.79}$$

Note that

$$\frac{d}{dt}(x^2) = 2x\dot{x} \tag{2.80}$$

$$\frac{d}{dt}(\dot{x}^2) = 2\dot{x}\ddot{x} \tag{2.81}$$

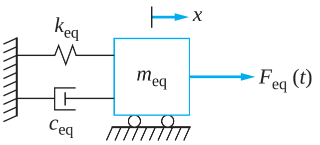


FIGURE 2.44
Equivalent mass-spring and viscous-damper system when a linear displacement x is chosen as the generalized coordinate.

and

$$\frac{d}{dt}\left(\int_{x_1}^{x} c_{eq}\dot{x}dx\right) = \frac{d}{dt}\left(\int_{t_1}^{t} c_{eq}\dot{x}^2dt\right) = c_{eq}\dot{x}^2 \tag{2.82}$$

Equation (2.79) becomes

$$F_{eq}\dot{x} - c_{eq}\dot{x} = m_{eq}\ddot{x}\dot{x} + k_{eq}x\dot{x} \tag{2.83}$$

Equation (2.80) has two solutions: $\dot{x} = 0$ (the static case) and x. This satisfies

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F_{eq}(t) \tag{2.84}$$

Equation (2.84) is the differential equation for any linear, single degree-of-freedom system. It only requires identification of m_{eq}, c_{eq}, k_{eq}, and $F_{eq}(t)$. That is, any linear SDOF system is modeled by a mass-spring and viscous-damper system with equivalent coefficients, as in Figure 2.44. The equivalent mass is identified from the quadratic form of kinetic energy in $T = \frac{1}{2}m_{eq}\dot{x}^2$. The equivalent stiffness is identified from the quadratic form of potential energy in $V = \frac{1}{2}k_{eq}x^2$. The equivalent viscous-damping coefficient is identified from the energy dissipation in $U_{1\to2} = -\int_{x_1}^{x_2} c_{eq}\dot{x}dt$. The work done by external forces, shown as $\int_{t_1}^{t_2} F_{eq}\dot{x}dt$, is used to calculate $F_{eq}(t)$.

If an angular coordinate is chosen as the generalized coordinate, the appropriate form of Equation (2.84) is

$$I_{eq}\ddot{\theta} + \mathrm{c}_{t,eq}\dot{\theta} + \mathrm{k}_{t,eq}\theta = M_{eq}(t) \tag{2.85}$$

The appropriate equivalent systems model is a thin disk of moment of inertia I_{eq} attached to a shaft of torsional stiffness $k_{t,eq}$ in parallel with a torsional viscous-damper coefficient $c_{t,eq}$ as shown in Figure 2.45.

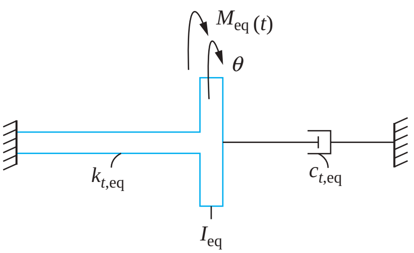


FIGURE 2.45
Equivalent torsional system used when an angular coordinate θ is chosen as the generalized coordinate.

EXAMPLE 2.25

Use the equivalent systems method to derive the differential equation governing the motion of the bar of Figure 2.43(a) and Example 2.24 using θ as the clockwise angular displacement of the bar from the system's equilibrium position and as the chosen generalized coordinate. Assume small θ.

SOLUTION

The kinetic energy of the bar at an arbitrary instant is

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 = \frac{1}{2}m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9}\,mL^2\right)\dot{\theta}^2 \qquad \text{(a)}$$

Thus, $I_{eq} = \frac{1}{9}\,mL^2$. The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k\left(\frac{L}{3}\,\theta\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\,\theta\right)^2 = \frac{1}{2}\left(\frac{5}{9}kL^2\right)\theta^2 \qquad \text{(b)}$$

The equivalent torsional stiffness is $k_{t,eq} = \frac{5}{9}kL^2$. The work done by the viscous damper between an initial position and an arbitrary position is

$$W_{1\to 2} = -\int_{\theta_1}^{\theta}\left(c\frac{L}{6}\,\dot{\theta}\right)d\left(\frac{L}{6}\,\theta\right) = -\int_{\theta_1}^{\theta}\left(c\frac{L^2}{36}\,\dot{\theta}\right)d\theta \qquad \text{(c)}$$

Hence, the equivalent torsional stiffness is $c_{t,eq} = c\frac{L^2}{36}$. The differential equation governing θ is

$$\frac{1}{9}\,mL^2\ddot{\theta} + c\frac{L^2}{36}\,\dot{\theta} + \frac{5}{9}kL^2\theta = 0 \qquad \text{(d)}$$

Equation (d) reduces to Equation (b) of Example 2.24.

EXAMPLE 2.26

Use the equivalent system method to derive the differential equation governing the free vibrations of the system of Figure 2.46. Use x, the displacement of the mass center of the disk from the system's equilibrium position, as the generalized coordinate. The disk rolls without slipping, no slip occurs at the pulley, and the pulley is frictionless. Include an approximation for the inertia effects of the springs. Each spring has a mass m_s.

SOLUTION

Let θ be the clockwise angular rotation of the pulley from the system's equilibrium position and x_B be the downward displacement of the block, also measured from equilibrium. Then

$$x = r\theta \quad x_B = 2r\theta \qquad \text{(a)}$$

Eliminating θ between these equations leads to $x_B = 2x$. Since the disk rolls without slip, its angular velocity is $\omega_D = \dot{x}/r_D$. The inertia effect of each spring is approximated by placing a particle of mass $m_s/3$ at the location where the spring is attached to the system.

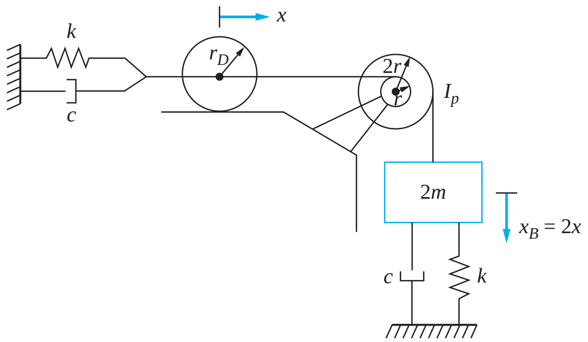


FIGURE 2.46
The system of Example 2.26 is modeled by the equivalent system of Figure 2.44.

To this end it is imagined that a particle of mass $m_s/3$ is attached to the center of the disk and a particle of mass $m_s/3$ is attached to the block. The total kinetic energy of the system, including the kinetic energies of the imagined attached particles is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_D\omega_D^2 + \frac{1}{2}I_p\dot{\theta}^2 + \frac{1}{2}(2m)\dot{x}_B^2 + T_{s_1} + T_{s_2}$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}\mathrm{m}r_D^2\right)\left(\frac{\dot{x}}{r_D}\right)^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}(2m)(2\dot{x})^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}(2\dot{x})^2$$

$$= \frac{1}{2}\left(\frac{19}{2}m + \frac{I_P}{r^2} + \frac{5}{3}m_s\right)\dot{x}^2 \tag{b}$$

The equivalent mass is

$$m_{\mathrm{eq}} = \frac{19}{2}m + \frac{1_P}{r^2} + \frac{5}{3}m_s \tag{c}$$

The potential energy of the system at an arbitrary instant is

$$V = \tfrac{1}{2}kx^2 + \tfrac{1}{2}k(2x)^2 = \tfrac{1}{2}(5k)x^2 \tag{d}$$

Comparison to the quadratic form of potential energy leads to $k_{\mathrm{eq}} = 5k$.

The work done by the viscous dampers between two arbitrary instants is

$$U_{1\to2} = -\int_{x_1}^{x_2} c\dot{x}\,dx - \int_{x_1}^{x_2} c(2\dot{x})\;d(2x) = -\int_{x_1}^{x_2} 5c\dot{x}\,dx$$

Comparison with the general form of work done by a viscous damper leads to $c_{\mathrm{eq}} = 5c$.

The differential equation governing free vibration of the system is

$$\left(\frac{19}{2}m + \frac{I_P}{r^2} + \frac{5}{3}m_s\right)\ddot{x} + 5c\dot{x} + 5kx = 0$$

EXAMPLE 2.27

The slender rod of Figure 2.47 will be subject only to small displacements from equilibrium. Use the equivalent systems method to derive the differential equation governing the motion of the rod using θ, the counterclockwise angular displacement of the rod from its equilibrium position, as the generalized coordinate.

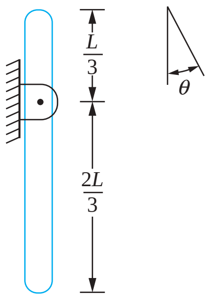


FIGURE 2.47
The compound pendulum is modeled by the equivalent torsional system of Figure 2.45.

SOLUTION

The kinetic energy of the bar at an arbitrary instant is

$$T = = \frac{1}{2}m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9}mL^2\right)\dot{\theta}^2 \tag{a}$$

Comparison with the quadratic form of kinetic energy leads to $I_{eq} = mL^2/9$.

The potential energy in the system is due to gravity. Choosing the plane of the pin support as the datum, the potential energy of the system at an arbitrary instant is

$$V = -mg\frac{L}{6}\cos\theta \tag{b}$$

For small θ, the Taylor series expansion for $\cos\theta$ truncated after the second term leads to an approximation for the potential energy as

$$V = -mg\frac{L}{6}\left(1 - \frac{1}{2}\theta^2\right) = \frac{1}{2}mg\frac{L}{6}\theta^2 - mg\frac{L}{6} \tag{c}$$

Comparison with the quadratic form of potential energy leads to $k_{t,eq} = mgL/6$. Since the datum was chosen as the plane of the pin support, the system has a potential energy of $V_0 = -mgL/6$ when it is in equilibrium.

Equation (2.84) is used to write the differential equation governing the motion of the system as

$$\frac{1}{9}mL^2\ddot{\theta} + \frac{1}{6}mgL\theta = 0 \tag{d}$$

EXAMPLE 2.28

A simplified model of a rack-and-pinion steering system is shown in Figure 2.48. A gear of radius r and polar moment of inertia J is attached to a shaft of torsional stiffness k_t. The gear rolls without slip on the rack of mass m. The rack is attached to a spring of stiffness k. Derive the differential equation governing the motion of the system using x, the horizontal displacement of the rack from the system's equilibrium position, as the generalized coordinate.

SOLUTION

Since there is no slip between the rack and the gear, $\theta = x/r$, where θ is the angular displacement of the gear from equilibrium. The kinetic energy of the system at an arbitrary instant is

$$T = = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\dot{x}^2 \tag{a}$$

from which the equivalent mass is determined as $m_{eq} = m + J/r^2$. The potential energy of the system at an arbitrary instant is

$$V = = \frac{1}{2}kx^2 + \frac{1}{2}k_t\left(\frac{x}{r^2}\right) = \frac{1}{2}\left(k + \frac{k_t}{r^2}\right)x^2 \tag{b}$$

from which the equivalent stiffness is determined as $k_{eq} = k + k_t/r^2$. The differential equation is

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + \left(k_t\frac{k_t}{r^2}\right)x = 0 \tag{c}$$

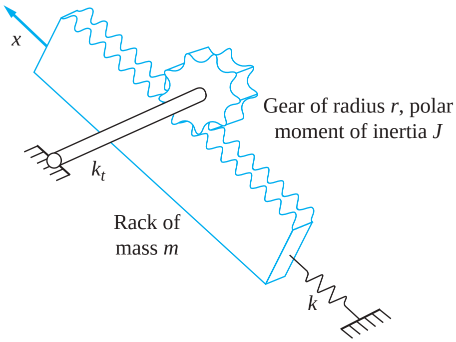


FIGURE 2.48
Model of the rack-and-pinion system of Example 2.28.

EXAMPLE 2.29

A simplified transmission system is shown in Figure 2.49. A motor supplies a torque, which turns a shaft. The shaft has a gear on it, which meshes to a second gear designed such that the speed of the second shaft is greater than the first. The shafts are mounted on identical bearings each with a torsional damping coefficient c_t. Let $\dot{\theta}_1$ be the angular velocity of the shaft directly connected to the motor. Derive a differential equation governing θ_1, which is angular displacement of the shaft directly connected to the motor.

SOLUTION

The meshing gears imply a relationship between the angular velocities of the shafts. The gear equation gives

$$n_1\omega_1 = n_2\omega_2 \tag{a}$$

The total kinetic energy of the shafts is

$$T = \;= \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\left(\frac{n_1}{n_2}\dot{\theta}_1\right)^2 = \frac{1}{2}\left[J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2\right]\dot{\theta}_1^2 \tag{b}$$

Thus, the equivalent moment of inertia is $I_{eq} = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2$. The work done by the torsional viscous dampers is

$$W_{1\to 2} = -\int_{\theta_1}^{\theta} c_t\dot{\theta}_1\,d\theta_1 - \int_{\theta_1}^{\theta} c_t\left(\frac{n_1}{n_2}\dot{\theta}_1\right) d\left(\frac{n_1}{n_2}\theta_1\right) = -\int_{\theta_1}^{\theta} c_t\left[1 + \left(\frac{n_1}{n_2}\right)^2\right]\dot{\theta}_1\,d\theta_1 \tag{c}$$

The equivalent viscous damping coefficient is $c_{t,eq} = c_t\left[1 + \left(\frac{n_1}{n_2}\right)^2\right]$.

The work done by the external moment supplied by the motor is

$$W_{1\to 2} = \int_{t_1}^{t} M(t)\dot{\theta}_1\,dt \tag{d}$$

The equivalent moment is $M_{eq}(t) = M(t)$.

Thus the differential equation governing the angular displacement of the shaft is

$$\left[J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2\right]\ddot{\theta}_1 + c_t\left[1 + \left(\frac{n_1}{n_2}\right)^2\right]\dot{\theta}_1 + M(t) \tag{e}$$

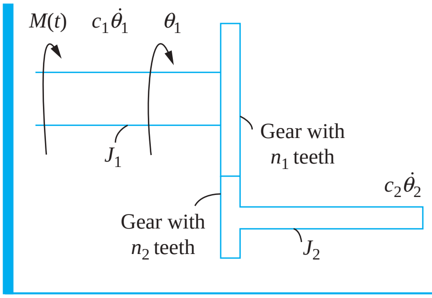


FIGURE 2.49
Model of the transmission system of Example 2.29.

2.13 BENCHMARK EXAMPLES

In this section, the benchmark examples introduced in Section 1.8 are considered. The free-body diagram method is used to derive the differential equations for the machine mounted on a beam and for the simplified vehicle suspension system.

2.13.1 MACHINE ON A FLOOR IN AN INDUSTRIAL PLANT

A machine is mounted on the floor of an industrial plant. The floor is modeled as a W14×30 steel fixed-pinned beam. The appropriate SDOF model is that of a mass suspended from a spring of appropriate stiffness, as shown in Figure 2.50(a). The stiffness is calculated using Appendix D. The equation for the deflection of a fixed-free beam due to a unit concentrated load at $x = a$ evaluated for $x < a$ is

$$w(x) = \frac{1}{2EI}\left(1 - \frac{a}{L}\right)\left[\left(\left(\frac{a^2}{L^2} - 2\frac{a}{L} - 2\right)\frac{x^3}{6} + a\left(2 - \frac{a}{L}\right)\frac{x^2}{2}\right]\tag{a}$$

The machine is located at $a = 0.6L$. Substituting this value into Equation (a) leads to

$$w(0.6L) = 0.00979\frac{L^3}{EI}\tag{b}$$

The stiffness is the reciprocal of $w(0.6\text{L})$

$$k = \frac{EI}{0.00979L^3} = \frac{(30 \times 10^6\,\text{psi})\,(291\,\text{in}^4)}{0.00979(20\ \text{ft})^3}\left(\frac{1\ \text{ft}}{12\ \text{in}}\right)^2 = 7.74 \times 10^5\ \text{lb/ft}\tag{c}$$

One model is a mass of 31.06 slugs (the mass of the machine) attached to a spring of stiffness 7.74×10^5 lb/ft.

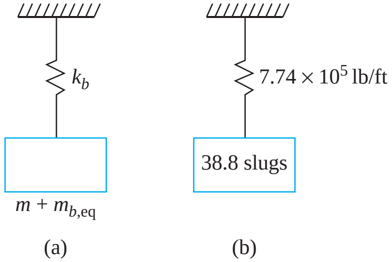


FIGURE 2.50
(a) SDOF model for system of the first benchmark problem. (b) Equivalent mass and equivalent stiffness are calculated for the model.

The inertia of the beam is included in the model by adding a particle of an appropriate mass to the mass of the machine. The expression for the displacement of the beam due to a concentrated load P applied at $x = 0.6\ L$ is obtained from Appendix D as

$$w(z) = \frac{P}{EI}\begin{cases} 0.84Lz - 0.0946t^3 & z < 0.6L \\ \frac{1}{6}(z - 0.6L) + 0.84Lz^2 - 0.0946z^3 & 0.6L < z \end{cases} \quad \text{(d)}$$

It takes a load of $P = \frac{102.14L^3}{EI}$ to cause a unit deflection at $z = 0.6$L. If x is the deflection where the machine is supported, the beam's kinetic energy is

$$T = \frac{1}{2}\dot{x}^2\left(\frac{102.14EI}{L^3}\right)^2\left\{\int_0^{0.6L} \rho A\left[\frac{1}{EI}(0.84Lz - 0.0946z^3)\right]^2 dz\right.$$

$$\left. + \int_{0.6L}^{L}\left(\frac{1}{EI}\right)^2 \rho A\left[\frac{1}{6}(z - 0.6L)^3 + 0.84Lz^2 - 0.0946z^3\right]^2 dz\right\}$$

$$= \frac{1}{2}(0.418)\rho AL\dot{x}^2 \quad \text{(e)}$$

Thus, the equivalent weight of the beam (noting that the weight per foot of a W14 $\times$ 30 steel beam is 30 lb/ft) is

$$W_{\mathrm{eq}} = 0.418W_b = 0.418(30\ \text{lb/ft})(20\ \text{ft}) = 250.8\ \text{lb} \quad \text{(f)}$$

Thus, the equivalent weight of the machine and the beam is 1250.8 lb. The mass of the machine must be expressed in slugs as

$$m = \frac{W}{g} = \frac{1250\ \text{lb}}{32.2\ \text{ft/s}^2} = 38.8\ \text{slugs} \quad \text{(g)}$$

The system is modeled by a machine of weight 1250.8 lb attached to a spring of stiffness 7.74×10^5 lb/ft as shown in Figure 2.50(b). The differential equation modeling the system is

$$38.8\ddot{x} + 7.74 \times 10^5 x = F(t) \quad \text{(h)}$$

2.10.2 SIMPLIFIED SUSPENSION SYSTEM

A single degree-of-freedom model of a simplified suspension system is shown in Figure 2.51(a).

The "sprung mass," which is the mass of the main vehicle, is modeled as a particle connected to the axle by the suspension system. The suspension system is modeled as a spring in parallel with a viscous damper. The wheel is assumed to be rigid (an assumption to be examined later) and it traverses the road contour. Let m be the mass of the vehicle, k the stiffness of the spring, and c the damping coefficient of the viscous damper. Let $y(\xi)$ be the road contour. If the vehicle travels with a constant horizontal velocity y, then the vehicle travels a distance $\xi = vt$ in time t. Thus, the wheel experiences $y(vt)$.

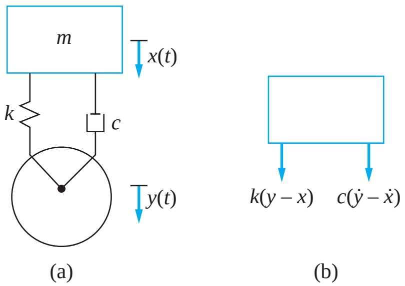


FIGURE 2.51
(a) SDOF model for simplified suspension system. Model ignores the stiffness of the tires and the mass of the axle. (b) FBD of the system at an arbitrary instant.

Applying Newton's law to a free-body diagram of the vehicle drawn at an arbitrary instant in Figure 2.51(b), we have

$$-k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x} \quad \text{(a)}$$

which is rearranged to

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \quad \text{(b)}$$

The model of the suspension system is that of a mass-spring and viscous-damper system subject to motion input.

Parameters for the suspension system may be $m = 300$ kg, $c = 1200$ N·s/m, and $k = 12{,}000$ N/m. Thus, the model for this suspension system is

$$300\ddot{x} + 1200\dot{x} + 12{,}000x = 1200\dot{y} + 12{,}000y \quad \text{(c)}$$

2.14 FURTHER EXAMPLES

The small angle assumption, where appropriate, is made in these problems. Assuming all systems are linear, the generalized coordinate is measured from the system's equilibrium position. Thus, the static forces in the spring cancel with the gravity forces, which cause them, and neither are included on the FBDs.

EXAMPLE 2.30

A mass of 30 kg (shown in Figure 2.52(a)) is hung from a spring of stiffness $k = 2.5 \times 10^5$ N/m, which is attached to an aluminum beam ($E = 71 \times 10^9$ N/m^2, $\rho = 2.7 \times 10^3$ kg/m^3) of moment of inertia $I = 3.5 \times 10^{-8}$ m^4 and of length 35 cm. The beam is supported at its free end and by a circular aluminum cable of diameter 1 mm and length 30 cm.

(a) Determine the equivalent stiffness of the assembly.

(b) Write the differential equation governing in the motion of the mass.

SOLUTION

The stiffness of the beam is

$$k_b = \frac{3EI}{L^3} = \frac{3(71 \times 10^9 \text{ N/m}^2)(3.5 \times 10^{-8} \text{ m}^4)}{(0.35 \text{ m})^3} = 1.74 \times 10^5 \text{ N/m} \quad \text{(a)}$$

The equivalent stiffness of the cable is

$$k_c = \frac{EA}{L} = \frac{(71 \times 10^9 \text{ N/m}^2)\pi(5 \times 10^{-4})^2}{0.30 \text{ m}} = 1.86 \times 10^5 \text{ N/m} \quad \text{(b)}$$

The beam and cable behave as two springs in parallel, because they have the same displacements at their end. The discrete spring is in series with the parallel combination, because

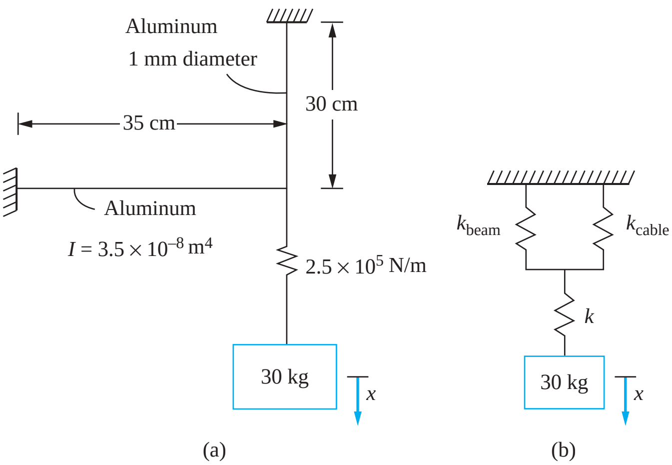


FIGURE 2.52
(a) System of Example 2.30. Mass is suspended from a beam supported by a column. (b) Beam and column are modeled by springs resulting in the equivalent systems model shown.

the displacement of the mass is the sum of the displacement of the spring and the displacement of the end of the beam. The equivalent model is shown in Figure 2.52(b). The equivalent stiffness of the combination is

$$k_{eq} = \frac{1}{\dfrac{1}{k} + \dfrac{1}{k_b + k_c}}$$

$$= \frac{1}{\dfrac{1}{2.5 \times 10^5 \text{ N/m}} + \dfrac{1}{(1.74 \times 10^5 \text{ N/m}) + (1.86 \times 10^5 \text{ N/m})}}$$

$$= 1.48 \times 10^5 \text{ N/m} \tag{c}$$

(b) The differential equation for a SDOF model of the motion of the mass (assuming the beam and the column are massless) is

$$30\ddot{x} + 1.48 \times 10^5 x = 0 \tag{d}$$

EXAMPLE 2.31

A schematic diagram of a compactor is shown in Figure 2.53(a). The compactor is a cylinder of mass 35 kg, radius 0.9 m, and length 1.5 m. To each end of the cylinder, a viscous damper of damping coefficient $c = 1000$ N·m/s is connected to the center, while a spring of stiffness $k = 1.4 \times 10^5$ N/m is connected to a point 0.2 m from the center.

(a) Derive a mathematical model for the unforced motion of the cylinder if it rolls without slipping.

(b) Derive a mathematical model for the unforced motion of the cylinder when it rolls and slips with a coefficient of friction of 0.25.

SOLUTION

(a) The free-body diagram method is used with projections of the diagrams showing the equivalent and effective forces in Figure 2.53(b). When the cylinder rolls without slipping, there is an unknown friction force between the cylinder and the ground. Additionally, a

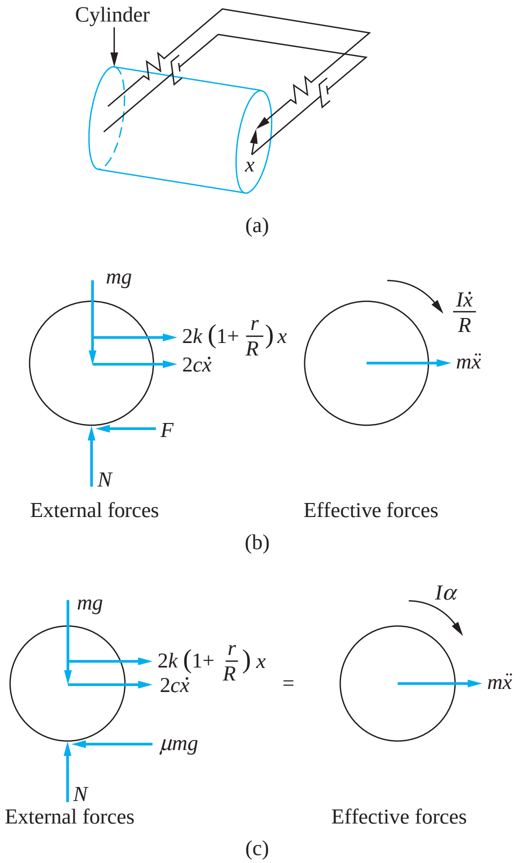


FIGURE 2.53
(a) System of Example 2.31. A compactor is modeled as a cylinder with viscous dampers attached at the center and springs attached at a point above the center. (b) FBDs of the compactor, assuming it rolls without slipping. (c) FBDs of the compactor in the case of slipping.

kinematic relationship exists between the displacement of the mass center and the angular acceleration $\overline{\alpha} = R\alpha$. When the mass center of the disk has moved a distance x from equilibrium, the spring has also changed in length $r\theta$ where r 5 0.2 m and θ is the angular rotation of the disk. Since $x = R\theta$, the change in length of the spring is $\left(1 + \frac{r}{R}\right) x$. Summing moments on these FBDs using $(\sum \mathrm{Mc})_{\mathrm{ext}} = (\sum \mathrm{Mc})_{\mathrm{eff}}$ gives

$$-(2c\dot{x})R - \left[2k\left(1 + \frac{r}{R}\right)x\right](r + R)x = I\left(\frac{\ddot{x}}{R}\right) + (m\ddot{x})R \qquad \text{(a)}$$

$$\left(\frac{1}{R^2} + m\right)\ddot{x} + 2c\dot{x} + 2k\left(1 + \frac{r}{R}\right)^2 x = 0 \qquad \text{(b)}$$

Substituting given values, noting the moment of inertia of a circular cylinder about the axis of rotation is $I = \frac{1}{2}mR^2$, leads to

$$52.5\ddot{x} + 2000\dot{x} + 4.18 \times 10^5 x = 0 \qquad \text{(c)}$$

(b) if the disk rolls and slips, the friction force is equal to the maximum allowable friction force equal to μN, and there is no kinematic relationship between the angular acceleration and the acceleration of the mass center. The appropriate FBDs are shown in Figure 2.53(c). Summing moments about the point contact using the FBDs and $(\Sigma M_C)_{ext} = (\Sigma M_C)_{eff}$, we have

$$-(2c\dot{x})R - \left[2k\left(1 + \frac{r}{R}\right)x\right](r + R)x = I\alpha + (m\ddot{x})R \quad \text{(d)}$$

Summing moments about the center of the disk using these FBDs and $(\Sigma M_G)_{ext} = (\Sigma M_G)_{eff}$, we have

$$-\left[2k\left(1 + \frac{r}{R}\right)x\right]r + \mu mg\, R = I\alpha \quad \text{(e)}$$

Substituting Equation (e) into Equation (d) leads to

$$m\ddot{x} + 2c\dot{x} + 2k\left(1 + \frac{r}{R}\right)R = -\mu mgR \quad \text{(f)}$$

Equation (f) is derived assuming $\dot{x} > 0$. The right-hand side is positive if $\dot{x} < 0$. Upon substitution of given values and taking into account the sign dependence of the right-hand side on $\dot{x}$ Equation (f) becomes

$$35\ddot{x} + 2000\dot{x} + 3.08x10^5 = \begin{cases} -77.25 & \dot{x} > 0 \\ 77.25 & \dot{x} < 0 \end{cases} \quad \text{(g)}$$

EXAMPLE 2.32

Consider the system shown in Figure 2.54(a). A thin rod of mass m is pinned at O at a distance of $\frac{3L}{10}$ from its left end is attached to a viscous damper of damping coefficient c at its left end. Attached to its right end is a cubic block of side d and mass m which is initially half submerged in a liquid of mass density ρ.

(a) Determine the value of d such that the equilibrium position is the horizontal configuration of the bar.

(b) Determine the equation of motion for small oscillations about the horizontal equilibrium position. Use θ as the chosen generalized coordinate.

SOLUTION

When the system is in equilibrium, the moment of the gravity force must balance with the moment of the buoyant force acting on the block. For the horizontal configuration whose free-body diagram is shown in Figure 2.54(b), summing moments about the pin support $\Sigma M_O = 0$, leads to

$$-mg\left(\frac{2L}{10}\right) + F_B\left(\frac{7L}{10}\right) = 0 \quad \text{(a)}$$

The buoyant force is equal to the weight of the fluid displaced by the block. For half of the cube to be submerged,

$$F_B = \rho d^2\left(\frac{d}{2}\right) = \rho\frac{d^3}{2} \quad \text{(b)}$$

Using Equation (b) in Equation (a) leads to

$$\left(\frac{7}{10}\right)\rho\frac{d^3}{2} = \frac{2}{10}mg \Rightarrow d = \left(\frac{4mg}{7\rho}\right)^{\frac{1}{3}} \quad \text{(c)}$$

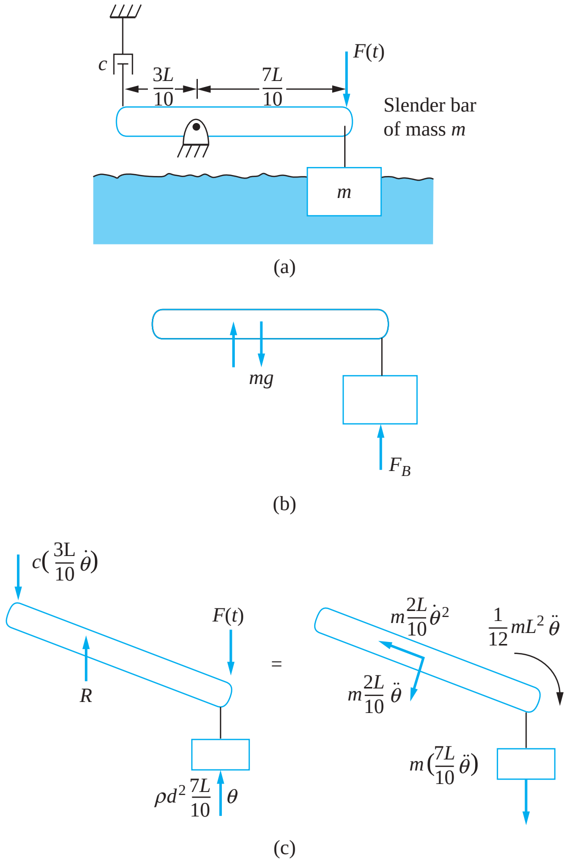


FIGURE 2.54 (a) System of Example 2.32. A cube is at the end of a thin bar and is partially submerged in a liquid when acted on by a time dependent force. (b) FBD of the equilibrium position. (c) FBDs at an arbitrary instant. The gravity force and static buoyancy force cancel with each other when deriving the differential equation.

(b) When the bar has an angular displacement θ from its, equilibrium position, the buoyant force acting on the block (assuming small θ) becomes

$$F_B = \rho d^2\left(\frac{d}{2} + \frac{7}{10}L\theta\right) \tag{d}$$

Summing moments about the point of support using the free-body diagrams of Figure 2.54(c), $(\Sigma M_O)_{\text{ext}} = (\Sigma M_O)_{\text{eff}}$ leads to

$$F(t)\frac{7L}{10} - \frac{3}{10}Lc\dot{\theta}\left(\frac{3}{10}L\right) + \frac{2}{10}mgL - \frac{7}{10}L\left[\rho d^2\left(\frac{d}{2} + \frac{7}{10}L\theta\right)\right]$$
$$= \frac{1}{12}mL^2\ddot{\theta} + \frac{2}{10}mL\ddot{\theta}\left(\frac{2}{10}L\right) + \frac{7}{10}mL\ddot{\theta}\left(\frac{7}{10}L\right) \tag{e}$$

After subtracting the equilibrium condition of Equation (a), Equation (d) becomes

$$\frac{184}{300}mL^2\ddot{\theta} + \frac{9}{100}cL^2\dot{\theta} + \frac{49}{100}\rho d^2L^2\theta = \frac{7L}{10}F(t). \tag{f}$$

$$184\ m\ddot{\theta} + 27c\dot{\theta} + 147\ \rho d^2\theta = \frac{210}{L}F(t) \tag{g}$$

EXAMPLE 2.33

Use the free-body diagram method to derive the differential equation governing the motion of the system shown in Figure 2.55(a). Use θ as the clockwise angular displacement of the bar measured from the system's equilibrium position and as the chosen generalized coordinate. Assume small θ.

SOLUTION

FBDs showing the external forces and the effective forces acting on the bar at an arbitrary instant are shown in Figure 2.55(b). The small angle assumption implies that $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and the springs remain vertical. Thus, a linear differential equation will be derived, and it can be assumed that static spring forces cancel with gravity when deriving the differential equation. Summing moments about the point of support $(\Sigma M_O)_{\text{ext}} = (\Sigma M_O)_{\text{eff}}$ and using the FBDs, we have

$$-c\left(\frac{2L}{3}\dot{\theta}\right)\left(\frac{2L}{3}\right) - k\left(\frac{L}{3}\theta\right)\left(\frac{L}{3}\right) - 2k\left(\frac{L}{3}\theta\right)\left(\frac{L}{3}\right) = \frac{1}{12}mL^2\ddot{\theta} + m\left(\frac{L}{6}\ddot{\theta}\right)\left(\frac{L}{6}\right) \tag{a}$$

which reduces to

$$m\ddot{\theta} + 4c\dot{\theta} + 3k\theta = 0 \tag{b}$$

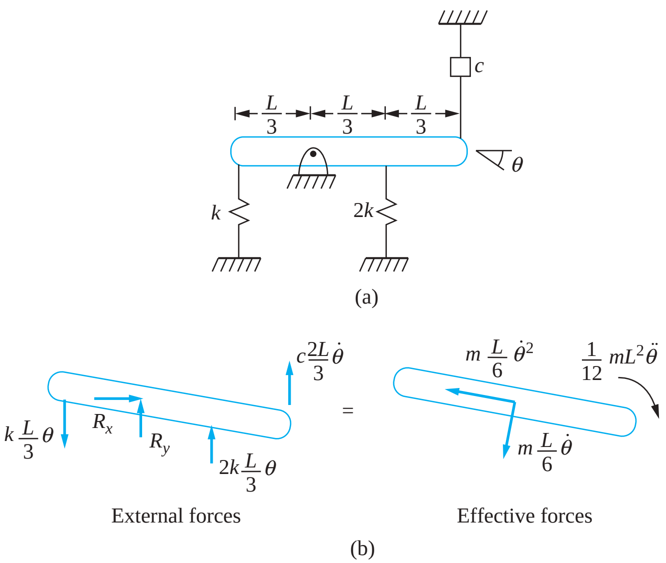


FIGURE 2.55
(a) System of Example 2.33. The small angle assumption is used to linearize the differential equation *a priori*. (b) FBDs of the system at an arbitrary instant.

EXAMPLE 2.34

Derive the differential equation governing the motion of the system of Figure 2.56. The system is in equilibrium when the bar is in the vertical position. Use the equivalent systems method using the angular coordinate θ as the counterclockwise angular displacement of the bar when it is in equilibrium and as the generalized coordinate. Assuming small θ, the disk rolls without slipping, and there is no friction between the cart and the surface.

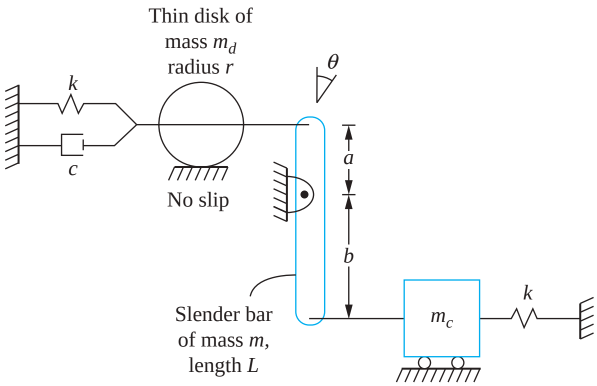


FIGURE 2.56
The thin rod connects the disk that rolls without slipping and the cart which moves on a surface without friction.

SOLUTION

The displacement of the center of the disk is $x = a\theta$, and the displacement of the cart is $y = b\theta$ with both assuming small θ. The appropriate equivalent systems model is the torsional system whose equation is

$$I_{eq}\ddot{\theta} + c_{t,eq}\dot{\theta} + k_{t,eq}\theta = 0 \quad \text{(a)}$$

The equivalent moment of inertia is obtained using kinetic energy. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m_d\dot{x}^2 + \frac{1}{2}I_d\omega^2 + \frac{1}{2}I_b\dot{\theta}^2 + \frac{1}{2}m_c\dot{y}^2 \quad \text{(b)}$$

Noting that, if the disk rolls without slipping, then $\omega = \frac{\dot{x}}{r}$, the moment of inertia of the thin disk is $I_d = \frac{1}{2}m_d r^2$, and the moment of inertia of the slender bar is $I_b = \frac{1}{12}mL^2$. Equation (b) becomes

$$\begin{aligned} T &= \frac{1}{2}m_d\left(a\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{2}m_d r^2\right)\left(\frac{a\dot{\theta}}{r}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m_c(b\dot{\theta})^2 \\ &= \frac{1}{2}\left(\frac{3}{2}m_d a^2 + \frac{1}{12}mL^2 + m_c b^2\right)\dot{\theta}^2 \end{aligned} \quad \text{(c)}$$

Hence, $I_{eq} = \frac{3}{2}m_d a^2 + \frac{1}{12}mL^2 + m_c b^2$.

The potential energy at an arbitrary instant is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 = \frac{1}{2}k(a^2 + b^2)\theta \quad \text{(d)}$$

Thus, $k_{t,eq} = k(a^2 + b^2)$. The work done by the viscous damping force is

$$U = -\int c\dot{x}\,dx = -\int c(a\dot{\theta})d(a\theta) = -\int ca^2\dot{\theta}\,d\theta \quad \text{(e)}$$

The equivalent viscous damping coefficient is $c_{t,\mathrm{eq}} = ca^2$. Hence, the governing differential equation is

$$\left(\frac{3}{2}m_d a^2 + \frac{1}{12}mL^2 + m_c b^2\right)\ddot{\theta} + ca^2\dot{\theta} + k(a^2 + b^2)\theta = 0 \tag{f}$$

EXAMPLE 2.35

The bar of Figure 2.57(a) is attached to a spring and viscous damper which is attached to a cam and follower system. The cam is designed such that it imparts a displacement $y(t)$ to the spring and viscous damper. The bar is designed to impart a linear motion to the cart. Derive the differential equation governing the motion using x as the displacement of the cart and as the generalized coordinate. The motion occurs in the horizontal plane.

SOLUTION

Assume the displacement of the cart is small. The angular rotation of the bar is related to the displacement of the cart by $x = a\theta$. The displacement of the end of the bar where the spring is attached is $y = b\theta = \frac{b}{a}x$. FBDs showing the external and effective force acting on the bar are shown in Figure 2.57(b). Summing moments about the mass center of the bar $(\Sigma M_G)_{\mathrm{ext}} = (\Sigma M_G)_{\mathrm{eff}}$ and using these FBDs leads to

$$k\left(y - \frac{b}{a}x\right)b + c\left(\dot{y} - \frac{b}{a}\dot{x}\right)b - (kx)a = \frac{1}{12}m_2L^2\left(\frac{\ddot{x}}{a}\right)$$
$$+ (m_1\ddot{x})a + m_2\left(\frac{b-a}{2}\right)\frac{\ddot{x}}{a}\left(\frac{b-a}{2}\right) \tag{a}$$

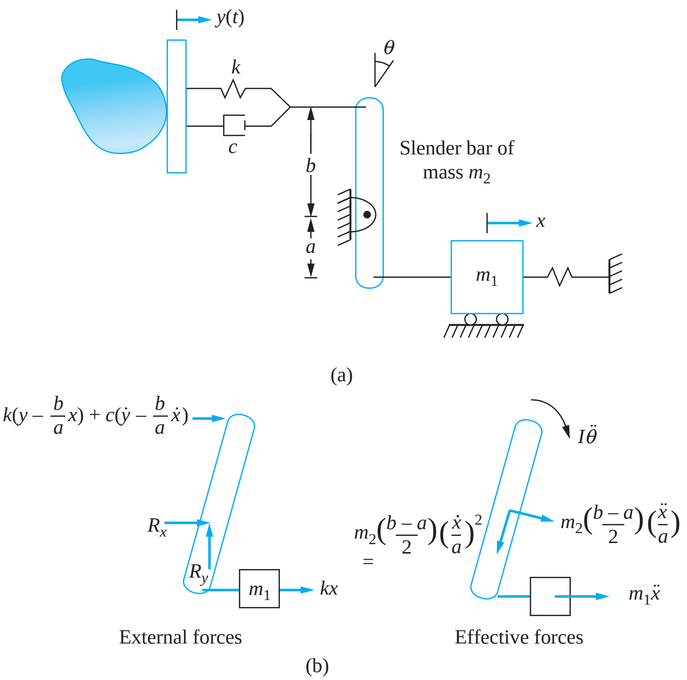


FIGURE 2.57
(a) The end of the bar is connected to a spring and viscous damper which is given motion input, perhaps from a cam and follower mechanism.
(b) FBDs of the bar at an arbitrary instant.

which is rearranged to

$$\left(m_1 a + \frac{m_2 L^2}{12a} + \frac{m_2}{4a}(b-a)^2\right)\ddot{x} + c\frac{b^2}{a}\dot{x} + k\left(a + \frac{b^2}{a}\right)x = c\frac{b^2}{a}\dot{y} + k\frac{b^2}{a}y \quad \text{(b)}$$

2.15 CHAPTER SUMMARY

2.15.1 IMPORTANT CONCEPTS

- A spring is a flexible link between two particles in a mechanical system.
- Structural elements may be used as springs.
- A combination of springs may be replaced by a single spring of equivalent stiffness for purposes of analysis.
- The magnitude of a spring force (drawn at an arbitrary instant on a FBD) is the stiffness of the spring times the change in length of the spring. If one end of the spring is fixed, the change in length of the spring is simply the displacement of the particle to which the spring is attached.
- The direction of a spring force (drawn on a FBD at an arbitrary instant) is consistent with the state of the spring for a positive value of the generalized coordinate. If the spring is stretched, the force is drawn acting away from the body. If the spring is compressed, the force is drawn acting on the body. The direction of the spring force takes care of itself as motion continues.
- Viscous damping is often used in mechanical systems because the addition of viscous damping leads to a linear term in the governing differential equation.
- The force from a viscous damper (drawn on a FDB at an arbitrary instant) is equal to the viscous-damping coefficient times the velocity of the particle to which it is attached and opposite to the direction of positive velocity of the particle.
- The viscous dampers in a system may be replaced (for analysis purposes) by a single viscous damper, such that the work done by the single damper is equivalent to the work done by all viscous dampers.
- All inertia elements in a system may be replaced by a particle (for analysis purposes) such that the kinetic energy of the particle is equal to the kinetic energy of all inertia elements.
- The inertia of a spring may be approximated by adding a particle of one third of the mass of the spring at the location in the system where the spring is attached.
- When a mass is vibrating in a liquid, the motion of the entrained liquid can be approximated by added mass. That is, a particle of an appropriate mass is added to the mass of the vibrating body.
- All external forces acting on a system can be replaced (for analysis purposes) by a single force whose work is equal to the work done by all external forces.
- The free-body diagram method can be used to derive the differential equation of any SDOF. The method consists of drawing FBDs of the system at an arbitrary instant. If the system can be modeled as a particle, the appropriate conservation law is $\Sigma\mathbf{F} = m\mathbf{a}$. If the system can be modeled as a rigid body undergoing planar motion with rotation about a fixed axis through O, the appropriate equations are $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$

and $\Sigma M_O = I_0\alpha$. If the system is composed of more than one body or involves planar motion of a rigid body, the conservation equations are $(\Sigma\mathbf{F})_{\text{ext}} = (\Sigma\mathbf{F})_{\text{eff}}$ and $(\Sigma\mathbf{M}_A)_{\text{ext}} = (\Sigma\mathbf{M}_A)_{\text{eff}}$ where A is any axis.

- For a linear system, if the generalized coordinate is measured from the system's equilibrium position, static forces developed in springs cancel with the gravity forces that cause them when the differential equation governing the motion is derived. Thus, neither are included on a FBD or in formulation of potential energy.
- The small angle assumption can be used to linearize a nonlinear differential equation. It can be applied *a priori* to deriving the differential equation governing the motion of the system.
- The equivalent systems method can be applied to any linear system. A generalized coordinate is selected. An equivalent mass is calculated using the kinetic energy of the system, an equivalent stiffness is calculated using the potential energy of the system, an equivalent viscous-damping coefficient is calculated using the work done by the viscous-damping forces, and an equivalent force is calculated using the work done by external forces. The differential equation governing the motion of is that of a mass-spring and viscous-damper system using the equivalent coefficients.

2.15.2 IMPORTANT EQUATIONS

Force-displacement relation for a linear spring

$$F = kx \tag{2.4}$$

Potential energy developed in a linear spring

$$V = \frac{1}{2}kx^2 \tag{2.6}$$

Stiffness of a helical coil spring

$$k = \frac{GD^4}{64Nr^3} \tag{2.11}$$

Stiffness of longitudinal bar

$$k = \frac{AE}{L} \tag{2.16}$$

Stiffness of a simply supported beam at its midspan

$$k = \frac{48EI}{L^3} \tag{2.18}$$

Stiffness of a cantilever beam at its end

$$k = \frac{3EI}{L^3} \tag{2.21}$$

Torsional stiffness of shaft

$$k_t = \frac{JG}{L} \tag{2.25}$$

Equivalent stiffness of n springs in parallel

$$k_{\text{eq}} = \sum_{i=1}^{n} k_i \tag{2.28}$$

Equivalent stiffness of n springs in series

$$k_{\text{eq}} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{k_i}} \tag{2.31}$$

Determination of equivalent stiffness for arbitrary combination of springs

$$V = \frac{1}{2} k_{eq} x^2 \tag{2.32}$$

Potential energy due to gravity

$$V = mgh \tag{2.34}$$

Force developed in viscous damper

$$F = cv \tag{2.37}$$

Work done by viscous damping forces

$$U_{1\to 2} = -\int_0^{x} c_{\text{eq}} \dot{x}\, dx \tag{2.47}$$

Equivalent mass when linear displacement is used as generalized coordinate

$$T = \frac{1}{2} m_{\text{eq}} \dot{x}^2 \tag{2.50}$$

Equivalent moment of inertia when angular coordinate is used as generalized coordinate

$$T = \frac{1}{2} I_{\text{eq}} \dot{\theta}^2 \tag{2.51}$$

Equivalent mass of a system including approximation of inertia effects in springs

$$m_{\text{eq}} = m + \frac{m_s}{3} \tag{2.57}$$

Work done by external sources

$$U_{1\to 2} = -\int_{t_1}^{t_2} F_{\text{eq}} \dot{x}\, dt \tag{2.64}$$

Small angle assumption

$$\sin\theta \approx \theta \tag{2.71}$$

$$\cos\theta \approx 1 \tag{2.73}$$

$$\tan\theta \approx \theta \tag{2.74}$$

Differential equation governing equivalent mass-spring and viscous-damper system

$$m_{\text{eq}} \ddot{x} + c_{\text{eq}} \dot{x} + k_{\text{eq}} x = F_{\text{eq}}(t) \tag{2.84}$$

Differential equation governing equivalent system when chosen generalized coordinate is an angular coordinate

$$I_{\text{eq}} \ddot{\theta} + c_{t,\,\text{eq}} \dot{\theta} + k_{t,\,\text{eq}} \theta = M_{\text{eq}}(t) \tag{2.85}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 2.1 through 2.15, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

2.1 The differential equation governing the free vibrations of a sliding mass-spring and viscous-damper system (without friction) is the same as the differential equation for a hanging mass-spring and viscous-damper system.

2.2 The differential equation governing the motion of a SDOF linear system is fourth order.

2.3 Springs in series have an equivalent stiffness that is the sum of the individual stiffnesses of these springs.

2.4 The equivalent stiffness of a uniform simply supported beam at its middle is $3EI/L^3$.

2.5 The term representing viscous damping in the governing differential equation for a system is linear.

2.6 When the equivalent systems method is used to derive the differential equation for a system with an angular coordinate used as the generalized coordinate, the kinetic energy is used to derive the equivalent mass of the system.

2.7 The equivalent systems method can be used to derive the differential equation for linear SDOF systems with viscous damping.

2.8 The inertia effects of a simply supported beam can be approximated by placing a particle of mass one-third of the mass of the beam at the midspan of the beam.

2.9 The static deflection of the spring in the system if Figure SP2.9 is mg/k.

2.10 The springs in the system of Figure SP2.10 are in series.

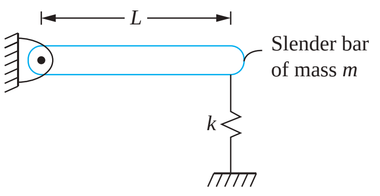


FIGURE SP 2.09

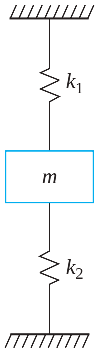


FIGURE SP 2.10

2.11 A shaft can be used as a spring of torsional stiffness JG/L.

2.12 Energy dissipation is used to calculate the equivalent viscous-damping coefficient for a combination of viscous dampers.

2.13 The added mass of a fluid entrained by a vibrating system is determined by calculating the potential energy developed in the fluid.

2.14 If it is desired to calculate the reactions at the support of Figure SP2.14, the effects of the static spring force and gravity cancel and do not need to be included on the FBD or in summing forces on the FBD.

2.15 Gravity cancels with the static spring force, and hence, the potential energy of neither is included in potential energy calculations for the system of Figure SP2.15.

Problems 2.16 through 2.25 require a short answer.

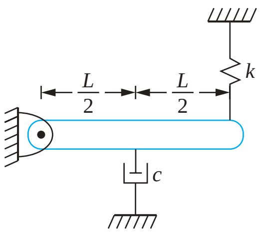


FIGURE SP 2.14

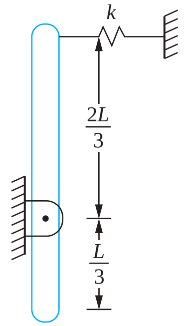


FIGURE SP 2.15

2.16 What is the small angle assumption and how is it used?

2.17 When are the free-body diagrams of a system drawn when they are used to derive the differential equation of a linear SDOF system?

2.18 What is meant by "quadratic forms"?

2.19 The inertia effects of the spring in a mass-spring and viscous-damper system can be approximated by adding a particle of what to the mass?

2.20 What is the same in each spring for a combination of springs in parallel?

2.21 In general, how is the equivalent stiffness of a combination of springs calculated?

2.22 Draw a FBD showing the spring forces applied to the system of Figure SP2.22 at an arbitrary instant. Label the forces in terms of $\dot{\theta}$.

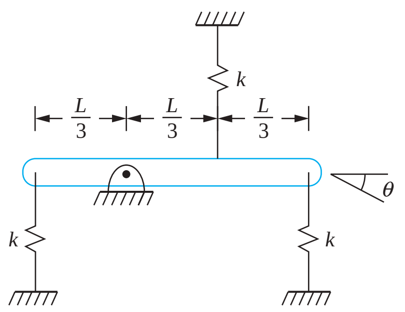


FIGURE SP 2.22

2.23 Draw a FBD showing the forces developed in the viscous dampers acting on the bar of Figure SP2.23 at an arbitrary instant. Label the forces in terms of $\dot{\theta}$.

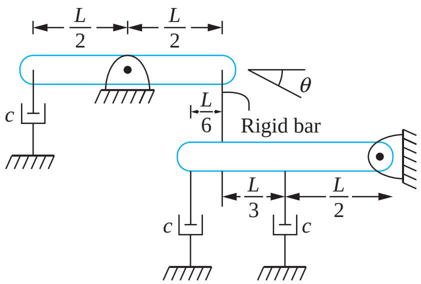


FIGURE SP 2.23

2.24 Describe the equivalent systems method.
2.25 When are static spring forces not drawn on the FBD of external forces?
2.26 Can the equivalent systems method be used to derive the differential equation of a nonlinear SDOF system? Explain.

Problems 2.27 through 2.44 require short calculations.

2.27 What is the equivalent stiffness of springs of individual stiffnesses k_1 and k_2 placed in series?
2.28 What is the equivalent stiffness of the springs in the system of Figure SP2.28?
2.29 What is the equivalent torsional stiffness of the shafts in Figure SP2.29?

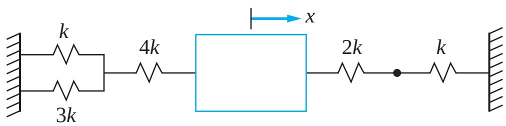


FIGURE SP 2.28

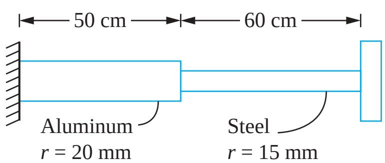


FIGURE SP 2.29

2.30 When a tensile force of 300 N is applied to an elastic element, it has an elongation of 1 mm. What is the stiffness of the element?
2.31 What is the potential energy developed in the elastic element of Short Problem 2.30 when a 300 N tensile force is applied?
2.32 What is the potential energy in the elastic element of Short Problem 2.30 when a 300 N compressive force is applied?
2.33 A spring of torsional stiffness 250 N·m/rad has a rotation of 2° when a moment is applied. Calculate the potential energy developed in the spring.
2.34 What is the torsional stiffness of an annular steel shaft ($G = 80 \times 10^9$ N/m^2) with a length of 2.5 m, inner radius of 10 cm, and outer radius of 15 cm?
2.35 What is the torsional stiffness of a solid aluminum shaft ($G = 40 \times 10^9$ N/m^2) with a length of 1.8 m and a radius of 25 cm?
2.36 What is the longitudinal stiffness of a steel bar ($E = 200 \times 10^9$ N/m^2) with a length of 2.3 m and a rectangular cross section of 5 cm × 6 cm?

2.37 What is the transverse stiffness of a cantilever steel beam ($E = 200 \times 10^9$ N/m^2) with a length of 10μ m and a rectangular cross section with a width of 1 μm and height of 0.5 μm?

2.38 Calculate the static deflection in a linear spring of stiffness 4000 N/m when a mass of 20 kg is hanging from it.

2.39 A spring of unstretched length of 10 cm has a linear density of 2.3 g/cm. The spring is attached between a fixed support and a block of mass of 150 g. What mass should be added to the block to approximate the inertia effects of the spring?

2.40 What is the kinetic energy of the system of Figure SP2.40 at an arbitrary instant in terms of x, which is the downward displacement of the block of mass m_1? Include an approximation of the inertia effects of the springs. The mass of each spring is ms.

2.41 Calculate an equivalent torsional-damping coefficient for the system of Figure SP2.41 when θ, which is the clockwise angular rotation of the bar, is used as the generalized coordinate.

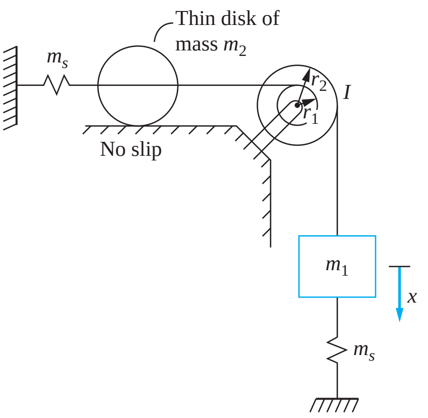


FIGURE SP2.40

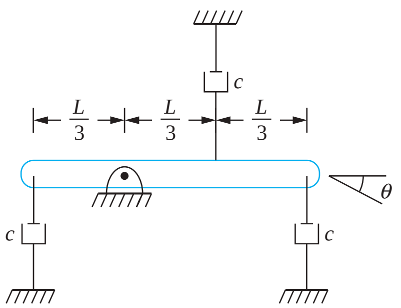


FIGURE SP2.41

2.42 Evaluate without using a calculator. The argument of the trigonometric function is in radians.

(a) sin 0.05 (b) cos 0.05
(c) 1-cos 0.05 (d) tan 0.05
(e) cot 0.05 (f) sec 0.05
(g) csc 0.05

2.43 Evaluate without using a calculator.

(a) sin 3° (b) cos 3°
(c) 1-cos 3° (d) tan 3°

2.44 Calculate the equivalent moment of inertia of the three shafts of Figure SP2.44 when θ_2 is used as the generalized coordinate. Assume the gears mesh perfectly and their moments of inertia are negligible.

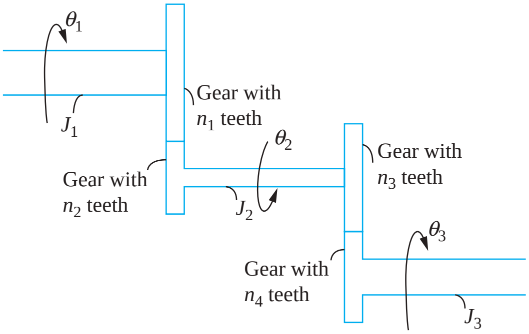


FIGURE SP 2.44

2.45 Match the quantity with the appropriate units

(a) spring stiffness, k		(i)	N·m
(b) torsional stiffness, k_t		(ii)	rad
(c) damping coefficient, c		(iii)	N·m/rad
(e) torsional damping coefficient, c_t		(iv)	N·m/s
(f) potential energy, V		(v)	kg·m^2
(g) power delivered by external force, P		(vi)	N/m
(h) moment of inertia, I		(vii)	N·m·s/rad
(i) angular displacement θ		(viii)	N·s/m

CHAPTER PROBLEMS

2.1–2.8 Determine the equivalent stiffness of a linear spring when a SDOF mass-spring model is used for the systems shown in Figures P2.1 through P2.8 with x being the chosen generalized coordinate.

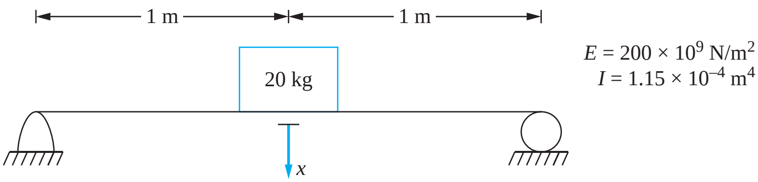


FIGURE P 2.1

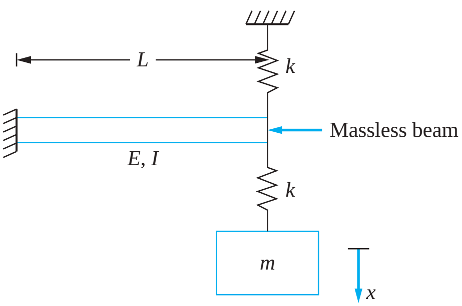


FIGURE P 2.2

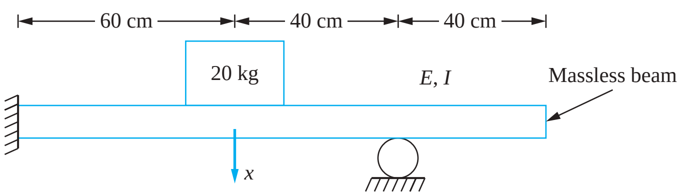


FIGURE P 2.3

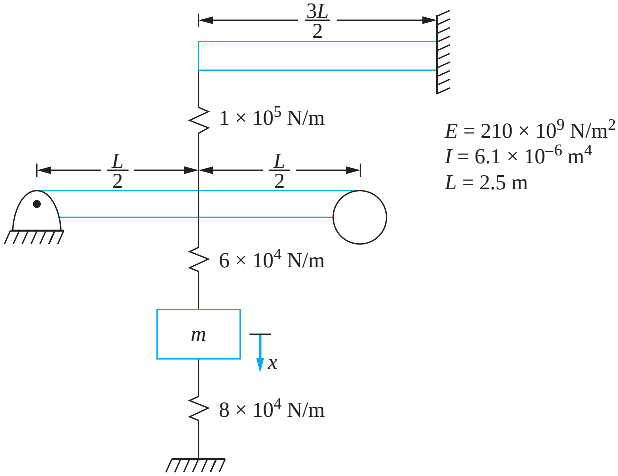


FIGURE P 2.4

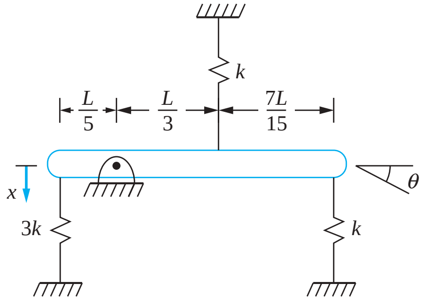


FIGURE P 2.5

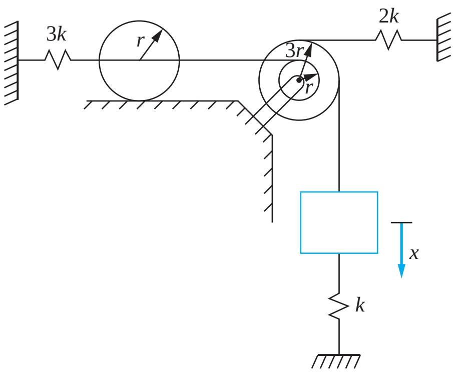


FIGURE P 2.6

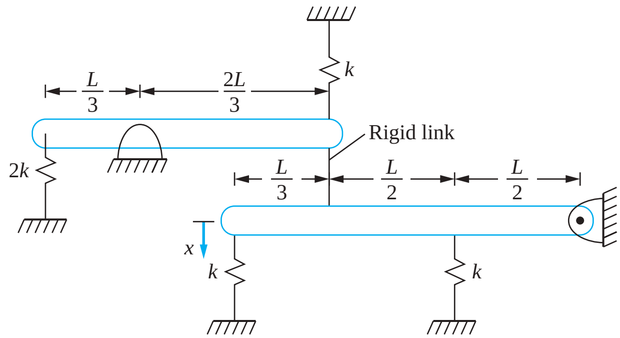


FIGURE P 2.7

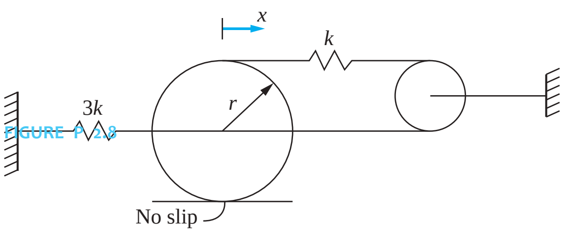


FIGURE P 2.8

2.9 Two helical coil springs are made from a steel ($E = 200 \times 10^9\ \text{N/m}^2$) bar with a radius of 20 mm. One spring has a coil diameter of 7 cm; the other has a coil diameter of 10 cm. The springs have 20 turns each. The spring with the smaller coil diameter is placed inside the spring with the larger coil diameter. What is the equivalent stiffness of the assembly?

2.10 A thin disk attached to the end of an elastic beam has three uncoupled modes of vibration. The longitudinal motion, the transverse motion, and the torsional oscillations are all kinematically independent. Calculate the following for the system of Figure P2.10.

(a) The longitudinal stiffness
(b) The transverse stiffness
(c) The torsional stiffness

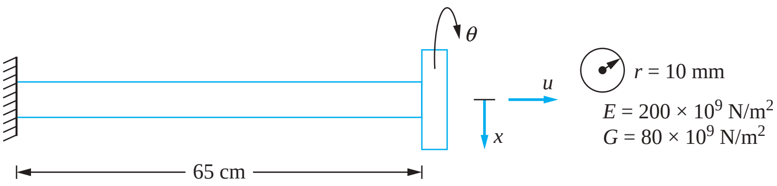


FIGURE P 2.10

2.11 Find the equivalent stiffness of the springs in Figure P2.11 in the x direction.

2.12 A bimetallic strip used as a MEMS sensor is shown in Figure P2.12. The strip, has a length of 20 μm. The width of the strip is 1 μm. It has an upper layer made of steel (E 5 210 3 109 N/m2) and a lower layer made of aluminum (E 5 80 3 109 N/m2) . Each layer is 0.1 μm thick. Determine the equivalent stiffness of the strip in the axial direction.

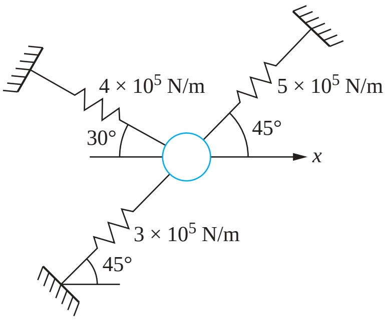


FIGURE P 2.11

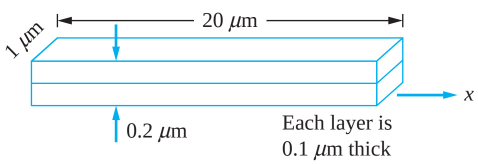


FIGURE P 2.12

2.13 A gas spring consists of a piston of area A moving in a cylinder of gas. As the piston moves, the gas expands and contracts, changing the pressure exerted on the piston. The process occurs adiabatically (without heat transfer), so

$$p = C\rho^{\gamma}$$

where p is the gas pressure, ρ is the gas density, γ is the constant ratio of specific heats, and C is a constant dependent on the initial state. Consider a spring when the initial pressure is p_0 and the initial temperature is T_0. At this pressure, the height of the gas column in the cylinder is h. let $F = \rho_0 A + \delta F$ be the pressure force acting on the piston when it has displaced a distance x into the gas from its initial height.

(a) Determine the relation between δF and x.

(b) Linearize the relationship of part (a) to approximate the air spring by a linear spring. What is the equivalent stiffness of the spring?

(c) What is the required piston area for an air spring ($\gamma = 1.4$) to have a stiffness of 300 N · m for a pressure of 150 kPa (absolute) with $h = 30$ cm.

2.14 A wedge is floating stably on an interface between a liquid of mass density ρ, as shown in Figure P2.14. Let x be the displacement of the wedge's mass center when it is disturbed from equilibrium.

(a) What is the buoyant force acting on the wedge?

(b) What is the work done by the buoyant force as the mass center of the wedge moves from x_1 to x_2?

(c) What is the equivalent stiffness of the spring if the motion of the mass center of the wedge is modeled as a mass attached to a linear spring?

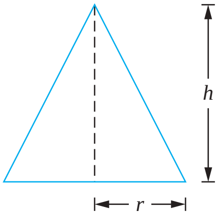


FIGURE P 2.14

2.15 Consider a solid circular shaft of length L and radius c made of an elastoplastic material whose shear stress–shear strain diagram is shown in Figure P2.15(a). If the applied torque is such that the shear stress at the outer radius of the shaft is less than τ_p, a linear relationship between the torque and the angular displacement exists. When the applied torque is large enough to cause plastic behavior, a plastic shell is developed around an elastic core of radius $r < c$, as shown in Figure 2.15(b). Let $T = \frac{\pi\tau_p c^2}{2} + \delta T$ be the applied torque which results in an angular displacement of $\theta = \frac{\tau_p L}{cG} + \delta\theta$

(a) The shear strain at the outer radius of the shaft is related to the angular displacement $\theta = \frac{\gamma_c L}{c}$. The shear strain distribution is linear over a given cross section. Show that this implies

$$\theta = \frac{L\tau_P}{rG}$$

(b) The torque is the resultant moment of the shear stress distribution over the cross section of the shaft,

$$T = \int_0^c 2\pi\tau\rho^2 d\rho$$

Use this to relate the torque to the radius of the elastic core.

(c) Determine the relationship between δT and $\delta\theta$.

(d) Approximate the stiffness of the shaft by a linear torsional spring. What is the equivalent torsional stiffness?

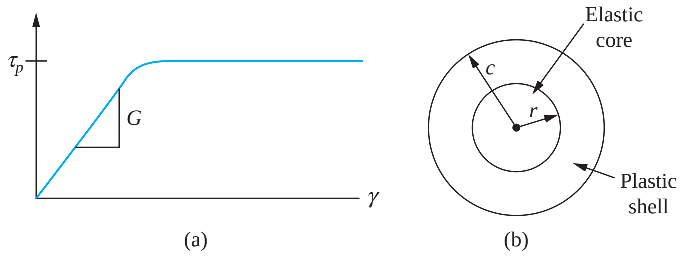


FIGURE P 2.15

2.16 A bar of length L and cross-sectional area A is made of a material whose stress-strain diagram is shown in Figure P2.16. If the internal force developed in the bar is such that $\sigma < \sigma_p$, the bar's stiffness for a SDOF model is $k = \frac{AE}{L}$. Consider the case where $\sigma > \sigma_p$. Let $P = \sigma_p A + \delta P$ be the applied load which results in a deflection of $\Delta = \frac{\sigma_p L}{E} + \delta\Delta$.

(a) The work done by the applied force is equal to the strain energy developed in the bar. The strain energy per unit volume is the area under the stress–strain curve. Use this information to relate δP to $\delta\Delta$.

(b) What is the equivalent stiffness when the bar is approximated as a linear spring for $\sigma > \sigma_p$?

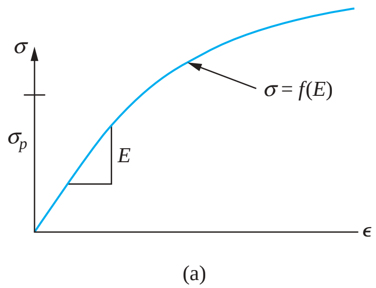


FIGURE P 2.16

2.17 Calculate the static deflection of the spring in the system of Figure P2.17.

2.18 Determine the static deflection of the spring in the system of Figure P2.18.

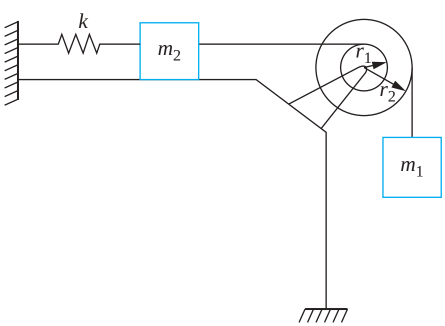


FIGURE P 2.17

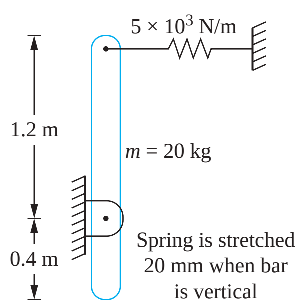


FIGURE P 2.18

2.19 A simplified SDOF model of a vehicle suspension system is shown in Figure P2.19. The mass of the vehicle is 500 kg. The suspension spring has a stiffness of 100,000 N/m. The wheel is modeled as a spring placed in series with the suspension spring. When the vehicle is empty, its static deflection is measured as 5 cm.
(a) Determine the equivalent stiffness of the wheel
(b) Determine the equivalent stiffness of the spring combination.

2.20 The spring of the system in Figure P2.20 is unstretched in the position shown. What is the deflection of the spring when the system is in equilibrium?

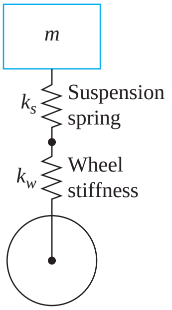


FIGURE P 2.19

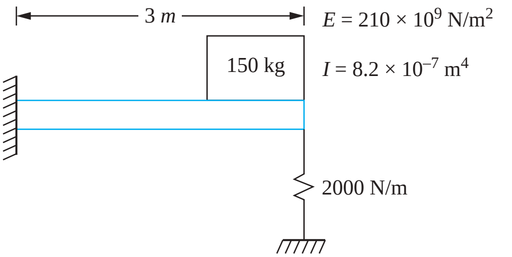


FIGURE P 2.20

2.21 Determine the static deflection of the spring in the system of Figure P2.21.
2.22 Determine the static deflections in each of the springs in the system of Figure P2.22.

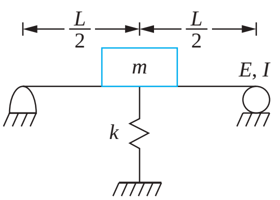


FIGURE P 2.21

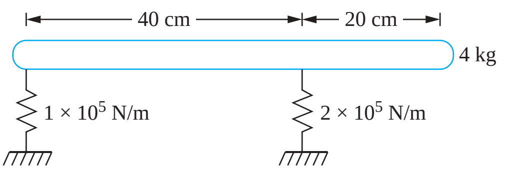


FIGURE P 2.22

2.23 A 30 kg compressor sits on four springs, each of stiffness 1×10^4 N/m. What is the static deflection of each spring.
2.24 The propeller of a ship is a tapered circular cylinder, as shown in Figure P2.24. When installed in the ship, one end of the propeller is constrained from longitudinal motion relative to the ship while a 500-kg propeller mass is attached to its other end.
(a) Determine the equivalent longitudinal stiffness of the shaft for a SDOF model.
(b) Assuming a linear displacement function along the shaft, determine the equivalent mass of the shaft to use in a SDOF model.

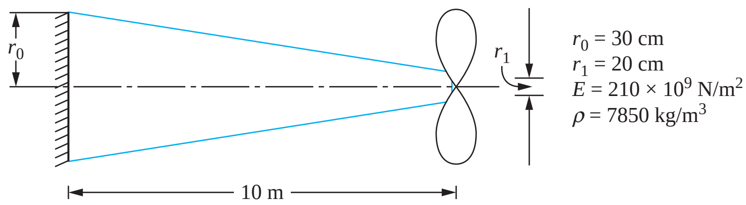


FIGURE P 2.24

2.25 (a) Determine the equivalent torsional stiffness of the propeller shaft of Problem 2.24.
(b) Determine an equivalent moment of inertia of the shaft of Problem 2.24 to be placed on the end of the shaft for a SDOF model of torsional oscillations.

2.26 A tightly wound helical coil spring is made from an 1.88-mm diameter bar made from 0.2 percent hardened steel ($G = 80 \times 10^9$ N/m^2, $\rho = 7600$ kg/m^3). The spring has a coil diameter of 1.6 cm with 80 active coils. Calculate
(a) the stiffness of the spring,
(b) the static deflection when a 100 g particle is hung from the spring, and
(c) the equivalent mass of the spring for a SDOF model.

2.27 One end of a spring of mass m_{s1} and stiffness k_1 is connected to a fixed wall, while the other end is connected to a spring of mass m_{s2} and stiffness k_2. The other end of the second spring is connected to a particle of mass m. Determine the equivalent mass of these two springs.

2.28 A block of mass m is connected to two identical springs in series. Each spring has a mass m and a stiffness k. Determine the equivalent mass of the two springs at the mass.

2.29 Show that the inertia effects of a torsional shaft of polar mass moment of inertia J can be approximated by adding a thin disk of moment of inertia $J/3$ at the end of the shaft.

2.30 Use the static displacement of a simply supported beam to determine the mass of a particle that should be added at the midspan of the beam to approximate inertia effects in the beam.

2.31–35 Determine the equivalent mass or equivalent moment of inertia of the system shown in Figures P2.31 through P2.35 when the indicated generalized coordinate is used.

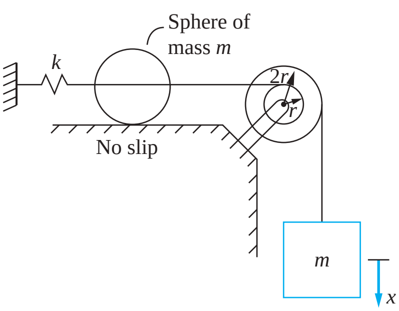


FIGURE P 2.31

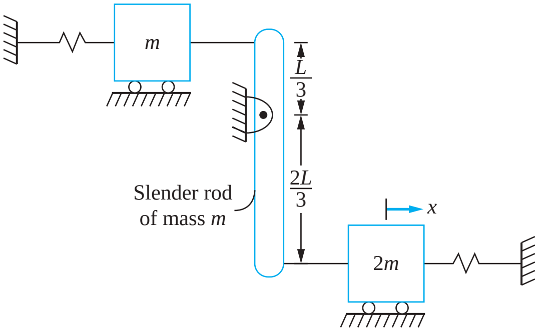


FIGURE P 2.32

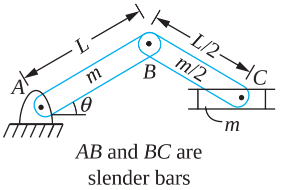


FIGURE P 2.33

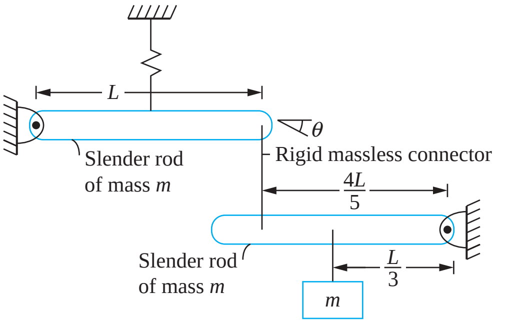


FIGURE P 2.34

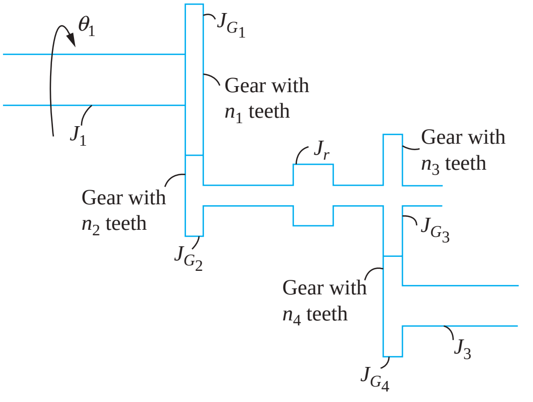


FIGURE P 2.35

2.36 Determine the kinetic energy of the system of Figure P2.36 at an arbitrary instant in terms of $\dot{x}$ including inertia effects of the springs.

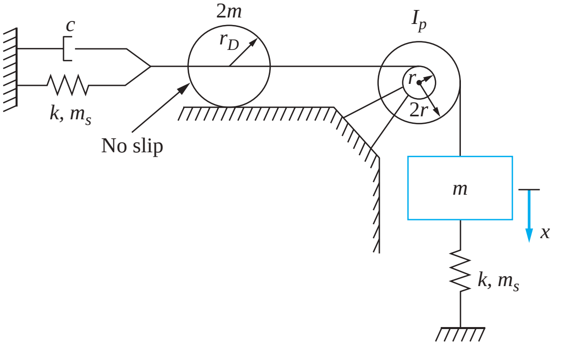


FIGURE P 2.36

2.37 The time-dependent displacement of the block of mass m of Figure P2.36 is $x(t) = 0.03e^{-1.35t} \sin(4t)$ m. Determine the time-dependent force in the viscous damper if $c = 125$ N·s/m.

2.38 Calculate the work done by the viscous damper of Problem 2.37 between $t = 0$ and $t = 1$ s.

2.39 Determine the torsional viscous-damping coefficient for the torsional viscous damper of Figure P2.39. Assume a linear velocity profile between the bottom of the dish and the disk.

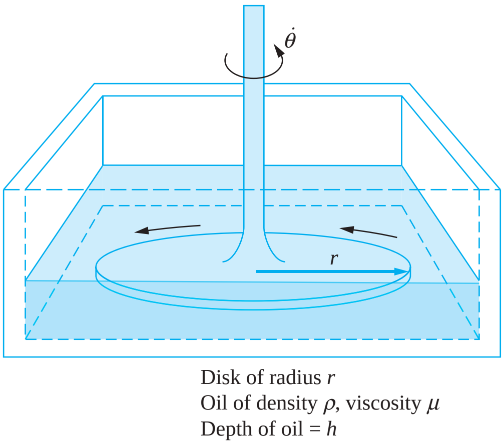


FIGURE P 2.39

2.40 Determine the torsional viscous-damping coefficient for the torsional viscous damper of Figure P2.40. Assume a linear velocity profile in the liquid between the fixed surface and the rotating cone.

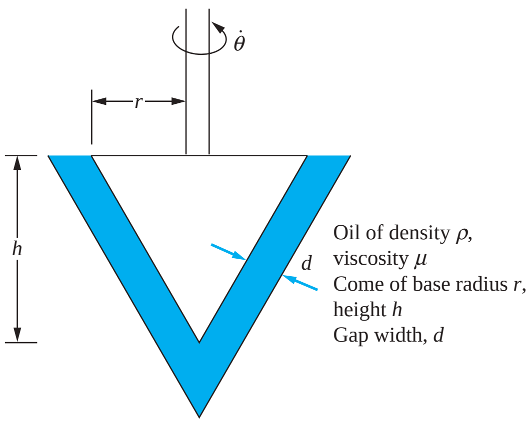


FIGURE P 2.40

2.41 Shock absorbers and many other forms of viscous dampers use a piston moving in a cylinder of viscous liquid as illustrated in Figure P2.41. For this configuration the force developed on the piston is the sum of the viscous forces acting on the side of the piston and the force due to the pressure difference between the top and bottom surfaces of the piston.

(a) Assume the piston moves with a constant velocity v_p. Draw a free-body diagram of the piston and mathematically relate the damping force, the viscous force, and the pressure force.

(b) Assume steady flow between the side of the piston and the side of the cylinder. Show that the equation governing the velocity profile between the piston and the cylinder is $\frac{dp}{dx} = \mu \frac{\partial v^2}{\partial r^2}$

(c) Assume the vertical pressure gradient is constant. Use the preceding results to determine the velocity profile in terms of the damping force and the shear stress on the side of the piston.

(d) Use the results of part (c) to determine the wall shear stress in terms of the damping force.

(e) Note that the flow rate between the piston and the cylinder is equal to the rate at which liquid is displaced by the piston. Use this information to determine the damping force in terms of the velocity and thus the damping coefficient.

(f) Use the results of part (e) to design a shock absorber for a motorcycle that uses SAE 1040 oil and requires a damping coefficient of 1000 N•m/s.

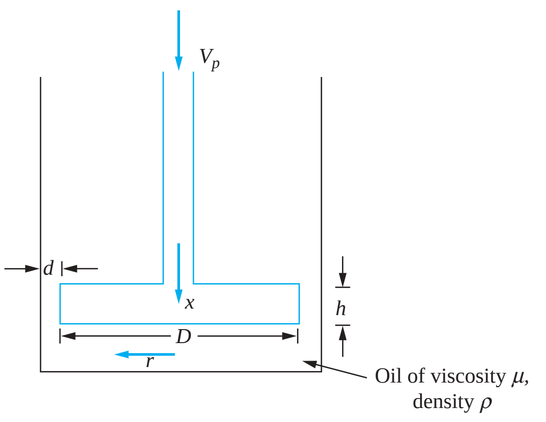


FIGURE P 2.41

2.42–51 Derive the differential equation governing the motion of the one degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinate shown in Figures P2.42 through P2.51. Linearize nonlinear differential equations by assuming small displacements.

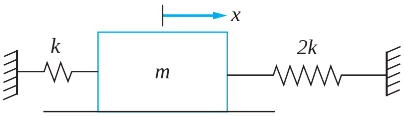


FIGURE P 2.42

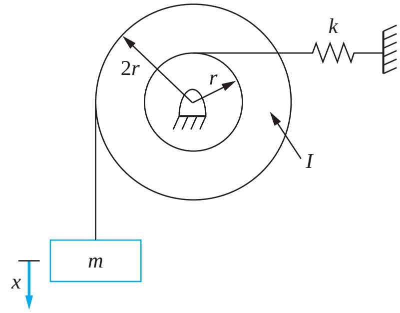


FIGURE P 2.43

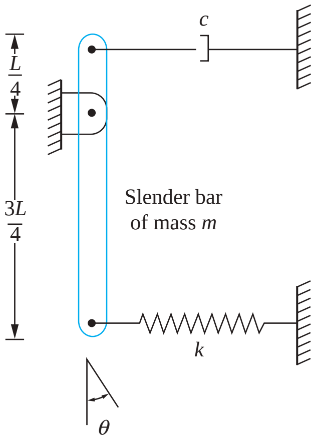


FIGURE P 2.44

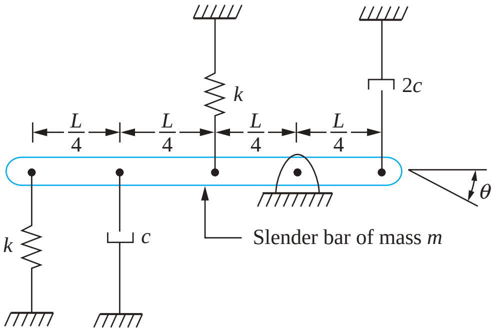
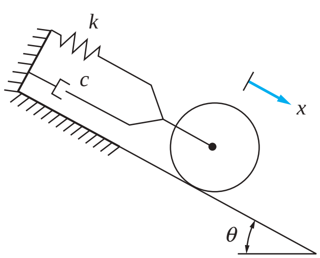
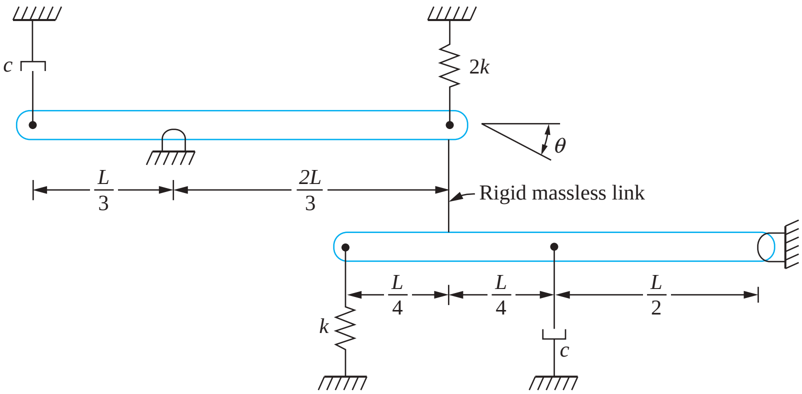


FIGURE P 2.45



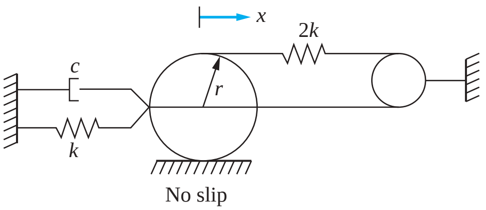
Thin disk of mass m
radius r rolls
without slip

FIGURE P 2.46



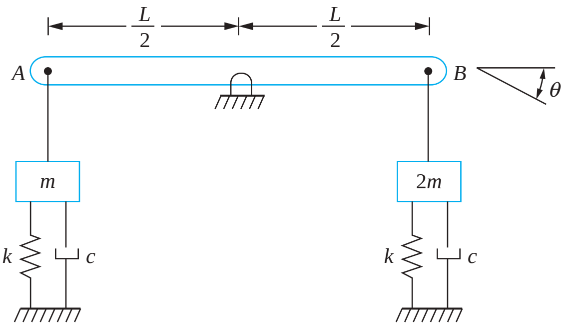
Identical slender bars of mass m, length L

FIGURE P 2.47



Thin disk of mass m, radius r

FIGURE P 2.48



Slender bar of mass m connected to blocks through rigid links at A and B

FIGURE P 2.49

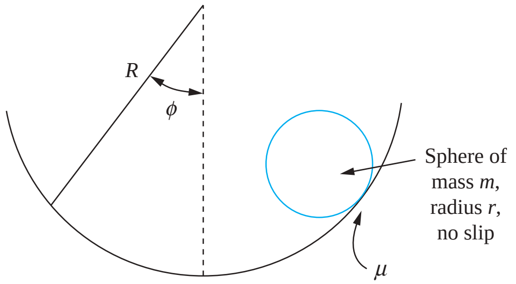


FIGURE P 2.50

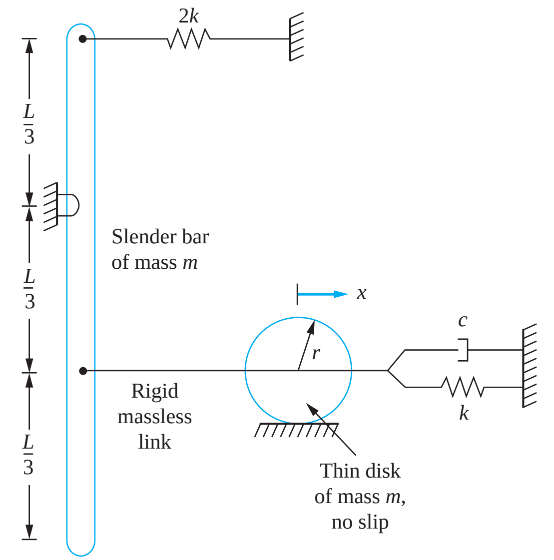


FIGURE P 2.51

2.52–61 Determine the differential equations governing the motion of the system by using the equivalent systems method. Use the generalized coordinates shown in Figures P2.52 through P2.61.

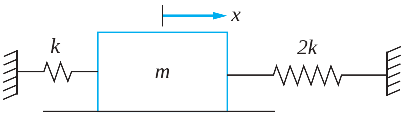


FIGURE P 2.52

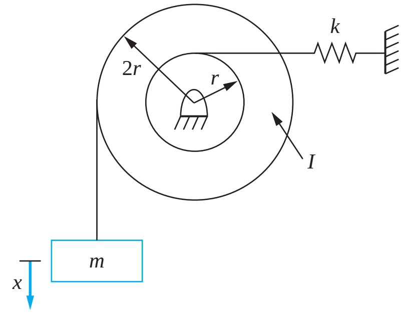


FIGURE P 2.53

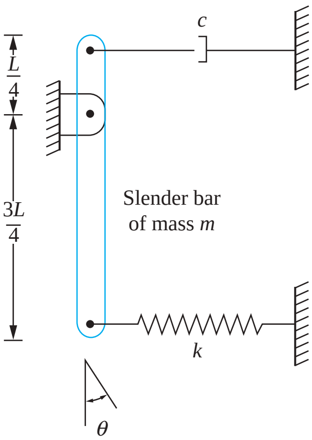


FIGURE P 2.54

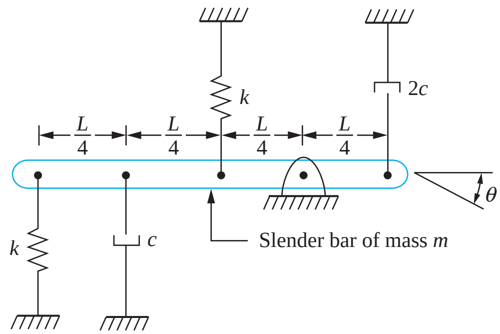


FIGURE P 2.55

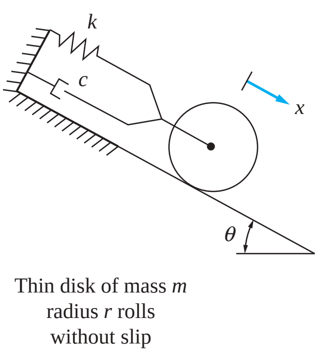


FIGURE P 2.56

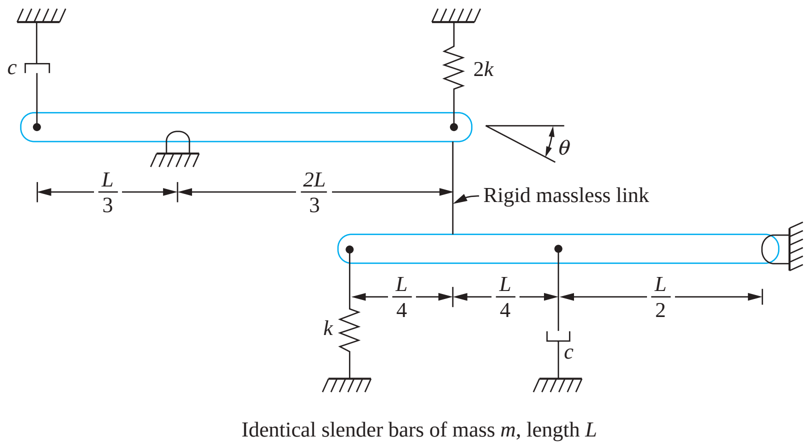


FIGURE P 2.57

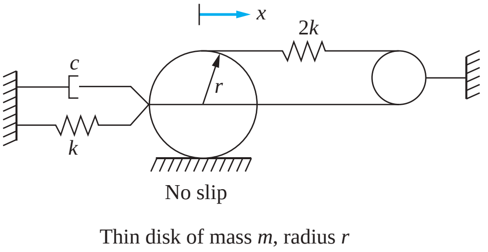


FIGURE P 2.58

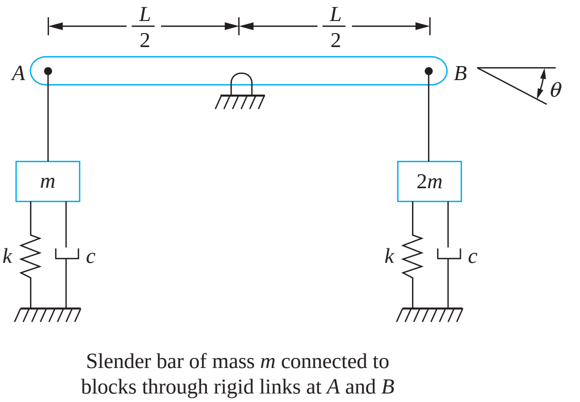


FIGURE P 2.59

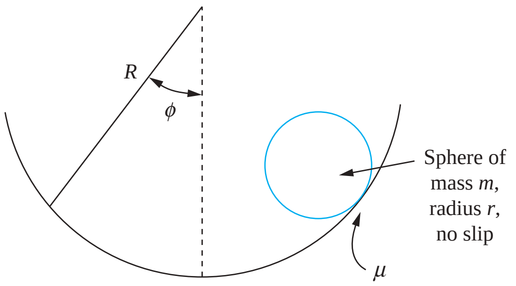


FIGURE P 2.60

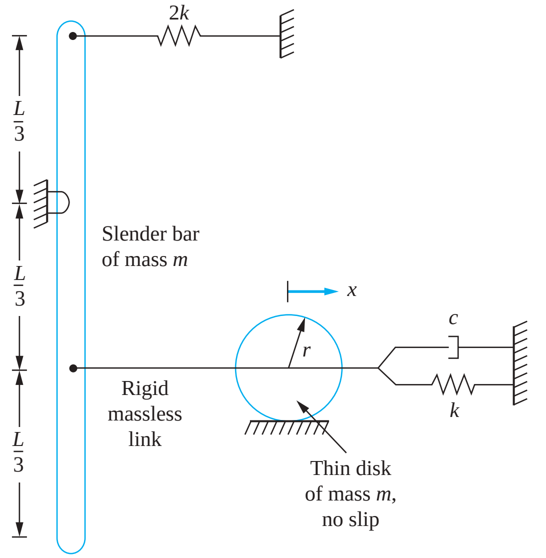


FIGURE P 2.61

Chapter 3

FREE VIBRATIONS OF SDOF SYSTEMS

3.1 INTRODUCTION

Free vibrations are oscillations about a system's equilibrium position that occur in the absence of an external excitation. Free vibrations are a result of a kinetic energy imparted to the system or of a displacement from the equilibrium position that leads to a difference in potential energy from the system's equilibrium position.

Consider the model single degree-of-freedom (SDOF) system of Figure 3.1. When the block is displaced a distance x_0 from its equilibrium position, a potential energy $kx_0^2/2$ is developed in the spring. When the system is released from equilibrium, the spring force draws the block toward the system's equilibrium position, with the potential energy being converted to kinetic energy. When the block reaches its equilibrium position, the kinetic energy reaches a maximum and motion continues. The kinetic energy is converted to potential energy until the spring is compressed a distance x_0. This process of transfer of potential energy to kinetic energy and vice versa is continual in the absence of nonconservative forces. In a physical system, such perpetual motion is impossible. Dry friction, internal friction in the spring, aerodynamic drag, and other nonconservative mechanisms eventually dissipate the energy.

Examples of free vibrations of systems that can be modeled using one degree of freedom include the oscillations of a pendulum about a vertical equilibrium position, the motion of a recoil mechanism of a firearm once it has been fired, and the motion of a vehicle suspension system after the vehicle encounters a pothole.

Free vibrations of a SDOF system are described by a homogeneous second-order ordinary differential equation. The independent variable is time, while the dependent variable is the chosen generalized coordinate. The chosen generalized coordinate represents the

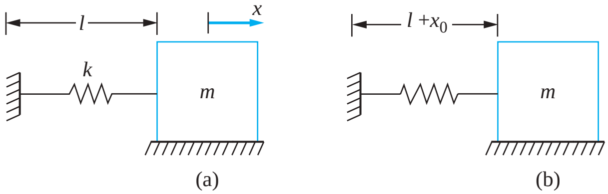


FIGURE 3.1
When the mass is displaced, a distance x_0, a force kx_0, and a potential energy $\frac{1}{2}kx_0^2$ develop in the spring. When released from rest, a cyclic motion occurs. In the absence of any dissipative mechanisms, the system returns to the same position at the end of every cycle.

displacement of a particle in the system or an angular displacement and is measured from the system's equilibrium position.

The differential equation governing free vibrations of a linear system are derived in Chapter 2 and is shown to have the form

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = 0 \tag{3.1}$$

when a linear displacement x is chosen as the generalized coordinate. The second derivative term is due to the inertia forces (effective forces) of the system, the first derivative term is present if there is viscous damping in the system, and the zeroth derivative term is from the elastic forces. If the energy method is used to derive the differential equation, the second derivative term is a result of the system's kinetic energy, the first derivative term is a result of the work done by the viscous friction forces, and the zeroth order derivative term is a result of the system's potential energy.

The general solution of the second-order differential equation is a linear combination of two linearly independent solutions. The arbitrary constants, called *constants of integration*, are uniquely determined upon application of two initial conditions. The necessary initial conditions are values of the generalized coordinate and its first time derivative at a specified time, usually $t = 0$.

The differential equation governing free vibration of a SDOF system is written in a standard form in terms of two parameters. The form of the solution of the differential equation depends upon the parameters. For example, the mathematical form of the solution for an undamped system is simple harmonic motion. The mathematical form of the solution for a damped system varies with a parameter called the *damping ratio*.

The response of a system under other forms of damping also is considered. *Dry sliding friction*, or *Coulomb damping*, leads to two differential equations that govern the motion: one for a positive velocity and another for a negative velocity. This leads to a nonlinear system, but one whose solution is available. The response of a system with *hysteretic damping* (the damping due to energy loss within a material) is characterized by an equivalent viscous-damping coefficient under certain conditions.

3.2 STANDARD FORM OF DIFFERENTIAL EQUATION

The differential equation governing any SDOF system was shown in Chapter 2 to have the form

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F_{eq} \tag{3.2}$$

If the generalized coordinate is an angular coordinate, then

$$I_{eq}\ddot{\theta} + c_{t,eq}\dot{\theta} + k_{t,eq}\theta k_{t,eq}\theta = M_{eq}(t) \tag{3.3}$$

Free vibrations occur in the absence of any forcing and as a result of an initial potential or kinetic energy present in the system at $t = 0$. Thus, for this chapter, $F_{eq} = 0$ or $M_{eq} = 0$. Without loss of generality, assume the generalized coordinate is a linear displacement and the differential equation is written in the form of Equation (3.1).

Dividing Equation (3.1) by m_{eq} leads to

$$\ddot{x} + \frac{c_{eq}}{m_{eq}}\dot{x} + \frac{k_{eq}}{m_{eq}}x = 0 \tag{3.4}$$

Equation (3.4) is written in terms of two parameters, $\frac{c_{eq}}{m_{eq}}$ and $\frac{k_{eq}}{m_{eq}}$, which have an effect on the solution. They are defined as

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} \tag{3.5}$$

which is the *natural frequency* of motion and

$$\zeta = \frac{c_{eq}}{2\sqrt{k_{eq}m_{eq}}} \tag{3.6}$$

which is the *damping ratio*. The reasons for the names of these parameters will become apparent later. The differential equation is written in terms of these parameters as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{3.7}$$

Equation (3.7) is called the standard form of the differential equation for SDOF systems. It is supplemented by two initial conditions:

$$x(0) = x_0 \tag{3.8}$$

and

$$\dot{x}(0) = \dot{x}_0 \tag{3.9}$$

Equation (3.7) is a linear, ordinary homogeneous differential equation with constant coefficients. A solution of Equation (3.7) is assumed to be of the form

$$x(t) = Ae^{\alpha t} \tag{3.10}$$

Substitution of Equation (3.10) into Equation (3.7) leads to

$$\left(\alpha^2 + 2\zeta\omega_n\alpha + \omega_n^2\right)Ae^{\alpha t} = 0 \tag{3.11}$$

The solution is obtained by setting $\alpha^2 + 2\zeta\omega_n\alpha + \omega_n^2 = 0$. Using the quadratic formula to obtain a solution, we have

$$\alpha = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \tag{3.12}$$

or

$$\alpha = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}) \tag{3.13}$$

The form of the solution of this differential equation depends upon the values of α, the roots of the characteristic equation. Defining $i = \sqrt{-1}$, there are four cases.

1. When $\zeta = 0$, the roots are purely imaginary, as $\alpha = \pm i\omega_n$. The free vibrations are *undamped*.
2. When $0 < \zeta < 1$, the roots are complex conjugates, as $\alpha = \omega_n(-\zeta \pm i\sqrt{1-\zeta^2})$. The free vibrations are *underdamped*.
3. When $\zeta = 1$, the characteristic equation has only one real root, $\alpha = -\omega_n$. The free vibrations are *critically damped*.
4. When $\zeta > 1$ the characteristic equation has two real roots $\alpha = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$. The free vibrations are *overdamped*.

The solution varies with ζ. The mathematical form of the solution is different for each case.

3.3 FREE VIBRATIONS OF AN UNDAMPED SYSTEM

When the system is undamped, the roots of the characteristic equation given by Equation (3.12) are purely imaginary, as $\pm\omega_n i$. The general solution is a linear combination of all possible solutions, thus

$$x(t) = B_1 e^{i\omega_n t} + B_2 e^{-i\omega_n t} \tag{3.14}$$

where B_1 and B_2 are constants of integration.

Euler's identity states

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3.15}$$

Application of Euler's identity to Equation (3.14) leads to

$$x(t) = B_1(\cos\omega_n t + i\sin\omega_n t) + B_2(\cos\omega_n t - i\sin\omega_n t) \tag{3.16}$$

or

$$x(t) = C_1\cos\omega_n t + C_2\sin\omega_n t \tag{3.17}$$

where $C_1 = B_1 + B_2$ and $C_2 = i(B_1 - B_2)$ are redefined constants of integration. As defined, C_1 and C_2 are real, while B_1 and B_2 are complex conjugates. Substituting the initial conditions, Equations (3.8) and (3.9), into Equation (3.17) leads to

$$x(t) = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \tag{3.18}$$

An alternate and more instructive form of Equation (3.18) is

$$x(t) = A\sin(\omega_n t + \phi) \tag{3.19}$$

Expanding Equation (3.19) using the trigonometric identity for the sine of the sum of angles

$$\sin(a + b) = \sin a\cos b + \cos a\sin b \tag{3.20}$$

gives

$$x(t) = A\cos\phi\sin\omega_n t + A\sin\phi\cos\omega_n t \tag{3.21}$$

Equating coefficients of like trigonometric terms of Equations (3.18) and (3.21) leads to

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \tag{3.22}$$

and

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{\dot{x}_0}\right) \tag{3.23}$$

Equation (3.19) is an example of the simple harmonic motion discussed in Section 1.6. The amplitude of the motion is A, the frequency is ω_n, its phase is ϕ, and its period is $\frac{2\pi}{\omega_n}$. The parameter ω_n is called the *natural frequency*, because it is the frequency at which the undamped free response occurs naturally.

The undamped motion of a SDOF system is simple harmonic motion. The initial conditions determine the energy initially present in the system. Potential energy is converted to kinetic energy and vice versa without dissipation. Since energy is conserved, the system eventually returns to its initial state with the original potential and kinetic energies, completing one full cycle of motion. The subsequent cycle duplicates the first cycle. The system takes the same amount of time to execute the second cycle as it does the first. Since no energy is dissipated, it executes subsequent cycles in the same amount of time. Thus, the motion is *cyclic* and *periodic*. Figure 3.2 illustrates simple harmonic motion of an undamped SDOF system.

The *amplitude A*, defined by Equation (3.22), is the maximum displacement from equilibrium. The amplitude is a function of the system parameters and the initial conditions. The amplitude is a measure of the energy imparted to the system through the initial conditions. For a linear system

$$A = \sqrt{\frac{2E}{k_{\text{eq}}}} \tag{3.24}$$

where E is the sum of kinetic and potential energies.

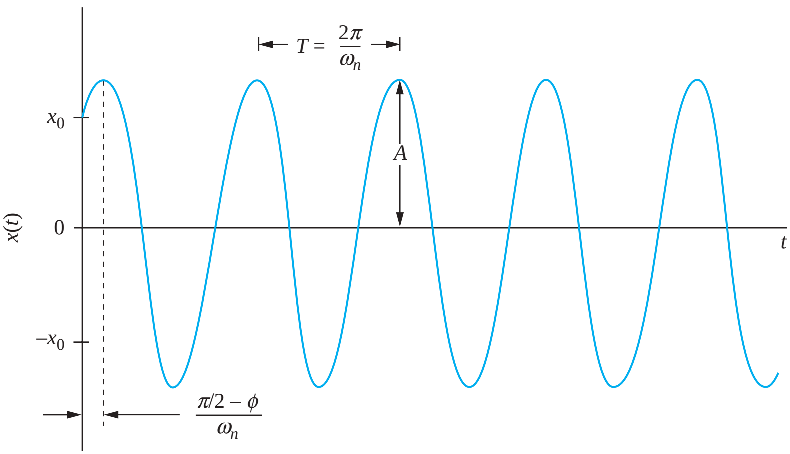


FIGURE 3.2
Illustration of free response of an undamped system. The motion is cyclic and periodic.

The phase angle ϕ, calculated from Equation (3.23) is an indication of the lead or lag between the response and a pure sinusoidal response. The response is purely sinusoidal with $\phi = 0$ if $x_0 = 0$. The response leads a pure sinusoidal response by $\pi/2$ rad if $\dot{x}_0 = 0$. The system takes a time of

$$t = \begin{cases} \dfrac{\pi - \phi}{\omega_n} & \phi > 0 \\ -\dfrac{\phi}{\omega_n} & \phi \leq 0 \end{cases} \tag{3.25}$$

to reach its equilibrium position from its initial position.

EXAMPLE 3.1

An engine of mass 500 kg is mounted on an elastic foundation of equivalent stiffness 7×10^5 N/m. Determine the natural frequency of the system.

SOLUTION

The system is modeled as a hanging mass-spring system. Equation (3.3) with $c_{eq} = 0$ governs the displacement of the engine from its static-equilibrium position. The natural frequency is determined by using Equation (3.5)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7 \times 10^5 \,\text{N/m}}{500\,\text{kg}}} = 37.4\,\text{rad/s} \tag{a}$$

or expressed in Hz.

$$f = \frac{\omega_n}{2\pi} = \frac{37.4\,\text{rad/s}}{2\pi\,\text{rad/cycle}} = 5.96\,\text{Hz} \tag{b}$$

EXAMPLE 3.2

A wheel is mounted on a steel shaft ($G = 83 \times 10^9$ N/m^2) of length 1.5 m and radius 0.80 cm. The wheel is rotated 5° and released. The period of oscillation is observed as 2.3 s. Determine the mass moment of inertia of the wheel.

SOLUTION

The oscillations of the wheel about its equilibrium position are modeled as the torsional oscillations of a disk on a massless shaft, as illustrated in Figure 3.3. The differential equation for such a system is derived in Example 2.17 as

$$I\ddot{\theta} + \frac{JG}{L}\theta = 0 \tag{a}$$

Equation (a) is written in the standard form by dividing by I, giving

$$\ddot{\theta} + \frac{JG}{IL}\theta = 0 \tag{b}$$

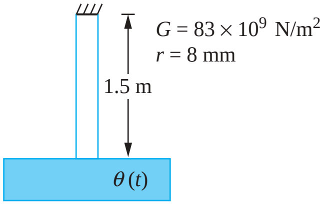


FIGURE 3.3
System of Example 3.2. A wheel is mounted on a shaft, and the period of oscillations is observed, which is used to calculate the moment of inertia of the wheel.

The natural frequency is obtained from Equation (b) as

$$\omega_n = \sqrt{\frac{JG}{IL}} \tag{c}$$

The natural frequency is calculated from the observed period by

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi \text{ rad/cycle}}{2.3 \text{ s/cycle}} = 2.73 \text{ rad/s} \tag{d}$$

The moment of inertia of the wheel is calculated using Equation (c) as

$$I = \frac{JG}{L\omega_n^2} = \frac{\frac{\pi}{2}(0.008 \text{ m})^4(83 \times 10^9 \text{ N/m}^2)}{(1.5 \text{ m})(2.73 \text{ rad/s})^2} = 47.7 \text{ kg} \cdot \text{m}^2 \tag{e}$$

EXAMPLE 3.3

A mass of 5 kg is dropped onto the end of a cantilever beam with a velocity of 0.5 m/s, as shown in Figure 3.4(a). The impact causes vibrations of the mass, which sticks to the beam. The beam is made of steel ($E = 210 \times 10^9$ N/m^2), is 2.1 m long, and has a moment of inertia $I = 3 \times 10^{-6}$ m^4. Neglect inertia of the beam and determine the response of the mass.

SOLUTION

Let $x(t)$ represent the displacement of the mass, which is measured positive downward from the equilibrium position of the mass after it is attached to the beam. As shown in Figure 3.4(b), the system is modeled as a 5 kg mass hanging from a spring of stiffness

$$k_{eq} = \frac{3EI}{L^3} = \frac{3(210 \times 10^9 \text{ N/m}^2)(3 \times 10^{-6} \text{ m}^4)}{(2.1 \text{ m})^3} = 2.04 \times 10^5 \text{ N/m} \tag{a}$$

The natural frequency of free vibration is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{2.04 \times 10^5 \text{ N/m}}{5 \text{ kg}}} = 202.0 \text{ rad/s} \tag{b}$$

The beam is in equilibrium at $t = 0$ when the particle hits. However, x is measured from the equilibrium position of the system with the particle attached. Thus,

$$x(0) = -\Delta_{st} = -\frac{mg}{k_{eq}} = -\frac{(5 \text{ kg})(9.81 \text{ m/s}^2)}{2.04 \times 10^5 \text{ N/m}} = -2.40 \times 10^{-4} \text{ m} \tag{c}$$

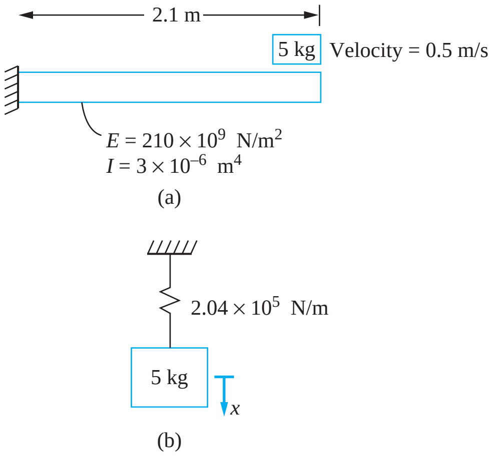


FIGURE 3.4
(a) System of Example 3.3. A mass is dropped onto a fixed-free beam. (b) The system is modeled as a mass hanging from a spring of equivalent stiffness. Since x is measured from the equilibrium position of the system, the initial displacement is the negative of the static deflection of the beam.

The initial velocity is $\dot{x}(0) = 0.5$ m/s. The time history of motion is calculated using Equation (3.19) as

$$x(t) = A \sin(202.0t + \phi) \tag{d}$$

where the amplitude A and the phase ϕ are determined using Equations (3.22) and (3.23), respectively:

$$A = \sqrt{(-2.40 \times 10^{-4}\text{ m})^2 + \left(\frac{0.5\text{ m/s}}{202.2\text{ rad/s}}\right)^2} = 2.48\text{ mm} \tag{e}$$

$$\phi = \tan^{-1}\left[\frac{(202.0\text{ rad/s})(-2.40 \times 10^{-4}\text{ m})}{0.5\text{ m/s}}\right] = -0.0968\text{ rad} = -5.59° \tag{f}$$

EXAMPLE 3.4

An assembly plant uses a hoist to raise and maneuver large objects. The hoist shown in Figure 3.5 is a winch attached to a beam that can move along a track. Determine the natural frequency of the system when the hoist is used to raise a 800-kg machine part at a cable length of 9 m.

SOLUTION

The beam is modeled as a pinned-pinned beam. If the hoist is at its midspan, its stiffness is

$$k_b = \frac{48EI}{L^3} = \frac{48(200 \times 10^9\text{ N/m}^2)(3.5 \times 10^{-4}\text{ m}^4)}{(3.1\text{ m})^3} = 1.13 \times 10^8\text{ N/m} \tag{a}$$

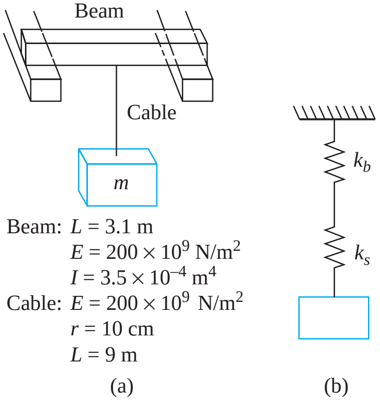


FIGURE 3.5
(a) System of Example 3.4 in which a hoisting mechanism consists of a cable attached to an overhead beam. (b) The system is modeled as a SDOF system with the stiffness of the beam and the stiffness of the cable acting as springs in series.

The stiffness of the cable is

$$k_c = \frac{AE}{L} = \frac{\pi(0.1\text{ m})^2(200 \times 10^9\text{ N/m}^2)}{9\text{ m}} = 6.98 \times 10^8\text{ N/m} \quad \text{(b)}$$

The beam and the cable act as springs in series with an equivalent stiffness of

$$k_{eq} = \frac{1}{\frac{1}{k_b} + \frac{1}{k_c}} = \frac{1}{\frac{1}{1.13 \times 10^8\text{ N/m}} + \frac{1}{6.98 \times 10^8\text{ N/m}}} = 9.71 \times 10^7\text{ N/m} \quad \text{(c)}$$

The system's natural frequency is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{9.71 \times 10^7\text{ N/m}}{800\text{ kg}}} = 3.48 \times 10^2\text{ rad/s} \quad \text{(d)}$$

EXAMPLE 3.5

The pendulum of a cuckoo clock consists of a slender rod on which an aesthetically designed mass slides. If the clock gains time, should the mass be moved closer to or farther away from the support to correct the tuning?

SOLUTION

The pendulum is modeled as a particle of mass m on a rigid, massless rod. The particle is assumed to be a distance l from its axis of rotation. Summing moments about the point of support on the free-body diagrams of Figure 3.6 leads to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad \text{(a)}$$

Application of the small-angle assumption yields the linearized equation of motion

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad \text{(b)}$$

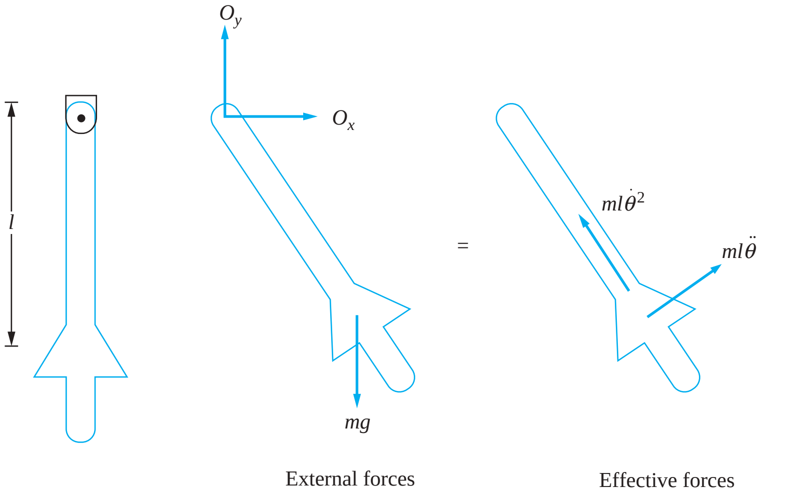


FIGURE 3.6
(a) System of Example 3.5 in which the pendulum of a cuckoo clock is a massless rod with a particle attached. (b) FBDs at an arbitrary instant.

from which the natural frequency is calculated as

$$\omega_n = \sqrt{\frac{g}{l}}$$

The period of oscillation is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Since the clock is running fast, the period of the pendulum needs to be increased. Thus l should be increased and the mass moved farther away from the axis of rotation.

The nonlinear differential equation derived in Example 3.5 is linearized by assuming small θ and replacing sin θ by θ. The exact nonlinear pendulum equation, Equation (a) of Example 3.5, is one of the few nonlinear equations for which an exact solution is known. The solution subject to $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$ is developed in terms of elliptic integrals, which are well-known tabulated functions.

The period of motion of a nonlinear system is dependent upon the initial conditions, while the period of a linear system is independent of initial conditions. One method of assessing the validity of the small-angle approximation for a given amplitude is to compare the period calculated using the exact solution to the period calculated using the linearized differential equations for different initial displacements. This comparison is given in Table 3.1, which shows that the small angle approximation leads to accurate prediction of the period for amplitudes as large as 40°. For an initial angular displacement of 40°, the error in the period from using the small angle approximation is only 3.1 percent.

The success of the use of the small-angle approximation in the pendulum example should give confidence to its use in other problems, where an exact solution is not available.

TABLE 3.1 Ratio of period of simple pendulum, T, calculated from exact nonlinear solution to period calculated from linearized equation as a function of initial angle, θ_o, $\frac{2\pi}{\sqrt{g/l}}$. Nonlinear period is $4K$ where K is the complete elliptic integral of the first kind with a parameter of $\sin(\theta_o/2)$

$\theta_o(°)$	$\frac{T}{2\pi}\sqrt{g/l}$	$\theta_o(°)$	$\frac{T}{2\pi}\sqrt{g/l}$
2	1.00007	48	1.04571
4	1.00032	50	1.04978
6	1.00070	52	1.05405
8	1.00120	54	1.05851
10	1.00191	56	1.06328
12	1.00274	58	1.06806
14	1.00376	60	1.07321
16	1.00490	62	1.07850
18	1.00618	64	1.08404
20	1.00764	66	1.08982
22	1.00930	68	1.09588
24	1.01108	70	1.10211
26	1.01305	72	1.10867
28	1.01515	74	1.11548
30	1.01738	76	1.12255
32	1.01987	78	1.12987
34	1.02248	80	1.13751
36	1.02528	82	1.14540
38	1.02821	84	1.15368
40	1.03132	86	1.16221
42	1.03463	88	1.17112
44	1.03814	90	1.18035
46	1.04183		

3.4 UNDERDAMPED FREE VIBRATIONS

When $0 < \zeta < 1$, the roots of the equation for α are complex conjugates, and the system is said to be underdamped. The general solution of the governing equation is

$$x(t) = B_1 e^{(-\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2})t} + B_2 e^{(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2})t} \tag{3.26}$$

which can be rewritten using Euler's identity as

$$x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\left(\omega_n\sqrt{1-\zeta^2}\right)t + C_2 \sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right] \tag{3.27}$$

The constants of integration are determined by applying the initial conditions, Equation (3.8) and (3.9), resulting in

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \tag{3.28}$$

An alternative form of the solution is developed by using the trigonometric identity, Equation (3.20)

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d) \tag{3.29}$$

where

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \tag{3.30}$$

$$\phi_d = \tan^{-1}\left(\frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}\right) \tag{3.31}$$

and

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{3.32}$$

Equation (3.29) is plotted in Figure 3.7. Once free oscillations of a viscously damped system commence, the nonconservative viscous damping force continually dissipates energy. Since no work is being done on the system, this leads to a continual decrease in the sum of the potential and kinetic energies. For underdamped free vibrations, the system oscillates about an equilibrium position. However, each time it reaches equilibrium, the system's total energy level is less than at the previous time. The maximum displacement on each cycle of motion is continually decreasing. Equation (3.29) and Figure 3.7 show that the amplitude decreases exponentially with time.

The free vibrations of an underdamped system are cyclic but not periodic. Even though the amplitude decreases between cycles, the system takes the same amount of time to execute each cycle. This time is called the *period of free underdamped vibrations* or the *damped period* and is given by

$$T_d = \frac{2\pi}{\omega_d} \tag{3.33}$$

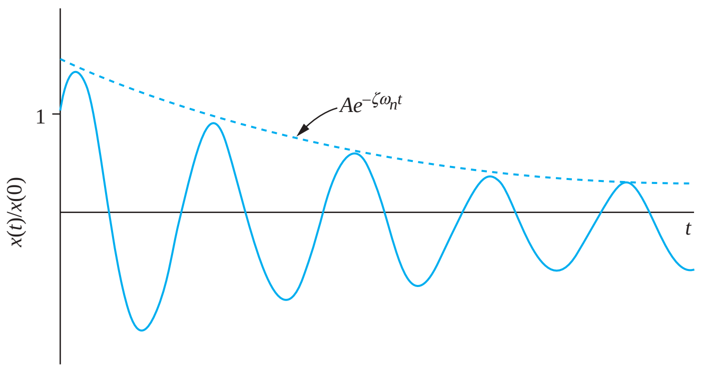


FIGURE 3.7
Free vibrations of an underdamped SDOF system decay exponentially.

Thus, ω_d is called the *damped natural frequency.* Note that $\omega_d < \omega_n$ and $T_d > T$. This is due to the viscous friction which resists the motion of the system and slows it down.

Consider a mass-spring and viscous-damper system with $x(0) = x_0$ and $\dot{x}(0) = 0$. Then

$$\phi_d = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{3.34}$$

Hence, $\sin\phi_d = \sqrt{1-\zeta^2}$, $\cos\phi_d = \zeta$, and

$$A = \frac{x_0}{\sqrt{1-\zeta^2}} \tag{3.35}$$

The total energy present in an underdamped system at time t is

$$\begin{aligned} E &= \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 \\ &= \frac{1}{2}\frac{kx_0^2 e^{-2\zeta\omega_n t}}{(1-\zeta^2)}\Big[(1+\zeta^2)\sin^2(\omega_d t + \phi_d) - 2\zeta\sqrt{1-\zeta^2}\sin(\omega_d t + \phi_d) \\ &\quad \cos(\omega_d t + \phi_d) + (1-\zeta^2)\cos^2(\omega_d t + \phi_d)\Big] \end{aligned} \tag{3.36}$$

The total energy in the system at the end of the nth cycle, $t = \frac{2n\pi}{\omega_d}$, is

$$E_n = E(nT_d) = \frac{1}{2}kx_0^2 e^{-4n\zeta\pi/\sqrt{1-\zeta^2}} \tag{3.37}$$

The energy dissipated as the system executes one cycle of motion is

$$\begin{aligned} \Delta E_n &= E_n - E_{n+1} \\ &= \frac{1}{2}kx_0^2 e^{-4n\zeta\pi/\sqrt{1-\zeta^2}}\left(1 - e^{-4\pi\zeta/\sqrt{1-\zeta^2}}\right) \end{aligned} \tag{3.38}$$

The ratio of the energy dissipated over a cycle compared to the total energy at the beginning of the cycle is

$$\frac{\Delta E_n}{E_n} = 1 - e^{4\pi\zeta/\sqrt{1-\zeta^2}} \tag{3.39}$$

Equations (3.38) and (3.39) show that the energy dissipated per cycle of motion is constant, and thus, it has a constant ratio. The sequence of energies at the beginning of each cycle is a geometric sequence with ratio $1 - e^{-4\pi\zeta/\sqrt{1-\zeta^2}}$. For example, if $\zeta = 0.1$, $\frac{\Delta E_n}{E_n} = 0.717$. The percentage of energy at the end of the nth cycle is $(0.717)^n$ times the initial energy. The larger the damping ratio, the smaller the ratio, and a larger fraction of energy is dissipated per cycle. Since the sequence of energies is a geometric sequence, the energy is never completely dissipated, thus indicating that the free vibrations of an underdamped system continues indefinitely with exponentially decreasing amplitude.

Taking the limit of the energy ratio as the damping ratio approaches one, $\lim_{\zeta\to 1}\frac{\Delta E_n}{E_n} = 1$. All of the energy would be dissipated within the first cycle. This is the origin of the term underdamped; the damping force is not large enough to ever dissipate all of the energy.

The *logarithmic decrement*, δ, is defined for underdamped free vibrations as the natural logarithm of the ratio of the amplitudes of vibration on successive cycles.

$$\delta = \ln\left(\frac{x(t)}{x(t+T_d)}\right) = \ln\left(\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d)}{Ae^{-\zeta\omega_n(t+T_d)}\sin[\omega_d(t+T_d)+\phi_d]}\right)$$

$$= \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{3.40}$$

For small ζ,

$$\delta = 2\pi\zeta \tag{3.41}$$

The logarithmic decrement is often measured by experiment and damping ratio determined from

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \tag{3.42}$$

It can be shown that the following equations can also be used to calculate the logarithmic decrement:

$$\delta = \frac{1}{n}\ln\left(\frac{x(t)}{x(t+nT_d)}\right) \tag{3.43}$$

for any integer n and

$$\delta = \ln\left(\frac{\dot{x}(t)}{\dot{x}(t+T_d)}\right) \tag{3.44}$$

$$\delta = \ln\left(\frac{\ddot{x}(t)}{\ddot{x}(t+T_d)}\right) \tag{3.45}$$

Equation (3.43) implies that the logarithmic decrement can be determined from amplitudes measured on nonsuccessive cycles, while Equations (3.44) and (3.45) imply that velocity and acceleration data can also be used to determine the logarithmic decrement.

The free vibrations of an underdamped system decay exponentially with time. When the initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$, the response of the system is shown in Figure 3.8.

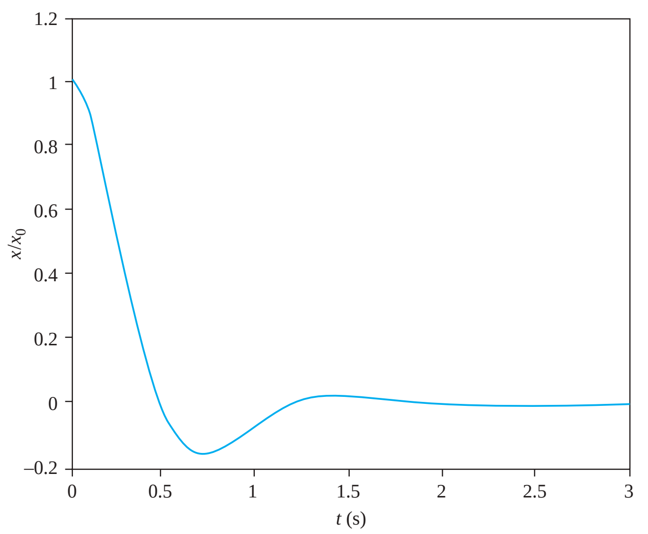


FIGURE 3.8
Underdamped response due to initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. The overshoot is the amplitude at the end of the first half-period.

The absolute value of the displacement after the first half-cycle is called the overshoot. The overshoot is calculated by

$$\eta = -x\left(\frac{T_d}{2}\right) = -\frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} \sin(\pi + \phi_d)$$
$$= x_0 e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{3.46}$$

The percent overshoot is $100\frac{\eta}{x_0} = 100\, e^{-\zeta\pi/\sqrt{1-\zeta^2}}$.

EXAMPLE 3.6

Determine (a) the response of the accelerometer of Example 2.20 if it has an initial velocity of 30 m/s and an initial displacement of 0 m. (b) What is the value of the displacement at $t = 1\ \mu$s?

SOLUTION

(a) The differential equation governing the free response of the accelerometer is

$$4.6 \times 10^{-12}\ddot{x} + 4.93 \times 10^{-7}\dot{x} + 0.380x = 0 \tag{a}$$

Putting the equation in standard form, we have

$$\ddot{x} + 1.07 \times 10^{5}\dot{x} + 8.26 \times 10^{10}x = 0 \tag{b}$$

The natural frequency is

$$\omega_n = \sqrt{8.26 \times 10^{10}} = 2.87 \times 10^{5}\ \text{rad/s} \tag{c}$$

and the damping ratio is determined as

$$\zeta = \frac{1.07 \times 10^{5}}{2(2.87 \times 10^{5})} = 0.186 \tag{d}$$

The system is underdamped and the response for the given initial conditions is

$$x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{e}$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.87 \times 10^{5}\ \text{rad/s}\sqrt{1-(0.187)^2} = 2.82 \times 10^{5}\ \text{rad/s} \tag{f}$$

Thus,

$$x(t) = \frac{30\ \text{m/s}}{2.82 \times 10^{5}\ \text{rad/s}} e^{-0.187(2.87\times 10^{5})t} \sin(2.82 \times 10^{5}t)$$
$$= 1.04 \times 10^{-4} e^{-5.36\times 10^{4}t}(\sin 2.82 \times 10^{5}t)\ \text{m} \tag{g}$$

(b) At $t = 1\ \mu$s,

$$x(10^{-6}\ \text{s}) = 1.04 \times 10^{-4} e^{-5.36\times 10^{4}(10^{-6})} \sin\left[2.82 \times 10^{5}(10^{-6})\right] = 3.07 \times 10^{-5}\ \text{m} \tag{h}$$

EXAMPLE 3.7

The slender bar of Figure 3.9(a) has a mass of 31 kg and a length of 2.6 m. A 50 N force is statically applied to the bar at P then removed. The ensuing oscillations of P are monitored, and the acceleration data is shown in Figure 3.9(b) where the time scale is calibrated but the acceleration scale is not.

(a) Use the data to find the spring stiffness k and the damping coefficient c.
(b) Calibrate the acceleration scale.

SOLUTION

FBDs of the system at an arbitrary instant are shown in Figure 3.9(c). Applying $(\Sigma M_O)_{\text{ext}} = (\Sigma M_O)_{\text{eff}}$ to these FBDs leads to the differential equation of motion:

$$\ddot{x} + \frac{3c}{7m}\dot{x} + \frac{27k}{7m}x = 0 \tag{a}$$

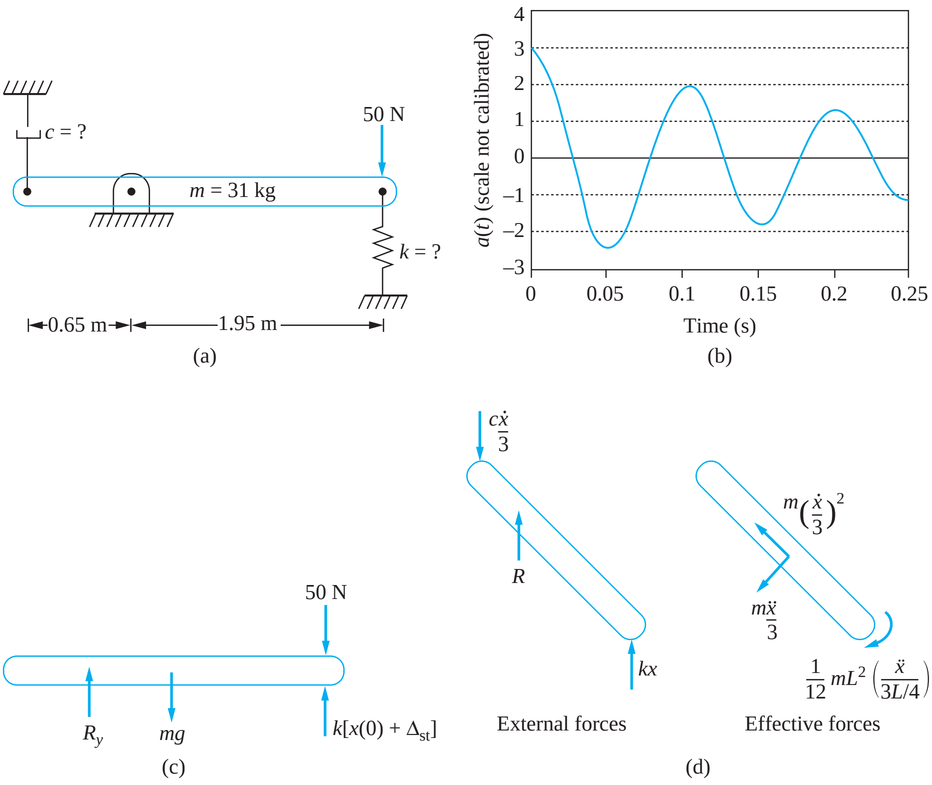


FIGURE 3.9
(a) System of Example 3.7. (b) Accelerometer data for free vibration response. (c) FBD when system is in equilibrium. (d) FBDs of system at an arbitrary instant.

The natural frequency and damping ratio are determined by comparing the preceding equation with the standard form of the differential equation for damped free vibrations as

$$\omega_n = \sqrt{\frac{27k}{7m}} \tag{b}$$

$$2\zeta\omega_n = \frac{3c}{7m} \Rightarrow \zeta = \frac{3c}{14m\omega_n} \tag{c}$$

The period of damped free vibrations is determined from the accelerometer data as 0.1 s. The value of the logarithmic decrement is determined from the accelerometer data and Equation (3.45) as

$$\delta = \ln\left[\frac{\ddot{x}(0)}{\ddot{x}(0.1\text{ s})}\right] = \ln\frac{3}{2} = 0.406 \tag{d}$$

The damping ratio is calculated using Equation (3.42) as

$$\zeta = \frac{0.406}{\sqrt{4\pi^2 + (0.406)^2}} = 0.0644 \tag{e}$$

The damped natural frequency is

$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{0.1\text{ s}} = 62.8\text{ rad/s} \tag{f}$$

from which the natural frequency is calculated as

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{62.8\text{ rad/s}}{\sqrt{1 - (0.0644)^2}} = 63.0\text{ rad/s} \tag{g}$$

(a) The stiffness is calculated from Equation (b) as

$$k = \frac{7m\omega_n^2}{27} = \frac{7(31\text{ kg})(63.0\text{ rad/s})^2}{27} = 3.19 \times 10^4\text{ N/m} \tag{h}$$

and the damping coefficient is calculated from Equation (c) as

$$c = \frac{14m\omega_n\zeta}{3} = \frac{14(31\text{ kg})(63.0\text{ rad/s})(0.0643)}{3} = 585.7\text{ N}\cdot\text{s/m} \tag{i}$$

(b) A static analysis of the equilibrium position in Figure 3.9(c) provides the initial displacement from equilibrium as

$$x(0) = \frac{F}{k} = \frac{50\text{ N}}{3.19 \times 10^4\text{ N/m}} = 1.6\text{ mm} \tag{j}$$

The initial acceleration is calculated using the governing differential equation as

$$\ddot{x}(0) = -2\zeta\omega_n\dot{x}(0) - \omega_n^2 x(0) = -(63.0)^2(0.0016\text{ m}) = -6.22\text{ m/s}^2 \tag{k}$$

The acceleration scale is then calibrated as

$$1 \text{ unit} = \frac{6.22 \text{ m/s}^2}{3} = 2.07 \text{ m/s}^2 \quad \text{(l)}$$

3.5 CRITICALLY DAMPED FREE VIBRATIONS

When $\zeta = 1$, the free vibrations are said to be *critically damped*. In this case, there is only one root of the quadratic equation defining α. The root is $-\omega_n$; thus, one solution of the differential equation is $e^{-\omega_n t}$. The second linearly independent solution is obtained by multiplying the first by t. Thus, the general solution is

$$x(t) = e^{-\omega_n t}(C_1 + C_2 t) \quad (3.47)$$

Application of the initial conditions leads to

$$x(t) = e^{-\omega_n t}[x_0 + (\dot{x}_0 + \omega_n x_0)t] \quad (3.48)$$

The response of a SDOF system subject to critical viscous damping is plotted in Figure 3.10 for different initial conditions. If the initial conditions are of opposite sign or if $\dot{x}_0 = 0$, the motion decays immediately. If both initial conditions have the same sign or if $x_0 = 0$, the absolute value of x initially increases and reaches a maximum of

$$x_{\max} = e^{-\dot{x}_0/(\dot{x}_0+\omega_n x_0)}\left(x_0 + \frac{\dot{x}_0}{\omega_n}\right) \quad (3.49)$$

at

$$t = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} \quad (3.50)$$

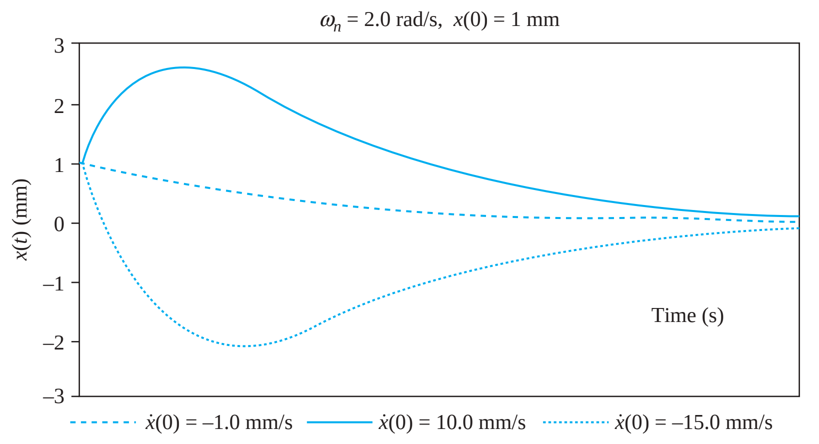


FIGURE 3.10
Free vibration response for a system with critical damping. The damping is just sufficient to dissipate the energy within one cycle. Depending on initial conditions, the response may overshoot the equilibrium position.

If the signs of the initial conditions are opposite and

$$\frac{x_0}{\dot{x}_0 + \omega_n x_0} < 0 \tag{3.51}$$

then the response overshoots the equilibrium position before eventually decaying and approaching equilibrium from the direction opposite that of the initial position. Equation (3.51) is equivalent to specifying that the initial conditions are opposite and the initial kinetic energy is greater than the initial potential energy.

Free vibrations with $\zeta = 1$ are called critically damped because the damping force is just sufficient to dissipate the energy within one cycle of motion. The system never executes a full cycle; it approaches equilibrium with exponentially decaying displacement.

A system with critical damping returns to equilibrium the fastest without oscillation. A system that is overdamped has a larger damping coefficient and offers more resistance to the motion.

EXAMPLE 3.8

The recoil mechanisms of large firearms are designed with critical damping to take advantage of the quickest return to the firing position without oscillation. A 52 kg cannon is to return to within 50 mm of its firing position 0.1 s after maximum recoil. The initial recoil velocity of the cannon is 2.5 m/s. Determine (a) the stiffness of the recoil mechanism, (b) the damping coefficient of the recoil mechanism, and (c) the maximum recoil.

SOLUTION

The maximum recoil of a critically damped system with a initial velocity $v = 2.5$ m/s and an initial displacement of zero is given by Equation (3.49) as

$$x_{\max} = \frac{2.5 \text{ m/s}}{e\omega_n} \tag{a}$$

Take $t = 0$ to occur at the maximum velocity of the mechanism when $\dot{x}(0) = 0$ and $x(0) = \frac{2.5}{e\omega_n}$. The response of the system is given by Equation (3.48) as

$$x(t) = \frac{2.5}{e\omega_n} e^{-\omega_n t}(1 + \omega_n t) \text{ m} \tag{b}$$

Requiring that the mechanism return to within 50 mm of equilibrium 0.1 s after maximum recoil leads to

$$0.050 = \frac{2.5}{e\omega_n} e^{-\omega_n(0.1)}[1 + 0.1\omega_n] \tag{c}$$

An iterative solution is used to solve Equation (c), for $\omega_n = 12.1$ rad/s.

(a) The stiffness of the recoil mechanism is

$$k = m\omega_n^2 = (52 \text{ kg})(12.1 \text{ rad/s})^2 = 7.61 \times 10^3 \text{ N/m} \tag{d}$$

(b) Since the mechanism is critically damped, we have

$$c = 2m\omega_n = 2(52 \text{ kg})(12.1 \text{ rad/s}) = 1.26 \times 10^3 \text{ N} \cdot \text{s/m} \tag{e}$$

(c) The maximum recoil given by Equation (a) is

$$x_{\max} = \frac{2.5 \text{ m/s}}{e\omega_n} = \frac{2.5 \text{ m/s}}{e(12.1 \text{ rad/s})} = 76.0 \text{ mm} \tag{f}$$

3.6 OVERDAMPED FREE VIBRATIONS

When $\zeta > 1$, the characteristic equation has two real roots as $\omega_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$. The general solution of the governing differential equation Equation (3.7) is

$$x(t) = C_1 e^{-\omega_n(\zeta + \sqrt{\zeta^2-1})t} + C_2 e^{-\omega_n(\zeta - \sqrt{\zeta^2-1})t} \tag{3.52}$$

Application of initial conditions from Equations (3.8) and (3.9) to Equation (3.52) leads to

$$x(t) = \frac{e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}} \left\{ \left[\frac{\dot{x}_0}{\omega_n} + x_0(\zeta + \sqrt{\zeta^2 - 1}) \right] e^{\omega_n \sqrt{\zeta^2-1}t} + \left[-\frac{\dot{x}_0}{\omega_n} + x_0(-\zeta + \sqrt{\zeta^2 - 1}) \right] e^{-\omega_n \sqrt{\zeta^2-1}t} \right\} \tag{3.53}$$

Equation (3.53) is plotted in Figure 3.11. The response of an overdamped SDOF system is not periodic. It attains its maximum either at $t = 0$ or at

$$t = -\frac{1}{2\omega_n\sqrt{\zeta^2 - 1}} \ln \left[\frac{\zeta - \sqrt{\zeta^2 - 1}}{\zeta + \sqrt{\zeta^2 - 1}} \, \frac{\dfrac{\dot{x}_0}{\omega_n} + x_0(\zeta + \sqrt{\zeta^2 - 1})}{\dfrac{\dot{x}_0}{\omega_n} + x_0(\zeta - \sqrt{\zeta^2 - 1})} \right] \tag{3.54}$$

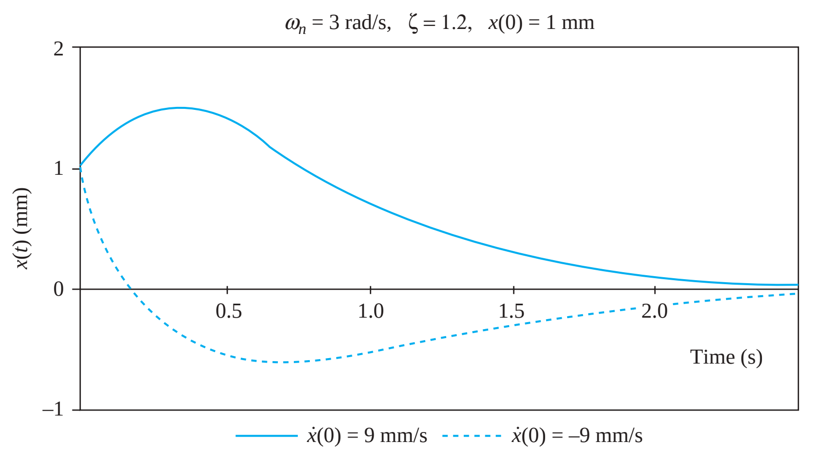


FIGURE 3.11
Free vibration response for a system that is overdamped. The damping force is sufficient to dissipate the energy within a full cycle.

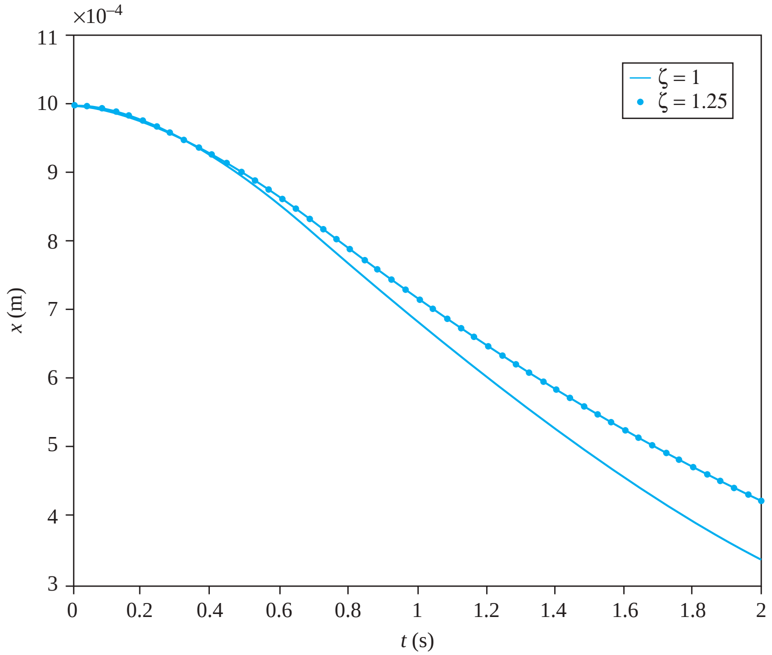


FIGURE 3.12
Comparison between the free response of a critically damped system and an overdamped system.

The response of a system that is overdamped is similar to a critically damped system. An *overdamped* system has more resistance to the motion than critically damped systems. Therefore, it takes longer to reach a maximum than a critically damped system, but the maximum is smaller. An overdamped system also takes longer than a critically damped system to return to equilibrium. Two systems with the same initial conditions are shown in Figure 3.12. One system has a damping ratio of 1 and the other of 1.25. It is obvious that the system that is overdamped is slower.

EXAMPLE 3.9

The restroom door of Figure 3.13 is equipped with a torsional spring and a torsional viscous damper so that it automatically returns to its closed position after being opened. The door has a mass of 60 kg and a centroidal moment of inertia about an axis parallel to the axis of the door's rotation of 7.2 kg · m^2. The torsional spring has a stiffness of 25 N · m/rad.

(a) What is the damping coefficient such that the system is critically damped?
(b) A man with an armload of packages, but in a hurry, kicks the door to cause it to open. What angular velocity must his kick impart to cause the door to open 70°?
(c) How long after his kick will the door return to within 5° of completely closing?
(d) Repeat parts a through c if the door is designed with a damping ratio, $\zeta = 1.3$.

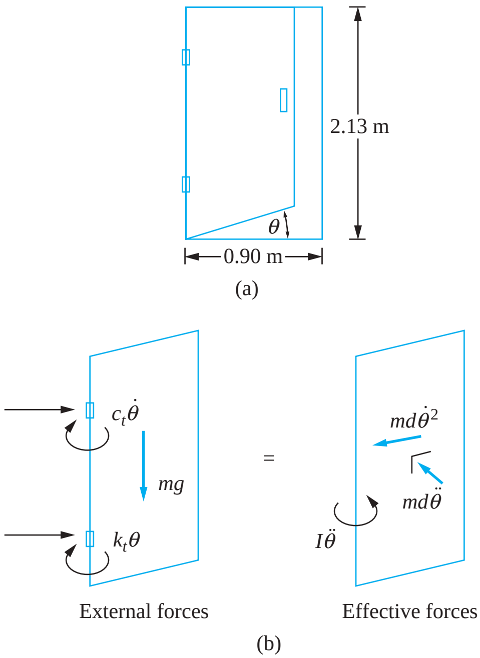


FIGURE 3.13
The restroom door of Example 3.9 is modeled as a SDOF system with a torsional spring and a torsional viscous damper. (b) FBDs at an arbitrary instant.

SOLUTION

The differential equation is derived from the free-body diagrams of Figure 3.13(b),

$$(\bar{I} + md^2)\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \tag{a}$$

Equation (a) is put in the standard form of Equation (3.7) by dividing by $\bar{I} + md^2$. Then it is evident that

$$\omega_n = \sqrt{\frac{k_t}{\bar{I} + md^2}} = \sqrt{\frac{25\ \text{N} \cdot \text{m/rad}}{7.2\ \text{Kg} \cdot \text{m}^2 + (60\ \text{kg})(0.45\ \text{m})^2}} = 1.14\ \text{rad/s} \tag{b}$$

and

$$\zeta = \frac{c_t}{2\omega_n(\bar{I} + md^2)} \tag{c}$$

(a) For critical damping, the damping ratio is 1. Thus,

$$c_t = 2\omega_n(\bar{I} + md^2) = 44.0\ \text{N} \cdot \text{m} \cdot \text{s} \tag{d}$$

(b) If the kick is given when the door is closed, $\theta(0) = 0$, the time the maximum displacement occurs is given by Equation (3.50)

$$t = \frac{1}{\omega_n} = 0.88\ \text{s} \tag{e}$$

and from Equation (3.49) is

$$\theta_{\max} = \frac{\dot{\theta}_0}{e\omega_n} \tag{f}$$

Requiring $\theta_{\max} = 70°$ yields

$$\dot{\theta}_0 = 70°\left(\frac{2\pi \text{ rad}}{360°}\right)(1.14 \text{ rad/s})e = 3.78 \text{ rad/s} \tag{g}$$

(c) Applying Equation (3.48) with $\theta = 5°$ gives

$$5°\left(\frac{2\pi \text{ rad}}{360°}\right) = e^{-(1.14 \text{ rad/s})t}(3.78 \text{ rad/s})t \tag{h}$$

which is solved by trial and error to yield $t = 4.658$ s.
(d) Setting $\zeta = 1.3$ yields

$$c_t = 2\zeta(\bar{I} + md^2)\omega_n = 57.2 \text{ N} \cdot \text{m} \cdot \text{s} \tag{i}$$

From Equation (3.54) the maximum displacement occurs at

$$t = -\frac{1}{2(1.14 \text{ rad/s})\sqrt{(1.3)^2 - 1}} \ln\left(\frac{1.3 - \sqrt{(1.3)^2 - 1}}{1.3 + \sqrt{(1.3)^2 - 1}}\right) = 0.80 \text{ s} \tag{j}$$

Substituting the preceding result in Equation (3.53) and setting $\theta = 70°$ yields

$$70°\left(\frac{2\pi \text{ rad}}{360°}\right) = \left(\frac{\dot{\theta}_0}{1.14 \text{ rad/s}}\right)\frac{1}{2\sqrt{(1.3)^2 - 1}}e^{-1.3(1.14 \text{ rad/s})(0.8 \text{ s})}$$
$$\times\left(\mathrm{e}^{1.14 \text{ rad/s}\sqrt{(1.3)^2-1}(0.8 \text{ s})} - e^{-1.14 \text{ rad/s}\sqrt{(1.3)^2-1}(0.8 \text{ s})}\right) \tag{k}$$

which gives

$$\dot{\theta}_0 = 4.54 \text{ rad/s} \tag{l}$$

Applying Equation (3.53) with $\theta = 5°$ yields

$$5°\left(\frac{2\pi \text{ rad}}{360°}\right) = \left(\frac{e^{-1.14(1.3)t}}{2\sqrt{(1.3)^2 - 1}}\right)\left(\frac{4.54 \text{ rad/s}}{1.14 \text{ rad/s}}\right)$$
$$\times\left(e^{1.14\sqrt{(1.3)^2-1}\,t} - e^{-1.14\sqrt{(1.3)^2-1}\,t}\right) \tag{m}$$

This equation could be solved by trial and error. However, a good approximation is obtained by neglecting the smaller exponential to give $t = 6.2$ s. The neglected term at this time is 0.00081 rad which is only 0.9% of the total angular displacement.

Note that a harder kick is required to open the door when the system is overdamped than when the system is critically damped even though the time required to open the door is approximately the same. This reflects the increase in the viscous resistance moment. The response of the critically damped system against the response of an overdamped system with $\zeta = 1.3$ is plotted in Figure 3.14.

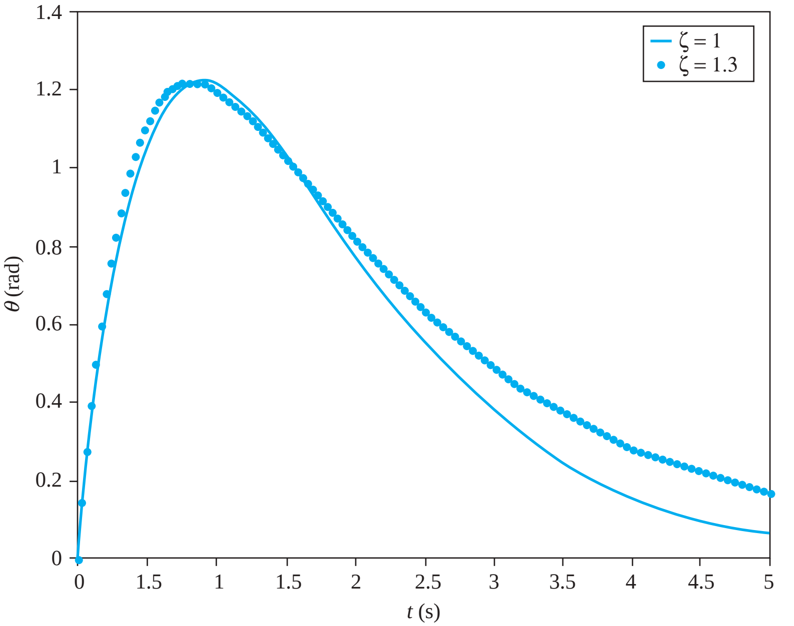


FIGURE 3.14
MATLAB plot of responses of the system of Example 3.8 for a critically damped system and an overdamped system.

3.7 COULOMB DAMPING

Coulomb damping is the damping that occurs due to dry friction when two surfaces slide against one another. Coulomb damping can be the result of a mass sliding on a dry surface, axle friction in a journal bearing, belt friction, or rolling resistance. The case of a mass sliding on a dry surface is analyzed here, but the qualitative results apply to all forms of Coulomb damping.

As the mass of Figure 3.15 (a) slides on a dry surface, a friction force that resists the motion develops between the mass and the surface. Coulomb's law states that the friction force is proportional to the normal force developed between the mass and the surface. The constant of proportionality μ, is called the *kinetic coefficient of friction.* Since the friction force always resists the motion, its direction depends on the sign of the velocity.

Application of Newton's law to the free-body diagrams of Figure 3.15(b) and (c) yields the following differential equations:

$$m\ddot{x} + kx = \begin{cases} -\mu mg & \dot{x} > 0 \\ \mu mg & \dot{x} < 0 \end{cases} \tag{3.55}$$

Equations (3.55) are generalized by using a single equation

$$m\ddot{x} + kx = -\mu mg \frac{|\dot{x}|}{\dot{x}} \tag{3.56}$$

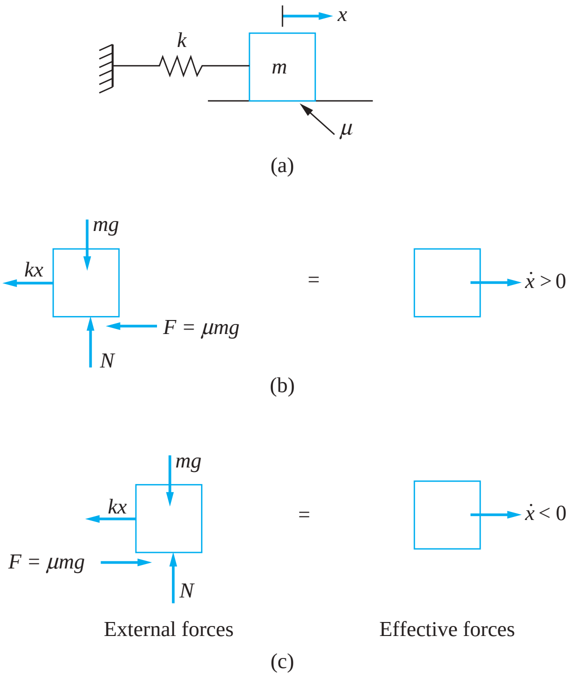


FIGURE 3.15
(a) A mass slides on a surface with a coefficient of friction μ. (b) FBDs at an arbitrary instant for $\dot{x} > 0$. (c) FBDs at an arbitrary instant for $\dot{x} < 0$.

The right-hand side of Equation (3.56) is a nonlinear function of the generalized coordinate. Thus the free vibrations of a one-degree-of-freedom system with Coulomb damping are governed by a nonlinear differential equation. However, an analytical solution exists and is obtained by solving Equation (3.55).

Without loss of generality, assume that free vibrations of the system of Figure 3.15 are initiated by displacing the mass a distance δ to the right, from equilibrium, and releasing it from rest. The spring force draws the mass toward equilibrium; thus the velocity is initially negative. Equation (3.55) applies over the first half-cycle of motion, until the velocity again becomes zero.

The solution of Equation (3.55) subject to $x(0) = \delta$ and $\dot{x}(0) = 0$ with μmg on the right-hand side is

$$x(t) = \left(\delta - \frac{\mu mg}{k}\right)\cos\omega_n t + \frac{\mu mg}{k} \tag{3.57}$$

Equation (3.57) describes the motion until the velocity changes sign at $t = \pi/\omega_n$ when

$$x\left(\frac{\pi}{\omega_n}\right) = -\delta + \frac{2\mu mg}{k} \tag{3.58}$$

Equation (3.55) with $-\mu mg$ on the right-hand side governs the motion until the velocity next changes sign. The solution of Equation (3.55) using Equation (3.58) and $\dot{x}\left(\frac{\pi}{\omega_n}\right) = 0$ as initial conditions is

$$x(t) = \left(\delta - \frac{3\mu mg}{k}\right)\cos\omega_n t - \frac{\mu mg}{k} \qquad \frac{\pi}{\omega_n} \le t \le \frac{2\pi}{\omega_n} \tag{3.59}$$

The velocity again changes sign at $t = 2\pi/\omega_n$ when

$$x\left(\frac{2\pi}{\omega_n}\right) = \delta - \frac{4\mu mg}{k} \tag{3.60}$$

The motion during the first complete cycle is described by Equations (3.57) and (3.59). The amplitude change between the beginning and the end of the cycle is

$$x(0) - x\left(\frac{2\pi}{\omega_n}\right) = \frac{4\mu mg}{k} \tag{3.61}$$

The motion is cyclic. The analysis of the subsequent and each successive cycle continues in the same fashion. The initial conditions used to solve for the displacement during a half-cycle are that the velocity is zero and the displacement is the displacement calculated at the end of the previous half-cycle.

The period of each cycle is

$$T = \frac{2\pi}{\omega_n} \tag{3.62}$$

Thus Coulomb damping has no effect on the natural frequency.

Mathematical induction is used to develop the following expressions for the displacement of the mass during each half-cycle:

$$x(t) = \left[\delta - (4n-3)\frac{\mu mg}{k}\right]\cos\omega_n t + \frac{\mu mg}{k}$$

$$2(n-1)\frac{\pi}{\omega_n} \le t \le 2\left(n - \frac{1}{2}\right)\frac{\pi}{\omega_n} \tag{3.63}$$

$$x(t) = \left[\delta - (4n-1)\frac{\mu mg}{k}\right]\cos\omega_n t - \frac{\mu mg}{k}$$

$$2\left(n - \frac{1}{2}\right)\frac{\pi}{\omega_n} \le t \le 2n\frac{\pi}{\omega_n} \tag{3.64}$$

$$x\left(2n\frac{\pi}{\omega_n}\right) = \delta - \left(\frac{4\mu mg}{k}\right)n \tag{3.65}$$

Equation (3.65) shows that the displacement at the end of each cycle is $4\mu mg/k$ less than the displacement at the end of the previous cycle. Thus the amplitude of free vibration decays linearly as shown, when Equations (3.63) and (3.64) are plotted in Figure 3.16.

The amplitudes on successive cycles form an arithmetic sequence. If x_n is the amplitude at the end of the nth cycle then

$$x_n - x_{n-1} = \frac{4\mu mg}{k} \tag{3.66}$$

with $x_0 = \delta$. The solution of this difference equation is Equation (3.65).

The motion continues with this constant decrease in amplitude as long as the restoring force is sufficient to overcome the resisting friction force. However, since the friction

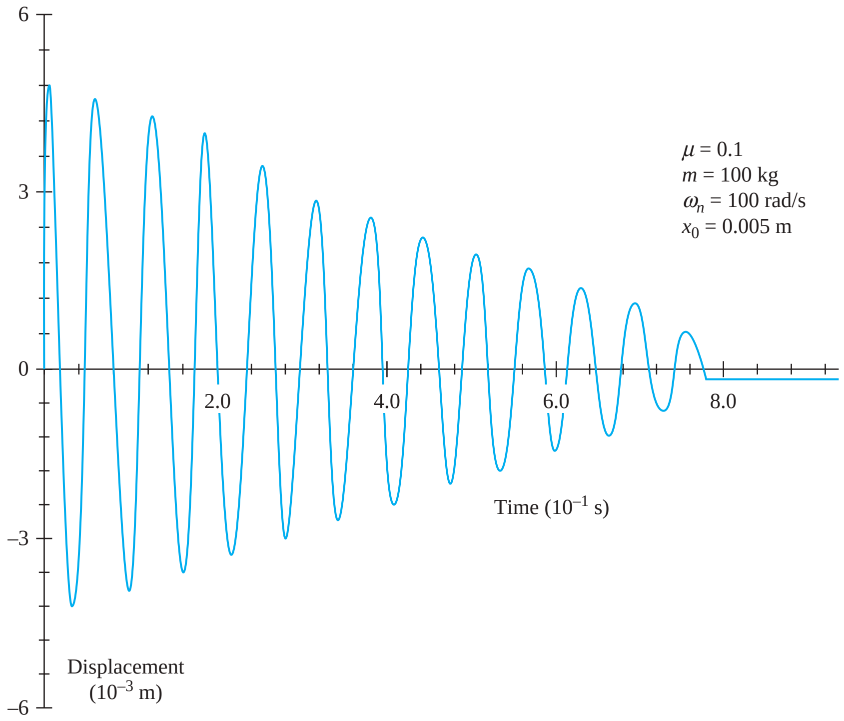


FIGURE 3.16
Free response of a system with Coulomb damping. The motion is cyclic with a linear decay of amplitude. The period is the same as the natural period with motion ceasing with a permanent displacement.

causes a decrease in amplitude, the restoring force eventually becomes less than the friction force. This occurs when

$$k\left|x\left(2n\frac{\pi}{\omega_n}\right)\right| \leq \mu mg \tag{3.67}$$

Motion ceases during the nth cycle, where n is the smallest integer such that

$$n > \frac{k\delta}{4\mu mg} - \frac{1}{4} \tag{3.68}$$

When motion ceases a constant displacement from equilibrium of $\mu mg/k$ is maintained.

The effect of Coulomb damping differs from the effect of viscous damping in these respects:

1. Viscous damping causes a linear term proportional to the velocity in the governing differential equation, while Coulomb damping gives rise to a nonlinear term.
2. The natural frequency of an undamped system is unchanged when Coulomb damping is added, but is decreased when viscous damping is added.
3. Motion is not cyclic if the viscous damping coefficient is large enough, whereas the motion is always cyclic when Coulomb damping is the only source of damping.
4. The amplitude decreases linearly because of Coulomb damping and exponentially because of viscous damping.

5. Coulomb damping leads to a cessation of motion with a resulting permanent displacement from equilibrium, while motion of a system with only viscous damping continues indefinitely with a decaying amplitude.

Since the motion of all physical systems ceases in the absence of continuing external excitation, Coulomb damping is always present. Coulomb damping appears in many forms, such as axle friction in journal bearings and friction due to belts in contact with pulleys or flywheels. The response of systems to these and other forms of Coulomb damping can be obtained in the same manner as the response for dry sliding friction.

The general form of the differential equation governing the free vibrations of a linear system where Coulomb damping is the only source of damping is

$$\ddot{x} + \omega_n^2 x = \begin{cases} \dfrac{F_f}{m_{\text{eq}}} & \dot{x} < 0 \\ -\dfrac{F_f}{m_{\text{eq}}} & \dot{x} > 0 \end{cases} \tag{3.69}$$

where F_f is the magnitude of the Coulomb damping force. The decrease in amplitude per cycle of motion is

$$\Delta A = \frac{4F_f}{m_{\text{eq}}\omega_n^2} \tag{3.70}$$

EXAMPLE 3.10

An experiment is run to determine the kinetic coefficient of friction between a block and a surface. The block is attached to a spring and displaced 150 mm from equilibrium. It is observed that the period of motion is 0.5 s and that the amplitude decreases by 10 mm on successive cycles. Determine the coefficient of friction and how many cycles of motion the block executes before motion ceases.

SOLUTION

The natural frequency is calculated as

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.5 \text{ s}} = 12.57 \text{ rad/s} \tag{a}$$

The decrease in amplitude is expressed as

$$\Delta A = \frac{4\mu mg}{k} = \frac{4\mu g}{\omega_n^2} \tag{b}$$

which is rearranged to yield

$$\mu = \frac{\Delta A}{4g}\omega_n^2 = \frac{(0.01 \text{ m})(12.57 \text{ rad/s})^2}{4(9.81 \text{ m/s}^2)} = 0.04 \tag{c}$$

From Equation (3.68) the motion ceases during the 15th cycle. The mass has a permanent displacement of 2.5 mm from its original equilibrium position.

EXAMPLE 3.11

A father builds a swing for his children. The swing consists of a board attached to two ropes, as shown in Figure 3.17. The swing is mounted on a tree branch, with the board 3.5 m below the branch. The diameter of the branch is 8.2 cm and the kinetic coefficient of friction between the ropes and the branch is 0.1. After the swing is installed and his child is seated, the father pulls the swing back 10° and releases. What is the decrease in angle of each swing and how many swings will the child receive before Dad needs to give another push?

SOLUTION

Because of the friction between the tree branch and the ropes, the tension on opposite sides of a rope will be different. These tensions can be related using the principles of belt friction. When the swing is swinging clockwise,

$$T_2 = T_1 e^{\mu\beta} \tag{a}$$

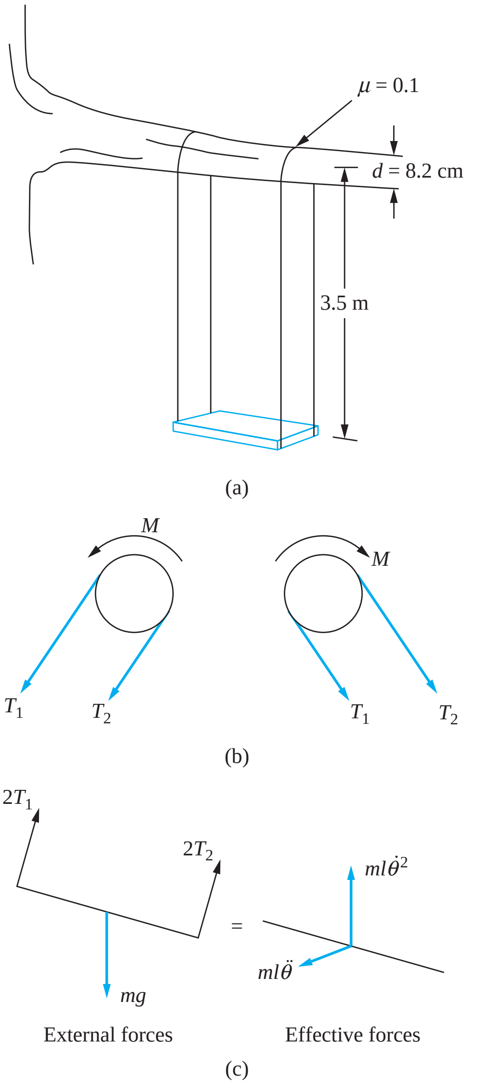


FIGURE 3.17
(a) Tree swing of Example 3.11. (b) The tension developed in opposite sides of a rope are unequal due to friction. (c) FBDs of swing at an arbitrary instant.

where β is the angle of contact between the tree branch and the rope. As the child swings the angle of contact may vary. However, this complication is too much to handle with a simplified analysis. A good approximation is to assume β is constant and $\beta = \pi$ rad. When the swing is swinging counterclockwise

$$T_1 = T_2 e^{\mu\beta} \tag{b}$$

Let θ be the clockwise angular displacement of the swing from equilibrium. Summing forces in the direction of the tensions gives $\sum F_{\text{ext}} = \sum F_{\text{eff}}$

$$2T_1 + 2T_2 - mg\cos\theta = ml\dot{\theta}^2 \tag{c}$$

The swing is pulled back only 10°. Thus the usual small-angle approximation is valid, with $\cos\theta \approx 1$ and the nonlinear inertia term ignored in comparison to the tensions and gravity. The belt friction relations and the normal force equation are solved simultaneously to yield

$$\dot{\theta} > 0, \qquad T_1 = \frac{mg}{2(1 + e^{\mu\pi})} \qquad T_2 = \frac{mge^{\mu\pi}}{2(1 + e^{\mu\pi})} \tag{d}$$

$$\dot{\theta} > 0, \qquad T_1 = \frac{mge^{\mu\pi}}{2(1 + e^{\mu\pi})} \qquad T_2 = \frac{mg}{2(1 + e^{\mu\pi})} \tag{e}$$

Summing moments about the center of the tree branch, using the free-body diagrams of Figure 3.17(c) and the small-angle assumption yields

$$\left(\sum M_O\right)_{\text{ext}} = \left(\sum M_O\right)_{\text{eff}}$$
$$(2T_1 - 2T_2)\frac{d}{2} - mgl\theta = ml^2\ddot{\theta} \tag{f}$$

Substituting for the tensions into the preceding equation and rearranging leads to

$$\ddot{\theta} + \frac{g}{l}\theta = \begin{cases} \dfrac{gd}{2l^2}\dfrac{1 - e^{\mu\pi}}{1 + e^{\mu\pi}} & \dot{\theta} > 0 \\ -\dfrac{gd}{2l^2}\dfrac{1 - e^{\mu\pi}}{1 + e^{\mu\pi}} & \dot{\theta} < 0 \end{cases} \tag{g}$$

The frequency of the swinging is

$$\omega_n = \sqrt{\frac{g}{l}} = 1.67 \text{ rad/s} \tag{h}$$

which is the same as it would be in the absence of friction.

The governing differential equation is of the form of Equation (3.69). Thus, from Equation (3.70), the decrease in amplitude per swing is

$$\frac{2d}{l}\frac{e^{\mu\pi}-1}{e^{\mu\pi}+1} = 2\left(\frac{0.082\text{ m}}{3.5\text{ m}}\right)\frac{e^{0.1\pi}-1}{e^{0.1\pi}+1} = 0.0073\text{ rad} = 0.42°$$

Motion ceases when, at the end of a cycle, the moment of the gravity force about the center of the branch is insufficient to overcome the frictional moment. This occurs when

$$mgl\theta < |T_2 - T_1|d$$

or

$$\theta < \frac{d}{2l}\frac{e^{\mu\pi}-1}{e^{\mu\pi}+1} = 0.10°$$

Thus, if Dad does not give the swing another push after 23 swings, the swing will come to rest with an angle of response of 0.1°.

3.8 HYSTERETIC DAMPING

The stress–strain diagram for a typical linearly elastic material is shown in Figure 3.18. Ideally, if the material is stressed below its yield point and then unloaded, the stress-strain curve for the unloading follows the same curve for the loading. However, in a real engineering material, internal planes slide relative to one another and molecular bonds are broken, causing conversion of strain energy into thermal energy and causing the process to

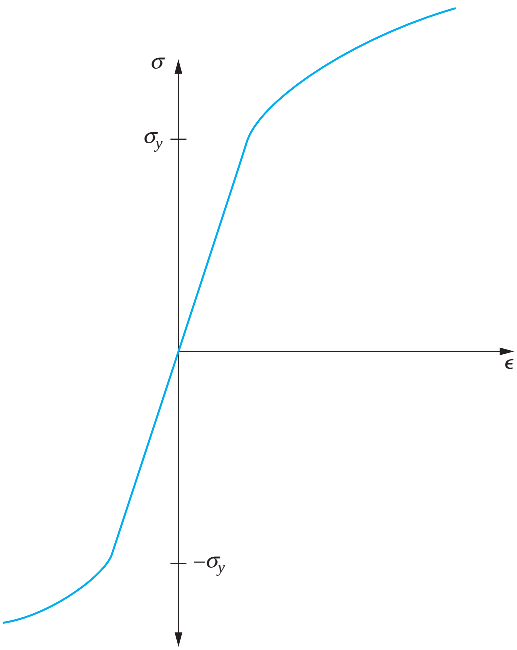


FIGURE 3.18
Stress–strain diagram for a linearly elastic isotropic material with the same behavior in compression and tension. Material behavior is linear for $|\sigma| < \sigma_y$.

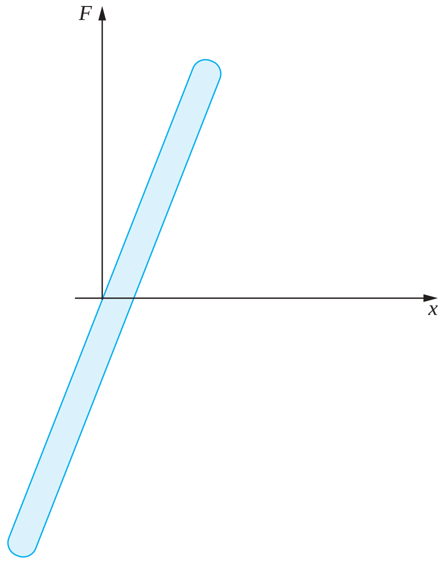


FIGURE 3.19
Behavior of a real engineering material as a system executes one cycle of motion. The area enclosed by the curve is the dissipated strain energy per unit volume. This dissipated energy is the basis for hysteretic damping.

be irreversible. A more realistic stress-strain curve for the loading-unloading process is shown in Figure 3.19 when $|\sigma| < \sigma_y$.

The curve in Figure 3.19 is a hysteresis loop. The area enclosed by the hysteresis loop from a force–displacement curve is the total strain energy dissipated during a loading–unloading cycle. In general, the area under a hysteresis curve is independent of the rate of the loading-unloading cycle.

In a vibrating mechanical system an elastic member undergoes a cyclic load-displacement relationship as shown in Figure 3.19. The loading is repeated over each cycle. The existence of the hysteresis loop leads to energy dissipation from the system during each cycle, which causes natural damping, called *hysteretic damping.* It has been shown experimentally that the energy dissipated per cycle of motion is independent of the frequency and proportional to the square of the amplitude. An empirical relationship is

$$\Delta E = \pi k h X^2 \tag{3.71}$$

where X is the amplitude of motion during the cycle and h is a constant, called the *hysteretic damping coefficient.*

The hysteretic damping coefficient cannot be simply specified for a given material. It is dependent upon other considerations such as how the material is prepared and the geometry of the structure under consideration. Existing data cannot be extended to apply to every situation. Thus it is usually necessary to empirically determine the hysteretic damping coefficient.

Mathematical modeling of hysteretic damping is developed from a work-energy analysis. Consider a simple mass-spring system with hysteretic damping. Let X_1 be the amplitude at a time when the velocity is zero and all energy is potential energy stored in the spring. Hysteretic damping dissipates some of that energy over the next cycle of motion. Let X_2 be the displacement of the mass at the next time when the velocity is zero, after the system executes one half-cycle of motion. Let X_3 be the displacement at the subsequent time when the velocity is zero, one full cycle later. Application of the work-energy principle over the first half-cycle of motion gives

$$T_1 + V_1 = T_2 + V_2 + \frac{\Delta E}{2} \tag{3.72}$$

The energy dissipated by hysteretic damping is approximated by Equation (3.71) with X as the amplitude at the beginning of the half-cycle.

$$\frac{1}{2}kX_1^2 = \frac{1}{2}kX_2^2 + \frac{1}{2}\pi khX_1^2 \tag{3.73}$$

This yields

$$X_2 = \sqrt{1 - \pi h}X_1 \tag{3.74}$$

A work-energy analysis over the second half-cycle leads to

$$X_3 = \sqrt{1 - \pi h}X_2 = (1 - \pi h)X_1 \tag{3.75}$$

Thus the rate of decrease of amplitude on successive cycles is constant, as it is for viscous damping. By analogy a logarithmic decrement is defined for hysteretic damping as

$$\delta = \ln\frac{X_1}{X_3} = -\ln(1 - \pi h) \tag{3.76}$$

which for small h is approximated as

$$\delta = \pi h \tag{3.77}$$

By analogy with viscous damping an equivalent damping ratio for hysteretic damping is defined as

$$\zeta = \frac{\delta}{2\pi} = \frac{h}{2} \tag{3.78}$$

and an equivalent viscous damping coefficient is defined as

$$c_{\text{eq}} = 2\zeta\sqrt{mk} = \frac{hk}{\omega_n} \tag{3.79}$$

The free vibrations response of a system subject to hysteric damping is the same as the response of the system when subject to viscous damping with an equivalent viscous damping coefficient given by Equation (3.79). This is true only for small hysteretic damping, as subsequent plastic behavior leads to a highly nonlinear system. The analogy between viscous damping and hysteretic damping is also only true for linearly elastic materials and for materials where the energy dissipated per unit cycle is proportional to the square of the amplitude. In addition, the hysteretic damping coefficient is a function of geometry as well as the material.

The response of a system subject to hysteretic or viscous damping continues indefinitely with exponentially decaying amplitude. However, hysteretic damping is significantly different from viscous damping in that the energy dissipated per cycle for hysteretic damping is independent of frequency, whereas the energy dissipated per cycle increases with frequency for viscous damping. Thus while the mathematical treatments of viscous damping and hysteretic damping are the same they have significant physical differences.

EXAMPLE 3.12

The force-displacement curve for a structure of Figure 3.20(a) modeled by the system of Figure 3.20(b) is shown in Figure 3.20(c). The structure is modeled as a one-degree-of-freedom system with an equivalent mass 500 kg located at the position where the measurements are made. Describe the response of this structure when a shock imparts a velocity of 20 m/s to this point on the structure.

SOLUTION

The area under the hysteresis curve is approximated by counting the squares inside the hysteresis loop. Each square represents $(1 \times 10^4 \text{ N})(0.002 \text{ m}) = 20 \text{ N} \cdot \text{m}$ of dissipated energy. There are approximately 38.5 squares inside the hysteresis loop resulting in $770 \text{ N} \cdot \text{m}$ dissipated over one cycle of motion with an amplitude of 20 mm.

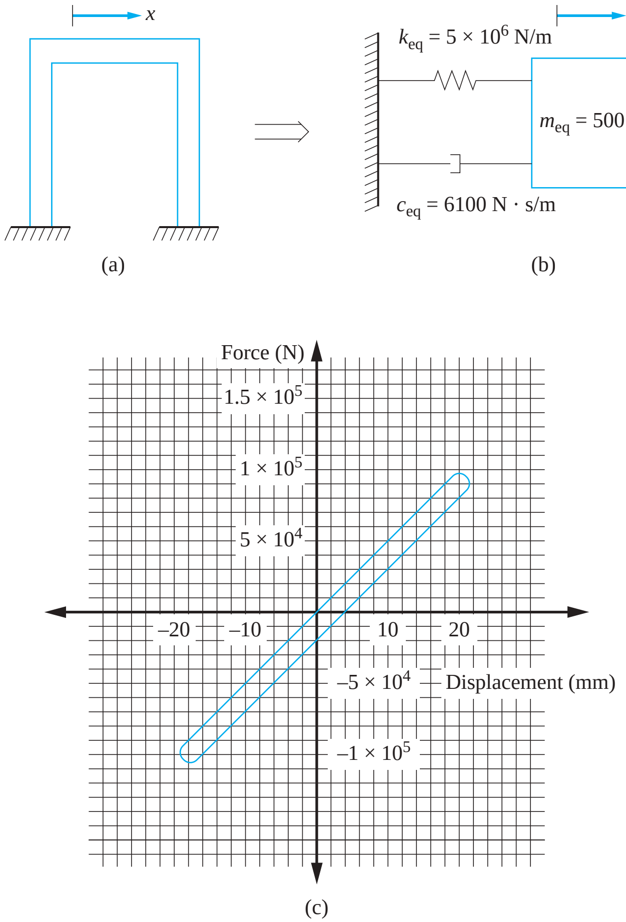


FIGURE 3.20
(a) One-story frame structure modeled as a SDOF system. (b) Hysteretic damping leads to an equivalent viscous-damping coefficient of 6100 N · s/m. (c) Force-displacement curve over one cycle for the system of Example 3.12.

The equivalent stiffness is the slope of the force deflection curve and is determined as 5×10^6 N/m. Application of Equation (3.71) leads to

$$h = \frac{\Delta E}{\pi k X^2} = \frac{770 \text{ N} \cdot \text{m}}{\pi (5 \times 10^6 \text{ N/m})(0.02 \text{ m})^2} = 0.123 \tag{a}$$

The logarithmic decrement, damping ratio, and natural frequency are calculated by using Equations (3.77) and (3.78)

$$\delta = \pi h = 0.385 \tag{b}$$

$$\zeta = \frac{h}{2} = 0.0613 \tag{c}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5 \times 10^6 \text{ N/m}}{500 \text{ kg}}} = 100 \text{ rad/s} \tag{d}$$

The response of this structure with hysteretic damping is approximately the same as the response of a simple mass-spring-dashpot system with a damping ratio of 0.0615 and a natural frequency of 100 rad/s. Then from Equation (3.28) with $\dot{x}_0 = 20$ m/s and $x_0 = 0$, the response is

$$x(t) = 0.20 e^{-6.13t} \sin(99.81t) \text{ m} \tag{e}$$

3.9 OTHER FORMS OF DAMPING

A mechanical or structural system may be subject to other forms of damping such as aerodynamic drag, radiation damping, or anelastic damping. However, these give rise to nonlinear terms in the governing differential equations. Exact solutions do not exist for these forms of damping. The periodic motion of systems subject to these forms of damping can be approximated by developing an equivalent viscous damping coefficient. The equivalent viscous damping coefficient is obtained by equating the energy dissipated over one cycle of motion, assuming harmonic motion at a specific amplitude and frequency, for the particular form of damping with the energy dissipated over one cycle of motion because of the force in a dashpot of the equivalent viscous damping coefficient.

For a harmonic motion of the form $x(t) = X \sin \omega t$, the energy dissipated over one cycle of motion due to a damping force F_D is

$$\Delta E = \int_0^{2\pi/\omega} F_D \dot{x}\, dt = \int_0^{2\pi/\omega} F_D X \omega \cos \omega t\, dt \tag{3.80}$$

For viscous damping, Equation (3.80) yields

$$\Delta E = \int_0^{2\pi/\omega} c \dot{x}^2\, dt = \int_0^{2\pi/\omega} c \omega^2 X^2 \cos^2 \omega t\, dt = c \omega \pi X^2 \tag{3.81}$$

Thus, by analogy, the equivalent viscous damping coefficient for another form of damping is

$$c_{eq} = \frac{\Delta E}{\pi \omega X^2} \tag{3.82}$$

Aerodynamic drag is present in all real problems. However, its effect is often ignored. The determination of the correct form of the drag force is a problem in fluid mechanics. At high Reynolds numbers, the drag is very nearly proportional to the square of the velocity and can be written as

$$F_D = C_D \dot{x} |\dot{x}| \tag{3.83}$$

where C_D is a coefficient that is a function of body geometry and air properties. For moderate Reynolds numbers, appropriate forms of the drag force have been proposed as

$$F_D = C_D |\dot{x}|^{\alpha} \dot{x} \tag{3.84}$$

where $0 < \alpha \leq 1$. In either case, the resulting differential equation is nonlinear.

Some materials (e.g., rubber) are viscoelastic and obey a constitutive equation in which stress is related to strain and strain rate. It is shown in Chapter 4 that for an undamped system the forced response is in phase with a harmonic excitation, whereas a phase lag occurs for a damped system. This phase lag also occurs for many viscoelastic materials. Indeed, many viscoelastic materials have constitutive equations that are derived by modeling the material as a spring in parallel with a dashpot. This is called a Kelvin model. The phase lag results in energy dissipation and the resulting damping is called anelastic damping.

Damping occurs when energy is dissipated from a vibrating body by any means. Another example is radiation damping that occurs for a body vibrating on the free surface between two fluids. The vibrating body causes pressure waves to be radiated outward, causing energy transfer from the body to the surrounding fluids.

Most physical systems are subject to a combination of forms of damping. Indeed, a simple mass-spring-dashpot system is subject to viscous damping from the dashpot, Coulomb damping from the dry sliding friction, hysteretic damping from the spring, and aerodynamic drag. The presence of Coulomb damping leads to cessation of free vibrations after a finite time. The aerodynamic drag is usually neglected in an analysis as its effect is negligible and it leads to a nonlinear differential equation. The hysteretic damping acts in parallel with the viscous damping. The equivalent damping coefficient is the sum of the viscous damping coefficient for the dashpot and the equivalent viscous damping coefficient for the hysteretic damping. For small amplitudes the effect of viscous damping is much greater than the effect of hysteretic damping. For large amplitudes the hysteretic damping can be dominant.

EXAMPLE 3.13

A block of mass 1 kg is attached to a spring of stiffness 3×10^5 N/m. The block is displaced 20 mm from equilibrium and released from rest. The block is in a fluid where the drag force is given by Equation (3.83) with $C_D = 0.86\ \text{N} \cdot \text{s}^2/\text{m}$. Approximate the number of cycles before the amplitude is reduced to 15 mm.

SOLUTION

The energy lost per cycle of motion due to aerodynamic drag is calculated from Equation (3.80)

$$\Delta E = \int_0^{2\pi/\omega} C_D X^3 \omega^3 \cos^2 \omega t |\cos \omega t| dt$$

$$= 4\int_0^{\pi/2\omega} C_D X^3 \omega^3 \cos^3 \omega t \, dt = \frac{8}{3} C_D \omega^2 X^3 \tag{a}$$

From Equation (3.82) the equivalent viscous damping coefficient is calculated as

$$c_{\text{eq}} = 0.730\omega X \tag{b}$$

If the equivalent viscous damping is small, the frequency is approximately equal to the natural frequency of free undamped vibrations

$$\omega = \sqrt{\frac{k}{m}} = 547.7 \text{ rad/s} \tag{c}$$

The damping ratio on a given cycle is

$$\zeta = \frac{c_{\text{eq}}}{2\sqrt{km}} = \frac{0.73(547.7 \text{ rad/s})X}{2\sqrt{(1\text{kg})(3 \times 10^5 \text{ N/m})}} \tag{d}$$

From Equation (3.41) the logarithmic decrement is

$$\delta = 2\pi\zeta = 2.29X \tag{e}$$

Since the equivalent viscous damping coefficient, and hence the damping ratio and the logarithmic decrement, depend on the amplitude, the decrease in amplitude is not constant on each cycle. Using an amplitude of 20 mm for the first cycle, the amplitude at the beginning of the second cycle is obtained using the logarithmic decrement, which in turn is used to predict the amplitude at the beginning of the third cycle. Table 3.2 is developed in this fashion. The amplitude of vibration is reduced to 15 mm in seven cycles.

TABLE 3.2 Viscous approximation used to predict decay in amplitude for Example 3.13

Cycle	Amplitude at beginning of cycle $X_n = X_{n-1} e^{-2.32X_{n-1}}$
1	20.0
2	19.09
3	18.26
4	17.50
5	16.81
6	16.16
7	15.56
8	15.00

3.10 BENCHMARK EXAMPLES

3.10.1 MACHINE ON THE FLOOR OF AN INDUSTRIAL PLANT

During operation, the machine is to be subject to an impulse of magnitude 50 lb · s. The effect of the impulse on the machine is to give the machine an initial velocity using the equivalent mass of the machine. Application of the principle of impulse and linear momentum to the machine leads to

$$v = \frac{I}{m} = \frac{50\,\text{lb}\cdot\text{s}}{38\,\text{slugs}} = 1.29\ \text{ft/s} \tag{a}$$

The ensuing free vibrations of the machine, accounting for the inertia of the beam, are modeled by

$$38.8\ddot{x} + 7.74 \times 10^5 x = 0 \tag{b}$$

with $x(0) = 0$ and $\dot{x}(0) = 1.29$ ft/s. Putting the differential equation in standard form leads to

$$\ddot{x} + 1.99 \times 10^5 x = 0 \tag{c}$$

from which the natural frequency is calculated as

$$\omega_n = \sqrt{1.99 \times 10^5} = 141.4\ \text{rad/s} \tag{d}$$

The system response due to the initial conditions is

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin \omega_n t = \frac{1.29\ \text{ft/s}}{141.4\ \text{rad/s}}\sin(141.4t) = 9.20 \times 10^{-3}\sin(141.4t)\ \text{ft} \tag{e}$$

Equation (e) predicts that the motion will continue indefinitely without amplitude decay. This is false, but it does predict closely the frequency of vibrations and their maximum amplitude. To explore the possible effects of energy dissipation through hysteretic damping, transverse vibrations of the floor are initiated and the history of the response is recorded using an accelerometer placed at the location where the machine is to be attached. The amplitude of vibration decays to half of its initial value in 10 cycles. The logarithmic decrement is calculated as

$$\delta = \frac{1}{10}\ln\left(\frac{2}{1}\right) = 0.0693 \tag{f}$$

from which a hysteretic damping coefficient is determined as

$$h = \frac{\delta}{2} = 0.0347 \tag{g}$$

The response thus is modeled with hysteretic damping as a system with an equivalent viscous-damping ratio

$$\zeta = \frac{\delta}{2\pi} = 0.0110 \tag{h}$$

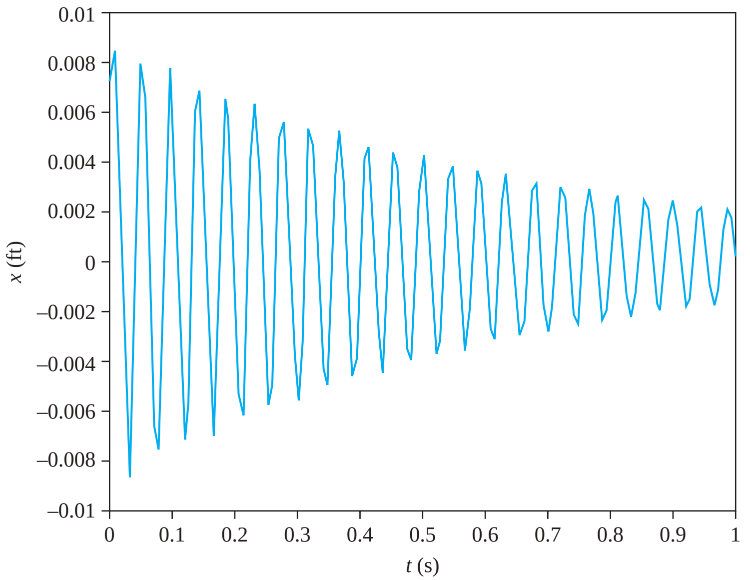


FIGURE 3.21
Plot of the free response of a machine attached to a fixed-free beam when hysteretic damping is included.

The response of the system with hysteretic damping is

$$\begin{aligned} x(t) &= \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) \\ &= \frac{1.29 \text{ ft/s}}{(141.4 \text{ rad/s})\sqrt{1-(0.0110)^2}} e^{-(0.0110)(141.4)t}\sin\left(141.4\sqrt{1-(0.0110)^2}t\right) \\ &= 9.22\times10^{-3}e^{-1.55t}\sin(141.4t)\text{ ft} \end{aligned} \quad \text{(i)}$$

Equation (i) is illustrated in Figure 3.21.

3.10.2 SIMPLIFIED SUSPENSION SYSTEM

The model for free vibrations of the vehicle suspension system with an empty vehicle is

$$300\ddot{x} + 1200\dot{x} + 12000x = 0 \quad \text{(a)}$$

Putting the differential equation in standard form, it becomes

$$\ddot{x} + 4\dot{x} + 40x = 0 \quad \text{(b)}$$

The vehicle has a natural frequency of

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12000 \text{ N/m}}{300 \text{ kg}}} = \sqrt{40\frac{1}{s^2}} = 6.32 \text{ rad/s} \quad \text{(c)}$$

and a damping ratio of

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{1200 \text{ N} \cdot \text{s/m}}{2\sqrt{(300 \text{ kg})(12000 \text{ N/m})}} = 0.316 \quad \text{(d)}$$

The vehicle encounters a sudden change in road contour of a drop of distance h. The system is modeled with the equilibrium position taken after the drop, which implies that the initial conditions are $x(0) = -h$ and $\dot{x}(0) = 0$. The solution of an underdamped system subject to these initial conditions is

$$x(t) = h\sqrt{1 + \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)^2} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi_d\right) \quad \text{(e)}$$

where

$$\phi_d = \tan^{-1}\left(\frac{-h\sqrt{1-\zeta^2}}{-h\zeta}\right) = \tan^{-1}\left(\frac{-\sqrt{1-(0.316)^2}}{-0.316}\right) = 4.39 \quad \text{(f)}$$

Note that the numerator and the denominator in the argument of the inverse tangent are both negative. The negative sign does not cancel; instead, a four-quadrant evaluation of the inverse tangent is used. Substituting numbers in $x(t)$ leads to

$$x(t) = 1.054he^{-2.00t} \sin(6.00t + 4.39) \quad \text{(g)}$$

One concept associated with the free response of a vehicle when it encounters a sudden contour change is overshoot, where the absolute value of maximum displacement at the end of the first half-cycle is

$$\gamma = \left|x\left(\frac{T_d}{2}\right)\right| = he^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad \text{(h)}$$

Expressed as a percentage, the overshoot is

$$\eta = 100\frac{\gamma}{h} = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad \text{(i)}$$

The mass of the vehicle varies with passengers and cargo from an empty value of 300 kg to a fully loaded value of 600 kg. The damping ratio is inversely proportional to the square root of the mass, and hence, the overshoot increases with increasing mass. The variation of overshoot with mass is shown in Figure 3.22.

Another important concept is the 2 percent settling time $t_{2\%}$, which is how long it takes for the system response to be permanently reduced to be within 2 percent of the initial displacement of equilibrium. It is calculated from the last time that $x(t) = |0.02h|$, which is calculated in term of the mass of the vehicle using Equation (e). The value of $\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi_d)$ ranges between –1 and 1 and does not have much effect on the solution for the 2 percent settling time. Ignoring this term and eliminating the absolute value (since the remainder of the terms are positive) leads to

$$0.02h = h\sqrt{1 + \frac{\zeta}{\sqrt{1-\zeta^2}}}\, e^{-\zeta\omega_n t_{2\%}} \quad \text{(j)}$$

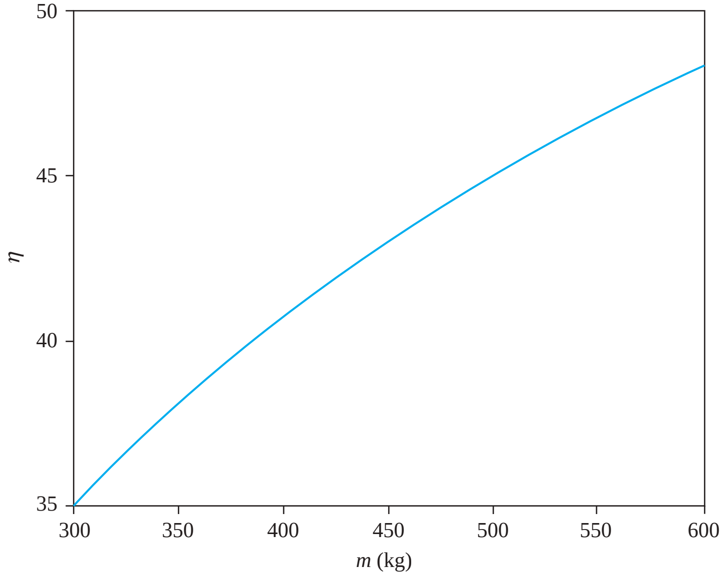


FIGURE 3.22
Percent overshoot as a function of mass of the vehicle for the simplified model of the vehicle suspension system.

which is solved, leading to

$$t_{2\%} = \frac{1}{\zeta\omega_n}\left[3.912 + \frac{1}{2}\ln\left(1 + \frac{\zeta}{\sqrt{1 - \zeta^2}}\right)\right] \quad \text{(k)}$$

Equation (j) is plotted in Figure 3.23 from an empty vehicle to a fully loaded vehicle.

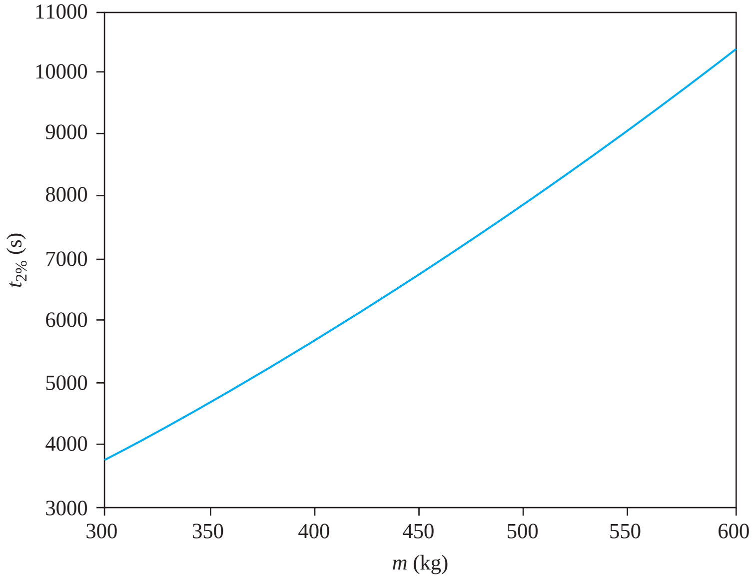


FIGURE 3.23
Two percent settling time as a function of the mass of the vehicle for the simplified model of the vehicle suspension system.

3.11 FURTHER EXAMPLES

EXAMPLE 3.14

A particle of mass of 50 g is to be attached along the length of a thin bar with a length of 25 cm, mass of 200 g, and centroidal moment of inertia of 9.0×10^{-3} kg · m^2. The assembly is suspended from a pin support attached at one end of the bar. The center of gravity of the bar is 15 cm from the pin support. The assembly is to be tuned such that it has a period of 1.25 s. Determine the length along the bar where the particle is to be placed.

SOLUTION

The assembly shown in Figure 3.24(a) is modeled as a compound pendulum with an attached particle. The generalized coordinate used in the modeling is θ, which is the counterclockwise angular displacement of the pendulum from equilibrium. It is assumed that θ is small, so that the small angle assumption applies. Free-body diagrams drawn for an arbitrary value of θ are shown in Figure 3.24(b). Using these free-body diagrams to sum moments about an axis through the pin support, $(\sum M_O)_{\text{ext}} = (\sum M_O)_{\text{eff}}$, yields

$$-m_1 ga\theta - m_2 gb\theta = \bar{I}\ddot{\theta} + (m_1 a\ddot{\theta})a + (m_2 b\ddot{\theta})b \tag{a}$$

where a is the distance from the pin support to the mass center of the ban.

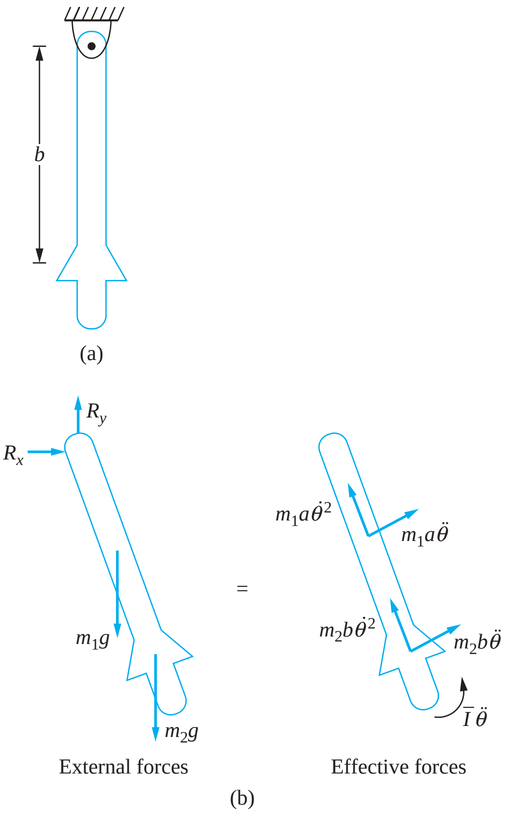


FIGURE 3.24
Pendulum composed of a mass which can slide along the rod. (b) FBDs at an arbitrary instant where θ is the chosen generalized coordinate.

Equation (a) is rearranged to

$$(\bar{I} + m_1a^2 + m_2b^2)\ddot{\theta} + (m_1a + m_2b)g\theta = 0 \quad \text{(b)}$$

Equation (b) is put into standard form, and the natural frequency identified as

$$\omega_n = \sqrt{\frac{(m_1a + m_2b)g}{\bar{I} + m_1a^2 + m_2b^2}} \quad \text{(c)}$$

The period of free oscillation is

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I + m_1a^2 + m_2b^2}{(m_1a + m_2b)g}} \quad \text{(d)}$$

Requiring the period to be 1.25 s and substituting in the given values leads to

$$1.25 \text{ s} = 2\pi\sqrt{\frac{9 \times 10^{-3}\,\text{kg}\cdot\text{m}^2 + (0.2\text{ kg})(0.15\text{ m})^2 + (0.05\text{ kg})b^2}{[(0.2\text{ kg})(0.15\text{ m}) + (0.05\text{ kg})b](9.81\text{ m/s}^2)}} \quad \text{(e)}$$

Dividing by 2π, squaring, multiplying by the denominator, and rearranging leads to

$$b^2 - 0.3882b + 0.03709 = 0 \quad \text{(f)}$$

The solution of the quadratic equation is $b = 0.169, 0.219$ m. The mass can be placed at either location.

EXAMPLE 3.15

The parameters in the system of Figure 3.25 have the following values: $I_D = 0.002\text{ kg}\cdot\text{m}^2$, $r = 100$ mm, $m = 1.2$ kg, and $k = 3 \times 10^4$ N/m.

(a) Let x be the displacement of the mass center of the cart as the generalized coordinate. Derive the differential equation for the system using the equivalent systems method. Assume there is no friction between the cart and the surface.
(b) For what value of c is the system critically damped? Call this value c_c.
(c) Suppose the cart is displaced 3 cm from equilibrium and released. Determine $x(t)$ if (i) $c = 0.25c_c$, (ii) $c = c_c$, and (iii) $c = 1.25c_c$.
(d) How long will it take for the response to be permanently within 1 mm of the equilibrium position if (i) $c = 0.25c_c$, (ii) $c = c_c$, and (iii) $c = 1.25c_c$?

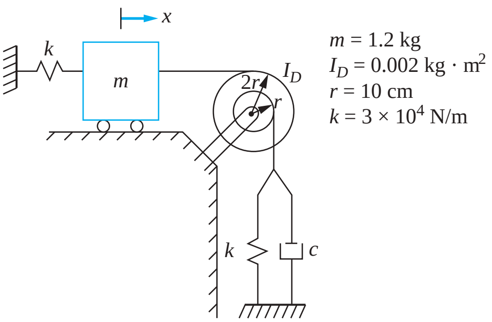


FIGURE 3.25
System of Example 3.15.

SOLUTION

(a) The kinetic energy of the system at an arbitrary instant is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_D\omega^2$ where ω is the angular velocity of the disk. Assuming the cables are inextensible, the velocity of the point on the disk where the cable is being taken up or let out is the same as the velocity of the cable, which also is the same as the velocity of the cart. Thus, $\dot{x} = 2r\dot{\theta}$. The kinetic energy becomes

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_D\left(\frac{\dot{x}}{2r}\right)^2 = \frac{1}{2}\left(m + \frac{I_D}{4r^2}\right)\dot{x}^2 = \frac{1}{2}\left(1.2\text{ kg} + \frac{0.002\text{ kg}\cdot\text{m}^2}{4(0.01\text{ m})^2}\right)\dot{x}^2$$

$$= \frac{1}{2}(6.2\text{ kg})\dot{x}^2 \tag{a}$$

Thus, the equivalent mass is $m_{eq} = 6.2$ kg. The potential energy at an arbitrary instant is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k(r\theta)^2 = \frac{1}{2}kx^2 + \frac{1}{2}k\left(\frac{x}{2}\right)^2 = \frac{1}{2}\left(\frac{5k}{4}\right)x^2 = \frac{1}{2}\left[\frac{5}{4}\left(3 \times 10^4\text{ N/m}\right)\right]x^2$$

$$= \frac{1}{2}(3.75 \times 10^4\text{ N/m})x^2 \tag{b}$$

which leads to $k_{eq} = 3.75 \times 10^4$ N/m. The work done by the viscous damper between $t = 0$ and an arbitrary instant is

$$U_{1\to 2} = -\int c\frac{\dot{x}}{2}d\left(\frac{x}{2}\right) = -\int \frac{c}{4}\dot{x}\,dx \tag{c}$$

Hence, the equivalent viscous-damping coefficient is $c_{eq} = c/4$. The differential equation governing the system is

$$6.2\ddot{x} + \frac{1}{4}c\dot{x} + 3.75 \times 10^4 x = 0 \tag{d}$$

(b) The natural frequency of the system is

$$\omega_n = \sqrt{\frac{3.75 \times 10^4\text{ N/m}}{6.2\text{ kg}}} = 77.8\text{ rad/s} \tag{e}$$

The form of the damping ratio is

$$\zeta = \frac{c}{8(6.2\text{ kg})(77.8\text{ rad/s})} = \frac{c}{3860\text{ N}\cdot\text{s/m}} \tag{f}$$

For critical damping, the damping ratio is 1, which leads to $c_c = 3860$ N · s/m.

(c) The initial conditions are $x(0) = 0.03$ m and $\dot{x}(0) = 0$ m/s. (i) If $c_c = 0.25$, the system is underdamped with $\zeta = 0.25$. The solution for an underdamped system is given by Equation 3.28 and is applied to this problem as

$$x(t) = \sqrt{(0.03\text{ m})^2 + \left[\frac{0\text{ m/s} + (0.25)(77.8\text{ rad/s})(0.03\text{ m})}{(77.8\text{ rad/s})\sqrt{1 - (0.25)^2}}\right]^2}$$

$$\sin\left\{(77.8\text{ rad/s})\sqrt{1 - (0.25)^2}t\right.$$

$$+\tan^{-1}\left[\frac{(0.03\text{ m})(77.8\text{ rad/s})\sqrt{1-(0.25)^2}}{0\text{ m/s}+(0.25)(77.8\text{ rad/s})(0.03\text{ m})}\right]\Bigg\}$$

$$= 0.0310\sin(75.3t + 1.32)\text{ m} \tag{g}$$

(ii) For $c = c_c$, the system is critically damped, and $\zeta = 1$. The free response of a critically damped system is given by Equation 3.48, which is applied to yield

$$x(t) = e^{-(77.8\text{ rad/s})t}\Big\{0.03\text{ m} + \big[0\text{ m/s} + (77.8\text{ rad/s})(0.03\text{ m})t\big]\Big\}$$

$$= e^{-(77.8\text{ rad/s})t}(0.03 + 2.33t)\text{ m} \tag{h}$$

(iii) For $c = 1.25\,c_c$, the system is overdamped with $\zeta = 1.25$. The free response of an overdamped system is given by Equation 3.53, which is applied to yield

$$x(t) = \frac{e^{-(1.25)(77.8\text{ rad/s})t}}{2\sqrt{(1.25)^2-1}}\Bigg\{\Bigg[\frac{0\text{ m/s}}{77.8\text{ rad/s}} + (0.03\text{ m})(1.25$$

$$+\sqrt{(1.25)^2-1})\Bigg]e^{(77.8\text{ rad/s})\sqrt{(1.25)^2-1}\,t}$$

$$+\left[\frac{0\text{ m/s}}{77.8\text{ rad/s}} + (0.03\text{ m})(-1.25+\sqrt{(1.25)^2-1})\right]e^{-(77.8\text{ rad/s})\sqrt{(1.25)^2-1}\,t}\Bigg\}$$

$$= (0.04e^{-38.9t} - 0.01e^{-155.6t})\text{m} \tag{i}$$

(d)

(i) For an underdamped system, the logarithmic decrement can be used to determine how long it will take for the system to be permanently within 1 mm of equilibrium. To this end,

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.25)}{\sqrt{1-(0.25)^2}} = 1.622 \tag{j}$$

From the requirements, the number of cycles is determined by

$$1.622 = \frac{1}{n}\ln\left(\frac{0.03\text{ m}}{0.001\text{ m}}\right) = \frac{3.410}{n} \Rightarrow n = \frac{3.410}{1.622} \Rightarrow 2.10 \tag{k}$$

The system will return to within 1 mm of equilibrium within 3 cycles. Thus,

$$t = 3T_d = 3\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 3\frac{2\pi}{(77.8\text{ rad/s})\sqrt{(1.25)^2-1}} = 0.250\text{ s} \tag{l}$$

(ii) For $\zeta = 1$, an iteration is performed on

$$0.001\text{ m} = e^{-(77.8\text{ rad/s})t}(0.03 + 2.33t)\text{ m} \tag{m}$$

leading to $t = 0.067$ s.

(iii) For $\zeta = 1.25$, the solution is composed of two exponential terms with negative exponents. The solution simply decays without crossing the axis. When the response is within 0.001 m from equilibrium, the term with the larger exponent (smaller absolute value) should be much greater than the term with the smaller exponent. Thus, a good approximation for the time to be permanently within 1 mm of equilibrium is approximated by

$$0.001 \text{ m} = 0.04e^{-38.9t} \text{ m} \tag{n}$$

which leads to $t = 0.0948$ s. The neglected term is $.01e^{-155.6(0.0948)} = 3.92 \times 10^{-9}$, which is much less than 0.001, and hence, $t = 0.0948$ is a good approximation.

EXAMPLE 3.16

A torsional pendulum shown in Figure 3.26(a) is composed of a thin disk with a moment of inertia I which is pinned at its mass center and allowed to rotate about the pin support. The pendulum is attached to a torsional spring of stiffness $k_t = 1.8$ N · m/rad. As the disk rotates, it moves through an electromagnet. A body moving through a magnetic field generates a force whose magnitude is qvB if the magnetic field is perpendicular to the velocity where q is the charge on the body, B is the magnitude of the magnetic field, and v is the velocity of the body. Since the force is proportional to the velocity, the pendulum behaves as if has viscous damping. The net result of the pendulum passing through the magnetic field is to generate a moment resisting the motion about the center of the disk. The magnetic field acts as a torsional viscous damper.

(a) When the magnetic field is off, the torsional pendulum is rotated 40° from its equilibrium position and released. It takes 2 s to complete one cycle of motion. Determine the moment of inertia of the pendulum.
(b) When the magnetic field is turned on, the amplitude of successive cycles of motion is observed as 30°, 25°, 20.8°, etc. What is the damping ratio of the system?

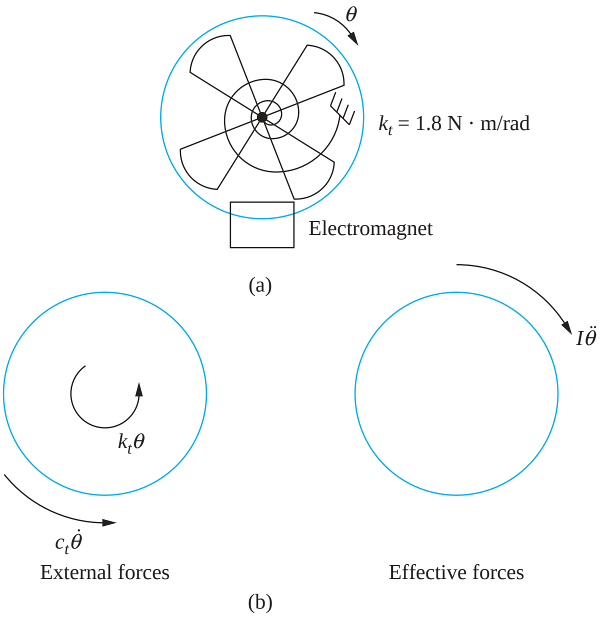


FIGURE 3.26
A torsional pendulum consists of a thin disk pinned at its center. The disk is attached to a torsional spring and rotates through a magnetic field which serves as a torsional damper. (b) FBDs of pendulum at an arbitrary instant, assuming viscous damping and ignoring Coulomb damping.

(c) When the magnetic field is turned on and the pendulum is given an initial amplitude of 30°, describe the resulting motion of the system.
(d) If the electromagnet is turned off and the amplitude of free, oscillations observed on successive cycles is 30°, 28°, and 26°. What frictional moment is generated at the pin support?

SOLUTION

(a) Summing moments on a FBD of the pendulum drawn at an arbitrary instant, Figure 3.26(b) yields

$$I\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \quad \text{(a)}$$

The differential equation is divided by I arriving at the standard form of

$$\ddot{\theta} + \frac{c_t}{I}\dot{\theta} + \frac{k_t}{I}\theta = 0 \quad \text{(b)}$$

from which the natural frequency is obtained as

$$\omega_n = \sqrt{\frac{k_t}{I}} \quad \text{(c)}$$

The period of free oscillations T is observed as 2 s. The pendulum's natural frequency is

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{2\text{ s}} = 3.14 \text{ rad/s} \quad \text{(d)}$$

Equating Equations (c) and (d) leads to

$$\sqrt{\frac{k_t}{I}} = 3.14 \Rightarrow I = \frac{1.8 \text{ N}\cdot\text{m/rad}}{(3.14 \text{ rad/s})^2} = 0.183 \text{ kg}\cdot\text{m}^2 \quad \text{(e)}$$

(b) The amplitudes on successive cycles are in a constant ratio. The logarithmic decrement is

$$\delta = \ln\frac{30°}{28°} = 0.690 \quad \text{(f)}$$

from which the damping ratio is calculated from

$$\zeta = \frac{\delta}{\sqrt{4\pi^4 + \delta^2}} = \frac{0.690}{\sqrt{4\pi^4 + (0.690)^2}} = 0.011 \quad \text{(g)}$$

(c) The damped natural frequency is

$$\omega_d = (3.14 \text{ rad/s})\sqrt{1 - (0.011)^2} = 2.85 \text{ rad/s}$$

The motion of an underdamped system with $\theta(0) = 30°$ and $\dot{\theta}(0) = 0$ rad/s is

$$\theta(t) = (30°)\sqrt{1 + \left(\frac{0.011}{\sqrt{1-(0.011)^2}}\right)^2} e^{-(0.011)(3.14)t}$$

$$\sin\left[3.14t + \tan^{-1}\left(\frac{\sqrt{1-(0.11)^2}}{0.11}\right)\right]$$

$$= 30.16° e^{-0.0345t} \sin(3.14t + 89.4°) \quad \text{(h)}$$

(d) The system is undergoing Coulomb damping. The differential equation governing the motion when system is under the effect of Coulomb damping is

$$I\ddot{\theta} + k_t\theta = \begin{cases} -M_f & \dot{\theta} > 0 \\ M_f & \dot{\theta} < 0 \end{cases} \tag{i}$$

where M_f is the resisting moment due to the friction at the pin support. The system loses 2° of amplitude every cycle of motion, which is given by

$$\Delta A = \frac{4M_f}{I\omega_n^2} \tag{j}$$

Thus,

$$\frac{4M_f}{I\omega_n^2} = (2°)\left(\frac{2\pi \text{ rad}}{360°}\right) = 0.0349 \text{ rad} \tag{k}$$

Equation (k) is solved to yield

$$M_f = \frac{0.0349(0.183 \text{ kg} \cdot \text{m}^2)(3.14 \text{ rad/s})^2}{4} = 0.0157 \text{ N} \cdot \text{m} \tag{l}$$

EXAMPLE 3.17

A MEMS system consists of a mass of 50 μg hanging from a silicon ($E = 73 \times 10^9$ N/m^2) cable with a diameter 0.2 μm and a length of 120 μm. The cable is suspended from a simply supported, circular silicon beam with a diameter of 1.6 μm and a length of 50 μm, as shown in Figure 3.27. The mass vibrates in a silicone oil such that its damping coefficient is 1.2×10^{-6} N · s/m. The mass is given as an initial displacement of 2 μm and released. Determine the response of the system.

SOLUTION

The stiffness of the beam is

$$k_b = \frac{48EI}{L^3} = \frac{48(73 \times 10^9 \text{ N/m}^2)(0.8\,\mu\text{m})^4\pi/4}{(50\,\mu\text{m})^3} = 9.018 \text{ N/m} \tag{a}$$

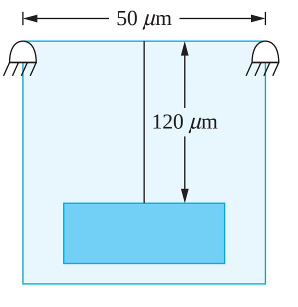


FIGURE 3.27
System of Example 3.17 is a MEMS system. The damping is provided by a surrounding fluid.

The stiffness of the cable is

$$k_c = \frac{AE}{L} = \frac{\pi(0.1\mu\text{m})^2(73 \times 10^9\ \text{N/m}^2)}{120\ \mu\text{m}} = 19.11\ \text{N/m} \quad \textbf{(b)}$$

The springs are in series with an equivalent stiffness as

$$k_{\text{eq}} = \frac{1}{\dfrac{1}{9.08\ \text{N/m}} + \dfrac{1}{19.11\ \text{N/m}}} = 6.13\ \text{N/m} \quad \textbf{(c)}$$

The undamped natural frequency is

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}} = \sqrt{\frac{6.14\ \text{N/m}}{50\ \mu\text{g}}} = 1.10 \times 10^4\ \text{rad/s} \quad \textbf{(d)}$$

The damping ratio is

$$\zeta = \frac{c}{2m\omega_n} = \frac{1.2 \times 10^{-6}\ \text{N}\cdot\text{s/m}}{2(50\ \mu\text{g})(1.10 \times 10^4\ \text{rad/s})} = 0.0011 \quad \textbf{(e)}$$

The damped natural frequency is

$$\omega_d = (1.10 \times 10^4\ \text{rad/s})\sqrt{1 - (0.0011)^2} = 1.10 \times 10^4\ \text{rad/s} \quad \textbf{(f)}$$

The response of an underdamped system with an initial displacement is

$$x(t) = (2\,\mu\text{m})\sqrt{1 + \left[\frac{0.0011}{\sqrt{1 - (0.0011)^2}}\right]^2} e^{-(0.0011)(1.10\times 10^4\ \text{rad/s})t}$$
$$\sin(1.10 \times 10^4 t + 1.57)$$
$$= 2e^{-12t}\sin(1.10 \times 10^4 t + 1.57)\mu\text{m} \quad \textbf{(g)}$$

3.12 CHAPTER SUMMARY

3.12.1 IMPORTANT CONCEPTS

The following refer to free vibrations of a linear SDOF system.

- The natural frequency of a one degree-of-freedom system is the frequency at which undamped free vibrations occur.
- The expression for the natural frequency is determined from the differential equation of motion. It is a function of the stiffness and inertia properties of the system.
- The damping ratio is a measure of the magnitude of the damping force on the system. If the damping ratio is between zero and one, the system is underdamped. If the damping ratio is exactly equal to one, the system is critically damped. If the damping ratio is greater than one, the system is overdamped.
- The free undamped vibrations of a one degree-of-freedom system are cyclic and periodic.

- A system with undamped free vibrations undergoes simple harmonic motion. For a linear system, the period of motion is independent of the initial conditions. The frequency of the motion is the natural frequency of the system.
- An underdamped system undergoes cyclic motion that is not periodic.
- The amplitude of an underdamped system is exponentially decaying.
- The mechanical energy present in an underdamped system at the end of a cycle is a constant fraction of the mechanical energy at the beginning of the cycle. The fraction is dependent upon the damping ratio.
- The logarithmic decrement, which is a measure of the natural logarithm of the ratio of amplitudes on successive cycles, can be used to determine the damping ratio.
- When a system is critically damped, the damping force is just sufficient to dissipate all of the initial energy within one cycle of motion.
- The response of a critically damped system is exponentially decaying. The response overshoots the equilibrium position if the initial conditions are of opposite signs and the initial kinetic energy is larger than the initial potential energy.
- The response of an overdamped system decays exponentially.
- Given the same initial conditions, a critically damped system returns to within a fraction of equilibrium quicker than an overdamped system.
- Coulomb damping results from two surfaces moving relative to one another.
- A system subject to Coulomb damping has the same natural frequency as an undamped system.
- Coulomb damped systems have a constant decrease in amplitude per cycle of motion.
- Motion eventually ceases for a system with Coulomb damping with a permanent displacement from equilibrium.
- Hysteretic damping is the loss of energy experienced by engineering materials due to bonds breaking between atoms and imperfections in the material.
- The energy loss per cycle of motion for a system with hysteretic damping is proportional to the square of the amplitude at the beginning of the cycle and is independent of the frequency of motion.
- The ratio of amplitudes on successive cycles is constant for hysteretic damping, leading to an equivalent viscous-damping model.
- An equivalent viscous-damping coefficient can be calculated for any form of damping by equating the energy dissipated by viscous damping over one cycle of motion to the energy dissipated by the actual damping over one cycle of motion, assuming the motion is harmonic.

3.12.2 IMPORTANT EQUATIONS

Natural frequency of SDOF system

$$\omega_n = \sqrt{\frac{k_{\mathrm{eq}}}{m_{\mathrm{eq}}}} \tag{3.5}$$

Damping ratio of SDOF system

$$\zeta = \frac{c_{\mathrm{eq}}}{2\sqrt{k_{\mathrm{eq}} m_{\mathrm{eq}}}} \tag{3.6}$$

Standard form of differential equation for free vibrations of a linear SDOF system with generalized coordinate x

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{3.7}$$

Roots of characteristic equation

$$\alpha = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}) \tag{3.13}$$

Free response of undamped system

$$x(t) = A \sin(\omega_n t + \phi) \tag{3.19}$$

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \tag{3.22}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{\dot{x}_0}\right) \tag{3.23}$$

Free response of underdamped system

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_d) \tag{3.29}$$

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \tag{3.30}$$

$$\phi_d = \tan^{-1}\left(\frac{\omega_d x_0}{\dot{x}_0 + \zeta\omega_n x_0}\right) \tag{3.31}$$

Damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{3.32}$$

Damped period

$$T_d = \frac{2\pi}{\omega_d} \tag{3.33}$$

Logarithmic decrement

$$\delta = \ln\left(\frac{x(t)}{x(t + T_d)}\right) = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \tag{3.40}$$

Logarithmic decrement over n cycles

$$\delta = \frac{1}{n}\ln\left(\frac{x(t)}{x(t + nT_d)}\right) \tag{3.43}$$

Response of critically damped system

$$x(t) = e^{-\omega_n t}\left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] \tag{3.48}$$

Response of overdamped system

$$x(t) = \frac{e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}\left\{\left[\frac{\dot{x}_0}{\omega_n} + x_0\left(\zeta + \sqrt{\zeta^2 - 1}\right)\right]e^{\omega_n\sqrt{\zeta^2-1}\,t} + \left[-\frac{\dot{x}_0}{\omega_n} + x_0\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\right]e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right\} \tag{3.53}$$

Differential equation for mass sliding on a surface with friction

$$m\ddot{x} + kx = \begin{cases} -\mu mg & \dot{x} > 0 \\ \mu mg & \dot{x} > 0 \end{cases} \tag{3.55}$$

Motion ceases due to Coulomb damping on the nth cycle

$$n > \frac{k\delta}{4\mu mg} - \frac{1}{4} \tag{3.68}$$

Change in amplitude per cycle of motion for system with Coulomb damping

$$\Delta A = \frac{4F_t}{m_{\mathrm{eq}}\omega_n^2} \tag{3.70}$$

Energy loss per cycle due to hysteretic damping

$$\Delta E = \pi k h X^2 \tag{3.71}$$

Equivalent viscous damping ratio for hysteretic damping

$$\zeta = \frac{h}{2} \tag{3.78}$$

Equivalent viscous damping coefficient for any form of damping

$$c_{\mathrm{eq}} = \frac{\Delta E}{\pi\omega X^2} \tag{3.82}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 3.1 through 3.15, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

3.1 The period of free vibration of a linear system is independent of initial conditions.

3.2 The natural frequency determined directly from the differential equation of motion has units of Hertz.

3.3 A system with a natural frequency of 10 rad/s has a shorter period than a system of natural frequency 100 rad/s.

3.4 The free vibrations of an overdamped SDOF system are cyclic.

3.5 An undamped SDOF system has free vibrations which are periodic.

3.6 A system with a damping ratio of 1.2 is overdamped.

3.7 The energy lost per cycle of motion for hysteretic damping is independent of the amplitude of motion but depends upon the square of the frequency.

3.8 The energy lost per cycle of motion for underdamped free vibrations is a constant fraction of the energy present at the beginning of the cycle.

3.9 Motion eventually ceases due to viscous damping for a system with underdamped free vibrations.

3.10 A system that has viscous damping with a damping coefficient such that it is overdamped is governed by two differential equations: one for positive velocity and another for negative velocity.

3.11 There is a permanent displacement from equilibrium when motion ceases for a system with Coulomb damping.

3.12 The period, measured in s, is the reciprocal of the natural frequency, measured in rad/s.

3.13 The differential equation governing the free vibrations of a SDOF system with viscous damping as the only form of friction is a second-order homogeneous differential equation.

3.14 The damping ratio for a SDOF system with viscous damping is always positive.

3.15 The amplitude of an undamped SDOF system is time dependent.

Problems 3.16 through 3.35 require a short answer.

3.16 Consider the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Define in words and in terms of system parameters m, c, and k for (a) ω_n and (b) ζ.

3.17 A critically damped system has a natural frequency of 10 rad/s. Which of the following sets of initial conditions leads to the system overshooting the equilibrium position?

(a) $x_0 = 1 \text{ mm}, \dot{x}_0 = 0 \text{ m/s}$ (b) $x_0 = 0 \text{ mm}, \dot{x}_0 = 1 \text{ m/s}$
(c) $x_0 = 1 \text{ mm}, \dot{x}_0 = 1 \text{ m/s}$ (d) $x_0 = 1 \text{ mm}, \dot{x}_0 = -1 \text{ m/s}$
(e) $x_0 = \text{ mm}, \dot{x}_0 = -0.2 \text{ m/s}$

3.18 Systems with a mass of 1 kg and stiffness of 100 N/m are given an initial displacement of 1 mm and released form rest. Match the plot of system displacement, shown in Figure SP3.18 on the next page, with the system that is (a) undamped, (b) underdamped, (c) critically damped, and (d) overdamped.

3.19 List four differences between the free vibrations of an underdamped system and a system with Coulomb damping.

3.20 An underdamped system is given an initial displacement and released from rest. The amplitudes of motion on successive cyclers form a (an) _____________ series.

3.21 A system with Coulomb damping is given an initial displacement and released from rest. The amplitudes of motion on successive cycles form a (an) _____________ series.

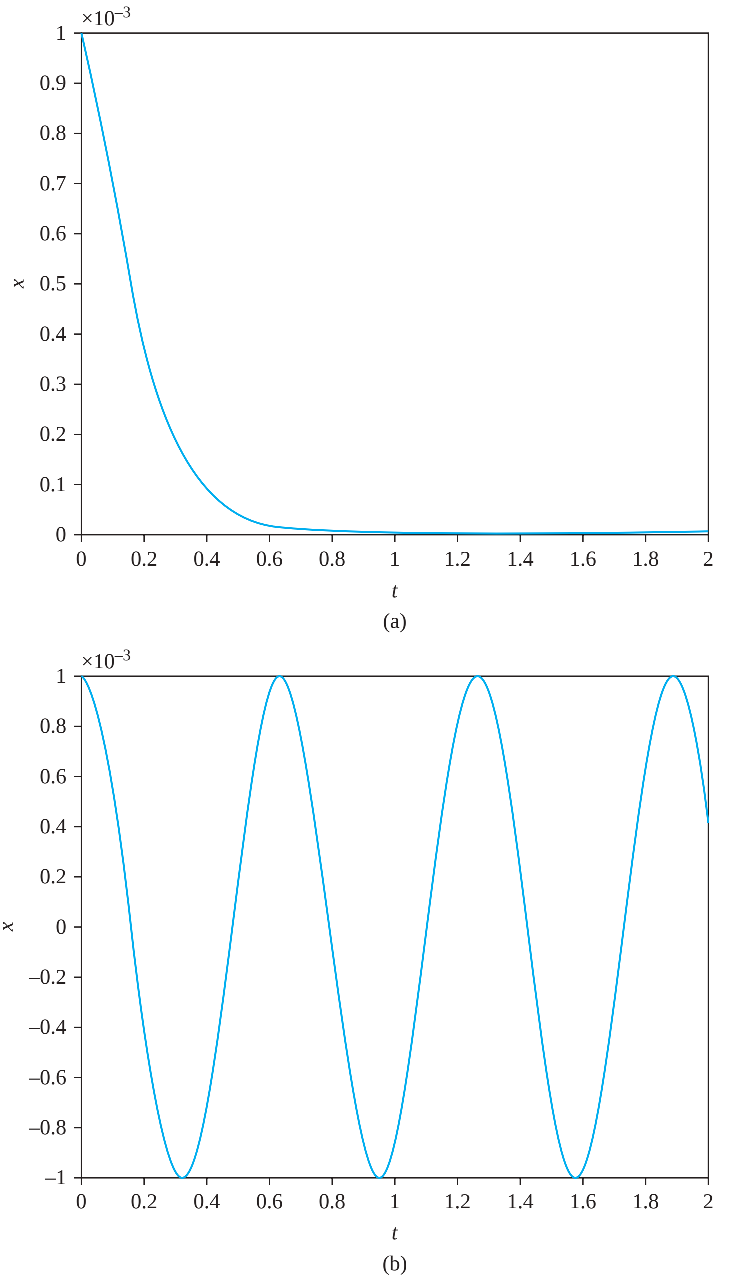
3.22 Identify the following equation and every parameter

$$x(t) = A\sin(\omega_n t + \phi)$$

3.23 Explain the concept of hysteresis? What is the area under a hysteresis cycle?

3.24 Why can't the concept of logarithmic decrement be used to measure viscous damping ratios greater than or equal to one.

FIGURE SP3.18

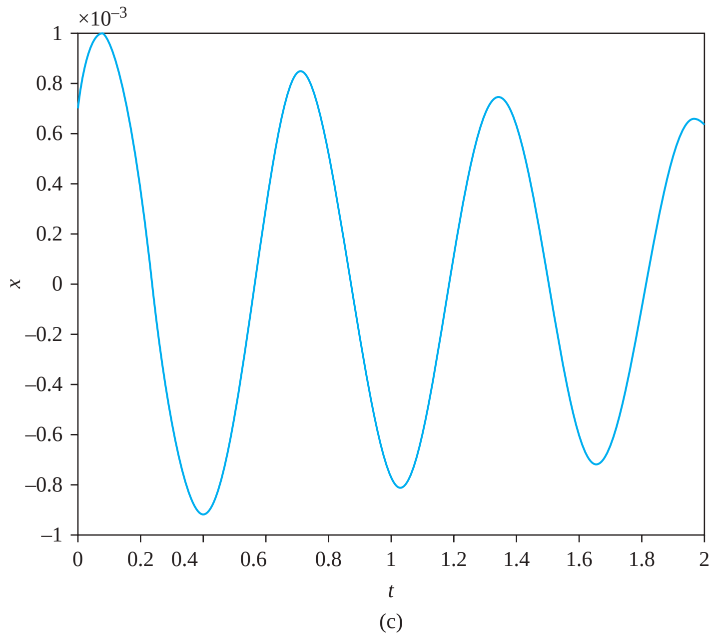


(a)

(b)

3.25 When given the same initial conditions a system that is critically damped returns to equilibrium faster than the same system that is overdamped. Why?

3.26 Two systems have the same stiffness and viscous damping coefficient, but one has an equivalent mass of 2 kg, the other has an equivalent mass of 3 kg. Which system has a higher damping ratio. Why?



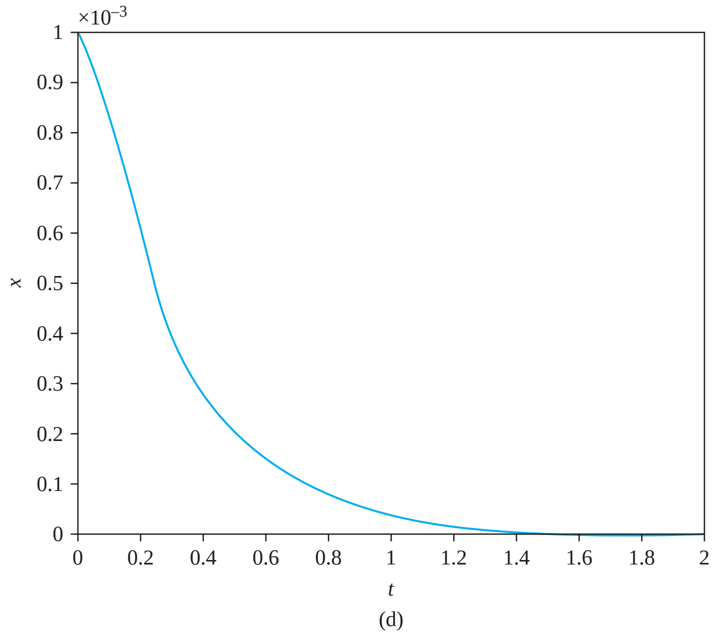


FIGURE SP3.18 (*Continued*)

3.27 A system with viscous damping has a (longer or shorter) period of free vibration than the corresponding undamped system. Why?

3.28 What are the two initial conditions which must be formulated for a SDOF system?

3.29 What are the initial conditions for a mass-spring-viscous damper system that is released from rest with an initial displacement δ.

3.30 What are the initial conditions for a mass-spring-viscous damper system subject to an impulse of magnitude I when in equilibrium?

3.31 What is meant by the term total energy?

3.32 Describe the process by which aerodynamic drag is modeled by viscous damping with an equivalent damping coefficient.

3.33 A pendulum consists of a particle of mass m along a massless rod that is pinned at the upper end of the rod. To lengthen the period of the pendulum should the mass be moved closer to the pin support of farther away?

3.34 A mass m is attached to a spring of stiffness k_1 given an initial displacement and released to slide on a surface. The number of cycles executed is recorded. The same mass m is attached to a spring of stiffness $k_2 > k_1$. Do you predict that the number of cycles executed by the mass will increase, remain the same, or decrease? Why?

3.35 A mass m is attached to a spring of stiffness k_1 and viscous damper of damping coefficient c_1 in parallel. The mass is given an initial displacement and released. The natural frequency of vibration is observed. The same mass is attached to another spring of stiffness $k_2 > k_1$ and viscous damper of damping coefficient $c_2 > c_1$ in parallel. When given the same initial displacement, the motion is still cyclic but with a smaller frequency. Explain.

Short calculations are required for Problems 3.36 through 3.48.

3.36 The free vibrations of a system are governed by the differential equation

$$2\ddot{x} + 40\dot{x} + 1800x = 0$$

with initial conditions $x(0) = 0.001$ m and $\dot{x}(0) = 3$ m/s. Calculate or specify the following.

(a) The natural frequency, ω_n
(b) The damping ratio, ζ
(c) Whether the system is undamped, underdamped, critically damped, or overdamped
(d) The undamped period, T
(e) The frequency in Hz, f
(f) The damped natural frequency (if appropriate), ω_d
(g) The logarithmic decrement (if appropriate), δ
(h) The amplitude, A
(i) The phase between the response and a pure sinusoid (if appropriate), ϕ
(j) The free response of the system

3.37 Repeat Short Problem 3.36 for the differential equation

$$2\ddot{x} + 600\dot{x} + 9800x = 0$$

subject to $x(0) = 0.001$ m and $\dot{x}(0) = 3$ m/s.

3.38 The free vibrations of a system are governed by

$$2\ddot{x} + 1800x = \begin{cases} 3 & \dot{x} < 0 \\ -3 & \dot{x} > 0 \end{cases}$$

with $x(0) = 0.02$ m and $\dot{x}(0) = 0$. Calculate or specify the following.

(a) The period of motion
(b) The change in amplitude per cycle of motion
(c) The permanent displacement when motion ceases
(d) The number of cycles before motion ceases

3.39–43 What is the natural frequency of the system shown when a SDOF model is used?

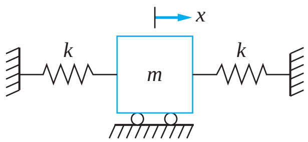


FIGURE SP3.39

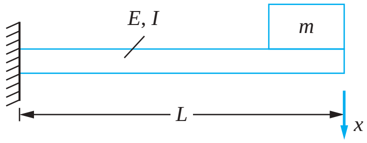


FIGURE SP3.40

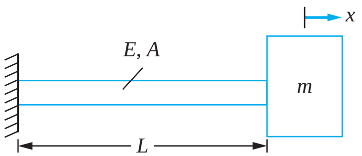


FIGURE SP3.41

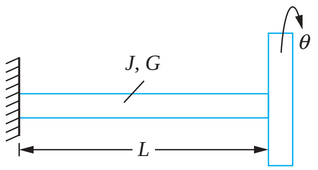


FIGURE SP3.42

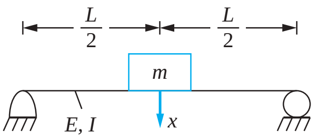


FIGURE SP3.43

3.44 A mass of 12 kg is attached to two springs each of stiffness 4000 N/m and mounted in parallel. What is the natural frequency of the system?

3.45 A mass of 30 g is attached to a spring of stiffness 150 N/m in parallel with a viscous damper. What is the damping coefficient such that the system is critically damped?

3.46 When an engine with a mass of 400 kg is mounted on an elastic foundation, the foundation deflects 5 mm. What is the natural frequency of the system?

3.47 A 2 kg mass is connected to a spring with a stiffness of 1000 N/m. When given an initial displacement of 25 mm, the area under the hysteresis curve of the spring is measured as 0.06 N · m. What is the equivalent viscous damping ratio of the motion?

3.48 What is the response of a system with a equivalent mass of 0.5 kg and a natural frequency of 100 rad/s that has a hysteretic damping coefficient of 0.06 to an initial velocity of 2 m/s?

3.49 Match the quantity with the appropriate units (units may used more than once, some units may not be used).

(a) The natural frequency, ω_n	(i) $N \cdot m$
(b) The damping ratio, ζ	(ii) rad
(c) Damped natural frequency, ω_d	(iii) None
(d) Logarithmic decrement, δ	(iv) rad/s
(e) Phase angle, ϕ	(v) Hz
(f) Change in amplitude per cycle, ΔA	(vi) m
(g) Energy loss under a hysteresis loop, ΔE	(vii) $N \cdot s$
(h) Hysteretic damping coefficient, h	(viii) m/s
(i) Initial angular velocity of torsional system, $\dot{\theta}(0)$	(ix) N/s

CHAPTER PROBLEMS

3.1 The mass of a pendulum bob of a cuckoo clock is 30 g. How far from the pin support should the bob be placed such that its period is 1.0 s?

3.2 A ceiling fan assembly of five blades is driven by a motor. The assembly is attached to the ceiling by a thin shaft fixed at the ceiling. What is the natural frequency of torsional oscillations of the fan of Figure P3.2.

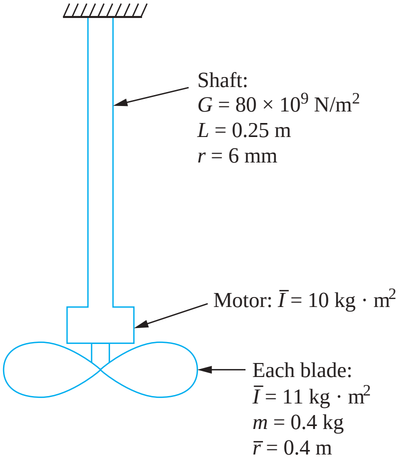


FIGURE P3.2

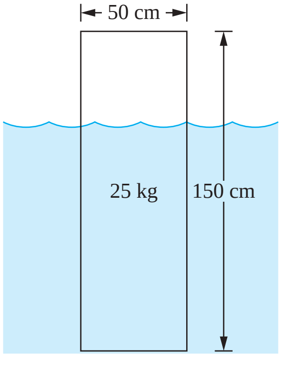


FIGURE P3.3

3.3 The cylindrical container of Figure P3.3 has a mass of 25 kg and floats stably on the surface of an unknown fluid. When disturbed, the period of free oscillations is measured as 0.2 s. What is the specific gravity of the liquid?

3.4 When the 5.1 kg connecting rod of Figure P3.4 is placed in the position shown, the spring deflects 0.5 mm. When the end of the rod is displaced and

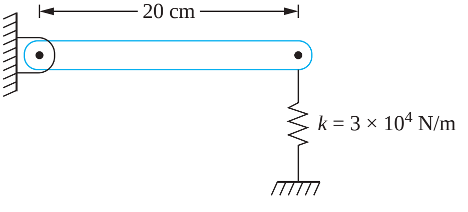


FIGURE P3.4

released, the resulting period of oscillation is observed as 0.15 s. Determine the location of the center of mass of the connecting rod and the centroidal mass moment of inertia of the rod.

3.5 When a 2000 lb vehicle is empty, the static deflection of its suspension system is measured as 0.8 in. What is the natural frequency of the vehicle when it is carrying 700 lb of passengers and cargo?

3.6 A 400 kg machine is placed at the midspan of a 3.2-m simply supported steel ($E = 200 \times 10^9$ N/m^2) beam. The machine is observed to vibrate with a natural frequency of 9.3 Hz. What is the moment of inertia of the beam's cross section about its neutral axis?

3.7 A one degree-of-freedom model of a 9-m steel flagpole ($\rho = 7400$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $G = 80 \times 10^9$ N/m^2) is that of a beam fixed at one end and free at one end. The flagpole has an inner diameter of 4 cm and an outer diameter of 5 cm.

(a) Approximate the natural frequency of transverse vibration.
(b) Approximate the natural frequency of torsional oscillation.

3.8 A 250 kg compressor is to be placed at the end of a 2.5-m fixed-free steel ($E = 200 \times 10^9$ N/m^2) beam. Specify the allowable moment of inertia of the beam's cross section about its neutral axis such that the natural frequency of the machine is outside the range of 100 to 130 Hz.

3.9 A 50 kg pump is to be placed at the midspan of a 2.8-m simply supported steel ($E = 200 \times 10^9$ N/m^2) beam. The beam is of rectangular cross section of width 25 cm. What are the allowable values of the cross-sectional height such that the natural frequency is outside the range of 50 to 75 Hz?

3.10 A diving board is modeled as a simply supported beam with an overhang. What is the natural frequency of a 140-lb diver at the end of the diving board of Figure P3.10

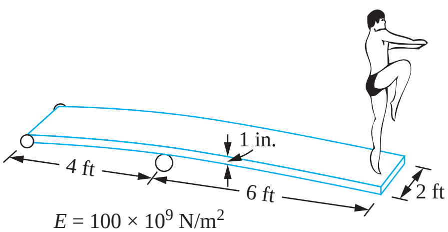


FIGURE P3.10

3.11 A diver is able to slightly adjust the location of the intermediate support on the diving board in Figure P3.10. What is the range of natural frequencies a 140 lb diver can attain if the distance between the supports can be adjusted between 4 and 6.5 ft?

3.12 A 60 kg drum of waste material is being hoisted by an overhead crane and winch system as illustrated in Figure P3.12. The system is modeled as a simply supported beam to which the cable is attached. The drum of waste material is attached to the end of the cable. When the length of the cable is 6 m, the

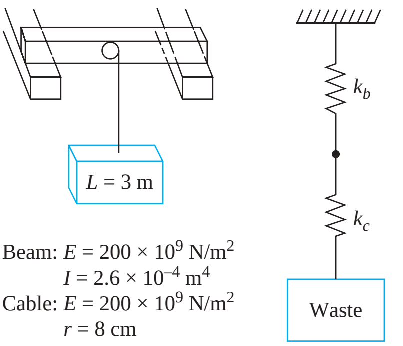


FIGURE P3.12

natural period of the system is measured as 0.3 s. What is the mass of the waste material?

3.13 A 200-kg package is being hoisted by a 120-mm-diameter steel cable ($E = 200 \times 10^9$ N/m^2) at a constant velocity v. What is the largest value of v such that the cable's elastic strength of 560×10^6 N/m^2 is not exceeded if the hoisting mechanism suddenly fails when the cable has a length of 10 m.

3.14 Determine the natural frequency of the system of Figure P2.43.

3.15 Determine the natural frequency and damping ratio of the system of Figure P2.45.

3.16 Determine the natural frequency and damping ratio for the system of Figure P2.47.

3.17 Determine the natural frequency and damping ratio for the system of Figure P2.49.

3.18 Determine the natural frequency and damping ratio for the system of Figure P2.53.

3.19–23 The inertia of the elastic elements is negligible. What is the natural frequency of the system assuming a SDOF model is used? See Figures P3.19 through P3.23.

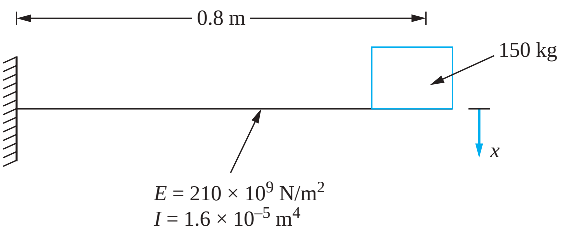


FIGURE P3.19

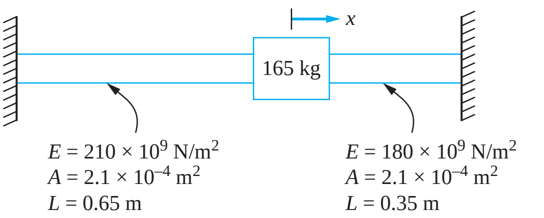


FIGURE P3.20

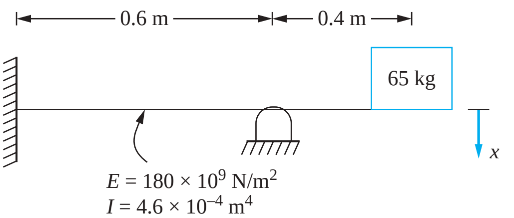


FIGURE P3.21

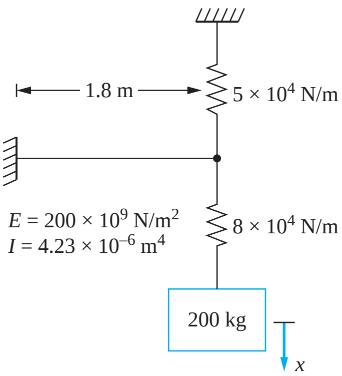


FIGURE P3.22

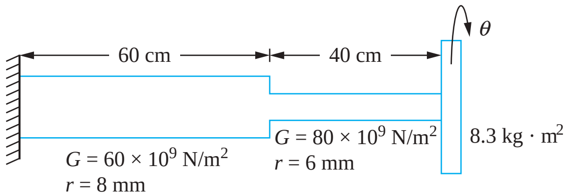


FIGURE P3.23

3.24 The center of the disk of Figure P3.24 is displaced a distance δ from its equilibrium position and released. Determine $x(t)$ if the disk rolls without slip.

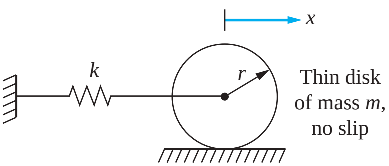


FIGURE P3.24

3.25 The coefficient of friction between the disk and the surface in Figure P3.24 is μ. What is the largest initial velocity of the mass center that can be imparted such that the disk rolls without slip for its entire motion?

3.26–3.31 For the systems shown in Figures P3.26 through P3.31.

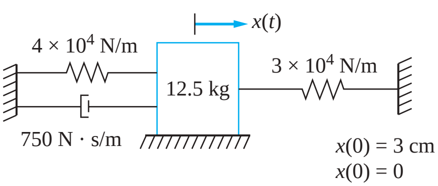


FIGURE P3.26

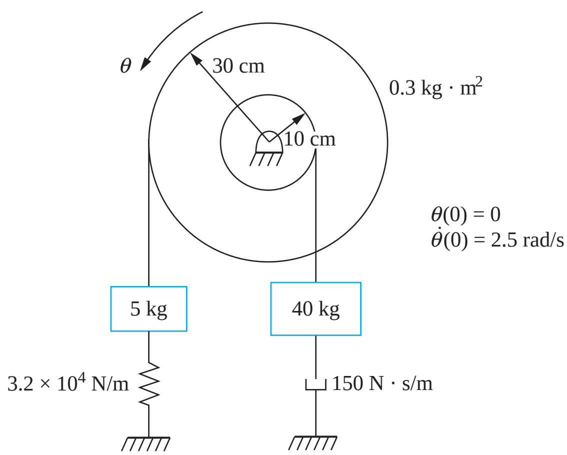


FIGURE P3.27

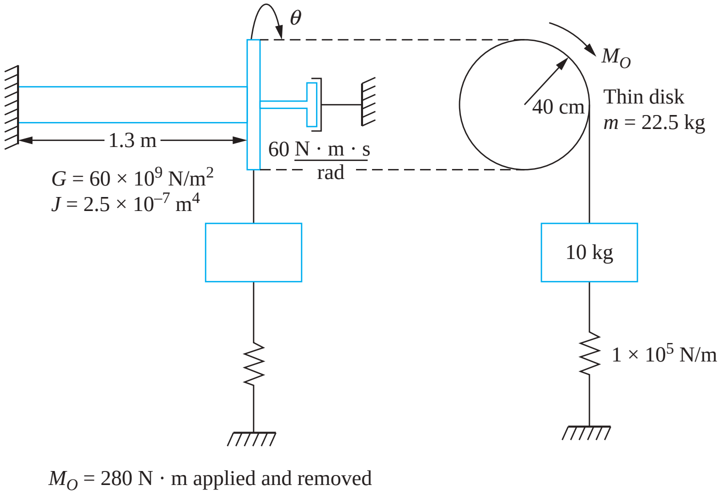


FIGURE P3.28

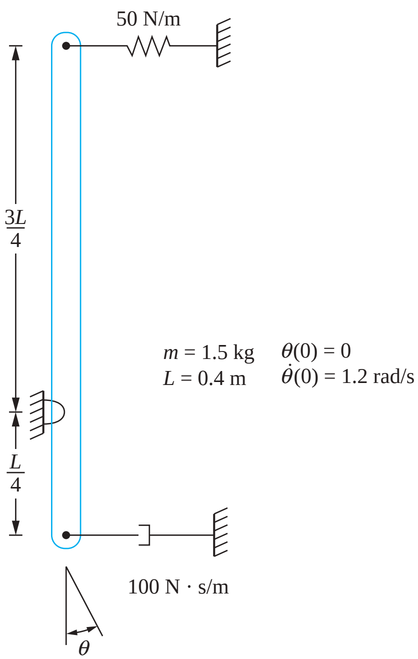


FIGURE P3.29

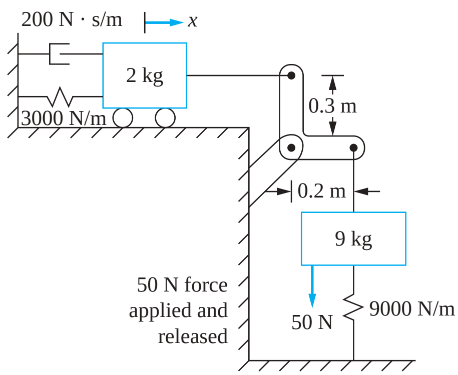


FIGURE P3.30

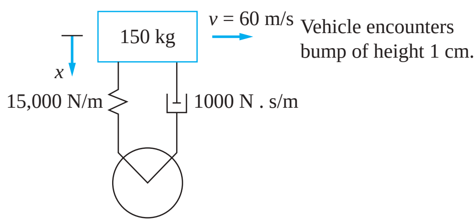


FIGURE P3.31

(a) Determine the damping ratio
(b) State whether the system is underdamped, critically damped, or overdamped
(c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

3.32 The amplitude of vibration of the system of Figure P3.32 decays to half of its initial value in 11 cycles with a period of 0.3 s. Determine the spring stiffness and the viscous damping coefficient.

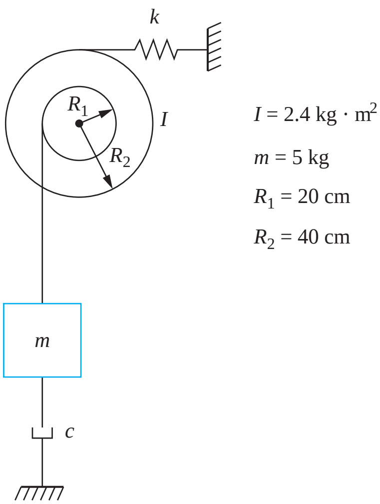


FIGURE P3.32

3.33 The damping ratio of the system of Figure P3.33 is 0.3. How long will it take for the amplitude of free oscillation to be reduced to 2 percent of its initial value?

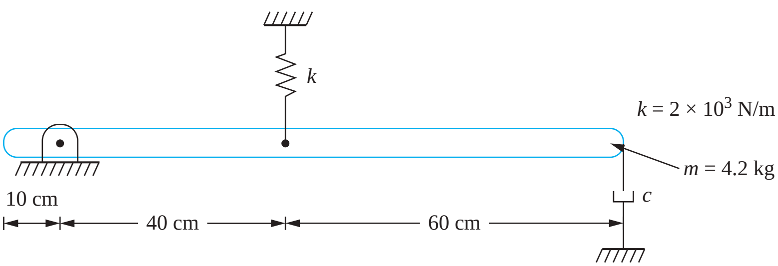


FIGURE P3.33

3.34 When a 40-kg machine is placed on an elastic foundation, its free vibrations appear to decay exponentially with a frequency of 91.7 rad/s. When a 60-kg machine is placed on the same foundation, the frequency of the exponentially decaying oscillations is 75.5 rad/s. Determine the equivalent stiffness and equivalent viscous damping coefficient for the foundation.

3.35 A suspension system is being designed for a 1300-kg vehicle. When the vehicle is empty, its static deflection is measured as 2.5 mm. It is estimated that the largest cargo carried by the vehicle will be 1000 kg. What is the minimum value of the damping coefficient such that the vehicle will be subject to no more than 5 percent overshoot, whether it is empty or fully loaded.

3.36 During operation a 500-kg press machine is subject to an impulse of magnitude 5000 N · s. The machine is mounted on an elastic foundation that can be modeled as a spring of stiffness 8×10^5 N/m in parallel with a viscous damper of damping coefficient 6000 N · s/m. What is the maximum displacement of the press after the impulse is applied. Assume the press is at rest when the impulse is applied.

3.37 For the press of Chapter Problem 3.36, determine (a) the force transmitted to the floor as a function of time, (b) the time at which the maximum transmitted force occurs, and (c) the value of the maximum transmitted force.

3.38 Repeat Chapter Problem 3.37 if the system has the same mass and stiffness but it is designed to be overdamped with a damping ratio of 1.3.

3.39 One end of the mercury filled U-tube manometer of Figure P3.39 is open to the atmosphere while the other end is capped an under a pressure of 20 psig. The cap is suddenly removed.

(a) Determine $x(t)$ as the displacement of the mercury-air interface from the column's equilibrium position if the column is undamped.
(b) Determine $x(t)$ if it is determined that the column of mercury has viscous damping with a damping ratio of 0.1.
(c) Determine $x(t)$ if it is observed that after 5 cycles of motion the amplitude has decreased to one-third of its initial value.

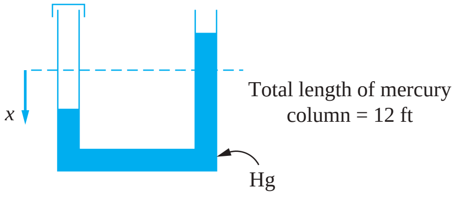


FIGURE P3.39

3.40 The disk of Figure P3.40 rolls without slip.
(a) What is the critical damping coefficient, c_c, for the system?
(b) If $c = c_c/2$, plot the response of the system when the center of the disk is displaced 5 mm from equilibrium and released from rest.
(c) Repeat part (b) if $c = 3c_c/2$.
(d) Repeat part (b) if $c = c_c$.

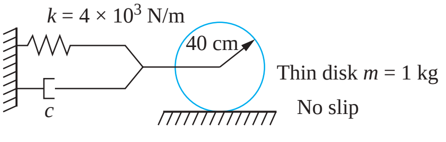


FIGURE P3.40

(e) If the coefficient of friction between the disk and surface is 0.15, is the no-slip assumption still valid for the systems of parts (b), (c), and (d).

3.41 A recoil mechanism of a gun is designed as a spring and viscous damper in parallel such that the system has critical damping. A 52-kg cannon has a maximum recoil of 50 cm after firing. Specify the stiffness and damping coefficient of the recoil mechanism such that the mechanism returns to within 5 mm of firing position within 0.5 s after firing.

3.42 The initial recoil velocity of a 1.4-kg gun is 2.5 m/s. Design a recoil mechanism that is critically damped such that the mechanism returns to within 0.5 mm of firing within 0.5 s after firing.

3.43 A railroad bumper is modeled as a linear spring in parallel with a viscous damper. What is the damping coefficient of a bumper of stiffness 2×10^5 N/m such that the system has a damping ratio of 1.15 when it is engaged by a 22,000-kg railroad car.

3.44 Plot the responses of the bumper of Chapter Problem 3.43 when it is engaged by railroad cars traveling at 20 m/s when the mass of the railroad car is (a) 1500 kg, (b) 22,000 kg, and (c) 30000 kg.

3.45 Reconsider the restroom door of Example 3.9. The man, instead of kicking the door, pushes it so that it opens to 80° and then lets go. How long will it take the door after he lets go to close to within 5° of being shut if it is designed (a) with critical damping and (b) with a damping ratio of 1.5?

3.46 A block of mass m is attached to a spring of stiffness k and slides on a horizontal surface with a coefficient of friction μ. At some time t, the velocity is zero and the block is displaced a distance δ from equilibrium. Use the principle of work-energy to calculate the spring deflection at the next instant when the velocity is zero. Can this result be generalized to determine the decrease in amplitude between successive cycles?

3.47 Reconsider Example 3.11 using a work-energy analysis. That is, assume the amplitude of the swing is θ at the end of an arbitrary cycle. Use the principle of work-energy to determine the amplitude at the end of the next half-cycle.

3.48 The center of the thin disk of Figure P3.48 is displaced a distance δ and the disk released. The coefficient of friction between the disk and the surface is μ. The initial displacement is sufficient to cause the disk to roll and slip.

(a) Derive the differential equation governing the motion when the disk rolls and slips.

(b) When the displacement of the mass center from equilibrium becomes small enough, the disk rolls without slip. At what displacement does this occur?

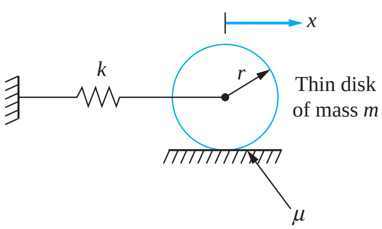


FIGURE P3.48

(c) Derive the differential equation governing the motion when the disk rolls without slip.
(d) What is the change in amplitude per cycle of motion?

3.49 A 10-kg block is attached to a spring of stiffness 3×10^4 N/m. The block slides on a horizontal surface with a coefficient of friction of 0.2. The block is displaced 30 mm and released. How long will it take before the block returns to rest?

3.50 The block of Chapter Problem 3.49 is displaced 30 mm and released. What is the range of values of the coefficient of friction such that the block comes to rest during the 14th cycle?

3.51 A 2.2-kg block is attached to a spring of stiffness 1000 N/m and slides on a surface that makes an angle of 7° with the horizontal. When displaced from equilibrium and released, the decrease in amplitude per cycle of motion is observed to be 2 mm. Determine the coefficient of friction.

3.52 A block of mass m is attached to a spring of stiffness k and viscous damper of damping coefficient c and slides on a horizontal surface with a coefficient of friction μ. Let $x(t)$ represent the displacement of the block from equilibrium.

(a) Derive the differential equation governing $x(t)$.
(b) Solve the equation and sketch the response over two periods of motion.

3.53 A connecting rod is fitted around a cylinder with a connecting rod between the cylinder and bearing. The coefficient of friction between the cylinder and bearing is 0.08. If the rod is rotated 12° counterclockwise and then released, how many cycles of motion will it execute before it comes to rest? The ratio of the diameter of the cylinder to the distance to the center of mass of the connecting rod from the center of the cylinder is 0.01.

3.54 A one-degree-of-freedom structure has a mass of 65 kg and a stiffness of 238 N/m. After 10 cycles of motion the amplitude of free vibrations is decreased by 75 percent. Calculate the hysteretic damping coefficient and the total energy lost during the first 10 cycles if the initial amplitude is 20 mm.

3.55 The end of a steel cantilever beam ($E = 210 \times 10^9$ N/m^2) of $I = 1.5 \times 10^{-4}$ m^4 is given an initial amplitude of 4.5 mm. After 20 cycles of motion the amplitude is observed as 3.7 mm. Determine the hysteretic damping coefficient and the equivalent viscous damping ratio for the beam.

3.56 A 500-kg press is placed at the midspan of a simply supported beam of length 3 m, elastic modulus 200×10^9 N/m^2, and cross-sectional moment of inertia 1.83×10^{-5} m^4. It is observed that free vibrations of the beam decay to half of the initial amplitude in 35 cycles. Determine the response of the press, $x(t)$, if it is subject to an impulse of magnitude 10,000 N · s.

3.58 Use the theory of Section 3.9 to derive the equivalent viscous damping coefficient for Coulomb damping. Compare the response of a one-degree-of-freedom system of natural frequency 35 rad/s and friction coefficient 0.12 using the exact theory to that obtained using the approximate theory with an equivalent viscous damping coefficient.

3.59 A 0.5-kg sphere is attached to a spring of stiffness 6000 N. The sphere is given an initial displacement of 8 mm from its equilibrium position and released. If aerodynamic drag is the only source of friction, how many cycles will the system execute before the amplitude is reduced to 1 mm?

3.60 A one-degree-of-freedom model of a suspension system is shown in Figure P3.60(a). For this model the mass of the vehicle is much greater than the axle mass, but the tire has characteristics which should be included in the analysis. In the model of Figure P3.60(b), the tire is assumed to be elastic with a stiffness k_t. The tire stiffness acts in series with the spring and viscous damper of the suspension system.

(a) Derive a third-order differential equation governing the displacement of the vehicle from the system's equilibrium position.

(b) Solve the differential equation to determine the response of the system when the wheel encounters a pothole of depth h.

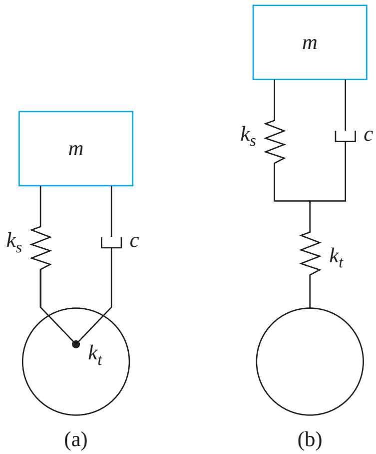


FIGURE P3.60

3.61 A one-degree-of-freedom model of a suspension system is shown in Figure P3.61(a). Consider a model in which the tire is modeled by a viscous

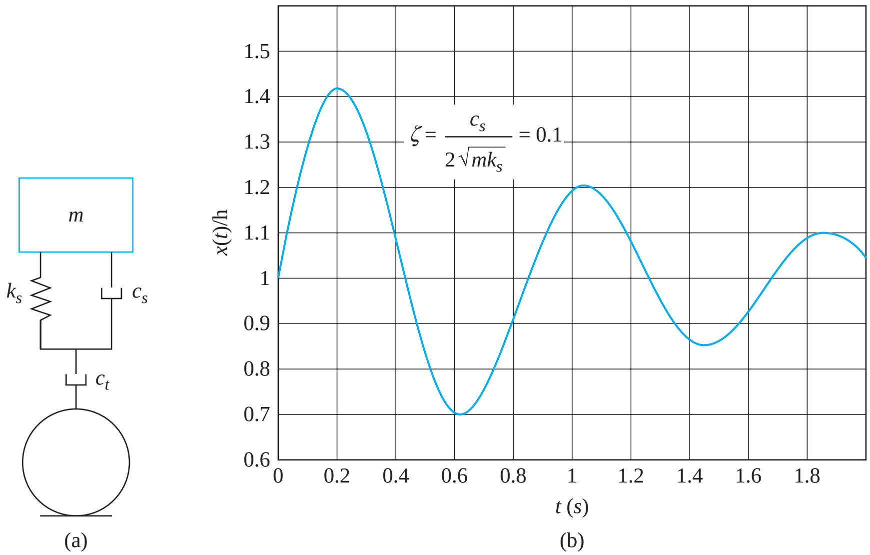


FIGURE P3.61

damper of damping coefficient c_t and is placed in series with the spring and viscous damper modeling the suspension system, as illustrated in Figure P3.61(a).

(a) Derive a third-order differential equation governing the displacement of the vehicle from the system's equilibrium position.
(b) A plot of the suspension system when the wheel encounters a pothole is given in Figure P3.61(b). The plot is made for a suspension system that is designed to have a damping ratio of 0.1. Use this information to find c_t.

Chapter 4

HARMONIC EXCITATION OF SDOF SYSTEMS

4.1 INTRODUCTION

Forced vibrations of a single degree-of-freedom (SDOF) system occur when work is being done on the system while the vibrations occur. Examples of forced vibration include the ground motion during an earthquake, the motion caused by unbalanced reciprocating machinery, or the ground motion imparted to a vehicle as its wheel traverses the road contour. Figure 4.1 illustrates an equivalent systems model for the forced vibrations of a SDOF system when a linear displacement is chosen as the generalized coordinate. The governing differential equation is

$$m_{\mathrm{eq}}\ddot{x} + c_{\mathrm{eq}}\dot{x} + k_{\mathrm{eq}}x = F_{\mathrm{eq}}(t) \tag{4.1}$$

Although, the derivations that follow use a linear displacement as a generalized coordinate they are also valid if an angular displacement is used as a generalized coordinate. The form of the differential equation, Equation (4.1) is used as a model equation.

Dividing Equation (4.1) by m_{eq} leads to

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m_{\mathrm{eq}}}F_{\mathrm{eq}}(t) \tag{4.2}$$

Equation (4.2) is the standard form of the differential equation governing linear forced vibrations of a SDOF system with viscous damping.

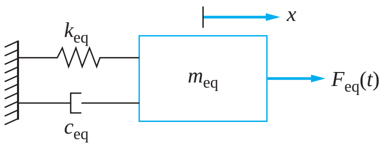


FIGURE 4.1
SDOF model for a linear system with forcing.

The general solution of Equation (4.2) is

$$x(t) = x_h(t) + x_p(t) \tag{4.3}$$

where $x_h(t)$ is the homogeneous solution, the solution obtained if $F_{eq}(t) = 0$, and $x_p(t)$ the particular solution, a solution that is specific to $F_{eq}(t)$. The homogeneous solution is in terms of two constants of integration. However the initial conditions are not imposed until the general solution of Equation (4.3) is developed. For an underdamped system

$$x_h(t) = e^{-\zeta\omega_n t}[C_1 \cos(\omega_d t) + C_1 \sin(\omega_d t)] \tag{4.4}$$

Many ways exist to solve the particular solution. These include the method of undetermined coefficients, variation of parameters, annihilator methods, Laplace transform methods, and numerical methods.

This chapter is concerned with the solution of Equation (4.2) subject to periodic excitations. An excitation is periodic of period T if

$$F_{eq}(t + T) = F_{eq}(t) \tag{4.5}$$

for all t. Figure 4.2 periodic shows examples of periodic excitations. A single-frequency periodic excitation is defined as

$$F_{eq}(t) = F_0 \sin(\omega t + \psi) \tag{4.6}$$

where F_0 is the amplitude of the excitation, ω is its frequency such that $\omega = \frac{2\pi}{T}$ and ψ is its phase. Note that ω is independent of ω_n, the natural frequency which is a function of the stiffness and mass properties of the system. They are independent, but the frequencies may coincide.

The steady-state response for a periodic excitation is defined as

$$x_{ss} = \lim_{t\to\infty} x(t) = \lim_{t\to\infty}[x_h(\mathrm{t}) + x_p(\mathrm{t})] \tag{4.7}$$

which for systems with viscous damping becomes

$$x_{ss} = \lim_{t\to\infty} x_p(t) \tag{4.8}$$

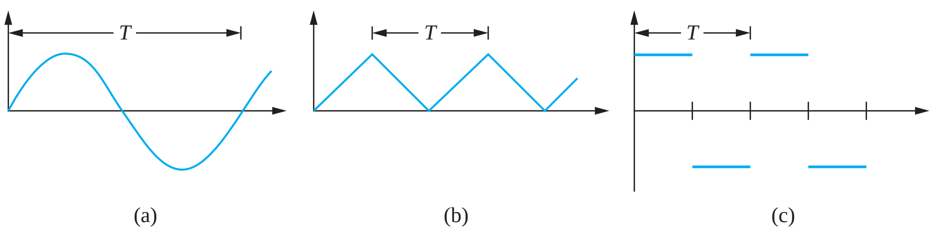


FIGURE 4.2
Examples of periodic excitations (a) a pure sinusoid; (b) a periodic triangular wave; and (c) a periodic square wave.

Beginning with Section 4.3, the "steady-state" will be dropped from steady-state response, and it will be understood that a response refers to a steady-state response.

For an undamped system, the limit of the homogenous solution as t approaches infinity is not zero. The homogeneous response is important if the frequency of excitation coincides or is close to the natural frequency. Otherwise it is assumed that some form of damping really occurs and the free response does decay leaving only the forced response as the long-term response.

When the system is undamped and the frequency of the excitation coincides with the natural frequency a condition of resonance exists. When the system is undamped and the excitation frequency is close, but not equal to, the natural frequency a phenomena called beating occurs.

When the system is undamped with the excitation frequency far enough away from the natural frequency or the system has viscous damping the particular solution of Equation (4.2) subject to the excitation of Equation (4.6) is determined in terms of terms of system parameters. The solution is characterized in term of a steady-state amplitude and a steady-state phase. The relations for these terms are non-dimensionalized resulting in a non-dimensional magnification factor as a function of the damping ratio and the non-dimensional frequency ratio. The phase is written as a function of the frequency ratio and the damping ratio. The concept of frequency response involves studying the behavior of these functions with the frequency ratio for different values of the damping ratio. The frequency response is studied from the equations defining the functions and their graphs.

A special case of a frequency squared excitation, when the amplitude of excitation is proportional to the square of its frequency, is considered. A new non-dimensional function representing the frequency response of such systems is introduced. The general theory is applied to a variety of physical problems including vibrations of reciprocating machines with an unbalanced rotating component and vibrations induced by vortex shedding from a circular cylinder.

Two important quantities in studying the response of a system due to harmonic motion of its base are the absolute acceleration of the system and the displacement of the system relative to its base. The latter is shown to be an application of the theory of frequency squared excitations while the former is an application of vibration isolation theory.

Vibration isolation is the insertion of an elastic member between an object, say a machine, and its foundation to protect either the foundation from large forces generated during operation of the machine or to protect the machine from large accelerations generated through motion of the foundation. A suspension system provides vibration isolation to a vehicle as it protects the vehicle from the accelerations generated by the wheels. Vibration isolation theory is developed for a SDOF system subject to harmonic input.

A Fourier series is a representation of a periodic function by an infinite series of sine and cosine terms. The series converges to the periodic function pointwise at every point where function is continuous. The Fourier series representation and the method of linear superposition are used to solve for the steady-state response of a system due to a general periodic excitation.

Seismic vibration measurement instruments use the vibrations of a seismic mass to measure the vibrations of a body. Because the seismic mass is attached to the instrument which is rigidly attached to the body whose vibrations are being measured the vibrations of the seismic mass relative to the body is actually measured. A seismometer measures this relative motion and requires a large frequency ratio for accuracy. An accelerometer converts the output so that it measures the acceleration and requires a small frequency ratio for accuracy.

The response of a system with Coulomb damping due to harmonic forcing is complicated by the possibility of stick-slip in which the motion ceases during a period when the spring force and the input force are insufficient to overcome the friction force. This makes the response of the system highly nonlinear. It is possible under certain assumptions to assume a steady-state response at the same frequency as the input and use the methods of Chapter 3 to determine an equivalent viscous damping coefficient. The frequency response is then studied. The same method is used to approximate the frequency response for a system with hysteretic damping.

4.2 FORCED RESPONSE OF AN UNDAMPED SYSTEM DUE TO A SINGLE-FREQUENCY EXCITATION

The differential equation for undamped forced vibrations of a SDOF system subject to a single-frequency harmonic excitation of the form of Equation (4.2) is

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m_{\text{eq}}} \sin(\omega t + \psi) \tag{4.9}$$

The method of undermined coefficients is used to find the particular solution of Equation (4.9). Assume a solution of

$$x_p(t) = U\cos(\omega t + \psi) + V\sin(\omega t + \psi) \tag{4.10}$$

Substitution of Equation (4.10) into Equation (4.9) leads to

$$(\omega_n^2 - \omega^2)\,U\cos(\omega t + \psi) + (\omega_n^2 - \omega^2)\,V\sin(\omega t + \psi) = \frac{F_0}{m_{\text{eq}}}\sin(\omega t + \psi) \tag{4.11}$$

The functions $\cos(\omega t + \psi)$ and $\sin(\omega t + \psi)$ are linearly independent. Thus, Equation (4.11) implies that

$$(\omega_n^2 - \omega^2)U = 0 \tag{4.12}$$

and

$$(\omega_n^2 - \omega^2)V = \frac{F_0}{m_{\text{eq}}} \tag{4.13}$$

if $\omega \neq \omega_n$, Equation (4.12) implies $U = 0$ and then from Equation (4.13)

$$V = \frac{F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)} \tag{4.14}$$

The particular solution for $\omega \neq \omega_n$ becomes

$$x_p(t) = \frac{F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)} \sin(\omega t + \psi) \tag{4.15}$$

or alternately,

$$x_p(t) = \left| \frac{F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)} \right| \sin(\omega t + \psi - \phi) \tag{4.16}$$

where the amplitude of the particular solution is positive and

$$\phi = \begin{cases} 0 & \omega_n > \omega \\ \pi & \omega_n < \omega \end{cases} \tag{4.17}$$

The response is in phase with the excitation if $\omega_n > \omega$ and 180 degrees out of phase if $\omega_n < \omega$.

The general solution is formed by adding the homogeneous solution to the particular solution. Then the initial conditions are applied yielding

$$x(t) = \left[x_0 - \frac{F_0 \sin\psi}{m_{eq}(\omega_n^2 - \omega^2)}\right]\cos(\omega_n t) + \frac{1}{\omega_n}\left[\dot{x}_0 - \frac{F_0\omega\cos\psi}{m_{eq}(\omega_n^2 - \omega^2)}\right]\sin(\omega_n t) + \left|\frac{F_0}{m_{eq}(\omega_n^2 - \omega^2)}\right|\sin(\omega t + \psi - \phi) \tag{4.18}$$

The response, plotted in Figure 4.3, is the sum of two trigonometric terms of different frequencies.

The case when $\omega = \omega_n$ is special. The nonhomogeneous term in Equation (4.9) and the homogeneous solution are not linearly independent. Thus, when the method of undetermined coefficients is used to determine the particular solution, Equation (4.12) is identically satisfied and Equation (4.13) cannot be satisfied unless $V = \infty$. A particular solution is assumed in this case as

$$x_p(t) = Ut\sin(\omega_n t + \psi) + Vt\cos(\omega_n t + \psi) \tag{4.19}$$

Substitution of Equation (4.19) in Equation (4.9) leads to

$$x_p(t) = -\frac{F_0}{2m_{eq}\omega_n}t\cos(\omega_n t + \psi) \tag{4.20}$$

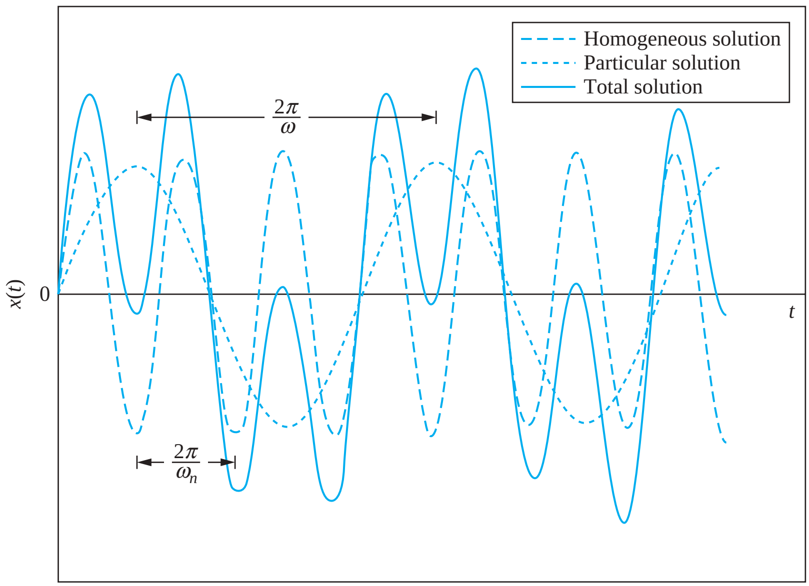


FIGURE 4.3
Response of an undamped SDOF system when $\omega < \omega_n$.

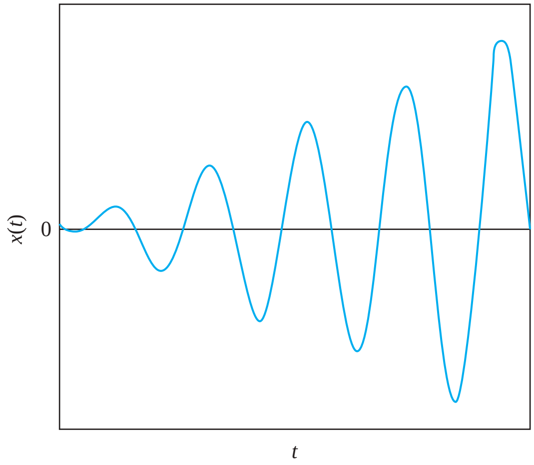


FIGURE 4.4
Undamped response when the excitation frequency equals the natural frequency. The response grows without bound producing resonance.

Application of initial conditions to the sum of the homogeneous and particular solution yields

$$x(t) = x_0 \cos(\omega_n t) + \left(\frac{\dot{x}_0}{\omega_n} + \frac{F_0 \cos\psi}{2m_{\text{eq}}\omega_n^2}\right)\sin(\omega_n t) - \frac{F_0}{2m_{\text{eq}}\omega_n} t \cos(\omega_n t + \psi) \tag{4.21}$$

The response of a system in for which the excitation frequency equals the natural frequency is illustrated in Figure 4.4. Since the amplitude of the response is proportional to t it grows without bound producing a condition called *resonance*. The resonance leads to an amplitude increase to a value where the assumptions used in modeling the physical system are no longer valid. For example in a system with a helical coil spring the proportional limit of the spring's material is exceeded as the amplitude increases. After this time the motion is governed by a nonlinear differential equation.

Resonance is a dangerous condition in a mechanical or structural system and will produce unwanted large displacements or lead to failure. Resonant torsional oscillations were partially the cause of the famous Tacoma Narrows Bridge disaster. It is suspected that the frequency at which vortices were shed from the bridge co-incided with a torsional natural frequency, leading to oscillations that grew without bound.

When vibrations of a conservative system are initiated, the motion is sustained at the system's natural frequency without additional energy input. Thus, when the frequency of excitation is the same as the natural frequency, the work done by the external force is not needed to sustain motion. The total energy increases because of the work input and leads to a continual increase in amplitude. When the frequency of excitation is different from the natural frequency, the work done by the external force is necessary to sustain motion at the excitation frequency.

When the excitation frequency is close, but not exactly equal, to the natural frequency, an interesting phenomenon called *beating* occurs. Beating is a continuous buildup and decrease of amplitude as shown in Figure 4.5. When ω is very close to ω_n and $x_0 = \dot{x}_0 = 0$ and $\psi = 0$, Equation (4.18) can be written as

$$x(t) = \frac{2F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)} \sin\left[\left(\frac{\omega - \omega_n}{2}\right)t\right] \cos\left[\left(\frac{\omega + \omega_n}{2}\right)t\right] \tag{4.22}$$

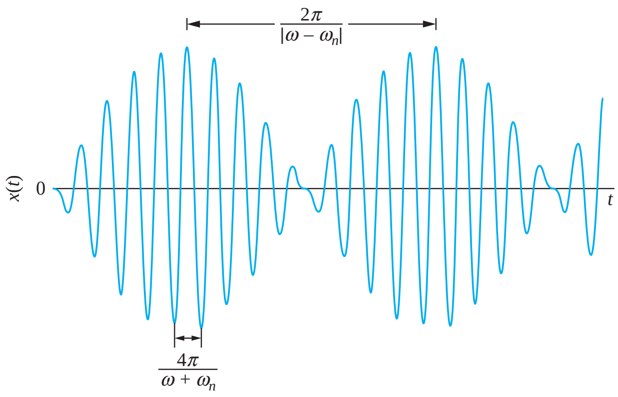


FIGURE 4.5
Beating, which occurs in an undamped system when $\omega \approx \omega_n$, is characterized by a continual build-up and decay of amplitude.

Since $|\omega - \omega_n|$ is small the solution, Equation (4.22) is viewed as a cosine wave with a slowly varying amplitude

$$x(t) = A(\varepsilon t)\cos\beta t \tag{4.23}$$

where

$$\beta = \frac{1}{2}(\omega + \omega_n) \tag{4.24}$$

is the frequency of the vibration and

$$\varepsilon = \frac{1}{2}|\omega - \omega_n| \tag{4.25}$$

is the frequency of the beating and

$$A(\varepsilon t) = \frac{2F_0}{m_{eq}\varepsilon\beta}\sin\varepsilon t \tag{4.26}$$

The amplitude reaches a maximum value of $\frac{2F_0}{m_{eq}\varepsilon\beta}$ when $\varepsilon t = \frac{1}{2}(2n-1)\pi$ for any integer $n = 1, 2, \ldots$

EXAMPLE 4.1

The equivalent mass of a SDOF of 10 kg. The system has a natural frequency of 80 rad/s. The system is at rest in equilibrium when it is subject to a time dependent force. Determine and plot the response of the system if it is subject to a force of (a) 10 sin(40t)N, (b) 10 sin(80t) N, and (c) 10 sin(82t) N.

SOLUTION

(a) The input is a single frequency excitation of frequency 40 r/s with $\psi = 0$. Since the excitation frequency is not equal to or close to the natural frequency the response of the system is given by Equation (4.18) which leads to

$$\begin{aligned} x(t) &= \frac{(10\text{ N})}{(10\text{ kg})\left[(80\text{ rad/s})^2 - (40\text{ rad/s})^2\right]}\left[\sin(40t) - \frac{40\text{ rad/s}}{80\text{ rad/s}}\sin(80t)\right] \\ &= 2.08\times10^{-4}\left[\sin(40t) - 0.5\sin(80t)\right]\text{ m} \end{aligned} \tag{a}$$

Equation (a) is plotted in Figure 4.6(a). Two distinct frequencies are shown.

(b) The natural frequency is equal to the excitation frequency, hence resonance occurs. The solution is for this case is given by Equation (4.21)

$$\begin{aligned} x(t) &= \frac{10\ \text{N}}{2(10\ \text{kg})(80\ \text{rad/s})}\left[\left[\frac{1}{80\ \text{rad/s}}\sin(80t) - t\cos(80t)\right]\right. \\ &= 6.25 \times 10^{-3}[0.125\sin(80t) - t\cos(80t)]\ \text{m} \end{aligned} \tag{b}$$

Equation (b) is shown in Figure 4.6(b). The unbounded growth in amplitude is evident.

(c) The excitation frequency is close to but not equal to the natural frequency. Thus, Equation (4.22) is the applicable solution

$$\begin{aligned} x(t) &= \frac{2(10\ \text{N})}{(10\ \text{kg})[(80\ \text{rad/s})^2 - (82\ \text{rad/s})^2]} \\ &\quad \times \left[\sin\left(\frac{82\ \text{rad/s} - 80\ \text{rad/s}}{2}t\right)\cos\left(\frac{82\ \text{rad/s} + 80\ \text{rad/s}}{2}t\right)\right] \\ &= -6.17 \times 10^{-3}\sin t\cos(81t)\ \text{m} \end{aligned} \tag{c}$$

Equation (c) is plotted in Figure 4.6(c) where the build up and decay of amplitude is obvious. The period of vibration is

$$T = \frac{2\pi}{81} = 00776\ \text{s} \tag{d}$$

and the period of beating is

$$T_b = 2\pi = 6.28\ \text{s} \tag{e}$$

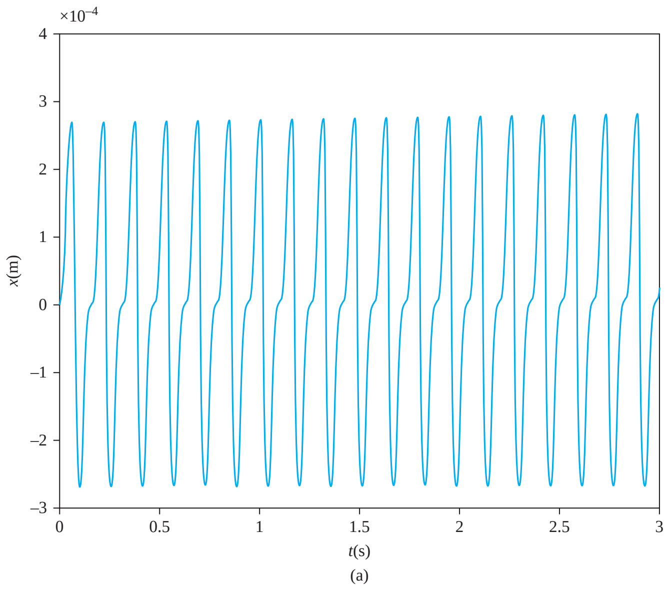
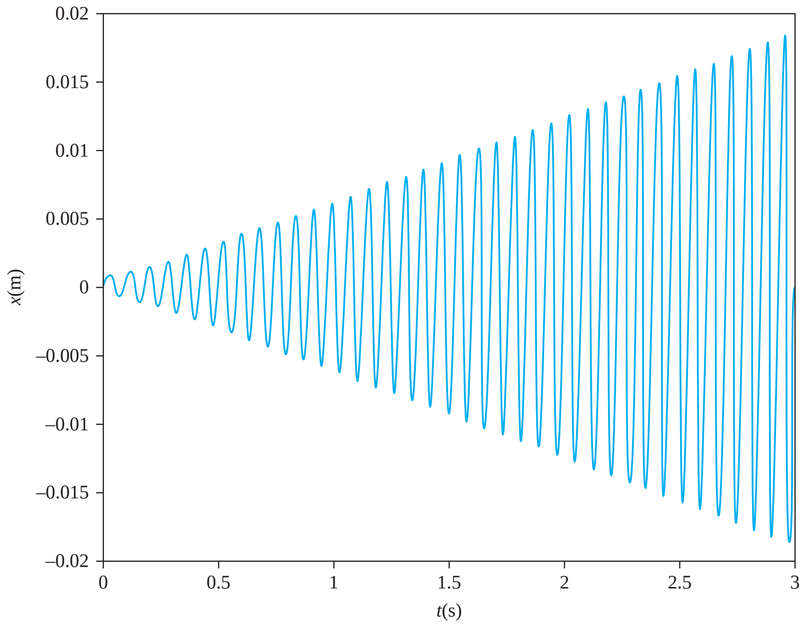
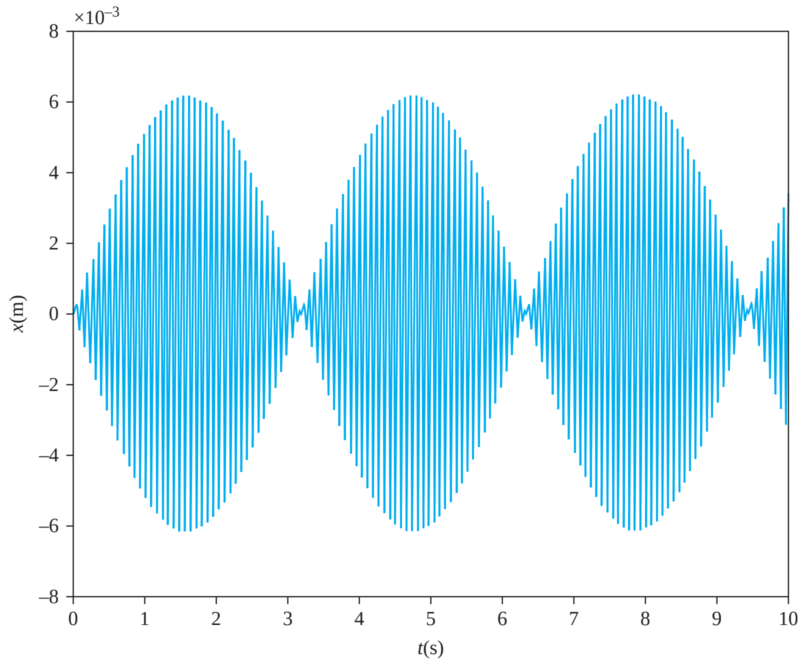


FIGURE 4.6
Response of system of Example 4.1 for (a) $\omega = 40$ rad/s, (b) $\omega = 80$ rad/s for which resonance occurs; and (c) $\omega = 82$ rad/s for which beating occurs with a period of $T = 6.28$ s.



(b)



(b)

FIGURE 4.6
(*Continued*)

4.3 FORCED RESPONSE OF A VISCOUSLY DAMPED SYSTEM SUBJECT TO A SINGLE-FREQUENCY HARMONIC EXCITATION

The standard form of the differential equation governing the motion of a viscously damped SDOF system with the single-frequency harmonic excitation of Equation (4.9) is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m_{\text{eq}}}\sin(\omega t + \psi) \tag{4.27}$$

A particular solution is assumed as

$$x_p(t) = U\cos(\omega t + \psi) + V\sin(\omega t + \psi) \tag{4.28}$$

Substitution of Equation (4.28) into Equation (4.27) leads to the following simultaneous equations for U and V

$$(\omega_n^2 - \omega^2)U + 2\zeta\omega\omega_n V = 0 \tag{4.29}$$

$$-2\zeta\omega\omega_n U + (\omega_n^2 - \omega^2)V = \frac{F_0}{m_{\text{eq}}} \tag{4.30}$$

Solving these equations and substituting the results into Equation (4.28) leads to

$$x_p(t) = \frac{F_0}{m_{\text{eq}}\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2\right]}\left[-2\zeta\omega\omega_n\cos(\omega t + \psi) + (\omega_n^2 - \omega_2)\sin(\omega t + \psi)\right] \tag{4.31}$$

Use of the trigonometric identity for the sine of the difference of angles and algebraic manipulation leads to the following alternate form of Equation (4.31)

$$x_p(t) = X\sin(\omega t + \psi - \phi) \tag{4.32}$$

where
$$X = \frac{F_0}{m_{\text{eq}}\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2\right]^{1/2}} \tag{4.33}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right) \tag{4.34}$$

X is the amplitude of the forced response and ϕ is the phase angle between the response and the excitation.

The amplitude and phase angle provide important information about the forced response. Formulation of Equations (4.33) and (4.34) in nondimensional form allows better qualitative interpretation of the response. It is noted from these equation that

$$X = f(F_0, m_{\text{eq}}, \omega, \omega_n, \zeta) \tag{4.35}$$

and

$$\phi = g(\omega, \omega_n, \zeta) \tag{4.36}$$

The parameters use three basic dimensions: mass, length, and time. The Buckingham Pi theorem (Section 1.5) implies that the formulation of the amplitude relationship is a function of $6 - 3 = 3$ non-dimensional parameters. One is a dependent parameter involving the amplitude and the other two independent parameters.

Multiplying Equation (4.33) by $m_{eq}\omega_n^2/F_0$ gives

$$\frac{m_{eq}\omega_n^2 X}{F_0} = \frac{1}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \tag{4.37}$$

where $$r = \frac{\omega}{\omega_n} \tag{4.38}$$

is the frequency ratio. The ratio

$$M = \frac{m_{eq}\omega_n^2 X}{F_0} \tag{4.39}$$

is dimensionless and is often called the *amplitude ratio* or *magnification factor*. The magnification factor has the interpretation that it is the ratio of the amplitude of response to the static deflection of a spring of stiffness k due to a constant force F_0,

$$M = \frac{X}{\Delta_{st}} \tag{4.40}$$

An alternate interpretation is that it is the maximum force developed in the spring of a mass-spring and viscous-damper system, $F_{max} = kX = m\omega_n^2 X$ to the maximum of the excitation. It represents how much the force is magnified by the system. The magnification factor is really a force ratio, necessary for dynamic similitude

$$M = \frac{F_{max}}{F_0} \tag{4.41}$$

Thus the nondimensional form of Equation (4.33) is

$$M(r, \zeta) = \frac{1}{\sqrt{(1 - r^2)^2 - (2\zeta r)^2}} \tag{4.42}$$

The magnification factor as a function of frequency ratio for different values of the damping ratio is shown in Figure 4.7. These curves are called frequency response curves. The following are noted about Equation 4.42 and Figure 4.7.

1. $M = 1$ when $r = 0$. In this case the excitation force is a constant and the maximum force developed in the spring of a mass-spring-dashpot system is equal to the value of the exciting force.
2. $\lim_{r\to\infty} M(r, \zeta) = \frac{1}{r^2}$. The amplitude of the forced response is very small for high-frequency excitations.
3. For a given value of r, M decreases with increasing ζ.
4. The magnification factor grows without bound only for $\zeta = 0$. For $0 < \zeta \leq 1/\sqrt{2}$, the magnification factor has a maximum for some value of ζ.

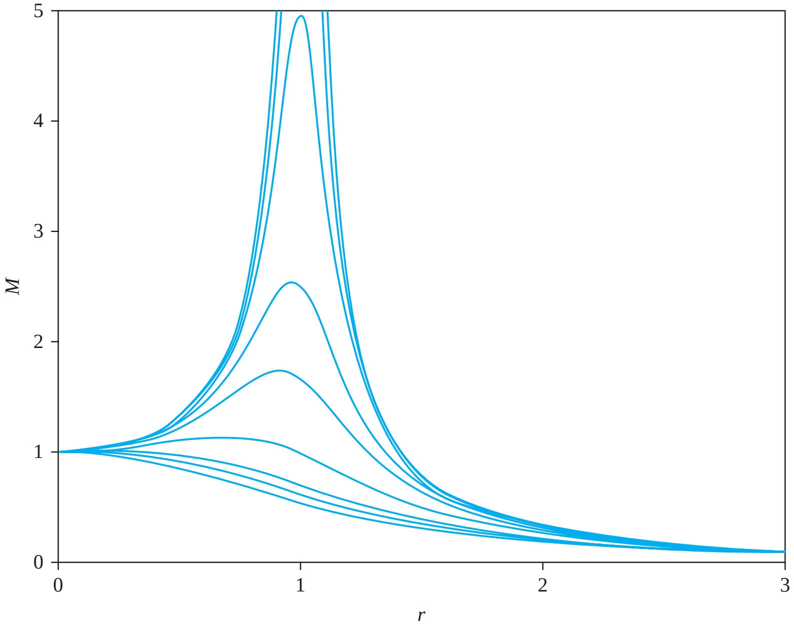


FIGURE 4.7
Magnification factor versus frequency ratio for different values of the damping ratio.

5. For $0 < \zeta \leq 1/\sqrt{2}$, the maximum value of the magnification factor occurs for a frequency ratio of

$$r_m = \sqrt{1 - 2\zeta^2} \tag{4.43}$$

Equation (4.43) is obtained from Equation (4.42) by determining the value of r such that $dM/dr = 0$.

6. The corresponding maximum value of M is

$$M_{\max} = \frac{1}{2\zeta(1 - \zeta^2)^{1/2}} \tag{4.44}$$

7. For $\zeta = 1/\sqrt{2}$, $dM/dr = 0$ for $r = 0$. For $\zeta \geq 1/\sqrt{2}$, there is no real value of r satisfying Equation (4.43). $M(r, \zeta)$ does not achieve a maximum. It monotonically decreases with increasing r and approaches zero as $1/r^2$ for large r.

The nondimensinoal form of Equation (4.34) is

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{4.45}$$

The phase angle from Equation (4.45) is plotted as a function of frequency ratio for different values of the damping ratio in Figure 4.8. The following are noted from Equation 4.45 and Figure 4.8:

1. The forced response and the excitation force are in phase for $\zeta = 0$. For $\zeta > 0$, the response and excitation are in phase only for $r = 0$.

2. If $\zeta = 0$ and $0 < r < 1$, then $0 < \phi < \pi/2$. The response lags the excitation.

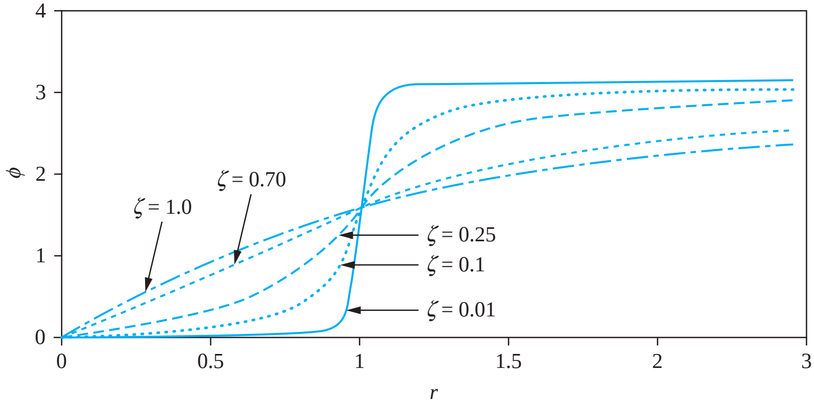


FIGURE 4.8
Phase angle versus frequency ratio for different values of the damping ratio.

3. If $\zeta < 0$ and $r = 1$, then $\phi = \pi/2$. If $\psi = 0$, then the excitation is a pure sine wave while the steady-state response is a pure cosine wave. The excitation is in phase with the velocity. The direction of the excitation is always the same as the direction of motion.
4. If $\zeta > 0$ and $r > 1$, then $\pi/2 < \phi < \pi$. The response leads the excitation as shown in Figure 4.9.
5. If $\zeta > 0$ and $r \gg 1$, then $\phi \approx \pi$. The sign of the steady-state response is opposite that of the excitation.
6. For $\zeta = 0$, the response is in phase with the excitation for $r < 1$ and π radians (180°) out of phase for $r > 1$.

Equation (4.42) and (4.45) constitute the frequency response of a SDOF system. The frequency response is the variation of the steady-state amplitude and the steady-state phase. The graphical representation of the frequency response is illustrated in Figures 4.7 and 4.8.

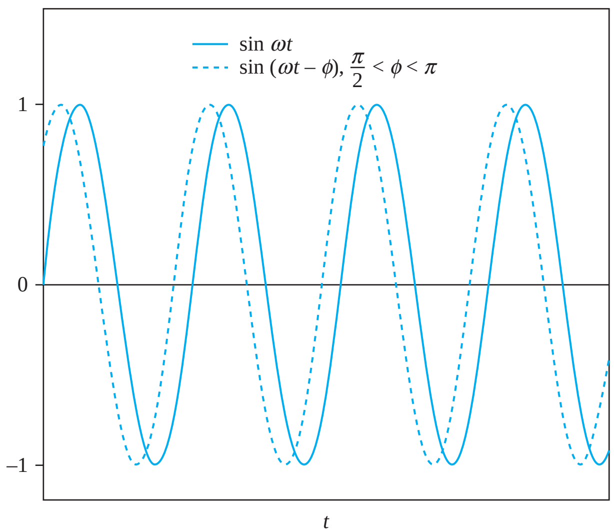


FIGURE 4.9
Response leads excitation when $r > 1$.

If the stiffness or damping ratio of a system is not known the frequency response may be determined experimentally and used to identify the system parameters.

The steady-state response of an SDOF system due to a single-frequency harmonic excitation is

$$x(t) = \frac{F_0}{m_{eq}\omega_n^2} M(r,\zeta) \sin(\omega t + \psi - \phi) \tag{4.46}$$

where $M(r, \zeta)$ is given by Equation (4.42) and ϕ is given by Equation (4.45). The theory can handle the undamped response covered in Section 4.2 by taking $\zeta = 0$ these equations yielding

$$M(r,0) = \frac{1}{\sqrt{(1-r^2)^2}} = \frac{1}{|1-r^2|} \tag{4.47}$$

and

$$\phi = \tan^{-1}\left(\frac{0}{1-r^2}\right) = \begin{cases} 0 & r < 1 \\ \pi & r > 1 \end{cases} \tag{4.48}$$

The value of the magnification factor $M(1, 0)$ does not exist, as there is no steady-state in the case of an undamped SDOF system under resonant conditions.

EXAMPLE 4.2

A moment, $M_0 \sin \omega t$, is applied to the end of the bar of Figure 4.10. Determine the maximum value of M_0 such that the steady-state amplitude of angular oscillation does not exceed 10° if $\omega = 500$ rpm, $k = 7000$ N/m, $c = 650$ N·s/m, $L = 1.2$ m, and the mass of the bar is 15 kg.

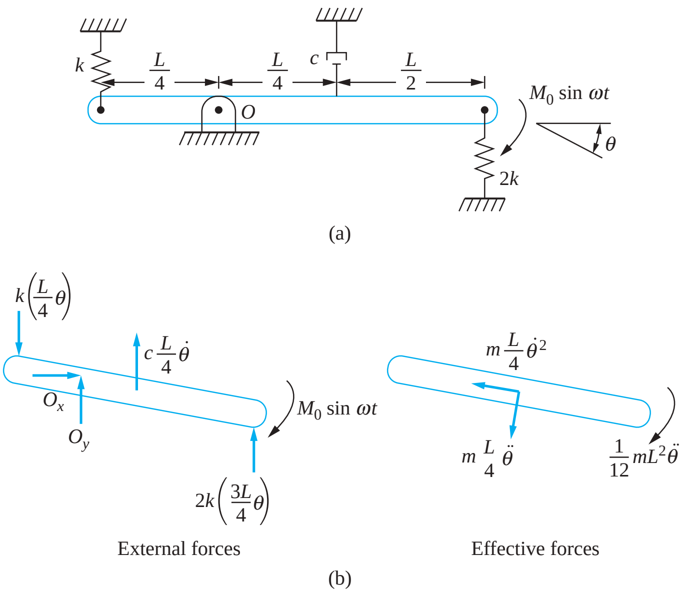


FIGURE 4.10
(a) System of Example 4.2. (b) FBDs at an arbitrary instant.

SOLUTION

The differential equation obtained by summing moments about 0 using the free-body diagrams of Figure 4.10(b) is

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{1}{16}cL^2\dot{\theta} + \frac{19}{16}kL^2\theta = M_0 \sin\omega t \tag{a}$$

Using the notation of Equation (4.1)

$$I_{\text{eq}} = \frac{7}{48}mL^2 = \frac{7}{48}(15\text{ kg})(1.2\text{ m})^2 = 3.15\text{ kg}\cdot\text{m}^2 \tag{b}$$

The differential equation is rewritten in the form of Equation (4.2) by dividing by I_{eq}:

$$\ddot{\theta} + \frac{3}{7}\frac{c}{m}\dot{\theta} + \frac{57}{7}\frac{k}{m}\theta = \frac{M_0}{I_{\text{eq}}}\sin\omega t \tag{c}$$

The preceding equation has a steady-state solution of the form

$$\theta(t) = \Theta \sin(\omega t - \phi) \tag{d}$$

The natural frequency and damping ratio are obtained by comparison to Equation (4.2)

$$\omega_n = \sqrt{\frac{57}{7}\frac{k}{m}} = \sqrt{\frac{(57)(7000\text{ N/m})}{(7)(15\text{ kg})}} = 61.6\ \frac{\text{rad}}{\text{s}} \tag{e}$$

$$\zeta = \frac{3}{14}\frac{c}{m\omega_n} = \frac{(3)(650\text{ N}\cdot\text{s/m})}{(14)(15\text{ kg})(61.6\text{ rad/s})} = 0.15 \tag{f}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{(500\text{ rev/min})(2\pi\text{ rad/rev})(1\text{ min/60 s})}{61.6\text{ rad/s}} = 0.85 \tag{g}$$

The magnification factor is calculated from Equation (4.42)

$$M(0.85, 0.15) = \frac{1}{\sqrt{[1-(0.85)^2]^2 + [2(0.15)(0.85)]^2}} = 2.64 \tag{h}$$

The maximum allowable magnitude of the applied moment is calculated using Equation (4.37),

$$\frac{I_{\text{eq}}\omega_n^2\Theta}{M_0} = M(0.85, 0.15) = 2.64 \tag{i}$$

Requiring $\Theta < 10°$ leads to

$$M_0 < \frac{(3.15\text{ kg}\cdot\text{m}^2)(61.6\text{ rad/s})^2(10°)(2\pi\text{ rad/360°})}{2.64} = 790.2\text{ N}\cdot\text{m} \tag{j}$$

EXAMPLE 4.3

A machine of mass 25.0 kg is placed on an elastic foundation. A sinusoidal force of magnitude 25 N is applied to the machine. A frequency sweep reveals that the maximum steady-state amplitude of 1.3 mm occurs when the period of response is 0.22 s. Determine the equivalent stiffness and damping ratio of the foundation.

SOLUTION

The system is modeled as a mass attached to a spring in parallel with a viscous damper with a applied sinusoidal force of amplitude 25 N. For a linear system the frequency of response is the same as the frequency of excitation. Thus the maximum response occurs for a period of 0.22 s which corresponds to a frequency of

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.22\,\mathrm{s}} = 28.6\ \mathrm{rad/s} \tag{a}$$

The frequency ratio at which the maximum response occurs is given by Equation (4.43)

$$r = \frac{\omega}{\omega_n} = \sqrt{1 - 2\zeta^2} \tag{b}$$

Solving Equation (b) for the natural frequency

$$\omega_n = \frac{\omega}{\sqrt{1-2\zeta^2}} = \frac{28.6\ \mathrm{rad/s}}{\sqrt{1-2\zeta^2}} \tag{c}$$

The maximum value of the response is given by Equation (4.44) which upon substitution and use of Equation (4.39) becomes

$$\frac{(25.0\ \mathrm{kg})\ (0.0013\ \mathrm{m})(28.6\ \mathrm{rad/s})^2}{(25\ \mathrm{N})\ (1-2\zeta^2)} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{d}$$

Squaring Equation (d) and rearranging leads to

$$\zeta^4 - \zeta^2 + 0.118 = 0 \tag{e}$$

which is a quadratic equation for ζ^2. Using the quadratic formula leads to $\zeta = 0.369$, 0.929. The larger value is discarded because a frequency sweep would only yield a maximum for a value of $\zeta < \frac{1}{\sqrt{2}}$. Thus $\zeta = 0.369$. The natural frequency is calculated from Equation (c) as

$$\omega_n = \frac{28.6\ \mathrm{rad/s}}{\sqrt{1-2(0.369)^2}} = 33.5\,\mathrm{rad/s} \tag{f}$$

The stiffness of the foundation is

$$k = m\omega_n^2 = (25.0\,\mathrm{kg})\ (33.5\ \mathrm{rad/s})^2 = 2.80 \times 10^4\,\mathrm{N/m} \tag{g}$$

4.4 FREQUENCY-SQUARED EXCITATIONS

4.4.1 GENERAL THEORY

Many SDOF system are subject to single-frequency harmonic excitation whose amplitude is proportional to the square of its frequency

$$F_{\mathrm{eq}}(t) = A\omega^2 \sin(\omega t + \psi) \tag{4.49}$$

where A is a constant of proportionality with dimensions of $F \cdot T^2$ or $M \cdot L$. When $F_{\mathrm{eq}}(t)$ represents a moment A it has dimensions of $F \cdot L \cdot T^2$ or $M \cdot L^2$. The steady-state response due to this type of excitation is developed by applying equations developed in Section 4.3 with

$$F_0 = A\omega^2 \tag{4.50}$$

Substitution of Equation (4.50) into Equation (4.37) yields

$$\left(\frac{m_{\mathrm{eq}}X}{A}\right)\left(\frac{\omega_n}{\omega}\right)^2 = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

or $$m_{\mathrm{eq}}\frac{X}{A} = \Lambda(r, \zeta) \tag{4.51}$$

where $$\Lambda(r, \zeta) = \frac{r^2}{\sqrt{(1 - r^2) + (2\zeta r)^2}} \tag{4.52}$$

Λ is, like M, a nondimensional function of the frequency ratio and the damping ratio. Λ is related to M by

$$\Lambda(r, \zeta) = r^2 M(r, \zeta) \tag{4.53}$$

The steady-state response is given by Equation (4.32) where X is determined from Equations (4.51) and (4.52), and ϕ is determined using Equation (4.45).

Λ is plotted as a function of r for various values of ζ in Figure 4.11. The following are noted from Equation (4.52) and Figure 4.11.

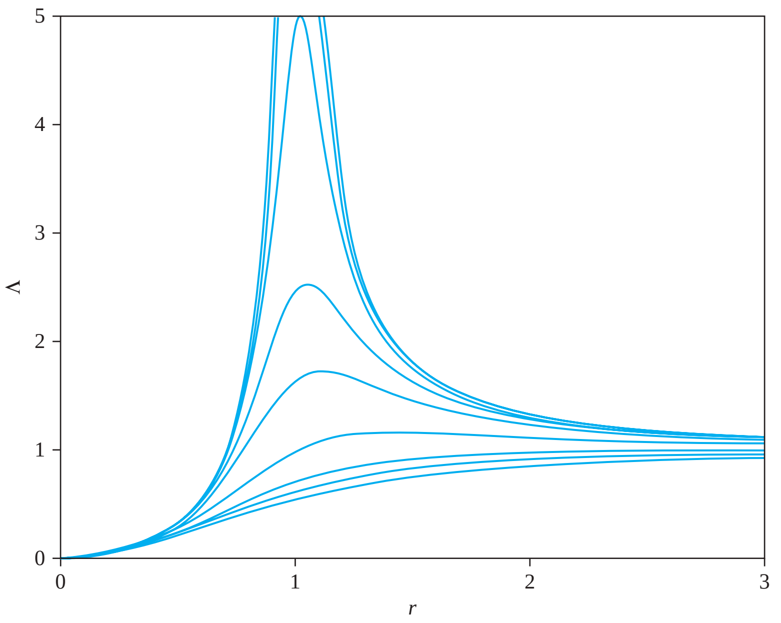


FIGURE 4.11
$\Lambda(r, \zeta)$ versus r for different values of ζ.

1. $\Lambda = 0$ if and only if $r = 0$ for all values of ζ.
2. $\lim_{r \to 0} \Lambda(r, \zeta) = 1$ for all values of ζ.
3. Λ grows without bound near $r = 1$ for $\zeta = 0$.
4. For $0 < \zeta < 1/\sqrt{2}$, Λ has a maximum for a frequency ratio of

$$r_m = \frac{1}{\sqrt{1 - 2\zeta^2}} \tag{4.54}$$

Equation (4.54) is derived by finding the value of r such that $d\Lambda/dr = 0$.

5. For a given $0 < \zeta < 1/\sqrt{2}$, the maximum value of Λ corresponds to the frequency ratio of Equation (4.54) and is given by

$$\Lambda_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{4.55}$$

6. For $\zeta > 1/\sqrt{2}$, Λ does not reach a maximum. Λ grows slowly from zero near $r = 0$, monotonically increases, and asymptotically approches one from below.

EXAMPLE 4.4

A one-degree-of-freedom system is subject to a harmonic excitation whose magnitude is proportional to the square of its frequency. The frequency of excitation is varied and the steady-state amplitude noted. A maximum amplitude of 8.5 mm occurs at a frequency of 200 Hz. When the frequency is much higher than 200 Hz, the steady-state amplitude is 1.5 mm. Determine the damping ratio for the system.

SOLUTION

From Figure 4.11, $\Lambda \to 1$ as $r \to \infty$. Thus, from Equation (4.51) and the given information,

$$\frac{m_{\text{eq}}}{A} = \frac{1}{1.5\,\text{mm}} \tag{a}$$

Substituting Equation (a) into Equation (4.55) yields

$$\Lambda_{\max} = \frac{m}{A} X_{\max} = \frac{8.5\,\text{mm}}{1.5\,\text{mm}} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{b}$$

Inverting, squaring, and rearranging leads to

$$\zeta^4 - \zeta^2 + 0.00778 = 0 \tag{c}$$

The roots of Equation (c) are $\zeta = \pm 0.089, \pm 0.996$. Since a maximum was attained, $0 < \zeta < \frac{1}{\sqrt{2}}$, the appropriate value of ζ is 0.089.

4.4.2 ROTATING UNBALANCE

The machine of Figure 4.12(a) has a component which rotates at a constant speed, ω. Its center of mass is located a distance e, called the eccentricity, from the axis of rotation. The mass of the rotating component is m_0, while the total mass of the machine, including the rotating component, is m. The machine is constrained to move vertically.

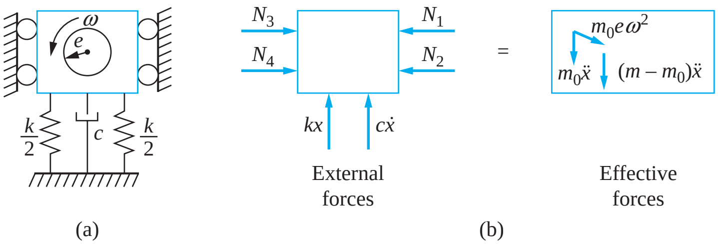


FIGURE 4.12
(a) Machine with a rotating unbalance produces a harmonic excitation whose amplitude is proportional to the square of its frequency. (b) FBDs of the machine at an arbitrary instant.

Let x represent the downward motion of the machine. The acceleration of the rotating component is obtained using the relative acceleration equation

$$\mathbf{a}_{\mathbf{r}} = \mathbf{a}_{\mathbf{c}} + \mathbf{a}_{\mathbf{r/c}} \tag{4.56}$$

where $|\mathbf{a}_{\mathbf{c}}| = \ddot{x}$ and is directed downward and $|\mathbf{a}_{\mathbf{r/c}}| = e\omega^2$ directed toward the center of rotation. The center of mass of the rotating component moves in a circular path about the center of rotation at a constant speed. Let θ represent the angle made by the line segment between the center of rotation and the center of mass at an arbitrary instant. Resolving the relative acceleration into horizontal and vertical components the vertical component of the absolute acceleration of the center of mass of the rotating component is

$$a_{r,\,x} = \ddot{x} + e\omega^2 \sin\theta \tag{4.57}$$

Summation of forces, $\sum F_{\text{ext}} = \sum F_{\text{eff}}$ applied in the vertical direction, positive downward to the FBDs of Figure 4.12(b) yields

$$-kx - c\dot{x} = m\ddot{x} + m_0 e\omega^2 \sin\theta \tag{4.58}$$

For constant ω,

$$\theta = \omega t + \theta_0 \tag{4.59}$$

where θ_0 is an angle between the initial position of the center of mass of the rotating component and the horizontal. Using Equation (4.59) in Equation (4.58), and rearranging yields

$$m\ddot{x} + c\dot{x} + kx = -m_0 e\omega^2 \sin(\omega t + \theta_0) \tag{4.60}$$

The negative sign is incorporated into the sine function by defining $\psi = \theta_0 + \pi$. Then Equation (4.60) becomes

$$m\ddot{x} + c\dot{x} + kx = m_0 e\omega^2 \sin(\omega t + \psi) \tag{4.61}$$

It is apparent from Equation (4.61) that the unbalanced rotating component leads to a harmonic excitation whose amplitude is proportional to the square of its frequency. The constant of proportionality is

$$A = m_0 e \tag{4.62}$$

Using Equation (4.51) gives

$$\frac{mX}{m_0 e} = \Lambda(r, \zeta) \tag{4.63}$$

EXAMPLE 4.5

A 150-kg electric motor has a rotating unbalance of 0.5 kg, 0.2 m from the center of rotation. The motor is to be mounted at the end of a steel ($E = 210 \times 10^9$ N/m^2) cantilever beam of length 1 m. The operating range of the motor is from 500 to 1200 rpm. For what values of I, the beam's cross-sectional moment of inertia, will the steady-state amplitude of vibration be less than 1 mm? Assume the damping ratio is 0.1.

SOLUTION

The maximum allowable value of Λ is

$$\Lambda_{\text{allow}} = \frac{mX_{\text{allow}}}{m_0 e} = \frac{(150 \text{ kg})(0.001 \text{ m})}{(0.5 \text{ kg})\ (0.2 \text{ m})} = 1.5 \qquad \text{(a)}$$

Since $\Lambda_{\text{allow}} > 1$ and $\zeta < 1/\sqrt{2}$, Figure 4.11 shows that two values of r correspond to $\Lambda = \Lambda_{\text{allow}}$. These are determined using Equation (4.52)

$$1.5 = \frac{r^2}{\sqrt{(1 - r^2) + (0.2r)^2}} \qquad \text{(b)}$$

Rearrangement leads to the following equation:

$$0.556r^4 - 1.96r^2 + 1 = 0 \qquad \text{(c)}$$

whose positive roots are

$$r = 0.787, \quad 1.71 \qquad \text{(d)}$$

However if $r = 0.787$ corresponds to $\omega = 1200$ rpm then $\Lambda < \Lambda_{\text{allow}}$ for all r in the operating range. Whereas if $r = 0.787$ corresponds to $\omega = 500$ rpm then $\Lambda > \Lambda_{\text{allow}}$ for r over part of the operating range. Thus requiring $r < 0.787$ over the entire operating range yields.

$$\frac{(1200 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{\omega_n} < 0.787 \qquad \text{(e)}$$

or $\omega_n > 159.7$ rad/s. The one degree-of-freedom approximation for the natural frequency of the motor attached to the end of a cantilever beam of negligible mass is

$$\omega_n = \sqrt{\frac{3EI}{mL^3}} \qquad \text{(f)}$$

Thus,

$$I > \frac{(159.7 \text{ rad/s})^2 L^3 m}{3E} = \frac{(159.7 \text{ rad/s})^2(1 \text{ m})^3(150 \text{ kg})}{3(210 \times 10^9 \text{ N/m}^2)} = 6.07 \times 10^{-6} \text{m}^4 \qquad \text{(g)}$$

Using a similar reasoning $r = 1.71$ should correspond to $\omega = 500$ rpm. Thus,

$$\frac{(500 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{\omega_n} > 1.71 \qquad \text{(h)}$$

or $\omega_n < 30.6$ rad/s. This requirement leads to $I < 2.23 \times 10^{-7}$ m^4.

Thus the amplitude of vibration will be limited to 1 mm if $I > 6.08 \times 10^{-6}$ m^4 or $I < 2.23 \times 10^{-7}$ m^4. However, other considerations limit the design of the beam. The smaller the moment of inertia, the larger the bending stress in the outer fibers of the beam at the support.

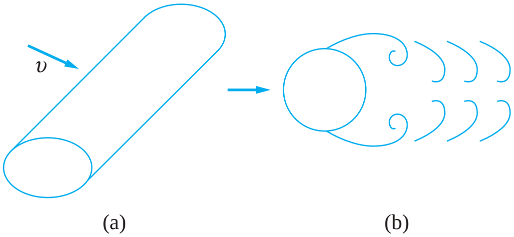


FIGURE 4.13
(a) Circular cylinder in steady flow.
(b) Cross section of cylinder showing vortices shed alternately from each surface of the cylinder, resulting in a wake behind the cylinder and a harmonic force acting on the cylinder.

4.4.3 VORTEX SHEDDING FROM CIRCULAR CYLINDERS

When a circular cylinder is placed in a steady uniform stream at sufficient velocity, flow separation occurs on the cylinder's surface, as illustrated in Figure 4.13. The separation leads to vortex shedding from the cylinder and the formation of a wake behind the cylinder. Vortices are shed alternately from the upper and lower surfaces of the cylinder at a constant frequency. The alternate shedding of vortices causes oscillating streamlines in the wake which, in turn, lead to an oscillating pressure distribution. The oscillating pressure distribution, in turn, gives rise to an oscillating force acting normal to the cylinder,

$$F(t) = F_0 \sin(\omega t) \tag{4.64}$$

where F_0 is the magnitude of the force and ω is the frequency of vortex shedding.

These parameters are dependent upon the fluid properties and the geometry of the cylinder. That is,

$$F_0 = F_0(v, \rho, \mu, D, L) \tag{4.65}$$

and $$\omega = \omega(v, \rho, \mu, D, L) \tag{4.66}$$

where v = the magnitude of fluid velocity, $[L]/[T]$
ρ = the fluid density, $[M]/[L]^3$
μ = the dynamic viscosity of fluid, $[M]/([L][T])$
D = the diameter of cylinder, $[L]$
L = the length of cylinder, $[L]$

The dependent parameters F_0 and ω are both functions of five independent parameters. Dimensional analysis theory implies that Equations (4.65) and (4.66) can be rewritten as relationships between three dimensionless parameters. Indeed, nondimensional forms of Equations (4.65) and (4.66) are

$$C_D = f\left(\mathrm{Re}, \frac{D}{L}\right) \tag{4.67}$$

$$S = f\left(\mathrm{Re}, \frac{D}{L}\right) \tag{4.68}$$

The dependent dimensionless parameters are the drag coefficient

$$C_D = \frac{F_0}{\frac{1}{2}\rho v^2 DL} \tag{4.69}$$

which is the ratio of the drag force to the inertia force, and the Strouhal number

$$S = \frac{\omega D}{2\pi v} \tag{4.70}$$

which is the ratio of the inertia force due to the local acceleration of the inertia force due to the convective acceleration of the inertia force.

The independent dimensionless parameters are the Reynolds number

$$\mathrm{R} = \frac{\rho v D}{\mu} \tag{4.71}$$

which is the ratio of the inertia force to the viscous force and the diameter-to-length ratio D/L.

For long cylinders ($D/L \ll 1$), a two-dimensional approximation is used. Then the effect of D/L on the drag coefficient and Strouhal number is negligible. Empirical data are used to determine the forms of Equations (4.67) and (4.68) assuming that both the drag coefficient and Strouhal number are independent of D/L.

The density and dynamic viscosity of air at 20°C are 1.204 kg/m^3 and 1.82×10^{-5} N · s/m, respectively. Thus, for air at 20°C, the Reynolds number for flow over a 10-cm-diameter circular cylinder at 20 m/s is

$$\mathrm{Re} = \frac{(1.204\ \mathrm{kg/m^3})(20\ \mathrm{m/s})(0.1\ \mathrm{m})}{1.82 \times 10^{-5}\ \mathrm{N \cdot s/m}} = 1.3 \times 10^5$$

The Reynolds number for many situations involving wind-induced oscillations is between 1×10^3 and 2×10^5. Over this Reynolds number regime, both the drag coefficient and the Strouhal number are approximately constant. For long cylinders ($D/L \ll 1$) empirical evidence suggests that

$$C_D \approx 1 \qquad 1 \times 10^3 < \mathrm{Re} < 2 \times 10^5 \tag{4.72}$$

$$S \approx 0.2 \qquad 1 \times 10^3 < \mathrm{Re} < 2 \times 10^5 \tag{4.73}$$

From Equation (4.73) and the definition of the Strouhal number, Equation (4.70),

$$v = \frac{\omega D}{0.4\pi} \tag{4.74}$$

Then from Equations (4.69), (4.72), and (4.74),

$$F_0 = 0.317\, \rho D^3 L \omega^2 \tag{4.75}$$

Hence the harmonic excitation to a circular cylinder provided by vortex shedding when the Reynolds number is between 1×10^3 and 2×10^5 has a magnitude that is proportional to the square of its frequency. Using the notation of Equations (4.50) and (4.51) gives

$$A = 0.317 \rho D^3 L \tag{4.76}$$

and
$$\frac{3.16\, mX}{\rho D^3 L} = \Lambda(r, \zeta) \tag{4.77}$$

The theory is presented for vortex shedding from circular cylinders. If the frequency at which the vortices are shed is near the natural frequency of the structure, then large-amplitude vibrations exist. The effects of vortex shedding must be taken into account when designing structures such as street lamp posts, transmission towers, chimneys, and tall buildings. Vortex shedding also occurs from noncircular structures such as buildings and bridges.

EXAMPLE 4.6

A street lamp consists of a 60-kg light fixture attached at the end of a 3-m-tall solid steel ($E = 210 \times 10^9$ N/m^2) cylinder with a diameter of 20 cm. Use a one degree-of-freedom model consisting of a cantilever beam with a concentrated mass at its end to analyze the response of the light fixture to wind excitation. Assume the beam has an equivalent viscous damping ratio of 0.2.

(a) At what wind speed will the maximum steady-state amplitude of vibration due to vortex shedding occur?

(b) What is the corresponding maximum amplitude?

(c) Redesign the light by changing its diameter such that the maximum amplitude of vibration does not exceed 0.10 mm for any wind speed.

SOLUTION

Before proceeding with the analysis, there are several questions associated with the modeling that must be addressed. Vortices are shed along the entire length of the cylinder. The two-dimensional assumption implies that the force per unit length is constant along the entire length of the light post. Thus the force given by Equation (4.64) is really the resultant of this force per unit length distribution. Its point of application should be the midpoint of the light post. However, the problem is not really two dimensional because of among other things, the boundary layer of the earth. The presence of a boundary layer causes a varying wind velocity over the length of the light post, which, in turn, causes a nonuniform force per unit length distribution, as shown in Figure 4.14(a). Thus the actual point of application of the resultant force will be somewhat higher than the midpoint of the light post. In addition, the mass is assumed to be lumped at the end of the beam, while the point of application of the applied force is elsewhere. The resultant force can be replaced by a force of the same magnitude located at the end of the beam and a moment. However, the moment causes rotational effects which are not adequately taken into account in a one-degree-of-freedom model. At least a two-degree-of-freedom model should be used. In order to attain an approximate result, these effects are neglected. A one degree-of-freedom model is used where the excitation is provided by a concentrated harmonic load located at the light of fixture, as shown in Figure 4.14(b).

Assume air at 20°C. The Rynolds number for a velocity of 20 m/s is

$$\mathrm{Re} = \frac{(1.204\,\mathrm{kg/m^3})(20\,\mathrm{m/s})(0.20\,\mathrm{m})}{(1.82 \times 10^{-5}\,\mathrm{N \cdot s/m})} = 2.6 \times 10^5 \qquad \textbf{(a)}$$

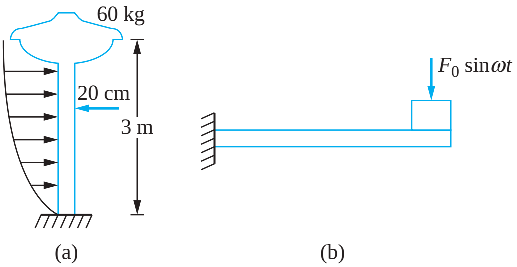


FIGURE 4.14
(a) Street light post in steady wind is subject to harmonic excitation whose amplitude is proportional to the square of the frequency because of vortex shedding. (b) The model of the system is a mass attached to the end of a cantilever beam.

This Reynolds number is higher than the 2×10^5 upper limit on the range of strict applicability of the theory presented previously. However, the Strouhal number is only slightly higher than 0.2. Using 0.2 as an approximation for the Strouhal number is in line with other approximations made in the modeling.

(a) Using a one degree-of-freedom model, the natural frequency of the cantilever beam is

$$\omega_n = \sqrt{\frac{3EI}{mL^3}} = \sqrt{\frac{3(210 \times 10^9 \text{ N/m}^2)(\pi/64)(0.2 \text{ m})^4}{(60 \text{ kg})(3 \text{ m})^3}} = 174.8 \text{ rad/s} \qquad \text{(b)}$$

The magnitude of the excitation force is proportional to the square of its frequency. Thus, from Equation (4.54), the maximum steady-state amplitude occurs for a frequency ratio of

$$r_{\max} = \frac{1}{\sqrt{1 - 2\zeta^2}} = 1.043 \qquad \text{(c)}$$

Thus the frequency at which the maximum amplitude occurs is

$$\omega = 1.043(174.8 \text{ rad/s}) = 182.2 \text{ rad/s} \qquad \text{(d)}$$

The wind velocity that gives rise to this frequency is calculated using the definition of the Strouhal number

$$v = \frac{\omega D}{2\pi S} = \frac{(182.2 \text{ rad/s})(0.2 \text{ m})}{2\pi(0.2)} = 29.0 \text{ m/s} \qquad \text{(e)}$$

(b) The value of Λ corresponding to this frequency ratio is calculated from Equation (4.55)

$$\Lambda_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} = 2.55 \qquad \text{(f)}$$

The corresponding maximum amplitude is calculated by using Equation (4.77)

$$X = \frac{\rho D^3 L\Lambda}{3.16m} = \frac{(1.204 \text{ kg/m}^3)(0.2 \text{ m})^3(3 \text{ m})(2.55)}{3.16(60 \text{ kg})} = 3.9 \times 10^{-4} \text{m} \qquad \text{(g)}$$

(c) The maximum value of Λ is a function of ζ only and does not change with ω_n. The steady-state amplitude can be limited to 0.1 mm for all wind speeds by requiring that $\Lambda = 2.55$ for $X = 0.1$ mm. This leads to

$$D = \left(\frac{3.16\, mX}{\rho L\Lambda}\right)^{1/3} = 12.7 \text{ cm} \qquad \text{(h)}$$

Thus, the maximum diameter of the light pole should be 12.7 cm.

4.5 RESPONSE DUE TO HARMONIC EXCITATION OF SUPPORT

Consider the mass-spring-dashpot system of Figure 4.15. The spring and dashpot are in parallel with one end of each connected to the mass and the other end of each connected to a moveable support. Let $y(t)$ denote the known displacement of the support and let $x(t)$

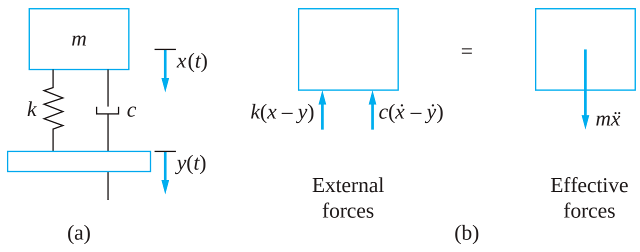


FIGURE 4.15
(a) Block is connected through parallel combination of spring and viscous damper to a moveable support.
(b) FBDs at an arbitrary instant. Spring and viscous-damper forces include effects of base motion.

denote the absolute displacement of the mass. Application of Newton's law to the free-body diagrams of Figure 4.15(b) yields

$$-k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x} \tag{4.78}$$

or
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{4.79}$$

Define

$$z(t) = x(t) - y(t) \tag{4.80}$$

as the displacement of the mass relative to the displacement of its support. Equation (4.79) is rewritten using z as the dependent variable

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4.81}$$

Dividing Equations (4.79) and (4.81) by m yields

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y \tag{4.82}$$

and
$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \tag{4.83}$$

If the base displacement is given by a single-frequency harmonic of the form

$$y(t) = Y\sin\omega t \tag{4.84}$$

then Equations (4.82) and (4.83) become

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega Y\cos\omega t + \omega_n^2 Y\sin\omega t \tag{4.85}$$

and
$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \omega^2 Y\sin\omega t \tag{4.86}$$

Equation (4.86) shows that a mass-spring-dashpot system subject to harmonic base motion is yet another example in which the magnitude of a harmonic excitation is proportional to the square of its frequency. Using the theory of Section 4.4,

$$z(t) = Z\sin(\omega t - \phi) \tag{4.87}$$

where
$$Z = Y\Lambda(r, \zeta) \tag{4.88}$$

where Λ is defined in Equation (4.52) and ϕ defined by Equation (4.45).

When Equations (4.87) and (4.88) are substituted into Equation (4.80) the absolute displacement becomes

$$x(t) = Y[\Lambda\sin(\omega t - \phi) + \sin\omega t] \tag{4.89}$$

Using the trigonometric relationship for the sine of the difference of two angles, it is possible to express Equation (4.89) in the form

$$x(t) = X\sin(\omega t - \lambda) \tag{4.90}$$

where $$\frac{X}{Y} = T(r, \zeta) \tag{4.91}$$

and $$\lambda = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right] \tag{4.92}$$

where $T(r, \zeta)$ is yet another nondimensional function of the frequency ratio and the damping ratio defined by

$$T(r, \zeta) = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.93}$$

X/Y is the amplitude of the absolute displacement of the mass to the amplitude of displacement of the base.

Multiplying the numerator and denominator by ω^2 leads to

$$\frac{\omega^2 X}{\omega^2 Y} = T(r, \zeta) \tag{4.94}$$

Thus $T(r, \zeta)$ is also the ratio of the acceleration amplitude of the body to the acceleration amplitude of the base.

Equation (4.93) is plotted in Figure 4.16. The following are noted about $T(r, \zeta)$:

1. $T(r, \zeta)$ is near one for small r.
2. $$\lim_{r \to \infty} T(r, \zeta) = \frac{2\zeta}{r} \tag{4.95}$$

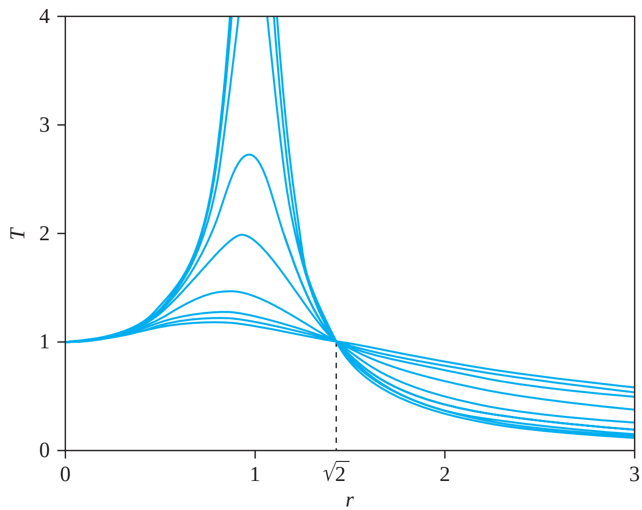


FIGURE 4.16
$T(r, \zeta)$ versus r for several values of ζ. The range for $r < \sqrt{2}$ is called the range of amplification, while the range for $r > \sqrt{2}$ is called the range of isolation.

3. For all ζ, $T(r, \zeta)$ grows until it reaches a maximum for a frequency ratio of

$$r_{\max} = \frac{1}{2\zeta}\left(\sqrt{1 + 8\zeta^2} - 1\right)^{1/2} \tag{4.96}$$

4. The maximum $T(r, \zeta)$ corresponding to the frequency ratio of Equation (4.96)

$$T_{\max} = 4\zeta^2\left[\frac{\sqrt{1 + 8\zeta^2}}{2 + 16\zeta^2 + (16\zeta^4 - 8\zeta^2 - 2)\sqrt{1 + 8\zeta^2}}\right]^{1/2} \tag{4.97}$$

5. $T(\sqrt{2}, \zeta) = 1$, independent of the value of ζ.
6. For $r < \sqrt{2}, T(r, \zeta)$ is larger for smaller values of ζ. However, for $r > \sqrt{2}, T(r, \zeta)$ is smaller for smaller values of ζ.
7. For all values of ζ, $T(r, \zeta)$ is less than one when and only when $r > \sqrt{2}$.

The body is isolated from large accelerations of the base only if $T(r, \zeta) < 1$. This occurs on when $r > \sqrt{2}$. For this reason the range $r > \sqrt{2}$ is called the range of isolation and $r < \sqrt{2}$ is called the range of amplification. When isolation occurs an increase in ζ hinders isolation. Better isolation occurs for smaller damping ratios. Some damping is still required to limit the amplitude of vibration during start up.

The function $T(r, \zeta)$ is called the transmissibility ratio. It is the ratio of the transmitted acceleration to the acceleration of the base. When $T > 1$ the presence of an elastic element between the base and the body actually amplifies the acceleration that is transmitted to the body. Only when $T < 1$ is the transmitted acceleration less than the acceleration of the body.

The amplitude of relative motion, $Z = Y\Lambda(r, \zeta)$ is the amplitude of the maximum displacement of the elastic element.

EXAMPLE 4.7

A 50 kg laboratory experiment is to be mounted onto a table in a laboratory. The table, which is rigidly attached to the floor is vibrating due operation of the other machinery. Measurements indicate that the floor's acceleration amplitude is 1.2 m/s^2 and it vibrates at 100 Hz. Accurate use of the equipment requires that its acceleration amplitude be limited to 0.6 m/s^2.

(a) What is the largest equivalent stiffness of a mounting of damping ratio 0.1 that can be used to limit the acceleration amplitude to 0.6 m/s^2?

(b) What is the maximum deflection of the mounting?

SOLUTION

(a) The transmissibility ratio is

$$T = \frac{\omega^2 X}{\omega^2 Y} = \frac{0.6 \text{ m/s}^2}{1.2 \text{ m/s}^2} = 0.5 \tag{a}$$

Requiring $T(r, 0.1) = 0.5$ leads to

$$T(r, 0.1) = 0.5 = \sqrt{\frac{1 + [2(0.1)r]^2}{(1 - r^2)^2 + [2(0.1)r]^2}} \tag{b}$$

Squaring Equation (b), multiplying the resulting equation by the denominator of the right hand side and rearranging gives

$$r^4 - 2.12r^2 - 3 = 0 \tag{c}$$

Equation (c) is solved leading to $r = 1.76$. Recalling $r = \frac{\omega}{\omega_n}$ and $\omega = 100$ Hz $=$ (100 cycles/s) (2π rad/cycle) $= 6.28 \times 10^2$ rad/s gives

$$\omega_n = \frac{\omega}{r} = \frac{6.28 \times 10^2 \text{ rad/s}}{1.76} = 3.57 \times 10^2 \text{ rad/s} \tag{d}$$

The maximum stiffness for an elastic mounting is

$$k = m\omega_n^2 = (50 \text{ kg})(3.57 \times 10^2 \text{rad/s}) = 6.39 \times 10^6 \text{ N/m} \tag{f}$$

(b) The displacement of the mounting is the relative displacement between the experiment and the table $z(t)$. The maximum displacement is the steady-state amplitude which is

$$Z = Y\Lambda(1.76, 0.1) \tag{g}$$

The steady-state amplitude of the table is

$$Y = \frac{\omega^2 Y}{\omega^2} = \frac{1.2 \text{ m/s}^2}{(6.28 \times 10^2 \text{ rad/s})^2} = 3.04 \times 10^{-6} \text{m} \tag{h}$$

and
$$\Lambda(1.76, 0.1) = \frac{(1.76)^2}{\sqrt{[1 - (1.76)^2]^2 + [2(0.1)(1.76)]^2}} = 1.46 \tag{i}$$

The maximum displacement of the mounting is obtained by substituting Equation (h) and Equation (i) into Equation (g) resulting in

$$Z = (3.04 \times 10^{-6} \text{ m})(1.46) = 4.43 \times 10^{-6} \text{ m} \tag{j}$$

Mechanisms can be used to produce harmonic base excitations. One simple example is the eccentric circular cam of Figure 4.17. When rotating at a constant speed, the cam produces a displacement of $e \sin \omega t$ to its follower, which, in turn, produces a harmonic base excitation in the arrangement shown. The Scotch yoke of Figure 4.18 is another mechanism that produces simple harmonic motion. When the crank is rotating at a constant speed the base is given a displacement of $l \sin \omega t$.

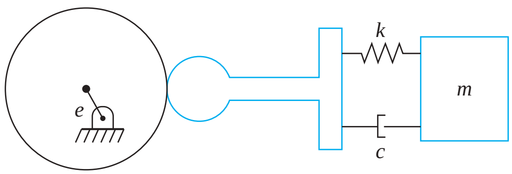


FIGURE 4.17
Eccentric circular cam produces harmonic motion of follower which provides support motion to the mass-spring-viscous damper system.

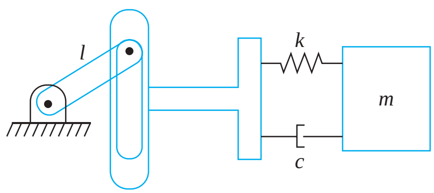


FIGURE 4.18
Scotch yoke mechanism produces simple harmonic motion and provides support excitation to mass-spring-viscous damper system.

EXAMPLE 4.8

A Scotch yoke mechanism provides a harmonic base excitation for the mass-spring-dashpot system of Figure 4.18. The crank arm is 80 mm long. The speed of rotation of the crank arm is varied and the resulting steady-state amplitude is recorded at each speed. The maximum recorded amplitude of the 14.73 kg block is 13 cm at 1000 rpm. Determine the spring stiffness and damping ratio.

SOLUTION

The amplitude of the base displacement is 0.08 m. The maximum displacement of the mass is 0.13 m. Thus,

$$T_{\max} = \frac{X_{\max}}{Y} = \frac{0.13 \text{ m}}{0.08 \text{ m}} = 1.625$$

The value of ζ which corresponds to this $T_{\max}$ is determined by solving Equation (4.97). However, algebraic manipulation of Equation (4.97) yields a fifth-order polynomial equation for ζ^2. A numerical method must be used to find ζ. An easier trial-and-error approach is outlined in the following discussion, and then used to find the value of ζ for this example.

Equation (4.96) is rearranged as

$$\zeta = \sqrt{\frac{1 - r_{\max}^2}{2r_{\max}^4}}$$

A value of $r_{\max} < 1$ is guessed and its corresponding value of ζ calculated from the preceding equation. Equation (4.93) or (4.97) is then used to calculate the value of $T_{\max}$ corresponding to the guessed value of $r_{\max}$. However, small changes in the accuracy of an intermediate calculation using Equation (4.97) lead to large changes in the result. Thus, Equation (4.93) is usually used. The calculated value of $T_{\max}$ is compared against the desired value of 1.625. If $T_{\max} > 1.625$ another guess for $r_{\max}$, smaller than the previous one, should be made. Other iteration schemes are possible, but the method presented is the most direct using the equations as presented. The trial-and-error scheme is illustrated in the following table:

$r_{\max}$ (guess)	ζ	$T_{\max}$ [from Equation (4.93)]
0.98	0.147	3.180
0.90	0.381	1.702
0.89	0.407	1.640
0.88	0.437	1.573

Then, for $r_{\max} = 0.89$,

$$\omega_n = \frac{\omega}{r_{\max}} = \left(1000\frac{\text{rev}}{\text{min}}\right)\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right)\frac{1}{0.89} = 117.7 \text{ rad/s}$$

and $\quad k = m\omega_n^2 = 2.04 \times 10^5$ N/m.

4.6 VIBRATION ISOLATION

Consider a machine bolted to its foundation. During operation the machine produces or is subject to large amplitude harmonic forces. The force is directly passed onto the foundation. This could lead to problems such as fatigue of the foundation and acoustic wave propagation in the foundation.

The remedy to this situation is to mount the machine on a vibration isolator, which can be discrete springs or elastic pads, as shown in Figure 4.19. The vibration isolator acts to reduce the amplitude of the harmonic force transmitted to the foundation. With an excitation force of $F(t) = F_0 \sin(\omega t)$, the transmitted force is

$$F_{TM} = kx + c\dot{x} \tag{4.98}$$

The steady-state response of the system is $x(t) = X \sin(\omega t - \phi)$, thus

$$F_{TM} = kX \sin(\omega t - \phi) + c\omega \cos(\omega t - \phi) \tag{4.99}$$

Let F_T represent the amplitude of the transmitted force

$$F_{TM} = F_T \sin(\omega t - \lambda) \tag{4.100}$$

and F_0 represent the amplitude of the excitation force. It can be shown that

$$\frac{F_T}{F_0} = T(r, \zeta) \tag{4.101}$$

and λ is as given in Equation (4.92).

The theory of vibration isolation to protect against large transmitted forces is the same as the theory to protect against large transmitted accelerations. To see this, consider the differential equation for the relative displacement, $z = x - y$, of a mass attached to a moveable support,

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4.102}$$

The acceleration of the base is given by $\ddot{x} = \ddot{z} + \ddot{y}$ or using Equation (4.97)

$$m\ddot{x} = -(c\dot{z} + kz) \tag{4.103}$$

where $F = c\dot{z} + kz$ is the force developed in the elastic element connecting the mass and the base.

Vibration isolation only occurs for $r > \sqrt{2}$. When isolation occurs it is negatively affected by damping. Damping is present to protect against large amplitude oscillations during start-up necessary to reach a value of $r > \sqrt{2}$

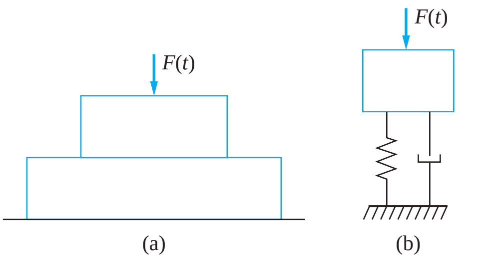


FIGURE 4.19
(a) Elastic mounting is used as a vibration isolator to protect foundation from large forces generated during operation of the machine. (b) SDOF model of machine mounted on isolator.

EXAMPLE 4.9

An air conditioner weighs 250 lb and is driven by a motor at 500 rpm. What is the required static deflection of an undamped isolator to achieve 80 percent isolation (a) if $\zeta = 0$ (b) if $\zeta = 0.1$?

SOLUTION

(a) Eighty percent isolation means that the transmitted force is reduced by 80 percent of that if the machine were directly bolted to the floor. It is 20 percent of the value of the excitation force,

$$\frac{F_T}{F_0} = 0.2 \tag{a}$$

For an undamped isolator

$$T(r,\ 0) = 0.2 \tag{b}$$

or

$$0.2 = \sqrt{\frac{1}{(1-r^2)^2}} \tag{c}$$

Since $r > \sqrt{2}$ to achieve isolation, and a positive result is required from the square root, the appropriate form of the preceding equation after the square root is taken is

$$0.2 = \frac{1}{r^2 - 1} \tag{d}$$

which yields $r = 2.45$. The maximum natural frequency for the air conditioner-isolator system to achieve 80 percent isolation is calculated as

$$\omega_n = \frac{\omega}{r} = \frac{(500\ \text{rev/min})(2\pi\ \text{rad/rev})(1\ \text{min}/60\ \text{s})}{2.45} = 21.4\ \text{rad/s} \tag{e}$$

The required static deflection is obtained from

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta_{\text{st}}}} \tag{f}$$

or

$$\Delta_{\text{st}} = \frac{g}{\omega_n^2} = \frac{32.2\ \text{ft/s}^2}{(21.4\ \text{rad/s})^2} = 0.07\ \text{ft} \tag{g}$$

(b) It is required to find r such that

$$T(r,\ 0.1) = 0.2 \tag{h}$$

or

$$\sqrt{\frac{1 + [2(0.1)r]^2}{(1-r^2)^2 + [2(0.1)r]^2}} = 0.2 \tag{i}$$

Squaring both sides of Equation (g), multiplying by the denominator of the left hand side and rearranging leads to

$$r^4 - 2.96r^2 - 24 = 0 \tag{j}$$

Equation (h) is a quadratic equation in r^2. Solution using the quadratic formula yields $r^2 = -3.64, 6.60$. Choosing the positive value and taking the square root leads to $r = 2.57$. Note that this value is greater than the value obtained for $\zeta = 0$. Thus

$$\omega_n < \frac{\omega}{2.56} = \frac{52.4 \text{ rad/s}}{2.57} = 20.4 \text{ rad/s} \tag{k}$$

The minimum static deflection is

$$\Delta_{st} = \frac{g}{\omega_n^2} = \frac{32.2 \text{ ft/s}^2}{(20.4 \text{ rad/s})^2} = 0.0775 \text{ ft} = 0.930 \text{ in.} \tag{l}$$

EXAMPLE 4.10

An industrial sewing machine has a mass of 430 kg and operates at 1500 rpm (157 rad/s). It appears to have a rotating unbalance of magnitude $m_0 e = 0.8$ kg · m. Structural engineers suggest that the maximum repeated force transmitted to the floor is 10,000 N. The only isolator available has a stiffness of 7×10^6 N/m and a damping ratio of 0.1. If the isolator is placed between the machine and the floor, will the transmitted force be reduced to an acceptable level? If not, what can be done?

SOLUTION

The maximum allowable transmissibility ratio is

$$T_{max} = \frac{F_{T_{max}}}{m_0 e w^2} = \frac{10{,}000 \text{ N}}{(0.8 \text{ kg} \cdot \text{m})(157 \text{ rad/s})^2} = 0.507 \tag{a}$$

The natural frequency with the isolator in place is

$$\omega_n = \sqrt{\frac{7 \times 10^6 \text{ N/m}}{430 \text{ kg}}} = 127.6 \text{ rad/s} \tag{b}$$

which leads to a frequency ratio of $1.24 < \sqrt{2}$. Use of this isolator actually amplifies the force transmitted to the floor.

Adequate isolation is achieved only by increasing the frequency ratio, thus decreasing the natural frequency. The maximum allowable natural frequency is obtained by solving for r from

$$T(r, 0.1) = 0.507 = \sqrt{\frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2}} \tag{c}$$

Equation (c) is squared and rearranged to yield the following quadratic equation for r^2:

$$r^4 - 2.12r^2 - 2.89 = 0 \tag{d}$$

The appropriate solution is $r = 1.75$. Thus the maximum natural frequency is

$$\omega_n = \frac{157 \text{ rad/s}}{1.75} = 89.7 \text{ rad/s} \tag{e}$$

If more than one of the described isolator were available, the natural frequency of the system can be decreased by placing isolators in series. The equivalent stiffness for n isolators in series is k/n. Further calculations show that at least two isolator pads in series are necessary to reduce the natural frequency below 89.7 rad/s.

If only one isolator pad is available, the natural frequency is decreased by adding mass to the machine. A mass of at least 440 kg must be rigidly attached to the machine and the assembly placed on the existing isolator.

EXAMPLE 4.11

A flow-monitoring device of mass 10 kg is to be installed to monitor the flow of a gas in a manufacturing process. Because of the operation of pumps and compressors, the floor of the plant vibrates with an amplitude of 4 mm at a frequency of 2500 rpm. Effective operation of the flow-monitoring device requires that its acceleration amplitude be limited to 5g. What is the equivalent stiffness of an isolator with a damping ratio of 0.05 to limit the transmitted acceleration to an acceptable level? What is the maximum displacement of the flow-monitoring device and what is the maximum deformation of the isolator?

SOLUTION

The acceleration amplitude of the floor is

$$\omega^2 Y = \left[\left(2500\frac{\text{rev}}{\text{min}}\right)\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\left(1\frac{\text{min}}{60\text{ s}}\right)^2\right](0.004\text{ m}) = 274.1\text{m/s}^2 = 27.95g \quad \text{(a)}$$

The maximum allowable transmissibility ratio is

$$T_{\max} = \frac{\omega^2 X}{\omega^2 Y} = \frac{5g}{27.95\ g} = 0.179 \quad \text{(b)}$$

Requiring $T(r, 0.05) = 0.179$, we have

$$0.179 < \sqrt{\frac{1 + 0.01r^2}{1 - 1.99r^2 + r^4}} \quad \text{(c)}$$

Solution of the preceding equation gives the minimum frequency ratio for which vibrations are sufficiently isolated. It yields $r > 2.60$. Thus

$$\omega_n < \frac{\omega}{2.60} = 100.6\text{ rad/s} \quad \text{(d)}$$

The maximum stiffness of the isolator is

$$k = m\omega_n^2 = 1.01 \times 10^5\text{ N/m} \quad \text{(e)}$$

When $T = 0.179$, Equation (4.91) is used to calculate the steady-state amplitude of the flow-monitoring device as

$$X = YT = (0.004\text{ m})(0.179) = 0.72\text{ mm} \quad \text{(f)}$$

Since the isolator is placed between the floor and the flow-monitoring device, its deformation is equal to the relative displacement between the floor and the device.

The steady-state amplitude of the relative displacement is calculated by using Equation (4.88).

$$Z = \Lambda Y = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = 4.69\text{ mm} \quad \text{(g)}$$

4.7 VIBRATION ISOLATION FROM FREQUENCY-SQUARED EXCITATIONS

A special case occurs when the amplitude of the excitation force is proportional to the square of the excitation frequency, as for the harmonic excitation due to a rotating unbalance. Since the maximum allowable force transmitted to the foundation is independent of the frequency of excitation, the percentage of isolation required varies with the frequency. When the excitation is caused by a rotating unbalance, Equation (4.101) becomes

$$\frac{F_T}{m_0 e \omega^2} = T(r, \zeta)$$

or

$$\frac{F_T}{m_0 e \omega_n^2} = r^2 T(r, \zeta) = R(r, \zeta) \tag{4.104}$$

The nondimensional function $R(r, \zeta)$ is defined as

$$R(r, \zeta) = r^2 \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.105}$$

$R(r, \zeta)$ is plotted in Figure 4.20. The following is noted about its behavior

1. $R(r, \zeta)$ is asymptotic to the line $f(r) = 2\zeta r$ for large r. That is,

$$\lim_{x \to \infty} R(r, \zeta) = 2\zeta r \tag{4.106}$$

2. For $\zeta < \sqrt{2}/4 = 0.354$, $R(r, \zeta)$ increases with increasing r, from 0 at $r = 0$ and reaches a maximum value. R then decreases and reaches a relative minimum. As r increases from the value where the minimum occurs, R grows without bound and approaches the asymptotic limit given by Equation (4.106). The values of r where the maximum and relative minimum occur are obtained by setting, $dR/dr = 0$, yielding

$$1 + (8\zeta^2 - 1)r^2 + 8\zeta^2(2\zeta^2 - 1)r^4 + 2\zeta^2 r^6 = 0 \tag{4.107}$$

Equation (4.107) is a cubic polynomial in r^2. It has three roots. One root is the value of r where the maximum occurs, another is the value of r where the relative minimum

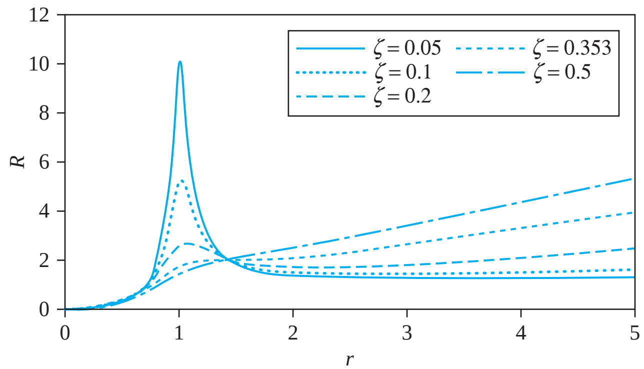


FIGURE 4.20
$R(r, \zeta)$ versus r for several values of ζ.

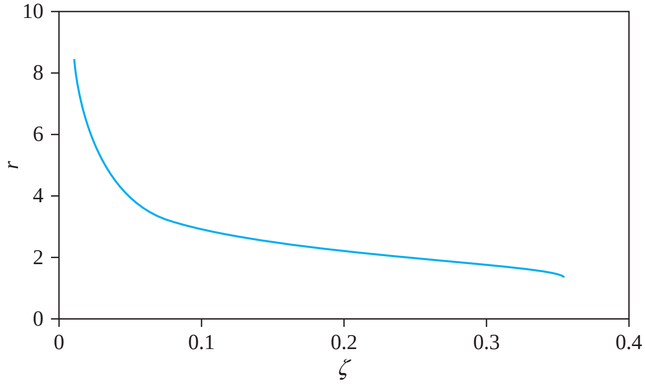


FIGURE 4.21
Value of r for which the minimum $R(r, \zeta)$ occurs as a function of ζ.

occurs, and one root is negative and irrelevant. Figure 4.21 shows the value of r for which the minimum occurs as a function of ζ. Figure 4.22 shows the corresponding value of R at its relative minimum.

3. $R = 2$ for $r = \sqrt{2}$ for all values of ζ.
4. Equation (4.107) has a double root of $r = \sqrt{2}$ for $\zeta = \sqrt{2}/4 = 0.354$. The maximum and minimum coalesce for this value of ζ. For $\zeta = 0.354$, $r = \sqrt{2}$ is an inflection point.
5. For $\zeta > \sqrt{2}/4$, Equation (4.107) has no positive roots. Thus R does not reach a maximum, but grows without bound from $R = 0$ at $r = 0$.

If the natural frequency of a system whose vibrations are due to a rotating unbalance is fixed, Figure 4.20 shows that the transmitted force has a minimum for some value of r. If r exceeds this value, the force increases without bound as r increases. If ζ is small, the curve in the vicinity of the relative minimum is flat. The transmitted force varies little over a range of r. This suggests that for situations where vibrations must be isolated over a range of excitation frequencies, it is best to chose ω_n such that the value of r at the center of the operating range is near the value of r for which the relative minimum occurs.

The limit process used to develop Equation (4.106) is performed for a fixed value of ω_n as ω is increased. Thus, for a fixed ω_n, the transmitted force approaches $m_0 e\omega\omega_n$.

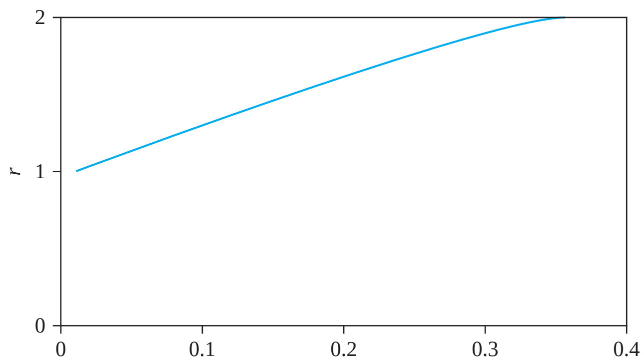


FIGURE 4.22
$R_{min}(\zeta)$.

The limit of F_T as ω_n goes to zero for a fixed ω is zero. Thus decreasing the natural frequency decreases the magnitude of the transmitted force for a specific excitation frequency. Decreasing the natural frequency such that the minimum is to the left of the operating range reduces the magnitude of the repeating component of the transmitted force over a portion of the operating range. However, the transmitted force may vary greatly over the operating range.

EXAMPLE 4.12

A 250-kg pump operates at speeds between 1000 and 2400 rpm and has a rotating unbalance of 2.5 kg · m. The pump is placed at a location in an industrial plant where it has been determined that the maximum repeated force that should be applied to the floor is F_{max}. Specify the stiffness of an isolator of damping ratio 0.1 that can be used to reduce the repeating component of the transmitted force to an acceptable level. Solve for (a) F_{max} = 15,000 N; (b) F_{max} = 10,000 N.

SOLUTION

If the pump is placed directly on the floor, the repeating component of the transmitted force is 27,400 N at 1000 rpm and 157,800 N at 2400 rpm. Thus isolation is necessary.

(a) From Figure 4.22, for $\zeta = 0.1$ the minimum value of R occurs for $r = 2.94$. If ω_n is chosen such that $r = 2.94$ is at the center of the operating range, then

$$\omega_n = \frac{1700 \text{ rpm}}{2.94} = 578.2 \text{ rpm} = 60.55 \text{ rad/s} \tag{a}$$

At the lower end of the operating range, the frequency ratio is 1.73 and the transmitted force is

$$\begin{aligned} F_T &= m_0 e \omega_n^2 R(1.73, 0.1) \\ &= 2.5 \text{ kg} \cdot \text{m } (60.55 \text{ rad/s})^2 (1.73)^2 \sqrt{\frac{1 + (0.346)^2}{[1-(1.73)^2]^2 + (0.346)^2}} \\ &= 14{,}350 \text{ N} \end{aligned} \tag{b}$$

At the upper end of the operating range, the frequency ratio is 4.15 and the transmitted force is

$$\begin{aligned} F_T &= m_0 e \omega_n^2 R(4.15, 0.1) \\ &= (2.5 \text{ kg} \cdot \text{m}) \ (60.55 \text{ rad/s})^2 (4.15)^2 \sqrt{\frac{1 + (0.830)^2}{[1 - (4.15)^2]^2 + (0.830)^2}} \\ &= 12{,}630 \text{ N} \end{aligned} \tag{c}$$

Thus, choosing an isolator such that $r = 2.94$ corresponds to 1200 rpm will reduce the transmitted force to less than 15,000 N at all speeds between 1000 and 2400 rpm. The stiffness of such an isolator is

$$k = m\omega_n^2 = (250 \text{ kg})(60.55 \text{ rad/s})^2 = 9.17 \times 10^5 \text{ N/m} \tag{d}$$

(b) The above analysis works for $F_{T_{max}}$ = 15,000 N but does not work for $F_{T_{max}}$ = 10,000 N, as the transmitted force at both ends of the operating range is larger than

10,000 N when the center of the operating range corresponds to the minimum value of R. Setting $F_{T_{\max}} = 10{,}000$ N for $\omega = 1000$ rpm leads to

$$T(r, 0.1) = \frac{F_{T_{\max}}}{m_0 e\omega^2} = \frac{10{,}000 \text{ N}}{(2.5 \text{ kg}\cdot\text{m}) \ (104.7 \text{ rad/s})^2} = 0.365 \tag{e}$$

which leads to $r = 2.012$. Then

$$\omega_n = \frac{104.7 \text{ rad/s}}{2.102} = 52.02 \text{ rad/s} \tag{f}$$

Then for $\omega = 2400$ rpm, $r = 4.83$ and

$$F_T = m_0 e\omega_n^2 R(4.83, 0.1) = 9810 \text{ N} \tag{g}$$

Thus, the transmitted force is less than 10,000 at all speeds within the operating range and

$$k = m\omega_n^2 = (250 \text{ kg}) \ (52.02 \text{ rad/s})^2 = 6.77 \times 10^5 \text{N/m} \tag{h}$$

4.8 PRACTICAL ASPECTS OF VIBRATION ISOLATION

Vibration isolation is required in a variety of military and industrial applications. Isolation is required to reduce the force transmitted between a machine and its foundation during ordinary operation or to isolate a machine from vibrations of its surroundings. Motors are often isolated to protect mountings from forces arising from harmonic variation of torque and unbalanced rotors. Electrical components such as transformers and circuit breakers are isolated to protect surroundings from electromagnetic forces generated in solenoids or as a result of alternating current. Large harmonic inertia forces are developed by rotating components of single-cylinder reciprocating engines. Isolation is required to protect the engine mounting from these forces. Other machines with rotating components such as fans, pumps, and presses are often isolated to protect against inherent rotating unbalance.

The maximum stiffness of an isolator required for a particular application is calculated by using the theory of Section 4.6. A SDOF system using an isolator is modeled as the simple mass-spring-dashpot system of Figure 4.19(b).

Specifications provided in catalogs of commercially available isolators include allowable static deflections. If the isolated system of Figure 4.19 has a minimum required natural frequency ω_n, the required minimum static deflection of the isolator is

$$\Delta_{\text{st}} = \frac{g}{\omega_n^2} \tag{4.108}$$

Isolation of low-frequency vibrations requires a small natural frequency, which leads to a large isolator static deflection.

The vibration amplitude of a machine during operation is calculated from Equation (4.39)

$$\frac{m\omega_n^2 X}{F_0} = M(r, \zeta) \tag{4.109}$$

Multiplying both sides of the preceding equation by r^2 leads to

$$\frac{m\omega^2 X}{F_0} = r^2 M(r, \zeta) = \Lambda(r, \zeta) \tag{4.110}$$

where $\Lambda(r, \zeta)$ is defined in Equation (4.52). Since vibration isolation requires $r > \sqrt{2}$ and $\Lambda(r, \zeta)$ decreases and approaches 1 as r increases, the steady-state amplitude decreases as isolation is improved. However, for fixed m, F_0, and ω the steady-state amplitude has a lower bound given by

$$X > \frac{F_0}{m\omega^2} \tag{4.111}$$

Equations (4.110) and (4.111) show that if an isolator is being designed to provide isolation over a range of frequencies, the steady-state amplitude is greatest at the lowest operating speed.

Since vibration isolation requires $r > \sqrt{2}$, the speeds at which the maximum vibration amplitude occurs must be passed during start-up and stopping. The maximum vibration amplitude for a fixed ω_n is obtained using Equation (4.44) as

$$X_{\max} = \frac{F_0}{m\omega_n^2} \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{4.112}$$

The smaller the natural frequency, the larger the maximum amplitude. In addition, the larger the damping ratio, the smaller the maximum amplitude.

A large vibration amplitude can lead to ineffective operation of machinery. Large-amplitude vibrations of machines which must be properly aligned with devices that feed materials to the machine can lead to improper alignment and improper operation. Many machine tools require a rigid foundation for effective operation. Equation (4.110) shows that one way to reduce the amplitude of vibration during operation and the maximum amplitude is to increase the mass of the isolated system. Equation (4.111) shows that the only way to reduce the amplitude below a calculated value at a given operating speed is to increase the system mass. Increasing the mass allows a proportional increase in the stiffness required to achieve sufficient isolation.

The mass of a system can be increased by rigidly mounting the machine on a block of concrete. A small machine can be mounted above ground, while a large machine is usually mounted in a specially designed pit. The static load applied to the isolator and the mounting is increased when the mass of the system is increased.

There are three important considerations in vibration isolator design: the maximum amplitude during start-up, the steady-state amplitude, and the amplitude of the transmitted force. There are three parameters which can be controlled: m, ω_n (or Δ_{st}), and ζ. The three parameters can be adjusted to provide the necessary isolation.

EXAMPLE 4.13

A milling machine of mass 450 kg operates at 1800 rpm and has an unbalance which causes a harmonic repeated force of magnitude 20,000 N. Design an isolation system to limit the transmitted force to 4000 N, the amplitude of vibration during operation to 1 mm, and the amplitude of vibration during start-up to 10 mm. Specify the required stiffness of the isolator and the minimum mass that should be added to the machine. Assume a damping ratio of 0.05.

SOLUTION

The maximum allowable transmissibility is

$$T = \frac{4000\ \text{N}}{20{,}000\ \text{N}} = 0.2 \tag{a}$$

The minimum frequency ratio is determined by solving

$$0.2 = \sqrt{\frac{1 + 0.01\ r^2}{1 - 1.99r^2 + r^4}} \tag{b}$$

which yields $r = 2.48$ and a maximum natural frequency of

$$\omega_n = \frac{\omega}{2.48} = 76.0\ \text{rad/s} \tag{c}$$

The maximum amplitude during start-up for the 450-kg machine mounted on an isolator such that the system natural frequency is 76.0 rad/s is

$$X_{\text{max}} = \frac{200{,}000\ \text{N}}{(450\ \text{kg})\ (76.0\ \text{rad/s})^2}\ \frac{1}{2(0.05)\ \sqrt{1 - (0.05)^2}} = 76.9\ \text{mm} \tag{d}$$

The resonant amplitude can be decreased to 10 mm only by increasing the mass to

$$m = \frac{20{,}000\ \text{N}}{(0.01\ \text{m})\ (76.0\ \text{rad/s})^2}\ \frac{1}{2(0.05)\ \sqrt{1 - (0.05)^2}} = 3460\ \text{kg} \tag{e}$$

When the mass is increased to 3460 kg, the amplitude of vibration of the milling machine when operating at 1800 rpm is

$$X = \frac{20{,}000\ \text{N}}{(3460\ \text{kg})\ (76.0\ \text{rad/s})^2}\ \frac{1}{\sqrt{[1-(2.48)^2]^2 + [2(0.05)(2.48)]^2}} = 0.19\ \text{mm} \tag{f}$$

The isolator stiffness is calculated by

$$k = m\omega_n^2 = (3460\ \text{kg})\ (76.0\ \text{rad/s})^2 = 2.0 \times 10^7\ \text{N/m} \tag{g}$$

The milling machine should be mounted on a concrete block of mass 3010 kg and the system isolated by springs with an equivalent stiffness of 2×10^7 N/m.

There are three classes of isolators in general use. The choice of an isolator for a particular application depends on the constraints noted previously, as well as other factors such as cost, weight limitations space limitations, the amount of damping required, and environmental conditions.

Helical coil steel springs are used as isolators when large static deflection (> 1 in. or 3 cm) are required and a flexible foundation is acceptable. This occurs when good isolation is required at low operating speeds. Hysteresis in steel springs is low, so discrete viscous dampers are used in parallel with the springs to provide adequate damping. Steel springs may be used in combination with other isolation methods when a machine must be mounted on a concrete block. These isolators can be designed for specific use or can be obtained commercially.

Isolators made of elastomers are used in applications where small static deflections are required. If used for larger static loads, the elastomers are subject to creep, reducing their effectiveness after a period of time. Caution should be taken in using these isolators in extreme temperatures. Hysteretic damping inherent in the isolators is usually sufficient. However, discrete dampers can be employed in conjunction with these isolators. The damping ratio of an isolator depends on the elastomeric material from which it is made, the steady-state frequency, and the amplitude. The damping ratio for isolators made of natural rubber varies little with amplitude but is highly dependent on frequency. The damping ratio of a natural rubber isolator at 200 Hz is $\zeta = 0.03$, while $\zeta = 0.09$ at 1200 Hz.

Pads made of materials such as cork, felt, or elastomeric resin are often used to isolate large machines. Pads used to isolate a specific machine can be cut from larger pads. Pads of prescribed thicknesses can be placed on top of one another, acting as springs in series, to provide increased flexibility.

4.9 MULTIFREQUENCY EXCITATIONS

A multifrequency excitation has the form

$$F(t) = \sum_{i=1}^{n} F_i \sin(\omega_i t + \psi_i) \tag{4.113}$$

Without loss of generality, it is assumed that $F_i > 0$ for each i. The steady-state response due to a multifrequency excitation is obtained using the response for a single-frequency excitation and the principle of linear superposition. The total response is the sum of the responses due to each of the individual frequency terms. Thus, the solution of Equation (4.2) with the excitation of Equation (4.113) is

$$x(t) = \sum_{i=1}^{n} X_i \sin(\omega_i t + \psi_i - \phi_i) \tag{4.114}$$

where $$X_i = \frac{M_i F_i}{m_{eq}\omega_n^2} \tag{4.115}$$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta r_i}{1 - r_i^2}\right) \tag{4.116}$$

$$r_i = \frac{\omega_i}{\omega_n} \tag{4.117}$$

and $$M_i = M(r_i, \zeta) = \frac{1}{\sqrt{(1 - r_i^2)^2 + (2\zeta r_i)^2}} \tag{4.118}$$

The maximum displacement from equilibrium is difficult to obtain. The maxima of the trigonometric terms in Equation (4.114) do not occur simultaneously. An upper bound on the maximum is

$$X_{max} \leq \sum_{i=1}^{n} X_i \tag{4.119}$$

EXAMPLE 4.14

A slider-crank mechanism is used to provide a base motion for the block shown in Figure 4.23. Plot the maximum absolute displacement of the block as a function of frequency ratio for a damping ratio of 0.05. The crank rotates with a constant speed, ω.

SOLUTION

The instantaneous position of the block relative to point O is

$$y(t) = \hat{r}\cos\omega t + l\cos\alpha \tag{a}$$

Application of the law of sines gives

$$\sin\alpha = \frac{\hat{r}}{l}\sin\omega t \tag{b}$$

Thus

$$y(t) = \hat{r}\cos\omega t + l\sqrt{1-\left(\frac{\hat{r}}{l}\sin\omega t\right)^2} \tag{c}$$

Assuming $\hat{r}/l$ is small, the binomial expansion is used to expand the square root

$$y(t) = l - \frac{l}{4}\left(\frac{\hat{r}}{l}\right)^2 + \hat{r}\cos\omega t + \frac{l}{4}\left(\frac{\hat{r}}{l}\right)^2\cos 2\,\omega t + \cdots \tag{d}$$

where the expansion has been terminated after the term proportional to $\sin^2\omega t$ and the double-angle formula is used to replace $\sin^2\omega t$. The principle of linear superposition and the theory of Section 4.6 are used to solve for the absolute displacement of the mass

$$x(t) = l\left[1 - -\frac{1}{4}\left(\frac{\hat{r}}{l}\right)^2\right] + \hat{r}T_1\sin\left(\omega t - \lambda_1 + \frac{\pi}{2}\right)$$

$$+ \frac{l}{4}\left(\frac{\hat{r}}{l}\right)^2 T_2\sin\left(2\omega t - \lambda_2 + \frac{\pi}{2}\right) \tag{e}$$

where $$T_i = T(r_i, \zeta) = \sqrt{\frac{1 + (2\zeta r_i)^2}{(1 - r_i^2)^2 + (2\zeta r_i)^2}} \tag{f}$$

and $$\lambda_i = \tan^{-1}\left[\frac{2\zeta r_i^3}{1 + (4\zeta^2 - 1)r_i^2}\right] \tag{g}$$

with $$r_1 = \frac{\omega}{\omega_n} \tag{h}$$

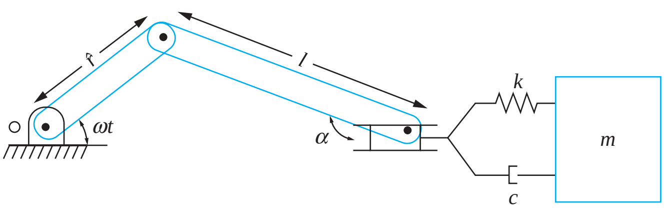


FIGURE 4.23
Slider crank mechanism produces multi-frequency base motion for SDOF system.

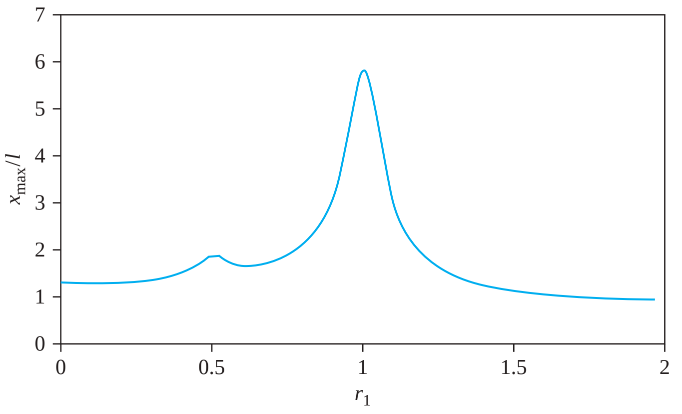


FIGURE 4.24
Upper bound on absolute displacement as a function of frequency ratio for system with base motion provided by slider crank mechanism.

$$\text{and} \quad r_2 = \frac{2\omega}{\omega_n} \tag{i}$$

The response is the sum of the responses due to each frequency term plus the response due to the constant term. The maximum displacement is difficult to attain. Instead an upper bound is calculated

$$x_{max} < l\left[1 - \frac{1}{4}\left(\frac{\hat{r}}{l}\right)^2\right] + \hat{r}T_1 + \frac{1}{4}\left(\frac{\hat{r}}{l}\right)^2 T_2 \tag{j}$$

x_{max}/l versus ω/ω_n is plotted in Figure 4.24 for $\hat{r}/l = \frac{1}{2}$ and $\zeta = 0.05$. The graph has two peaks. The first peak near $\omega/\omega_n = \frac{1}{2}$ is smaller than the second peak near $\omega/\omega_n = 1$. If additional terms from the binomial expansion were used, higher harmonics would appear in the solution. Small peaks on the frequency response curve will appear near values of $\omega/\omega_n = 1/i$ where i is an even integer. The magnitude of the peaks grows smaller with increasing i.

4.10 GENERAL PERIODIC EXCITATIONS

4.10.1 FOURIER SERIES REPRESENTATION

Consider the function $H(t)$ of Figure 4.25. It is periodic of period T. The function is constructed such that it is an odd function; that is, if a periodic extension of the function were performed backward in time (Figure 4.26) and it existed for negative time, then

$$H(-t) = -H(t) \tag{4.120}$$

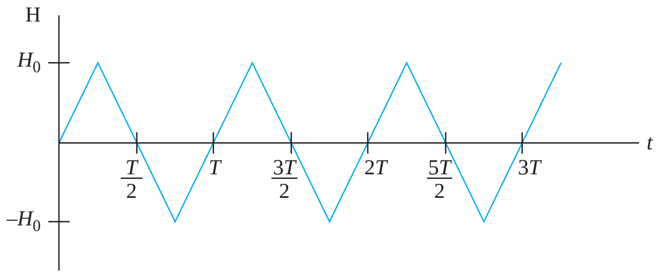


FIGURE 4.25
Odd periodic function.

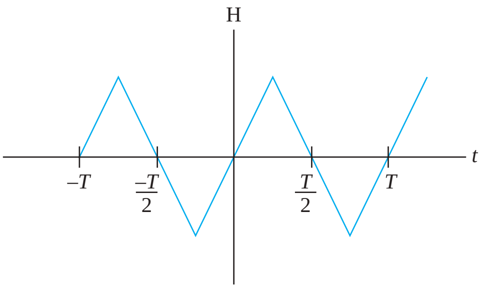


FIGURE 4.26
Periodic extension of $F(t)$ one period into negative time.

for all t, $0 \leq t \leq T$. Now consider the function

$$H_1(t) = \sin\left(\frac{2\pi}{T}t\right) = \sin(\omega_1 t) \tag{4.121}$$

$H_1(t)$ is also a periodic function of period T. Now consider the function

$$H_2(t) = \sin\left(\frac{4\pi}{T}t\right) = \sin(2\omega_1 t) \tag{4.122}$$

$H_2(t)$ is a periodic function of period $T/2$. However, a function of period $T_2 = T/2$ is also periodic of period T, as

$$H_2(t + T) = H_2\left(t + 2\frac{T}{2}\right) = H_2(t + 2T_2) = H_2(t) \tag{4.123}$$

Consider the sequence of functions $H_i(t)$ where

$$H_i(t) = \sin\left(\frac{2\pi i}{T}t\right) = \sin(i\omega_1 t) \tag{4.124}$$

The ith function in the sequence $H_i(t)$ is a periodic function of period $T_i = T/i$. But a function of period T/i is also periodic of period T, as

$$H_i(t + T) = H_i\left(t + i\frac{T}{i}\right) = H_i(t + iT_i) = H_i(t) \tag{4.125}$$

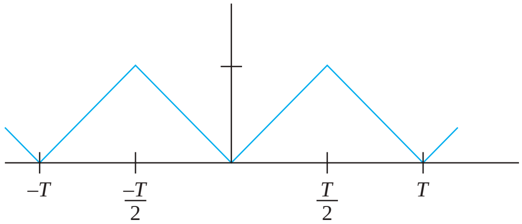


FIGURE 4.27
An even function.

The sequence of functions $H_i(t)$, for $i = 1, 2, 3, \ldots$ is said to be complete over the set of periodic odd functions, which means that any odd periodic function can be written as a linear combination of elements of the sequence. That is, there exists constants b_i such that

$$H(t) = \sum_{i=1}^{\infty} b_i \sin(i\omega_1 t) \tag{4.126}$$

The sequence of partial sums $z_n = \sum_{i=1}^{n} b_i \sin(i\omega_1 t)$ (with appropriate constants) converges to the function of Figure 4.25.

An even function $G(t)$, illustrated in Figure 4.27, is one where if a periodic extension were made into negative time

$$G(-t) = G(t) \tag{4.127}$$

for all t, $0 \leq t \leq T$. The function $G_0(t) = 1$ is an even function that is periodic of any period. The function $G_1(t) = \cos\left(\frac{2\pi}{T}t\right) = \cos(\omega t)$ is an even periodic function of period T. Define the sequence of functions $G_i(t) = \cos(i\omega t)$, $i = 1, 2, 3, \ldots$. The function $G_i(t)$ is an even function that is periodic of period T/i, and thus, it is also periodic of period T. The sequence is complete over the set of even periodic functions, which implies there exists constants a_i such that

$$G(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i \cos(i\omega_i t) \tag{4.128}$$

A general periodic function is composed of an odd function and an even function, as in Figure 4.28:

$$F(t) = G(t) + H(t) \tag{4.129}$$

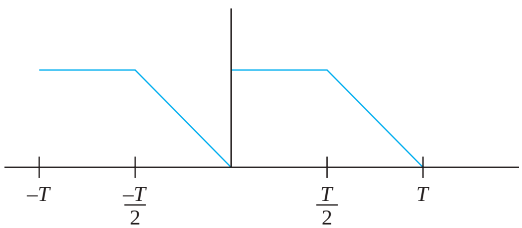


FIGURE 4.28
A function that is neither even or odd.

Which implies that $F(t)$ can be written as

$$F(t) = \frac{a_0}{2} + \sum_{t=1}^{\infty}[a_i \cos(\omega_i t) + b_i \sin(\omega_i t)] \tag{4.130}$$

where

$$\omega_i = i\omega_1 = \frac{2\pi i}{T} \tag{4.131}$$

Equation (4.130) is called the *Fourier series representation* of $F(t)$. The coefficients in the expansion are called the *Fourier coefficients.* They are

$$a_0 = \frac{2}{T}\int_0^T F(t)\,dt \tag{4.132}$$

$$a_i = \frac{2}{T}\int_0^T F(t)\cos(\omega_i t)\,dt \quad i = 1, 2, \ldots \tag{4.133}$$

$$b_i = \frac{2}{T}\int_0^T F(t)\sin(\omega_i t)\,dt \quad i = 1, 2, \ldots \tag{4.134}$$

The Fourier series for $F(t)$ has the following properties:

1. The Fourier series representation converges to $F(t)$ at all t where $F(t)$ is continuous for $0 \leq t \leq T$.
2. If $F(t)$ has a finite jump discontinuity at t, the Fourier series representation converges to $\frac{1}{2}[F(t^-) + F(t^+)]$, which is the average value of $F(t)$.
3. The Fourier series representation converges to the periodic extension of $F(t)$ for $t > T$.
4. If $F(t)$ is an odd function defined by Equation (4.120), then the Fourier coefficients $a_i = 0$ for $i = 0, 1, 2, \ldots$.
5. If $F(t)$ is an even function defined by Equations (4.127), then the Fourier coefficients $b_i = 0$ for $i = 1, 2, \ldots$.

EXAMPLE 4.15

One period of a periodic excitation is shown in Figures 4.29(a) through (c). Draw the function that the Fourier series representations for each of these excitations converge to for the interval $[-2T, 2T]$.

SOLUTION

(a) The function for the convergence of the Fourier series representation is shown in Figure 4.29(d). The excitation is even and continuous everywhere.

(b) The function for the convergence of the Fourier series representation for Figure 4.29(b) is shown in Figure 4.29(e). The function is neither even or odd. It converges to $[2 + (-1)]/2 = 1/2$ at $t = -2, -1, 0, 1$, and 2.

(c) The function for the convergence of the Fourier series representation for Figure 4.29(c) is shown in Figure 4.29(f). The function is odd. It converges to $[2 + (-2)]/2 = 0$ at $t = -6$, -3, 0, 3, and 6. At $t = -4, -1, 2$, and 5, the Fourier series converges to $[0 + 2]/2 = 1$. At $t = -5, -2, 1$, and 4, the Fourier series converges to $[0 + (-2)]/2 = -1$.

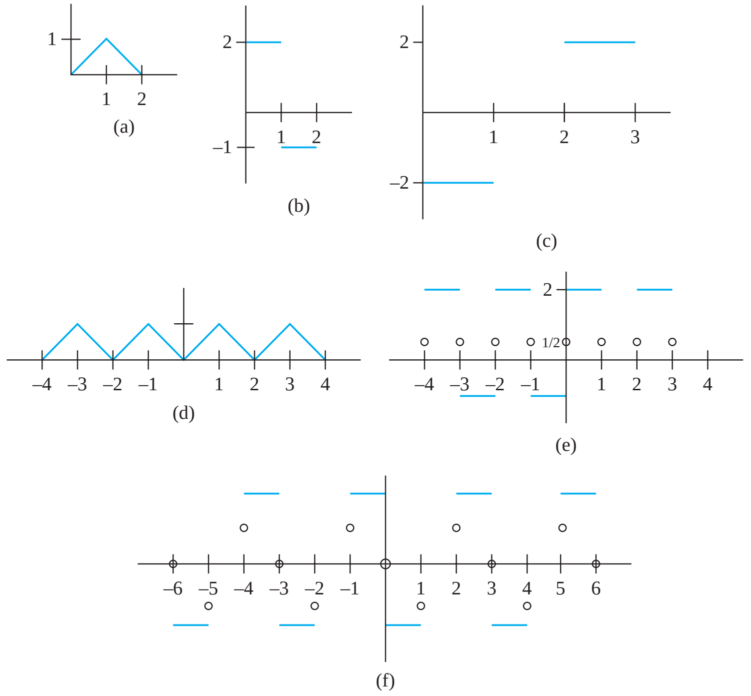


FIGURE 4.29
(a), (b), and (c) One period of periodic excitations for Example 4.15 parts (a), (b), and (c). (d), (e), and (f) Functions that Fourier series converges to over $[-2T, 2T]$.

Use of the trigonometric identity for the sine of the sum of two angles and algebraic manipulation leads to an alternative form for the Fourier series representation

$$F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} c_i \sin(\omega_i t + \kappa_i) \tag{4.135}$$

where $c_i = \sqrt{a_i^2 + b_i^2}$ **(4.136)**

and $\kappa_i = \tan^{-1}\dfrac{a_i}{b_i}$ **(4.137)**

Note that $\tan^{-1}\left(\frac{-0.5}{0.866}\right) = \frac{2\pi}{3}$, but $\tan^{-1}\left(\frac{0.5}{-0.866}\right) = -\frac{\pi}{6}$, or $\frac{11\pi}{6}$. The inverse tangent function has the same argument, but it is multi-valued. A calculator typically evaluates the inverse tangent between $-\pi/2$ and $\pi/2$. The calculation for κ_i must be carried out using the four quadrant evaluation of the inverse tangent. Using MATLAB, this involves using the function `atan2(a, b)`, where `a` is the numerator of the inverse tangent function, and `b` is in the denominator.

4.10.2 RESPONSE OF SYSTEMS DUE TO GENERAL PERIODIC EXCITATION

If $F(t)$ is a periodic excitation for a SDOF system with viscous damping, the differential equation governing the response of the system is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m_{\text{eq}}}\left[\frac{a_0}{2} + \sum_{l=1}^{\infty} c_i \sin(\omega_i t + \kappa_i)\right] \tag{4.138}$$

The principle of linear superposition is used to determine the response as

$$x(t) = \frac{1}{m_{\text{eq}}\omega_n^2}\left[\frac{a_0}{2} + \sum_{i=1}^{\infty} c_i\, M_i \sin(\omega_i^2 t + \kappa_i - \phi_i)\right] \tag{4.139}$$

where M_i and ϕ_i are defined in Equation (4.118) and (4.116), respectively.

The principle of linear superposition used to find the steady-state solution of Equation (4.139) applies, because the Fourier series converges to something at every value of t. Under this condition, the method applies and the response converges. While the excitation may be discontinuous, the response of the system must be continuous.

EXAMPLE 4.16

A punch press of mass 500 kg sits on an elastic foundation of stiffness $k = 1.25 \times 10^6$ N/m and damping ratio $\zeta = 0.1$. The press operates at a speed of 120 rpm. The punching operation occurs over 40 percent of each cycle and provides a force of 5000 N to the machine. The excitation force is approximated as the periodic function of Figure 4.30. Estimate the maximum displacement of the elastic foundation.

SOLUTION

From the given information, the period of one cycle is 0.5 s and the natural frequency of the system is 50 rad/s.

The excitation force is periodic, but it is neither an even function nor an odd function. Its mathematical representation is

$$F(t) = \begin{cases} 5000\text{ N} & 0 < t < 0.2\text{ s} \\ 0 & 0.2\text{ s} < t < 0.5\text{ s} \end{cases} \tag{a}$$

The Fourier coefficients for the Fourier series representation for $F(t)$ are

$$a_0 = \frac{2}{0.5\text{ s}}\left(\int_0^{0.2\text{ s}} 5000\text{ N}\, dt + \int_{0.2\text{ s}}^{0.5\text{ s}} (0)\, dt\right) = 4000\text{ N} \tag{b}$$

$$a_i = \frac{2}{0.5\text{ s}}\left(\int_0^{0.2\text{ s}} 5000\text{ N} \cos 4\pi it\, dt\right)$$

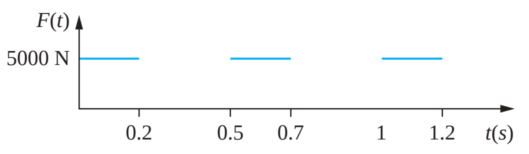


FIGURE 4.30
Force developed during punching operation of Example 4.16 is periodic.

$$= \frac{5000}{\pi i} \sin 4\pi it \bigg|_0^{0.2\,\text{s}} \text{N} = \frac{5000}{\pi i} \sin 0.8\pi i \text{ N} \qquad \text{(c)}$$

and $$b_i = \frac{2}{0.5\text{ s}}\left(\int_0^{0.2\text{ s}} 5000\text{ N} \sin 4\pi it\, dt\right) \qquad \text{(d)}$$

$$= \frac{5000}{\pi i} \cos 4\pi it \bigg|_0^{0.2\text{ s}} \text{N} = \frac{5000}{\pi i}(1 - \cos 0.8\pi i)\text{ N} \qquad \text{(e)}$$

The Fourier series representation of the excitation force is

$$F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} c_i \sin(4\pi it + \kappa_i) \qquad \text{(f)}$$

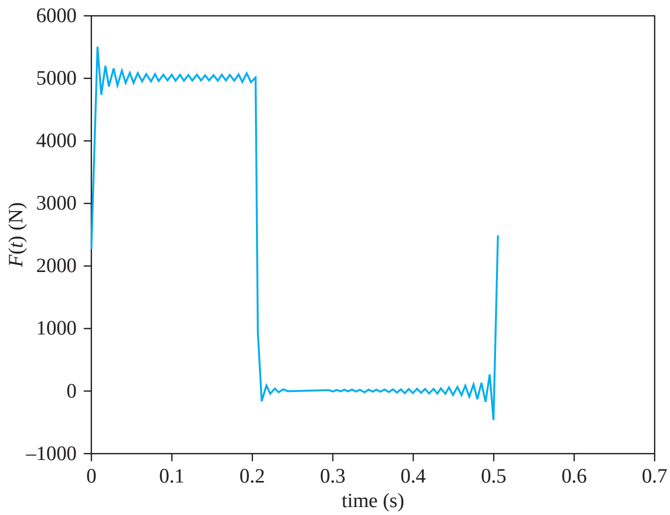
where $$\kappa_i = \frac{5000}{\pi i}\sqrt{2(1 - \cos 0.8\pi i)}\text{ N} \qquad \text{(g)}$$

and $$\kappa_i = \tan^{-1}\left(\frac{\sin 0.8\,\pi i}{1 - \cos 0.8\,\pi i}\right) \qquad \text{(h)}$$

An upper bound on the displacement is

$$x_{\max} < \frac{1}{m\omega_n^2}\left(\frac{a_0}{2} + \sum_{i=1}^{\infty} c_i M_i\right) \qquad \text{(i)}$$

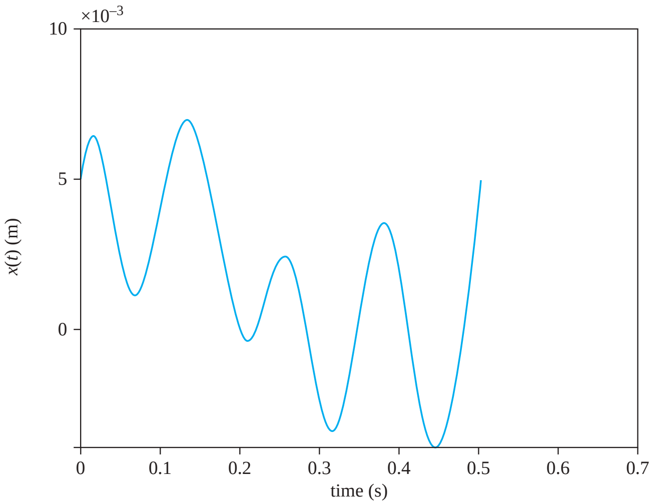
A MATLAB program was written to develop the Fourier series representation for $F(t)$ and the response of the system, $x(t)$. Figure 4.31 shows the MATLAB generated plots from which the maximum displacement is determined.



(a)

FIGURE 4.31

(a) Fourier series representation for $F(t)$ with 50 terms. (b) $x(t)$ over one period from 50 terms in the Fourier series representation.



(b)

FIGURE 4.31
(Continued)

4.10.3 VIBRATION ISOLATION FOR MULTI-FREQUENCY AND PERIODIC EXCITATIONS

Vibration isolation of a system subject to a multifrequency excitation can be difficult, especially if the lowest frequency is very low. Consider a system subject to an excitation composed of n harmonics

$$F(t) = \sum_{i=1}^{n} F_i \sin(\omega_i t + \psi_i) \tag{4.140}$$

The principle of linear superposition is used to calculate the total response of the system due to this excitation. The principle of linear superposition is also used to calculate the transmitted force leading to

$$F_T(t) = \sum_{i=1}^{n} T(r_i, \zeta) F_i \sin(\omega_i t + \psi_i - \lambda_i) \tag{4.141}$$

where $r_i = \frac{\omega_i}{\omega_n}$. Since the harmonic terms of Equation (4.141) are out of phase, their maxima occur at different times. A closed-form expression for the absolute maximum is difficult to attain. The following is used as an upper bound:

$$F_{T_{\max}} < \sum_{i=1}^{n} F_i T(r_i, \zeta) \tag{4.142}$$

An initial guess for the upper bound is obtained by determining the natural frequency such that the transmitted force due to the lowest-frequency harmonic only is reduced to F_T. Since additional forces at higher frequencies are present, greater isolation is required. The natural frequency can be systematically reduced from this initial guess, checking Equation (4.142), until an upper bound is obtained.

EXAMPLE 4.17

The 500-kg punch press of Example 4.16 is to be mounted on an isolator such that the maximum of the repeating force transmitted to the floor is 1000 N. Determine the required static deflection of an isolator, assuming a damping ratio of 0.1. What is the resulting maximum deflection of the isolator during the punching operation?

SOLUTION

From Example 4.16, the excitation force is periodic and is expressed by a Fourier series as

$$F(t) = 2000 + \frac{5000\sqrt{2}}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{1 - \cos 0.8\pi i}\,\sin(4\pi it + \kappa_i)\text{ N} \tag{a}$$

The 2000 N term is the average force applied to the punch during one cycle. It contributes to the total static load applied to the floor and is not part of the repeating load. Application of Equation (4.142) to the repeating components of loading gives

$$1000 > \frac{5000\sqrt{2}}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{1 - \cos 0.8\pi i}\;T(r_i, \zeta) \tag{b}$$

where $$r_i = \frac{4\pi i}{\omega_n} = ir_1 \tag{c}$$

An initial guess for an upper bound for the natural frequency is obtained by calculating r_1 such that the transmitted force due to the lowest-frequency harmonic is less than 1000 N. This leads to

$$1000 = \frac{5000}{\pi}\sqrt{2(1 - \cos 0.8\pi)}\sqrt{\frac{1 + (0.2r_1)^2}{(1 - r_1^2)^2 + (0.2r_1)^2}} \tag{d}$$

which gives $r_1 = 2.06$. Defining

$$f(r_1) = \frac{5000\sqrt{2}}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{1 - \cos 0.8\pi i}\;T(ir_1, \zeta) \tag{e}$$

it is desired to solve

$$f(r_1) = 1000 \tag{f}$$

A lower bound on the value of r_1 that solves the preceding equation is 2.06. A trial-and-error solution using ten terms in the summation is used to determine r_1, leading to $r_1 = 2.19$. For $r_1 = 2.19$, an upper bound for the natural frequency is calculated as

$$\omega_n = \frac{\omega_1}{2.19} = \frac{4\pi}{2.19} = 5.74\text{ rad/s} \tag{g}$$

The required static deflection of the isolator is $\Delta_{st} = g/\omega_n^2 = 298$ mm. The static deflection is excessive, and a flexible foundation is required. The total static load on the isolator is the weight of the machine plus the average value of the excitation force, $a_0/2 = 2000$ N. Thus, the total static load to be supported is

$$F_{\text{static}} = (500\text{ kg})(9.81\text{ m/s}^2) + 2000\text{ N} = 6905\text{ N} \tag{h}$$

4.11 SEISMIC VIBRATION MEASURING INSTRUMENTS

Time histories of vibrations are sensed using seismic transducers. A *transducer* is a device that converts mechanical motion into voltage. A schematic of a piezoelectric transducer is shown in Figure 4.32. The transducer is mounted on a body whose vibrations are to be measured. As the vibrations occur, the seismic mass moves relative to the transducer, causing deformation in the piezoelectric crystal. A charge is produced in the piezoelectric crystal that is proportional to its deformation. The charge is amplified and displayed on an output device. The measured signal is the motion of the seismic mass relative to the transducer housing.

4.11.1 SEISMOMETERS

A model of the transducer is shown in Figure 4.33. The piezoelectric crystal is assumed to provide viscous damping. The purpose of the transducer is to measure the motion of the body, $y(t)$. However, it actually measures $z(t)$, which is the displacement of the seismic mass relative to the body. Assume the vibrations of the body are a single-frequency harmonic of the form

$$y(t) = Y \sin \omega t \tag{4.143}$$

The displacement of the seismic mass relative to the vibrating body is

$$z(t) = Z \sin(\omega t - \phi) \tag{4.144}$$

where $Z = Y\Lambda(r, \zeta) \qquad \phi = \tan^{-1}\left(\dfrac{2\zeta r}{1 - r^2}\right)$

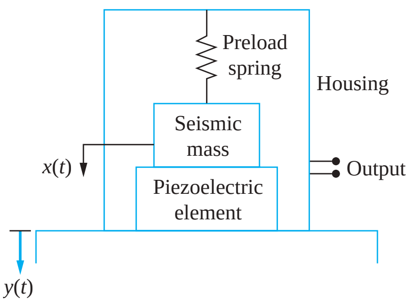


FIGURE 4.32
Diagram of a piezoelectric crystal transducer. As seismic mass moves, a charge is produced in the piezoelectric element that is proportional to its deflection. The transducer actually measures $z(t) = x(t) = y(t)$.

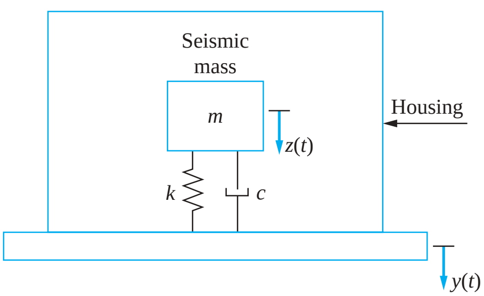


FIGURE 4.33
Schematic representation of the transducer. The piezoelectric crystal provides viscous damping and stiffness.

and $\Lambda(r, \zeta)$ is defined by Equation (4.53) and $r = \omega/\omega_n$, where ω_n and ζ are the natural frequency and damping ratio of the transducer.

Figure 4.11 shows that Λ is approximately 1 for large $r(r > 3)$. In this case the amplitude of the relative displacement which is monitored by the transducer is approximately the same as the vibration amplitude of the body. From Figure 4.8, it is noted that for large r, ϕ is approximately π. Thus for large r, the transducer response is approximately that of the response to be measured, but out of phase by π radians.

A seismic transducer that requires a large frequency ratio for accurate measurement is called a *seismometer*. A large frequency ratio requires a small natural frequency for the transducer. This, in turn, requires a large seismic mass and a very flexible spring. Because of the required size for accurate measurement, seismometers are not practical for many applications.

The percentage error in using a seismic transducer is

$$E = 100\left|\frac{Y_{\text{actual}} - Y_{\text{measured}}}{Y_{\text{actual}}}\right| \tag{4.145}$$

When using a seismometer the percentage error is

$$E = 100\left|\frac{Y - Z}{Y}\right| = 100|1 - \Lambda| \tag{4.146}$$

4.11.2 ACCELEROMETERS

The acceleration of the body is

$$\ddot{y}(t) = -\omega^2 Y \sin \omega t \tag{4.147}$$

Noting that $Z/Y = \Lambda(r, \zeta)$ and $\Lambda = r^2 M(r, \zeta)$ leads to

$$\ddot{y}(t) = -\omega^2 \frac{Z}{\Lambda(r,\zeta)} \sin \omega t = -\omega^2 \frac{Z}{r^2 M(r,\zeta)} \sin \omega t = -\omega_n^2 \frac{Z}{M} \sin \omega t \tag{4.148}$$

Comparing Equation (4.144) to Equation (4.148) makes it apparent that

$$\ddot{y}(t) = \frac{\omega_n^2}{M(r,\zeta)} z\left(t - \frac{\phi}{\omega} - \frac{\pi}{\omega}\right) \tag{4.149}$$

The negative sign in Equation (4.148) is taken into account in Equation (4.149) by subtracting π from the phase. For small r, $M(r, \zeta)$ is approximately 1, and

$$\ddot{y}(t) \approx \omega_n^2 z\left(t - \frac{\phi}{\omega} - \frac{\pi}{\omega}\right) \tag{4.150}$$

Thus, for small r, the acceleration of the particle to which the seismic instrument is attached is approximately proportional to the relative displacement between the particle and the seismic mass, but on a shifted time scale. A vibration measuring instrument that works on this principle is called an *accelerometer*. The transducer in an accelerometer records the relative displacement, which is electronically multiplied by ω_n^2, which is the square of the natural frequency of the accelerometer. The acceleration is integrated twice to yield the displacement.

The natural frequency of an accelerometer must be high to measure vibrations accurately over a wide range of frequencies. The seismic mass must be small and the spring stiffness must be large. The error in using an accelerometer is

$$E = 100\left|\frac{\omega^2 Y - \omega_n^2 Z}{\omega^2 Y}\right| = 100\left|1 - \frac{1}{r^2}\Lambda(r, \zeta)\right| = 100|1 - M(r, \zeta)| \tag{4.151}$$

Consider the measurement of the vibration of a multifrequency vibration,

$$y(t) = \sum_{i=1}^{n} Y_i \sin(\omega_i t + \psi_i) \tag{4.152}$$

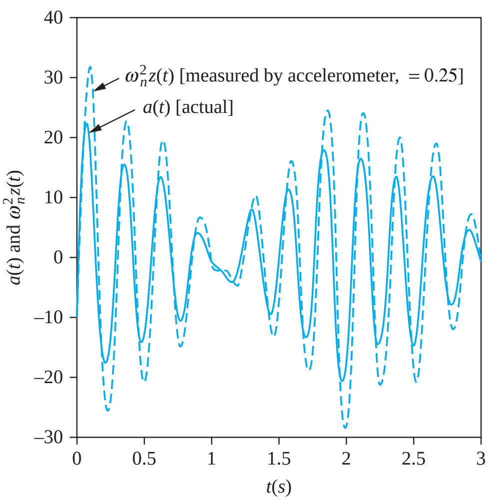
According to the theory of Section 4.9 (the principle of linear superposition), the displacement of a seismic mass relative to the housing of a seismic instrument is

$$\begin{aligned} z(t) &= \sum_{i=1}^{n} \Lambda(r_i, \zeta) Y_i \sin(\omega_i t + \psi_i - \phi_i) \\ &= \frac{1}{\omega_n^2}\sum_{i=1}^{n} \omega_i^2 M(r_i, \zeta) Y_i \sin(\omega_i t + \psi_i - \phi_i) \end{aligned} \tag{4.153}$$

The accelerometer measures $-\omega_n^2 z(t)$. Note that each term in the summation of Equation (4.153) has a different phase shift. When summed, the accelerometer output will be distorted from the true measurement. This phase distortion is illustrated in Figure 4.34(a), which compares the accelerometer output to the signal to be measured for a 10-frequency vibration. The damping ratio of the accelerometer is 0.25, and the largest frequency ratio in the measurement is 0.66.

Accelerometers are used only when $r < 1$. In this frequency range, the phase shift is approximately linear with r for $\zeta = 0.7$ (See Figure 4.8). Then

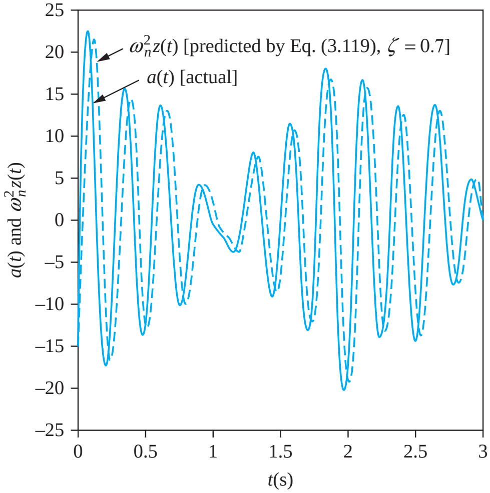
$$\phi_i = \alpha\frac{\omega_i}{\omega_n} \tag{4.154}$$



(a)

FIGURE 4.34
Comparison of $a(t)$, which is the acceleration to be measured, and $V_n^2 z(t)$, which is the acceleration actually measured or predicted, for a vibration composed of 10 different frequencies. (a) The phase distortion is obvious with an accelerometer damping ratio 0.25. (b) The accelerometer damping ratio is 0.7, which eliminates the phase distortion, giving a phase shift.

FIGURE 4.34
(Continued)



(b)

where α is the constant of proportionality. Using Equation (4.154) in Equation (4.153) leads to

$$z(t) = -\frac{1}{\omega_n^2}\sum_{i=1}^{n} M(r_i, \zeta)\, Y_i \sin\left[\omega_i\left(t - \frac{\alpha}{\omega_n}\right) + \psi_i\right] \tag{4.155}$$

If $r_i \ll 1$, then $M(r_i, \zeta) \approx 1$ for $i = 1, 2, \ldots, n$ and

$$z(t) \approx -\frac{1}{\omega_n^2}\, \ddot{y}\left(t - \frac{\alpha}{\omega_n}\right) \tag{4.156}$$

Thus, when an accelerometer with $\zeta = 0.7$ is used, its output device duplicates the actual acceleration, but on a shifted time scale. This is illustrated in Figure 4.34(b), which compares the use of Equation (4.153) with $\zeta = 0.7$ to the actual acceleration for the example of Figure 4.34(a).

EXAMPLE 4.18

What is the smallest natural frequency of an accelerometer of damping ratio 0.2 that measures to vibrations of a body vibrating at 200 Hz with an error of a 2 percent?

SOLUTION

Requiring that the error in the measurement is less than 2 percent is equivalent to requiring that

$$100|1 - M(r, 0.2)| < 2 \tag{a}$$

Since the damping ratio is 0.2, which is less than $1/\sqrt{2}$, $M(r, 0.2) > 1$ near $r = 0$. Thus, Equation (a) is equivalent to

$$M(r, 0.2) < 1.02 \tag{b}$$

or

$$\frac{1}{\sqrt{(1 - r^2)^2 + [2(0.2)r]^2}} < 1.02 \quad \text{(c)}$$

Equation (c) is solved leading to $r < 0.146$ or $r > 1.349$. However, the accelerometer works on the principle of small r, so the second solution is rejected. It is also rejected because for some $r > 1.349$, $M(r, 0.2) < 0.98$ and when the error in the accelerometer measurement is greater than 2 percent. Thus, it is required that $r < 0.146$, leading to

$$\frac{\omega}{\omega_n} < 0.146 \Rightarrow \omega_n > \frac{\omega}{0.146} = \frac{\left(200 \frac{\text{cycles}}{\text{s}}\right)\left(\frac{2\pi \text{ rad}}{\text{cycle}}\right)}{0.146} = 8.60 \times 10^3 \frac{\text{rad}}{\text{s}} \quad \text{(d)}$$

4.12 COMPLEX REPRESENTATIONS

The use of complex algebra provides an alternative method to the solution of the differential equations governing the forced response of systems subject to harmonic excitation. It can prove to be less tedious than the use of trigonometric solutions. Recall that if Q is a complex number, it has the representation

$$Q = Q_r + iQ_i \tag{4.157}$$

where $Q_r = \text{Re}\,(Q)$ is the real part of Q and $Q_i = \text{Im}\,(Q)$ is the imaginary part of Q. The complex number also has the polar form

$$Q = Ae^{i\phi} \tag{4.158}$$

where A is the magnitude of Q and ϕ is the phase of Q. Euler's identity

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{4.159}$$

leads to

$$A = \sqrt{Q_r^2 + Q_i^2} \tag{4.160}$$

and $$\phi = \tan^{-1}\left(\frac{Q_i}{Q_r}\right) \tag{4.161}$$

In view of Euler's identity, it is noted that

$$\cos(\omega t) = \text{Re}\,(e^{i\omega t}) \qquad \sin(\omega t) = \text{Im}\,(e^{i\omega t}) \tag{4.162}$$

Thus the standard form of the differential equation governing the motion of a linear one degree-of-freedom system subject to a single-frequency sinusoidal excitation can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\text{Im}(e^{i\omega t}) \tag{4.163}$$

Then the solution of Equations (4.163) is the imaginary part of the solution of

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}e^{i\omega t} \tag{4.164}$$

A solution of Equation (4.164) is assumed as

$$x(t) = He^{i\omega t} \tag{4.165}$$

where H is complex. Substitution of Equation (4.165) into Equation (4.164) leads to

$$H = \frac{F_0}{m(\omega_n^2 - \omega^2 + 2i\zeta\omega\omega_n)} \tag{4.166}$$

Equation (4.166) can be rewritten by using the definition of the frequency ratio $r = \omega/\omega_n$:

$$H = \frac{F_0}{m\omega_n^2(1 - r^2 + 2i\zeta r)} \tag{4.167}$$

Multiplying the numerator and denominator by the complex conjugates of the denominator puts H in its proper form as

$$H = \frac{F_0}{m\omega_n^2[(1 - r^2)^2 + (2\zeta r)^2]}(1 - r^2 - 2i\zeta r) \tag{4.168}$$

Then, from Equations (4.160) and (4.161), H can be written as

$$H = Xe^{-i\phi} \tag{4.169}$$

where $$X = \frac{F_0}{m\omega_n^2}\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.170}$$

and $$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{4.171}$$

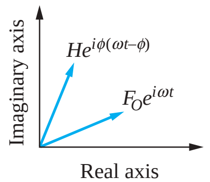


FIGURE 4.35 Graphical representation of excitation and response in complex plane.

Equations (4.170) and (4.171) are the same as those derived by using a trigonometric solution. The system response is

$$x(t) = \mathrm{Im}(Xe^{-i\phi}e^{\omega t}) = X\sin(\omega t - \phi) \tag{4.172}$$

A graphical interpretation of the complex representation of the excitation and response is shown in Figure 4.35.

4.13 SYSTEMS WITH COULOMB DAMPING

The differential equations derived using the free-body diagram of Figure 4.36 governing the response of a one degree-of-freedom system with Coulomb damping due to a harmonic excitation are

$$m\ddot{x} + kx = F_0\sin(\omega t + \psi) - F_f \qquad \dot{x} > 0 \tag{4.173a}$$

$$m\ddot{x} + kx = F_0\sin(\omega t + \psi) + F_f \qquad \dot{x} < 0 \tag{4.173b}$$

where $F_f = \mu mg$ is the magnitude of the friction force.

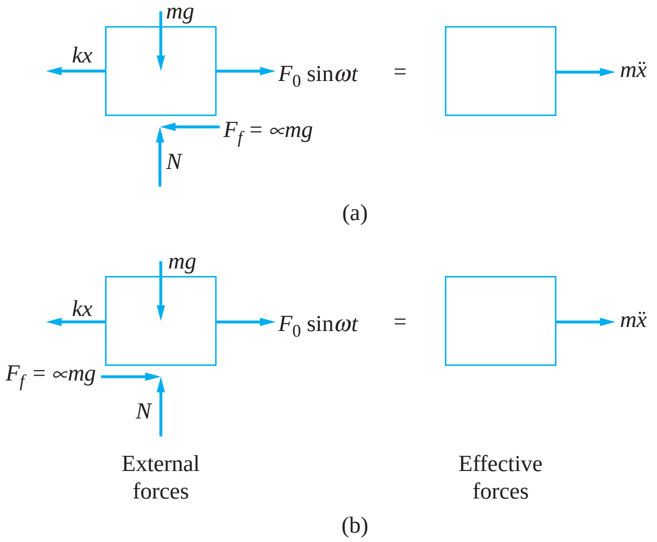


FIGURE 4.36
FBDs for systems subject to Coulomb damping and a harmonic excitation at an arbitrary instant for (a) $\dot{x} > 0$ and (b) $\dot{x} < 0$.

If the initial displacement and velocity are both zero, motion commences only when the excitation force is as large as the friction force. Motion will continue until the resultant of the spring force and the excitation force is less than the friction force,

$$|kx - F_0 \sin \omega t| < F_f \Rightarrow \dot{x} = 0 \tag{4.174}$$

The resultant eventually grows large enough such that the inequality in Equation (4.174) is no longer satisfied, when motion again commences. This process is known as stick-slip and can occur several times during one cycle of motion.

Equation (4.173) is nonlinear. Thus, the principles guiding the solution of linear differential equations are not applicable. Specifically, the general solution cannot be written as a homogeneous solution independent of the excitation plus a particular solution. Thus, even though free vibrations of a system with Coulomb damping decay linearly and eventually cease, it is not possible to predict the particular solution as a steady-state solution. Indeed, from the preceding discussion, the stick-slip process should occur for large time and cannot be predicted by a particular solution.

The analytical solution to Equation (4.173) can be attained using a procedure similar to that of Section 3.7 used to obtain the free-vibration response of a system subject to Coulomb damping. The solution of Equations (4.173a and b) are readily available over the time that the equation governs. The constants of integration are determined by noting that the velocity is zero and the displacement is continuous at the time when the equation first begins to govern. Equation (4.174) must be checked over each half-cycle to determine if and when the mass sticks.

The analytical solution is very involved and difficult to use to predict long-term behavior. In many applications only the maximum displacement is of interest. It is a function of five parameters

$$X = f(m, \omega, \omega_n, F_0, F_f) \tag{4.175}$$

Using [M], [L], and [T] as basic dimensions, the Buckingham Pi theorem implies that the nondimensional formulation involves $6 - 3 = 3$ dimensionless groups. The nondimensional formulation of Equation (4.176) is

$$\frac{m\omega_n^2 X}{F_0} = f(r, \iota) \tag{4.176}$$

where

$$\iota = \frac{F_f}{F_0} \tag{4.177}$$

For small ι, the friction force is much less than the magnitude of the excitation force, and it is expected that the transient solution will decrease as t increases and a harmonic steady state of the form

$$x(t) = X_c \sin(\omega t - \phi_c) \tag{4.178}$$

exists for large t. In this case the effects of Coulomb damping can be reasonably approximated by an equivalent viscous damping model as discussed in Section 3.9. The equivalent viscous damping coefficient for Coulomb damping is

$$c_{\text{eq}} = \frac{4F_f}{\pi\omega X_c} \tag{4.179}$$

An equivalent damping ratio is defined by

$$\zeta_{\text{eq}} = \frac{c_{\text{eq}}}{2m\omega_n} = \frac{2F_f}{\pi m\omega\omega_n X_c} \tag{4.180}$$

Rearrangement of Equation (4.180) leads to

$$\zeta_{\text{eq}} = \frac{2\iota F_0}{\pi r m\omega_n^2 X} = \frac{2\iota}{\pi r M_c} \tag{4.181}$$

where M_c, the magnification factor for Coulomb damping, is

$$M_c = \frac{m\omega_n^2 X}{F_0} \tag{4.182}$$

Using ζ_{eq} in place of ζ in Equation (4.42) leads to

$$M_c(r, \iota) = \frac{1}{\sqrt{(1 - r^2)^2 + \left(\dfrac{4\iota}{\pi M_c}\right)^2}} \tag{4.183}$$

which is solved for M_c, yielding

$$M_c(r, \iota) = \sqrt{\frac{1 - \left(\dfrac{4\iota}{\pi}\right)^2}{(1 - r^2)^2}} \tag{4.184}$$

The magnification factor for Coulomb damping is plotted in Figure 4.37 as a function of r for several values of ι. The following are noted from Equation (4.184) and Figure 4.37.

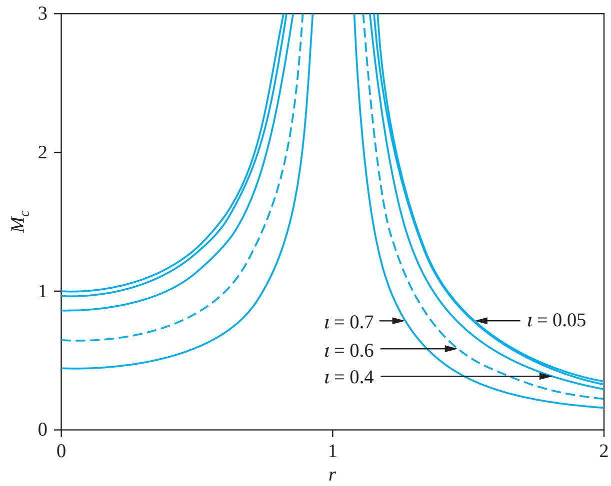


FIGURE 4.37
$M_c(r, \iota)$ versus r for different values of ι using an equivalent viscous-damping coefficient.

1. The small ι theory predicts that $M_c(r, \iota)$ exists only for $\iota < \pi/4$. The equivalent viscous damping theory cannot be used to predict the maximum displacement for $\iota > \pi/4$.
2. $\lim_{r\to\infty} M_c(r, \iota) = \dfrac{1}{r^2}$ **(4.185)**
3. Resonance occurs for systems with Coulomb damping with small ι when $r = 1$. Resonance occurs because, for small ι, the excitation provides more energy per cycle of motion than is dissipated by the friction. Since free vibrations sustain themselves at the natural frequency, the extra energy leads to an amplitude buildup.
4. For all values of r, M_c is smaller for larger ι.

When Equation (4.181) is substituted into Equation (4.45) and the resulting equation manipulated, the following result for the phase angle occurs:

$$\phi_c = \tan^{-1}\left[\frac{\dfrac{4\iota}{\pi}}{\sqrt{1-\left(\dfrac{4\iota}{\pi}\right)^2}}\right] \qquad r < 1 \tag{4.185a}$$

$$\phi_c = -\tan^{-1}\left[\frac{\dfrac{4\iota}{\pi}}{\sqrt{1-\left(\dfrac{4\iota}{\pi}\right)^2}}\right] \qquad r > 1 \tag{4.185b}$$

The phase angle is constant with r, except that it is positive for $r < 1$ and negative for $r > 1$.

The preceding theory is sufficient for small ι. For larger ι, the equation is truly nonlinear and the results more complex. However, it is expected that larger ι leads to smaller-amplitude vibrations and less serious problems. In the absence of initial energy, vibrations will not be initiated for $\iota > 1$.

EXAMPLE 4.19

A Scotch yoke mechanism operating at 30 rad/s is used to provide base excitation to a block as shown in Figure 4.38. The block has a mass of 1.5 kg and is connected to the Scotch yoke through a spring of stiffness 500 N/m. The coefficient of friction between the block and the surface is 0.13. Approximate the steady-state response of the block.

SOLUTION

The differential equation governing the motion of the block is

$$m\ddot{x} + kx = kl \sin \omega t \mp \mu mg \tag{a}$$

The amplitude of the excitation is kl. Thus

$$\iota = \frac{\mu mg}{kl} = \frac{(0.13)(1.5 \text{ kg})(9.81 \text{ m/s}^2)}{(500 \text{ N/m})(0.1 \text{ m})} = 0.038 \tag{b}$$

The system's natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = 18.26 \text{ rad/s} \qquad r = \frac{\omega}{\omega_n} = 1.64 \tag{c}$$

The Coulomb damping magnification factor is

$$M_c(1.64,\ 0.038) = \sqrt{\frac{1 - \left[\dfrac{4(0.038)}{\pi}\right]^2}{[1 - (1.64)^2]^2}} = 0.587 \tag{d}$$

The steady-state response is calculated from

$$\frac{m\omega_n^2 X}{kl} = \frac{X}{l} = M_c(1.64,\ 0.038) \tag{e}$$

$$X = (0.1 \text{ m})(0.587) = 0.0588 \text{ m} \tag{f}$$

The phase angle is calculated from Equation (4.185b) as

$$\phi_c = -\tan^{-1}\left[\frac{\dfrac{4(0.038)}{\pi}}{1 - \left(\dfrac{4(0.038)}{\pi}\right)^2}\right] = -0.0488 \tag{g}$$

The response of the system is

$$x(t) = 0.0588 \sin (18.26t + 0.0488)m \tag{h}$$

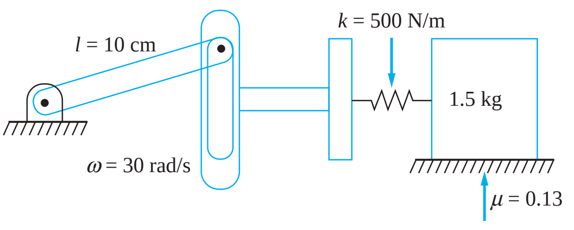


FIGURE 4.38 Scotch yoke mechanism providing base displacement for system with Coulomb damping.

4.14 SYSTEMS WITH HYSTERETIC DAMPING

Recall from Section 3.8 that the energy dissipated per cycle of motion for a system with hysteretic damping is independent of frequency but proportional to the square of the amplitude. This leads to the direct analogy between viscous damping and hysteretic damping and the development of an equivalent viscous damping coefficient

$$c_{eq} = \frac{hk}{\omega} \tag{4.186}$$

The true forced response of a mass-spring system with hysteretic damping is non-linear. The equivalent viscous damping coefficient of Equation (4.186) is valid only when the excitation consists of a single-frequency harmonic. During the initial part of the response, the transient solution and the particular solution have harmonic terms with different frequencies. On the basis of the viscous damping analogy, it is suspected that the transient solution decays leaving only the steady-state solution after a long time. The differential equation governing the steady-state response of a mass-spring system with hysteretic damping due to a single-frequency harmonic excitation is assumed to be

$$m\ddot{x} + \frac{kh}{\omega}\dot{x} + kx = F_0 \sin(\omega t + \psi) \tag{4.187}$$

It is noted that the generalization of Equation (4.187) to a more general excitation is not permissible because the damping approximation is valid only for a single-frequency harmonic excitation. The equation is also nonlinear so that the method of superposition is not applicable to determine particular solutions for multifrequency excitations.

The steady-state solution of Equation (4.187) is obtained by comparison with Equation (4.2). The equivalent damping ratio is

$$\zeta_{eq} = \frac{h}{2r} \tag{4.188}$$

The steady-state response is

$$x(t) = X_h \sin(\omega t - \phi_h) \tag{4.189}$$

where X_h and ϕ_h are obtained by analogy with Equations (4.37), (4.42), and (4.45)

$$\frac{m\omega_n^2 X_h}{F_0} = M_h(r,\ h) \tag{4.190}$$

$$M_h(r, h) = \frac{1}{\sqrt{(1 - r^2)^2 + h^2}} \tag{4.191}$$

$$\phi_h = \tan^{-1}\left(\frac{h}{1 - r^2}\right) \tag{4.192}$$

Equations (4.191) and (4.192) are plotted in Figures 4.39 and 4.40. The following are noted from these equations and figures:

1. $M_h(0, h) = \frac{1}{\sqrt{1 + h^2}}$ **(4.193)**

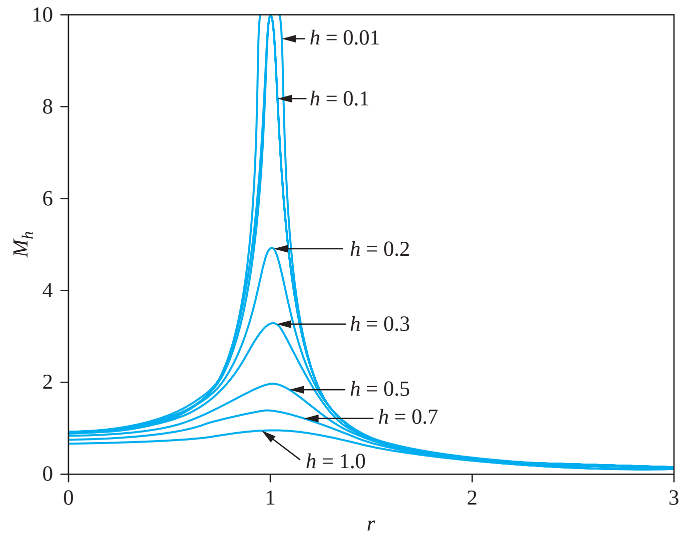


FIGURE 4.39
Magnification factor for hysteretic damping for different values of h.

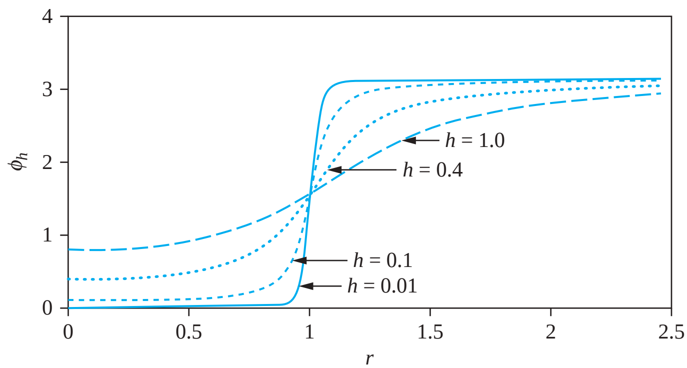


FIGURE 4.40
f_h versus r for different values of h. The response of a system with hysteretic damping is never in phase with the excitation.

2. $\lim_{r\to\infty} M_h(r, h) = \dfrac{1}{r^2}$ **(4.194)**
3. For a given h, $\dfrac{dM_h}{dr} = 0$ when $r = 1$ and the maximum value of $M_h(r, h) = \dfrac{1}{h}$.
4. The phase angle is non-zero for $r = 0$. The response is never in phase with the excitation.
5. $\lim_{r\to\infty} \phi_h = \pi$

Most damping is not viscous, but hysteretic. The differences are slight, but noticeable. Viscous damping is often assumed, even when hysteretic damping is present. The viscous damping assumption is easier to use because the damping ratio is independent of frequency. For hysteretic damping, the damping ratio is higher for lower frequencies.

If the concept of complex frequency from Section 4.13 is used, the differential equation for the forced response with hysteretic damping becomes

$$m\ddot{x} + \frac{hk}{\omega}\dot{x} + kx = F_0 e^{i\omega t} \tag{4.195}$$

Assuming a solution of the form, $x(t) = He^{i\omega t}$ results in

$$H = \frac{F_0}{-m\omega^2 + k(1 + ih)} \tag{4.196}$$

which is the same response obtained from the differential equation as

$$m\ddot{x} + k(1 + ih)x = F_0 e^{i\omega t} \tag{4.197}$$

Thus, the forced response of a system with hysteretic damping can be modeled by a system with a complex stiffness of $k(1 + ih)$.

EXAMPLE 4.20

A 100-kg lathe is mounted at the midspan of a 1.8-m simply supported beam ($E = 200 \times 10^9$ N/m, $I = 4.3 \times 10^{-6}$ m^4). The lathe has a rotating unbalance of 0.43 kg · m and operates at 2000 rpm. When a free vibrations test is performed on the system it is found that the ratio of amplitudes on successive cycles is 1.8 to 1. Determine the steady-state amplitude of vibration induced by the rotating unbalance. Assume the damping is hysteretic.

SOLUTION

The beam's stiffeness is

$$k = \frac{48EI}{L^3} = \frac{48(200 \times 10^9 \text{ N/m}^2)(4.3 \times 10^{-6} \text{ m}^4)}{(1.8 \text{ m})^3} = 7.08 \times 10^6 \text{ N/m} \tag{a}$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7.08 \times 10^6 \text{ N/m}}{100 \text{ kg}}} = 266.1 \text{ rad/s} \tag{b}$$

$$r = \frac{\omega}{\omega_n} = \frac{(2000 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{266.1 \text{ rad/s}} = 0.787 \tag{c}$$

The logarithmic decrement and hysteretic damping coefficient are calculated as

$$\delta = \ln 1.8 = 0.588 \qquad h = \frac{\delta}{\pi} = 0.187 \tag{d}$$

The appropriate form of Λ for hysteretic damping is

$$\Lambda_h(r, h) = \frac{r^2}{\sqrt{(1 - r^2)^2 + h^2}} \tag{e}$$

$$\Lambda_h(0.787, 0.187) = \frac{(0.787)^2}{\sqrt{[1 - (0.787)^2]^2 + (0.187)^2}} = 1.46 \tag{f}$$

The lathe's steady-state amplitude is

$$X = \frac{m_0 e}{m}\Lambda_h(0.787, 0.187) = \frac{0.43 \text{ kg} \cdot \text{m}}{100 \text{ kg}}(1.46) = 6.3 \text{ mm} \tag{g}$$

4.15 ENERGY HARVESTING

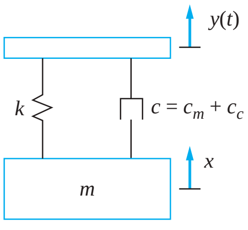


FIGURE 4.41
An energy harvester captures the vibrations of a body and converts the energy of the vibration to electrical energy.

In MEMS systems, the desire is to harvest energy from vibration: that is, to capture the energy from unwanted vibrations. A crude energy harvester, shown in Figure 4.41, consists of a seismic mass attached through an elastic element to the body whose vibrations are to be harvested (say, a machine). In addition to the stiffness which is necessary to generate vibrations of the harvester, a damping element must be present. The damping is to facilitate power transfer from the harvester and convert the power to electrical energy.

The harvester is subject to the vibrations of its base, which excites the harvester. The relative vibration between the harvester and the machine is

$$z(t) = Z \sin(\omega t - \phi) \tag{4.198}$$

The energy harvested by the viscous damper over one cycle of motion is the work done by the force in the viscous damper as $c\dot{z} = c\omega Z \cos(\omega t - \phi)$, leading to

$$E = \int_0^T c\dot{z}^2\, dt = \int_0^{\frac{2\pi}{\omega}} c\omega^2 Z^2 \cos^2(\omega t - \phi)\, dt = \pi c \omega Z^2 \tag{4.199}$$

The average power is

$$\bar{P} = \frac{\omega E}{2\pi} = \frac{\omega}{2\pi}(\pi c \omega Z^2) = \frac{1}{2} c\omega^2 Z^2 \tag{4.200}$$

Substituting $Z = Y\Lambda(r, \zeta)$, $c = 2\zeta\omega m_n$, and $r = \frac{\omega}{\omega_n}$ yields

$$\bar{P} = \zeta m \omega_n^3 r^2 \Lambda^2(r, \zeta) Y^2 \tag{4.201}$$

A nondimensional average power is defined as

$$\frac{\bar{P}}{m\omega_n^3 Y^2} = \zeta r^2 \Lambda^2(r, \zeta) = \Phi(r, \zeta) \tag{4.202}$$

Equation (4.202) is a nondimensional relationship for the average power generated by a specific energy harvester over a range of frequencies. The nondimensional function $\Psi\,(r, \zeta)$ is plotted in Figure 4.42 for several values of ζ.

The maximum average power is obtained from

$$\begin{aligned} \frac{d\Psi}{dr} &= 0 = \frac{d}{dr}\left[\frac{\zeta r^6}{(1 - r^2)^2 + (2\zeta r)^2}\right] \\ &= \frac{\zeta}{[(1 - r^2)^2 + (2\zeta r)^2]^2}\{5r^5[(1 - r^2)^2 + (2\zeta r)^2] + r^6[4r^3 - 2(2 - 4\zeta^2)r]\} \end{aligned} \tag{4.203}$$

Evaluation of Equation (f) leads to

$$r^4 - 3(2 - 4\zeta^2)r^2 + 1 = 0 \tag{4.204}$$

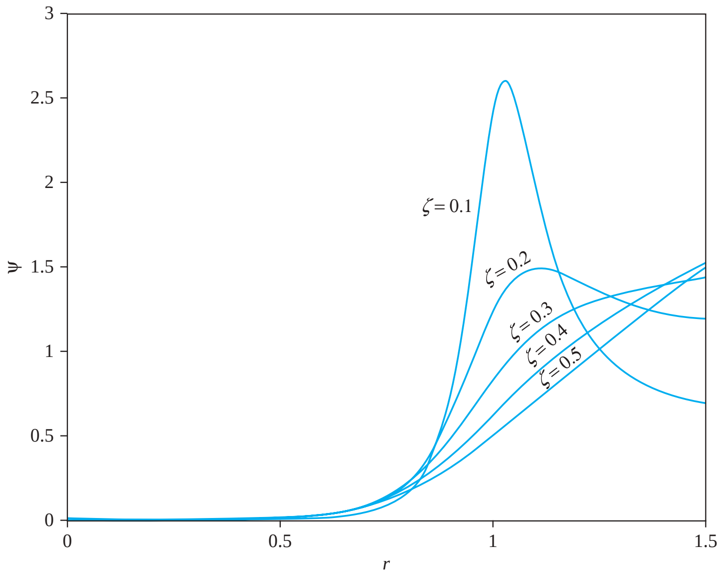


FIGURE 4.42
$\Psi(r, \zeta)$ versus r for several values of ζ. For $\zeta < 0.577$, the function has a maximum.

The solutions to Equation (g) are

$$r = \pm\left[\frac{3}{2}(2 - 4\zeta^2) \pm \frac{1}{2}\sqrt{32 - 144\zeta^2 + 144\zeta^4}\right]^{0.5} \tag{4.205}$$

The maximum average power is obtained by substituting Equation (4.205) into Equation (4.205). Equation (4.205), which is plotted in Figure 4.43, shows that for $\zeta > \frac{\sqrt{3}}{3} = 0.577$ a real value of r that solves Equation (4.204) does not exist. The value

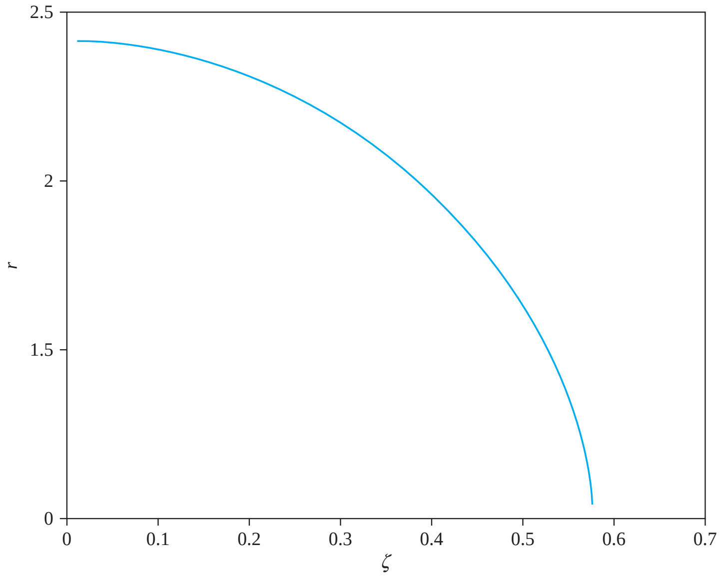


FIGURE 4.43
Solution of Equation (4.204) as a function of ζ. $\Psi(r, 0.577)$ has a maximum value at $r = 1$.

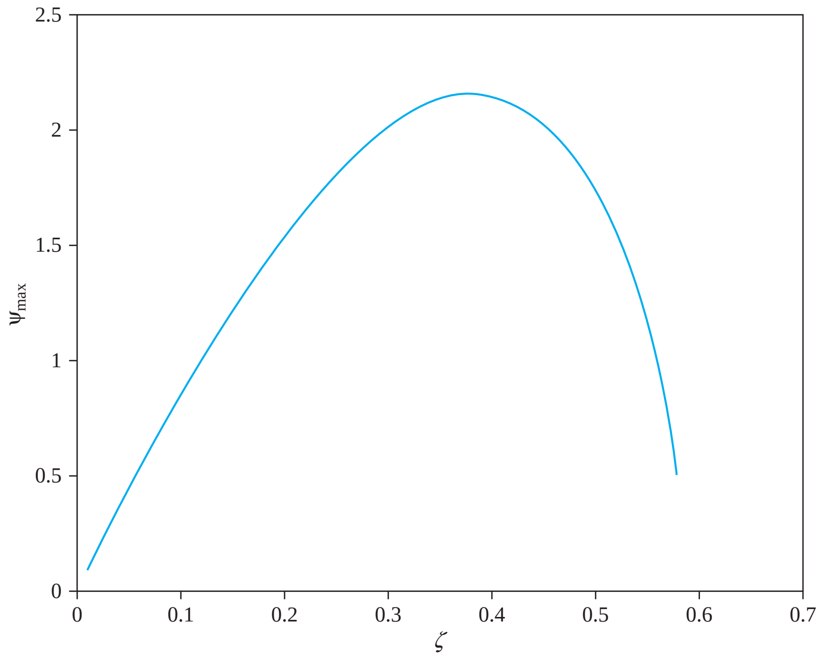


FIGURE 4.44
Ψ_{max} versus ζ.

of r for which the power has a maximum only exists for $\zeta = 0.577$. The plot of the maximum average power over the range $0 < \zeta < 0.577$ is plotted in Figure 4.44. The maximum average power reaches a maximum around $\zeta = 0.45$.

Figure 4.44 is the plot of maximum power versus ζ for an energy harvester of a given natural frequency; the natural frequency appears in the nondimensionalization of Ψ. In energy harvesting, the task is to decide upon the best natural frequency ω_n to harvest the energy at the vibration frequency ω. A reformulation yields of the average power dissipated by the viscous damper such that ω is a parameter in the non-dimensionalization of $\bar{P}$ and yields

$$\frac{\bar{P}}{m\omega^3 Y^2} = \frac{\zeta}{r}\Lambda^2(r, \zeta) = \Phi(r, \zeta) \tag{4.206}$$

Figure 4.45 shows $\Phi(r, \zeta)$versus r for several values of ζ. The maximum of $\Phi(r, \zeta)$ over all r is obtained from

$$\frac{d\Phi}{dr} = \frac{d}{dr}\left[\frac{\zeta r}{(1 - r^2)^2 + (2\zeta r)^2}\right] = \frac{\zeta[-3r^4 + (2 - 4\zeta^2)r^2 + 1]}{[(1 - r^2)^2 + (2\zeta r)^2]^2} = 0 \tag{4.207}$$

which yields

$$r = \left[\frac{1}{3}\left(1 - 2\zeta^2 \pm \sqrt{4 - \zeta^2 + \zeta^4}\right)\right]^{0.5} \tag{4.208}$$

The real solution of Equation (4.208) is plotted in Figure 4.46, and the maximum average power from Equation (4.206) is plotted in Figure 4.47 on page 272.

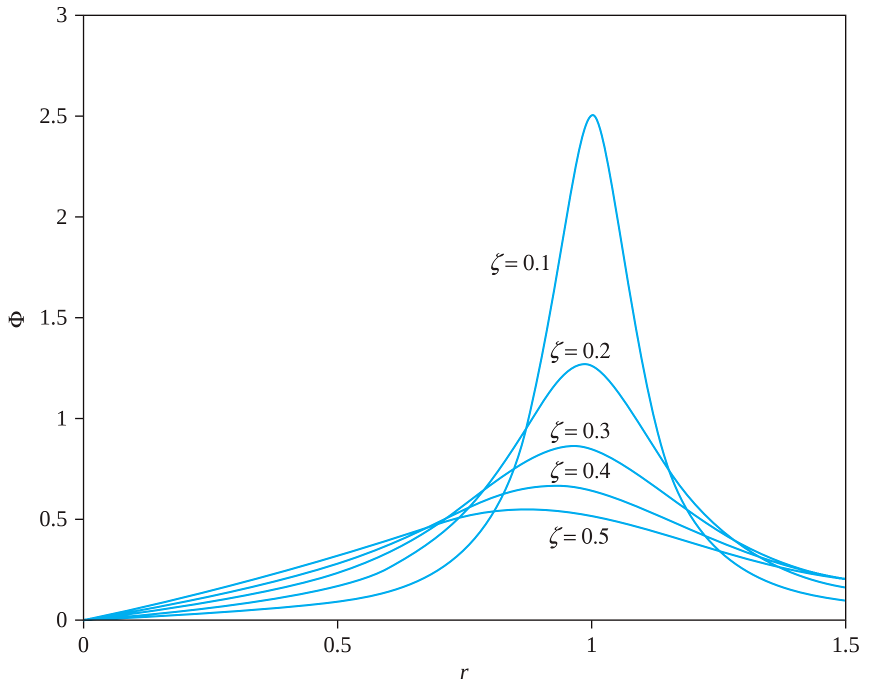


FIGURE 4.45
$\Phi(r, \zeta)$ versus r for several values of ζ.

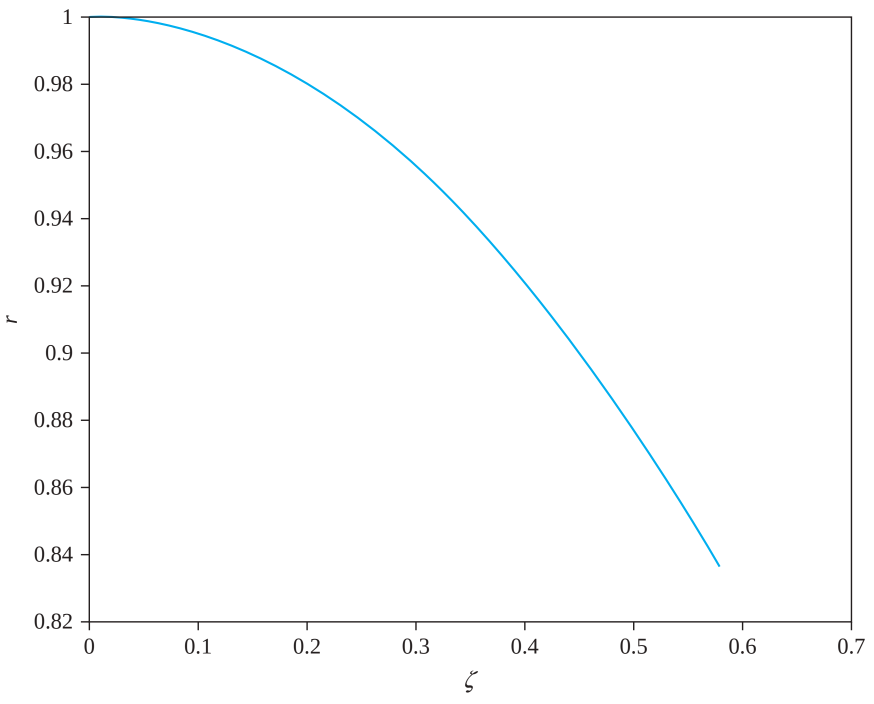


FIGURE 4.46
Solution of Equation (4.207) as a function of ζ.

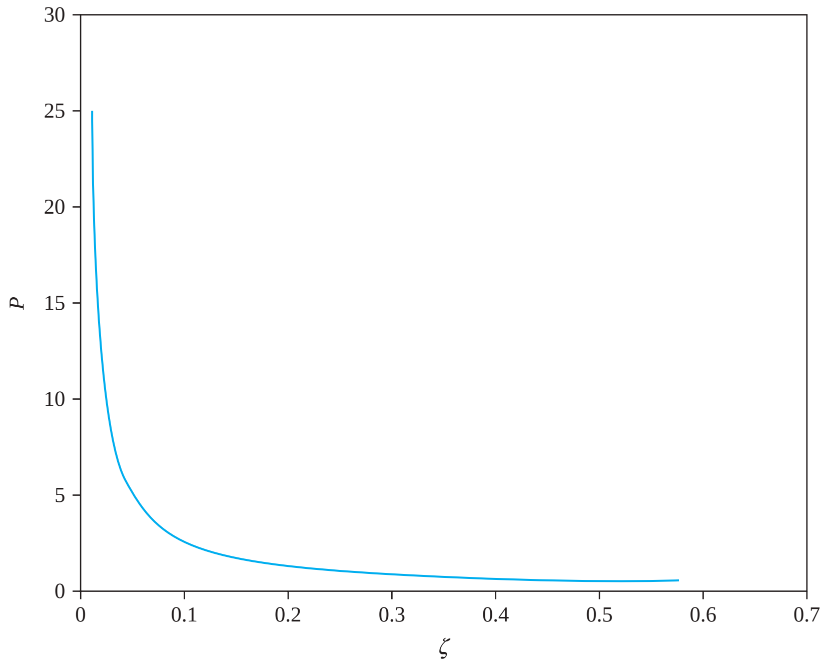


FIGURE 4.47
Φ_{max} versus ζ.

The maximum power is predicted to approach infinity for $\zeta = 0$, but this is the resonance condition. A steady-state is not reached, so the solution is not applicable. Figure 4.47 suggests that the optimal damping ratio is small. However, part of the damping ratio is from the electrical circuit that captures the energy. Thus, damping is required. However, from Figure 4.45, it is clear that a larger damping ratio gives a wider range of frequencies over which the harvester can be used.

EXAMPLE 4.21

An energy harvester is being designed with a damping ratio of 0.1 to harvest vibrations at an amplitude of 0.1 mm 30 Hz. The mass of the harvester is 1.5 g. What is the theoretical power harvested in one hour of operation?

SOLUTION

Equation (4.208) implies that $r = 0.9962$, and the natural frequency of the harvester should be

$$\omega_n = 0.9962\left(30\frac{\text{cycles}}{\text{s}}\right)\left(\frac{2\pi\ \text{rad}}{\text{cycle}}\right) = 187.8\ \text{rad/s} \tag{a}$$

The nondimensional function Φ is

$$\Phi(0.9962,\ 0.1) = \frac{(0.1)(0.9962)}{[1-(0.9962)^2] + [2(0.1)(0.9962)]^2} = 2.50 \tag{b}$$

The average power harvested over one cycle is obtained from Equation (4.206) as

$$\bar{P} = m\omega^3 Y^2 \Phi(0.9962, 0.1) = (0.0015 \text{ kg})(188.5 \text{ rad/s})^3 (0.0001\text{m})^2(2.50) = 0.2517 \text{ mW} \tag{c}$$

The number of cycles executed in one hour is

$$n = (1 \text{ hr})(3600 \text{ s/hr})(30 \text{ cycles/s}) = 108{,}000 \text{ cycles} \tag{d}$$

The power harvested in one hour is

$$P = n\bar{P} = 108{,}000(0.2511 \text{ mW}) = 27.2 \text{ W} \tag{e}$$

4.16 BENCHMARK EXAMPLES

4.16.1 MACHINE ON FLOOR OF INDUSTRIAL PLANT

During operation, the machine develops a sinusoidal force of amplitude of 20,000 lb at a speed of 80 rad/s. The ratio of the excitation frequency to the natural frequency is

$$r = \frac{\omega}{\omega_n} = \frac{80 \text{ rad/s}}{141.4 \text{ rad/s}} = 0.566 \tag{a}$$

Assuming the system is undamped, the steady-state amplitude of the machine is

$$X = \frac{F_0}{m\omega_n^2} M(0.566, 0) = \frac{20{,}000 \text{ lb}}{(38.8 \text{ slugs})(141.4 \text{ rad/s})^2} \frac{1}{1 - (0.566)^2} = 0.0379 \text{ ft} \tag{b}$$

Assuming viscous damping with a damping ratio of 0.0110, the steady-state amplitude is

$$\begin{aligned} X &= \frac{F_0}{m\omega_n^2} M(0.566, 0.0110) \\ &= \frac{20{,}000 \text{ lb}}{(38.8 \text{ slugs})(141.4 \text{ rad/s})^2} \frac{1}{\sqrt{[1 - (0.566)^2]^2 + [2(0.0110)(0.566)]^2}} \\ &= 0.0379 \text{ ft} \end{aligned} \tag{c}$$

The amplitude of the machine assuming hysteretic damping of the hysteretic damping coefficient 0.0347 is

$$\begin{aligned} X &= \frac{F_0}{m\omega_n^2} M_h(0.566, 0.0347) \\ &= \frac{20{,}000 \text{ lb}}{(38.8 \text{ slugs})(141.4 \text{ rad/s})^2} \frac{1}{\sqrt{[1 - (0.566)^2]^2 + (0.0347)^2}} \\ &= 0.0379 \text{ ft} \end{aligned} \tag{d}$$

The force transmitted to the floor is too large. A vibration isolator is designed to protect the floor from large transmitted forces generated during operation of the machine. An isolator modeled as a spring in parallel with a viscous damper is placed between the machine and the foundation. If the mass of the beam is ignored, the isolator is in series

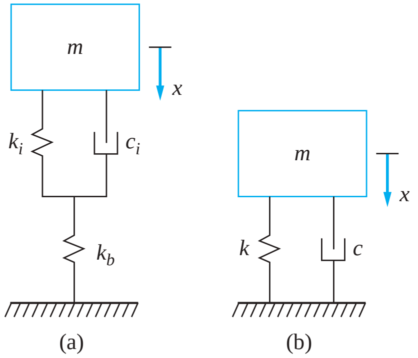


FIGURE 4.48
(a) When the mass of the beam is ignored, the beam is in series with the isolator. As an approximation, when a series combination is used to calculate the equivalent stiffness of the isolator and the beam, the stiffness of the beam is much larger than the stiffness of the isolator and can be ignored. (b) SDOF model of isolator between the machine and the beam.

with the beam, as illustrated in Figure 4.48(a), but the stiffness of the beam is much larger than the stiffness of the isolator. The equivalent stiffness is approximately that of the isolator. Thus, the flexibility of the beam is ignored, and the isolator is designed based upon a SDOF model, as illustrated in Figure 4.48(b).

To limit the transmitted force to 5000 lb,

$$T(r,\zeta) = \frac{F_T}{F_0} = \frac{5000 \text{ lb}}{20{,}000 \text{ lb}} = 0.25 \tag{e}$$

which is equivalent to

$$0.25 = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r)^2 + (2\zeta r)^2}} \tag{f}$$

The required value of r is obtained by solving Equation (f) for a specific value of ζ. The maximum natural frequency is $\omega_n = \frac{\omega}{r}$ with $\omega = 80$ rad/s. The maximum stiffness is determined from $k = m\omega_n^2$, recalling that the weight of the machine is 1000 lb. The results of the calculation for $\zeta = 0$ are r = 2.24, $\omega_n = 35.6$ rad/s, and $k = 3.93 \times 10^4$ lbf/ft.

The mass of the machine without the added inertia effects of the beam was used in the calculation of the stiffness.

The assumption that the stiffness of the beam is much larger than the stiffness of the isolator is checked. The maximum isolator stiffness is 3.93×10^4 lbf/ft, whereas the stiffness of the beam is 7.74×10^5 lbf/ft, which is 19.7 times the stiffness of the isolator. Thus, the assumption is valid.

Allowing the maximum transmitted force to vary, Figure 4.49 shows the maximum stiffness as a function of maximum transmitted force for $\zeta = 0$ and $\zeta = 0.1$.

4.16.2 SIMPLIFIED SUSPENSION SYSTEM

The differential equation of the vehicle as it traverses a road is

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{a}$$

The displacement of the vehicle relative to the road is $z = x - y$ and is governed by the equation

$$m\ddot{z} + c\dot{z} + kz = m\ddot{y} \tag{b}$$

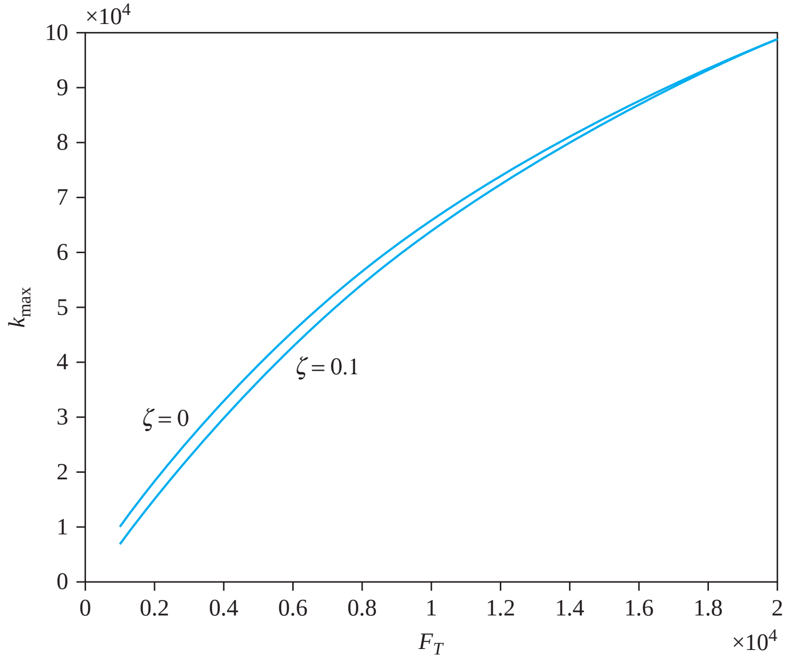


FIGURE 4.49 Maximum stiffness of isolator as a function of maximum transmitted force for $\zeta = 0$ and $\zeta = 0.1$.

or

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \ddot{y} \tag{c}$$

Consider the vehicle having a constant horizontal speed v as it traverses a road with a sinusoidal road contour $y(\xi) = Y\sin\left(\frac{2\pi\xi}{d}\right)$. Since the vehicle is traveling at a constant horizontal speed, it traverses a distance ξ in time vt. Thus, the time-dependent displacement imparted to the vehicle is $y(t) = Y\sin\left(\frac{2\pi v}{d}t\right)$. Thus, the input is a sinusoidal input of frequency $\omega = \frac{2\pi v}{d}$. The input to the relative displacement equation is a frequency-squared excitation of amplitude $m\omega^2 Y$. The key steady-state quantities are the steady-state amplitude of relative displacement

$$Z = Y\Lambda(r, \zeta) \tag{d}$$

and the amplitude of absolute acceleration

$$A = \omega^2 X = \omega^2 YT(r, \zeta) \tag{e}$$

The amplitude of absolute acceleration can be written as

$$\frac{A}{\omega_n^2 Y} = r^2 T(r,\zeta) = R(r, \zeta) \tag{f}$$

Plots of Z versus vehicle speed and A versus speed of the empty vehicle (for a half-loaded vehicle and a fully loaded vehicle for $d = 5$ m and $Y = 0.02$) are given in Figures 4.50 and 4.51, respectively. The plots are made for a vehicle with $\omega_n = 6.32$ rad/s and a damping ratio of 0.316.

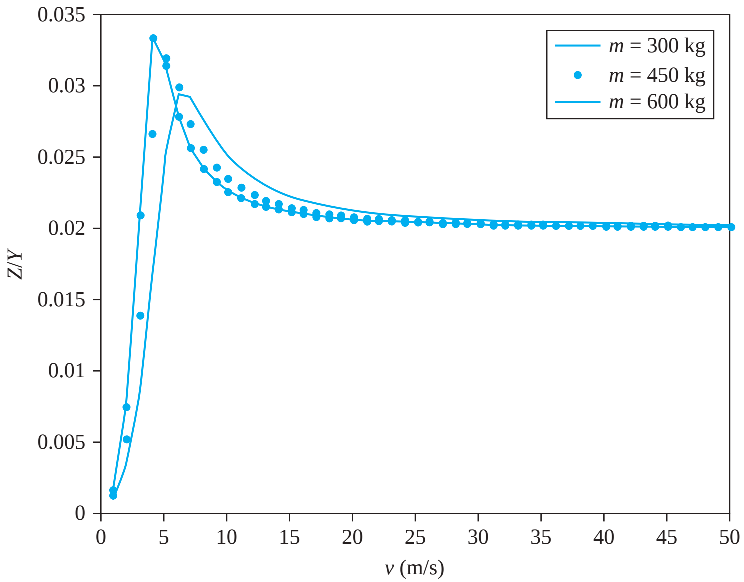


FIGURE 4.50
Z/Y versus speed for a vehicle that is empty, half-loaded, and fully loaded.

Next, consider the vehicle as it traverses a periodic road whose contour is shown in Figure 4.52, which models a road with expansion joints every 3 m. The Fourier series for the road contour is

$$y(\xi) = \frac{a_0}{2} + \sum_{i=1}^{\infty}(a_i \cos \beta_i \xi + b_i \sin \beta_i \xi) \tag{g}$$

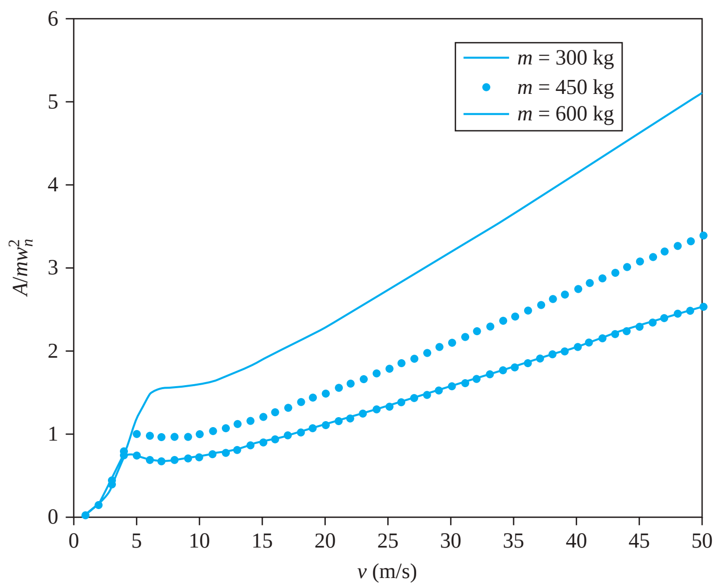


FIGURE 4.51
A/w²Y versus speed for the empty vehicle, a half-loaded vehicle, and a fully loaded vehicle.

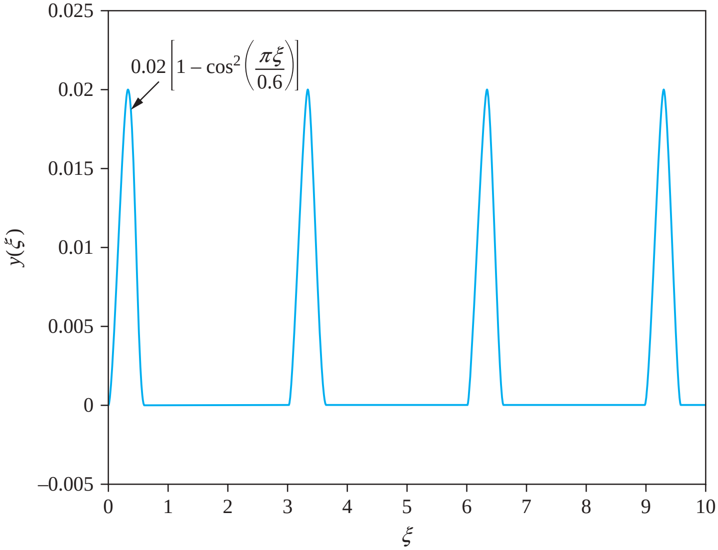


FIGURE 4.52 Periodic road contour with expansion joints every 3 m.

where

$$\lambda_i = \frac{2\pi i}{3} \tag{h}$$

The function defining the road joints is expressed as

$$y(\xi) = \begin{cases} 0.02\left(1 - \cos^2\frac{\pi}{0.6}\xi\right) & 0 \le \xi \le 0.6\,\text{m} \\ 0 & 0.6 \le \xi \le 3\,\text{m} \end{cases} \tag{i}$$

The Fourier coefficients are

$$a_0 = \frac{2}{3\,\text{m}}\int_0^T y(\xi)\,d\xi = \frac{2}{3}\int_0^{0.6} 0.02\left(1 - \cos^2\frac{\pi}{0.6}\xi\right)d\xi = 0.004 \tag{j}$$

$$a_i = \frac{2}{3\,\text{m}}\int_0^T y(\xi)\cos(\beta_i\xi)\,dt = \frac{2}{3}\int_0^{0.6} 0.02\left(1 - \cos^2\frac{\pi}{0.6}\xi\right)\cos\left(\frac{2}{3}\pi i\xi\right)d\xi$$

$$= \begin{cases} \dfrac{0.01}{\pi i}\left\{1 + \dfrac{i^2}{25[1-(0.2i)^2]}\right\}\sin(0.4\pi i) & i \ne 5 \\ 0.0020 & i = 5 \end{cases} \tag{k}$$

and

$$b_i = \frac{2}{3\,\text{m}}\int_0^T y(\xi)\sin(\beta_i\xi)dt = \frac{2}{3}\int_0^{0.6} 0.02\left(1 - \cos^2\frac{\pi}{0.6}\xi\right)\sin\left(\frac{2}{3}\pi i\xi\right)d\xi$$

$$= \begin{cases} -\dfrac{0.01}{\pi i}\left\{\left[1 + \dfrac{i^2}{25[1-(0.2i)^2]}\right][\cos(0.4\pi i) - 1]\right\} & i \ne 5 \\ 0 & i = 5 \end{cases} \tag{l}$$

The Fourier series converges $y(\xi)$, as illustrated in Figure 4.53. Rewriting the Fourier series as

$$y(\xi) = \frac{a_0}{2} + \sum_{i=1}^{\infty} c_i \sin(\beta_i \xi + \kappa_i) \tag{m}$$

where

$$c_i = (a_i^2 + b_i^2)^{1/2} = \begin{cases} \dfrac{0.01}{\pi i}\left\{1 + \dfrac{i^2}{25[1-(0.2i)^2]}\right\}\sqrt{2(1-\cos 0.4\pi i)} & i \neq 5 \\ 0.02 & i = 5 \end{cases} \tag{n}$$

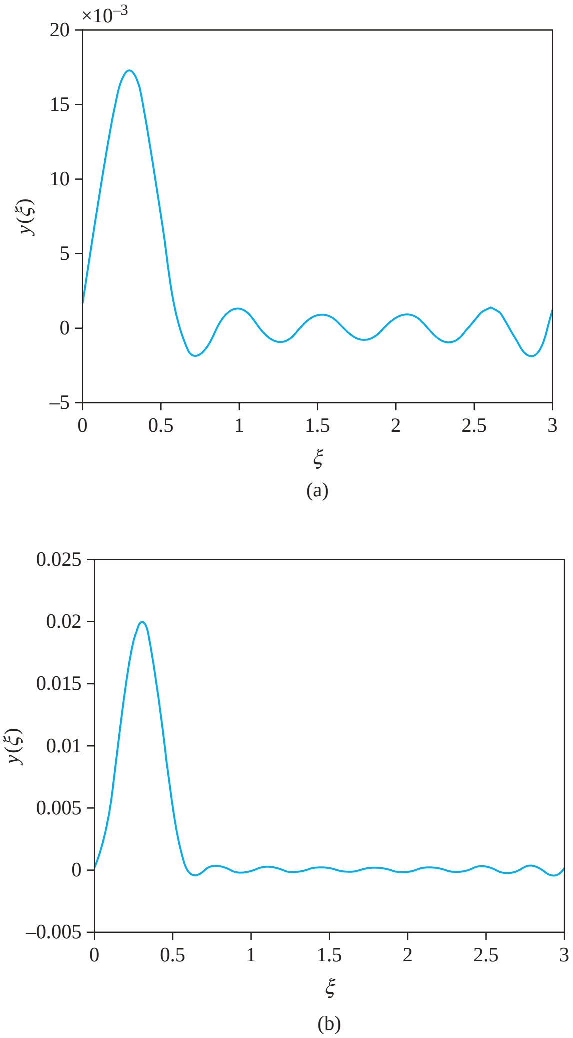
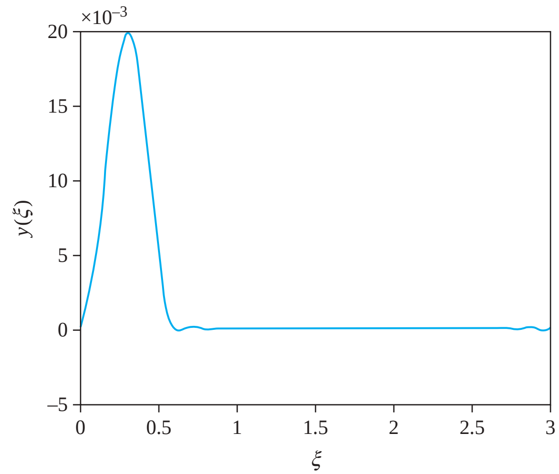
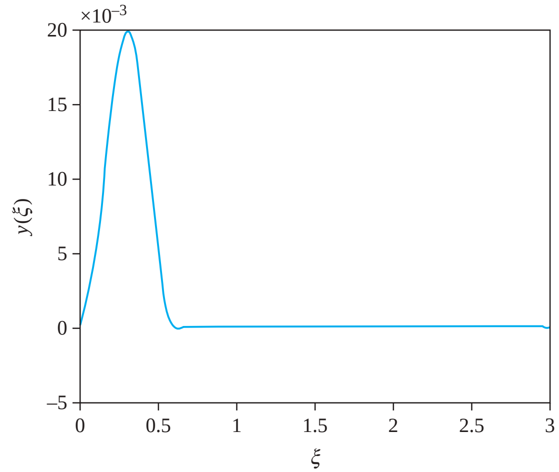


FIGURE 4.53
Convergence of Fourier series representation to $y(\xi)$ with (a) 5 terms, (b) 8 terms, (c) 15 terms, and (d) 25 terms.



(c)



(d)

FIGURE 4.53
(Continued)

and

$$\kappa_i = \tan^{-1}\frac{a_i}{b_i} = \begin{cases} \tan^{-1}\left[\dfrac{\sin 0.4\pi i}{-(\cos 0.4\pi i - 1)}\right] & i \neq 0 \\ -\dfrac{\pi}{2} & i = 5 \end{cases} \tag{o}$$

Since the vehicle is traveling at a constant horizontal speed, it traverses a distance ξ in time vt. Thus, the motion excitation applied to the wheels is $y(vt)$ or

$$y(t) = \frac{a_0}{2} + \sum_{t=1}^{\infty} c_i \sin(\beta_i vt + \kappa_i) \tag{p}$$

The differential equation governing the displacement of the body of the vehicle is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y \tag{q}$$

or

$$\begin{aligned}\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = {} & 2\zeta\sum_{t=1}^{\infty} c_i\beta_i v\cos(\beta_i vt + \kappa_i) \\ & + \omega_n^2\left[\frac{a_0}{2} + \sum_{i=1}^{\infty} c_i\sin(\beta_i vt + \kappa_i)\right]\end{aligned} \tag{r}$$

Noting that the solution of Equation (q) with a single-frequency term on the right-hand side with magnitude Y is $y(t) = YT(r, \zeta)\sin(\omega t + \lambda)$, the principle of linear superposition is applied yielding

$$x(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} T(r_i, \zeta)c_i\sin(\beta_i vt + \kappa_i - \lambda_i) \tag{s}$$

where

$$r_i = \frac{vB_i}{\omega_n} \tag{t}$$

The plot of the steady-state response over on period is given in Figure 4.54 for $v = 30$ m/s. The acceleration is

$$a(t) = \sum_{i=1}^{\infty}(\lambda_i v)^2 T(r_i, \zeta)c_i\sin(\lambda_i vt + \kappa_i - \lambda_i) \tag{u}$$

The steady-state acceleration is plotted in Figure 4.55 for $v = 30$ m/s.

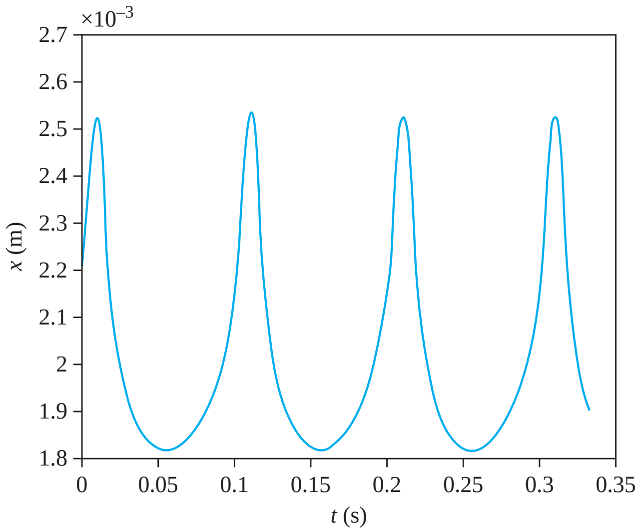


FIGURE 4.54
Displacement of vehicle as a function of time for $v = 30$ m/s.

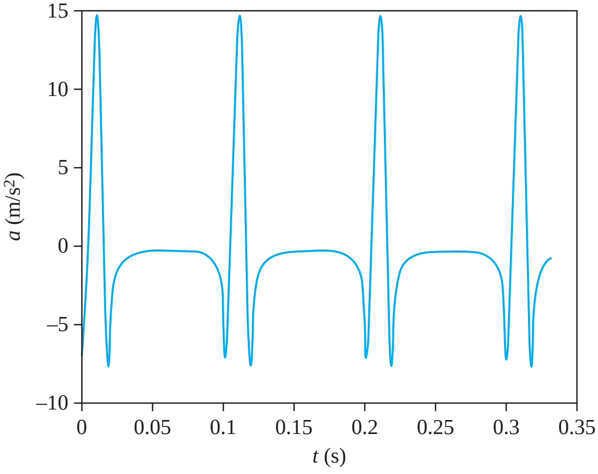


FIGURE 4.55
Acceleration of vehicle as a function of time for $v = 30$ m/s.

4.17 FURTHER EXAMPLES

EXAMPLE 4.22

A 50-kg machine tool is mounted on an elastic foundation that is modeled as a spring and viscous damper in parallel. In order to determine the properties of the foundation, a force with a magnitude of 8000 N is applied to the machine tool at a variety of speeds. It is observed that the maximum steady-state amplitude is 2.5 mm, which occurs at 35 Hz. Determine the equivalent stiffness and equivalent damping coefficient of the foundation.

SOLUTION

The maximum steady-state amplitude occurs for a frequency ratio of $r_m = \omega_m/\omega_n = \sqrt{1-2\zeta^2}$ and corresponds to a magnification factor $M_{\max} = \dfrac{m\omega_n^2 X_{\max}}{F_0} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$. Substituting given numbers leads to

$$\frac{(35\text{ cycles/s})(2\pi\text{ rad/cycle})}{\omega_n} = \sqrt{1-2\zeta^2} \tag{a}$$

and

$$\frac{(50\,\text{kg})\ \omega_n^2(0.0025\text{ m})}{8000\text{ N}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{b}$$

Eliminating ω_n between Equations (a) and (b) yields

$$0.756 = \frac{1-2\zeta^2}{2\zeta\sqrt{1-\zeta^2}} \tag{c}$$

Rearranging Equation (c) leads to

$$6.286\zeta^4 - 6.286\zeta^2 + 1 = 0 \tag{d}$$

whose solutions are $\zeta = 0.446, 0.895$. The smaller value of ζ is the appropriate solution, as it is less than $1/\sqrt{2}$ for which M reaches a maximum. Thus,

$$\omega_n = \frac{\omega}{\sqrt{1 - 2\zeta^2}} = \frac{70\pi \ \text{rad/s}}{\sqrt{1 - 2(0.446)^2}} = 283.2 \ \text{rad/s} \tag{e}$$

The stiffness is calculated is

$$k = m\omega_n^2 = (50 \ \text{kg})(245.7 \ \text{rad/s})^2 = 4.0 \times 10^6 \ \text{N/m} \tag{f}$$

and the damping coefficient is

$$c = 2\zeta m\omega_n = 2(0.446)(50 \ \text{kg}) \ (245.7 \ \text{rad/s}) = 1.26 \times 10^4 \ \text{N} \cdot \text{s/m} \tag{g}$$

EXAMPLE 4.23

A 65 kg industrial sewing machine operates at 125 Hz and has a rotating unbalance of 0.15 kg·m. The machine is mounted on a foundation with a stiffness of 2×10^6 N/m and a damping ratio of 0.12. Determine the machine's steady amplitude.

SOLUTION

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \times 10^6 \ \text{N/m}}{65 \ \text{kg}}} = 175.4 \ \text{r/s} \tag{a}$$

The frequency ratio for the excitation is

$$r = \frac{\omega}{\omega_n} = \frac{(125 \ \text{cycles/s})(2\pi \ \text{rad/cycle})}{175.5 \ \text{rad/s}} = 4.48 \tag{b}$$

The steady-state amplitude is found from

$$\frac{mX}{m_0 e} = \Lambda(4.48, \ 0.12) = \frac{(4.48)^2}{\sqrt{(1 - 4.48^2)^2 + [2(0.12)(4.48)]^2}} = 1.051 \tag{c}$$

Equation (c) is solved, yielding

$$X = \frac{m_0 e}{m}\Lambda(4.48, \ 0.12) = \left(\frac{0.15 \ \text{kg} \cdot \text{m}}{65 \ \text{kg}}\right)1.051 = 2.43 \ \text{mm} \tag{d}$$

EXAMPLE 4.24

A 500 kg tumbler has a rotating unbalance of 12.6 kg, which is 5 cm from its axis of rotation. For what stiffnesses of an elastic mounting of damping ratio 0.06 will the tumbler's steady–state amplitude be less than 2 mm for all speeds of operation between 200 rpm and 600 rpm?

SOLUTION

From the given information, the allowable value of the nondimensional parameter Λ is

$$\Lambda_{\text{all}} = \frac{mX_{\text{all}}}{m_0 e} = \frac{(500 \ \text{kg})(0.002 \ \text{m})}{(12.6 \ \text{kg})(0.05 \ \text{m})} = 1.587 \tag{a}$$

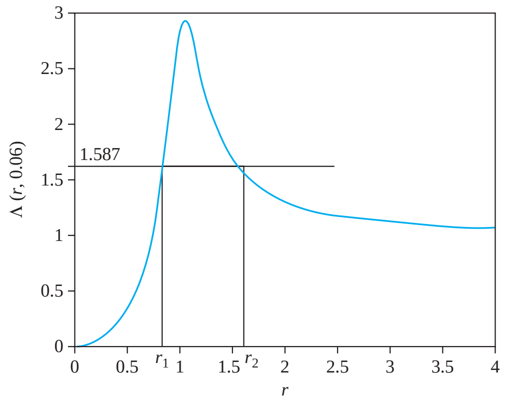


FIGURE 4.56
$\Lambda(r, 0.06)$ versus r.

The curve of $\Lambda(r, 0.06)$ versus r is shown in Figure 4.56. Since $\Lambda_{\text{all}} > 1$ there are two values of r for which $\Lambda(r, 0.06) = \Lambda_{\text{all}}$. These can be found by solving

$$\frac{r^2}{\sqrt{(1 - r^2)^2 + [2(0.06)r]^2}} = 1.587 \tag{b}$$

The solutions are $r = 0.788, 1.635$. Consider first the lower value of r, $\Lambda < 1.587$ for $r < 0.788$. Thus, if $r = 0.788$ corresponds to $\omega = 600$ rpm, the steady-state amplitude is less than 2 mm for all speeds less than 600 rpm. Thus, requiring $r < 0.788$ or equivalently $\frac{600 \text{ rpm}}{\omega_n} < 0.788$, this implies $\omega_n > 761.4$ rpm or $\omega_n > \left(761.4 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 79.73$ rad/s. This leads to

$$k > (500 \text{ kg})\left(79.73 \frac{r}{s}\right)^2 = 3.18 \times 10^6 \text{ N/m} \tag{c}$$

If $r = 1.635$ corresponds to $\omega = 200$ rpm, then $\Lambda < 1.537$ or $X < 2$ mm for all $\omega > 200$ rpm. Thus, $r > 1.635$ implies that $\frac{200 \text{ rpm}}{\omega_n} > 1.635$, which leads to $\omega_n < 122.3$ rpm or $\omega_n < 12.81$ rad/s. The allowable stiffnesses are

$$k > (500 \text{ kg})\ (12.81 \text{ rad/s})^2 = 8.21 \times 10^4 \text{ N/m} \tag{d}$$

Thus, the steady-state amplitude of the machine is less than 2 mm at all speeds between 200 rpm and 600 rpm if $k > 3.18 \times 10^6$ N/m or $k < 8.21 \times 10^4$ N/m.

EXAMPLE 4.25

What is the minimum static deflection of an isolator to provide 85 percent isolation to a fan that operates at speeds between 1500 rpm and 2200 rpm if (a) the isolator is undamped and (b) the isolator has a damping ratio $\zeta = 0.1$?

SOLUTION

Eighty-five percent isolation leads to a transmissibility ratio of $T = 0.15$.

(a) If the isolator is undamped, the appropriate equation to use is

$$T(r, 0) = \frac{1}{r^2 - 1} \tag{a}$$

which leads to $0.15 = \frac{1}{r^2 - 1}$ and $r = 2.77$. Since $T(r, 0) < 0.15$ for $r > 2.77$, it is required that $r = 2.77$ corresponds to the lowest allowable frequency at $\omega = 1500$ rpm = 157.1 rad/s. To this end,

$$\frac{157.1 \text{ rad/s}}{\omega_n} = 2.77 \tag{b}$$

which gives $\omega_n = 56.7$ rad/s. The required static deflection is

$$\Delta_s = \frac{mg}{k} = \frac{g}{\omega_n^2} = \frac{9.81 \text{ m/s}^2}{(56.7 \text{ rad/s})^2} = 3.1 \text{ mm} \tag{c}$$

(b) If the isolator has a damping ratio of 0.1, then

$$T(r, 0.1) = 0.15 = \sqrt{\frac{1 + [2(0.1)r]^2}{(1 - r^2)^2 + [2(0.1)r]^2}} \tag{d}$$

Squaring both sides and rearranging leads to

$$r^4 - 3.737r^2 - 43.44 = 0 \tag{e}$$

whose solution is $r = 2.953$. Following the procedure in part (a), the required natural frequency is calculated as $\omega_n = 53.2$ rad/s and $\Delta_s = 3.5$ mm. The increased damping ratio leas to a lower natural frequency and a higher required static deflection.

EXAMPLE 4.26

A 50 kg machine has a rotating unbalance. The machine is mounted on an elastic foundation with a stiffness of 1.3×10^5 N/m, and damping ratio of 0.04 and operates at 1500 rpm. An accelerometer is mounted on the machine to monitor its steady-state vibrations.

(a) What is the minimum natural frequency of an accelerometer of damping ratio 0.2 such that it measures the vibrations of the machine with no more than 2 percent error?
(b) When the accelerometer of part (a) is used, it measures a steady-state amplitude of 14.8 m/s^2. What is the magnitude of the rotating unbalance?
(c) What is the accelerometer output if the machine operates at 1200 rpm?

SOLUTION

(a) The percent error in the accelerometer measurement is $E = 100|1 - M(r, \zeta)|$ where the frequency ratio refers to the ratio of the frequency of excitation to the natural frequency of the accelerometer. The accelerometer works in the range of small r and $\zeta < \frac{1}{\sqrt{2}}$. Thus, $M(r, \zeta) > 1$. In order for the error to be less than 2 percent,

$$100[M(r, 0.2) - 1] < 2 \tag{a}$$

or $M(r, 0.2) < 1.02$, which implies that

$$\frac{1}{\sqrt{(1 - r^2)^2 + [2(0.2)r]^2}} < 1.02 \tag{b}$$

The solutions of Equation (a) are $r < 0.146$ and $r > 1.35$. However, requiring $r > 1.35$ will lead to the error being greater than 2 percent for when $100[1 - M(r, 0.2)] < 0.98$.

Thus, the minimum natural frequency for the error to be less than 2 percent requires that $r = 1.46$ corresponds to $\omega = 1500$ rpm. To this end

$$\frac{\left(1500\frac{\text{rev}}{\text{min}}\right)\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)}{\omega_n} = 0.146 \tag{c}$$

which leads to $\omega_n = 1076$ rad/s.

(b) The error in the measurement is 2 percent. Thus, if A is the actual acceleration and B is the measurement, then $B = 1.02A$. With $B = 14.8\ \text{m/s}^2$, this gives $A = 14.5\ \text{m/s}^2$. Then the amplitude of the steady-state vibration is related to the acceleration amplitude by $A = \omega^2 X$. With $\omega = 1500$ rpm $= 157.1$ rad/s, the steady-state amplitude is $X = 5.87 \times 10^{-4}$ m. For the machine with a rotating unbalance,

$$\frac{mX}{m_0 e} = \Lambda(r_m, 0.04)$$

where r_m is the ratio of the excitation frequency to the natural frequency of the machine. Performing the necessary calculations, the natural frequency of the machine is

$$\omega_n = \sqrt{\frac{k}{m_m}} = \sqrt{\frac{1.3 \times 10^5 \frac{\text{N}}{\text{m}}}{50\ \text{kg}}} = 51.0\ \text{rad/s} \tag{d}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{157.1\ \text{rad/s}}{51.0\ \text{rad/s}} = 3.08 \tag{e}$$

Then

$$\Lambda(3.08, 0.04) = \frac{(3.08)^2}{\sqrt{[1 - (3.08)^2]^2 + [2(0.04)(3.08)]^2}} = 1.12 \tag{f}$$

and the magnitude of the rotating unbalance is

$$m_0 e = \frac{mX}{\Lambda(3.08,\ 0.04)} = \frac{(50\ \text{kg})(5.9 \times 10^{-4}\text{m})}{1.12} = 0.0264\ \text{kg} \cdot \text{m} \tag{g}$$

(c) The machine now rotates at $\omega = 1200$ rpm $= 125.7$ rad/s. Thus, $r = \frac{125.7\ \text{rad/s}}{51.0\ \text{rad/s}} = 2.46$ and $\Lambda(2.46,\ 0.04) = 1.197$. The steady-state response of the machine is $x(t) = X \sin\ (\omega t - \phi)$ where

$$X = \frac{m_0 e}{m_m}\Lambda(r,\ 0.04) = \frac{0.0264\ \text{kg} \cdot \text{m}}{50\ \text{kg}}(1.197) = 6.32 \times 10^{-4}\ \text{m} \tag{h}$$

and

$$\phi = \tan^{-1}\left[\frac{2(0.04)r}{1 - r^2}\right] = \tan^{-1}\left[\frac{(0.08)(2.46)}{1 - (2.46)^2}\right] = -0.0389\ \text{rad} \tag{i}$$

Thus, the steady-state response of the machine is

$$x(t) = 6.32 \times 10^{-4}\ \sin\ (125.7t + 0.0389)\ \text{m} \tag{j}$$

The accelerometer output is $-\frac{\omega_n^2}{M(r_a, 0.2)}z(t)$ where $r_a = \frac{125.7 \text{ rad/s}}{1076 \text{ rad/s}} = 0.117$ and $M(0.117,0.2) = 1.013$. The error in the accelerometer measurement is 1.3 percent. $z(t)$ is the displacement of the seismic mass relative to the machine and is given as

$$z(t) = Z_a \sin(125.7t + 0.0389 - \phi_a) \quad \text{(k)}$$

where

$$Z_a = X\Lambda(0.117, 0.2) = (6.32 \times 10^{-4} \text{ m})(0.013) = 8.78 \times 10^{-6} \text{ m} \quad \text{(l)}$$

and

$$\phi_a = \tan^{-1}\left[\frac{2(0.2)(0.117)}{1-(0.117)^2}\right] = 0.0461 \text{ rad} \quad \text{(m)}$$

Thus, the accelerometer output is

$$\begin{aligned} a(t) &= -\frac{(1076 \text{ rad/s})^2}{1.013}(8.78 \times 10^{-6} \text{ m}) \sin(125.7t + 0.0389 - 0.0461) \\ &= 10.03 \sin(125.7t - 0.0072) \text{ m/s}^2 \end{aligned} \quad \text{(n)}$$

EXAMPLE 4.27

An energy harvester is being designed to harvest energy from a MEMS system whose vibrations are given by

$$y(t) = (10 \sin 400t + 15 \sin 500t) \ \mu\text{m} \quad \text{(a)}$$

The harvester is to have damping ratio 0.2 and a mass of 0.002 g.

(a) What is the best natural frequency for the harvester?
(b) How much power is harvested in one hour?

SOLUTION

(a) Since the periods of both terms in the vibration are not the same, it is difficult to define the average power over one cycle. The period over which both vibrations repeat is

$$T_c = \frac{2\pi(900)}{(400)(500)} = 0.0282 \text{ s} \quad \text{(b)}$$

The relative response between the harvester and the machine is

$$z(t) = 10\Lambda(r_1, \zeta)\sin(400t - \phi_1) + 15\Lambda(r_2, \zeta)\sin(500t - \phi_2) \quad \text{(c)}$$

The power dissipated by the viscous damper over this period is

$$\begin{aligned} P &= 10^{-12}\int_0^{0.0282} c[(10)(400)\Lambda(r_1,\zeta)\cos(400t - \phi_1) \\ &\quad + (15)(500)\Lambda(r_1, \zeta)\cos(500t - \phi_2)]^2 \, dt \\ &= 2\zeta m\omega_n 10^{-6}\{0.226\Lambda^2(r_1, \zeta) + 0.763\Lambda^2(r_1, \zeta) \\ &\quad + 0.3\Lambda(r_1, \zeta)\Lambda(r_2, \zeta)[\sin(\phi_2 - \phi_1) - \sin(2.821 + \phi_2 - \phi_1)]\} \end{aligned} \quad \text{(d)}$$

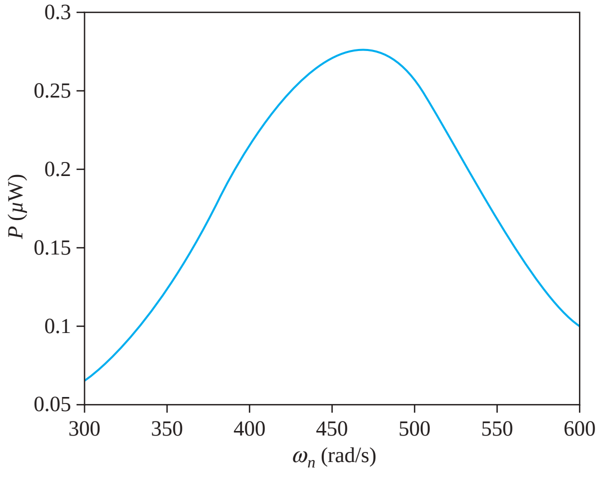


FIGURE 4.57
Plot of power harvested versus ω_n for system of Example 4.27.

Equation (d) is plotted against ω_n in Figure 4.57. The largest power harvested is 0.277 μW and occurs for $\omega_n = 468$ rad/s.

(b) The number of cycles in one hour is

$$n = \left(\frac{3600 \text{ s/hr}}{0.0282 \text{ s/cycle}}\right)(1 \text{ hr}) = 1.27 \times 10^5 \text{ cycles} \tag{e}$$

The power captured in one hour is

$$P = \left(0.277 \frac{\mu\text{W}}{\text{cycle}}\right)(1.27 \times 10^5 \text{ cycles}) = 3.52 \times 10^{-2} \text{ W} \tag{f}$$

EXAMPLE 4.28

The torsional spring of the system of Example 3.16 is attached to an actuator which provides a harmonic displacement of $\Phi \sin \omega t$ to the system as shown in Figure 4.58. Take $\Phi = 10°$.

(a) If the electromagnet is turned off determine the form of the magnification factor for the pendulum (M_c), assuming Coulomb damping. What is the steady-state amplitude of the pendulum if $\omega = 4$ rad/s?
(b) If the electromagnet is turned on, predict the steady-state amplitude of the pendulum if $\omega = 4$ rad/s.

SOLUTION

If the electromagnet is turned off the pendulum is subject to Coulomb damping with a resisting moment of 0.0629 N · m (Example 3.16). The differential equation governing the forced oscillations of the pendulum is

$$I\ddot{\theta} + k_t\theta = k_t\Phi \sin\omega t + \begin{cases} -M_f & \dot{\theta} > 0 \\ M_f & \dot{\theta} < 0 \end{cases} \tag{a}$$

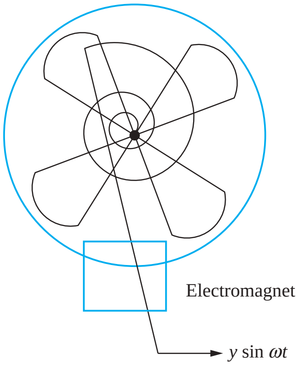


FIGURE 4.58
System of Example 4.28.

where $I = 0.183\ \text{kg} \cdot \text{m}^2$ and $k_t = 1.8\ \text{N} \cdot \text{m/rad}$. The theory regarding steady-state vibration of systems with Coulomb damping applies with (in Example 3.16 if was found that $M_f = 0.0157\ \text{N} \cdot \text{M}$)

$$\iota = \frac{M_f}{k_t \Phi} = \frac{0.0157\ \text{N} \cdot \text{m}}{\left(1.8 \dfrac{\text{N} \cdot \text{m}}{\text{rad}}\right)(10°)\left(\dfrac{2\pi\ \text{rad}}{360°}\right)} = 0.050 \tag{b}$$

The magnification factor is

$$M_c(r, 0.2) = \sqrt{\frac{1 - \left[\dfrac{4(0.05)}{\pi}\right]^2}{(1 - r^2)^2}} = \sqrt{\frac{0.996}{(1 - r^2)^2}} = \frac{0.998}{|1 - r^2|} \tag{c}$$

For $\omega = 4$ rad/s, $r = \frac{4\ \text{rad/s}}{3.14\ \text{rad/s}} = 1.27$ and $M_c\ (1.27, 0.2) = 1.63$. The steady-state amplitude is

$$\Theta = \frac{k_t \Phi}{I \omega_n^2} M_c(1.27{,}0.2) = \frac{(1.8\ \text{N} \cdot \text{m/rad})(10°)}{(0.183\ \text{kg} \cdot \text{m}^2)(3.14\ \text{rad/s})^2}(1.63) = 16.26° \tag{d}$$

(b) If the electromagnet is turned on, the system has viscous damping which dominates the Coulomb damping. The differential equation governing the motion of the system is

$$I\ddot{\theta} + c_t \dot{\theta} + k_t \theta = k_t \Phi \sin \omega t \tag{e}$$

which is written in standard form as

$$\ddot{\theta} + 2\zeta\omega_n \dot{\theta} + \omega_n^2 \theta = \omega_n^2 \Phi \sin \omega t \tag{f}$$

The steady-state amplitude is given by

$$\Theta = \frac{k_t \Phi}{k_t} M(r, \zeta) = \Phi M(r, \zeta) \tag{g}$$

The demping ratio is 0.011 (Example 3.16) and for $\omega = 4$ rad/s, $r = \frac{4\ \text{rad/s}}{3.14\ \text{rad/s}} = 1.27$. Thus,

$$\Theta = (10°)M(1.27{,}0.011) = (10°)\frac{1}{\sqrt{[1-(1.27)^2]^2 + [2(0.011)(1.27)]^2}} = 16.29° \tag{h}$$

4.18 CHAPTER SUMMARY

4.18.1 IMPORTANT CONCEPTS

The topics covered in this chapter included steady-state vibrations of SDOF systems. The following refer to these topics.

- Resonance, which is characterized by an unbounded growth in amplitude, occurs in an undamped system when the input frequency coincides with the natural frequency.
- Resonance occurs because the work done by the external force is not necessary to sustain the vibrations at the natural frequency.
- Beating, which occurs in an undamped system when the input frequency is near but not equal to the natural frequency, is characterized by a continual build up and decay of amplitude.
- Free vibrations of a damped system die out after a period of time leaving only the particular solution, which is the steady–state solution.
- The steady–state response of a system with viscous damping due to a single-frequency harmonic excitation is at the same frequency as the input but at a different phase angle.
- The amplitude of the response is affected by the stiffness, inertia, and damping properties of the system.
- The nondimensional magnification factor, which is the ratio maximum force developed in the spring to the maximum of the excitation force, is a function of the frequency ratio and the damping ratio $M(r, \zeta)$.
- The frequency response is studied by considering the behavior of $M(r, \zeta)$ for varying r for different values of ζ where $M(0, \zeta) = 1$ and $\lim_{r\to\infty} M(r, \zeta) = 0$. For $\zeta < \frac{1}{\sqrt{2}}$, $M(r, \zeta)$ increases as r increases from zero and reaches a maximum before it starts decreasing. For $\zeta > \frac{1}{\sqrt{2}}$, $M(r, \zeta)$ decreases monotonically with increasing r.
- Frequency-squared excitations occur when the amplitude of excitation is proportional to the square of the frequency. A machine with a rotating unbalance is an example of a system with frequency-squared excitation.
- The frequency response for frequency-squared excitations is given by a nondimensional function $\Lambda(r, \zeta)$ where $\Lambda(r, 0) = 0$ and $\lim_{r\to\infty} \Lambda(r, \zeta) = 1$. For $\zeta < \frac{1}{\sqrt{2}}$, $\Lambda(r, \zeta)$ reaches a maximum and then approaches 1 from above. For $\zeta > \frac{1}{\sqrt{2}}$, $\Lambda(r, \zeta)$ has no maximum and approaches 1 from below.
- Harmonic-based motion is analyzed by considering the displacement of the mass relative to the base. The relative displacement is governed by the standard differential equation in which the mass times acceleration of the base replaces the forcing term. The steady-state amplitude of relative displacement is given by the amplitude of the base motion times $\Lambda(r, \zeta)$.
- The ratio of the amplitude of acceleration of the mass to the amplitude of acceleration of the base is given by a nondimensional function $T(r, \zeta)$, which is only less than 1 for $r > \sqrt{2}$.

- The range $r > \sqrt{2}$ is called the range of isolation; $r < \sqrt{2}$ is called the range of amplification.
- An increase in damping ratio leads to an increase in $T(r, \zeta)$ in the range of isolation. Damping hurts isolation.
- Vibration isolation theory includes protection of machines from large amplitude accelerations of their bases and the protection of foundations from large amplitude forces developed in machines.
- The steady–state response due to multi-frequency excitations is obtained using the principle of linear superposition.
- Any periodic excitation has a Fourier series representation which converges pointwise to the function at all times where it is continuous.
- All Fourier cosine coefficients are zero for an odd function. All Fourier sine coefficients are zero for an even function.
- Seismic vibration measuring instruments have a seismic mass which moves relative to the body whose vibrations are being measured.
- Seismometers measure the motion of the seismic mass relative to its housing and operate with a large frequency ratio where $\Lambda(r, \zeta)$ is close to l.
- Accelerometers measure the acceleration of the body whose vibrations are to be measured and operate with a small frequency ratio where $M(r, \zeta)$ is close to 1.
- An equivalent viscous-damping ratio is used to formulate a magnification factor for Coulomb damping.
- The steady-state behavior of a system with hysteretic damping can be obtained using a complex stiffness.
- An energy harvester has a seismic mass which vibrates relative to the body whose vibrations are being harvested. The average power harvested per cycle of steady-state motion increases with the decreasing damping ratio of the harvester.

4.18.2 IMPORTANT EQUATIONS

Standard form of differential equation governing forced vibrations of linear, single degree-of-freedom systems

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m_{\text{eq}}} F_{\text{eq}}(t) \tag{4.2}$$

Particular solution for undamped system when excitation frequency coincides with natural frequency

$$x_p(t) = -\frac{F_0}{2m_{\text{eq}}\omega_n} t \cos(\omega_n t + \psi) \tag{4.20}$$

Response when beating occurs

$$x(t) = \frac{2F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)} \sin\left[\left(\frac{\omega - \omega_n}{2}\right)t\right] \cos\left[\left(\frac{\omega + \omega_n}{2}\right)t\right] \tag{4.22}$$

Steady–state response of system with viscous damping

$$x_p(t) = X \sin(\omega t + \psi - \phi) \tag{4.32}$$

Frequency ratio

$$r = \frac{\omega}{\omega_n} \tag{4.38}$$

Magnification factor

$$M = \frac{m_{\mathrm{eq}}\omega_n^2 X}{F_0} \tag{4.39}$$

Functional form of magnification factor

$$M(r, \zeta) = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.42}$$

Phase angle

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{4.45}$$

Frequency-squared excitation

$$F_0 = A\omega^2 \tag{4.50}$$

Amplitude of response due to frequency-squared excitation

$$\frac{m_{\mathrm{eq}}X}{A} = \Lambda(r, \zeta) \tag{4.51}$$

Functional form of $\Lambda(r, \zeta)$

$$\Lambda(r, \zeta) = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.52}$$

Rotating unbalance as frequency-squared excitation

$$A = m_0 e \tag{4.62}$$

Frequency response due to rotating unbalance

$$\frac{mX}{m_0 e} = \Lambda(r, \zeta) \tag{4.63}$$

Displacement of mass relative to base

$$z(t) = x(t) - y(t) \tag{4.80}$$

Differential equation for relative motion of mass to base due to harmonic-base excitation

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = \omega^2 Y \sin\omega t \tag{4.86}$$

Amplitude of motion of mass relative to base

$$Z = Y\Lambda(r, \zeta) \tag{4.88}$$

Steady–state response of absolute displacement

$$x(t) = X \sin(\omega t - \lambda) \tag{4.90}$$

Amplitude of absolute displacement

$$\frac{X}{Y} = T(r, \zeta) \tag{4.91}$$

Functional form of $T(r, \zeta)$

$$T(r, \zeta) = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.93}$$

Ratio of acceleration amplitudes

$$\frac{\omega^2 X}{\omega^2 Y} = T(r, \zeta) \tag{4.94}$$

Ratio of amplitude of transmitted force to amplitude of excitation

$$\frac{F_T}{F_0} = T(r, \zeta) \tag{4.101}$$

Vibration isolation due to rotating unbalance

$$\frac{F_T}{m_0 e \omega_n^2} = r^2 T(r, \zeta) = R(r, \zeta) \tag{4.104}$$

Functional form of $R(r, \zeta)$

$$R(r, \zeta) = r^2 \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4.105}$$

Fourier series representation of periodic functions

$$F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cos \omega_i t + b_i \sin \omega_i t) \tag{4.130}$$

$$\omega_i = \frac{2\pi i}{T} \tag{4.131}$$

$$a_0 = \frac{2}{T} \int_0^T F(t)\, dt \tag{4.132}$$

$$a_i = \frac{2}{T} \int_0^T F(t) \cos \omega_i t\, dt \qquad i = 1, 2, \ldots \tag{4.133}$$

$$b_i = \frac{2}{T} \int_0^T F(t) \sin \omega_i t\, dt \qquad i = 1, 2, \ldots \tag{4.134}$$

Alternate form of Fourier series

$$F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} c_i \sin(\omega_i t + \kappa_i) \tag{4.135}$$

Response due to general periodic excitation

$$x(t) = \frac{1}{m_{\text{eq}} \omega_n^2} \left[\frac{a_0}{2} + \sum_{i=1}^{\infty} c_i M_i \sin(\omega_i t + \kappa_i - \phi_i) \right] \tag{4.139}$$

Percent error in using seismometer

$$E = 100|1 - \Lambda| \tag{4.146}$$

Percent error in using accelerometer

$$E = 100|1 - M(r, \zeta)| \tag{4.151}$$

Magnification factor for Coulomb damping

$$M_c(r, \iota) = \sqrt{\frac{1 - (\frac{4\iota}{\pi})^2}{(1 - r^2)^2}} \tag{4.184}$$

Magnification factor for hysteretic damping

$$M_h(r, h) = \frac{1}{\sqrt{(1 - r^2)^2 + h^2}} \tag{4.191}$$

Average power harvested during cycle

$$\frac{\bar{P}}{m\omega^3 Y^2} = \frac{\zeta}{r}\Lambda^2(r, \zeta) = \Phi(r, \zeta) \tag{4.206}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 4.1 through 4.16, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

4.1 The steady–state response of a linear SDOF system occurs at the same frequency as the excitation.

4.2 Beating is characterized by a continual build-up of amplitude.

4.3 The amplitude of a machine subject to a rotating unbalance approaches one for large frequencies.

4.4 An increase in damping leads to an increase in the percentage of isolation.

4.5 The phase angle for an undamped system is always π.

4.6 The phase angle depends upon F_0, which is the amplitude of excitation.

4.7 If ϕ is positive in the equation $x(t) = X\sin(\omega t - \phi)$, the response lags the excitation.

4.8 $M(r, \zeta)$ approaches 0 for large r for all values of ζ.

4.9 $\Lambda(r, \zeta)$ approaches 0 for large r for all values of ζ.

4.10 $T(r, \zeta)$ approaches 1 for large r for all values of ζ.

4.11 The amplitude of the response of a system relative to the motion of its base is given by $R(r, \zeta)$ if the base is subject to a single-frequency harmonic excitation.

4.12 The phase angle for the response of a system with Coulomb damping is independent of the frequency of excitation.

4.13 The equation for the response of a system with hysteretic damping is nonlinear in general but is linear when the system is subject to a single-frequency excitation.

4.14 A seismometer actually measures the displacement of the seismic mass relative to the displacement of the body the instrument is set up to measure.
4.15 Hysteretic damping can be modeled using a differential equation with a complex stiffness.
4.16 $M(r, \zeta)$ has a maximum when $\zeta < \frac{1}{\sqrt{2}}$.

Problems 4.17 through 4.38 require a short answer.

4.17 Explain why resonance occurs for undamped systems when the natural frequency coincides with the excitation frequency.
4.18 Why doesn't the amplitude grow unbounded when the frequency of excitation coincides with the natural frequency for systems with viscous damping?
4.19 For an undamped system, when is the response out of phase with the excitation?
4.20 In the equation $x(t) = X\sin(\omega t - \phi)$, when is ϕ negative?
4.21 How many real positive values of r satisfy the following.

(a) $M(r, 0.3) > 3$
(b) $M(r, 0.8) = 1.2$
(c) $M(r, 0.1) = 1.3$

4.22 How many real positive values of r satisfy the following.

(a) $\Lambda(r, 0) = 1$
(b) $\Lambda(r, 0.1) = 1.5$
(c) $\Lambda(r, 0.9) = 1.3$
(d) $\Lambda(r, 0.3) < 3$

4.23 How many real positive values of r satisfy the following.

(a) $T(r, 0.1) = 1$
(b) $T(r, 0.5) = 0.5$
(c) $T(r, 0) = 3$

4.24 How many real positive values of r satisfy the following.

(a) $\frac{dR}{dr}(r, 0.05) = 0$
(b) $\frac{dR}{dr}(r, 0.4) = 0$
(c) $\frac{dR}{dr}(r, 0.8) = 0$

4.25 Explain the concept of frequency response.
4.26 How is frequency response determined for a machine with a rotating unbalance?
4.27 How is frequency response determined for the motion of a machine on a moveable foundation?
4.28 Explain why vibration isolation is difficult at low speeds.
4.29 What is percentage isolation?
4.30 Explain why protecting a foundation from large forces generated by a machine is similar to protecting a body from large accelerations by its base.

4.31 Seismometers have a ____________ natural frequency and thus operate only for ____________ frequency ratios.
4.32 Explain the concept of phase distortion. Why is it a problem for accelerometers and not seismometers?
4.33 Explain the principle of linear superposition and how it applies to systems with multiple frequency input.
4.34 Why does the principle of linear superposition apply to general periodic input?
4.35 Explain the concept of stick-slip.
4.36 What are the limitations on ι, which is the nondimensional value of the ratio of the force causing Coulomb friction to the amplitude of the excitation force?
4.37 Why is viscous damping used in vibration isolation, since it has a negative effect on vibration isolation?
4.38 Does a steady–state response of the differential equation exist for the following?

(a) $3\ddot{x} + 2700x = 20 \sin 30t$

(b) $3\ddot{x} + 40\dot{x} + 2700x = 20 \sin 30t$

(c) $3\ddot{x} + 2700x = 20 \sin 10t$

Problems 4.39 through 4.59 require short calculations.

4.39 Find all real positive values of r that satisfy the following.

(a) $M(r, 0) = 1.4$
(b) $M(r, 0.4) > 3$
(c) $M(r, 0.8) < 1.2$

4.40 Find all positive values of r that satisfy the following.

(a) $T(r, 0.1) < 1$
(b) $T(r, 0.8) > 1$
(c) $T(r, 0.4) > T(r, 0.3)$

4.41 A machine with a mass of 30 kg is operating at a frequency of 60 rad/s. What equivalent stiffness of the machine's mounting leads to resonance?
4.42 An undamped SDOF system with a natural frequency of 98 rad/s is subject to a excitation of frequency 100 rad/s. (a) What is the period of response? (b) What is the period of beating?
4.43 A machine operates at 100 rad/s and has a rotating component of mass 5 kg whose center of mass is 3 cm from the axis of rotation. What is the amplitude of the harmonic excitation experienced by the machine?
4.44 Convert 1000 rpm to rad/s.
4.45 A machine is subject to a harmonic excitation with an amplitude of 15,000 N. The force transmitted to the floor through an isolator has an amplitude of 3000 N. What percentage isolation is achieved by the isolator?
4.46 A 50 kg machine is mounted on an isolator with a stiffness of 6×10^5 N/m. During operation, the machine is subject to a harmonic excitation with a frequency of 140 rad/s. (a) What is the frequency ratio? (b) Does this isolator actually isolate the vibrations?

4.47 Recall that the Fourier series representation of a periodic function is

$$F = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cos \omega_i t + b_i \sin \omega_i t)$$

Describe which of the Fourier coefficients (a_0, a_i, b_i, or none) are zero for each of the functions (illustrated over one period) shown in Figure SP4.46.

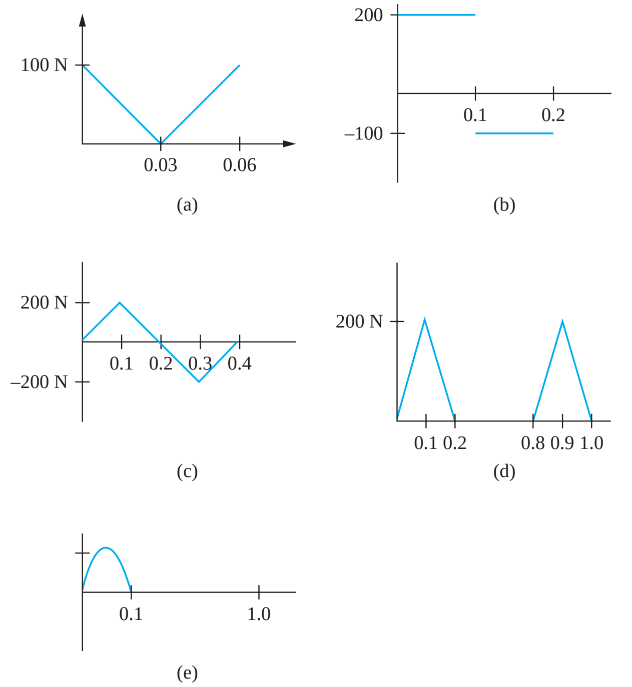


FIGURE SP4.46

4.48 Draw the function that the Fourier series representation of the function shown in Figure SP4.47 converges to on the interval [−5, 5].

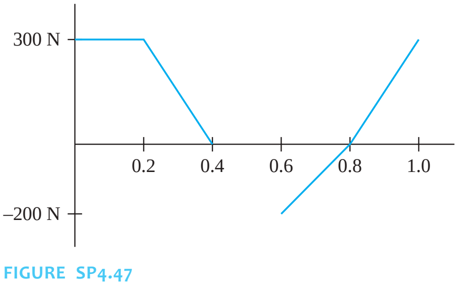


FIGURE SP4.47

4.49 What is the largest frequency whose vibrations can be measured by an undamped accelerometer of natural frequency 200 rad/s if the error is no more than 1 percent?

4.50 What is the smallest frequency whose vibrations can be measured by an undamped seismometer of natural frequency 20 rad/s if the error is no more than 1.5 percent?

Find the steady-state solution of the differential equation for Problems 4.51 through 4.59.

4.51 $3\ddot{x} + 2700x = 20 \sin 10t$

4.52 $3\ddot{x} + 2700x = 20 \sin 60t$

4.53 $3\ddot{x} + 30\dot{x} + 2700x = 20 \sin 10t$

4.54 $3\ddot{x} + 30\dot{x} + 2700x = 0.01\omega^2 \sin \omega t$

4.55 $3\ddot{x} + 30\dot{x} + 2700x = 30(0.002)(40) \cos 40t + 2700(0.002) \sin 40t$

4.56 $3\ddot{x} + \dfrac{2700(0.002)}{\omega}\dot{x} + 2700x = 20 \sin \omega t$

4.57 $3\ddot{x} + 30\dot{x} + 2700x = 30 \sin 50t + 20 \sin 20t$

4.58 $3\ddot{x} + 2700x = \begin{cases} 50 \sin 20t - 5 & \dot{x} > 0 \\ 50 \sin 20t + 5 & \dot{x} < 0 \end{cases}$

4.59 Match the quantity with the appropriate units (units may used more than once, some units may not be used).

(a) Steady-state amplitude, X	(i) m
(b) Steady-state amplitude of torsional oscillations, Θ	(ii) none
(c) Magnification factor, $M(r, \zeta)$	(iii) N
(d) Transmissibility ratio, $T(r, \zeta)$	(iv) N/m^2
(e) Acceleration amplitude, $\omega^2 X$	(v) rad
(f) Relative displacement amplitude, Z	(vi) $N \cdot s/m$
(g) Frequency ratio, r	(viii) $N \cdot s \cdot m/rad$
(h) Equivalent viscous-damping coefficient for Coulomb damping, c_{eq}	(ix) $N \cdot s$
(i) Ratio of friction force to excitation force, ι	(x) $N \cdot m$
(j) Hysteretic-damping coefficient, h	(xi) m/s^2
(k) Energy captured by energy harvester, E	(xii) W/cycle
(l) Average power captured by energy harvester, $\overline{P}$	(xiii) N/m

CHAPTER PROBLEMS

4.1 A 40 kg mass hangs from a spring with a stiffness of 4×10^4 N/m. A harmonic force with a magnitude of 120 rad/s is applied. Determine the amplitude of the forced response.

4.2 Determine the amplitude of forced oscillations of the 30 kg block of Figure P4.2.

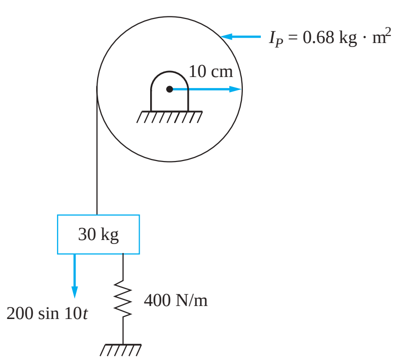


FIGURE P4.2

4.3 For what values of M_0 will the forced amplitude of angular displacement of the bar in Figure P4.3 be less than 3° if $\omega = 25$ rad/s?

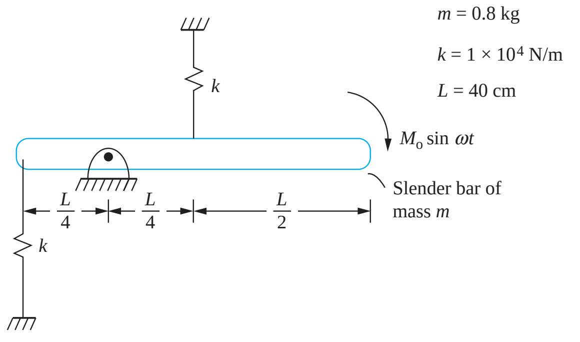


FIGURE P4.3

4.4 For what values of ω will the forced amplitude of the bar in Figure P4.3 be less than 3° if $M_0 = 300$ N · m?

4.5 A 2 kg gear with a radius of 20 cm is mounted to the end of a 1-m long steel ($G = 80 \times 10^9$ N/m^2) shaft. A moment $M = 100 \sin 150t$ is applied to the gear. For What shaft radii is the value of the forced amplitude of torsional oscillations less than 4°?

4.6 During operation, a 100 kg reciprocating machine is subject to a force $F(t) = 200 \sin 60t$ N. The machine is mounted on springs of an equivalent stiffness of 4.3×10^6 N/m. What is the machine's steady-state amplitude?

4.7 A 40 kg pump is to be placed at the midspan of a 2.5-m long steel ($E = 200 \times 10^9$ N/m^2) beam. The pump is to operate at 3000 rpm. For what values of the cross-sectional moment of inertia will the oscillations of the pump be within 3 Hz of resonance?

4.8 To determine the equivalent moment of inertia of a rigid helicopter component, an engineer decides to run a test in which she pins the component a distance of 40 cm and mounts the component on two springs of stiffness 3.6×10^5 N/m, as shown in Figure P4.8. She then provide a harmonic excitation to the component at different frequencies and finds that the maximum amplitude occurs at 50 rad/s. What is the equivalent centroidal moment of inertia predicted by the test?

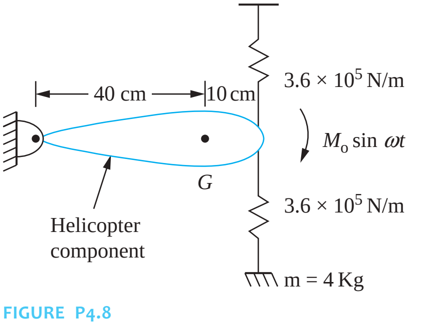


FIGURE P4.8

4.9 The modeling of an airfoil requires at least two degrees-of-freedom. However, its torsional stiffness is unknown, so an engineer devises a test. She prevents the airfoil from motion in the transverse direction at A but still allows it to rotate as shown in Figure P4.9. She then places two springs with a stiffness of 3×10^4 N/m at the tip of the airfoil and excites the airfoil with a harmonic excitation at the tip. She notices that the maximum amplitude of the tip occurs at a frequency of 150 rad/sec. The mass of the airfoil is 15 kg, and the moment of inertia of the airfoil about its mass center is 4.4 kg · m². The distance between the mass center and A is 20 cm, and the tip is 60 cm from A.

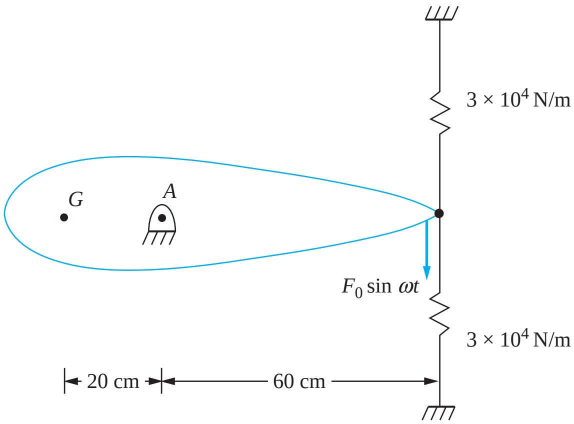


FIGURE P4.9

4.10 A machine with a mass of 50 kg is mounted on springs of equivalent stiffness 6.10×10^4 N/m and subject to a harmonic force of 370 sin 35t N while operating. The natural frequency is close enough to the excitation frequency for beating to occur.

(a) Write the overall response of the system, including the free response.
(b) Plot the response of the system.
(c) What is the maximum amplitude?
(d) What is the period of beating?

4.11 A machine with a mass of 30 kg is mounted on springs with an equivalent stiffness of 4.8×10^4 N/m. During operation, it is subject to a force of 200 sin ωt. Determine and plot the response of the system if the machine is at rest in equilibrium when the forcing starts and at (a) $\omega = 20$ rad/s, (b) $\omega = 40$ rad/s, and (c) $\omega = 41$ rad/s.

4.12 A 5 kg block is mounted on a helical coil spring such that the system's natural frequency is 50 rad/s. The block is subject to a harmonic excitation of amplitude 45 N at a frequency of 50.8 rad/s. What is the maximum displacement of the block from its equilibrium positions?

4.13 A 50-kg turbine is mounted on four parallel springs, each with a stiffness of 3×10^5 N/m. When the machine operates at 40 Hz, its steady–state amplitude is observed as 1.8 mm. What is the magnitude of the excitation?

4.14 A system with an equivalent mass of 30 kg has a natural frequency of 120 rad/s and a damping ratio of 0.12 and is subject to a harmonic excitation of amplitude 2000 N and frequency 150 rad/s. What are the steady–state amplitude and phase angle of the response?

4.15 A 30-kg block is suspended from a spring with a stiffness of 300 N/m and attached to a dashpot of damping coefficient of 120 N · s/m. The block is subject to a harmonic excitation of amplitude 1150 N at a frequency of 20 Hz. What is the block's steady–state amplitude?

4.16 What is the amplitude of steady–state oscillation of the 30 kg block of the system of Figure P4.16?

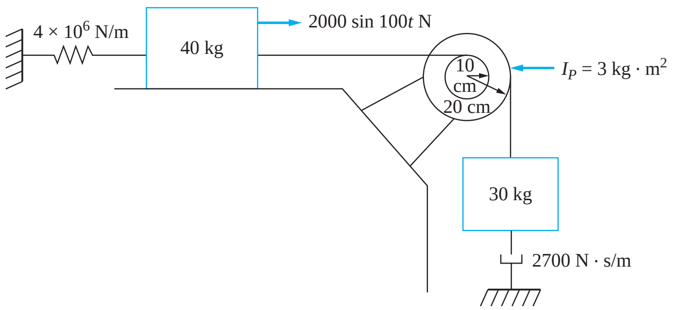


FIGURE P4.16

4.17 If $\omega = 16.5$ rad/s, what is the maximum value of M_0 such that the disk of Figure P4.17 rolls without slip?

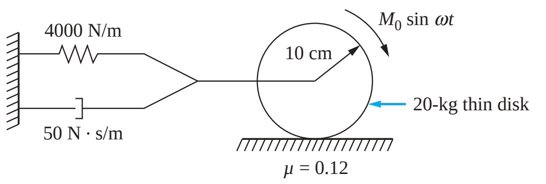


FIGURE P4.17

4.18 If $M_0 = 2$ N · m, for what values of ω will the disk of Figure P4.17 roll without slip?

4.19 For what values of d will the steady–state amplitude of angular oscillations be less than 1° for the rod of Figure P4.19?

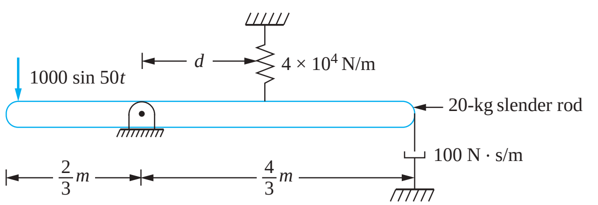


FIGURE P4.19

4.20 A 30-kg compressor is mounted on an isolator pad of stiffness 6×10^5 N/m. When subject to a harmonic excitation of magnitude 350 N and frequency 100 rad/s, the phase difference between the excitation and steady–state response is 24.3°. What is the damping ratio of the isolator and its maximum deflection due to this excitation?

4.21 A thin disk with a mass of 5 kg and a radius of 10 cm is connected to a torsional damper of coefficient 4.1 N · s · m/rad and a solid circular shaft with a radius of 10 mm, length 40 cm, and shear modulus 80×10^9 N/m^2. The disk is subject to a harmonic moment of magnitude 250 N · m and frequency 600 Hz. What is the amplitude of steady–state torsional oscillations?

4.22 A 50-kg machine tool is mounted on an elastic foundation. An experiment is run to determine the stiffness and damping properties of the foundation. When the tool is excited with a harmonic force of magnitude 8000 N at a variety of frequencies, the maximum steady–state amplitude obtained is 2.5 mm, occurring at a frequency of 32 Hz. Use this information to determine the stiffness and damping ratio of the foundation.

4.23 A machine with a mass of 30 kg is placed on an elastic mounting of unknown properties. An engineer excites the machine with a harmonic force with a magnitude of 100 N at a frequency of 30 Hz. He measures the steady–state response as having an amplitude of 0.2 mm with a phase lag of 20°. Determine the stiffness and damping coefficient of the mounting.

4.24 A 80-kg machine tool is placed on an elastic mounting. The phase angle is measured as 35.5° when the machine is excited at 30 Hz. When the machine is excited at 60 Hz, the phase angle is 113°. Determine the equivalent damping coefficient and equivalent stiffness of the mounting.

4.25 A 100-kg machine tool has a 2-kg rotating component. When the machine is mounted on an isolator and its operating speed is very large, the steady–state vibration amplitude is 0.7 mm. How far is the center of mass of the rotating component from its axis of rotation?

4.26 A 1000 kg turbine with a rotating unbalance is placed on springs and viscous dampers in parallel. When the operating speed is 20 Hz, the observed steady–state amplitude is 0.08 mm. As the operating speed is increased, the steady–state amplitude increases with an amplitude of 0.25 mm at 40 Hz and an amplitude of 0.5 mm for much larger speeds. Determine the equivalent stiffness and damping coefficient of this system.

4.27 A 120-kg fan with a rotating unbalance of 0.35 kg · m is to be placed at the midspan of a 2.6-m simply supported beam. The beam is made of steel ($E = 210 \times 10^9$ N/m^2) with a uniform rectangular cross section of height of 5 cm. For what values of the cross-sectional depth will the steady–state amplitude of the machine be limited to 5 mm for all operating speeds between 50 and 125 rad/s?

4.28 Solve Chapter Problem 4.27 assuming the damping ratio of the beam is 0.04.

4.29 A 620-kg fan has a rotating unbalance of 0.25 kg · m. What is the maximum stiffness of the fan's mounting such that the steady–state amplitude is 0.5 mm or less at all operating speeds greater than 100 Hz? Assume a damping ratio of 0.08.

Problems 4.30 and 4.31 refer to the following situation: The tail rotor section of the helicopter of Figure P4.30 consists of four blades, each of mass 2.1 kg, and an engine box of mass 25 kg. The center of gravity of each blade is 170 mm from the rotational axis. The tail section is connected to the main body of the helicopter by an elastic structure. The natural frequency of the tail section has been observed as 150 rad/s. During flight the rotor operates at 900 rpm. Assume the system has a damping ratio of 0.05.

4.30 During flight a 75-g particle becomes stuck to one of the blades, 25 cm from the axis of rotation. What is the steady–state amplitude of vibration caused by the resulting rotating unbalance?

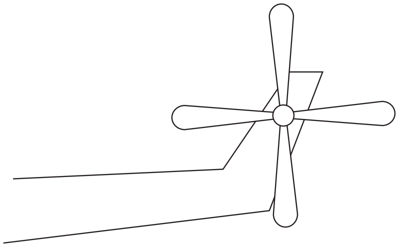


FIGURE P4.30

4.31 Determine the steady–state amplitude of vibration if one of the blades in Figure P4.30 snaps off during flight.

4.32 Whirling is a phenomenon that occurs in a rotating shaft when an attached rotor is unbalanced. The motion of the shaft and the eccentricity of the rotor cause an unbalanced inertia force, pulling the shaft away from its centerline, causing it to bow. Use Figure P4.32 and the theory of Section 4.5 to show that the amplitude of whirling is

$$X = e\Lambda(r, \zeta)$$

where e is the distance from the center of mass of the rotor to the axis of the shaft.

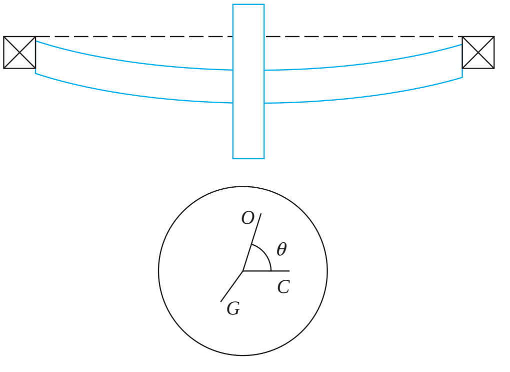


FIGURE P4.32

4.33 A 30-kg rotor has an eccentricity of 1.2 cm. It is mounted on a shaft and bearing system whose stiffness is 2.8×10^4 N/m and damping ratio is 0.07. What is the amplitude of whirling when the rotor operates at 850 rpm? Refer to Chapter Problem 4.32 for an explanation of whirling.

4.34 An engine flywheel has an eccentricity of 0.8 cm and mass 38 kg. Assuming a damping ratio of 0.05, what is the necessary stiffness of the bearings to limit its whirl amplitude to 0.8 mm at all speeds between 1000 and 2000 rpm? Refer to Chapter Problem 4.32 for an explanation of whirling.

4.35 It is proposed to build a 6-m smokestack on the top of a 60-m factory. The smokestack will be made of steel ($\rho = 7850$ kg/m^3) and will have an inner radius of 40 cm and an outer radius of 45 cm. What is the maximum amplitude of vibration due to vortex shedding and at what wind speed will it occur? Use a SDOF model for the smokestack with a concentrated mass at its end to account for inertia effects. Use $\zeta = 0.05$.

4.36 What is the steady–state amplitude of oscillation due to vortex shedding of the smokestack of Chapter Problem. P4.35 if the wind speed is 22 mph?

4.37 A factory is using the piping system of Figure P4.37 to discharge environmentally safe waste-water into a small river. The velocity of the river is estimated as 5.5 m/s. Determine the allowable values of l such that the amplitude of torsional oscillations of the vertical pipe due to vortex shedding is less than 1°. Assume the vertical pipe is rigid and rotates about an axis perpendicular to the page through the elbow. The horizontal pipe is restrained from rotation at the river bank. Assume a damping ratio of 0.05.

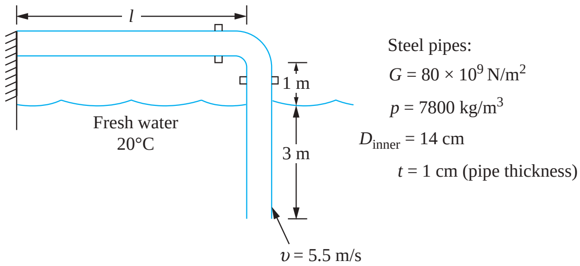


FIGURE P4.37

4.38–4.42 Determine the amplitude of steady–state vibration for the systems shown in Figures P4.38 through P4.42. Use the indicated generalized coordinate.

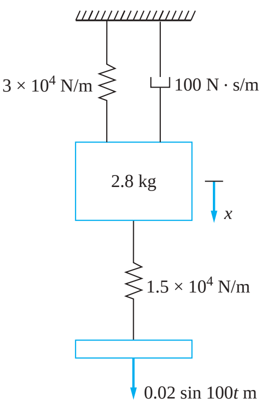


FIGURE P4.38

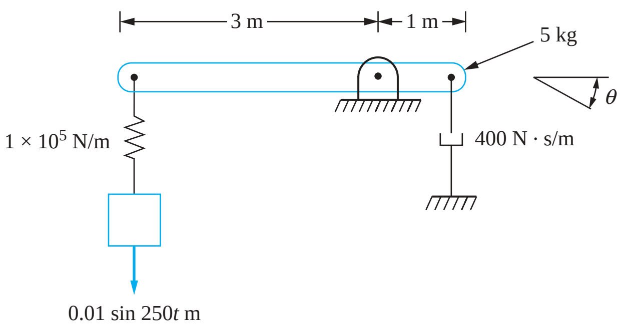


FIGURE P4.39

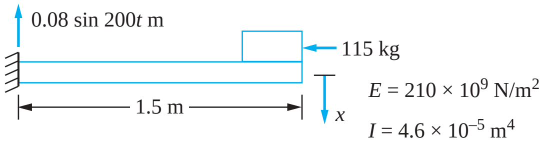


FIGURE P4.40

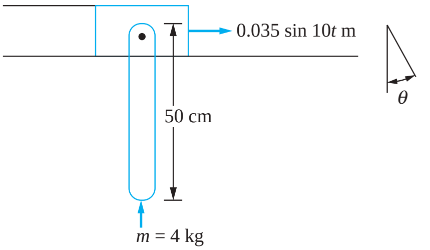


FIGURE P4.41

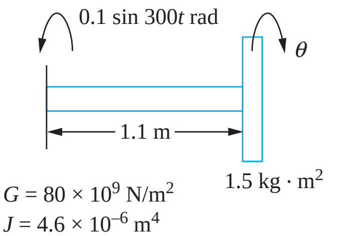


FIGURE P4.42

4.43 A 40 kg machine is attached to a base through a spring of stiffness 2×10^4 N/m in parallel with a dashpot of damping coefficient 150 N · s/m. The base is given a time-dependent displacement 0.15 sin 30.1t m. Determine the amplitude of the absolute displacement of the machine and the amplitude of displacement of the machine relative to the base.

4.44 A 5-kg rotor-balancing machine is mounted on a table through an elastic foundation of stiffness 3.1×10^4 N/m and damping ratio 0.04. Transducers indicate that the table on which the machine is placed vibrates at a frequency of 110 rad/s with an amplitude of 0.62 mm. What is the steady–state amplitude of acceleration of the balancing machine?

4.45 During a long earthquake the one-story frame structure of Figure P4.45 is subject to a ground acceleration of amplitude 50 mm/s^2 at a frequency of 88 rad/s. Determine the acceleration amplitude of the structure. Assume the girder is rigid and the structure has a damping ratio of 0.03.

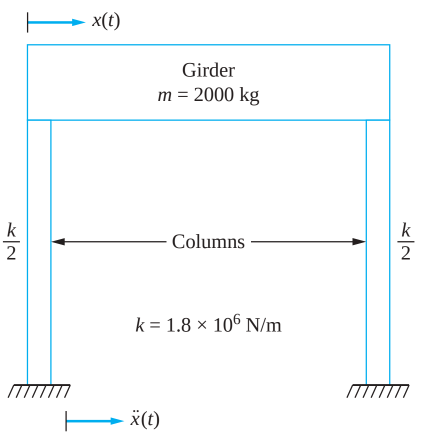


FIGURE P4.45

4.46 What is the required column stiffness of a one-story structure to limit its acceleration amplitude to 2.1 m/s^2 during an earthquake whose acceleration amplitude is 150 mm/s^2 at a frequency of 50 rad/s? The mass of structure is 1800 kg. Assume a damping ratio of 0.05.

4.47 In a rough sea, the heave of a ship is approximated as harmonic of amplitude 20 cm at a frequency of 1.5 Hz. What is the acceleration amplitude of a 20-kg computer workstation mounted on an elastic foundation in the ship of stiffness 700 N/m and damping ratio 0.04?

4.48 In the rough sea of Chapter Problem 4.47, what is the required stiffness of an elastic foundation of damping ratio 0.05 to limit the acceleration amplitude of a 5-kg radio set to 1.5 m/s^2?

4.49 Consider the one degree-of-freedom model of a vehicle suspension system of Figure P4.49. Consider a motorcycle of mass 250 kg. The suspension stiffness is 70,000 N/m and the damping ratio is 0.15. The motorcycle travels over a terrain that is approximately sinusoidal with a distance between peaks of 10 m and the distance from peak to valley is 10 cm. What is the acceleration amplitude felt by the motorcycle rider when she is traveling at

(a) 30 m/s
(b) 60 m/s
(c) 120 m/s

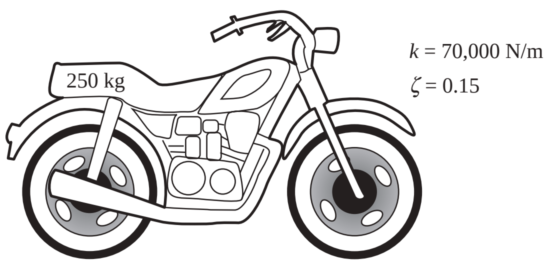


FIGURE P4.49

4.50 For the motorcycle of Chapter Problem 4.49 determine (a) the "frequency response" of the motorcycle's suspension system by plotting the amplitude of acceleration versus motorcycle speed and (b) determine and plot the amplitude of displacement of the motorcycle versus its speed.

4.51 What is the minimum static deflection of an undamped isolator that provides 75 percent islolation to a 200-kg washing machine at 5000 rpm?

4.52 What is the maximum allowable stiffness of an isolator of damping ratio 0.05 that provides 81 percent isolation to a 40-kg printing press operating at 850 rpm?

4.53 When set on a rigid foundation and operating at 800 rpm, a 200-kg machine tool provides a harmonic force with a magnitude of 18,000 N to the foundation. An engineer has determined that the maximum magnitude of a harmonic force to which the foundation should be subjected to is 2600 N.

(a) What is the maximum stiffness of an undamped isolator that provides sufficient isolation between the tool and the foundation?
(b) What is the maximum stiffness of an isolator with a damping ratio of 0.11?

4.54 A 150-kg engine operates at 1500 rpm.

(a) What percent isolation is achieved if the engine is mounted on four identical springs each of stiffness 1.2×10^5 N/m?
(b) What percent isolation is achieved if the springs are in parallel with a viscous damper of damping coefficient 1000 N · s/m?

4.55 A 150 kg engine operates at speeds between 1000 and 2000 rpm. It is desired to achieve at least 85 percent isolation at all speeds. The only readily available isolator has a stiffness of 5×10^5 N/m. How much mass must be added to the engine to achieve the desired isolation?

4.56 Cork pads with a stiffness of 6×10^5 N/m and a damping ratio of 0.2 are used to isolate a 40-kg machine tool from its foundation. The machine tool operates at 1400 rpm and produces a harmonic force of magnitude 80,000 N. If the pads are placed in series, how many are required such that the magnitude of the transmitted force is less than 10,000 N?

4.57 A 100-kg machine operates at 1400 rpm and produces a harmonic force of magnitude 80,000 N. The magnitude of the force transmitted to the foundation is to be reduced to 20,000 N by mounting the machine on four identical undamped isolators in parallel. What is the minimum stiffness of each isolator?

4.58 A 10-kg laser flow-measuring device is used on a table in a laboratory. Because of operation of other equipment, the table is subject to vibration. Accelerometer measurements show that the dominant component of the table vibrations is at 300 Hz and has an amplitude of 4.3 m/s^2. For effective operation, the laser can be subject to an acceleration amplitude of 0.7 m/s^2.

(a) Design an undamped isolator to reduce the transmitted acceleration, to an acceptable amplitude.
(b) Design the isolator such that it has a damping ratio of 0.04.

4.59 Rough seas cause a ship to heave with an amplitude of 0.4 m at a frequency of 20 rad/s. Design an isolation system with a damping ratio of 0.13 such that a 45 kg navigational computer is subject to an acceleration of only 20 m/s^2.

4.60 A sensitive computer is being transported by rail in a boxcar. Accelerometer measurements indicate that when the train is traveling at its normal speed of 85 m/s the dominant component of the boxcar's vertical acceleration is 8.5 m/s^2 at a frequency of 36 rad/s. The crate in which the computer is being transported is tied to the floor of the boxcar. What is the required stiffness of an isolator with a damping ratio of 0.05 such that the acceleration amplitude of the 60 kg computer is less than 0.5 m/s^2? With this isolator, what is the displacement of the computer relative to the crate?

4.61 A 200 kg engine operates at 1200 rpm. Design an isolator such that the transmissibility ratio during start-up is less than 4.6 and the system achieves 80 percent isolation.

4.62 A 150 kg machine tool operates at speeds between 500 and 1500 rpm. At each speed a harmonic force of magnitude 15,000 N is produced. Design an isolation system such that the maximum transmitted force during start-up is 60,000 N and the maximum transmitted steady–state force is 2000 N.

4.63 A 200 kg testing machine operates at 500 rpm and produces a harmonic force of magnitude 40,000 N. An isolation system for the machine consists of a damped isolator and a concrete block for mounting the machine. Design the isolation system such that all of the following are met.

(i) The maximum transmitted force during start-up is 100,000 N.
(ii) The maximum transmitted force in the steady–state is 5000 N.
(iii) The maximum steady–state amplitude of the machine is 2 cm.

4.64 A 150-kg washing machine has a rotating unbalance of 0.45 kg · m. The machine is placed on isolators of equivalent stiffness 4×10^5 N/m and damping ratio 0.08. Over what range of operating speeds will the transmitted force between the washing machine and the floor be less than 3000 N?

4.65 A 54-kg air compressor operates at speeds between 800 and 2000 rpm and has a rotating unbalance of 0.23 kg · m. Design an isolator with a damping ratio of 0.15 such that the transmitted force is less than 1000 N at all operating speeds.

4.66 A 1000 kg turbomachine has a rotating unbalance of 0.1 kg · m. The machine operates at speeds between 500 and 750 rpm. What is the maximum isolator stiffness of an undamped isolator that can be used to reduce the transmitted force to 300 N at all operating speeds?

4.67 A motorcycle travels over a road whose contour is approximately sinusoidal, $y(z) = 0.2 \sin(0.4z)$ m where z is measured in meters. Using a SDOF model, design a suspension system with a damping ratio of 0.1 such that the acceleration felt by the rider is less than 15 m/s^2 at all horizontal speeds between 30 and 80 m/s. The mass of the motorcycle and the rider is 225 kg.

4.68 A suspension system is being designed for a 1000 kg vehicle. A first model of the system used in the design process is a spring of stiffness k in parallel with a viscous damper of damping coefficient c. The model is being analyzed as the vehicle traverses a road with a sinusoidal contour, $y(z) = Y \sin(2\pi z/d)$ when the vehicle has a constant horizontal speed v. The suspension system is to be designed such that the maximum acceleration of the passengers is 2.5 m/s^2 for all vehicle speeds less than 60 m/s for all reasonable road contours. It is estimated that for such contours, $Y < 0.01$ m and $0.2 \text{ m} < d < 1$ m. Specify k and c for such a design.

4.70 A 20 kg block is connected to a spring of stiffness 1×10^5 N/m and placed on a surface which makes an angle of 30° with the horizontal. A force of $300 \sin 80t$ N is applied to the block. The steady–state amplitude is measured as 10.6 mm. What is the coefficient of friction between the block and the surface?

4.71 A 40 kg block is connected to a spring of stiffness 1×10^5 N/m and slides on a surface with a coefficient of friction 0.2. When a harmonic force of frequency 60 rad/s is applied to the block, the resulting amplitude of steady–state vibrations is 3 mm. What is the amplitude of the excitation?

4.72–4.73 Determine the steady–state amplitude of motion of the 5-kg block. The coefficient of friction between the block and surface is 0.11. (See Figures P4.72 and P4.73.)

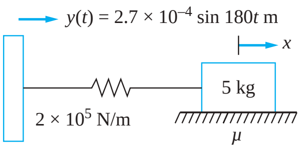


FIGURE P4.72

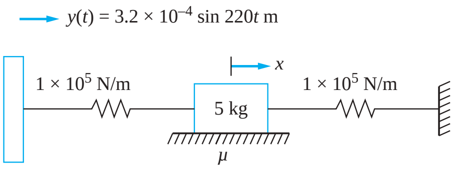


FIGURE P4.73

4.74 Use the equivalent viscous damping approach to determine the steady–state response of a system subject to both viscous damping and Coulomb damping.

4.75 The area under the hysteresis curve for a particular helical coil spring is 0.2 N · m when subject to a 350 N load. The spring has a stiffness of 4×10^5 N/m. If a 44 kg block is hung from the spring and subject to an excitation force of $350 \sin 35t$ N, what is the amplitude of the resulting steady–state oscillations?

4.76 When a free-vibration test is run on the system of Figure P4.76, the ratio of amplitudes on successive cycles is 2.8 to 1. Determine the response of the pump when it has an excitation force of magnitude 3000 N at a frequency of 2000 rpm. Assume the damping is hysteretic.

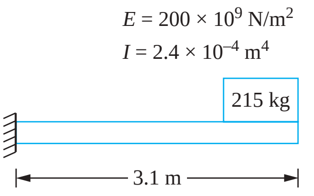


FIGURE P4.76

4.77 When a free-vibration test is run on the system of Figure P4.76, the ratio of amplitudes on successive cycles is 2.8 to 1. When operating, the engine has a rotating unbalance of magnitude 0.25 kg · m. The engine operates at speeds between 500 and 2500 rpm. For what value of ω within the operating range will the pump's steady–state amplitude be largest? What is the maximum amplitude? Assume the damping is hysteretic.

4.78 When the pump at the end of the beam of Figure P4.76 operates at 1860 rpm, it is noted that the phase angle between the excitation and response is 18°. What is the steady–state amplitude of the pump if it has a rotating unbalance of 0.8 kg · m and operates at 1860 rpm? Assume hysteretic damping.

4.79 A schematic of a single-cylinder engine mounted on springs and a viscous damper is shown in Figure P4.79. The crank rotates about O with a constant speed ω. The connecting rod of mass m_r connects the crank and the piston of mass m_p such that the piston moves in a vertical plane. The center of gravity of the crank is at its axis of rotation.

(a) Derive the differential equation governing the absolute vertical displacement of the engine including the inertia forces of the crank and piston, but ignoring forces due to combustion. Use an exact expression for the inertia forces in terms of m_r, m_p, ω, the crank length r, and the connecting rod length l.

(b) Since $F(t)$ is periodic, a Fourier series representation can be used. Set up, but do not evaluate, the integrals required for a Fourier series expansion for $F(t)$.

(c) Assume $r/l \ll 1$. Rearrange $F(t)$ and use a binomial expansion such that

$$F(t) = \sum_{i=1}^{\infty} a_i \left(\frac{r}{l}\right)^i$$

(d) Truncate the preceding series after $i = 3$. Use trigonometric identities to approximate

$$F(t) \approx b_1 \cos \omega t + b_2 \cos 2\omega t + b_3 \cos 3\omega t$$

(e) Find an approximation to the steady–state form of $x(t)$.

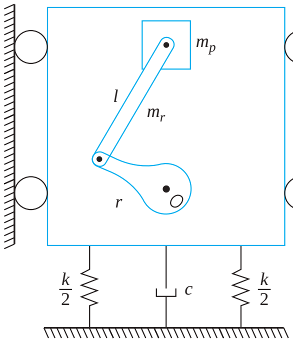


FIGURE P4.79

4.80 Using the results of Problem P4.79, determine the maximum steady–state response of a single-cylinder engine with $m_r = 1.5$ kg, $m_p = 1.7$ kg, $r = 5.0$ cm, $l = 15.0$ cm, $\omega = 800$ rpm, $k = 1 \times 10^5$ N/m, $c = 500$ N · s/m, and total mass 7.2 kg.

4.81 A 5-kg rotor-balancing machine is mounted to a table through an elastic foundation of stiffness 10,000 N/m and damping ratio 0.04. Use of a transducer reveals that the table's vibration has two main components: an amplitude of 0.8 mm at a frequency of 140 rad/s and an amplitude of 1.2 mm at a frequency of 200 rad/s. Determine the steady–state response of the rotor balancing machine.

4.82–4.86 During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady–state response of the press and approximate its maximum displacement from equilibrium. Each excitation is shown over one period.

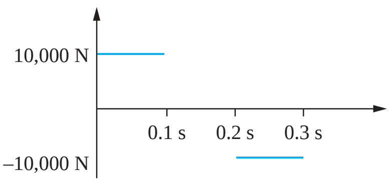


FIGURE P4.82

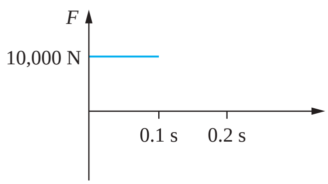


FIGURE P4.83

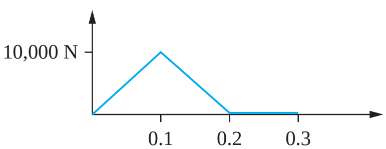


FIGURE P4.84

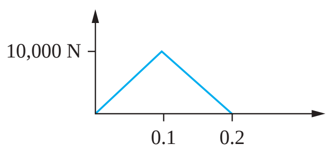


FIGURE P4.85

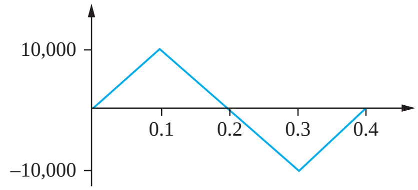


FIGURE P4.86

4.87 Use of an accelerometer of natural frequency 100 Hz and damping ratio 0.15 reveals that an engine vibrates at a frequency of 20 Hz and has an acceleration amplitude of 14.3 m/s^2. Determine

(a) The percent error in the measurement
(b) The actual acceleration amplitude
(c) The displacement amplitude

4.88 An accelerometer with a natural frequency of 200 Hz and damping ratio of 0.7 is used to measure the vibrations of a system whose actual displacement is $x(t) = 1.6 \sin 45.1t$ mm. What is the accelerometer output?

4.89 An accelerometer with a natural frequency of 200 Hz and damping ratio of 0.2 is used to measure the vibrations of an engine operating at 1000 rpm. What is the percent error in the measurement?

4.90 When a machine tool is placed directly on a rigid floor, it provides an excitation of the form

$$F(t) = (4000 \sin 100t + 5100 \sin 150t) \text{ N}$$

to the floor. Determine the natural frequency of the system with an undamped isolator with the minimum possible static deflection such that when the machine is mounted on the isolator the amplitude of the force transmitted to the floor is less than 3500 N.

4.91 Use the force shown in Figure P4.91 as an approximation to the force provided by the punch press during its operation. Rework Example 4.17 for the excitation.

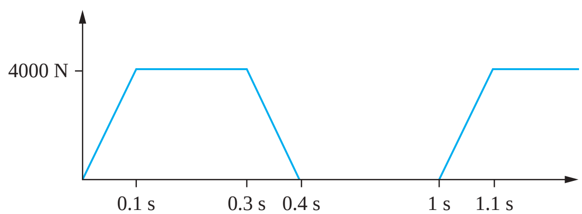


FIGURE P4.91

4.92 A 550-kg industrial sewing machine has a rotating unbalance of 0.24 kg · m. The machine operates at speeds between 2000 and 3000 rpm. The machine is placed on an isolator pad of stiffness 5×10^6 N/m and damping ratio 0.12. What is the maximum natural frequency of an undamped seismometer that can be used to measure the steady–state vibrations at all operating speeds with an error less than 4 percent. If this seismometer is used, what is its output when the machine is operating at 2500 rpm?

4.93 The system of Figure P4.93 is subject to the excitation

$$F(t) = 1000 \sin 25.4t + 800 \sin (48t + 0.35) - 300 \sin(100t + 0.21) \text{ N}$$

What is the output in mm/s^2 of an accelerometer of natural frequency 100 Hz and damping ratio 0.7 placed at A?

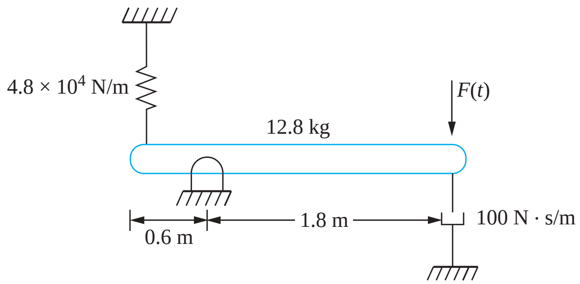


FIGURE P4.93

4.94 What is the output, in mm, of a seismometer with a natural frequency of 2.5 Hz and a damping ratio of 0.05 placed at point A for the system of Figure P4.93?

4.95 A 20 kg block is connected to a moveable support through a spring of stiffness 1×10^5 N/m in parallel with a viscous damper of damping coefficient 600 N · s/m. The support is given a harmonic displacement of amplitude 25 mm and frequency 40 rad/s. An accelerometer of natural frequency 25 Hz and damping ratio 0.2 is attached to the block. What is the output of the accelerometer in mm/s^2?

4.96 An accelerometer has a natural frequency of 80 Hz and a damping coefficient of 8.0 N · s/m. When attached to a vibrating structure, it measures an amplitude of 8.0 m/s^2 and a frequency of 50 Hz. The true acceleration of the structure is 7.5 m/s^2. Determine the mass and stiffness of the accelerometer.

4.97 Vibrations of a 30 kg machine occur at 150 rad/s with an amplitude of 0.003 mm.

(a) Design an energy harvester with a damping ratio of 0.2 that harvests theoretical maximum power over one cycle of vibrations from the body.
(b) What is the power harvested by this harvester in one hour?

4.98 An energy harvester is being designed to harvest the vibrations form a 200 kg machine that has a rotating unbalance of 0.1 kg · m which operates at 1000 rpm. The harvester is to have a mass of 1 kg and a damping ratio of 0.1.

(a) What is the stiffness of the harvester?
(b) What is the power harvested from the machine if it operates continuously in one day.

4.99 An energy harvester is being designed for a vehicle with a simplified suspension system similar to that in the benchmark examples. The harvester, which is to be mounted on the vehicle, is to harvest energy as the vehicle vibrates while traveling. The harvester will have a mass of 0.1 kg, damping ratio of 0.1, and natural frequency of 30 rad/s. Estimate how much power is harvested over one cycle of a sinusoidal road with a spatial period of 10 m and amplitude of 5 mm while the vehicle is traveling at 50 m/s.

4.100 How much energy is harvested over one period by the energy harvester of Problem 4.99 if the vehicle is traveling at 50 m/s over a road whose contour is shown in Figure P4.100.

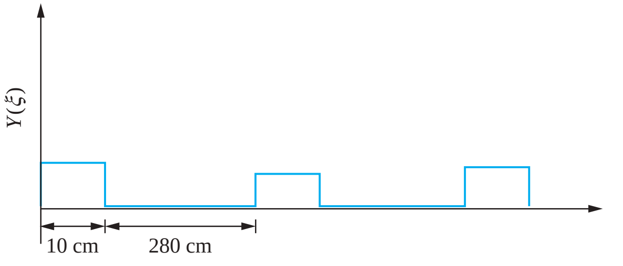


FIGURE P4.100

4.101 An energy harvester is being designed to harvest energy from a MEMS system. The harvester consists of a micro-cantilever beam vibrating in a viscous liquid such that its damping ratio is 0.2. The micro-cantilever beam is made of silicon ($E = 1.9 \times 10^{11}$ N/m^2) is 30 μm long, is rectangular in cross section, has a base width of 2 μm, and a height of 0.5 μm. The mass density of silicon is 2.3 g/cm^3.

(a) What is the natural frequency of the energy harvester using a SDOF model? Use the equivalent mass of a cantilever beam at its end.
(b) What energy is harvested over one cycle of motion if the harvesting occurs at the natural frequency with a vibration amplitude of 1 μm?
(c) What is the average power harvested over one cycle?
(d) What is the power harvested over one hour?

Chapter 5

TRANSIENT VIBRATIONS OF SDOF SYSTEMS

5.1 INTRODUCTION

When vibrations of a mechanical or structural system are initiated by a periodic excitation, an initial transient period occurs where the free-vibration response is as large as the forced response. The free-vibration response quickly decays, resulting in a steady-state motion. In many cases, when a system is subject to a nonperiodic excitation, the free vibration response interacts with the forced response and is important throughout the duration of the motion of the system. Such is the case when a system is subject to a pulse of finite duration where the period of free vibration is greater than the pulse duration.

One example of a nonperiodic excitation is the ground motion of an earthquake. The response of structures due to ground motion is obtained by using the methods of this chapter. An earthquake is usually of short duration, but maximum displacements and stresses occur while the earthquake takes place. The terrain traveled by a vehicle is usually nonperiodic. Suspension systems must be designed to protect passengers from sudden changes in road contour. Forces produced in operation of machines in manufacturing processes are often nonperiodic. Sudden changes in forces occur in presses and milling machines.

Forced vibrations of SDOF systems are described by the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_{\mathrm{eq}}(t)}{m_{\mathrm{eq}}} \tag{5.1}$$

Initial conditions, values of $x(0)$ and $\dot{x}(0)$, complete the problem formulation. Solution of Equation (5.1) for periodic forms of $F_{\text{eq}}(t)$ is discussed in Chapter 4.

The purpose of this chapter is to analyze the motion of systems undergoing transient vibrations. Equation (5.1) is a second-order linear nonhomogeneous ordinary differential equation. For certain forms of $F_{eq}(t)$, the method of undetermined coefficients, as applied in Chapter 4, can be used to determine the particular solution. The homogeneous solution is added to the particular solution, resulting in a general solution involving two constants of integration. Initial conditions are applied to evaluate the constants of integration. If damping is present the homogeneous solution dies out, leaving the particular solution as a steady-state solution. The method of undetermined coefficients is best suited for harmonic, polynomial, or exponential excitations and not useful for most excitations studied in this chapter.

The initial conditions and the homogeneous solution have an important effect on the short-term transient motion of vibrating systems. For these problems, it is convenient to use a solution method in which the homogeneous solution and particular solution are obtained simultaneously and the initial conditions are incorporated in the solution.

Many excitations are of short duration. For short-duration responses, the maximum response may occur after the excitation has ceased. Thus it is necessary to develop a solution method which determines the response of a system for all time, even after the excitation is removed. In addition, many excitations change form at discrete times. For these excitations a solution method in which a unified mathematical form of the response is determined is a great convenience.

The primary method of solution presented in this chapter is use of the convolution integral. The convolution integral is derived using the principle of impulse and momentum and linear superposition. It can also be derived by application of the method of variation of parameters. The convolution integral provides the most general closed-form solution of Equation (5.1). The initial conditions are applied in the derivation of the integral, and need not be applied during every application. The convolution integral can be used to generate a unified mathematical response for excitations whose form changes at discrete times. Since it only requires evaluation of an integral, it is easy to apply.

A second method presented in this chapter is the Laplace transform method. Initial conditions are applied during the transform procedure and the Laplace transform can be used to develop a unified mathematical response for excitations whose form changes at discrete times. Use of tables of transforms makes application of the method convenient. The algebraic effort can be less than that using the convolution integral for damped systems, if appropriate transforms are available in a table. However, if the appropriate transforms are not available in a table, determination of the response is difficult.

The system's *transfer function* is the ratio of the Laplace transform of its output to the Laplace transform of its input. Thus, the transfer function is independent of the input. It is a property of the system itself and contains information regarding the system's dynamics. If the transfer function for a system is known, multiplication by the transform of the input leads to the transform of the system response, which can be inverted. The transfer function is also the Laplace transform of its impulsive response, which is the response due to a unit impulse.

There are some excitations in which a closed-form solution of Equation (5.1) does not exist. In these cases, the convolution integral does not have a closed-form evaluation, and application of the Laplace transform method leads only to the convolution integral. In addition, situations exist when the excitation is not known explicitly at all values of time.

The excitation may be obtained empirically. In these situations, numerical methods must be used to develop approximations to the response at discrete times. These numerical methods include numerical evaluation of the convolution integral and direct numerical solution of Equation (5.1).

Whether the solution is obtained using the convolution integral, Laplace transforms, or numerical methods, questions arise regarding maximum displacement, maximum transmitted force, and design used to reduce maximum vibration. These questions are answered for pulses of finite duration. The *response spectrum*, which is a nondimensional plot of maximum displacement versus duration of the pulse, is drawn when the shape of the pulse matters. For short-duration pulses, the shape of the pulse does not matter (only the total impulse imparted to the system matters), and the design of the system to minimize the maximum displacement is based upon the concept of isolator efficiency.

5.2 DERIVATION OF CONVOLUTION INTEGRAL

5.2.1 RESPONSE DUE TO A UNIT IMPULSE

The impulse delivered to a system by a force $F(t)$ between times t_1 and t_2 is defined as

$$I = \int_{t_1}^{t_2} F(\tau)\,d\tau \tag{5.2}$$

An *impulsive force* is a very large force applied over a very short interval of time. The principle of impulse and momentum (a form of Newton's second law integrated over time) is

$$mv(t_1) + I = mv(t_2) \tag{5.3}$$

where $v(t)$ is the system's velocity at time t. If the limit of the time over which the force is applied approaches zero and the impulse remains finite, it is said that an impulse is applied to the system. In this context, *impulse* refers to an impulsive force which is applied instantaneously.

Consider a SDOF system at rest in equilibrium. Let $x(t)$ be a generalized coordinate representing the displacement of a particle. A linear SDOF system has the equivalent systems model of Figure 5.1(a). An impulse of magnitude I is applied to a system at rest at

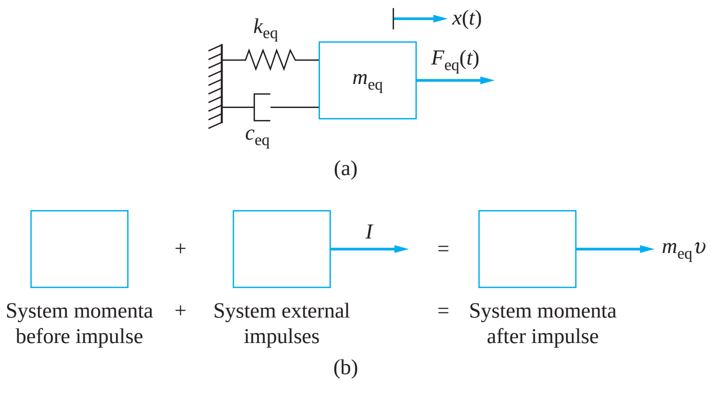


FIGURE 5.1
(a) Equivalent system model of a linear SDOF system. (b) Impulse and momentum diagrams used to obtain velocity immediately after application of an impulse.

$t = 0$ as shown in Figure 5.1(b). The principle of impulse and momentum is used to calculate the velocity of the particle immediately after application of the impulse as

$$v = \frac{I}{m_{\mathrm{eq}}} \tag{5.4}$$

Application of an impulse leads to a discrete change in velocity. The velocity immediately after application of the impulse is I/m. Thus, the response of the system is the same as the solution initial value problem

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{5.5}$$

with

$$x(0) = 0 \tag{5.6}$$

and

$$\dot{x}(0) = \frac{I}{m} \tag{5.7}$$

For a system whose free vibrations are underdamped, the solution of Equations (5.5) through (5.7) is

$$x(t) = \frac{I}{m_{\mathrm{eq}}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{5.8}$$

Equation (5.8) can be written as

$$x(t) = Ih(t) \tag{5.9}$$

where

$$h(t) = \frac{1}{m_{\mathrm{eq}}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{5.10}$$

is the response due to a unit impulse applied at $t = 0$.

For a system that is *critically damped*,

$$h(t) = \frac{1}{m_{\mathrm{eq}}} te^{-\omega_n t} \tag{5.11}$$

and for an *overdamped* system,

$$\begin{aligned} h(t) &= \frac{e^{-\zeta\omega_n t}}{2m_{\mathrm{eq}}\omega_n\sqrt{\zeta^2 - 1}}\left(e^{\omega_d\sqrt{\zeta^2-1}t} - e^{-\omega_d\sqrt{\zeta^2-1}t}\right) \\ &= \frac{e^{-\zeta\omega_n t}}{m_{\mathrm{eq}}\omega_n\sqrt{\zeta^2 - 1}} \sinh\left(\omega_d\sqrt{\zeta^2 - 1}t\right) \end{aligned} \tag{5.12}$$

If the unit impulse is not applied at $t = 0$ but at a time t_0, the response at time t is shifted by t_0 such that

$$x(t) = h(t - t_0)u(t - t_0) \tag{5.13}$$

where $u(t - t_0)$ is the unit step function of argument $t - t_0$, which takes on a value of 0 for $t < t_0$ and a value of 1 for $t > t_0$. The unit step function's presence in Equation (5.13) guarantees that the response does not occur until the impulse has been applied. Actually, the response for an impulse applied at $t = 0$ should be multiplied by $u(t)$, but t is measured from 0. For an *underdamped* system,

$$h(t - t_0) = \frac{1}{m_{\text{eq}}\omega_d} e^{-\zeta\omega_n(t - t_0)} \sin[\omega_d(t - t_0)] \tag{5.14}$$

An alternative to using a non-zero initial velocity to determine the response of a system to a unit impulse is to use a unit impulse function (see Appendix A) as the forcing function in the differential equation. The *unit impulse function* $\delta(t)$ is the mathematical representation of a force required to provide a unit impulse to a system. It possesses the properties of an impulsive force. It is zero except at $t = 0$, where it is infinite; yet its integral over time is equal to 1. Use of the unit impulse function as the forcing function in the differential equation gives

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m}\delta(t) \tag{5.15}$$

The solution of the differential equation is $h(t)$, which is called the *impulsive response.*

If the impulse is applied at a time other than zero (say t_0), the force required to cause the impulse is $\delta(t - t_0)$, and the differential equation governing the response of the system is

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m}\delta(t - t_0) \tag{5.16}$$

The solution of Equation (5.16) is $h(t - t_0)u(t - t_0)$. If the magnitude of the applied impulse is other than one (say I), the differential equation becomes

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{I}{m}\delta(t - t_0) \tag{5.17}$$

The solution to Equation (5.17) is $Ih(t - t_0)u(t - t_0)$.

EXAMPLE 5.1

During its operation, a punch press is subject to impulses of magnitude $5\ \text{N} \cdot \text{s}$ at $t = 0$ and at $t = 1.5$ sec. The mass of the press is 10 kg, and it is mounted on an elastic pad with a stiffness of 2×10^4 N/m and damping ratio of 0.1. Determine the response of the press.

SOLUTION

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \times 10^4\ \text{N/m}}{10\ \text{kg}}} = 44.7\ \text{rad/s} \tag{a}$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 44.7\ \text{rad/s}\sqrt{1 - (0.1)^2} = 44.5\ \text{rad/s} \tag{b}$$

The differential equation governing the response of the press is

$$\ddot{x} + 8.94\dot{x} + 2000x = \frac{1}{10}[5\delta(t) + 5\delta(t - 1.5)] \tag{c}$$

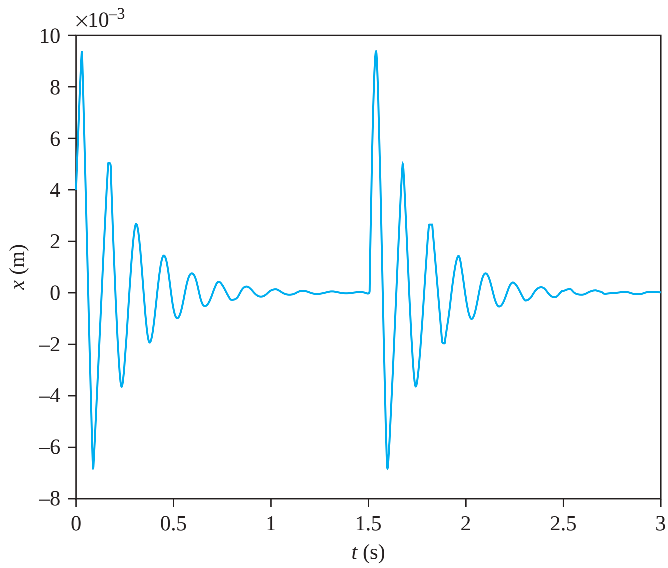


FIGURE 5.2
Time dependent response of a punch press subject to two impulses.

The principle of linear superposition is used to find the response of the system as

$$
\begin{aligned}
x(t) &= \frac{5\ \mathrm{N\cdot s}}{(10\ \mathrm{kg})(44.5\ \mathrm{rad/s})} e^{-4.47t}\sin(44.5t)u(t) \\
&\quad + \frac{5\ \mathrm{N\cdot s}}{(10\ \mathrm{kg})(44.5\ \mathrm{rad/s})} e^{-4.47(t-1.5)}\sin[44.5(t-1.5)]u(t-1.5) \\
&= 0.0112[e^{-4.47t}\sin(44.5t)u(t) \\
&\quad + e^{-4.47t+6.705}\sin(44.5t-66.75)u(t-1.5)]\ \mathrm{m}
\end{aligned} \tag{d}
$$

The graph of the time response is shown in Figure 5.2

5.3 RESPONSE DUE TO A GENERAL EXCITATION

Consider a SDOF system subject to an arbitrary external force, as illustrated in Figure 5.3(a). The time scale is written as τ, because t is reserved for the time at which the response is to be calculated. The interval from 0 to t is broken into n subintervals each of duration $\Delta\tau$ as illustrated in Figure 5.3(b). An effect of the force on the interval from $k\Delta\tau$ to $(k+1)\Delta\tau$ is to provide an impulse with a magnitude of

$$I_k^n = \int_{k\Delta\tau}^{(k+1)\Delta\tau} F(\tau)\,d\tau \tag{5.18}$$

to the system as shown in Figure 5.3(c). The mean value theorem of integral calculus implies that there exists a τ_k^* where $k\Delta\tau \le \tau_k^* \le (k+1)\Delta\tau$ such that

$$I_k^n = F(\tau_k^*)\Delta\tau \tag{5.19}$$

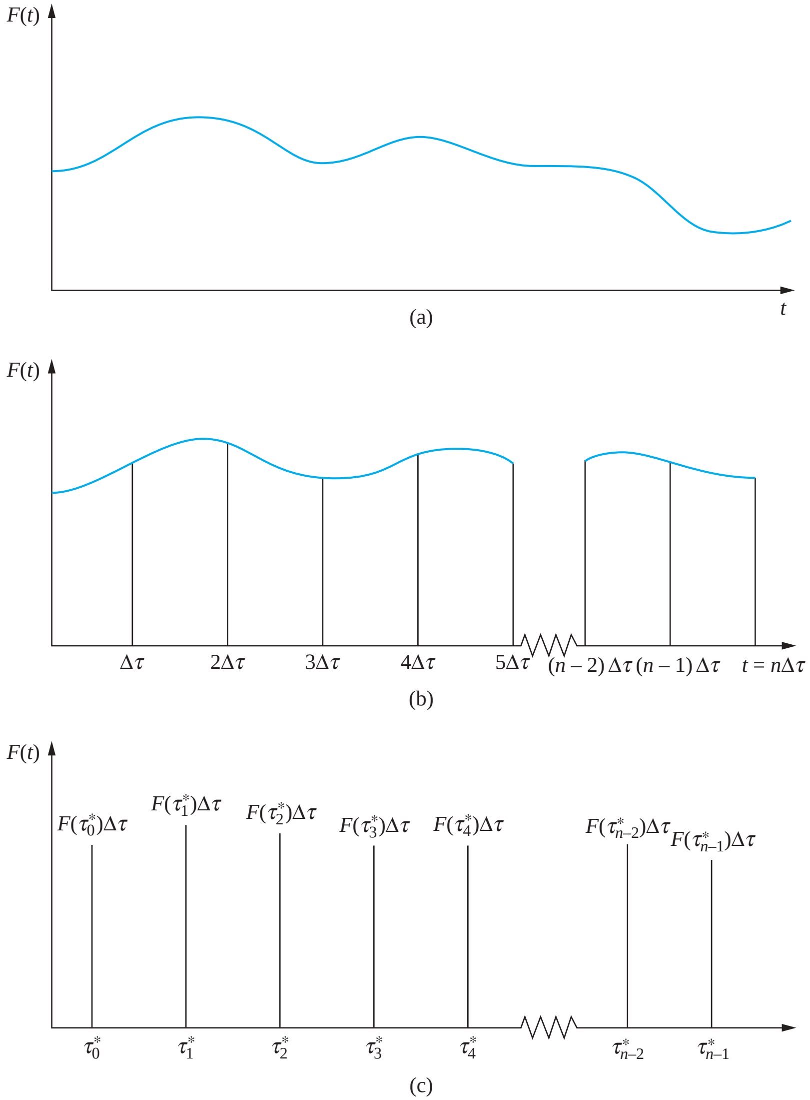


FIGURE 5.3
(a) Arbitrary excitation applied to a SDOF system. (b) The interval from 0 to t is divided into n equal intervals of duration $\Delta\tau = t/n$. (c) The effect of the force applied during the kth interval is approximated by the effect at time t due to an impulse of an appropriate magnitude. In the limit as n approaches infinity, the approximation becomes exact.

If $\Delta\tau$ is small, the effect of the force applied between $k\Delta\tau$ and $(k+1)\Delta\tau$ can be approximated by an impulse of magnitude I_k^n applied at $\tau_k = (k + 1/2)\Delta\tau$. Thus, as illustrated in Figure 5.2(b), the excitation $F(t)$ applied between 0 and t is approximated by the sequence of impulses I_k^n, $k = 0, 1, 2, \ldots, n-1$.

The response of the system at time t due to an impulse with a magnitude of I_k^n applied at time τ_k is obtained using Equations (5.8) and (5.13):

$$x_k^n(t) = I_k^n h(t - \tau_k)u(t - \tau_k) \tag{5.20}$$

The force $F(\tau)$ from 0 to t is approximated by

$$F(\tau) = \sum_{k=1}^{n} I_k^n \delta(\tau - \tau_k) \tag{5.21}$$

The system is aware of the time history of the applied force, but it cannot predict the future. Thus, since Equation (5.1) is linear and has $F(\tau)$ as expressed in Equation (5.21) on the right-hand side, the principle of linear superposition is applied to determine the response at time t as

$$x^n(t) = \sum_{k=0}^{n-1} x_k^n(t) = \sum_{k=0}^{n-1} F(\tau_k^*)\, h(t-\tau_k)\, u(t-\tau_k)\Delta\tau \tag{5.22}$$

The approximation of Equation (5.21) becomes exact in the limit as $n \to \infty$ or $\Delta\tau \to 0$. To this end,

$$x(t) = \lim_{\substack{n\to\infty \\ \Delta\tau\to 0}} x^n(t) = \lim_{\substack{n\to\infty \\ \Delta\tau\to 0}} \sum_{k=0}^{n-1} F(\tau_k^*)\, h(t-\tau_k)\, u(t-\tau_k)\Delta\tau \tag{5.23}$$

In the limit as $n \to \infty$, τ_k and τ_k^* become a continuous variable τ. Also, in the limit, the sum becomes a Riemann sum and

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)d\tau \tag{5.24}$$

For a system whose free vibrations are underdamped, Equation (5.10) is used in Equation (5.24), leading to

$$x(t) = \frac{1}{m_{\text{eq}}\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau \tag{5.25}$$

The integral representation of Equation (5.24) is called the *convolution integral.* It can be used to determine the response of a SDOF system initially at rest in equilibrium subject to any form of excitation. The convolution integral solution is valid for all linear systems where $h(t)$ is viewed as the response of the system due to a unit impulse at $t = 0$. It is the solution of the differential equation of Equation (5.1) that is subject to $x(0) = 0$ and $\dot{x}(0) = 0$.

The response of a system with a nonzero initial velocity is obtained by adding to the convolution integral of Equation (5.24) the response of the system due to a unit impulse at $t = 0$ necessary to cause the initial velocity. The response of a system that is not in its equilibrium position at $t = 0$ is obtained by defining a new independent variable as $y = x - x(0)$. The differential equation governing $y(t)$ is

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = -\frac{k_{\text{eq}}}{m_{\text{eq}}}x(0) + \frac{F_{\text{eq}}(t)}{m_{\text{eq}}} \tag{5.26}$$

The convolution integral is used to obtain

$$y(t) = \int_0^t [-k_{\text{eq}}x(0) + F_{\text{eq}}(\tau)]\, h(t-\tau)d\tau \tag{5.27}$$

The resulting general solution for a system whose free vibrations are underdamped is

$$x(t) = x(0)\, e^{-\zeta\omega_n t}\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$
$$+ \frac{1}{m_{\text{eq}}\omega_d}\int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau \tag{5.28}$$

EXAMPLE 5.2

Find the response of an underdamped SDOF mass-spring-dashpot system initially at rest in equilibrium when the force

$$F(t) = F_0 e^{-\alpha t} \tag{a}$$

is applied.

SOLUTION

Application of Equation (5.25) for this particular form of $F(t)$ gives

$$\begin{aligned} x(t) &= \int_0^t \frac{F_0 e^{-\alpha t}}{m_{eq}\omega_d} e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)d\tau \\ &= \frac{F_0}{m_{eq}\omega_d(\omega_n^2 - 2\zeta\omega_n\alpha + \alpha^2)} \\ &\quad \times \left\{ e^{-\zeta\omega_n t}[(\alpha - \zeta\omega_n)\sin\omega_d t - \omega_d\cos\omega_d t] - \omega_d e^{-\alpha t} \right\} \end{aligned} \tag{b}$$

EXAMPLE 5.3

A press of mass m is mounted on an elastic foundation of stiffness k. During operation, the force applied to the press builds up to its final value F_0 in a time t_0, as illustrated in Figure 5.4. Determine the response of the press for (a) $t < t_0$, and (b) $t > t_0$.

SOLUTION

The force applied to the press can be expressed as

$$F(t) = \begin{cases} F_0 \dfrac{t}{t_0} & t < t_0 \\ F_0 & t \geq t_0 \end{cases} \tag{a}$$

For an undamped system, the convolution integral of Equation (5.25) becomes

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)d\tau \tag{b}$$

(a) For $t < t_0$, the convolution integral yields

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\int_0^t F_0\frac{\tau}{t_0}\sin\omega_n(t-\tau)d\tau \\ &= \frac{F_0}{m\omega_n t_0}\left[\frac{\tau}{\omega_n}\cos\omega_n(t-\tau) + \frac{1}{\omega_n^2}\sin\omega_n(t-\tau)\right]_{\tau=0}^{\tau=t} \\ &= \frac{F_0}{m\omega_n^2 t_0}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right) \end{aligned} \tag{c}$$

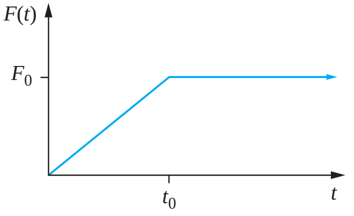


FIGURE 5.4
Excitation of Example 5.3.

(b) For $t > t_0$, application of the convolution integral leads to

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} F_0 \frac{\tau}{t_0}\sin\omega_n(t-\tau)d\tau + \int_{t_0}^{t} F_0 \sin\omega_n(t-\tau)d\tau\right]$$

$$= \frac{F_0}{m\omega_n}\left\{\left[\frac{\tau}{\omega_n}\cos\omega_n(t-\tau) + \frac{1}{\omega_n^2}\sin\omega_n(t-\tau)\right]_{\tau=0}^{\tau=t_0}\right.$$

$$\left. + \left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{\tau=t_0}^{\tau=t}\right\}$$

$$= \frac{F_0}{m\omega_n^2 t_0}\left[t_0\cos\omega_n(t-t_0) + \frac{1}{\omega_n}\sin\omega_n(t-t_0) - \frac{1}{\omega_n}\sin\omega_n t + \frac{1}{\omega_n}\right.$$

$$\left. - \frac{1}{\omega_n}\cos\omega_n(t-t_0)\right] \quad \text{(d)}$$

EXAMPLE 5.4

The restroom door of Example 3.9 is designed such that it is critically damped. The door is closed when a man applies a force of 10 N for a duration of 2 s to the knob. What is the time dependent response of the door?

SOLUTION

Using data from Example 3.9, the force applied to the knob results in a moment applied to the door of

$$M = (10\text{ N})(0.90\text{ m}) = 9.0\text{ N}\cdot\text{m} \quad \text{(a)}$$

The differential equation governing the motion of the door is

$$19.35\ddot{\theta} + 44.1\dot{\theta} + 25\theta = \begin{cases} 9.0 & t < 2 \\ 0 & t > 2 \end{cases} \quad \text{(b)}$$

The convolution integral solution of Equation (b) subject to $\theta(0) = 0$ and is

$$\theta(t) = \int_0^t M(\tau)\frac{1}{I_{eq}}(t-\tau)e^{-\omega_n(t-\tau)}\, d\tau \quad \text{(c)}$$

For $t < 2$ s, the integral becomes

$$\theta(t) = \frac{9.0}{19.35}\int_0^t (t-\tau)e^{-1.14(t-\tau)}d\tau \quad \text{(d)}$$

The integral is evaluated by letting $u = t - \tau$, leading to

$$\theta(t) = 0.465\int_t^0 ue^{-1.14u}(-du) = 0.465\int_0^t ue^{-1.14u}du$$

$$= -0.465\left[\frac{u}{1.14}e^{-1.14u} + \frac{1}{(1.14)^2}e^{-1.14u}\right]_{u=0}^{u=t}$$

$$= 0.357 - 0.357e^{-1.14t} - 0.408te - 1.14t \quad \text{(e)}$$

For $t > 2$ s, the convolution integral leads to

$$\theta(t) = \frac{9.0}{19.35}\int_0^2 (t-\tau)e^{-1.14(t-\tau)}d\tau \tag{f}$$

Let $u = t - \tau$, then

$$\begin{aligned}\theta(t) &= 0.470\int_{t-2}^{t} ue^{-1.14u}du \\ &= 0.357e^{-1.14(t-2)} + 0.408(t-2)e^{-1.14(t-2)} - 0.357e^{-1.14t} - 0.408te^{-1.14t} \\ &= 3.58te^{-1.14t} - 4.84e^{-1.14t}\end{aligned} \tag{g}$$

Thus,

$$\theta(t) = \begin{cases} 0.361 - 0.361e^{-1.14t} - 0.412te^{-1.14t} & t < 2 \text{ s} \\ 3.58te^{-1.14t} - 4.84e^{-1.14t} & t > 2 \text{ s}\end{cases} \tag{h}$$

5.4 EXCITATIONS WHOSE FORMS CHANGE AT DISCRETE TIMES

Many engineering systems are subject to a force whose mathematical form changes at discrete-values of time. Such is the case with the force applied to the press in Example 5.3. The force linearly increases to its maximum value in a time t_0. The mathematical form of the response of the press is different for $t < t_0$ than it is for $t > t_0$. It is more convenient to have unified mathematical forms for the excitation and response. To this end, the unit step function, introduced in Appendix A, is used.

If a constant force F_0 is not applied until time t_0, it can be represented using a delayed unit step function

$$F(t) = \begin{cases} 0 & t \le t_0 \\ F_0 & t > t_0 \end{cases} = F_0 u(t - t_0) \tag{5.29}$$

EXAMPLE 5.5

Use the unit step function to write a unified mathematical expression for each of the forces of Figure 5.5.

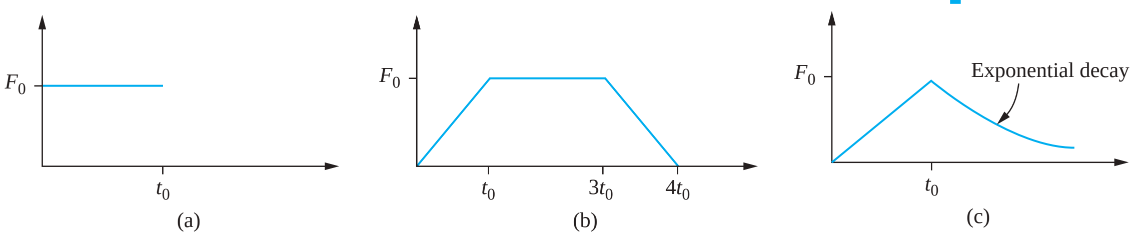


FIGURE 5.5
Excitations of Example 5.5.

SOLUTION

Each of the forces of Figure 5.5 can be written as the sum and/or difference of functions that are nonzero only after a discrete time. The graphical breakdown for each function is shown in Figure 5.6. The unit step function is used to write a mathematical expression for each term in the forcing functions, leading to

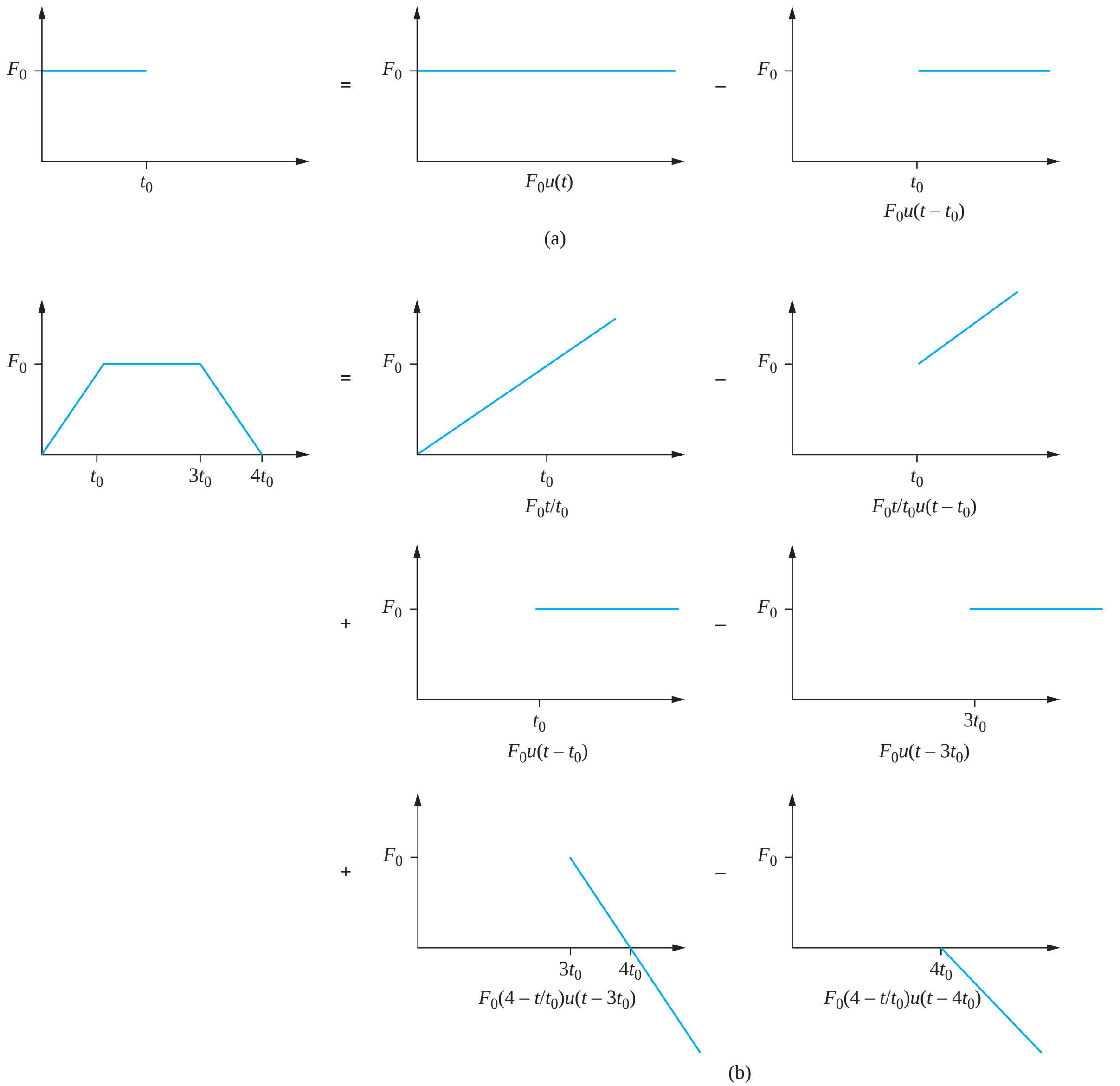
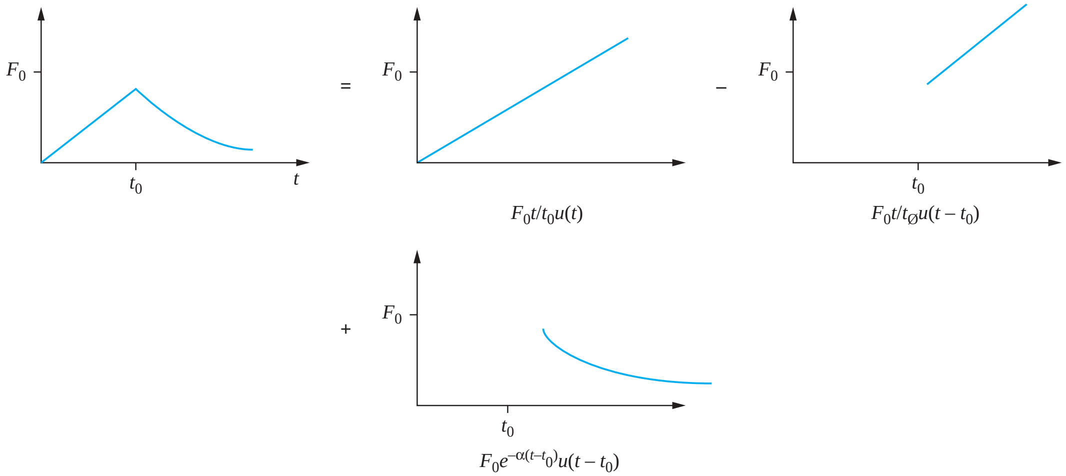


FIGURE 5.6
Graphical breakdown of excitations of Example 5.5 into functions that can be written by using unit step functions.



(c)

FIGURE 5.6
(*Continued*)

(a) $F(t) = F_0[u(t) - u(t - t_0)]$ **(a)**

$$\text{(b)}\quad F(t) = \frac{F_0 t}{t_0}[u(t) - u(t - t_0)] + F_0[u(t - t_0) - u(t - 3t_0)]$$
$$+ F_0\left(4 - \frac{t}{t_0}\right)[u(t - 3t_0) - u(t - 4t_0)]$$
$$= \frac{F_0}{t_0}tu(t) - \frac{F_0}{t_0}(t - t_0)u(t - t_0) - \frac{F_0}{t_0}(t - 3t_0)u(t - 3t_0)$$
$$+ \frac{F_0}{t_0}(t - 4t_0)u(t - 4t_0) \qquad \textbf{(b)}$$

$$\text{(c)}\quad F(t) = \frac{F_0 t}{t_0}[u(t) - u(t - t_0)] + F_0 e^{-\alpha(t - t_0)}u(t - t_0) \qquad \textbf{(c)}$$

Many functions found in practice can be written as combinations of impulses, step functions, ramp functions, exponentially decaying functions, and sinusoidal pulses. Many functions which cannot be mathematically defined in terms of these functions are often approximated by these functions for estimation purposes.

Table 5.1 provides the response of an undamped SDOF system to common excitation terms delayed by a time t_0. The responses are derived from the convolution integral making use of the following formula:

$$\int_0^t F(\tau)u(\tau - t_0)d\tau = u(t - t_0)\int_{t_0}^t F(\tau)d\tau \qquad \textbf{(5.30)}$$

EXAMPLE 5.6

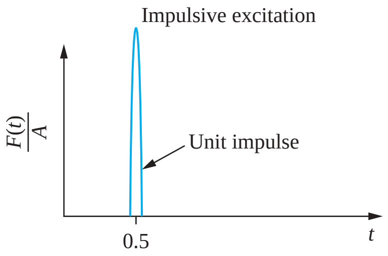
Use the convolution integral to derive the responses of an undamped linear SDOF system of mass m and natural frequency ω_n when subject to the delayed exponential excitation illustrated in Table 5.1.

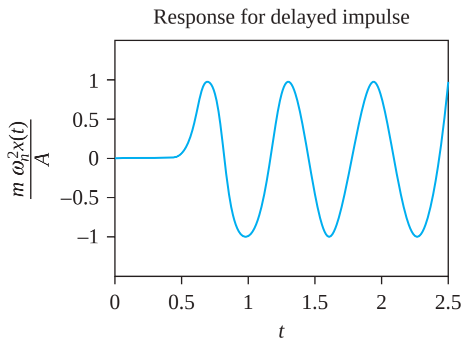
TABLE 5.1 Response of an undamped SDOF to common forms of excitation

Delayed impulse

Excitation: $F(t) = A\delta(t - t_0)$

Response: $m_{eq}\omega_n^2 x(t)/A = \omega_n \sin \omega_n (t - t_0) u(t - t_0)$

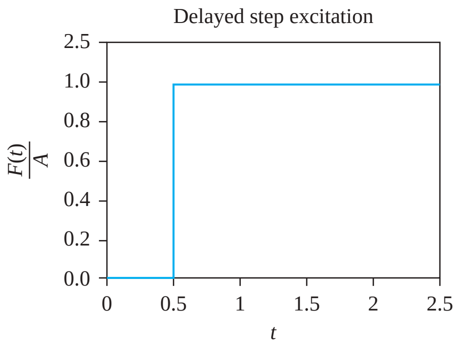


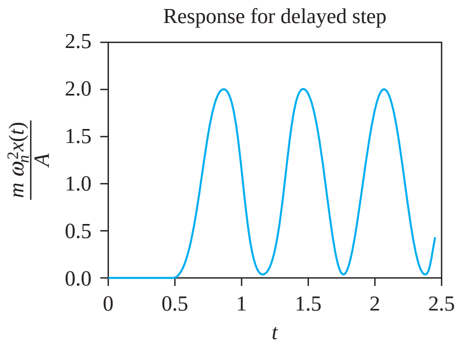


Delayed step function

Excitation: $F(t) = Au(t - t_0)$

Response: $m_{eq}\omega_n^2 x(t)/A = [1 - \cos \omega_n (t - t_0)]u(t - t_0)$

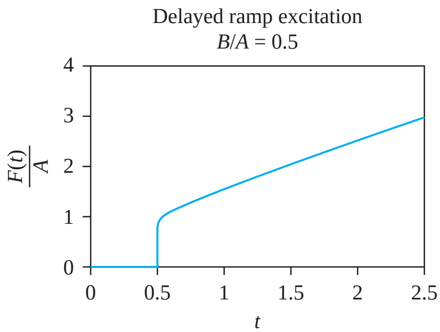




Delayed ramp function

Excitation: $F(t) = (At + B)u(t - t_0)$

Response: $m_{eq}\omega_n^2 x(t)/A = \left[t + B/A - (t_0 + B/A) \cos \omega_n (t - t_0) - \dfrac{1}{\omega_n} \sin \omega_n (t - t_0)\right] u(t - t_0)$



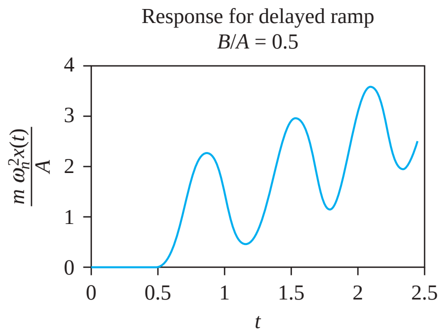
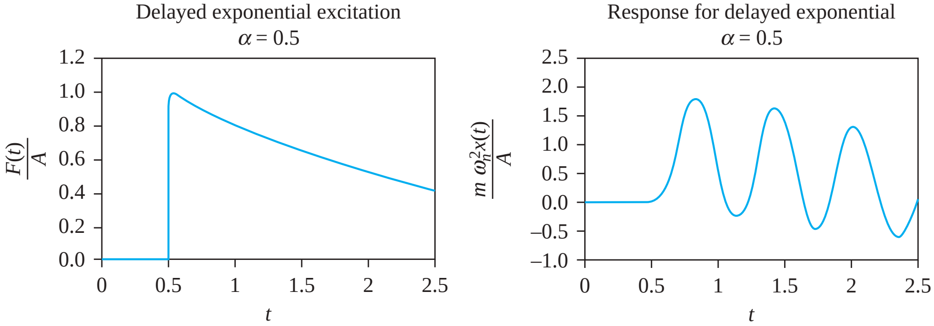


TABLE 5.1 (CONTINUED)

Delayed exponential function

Excitation: $F(t) = Ae^{-\alpha(t-t_0)}u(t-t_0)$

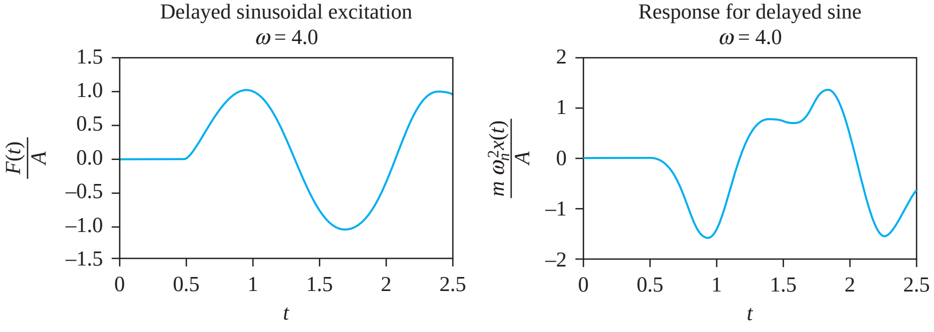
Response: $m_{eq}\omega_n^2 x(t)/A = [e^{-\alpha(t-t_0)} + \alpha/\omega_n \sin\omega_n(t-t_0) - \cos\omega_n(t-t_0)]/(1+\alpha^2/\omega_n^2)u(t-t_0)$



Delayed sine function:

Excitation: $F(t) = A\sin[\omega(t-t_0)]u(t-t_0)$

Response:
$$\frac{m_{eq}\omega_n^2 x(t)}{A} = \frac{1}{2}\left\{\left(\frac{1}{\omega/\omega_n - 1}\right)[\sin\omega(t-t_0) - \sin\omega_n(t-t_0)] - \left(\frac{1}{\omega/\omega_n + 1}\right)[\sin\omega(t-t_0) + \sin\omega_n(t-t_0)]\right\}u(t-t_0)$$



This table provides the response of an undamped SDOF system to common forms of excitation. Many forms of excitation can be written as combinations of the excitations whose system responses are provided in the table. Superposition can be used to determine the response due to these excitations. In other cases, excitations can be approximated by combinations of excitations in this table. Then this table and superposition is used to approximate the response of an undamped SDOF system.

The table provides the mathematical form of the excitation and response as well as graphical representations. In all cases, values of $\omega_n = 10$ rad/s and $r_o = 0.5$ s were used to generate the graphs. The values of specific parameters used for specific excitations are given.

SOLUTION

The mathematical representation of the forcing function is

$$F(t) = F_0 e^{-\alpha(t - t_0)} u(t - t_0) \tag{a}$$

The convolution integral of Equation (5.25) is used to write the solution as

$$x(t) = \frac{F_0}{m_{\text{eq}}\omega_n} \int_0^t e^{-\alpha(\tau - t_0)} u(\tau - t_0) \sin \omega_n (t - \tau) d\tau \tag{b}$$

which using Equation (5.30) is rearranged as

$$\begin{aligned} x(t) &= u(t - t_0) \frac{F_0}{m_{\text{eq}}\omega_n} \int_0^t e^{-\alpha(\tau - t_0)} \sin \omega_n (t - t_\tau) d\tau \\ &= u(t - t_0) \frac{F_0}{m_{\text{eq}}\omega_n(\alpha^2 + \omega_n^2)} [\omega_n e^{-\alpha(t - t_0)} + \alpha \sin \omega_n (t - t_0) \\ &\qquad - \omega_n \cos \omega_n (t - t_0)] \end{aligned} \tag{c}$$

Often, excitations are linear combinations of the function whose responses are presented in Table 5.1. The general form of an excitation that changes form at discrete times $t_1, t_2, \ldots, t_n$ is

$$F(t) = \sum_{i=1}^{n} f_i(t) u(t - t_i) \tag{5.31}$$

Application of the convolution integral to the excitation of Equation (5.31), using Equation (5.30), yields

$$x(t) = \sum_{i=1}^{n} u(t - t_i) \int_{t_i}^{t} f_i(\tau) h(t - \tau)\, d\tau \tag{5.32}$$

Equation (5.32) shows that the total response is the sum of the responses due to the individual terms of the excitation. This result is due to the linearity of Equation (5.1). The effects of any nonzero initial conditions are included with the response due to $f_1(t)$.

EXAMPLE 5.7

Use Table 5.1 to develop the response of a linear, SDOF system of mass m and natural frequency ω_n when subject to the triangular pulse excitation of Figure 5.7.

SOLUTION

The triangular pulse can be written as the sum and difference of ramp functions as shown. The response due to the triangular pulse is obtained by adding and subtracting the responses due to each ramp function according to

$$x(t) = x_a(t) - x_b(t) + x_c(t) - x_d(t) \tag{a}$$

where the individual responses are determined from Table 5.1.

For $x_a(t)$, the ramp function entry of Table 5.1 is used with $A = F_0/t_1$, $B = 0$, and $t_0 = 0$ leading to

$$x_a(t) = \frac{F_0}{m\omega_n^2}\left[\frac{t}{t_1} - \frac{1}{\omega_n t_1} \sin \omega_n t\right] \tag{b}$$

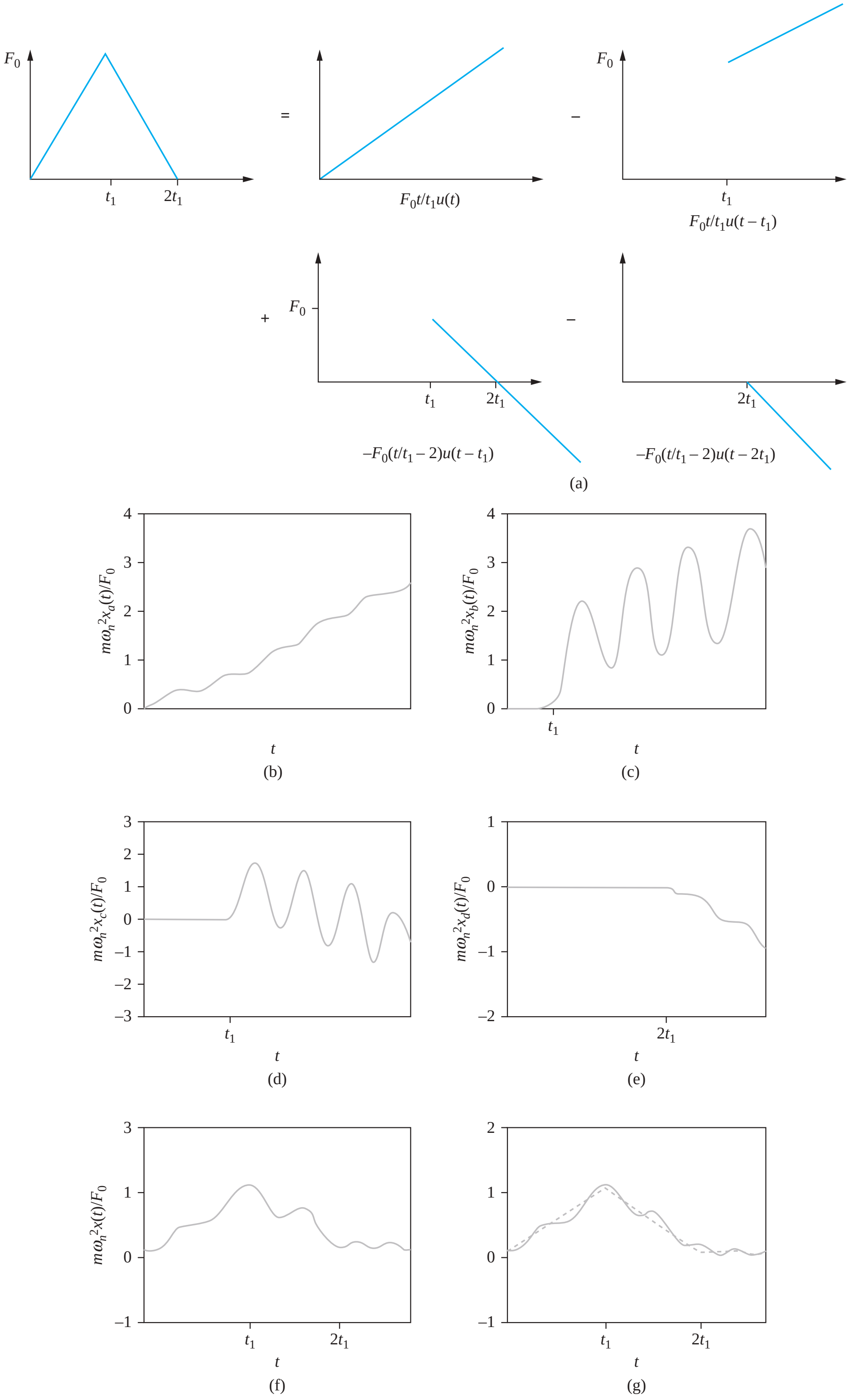


FIGURE 5.7
(a) Triangular pulse of Example 5.7 and its graphical breakdown. (b)–(e) Response of a SDOF undamped system due to the component parts of a triangular pulse excitation obtained using Table 5.1. (f) Response of a SDOF system due to triangular pulse excitation obtained using the principle of linear superposition. (g) Comparison of triangular pulse and the resulting excitation.

$x_b(t)$ is determined from the ramp function entry of Table 5.1 with $A = F_0/t_1$, $B = 0$, $t_0 = t_1$. This gives

$$x_b(t) = \frac{F_0}{m\omega_n^2}\left[\frac{t}{t_1} - \cos\omega_n(t - t_1) - \frac{1}{\omega_n t_1}\sin\omega_n(t - t_1)\right]u(t - t_1) \quad \text{(c)}$$

For $x_c(t)$, the ramp function entry of Table 5.1 is used with $A = -F_0/t_1$, $B = 2F_0$, and $t_0 = t_1$. This leads to

$$x_c(t) = \frac{F_0}{m\omega_n^2}\left[\left(2 - \frac{t}{t_1}\right) - \cos\omega_n(t - t_1) + \frac{1}{\omega_n t_1}\sin\omega_n(t - t_1)\right]u(t - t_1) \quad \text{(d)}$$

$x_d(t)$ is determined using the ramp function entry of Table 5.1 with $A = -F_0/t_1$, $B = 2F_0$, and $t_0 = 2t_1$. This gives

$$x_d(t) = \frac{F_0}{m\omega_n^2}\left[\left(2 - \frac{t}{t_1}\right) + \frac{1}{\omega_n t_1}\sin\omega_n(t - 2t_1)\right]u(t - 2t_1) \quad \text{(e)}$$

Simplifying the resulting expression in each interval of time yields

$$x(t) = \frac{F_0}{m\omega_n^2}\begin{cases} \dfrac{t}{t_1} - \dfrac{1}{\omega_n t_1}\sin\omega_n t & 0 \le t \le t_1 \\ 2 - \dfrac{t}{t_1} + \dfrac{1}{\omega_n t_1}[2\sin\omega_n(t - t_1) - \sin\omega_n t] & t_1 \le t \le 2t_1 \\ \dfrac{1}{\omega_n t_1}[2\sin\omega_n(t - t_1) - \sin\omega_n t - \sin\omega_n(t - 2t_1)] & t_1 > 2t_1 \end{cases} \quad \text{(f)}$$

The response of each component part and the total response is shown in Figure 5.7(b) through (g).

5.5 TRANSIENT MOTION DUE TO BASE EXCITATION

Many mechanical systems and structures are subject to nonperiodic base excitation. A rigid wheel traveling along a road contour excites motion of a vehicle through the suspension system. Earthquakes excite structures through base motion.

Recall the governing equation for the relative displacement between a mass and its base when the mass is connected to the base through a spring and viscous damper in parallel

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \quad (5.33)$$

where y is the prescribed base motion. If $z(0) = 0$ and $\dot{z}(0) = 0$, the convolution integral is used to solve Equation (5.33), yielding

$$z(t) = -m_{\text{eq}}\int_0^t \ddot{y}(\tau)h(t - \tau)\,d\tau \quad (5.34)$$

Equation (5.34) is integrated by parts to write the solution in terms of the base velocity

$$z(t) = m_{\text{eq}}[\dot{y}(0)h(t) - \int_0^t \dot{y}(\tau)\dot{h}(t - \tau)d\tau \tag{5.35}$$

where

$$\dot{h}(t) = -\frac{e^{-\zeta\omega_n t}}{m_{\text{eq}}\sqrt{1-\zeta^2}}\sin(\omega_d t - \chi) \tag{5.36}$$

$$\chi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{5.37}$$

If the base displacement is known, it can be differentiated to calculate the velocity and Equation (5.35) can be used to determine the relative displacement. Alternatively, the absolute displacement of the base can be attained by solving

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -2\zeta\omega_n\dot{y} - \omega_n^2 y \tag{5.38}$$

When applied to Equation (5.38), the convolution integral yields

$$x(t) = -m_{\text{eq}}\int_0^t [2\zeta\omega_n\dot{y}(\tau) + \omega_n^2 y(\tau)]h(t-\tau)d\tau \tag{5.39}$$

EXAMPLE 5.8

Determine the response of a block of mass m connected through a spring of stiffness k to a base when the base is subject to the rectangular velocity pulse of Figure 5.8. Use (a) Equation (5.35) and (b) Equation (5.34).

SOLUTION

The mathematical expression for the velocity pulse is

$$\dot{y}(t) = v[u(t) - u(t - t_0)]$$

(a) By definition $u(0) = 0$, thus $\dot{y}(0) = 0$. In using Equation (5.35) for an undamped, system, note that $\chi = \pi/2$ and $\sin(\omega_n t - \pi/2) = \cos\omega_n t$. Application of Equation (5.35) then yields

$$z(t) = -v\int_0^t [u(\tau) - u(\tau - t_0)]\cos\omega_n(t-\tau)d\tau \tag{a}$$

Using Equation (5.30) to evaluate the integral leads to

$$\begin{aligned} z(t) &= -v[u(t)\int_0^t \cos\omega_n(t-\tau)d\tau - u(t-t_0)\int_{t_0}^t \cos\omega_n(t-\tau)d\tau] \\ &= \frac{v}{\omega_n}[\sin\omega_n(t-t_0)u(t-t_0) - \sin(\omega_n t)u(t)] \end{aligned} \tag{b}$$

(b) The base acceleration is obtained by differentiating the base velocity with respect to time. Noting that the derivative of the unit step function is the unit impulse function, differentiation gives

$$\ddot{y}(t) = v[\delta(t) - \delta(t - t_0)] \tag{c}$$

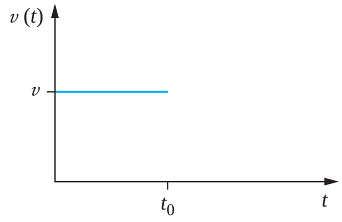


FIGURE 5.8
Velocity pulse for Example 5.8.

The base velocity changes instantaneously at $t = 0$ and $t = t_0$. Instantaneous velocity changes result only from applied impulses.

Substituting the base acceleration into Equation (5.34) gives

$$z(t) = -\frac{v}{\omega_n}\int_0^t [\delta(\tau) - \delta\,(\tau - t_0)]\ \sin\omega_n(t-\tau)d\tau \tag{d}$$

The integrals are evaluated after noting

$$\int_0^t \delta(\tau - t_0)f(\tau)d\tau = f(t_0)\,u(t - t_0) \tag{e}$$

The relative displacement is determined as

$$z(t) = \frac{v}{\omega_n}[\ \sin\omega_n\ (t - t_0)u(t - t_0) - \sin\,(\omega_n t)u(t)] \tag{f}$$

5.6 LAPLACE TRANSFORM SOLUTIONS

The Laplace transform method is a convenient method for finding the response of a system due to any excitation. The basic method is to use known properties of the transform to transform an ordinary differential equation into an algebraic equation, using the initial conditions. The algebraic equation is solved to find the transform of the solution. This transform is inverted by using properties of the transform and a table of known transform pairs.

The Laplace transform can be used to solve linear ordinary differential equations with constant or polynomial coefficients. The method easily handles excitations whose form changes with time. Such excitations are written in a unified mathematical expression by using the unit step functions. The shifting theorems help perform the transform and evaluate the inversions.

The Laplace transform is not as easy to apply as the convolution integral unless one has extensive experience in its use. The main drawback of the method is the difficulty in inverting the transform. A formal inversion theorem, involving contour integration in the complex plane, is available, but is beyond the scope of this text.

The transform pairs and properties used in the following discussion are summarized and explained in Appendix B.

Let $X(s)$ be the Laplace transform of the generalized coordinate for a SDOF system. That is,

$$X(s) = \int_0^\infty x(t)e^{-st}dt \tag{5.40}$$

Let $F(s)$ be the Laplace transform of the known forcing function which, for a specific form of $F_{\text{eq}}(t)$, is calculated from the transform definition, referring to a table of transform pairs, or using basic properties in conjunction with a table.

Taking the Laplace transform of Equation (5.1) and using linearity of the transform,

$$\mathcal{L}\{\ddot{x}\} + 2\zeta\omega_n \mathcal{L}\{\dot{x}\} + \omega_n^2 X(s) = \frac{F(s)}{m_{\text{eq}}} \tag{5.41}$$

The property for transform of derivatives allows the transform of the differential equation for $x(t)$ into an algebraic equation for $X(s)$. Its application to Equation (5.41) gives

$$s^2X(s) - sx(0) - \dot{x}(0) + 2\zeta\omega_n[sX(s) - x(0)] + \omega_n^2 X(s) = \frac{F(s)}{m_{\text{eq}}}$$

which rearranges to

$$X(s) = \frac{\dfrac{F(s)}{m_{\text{eq}}} + (s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.42}$$

The definition and linearity of the inverse transform is used to find $x(t)$,

$$x(t) = \frac{1}{m_{\text{eq}}}\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \tag{5.43}$$

The inverse transform of each term of Equation (5.43) depends on the types of roots in the denominator, which, in turn, depend on the value of ζ. For a given ζ, the inverse transform of the last term of Equation (5.43) is directly determined. The inverse transform of the first term is determined only after specifying $F_{\text{eq}}(t)$ and taking its Laplace transform.

If the system is undamped, $\zeta = 0$, and the inverse transform of the second term becomes

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{sx(0) + \dot{x}(0)}{s^2 + \omega_n^2}\right\} \\ &= x(0)\,\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega_n^2}\right\} + \dot{x}(0)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega_n^2}\right\}\end{aligned} \tag{5.44}$$

Using transform pairs $B4$ and $B5$ to invert the transforms for an undamped system

$$\mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t \tag{5.45}$$

If the free vibrations are underdamped, then the denominator has two complex roots. In this case, it is convenient to complete the square of the denominator as

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + 2\zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) \tag{5.46}$$

Substituting Equation (5.46) into the last term of Equation (5.43) yields

$$\mathcal{L}^{-1}\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}=\mathcal{L}^{-1}\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}\right\} \tag{5.47}$$

Equation (5.47) is written in a form for use in the first shifting theorem (that is, wherever s appears, it appears as $s+\zeta\omega_n$ in the denominator). Using linearity of the inverse transform, we have

$$\begin{aligned}\mathcal{L}^{-1}&\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}\\ &=x(0)\,\mathcal{L}^{-1}\left\{\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}\right\}\\ &\quad+(\dot{x}(0)+\zeta\omega_n x(0))\mathcal{L}^{-1}\left\{\frac{1}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}\right\}\end{aligned} \tag{5.48}$$

The first shifting theorem along with transform pair $B5$ are used to invert the first term, while the first shifting theorem and transform pair $B4$ are used to invert the second term, yielding for an underdamped system:

$$\begin{aligned}\mathcal{L}^{-1}&\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}\\ &=x(0)e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\\ &\quad+\left[\dot{x}(0)+\zeta\omega_n x(0)\right]e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\end{aligned} \tag{5.49}$$

If the free vibrations are critically damped, the denominator of Equation (5.43) is a perfect square as $(s+\omega_n)^2$ and it yields

$$\mathcal{L}^{-1}\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}=\mathcal{L}^{-1}\left\{\frac{(s+2\omega_n)x(0)+\dot{x}(0)}{(s+\omega_n)^2}\right\} \tag{5.50}$$

Using linearity of the inverse transform, the right-hand side of Equation (5.50) is rewritten as

$$\begin{aligned}\mathcal{L}^{-1}&\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}\\ &=x(0)\,\mathcal{L}^{-1}\left\{\frac{1}{s+\omega_n}\right\}+(\omega_n x(0)+\dot{x}(0))\mathcal{L}^{-1}\left\{\frac{1}{(s+\omega_n)^2}\right\}\end{aligned} \tag{5.51}$$

Inverting using transform pairs $B3$ on the first term and the first shifting theorem and transform pair $B2$ on the second term leads to:

$$\mathcal{L}^{-1}\left\{\frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}=x(0)e^{-\omega_n t}+(\omega_n x(0)+\dot{x}(0))te^{-\omega_n t} \tag{5.52}$$

When the free vibrations are overdamped, the denominator of Equation (5.43) can be factored into two linear factors $(s-s_1)(s-s_2)$ where $s_1=-\omega_n(\zeta+\sqrt{\zeta^2-1})$ and

$s_1 = -\omega_n(\zeta - \sqrt{\zeta^2 - 1})$. A partial fraction decomposition of the transform leads to

$$\mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = \frac{[(s_1 + 2\zeta\omega_n)x(0) + \dot{x}(0)]}{s_1 - s_2}\mathcal{L}^{-1}\left\{\frac{1}{s - s_1}\right\} + \frac{[(s_2 + 2\zeta\omega_n)x(0) + \dot{x}(0)]}{s_2 - s_1}\mathcal{L}^{-1}\left\{\frac{1}{s - s_2}\right\} \tag{5.53}$$

The transform is inverted using transform pair *B*3, yielding

$$\mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = \frac{[(s_1 + 2\zeta\omega_n)x(0) + \dot{x}(0)]}{s_1 - s_2}e^{s_1 t} + \frac{[(s_2 + 2\zeta\omega_n)x(0) + \dot{x}(0)]}{s_2 - s_1}e^{s_2 t} \tag{5.54}$$

The inverse transform of the first term of Equation (5.43) is found by finding $F(s)$ for the particular form of $F(t)$, forming $F(s)/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, and inverting using algebraic manipulations, transform properties, and a table of known transform pairs.

EXAMPLE 5.9

A 200-kg machine is to be mounted on an elastic surface of equivalent stiffness 2×10^5 N/m with no damping. During operation, the machine is subject to a constant force of 2000 N for 3 s. Can vibrations be eliminated without adding damping? If so, what is the maximum deflection of the machine?

SOLUTION

The differential equation governing motion of the machine is

$$\ddot{x} + \omega_n^2 x = F_0[u(t) - u(t - 3)] \tag{a}$$

where $F_0 = 2000$ N and $\omega_n = 31.63$ rad/s. The Laplace transform of $F(t)$ is obtained by using the second shifting theorem

$$\mathcal{L}\{F_0[u(t) - u(t - 3)]\} = \frac{F_0}{s}(1 - e^{-3s}) \tag{b}$$

Then from Equation (5.43) with $x(0) = 0$ and $\dot{x}(0) = 0$,

$$X(s) = \frac{F_0}{m}\mathcal{L}^{-1}\left\{\frac{1 - e^{-3s}}{s(s^2 + \omega_n^2)}\right\} \tag{c}$$

Partial fraction decomposition yields

$$X(s) = \frac{F_0}{m\omega_n^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right)(1 - e^{-3s}) \tag{d}$$

The second shifting theorem is used to help invert the transform

$$x(t) = \frac{F_0}{m\omega_n^2}[1 - \cos\omega_n t - u(t-3)(1 - \cos\omega_n(t-3))] \tag{e}$$

The solution for $t > 3$ s is

$$x(t) = \frac{F_0}{m\omega_n^2}[\cos\omega_n(t-3) - \cos\omega_n t] \qquad t > 3 \text{ s} \tag{f}$$

For no steady-state motion,

$$\cos\omega_n t = \cos\omega_n(t-3) \tag{g}$$

which is satisfied if $3\omega_n = 2n\pi$ for any positive integer n. Thus steady-state vibrations are eliminated by requiring

$$\omega_n = \frac{2n\pi}{3} = 2.09n \text{ rad/s} \tag{h}$$

For $n = 15$, $\omega_n = 31.35$ rad/s, which is attained if $m = 203.5$ kg. Thus steady-state vibrations are eliminated if 3.5 kg is rigidly added to the machine.

The machine undergoes 15 cycles while the force is applied, and motion ceases when the force is removed. The maximum displacement during operation is

$$x_{\max} = \frac{2F_0}{m\omega_n^2} = \frac{2F_0}{k} = 0.02 \text{ m} \tag{i}$$

EXAMPLE 5.10

Use the Laplace transform method to determine the response of an underdamped SDOF system to the rectangular velocity pulse of Figure 5.8.

SOLUTION

From the analysis in Example 5.8, the differential equation governing displacement of the mass relative to its base when the base is subject to a rectangular velocity pulse is

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -v[\delta(t) - \delta(t - t_0)]$$

Using transform pair $B1$, and assuming $z(0) = 0$ and $\dot{z}(0) = 0$, Equation (5.42) becomes

$$Z(s) = \frac{-v(1 - e^{-st_0})}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The transform is inverted by completing the square in the denominator and using both the first shifting theorem and the second shifting theorem to obtain

$$z(t) = \frac{-v}{\omega_n}[e^{-\zeta\omega_n t}\sin\omega_d t - e^{-\zeta\omega_n(t-t_0)}\sin\omega_d(t - t_0)u(t - t_0)]$$

5.7 TRANSFER FUNCTIONS

Taking the Laplace transform of Equation (5.1), assuming $x(0) = 0$ and $\dot{x}(0) = 0$, leads to an equation of the form

$$X(s) = F(s)G(s) \tag{5.55}$$

where $X(s)$ is the Laplace transform of $x(t)$, $F(s)$ is the Laplace transform of $F(t)$, and $G(s)$ is called the transfer function. The *transfer function* is always defined assuming the initial conditions are zero. Since

$$G(s) = \frac{X(s)}{F(s)} \tag{5.56}$$

the transfer function is independent of the input to the system. It is a function of only the system and its parameters. For a SDOF system, the transfer function is dependent upon the mass, damping ratio, and natural frequency.

EXAMPLE 5.11

(a) Determine the transfer function for a SDOF system of natural frequency 10 rad/s and a damping ratio of 1.5 due to a force input. The mass of the system is 2 kg.
(b) Find the response of the system due to a force $F(t) = 10e^{-3t}$.

SOLUTION

(a) The differential equation governing the motion of the system is

$$\ddot{x} + 30\dot{x} + 100x = \frac{1}{2}F(t) \tag{a}$$

Taking the Laplace transform of Equation (a) and setting both initial conditions to zero leads to

$$(s^2 + 30s + 100)X(s) = \frac{1}{2}F(s) \tag{b}$$

Rearranging Equation (b) leads to

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{2(s^2 + 30s + 100)} \tag{c}$$

(b) The Laplace transform of $F(t) = 10e^{-3t}$ is $F(s) = \dfrac{10}{s+3}$. From Equation (5.49),

$$X(s) = F(s)G(s) = \frac{10}{2(s^2 + 30s + 100)(s + 3)} \tag{d}$$

The system is overdamped, so the denominator of its transfer function is factorable with real factors as

$$X(s) = \frac{5}{(s + 3.82)(s + 26.18)(s + 3)} \tag{e}$$

Performing a partial fraction decomposition on the right-hand side of Equation (e), we have

$$X(s) = \frac{-0.244}{s + 3.82} + \frac{9.69 \times 10^{-3}}{s + 26.18} + \frac{0.234}{s + 3} \tag{f}$$

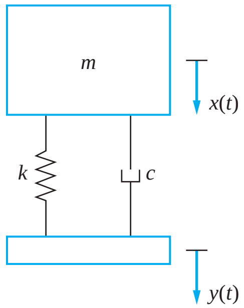
Inverting Equation (f) leads to

$$x(t) = 0.234e^{-3t} + 9.69 \times 10^{-3}e^{-26.18t} - 0.244e^{-3.82t} \tag{g}$$

EXAMPLE 5.12

Determine the transfer function for the system of Figure 5.9, which has motion input.

FIGURE 5.9
Mechanical system with motion input.



SOLUTION

The differential equation is derived in Section 4.5 as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y \tag{a}$$

The transfer function for this system is defined as

$$G(s) = \frac{X(s)}{Y(s)} \tag{b}$$

where $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of Equation (a), we have

$$\mathcal{L}\{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x\} = \mathcal{L}\{2\zeta\omega_n\dot{y} + \omega_n^2 y\} \tag{c}$$

Using the properties of linearity of the transform and the transform of derivatives with the initial conditions taken to be zero leads to

$$s^2 X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = 2\zeta\omega_n sY(s) + \omega_n^2 Y(s) \tag{d}$$

Rearranging Equation (d) and solving for the transfer function leads to

$$G(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{e}$$

The transfer function for SDOF systems are as follows:

- System with force input

$$G(s) = \frac{\frac{1}{m}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.57}$$

- System with motion input

$$G(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.58}$$

The impulsive response of a system $x_I(t)$ is the response due to a unit impulse function:

$$\ddot{x}_I + 2\zeta\omega_n\dot{x}_I + \omega_n^2 x_I = \frac{1}{m}\delta(t) \tag{5.59}$$

Noting that $\mathcal{L}\{\delta(t)\} = 1$, the Laplace transform of the impulsive response $H(s)$ obtained from Equation (5.55) is

$$H(s) = G(s) \tag{5.60}$$

Thus, the transfer function is the transform of the system's impulsive response. Using the notation of previous sections, we have

$$h(t) = \mathcal{L}^{-1}G\{(s)\} \tag{5.61}$$

Use of the convolution theorem on Equation (5.55) and noting Equation (5.61) yields

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau \tag{5.62}$$

The response of a system due to a unit step function is given by

$$\ddot{x}_s + 2\zeta\omega_n\dot{x}_s + \omega_n^2 x_s = \frac{1}{m}u(t) \tag{5.63}$$

Noting that $\mathcal{L}\{u(t)\} = 1/s$, the Laplace transform of the step response is

$$X_s(s) = \frac{1}{s}G(s) \tag{5.64}$$

Taking the inverse of Equation (5.64) and using the property of transforms of integrals yields

$$x_s(t) = \int_0^t u(\tau)h(t-\tau)d\tau = \int_0^t h(t-\tau)d\tau \tag{5.65}$$

Changing the variable of integration in Equation (5.58) by letting $v = t - \tau$ leads to

$$x_s(t) = \int_0^t h(v)dv \tag{5.66}$$

Writing Equation (5.66) as

$$X(s) = [sF(s)]\left[\frac{1}{s}G(s)\right] \tag{5.67}$$

leads to a convolution integral solution of

$$x(t) = \int_0^t [\dot{F}(\tau) + F(0)]x_s(t-\tau)d\tau \tag{5.68}$$

EXAMPLE 5.13

Find the step response of a critically damped SDOF system.

SOLUTION

The impulsive response of a critically damped SDOF system is

$$h(t) = \frac{1}{m}te^{-\omega_n t} \tag{a}$$

Use of Equation (5.66) gives

$$x_s(t) = \frac{1}{m}\int_0^t ve^{-\omega_n v}dv \tag{b}$$
$$= \frac{1}{m\omega_n^2}(1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t})$$

5.8 NUMERICAL METHODS

The convolution integral and Laplace transform methods are easy methods of solving Equation (5.1) for any excitation. However, closed-form solutions using these methods are limited to cases where the forcing function has an explicit mathematical formulation and closed-form evaluation of the convolution integral is possible. In addition, there are explicitly defined forcing functions such as those proportional to non-integral powers of time where a closed-form evaluation of the convolution integral or evaluation of the inverse Laplace transform is very difficult. When these situations occur, numerical methods must be used to obtain an approximate solution to the differential equation at discrete values of time.

Numerical solutions of forced SDOF vibrations problems are of two classes: numerical evaluation of the convolution integral and direct numerical evaluation of Equation (5.1).

5.8.1 NUMERICAL EVALUATION OF CONVOLUTION INTEGRAL

Many numerical integration techniques are available for evaluation of integrals. Most numerical integration techniques use piecewise defined functions to interpolate the integrand. A closed-form integration of the interpolated integrand is performed. The method described here uses an interpolation for $F_{\text{eq}}(t)$ from which an approximation to the convolution integral is obtained. The discretization of a time interval and possible interpolations to $F_{\text{eq}}(t)$ are shown in Figure 5.10.

Let $t_1, t_2, \ldots$ be values of time at which an approximate solution is to be obtained. Let $F_1(t)$, $F_2(t), \ldots$ be the interpolating functions such that $F_k(t)$ interpolates $F_{\text{eq}}(t)$ on the interval $t_{k-1} < t < t_k$. Let x_k be the numerical approximation for $x(t_k)$. Also define

$$\Delta j = t_j - t_{j-1}$$

The convolution integral is used to obtain the response of an underamped SDOF system as

$$x(t) = x(0)e^{-\zeta\omega_n t}\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$
$$+ \int_0^t \frac{F_{\text{eq}}(\tau)}{m_{\text{eq}}\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \tag{5.69}$$

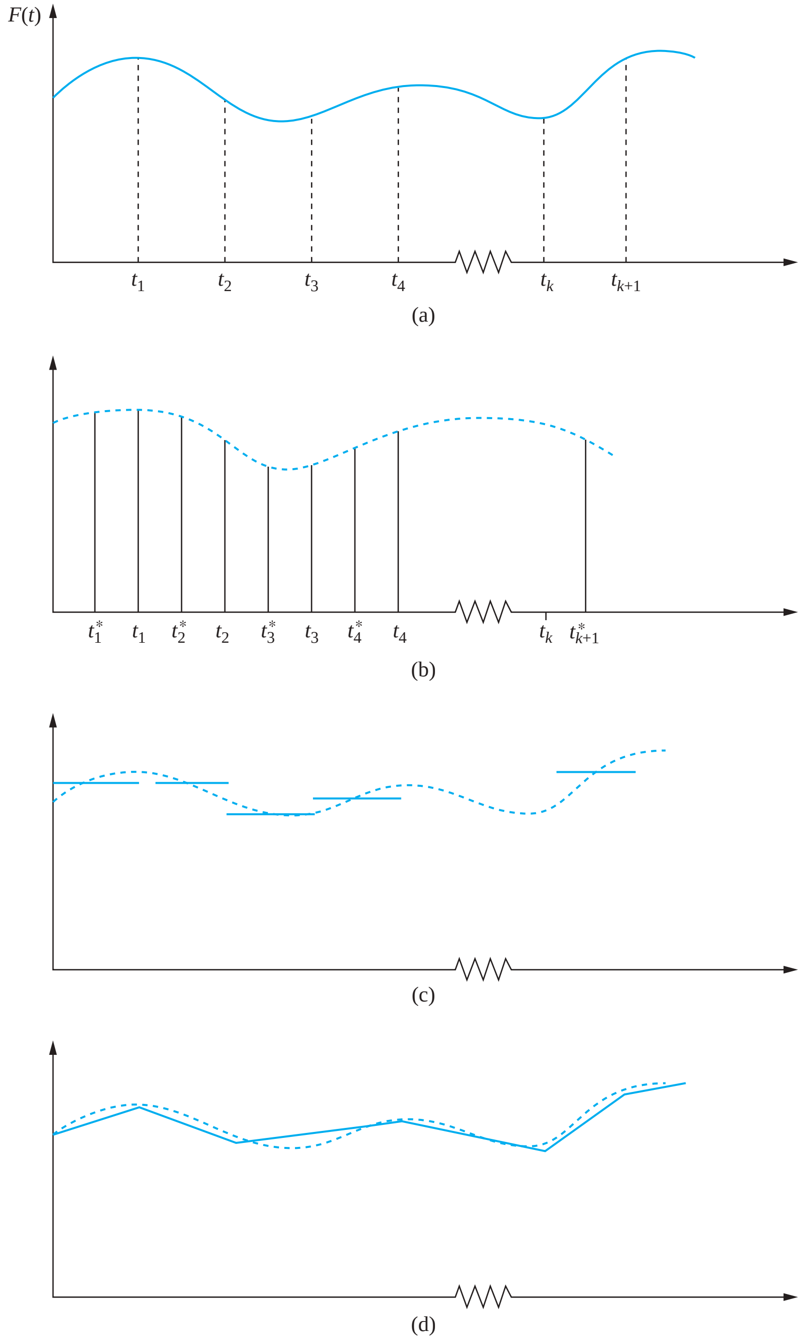


FIGURE 5.10
(a) Discretization of time for numerical integration of convolution integral. (b) Interpolation of $F(t)$ by a series of impulses. (c) Interpolation of $F(t)$ by piecewise constants. (d) Piecewise linear interpolation for $F(t)$.

The trigonometric identity for the sine of the difference of angles is used to rewrite Equation (5.69) as

$$x(t) = e^{-\zeta\omega_n t}\left[x(0)\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\sin\omega_d t\right] + \frac{1}{m_{eq}\omega_d}e^{-\zeta\omega_n t}\left[\sin\omega_d t\int_0^t F_{eq}(\tau)e^{\zeta\omega_n\tau}\cos\omega_d\tau - \cos\omega_d t\int_0^t F_{eq}(\tau)e^{\zeta\omega_n\tau}\cos\omega_d\tau\,d\tau\right] \quad (5.70)$$

Define

$$G_{1j} = \int_{t_{j-1}}^{t_j} F_{eq}(\tau)e^{\zeta\omega_n\tau}\cos\omega_d\tau\,d\tau \quad (5.71)$$

and

$$G_{2j} = \int_{t_{j-1}}^{t_j} F_{eq}(\tau)e^{\zeta\omega_n\tau}\cos\omega_d\tau\,d\tau \quad (5.72)$$

Using the definitions in Equations (5.71) and (5.72) in Equation (5.69) leads to

$$x_k = e^{-\zeta\omega_n t_k}\left[x(0)\cos\omega_d t_k + \frac{\zeta\omega_n x(0) + \dot{x}(0)}{\omega_d}\sin\omega_d t_k\right] + \frac{1}{m_{eq}\omega_d}e^{-\zeta\omega_n t_k}\left[\sin\omega_d t_k\sum_{j=1}^{k} G_{1j} - \cos\omega_d t_k\sum_{j=1}^{k} G_{2j}\right] \quad (5.73)$$

Interpolating functions are chosen for $F_{eq}(t)$ such that Equations (5.71) and (5.72) have closed-form evaluations when the interpolating function is used in place of $F_{eq}(t)$. Then Equation (5.73) is used to calculate approximations to the solution at discrete times.

First, consider the case where $F_{eq}(t)$ is interpolated by a series of impulses, as illustrated in Figure 5.10(b). During the interval between t_{j-1} and t_j, application of $F_{eq}(t)$ results in an impulse of magnitude

$$I_j = \int_{t_{j-1}}^{t_j} F_{eq}(\tau)\,d\tau \quad (5.74)$$

The mean value theorem of integral calculus implies that there exists a t_j^*, $t_{j-1} < t_j^* < t_j$, such that

$$I_j = F_{eq}(t_j^*)\Delta_j \quad (5.75)$$

For the sake of interpolation, approximate t_j^* by

$$t_j^* \approx \frac{t_j + t_{j-1}}{2} \quad (5.76)$$

Thus, on the interval $t_{j-1} < t < t_j$, $F(t)$ is interpolated by an impulse of magnitude I_j applied at the midpoint of the interval. With this choice of interpolation, Equations (5.71) and (5.72) are evaluated as

$$G_{1j} = F_{eq}(t_j^*)\Delta_j e^{\zeta\omega_n t_j^*}\cos\omega_d t_j^* \quad (5.77)$$

$$G_{2j} = F_{eq}(t_j^*)\Delta_j e^{\zeta\omega_n t_j^*}\sin\omega_d t_j^* \quad (5.78)$$

It is also possible to interpolate $F_{eq}(t)$ with piecewise constants. Over the interval from t_{j-1} to t_j, the interpolate for $F_{eq}(t)$ assumes the value of $F_{eq}(t)$ at the interval's midpoint, as illustrated in Figure 5.10(c). Call the value of the interpolate f_j. Then

$$G_{1j} = f_j C_j \tag{5.79}$$

$$G_{2j} = f_j D_j \tag{5.80}$$

where
$$C_j = \frac{1-\zeta^2}{\omega_d}\left[e^{\zeta\omega_n t_j}\left(\sin\omega_d t_j + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_j\right) - e^{\zeta\omega_n t_{j-1}}\left(\sin\omega_d t_{j-1} + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_{j-1}\right)\right] \tag{5.81}$$

$$D_j = \frac{1-\zeta^2}{\omega_d}\left[e^{\zeta\omega_n t_j}\left(-\cos\omega_d t_j + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_j\right) - e^{\zeta\omega_n t_{j-1}}\left(-\cos\omega_d t_{j-1} + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_{j-1}\right)\right] \tag{5.82}$$

Finally, consider the case where $F_{eq}(t)$ is interpolated linearly between t_{j-1} and t_j, as illustrated in Figure 5.10(d). Then if $g_j = f(t_j)$,

$$G_{1j} = \frac{1}{\Delta_j}\left[(g_j - g_{j-1})A_j + (g_{j-1}t_j - g_j t_{j-1})C_j\right] \tag{5.83}$$

$$G_{2j} = \frac{1}{\Delta_j}\left[(g_j - g_{j-1})B_j + (g_{j-1}t_j - g_j t_{j-1})D_j\right] \tag{5.84}$$

where C_j and D_j are given by Equations (5.81) and (5.82), respectively, and

$$A_j = \frac{1-\zeta^2}{\omega_d}\left[t_j e^{\zeta\omega_n t_j}\left(\sin\omega_d t_j + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_j\right) - t_{j-1}e^{\zeta\omega_n t_{j-1}}\left(\sin\omega_d t_{j-1} + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_{j-1}\right) - \left(D_j + \frac{\zeta\omega_n}{\omega_d}C_j\right)\right] \tag{5.85}$$

$$B_j = \frac{1-\zeta^2}{\omega_d}\left[t_j e^{\zeta\omega_n t_j}\left(\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_j - \cos\omega_d t_j\right) - t_{j-1}e^{\zeta\omega_n t_{j-1}}\left(\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_{j-1} - \cos\omega_d t_{j-1}\right) + \left(C_j - \frac{\zeta\omega_n}{\omega_d}D_j\right)\right] \tag{5.86}$$

Other choices for interpolating functions for $F_{eq}(t)$ are possible. Higher-order piecewise polynomials may be used, as well as interpolates which require more smoothness at each t_j, such as splines. Any form of interpolating function can be chosen as long as Equations (5.71) and (5.72) have closed-form evaluations. However, the more complicated the interpolating function, the more algebra is involved in the evaluation of G_{1j} and G_{2j}. The numerical evaluation of the convolution integral also requires more computations for more complicated interpolating functions.

If $F_{eq}(t)$ is known empirically, any of the methods presented may be used to evaluate the convolution integral. If piecewise impulses or piecewise constants are used, the times where

$F_{eq}(t)$ is known are taken as midpoints of the intervals. If piecewise linear interpolates are used, the times where $F_{eq}(t)$ is known are taken as the t_j's.

Error analysis of the preceding methods is beyond the scope of this text. Better accuracy for the response is, of course, obtained with better accuracy of the interpolate. Error analysis usually involves comparing the interpolation with a Taylor series expansion to estimate the error in the interpolation. The error is usually expressed as being the order of some power of Δ_j. Bounds on the error in using the convolution integral are obtained. Integration tends to smooth errors.

Determination of the response using these methods requires evaluation of the convolution integral at discrete values of time. Since errors are introduced in the evaluation of G_{1j} and G_{2j}, the more of these terms used in the evaluation, the larger is the error. Hence the error in approximation grows with increasing t. Reduction of error can be achieved by using smaller time intervals, if possible, or by using more accurate interpolates.

5.8.2 NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

An alternative to numerical evaluation of the convolution integral is to approximate the solution of Equation (5.1) by direct numerical integration. Many methods are available for numerical solution of ordinary differential equations.

Since vibrations of discrete systems are governed by initial value problems, it is best to use a numerical method that is self-starting. That is, previous knowledge of the solution at only one time is required to start the procedure.

Best application of self-starting methods required the rewriting of an nth-order differential equation as n first-order differential equations. This is done for Equation (5.1) by defining

$$y_1(t) = x(t) \tag{5.87a}$$

$$y_2(t) = \dot{x}(t) \tag{5.87b}$$

Thus,

$$\dot{y}_1(t) = y_2(t) \tag{5.88a}$$

and from Equation (5.1)

$$\dot{y}_2(t) = \frac{F_{eq}}{m_{eq}} - 2\zeta\omega_n y_2(t) - \omega_n^2 y_1(t) \tag{5.88b}$$

Equations (5.88a) and (5.88b) are two simultaneous linear first-order ordinary differential equations whose numerical solution yields the values of displacement and velocity at discrete times.

In the following let t_i, $i = 1, 2, \ldots$, be the discrete times at which the solution is obtained and let $y_{1,i}$ and $y_{2,i}$ be the displacements and velocities at these times and define

$$\Delta_j = t_{j+1} - t_j \tag{5.89}$$

The recurrence relations for the simplest self-starting method, called the Euler method, are obtained from truncating Taylor series expansions for $y_{k,i+1}$ about $y_{k,i}$ after the linear terms. These recurrence relations are

$$y_{1,i+1} = y_{1,i} + (t_{i+1} - t_i)y_{2,i} \tag{5.90a}$$

$$y_{2,i+1} = y_{2,i} + (t_{i+1} - t_i)\left[\frac{F_{eq}(t_i)}{m_{eq}} - 2\zeta\omega_n y_{2,i} - \omega_n^2 y_{1,i}\right] \tag{5.90b}$$

Given initial values of y_1 and y_2, Equations (5.90a) and (5.90b) are used to calculate recursively the displacement and velocity at increasing times. The Euler method is first-order accurate meaning that the error is of the order of Δ_j.

Runge-Kutta methods are more popular than the Euler method because of their better accuracy, while still being easy to use. A Runge-Kutta formula for the solution of the first-order differential equation

$$\dot{y} = f(y, t) \tag{5.91}$$

is of the form

$$y_{i+1} = y_i + \sum_{j=1}^{n} a_j k_j \tag{5.92}$$

where

$$\begin{aligned}
k_1 &= (t_{i+1} - t_i) f(y_i, t_i) \\
k_2 &= (t_{i+1} - t_i) f(y_1 + q_{1,1} k_1, t_i + p_1) \\
k_3 &= (t_{i+1} - t_i) f(y_i + q_{2,1} k_1 + q_{2,2} k_2, t_i + p_2) \\
&\vdots \\
k_n &= (t_{i+1} - t_i) f(y_i + q_{n-1,1} k_1 + q_{n-2,2} k_2 + \cdots \\
&\qquad + q_{n-1,n-1} k_{n-1}, t_i + p_{n-1})
\end{aligned} \tag{5.93}$$

and the a, q, and p coefficients are chosen by using Taylor series expansions to approximate the differential equation to the desired accuracy.

The error for a fourth-order Runge-Kutta formula is proportional to Δ_j^4. A fourth-order Runge-Kutta formula is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.94}$$

where

$$\begin{aligned}
k_1 &= (t_{i+1} - t_i) f(y_i, t_i) \\
k_2 &= (t_{i+1} - t_i) f\left(y_i + \frac{1}{2} k_1, \frac{1}{2}(t_i + t_{i+1}) \right) \\
k_3 &= (t_{i+1} - t_i) f\left(y_i + \frac{1}{2} k_2, \frac{1}{2}(t_i + t_{i+1}) \right) \\
k_4 &= (t_{i+1} - t_i) f(y_i + k_3, t_{i+1})
\end{aligned} \tag{5.95}$$

Equation (5.94) can be used for higher-order differential equations by rewriting it as a system of first-order equations as has been done in Equation (5.90) for a SDOF system. The result is

$$y_{1,i+1} = y_{1,i} + \frac{1}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + 2k_{1,4}) \tag{5.96a}$$

$$y_{2,i+1} = y_{2,i} + \frac{1}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + 2k_{2,4}) \tag{5.96b}$$

where

$$k_{1,1} = (t_{i+1} - t_i) y_{2,i} \tag{5.97a}$$

$$k_{1,2} = (t_{i+1} - t_i)\left(y_{2,i} + \frac{1}{2} k_{2,1} \right) \tag{5.97b}$$

$$k_{1,3} = (t_{i+1} - t_i)\left(y_{2,i} + \frac{1}{2}k_{2,2}\right) \tag{5.97c}$$

$$k_{1,4} = (t_{i+1} - t_i)(y_{2,i} + k_{2,3}) \tag{5.97d}$$

$$k_{2,1} = (t_{i+1} - t_i)\left[\frac{F_{\text{eq}}(t_i)}{m_{\text{eq}}} - 2\zeta\omega_n y_{2,i} - \omega_n^2 y_{1,i}\right] \tag{5.97e}$$

$$k_{2,2} = (t_{i+1} - t_i)\left[\frac{F_{\text{eq}}\left(\frac{1}{2}(t_i + t_{i+1})\right)}{m_{\text{eq}}} - 2\zeta\omega_n\left(y_{2,i} + \frac{1}{2}k_{2,1}\right) - \omega_n^2\left(y_{1,i} + \frac{1}{2}k_{1,1}\right)\right] \tag{5.97f}$$

$$k_{2,3} = (t_{i+1} - t_i)\left[\frac{F_{\text{eq}}\left(\frac{1}{2}(t_i + t_{i+1})\right)}{m_{\text{eq}}} - 2\zeta\omega_n\left(y_{2,i} + \frac{1}{2}k_{2,2}\right) - \omega_n^2\left(y_{1,i} + \frac{1}{2}k_{1,2}\right)\right] \tag{5.97g}$$

$$k_{2,4} = (t_{i+1} - t_i)\left[\frac{F_{\text{eq}}(t_{i+1})}{m_{\text{eq}}} - 2\zeta\omega_n(y_{2,i} + k_{2,3}) - \omega_n^2(y_{1,i} + k_{1,3})\right] \tag{5.97h}$$

The Runge-Kutta method is often used because it is easy to program for a digital computer. Its most restrictive limitation is that extension of the approximation between two discrete times requires evaluation of the excitation at an intermediate time. If the forcing function is known only at discrete times, evaluation at the appropriate intermediate times is often impossible. In addition, a large number of function evaluations can lead to large computer times.

Adams' formulas provide more accurate approximations of ordinary differential equations. An open Adams formula requires knowledge of the functions at the two previous time steps to calculate the approximation at the desired time. A closed Adams formula requires knowledge of the function at only the previous time step, but the formula involves the evaluation of the function at the time step of interest. Thus a closed Adams formula requires an iterative solution at each time step. The closed Adams formula is much more accurate than an open formula of the same order. The closed formula is self-starting, whereas the open formula is not self-starting.

A predictor-corrector method is a compromise that uses the closed formula for increased accuracy, but uses the open formula to reduce computation time. The open formula is used to "predict" the solution at the desired time, then the closed formula is used to "correct" by using the predicted value as an initial guess. Iterations are not necessary as the first correction is very accurate. Since the open Adams formulas are not self-starting, a self-starting method such as the Runge-Kutta method of the same order is used to calculate the solution at the first time. The predictor-corrector method is used for the remainder of the calculations.

EXAMPLE 5.14

A 200-kg milling machine is subject to the versed sine pulse of Figure 5.11 during operation. The machine is mounted on an elastic foundation of stiffness 1×10^6 N/m and damping ratio of 0.2. Write a MATLAB script that uses piecewise constants as interpolating functions to numerically integrate the convolution integral to obtain the response of the machine up to $t = 0.5$ s.

SOLUTION

The MATLAB script and the resulting plot of displacement are illustrated in Figure 5.12. The MATLAB script is written in a general form. When the script is run by MATLAB, the user will be prompted for input. The form of the excitation is provided in a separate MATLAB m file.

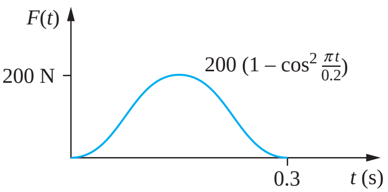


FIGURE 5.11
Versed sine pulse of Examples 5.14 and 5.15.

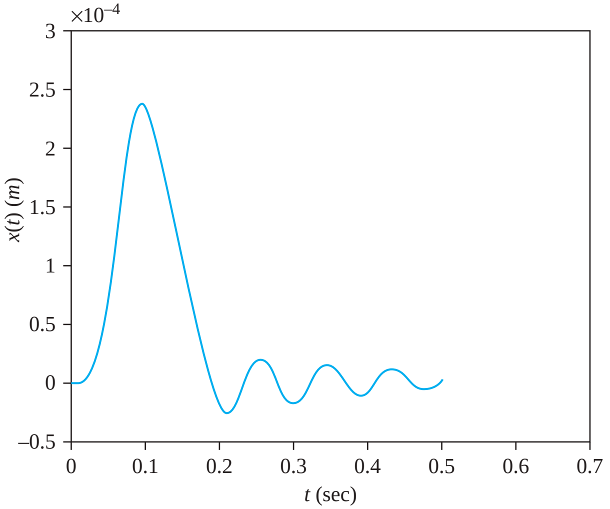
```
% Example 5.14
% Numerical integration of convolution integral using
% piecewise constants to interpolate excitation
m=200;                            % Mass of system
k=l.*10^6;                        % Stiffness
zeta=0.06;                        % Damping ratio
omega_n=sgrt (k/m);               % Natural frequency
omega_d=omega_n*sqrt (l-zeta^2);  % Damped natural frequency
F0=200;                           % Magnitude of pulse
t0=0.2;                           % Duration of pulse
x0=0;                             % Initial displacement
xdot0=0;                          % Initial velocity
t=linspace(0, .5, 1001);          % Discretization of time scale
suml=0;                           % Initialization of sum for Gl
sum2=0;                           % Initialization of sum for G2
x(l)=x0;                          % Initialization of x
Cl=(l-zeta^2)/omega_d;
C2=zeta*omega_n;
C3=C2/omega_d;
for k=2: 1001
   % Calculating F(t)
   if t(k) < = t0
      F=F0*(l-(cos(pi*t(k)/t0)^2)); % F(t)
   else
      F=0
   end
```

(Continued)

FIGURE 5.12
(a) MATLAB script for Example 5.14, numerical integration of convolution integral using piecewise constants for interpolation of excitation force. (b) Plot of displacement versus time obtained by running the script.

```
    % Numerical integration formula Eqs. (5.79) – (5.82)
    EK=exp (C2*t(k));
    EK1=exp(C2*t(k-1));
    SK=sin(omega_d*t(k));
    SK1=sin(omega_d*t(k-1));
    CK=cos(omega_d*t(k));
    CK1=cos(omega_d*t(k-1));
    G1=F*C1*(EK*(SK+C3*CK)-EK1*(SK1+C3*CK1));
    G2=F*C1*(EK*(-CK+C3*SK)–EK1*(-CK1+C3*SK1));
    sum1=sum1+G1;
    sum2=sum2+G2;
    % Eq.(5.73)
    xK=(x0*CK+(C2*x0+xdot0)/omega_d/*SK)/EK;
    x(k)=xK+(SK*sum1-CK*sum2)/(EK*m*omega_d);
end
plot(t,x)
xlabel('t (sec)')
ylabel('x(t)(m)')
```

(a)



(b)

FIGURE 5.12
(Continued)

EXAMPLE 5.15

Write a MATLAB script using the program ODE45 to determine the response of the system of Example 5.14.

SOLUTION

The MATLAB script for the development of the response is given in Figure 5.13(a). The script uses the MATLAB function ODE45, which uses a Runge-Kutta-Fehlberg method to numerically approximate the response.

The resulting response generated from MATLAB is shown in Figure 5.13(b). The response is very close to that generated in Example 5.14 by numerical integration of the convolution integral.

```
% Runge-Kutta solution to Example 5.15 using
MATLAB program ODE45
% Initial conditions
x0=0;
xdot0=0;
% y(1)=x;  y(2)=xdot
y0(1)=x0;
y0(2)=xdot0;
y0=[y0(1);y0(2)];
TSPAN=[0 0.5];
[T,Y]=ode45('fun412',TSPAN,y0);
plot(T,Y(:,1))
xlabel('time (s)')
ylabel('x(t) (m)')
```

(a)

```
% Defining file for function of Example 5.15
function F=fun412 (T,Y)
m=200;                          % Mass of system
k=1.*10^6;                      % Stiffness
zeta=0.06;                      % Damping ratio
omega_n=sqrt(k/m);              % Natural frequency
F0=200;                         % Magnitude of pulse
t0=0.2;                         % Duration of pulse
F(1)=Y(2);
% Calculating F(T)
if T<=t0
   f1=F0/m*(1-(cos(pi*T/t0))^2);
else
   f1=0;
end
% xdot=F(1), xddot=F(2)
F(2)-2*zeta*Y(2)-omega_n^2*Y(1)+f1;
F=[F(1); F(2)];
```

(b)

FIGURE 5.13
(a) MATLAB script for solving differential equation for Example 5.15 using ODE45, a Runge-Kutta solution. (b) User provided function for Example 5.15.

5.9 SHOCK SPECTRUM

Design problems often require the determination of system parameters such that constraints are satisfied. In many problems, the design criteria involve limiting maximum displacements and/or maximum stresses for a given type of excitation. For example, if it is determined that all earthquakes in a given area have similar forms of excitations, only with different levels of severity, then knowledge of the maximum displacement as a function of system parameters is useful in the design of a structure to withstand a certain level of earthquake. The structure's ability to withstand the earthquake depends on the maximum displacement developed in the structure during the earthquake and the maximum stresses developed. A structure in California along the San Andreas fault will usually be designed to withstand a more severe earthquake than a structure in Ohio. This, of course, depends on the use of the structure.

Thus it is useful for the designer to know the maximum response of a structure as a function of system parameters. The transmissibility curves presented in Chapter 4 actually do this for the steady-state response due to harmonic excitations. For a given value of the damping ratio, the transmissibility curve plots the nondimensional ratio of the amplitude of the transmitted force to the maximum amplitude of the excitation force against the nondimensional frequency ratio.

Similar curves are useful for analysis and design of systems that are subject to shock excitations. A shock is a large force applied over a short interval resulting in transient vibration. The maximum response is a function of the type of shock and system parameters.

A *shock spectrum* (*response spectrum*) is a nondimensional plot of the maximum response of a SDOF system for a specified excitation as a function of a nondimensional time ratio. The vertical axis of the plot is the maximum value of the force developed in the spring divided by the maximum of the excitation force. The horizontal axis is the ratio of a characteristic time for the excitation divided by the natural period. For a shock excitation, the characteristic time is usually taken as the duration of the shock.

Shock spectra are often plotted only for undamped systems as damping tends to act favorably to reduce the maximum response. Also, a shock spectrum is very tedious to calculate and plot. Inclusion of damping in the development of a shock spectrum greatly increases the amount of algebra performed. The resulting complexity may obscure the usefulness of the results.

EXAMPLE 5.16

A one-story frame structure is to be built to house a chemical laboratory. The experiments performed in the laboratory involve highly volatile chemicals and the possibility of explosion is great. It is estimated that the worst explosion will generate a force of 5×10^6 N lasting 0.5 s. The structure is to be designed such that the maximum displacement due to such an explosion is 10 mm. The equivalent mass of the structure is 500,000 kg. Draw the shock spectrum for the structure subject to a rectangular pulse and determine the minimum allowable stiffness for the structure.

SOLUTION

The laboratory frame structure of Figure 5.14 is modeled as an undamped SDOF system with $x(t)$ representing the displacement at the rigid girder. The excitation is modeled as a rectangular pulse of magnitude $F_0 = 5 \times 10^6$ N and duration $t_0 = 0.5$ s. The response of an undamped SDOF system to a rectangular pulse with zero initial conditions is calculated using superposition and Table 5.1 as

$$x(t) = \frac{F_0}{k}\{1 - \cos\omega_n t - u(t - t_0)[1 - \cos\omega_n(t - t_0)]\} \quad \text{(a)}$$

For $t < t_0$, the nondimensional force ratio is

$$\frac{kx}{F_0} = 1 - \cos\omega_n t \quad \text{(b)}$$

The preceding function increases until $t = \pi/\omega_n$ when it reaches a maximum value of 2. If $t_0 < \pi/\omega_n$, the maximum nondimensional force ratio in this interval is

$$\frac{kx_{\max}}{F_0} = 1 - \cos\omega_n t_0 \quad \text{(c)}$$

However, since the response is continuous, the maximum response for $t > \pi/\omega_n$ must be at least this large. For $t > t_0$, the nondimensional force ratio is

$$\frac{kx}{F_0} = \cos\omega_n(t - t_0) - \cos\omega_n t \quad \text{(d)}$$

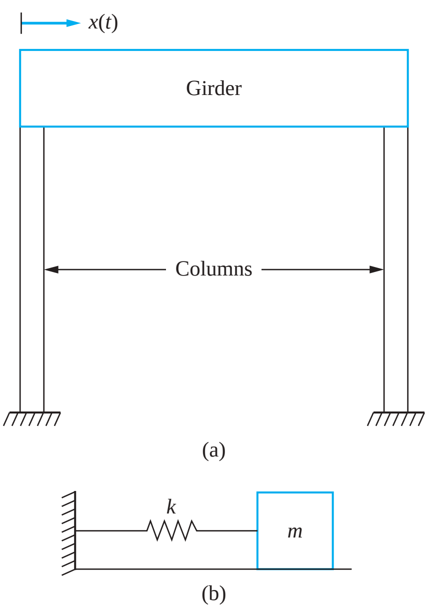


FIGURE 5.14
(a) The one-story chemical laboratory of Example 5.16 is modeled as a frame structure. (b) The frame structure is modeled as a SDOF mass-spring system, assuming the girder is very rigid compared to the columns.

Trigonometric identities are used on the above equation to obtain

$$\frac{kx}{F_0} = 2\sin\frac{\omega_n t_0}{2}\sin(\omega_n t - \alpha) \tag{e}$$

where

$$\tan\alpha = \frac{\cos\omega_n t_0 - 1}{\sin\omega_n t_0} \tag{f}$$

Thus,

$$\frac{kx_{\max}}{F_0} = 2\sin\frac{\omega_n t_0}{2} \qquad t_0 < \frac{\pi}{\omega_n} \tag{g}$$

In summary,

$$\frac{kx_{\max}}{F_0} = \begin{cases} 2\sin\dfrac{\omega_n t_0}{2} & t_0 < \dfrac{\pi}{\omega_n} \quad \left(\dfrac{t_0}{\tau} \leq \dfrac{1}{2}\right) \\ 2 & t_0 > \dfrac{\pi}{\omega_n} \quad \left(\dfrac{t_0}{\tau} > \dfrac{1}{2}\right) \end{cases} \tag{h}$$

The shock spectrum is plotted in Figure 5.15.

Returning to the specific problem, $t_0 = 0.5$ s, $F_0 = 5 \times 10^6$ N, $x_{\max} = 0.01$ m, and $m = 500{,}000$ kg. The natural frequency is $\omega_n = \sqrt{k/m}$, and the problem is to determine appropriate values of k. The natural period is $T = 2\pi/\omega_n$. First assume $t_0/T < 0.5$, which is equivalent to

$$\frac{\omega_n t_0}{2\pi} < 0.5 \tag{i}$$

or

$$\omega_n < 2\pi \text{ rad/s} \tag{j}$$

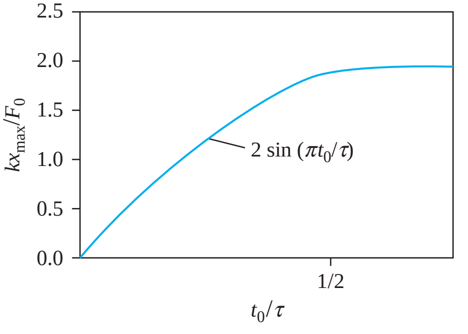


FIGURE 5.15 Shock spectrum of an undamped SDOF system for a rectangular pulse.

The equation to solve for k is

$$\frac{k(0.01\text{ m})}{5 \times 10^6\text{ N}} = 2\sin\left(\sqrt{\frac{k}{500{,}000\text{ kg}}}\frac{0.5\text{ s}}{2}\right) \quad \text{(k)}$$

Equation (k) becomes

$$1 \times 10^{-8}k = \sin\left(3.54 \times 10^{-4}\sqrt{k}\right) \quad \text{(l)}$$

Equation (l) is a transcendental equation to solve for k with the smallest positive solution being $k = 5.33 \times 10^7$ N/m. The natural frequency with this value of k is

$$\omega_n = \sqrt{\frac{5.33 \times 10^7\text{ N/m}}{500{,}000\text{ kg}}} = 10.32\text{ rad/s} > 2\pi\text{ rad/s} \quad \text{(m)}$$

Thus, there is no solution for $\frac{\omega_n t_0}{2\pi} < 0.5$. Hence, $\omega_n > 2\pi$ rad/s and $\frac{kx_{\max}}{F_0} = 2$, which leads to

$$\frac{k(0.01\text{ m})}{5 \times 10^6\text{ N}} = 2 \quad \text{(n)}$$

which is solved yielding

$$k = 1 \times 10^9\text{ N/m} \quad \text{(o)}$$

If $k > 1 \times 10^9$ N/m, the maximum displacement will be less than 0.01 m.

The important question in Example 5.16 is whether the duration of the pulse is long enough so that the maximum response occurs when the excitation is occurring. If the pulse is too short, the maximum displacement occurs after the pulse is removed. The rectangular pulse is the simplest pulse for analysis of the response of a SDOF system. Its response spectrum is also the simplest to draw.

Shock spectra are often calculated only for undamped systems. Algebraic complexity usually prevents analytical determination of shock spectra for damped systems. The maximum response is obtained either by numerical evaluation of the exact expression for the displacement or by numerical solution of the differential equation. Damping does not have as much effect on the transient response due to a pulse of longer duration as it does on the steady-state response due to a harmonic excitation or on the response due to a short-duration pulse.

Since shock isolation often involves minimizing the force transmitted between a system and its support, a plot similar to the shock spectrum, but involving the maximum value of the transmitted force, is useful. The vertical coordinate of the force spectrum is the ratio of the maximum value of the transmitted force to the maximum value of the excitation force. When the system is undamped, the force spectrum is the same as the shock spectrum.

Figures 5.16 through 5.21 present displacement spectra and force (acceleration) spectra for common pulse shapes. These spectra were obtained by using a Runge-Kutta solution of the governing differential equation. A system with $\omega_n = 1$ and $m = 1$ was arbitrarily used. A time increment of the smaller of $t_0/50$ and $T/50$ was used. The Runge-Kutta solution was carried out until the larger of $4t_0$ or $4T$. The displacement and transmitted force were calculated at each time step and compared to maxima from the previous times. The spectra were developed for several values of ζ.

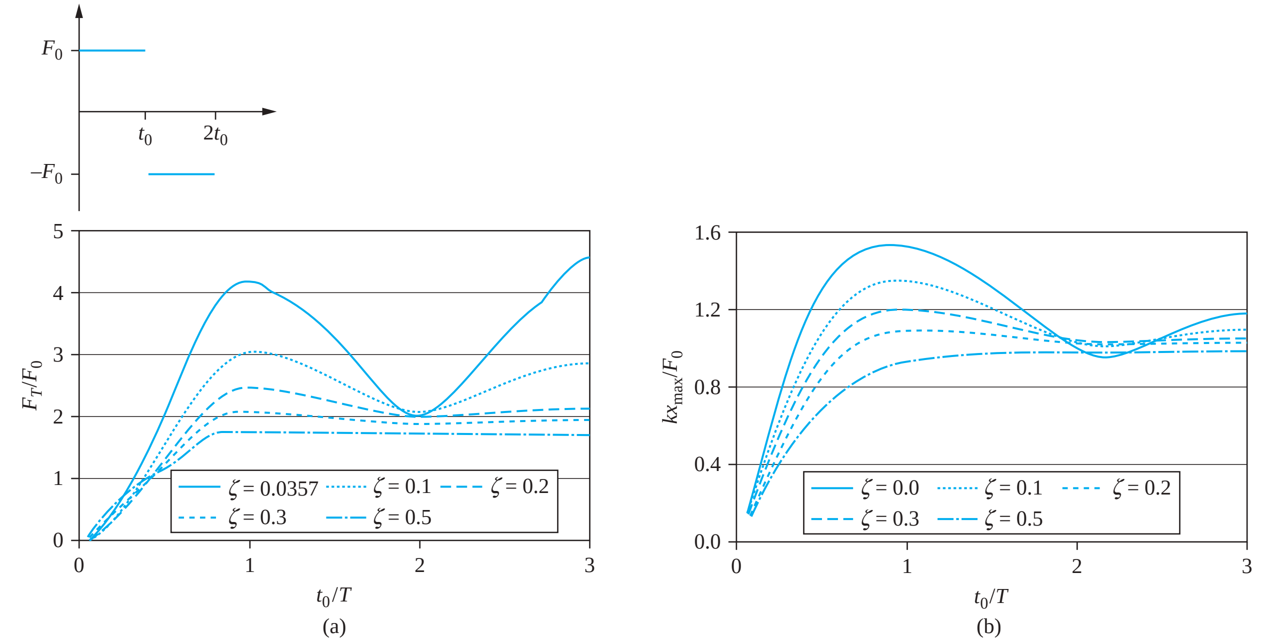


FIGURE 5.16
(a) Force spectrum for a triangular pulse. (b) Response spectrum for a triangular pulse.

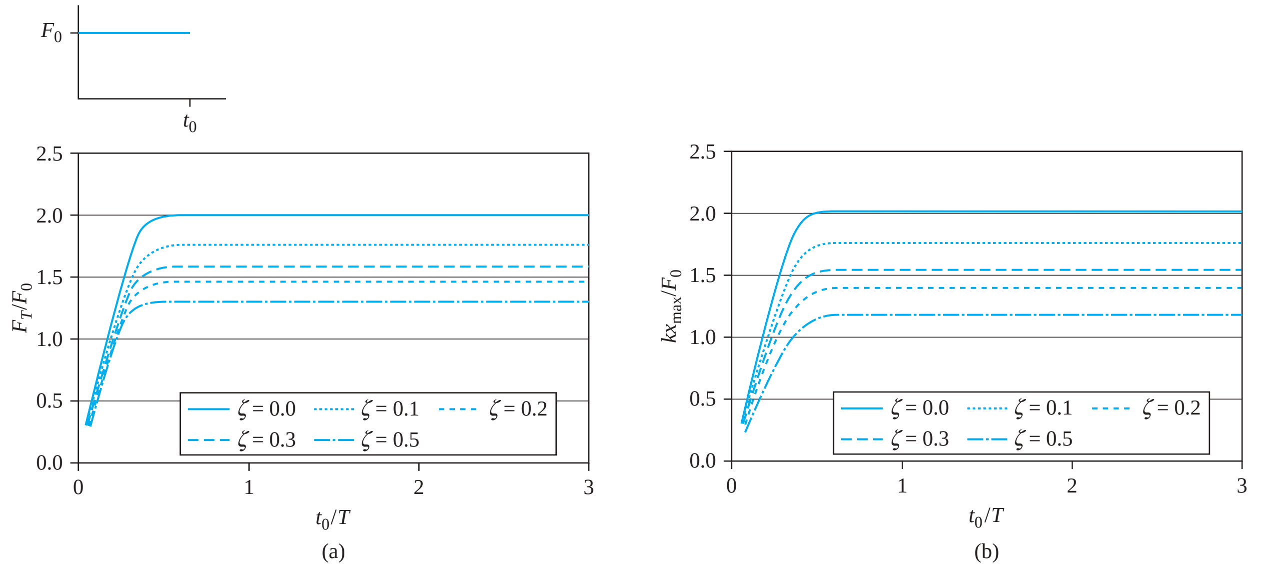


FIGURE 5.17
(a) Force spectrum for a rectangular pulse. (b) Response spectrum for a rectangular pulse.

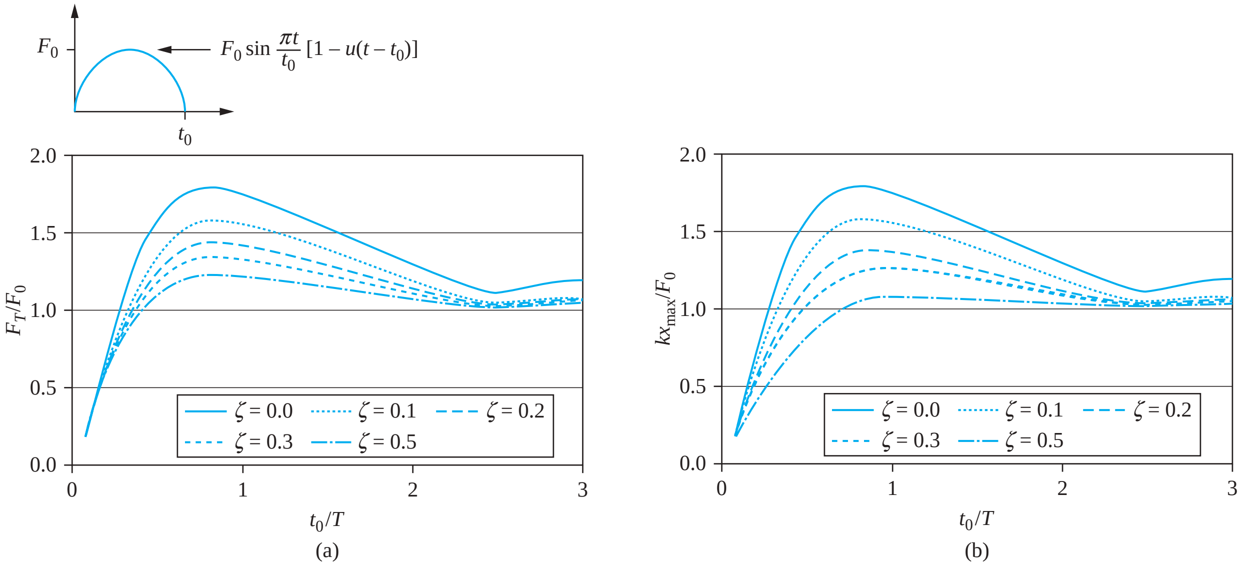


FIGURE 5.18
(a) Force spectrum for a sinusoidal pulse. (b) Response spectrum for a sinusoidal pulse.

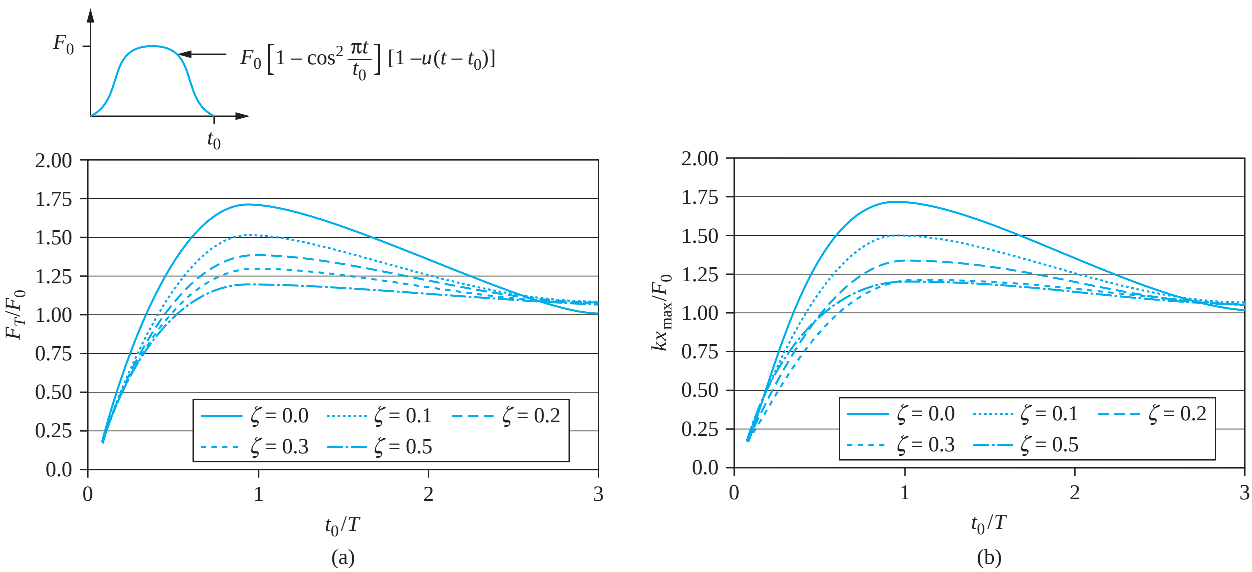


FIGURE 5.19
(a) Force spectrum for a versed sine pulse. (b) Response spectrum for a versed sine pulse.

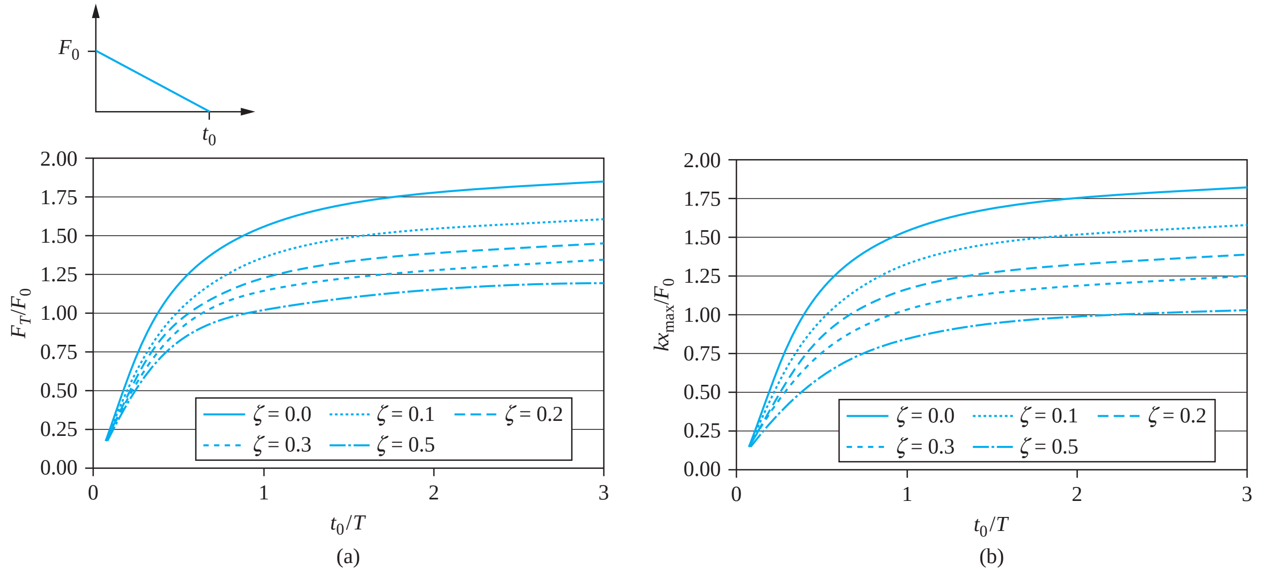


FIGURE 5.20
(a) Force spectrum for a negative slope pulse. (b) Response spectrum for a negative slope pulse.

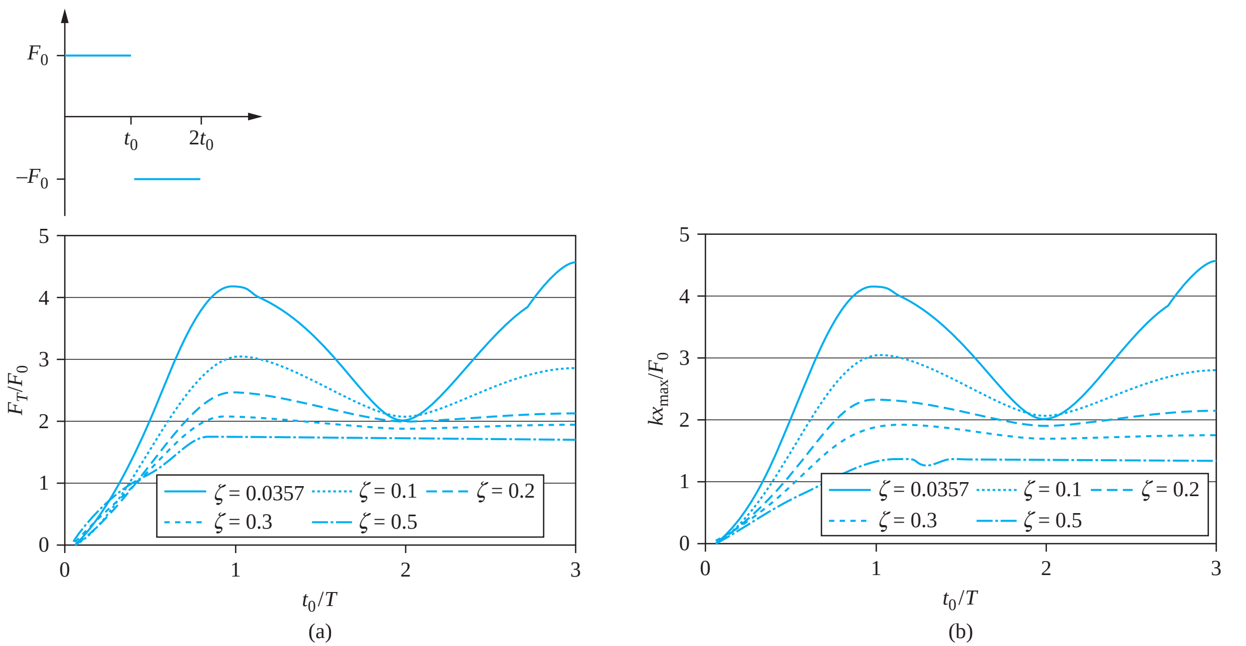


FIGURE 5.21
(a) Force spectrum for a reversed loading pulse. (b) Response spectrum for a reversed loading pulse.

The force spectra for the rectangular pulse, the triangular pulse, the sinusoidal pulse, the versed sine pulse, the negative-slope ramp pulse, and the reversed loading pulse show that shock isolation is achieved only for small natural frequencies. The shock spectra for these excitations show that the nondimensional displacement is small for small natural frequencies. However, the dimensional displacement is calculated by using the nondimensional displacement from

$$x_{\max} = \frac{F_0}{m\omega_n^2}\left(\frac{m\omega_n^2 x_{\max}}{F_0}\right) \tag{5.98}$$

Thus, a small natural frequency leads to a large displacement.

EXAMPLE 5.17

A 1000-kg machine is subject to a triangular pulse of duration 0.05 s and peak of 20,000 N. What is the range of isolator stiffness for an undamped isolator such that the maximum transmitted force is less than 8000 N and the maximum displacement is less than 2.8 cm?

SOLUTION

The force spectrum for the triangular pulse shows that for $F_T/F_0 < 0.4$, $\omega_n t_0/(2\pi) < 0.16$, which gives

$$\omega_n < \frac{2\pi(0.16)}{0.05\ \text{s}} = 20.1\ \text{rad/s}$$

The lower bound on the natural frequency is obtained by trial and error, using the displacement spectrum for the triangular pulse. For a guessed value of ω_n, $\omega_n t_0/(2\pi)$ is calculated and the corresponding value of the maximum nondimensional displacement is found from the displacement spectrum. The maximum dimensional displacement is calculated from Equation (5.98). If the displacement is greater than the allowable displacement, the guess for the lower bound must be increased. The calculations for this example are given in Table 5.2. The lower bound is calculated as 17 rad/s. Thus the allowable stiffness range is

$$2.89 \times 10^5\ \text{N/m} < k < 4.04 \times 10^5\ \text{N/m}$$

TABLE 5.2

ω_n, rad/s	$\dfrac{\omega_n t_0}{2\pi}$	$\dfrac{m\omega_n^2 x_{\max}}{F_0}$	$x_{\max}$, cm
10	0.08	0.25	5.0
15	0.12	0.38	3.4
18	0.14	0.42	2.6
17	0.135	0.40	2.8

5.10 VIBRATION ISOLATION FOR SHORT DURATION PULSES

If the forge hammer of Figure 5.22 is rigidly mounted to the foundation, the foundation is subject to a large impulsive force when the hammer impacts the anvil. An isolation system modeled as a spring and viscous damper in parallel can be designed to reduce the

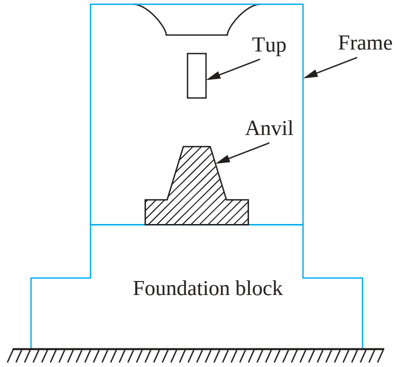


FIGURE 5.22
Schematic of a forge hammer. When the tup impacts the anvil, an impulsive force is developed.

magnitude of the force to which the foundation is subject. The principles used in the design of a shock isolation system are similar to the principles used to design an isolation system to protect against harmonic excitation, but the equations are different.

If the duration t_0 of a transient excitation $F(t)$ is small, say $t_0 < T/5$ where T is the natural period of the system, then the system response can be adequately approximated by the response due to an impulse of magnitude

$$I = \int_0^{t_0} F(t)\,dt \tag{5.99}$$

If the system is at rest in equilibrium when a pulse of short duration is applied, the principle of impulse-momentum is used to calculate the velocity imparted to the mass as

$$v = \frac{I}{m} \tag{5.100}$$

The impulse provides external energy to initiate vibrations. Time is measured beginning immediately after the excitation is removed. The ensuing response is the free-vibration response due to an impulse providing the mass with an initial velocity v.

$$x(t) = \frac{v}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{5.101}$$

The maximum displacement occurs at a time

$$t_m = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{5.102}$$

and is equal to

$$x_{\max} = \frac{v}{\omega_n} \exp\left[-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \tag{5.103}$$

Equation 5.101 and trigonometric identities are used to calculate the force transmitted to the foundation through the isolator as

$$F_T(t) = \widetilde{F} e^{-\zeta\omega_n t} \sin(\omega_d t - \beta) \tag{5.104}$$

where

$$\tilde{F} = \frac{m\omega_n v}{\sqrt{1-\zeta^2}} \tag{5.105}$$

and

$$\beta = -\tan^{-1}\left(\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}\right) \tag{5.106}$$

The maximum value of the transmitted force is obtained by differentiating Equation (5.104) with respect to time, solving for the smallest time for which the derivative is zero, and finding the transmitted force at this time. The time for which the maximum transmitted force occurs is

$$t_{m_F} = \frac{1}{\omega_d}\tan^{-1}\left[\frac{\sqrt{1-\zeta^2}(1-4\zeta^2)}{\zeta(3-4\zeta^2)}\right] \tag{5.107}$$

The corresponding maximum transmitted force is

$$F_{T_{\max}} = mv\omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left[\frac{\sqrt{1-\zeta^2}(1-4\zeta^2)}{\zeta(3-4\zeta^2)}\right]\right) \tag{5.108}$$

Equation (5.107) shows that the maximum transmitted force occurs at $t = 0$ for $\zeta = 0.5$. For $\zeta > 0.5$, the first time where $dF/dt = 0$ corresponds to a minimum. Thus, for $\zeta \geq 0.5$, the maximum transmitted force occurs at $t = 0$ and is given by

$$F_T(0) = cv = 2\zeta m\omega_n v \tag{5.109}$$

Equations (5.108) and (5.109) are combined to develop a nondimensional function $Q(\zeta)$ that is a measure of the maximum transmitted force, which is defined by

$$Q(\zeta) = \frac{F_{T_{\max}}}{mv\omega_n} = \begin{cases} \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left[\frac{\sqrt{1-\zeta^2}(1-4\zeta^2)}{\zeta(3-4\zeta^2)}\right]\right) & \zeta < 0.5 \\ 2\zeta & 0.5 \leq \zeta < 1 \end{cases} \tag{5.110}$$

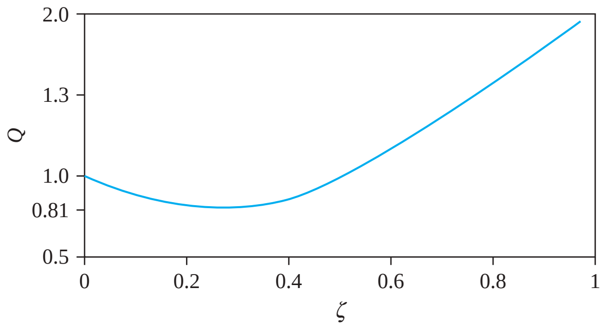
Figure 5.23 shows that $Q(\zeta)$ is flat and approximately equal to 0.81 for $0.23 < \zeta < 0.30$. If minimization of the transmitted force is the sole criterion for the isolator design, the isolator should have a damping ratio near 0.25.

Equation (5.110) shows that, for a given ζ, the transmitted force is proportional to the natural frequency. Thus a low natural frequency and large natural period is necessary and the short-duration assumption is often valid.

Equation (5.103) shows that the maximum displacement varies inversely with the natural frequency. Thus, requiring a small transmitted force leads to a large displacement. The natural frequency is eliminated between Equations (5.103) and (5.110), yielding

$$\frac{F_{T_{\max}} x_{\max}}{\frac{1}{2}mv^2} = S(\zeta) \tag{5.111}$$

FIGURE 5.23
$Q(\zeta)$ has a minimum of 0.81 for $\zeta \approx 0.25$.



where

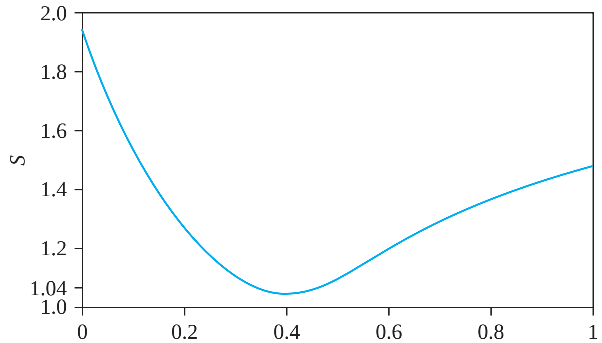
$$S(\zeta) = \begin{cases} 2\exp\left(-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left[\dfrac{\zeta\sqrt{1-\zeta^2}(4-8\zeta^2)}{8\zeta^2-8\zeta^4-1}\right]\right) & \zeta < 0.5 \\ 4\zeta\exp\left[-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] & 0.5 \le \zeta < 1 \end{cases} \tag{5.112}$$

The denominator of the nondimensional ratio of Equation (5.111) is the initial kinetic energy of the system. The numerator is a measure of the work done by the transmitted force. The inverse of this ratio is the fraction of energy absorbed by the isolator, the isolator efficiency. Figure 5.24 shows that the maximum isolator efficiency occurs for $\zeta = 0.40$ where $S = 1.04$.

If the idea of an isolator design is to set the maximum transmitted force to a given value while minimizing the maximum displacement, the damping ratio should be set at $\zeta = 0.4$, and the natural frequency should be calculated using $Q(\zeta)$ with $Q(0.4) = 0.886$. The maximum displacement is calculated from $S(\zeta)$. This maximizes the isolator efficiency.

In calculating $Q(\zeta)$ from Equation (5.111) and $S(\zeta)$ from Equation (5.112), the exponent must be negative. Therefore, the argument of the inverse tangent functions must be positive. That is, the range of evaluation of the inverse tangent functions must be between 0 and π rad. If evaluation leads to a negative argument, recall that the tangent function repeats every π rad, so simply add π rad to the evaluation.

FIGURE 5.24
$S(\zeta)$ has a minimum of 1.04 for $\zeta = 0.4$.



EXAMPLE 5.18

The 200 kg hammer of a 1000-kg forge hammer is dropped from a height of 1 m. Design an isolator to minimize the maximum displacement when the maximum force transmitted to the foundation is 20,000 N. What is the maximum displacement of the hammer when placed on this isolator?

SOLUTION

The excitation is a result of the impact of the hammer with the anvil and, thus, is of short duration. The velocity of the anvil at the time of impact is

$$v = \sqrt{2(9.81 \text{ m/s}^2)(1 \text{ m})} = 4.43 \text{ m/s}$$

The velocity of the machine after impact is determined by using the principle of impulse and momentum

$$v = \frac{(200 \text{ kg})(4.43 \text{ m/s})}{1000 \text{ kg}} = 0.886 \text{ m/s}$$

The product of the maximum transmitted force and the maximum displacement is minimized by selecting $\zeta = 0.4$. Then if the transmitted force is limited to 20,000 N, the maximum displacement is obtained by using Equation (5.111)

$$x_{\max} = \frac{\frac{1}{2}mv^2}{F_{T_{\max}}} S(0.4) = \frac{\frac{1}{2}(1000 \text{ kg})(0.886 \text{ m/s})^2}{20{,}000 \text{ N}} 1.04 = 0.02 \text{ m}$$

The natural frequency of the isolator is calculated by using Equation (5.110)

$$\omega_n = \frac{F_{T_{\max}}}{mvQ(0.4)} = \frac{20{,}000 \text{ N}}{(1000 \text{ kg})(0.886 \text{ m/s})(0.88)} = 25.65 \text{ rad/s}$$

and the maximum isolator stiffness is calculated as

$$k = m\omega_n^2 = (1000 \text{ kg})(25.65 \text{ rad/s})^2 = 6.58 \times 10^5 \text{ N/m}$$

5.11 BENCHMARK EXAMPLES

5.11.1 MACHINE ON FLOOR OF INDUSTRIAL PLANT

The machine is subject to a sinusoidal pulse with a magnitude of 20,000 lb and a duration of 0.1 s, as shown in Figure 5.25(a). It is desired to design an isolator to protect the beam from the large force that is transmitted to the foundation. The specifications are that the transmitted force is limited to 10,000 lb, and the maximum displacement is 0.1 ft.

The ratio of the maximum value of the allowable transmitted force to the magnitude of the excitation force is

$$\frac{F_T}{F_0} = \frac{10{,}000 \text{ lb}}{20{,}000 \text{ lb}} = 0.5 \quad \text{(a)}$$

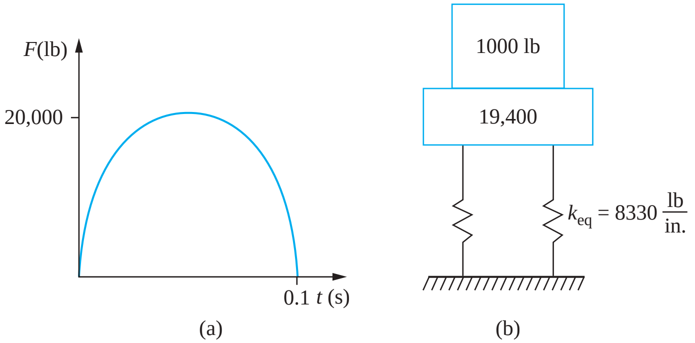


FIGURE 5.25
(a) Sinusoidal pulse excitation for machine of benchmark problem. (b) Isolation system for machine consists of the mass attached to a 19,400 lb concrete block and an elastic pad with an equivalent stiffness of 8330 lb/in.

The response spectrum for a sinusoidal pulse is given in Figure 5.18. For $F_T/F_0 = 0.5$, the value of t_0/T is read as 0.2, so

$$\frac{t_0}{T} = \frac{\omega_n t_0}{2\pi} = 0.2 \tag{b}$$

The natural frequency is calculated from Equation (b) as

$$\omega_n = \frac{(0.2)(2\pi)}{0.1 \text{ s}} = 12.6 \text{ rad/s} \tag{c}$$

For $t_0/T = 0.2$, the value of $\frac{m\omega_n^2}{F_0}x_{\max}$ is read as 0.5, which implies

$$x_{\max} = \frac{0.5gF_0}{W\omega_n^2} = \frac{(0.5)(32.2 \text{ ft/s}^2)(20{,}000 \text{ lb})}{(1000 \text{ lb})(12.6 \text{ rad/s})^2} = 2.02 \text{ ft} \tag{d}$$

The maximum displacement is too large. The only way to reduce the maximum displacement to an acceptable value is to add mass to the machine. The added mass must be sufficient to reduce the maximum displacement to 0.1 ft:

$$W = \frac{0.5gF_0}{x_{\max}\omega_n^2} = \frac{(0.5)(32.2 \text{ ft/s}^2)(20{,}000 \text{ lb})}{(0.1 \text{ ft})(12.6 \text{ rad/s})^2} = 2.04 \times 10^4 \text{ lb} \tag{e}$$

Mount the machine on a concrete block of weight:

$$W_c = W - W_m = 2.04 \times 10^4 \text{ lb} - 1 \times 10^3 \text{ lb} = 1.94 \text{ lb} \times 10^4 \tag{f}$$

The stiffness of the mounting is

$$k = \frac{W}{g}\omega_n^2 = \left(\frac{2.04 \times 10^4 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(12.6 \text{ rad/s})^2 = 1.00 \times 10^5 \text{ lb/ft} = 8.33 \times 10^3 \text{ lb/in.} \tag{g}$$

The SDOF model of the machine with this isolation system is illustrated in Figure 5.25(b).

5.11.2 SIMPLIFIED SUSPENSION SYSTEM

The vehicle encounters a bump in the road that is modeled as a versed sine pulse, as shown in Figure 5.26. The height of the pulse is 0.02 m and the length of the pulse is 0.6 m. Thus, the equation for the versed sine pulse is

$$y(\xi) = 0.02\left[1 - \cos^2\left(\frac{10\pi}{6}\xi\right)\right][1 - u(\xi - 0.6)] \tag{a}$$

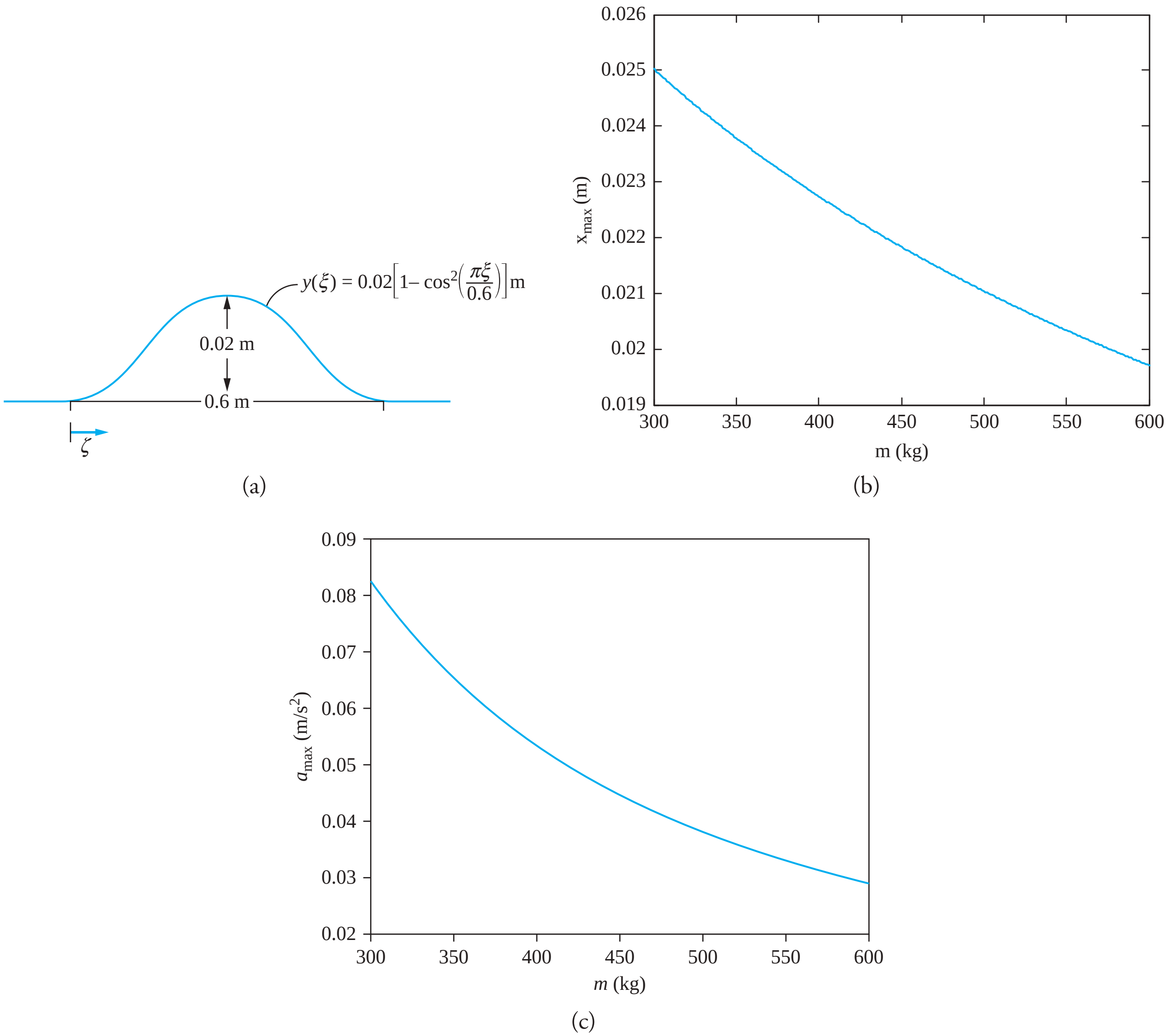


FIGURE 5.26
(a) Bump in road is modeled as a versed sine pulse. (b) x_{max} versus m. (c) a_{max} versus m.

The vehicle traverses the bump at a constant horizontal speed v, which leads to $\xi = vt$.

The differential equation modeling the system is

$$
\begin{aligned}
m\ddot{x} + 1200\dot{x} + 12{,}000x &= 1200\dot{y} + 12{,}000y \\
&= \left[1200\left(\frac{10\pi v}{6}\right)[0.02]\sin\left(\frac{20\pi v}{6}t\right)\right. \\
&\quad \left. + 12{,}000\left\{0.02\left[1 - \cos^2\left(\frac{10\pi v}{6}t\right)\right]\right\}\right]\left[1 - u\left(t - \frac{0.6}{v}\right)\right] \qquad \text{(b)}
\end{aligned}
$$

Let $z = x - y$ be the relative displacement of the vehicle with respect to the wheel. The differential equation for the relative displacement is

$$m\ddot{z} + 1200\dot{z} + 12{,}000z = -m\ddot{y} = 0.02\left(\frac{10\pi v}{6}\right)\left(\frac{20\pi v}{6}\right)\cos\left(\frac{20\pi v}{6}t\right)\left[1 - u\left(t - \frac{0.6}{v}\right)\right] \quad \text{(c)}$$

Equation (c) can be solved using the Laplace transform method.

Equation (c) is rearranged to

$$m\ddot{z} + 1200\dot{z} + 12{,}000z = -1.10mv^2\cos(10.48vt)\left[1 - u\left(t - \frac{0.6}{v}\right)\right] \quad \text{(d)}$$

The Laplace transform method or the convolution integral can be applied to solve Equation (d) for a specific value of m. For a fully loaded vehicle ($m = 600$ kg), Equation (d) becomes

$$\ddot{z} + 2\dot{z} + 20z = -1.10v^2\cos(10.48vt)\left[1 - u\left(t - \frac{0.6}{v}\right)\right] \quad \text{(e)}$$

The natural frequency for a fully loaded vehicle is $\omega_n = 4.47$ rad/s and the system has a damping ratio of $\zeta = 0.224$. The damped natural frequency is $\omega_d = 4.36$ rad/s. Application of the convolution integral leads to

$$z(t) = \frac{-1.10v^2}{10}\int_0^t \cos(10.48vt)\left[1 - u\left(\tau - \frac{0.6}{v}\right)\right]e^{-10(t-\tau)}\sin[10(t-\tau)]\,d\tau \quad \text{(f)}$$

Application of the Laplace transform method leads to

$$Z(s) = \frac{-1.10v^2 s\left(1 - e^{-\frac{0.6}{v}s}\right)}{(s^2 + 2s + 20)(s^2 + 109.8v^2)} \quad \text{(g)}$$

The response spectrum for a versed sine pulse is given in Figure 5.19. For an empty vehicle, $m = 300$ kg, the natural frequency is 6.32 rad/s, the damping ratio is 0.316, and the period is 1.0 s. The speed of the vehicle is important in this problem, as it defines t_0, which is the duration of the pulse. The driver, of course, slows down when he sees the bump. For a speed of 15 m/s, the vehicle is traversed in 0.6 m, 15 m/s, or 0.04 s. For an empty vehicle, $t_0/T = 0.04$. Thus, the pulse is truly a short-duration pulse. The total impulse provided by the bump is

$$I = \int_0^{0.6/v}\left[1200\left(\frac{10\pi v}{6}\right)[0.02]\sin\left(\frac{20\pi v}{6}t\right) + 12{,}000\left\{0.02\left[1 - \cos^2\left(\frac{10\pi v}{6}t\right)\right]\right\}\right]dt = \frac{72}{v}\text{ N}\cdot\text{s} \quad \text{(h)}$$

The maximum displacement due to this impulse is given by Equation (5.103). Application of Equation (5.103) leads to

$$x_{\max} = \frac{72}{m\omega_n}\exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{(i)}$$

The maximum acceleration is given by Equation (5.110) with $a_{\max} = F_{T\max}/m$

$$a_{\max} = \frac{72\omega_n}{mv}\exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{[1-4\zeta^2\sqrt{1-\zeta^2}}{\zeta[3-4\zeta^2]}\right) \quad \text{(j)}$$

The maximum displacement and the maximum acceleration plotted against the mass are plotted in Figure 5.26(b) and Figure 5.26(c), respectively.

5.12 FURTHER EXAMPLES

EXAMPLE 5.19

A one-story frame structure serves as a laboratory. The structure is composed of two beams and a rigid girder. The structure is modeled as a SDOF system with $m = 1000$ kg and $k = 9 \times 10^6$ N/m ($\omega_n = 94.9$ rad/s). The force from an explosion is modeled by the pulse shown in Figure 5.27(a). Unfortunately, an explosion occurs, and that explosion triggers a second explosion at $t = 0.07$ s, later, which lasts twice as long. The force is approximately that of Figure 5.27(b). What is the maximum displacement of the structure?

SOLUTION

The mathematical model for the dual explosions is

$$\begin{aligned} F(t) &= 50{,}000(1-20t)[u(t)-u(t-0.05)] \\ &\quad + 50{,}000(1.7-10t)[u(t-0.07)-u(t-0.17)] \end{aligned} \quad \text{(a)}$$

The response of the system can be obtained using the convolution integral or Table 5.1 and the superposition formula

$$x(t) = F_0[x_a(t) - x_b(t) + x_c(t) - x_d(t)] \quad \text{(b)}$$

where $F_0 = 50{,}000$ N and $x_a(t)$ is the response due to $(1-20t)u(t)$ or the response due to a delayed ramp function with $A = -20$, $B = 1$, and $t_0 = 0$.

$$x_a(t) = \frac{-20}{m\omega_n^2}\left(t - \frac{1}{20} + \frac{1}{20}\cos\omega_n t - \frac{1}{\omega_n}\sin\omega_n t\right) \quad \text{(c)}$$

- $x_b(t)$ is the response due to $(1-20t)u(t-0.05)$ or the response due to a delayed ramp function with $A = -20$, $B = 1$, and $t_0 = 0.05$.

$$x_b(t) = \frac{-20}{m\omega_n^2}\left[t - \frac{1}{20} - \frac{1}{\omega_n}\sin\omega_n(t-0.05)\right]u(t-0.05) \quad \text{(d)}$$

- $x_c(t)$ is the response due to $(1.7-10t)u(t-0.07)$ or due to a delayed ramp function with $A = -10$, $B = 1.7$, and $t_0 = 0.07$:

$$\begin{aligned} x_c(t) = \frac{-10}{m\omega_n^2}\bigg[& t - \frac{1.7}{10} - (0.07-0.17)\cos\omega_n(t-0.07) \\ & - \frac{1}{\omega_n}\sin\omega_n(t-0.07)\bigg]u(t-0.07) \end{aligned} \quad \text{(e)}$$

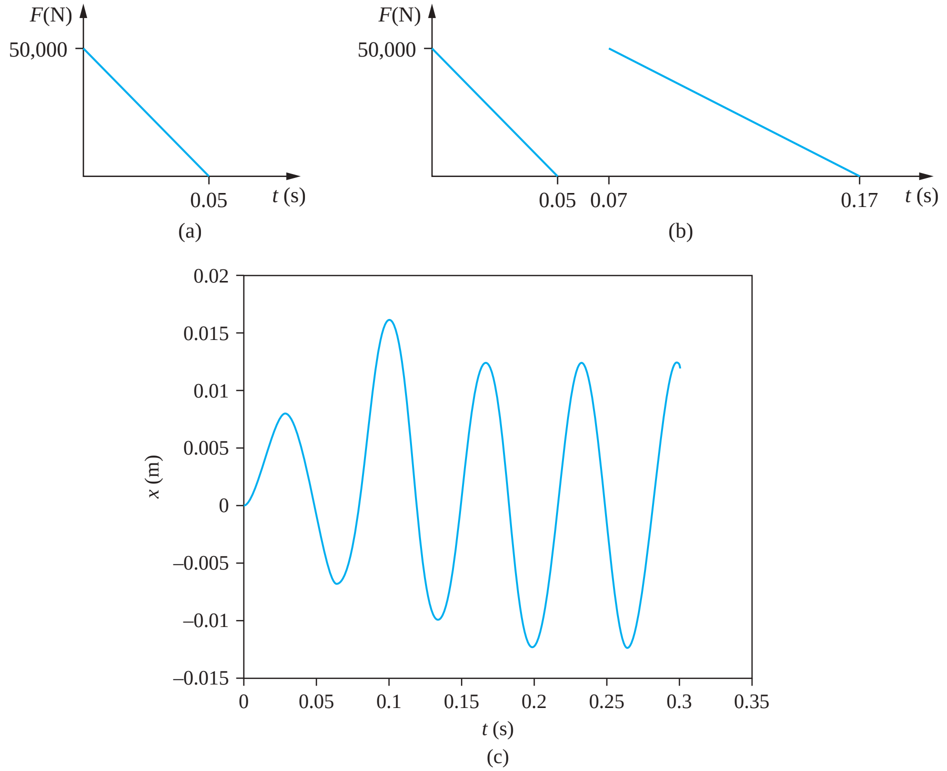


FIGURE 5.27
(a) Model of force provided to a chemical laboratory during an explosion. (b) First explosion triggers a second explosion, resulting in the excitation applied to system of Example 5.19. (c) Response of structure as a function of time.

- $x_d(t)$ is the response due to $(1.7 - 10t)u(t - 0.17)$ or the response due to a delayed ramp function with $A = -10$, $B = 1.7$, and $t_0 = 0.17$:

$$x_d(t) = \frac{-10}{m\omega_n^2}\left[t - \frac{1.7}{10} - \frac{1}{\omega_n}\sin\omega_n(t - 0.17)\right]u(t - 0.17) \tag{f}$$

Thus,

$$\begin{aligned} x(t) = &-0.0555\{2(t - 0.05 + 0.05\cos 94.9t - 0.0105\sin 94.9t)u(t) \\ &- 2[t - 0.05 - 0.0105\sin(94.9t - 4.745)] \\ &+ [t - 0.17 + 0.1\cos(94.9t - 6.643) \\ &- 0.0105\sin(94.9t - 6.643)]\,u(t - 0.07) \\ &- [t - 0.17 - 0.0105\sin(94.9t - 16.133)]\,u(t - 0.17)\} \end{aligned} \tag{g}$$

The maximum of the absolute value of the displacement is determined as 16.0 mm, as shown in Figure 5.27(c).

EXAMPLE 5.20

Determine the response of a SDOF system with a mass of 10 kg and natural frequency of $\omega_n = 10$ rad/s to the excitation of Figure 5.28(a).

SOLUTION

The excitation of Figure 5.28(a) can be broken down as shown in Figure 5.28(b). Mathematically, the function can be written as

$$F(t) = 100tu(t) - 100tu(t-0.1) + 10u(t-0.1) - 10u(t-0.5) + (35-50t)u(t-0.5) - (35-50t)u(t-0.7) \quad \text{(a)}$$

which is simplified to

$$F(t) = 100tu(t) + 10(1-10t)u(t-0.1) + 25(1-2t)\,u(t-0.5) + 5(7-10t)\,u(t-0.7) \quad \text{(b)}$$

The solution is a superposition of four functions, each of which is represented in Table 5.1,

$$x(t) = x_a(t) + x_b(t) + x_c(t) + x_d(t) \quad \text{(c)}$$

- $x_a(t)$: Ramp function, $A = 100$, $B = 0$, and $t_0 = 0$:

$$x_a(t) = \left(\frac{100 \text{ N}}{1000 \text{ N/m}}\right)\left(t - \frac{1}{10}\sin 10t\right) = 0.1(t - 0.1\sin 10t) \quad \text{(d)}$$

- $x_b(t)$: Delayed ramp function, $A = -100$, $B = 10$, and $t_0 = 0.1$:

$$x_b(t) = \left(\frac{-100 \text{ N}}{1000 \text{ N/m}}\right)\left[t - \frac{10}{100} - \left(0.1 - \frac{10}{100}\right)\cos 10(t-0.1) - \frac{1}{10}\sin 10(t-0.1)\right]u(t-0.1)$$
$$= -0.1[t - 0.1 - 0.1\sin(10t-1)]u(t-0.1) \quad \text{(e)}$$

- $x_c(t)$: Delayed ramp function, $A = -50$, $B = 25$, and $t_0 = 0.5$:

$$x_c(t) = \left(\frac{-50 \text{ N}}{1000 \text{ N/m}}\right)\left[t - \frac{25}{50} - \left(0.5 - \frac{25}{50}\right)\cos 10(t-0.5) - \frac{1}{10}\sin 10(t-0.5)\right]u(t-0.5)$$
$$= -0.05[t - 0.5 - 0.1\sin(10t-5)]u(t-0.5) \quad \text{(f)}$$

- $x_d(t)$: Delayed ramp function, $A = -50$, $B = 35$, and $t_0 = 0.7$:

$$x_d(t) = \left(\frac{-50 \text{ N}}{1000 \text{ N/m}}\right)\left[t - \frac{35}{50} - \left(0.7 - \frac{35}{50}\right)\cos 10(t-0.7) - \frac{1}{10}\sin 10(t-0.7)\right]u(t-0.7)$$
$$= -0.05\,[t - 0.7 - 0.1\sin(10t-7)]\,u(t-0.7) \quad \text{(g)}$$

The response is plotted in Figure 5.28(c). The maximum of the response is 1.96 cm.

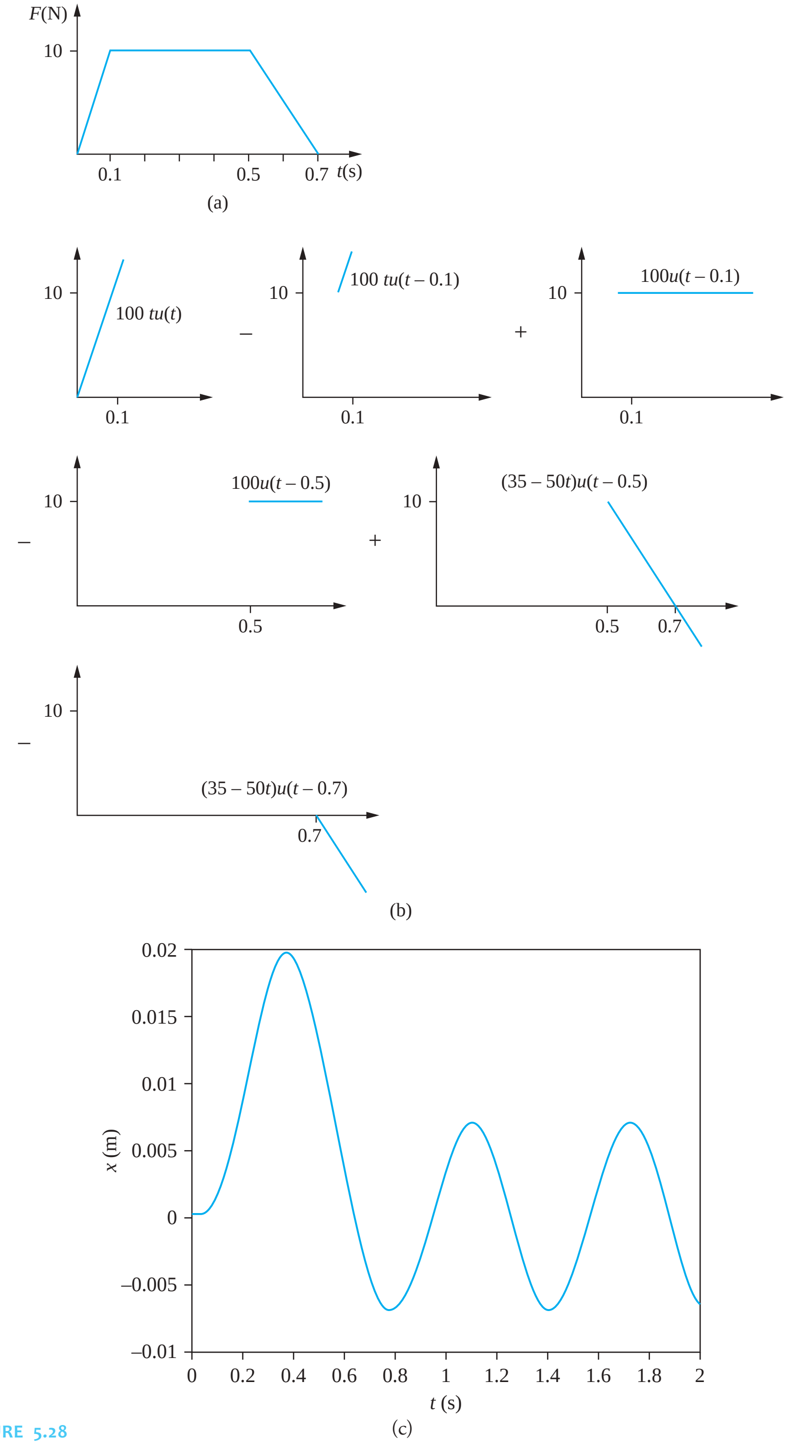


FIGURE 5.28

(a) Excitation applied to Example 5.20. (b) Graphical breakdown of excitation. (c) Response of the system.

EXAMPLE 5.21

During operation, a 200 kg machine is subject to a 1000 N reversed loading, as shown in Figure 5.29.

(a) If the machine is mounted on an elastic pad with a stiffness of 3×10^5 N/m and damping ratio of 0.1, what is the maximum displacement of the machine? What is its maximum transmitted force?

(b) It is desired to hold the amplitude of vibration of the machine to 1.5 cm and limit the transmitted force to 5000 N. Design an isolation system with a damping ratio of 0.1 to achieve these goals.

SOLUTION

(a) The loading is a reversed rectangular pulse with $F_0 = 2000$ N and $t_0 = 0.2$ s. The response spectrum for this force is given in Figure 5.21. The natural period of the machine is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{200 \text{ kg}}{3 \times 10^5 \text{ N/m}}} = 0.162 \text{ s} \tag{a}$$

The value of the nondimensional parameter on the horizontal scale of the response spectrum is

$$\frac{t_0}{T} = \frac{0.2 \text{ s}}{0.162 \text{ s}} = 1.23 \tag{b}$$

The corresponding value of $\frac{kx_{\max}}{F_0}$ read off the vertical scale of Figure 5.21(b) is 2.95. Thus,

$$x_{\max} = 2.95\frac{F_0}{k} = 2.95\frac{2000 \text{ N}}{3 \times 10^5 \text{ N/m}} = 0.020 \text{ m} \tag{c}$$

The corresponding value of $\frac{F_{T,\max}}{F_0}$ read off the vertical scale of Figure 5.21(a) is also 2.95. Thus,

$$F_{T,\max} = 2.95F_0 = 2.95(2000 \text{ N}) = 5900 \text{ N}$$

(b) The upper bound on the natural frequency is determined from

$$\frac{F_{T,\max}}{F_0} < \frac{5000 \text{ N}}{2000 \text{ N}} = 2.5 \tag{d}$$

which from Figure 5.21(a) occurs for

$$\frac{t_0}{T} = \frac{\omega_n t_0}{2\pi} < 0.8 \Rightarrow \omega_n < \frac{2\pi(0.8)}{(0.2 \text{ s})} = 25.1 \text{ rad/s} \tag{e}$$

$$k = m\omega_n^2 \Rightarrow k < (200 \text{ kg})(2.51 \text{ rad/s})^2 = 1.26 \times 10^5 \text{ N/m} \tag{f}$$

For this value of t_0/T, $\frac{kx_{\max}}{F_0} = 2.5 \Rightarrow x_{\max} = \frac{2.5(2000 \text{ N})}{1.26 \times 10^5 \text{ N/m}} = 0.040$ m. Thus, it is not possible to design an isolator such that the maximum force is less than 5000 N and the maximum displacement is less than 0.040 m. However, the mass of the machine can be increased without changing the natural frequency. Setting $x_{\max} = 0.015$ leads to

$$m = \frac{2.5(2000 \text{ N})}{(25.1 \text{ rad/s})^2(0.015 \text{ m})} = 527.7 \text{ kg} \tag{g}$$

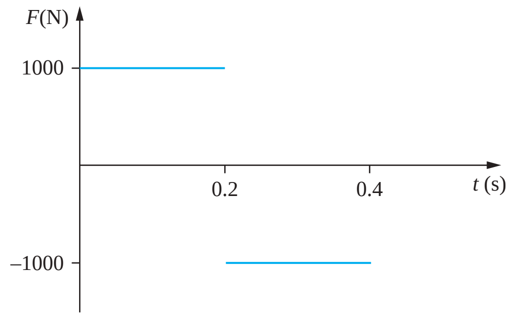


FIGURE 5.29
Pulse loading for Example 5.19.

Thus, to achieve a maximum displacement of 1.5 cm and a maximum transmitted force of 5000 N, mount the machine on a concrete block with a mass of 327.7 kg and an elastic pad with a stiffness of 3.33×10^5 N/m.

5.13 CHAPTER SUMMARY

5.13.1 IMPORTANT CONCEPTS

- The response of a system due to a unit impulse can be determined as the free response with zero initial displacement and an initial velocity equal to velocity imparted by the impulse.
- The convolution integral solution is derived using the principle of linear superposition and the response due to an impulse applied at a previous time.
- The convolution integral provides the response of a linear, SDOF system due to any form of excitation.
- The use of the unit step function allows excitations whose mathematical form changes at discrete values of time to be represented by a unified mathematical function.
- The principle of linear superposition and the representation of excitations that have changes at discrete values of time by unit step functions allow a unified mathematical response for all systems.
- Arbitrary base motion can be handled by the convolution integral.
- The Laplace transform method can be used to determine the response of a linear, SDOF system due to an arbitrary input.
- The transfer function for a system is the Laplace transform of its output divided by the Laplace transform of its input. The transfer function is dependent on the inertia, damping, and stiffness properties of a system.
- The transfer function for a system is the Laplace transform of the system's impulsive response.
- Numerical solutions for the response of a SDOF system are developed through numerical integration of the convolution integral or direct numerical simulation of the governing differential equation.

- Numerical integration of the convolution integral is obtained by interpolation of the excitation force and then integrating exactly the interpolation times the trigonometric function. Interpolating functions are piecewise impulses, piecewise constants or piecewise linear functions.
- Numerical simulation of the governing differential equation is best carried out using a self-starting method, such as Runge-Kutta.
- The response spectrum (shock spectrum) for the shape of a transient excitation is a nondimensional plot of the ratio of the maximum force in the spring to the maximum displacement versus the ratio of the duration of the force (or a characteristic time for the excitation) to the natural undamped period of the system. Numerical simulation of the governing equation is used to develop the response spectrum for different damping ratios.
- Vibration isolation protects foundations from large transient forces generated during operation of a machine is analyzed using the response spectrum for the form of the excitation.
- Vibration isolation for short-duration pulses $[t_0/T < 0.2]$ is analyzed using $Q(\zeta)$ and $S(\zeta)$. To minimize the maximum transmitted force, use a damping ratio of $0.23 < \zeta < 0.3$. To minimize the maximum displacement for a specified transmitted force use a damping ratio, $\zeta = 0.4$.

5.13.2 IMPORTANT EQUATIONS

Impulse delivered by a force

$$I = \int_{t_1}^{t_2} F(\tau)d\tau \tag{5.2}$$

Impulsive response of an underdamped system

$$h(t) = \frac{1}{m_{\text{eq}}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{5.10}$$

Convolution integral solution for differential equation

$$x(t) = \int_0^t F(\tau)\, h(t - \tau)d\tau \tag{5.24}$$

Convolution integral response for an underdamped system

$$x(t) = \frac{1}{m_{\text{eq}}\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau)d\tau \tag{5.25}$$

Convolution integral for relative displacement in base motion problems

$$z(t) = -m_{\text{eq}}\int_0^t \ddot{y}(\tau)h(t - \tau)d\tau \tag{5.34}$$

Laplace transform of a function

$$X(s) = \int_0^\infty x(t)e^{-st}dt \tag{5.40}$$

Laplace transform solution to differential equation

$$x(t) = \frac{1}{m_{\mathrm{eq}}}\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \tag{5.43}$$

Transfer function

$$G(s) = \frac{X(s)}{F(s)} \tag{5.56}$$

Impulsive response

$$h(t) = \mathcal{L}^{-1}\{G(s)\} \tag{5.61}$$

Convolution integral for step response

$$x(t) = \int_0^t [\dot{F}(\tau) + F(0)]x_s(t - \tau)d\tau \tag{5.68}$$

Numerical evaluation of convolution integral

$$x_k = e^{-\zeta\omega_n t_k}\left[x(0)\cos\omega_d t_k + \frac{\zeta\omega_n x(0) + \dot{x}(0)}{\omega_d}\sin\omega_d t_k\right] + \frac{1}{m_{\mathrm{eq}}\omega_d}\left[\sin\omega_d t_k\sum_{j=1}^{n} G_{1j} - \cos\omega_d t_k\sum_{j=1}^{n} G_{2j}\right] \tag{5.73}$$

Maximum transmitted force for short-duration pulse

$$Q(\zeta) = \frac{F_{T_{\max}}}{mv\omega_n} \tag{5.110}$$

Reciprocal of isolator efficiency for short-duration pulses

$$\frac{F_{T_{\max}}x_{\max}}{\frac{1}{2}mv^2} = S(\zeta) \tag{5.111}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 5.1 through 5.10, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

5.1 The convolution integral is the solution to the differential equation governing the motion of a SDOF system with initial conditions equal to zero.

5.2 The convolution integral can be derived using Laplace transforms or variation of parameters.

5.3 The effect of an impulse applied to a SDOF system is to cause a discrete change in displacement.

5.4 The Laplace transform method derives a solution in terms of constants of integration and the determination of the constants is obtained through application of initial conditions.

5.5 Numerical integration of the convolution integral can be obtained by interpolating the forcing function and exactly integrating the interpolation times $h(t - \tau)$.

5.6 Self-starting methods are best for numerical integration of the equation of motion.

5.7 The transfer function for a SDOF system is the ratio of the Laplace transform of the input to the Laplace transform of the output.

5.8 The transfer function is the Laplace transform of the step response of a system.

5.9 The maximum displacement of a machine mounted on an isolator due to an impulsive force is minimized by selecting the damping ratio of the system to be 0.25.

5.10 The maximum transmitted force of a machine mounted on an isolator due to an impulsive force is minimized by selecting the damping ratio of the system to be 0.25.

Problems 5.11 through 5.17 require a short answer.

5.11 What is the physical meaning of the function $h(t)$?

5.12 What pre-integrated form of Newton's second law is used in the derivation of $h(t)$?

5.13 What does the convolution integral represent?

5.14 Explain the meaning of

$$x(1) = \int_0^1 F(\tau)h(1 - \tau)d\tau$$

5.15 What is meant by the approximation of a pulse being short duration?

5.16 What is the response spectrum of a pulse?

5.17 Why is the impulsive response of a system with motion input not defined?

Problems 5.18 through 5.23 require a short calculation.

5.18 A mass-spring system with $m = 2$ kg and $k = 1000$ N/m is subject to an impulse of magnitude 12 N • s. What is the velocity imparted to the system?

5.19 A mass-spring and viscous-damper system is shown in Figure SP5.19. What is the transfer function for the system?

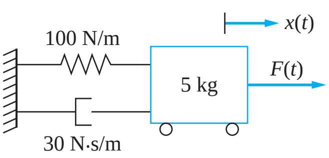


FIGURE SP5.19

5.20 A mass-spring and viscous-damper system with motion input is shown in Figure SP5.20. What is the transfer function for the system?

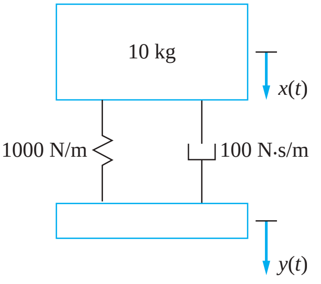


FIGURE SP5.20

5.21 A mass-spring and viscous-damper system is shown in Figure SP5.21. What is the Laplace transform of the system's impulsive response?

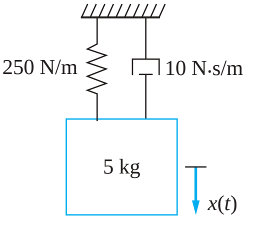


FIGURE SP5.21

5.22 Determine the impulsive response of an undamped mass-spring system with a mass of 5 kg and stiffness of 1000 N/m.

5.23 An impulse with a magnitude of 15 N • s is applied to a mass-spring system and removed. The mass of the system is 0.5 kg, and the stiffness is 200 N/m. Determine the response of the system.

5.24 Match the quantity with the appropriate units (units may used more than once, some units may not be used).

(a) Impulse, I	(i) N • m
(b) Maximum displacement, x_{max}	(ii) rad/s
(c) Initial kinetic energy, $1/2\, mv^2$	(iii) m
(d) Energy absorbed by isolator, $F_{T,\,max}\, x_{max}$	(iv) kg/s
(e) Impulsive response, $h(t)$	(v) s/kg
(f) Damped natural frequency, ω_d	(vi) N • s

CHAPTER PROBLEMS

5.1 A SDOF system with $m = 20$ kg, $k = 10{,}000$ N/m, and $c = 540$ N • s/m is at rest in equilibrium when a 50 N • s impulse is applied. Determine the response of the system.

5.2 A SDOF system with $m = 10$ kg, $k = 40{,}000$ N/m, and $c = 300$ N • s/m is at rest in equilibrium when a 80 N • s impulse is applied. This is followed by a 40 N • s impulse 0.02 s later. Determine the response of the system.

5.3 A SDOF system with $m = 1.3$ kg, $k = 12{,}000$ N/m, and $c = 400$ N • s/m is at rest in equilibrium when a 100 N • s impulse is applied. This is followed by a 150 N • s impulse 0.12 s later. Determine the response of the system.

5.4 Use the method of variation of parameters to obtain the general solution of Equation (5.1) and show that it can be written in the form of the convolution integral, Equation (5.25).

5.5 Use the convolution integral to determine the response of an underdamped SDOF system of mass m and natural frequency ω_n when the excitation is the unit step function, $u(t)$.

5.6 Let $g(t)$ be the response of an underdamped system to a unit step function and $h(t)$ the response of an underdamped system to a unit impulse function. Show

$$h(t) = \frac{dg}{dt}$$

5.7 Use the convolution integral and the notation and results of Chapter Problem 5.6 to derive the following alternative expression for the response of a system subject to an excitation, $F(t)$:

$$x(t) = F(0)g(t) + \int_0^t \frac{dF(\tau)}{d\tau} g(t-\tau)\,d\tau$$

5.8 A SDOF undamped system is initially at rest in equilibrium and subject to a force $F(t) = F_0 t e^{-t/2}$. Use the convolution integral to determine the response of the system.

5.9 The mass of Figure P5.9 has a velocity v when it engages the spring-dashpot mechanism. Let $x(t)$ be the displacement of the mass from the position where the mechanism is engaged. Use the convolution integral to determine $x(t)$. Assume the system is underdamped.

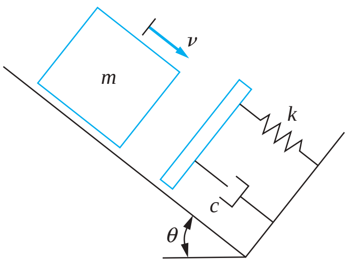


FIGURE P5.9

5.10 Use the convolution integral to determine the response of the system of Figure P5.10.

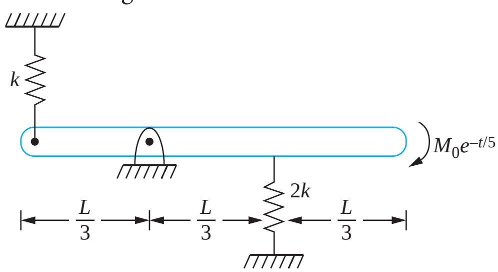


FIGURE P5.10

5.11 Use the convolution integral to determine the response of an underdamped SDOF system of natural frequency ω_n and damping ratio ζ when subject to a harmonic excitation $F(t) = F_0 \sin \omega t$.

5.12–5.18 A machine tool with a mass of 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation, it is subject to one of the machining force shown in Figures P5.12 through P5.18. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

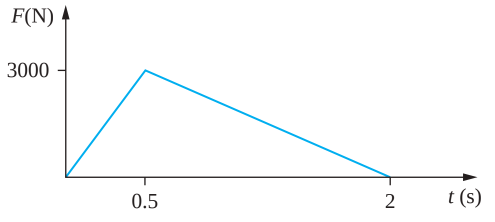


FIGURE P5.12

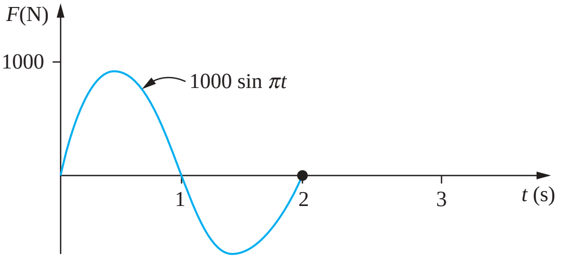


FIGURE P5.13

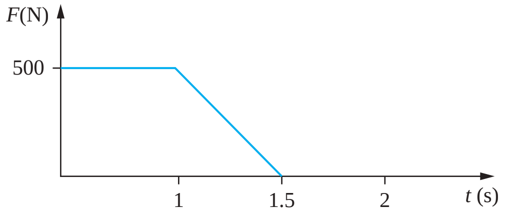


FIGURE P5.14

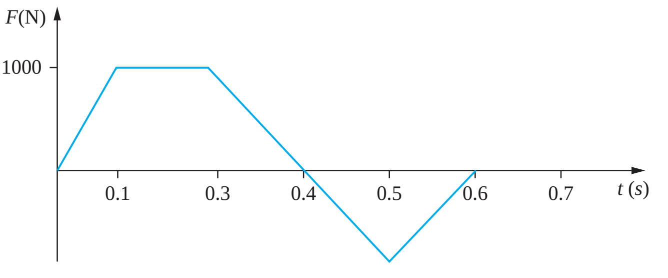


FIGURE P5.15

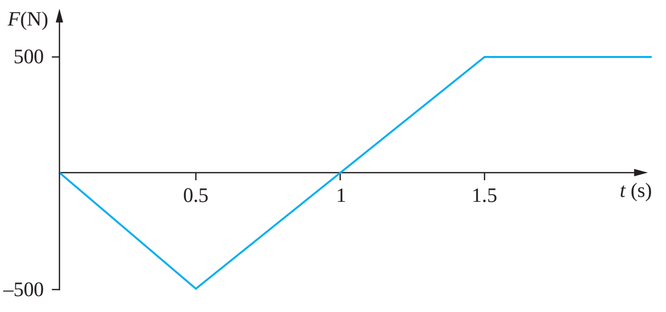


FIGURE P5.16

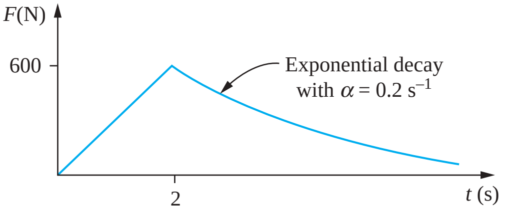


FIGURE P5.17

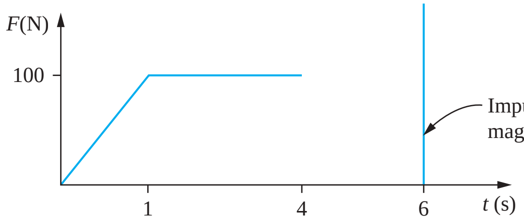


FIGURE P5.18

5.19 The force applied to the 120 kg anvil of a forge hammer during operation is approximated as a rectangular pulse of magnitude 2000 N for a duration of 0.3 s. The anvil is mounted on a foundation of stiffness 2000 N/m and damping ratio 0.4. What is the maximum displacement of the anvil?

5.20 A one-story frame structure houses a chemical laboratory. Figure P5.20 shows the results of a model test to predict the transient force to which the structure would be subject if an explosion would occur. The equivalent mass of the structure is 2000 kg and its equivalent stiffness is 5×10^6 N/m. Approximate the maximum displacement of the structure due to this blast.

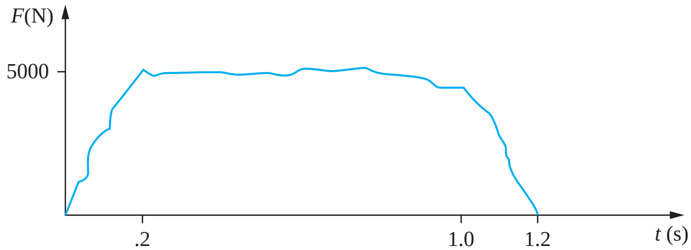


FIGURE P5.20

5.21 A 20 kg radio set is mounted in a ship on an undamped foundation of stiffness 1000 N/m. The ship is loosely tied to a dock. During a storm, the ship experiences the displacement of Figure P5.21. Determine the maximum acceleration of the radio.

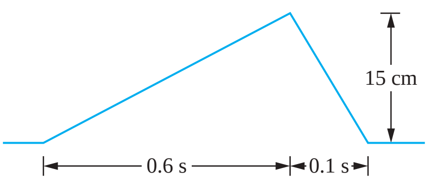


FIGURE P5.21

5.22 A personal computer of mass m is packed inside a box such that the stiffness and damping coefficient of the packing material are k and c, respectively. The package is accidentally dropped from a height h and lands on a hard surface without rebound. Set up the convolution integral whose evaluation leads to the displacement of the computer relative to the package.

5.23 Use the Laplace transform method to determine the response of a system at rest in equilibrium when subject to

$$F(t) = F_0 \cos \omega t[1 - u(t - t_0)]$$

for (a) $\zeta = 0$, (b) $0 < \zeta < 1$, (c) $\zeta = 1$, (d) $\zeta > 1$.

5.24 Use the Laplace transform method to determine the response of an undamped SDOF system initially at rest in equilibrium when subject to a symmetric triangular pulse of magnitude F_0 and total duration t_0.

5.25 Use the Laplace transform method to determine the response of an underdamped SDOF system to a rectangular pulse of magnitude F_0 and time t_0.

5.26 Use the Laplace transform method to derive the response of a SDOF system initially at rest in equilibrium when subject to a harmonic force $F_0 \sin \omega t$, when (a) $\omega \neq \omega_n$, and (b) $\omega = \omega_n$.

5.27 Determine the transfer function for the relative displacement of a SDOF system with base motion defined as $G(s) = \frac{Z(s)}{Y(s)}$ where $Z(s)$ is the Laplace transform of the relative displacement and $Y(s)$ is the Laplace transform of the motion of the base.

5.28 Determine the transfer function for the force transmitted to the foundation for a SDOF system. The transfer function is defined as $G(s) = \frac{F_t(s)}{F(s)}$ where $F_t(s)$ is the Laplace transform of the transmitted force and $F(s)$ is the Laplace transform of the applied force.

5.29 Use the transfer function to determine the response of a SDOF system excited by motion of its base with $m = 3$ kg and $k = 18{,}000$ N/m where the base motion is shown in Figure P5.29.

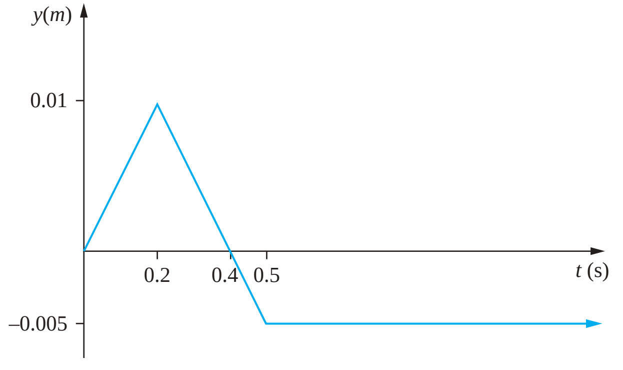


FIGURE P5.29

5.30 Use the transfer function to determine the response of a SDOF system with $m = 1$ kg, $k = 100$ N/m, and $c = 6$ N • s/m when the system is subject to motion of its base shown in Figure P5.30.

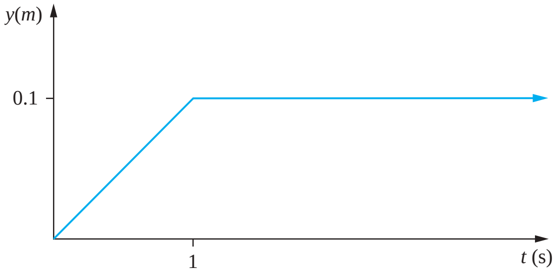


FIGURE P5.30

5.31 Repeat Chapter Problem 5.30 if the system parameters are $m = 1$ kg, $k = 200$ N/m, and $c = 30$ N • s/m.

5.32 For the system of Figure P5.32(a), complete the following.

(a) Determine its transfer function defined as $G(s) = \frac{X(s)}{Y(s)}$.

(b) Use the transfer function to find the response of the system due to $y(t)$ as shown in Figure P5.32(b). Use $m = 1$ kg, $k = 100$ N/m, and $c = 30$ N • s/m.

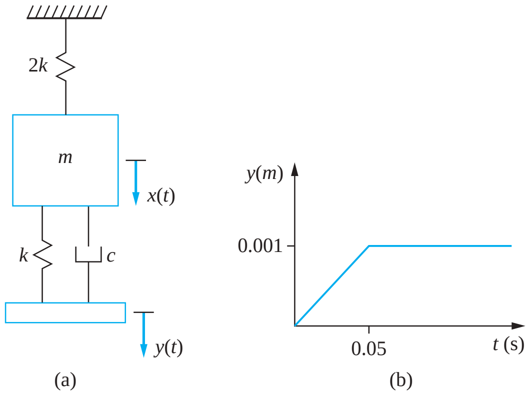


FIGURE P5.32

5.33 For the system of Figure P5.33(a), complete the following.

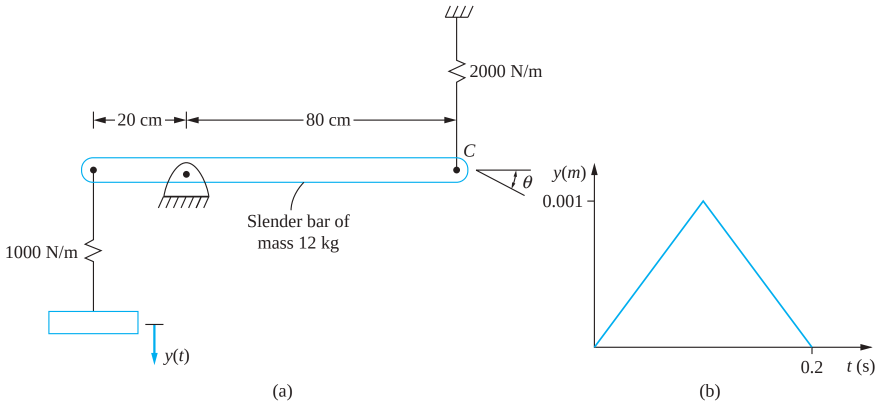


FIGURE P5.33

(a) Determine its transfer function defined as $G(s) = \frac{\theta(s)}{Y(s)}$ where $\theta(s)$ is the Laplace transform of the angular displacement of the bar.
(b) Use the transfer function to determine $\theta(t)$ due to $y(t)$, as shown in Figure P5.33(b).

5.34 During its normal, operation, a 144-kg machine tool is subject to a 15,000 N • s impulse. Design an efficient isolator such that the maximum force transmitted through the isolator is 2500 N and the maximum displacement is minimized.

5.35 A 110 kg pump is mounted on an isolator of stiffness 4×10^5 N/m and a damping ratio of 0.15. The pump is given a sudden velocity of 30 m/s. What is the maximum force transmitted through the isolator and what is the maximum displacement of the pump?

5.36 During operation, a 50-kg machine tool is subject to the short-duration pulse of Figure P5.36. Design an isolator that minimizes the maximum displacement and reduces the maximum transmitted force to 5000 N. What is the maximum displacement of the machine tool when this isolator is used?

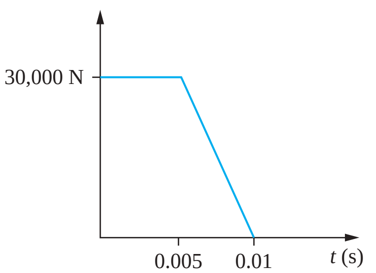


FIGURE P5.36

5.37 Repeat Chapter Problem 5.36 for the short-duration pulse of Figure P5.37.

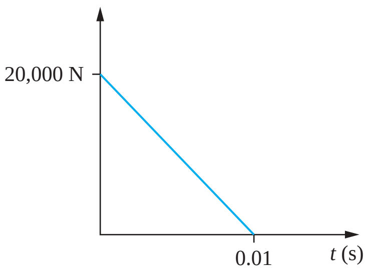


FIGURE P5.37

5.38 A ship is moored at a dock in rough seas and frequently impacts the dock. The maximum velocity change caused by the impact is 15 m/s. Design an isolator to protect a sensitive 80-kg navigational control system such that its maximum acceleration is 30 m/s^2.

5.39 A one-story frame structure with an equivalent mass of 12,000 kg and stiffness of 1.8×10^6 N/m is subject to a blast whose force is given in Figure P5.39. What is the maximum deflection of the structure?

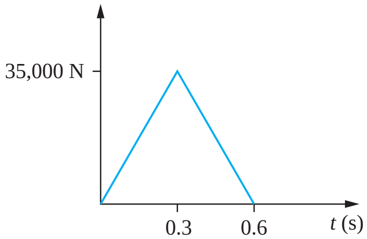


FIGURE P5.39

5.40 A 20 kg machine tool is on a foundation that is subject to an acceleration that is modeled as a versed sine pulse with a magnitude of 20 m/s^2 and duration of 0.4 s. Design an undamped isolator such that the maximum acceleration felt by the machine is 15 m/s^2. What is the maximum displacement of the machine tool relative to its foundation when this isolator is used?

5.41 During operation, a 100 kg machine tool is exposed to a force that is modeled as a sinusoidal pulse with a magnitude of 3100 N and duration of 0.05 s. Design an isolator with a damping ratio 0.1 such that the maximum force transmitted through the isolator is 2000 N and the maximum displacement of the machine tool is 3 cm.

5.42 During operation a 80 kg machine tool is subject to a triangular pulse with a magnitude of 30,000 N and duration of 0.15 s. What is the range of undamped isolator stiffness such that the maximum transmitted force is 15,000 N and the maximum displacement is 5 cm?

TWO DEGREE-OF-FREEDOM SYSTEMS

6.1 INTRODUCTION

Two degree-of-freedom systems require two generalized coordinates to describe the motion of every particle in the system. The system requires two (in general) coupled differential equations governing the motion of the system. The general form of the differential equations for a linear system with viscous damping is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{6.1}$$

or

$$\begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{6.2}$$

The matrix **M** is a 2×2 mass matrix, **C** is a 2×2 damping matrix, **K** is a 2×2 stiffness matrix, **F** is a 2×1 force vector and **x** is a 2×1 vector of generalized coordinates. The forms of the matrix are determined by deriving the differential equations of motion.

Two degree-of-freedom systems are considered before n degree-of-freedom systems because

- Many systems only require two degrees of freedom when modeling.
- While the equations are formulated in a matrix form, matrix algebra is not required to formulate a solution.
- Physical insight is gained by studying two degree-of-freedom systems.
- Viscous damping can be more easily handled.

The differential equations governing two degree-of-freedom systems are derived. A normal-mode solution for the free response for undamped systems is assumed in which both generalized coordinates are assumed to vibrate synchronously with different amplitudes. The normal-mode solution is used to obtain the natural frequencies and mode shapes, which are the relative amplitudes of vibration, for the two degree-of-freedom system. The two mode shapes are combined to formulate the free response for undamped systems. The solution is in terms of four constants of integration, which are determined through application of initial conditions.

An exponential solution is assumed for systems with viscous damping. This leads to a fourth-order algebraic equation for a parameter. The fourth-order equation includes odd powers, so it cannot be reduced to a quadratic and must be solved numerically. The modes of vibration can be underdamped, critically damped, or overdamped. The free response is obtained in terms of constants of integration. Initial conditions are applied to determine the constants.

When the differential equations are written using principal coordinates as the dependent variables, they are uncoupled. However, the principal coordinates are not obvious; sometimes a principal coordinate does not represent the displacement of a particle in the system.

The forced response of systems with harmonic excitations is developed. Both undamped systems and damped systems are considered. The sinusoidal transfer functions are developed as a means of determining the harmonic response. The concept of frequency response is considered.

An application of harmonic response of two degree-of-freedom systems is the vibration absorber. A *vibration absorber* is an auxiliary mass-spring system that is attached to a machine that is experiencing large amplitude vibrations due to near-resonance conditions. The addition of a vibration absorber changes a SDOF system to a two degree-of-freedom system. When the vibration absorber is properly "tuned," the steady-state vibrations of the machine are eliminated. One problem with vibration absorbers is that the lower natural frequency of the two degree-of-freedom system is lower than the tuned speed. Thus, the lower natural frequency is passed through during start-up, which leads to large amplitude vibrations. When damping is added to the vibration absorber to control the vibrations during start-up, the ability to eliminate steady-state vibrations of the machine is lost. An optimum damped vibration absorber is determined.

6.2 DERIVATION OF THE EQUATIONS OF MOTION

The equations of motion for a two degree-of-freedom system are derived using the free-body diagram method or an energy method. However, the energy method is delayed until Chapter 7. The free-body diagram method is the same as for SDOF systems, except that multiple free-body diagrams or equations may be used. Newton's law ($\sum F = ma$) is applied to the free-body diagram of a particle. The equations $\sum F = m\bar{a}$ and $\sum M_0 = I_0\alpha$ are applied to a free-body diagram of a rigid body undergoing planar motion with rotation about a fixed axis through 0. For a rigid body undergoing planar motion, D'Alembert's principle can be applied as $\sum F_{\text{ext}} = \sum F_{\text{eff}}$ and $(\sum M_A)_{\text{ext}} = (\sum M_A)_{\text{eff}}$ where A is any point. The system of effective forces is a force equal to $m\bar{a}$ applied at the mass center and a moment equal to $\bar{I}\alpha$.

EXAMPLE 6.1

Derive the differential equations governing the motion of the two degree-of-freedom system of Figure 6.1 using x_1 and x_2 as generalized coordinates. Both are measured from the system's equilibrium position.

SOLUTION

The free-body diagrams of the blocks drawn at an arbitrary instant are shown in Figure 6.1(b). The forces from gravity of the blocks cancel with the static spring forces, as in single degree-of-freedom systems. The bottom end of the spring connecting the two blocks has a displacement of x_2 from equilibrium, while the upper end of the spring has a displacement of x_1. Therefore, the change in length of the spring is $x_2 - x_1$, and the force developed in the spring is $k(x_2 - x_1)$. If $x_2 > x_1$, the spring is stretched, and the spring force is drawn acting away from the blocks.

Applying Newton's second law ($\sum F = ma$) to the first block yields

$$-kx_1 - c\dot{x}_1 + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) = m\ddot{x}_1 \tag{a}$$

or

$$m\ddot{x}_1 + 2c\dot{x}_1 + 2kx_1 - c\dot{x}_2 - kx_2 = 0 \tag{b}$$

Application of Newton's second law to the lower block leads to

$$-2kx_2 - 2c\dot{x}_2 - k(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) + F(t) = 2m\ddot{x}_2 \tag{c}$$

or

$$2m\ddot{x}_2 + 3c\dot{x}_2 + 3kx_2 - c\dot{x}_1 - kx_1 = F(t) \tag{d}$$

Rewriting Equations (b) and (d) in a matrix form gives

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 3c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{e}$$

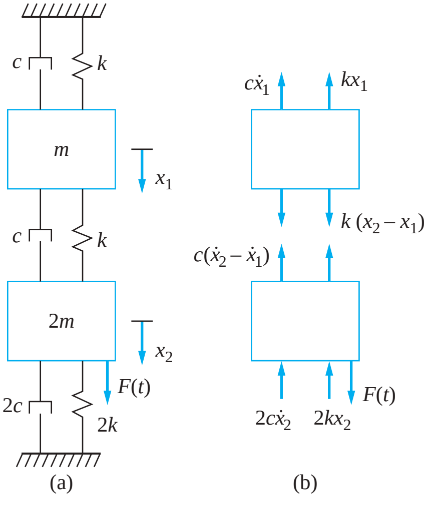


FIGURE 6.1
(a) System of Example 6.1 showing the chosen generalized coordinates. (b) FBDs at an arbitrary instant. Static spring forces cancel with gravity.

EXAMPLE 6.2

Consider the system shown in Figure 6.2 in which the slender bar of mass m and moment of inertia $1/12(mL^2)$ is attached to springs of stiffness k at its left end and three-quarters of the way across the bar. Derive the differential equations for the system of Figure 6.2 using the following.

(a) x is as generalized coordinates: the displacement of the mass center of the bar from equilibrium, and θ is the clockwise angular displacement of the bar.

(b) x_1 and x_2 are the vertical displacements of particles where the springs are attached and measured from equilibrium. Assume small θ.

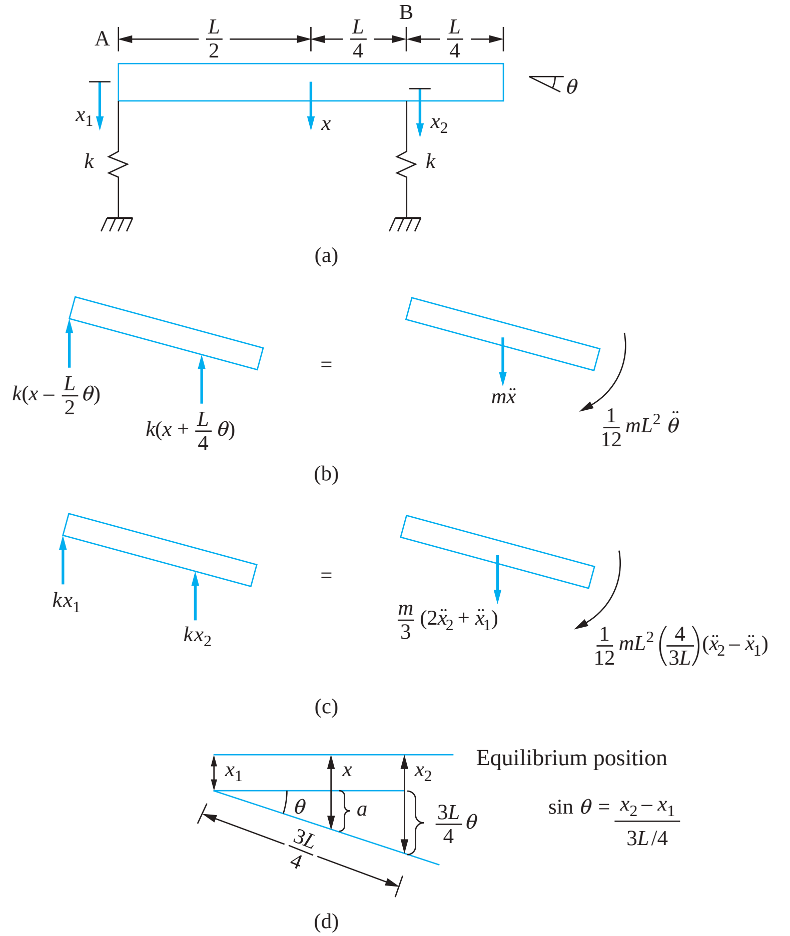


FIGURE 6.2

(a) System of Example 6.2. One choice of generalized coordinates is the displacement of the mass center x and the angular rotation of the bar θ. Another choice is x_1 and x_2, which are the points where the springs are attached. (b) FBDs of the system at an arbitrary instant using x and θ as generalized coordinates. (c) FBDs of the system at an arbitrary instant using x_1 and x_2 as generalized coordinates. (d) Geometry used to determine x and θ in terms of x_1 and x_2.

SOLUTION

(a) A free-body diagram of the bar drawn at an arbitrary instant using x and θ as generalized coordinates is shown in Figure 6.2(b). Rotation does not occur about a fixed axis; thus, the effective force method is used. Application of $\sum F_{ext} = \sum F_{eff}$ leads to

$$-k\left(x - \frac{L}{2}\theta\right) - k\left(x + \frac{L}{4}\theta\right) = m\ddot{x} \tag{a}$$

Application of the moment equation $(\sum M_G)_{ext} = (\sum M_G)_{eff}$ leads to

$$k\left(x - \frac{L}{2}\theta\right)\frac{L}{2} - k\left(x + \frac{L}{4}\theta\right)\frac{L}{4} = \frac{1}{12}mL^2\ddot{\theta} \tag{b}$$

Rearranging Equations (a) and (b) and writing them in a matrix form leads to

$$\begin{bmatrix} m & 0 \\ 0 & \frac{1}{12}mL^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & -k\frac{L}{4} \\ -k\frac{L}{4} & k\frac{5L^2}{16} \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{c}$$

(b) Free-body diagrams drawn at an arbitrary instant when x_1 and x_2 are used as generalized coordinates, as shown in Figure 6.2(c). The geometry used to calculate the displacement of the mass center and the angular rotation of the bar, as illustrated in Figure 6.2(d), is consistent with the small angle assumption. The angular rotation of the bar is

$$\theta = \frac{x_2 - x_1}{\frac{3L}{4}} = \frac{4(x_2 - x_1)}{3L} \tag{d}$$

$$x = x_1 + a = x_1 + \frac{L}{2}\theta = x_1 + \left(\frac{L}{2}\right)\frac{4(x_2 - x_1)}{3L} = \frac{2x_2 + x_1}{3} \tag{e}$$

Summation of moments about an axis through B, $(\sum M_B)_{ext} = (\sum M_B)_{eff}$, leads to

$$(kx_1)\left(\frac{3L}{4}\right) = \frac{1}{12}mL^2\left(\frac{4}{3L}\right)(\ddot{x}_2 - \ddot{x}_1) - m\left(\frac{2\ddot{x}_2 + \ddot{x}_1}{3}\right)\left(\frac{L}{4}\right) \tag{f}$$

Summation of moments about an axis through A, $(\sum M_A)_{ext} = (\sum M_A)_{eff}$, yields

$$-kx_2\left(\frac{3L}{4}\right) = \frac{1}{12}mL^2\left(\frac{4}{3L}\right)(\ddot{x}_2 - \ddot{x}_1) + m\left(\frac{2\ddot{x}_2 + \ddot{x}_1}{3}\right)\left(\frac{L}{2}\right) \tag{g}$$

Rewriting Equations (f) and (g) and writing them in matrix form leads to

$$\begin{bmatrix} \frac{7}{36}mL & \frac{1}{18}mL \\ \frac{1}{18}mL & \frac{4}{9}mL \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{3L}{4}k & 0 \\ 0 & \frac{3L}{4}k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{h}$$

6.3 NATURAL FREQUENCIES AND MODE SHAPES

Natural frequencies for two degree-of-freedom systems are the frequencies at which undamped vibrations naturally occur. They are determined by assuming that the free response is periodic with a specified frequency. Recalling that $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, the free response of a two degree-of-freedom system with $\mathbf{C} = 0$ is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{X}e^{i\omega t} \tag{6.3}$$

where $\mathbf{X} = [\chi_1\ \chi_2]^T$ is the mode shape vector. Equation (6.3) is called the *normal mode solution.* The normal mode solution assumes the generalized coordinates are synchronous; that is, they vibrate at the same frequency. Substituting Equation (6.3) into Equation (6.2) with $\mathbf{C} = 0$ leads to

$$-\omega^2\begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} + \begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.4}$$

which can be written as

$$-\omega^2\mathbf{MX} + \mathbf{KX} = 0 \tag{6.5}$$

Equation (6.5) represents a system of equations for $\mathbf{X}$, but it is homogeneous. Using Cramer's rule to determine the components of the solution vector leads to

$$\chi_1 = \frac{\begin{vmatrix} 0 & -\omega^2 m_{1,2} + k_{1,2} \\ 0 & -\omega^2 m_{2,2} + k_{2,2} \end{vmatrix}}{\det(-\omega^2\mathbf{M} + \mathbf{K})} \tag{6.6}$$

$$\chi_2 = \frac{\begin{vmatrix} -\omega^2 m_{1,1} + k_{1,1} & 0 \\ -\omega^2 m_{2,1} + k_{2,1} & 0 \end{vmatrix}}{\det(-\omega^2\mathbf{M} + \mathbf{K})} \tag{6.7}$$

The determinant of a matrix with a column of zeroes is zero. Thus, the solution to Equation (6.5) is the trivial solution $\chi_1 = 0$ and $\chi_2 = 0$, unless the denominator is zero. Thus, to obtain a non-trivial solution,

$$\det(-\omega^2\mathbf{M} + \mathbf{K}) = 0 \tag{6.8}$$

Equation (6.8) leads to a quadratic equation with two possible natural frequencies; both real and non-negative. The natural frequencies are ordered such that $\omega_1 \leq \omega_2$.

The mode shape vector corresponding to a natural frequency ω is the non-trivial solution of Equation (6.4) with that value of ω, as

$$\begin{bmatrix} -\omega^2 m_{1,1} + k_{1,1} & -\omega^2 m_{1,2} + k_{1,2} \\ -\omega^2 m_{2,1} + k_{2,1} & -\omega^2 m_{2,2} + k_{2,2} \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.9}$$

If ω satisfies Equation (6.8), then $-\omega^2\mathbf{M} + \mathbf{K}$ is singular, and the equations in Equation (6.9) are multiples of one another. A solution exists, but it not unique. Using the first of Equation (6.9), the solution has

$$\chi_2 = \frac{\omega^2 m_{1,1} - k_{1,1}}{-\omega^2 m_{1,2} + k_{1,2}}\chi_1 \tag{6.10}$$

Traditionally, $\chi_1 = 1$ for determining χ_2, and Equation (6.8) becomes

$$\chi_2 = \frac{\omega^2 m_{1,1} - k_{1,1}}{-\omega^2 m_{1,2} + k_{1,2}} \tag{6.11}$$

The value of χ_2, calculated by Equation (6.11), is called the *modal fraction* for the frequency. There are two modal fractions, one for the first mode shape, which we will label χ_1, and one for the second mode shape, which we will label χ_2. We will refer to the mode shape in general as $[1\ \chi]^T$.

The nodes are the particles in a system which has zero displacement when the system is vibrating at one of the natural frequencies. These can be determined from the mode shapes. For a two degree-of-freedom system, there are no nodes associated with the lowest natural frequency and one node associated with the higher natural frequency.

EXAMPLE 6.3

Consider the two degree-of-freedom system shown in Figure 6.3(a). Determine (a) the natural frequencies, (b) the modes shapes, and (c) the nodes for the system.

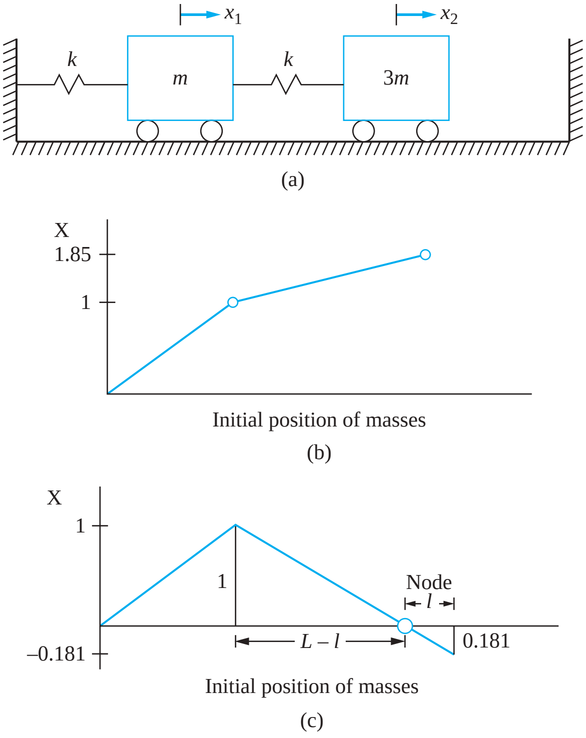


FIGURE 6.3
(a) System of Example 6.3. (b) Mode shape corresponding to first mode. (c) Mode shape corresponding to second mode.

SOLUTION

The differential equations governing the system are

$$\begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{a}$$

(a) The natural frequencies and mode shapes are determined using by Equation (6.7),

$$-\omega^2\begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b}$$

Setting $\det(-\omega^2\mathbf{M} + \mathbf{K}) = 0$ as in Equation (6.6) leads to

$$\begin{bmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 3m + k \end{bmatrix} = 0 \tag{c}$$

Evaluation of Equation (c) leads to

$$(-\omega^2 m + 2k)(-\omega^2 3m + k) - (-k)(-k) = 0 \tag{d}$$

When expanded, Equation (d) becomes

$$(3m)\omega^4 - (7mk)\omega^2 + (k^2) = 0 \tag{e}$$

Dividing Equation (e) by m and defining $\phi = k/m$ and $\lambda = \omega^2$, Equation (e) becomes

$$3\lambda^2 - 7\phi\lambda + \phi^2 = 0 \tag{f}$$

Using the quadratic formula $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve Equation (f) leads to

$$\lambda = \frac{7\phi \pm \sqrt{(7\phi)^2 - 4(3)(\phi)^2}}{2(3)} \tag{g}$$

or

$$\lambda_1 = \left(\frac{7 - \sqrt{37}}{6}\right)\phi \qquad \lambda_2 = \left(\frac{7 + \sqrt{37}}{6}\right)\phi \tag{h}$$

Realizing that $\omega = \sqrt{\lambda}$ and $\phi = k/m$, the natural frequencies are

$$\omega_1 = \sqrt{\left(\frac{7 - \sqrt{37}}{6}\right)\phi} = 0.391\sqrt{\frac{k}{m}} \tag{i}$$

and

$$\omega_2 = \sqrt{\left(\frac{7 + \sqrt{37}}{6}\right)\phi} = 1.47\sqrt{\frac{k}{m}} \tag{j}$$

(b) The mode shapes are determined using Equation (6.9). For $\omega_1^2 = 0.153\frac{k}{m}$, substitution in Equation (6.9) leads to a modal fraction of

$$\chi_1 = \frac{-0.153\frac{k}{m}(m) + 2k}{k} = 1.85 \tag{k}$$

Application of Equation (6.9) for the second mode leads to the modal fraction of

$$\chi_2 = \frac{-2.16\frac{k}{m}(m) + 2k}{k} = -0.181 \quad \text{(l)}$$

The mode shapes for the first mode and second mode are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.85 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.181 \end{bmatrix} \quad \text{(m)}$$

(c) The mode shape diagrams, which are plots of relative displacements for each mode drawn horizontally, are given in Figure 6.3(b) and Figure 6.3(c). The mode shape diagram for the first mode shows no point where the displacement is negative. Thus, the mode shape for the first mode has no nodes. The mode shape diagram for the second mode has one node. Assuming the spring is linear, similar triangles applied to the mode shape shown in Figure 6.3(c) leads to

$$\frac{\ell}{0.181} = \frac{L - \ell}{1} \quad \text{(n)}$$

or

$$\ell = 0.153L \quad \text{(o)}$$

where L is the length of the spring.

EXAMPLE 6.4

Determine the natural frequencies and mode shapes for the bar of Figure 6.2. Identify any nodes.

SOLUTION

The differential equation of the system is derived in Example 6.2. The natural frequencies do not depend on the choice of generalized coordinates, but the mode shape vectors are specific to the choice of generalized coordinates. The nodes are not dependent on the choice of generalized coordinates. Using x and θ as generalized coordinates, the natural frequencies are determined through application of Equation (6.7).

$$\begin{vmatrix} -\omega^2 m + 2k & -k\frac{L}{4} \\ -k\frac{L}{4} & -\omega^2\frac{1}{12}mL^2 + k\frac{5L^2}{16} \end{vmatrix} = 0 \quad \text{(a)}$$

Evaluation of the determinant leads to

$$(-\omega^2 m + 2k)\left(-\omega^2\frac{1}{12}mL^2 + k\frac{5L^2}{16}\right) - \left(-k\frac{L}{4}\right)\left(-k\frac{L}{4}\right) = 0 \quad \text{(b)}$$

Expansion of the above gives

$$\frac{1}{12}m^2L^2\omega^4 - \frac{46}{96}mkL^2\omega^2 + \frac{9}{16}k^2L^2 = 0 \tag{c}$$

Multiplying Equation (c) by $12/(m^2L^2)$ and defining $\phi = k/m$ and $\lambda = \omega^2$ leads to

$$\lambda^2 - \frac{23}{4}\phi\lambda + \frac{27}{4}\phi = 0 \tag{d}$$

Using the quadratic formula to solve Equation (d) gives

$$\lambda = \left(\frac{\frac{23}{4} \pm \sqrt{\left(\frac{23}{4}\right)^2 - 4\left(\frac{27}{4}\right)}}{2}\right)\phi = 1.64\phi,\ 4.11\phi \tag{e}$$

Recalling that $\omega = \sqrt{\lambda}$ and $\phi = k/m$ yields

$$\omega_1 = 1.28\sqrt{\frac{k}{m}} \qquad \omega_2 = 2.02\sqrt{\frac{k}{m}} \tag{f}$$

The mode shapes are calculated using Equation (6.9). For $\omega_1 = 1.28\sqrt{\phi}$, this yields

$$\chi_1 = \frac{\left(1.64\frac{k}{m}\right)m - 2k}{-k\frac{L}{4}} = \frac{1.42}{L} \tag{g}$$

For $\omega_2 = 2.07\sqrt{\phi}$, Equation (6.9) gives

$$\chi_2 = \frac{\left(4.11\frac{k}{m}\right)m - 2k}{-k\frac{L}{4}} = -\frac{8.42}{L} \tag{h}$$

The mode shape vectors are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ \frac{1.42}{L} \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ \frac{-8.42}{L} \end{bmatrix} \tag{i}$$

The mode shapes are illustrated in Figure 6.4. The first mode has no nodes on the bar, but it represents rigid-body motion about an axis through point O, which is not on the bar. Point O is a distance $0.19L$ from the end of the bar. The second mode has one node and represents a rigid-body motion about an axis through point P, which is a distance of 0.118 to the right of the mass center.

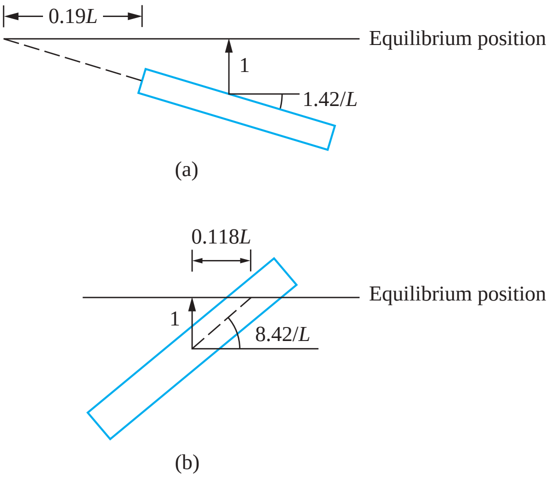


FIGURE 6.4
Mode shapes of Example 6.4. (a) First mode is a rigid-body rotation about point O, which is a point a distance $0.19L$ from the left end of the bar. (b) Second mode is a rigid-body rotation about point P, which is a distance of $0.118L$ to the right of the mass center.

6.4 FREE RESPONSE OF UNDAMPED SYSTEMS

The most general solution of a linear homogeneous problem is a linear combination of all possible solutions. The free response of a linear, undamped two degree-of-freedom system has two natural frequencies and two mode shapes. However, each natural frequency satisfies a fourth-order equation which only contains even powers of ω. It can be converted to a quadratic equation in ω^2. Thus, $+\omega$ and $-\omega$ are both solutions of the fourth-order equation. However, $-\omega$ has the same mode shape as $+\omega$. Thus, there are four solutions of the homogeneous equation: $e^{i\omega_i t}\boldsymbol{X}_1$, $e^{-i\omega_i t}\boldsymbol{X}_1$, $e^{i\omega_2 t}\boldsymbol{X}_2$, and $e^{-i\omega_2 t}\boldsymbol{X}_2$ where ω_1 and ω_2 are the natural frequencies and $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are their corresponding mode shape vectors. The general solution is

$$\boldsymbol{x}(t) = C_1 e^{i\omega_1 t}\boldsymbol{X}_1 + C_2 e^{-i\omega_1 t}\boldsymbol{X}_1 + C_3 e^{i\omega_2 t}\boldsymbol{X}_2 + C_4 e^{-i\omega_2 t}\boldsymbol{X}_2 \tag{6.12}$$

Euler's identity is used in the above to replace the exponentials with complex exponents by trigonometric functions

$$\boldsymbol{x}(t) = [C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)]\boldsymbol{X}_1 + [C_3 \cos(\omega_2 t) + C_4 \sin(\omega_2 t)]\boldsymbol{X}_2 \tag{6.13}$$

The system has four initial conditions to satisfy $x_1(0) = x_{1,0}$, $x_2(0) = x_{2,0}$, $\dot{x}_1(0) = \dot{x}_{1,0}$, and $\dot{x}_2(0) = \dot{x}_{2,0}$. Their application yields

$$x_{1,0} = C_1 + C_3 \tag{6.14a}$$

$$x_{2,0} = C_1\chi_1 + C_3\chi_2 \tag{6.14b}$$

$$\dot{x}_{1,0} = \omega_1 C_2 + \omega_2 C_4 \tag{6.14c}$$

$$\dot{x}_{2,0} = \omega_1 C_2 \chi_1 + \omega_2 C_4 \chi_2 \tag{6.14d}$$

The equations are two sets of two simultaneous equations whose solutions are

$$C_1 = \frac{x_{1,0}\chi_2 - x_{2,0}}{\chi_2 - \chi_1} \tag{6.15a}$$

$$C_2 = \frac{\dot{x}_{1,0}\omega_2\chi_2 - \dot{x}_{2,0}\omega_1}{\omega_1\omega_2(\chi_2 - \chi_1)} \tag{6.15b}$$

$$C_3 = \frac{x_{2,0} - x_{1,0}\chi_2}{\chi_2 - \chi_1} \tag{6.15c}$$

$$C_4 = \frac{\dot{x}_{2,0}\omega_1 - \dot{x}_{1,0}\omega_2\chi_2}{\omega_1\omega_2(\chi_2 - \chi_1)} \tag{6.15d}$$

Trigonometric identities can be used to write Equation (6.13) as

$$\mathbf{x}(t) = A_1\mathbf{X}_1 \sin(\omega_1 t + \phi_1) + A_2\mathbf{X}_2 \sin(\omega_2 t + \phi_2) \tag{6.16}$$

where

$$A_1 = (C_1^2 + C_2^2)^{1/2} \tag{6.17a}$$

$$A_2 = (C_3^2 + C_4^2)^{1/2} \tag{6.17b}$$

$$\phi_1 = \tan^{-1}(C_2 \backslash C_1) \tag{6.17c}$$

$$\phi_2 = \tan^{-1}(C_4 \backslash C_3) \tag{6.17d}$$

EXAMPLE 6.5

The system of Example 6.3 is given initial displacements of $x_1(0) = \delta$ and $x_2(0) = -\delta$ and is released from rest. Determine the resulting response of the system.

SOLUTION

The natural frequencies are determined in the solution of Example 6.3 as $\omega_1 = 0.391\sqrt{\frac{k}{m}}$ and $\omega_2 = 1.47\sqrt{\frac{k}{m}}$. The mode shapes are $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.85 \end{bmatrix}$ and $\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.181 \end{bmatrix}$. The general form of the response is given by Equation (6.16) as

$$\mathbf{x}(t) = A_1\begin{bmatrix} 1 \\ 1.85 \end{bmatrix} \sin\left(0.391\sqrt{\frac{k}{m}}t + \phi_1\right) + A_2\begin{bmatrix} 1 \\ -0.181 \end{bmatrix} \sin\left(1.47\sqrt{\frac{k}{m}}t + \phi_2\right) \tag{a}$$

Application of initial conditions leads to

$$x_1(0) = \delta = A_1 \sin\phi_1 + A_2 \sin\phi_2 \tag{b}$$

$$x_2(0) = -\delta = 1.85A_1 \sin\phi_1 - 0.181A_2 \sin\phi_2 \tag{c}$$

$$\dot{x}_1(0) = 0 = 0.391A_1 \cos\phi_1 + 1.47A_2 \cos\phi_2 \tag{d}$$

$$\dot{x}_2(0) = 0 = (1.85)(0.391)A_1 \cos\phi_1 + (-0.181)(1.47)A_2 \cos\phi_2 \tag{e}$$

Equations (d) and (e) are satisfied if $\cos\phi_1 = \cos\phi_2 = 0$, which implies $\phi_1 = \phi_2 = \frac{\pi}{2}$. Then equations (b) and (c) become

$$A_1 + A_2 = \delta \tag{f}$$

$$1.85A_1 - 0.181A_2 = -\delta \tag{g}$$

Equations (f) and (g) are solved to yield $A_1 = -0.4038$ and $A_2 = 1.4038$, leading to a response of

$$\mathbf{x}(t) = -\delta\begin{bmatrix}0.403\\0.746\end{bmatrix}\sin\left(0.391\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right) + \delta\begin{bmatrix}1.403\\-0.254\end{bmatrix}\sin\left(1.47\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right) \tag{h}$$

EXAMPLE 6.6

For what initial conditions will the system of Example 6.4 vibrate as if it were a rigid-body rotation about point P, which is a distance $0.118L$ to the right of the mass center?

SOLUTION

The point P is determined to be a node for the second mode. Thus, only the first mode is represented in the solution

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}1\\\frac{1.42}{L}\end{bmatrix}\left\{C_1\cos\left(1.28\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(1.28\sqrt{\frac{k}{m}}t\right)\right\} \tag{a}$$

Application of initial conditions leads to

$$x_{1,0} = C_1 \tag{b}$$

$$x_{2,0} = \frac{1.42}{L}C_1 \tag{c}$$

$$\dot{x}_{1,0} = 1.28C_2 \tag{d}$$

$$\dot{x}_{2,0} = (1.28)\left(\frac{1.42}{L}\right)C_2 \tag{e}$$

Dividing Equation (a) by Equation (b) yields

$$\frac{x_{1,0}}{x_{2,0}} = 0.694L \tag{f}$$

Dividing Equation (d) by Equation (e) yields

$$\frac{\dot{x}_{1,0}}{\dot{x}_{2,0}} = 0.694L \tag{g}$$

Any boundary conditions satisfying Equation (f) and Equation (g) will eliminate the second mode from the response.

6.5 FREE VIBRATIONS OF A SYSTEM WITH VISCOUS DAMPING

Free vibrations of a system with viscous damping cannot be qualitatively defined as for SDOF systems. Assuming a normal-mode solution of $\mathbf{x} = \mathbf{X}e^{i\omega t}$ leads to an algebraic equation with complex coefficients to determine ω. Instead, a solution of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{\lambda t} \tag{6.18}$$

is assumed. Substitution of Equation (6.18) into Equation (6.1) leads to

$$\lambda^2 \boldsymbol{MX} + \lambda \boldsymbol{CX} + \boldsymbol{KX} = 0 \tag{6.19}$$

Equation (6.19) is viewed as a system of simultaneous algebraic equations to solve for χ. Equation (6.19) has a non-trivial solution if and only if

$$\det(\lambda^2 \boldsymbol{MX} + \lambda \boldsymbol{CX} + \boldsymbol{KX}) = 0 \tag{6.20}$$

Expansion of the determinant leads to a fourth-order polynomial equation for λ. The four roots for λ can be all real, two real, and one pair of complex conjugates or two pairs of complex conjugates. The real roots correspond to overdamped modes of vibration. The complex roots correspond to underdamped modes of vibration. The real roots can be repeated, in which case they correspond to vibrations that are critically damped.

For specific real values of λ, substitution into Equation (6.20) leads to real-mode shape vectors. Hence, the solution for four real values of λ is

$$\mathbf{x}(t) = C_1\mathbf{X}_1 e^{\lambda_1 t} + C_2\mathbf{X}_2 e^{\lambda_2 t} + C_3\mathbf{X}_3 e^{\lambda_3 t} + C_4\mathbf{X}_4 e^{\lambda_4 t} \tag{6.21}$$

For complex conjugate values of λ, Equation (6.20) leads to complex conjugate mode shapes. The solution corresponding to a pair of complex conjugate values of λ is

$$\mathbf{x}(t) = C_1\mathbf{X}e^{\lambda t} + C_2\overline{\mathbf{X}}e^{\overline{\lambda} t} \tag{6.22}$$

Writing $\lambda = \lambda_r + i\lambda_i$ and $\mathbf{X} = \mathbf{X}_\mathrm{r} + \mathbf{iX}_\mathrm{i}$ and using Euler's identity on the exponentials with complex exponents leads to

$$\begin{aligned}\mathbf{x}(t) &= e^{\lambda_r t}[C_1(\mathbf{X}_\mathrm{r} + \mathbf{X}_\mathrm{i})(\cos\lambda_i t + i\sin\lambda_i t) + C_2(\mathbf{X}_\mathrm{r} - \mathbf{iX}_\mathrm{i})(\cos\lambda_i t - i\sin\lambda_i t)] \\ &= e^{\lambda_r t}[A_1(\mathbf{X}_\mathrm{r}\cos\lambda_i t - \mathbf{X}_\mathrm{i}\sin\lambda_i t) + A_2(\mathbf{X}_\mathrm{r}\sin\lambda_i t + \mathbf{X}_\mathrm{i}\cos\lambda_i t)]\end{aligned} \tag{6.23}$$

where $A_1 = C_1 + C_2$ and $A_2 = i(C_1 - C_2)$ are redefined constants of integration.

EXAMPLE 6.7

Determine the response of the system of Figure 6.5 when using x_1 and x_2 as generalized coordinates when $\dot{x}_2(0) = 2\frac{m}{s}$ and all other initial conditions are zero.

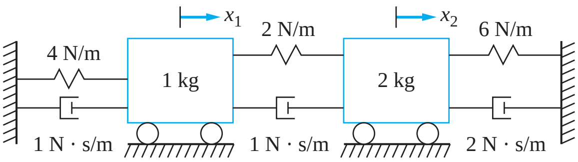


FIGURE 6.5
System of Example 6.7. Motion is initiated by giving the second mass an initial velocity of 2 m/s.

SOLUTION

The differential equations of motion for the system are

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{a}$$

Assume a solution of

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{\lambda t} \tag{b}$$

The values of λ which lead to a non-trivial solution of Equation (b) are the roots of

$$\begin{vmatrix} \lambda^2 + 2\lambda + 6 & -\lambda - 2 \\ -\lambda - 2 & 2\lambda^2 + 3\lambda + 8 \end{vmatrix} = 0 \tag{c}$$

Evaluation of the determinant leads to

$$(\lambda^2 + 2\lambda + 6)(2\lambda^2 + 3\lambda + 8) - (\lambda + 2)^2 = 0 \tag{d}$$

The roots of the fourth-order equation are $\lambda = -0.5122 \pm 1.7436i, -1.2378 \pm 2.2648i$. The system vibrates at frequencies $\omega_1 = 1.7436$ and $\omega_2 = 2.2468$. The complex modal fraction is determined from

$$\begin{bmatrix} \lambda^2 + 2\lambda + 6 & -\lambda - 2 \\ -\lambda - 2 & 2\lambda^2 + 3\lambda + 8 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{e}$$

The two equations represented by Equation (e) for the values of λ obtained previously are dependent. Thus, only the first equation is used, as

$$(\lambda^2 + 2\lambda + 6) - (\lambda + 2)\chi = 0 \tag{f}$$

or

$$\chi = \frac{\lambda^2 + 2\lambda + 6}{\lambda + 2} \tag{g}$$

For $\lambda = -0.5122 - 1.7436i$, the evaluation of Equation (g) becomes

$$\chi = \frac{(-0.5122 - 1.7436i)^2 + 2(-0.5122 - 1.7436i) + 6}{2 - 0.5122 - 1.7436i}$$
$$= (1.817 + 0.248i) \tag{h}$$

For $\lambda = -0.5122 + 1.7436i$, the evaluation leads to $\chi = (1.817 - 0.248i)$. For $-1.2378 \pm 2.2648i$, the evaluation of Equation (g) leads to $\chi = (-0.435 \mp 0.115i)$.

Using Equation (6.21), the response can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-0.5122t}\left(C_1\begin{bmatrix} 1 \\ 1.817 - 0.248i \end{bmatrix}e^{i1.7436t} + C_2\begin{bmatrix} 1 \\ 1.817 + 0.248i \end{bmatrix}e^{-i1.7436t}\right)$$
$$+ e^{-1.2378t}\left(C_3\begin{bmatrix} 1 \\ -0.435 - 0.115i \end{bmatrix}e^{i2.2468}C_4\begin{bmatrix} 1 \\ -0.435 + 0.115i \end{bmatrix}e^{-i2.2468}\right) \quad \text{(i)}$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-0.5122t}\left\{A_1\left(\begin{bmatrix} 1 \\ 1.817 \end{bmatrix}\cos 1.744t - \begin{bmatrix} 0 \\ 0.248 \end{bmatrix}\sin 1.744t\right)\right.$$
$$\left. + A_2\left(\begin{bmatrix} 1 \\ 1.817 \end{bmatrix}\sin 1.744t + \begin{bmatrix} 0 \\ 0.248 \end{bmatrix}\cos 1.744t\right)\right\}$$
$$+ e^{-1.2378t}\left\{A_3\left(\begin{bmatrix} 1 \\ -0.435 \end{bmatrix}\cos 2.247t - \begin{bmatrix} 0 \\ -0.115 \end{bmatrix}\sin 2.247t\right)\right.$$
$$\left. + A_4\left(\begin{bmatrix} 1 \\ -0.435 \end{bmatrix}\sin 2.247t + \begin{bmatrix} 0 \\ -0.115 \end{bmatrix}\cos 2.247t\right)\right\} \quad \text{(j)}$$

Applying the initial conditions leads to

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1.817 & 0.248 & -0.435 & -0.115 \\ -0.5122 & 1.744 & -1.238 & 2.247 \\ -1.390 & 3.295 & 0.871 & -0.258 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad \text{(k)}$$

Solution of Equation (k) leads to $A_1 = 4.49$, $A_2 = -1.95$, $A_3 = -2.12$ and $A_4 = 3.29$. Substitution of these results into Equation (j) leads to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-0.512t}\left(\begin{bmatrix} 4.49 \\ 7.68 \end{bmatrix}\cos 1.74t + \begin{bmatrix} -1.95 \\ -4.66 \end{bmatrix}\sin 1.74t\right)$$
$$+ e^{-1.237t}\left(\begin{bmatrix} -2.13 \\ 0.54 \end{bmatrix}\cos 2.25t + \begin{bmatrix} 3.29 \\ -1.67 \end{bmatrix}\sin 2.25t\right) \quad \text{(l)}$$

6.6 PRINCIPAL COORDINATES

It would be easier to solve uncoupled differential equations, but the coupling between coordinates is inevitable in most systems. The choice of generalized coordinates to derive the differential equations affects the coupling. If the coupling is through the stiffness matrix as in Example 6.1, the system is said to be statically coupled. If the coupling is through the mass matrix as in Example 6.2(b), the system is said to be dynamically coupled. Using the coordinates x and θ, the system of Example 6.2 is statically coupled and is not dynamically coupled. Using the coordinates x_1 and x_2, the differential equations are dynamically coupled but not statically coupled. A system can be statically coupled, dynamically coupled, statically coupled and dynamically coupled, or neither statically or dynamically coupled, depending on the choice of generalized coordinates. The choice of generalized coordinates does not affect the natural frequencies.

Suppose the differential equations are neither statically coupled nor dynamically coupled using a set of coordinates p_1 and p_2, called the principal coordinates. Then the differential equations are written as

$$\ddot{p}_1 + \omega_1^2 p_1 = 0 \tag{6.24}$$

$$\ddot{p}_2 + \omega_2^2 p_2 = 0 \tag{6.25}$$

The solutions of Equation (6.24) and (6.25) are simply

$$p_1(t) = P_1 \sin(\omega_1 t + \phi_1) \tag{6.26}$$

$$p_2(t) = P_2 \sin(\omega_2 t + \phi_2) \tag{6.27}$$

The decoupled system behaves as two SDOF systems. Since the choice of generalized coordinates does not affect the natural frequencies of the system, ω_1 and ω_2 are properties of the system. When written using coordinates x_1 and x_2,

$$\begin{aligned}\mathbf{x}(t) &= A_1\mathbf{X}_1 \sin(\omega_1 t + \phi_1) + A_2\mathbf{X}_2 \sin(\omega_2 t + \phi_2) \\ &= \frac{A_1}{P_1}\mathbf{X}_1 p_1(t) + \frac{A_1}{P_2}\mathbf{X}_2 p_2(t)\end{aligned} \tag{6.28}$$

Taking $\frac{A_1}{P_1} = \frac{A_1}{P_2} = 1$, Equation (6.28) becomes

$$\mathbf{x}(t) = \mathbf{X}_1 p_1(t) + \mathbf{X}_2 p_2(t) \tag{6.29}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \chi_1 \end{bmatrix} p_1 + \begin{bmatrix} 1 \\ \chi_2 \end{bmatrix} p_2 \tag{6.30}$$

Equation (6.30) is solved for the principal coordinates in terms of the original generalized coordinates yielding

$$p_1 = \frac{1}{\chi_2 - \chi_1}(\chi_2 x_1 - x_2) \tag{6.31}$$

$$p_2 = \frac{1}{\chi_2 - \chi_1}(x_2 - \chi_1 x_1) \tag{6.32}$$

Without loss of generality, since the generalized coordinates can represent points that have zero displacement for z, given mode $\chi_2 - \chi_1$ can be ignored and

$$p_1 = \chi_2 x_1 - x_2 \tag{6.33}$$

$$p_2 = x_2 - \chi_1 x_1 \tag{6.34}$$

The principal coordinates for a two degree-of-freedom system can be examined by looking at the nodes for a system. The second mode shape has a node which is in the system. This is a point of zero displacement for that node, and the response of that point only includes the first mode. This point can be taken to be a principal coordinate representing the first mode. The first mode does not have a node that is a particle on the system. Thus, the second mode does not represent the motion of a particle in the system.

EXAMPLE 6.8

Describe the principal coordinates for the system of Example 6.4. Write the differential equations for the principal coordinates.

SOLUTION

Recall that the natural frequency and modal fraction for the first mode using x and θ as generalized coordinates are $\omega_1 = 1.28\sqrt{\frac{k}{m}}$ and $\chi_1 = \frac{1.42}{L}$. The natural frequency and modal fraction for the second mode are $\omega_2 = 2.07\sqrt{\frac{k}{m}}$ and $\chi_2 = -\frac{8.44}{L}$. Using Equations (6.33) and (6.34), the principal coordinates are

$$p_1(t) = -\frac{8.44}{L}x(t) - \theta(t) \tag{a}$$

$$p_2(t) = \theta(t) - \frac{1.42}{L}x(t) \tag{b}$$

Equation (a) is the negative of the displacement of the node for the second mode, which as noted in Example 6.4 represents a rigid-body rotation about a point $0.118L$ to the right of the midspan of the bar. Equation (b) represents the negative of the rigid-body rotation $0.19L$ from the left end of the bar.

The differential equations the principal coordinates satisfy are

$$\ddot{p}_1 + 1.64\frac{k}{m}p_1 = 0 \tag{c}$$

$$\ddot{p}_2 + 4.28\frac{k}{m}p_2 = 0 \tag{d}$$

It is not possible to find principal coordinates for a system with a general form of viscous damping. However, if the damping matrix is proportional to a linear combination of the stiffness matrix and the damping matrix, the principal coordinates for the undamped system uncouple the system. The differential equations governing the principal coordinates become

$$\ddot{p}_1 + 2\zeta_1\omega_1\dot{p}_1 + \omega_1^2 p_1 = 0 \tag{6.35}$$

$$\ddot{p}_2 + 2\zeta_2\omega_2\dot{p}_2 + \omega_2^2 p_2 = 0 \tag{6.36}$$

where ζ_1 and ζ_2 are called modal damping ratios. This is covered in more detail in Chapter 8.

6.7 HARMONIC RESPONSE OF TWO DEGREE-OF-FREEDOM SYSTEMS

The harmonic response of two degree-of-freedom systems is determined using the method of *undetermined coefficients*. First, consider undamped systems whose differential equations are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}\sin(\omega t) \tag{6.37}$$

where $\mathbf{F} = [f_1 \quad f_2]^T$ is a vector of constants.

The method of undetermined coefficients can be used to find the steady-state solution. Assume a steady-state response of

$$\mathbf{x} = \mathbf{U}\sin(\omega t) \tag{6.38}$$

where $\mathbf{U} = [u_1 \quad u_2]^T$. Substitution of Equation (6.38) into Equation (6.37) leads to

$$-\omega^2\mathbf{MU}\sin(\omega t) + \mathbf{KU}\sin(\omega t) = \mathbf{F}\sin(\omega t) \tag{6.39}$$

from which the equation to solve for the components of $\mathbf{U}$ is

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{U} = \mathbf{F} \tag{6.40}$$

The component equations represented by Equation (6.40) are

$$(-\omega^2 m_{1,1} + k_{1,1})u_1 + (-\omega^2 m_{1,2} + k_{1,2})u_2 = f_1 \tag{6.41}$$

$$(-\omega^2 m_{2,1} + k_{2,1})u_1 + (-\omega^2 m_{2,2} + k_{2,2})u_2 = f_2 \tag{6.42}$$

The solution of Equation (6.41) and Equation (6.42) provide the values of u_1 and u_2.

The steady-state amplitudes are chosen to be positive. If a negative value is obtained (say $u_2 < 0$), the response of the system is written as $|u_2|\sin(\omega t - \pi)$.

EXAMPLE 6.9

Consider the two degree-of-freedom system of Figure 6.6. Determine the steady-state response of the system.

SOLUTION

The differential equations governing the motion of the system are

$$\ddot{x}_1 + 2x_1 - x_2 = 0 \tag{a}$$

$$2\ddot{x}_2 - x_1 + 3x_2 = 10\sin(2t) \tag{b}$$

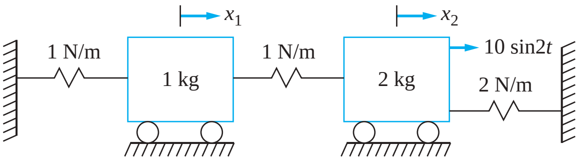


FIGURE 6.6
System of Example 6.9.

The steady-state response is determined by assuming

$$x_1 = u_1 \sin(2t) \tag{c}$$

$$x_2 = u_2 \sin(2t) \tag{d}$$

Substituting the solution into the differential equations leads to

$$-4u_1 + 2u_1 - u_2 = 0 \tag{e}$$

$$-8u_2 - u_1 + 3u_2 = 10 \tag{f}$$

or

$$-2u_1 - u_2 = 0 \tag{g}$$

$$-u_1 - 5u_2 = 10 \tag{h}$$

The solution to Equation (g) and Equation (h) is $u_1 = \frac{10}{9}$ and $u_2 = -\frac{20}{9}$. The steady-state responses of the two masses are

$$u_1(t) = \frac{10}{9}\sin(2t) \tag{i}$$

$$u_2(t) = \frac{20}{9}\sin(2t - \pi) \tag{j}$$

Now consider the steady-state responses for systems with viscous damping. The general form of the equation for systems that are viscously damped is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}\sin(\omega t) \tag{6.43}$$

or

$$\begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\sin(\omega t) \tag{6.44}$$

A steady-state response of

$$x_1 = u_1 \sin(\omega t) + v_1 \cos(\omega t) \tag{6.45}$$

$$x_2 = u_2 \sin(\omega t) + v_2 \cos(\omega t) \tag{6.46}$$

is assumed. Substituting into Equation (6.43) leads to four equations for four unknowns. The steady-state responses for x_1 and x_2 are written as

$$x_1 = X_1 \sin(\omega t - \phi_1) \tag{6.47}$$

and

$$x_2 = X_2 \sin(\omega t - \phi_2) \tag{6.48}$$

where

$$X_i = \sqrt{u_i^2 + v_i^2} \tag{6.49}$$

and

$$\phi_i = \tan^{-1}\left(\frac{v_i}{u_i}\right) \tag{6.50}$$

EXAMPLE 6.10

Find the steady-state response for the system of Figure 6.7.

SOLUTION

The differential equations governing the motion of the two degree-of-freedom system shown are

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -20 \\ -20 & 20 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 300 & -200 \\ -200 & 400 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\sin 5t \tag{a}$$

Assume a steady-state response of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\sin 5t + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cos 5t \tag{b}$$

Substitution of Equation (b) into Equation (a) gives

$$\begin{aligned} &\begin{bmatrix} -50 & 0 \\ 0 & -25 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\sin 5t + \begin{bmatrix} -50 & 0 \\ 0 & -25 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cos 5t \\ &\quad + \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\cos 5t + \begin{bmatrix} -150 & 100 \\ 100 & -100 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\sin 5t \\ &\quad + \begin{bmatrix} 300 & -200 \\ -200 & 400 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\sin 5t + \begin{bmatrix} 300 & -200 \\ -200 & 400 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cos 5t \\ &\quad = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\sin 5t \end{aligned} \tag{c}$$

Collecting coefficients of $\sin 5t$ and $\cos 5t$ from each equation leads to

$$\begin{bmatrix} 250 & -200 & -150 & 100 \\ -200 & 375 & 100 & -100 \\ 150 & -100 & 250 & -200 \\ -100 & 100 & -200 & 375 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \tag{d}$$

The solution to Equation (c) is $u_1 = 0.0212$, $u_2 = 0.0203$, $v_1 = -0.0077$, and $v_2 = -0.0039$. Substitution into Equation (b) gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.0212 \\ 0.0203 \end{bmatrix}\sin 5t + \begin{bmatrix} -0.0077 \\ -0.0039 \end{bmatrix}\cos 5t \tag{e}$$

or

$$x_1(t) = 0.0225\sin(5t + 0.348) \tag{f}$$

$$x_2(t) = 0.0207\sin(5t + 0.188) \tag{g}$$

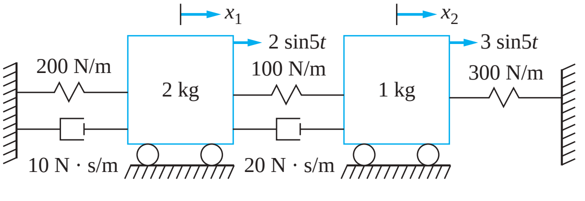


FIGURE 6.7
System of Example 6.10.

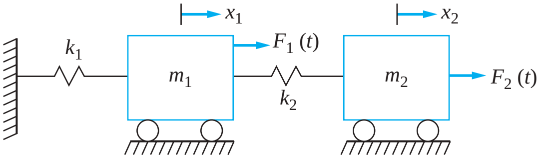
6.8 TRANSFER FUNCTIONS

Transfer functions are the ratio of the Laplace transform of a system output to the Laplace transform of a system input. When the system has multiple input and multiple outputs, a matrix of transfer functions is defined. A two degree-of-freedom system has two outputs and possibly two inputs, as illustrated in Figure 6.8. The transfer function matrix for this system is

$$\mathbf{G}(s) = \begin{bmatrix} G_{1,1}(s) & G_{1,2}(s) \\ G_{2,1}(s) & G_{2,2}(s) \end{bmatrix} \tag{6.51}$$

where $G_{i,j}(s)$ is the transfer function for x_i due to a force applied at x_j. Recalling the physical meaning of the transfer function from Chapter 5, it also represents the transform of the response due to a unit impulse. Thus, $G_{i,j}(s)$ also is the Laplace transform of the response of x_i due to an unit impulse applied at the location which is described by x_j.

FIGURE 6.8
A two degree-of-freedom system with two inputs.



EXAMPLE 6.11

The system of Figure 6.9 is at rest in equilibrium when a unit impulse is applied to the 2 kg block. Determine the resulting response of the 1 kg block.

SOLUTION

The differential equations governing the motion of the system are

$$\ddot{x}_1 + 1000x_1 - 500x_2 = 0 \tag{a}$$

$$2\ddot{x}_2 - 500x_1 + 1000x_2 = F(t) \tag{b}$$

Taking the Laplace transform of Equations (a) and (b) and using the principle of linearity leads to

$$\mathcal{L}\{\ddot{x}_1\} + 1000\mathcal{L}\{x_1\} - 500\mathcal{L}\{x_2\} = 0 \tag{c}$$

$$2\mathcal{L}\{\ddot{x}_2\} - 500\mathcal{L}\{x_1\} + 1000\mathcal{L}\{x_2\} = \mathcal{L}\{F(t)\} \tag{d}$$

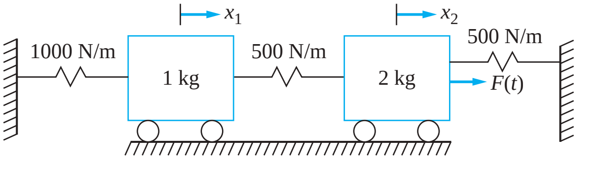


FIGURE 6.9
System of Example 6.11.

Letting $X_1(s) = \mathcal{L}\{x_1(t)\}$, $X_2(s) = \mathcal{L}\{x_2(t)\}$, and $F(s) = \mathcal{L}\{F(t)\}$ and using the property of transform of derivatives leads to

$$(s^2 + 1000)X_1(s) - 500X_2(s) = 0 \tag{e}$$

$$-500X_1(s) + (2s^2 + 1000)X_2(s) = F(s) \tag{f}$$

Writing Equations (e) and (f) in matrix form, we have

$$\begin{bmatrix} s^2 + 1000 & -500 \\ -500 & 2s^2 + 1000 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix} \tag{g}$$

Cramer's rule is used to solve for $X_1(s)$, leading to

$$X_1(s) = \frac{\begin{vmatrix} 0 & -500 \\ F(s) & 2s^2 + 1000 \end{vmatrix}}{\begin{vmatrix} s^2 + 1000 & -500 \\ -500 & 2s^2 + 1000 \end{vmatrix}} \tag{h}$$

Evaluation of the determinants leads to

$$X_1(s) = \frac{500F(s)}{2s^4 + 3000s^2 + 750{,}000} \tag{i}$$

The appropriate transfer function is

$$G_{1,2}(s) = \frac{X_1(s)}{F(s)} = \frac{250}{s^4 + 1500s^2 + 375{,}000} \tag{j}$$

The impulsive response is obtained by inverting the transfer function. To this end, the transfer function is factored as

$$G_{1,2}(s) = \frac{250}{(s^2 + 1183)(s^2 + 317)} \tag{k}$$

A partial fraction decomposition of Equation (k) leads to

$$G_{1,2}(s) = \frac{0.2887}{s^2 + 317} - \frac{0.2887}{s^2 + 1183} \tag{l}$$

Inversion of the transform leads to

$$x_{i_{1,2}} = 0.0162 \sin 17.8t - 0.0084 \sin 34.4t \tag{m}$$

EXAMPLE 6.12

Determine the transfer function for the 20 kg block of the system in Figure 6.10 due to a force applied to the 20 kg block.

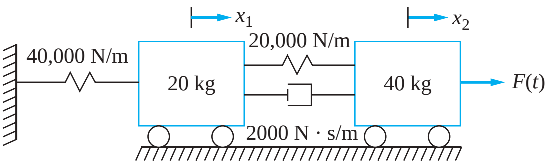


FIGURE 6.10
System of Example 6.12.

SOLUTION

The differential equations governing the system are

$$20\ddot{x}_1 + 2000\dot{x}_1 - 2000\dot{x}_2 + 60{,}000x_1 - 20{,}000x_2 = 0 \tag{a}$$

$$40\ddot{x}_2 - 2000\dot{x}_1 + 2000\dot{x}_2 - 20{,}000x_1 + 20{,}000x_2 = F(t) \tag{b}$$

Taking the Laplace transform of both equations and using the properties of the transform of derivatives and linearity yields

$$(20s^2 + 2000s + 60{,}000)X_1(s) - (2000s + 20{,}000)X_2(s) = 0 \tag{c}$$

$$-(2000s + 20{,}000)X_1(s) + (40s^2 + 2000s + 20{,}000)X_2(s) = F(s) \tag{d}$$

Rewriting Equations (c) and (d) in matrix form

$$\begin{bmatrix} 20s^2 + 2000s + 60{,}000 & -2000s - 20{,}000 \\ -2000s - 20{,}000 & 40s^2 + 2000s + 20{,}000 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix} \tag{e}$$

Cramer's rule is used to solve for $X_2(s)$, leading to

$$X_1(s) = \frac{\begin{vmatrix} 20s^2 + 2000s + 60{,}000 & 0 \\ -2000s + 20{,}000 & F(s) \end{vmatrix}}{\begin{vmatrix} 20s^2 + 2000s + 60{,}000 & -2000s - 20{,}000 \\ -2000s - 20{,}000 & 40s^2 + 2000s + 20{,}000 \end{vmatrix}} \tag{f}$$

Evaluation of the determinants leads to

$$X_2(s) = \frac{(20s^2 + 2000s + 60{,}000)F(s)}{800s^4 + 1.2 \times 10^5 s^3 + 2.8 \times 10^6 s^2 + 8 \times 10^7 s + 8 \times 10^8} \tag{g}$$

The appropriate transfer function is

$$G_{22}(s) = \frac{20s^2 + 2000s + 60{,}000}{800s^4 + 1.2 \times 10^5 s^3 + 2.8 \times 10^6 s^2 + 8 \times 10^7 s + 8 \times 10^8} \tag{h}$$

The transfer function may be used to derive a convolution integral response for the system. Note that

$$X_i(s) = F_j(s)G_{i,j}(s) \tag{6.52}$$

where $X_{i,j}(s)$ is the response of the system for $x_i(t)$ due to a force $F_j(t)$ applied at the location specified by $x_j(t)$. Using property *B7* (transform of convolution), we have

$$x_i(s) = \int_0^t F_j(\tau)h_{i,j}(t - \tau)d\tau \tag{6.53}$$

where $h_{i,j}(t)$ is the impulsive response $h_{i,j}(t) = \mathcal{L}^{-1}\{G_{i,j}(s)\}$.

Equation (6.53) is the convolution integral solution for the response of a two degree-of-freedom system. It is similar to that of a SDOF system.

EXAMPLE 6.13

Determine the response of the 1 kg mass of Figure 6.9 when the time-dependent force of Figure 6.11 is applied to the 2 kg block.

SOLUTION

The mathematical form of the force shown in Figure 6.11 is

$$F(t) = 10(1 - 10t)[u(t) - u(t - 0.1)] \tag{a}$$

The impulsive response of the 1 kg block due to a unit impulse applied to the 2 kg block is calculated in Example 6.11. The convolution integral of Equation (6.53) is used to determine the response of the system of Figure 6.10 as

$$\begin{aligned} x_1(t) = \int_0^t & 10(1 - 10\tau)[u(\tau) - u(\tau - 0.1)][0.0162 \sin 17.8(t - \tau) \\ & - 0.0084 \sin 34.4(t - \tau)]\, d\tau \end{aligned} \tag{b}$$

Equation (b) is written as

$$\begin{aligned} x_1(t) = 10\Bigg\{0.0162\Bigg[& \int_0^t (1 - 10\tau) \sin 17.8(t - \tau) u(\tau)\, d\tau \\ & - \int_0^t (1 - 10\tau) \sin 17.8(t - \tau) u(\tau - 0.1)\, d\tau\Bigg] \\ & - 0.0084\Bigg[\int_0^t (1 - 10\tau) \sin 34.4(t - \tau) u(\tau)\, d\tau \\ & - \int_0^t (1 - 10\tau) \sin 34.4(t - \tau) u(\tau - 0.1)\, d\tau\Bigg]\Bigg\} \end{aligned} \tag{c}$$

The integrals of Equation (c) are evaluated using the entries of Table 5.1. Use the table for the delayed ramp excitation with $A = -10$ and $B = 1$ with $\omega_n = 17.8$ for the first two integrals. Use $t_0 = 0$ for the first integral and $t_0 = 0.1$ for the second. The third and fourth integrals are evaluated using $\omega_n = 34.4$. Use $m_{eq} = 1$ when evaluating the integrals. For example, the second integral is evaluated as

$$\begin{aligned} & \int_0^t (1 - 10\tau) \sin 17.8(t - \tau) u(\tau - 0.1)\, d\tau \\ & \quad = \frac{-10}{317}\left[t + \frac{1}{-10} - \left(0.1 + \frac{1}{-10}\right) \cos 17.8(t - 0.1) \right. \\ & \qquad \left. - \frac{1}{17.8} \sin 17.8(t - 0.1)\right] u(t - 0.1) \\ & \quad = 0.0315[t - 0.1 - 0.0562 \sin 17.8(t - 0.1)]u(t - 0.1) \end{aligned} \tag{d}$$

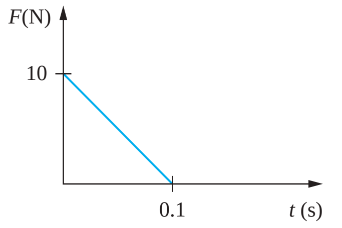


FIGURE 6.11
Excitation of Example 6.13.

The resulting solution is

$$\begin{aligned} x_1(t) = 10\{&(0.0162)(0.0315)[t - 0.1 - 0.1\cos 17.8t - 0.0562\sin 17.8t]u(t) \\ &- (0.0162)(0.0315)[t - 0.1 - 0.0562\sin 17.8(t - 0.1)]u(t - 0.1) \\ &- (0.0084)(0.0085)[t - 0.1 - 0.1\cos 34.8t - 0.0287\sin 17.8t]u(t) \\ &+ (0.0084)(0.0085)[t - 0.1 - 0.0287\sin 34.88(t - 0.1)]u(t - 0.1)\} \end{aligned} \tag{e}$$

Simplification results in

$$\begin{aligned} x_1(t) = (&0.0044t - 0.00044 - 0.0051\cos 17.8t + 7.14 \times 10^{-4}\cos 34.8t \\ &- 2.87 \times 10^{-4}\sin 17.8t + 2.05 \times 10^{-5}\sin 34.8t)u(t) \\ &- [0.0044t - 0.00044 - 2.87 \times 10^{-4}\sin 17.8(t - 0.1) \\ &+ 2.05 \times 10^{-5}\sin 34.8(t - 0.1)]u(t - 0.1) \end{aligned} \tag{f}$$

6.9 SINUSOIDAL TRANSFER FUNCTION

The use of the method of undetermined coefficients is fine for calculation of the steady-state amplitudes for a specific frequency, but the determination of the frequency response using this method leads to much unnecessary algebra. An alternate method is to use the *Laplace transform method.*

Consider the Laplace transform of a system subject to a sinusoidal input of $F(t) = F_0 \sin \omega t$:

$$X(s) = G(s)F(s) = \frac{F_0\omega}{s^2 + \omega^2}G(s) \tag{6.54}$$

where $G(s)$ is the transfer function. For an nth order system, the denominator of $G(s)$ is of order n. Let $s_1, s_2, \ldots, s_n$ where $Re\,(s_j) < 0$ for $j = 1, 2, \ldots, n$ is the zeros of the denominator of the transfer function. A partial fraction decomposition leads to

$$X(s) = \frac{A_1}{s + i\omega} + \frac{A_2}{s - i\omega} + \frac{B_1}{s - s_1} + \frac{B_2}{s - s_2} + \cdots + \frac{B_n}{s - s_n} \tag{6.55}$$

The steady-state response is obtained by inverting the first two terms in $X(s)$ as

$$\lim_{t\to\infty} \mathcal{L}^{-1}\left\{\frac{B_1}{s-s_1}+\frac{B_2}{s-s_2}+\cdots+\frac{B_n}{s-s_n}\right\}$$

$$= \lim_{t\to\infty}(B_1 e^{s_1 t} + B_2 e^{s_2 t} + \cdots B_n e^{s_n t}) = 0 \tag{6.56}$$

The steady-state response is

$$x(t) = \mathcal{L}^{-1}\left\{\frac{A_1}{s+i\omega}+\frac{A_2}{s-i\omega}\right\} \tag{6.57}$$

where

$$A_1 = \lim_{s\to -i\omega}\frac{F_0\omega G(s)(s+i\omega)}{s^2+\omega^2} = \frac{F_0}{-2i}G(-i\omega) \tag{6.58}$$

and

$$A_2 = \lim_{s\to i\omega}\frac{F_0\omega G(s)(s+i\omega)}{s^2+\omega^2} = \frac{F_0}{2i}G(i\omega) \tag{6.59}$$

The steady-state response becomes

$$\begin{aligned} x(t) &= A_1 e^{-i\omega t} + A_2 e^{i\omega t} \\ &= \frac{F_0[G(-i\omega)e^{-i\omega t} - G(i\omega)e^{i\omega t}]}{-2i} \end{aligned} \tag{6.60}$$

Since $G(i\omega)$ is a complex number, it can be expressed as

$$G(i\omega) = |G(i\omega)|e^{i\phi} \tag{6.61}$$

where

$$|G(i\omega)| = \sqrt{\mathrm{Re}[G(i\omega)]^2 + \mathrm{Im}[G(i\omega)]^2} \tag{6.62}$$

and

$$\phi = \tan^{-1}\left\{\frac{\mathrm{Im}[G(i\omega)]}{\mathrm{Re}[G(i\omega)]}\right\} \tag{6.63}$$

Substituting Equation (6.61) into Equation (6.60) and noting that $G(-i\omega) = \overline{G(i\omega)} = |G(i\omega)|e^{-i\phi}$ yields

$$x(t) = F_0|G(i\omega)|\frac{e^{i(\omega t+\phi)} - e^{-i(\omega t+\phi)}}{2i} \tag{6.64}$$

or

$$x(t) = F_0|G(i\omega)|\sin(\omega t + \phi) \tag{6.65}$$

The steady-state amplitude of any system is the magnitude of the excitation times the magnitude of the *sinusoidal transfer function* $G(i\omega)$. This is the frequency response of the system. The full power of the sinusoidal transfer function is not needed for SDOF systems because there exists only one steady-state amplitude. The steady-state amplitude in Equation (6.65) is non-dimensionalized by

$$\frac{k_1 X_1}{F_0} = k_1|G(i\omega)| \tag{6.66}$$

EXAMPLE 6.14

Determine the steady-state response of the 40 kg mass of Figure 6.12 when subject to a sinusoidal force of magnitude 200 N at a frequency of 50 rad/s.

SOLUTION

The transfer function for the system is determined in Example 6.12 as

$$G(s) = \frac{20s^2 + 2 \times 10^3 s + 6 \times 10^4}{8 \times 10^2 s^4 + 1.2 \times 10^5 s^3 + 2.8 \times 10^6 s^2 + 8 \times 10^7 s + 8 \times 10^8} \tag{a}$$

which becomes

$$G(s) = \frac{0.025s^2 + 2.5s + 75}{s^4 + 1.5 \times 10^2 s^3 + 3500s^2 + 1 \times 10^5 s + 1 \times 10^6} \tag{b}$$

when the numerator and denominator are divided by 8×10^2.

Use of the sinusoidal transfer function yields

$$x(t) = 200|G(50i)|\sin(\omega t + \phi) \tag{c}$$

where

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}(50i)}{\mathrm{Re}(50i)}\right) \tag{d}$$

Performing the calculations leads to

$$\begin{aligned} G(50i) &= \frac{0.025(50i)^2 + 2.5(50i) + 75}{(50i)^4 + 1.5 \times 10^2(50i)^3 + 3500(50i)^2 + 1 \times 10^5(50i) + 1 \times 10^6} \\ &= \frac{12.5 - 125i}{-(1.5 + 1.375i)10^6} = -(9.08 + 0.00817i)10^6 = 9.08e^{-3.13i} \end{aligned} \tag{e}$$

Thus the steady-state response of the system is

$$\begin{aligned} x(t) &= 200(9.08 \times 10^6)\sin(50t - 3.13) \\ &= 0.0018\sin(50t - 3.13)\text{ m} \end{aligned} \tag{f}$$

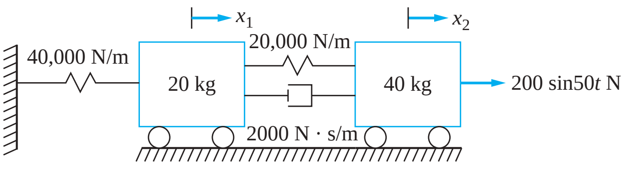


FIGURE 6.12
System of Example 6.14.

6.10 FREQUENCY RESPONSE

The frequency response refers to the variation of steady-state amplitude with a frequency of excitation. It is often described nondimensionally. A general two degree-of-freedom system is illustrated in Figure 6.13. The steady-state amplitudes are functions of the eleven parameters shown as

$$X_1 = X_1(m_1, m_2, k_1, k_2, k_3, c_1, c_2, c_3, F_{01}, F_{02}, \omega) \tag{6.67}$$

$$X_2 = X_2(m_1, m_2, k_1, k_2, k_3, c_1, c_2, c_3, F_{01}, F_{02}, \omega) \tag{6.68}$$

The Buckingham Pi theorem implies that a nondimensional formulation of the relationship between a steady-state amplitude and all parameters involves twelve (11 independent + 1 dependent) parameters minus three dimensions for nine nondimensional parameters. Many of the parameters would simply be mass, stiffness, and damping coefficient ratios. Unlike a SDOF system where the nondimensional relationship can be summarized on one set of coordinate axes (M versus r for different values of ζ), it is almost impossible to determine the effect of every parameter independently. The system has two parameters: the natural frequencies, which are determined from a quadratic equation. The modal fractions are determined from the solution of the resulting equation when the normal mode solution is assumed at a natural frequency.

Instead of having a general equation for the frequency response, each system configuration is studied individually. Consider the system of Figure 6.14. The differential equations governing the motion of this system are

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1(t) \tag{6.69}$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = F_2(t) \tag{6.70}$$

The matrix of transfer functions is determined as

$$\mathbf{G}(s) = \frac{1}{m_1m_2s^4 + (m_1k_2 + m_2k_1 + m_2k_2)s^2 + k_1k_2} \times \begin{bmatrix} m_2s^2 + k_2 & k_2 \\ k_2 & m_1s^2 + k_1 + k_2 \end{bmatrix} \tag{6.71}$$

The sinusoidal transfer functions are determined by substituting $s = i\omega$,

$$\mathbf{G}(i\omega) = \frac{1}{m_1m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} \times \begin{bmatrix} -m_2\omega^2 + k_2 & k_2 \\ k_2 & -m_1\omega^2 + k_1 + k_2 \end{bmatrix} \tag{6.72}$$

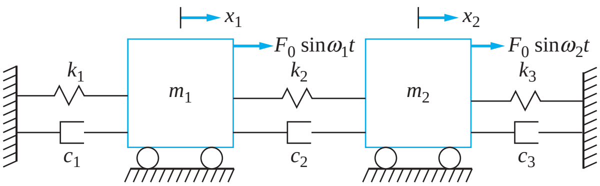


FIGURE 6.13
A general two degree-of-freedom system.

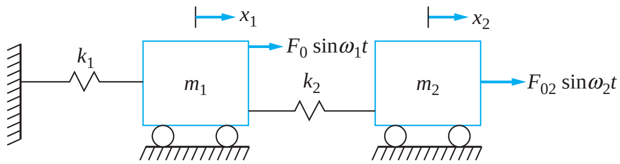


FIGURE 6.14
Two degree-of-freedom system with parameters m_1, m_2, k_1, k_2, F_{01}, F_{02}, and ω.

The steady-state amplitudes due to a harmonic force $F_1(t) = F_0 \sin\omega t$ is determined using the sinusoidal transfer functions as

$$X_1 = F_0|G_{1,1}(i\omega)| = \left|\frac{F_0(-m_2\omega^2 + k_2)}{m_1 m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2}\right| \tag{6.73}$$

$$X_2 = F_0|G_{2,1}(i\omega)| = \left|\frac{F_0k_2}{m_1 m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2}\right| \tag{6.74}$$

There are seven parameters, six independent parameters (m_1, m_2, k_1, k_2, F_0, ω) and one dependent parameter (X_1) in Equation (6.73) involving three independent dimensions (M, L, T). The Buckingham Pi theorem suggests there are $7 - 3 = 4$ independent dimensionless parameters involved in a nondimensional formulation. Equations (6.73) and (6.74) are nondimensionalized by dividing by F_0 and multiplying by something that has dimensions of stiffness (say k_1) as

$$\frac{k_1X_1}{F_0} = \left|\frac{-m_2\omega^2 + k_2}{\frac{m_1}{k_1}m_2\omega^4 - \left(\frac{m_1}{k_1}k_2 + m_2 + \frac{m_2k_2}{k_1}\right)\omega^2 + k_2}\right| \tag{6.75}$$

Defining

$$\omega_{1,1} = \sqrt{\frac{k_1}{m_1}} \tag{6.76}$$

$$\omega_{2,2} = \sqrt{\frac{k_2}{m_2}} \tag{6.77}$$

as parameters that have dimensions of $1/T$. Note that these are not the natural frequencies of the two degree-of-freedom system, they are just defined for convenience. Factoring out k_2 from the numerator and denominator of Equation (6.75) and rewriting the resulting equation in terms of $\omega_{1,1}$ and $\omega_{2,2}$ leads to

$$\frac{k_1X_1}{F_0} = \left|\frac{-\frac{\omega^2}{\omega_{2,2}^2} + 1}{\frac{\omega^4}{\omega_{1,1}^2\omega_{2,2}^2} - \left(\frac{1}{\omega_{1,1}^2} + \frac{1}{\omega_{2,2}^2} + \frac{m_2}{m_1\omega_{1,1}^2}\right)\omega^2 + 1}\right| \tag{6.78}$$

Defining

$$\mu = \frac{m_2}{m_1} \tag{6.79}$$

$$r_1 = \frac{\omega}{\omega_{1,1}} \tag{6.80}$$

$$r_2 = \frac{\omega}{\omega_{2,2}} \tag{6.81}$$

the right-hand side of Equation (6.78) is written as

$$M_{1,1}(r_1, r_2, \mu) = \left|\frac{1 - r_2^2}{r_1^2 r_2^2 - r_2^2 - (1+\mu)r_1^2 + 1}\right| \tag{6.82}$$

In a similar fashion, it is shown that

$$\frac{k_1 X_2}{F_0} = M_{2,1}(r_1, r_2, \mu) = \left|\frac{1}{r_1^2 r_2^2 - r_2^2 - (1+\mu)r_1^2 + 1}\right| \tag{6.83}$$

The frequency responses are plotted against r_1 for $r_2 = 0.5$ and $\mu = 0.5$. Both are shown in Figure 6.15.

Frequency-response equations for the force applied to the mass m_2 are

$$\frac{k_1 X_1}{F_0} = M_{1,2}(r_1, r_2, \mu) = \left|\frac{1}{r_1^2 r_2^2 - r_2^2 - (1+\mu)r_1^2 + 1}\right| \tag{6.84}$$

and

$$\frac{k_1 X_2}{F_0} = M_{2,2}(r_1, r_2, \mu) = \left|\frac{r_1^2\left(1 + \dfrac{\mu}{r_2^2}\right) + 1}{r_1^2 r_2^2 - r_2^2 - (1+\mu)r_1^2 + 1}\right| \tag{6.85}$$

Equations (6.84) and (6.85) versus r_1 for specific values of r_2, μ, and v are plotted in Figure 6.16 on page 415.

The frequency response of an undamped two degree-of-freedom system has two asymptotes corresponding to the natural frequencies of the system. These are the values of ω for which the denominator of the frequency response is zero. From Equation (6.73), this becomes

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0 \tag{6.86}$$

whose solutions are

$$\omega = \left(\frac{m_1 k_2 + m_2 k_1 + m_2 k_2 \pm \sqrt{(m_1 k_2 + m_2 k_1 + m_2 k_2)^2 - 4 m_1 m_2 k_1 k_2}}{2}\right)^{1/2} \tag{6.87}$$

Equation (6.87) is written as

$$\omega = \frac{\omega_{1,1}}{\sqrt{2}}\sqrt{1 + \left(\frac{\omega_{2,2}}{\omega_{1,1}}\right)^2(1+\mu) \pm \sqrt{\left(\frac{\omega_{2,2}}{\omega_{1,1}}\right)^4(1+\mu)^2 + 2\left(\frac{\omega_{2,2}}{\omega_{1,1}}\right)^2(\mu - 1) + 1}} \tag{6.88}$$

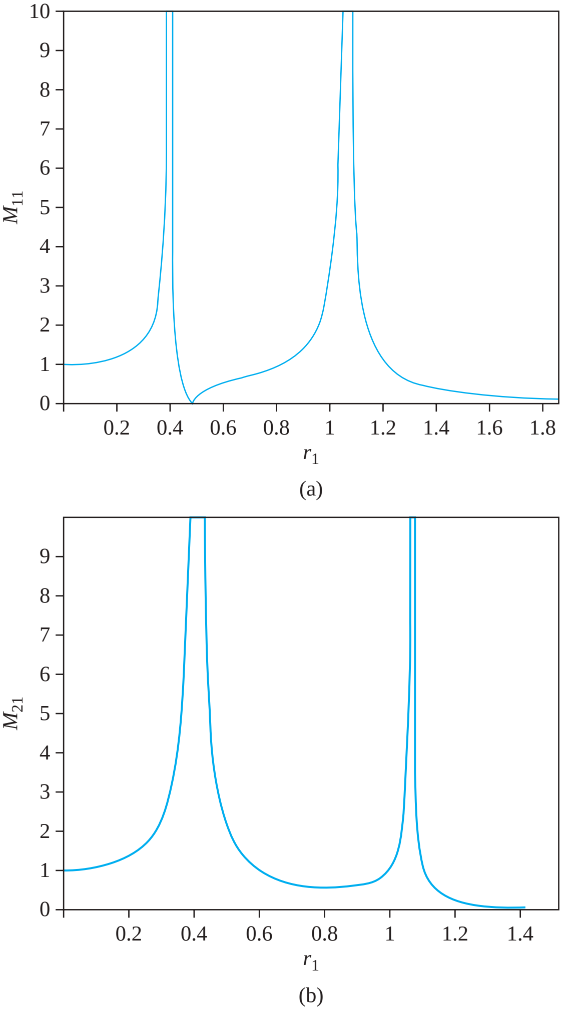


FIGURE 6.15
Frequency response curves: (a) $M_{1,1}$ versus r_1 for $r_2 = 0.5$ and $\mu = 0.5$. (b) $M_{2,1}$ versus r_1 for $r_2 = 0.5$ and $\mu = 0.5$.

or in nondimensional form as

$$r_1 = \frac{1}{\sqrt{2}}\sqrt{1 + \left(\frac{r_1}{r_2}\right)^2(1+\mu) \pm \sqrt{\left(\frac{r_1}{r_2}\right)^4(1+\mu)^2 + 2\left(\frac{r_1}{r_2}\right)^2(\mu - 1) + 1}} \qquad (6.89)$$

6.11 DYNAMIC VIBRATION ABSORBERS

When the machine of Figure 6.17 is subject to a harmonic excitation at a frequency near its natural frequency, large amplitude steady-state vibrations are a result. One remedy is to change the properties of the system such that the natural frequency is away from the excitation frequency. An alternate remedy is to add an auxiliary mass-spring system such that the system has two natural frequencies both of which are away from the excitation frequency.

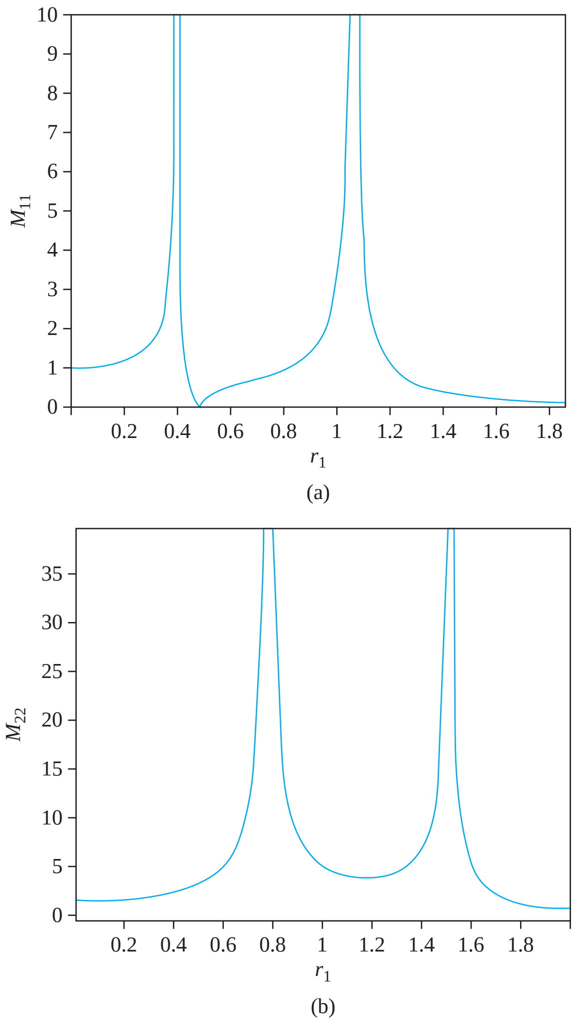


FIGURE 6.16
Frequency response curves when a force is applied to mass m_1: (a) $M_{1,2}$ versus r_1 for $r_2 = 0.5$ and $\mu = 0.75$. (b) $M_{2,2}$ versus r_1 for $r_2 = 0.5$ and $\mu = 0.75$.

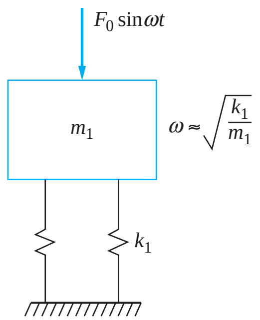


FIGURE 6.17
Large amplitude steady-state vibrations occur when the excitation frequency is close to the natural frequency of the machine.

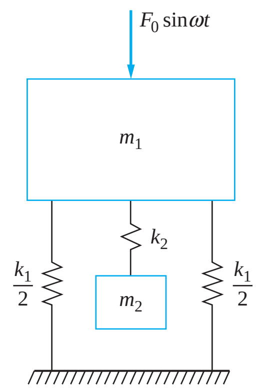


FIGURE 6.18
A vibration absorber is an auxiliary mass-spring system which is added to the primary system (the machine) to add one degree of freedom to the system and change its natural frequencies.

A vibration absorber is the auxiliary system. The original machine is termed the primary system. The resulting two degree-of-freedom system is illustrated in Figure 6.18. This is the configuration that was analyzed in Section 6.10, and its frequency response is

$$\frac{k_1 X_1}{F_0} = \left| \frac{1 - r_2^2}{r_1^2 r_2^2 - r_2^2 - (1 + \mu) r_1^2 + 1} \right| \tag{6.90}$$

The parameter $\omega_{1,1}$ is the natural frequency of the primary system, and the parameter $\omega_{2,2}$ is the natural frequency of the absorber if it were grounded (that is, directly connected to the ground). The system composed of the primary system attached to the auxiliary system is a two degree-of-freedom system with natural frequencies given by Equation (6.88).

The steady-state amplitude of the absorber is given by

$$\frac{k_1 X_2}{F_0} = \left| \frac{1}{r_1^2 r_2^2 - r_2^2 - (1 + \mu) r_1^2 + 1} \right| \tag{6.91}$$

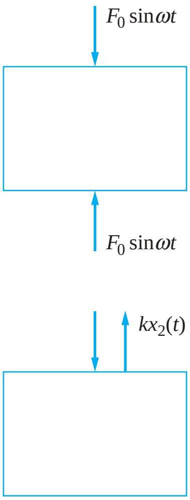


FIGURE 6.19
FBD of the primary system and the auxiliary system when the absorber is tuned to the excitation frequency.

The steady-state amplitude of the primary system is zero when the absorber is tuned such that $r_2 = 1$ or that

$$k_2 = m_2\omega^2 \tag{6.92}$$

When $r_2 = 1$, the steady-state vibrations of the primary system are zero. Thus, the excitation force is transmitted directly to the absorber system. Using the FBD of Figure 6.19, the steady-state behavior of the auxiliary system is

$$x_2(t) = -\frac{F_0}{k_2}\sin\omega t \tag{6.93}$$

Hence, the steady-state amplitude of the absorber mass when it is tuned such that $k_2 = m_2\omega^2$ is

$$X_2 = \frac{F_0}{k_2} \tag{6.94}$$

The frequency response for the primary system as a function of r_2 for $\omega_{2,2} = \omega$ is illustrated in Figure 6.20. Note that one of the system's two natural frequencies is less than the tuned frequency while the other is greater.

If the excitation speed varies slightly from the tuned speed, the larger the separation in natural frequencies the smaller the steady-state amplitude of the primary system. Defining

$$q = \frac{\omega_{22}}{\omega_{11}} \tag{6.95}$$

the separation in natural frequencies is a function of μ, as shown in Figure 6.21, and by the equation

$$\omega_2^2 - \omega_1^2 = \omega_{1,1}^2\sqrt{q^4(1+\mu)^2 + 2(\mu-1)q^2 + 1} \tag{6.96}$$

In situations where absorbers are employed, $q \approx 1$. Setting $q = 1$ in Equation (6.96) leads to

$$\omega_2^2 - \omega_1^2 = \omega_{1,1}^2\sqrt{\mu(4+\mu)} \tag{6.97}$$

The separation in natural frequencies is larger for larger μ. For $\mu = 0.25$, $\omega_2^2 - \omega_1^2 \approx \omega_{1,1}^2$.

The denominator in Equation (6.90) is positive for $\omega < \omega_1$ and $\omega > \omega_2$. It is negative in the range $\omega_1 < \omega < \omega_2$. The numerator is positive for $\omega < \omega_{2,2}$ and negative otherwise. When the ratio of the numerator to denominator is negative, the response of the primary system is 180° out of phase with the excitation. When the denominator is negative, the response of the auxiliary system is 180° with the excitation.

A dynamic vibration absorber is used to eliminate steady-state vibrations of a particle where the absorber is attached if the natural frequency of the absorber is tuned to the excitation frequency. The absorber has many applications in industrial processes. When the absorber is used on a SDOF system, it converts the system to two degrees of freedom. The following must be kept in mind when using an absorber:

- The steady-state amplitude of the primary system is zero when the auxiliary system (the absorber) is tuned such that $\omega_{2,2} = \omega$.
- One of the natural frequencies of the resulting two degree-of-freedom system is less than the tuned frequency, and one is higher than the tuned frequency. The lower

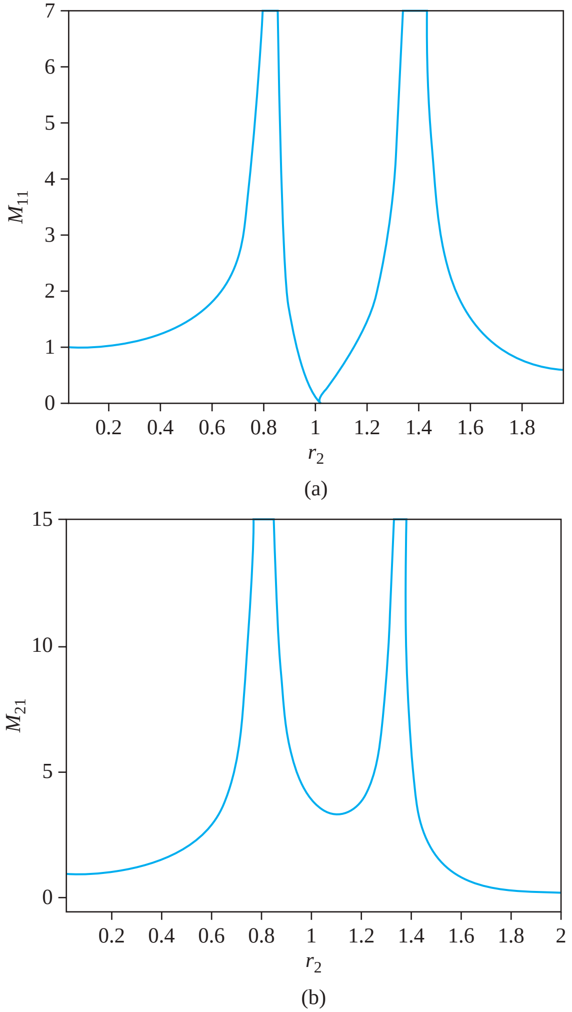


FIGURE 6.20
(a) Frequency response curve for primary system with absorber tuned to frequency of excitation and $\mu = 0.25$. (b) Frequency response of auxiliary system under same conditions.
$\mu = 0.3$
$q = 1.1$

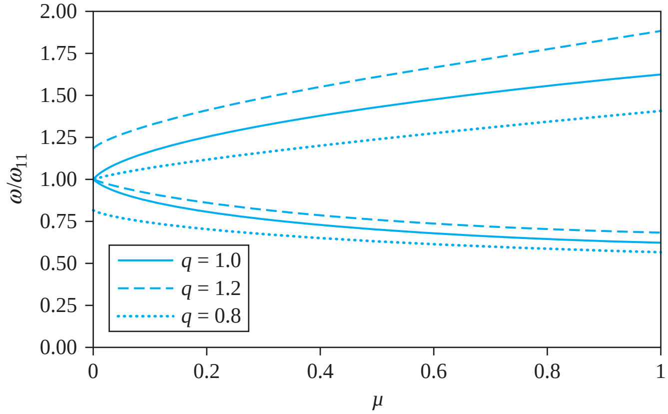


FIGURE 6.21
Natural frequencies of two degree-of-freedom system as a function of the mass ratio μ.

natural frequency must be passed during start-up and stopping, leading to large-amplitude vibrations during these transient periods.

- The steady-state vibrations of the primary system are eliminated only at a single frequency. If the system operates over a wide range of frequencies, the steady-state amplitudes at frequencies away from the tuned frequency may be large. An effective operating range should be defined for each application by limiting the amplitude of vibrations to an acceptable maximum.
- If the absorber is tuned to the excitation frequency and a given mass ratio μ is not to be exceeded, the maximum value of the absorber stiffness is

$$k_{2\,\text{max}} = \mu m_1 \omega^2 \tag{6.98}$$

and the minimum steady-state amplitude of the absorber mass is

$$X_{2\,\text{min}} = \frac{F_0}{\mu m_1 \omega^2} \tag{6.99}$$

- The analysis is valid only for undamped systems. If damping is present either in the primary system or in the absorber, it is not possible to eliminate steady-state vibrations of the primary system.

EXAMPLE 6.15

A machine of mass 150 kg with a rotating unbalance of 0.5 kg·m is paced at the midspan of a 2-m-long simply supported beam. The machine operates at a speed of 1200 rpm. The beam has an elastic modulus of 210×10^9 N/m^2 and a cross-sectional moment of inertia of 2.1×10^{-6} m^4.

(a) What is the steady-state amplitude of the primary system without an absorber?
(b) Design the dynamic vibration absorber of minimum mass such that, when attached to the midspan of the beam, the vibrations of the beam will cease and the steady-state amplitude of the absorber will be less than 20 mm.
(c) What are the system's natural frequencies when the absorber is in place?
(d) What is the effective operating range such that the midspan deflection does not exceed 5 mm when the absorber is in place?

SOLUTION

Modeling the vibrations of the machine on the beam using a SDOF system model and ignoring the mass of the beam, the stiffness and natural frequency of the primary system are calculated as

$$k_1 = \frac{48EI}{L^3} = \frac{48(210 \times 10^9 \text{ N/m}^2)(2.1 \times 10^{-6} \text{ m}^4)}{(2 \text{ m})^3} = 2.65 \times 10^6 \text{ N/m} \tag{a}$$

and

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{2.65 \times 10^6 \text{ N/m}}{150 \text{ kg}}} = 132.9 \text{ rad/s} \tag{b}$$

The operating speed is

$$\omega = (1200 \text{ rpm})\left(2\pi \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 125.7 \text{ rad/s} \tag{c}$$

(a) Since the excitation speed is near the natural frequency of the primary system, it will have large amplitude vibrations without an absorber. The frequency ratio is

$$r = \frac{\omega}{\omega_{11}} = \frac{125.7 \text{ rad/s}}{132.9 \text{ rad/s}} = 0.945 \quad \text{(d)}$$

Steady-state amplitude of the machine is

$$X_1 = \frac{m_0 e}{m}\Lambda(0.945, 0) = \left(\frac{0.5 \text{ kg} \cdot \text{m}}{150 \text{ kg}}\right)\frac{(0.945)^2}{1 - (0.945)^2} = 0.285 \text{ m} \quad \text{(e)}$$

(b) Steady-state vibrations of the primary system are eliminated when the absorber is tuned to the excitation frequency using

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = 125.7 \text{ rad/s} \quad \text{(f)}$$

Since the ratio of the absorber stiffness to absorber mass is fixed, the absorber with the minimum mass is also the absorber with the minimum stiffness. The amplitude of the absorber is to be limited to 20 mm, which from Equation (6.94) leads to

$$X_2 = \frac{F_0}{k_2} \Rightarrow k_2 \geq \frac{F_0}{X_2} = \frac{(0.5 \text{ kg} \cdot \text{m})(125.7 \text{ rad/s})^2}{0.002 \text{ m}} = 3.95 \times 10^5 \text{ N/m} \quad \text{(g)}$$

The minimum absorber stiffness is 3.95×10^5 N/m, leading to an absorber mass of

$$m_2 = \frac{k_2}{\omega_{22}^2} = \frac{3.95 \times 10^5 \text{ N/m}}{(125.7 \text{ rad/s})^2} = 25 \text{ kg} \quad \text{(h)}$$

(c) The natural frequencies of the two degree-of-freedom system are calculated from Equation (6.88) using $\mu = \frac{25 \text{ kg}}{150 \text{ kg}} = 0.167$ as

$$\omega_1 = 105.8 \text{ rad/s} \qquad \omega_2 = 157.6 \text{ rad/s} \quad \text{(i)}$$

(d) The effective operating range is obtained by setting $F_0 = 0.5\omega^2$ and using Equation (6.90). The denominator is negative between the two natural frequencies, and the numerator is positive for $r_2 < 1$. Take away the absolute value symbol and set $X_1 = -0.005$ m in this case. Rearrange the equation to

$$\omega^4 - 7.63 \times 10^4\omega^2 + 8.28 \times 10^8 = 0 \quad \text{(j)}$$

which (when solved for ω) leads to a lower bound on the operating range of 114.8 rad/s. For $r_2 > 1$, set $X_1 = 0.005$ m, leading to

$$\omega^4 - 2.79 \times 10^4\omega^2 + 1.67 \times 10^8 = 0 \quad \text{(k)}$$

and a upper bound on the operating range of 138.5 rad/s. Thus, the effective operating range is

$$114.8 \text{ rad/s} < \omega < 138.5 \text{ rad/s} \quad \text{(l)}$$

6.12 DAMPED VIBRATION ABSORBERS

Two problems exist when a vibration absorber is used. The lowest natural frequency of the two degree-of-freedom system must be passed through in order to build up to the operating speed. If the absorber is slightly mistuned, the vibration amplitude of the primary system can be large. Perhaps the addition of damping to the absorber can help with these issues.

Consider the configuration of the system of Figure 6.22 in which viscous damping is added in parallel with the stiffness in the auxiliary system. This is known as a damped vibration absorber. The steady-state amplitude of the primary system is given by

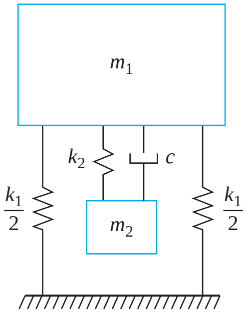


FIGURE 6.22
The auxiliary system of a damped vibration absorber consists of a mass attached to a spring in parallel with a viscous damper.

$$\frac{k_1 X_1}{F_0} = M_{1d}(r_1, q, \mu, \zeta)$$
$$= \sqrt{\frac{(2\zeta r_1 q)^2 + (r_1^2 - q^2)^2}{\{r_1^4 - [1 + (1+\mu)q^2 r_1^2] + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}} \tag{6.100}$$

The steady-state amplitude of the auxiliary system is

$$\frac{k_1 X_2}{F_0} = M_{2d}(r_1, q, \mu, \zeta)$$
$$= \sqrt{\frac{q^4 + (2\zeta q)^2}{\{r_1^4 - [1 + (1+\mu)q^2 r_1^2] + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}} \tag{6.101}$$

where

$$\zeta = \frac{c}{2\sqrt{m_2 k_2}} \tag{6.102}$$

is the damping ratio of the auxiliary system if it were grounded. The nondimensional steady-state amplitude of the primary system, given by Equation (6.100), is illustrated in Figure 6.23 for $\mu = 0.25$ and $q = 1$ for several values of ζ. The steady-state amplitude of the primary system is not zero for any value of r_1. A minimum amplitude is reached for r_1 near one between the peaks. The absorber was successful in significantly reducing the peak near the second natural frequency, but not very successful in reducing the peak amplitude near the first natural frequency. An investigation of the parameters affecting the damped vibration absorber is necessary. It is noted that each curve, for different ζ, passes through the same two points.

M_{1d} is plotted in Figure 6.24 for $\mu = 0.25$ and $q = 0.8$. The peak at the lower resonant frequency is smaller than the peak at the higher resonant frequency. However, the higher peak occurs near $r_1 = 1$, which is the region where an absorber is usually needed. Also, the effective operating range is still small. It is noted again that there are two fixed points through which each curve passes. These fixed points are different than those in Figure 6.23.

Since it is not possible to eliminate steady-state motion of the original system when damping is present, a damped vibration absorber must be designed to reduce the peak at the lower resonant frequency and to widen the effective operating range. Absorbers using the parameters used to generate Figure 6.23 and Figure 6.24 are not suitable for these purposes.

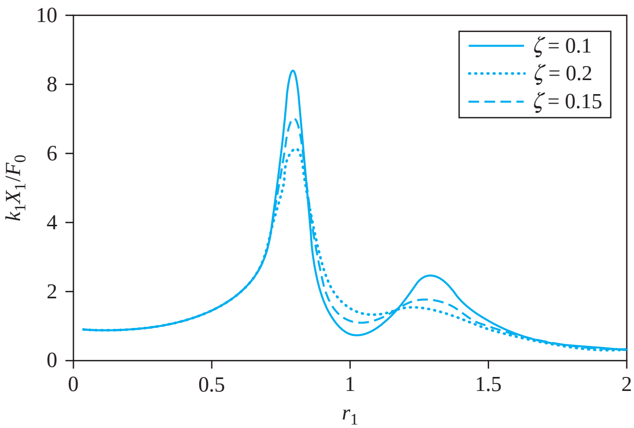


FIGURE 6.23
Response of primary system when a damped vibration absorber is used with $\mu = 0.25$ and $q = 1$ for several values of ζ.

Widening the operating range requires that the two peaks have approximately the same magnitude. Since the locations of the fixed points are dependent on q, it should be possible to tune the absorber such that the values of M_{1d} at the fixed points are the same. Since curves for all values of ζ pass through the fixed points, it should be possible to find a value of ζ such that the fixed points are near the peaks.

For fixed values of μ and q, there are two values of r_1 which yield a value of M_{1d}, independent of ζ. The value of M_{1d} at these points is written as

$$M_{1d} = \sqrt{\frac{A(\mu, q)\zeta^2 + B(\mu, q)}{C(\mu, q)\zeta^2 + D(\mu, q)}} \tag{6.103}$$

Since Equation (6.103) holds for all ζ and powers of ζ are linearly independent,

$$\frac{A}{C} = \frac{B}{D} \tag{6.104}$$

Using Equation (6.100) to determine the forms of A, B, C, and D, substituting into Equation (6.104), and rearranging leads to

$$r_1^4\left(1 + \frac{\mu}{2}\right) - [1 + q^2(1 + \mu)]r_1^2 + q^2 = 0 \tag{6.105}$$

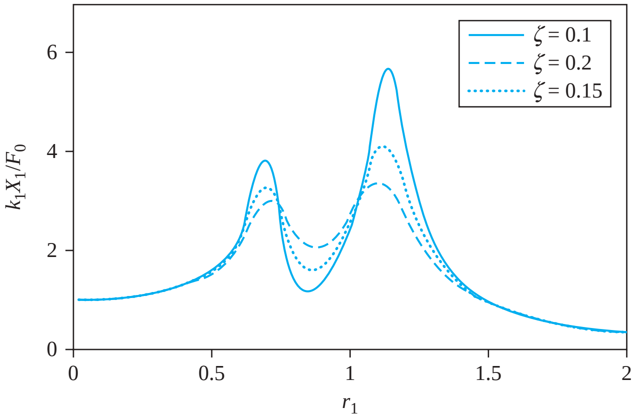


FIGURE 6.24
Response of primary system when an optimum damped vibration absorber is used with $\mu = 0.25$ and $q = 0.8$.

The solution of Equation (6.105) places the fixed points at

$$r_1 = \sqrt{\frac{1 + (1 + \mu)q^2 \pm \sqrt{1 - 2q^2 + (1 + \mu)^2 q^4}}{2 + \mu}} \tag{6.106}$$

Since Equation (6.103) yields the same value of M_{1d}, independent of ζ for r_1 given by Equation (6.106), letting $\zeta \rightarrow \infty$ gives

$$M_{1d} = \sqrt{\frac{1}{[1 - r_1^2(1 + \mu)]^2}} \tag{6.107}$$

Requiring M_{1d} to be the same at both fixed points leads to

$$q = \frac{1}{1 + \mu} \tag{6.108}$$

An optimum absorber could be designed with an appropriate value of ζ such that the smaller r_1 given by Equation (6.106) corresponds to both a fixed point and a peak on the frequency response curve. The appropriate value of ζ is obtained by setting $dM_{1d}/d\zeta = 0$, using q from Equation (6.108). The same procedure can be followed to yield the value of ζ such that the larger value of r_1 given by Equation (6.106) corresponds to both a fixed point and a peak. Since the values of ζ are not equal, their average is usually used to define the optimum damping ratio

$$\zeta_{\text{opt}} = \sqrt{\frac{3\mu}{8(1 + \mu)}} \tag{6.109}$$

In summary, the optimum design of a damped vibration absorber requires that the absorber be tuned to the frequency calculated from Equation (6.108) with the damping ratio of Equation (6.109). For $\mu = 0.25$, Equation (6.109) gives an optimum damping ratio of $\zeta = 0.2379$ and an optimum $q = 0.80$. Figure 6.25 shows M_{1d} for these values as a function of r_1. This figure also shows M_{1d} for the same μ and ζ but with values of q, one on each side of the optimum. The curve corresponding to the optimum value of q has smaller resonant peaks and the value of M_{1d} does not vary much between the peaks.

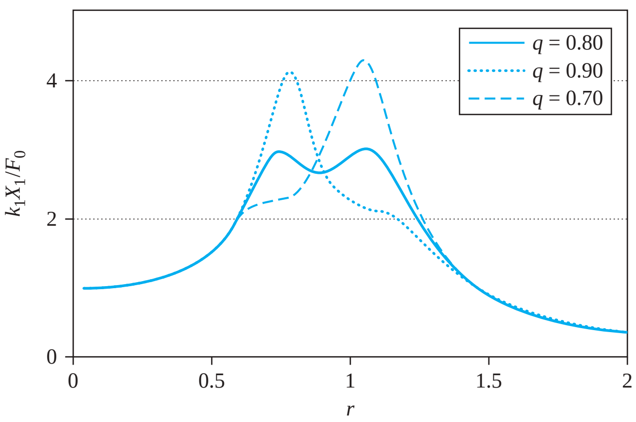


FIGURE 6.25
Steady-state amplitude of primary system for $\mu = 0.25$, $\zeta_{\text{opt}} = 0.2739$, and $q_{\text{opt}} = 0.80$.

EXAMPLE 6.16

A milling machine has a mass of 250 kg and a natural frequency of 120 rad/s and is subject to a harmonic excitation of magnitude 10,000 N at speeds between 95 rad/s and 120 rad/s. Design a damped vibration absorber of mass 50 kg such that the steady-state amplitude is no greater than 15 mm at all operating speeds.

SOLUTION

The mass ratio is

$$\mu = \frac{50 \text{ kg}}{250 \text{ kg}} = 0.2 \tag{a}$$

Since a wide operating range is required, the optimum absorber design is tried. From Equations (6.108) and (6.109),

$$q = \frac{1}{1.2} = 0.833 \qquad \zeta = \sqrt{\frac{3(0.2)}{8(1.2)}} = 0.25 \tag{b}$$

The steady-state amplitude at any operating speed for this absorber design is calculated by Equations (6.100) and (6.101). The results are used to generate the frequency response curve of Figure 6.26.

The fixed-points are calculated from Equation (6.106) as

$$r_1 = \sqrt{\frac{1 + (1 + 0.2)(0.833)^2 \pm \sqrt{1 - 2(0.833)^2 + (1 + 0.2)^2(0.833)^4}}{2 + 0.2}}$$

$$= 0.7629, 1.0414 \tag{c}$$

which leads to $\omega = 91.5$ rad/s, 125.0 rad/s.

Since the extremes of the operating range lie between the peaks and the steady-state amplitudes at the extremes are

$$X(\omega = 95 \text{ rad/s}) = 10.1 \text{ mm} \qquad X(\omega = 120 \text{ rad/s}) = 12.7 \text{ mm} \tag{d}$$

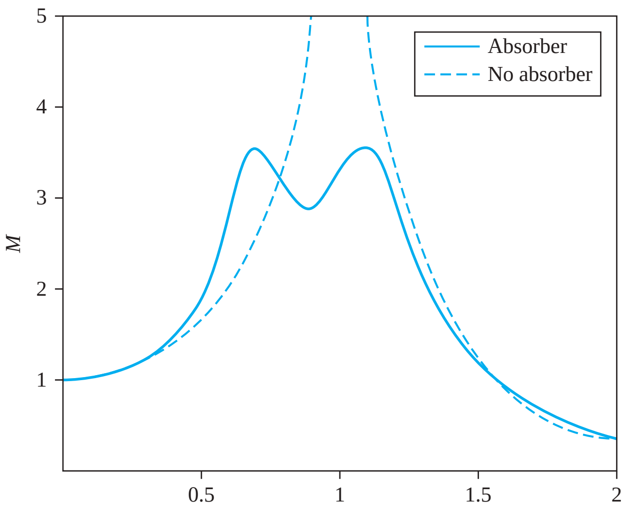


FIGURE 6.26
Frequency response for primary system of Example 6.16 with optimum damped absorber with $\mu = 0.25$ attached.

and both are less than 15 mm, the optimum design is acceptable. The absorber stiffness and damping ratio are calculated as

$$k_2 = m_2\omega_{22}^2 = \mu q^2 k_1 = (0.2)(0.833)^2(3.6 \times 10^6 \text{ N/m}) = 5.08 \times 10^5 \text{ N/m} \quad \text{(e)}$$

$$c = 2\zeta\sqrt{k_2 m_2} = 2500 \text{ N} \cdot \text{s/m} \quad \text{(f)}$$

6.13 VIBRATION DAMPERS

A *vibration damper* is an auxiliary system composed of an inertia element and a viscous damper that is connected to a primary system as a means of vibration control. Vibration dampers are used in situations where vibration control is required over a range of frequencies.

The Houdaille damper of Figure 6.27 is an example of a vibration damper that is used for vibration control of rotating devices such as engine crankshafts. The damper is inside a casing that is attached to the end of the shaft. The casing contains a viscous fluid and a mass that is free to rotate in the casing. The differential equations governing the motion of the two degree-of-freedom torsional system are

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_0 \sin \omega t \\ 0 \end{bmatrix} \quad \text{(6.110)}$$

The steady-state amplitude of the primary system is obtained by the methods of Section 6.10 as

$$\Theta_1 = \frac{M_0}{k}\sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} \quad \text{(6.111)}$$

where $$r = \frac{\omega}{\sqrt{\dfrac{k}{J_1}}} \qquad \zeta = \frac{c}{2J_2\sqrt{\dfrac{k}{J_1}}} \qquad \mu = \frac{J_2}{J_1} \quad \text{(6.112)}$$

The optimum damping ratio is defined as the damping ratio for which the maximum value of Θ_1 is smallest. The peak amplitude, $\Theta_{1p}(\zeta)$ is the value of $\Theta_1(r_m)$ where r_m is the value of r that yields $d\Theta_1/dr = 0$. The optimum damping ratio is the value of ζ such that $d\Theta_{1p}/d\zeta = 0$. Extensive algebra leads to

$$\zeta_{\text{opt}} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} \quad \text{(6.113)}$$

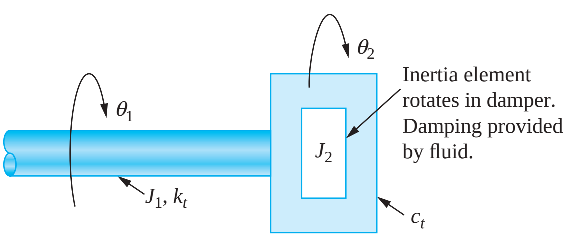


FIGURE 6.27
Houdaille damper.

If the optimum damping ratio is used in the design of a Houdaille damper then

$$r_m = \sqrt{\frac{2}{2+\mu}} \tag{6.114}$$

and

$$\Theta_{1p} = \frac{M_0}{k}\frac{2+\mu}{\mu} \tag{6.115}$$

6.14 BENCHMARK EXAMPLES

6.14.1 MACHINE ON FLOOR OF INDUSTRIAL PLANT

In Chapter 4, vibration isolation of the machine was considered by ignoring the mass and flexibility of the beam. They are taken into account using the model of Figure 6.28. The mass of the beam is lumped at the midspan using the equivalent mass of the beam. The stiffness of the beam is the stiffness used in the SDOF model. The force transmitted through the isolator to the beam is $k(x_2 - x_1)$.

The differential equations governing the two degree-of-freedom system are

$$31.1\ddot{x}_1 + kx_1 - kx_2 = F_0 \sin \omega t \tag{a}$$

$$7.79\ddot{x}_2 - kx_1 + (k + 7.74 \times 10^5)x_2 = 0 \tag{b}$$

which are written in matrix form as

$$\begin{bmatrix} 31.1 & 0 \\ 0 & 7.79 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k + 7.74 \times 10^5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix} \tag{c}$$

Consider the system with an isolator designed such that the transmitted force is 5000 N. The stiffness of the isolator is 3.93×10^4 lbf/ft, and the equations become

$$\begin{bmatrix} 31.1 & 0 \\ 0 & 7.79 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3.93 \times 10^4 & -3.93 \times 10^4 \\ -3.93 \times 10^4 & 8.13 \times 10^5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix} \tag{d}$$

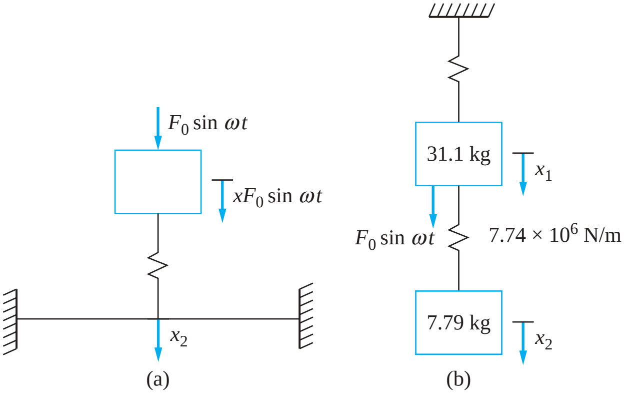


FIGURE 6.28
(a) Machine attached by isolator to beam. (b) Two degree-of-freedom model with inertia of beam included.

A normal-mode solution is used to calculate the natural frequencies and mode shapes resulting in

$$\begin{vmatrix} -31.1\omega^2 + 3.93 \times 10^4 & -3.93 \times 10^4 \\ -3.93 \times 10^4 & -7.79\omega^2 + 8.13 \times 10^5 \end{vmatrix} = 0 \tag{e}$$

which leads to

$$\omega_1 = 34.6 \text{ rad/s} \qquad \omega_2 = 323.9 \text{ rad/s} \tag{f}$$

For comparison purposes, the natural frequency of the machine on a rigid beam is 35.6 rad/s, and the natural frequency of the machine mounted directly to the flexible beam is 141.4 rad/s.

Since the force transmitted to the beam is $k(x_2 - x_1)$, define a new variable $z = x_2 - x_1$. The differential equations written using x_1 and z as generalized coordinates become

$$\begin{bmatrix} 31.1 & 0 \\ 7.79 & 7.79 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} 0 & -k \\ 7.74 \times 10^5 & 7.74 \times 10^5 + k \end{bmatrix}\begin{bmatrix} x_1 \\ z \end{bmatrix} = \begin{bmatrix} F_0 \sin\omega t \\ 0 \end{bmatrix} \tag{g}$$

The steady-state amplitude of z is determined using the sinusoidal transfer function. To this end, determine the transfer function $G(s) = \frac{z(s)}{F(s)}$. Taking the Laplace transform of the two equations with an arbitrary $F(t)$ in place of $F_0 \sin\omega t$, we have

$$\begin{bmatrix} 31.1s^2 & -k \\ 7.79s^2 + 7.74 \times 10^5 & 7.79s^2 + 7.74 \times 10^5 + k \end{bmatrix}\begin{bmatrix} X_1(s) \\ Z(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix} \tag{h}$$

Using Cramer's rule to solve for $Z(s)$, we have

$$Z(s) = \frac{\begin{vmatrix} 31.1s^2 & F(s) \\ 7.79s^2 + 7.74 \times 10^5 & 0 \end{vmatrix}}{\begin{vmatrix} 31.1s^2 & -k \\ 7.79s^2 + (7.74 \times 10^5) & 7.79s^2 + (7.74 \times 10^5) + k \end{vmatrix}}$$

$$= \frac{-(7.79s^2 + 7.74 \times 10^5)\,F(s)}{(31.1s^2)(7.79s^2 + 7.74 \times 10^5 + k) - (-k)(7.79s^2 + 7.74 \times 10^5)} \tag{i}$$

The transfer function is

$$G(s) = \frac{-(7.79s^2 + 7.74 \times 10^5)}{242.3s^4 + (2.41 \times 10^7 + 38.8k)s^2 + 7.74 \times 10^5 k} \tag{j}$$

The sinusoidal transfer function $G(80i)$ is

$$G(80i) = \frac{-(7.79(80i)^2 + 7.74 \times 10^5)}{242.3(80i)^4 + (2.41 \times 10^7 + 38.8k)(80i)^2 + 7.74 \times 10^5 k} \tag{k}$$

The magnitude of the sinusoidal transfer function is

$$|G(80i)| = \left|\frac{-7.24 \times 10^5}{5.26 \times 10^5 k - 1.4432 \times 10^{11}}\right| \tag{l}$$

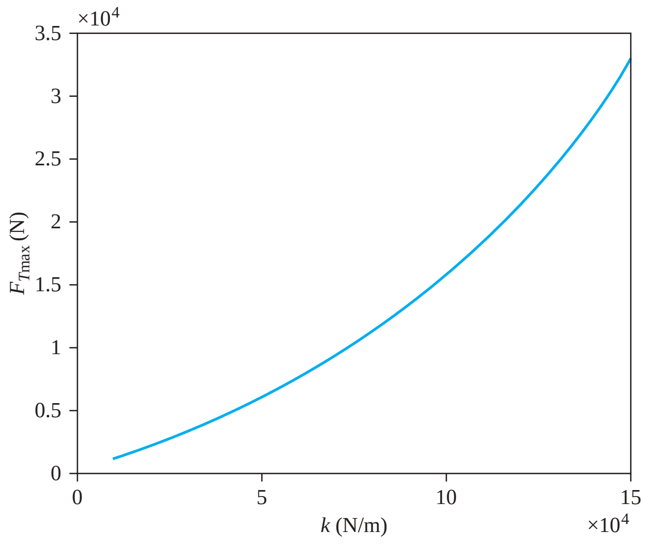


FIGURE 6.29
Amplitude of transmitted force as function of absorber stiffness.

Thus, the amplitude of kz or the amplitude of the force transmitted between the machine and the beam is $kF_0|G(80i)|$, so

$$kZ = \left|\frac{1.44 \times 10^{10}k}{5.26 \times 10^5 k - 1.4432 \times 10^{11}}\right| \tag{m}$$

Figure 6.29 shows the transmitted force as a function of k. A value of $k = 1 \times 10^5$ lb/ft leads to $F_T = 16{,}000$ lb, which is slightly less than the value of 20,000 lb predicted by the SDOF system with the rigid beam.

6.14.2 SIMPLIFIED SUSPENSION SYSTEM

The two degree-of-freedom model shown in Figure 6.30(a) is used for the vehicle suspension system. The "unsprung" mass represents the mass of the axle and wheel, and the additional stiffness represents the tire. The unsprung mass is 50 kg, which is much less than the mass of the vehicle, while the stiffness of the tire is 200,000 N/m, which is much greater than the stiffness of the suspension spring. A quick calculation reveals that lumping the unsprung and sprung masses together and assuming the two spring are in series, as shown in Figure 6.30(b), gives a natural frequency of

$$\omega_{n.\ell} = \sqrt{\frac{\dfrac{1}{\dfrac{1}{200{,}000 \text{ N/m}} + \dfrac{1}{12{,}000 \text{ N/m}}}}{350 \text{ kg}}} = 5.69 \text{ rad/s} \tag{a}$$

The differential equations governing the two degree-of-freedom model (assuming the sprung mass can vary) is

$$m_s\ddot{x}_1 + 1200\dot{x}_1 - 1200\dot{x}_2 + 12{,}000x_1 - 12{,}000x_2 = 0 \tag{b}$$

$$50\ddot{x}_2 - 1200\dot{x}_1 + 1200\dot{x}_2 + 12{,}000x_1 - 212{,}000x_2 = 200{,}000y \tag{c}$$

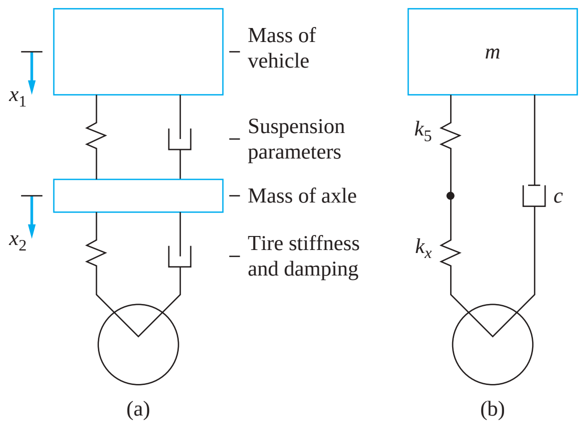


FIGURE 6.30
(a) Two degree-of-freedom model of vehicle suspension system. The mass of the axle is included in the model. (b) The stiffness of the wheel is imagined to be in series with the stiffness of the suspension system.

Consider free vibrations of an empty vehicle as $m_s = 300$ kg. The differential equations are summarized in matrix form as

$$\begin{bmatrix} 300 & 0 \\ 0 & 50 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 1200 & -1200 \\ -1200 & 1200 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 12{,}000 & -12{,}000 \\ -12{,}000 & 212{,}000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{d}$$

The free response is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{\lambda t} \tag{e}$$

Substituting Equation (e) into Equation (d) leads to

$$\begin{vmatrix} 300\lambda^2 + 1200\lambda + 12{,}000 & -1200\lambda - 12{,}000 \\ -1200\lambda - 12{,}000 & 50\lambda^2 + 1200\lambda + 212{,}000 \end{vmatrix} = 0 \tag{f}$$

Evaluation of the determinant leads to

$$15{,}000\lambda^4 + 4.2 \times 10^5\lambda^3 + 6.42 \times 10^7\lambda^2 + 2.40 \times 10^8\lambda + 2.4 \times 10^9 = 0 \tag{g}$$

whose roots are

$$\lambda_{1,2} = -1.88 \pm 5.94i, \ -12.2 \pm 63.28i \tag{h}$$

The modal fractions are given by

$$\chi = \frac{300\lambda^2 + 1200\lambda + 12{,}000}{1200\lambda + 12{,}000} \tag{i}$$

from which

$$\chi_1 = 0.0481 \pm 0.0328i \qquad \chi_2 = -4.56 \pm 15.43i \tag{j}$$

The general solution of the differential equations is obtained using Equation (6.21) as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-1.88t}\left\{A_1\left(\begin{bmatrix} 1 \\ -0.0481 \end{bmatrix}\cos 5.94t - \begin{bmatrix} 0 \\ -0.0328 \end{bmatrix}\sin 5.94t\right)\right.$$
$$\left. + A_2\left(\begin{bmatrix} 1 \\ -0.0481 \end{bmatrix}\sin 5.94t + \begin{bmatrix} 0 \\ -0.0328 \end{bmatrix}\cos 5.94t\right)\right\}$$
$$+ e^{-12.2t}\left\{A_3\left(\begin{bmatrix} 1 \\ -4.56 \end{bmatrix}\cos 63.28t - \begin{bmatrix} 0 \\ -15.43 \end{bmatrix}\sin 63.28t\right)\right.$$
$$\left. + A_4\left(\begin{bmatrix} 1 \\ -4.56 \end{bmatrix}\sin 63.28t + \begin{bmatrix} 0 \\ -15.43 \end{bmatrix}\cos 63.28t\right)\right\} \quad \text{(k)}$$

The initial conditions are assumed as

$$\mathbf{x}(0) = \begin{bmatrix} h \\ h \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(l)}$$

Substitution of the initial conditions into the solution yields

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -0.0481 & -0.0328 & -4.56 & 15.43 \\ -1.88 & 5.94 & -12.2 & 63.28 \\ 0.2853 & -0.2241 & -920.77 & -476.81 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} h \\ h \\ 0 \\ 0 \end{bmatrix} \quad \text{(m)}$$

The constants of integration are obtained as $A_1 = 1.029h$, $A_2 = -0.3579h$, $A_3 = -0.029h$ and $A_4 = 0.0584h$. The solution obtained from substitution of the values of the constants of integration into Equation (k) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = h\left\{e^{-1.88t}\left(\begin{bmatrix} 1.029 \\ -0.0378 \end{bmatrix}\cos 5.94t + \begin{bmatrix} -0.3579 \\ 0.0510 \end{bmatrix}\sin 5.94\,t\right)\right.$$
$$\left. + e^{-12.2t}\left(\begin{bmatrix} -0.029 \\ 1.0378 \end{bmatrix}\cos 63.28t + \begin{bmatrix} 0.0584 \\ 0.1942 \end{bmatrix}\sin 63.28\,t\right)\right\} \quad \text{(n)}$$

The time-dependent response of the system is plotted in Figure 6.31.

Now consider the response of the vehicle due to a sinusoidal road contour as $y(\xi) = Y\sin\left(\frac{2\pi\xi}{d}\right)$. The vehicle travels with a constant horizontal speed v. The differential equations expressing the motion of the vehicle are

$$\begin{bmatrix} m_s & 0 \\ 0 & 50 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 1200 & -1200 \\ -1200 & 1200 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 12{,}000 & -12{,}000 \\ -12{,}000 & 212{,}000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 200{,}000y(t)\sin\left(\frac{2\pi v}{d}t\right) \end{bmatrix} \quad \text{(o)}$$

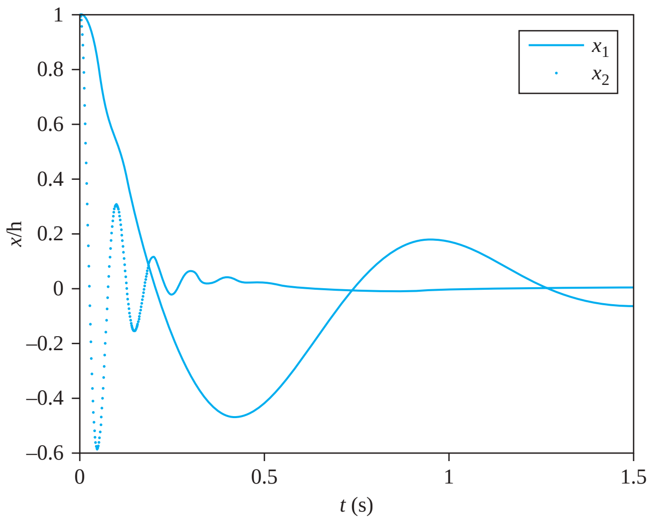


FIGURE 6.31
Time dependent response of the vehicle suspension system when it is subject to a bump in the road.

The frequency responses for x_1 and x_2 are derived using the sinusoidal transfer functions. The determination for $x_2(t)$ is detailed, and the transfer function for $x_1(t)$ is simply presented. Taking the Laplace transform of each of the differential equations with an arbitrary $y(t)$ on the right-hand side yields

$$\begin{bmatrix} m_s s^2 + 1200s + 12{,}000 & -1200s - 12{,}000 \\ -1200s - 12{,}000 & 50s^2 + 1200s + 212{,}000 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 200{,}000\,Y(s) \end{bmatrix} \tag{p}$$

The transfer function $G_2(s) = \frac{X_2(s)}{Y(s)}$ is determined from

$$X_2(s) = \frac{\begin{vmatrix} m_s s^2 + 1200s + 12{,}000 & 0 \\ -1200s + 12{,}000 & Y(s) \end{vmatrix}}{\begin{vmatrix} m_s s^2 + 1200s + 12{,}000 & -1200s - 12{,}000 \\ -1200s - 12{,}000 & 50s^2 + 1200s + 212{,}000 \end{vmatrix}} \tag{q}$$

from which the transfer function is calculated as

$$G(s) = \frac{m_s s^2 + 1200s + 12{,}000}{50m_s s^4 + (1200m_s + 60{,}000)s^3 + (212{,}000m_s + 600{,}000)s^2 + 2.4 \times 10^8 s + 2.4 \times 10^9} \tag{r}$$

The sinusoidal transfer function is

$$G(i\omega) = \frac{(12{,}000 - m_s\omega^2) + 1200\omega i}{[50m_s\omega^4 - (212{,}000m_s + 600{,}000)\omega^2 + 2.4 \times 10^9] + [2.4 \times 10^8\omega - (1200m_s + 60{,}000)\omega^3]i} \tag{s}$$

Defining

$$A = 50m_s\omega^4 - (212{,}000m_s + 60{,}000)\omega^2 + 2.4 \times 10^9 \tag{t}$$

$$B = (2.4 \times 10^8)\omega - (1200m_s + 60{,}000)\omega^3 \tag{u}$$

The steady-state amplitude is

$$X_2 = 200{,}000Y\,|G(i\omega)| = 200{,}000Y\frac{\sqrt{[(12{,}000 - m_s\omega^2)A - 1200\omega B]^2 + [(12{,}000 - m_s\omega^2)B + 1200\omega A]^2}}{A^2 + B^2} \tag{v}$$

The amplitude for $x_1(t)$ is

$$X_1 = 200{,}000Y\frac{\sqrt{(1200)^2[(A + \omega B)^2 + (\omega A + B)^2]}}{A^2 + B^2} \tag{w}$$

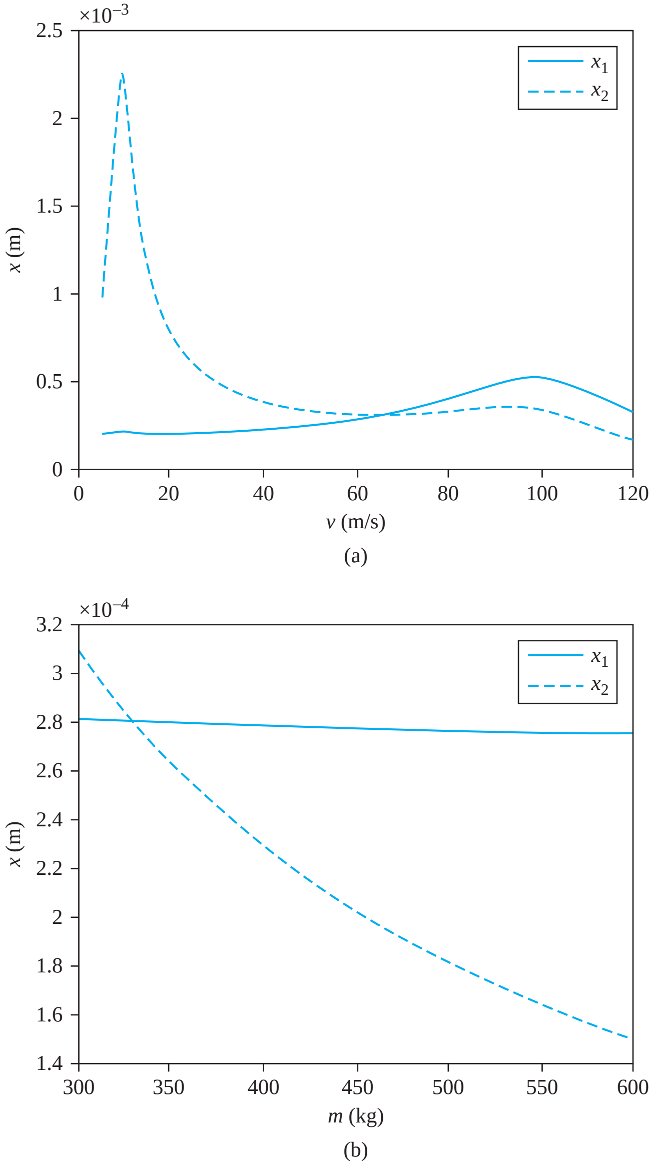


FIGURE 6.32
(a) Steady-state amplitude of vehicle and axle versus vehicle speed for empty vehicle (m_s = 300 kg).
(b) Steady-state amplitude of vehicle and axle versus mass for v = 60 m/s.

Equations (v) and (w) are illustrated in Figure 6.32(a) by plotting the steady-state amplitudes versus vehicle speed for an empty vehicle ($m_s = 300$ kg) and in Figure 6.32(b) by plotting steady-state amplitude versus m for $v = 60$ m/s. The frequency is substituted as $\omega = \frac{2\pi v}{d}$, the vehicle speed is the horizontal axis, d is taken as 10 m, and Y is 0.002 m.

6.15 FURTHER EXAMPLES

EXAMPLE 6.17

Determine the natural frequencies and mode shapes for the two degree-of-freedom system shown in Figure 6.33.

SOLUTION

The differential equations governing the motion of this system are

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 4k & -3k \\ -3k & 5k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{a}$$

Assuming a normal mode solution $\mathbf{x} = \mathbf{X}e^{i\omega t}$ and substituting into the differential equations leads to

$$\begin{bmatrix} -\omega^2 m + 4k & -3k \\ -3k & -\omega^2 2m + 5k \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b}$$

A non-trivial solution to Equation (b) exists only if

$$\begin{vmatrix} -\omega^2 m + 4k & -3k \\ -3k & -\omega^2 2m + 5k \end{vmatrix} = 0 \tag{c}$$

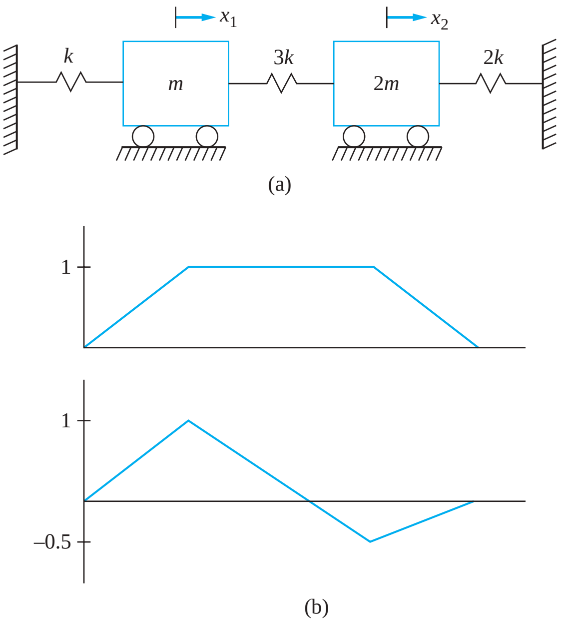


FIGURE 6.33
(a) System of Example 6.17. (b) Mode shapes for system.

Expansion of Equation (c) yields

$$(-\omega^2 m + 4k)(-\omega^2 2m + 5k) - (-3k)(-3k) = 0 \quad \text{(d)}$$

which is simplified to

$$2m^2\omega^4 - 13km\omega^2 + 11k^2 = 0 \quad \text{(e)}$$

Dividing Equation (e) by m^2 and letting $\phi = \frac{k}{m}$ leads to

$$2\omega^4 - 13\phi\omega^2 + 11\phi^2 = 0 \quad \text{(f)}$$

The quadratic formula is used to determine the roots of the quadratic equation as $\omega^2 = \phi$, -5.5ϕ, which leads to

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = 2.35\sqrt{\frac{k}{m}} \quad \text{(g)}$$

The mode shapes vectors are the solutions of Equation (b) for each value of ω as given in Equation (f). For ω_1, the equations become

$$\begin{bmatrix} -\phi m + 4k & -3k \\ -3k & -\phi 2m + 5k \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(h)}$$

The first of the equations in Equation (g) gives

$$(-\phi m + 4k) - 3k\chi = 0 \quad \text{(i)}$$

Dividing Equation (h) by m and rearranging leads to $\chi = 1$. The second equation only confirms the first equation and yields no new information. Thus, the mode shape vector corresponding to the first mode is any vector proportional to

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{(j)}$$

The second mode shape vector is determined by substituting ω_2 in Equation (b), leading to

$$\begin{bmatrix} -5.5\phi m + 4k & -3k \\ -3k & -(5.5\phi)2m + 5k \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(k)}$$

The first equation represented by Equation (j) is divided by m and rearranged to $\chi = -\frac{1}{2}$. The second mode shape vector is any vector proportional to

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \quad \text{(l)}$$

The mode shape vectors are illustrated graphically in Figure 6.33(b). There is a node for the second mode located in the spring.

EXAMPLE 6.18

The two degree-of-freedom system shown in Figure 6.34 is subject to the periodic force shown. Determine the steady-state response of the system.

SOLUTION

The differential equations of motion are

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sin 2t \end{bmatrix} \quad \text{(a)}$$

A solution to the differential equations is assumed as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\cos(2t) + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\sin(2t) \quad \text{(b)}$$

Substituting Equation (b) into Equation (a) leads to

$$\begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\cos(2t) + \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\sin(2t)$$
$$+ \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\sin(2t) + \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\cos(2t)$$
$$+ \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\cos(2t) + \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\sin(2t) = \begin{bmatrix} 0 \\ 2\sin 2t \end{bmatrix} \quad \text{(c)}$$

which is rearranged to

$$\left(\begin{bmatrix} 0 & -3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right)\cos 2t$$
$$+ \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right)\sin 2t$$
$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix}\sin 2t \quad \text{(d)}$$

Equating coefficients of $\sin 2t$ and $\cos 2t$, four equations for four unknowns are obtained

$$\begin{bmatrix} 0 & -3 & 1 & -1 \\ -3 & -3 & -1 & 2 \\ -1 & 1 & 0 & -3 \\ 1 & -2 & -3 & -3 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \quad \text{(e)}$$

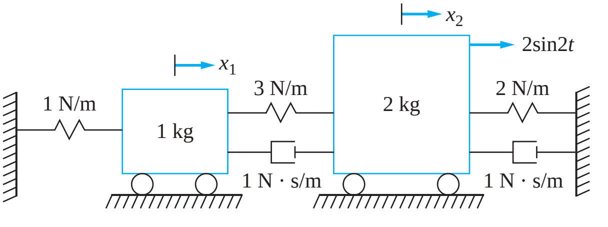


FIGURE 6.34
System of Example 6.18.

The solution to Equation (e) is $U_1 = 0.188$, $U_2 = -0.110$, $V_1 = -0.431$, and $V_2 = -0.094$. Thus,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.188 \\ -0.110 \end{bmatrix} \cos(2t) + \begin{bmatrix} -0.431 \\ -0.094 \end{bmatrix} \sin(2t) \tag{f}$$

The steady-state responses can be converted to a form with an amplitude and a phase by use of a trigonometric identity which leads to

$$x_1(t) = 0.470 \sin(2t + 2.70) \tag{g}$$

$$x_2(t) = 0.149 \sin(2t - 2.31) \tag{h}$$

EXAMPLE 6.19

A two-story frame structure, shown in Figure 6.35(a), can be modeled as the two degree-of-freedom system shown in Figure 6.35(b). The second story of the structure is subject to an explosion that leads to a force of the form of Figure 6.35(c). What is the maximum displacement of each story due to the explosion?

SOLUTION

The differential equations modeling the vibrations of each floor due to an explosion on the second floor are

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \tag{a}$$

$$m\ddot{x}_2 - kx_1 + kx_2 = F(t) \tag{b}$$

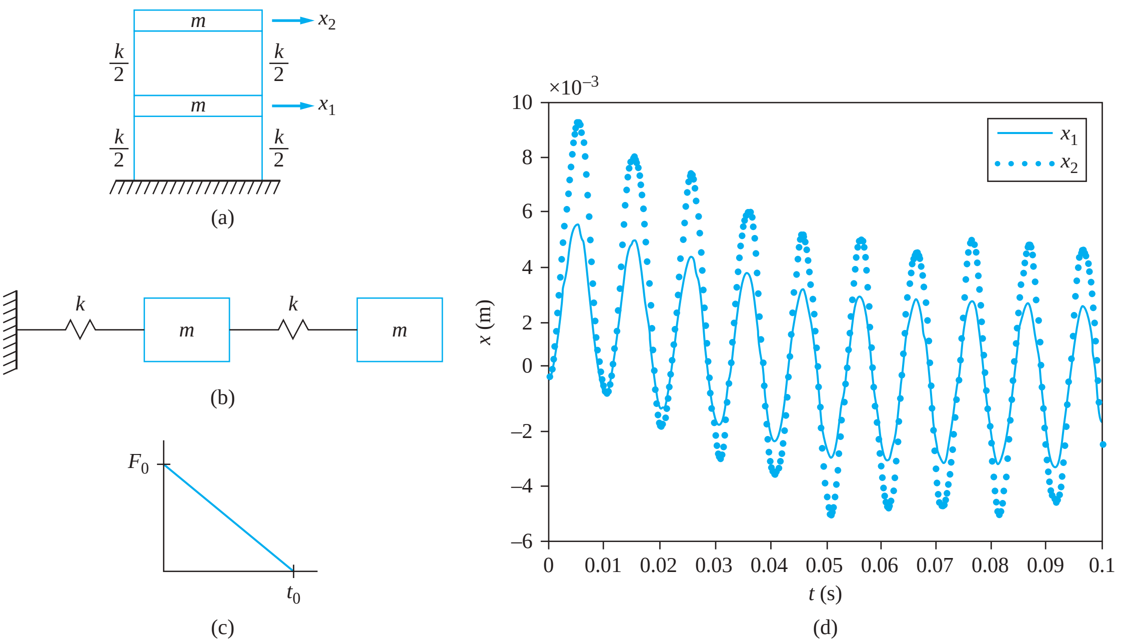


FIGURE 6.35
(a) Two-story frame structure of Example 6.19. (b) Two degree-of-freedom model of frame structure. (c) Force applied to second floor of structure. (d) Response of structure.

Taking the Laplace transform of both equations and summarizing the results in matrix form lead to

$$\begin{bmatrix} ms^2 + 2k & -k \\ -k & ms^2 + k \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix} \tag{c}$$

The transfer functions due to a force applied to the second story are obtained from

$$G_{12}(s) = \frac{\begin{vmatrix} 0 & -k \\ 1 & ms^2 + k \end{vmatrix}}{\begin{vmatrix} ms^2 + 2k & -k \\ -k & ms^2 + k \end{vmatrix}} = \frac{k}{m^2 s^4 + 3km + k^2} = \frac{\dfrac{k}{m^2}}{s^4 + 3\dfrac{k}{m}s^2 + \dfrac{k^2}{m^2}}$$

$$= \frac{\dfrac{k}{m^2}}{\left(s^2 + 0.382\dfrac{k}{m}\right)\left(s^2 + 2.618\dfrac{k}{m}\right)}$$

$$= \frac{0.447}{m}\left(\frac{1}{s^2 + 0.382\dfrac{k}{m}} - \frac{1}{s^2 + 2.618\dfrac{k}{m}}\right) \tag{d}$$

$$G_{22}(s) = \frac{\begin{vmatrix} ms^2 + 2k & 0 \\ -k & 1 \end{vmatrix}}{\begin{vmatrix} ms^2 + 2k & -k \\ -k & ms^2 + k \end{vmatrix}} = \frac{ms^2 + 2k}{m^2 s^4 + 3km + k^2} = \frac{\dfrac{1}{m}\left(s^2 + 2\dfrac{k}{m}\right)}{s^4 + 3\dfrac{k}{m}s^2 + \dfrac{k^2}{m^2}}$$

$$= \frac{\dfrac{1}{m}\left(s^2 + 2\dfrac{k}{m}\right)}{\left(s^2 + 0.382\dfrac{k}{m}\right)\left(s^2 + 2.618\dfrac{k}{m}\right)}$$

$$= \frac{1}{m}\left(\frac{0.724}{s^2 + 0.382\dfrac{k}{m}} + \frac{0.276}{s^2 + 2.618\dfrac{k}{m}}\right) \tag{e}$$

The impulsive responses are the inverses of the transfer functions, given here as

$$h_{12}(t) = \mathcal{L}^{-1}\{G_{12}(s)\} = \mathcal{L}^{-1}\left\{\frac{0.447}{m}\left(\frac{1}{s^2 + 0.382\dfrac{k}{m}} - \frac{1}{s^2 + 2.618\dfrac{k}{m}}\right)\right\}$$

$$= \frac{0.447}{m}\left[\frac{1}{0.618\sqrt{\dfrac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}t\right) - \frac{1}{1.618\sqrt{\dfrac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}t\right)\right] \tag{f}$$

$$h_{22}(t) = \mathcal{L}^{-1}\{G_{22}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{m}\left(\frac{0.724}{s^2 + 0.382\dfrac{k}{m}} + \frac{0.276}{s^2 + 2.618\dfrac{k}{m}}\right)\right\}$$

$$= \frac{1}{m}\left[\frac{0.724}{0.618\sqrt{\dfrac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}t\right) + \frac{0.276}{1.618\sqrt{\dfrac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}t\right)\right] \tag{g}$$

The forced response is the convolution integral of the impulsive response and the forcing function, given as

$$x_1(t) = \int_0^t F_0\left(1 - \frac{\tau}{t_0}\right)[u(\tau) - u(\tau - t_0)]\frac{0.447}{m}\left\{\frac{1}{0.618\sqrt{\dfrac{k}{m}}}\sin\left[0.618\sqrt{\frac{k}{m}}(t-\tau)\right]\right.$$

$$\left. - \frac{1}{1.618\sqrt{\dfrac{k}{m}}}\sin\left[1.618\sqrt{\frac{k}{m}}(t-\tau)\right]\right\}d\tau \tag{h}$$

Table 5.1 can help with the convolution integral evaluation. Use the delayed ramp function with $A = -1/t_0$, $B = 1$, and t_0 equal to either 0 or t_0 to evaluate an integral. The result for $x_1(t)$ is

$$x_1(t) = -\frac{0.447F_0}{m}\left\{\frac{1}{0.382\dfrac{k}{m}}\left[t - t_0 + t_0\cos\left(0.618\sqrt{\frac{k}{m}}t\right)\right.\right.$$

$$\left. - \frac{1}{0.618\sqrt{\dfrac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}t\right)\right]u(t)$$

$$- \frac{1}{0.382\dfrac{k}{m}}\left[t - t_0 - \frac{1}{0.618\sqrt{\dfrac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}(t - t_0)\right)\right]u(t - t_0)$$

$$- \frac{1}{2.618\dfrac{k}{m}}\left[t - t_0 + t_0\cos\left(1.618\sqrt{\frac{k}{m}}t\right) - \frac{1}{1.618\sqrt{\dfrac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}t\right)\right]u(t)$$

$$\left. + \frac{1}{2.618\dfrac{k}{m}}\left[t - t_0 - \frac{1}{1.618\sqrt{\dfrac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}(t - t_0)\right)\right]u(t - t_0)\right\} \tag{i}$$

The solution for $x_2(t)$ is

$$x_2(t) = \int_0^t F_0\left(1 - \frac{\tau}{t_0}\right)[u(\tau) - u(\tau - t_0)]\frac{1}{m}\left[\frac{0.724}{0.618\sqrt{\frac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}t\right) + \frac{0.276}{1.618\sqrt{\frac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}t\right)\right]d\tau \tag{j}$$

Using the same method to evaluate the convolution integral for $x_2(t)$, we have

$$\begin{aligned} x_2(t) = -\frac{F_0}{m}\Bigg\{ & \frac{0.724}{0.382\frac{k}{m}}\left[t - t_0 + t_0\cos\left(0.618\sqrt{\frac{k}{m}}t\right) - \frac{1}{0.618\sqrt{\frac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}t\right)\right]u(t) \\ & - \frac{0.724}{0.382\frac{k}{m}}\left[t - t_0 - \frac{1}{0.618\sqrt{\frac{k}{m}}}\sin\left(0.618\sqrt{\frac{k}{m}}(t - t_0)\right)\right]u(t - t_0) \\ & + \frac{0.276}{2.618\frac{k}{m}}\left[t - t_0 + t_0\cos\left(1.618\sqrt{\frac{k}{m}}t\right) - \frac{1}{1.618\sqrt{\frac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}t\right)\right]u(t) \\ & - \frac{0.276}{2.618\frac{k}{m}}\left[t - t_0 - \frac{1}{1.618\sqrt{\frac{k}{m}}}\sin\left(1.618\sqrt{\frac{k}{m}}(t - t_0)\right)\right]u(t - t_0)\Bigg\} \end{aligned} \tag{k}$$

Equations (j) and (k) are plotted in Figure 6.35(d) for $m = 1000$ kg, $k = 1 \times 10^6$ N/m, $t_0 = 0.05$ s and $F_0 = 50{,}000$ N.

EXAMPLE 6.20

A large machine has a mass of 200 kg and is mounted on an undamped elastic foundation of stiffness 2.5×10^6 N/m as shown in Figure 6.36(a). During operation at 110 r/s, the machine is subject to a harmonic force of magnitude 2200 N.

(a) Determine the steady-state amplitude of the machine as it operates.
(b) Determine the required stiffness of an undamped vibration absorber of mass 20 kg such that steady-state vibrations of the machine are eliminated during operation.

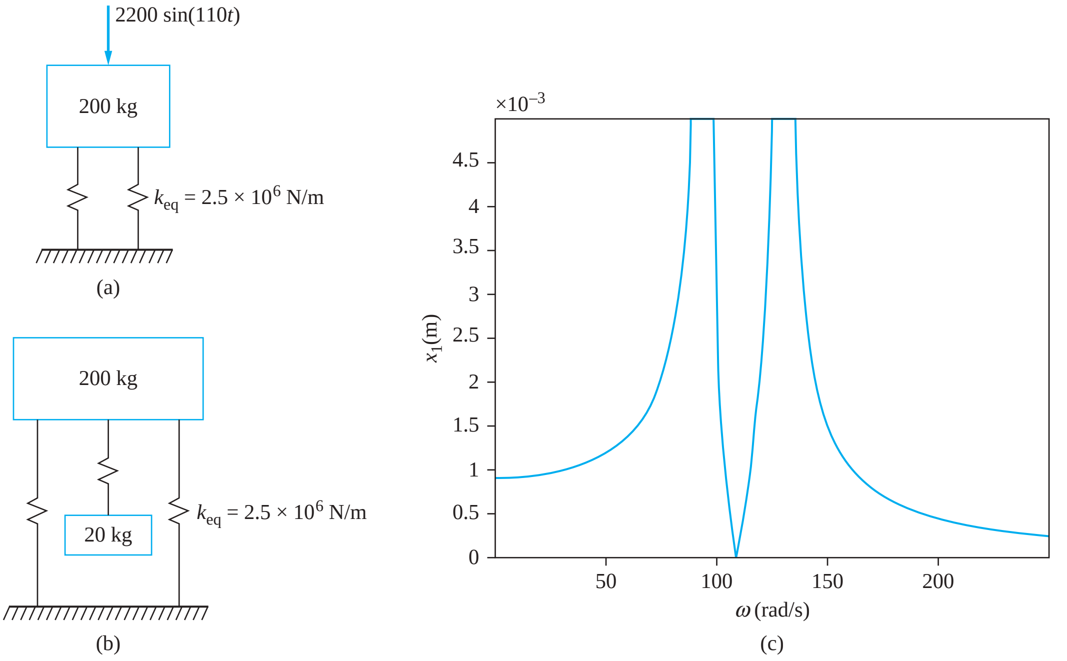


FIGURE 6.36
(a) Machine is mounted on an elastic foundation at an excitation frequency of 110 rad/s. (b) Vibration absorber of mass 20 kg is designed to eliminate steady-state vibrations of the machine. (c) Frequency response of machine with absorber in place.

(c) Determine the amplitude of the absorber mass when the vibration absorber of part (b) is used.
(d) What are the natural frequencies of the resulting two degree-of-freedom system?
(e) When this absorber is used, what is the frequency range such that the machine's steady-state amplitude is less than 1.2 mm?

SOLUTION

(a) The natural frequency of the machine mounted on the elastic foundation is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.5 \times 10^6 \text{ N/m}}{200 \text{ kg}}} = 111.8 \text{ rad/s} \tag{a}$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{110 \text{ rad/s}}{111.8 \text{ rad/s}} = 0.984 \tag{b}$$

The steady-state amplitude of the machine is

$$X = \frac{F_0}{m\omega_n^2} M(0.984, 0) = \frac{2200 \text{ N}}{2.5 \times 10^6 \text{ N/m}} \frac{1}{1 - (0.984)^2} = 2.75 \text{ cm} \tag{c}$$

(b) To eliminate steady-state vibrations at the excitation speed, the absorber is tuned to the excitation speed

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = \omega \tag{d}$$

Thus

$$k_2 = m_2\omega^2 = (20 \text{ kg})(110 \text{ rad/s})^2 = 2.42 \times 10^5 \text{ N/m} \tag{e}$$

(c) The steady-state amplitude of the absorber when the system operates at the frequency to which the absorber is tuned is

$$X_2 = \frac{F_0}{k_2} = \frac{2200 \text{ N}}{2.42 \times 10^5 \text{ N/m}} = 9.1 \text{ mm} \tag{f}$$

The absorber attached to the machine is illustrated in Figure 6.36(b).

(d) The ratio of the absorber mass to the mass of the machine is $\mu = (20 \text{ kg})/(200 \text{ kg}) = 0.1$. The ratio of the tuned frequency to the natural frequency of the machine is the same as the original frequency ratio $q = 0.984$. Natural frequencies of the two degree-of-freedom system with the absorber in place are

$$\omega_{1,2} = \frac{\omega_{11}}{\sqrt{2}}\sqrt{1 + q^2(1+\mu) \pm \sqrt{q^4(1+\mu^2) + 2(\mu - 1)q^2 + 1}}$$

$$= \frac{111.8 \frac{r}{s}}{\sqrt{2}}\sqrt{1+(0.984)^2(1+0.1) \pm \sqrt{(0.984)^4(1+0.1)^2+2(0.1-1)(0.984)^2+1}}$$

$$= 94.8 \text{ rad/s}, 129.7 \text{ rad/s} \tag{g}$$

(e) Let ω be a varying frequency. Define $r_1 = \frac{\omega}{111.8 \text{ rad/s}}$ and $r_2 = \frac{\omega}{110 \text{ rad/s}}$. The frequency response of the machine is given by

$$X_1 = \frac{F_0}{k_1}\left|\frac{1 - r_2^2}{r_1^2 r_2^2 - r_2^2 - (1+\mu)r_1^2 + 1}\right|$$

$$= \frac{2200 \text{ N}}{2.5\times10^6 \text{ N/m}}\left|\frac{1 - \left(\frac{\omega}{110 \text{ rad/s}}\right)^2}{\left(\frac{\omega}{111.8 \text{ rad/s}}\right)^2\left(\frac{\omega}{110 \text{ rad/s}}\right)^2 - \left(\frac{\omega}{110 \text{ rad/s}}\right)^2 - (1+0.1)\left(\frac{\omega}{111.8 \text{ rad/s}}\right)^2 + 1}\right| \tag{h}$$

The values of ω for which the steady-state amplitude of the machine is less 1.2 mm are obtained by setting $X_1 < 0.0012$ m in Equation (h) and solving for ω. There are two values of ω which satisfy $X_1 < 0.0012$ m: one value less than ω_{22} and one value greater than ω_{22}. In performing the calculations, note that the numerator is positive for $\omega < \omega_{22}$ and negative for $\omega > \omega_{22}$, but the denominator is always positive in the operating range. The equation can be rearranged into a quadratic equation in ω^2, resulting in an operating range of

$$104.3 \text{ rad/s} < \omega < 117.0 \text{ rad/s} \tag{i}$$

The frequency response of the pump is illustrated in Figure 6.36(c).

EXAMPLE 6.21

It is decided to place a damped vibration absorber on the machine of Example 6.21. In addition to changing the frequency-response curve of the primary system, it can serve as an energy harvester (see Section 4.15). Assume that an optimal damped vibration absorber of mass 20 kg is used. What is the average power harvested by the absorber over one cycle?

SOLUTION

The mass ratio of the absorber is $\mu = \frac{m_2}{m_1} = 0.1$. The optimum damping ratio of the absorber is

$$\zeta_{\text{opt}} = \sqrt{\frac{3\mu}{8(1+\mu)}} = \sqrt{\frac{3(0.1)}{8(1.1)}} = 0.184 \tag{a}$$

The absorber is tuned such that

$$q = \frac{1}{1+\mu} = 0.909 \tag{b}$$

or

$$\omega_{22} = 0.909\omega_{11} = 0.909(110 \text{ rad/s}) = 100.0 \text{ rad/s} \tag{c}$$

The average power harvested by the absorber is

$$\overline{P} = \frac{c\omega^2 Z^2}{2} = \zeta m_2 \omega_{22}^4 r_2^2 Z^2 \tag{d}$$

where Z is the amplitude of the relative displacement between the absorber and the primary system. If $x_1(t) = X_1 \sin(\omega t - \phi_1)$ and $x_2(t) = X_2 \sin(\omega t - \phi_2)$, then

$$z(t) = X_2 \sin(\omega t - \phi_2) - X_1 \sin(\omega t - \phi_1) = Z \sin(\omega t - \phi_3) \tag{e}$$

where

$$Z = \sqrt{X_1^2 - 2X_1X_2 \sin(\phi_1 + \phi_2) + X_2^2} \tag{f}$$

Defining

$$M = r_1^4 - [1 + (1+\mu)q^2 r_1^2] + q^2 \tag{g}$$

and

$$N = 2\zeta r_1 q[1 - r_1^2(1+\mu)] \tag{h}$$

analysis of the two degree-of-freedom system gives

$$X_1 = \frac{F_0}{k_1}\sqrt{\frac{(2\zeta r_1 q)^2 + (r_1^2 - q^2)^2}{\{r_1^4 - [1 + (1+\mu)q^2 r_1^2] + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}} = 0.0057\,\text{m} \tag{i}$$

$$\phi_1 = \tan^{-1}\left[\frac{2\zeta r_2 M - (1 - r_2^2)N}{(1 - r_2^2)q^2\mu M + 2\zeta N r_2\sqrt{\mu}}\right] = -1.784 \tag{j}$$

$$X_2 = \frac{F_0}{k_1}\sqrt{\frac{q^4 + (2\zeta q)^2}{\{r_1^4 - [1 + (1 + \mu)q^2 r_1^2] + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1 + \mu)]^2}} = 0.0027 \text{ m} \quad \text{(k)}$$

$$\phi_2 = \tan^{-1}\left[\frac{M - 2\zeta r_2 N}{2\zeta r_2 M + N}\right] = -2.278 \quad \text{(l)}$$

The value of Z using Equation (f) is $Z = 0.0039$ m. Thus, from Equation (d), the average power harvested over one cycle is

$$\overline{P} = (0.184)(10 \text{ kg})(100 \text{ rad/s})^4(0.909)^2(0.0039 \text{ m})^2 = 4.64 \text{ kW} \quad \text{(m)}$$

6.16 CHAPTER SUMMARY

6.16.1 IMPORTANT CONCEPTS

- Two degree-of-freedom systems are governed by two coupled differential equations.
- FBD method is used to derive differential equation governing the motion of two degree-of-freedom systems.
- A normal mode solution in which synchronous motion occurs is assumed for the free response of undamped systems.
- The natural frequencies are obtained by solution of a fourth-order algebraic equation for ω with only even powers of ω.
- The modal fraction for each mode is the second element of the mode shape vector when the first element is set equal to one.
- The mode shape vectors can be illustrated graphically.
- A node is a point of zero displacement for a mode.
- The general free response is a linear combination of the modes. The constants in the linear combination are determined from application of the initial conditions.
- An exponential solution is assumed for the free response of system with viscous damping. The exponents are obtained by solving a fourth order algebraic equation with odd powers.
- Every undamped system has a set of principal coordinates which when the differential equations are written in terms of the principal coordinates they are uncoupled.
- The harmonic response of two degree-of-freedom systems is obtained by the method of undetermined coefficients or use of the sinusoidal transfer function.
- A transfer function matrix can be defined when its elements are $G_{i,j}(s)$ where $G_{i,j}(s)$ is the transform of the response at x_i due to a unit impulse applied at x_j.
- A convolution integral solution provides the response of the system due to any forcing function.
- The frequency response is the variation of steady-state amplitude with frequency.
- A vibration absorber, when tuned to the excitation frequency, can be used to eliminate steady-state vibrations of the primary system.

- The vibration absorber works by changing a SDOF system to a two degree-of-freedom system. The natural frequencies of the resulting two degree-of-freedom system are away from the excitation frequency.
- Damped absorbers are designed to reduce the amplitude during start-up and to widen the operating range of the absorber.

6.16.2 IMPORTANT EQUATIONS

Matrix formulation of differential equations

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{6.1}$$

Normal mode solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{X}e^{i\omega t} \tag{6.3}$$

Determination of natural frequencies for undamped system

$$\det(-\omega^2\mathbf{M} + \mathbf{K}) = 0 \tag{6.8}$$

Modal fraction

$$\chi_2 = \frac{-\omega^2 m_{1,1} - k_{1,1}}{-\omega^2 m_{1,2} + k_{1,2}} \tag{6.11}$$

Free response of an undamped system

$$\boldsymbol{x}(t) = [C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)]\boldsymbol{X}_1 + [C_3\cos(\omega_2 t) + C_4\sin(\omega_2 t)]\boldsymbol{X}_2 \tag{6.13}$$

$$\mathbf{x}(t) = A_1\mathbf{X}_1\sin(\omega_1 t + \phi_1) + A_2\mathbf{X}_2\sin(\omega_2 t + \phi_2) \tag{6.16}$$

Solution for system with viscous damping

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \chi \end{bmatrix} e^{\lambda t} \tag{6.18}$$

Determination of free response for damped system

$$\det(\lambda^2\boldsymbol{MX} + \lambda\boldsymbol{CX} + \boldsymbol{KX}) = 0 \tag{6.20}$$

Differential equations for the principal coordinates

$$\ddot{p}_1 + \omega_1^2 p_1 = 0 \tag{6.24}$$

$$\ddot{p}_2 + \omega_2^2 p_2 = 0 \tag{6.25}$$

Steady-state vibrations of an undamped system due to single frequency excitation

$$\mathbf{x} = \mathbf{U}\sin(\omega t) \tag{6.38}$$

Steady-state response for system with viscous damping due to single frequency excitation

$$x_1 = u_1\sin(\omega t) + v_1\cos(\omega t) \tag{6.45}$$

$$x_2 = u_2\sin(\omega t) + v_2\cos(\omega t) \tag{6.46}$$

Steady-state amplitudes and phases

$$x_1 = X_1 \sin(\omega t - \phi_1) \tag{6.47}$$

$$x_2 = X_2 \sin(\omega t - \phi_2) \tag{6.48}$$

$$X_i = \sqrt{u_i^2 + v_i^2} \tag{6.49}$$

$$\phi_i = \tan^{-1}\left(\frac{v_i}{u_i}\right) \tag{6.50}$$

Convolution integral solution for x_i due to a force applied at x_j

$$x_i(t) = \int_0^t F_j(\tau) h_{i,j}(t - \tau) d\tau \tag{6.53}$$

Forced response of system

$$x(t) = F_0 |G(i\omega)| \sin(\omega t + \phi) \tag{6.65}$$

Frequency response for primary system when vibration absorber is used

$$\frac{k_1 X_1}{F_0} = \left| \frac{1 - r_2^2}{r_1^2 r_2^2 - r_2^2 - (1 + \mu) r_1^2 + 1} \right| \tag{6.90}$$

Tuning of absorber

$$k_2 = m_2 \omega^2 \tag{6.92}$$

Steady-state amplitude of tuned absorber

$$X_2 = \frac{F_0}{k_2} \tag{6.94}$$

Optimally damped absorber

$$q = \frac{1}{1 + \mu} \tag{6.108}$$

$$\zeta_{\text{opt}} = \sqrt{\frac{3\mu}{8(1 + \mu)}} \tag{6.109}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 6.1 through 6.15 indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

6.1 A two degree-of-freedom system has two natural frequencies.

6.2 The natural frequencies are determined by setting $|\omega^2 \mathbf{K} - \mathbf{M}| = 0$.

6.3 The natural frequencies of a two degree-of-freedom system depend upon the choice of generalized coordinates used to model the system.

6.4 The natural frequencies for an undamped two-degree-of-freedom system are determined by solving for the roots of a fourth-order polynomial that only has even powers of the frequency.
6.5 The modal fraction represents the damping of each mode.
6.6 The principal coordinates are the generalized coordinates for which the mass matrix and the stiffness matrix are symmetric matrices.
6.7 The free response of a damped two degree-of-freedom system has two modes of vibration, both of which are underdamped.
6.8 A displacement of a node for a mode of a two degree-of-freedom system can serve as a principal coordinate.
6.9 The modal fractions for a two degree-of-freedom system depend upon the choice of generalized coordinates used to model the system.
6.10 The sinusoidal transfer function can be used to determine the steady-state response of a two degree-of-freedom system.
6.11 Addition of an undamped vibration absorber transforms a SDOF system into a system with two degrees of freedom.
6.12 The undamped vibration absorber is tuned to the natural frequency of the primary system to eliminate steady-state vibrations of the absorber.
6.13 An optimally tuned damped vibration absorber is tuned such that only the amplitude of vibration during start-up is minimized.
6.14 Addition of a dynamic vibration absorber to a damped primary system will eliminate the steady-state vibrations of the primary system if the absorber is tuned to the excitation frequency.
6.15 A Houdaille damper is used for vibration control in engine crankshafts.

Problems 6.16 through 6.37 require a short answer.

6.16 Draw a FBD of the block whose displacement is x_1 of Figure SP6.16 at an arbitrary instant of time, appropriately labeling the forces.
6.17 Draw a FBD of the block whose displacement is x_2 of Figure SP6.17 at an arbitrary instant of time, appropriately labeling the forces.

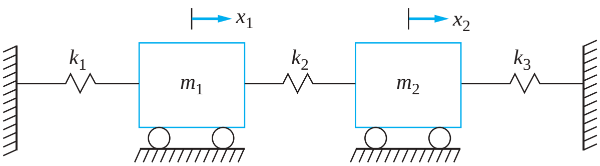


FIGURE SP6.16

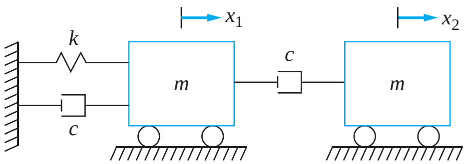


FIGURE SP6.17

6.18 What is the normal-mode solution and how is it used?
6.19 Discuss the difference in the assumed solution for free vibrations of an undamped two degree-of-freedom system and one with viscous damping.
6.20 What does a real solution of the fourth-order equation for a system with viscous damping to solve for λ mean regarding the mode of vibration?
6.21 What does a complex solution of the fourth-order equation for a system with viscous damping to solve for λ mean regarding the mode of vibration?
6.22 What is the meaning of the transfer function $G_{1,2}(s)$?

6.23 Define the sinusoidal transfer function.

6.24 Write the differential equations for the principal coordinates of free undamped vibrations of a two degree-of-freedom system with natural frequencies ω_1 and ω_2.

6.25 A two degree-of-freedom system has a mode with a modal fraction equal to zero. What does this imply?

6.26 A two degree-of-freedom system has a mode with a modal fraction equal to one. What does this imply?

6.27 How many nodes are there for the mode corresponding to the lowest natural frequency of a two degree-of-freedom system?

6.28 If the differential equations governing a two degree-of-freedom system are uncoupled when a certain set of generalized coordinates are used, the coordinates must be ____________ coordinates of the system.

6.29 The general form of the transfer function is

$$G(s) = \frac{N(s)}{D(s)}$$

The transfer functions $G_{1,1}(s)$ and $G_{2,1}(s)$, defined for a two degree-of-freedom system, have which in common (choose one)?

(a) The numerator $N(s)$
(b) The denominator $D(s)$
(c) Neither the numerator or the denominator
(d) Both the numerator and the denominator

6.30 State the convolution integral solution for the forced response of the generalized coordinate $x_1(t)$ when due to a force $F(t)$ applied at the location where the second generalized coordinate $x_2(t)$ is defined.

6.31 How are the amplitudes and phases determined for free vibrations of a two degree-of-freedom system?

6.32 How is $G(i\omega)$ resolved into polar coordinates?

6.33 What is the vibration amplitude of the primary system when a dynamic vibration absorber tuned to the excitation frequency is added to the system?

6.34 How does a dynamic vibration absorber work?

6.35 When is a vibration damper used?

6.36 What two problems does the addition of damping address when added to a vibration absorber?

6.37 How is the optimum damping ratio of a Houdaille damper defined?

Problems 6.38 through 6.47 require short calculations.

6.38 The equation

$$6\omega^4 - 27\omega^2 + 21 = 0$$

is an equation developed to determine the natural frequencies of a system. Solve the equation to determine the natural frequencies.

6.39 The equations for the natural frequencies and mode shape vectors of a two degree-of-freedom system are

$$\begin{bmatrix} -\omega^2 + 3 & -2 \\ -2 & -\omega^2 + 2 \end{bmatrix} \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) Define a system that would yield this equation.
(b) Calculate the natural frequencies of the system.
(c) Calculate the mode shape corresponding to the lower natural frequency.
(d) Draw a diagram illustrating the mode shape vector.

6.40 A two degree-of-freedom system has a modal fraction for one of its mode shapes of -1. (a) Draw the mode shape diagram corresponding to that mode. (b) Does the mode shape correspond to the lower or higher natural frequency?

6.41 The transfer function for one generalized coordinate of a two degree-of-freedom system is

$$G(s) = \frac{1}{s^4 + 3s^2 + 2}$$

(a) Calculate $G(3i)$.
(b) What are the natural frequencies of the system?
(c) If this system were excited by a force equal to $5 \sin 3t$, what is the steady-state response of the generalized coordinate?

6.42 The transfer function for a generalized coordinate, x_1, of a two degree-of-freedom system, due to a force at the other generalized coordinate, x_2, is

$$G(s) = \frac{1}{s^4 + 2s^3 + 4s^2 + 10s + 25}$$

If x_2 is subject to a force $2.5 \sin 4t$, what is the steady-state response of x_1?

6.43 A machine vibrates at a frequency ratio of 1.05. A vibration absorber tuned to the excitation frequency is added to the machine. What is the value of (a) r_2, (b) r_1, (c) q?

6.44 If the mass ratio of the absorber of Short Problem 6.43 is 0.2 and the natural frequency of the primary system is 100 rad/s, what are the natural frequencies with the absorber in place?

6.45 A machine is excited at a frequency of 30 Hz by a force with an amplitude of 200 N. It is desired to eliminate steady-state vibrations of the machine by addition of a vibration absorber.

(a) What frequency should the absorber be tuned?
(b) If the mass of the absorber is 3 kg, what is the stiffness of the absorber?
(c) When the machine is excited at 30 Hz, what is the amplitude of vibration of the absorber?
(d) What is the frequency of the absorber vibrations?

6.46 An optimally damped vibration absorber is being designed for a primary system of natural frequency 100 rad/s. The mass of the machine is 50 kg and the mass of the absorber is to be 10 kg.

(a) What is the natural frequency of the absorber?
(b) What damping ratio is to be used for the absorber?

6.47 An optimally designed Houdaille damper is to be used to absorb the vibrations of a rotational system. The moment of inertia of the primary system is $0.1\ \text{kg} \cdot \text{m}^2$ and the moment of inertia of the damper is to be is $0.01\ \text{kg} \cdot \text{m}^2$.

(a) What is the optimum damping ratio?
(b) What is the steady-state amplitude of the primary system if $\frac{M_0}{k} = 0.002$?

CHAPTER PROBLEMS

6.1 Derive the differential equation governing the two degree-of-freedom system shown in Figure P6.1 using x_1 and x_2 as generalized coordinates.

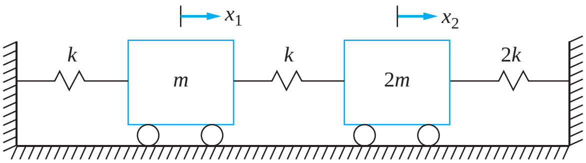


FIGURE P6.1

6.2 Derive the differential equation governing the two degree-of-freedom system shown in Figure P6.2 using x and θ as generalized coordinates.

6.3 Derive the differential equations governing the two degree-of-freedom system shown in Figure P6.3 using θ_1 and θ_2 as generalized coordinates.

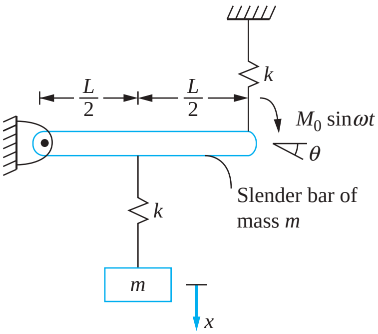


FIGURE P6.2

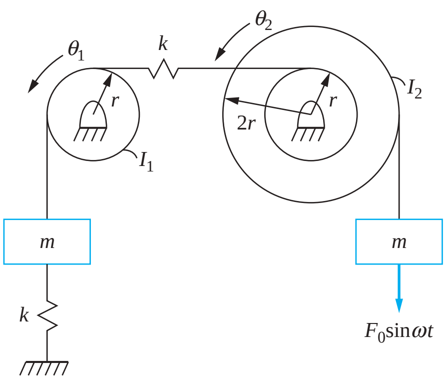


FIGURE P6.3

6.4 Derive the differential equations governing the two degree-of-freedom system shown in Figure P6.4 using θ_1 and θ_2 as generalized coordinates.

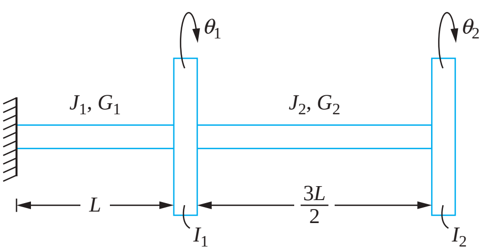


FIGURE P6.4

6.5 A two degree-of-freedom model of an airfoil shown in Figure P6.5 is used for flutter analysis. Derive the governing differential equations using h and θ as generalized coordinates.

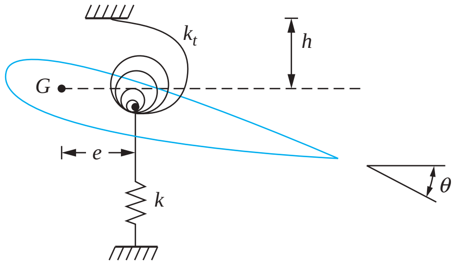


FIGURE P6.5

6.6 Derive the differential equations governing the damped two degree-of-freedom system shown in Figure P6.6 using x_1 and x_2 as generalized coordinates.

6.7 Derive the differential equations governing the damped two degree-of-freedom system shown in Figure P6.7 using x_1 and x_2 as generalized coordinates.

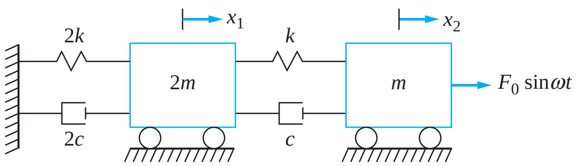


FIGURE P6.6

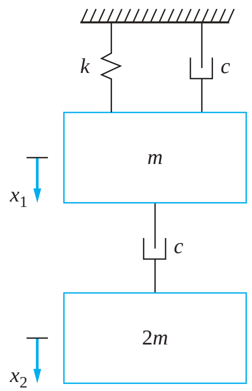


FIGURE P6.7

6.8 A two degree-of-freedom model of a machine tool is illustrated in Figure P6.8. Using x_1 and x_2 as generalized coordinates, derive the differential equations governing the motion of the system.

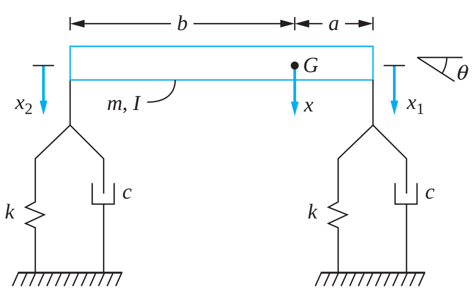


FIGURE P6.8

6.9 Derive the differential equation of the two degree-of-freedom model of the machine tool of Chapter Problem 6.8 using x and θ as generalized coordinates.

6.10 Determine the natural frequencies of the system of Figure P6.1 if $m = 10$ kg and $k = 1 \times 10^5$ N/m. Determine and graphically illustrate the mode shapes. Identify any nodes.

6.11 Determine the natural frequencies of the system of Figure P6.2 if $m = 2$ kg, $L = 1$ m and $k = 1000$ N/m. Determine the modal fractions for each mode.

6.12 Determine the natural frequencies of the system of Figure P6.3 if $m = 30$ g, $I_1 = 8 \times 10^{-6}$ kg · m^2, $I_2 = 2 \times 10^{-5}$ kg · m^2, $r = 5$ mm, and $k = 10$ N/m. Determine the modal fraction for each mode.

6.13 Determine the natural frequencies of the system of Figure P6.4 if $I_1 = 0.3$ kg · m^2, $I_2 = 0.4$ kg · m^2, $J_1 = J_2 = 1.6 \times 10^{-8}$ m^4, $G_1 = G_2 = 80 \times 10^9$ N/m^2, and $L = 30$ cm. Determine the modal fractions for each mode. Identify any nodes.

6.14 An overhead crane is modeled as a two degree-of-freedom system as shown in Figure P6.14. The crane is modeled as a mass of 1000 kg on a steel ($E = 200 \times 10^9$ N/m^2) fixed-fixed beam with a moment of inertia of 4.2×10^{-3} m^4 and length of 12 m. The crane has an elastic steel rope of diameter 20 cm. At a specific instant, the length of the rope is 10 m and is carrying a 300 kg load. What are the two natural frequencies of the system?

6.15 A seismometer of mass 30 g and stiffness 40 N/m is used to measure the vibrations of a SDOF system of mass 60 g and natural frequency 150 rad/s. It is feared that the mass of the seismometer may affect the vibrations that are to be measured. Check this out by calculating the natural frequencies of the two degree-of-freedom system with the seismometer attached.

6.16 Calculate the natural frequencies and modal fractions for the system of Figure P6.16.

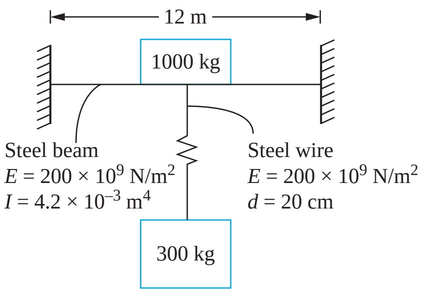


FIGURE P6.14

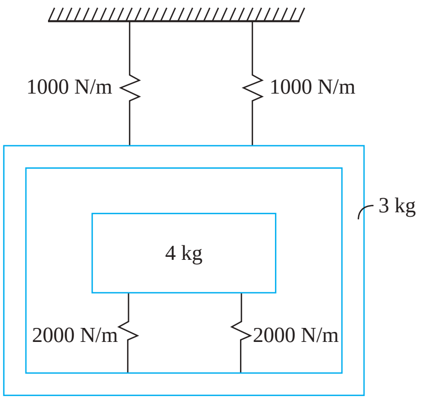


FIGURE P6.16

6.17 Determine the forced response to the system of Figure P6.1 and Chapter Problems 6.1 and 6.10 if the left-hand mass is given an initial displacement of 0.001 m while the right-hand mass is held in equilibrium and the system is released from rest.

6.18 Determine the response of the system of Figure P6.2 and Chapter Problems 6.2 and 6.11 if the particle is given an initial velocity of 2 m/s when the system is in equilibrium.

6.19 Determine the response of the system of Figure P6.4 and Chapter Problems 6.4 and 6.13 if the right-hand disk is given an angular displacement of 2° clockwise from equilibrium and the left-hand disk is given an angular displacement of 2° counterclockwise.

6.20 Determine the response of the system of Chapter Problem 6.14 if the crane is disturbed resulting in an initial velocity of 10 m/s downward.

6.21 Determine the output from the seismometer of Chapter Problem 6.15 if the 60 g mass is given an initial velocity of 15 m/s. Use a two degree-of-freedom system, remembering that the seismometer records the relative displacement between the seismic mass and the body whose vibrations are to be measured.

6.22 Determine the free response of the system of Figure P6.6 if the left-hand mass is given an initial displacement of 0.001 m while the right-hand mass is held in equilibrium and the system is released from rest. Use $m = 1$ kg, $k = 10{,}000$ N/m, and $c = 100$ N · s/m.

6.23 Determine the response of the system of Figure P6.7 if the lower mass is given a displacement from equilibrium of 0.004 m and the upper mass is held in its equilibrium position and the system is released. Use $m = 5$ kg, $k = 4000$ N/m, and $c = 30$ N · s/m.

6.24 Determine the free response of the system of Figure P6.8 if the machine tool has initial velocities of $\dot{x}(0) = 0.8$ m/s and $\dot{\theta}(0) = 5$ rad/s. if $I = 0.03$ kg · m^2, $c = 100$ N · s/m, $m = 3$ kg, $a = 0.3$ m, $b = 0.4$ m and $k = 3000$ N/m.

6.25 Determine the principal coordinates for the system of Figure P6.1 and Chapter Problem 6.10. Write the differential equations which the principal coordinates satisfy.

6.26 Determine the principal coordinates for the system of Figure P6.2 and Chapter Problem 6.11. Write the differential equations which the principal coordinates satisfy.

6.27 Determine the principal coordinates for the system of Figure P6.3 and Chapter Problem 6.12. Write the differential equations which the principal coordinates satisfy.

6.28 Determine the principal coordinates for the system of Figure P6.4 and Chapter Problem 6.13. Write the differential equations which the principal coordinates satisfy.

6.29 Determine the principal coordinates for the system of Figure P6.8 if it had no damping. Write the differential equations which the principal coordinates satisfy. Use $I = 0.03$ kg · m^2, $m = 3$ kg, $a = 0.03$ m, $b = 0.3$ m and $k = 3000$ N/m.

6.30 Determine the principal coordinates for the system of Chapter Problem 6.9. Write the differential equations which the principal coordinates satisfy. if $I = 0.03$ kg · m^2, $c = 0$ N · s/m, $m = 3$ kg, $a = 0.3$ m, $b = 0.4$ m and $k = 3000$ N/m.

6.31 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $200 \sin 110t$ N applied to the block whose displacement is x_1 using the method of undetermined coefficients.

6.32 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $200 \sin 80t$ applied to the block whose displacement is x_2 using the Laplace transform method.

6.33 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a sinusoidal force $100 \sin 70t$ N applied to the particle using the method of undetermined coefficients.

6.34 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a sinusoidal moment $50 \sin 90t$ N · m applied to the bar using the method of undetermined coefficients.

6.35 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to (a) a unit impulse applied to the block whose displacement is x_1, and (b) a unit impulse applied to the block whose displacement is x_2.

6.36 Determine the response of the system of Figure P6.1 and Chapter Problem 6.10 due to the force of Figure P6.36 applied to the block whose displacement is x_1.

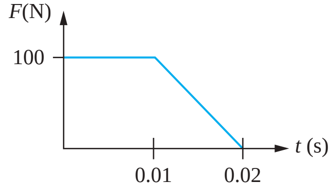


FIGURE P6.36

6.37 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a unit impulse applied to the particle.

6.38 Determine the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a unit impulsive moment applied to the bar.

6.39 Derive the response of the system of Figure P6.2 and Chapter Problem 6.11 due to the force of Figure P6.39 applied downward to the end of the bar.

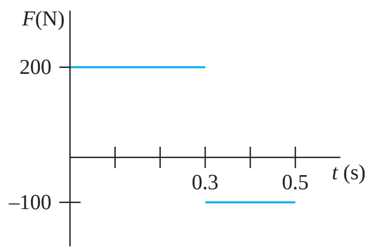


FIGURE P6.39

6.40 Derive the response of the system of Figure P6.2 and Chapter Problem 6.11 due to a moment $M(t) = 10e^{-2t}$ N · m applied to the bar.

6.41 Determine the response of the system of Figure P6.6 due to a force $F(t) = 20 \sin 20t$ N applied to the block whose displacement is x_2 using the method of undetermined coefficients. Use $m = 10$ kg, $k = 90{,}000$ N/m, and $c = 100$ N · s/m.

6.42 Determine the response of the system of Figure 6.7 due to a force $F(t) = 40 \sin 60t$ N applied to the block whose displacement is x_1 using the method of undetermined coefficients. Use $m = 20$ kg, $k = 200{,}000$ N/m, and $c = 400$ N · s/m.

6.43 Determine the response of the system of Figure P6.8 due to a unit impulse applied at the mass center. Use $m = 100$ kg, $I = 4.5$ kg · m^2, $k = 200{,}000$ N/m, $c = 500$ N · s/m, $b = 2$ m, and $a = 1$ m.

6.44 Determine the response of the system of Figure P6.8 and Chapter Problem 6.43 to a unit impulse applied t to the right end or the machine tool using x and θ as generalized coordinates.

6.45 Determine the response of the system of Figure P6.8 and Chapter Problem 6.43 to the force shown in Figure P6.45 applied at the right end of the machine tool.

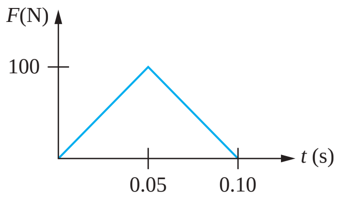


FIGURE P6.45

6.46 A schematic of part of a power transmission system is shown in Figure P6.46. A motor of moment of inertia $I = 100$ kg · m^2 is mounted on a shaft of shear modulus $G = 80 \times 10^9$ N/m^2, polar moment of inertia $J = 2.3 \times 10^{-4}$ m^4, and length 10 cm. Gear A, of moment of inertia 50 kg · m^2 with 40 teeth is at the end of the shaft which meshes with a gear, gear B, of moment of inertia 25 kg · m^2 with 20 teeth. Gear B is on a shaft of elastic modulus $G = 80 \times 10^9$ N/m^2, polar moment of inertia $J = 1.2 \times 10^{-5}$ m^4, and length 60 cm. At the end of the shaft is a large industrial fan of moment of inertia 300 kg · m^2. Determine the natural frequencies of the system and the modal fractions.

6.47 Determine the natural frequencies and modal fractions for the two degree-of-freedom system of Figure P6.47.

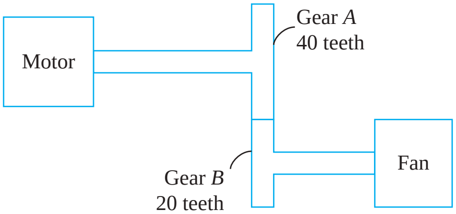


FIGURE P6.46

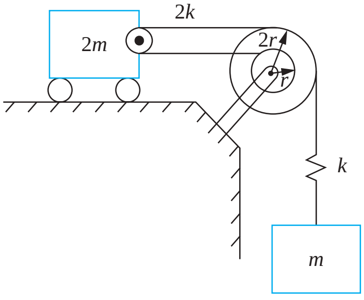


FIGURE P6.47

6.48 Determine the frequency response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $F_0 \sin \omega t$ applied to the block whose displacement is x_1.

6.49 Determine the frequency response of the system of Figure P6.1 and Chapter Problem 6.10 due to a sinusoidal force $F_0 \sin \omega t$ applied to the block whose displacement is x_2.

6.50 Determine the frequency response of the system of Figure P6.2 and Chapter Problem 6.11 due to a sinusoidal force $F_0 \sin \omega t$ applied to the particle.

6.51 Determine the frequency response of the system of Figure P6.7 and Chapter Problem 6.42 due to a sinusoidal force $F_0 \sin \omega t$ applied to the block whose displacement is x_1.

6.52 Determine the frequency response of the system of Figure P6.8 and Chapter Problem 6.43 due to a sinusoidal force $F_0 \sin \omega t$ applied to the mass center of the machine tool.

6.53 Determine the frequency response of the system of Figure P6.8 and Chapter Problem 6.43 due to a sinusoidal force $F_0 \sin \omega t$ applied to the right end of the machine tool.

6.54 A 50 kg lathe mounted on an elastic foundation of stiffness 4×10^5 N/m has a vibration amplitude of 35 cm when the motor speed is 95 rad/s. Design an undamped dynamic vibration absorber such that steady-state vibrations are completely eliminated at 95 rad/s and the maximum displacement of the absorber mass at this speed is 5 cm.

6.55 What is the required stiffness of an undamped dynamic vibration absorber whose mass is 5 kg to eliminate vibrations of a 25 kg machine of natural frequency 125 rad/s when the machine operates at 110 rad/s?

6.56 A 35 kg machine is attached to the end of a cantilever beam of length 2 m, elastic modulus 210×10^9 N/m^2, and moment of inertia 1.3×10^{-7} m^4. The machine operates at 180 rpm and has a rotating unbalance of 0.3 kg · m.

(a) What is the required stiffness of an undamped absorber of mass 5 kg such that steady-state vibrations are eliminated at 180 rpm?
(b) With the absorber in place, what are the natural frequencies of the system?
(c) For what range of operating speeds will the steady-state amplitude of the machine be less than 8 mm?

6.57 A 150 kg pump experiences large-amplitude vibrations when operating at 1500 rpm. Assuming this is the natural frequency of a SDOF system, design a dynamic vibration absorber such that the lower natural frequency of the two degree-of-freedom system is less than 1300 rpm and the higher natural frequency is greater than 1700 rpm.

6.58 A solid disk of diameter 30 cm and mass 10 kg is attached to the end of a solid 3-cm-diameter, 1-m-long steel shaft ($G = 80 \times 10^9$ N/m^2). A torsional vibration absorber consists of a disk attached to a shaft that is then attached to the primary system. If the absorber disk has a mass of 3 kg and a diameter of 10 cm, what is the required diameter of a 50-cm-long absorber shaft to eliminate steady-state vibrations of the original system when excited at 500 rad/s?

6.59 A 200 kg machine is placed on a massless simply supported beam as shown in Figure P6.59. The machine has a rotating unbalance of 1.41 kg · m and operates at 3000 rpm. The steady-state vibrations of the machine are to be absorbed by hanging a mass attached to a 40 cm steel cable from the location on the beam where the machine is attached. What is the required diameter of the cable such that machine vibrations are eliminated at 3000 rpm and the amplitude of the absorber mass is less than 50 mm?

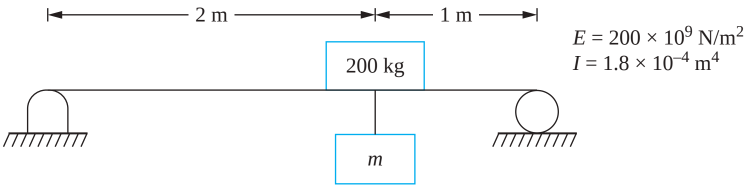


FIGURE P6.59

6.60 The disk in Figure P6.60 rolls without slip and the pulley is massless. What is the mass of the block that should be hung from the cable such that steady-state vibrations of the cylinder are eliminated when $\omega = 120$ rad/s?

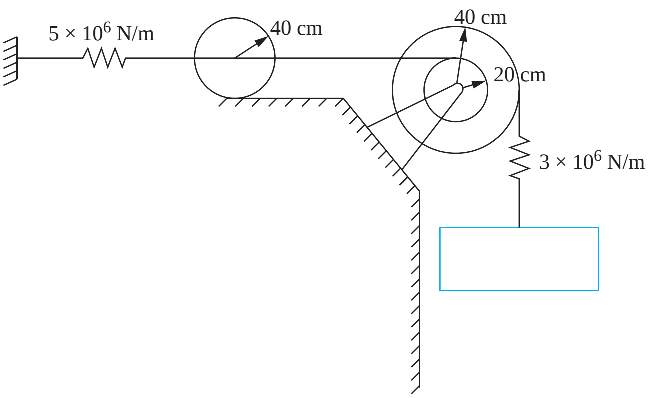


FIGURE P6.60

6.61 Vibration absorbers are used in boxcars to protect sensitive cargo from large accelerations due to periodic excitations provided by rail joints. For a particular railway, joints are spaced 5 m apart. The boxcar, when empty, has a mass of 25,000 kg. Two absorbers, each of mass 12,000 kg, are used. Absorbers for a particular boxcar are designed to eliminate vibrations of the main mass when the boxcar is loaded with a 12,000 kg cargo and travels at 100 m/s. The natural frequency of the unloaded boxcar is 165 rad/s.

(a) At what speeds will resonance occur for the boxcar with a 12,000 kg cargo?
(b) What is the best speed for the boxcar when it is loaded with a 25,000 kg cargo?

6.62 A 500 kg reciprocating machine is mounted on a foundation of equivalent stiffness 5×10^6 N/m. When operating at 800 rpm, the machine produces an unbalanced harmonic force of magnitude 50,000 N. Two cantilever beams with end masses are added to the machine to act as absorbers. The beams are made of steel ($E = 210 \times 10^9$ N/m^2) and have a moment of inertia of 4×10^{-6} m^4. A 10 kg mass is attached to each beam. The absorbers are adjustable in that the location of the mass on the absorber can be varied.

(a) How far away from the support should the masses be located when the machine is operating at 800 rpm? What is the amplitude of the absorber mass?

(b) If the machine operates at 1000 rpm and produces a harmonic force of amplitude 100,000 N, where should the absorber masses be placed and what is their vibration amplitude?

6.63 A 100 kg machine is placed at the midspan of a 2-m-long cantilever beam ($E = 210 \times 10^9$ N/m^2, $I = 2.3 \times 10^{-6}$ m^4). The machine produces a harmonic force of amplitude 60,000 N. Design a damped vibration absorber of mass 30 kg such that when hung from the beam at midspan, the steady-state amplitude of the machine is less than 8 mm at all speeds between 1300 and 2000 rpm.

6.64 Repeat Chapter Problem 6.63 if the excitation is due to a rotating unbalance of magnitude 0.33 kg · m.

6.65 For the absorber designed in Chapter Problem 6.63, what is the minimum steady-state amplitude of the machine and at what speed does it occur?

6.66 Determine values of k and c such that the steady-state amplitude of the center of the cylinder in Figure P6.66 is less than 4 mm for 60 rad/s $< \omega <$ 110 rad/s?

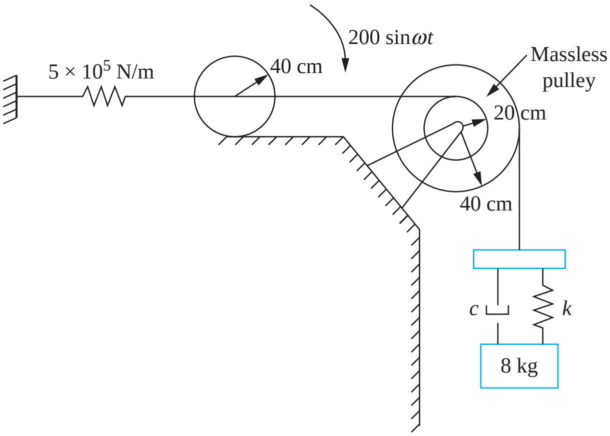


FIGURE P6.66

6.67 Use the Laplace transform method to analyze the situation of an undamped absorber attached to a viscously damped system, as shown in Figure P6.67.

(a) Determine the steady-state amplitude of the mass m_1.

(b) Use the results of part (a) to design an absorber for a 123 kg machine of natural frequency 87 rad/s and damping ratio of 0.13. Use an absorber mass of 35 kg.

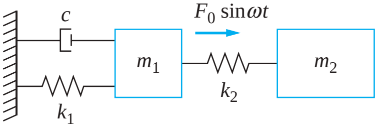


FIGURE P6.67

6.68 Design an undamped absorber such that the steady-state motion of the 25 kg machine component in Figure P6.68 ceases when the absorber is added. What is the steady-state amplitude of the 31 kg component?

6.69 A 300 kg compressor is placed at the end of a cantilever beam of length 1.8 m, elastic modulus 200×10^9 N/m^2, and moment of inertia 1.8×10^{-5} m^4. When the compressor operates at 1000 rpm, it has a steady-state amplitude of 1.2 mm. What is the compressor's steady-state amplitude when a 30 kg absorber of damping coefficient 500 N · s/m and stiffness 1.3×10^5 N/m is added to the end of the beam?

6.70 An engine has a moment of inertia of 7.5 kg · m^2 and a natural frequency of 125 Hz. Design a Houdaille damper such that the engine's maximum magnification factor is 4.8. During operation, the engine is subject to a harmonic torque of magnitude 150 N · m at a frequency of 120 Hz. What is the engine's steady-state amplitude when the absorber is used?

6.71 A 200 kg machine is subjected to an excitation of magnitude 1500 N. The machine is mounted on a foundation of stiffness 2.8×10^6 N/m. What are the mass and damping coefficient of an optimally designed vibration damper such that the maximum amplitude is 3 mm?

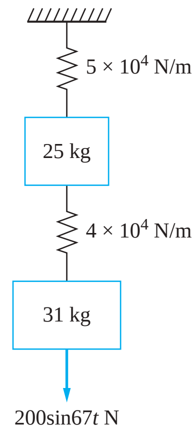


FIGURE P6.68

Chapter 7

MODELING OF MDOF SYSTEMS

7.1 INTRODUCTION

The number of degrees of freedom used to analyze a system is the number of kinematically independent coordinates necessary to describe the motion of every particle in the system. The system of Figure 7.1(a) has only one degree of freedom. If θ is chosen as the generalized coordinate, using the small angle approximation, $x = a\theta$ where x is displacement of a particle located a distance a from the pin support. If the pin support is removed as in Figure 7.1(b), using the small displacement approximation, the analysis of the system requires two coordinates. These could be chosen as x, as the displacement of the mass center and θ and as the clockwise angular rotation of the bar, all of which are measured from the system's equilibrium position. If a mass-spring system is hung from the mass center of the bar, as illustrated in Figure 7.1(c), the system has three degrees of freedom. A suitable choice of generalized coordinates is x_1 (the displacement of the left end of the bar), x_2 (the displacement of the right end of the bar), and x_3 (the displacement of the mass). All are measured from equilibrium.

Recall that for linear systems with static spring forces, the static spring forces cancel with the source of the spring forces when the differential equation is derived. Neither is included on a FBD when the objective is to derive the differential equation of motion. The potential energy of springs with static forces is calculated from energy that is calculated from the system's equilibrium position. The total potential energy is expressed as $V + V_0$ where V_0 is the potential energy in the spring when the system is in equilibrium. Since V_0

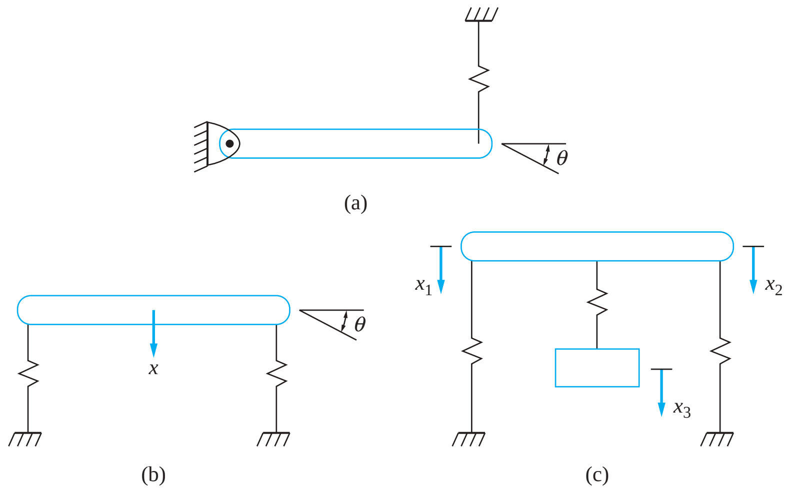


FIGURE 7.1
(a) The system is a SDOF system with θ as the chosen generalized coordinate.
(b) The system has two degrees of freedom with x and θ chosen as generalized coordinates. (c) A three degree-of-freedom system with x_1, x_2, and x_3 as generalized coordinates.

is a constant, it is not considered when calculating the equivalent stiffness. The same is true for multiple degree-of-freedom (MDOF) systems. The static forces in the springs cancel with the source of these spring forces and are not included on FBDs or in potential energy terms.

The analysis of an n degree-of-freedom (nDOF) system requires n independent differential equations. The differential equations for systems with two degrees of freedom, discussed in Chapter 6, were derived using the free-body diagram method. The method is used again in this chapter for systems with more than two degrees of freedom, but the energy method is the favored method. Lagrange's equations, which are a result of an energy method, are specified and used to derive the differential equations governing the vibrations of MDOF systems. The advantage of using Lagrange's equations is that, when the differential equations are linear and to to be expressed in matrix form, the mass matrix and the stiffness matrix are symmetric. This imposes appropriate orthogonality conditions on the mode shapes (Chapter 8) and leads to the derivation of the modal analysis method (Chapter 9) for determining the forced response. When viscous damping is present, application of Lagrange's equations also leads to a symmetric damping matrix which is crucial to developing the forced response to systems with viscous damping.

Application of Lagrange's equations requires that the kinetic energy is calculated in terms of the generalized coordinates and their time derivatives at an arbitrary instant. The potential energy is calculated in terms of the generalized coordinates at an arbitrary instant. Lagrange's equations may be used to derive the differential equations for linear systems and nonlinear systems. When viscous damping is present, Rayleigh's dissipation function is used to determine the energy dissipated by the damping forces. Linear equations can be expresses in a matrix form similar to those in Equation (6.1), as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.1}$$

When the equations are linear, the kinetic energy, potential energy, and Rayleigh's dissipation function all can be written in their *quadratic form*. The quadratic form of kinetic energy is used to directly determine the elements of the mass matrix. The quadratic form of Rayleigh's dissipation function is used to directly determine the elements of the damping matrix. The quadratic form of potential energy is used to directly determine the elements of the stiffness matrix. The force vector is determined by using the method of virtual work.

Since the *potential energy* of a system depends only upon the forces and the position of the system (not the time history of motion), it can be calculated by any method which leads to the instantaneous position. This is the basis of the stiffness influence coefficients. A unit deflection for a generalized coordinate is assumed, and the deflection of all other generalized coordinates is assumed to be zero. The forces needed to maintain this as an equilibrium position, which are the *stiffness influence coefficients*, are calculated. It is shown that these are the coefficients in the quadratic form of the potential energy and the elements of the stiffness matrix. A similar method with inertia influence coefficients and the elements of the mass matrix is developed.

The inverse of the stiffness matrix, when it exists, is the *flexibility matrix* **A**. Premultiplying Equation (6.1) by **A** leads to

$$\mathbf{AM\ddot{x}} + \mathbf{AC\dot{x}} + \mathbf{x} = \mathbf{AF} \tag{7.2}$$

Thus, **A** can be used to formulate the differential equations. A column of *flexibility influence coefficients* are the deflections of the generalized coordinates when a unit force is placed at the location described by one generalized coordinate. Flexibility influence coefficients are the elements of **A**.

Continuous systems are often modeled as discrete systems. Recall that a SDOF model of a machine at the end of a cantilever beam neglects the mass of the beam and models the stiffness of the beam as $3EI/L^3$. But this only leads to an approximation of the lowest natural frequency of the continuous system, which has an infinite number of natural frequencies. A MDOF model of the beam leads to approximations of higher natural frequencies. The finite-element method, discussed in Chapter 11, provides a discrete system model of a continuous system. The introduction of discrete modeling of continuous systems discussed in this chapter is developed using flexibility influence coefficients.

This chapter is concerned with the derivation of differential equations for discrete systems. Chapter 8 is concerned with the free response of discrete systems, and Chapter 9 is concerned with the forced response.

7.2 DERIVATION OF DIFFERENTIAL EQUATIONS USING THE FREE-BODY DIAGRAM METHOD

Governing differential equations for SDOF systems derived using the free-body diagram method require drawing a free-body diagram of the system at an arbitrary instant of time and applying the basic conservation laws to the free-body diagrams. Newton's second law ($\sum \mathbf{F} = m\mathbf{a}$), is applied to a particle, while rigid bodies undergoing planar motion also require $\sum M_0 = I_o \alpha$ where 0 is an axis of fixed rotation. If the rigid body does not have an axis of fixed rotation, it is best to draw two free-body diagrams of the system at an arbitrary instant: one showing the external forces and one showing the effective forces. Recall that the *effective forces* are defined as a force equal to $m\bar{\mathbf{a}}$ applied at the mass center and a couple equal to $\bar{I}\alpha$. Then the conservation laws are written as $(\sum F)_{\text{ext}} = (\sum F)_{\text{eff}}$ and $(\sum M_Q)_{\text{ext}} = (\sum M_Q)_{\text{eff}}$ where Q is any axis.

The first example illustrates the former procedure, while the second and third examples illustrate the latter.

EXAMPLE 7.1

The three blocks slide on a frictionless surface, as shown in Figure 7.2(a). Derive the differential equations governing the vibrations of the system using x_1, x_2, and x_3 as generalized coordinates.

SOLUTION

Free-body diagrams illustrating the forces acting on the blocks at an arbitrary instant are shown in Figure 7.2(b). Consider the force in the spring connecting the blocks whose displacements are x_1 and x_2. The spring force is the stiffness $2k$ times the change in length of the spring, which is $x_2 - x_1$, drawn in a direction such that when $x_2 - x_1$, the force is tensile. Therefore, the spring force is acting away from the blocks. The spring is assumed to be massless. Thus, the force in the spring is the same at both ends, and the force acting on the block from the spring whose displacement is x_2 is equal to and opposite the force acting on the block whose displacement is x_1. The determination of the other spring forces is made in the same manner.

Applying $\sum \mathbf{F} = m\mathbf{a}$ in the horizontal direction to the FBDs of each of the blocks leads to

$$-kx_1 + 2k(x_2 - x_1) = m\ddot{x}_1 \tag{a}$$

$$-2k(x_2 - x_1) + k(x_3 - x_2) = 2m\ddot{x}_2 \tag{b}$$

$$-k(x_3 - x_2) - 3kx_3 + F(t) = m\ddot{x}_3 \tag{c}$$

Taking everything involving the generalized coordinates to one side of the equations and everything not involving the generalized coordinates to the other side and rewriting the equations in a matrix form leads to

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix} \tag{d}$$

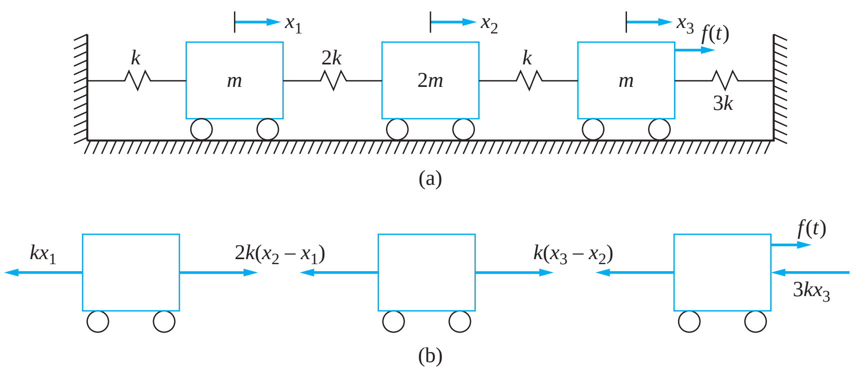


FIGURE 7.2
(a) System of Example 7.1. (b) FBDs of the blocks at an arbitrary instant.

EXAMPLE 7.2

A three degree-of-freedom model of an automobile suspension system and passenger is illustrated in Figure 7.3(a). The bar of mass m has its mass center at G, which is a distance a from the front springs. The attached mass-spring models a seat with a passenger strapped inside. The wheels provide a displacements of $y_1(t)$ and $y_2(t)$, as illustrated. Using x_1, θ, and x_2 as generalized coordinates, derive the equations of motion for the system. Assume small θ.

SOLUTION

Free-body diagrams of the body of the vehicle and the seat drawn at an arbitrary instant are shown in Figure 7.3(b). The geometry used in writing the force applied to the rear wheel is illustrated in Figure 7.3(c). The spring force is the stiffness times the change in length of the spring. One end of the spring is displaced at $y_2(t)$; the other end is displaced

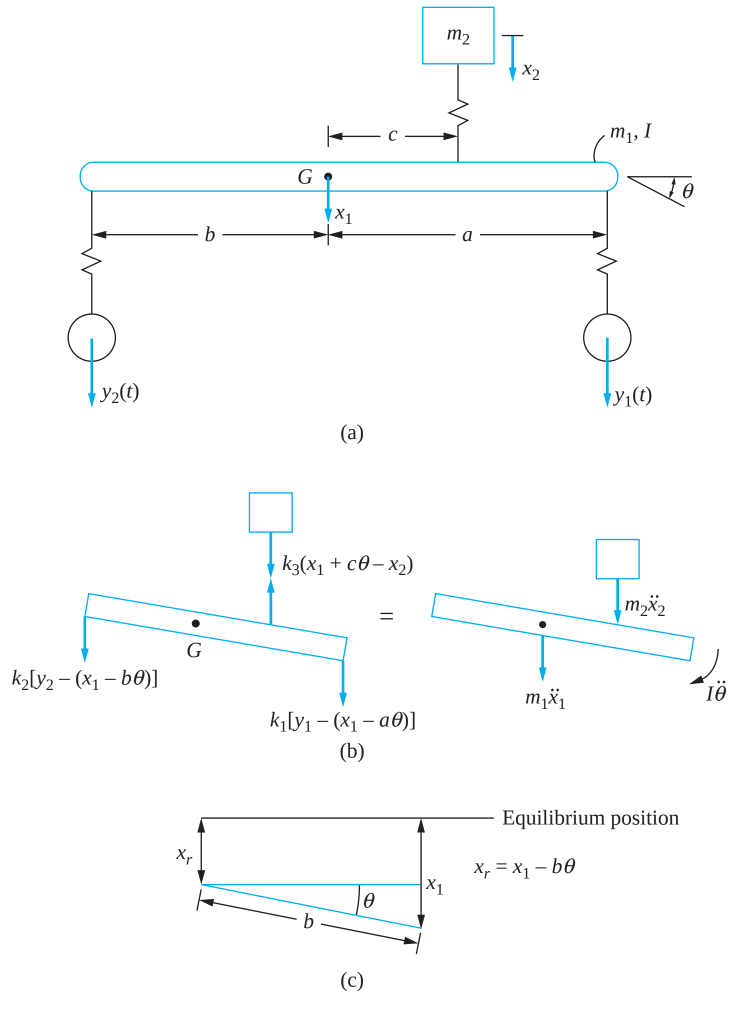


FIGURE 7.3
Three degree-of-freedom model of suspension system of Example 7.2. (b) FBDs of system drawn at an arbitrary instant. (c) Geometry used in calculation of spring force applied to rear wheel.

at $x_1 - b\theta$. Thus, the change in length of the spring is $y_2(t) - (x_1 - b\theta)$. Applying $(\sum F)_{ext} = (\sum F)_{eff}$ to the FBD of the vehicle yields

$$k_1[y_1(t) - (x_1 + a\theta)] + k_2[y_2(t) - (x_1 - b\theta)] - k_3[x_1 + c\theta - x_2] = m_1\ddot{x}_1 \quad \text{(a)}$$

Application of the moment equation $(\sum M_G)_{ext} = (\sum M_G)_{eff}$ to the FBD of the vehicle gives

$$k_1[y_1(t) - (x_1 + a\theta)](a) - k_2[y_2(t) - (x_1 - b\theta)](b) - k_3[x_1 + c\theta - x_2](c) = I\ddot{\theta} \quad \text{(b)}$$

Application of $(\sum F)_{ext} = (\sum F)_{eff}$ to the FBD of the seat yields

$$k_3[x_1 + c\theta - x_2] = m_2\ddot{x}_2 \quad \text{(c)}$$

Rearranging the equations such that everything involving the generalized coordinates is on one side and everything else is on the other, and writing the equations in a matrix form leads to

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + k_3 & k_1a - k_2b + k_3c & -k_3 \\ k_1a - k_2b + k_3c & k_1a^2 + k_2b^2 + k_3c^2 & -k_3c \\ -k_3 & -k_3c & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} k_1y_1(t) + k_2y_2(t) \\ k_1ay_1(t) - k_2by_2(t) \\ 0 \end{bmatrix} \quad \text{(d)}$$

EXAMPLE 7.3

The cart of Figure 7.4(a) rolls on a frictionless surface. A double pendulum consisting of two slender bars which can move freely is pinned to the cart. Using x, θ_1, and θ_2 as generalized coordinates, derive the equations of motion. Assume small θ_1 and θ_2.

SOLUTION

First consider the kinematics and the acceleration of the mass center of the bar AB.

$$\begin{aligned}\bar{\mathbf{a}}_{AB} &= \mathbf{a}_A + \boldsymbol{\alpha} x \mathbf{r}_{G/A} + \boldsymbol{\omega} x(\boldsymbol{\omega} x \mathbf{r}_{G/A}) \\ &= \ddot{x}\mathbf{i} + \ddot{\theta}_1\mathbf{k}x\left(\frac{L}{2}\sin\theta_1\mathbf{i} - \frac{L}{2}\cos\theta_1\mathbf{j}\right) + \dot{\theta}_1\mathbf{k}x\left[\dot{\theta}_1\mathbf{k}x\left(\frac{L}{2}\sin\theta_1\mathbf{i} - \frac{L}{2}\cos\theta_1\mathbf{j}\right)\right] \\ &= \left(\ddot{x} + \frac{L}{2}\ddot{\theta}_1\cos\theta_1 - \frac{L}{2}\dot{\theta}_1^2\sin\theta_1\right)\mathbf{i} + \left(\frac{L}{2}\ddot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_1^2\cos\theta_1\right)\mathbf{j} \end{aligned} \quad \text{(a)}$$

In a similar fashion, it is determined that

$$\mathbf{a}_B = (\ddot{x} + L\ddot{\theta}_1\cos\theta_1 - L\dot{\theta}_1^2\sin\theta_1)\mathbf{i} + (L\ddot{\theta}_1\sin\theta_1 + L\dot{\theta}_1^2\cos\theta_1)\mathbf{j} \quad \text{(b)}$$

The relative acceleration equation is applied between B and the mass center of bar BC:

$$\begin{aligned}\bar{\mathbf{a}}_{BC} &= \mathbf{a}_B + \boldsymbol{\alpha} x \mathbf{r}_{G/B} + \boldsymbol{\omega} x(\boldsymbol{\omega} x \mathbf{r}_{G/B}) \\ &= (\ddot{x} + L\ddot{\theta}_1\cos\theta_1 - L\dot{\theta}_1^2\sin\theta_1)\mathbf{i} \\ &\quad + (L\ddot{\theta}_1\sin\theta_1 + L\dot{\theta}_1^2\cos\theta_1)\mathbf{j} + \ddot{\theta}_2\mathbf{k}x\left(\frac{L}{2}\sin\theta_2\mathbf{i} - \frac{L}{2}\cos\theta_2\mathbf{j}\right)\end{aligned}$$

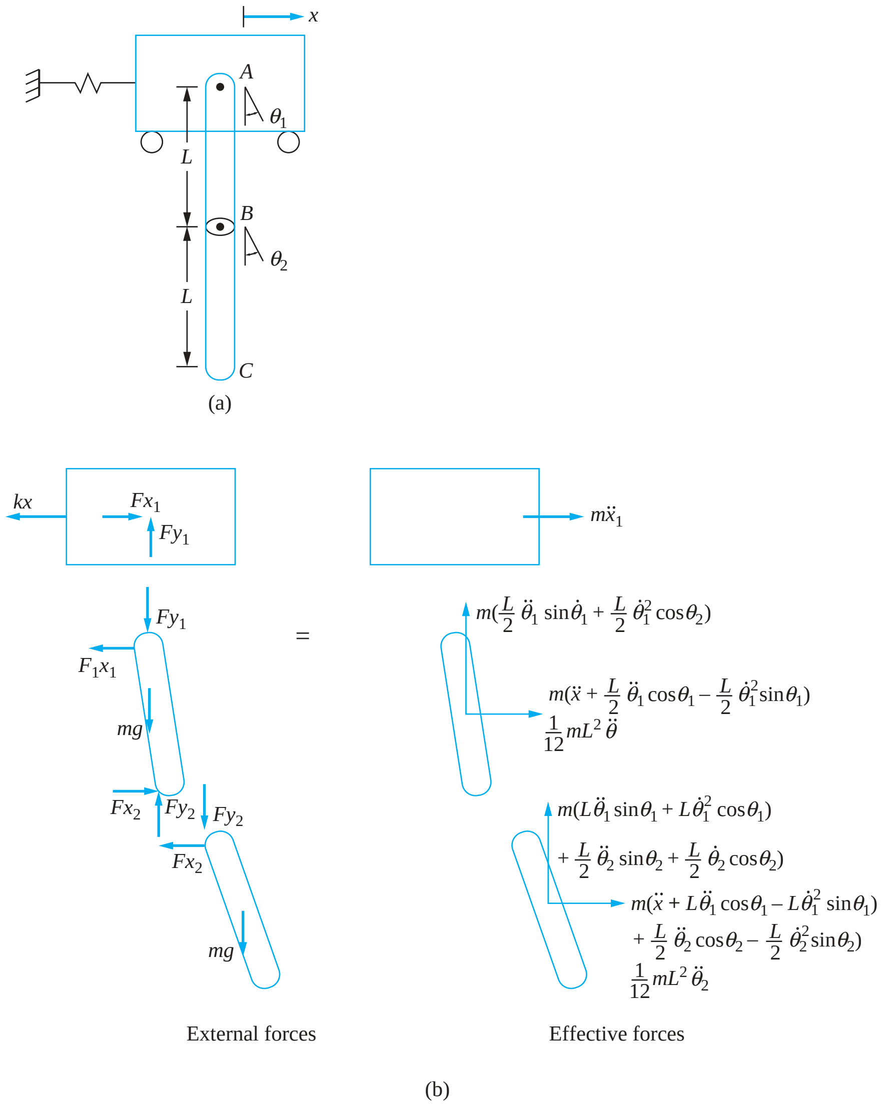


FIGURE 7.4
System of Example 7.3. (a) The cart rolls on a frictionless surface and the double pendulum is free to rotate about the center of the cart. (b) FBDs at an arbitrary instant.

$$+\dot{\theta}_2\mathbf{k}x\left[\dot{\theta}_2\mathbf{k}x\left(\frac{L}{2}\sin\theta_2\,\mathbf{i}-\frac{L}{2}\cos\theta_2\,\mathbf{j}\right)\right]$$

$$=\left(\ddot{x}+L\ddot{\theta}_1\cos\theta_1-L\dot{\theta}_1^2\sin\theta_1+\frac{L}{2}\ddot{\theta}_2\cos\theta_2-\frac{L}{2}\dot{\theta}_2^2\sin\theta_2\right)\mathbf{i}$$

$$+\left(L\ddot{\theta}_1\sin\theta_1+L\dot{\theta}_1^2\cos\theta_1+\frac{L}{2}\ddot{\theta}_2\sin\theta_2+\frac{L}{2}\dot{\theta}_2^2\cos\theta_2\right)\mathbf{j} \quad \text{(c)}$$

FBDs of the cart and the two bars, drawn at an arbitrary instant, are illustrated in Figure 7.4(b). Application of $(\sum F_x)_{\text{ext}} = (\sum F_x)_{\text{eff}}$ to the free-body diagram of the cart leads to

$$-kx + F_{x1} = m\ddot{x}_1 \tag{d}$$

Summing moments $(\sum M_B)_{\text{ext}} = (\sum M_B)_{\text{eff}}$ using the FBDs of bar AB leads to

$$\begin{aligned} &F_{x_1}(L\cos\theta_1) + F_{y_1}(L\sin\theta_1) + mg\frac{L}{2}\sin\theta_1 \\ &= m\left(\ddot{x} + \frac{L}{2}\ddot{\theta}_1\cos\theta_1 - \frac{L}{2}\dot{\theta}_1^2\sin\theta_1\right)\left(-\frac{L}{2}\cos\theta_1\right) \\ &\quad + m\left(\frac{L}{2}\ddot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_1^2\cos\theta_1\right)\left(-\frac{L}{2}\sin\theta_1\right) + \frac{1}{12}mL^2\ddot{\theta}_1 \end{aligned} \tag{e}$$

Summing moments $(\sum M_B)_{\text{ext}} = (\sum M_B)_{\text{eff}}$ using the FBDs of bar BC leads to

$$\begin{aligned} -mg\frac{L}{2}\sin\theta_2 &= m\left(\ddot{x} + L\ddot{\theta}_1\cos\theta_1 - L\dot{\theta}_1^2\sin\theta_1 + \frac{L}{2}\ddot{\theta}_2\cos\theta_2 - \frac{L}{2}\dot{\theta}_2^2\sin\theta_2\right)\left(\frac{L}{2}\cos\theta_2\right) \\ &+ m\left(L\ddot{\theta}_1\sin\theta_1 + L\dot{\theta}_1^2\cos\theta_1 + \frac{L}{2}\ddot{\theta}_2\sin\theta_2 + \frac{L}{2}\dot{\theta}_2^2\cos\theta_2\right)\left(\frac{L}{2}\sin\theta_2\right) + \frac{1}{12}mL^2\ddot{\theta}_2 \end{aligned} \tag{f}$$

Summation of forces $(\sum F_x)_{\text{ext}} = (\sum F_x)_{\text{eff}}$ on the FBDs of the bars gives

$$-F_{x1} + F_{x2} = m\left(\ddot{x} + \frac{L}{2}\ddot{\theta}_1\cos\theta_1 - \frac{L}{2}\dot{\theta}_1^2\sin\theta_1\right) \tag{g}$$

and

$$-F_{x2} = m\left(\ddot{x} + L\ddot{\theta}_1\cos\theta_1 - L\dot{\theta}_1^2\sin\theta_1 + \frac{L}{2}\ddot{\theta}_2\cos\theta_2 - \frac{L}{2}\dot{\theta}_2^2\sin\theta_2\right) \tag{h}$$

Summation of forces $(\sum F_y)_{\text{ext}} = (\sum F_y)_{\text{eff}}$ applied to the FBDs of the bars gives

$$-F_{y1} + F_{y2} - mg = m\left(\frac{L}{2}\ddot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_1^2\cos\theta_1\right) \tag{i}$$

and

$$-F_{y2} - mg = m\left(L\ddot{\theta}_1\sin\theta_1 + L\dot{\theta}_1^2\cos\theta_1 + \frac{L}{2}\ddot{\theta}_2\sin\theta_2 + \frac{L}{2}\dot{\theta}_2^2\cos\theta_2\right) \tag{j}$$

Use of Equations (g) through (j) in Equations (d) through (g) leads to

$$3m\ddot{x} + \frac{3}{2}mL\ddot{\theta}_1\cos\theta_1 - \frac{3m}{2}L\dot{\theta}_1^2\sin\theta_1 + m\frac{L}{2}\ddot{\theta}_2\cos\theta_2 - m\frac{L}{2}\dot{\theta}_2^2\sin\theta_2 + kx = 0 \tag{k}$$

$$\begin{aligned} &\frac{3}{2}mL\cos\theta_1\ddot{x} + \frac{13}{12}mL^2\ddot{\theta}_1 + m\frac{L^2}{4}\ddot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &\qquad + m\frac{L^2}{2}\dot{\theta}_2^2(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) + \frac{5}{2}mgL\sin\theta_1 = 0 \end{aligned} \tag{l}$$

$$m\ddot{x} + m\frac{L^2}{2}\ddot{\theta}_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + m\frac{L^2}{4}\dot{\theta}_1^2(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)$$
$$+ m\frac{L^2}{3}\ddot{\theta}_2 + mg\frac{L}{2}\sin\theta_2 = 0 \quad \text{(m)}$$

Assuming small θ_1 and θ_2 (which implies $\sin\theta_1 \approx \theta_1$, $\cos\theta_1 \approx 1$, $\sin\theta_2 \approx \theta_2$, and $\cos\theta_2 \approx 1$, along with products of generalized coordinates are small), Equations (k) through (m) are written (respectively) as

$$3m\ddot{x} + \frac{3}{2}mL\ddot{\theta}_1 + m\frac{L}{2}\ddot{\theta}_2 + kx = 0 \quad \text{(n)}$$

$$\frac{3}{2}mL\ddot{x} + \frac{13}{12}mL^2\ddot{\theta}_1 + m\frac{L^2}{4}\ddot{\theta}_2 + \frac{5}{2}mgL\theta_1 = 0 \quad \text{(o)}$$

$$m\ddot{x} + m\frac{L^2}{2}\ddot{\theta}_1 + m\frac{L^2}{3}\ddot{\theta}_2 + mg\frac{L}{2}\theta_2 = 0 \quad \text{(p)}$$

7.3 LAGRANGE'S EQUATIONS

Energy methods are more useful than the free-body diagram method for deriving differential equations governing MDOF systems. Lagrange's equations are derived using energy methods. The equivalent systems method, discussed in Chapter 2, is actually Lagrange's equations written for a linear SDOF system. Lagrange's equations can be applied to linear and nonlinear MDOF systems to derive the governing differential equations. When applied to linear systems, application of Lagrange's equations leads to symmetric mass and stiffness matrices.

However, the derivation of Lagrange's equations requires calculus of variations, and a formal derivation is beyond the scope of this book. The basis for the derivation of Lagrange's equations is the principle of work and energy. Instead of taking the dot product of Newton's law with a differential displacement vector, the dot product is taken with a variation of the displacement vector. Whereas a *differential*, dx, is a change in the dependent variable due to a change in the independent variable, (a variation written as δx is due to a change in the dependent variable, as show in Figure 7.5).

The independent variable is time t and the dependent variable is y. Imagine following a particle as it travels throughout space along a path $y(t)$. The actual path that the particle follows between time t_1 and time t_2 is $y(t)$. The varied path is $y(t) + \delta y$ as shown in Figure 7.5(a). The variation is an arbitrary function that the varied path could follow. The variation must be the same as the actual path at t_1 and t_2. That is, $\delta y(t_1) = 0$ and $\delta y(t_2) = 0$. Figure 7.5(b) illustrates the difference between a variation and a differential by examining both the function $y(t)$ and the variation $y(t) + \delta y$ during the time dt. The geometry of this illustration shows that $\delta(dy) = d(\delta y)$.

The actual path that the particle follows is not known. It is the job of calculus of variations to specify the actual path (or to derive an equation that specifies the actual path) by considering all possible variations. This is the purpose of Lagrange's equations. Application of Lagrange's equations specifies the equations for the actual path.

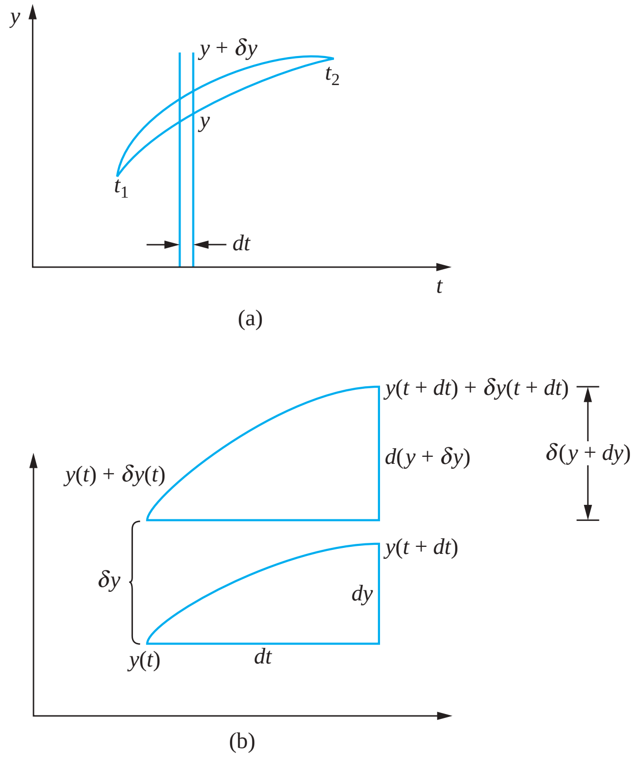


FIGURE 7.5
(a) Illustration of $y(t)$ and $y + \delta y$. (b) Enlargement of section of curve in part (a) showing detail of variation.

The discussion thus far has been for a particle with a one-dimensional motion. The particle has a position vector $\mathbf{r}(t)$ and the variation of the position vector is $\delta\mathbf{r}(t)$.

The expression $\sum \mathbf{F} \cdot \delta\mathbf{r}$ is referred to as the *virtual work* δW. Consider a system with nDOF with generalized coordinates of $x_1, x_2, \ldots, x_n$. The virtual work δW is the work done by external forces as the system's position changes from $(x_1, x_2, \ldots, x_n)$ to $(x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_n + \delta x_n)$. The virtual work is

$$\delta W = \sum \mathbf{F} \cdot \delta\mathbf{r} \tag{7.3}$$

where

$$\delta\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x_1}\delta x_1 + \frac{\partial \mathbf{r}}{\partial x_2}\delta x_2 + \cdots \frac{\partial \mathbf{r}}{\partial x_n}\delta x_n \tag{7.4}$$

The virtual work is broken down into the work done by conservative forces δW_c and the work done by non-conservative forces δW_{nc}. The work done by conservative forces is written as

$$\delta W_c = -\delta V \tag{7.5}$$

where δV is the variation of the potential energy.

The term $m\mathbf{a} \cdot \delta\mathbf{r}$ is manipulated into the variation of kinetic energy δT. Just like the principle of work and energy, the result is integrated between two times t_1 and t_2 with the requirement that the variation of the position vector is zero at these times. The result is Hamilton's principle, which is stated as

$$\delta \int_{t_1}^{t_2} (T - V + \delta W_{nc})\, dt = 0 \tag{7.6}$$

The Lagrangian is defined as

$$L = T - V \tag{7.7}$$

and if all forces are conservative, Hamilton's principle becomes

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \tag{7.8}$$

For a nDOF system with generalized coordinates $x_1, x_2, \ldots, x_n$, the Lagrangian L is a function of $2n$ variables. The potential energy is written at an arbitrary instant and is a function of n variables, which are the generalized coordinates. The kinetic energy is written at an arbitrary instant and is a function of $2n$ variables: the generalized coordinates and their time derivatives. In general,

$$L = L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \tag{7.9}$$

The integral $\int_{t_1}^{t_2} L dt$ is a functional or a function of variables whose result is a scalar. It takes on a variety of values for arbitrary choices of the generalized coordinates and their time derivatives, but only for the exact choice is its variation zero. Using a theorem of calculus of variations, $\delta \int_{t_1}^{t_2} L dt = 0$ if

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}_\iota}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n \tag{7.10}$$

Equations (7.10) are called Lagrange's equations and can be used to derive the differential for conservative nDOF systems.

EXAMPLE 7.4

Use Lagrange's equations to derive the differential equations governing the motion of the system of Example 7.1 using x_1, x_2, and x_3 as generalized coordinates.

SOLUTION

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} 2m\dot{x}_2^2 + \frac{1}{2} m\dot{x}_3^2 \tag{a}$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2} kx_1^2 + \frac{1}{2} 2k(x_2 - x_1)^2 + \frac{1}{2} k(x_3 - x_2)^2 + \frac{1}{2} 3kx_3^2 \tag{b}$$

The Lagrangian is

$$L + \frac{1}{2}[m\dot{x}_1^2 + 2m\dot{x}_2^2 + m\dot{x}_3^2 - kx_1^2 - 2k(x_2 - x_1)^2 - k(x_3 - x_2)^2 - 3kx_3^2] \tag{c}$$

Application of Lagrange's equations leads to

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \tag{d}$$

$$\frac{d}{dt}(m\dot{x}_1) - [-kx_1 - 2k(x_2 - x_1)(-1)] = 0 \tag{e}$$

$$m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0 \tag{f}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{g}$$

$$\frac{d}{dt}(2m\dot{x}_2) - [-2k(x_2 - x_1) - k(x_3 - x_2)(-1)] = 0 \tag{h}$$

$$2m\ddot{x}_2 - 2kx_1 + 3kx_2 - kx_3 = 0 \tag{i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = 0 \tag{j}$$

$$\frac{d}{dt}(m\dot{x}_3) - [-k(x_3 - x_2) - 3kx_3] = 0 \tag{k}$$

$$m\ddot{x}_3 - kx_2 + 4kx_3 = 0 \tag{l}$$

The differential equations derived from Lagrange's equations are identical to those obtained in Example 7.1 by the free-body diagram method.

EXAMPLE 7.5

Use Lagrange's equations to derive the differential equations governing the motion of the system of Figure 7.3(a) and Example 7.2.

SOLUTION

The kinetic energy of the system of Figure 7.3 is the sum of the kinetic energies of the body of the vehicle and the seat. The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}I\omega^2 + T_{\text{seat}} \\ &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}_2^2 \end{aligned} \tag{a}$$

The potential energy is the sum of the potential energies in the three springs. The change in lengths of the springs are measured from the system's equilibrium position and are determined in the solution of Example 7.2, resulting in

$$V = \frac{1}{2}k_1[y_1(t) - (x_1 + a\theta)]^2 + \frac{1}{2}k_2[y_2(t) - (x_1 - b\theta)]^2 + \frac{1}{2}k_3[x_1 + c\theta - x_2]^2 \tag{b}$$

The Lagrangian is

$$L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}I\dot{\theta}^2+\frac{1}{2}m_2\dot{x}_2^2-\frac{1}{2}k_1[y_1(t)-(x_1+a\theta)]^2-\frac{1}{2}k_2[y_2(t)-(x_1-b\theta)]^2$$
$$-\frac{1}{2}k_3[x_1+c\theta-x_2]^2 \tag{c}$$

Application of Lagrange's equation for x_1 leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0$$

$$\frac{d}{dt}\left[\frac{1}{2}m_1(2\dot{x}_1)\right]-\left\{\frac{1}{2}k_1(2)[y_1(t)-(x_1+a\theta)](-1)-\frac{1}{2}k_2(2)[y_2(t)-(x_1-b\theta)](-1)\right.$$
$$\left.-\frac{1}{2}k_3(2)[x_1+c\theta-x_2](1)\right\}=0 \tag{d}$$

Application of Lagrange's equations for θ leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

$$\frac{d}{dt}\left[\frac{1}{2}I(2\dot{\theta})\right]-\left\{-\frac{1}{2}k_1(2)[y_1(t)-(x_1+a\theta)](-a)-\frac{1}{2}k_2(2)[y_2(t)-(x_1-b\theta)](b)\right.$$
$$\left.-\frac{1}{2}k_3(2)[x_1+c\theta-x_2](c)=0\right. \tag{e}$$

Application of Lagrange's equations for x_2 leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=0$$

$$\frac{d}{dt}\left[\frac{1}{2}m_2(2\dot{x}_2)\right]-\left\{-\frac{1}{2}k_3(2)[x_1+c\theta-x_2](1)\right\}=0 \tag{f}$$

Equations (d) through (f) are rearranged and written in a matrix form leading to

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2+k_3 & k_1a-k_2b+k_3c & -k_3 \\ k_1a-k_2b+k_3b & k_1a^2+k_2b^2+k_3c^2 & -k_3c \\ -k_3 & -k_3c & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix}$$
$$=\begin{bmatrix} k_1y_1(t)+k_2y_2(t) \\ k_1ay_1(t)-k_2by_2(t) \\ 0 \end{bmatrix} \tag{g}$$

EXAMPLE 7.6

Derive the nonlinear equations governing the motion of Example 7.3 and Figure 7.4.

SOLUTION

The velocity of the mass center of bar AB is given by

$$\bar{\mathbf{v}}_{\mathbf{AB}} = \mathbf{v}_A + \omega \mathrm{x} \mathbf{r}_{G/A}$$
$$= \dot{x}\mathbf{i} + \dot{\theta}_1 \mathbf{k}\mathrm{x}\left(\frac{L}{2}\sin\theta_1\,\mathbf{i} - \frac{L}{2}\cos\theta_1\,\mathbf{j}\right)$$
$$= \left(\dot{x} + \frac{L}{2}\dot{\theta}_1\cos\theta_1\right)\mathbf{i} + \frac{L}{2}\dot{\theta}_1\sin\theta_1\,\mathbf{j} \quad \text{(a)}$$

Using a similar analysis, the velocity of particle B is

$$v_B = (\dot{x} + L\dot{\theta}_1\cos\theta_1)\,\mathbf{i} + L\dot{\theta}_1\sin\theta_1\,\mathbf{j} \quad \text{(b)}$$

The velocity of the mass center of bar BC is

$$\bar{\mathbf{v}}_{BC} = \mathbf{v}_B + \omega \mathrm{x} \mathbf{r}_{G/B}$$
$$= (\dot{x} + L\dot{\theta}_1\cos\theta_1)\mathbf{i} + L\dot{\theta}_1\sin\theta_1\,\mathbf{j} + \dot{\theta}_2\,\mathbf{k}\,\mathrm{x}\left(\frac{L}{2}\sin\theta_2\,\mathbf{i} - \frac{L}{2}\cos\theta_2\,\mathbf{j}\right)$$
$$= (\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2)\,\mathbf{i} + \left(L\dot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_2\sin\theta_2\right)\mathbf{j} \quad \text{(c)}$$

The kinetic energy of the system at an arbitrary position is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} + \frac{L}{2}\dot{\theta}_1\cos\theta_1\right)^2 + \left(\frac{L}{2}\dot{\theta}_1\sin\theta_1\right)^2\right] + \frac{1}{12}mL^2\dot{\theta}_1^2$$
$$+ \frac{1}{2}m\left[\left(\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2\right)^2 + \left(L\dot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_2\sin\theta_2\right)^2\right]$$
$$+ \frac{1}{12}mL^2\dot{\theta}_2^2 \quad \text{(d)}$$

The potential energy of the system at an arbitrary instant, using the plane of the cart as the datum, is

$$V = \frac{1}{2}kx^2 + mg\frac{L}{2}\cos\theta_1 + mg\left(L\cos\theta_1 + \frac{L}{2}\cos\theta_2\right) \quad \text{(e)}$$

The Lagrangian for the system is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} + \frac{L}{2}\dot{\theta}_1\cos\theta_1\right)^2 + \left(\frac{L}{2}\dot{\theta}_1\sin\theta_1\right)^2\right] + \frac{1}{12}mL^2\dot{\theta}_1^2$$
$$+ \frac{1}{2}m\left[\left(\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2\right)^2 + \left(L\dot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_2\sin\theta_2\right)^2\right] + \frac{1}{12}mL^2\dot{\theta}_2^2$$
$$-\left[\frac{1}{2}kx^2 + mg\frac{3L}{2}\cos\theta_1 + mg\frac{L}{2}\cos\theta_2\right] \quad \text{(f)}$$

Application of Lagrange's equations for x leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}m\,(2)\dot{x} + \frac{1}{2}m(2)\left(\dot{x} + \frac{L}{2}\dot{\theta}_1\cos\theta_1\right) + \frac{1}{2}m(2)\left(\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2\right)\right]$$

$$-\left[-\frac{1}{2}k(2)x\right] = 0 \qquad \text{(g)}$$

$$3m\ddot{x} + m\frac{3L}{2}\ddot{\theta}_1\cos\theta_1 - m\frac{3L}{2}\dot{\theta}_1^2\sin\theta_1 + m\frac{L}{2}\ddot{\theta}_2\cos\theta_2 - m\frac{L}{2}\dot{\theta}_2^2\sin\theta_2 + kx = 0 \qquad \text{(h)}$$

Application of Lagrange's equations for θ_1 yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left\{\frac{1}{2}m\left[(2)\left(\dot{x} + \frac{L}{2}\dot{\theta}_1\cos\theta_1\right)\left(\frac{L}{2}\cos\theta_1\right) + (2)\left(\frac{L}{2}\dot{\theta}_1\sin\theta_1\right)\left(\frac{L}{2}\sin\theta_1\right)\right]\right.$$

$$+\frac{1}{12}mL^2(2)\dot{\theta}_1 + \frac{1}{2}m\left[(2)\left(\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2\right)(L\cos\theta_1)\right.$$

$$\left.\left.+(2)\left(L\dot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_2\sin\theta_2\right)(L\sin\theta_1)\right]\right\} - \left[-mg\frac{3L}{2}\sin\theta_1\right] = 0 \qquad \text{(i)}$$

and

$$2m\ddot{x} + \frac{4}{3}mL^2\ddot{\theta}_1 - \frac{3}{2}mL\dot{x}\dot{\theta}_1\sin\theta_1 + m\frac{L}{2}\ddot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$-m\frac{L}{2}\dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2) + mg\frac{3L}{2}\sin\theta_1 = 0 \qquad \text{(j)}$$

Application of Lagrange's equations for θ_2 yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{dt}\left\{\frac{1}{2}m\left[(2)\left(\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2\right)\left(\frac{L}{2}\cos\theta_2\right)\right.\right.$$

$$\left.\left.+(2)\left(L\dot{\theta}_1\sin\theta_1 + \frac{L}{2}\dot{\theta}_2\sin\theta_2\right)\left(\frac{L}{2}\sin\theta_2\right)\right] + \frac{1}{12}mL^2(2)\dot{\theta}_2\right\} - \left[-mg\frac{L}{2}\sin\theta_2\right] = 0 \qquad \text{(k)}$$

and

$$\left(m\ddot{x} + mL\ddot{\theta}_1\cos\theta_1 - mL\dot{\theta}_1^2\sin\theta_1 + m\frac{L}{2}\ddot{\theta}_2\cos\theta_2 - m\frac{L}{2}\dot{\theta}_2^2\sin\theta_2\right)\left(\frac{L}{2}\cos\theta_2\right)$$

$$-m\frac{L}{2}\left(\dot{x} + L\dot{\theta}_1\cos\theta_1 + \frac{L}{2}\dot{\theta}_2\cos\theta_2\right)\dot{\theta}_2\sin\theta_2$$

$$+ m\left(L\ddot{\theta}_1 \sin\theta_1 + L\dot{\theta}_1^2 \cos\theta_1 + \frac{L}{2}\ddot{\theta}_2 \sin\theta_2 + m\frac{L}{2}\dot{\theta}_2^2 \cos\theta_2\right)\left(\frac{L}{2}\sin\theta_2\right)$$

$$+ m\frac{L}{2}\left(L\dot{\theta}_1 \sin\theta_1 + \frac{L}{2}\dot{\theta}_2 \sin\theta_2\right)\dot{\theta}_2 \cos\theta_2 + \frac{1}{12}mL^2\ddot{\theta}_2 + mg\frac{L}{2}\sin\theta_2 = 0 \quad \text{(l)}$$

Equations (g), (h), and (i) are the nonlinear differential equations that govern the motion of the system.

Using the small angle assumption ($\sin\theta_1 \approx \theta_1$, $\cos\theta_1 \approx 1$, $\sin\theta_2 \approx \theta_2$ and $\cos\theta_2 \approx 1$, and assuming terms involving higher powers or products of θ_1 and θ_2 are small), Equation (k) reduces to Equation (n) of Example 7.3 while Equations (l) and (m) are multiples of Equations (o) and (p) of Example 7.3.

If the system is non-conservative, Lagrange's equations are modified to taken the non-conservative forces into account and are written as

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}_\iota}\right) - \frac{\partial L}{\partial x_i} = Q_i \qquad i = 1, 2, \ldots, n \quad \textbf{(7.11)}$$

where the Q_i are referred to as *generalized forces*. The virtual work done by all nonconservative forces δW_{nc} is written as

$$\delta W_{\text{nc}} = \sum_{i=1}^{n} Q_i \delta x_i \quad \textbf{(7.12)}$$

The power dissipated by a viscous damper is the force in the viscous damper times the displacement of the particle to which the damper is attached. Rayleigh's dissipation function $\mathfrak{J}$ is the negative one-half of the total power dissipated in all viscous dampers.

$$\mathfrak{J} = -\frac{1}{2}P \quad \textbf{(7.13)}$$

Recall that the work done by the viscous damping force as the particle to which it is attached moves from x_1 to x_2 is $W = -\int_{x_1}^{x_2} c\dot{x}\, dx$, where c is the viscous-damping coefficient and $\dot{x}$ is the velocity of the particle to which it is attached. The power dissipated is

$$\begin{aligned} P = \frac{dW}{dt} &= -\frac{d}{dt}\int_{x_1}^{x_2} c\dot{x}\, dx \\ &= -\frac{d}{dt}\int_{t_1}^{t_2} c\dot{x}^2\, dt \\ &= -c\dot{x}^2 \end{aligned} \quad \textbf{(7.14)}$$

Now consider a viscous damper connected between two masses with displacements x_1 and x_2. The force in the viscous damper is $c(\dot{x}_2 - \dot{x}_1)$. The work done by the viscous-damping force is

$$W = -\int_{x_{2a}}^{x_{2b}} c(\dot{x}_2 - \dot{x}_1)\, dx_2 + \int_{x_{1a}}^{x_{1b}} c(\dot{x}_2 - \dot{x}_1)\, dx_1 \quad \textbf{(7.15)}$$

The power dissipated during this time is

$$P = -\frac{dW}{dt} = \frac{d}{dt}\left[\int_{x_{2a}}^{x_{2b}} c(\dot{x}_2 - \dot{x}_1)\, dx_2\right] - \frac{d}{dt}\left[\int_{x_{1a}}^{x_{1b}} c(\dot{x}_2 - \dot{x}_1)\, dx_1\right] \tag{7.16}$$

Changing the variables of integration to time leads to

$$\begin{aligned} P &= \frac{d}{dt}\left[\int_{t_1}^{t_2} c(\dot{x}_2 - \dot{x}_1)\dot{x}_2\, dt\right] - \frac{d}{dt}\left[\int_{t_1}^{t_2} c(\dot{x}_2 - \dot{x}_1)\dot{x}_1\, dt\right] \\ &= c(\dot{x}_2 - \dot{x}_1)^2 \end{aligned} \tag{7.17}$$

The generalized force due to viscous damping is

$$Q_i = \frac{\partial \mathfrak{F}}{\partial \dot{x}_\iota} \tag{7.18}$$

Then

$$Q_i = \frac{\partial \mathfrak{F}}{\partial \dot{x}_\iota} + Q_{i,\mathrm{nv}} \tag{7.19}$$

where $Q_{i,\mathrm{nv}}$ is the generalized forced due to nonviscous forces. Lagrange's equations then become

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}_\iota}\right) - \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_\iota} = Q_{i,\mathrm{nv}} \qquad i = 1, 2, \ldots, n \tag{7.20}$$

EXAMPLE 7.7

Derive the differential equations for the system of Figure 7.6 using x_1, x_2, and x_3 as generalized coordinates.

SOLUTION

The Lagrangian for this system is developed in Equation (c) of Example 7.4. Rayleigh's dissipation function is

$$\mathfrak{F} = -\frac{1}{2}c\dot{x}_1^2 - \frac{1}{2}2c(\dot{x}_2 - \dot{x}_1)^2 - \frac{1}{2}c(\dot{x}_3 - \dot{x}_2)^2 - \frac{1}{2}3c\dot{x}_3^2 \tag{a}$$

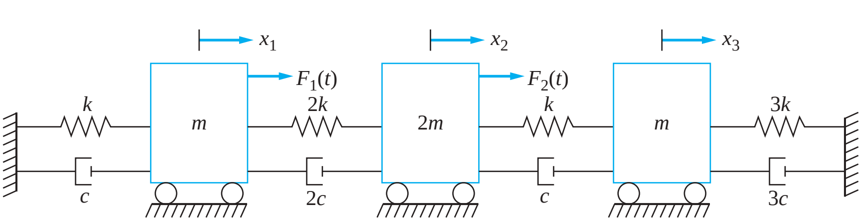


FIGURE 7.6
System of Example 7.7

The work done by the external forces is

$$\delta W = F_1(t)\,\delta x_1 + F_2(t)\,\delta x_2 \tag{b}$$

Thus, $Q_{1,\mathrm{nv}} = F_1(t)$, $Q_{2,\mathrm{nv}} = F_2(t)$ and $Q_{3,\mathrm{nv}} = 0$. Application of Lagrange's equation for x_1 leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_1} = Q_{1,\mathrm{nv}}$$

$$\frac{d}{dt}\left[\frac{1}{2}m(2)\dot{x}_1\right] - \left[-\frac{1}{2}k(2)x_2 - \frac{1}{2}2k(2)(x_2 - x_1)(-1)\right]$$

$$-\left[-\frac{1}{2}c(2)\dot{x}_2 - \frac{1}{2}2c(2)(\dot{x}_2 - \dot{x}_1)(-1)\right] = F_1(t) \tag{c}$$

Application of Lagrange's equation for x_2 leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_2} = Q_{2,\mathrm{nv}}$$

$$\frac{d}{dt}\left[\frac{1}{2}2m(2)\dot{x}_2\right] - \left[-\frac{1}{2}2k(2)(x_2 - x_1) - \frac{1}{2}k(2)(x_3 - x_2)(-1)\right]$$

$$-\left[-\frac{1}{2}2c(2)(\dot{x}_2 - \dot{x}_1) - \frac{1}{2}c(2)(\dot{x}_3 - \dot{x}_2)(-1)\right] = F_2(t) \tag{d}$$

Application of Lagrange's equation for x_3 gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_3} = Q_{3,\mathrm{nv}}$$

$$\frac{d}{dt}\left[\frac{1}{2}m(2)\dot{x}_3\right] - \left[-\frac{1}{2}k(2)(x_3 - x_2) - \frac{1}{2}3k(2)x_3\right]$$

$$-\left[-\frac{1}{2}c(2)(\dot{x}_3 - \dot{x}_2) - \frac{1}{2}3c(2)\dot{x}_3\right] = 0 \tag{e}$$

Rearranging Equations (c), (d), and (e) and summarizing in matrix form leads to

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3c & -2c & 0 \\ -2c & 3c & -c \\ 0 & -c & 4c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$+\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ 0 \end{bmatrix} \tag{f}$$

EXAMPLE 7.8

Derive the differential equations of the vehicle damping, as illustrated in Figure 7.7. Note this system was used in Example 7.5 without damping.

SOLUTION

The forms of the kinetic energy and potential energy are as in Example 7.5. The form of Rayleigh's dissipation functions for this example is

$$\Im = -\frac{1}{2} c_1 [\dot{y}_1 - (\dot{x}_1 + a\dot{\theta})]^2 - \frac{1}{2} c_2 [\dot{y}_2 - (\dot{x}_1 - b\dot{\theta})]^2 - \frac{1}{2} c_3 [(\dot{x}_1 - c\dot{\theta}) - \dot{x}_2]^2 \tag{a}$$

Using the Lagrangian of Equation (c) of Example 7.5, application of the nonconservative form of Lagrange's equations Equation (7.19) yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} - \frac{\partial \Im}{\partial \dot{x}_1} = 0$$

$$\begin{aligned}
\frac{d}{dt}\left[\frac{1}{2} m_1 (2\dot{x}_1)\right] - \Big\{\frac{1}{2} k_1 (2)[y_1(t) - (x_1 + a\theta)](-1) - \frac{1}{2} k_2 (2)[y_2(t) - (x_1 - b\theta)](-1)& \\
- \frac{1}{2} k_3 (2)[x_1 + c\theta - x_2](1)\Big\}& \\
- \Big\{- \frac{1}{2} c_1 (2)[\dot{y}_1 - (\dot{x}_1 + a\dot{\theta})](-1) - \frac{1}{2} c_2 (2)[\dot{y}_2 - (\dot{x}_1 - b\dot{\theta})](-1)& \\
- \frac{1}{2} c_3 (2)[(\dot{x}_1 + c\dot{\theta}) - \dot{x}_2]\Big\} = 0&
\end{aligned} \tag{b}$$

Application of Lagrange's equations for θ leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \frac{\partial \Im}{\partial \dot{\theta}} = 0$$

$$\frac{d}{dt}\left[\frac{1}{2} I (2\dot{\theta})\right] - \Big\{-\frac{1}{2} k_1 (2)[y_1(t) - (x_1 + a\theta)](-a) - \frac{1}{2} k_2 (2)[y_2(t) - (x_1 - b\theta)](b)$$

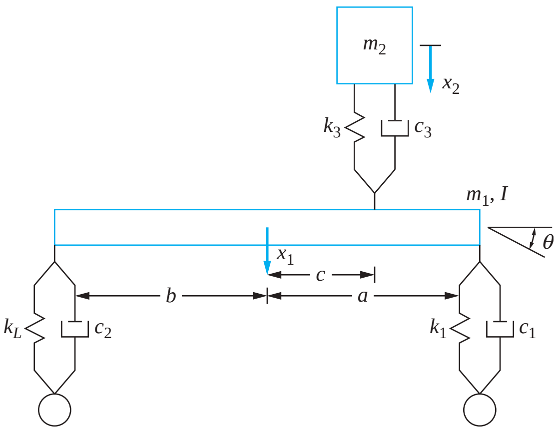


FIGURE 7.7
Two degree-of-freedom system of Example 7.8. The nature of the coupling depends upon the choice of generalized coordinates.

$$-\frac{1}{2}k_3(2)x_1 + c\theta - x_2](c)\Big\} - \Big\{-\frac{1}{2}c_1(2)[\dot{y}_1 - (\dot{x}_1 + a\dot{\theta})](-a)$$

$$-\frac{1}{2}c_2(2)[\dot{y}_2 - (\dot{x}_1 - b\dot{\theta})](b)\Big\} = 0 \quad \text{(c)}$$

Application of Lagrange's equations for x_2 leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_2} = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}m_2(2\dot{x}_2)\right] - \left\{-\frac{1}{2}k_3(2)[x_1 + c\theta - x_2](1)\right\}$$

$$-\left[-\frac{1}{2}c_3(2)(\dot{x}_1 + c\dot{\theta} - \dot{x}_2(-1)\right] = 0 \quad \text{(d)}$$

Equations (b) through (d) are rearranged and written in a matrix form leading to

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 + c_3 & c_1 a - c_2 b + c_3 c & -c_3 \\ c_1 a - c_2 b + c_3 c & c_1 a^2 + c_2 b^2 + c_3 c^2 & -c_3 c \\ -c_3 & -c_3 c & c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{\theta} \\ \dot{x}_2 \end{bmatrix}$$

$$+\begin{bmatrix} k_1 + k_2 + k_3 & k_1 a - k_2 b + k_3 c & -k_3 \\ k_1 a - k_2 b + k_3 c & k_1 a^2 + k_2 b^2 + k_3 c^2 & -k_3 c \\ -k_3 & -k_3 c & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 y_1(t) + k_2 y_2(t) + c_1\dot{y}_1(t) + c_2\dot{y}_2(t) \\ k_1 a y_1(t) - k_2 b y_2(t) + c_1 a\dot{y}_1(t) + c_2 b\dot{y}_2(t) \\ 0 \end{bmatrix} \quad \text{(e)}$$

7.4 MATRIX FORMULATION OF DIFFERENTIAL EQUATIONS FOR LINEAR SYSTEMS

It can be shown that for an nDOF linear system the potential and kinetic energies must have the quadratic forms

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j \quad \text{(7.21)}$$

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{x}_i\dot{x}_j \quad \text{(7.22)}$$

The Lagrangian for a linear system becomes

$$L = \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(m_{ij}\dot{x}_i\dot{x}_j - k_{ij}x_i x_j)\right] \quad \text{(7.23)}$$

Application of Lagrange's equations for a nonconservative system without viscous damping for generalized coordinate x_l leads to

$$Q_l = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_l}\right) - \frac{\partial L}{\partial x_l} \qquad l = 1, 2, \ldots, n$$

$$Q_l = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m_{ij}\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}_l}(\dot{x}_i\dot{x}_j)\right] + k_{ij}\frac{\partial}{\partial x_l}(x_i x_j)\right\}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m_{ij}\frac{d}{dt}\left[\dot{x}_i\frac{\partial \dot{x}_j}{\partial \dot{x}_l} + \dot{x}_j\frac{\partial \dot{x}_i}{\partial \dot{x}_l}\right] + k_{ij}\left(x_i\frac{\partial x_j}{\partial x_l} + x_j\frac{\partial x_i}{\partial x_l}\right)\right\} \tag{7.24}$$

Since

$$\frac{\partial x_i}{\partial x_l} = \delta_{il} = \begin{cases} 0 & i \neq l \\ 1 & i = l \end{cases} \tag{7.25}$$

Equation (7.24) becomes

$$Q_l = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[m_{ij}\frac{d}{dt}(\dot{x}_i\delta_{jl} + \dot{x}_j\delta_{il}) + k_{ij}(x_i\delta_{jl} + x_j\delta_{il})\right] \tag{7.26}$$

The right-hand side of the preceding equation is broken into four terms and the order of summation interchanged on the second and fourth terms. Then because of the presence of the δ's, the value of the term on the inner summation is nonzero only for one value of the summation index. Thus, the preceding equation can be rewritten using single summations as

$$Q_l = \frac{1}{2}\left(\sum_{i=1}^{n} m_{il}\ddot{x}_i + \sum_{j=1}^{n} m_{lj}\ddot{x}_j + \sum_{i=1}^{n} k_{il}x_i + \sum_{j=1}^{n} k_{lj}x_j\right) \tag{7.27}$$

The name of a summation index is arbitrary. Thus, these summations are combined, yielding

$$Q_l = \frac{1}{2}\left[\sum_{i=1}^{n}(m_{il} + m_{li})\ddot{x}_i + \sum_{i=1}^{n}(k_{il} + k_{li})x_i\right] \tag{7.28}$$

Note that in Equation (7.21), k_{il} and k_{li} both multiply $x_i x_l$. It seems reasonable that, without loss of generality, they can be set equal to one another (the formal proof of this fact will be given in Section 7.5. The same reasoning leads to $m_{il} = m_{li}$. Thus,

$$\sum_{i=1}^{n} m_{li}\ddot{x}_i + \sum_{i=1}^{n} k_{li}x_i = Q_l \qquad l = 1, \ldots, n \tag{7.29}$$

Equation (7.29) represents a system of n simultaneous linear differential equations.

The matrix formulation of Equation (7.29) is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.30}$$

where $\mathbf{M}$ is the $n \times n$ *mass matrix*, $\mathbf{K}$ is the $n \times n$ *stiffness matrix*, $\mathbf{F}$ is the $n \times 1$ *force vector*, $\mathbf{x}$ is the $n \times 1$ *displacement vector*, and $\ddot{\mathbf{x}}$ is the $n \times 1$ *acceleration vector*. Note from

Equation (7.28) that for the lth equation, the coefficient multiplying $\ddot{x}_i$ is $(m_{il} + m_{li})/2$, which is m_{li}, the element in the lth row and ith column of $\mathbf{M}$. Similarly m_{il}, the element in the ith row and lth column is determined as $(m_{li} + m_{il})/2$. Hence $m_{il} = m_{li}$ for each $i, l = 1, 2, \ldots, n$. Thus, the mass matrix is *symmetric*. The element in the ith row and jth column of the mass matrix is m_{ij}, the same coefficient that multiplies $\dot{x}_i \dot{x}_j$ in the quadratic form of the kinetic energy, Equation (7.22).

A similar argument can be used to show that the stiffness matrix is symmetric and that the element in the ith row and jth column of $\mathbf{K}$ is the coefficient that multiplies $x_i x_j$ in the quadratic form of the potential energy, Equation (7.21). The ith element of the force vector is the generalized force Q_i, as determined by the method of virtual work.

The matrix formulation of the differential equations governing the motion of a linear n degree-of-freedom system is used in deriving the free and forced responses of the system. If the mass and stiffness matrices and the force vector are known for a chosen set of generalized coordinates, differential equations of the form of Equation (7.30) can be directly written. Thus, if the quadratic forms of the kinetic and potential energies can be determined, the elements of the mass and stiffness matrices are the coefficients in these quadratic forms. Formal application of Lagrange's equations to derive the differential equations governing the motion of a linear system is not necessary.

The coupling of a system relative to the choice of generalized coordinate is specified according to how the mass and stiffness matrices are populated. A *diagonal matrix* is a matrix in which the only nonzero elements are along the main diagonal of the matrix. If the stiffness matrix is not a diagonal matrix, the system is said to be *statically coupled* relative to the choice of generalized coordinates. If the system is statically coupled with respect to a set of generalized coordinates x_i, $i = 1, 2, \ldots, n$, then there is at least one i such that application of a static force to the particle whose displacement is x_i results in a static displacement of the particle whose displacement is x_j, for some $j \neq i$.

If the mass matrix is not a diagonal matrix, the system is said to be *dynamically coupled.* If the system is dynamically coupled, then there exists at least one i such that application of an impulse to the particle whose displacement is x_i instantaneously induces a velocity $\dot{x}_j$, for some $j \neq i$.

EXAMPLE 7.9

Use the quadratic forms of kinetic and potential energy to derive the differential equations governing free vibration of the system of Figure 7.8 and discuss the coupling using (a) x and θ as generalized coordinates, and (b) x_A, the vertical displacement of particle A, and x_B, the vertical displacement of particle B, as generalized coordinates.

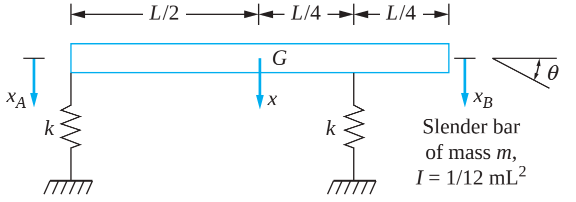


FIGURE 7.8
System of Example 7.9.

SOLUTION

(a) With x and θ as generalized coordinates, the kinetic and potential energies of the system at an arbitrary instant are

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12} mL^2\right)\dot{\theta}^2 \qquad \text{(a)}$$

$$V = \frac{1}{2} k\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2} k\left(x + \frac{L}{4}\theta\right)^2 = \frac{1}{2}\left(2kx^2 - k\frac{L}{2}x\theta + \frac{5}{16}kL^2\theta^2\right) \qquad \text{(b)}$$

Comparing the above equations with the quadratic forms of kinetic and potential energies, Equations (7.22) and (7.21), respectively, using x for x_1 and θ for x_2 leads to

$$m_{11} = m \qquad m_{12} = m_{21} = 0 \qquad m_{22} = \frac{1}{12} mL^2 \qquad \text{(c)}$$

$$k_{11} = 2k \qquad k_{12} = k_{21} = -k\frac{L}{4} \qquad k_{22} = \frac{5}{16} kL^2 \qquad \text{(d)}$$

Note that the term multiplying $x\theta$ in the quadratic form of potential energy is $2k_{12} = 2k_{21}$. Thus, the governing differential equations are

$$\begin{bmatrix} m & 0 \\ 0 & \frac{1}{12}ml^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & -k\frac{L}{4} \\ -k\frac{L}{4} & \frac{5}{16}kL^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(e)}$$

Since the stiffness matrix is not a diagonal matrix and the mass matrix is a diagonal matrix the system is statically coupled, but not dynamically coupled.

(b) With x_A and x_B as generalized coordinates, the quadratic forms of kinetic and potential energies at an arbitrary instant are

$$T = \frac{1}{2} m\left(\frac{\ddot{x}_A}{3} + \frac{2\ddot{x}_B}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{12} mL^2\right)\left(\frac{\ddot{x}_B - \ddot{x}_A}{\frac{3L}{4}}\right)^2$$

$$= \frac{1}{2}\left(\frac{7}{27}\dot{x}_A^2 + \frac{4}{27}\dot{x}_A\dot{x}_B + \frac{16}{27}\dot{x}_B^2\right) \qquad \text{(f)}$$

$$V = \frac{1}{2} kx_A^2 + \frac{1}{2} kx_B^2 \qquad \text{(g)}$$

The elements of the mass and stiffness matrices are obtained by comparing the above equations to Equations (7.22) and (7.21) respectively, leading to the following differential equations

$$\begin{bmatrix} \frac{7}{27}m & \frac{2}{27}m \\ \frac{2}{27}m & \frac{16}{27}m \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(h)}$$

Thus, the system is dynamically coupled, but not statically coupled, when x_A and x_B are used as generalized coordinates.

The method presented in this section to determine the mass and stiffness matrices for linear systems is the MDOF analogy to the equivalent systems method presented in Section 2.12 to derive the differential equations governing the motion of a linear SDOF system. The equivalent systems method uses the kinetic energy to determine an equivalent mass and the potential energy to determine an equivalent stiffness. The mass and stiffness matrices are analogous to the equivalent mass and the equivalent stiffness.

The differential equations governing the motion of a linear nDOF system when viscous damping is included are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.31}$$

where **C** is the nxn damping matrix. Rayleigh's dissipation function can be used to directly determine the elements of the damping matrix. Recall that the dissipation function is the negative of one-half of the power dissipated by all the viscous dampers. It can be shown to have a quadratic form of

$$\mathfrak{F} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}\dot{x}_i\dot{x}_j \tag{7.32}$$

The damping matrix is symmetric; that is, $c_{i,j} = c_{j,i}$.

When using the quadratic form of Rayleigh's dissipation function to determine the damping matrix, remember that like the mass matrix and the stiffness matrix, the diagonal terms are the terms multiplying $-\frac{1}{2}\dot{x}_i^2$, but that due to the dissipation function, including both $c_{i,j}\dot{x}_i\dot{x}_j$ and $c_{j,i}\dot{x}_j\dot{x}_i$, the off diagonal term $c_{i,j}$ is the negative of the coefficient multiplying $\dot{x}_i\dot{x}_j$. Unlike the quadratic forms of kinetic and potential energy, the definition of Rayleigh's dissipation function leads to the quadratic form being defined with a negative sign.

EXAMPLE 7.10

Determine the damping matrix for the three degree-of-freedom system shown in Figure 7.9.

SOLUTION

The power dissipated by viscous damping is

$$P = (c\dot{x}_1)\dot{x}_1 + [2c(\dot{x}_2 - \dot{x}_1)](\dot{x}_2 - \dot{x}_1) + [3c(\dot{x}_3 - \dot{x}_2)](\dot{x}_3 - \dot{x}_2) + (c\dot{x}_3)\dot{x}_3 \tag{a}$$

The energy dissipation function is calculated as

$$\mathfrak{F} = -\frac{1}{2}c\dot{x}_1^2 - \frac{1}{2}2c(\dot{x}_2 - \dot{x}_1)^2 - \frac{1}{2}3c(\dot{x}_3 - \dot{x}_2)^2 - \frac{1}{2}c\dot{x}_3^2 \tag{b}$$

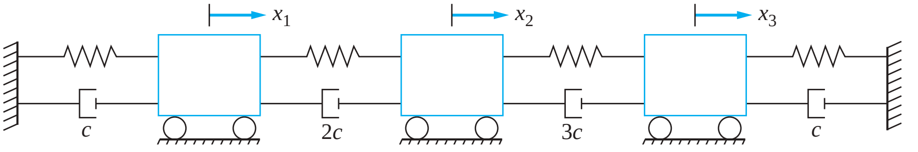


FIGURE 7.9
System of Example 7.10.

which is rearranged to

$$\Im = -\frac{3}{2}c\dot{x}_1^2 + 2c\dot{x}_1\dot{x}_2 - \frac{5}{2}c\dot{x}_2^2 + 3c\dot{x}_2\dot{x}_3 - 2c\dot{x}_3^2 \quad \text{(c)}$$

The diagonal element of the damping matrix $c_{i,i}$ is the negative of twice the coefficient of $\dot{x}_i^2$, while an off diagonal element $m_{i,j}$ for $i \neq j$ is the negative of the coefficient of $\dot{x}_i\dot{x}_j$. The damping matrix is

$$\mathbf{C} = \begin{bmatrix} 3c & -2c & 0 \\ -2c & 5c & -3c \\ 0 & -3c & 4c \end{bmatrix} \quad \text{(d)}$$

7.5 STIFFNESS INFLUENCE COEFFICIENTS

It is shown in Section 7.4 that the elements of the stiffness matrix for a linear system can be determined as the coefficients in the quadratic form of the potential energy. The work done by a conservative force is independent of path and can be expressed as the difference in potential energy between the initial position and the final position of the system. The potential energy function is a function only of the position of the system. Thus, when evaluating the potential energy for a specific system configuration, one can look at any means of arriving at that configuration, even if the configuration is obtained statically.

Stiffness influence coefficients provide an alternate means of determining the elements of the stiffness matrix. It is based on determining the potential energy for a system configuration that is obtained through static application of concentrated forces. To illustrate the development of the method, consider three particles along the span of a fixed-free beam as illustrated in Figure 7.10(a). The beam is initially in its static equilibrium configuration. Let x_1, x_2, and x_3 be the chosen generalized coordinates which represent the displacements of the particles.

Consider the static application of a set of concentrated loads with f_{11} applied to particle 1, f_{21} applied to particle 2, and f_{31} applied to particle 3 such that after their application, $x_1 = x_1$, $x_2 = 0$, and $x_3 = 0$ as illustrated in Figure 7.10(b). Since particles 2 and 3 do not change position during application of these loads, the forces applied to these particles do no work. The total work done by the external loads during this application is

$$U_{0\to 1} = \frac{1}{2}f_{11}x_1 \quad \text{(7.33)}$$

Now add a second set of forces with f_{12} applied to particle 1, f_{22} applied to particle 2, and f_{32} applied to particle 3 such that after static application of these loads, $x_1 = x_1$, $x_2 = x_2$, and $x_3 = 0$ as illustrated in Figure 7.10(c). Since particles 1 and 3 do not change position during application of these loads, only the forces applied to particle 2 do work. Note that the force f_{21} was already fully applied when the displacement occurred and the displacement occurred as f_{22} was being applied. Hence, the work done during application of these forces is

$$U_{1\to 2} = f_{21}x_2 + \frac{1}{2}f_{22}x_2 \quad \text{(7.34)}$$

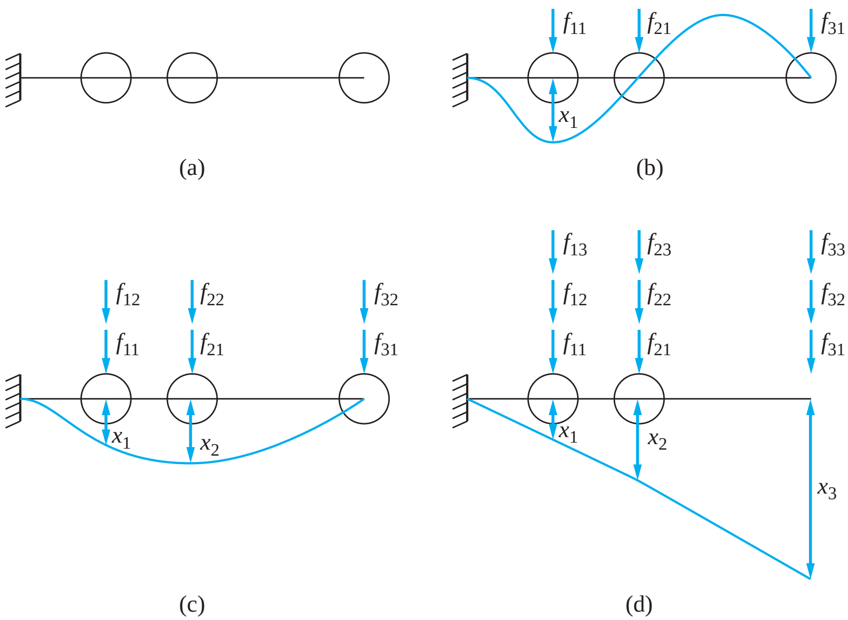


FIGURE 7.10
(a) Fixed-fixeds beam with three particles along its span. (b) Configuration of beam after first set of loads. (c) Configuration of beam after second set of loads. (d) Configuration of beam after third set of loads.

Next add a third set of forces f_{13} applied to particle 1, f_{23} applied to particle 2, and f_{33} applied to particle 3 such that after static application of these loads $x_1 = x_1$, $x_2 = x_2$, and $x_3 = x_3$ as illustrated in Figure 7.10(d). The work done during application of these forces is

$$U_{2\rightarrow 3} = f_{31}x_3 + f_{32}x_3 + \frac{1}{2}f_{33}x_3 \tag{7.35}$$

Thus, after application of the three sets of forces, the particles have arbitrary displacements. According to the principle of work and energy, the potential energy in the system is equal to the work done by the external forces between configuration 0 and configuration 3,

$$V = \frac{1}{2}f_{11}x_1 + f_{21}x_2 + \frac{1}{2}f_{22}x_2 + f_{31}x_3 + f_{32}x_3 + \frac{1}{2}f_{33}x_3 \tag{7.36}$$

The system is linear, thus a proportional change in the system of forces applied on any step leads to a proportional change in displacements. Define k_{11}, k_{21}, and k_{31} as the set of forces required to cause a unit displacement for the first particle. Then due to the linearity of the system

$$f_{11} = k_{11}x_1 \qquad f_{21} = k_{21}x_1 \qquad f_{31} = k_{31}x_1 \tag{7.37}$$

Similarly define k_{12}, k_{22}, and k_{32} as the set of forces required to cause a unit displacement for particle 2 and k_{13}, k_{23}, and k_{33} as the set of forces required to cause a unit displacement for particle 3. Then in general,

$$f_{ij} = k_{ij}x_j \tag{7.38}$$

Using Equation (7.38) in Equation (7.36) leads to

$$V = \frac{1}{2}k_{11}x_1x_1 + k_{21}x_1x_2 + \frac{1}{2}k_{22}x_2x_2 + k_{31}x_1x_3 + k_{32}x_2x_3 + \frac{1}{2}k_{33}x_3x_3 \tag{7.39}$$

The potential energy is a function only of the beam's configuration, not of how the configuration is attained. Thus, the potential energy would be the same if the order of the loading were reversed. Suppose the forces f_{12}, f_{22}, and f_{32} are applied first, resulting in $x_1 = 0$, $x_2 = x_2$, and $x_3 = 0$. Then the forces f_{21}, f_{22}, and f_{32} are applied such that after their static application, the beam's configuration is defined by $x_1 = x_1$, $x_2 = x_2$, and $x_3 = 0$. Then using Equation (7.38), the potential energy is calculated as

$$V = \frac{1}{2}k_{22}x_2x_2 + k_{12}x_2x_1 + \frac{1}{2}k_{11}x_1x_1 + k_{31}x_3x_1 + k_{32}x_3x_2 + \frac{1}{2}k_{33}x_3x_3 \tag{7.40}$$

Since the potential energy calculated by Equation (7.39) must be the same as that calculated by Equation (7.40) for arbitrary values of x_1, x_2, and x_3, $k_{12} = k_{21}$. Other combinations of the order of loading can be studied to show that in general,

$$k_{ij} = k_{ji} \tag{7.41}$$

This result, which guarantees that the stiffness matrix is symmetric, is known as *Maxwell's reciprocity relation*.

Then using Equation (7.41) in Equation (7.39) leads to

$$V = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij}x_ix_j \tag{7.42}$$

Equation (7.42) is identical to the quadratic form of the potential energy for this three degree-of-freedom system. Thus, the coefficients k_{ij}, $i, j = 1, 2, 3$ are the elements of the stiffness matrix. The k_{ij} calculated in this fashion are called *stiffness influence coefficients*. Equation (7.41) shows that the stiffness matrix is symmetric when stiffness influence coefficients are used in its determination.

The concept of stiffness influence coefficients can be generalized to any linear system. Each column of the stiffness matrix has a physical interpretation. The jth column of the stiffness matrix is the set of forces acting on the particles whose displacements are described by the chosen generalized coordinates such that after static application of these forces, $x_j = 1$ and $x_i = 0$ for $i \neq j$.

In summary, the influence coefficient method for determining the elements of an n degree-of-freedom system is as follows:

1. Assign a unit displacement for x_1, maintaining $x_2, x_3, \ldots, x_n$ in their static-equilibrium position. Calculate the system of forces required to maintain this as an equilibrium. position. The forces, k_{i1}, are applied at the locations whose displacements define the generalized coordinates in the directions of the positive values of the generalized coordinates. This set of forces yields the first column of the stiffness matrix.

2. Continue this procedure to find all columns of the stiffness matrix. The jth column is found by prescribing $x_j = 1$ and $x_i = 0$, $i \neq j$, and calculating the system of forces necessary to maintain this as an equilibrium position.

3. If x_j is an angular coordinate, then k_{ji} is an applied moment. When calculating the jth column of the stiffness matrix, a unit rotation in radians must be applied to the angle defined by x_j in the direction of the positive value of the angular coordinate. If the

small angle assumption is necessary to achieve a linear system, it is also used to calculate the stiffness influence coefficients.

4. Reciprocity implies the stiffness matrix must be symmetric: $k_{ij} = k_{ji}$. The symmetry can be used as a check.
5. When deriving differential equations for linear systems, note that static deflections in springs cancel with the gravity forces or other conservative forces that cause the static deflections. Thus, static deflections and their sources do not need to be considered in determining stiffness influence coefficients.

EXAMPLE 7.11

Use the stiffness influence coefficient method to calculate the stiffness matrix for the system of Figure 7.2 in Example 7.1.

SOLUTION

The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and calculating the system of applied forces necessary to maintain this position in equilibrium. Free-body diagrams of the blocks are shown in Figure 7.11. Setting $\sum F = 0$ yields

$$\text{Block } a\text{:} \quad -k - 2k + k_{11} = 0 \Rightarrow k_{11} = 3k$$

$$\text{Block } b\text{:} \quad 2k + k_{21} = 0 \Rightarrow k_{21} = -2k$$

$$\text{Block } c\text{:} \quad \Rightarrow k_{31} = 0$$

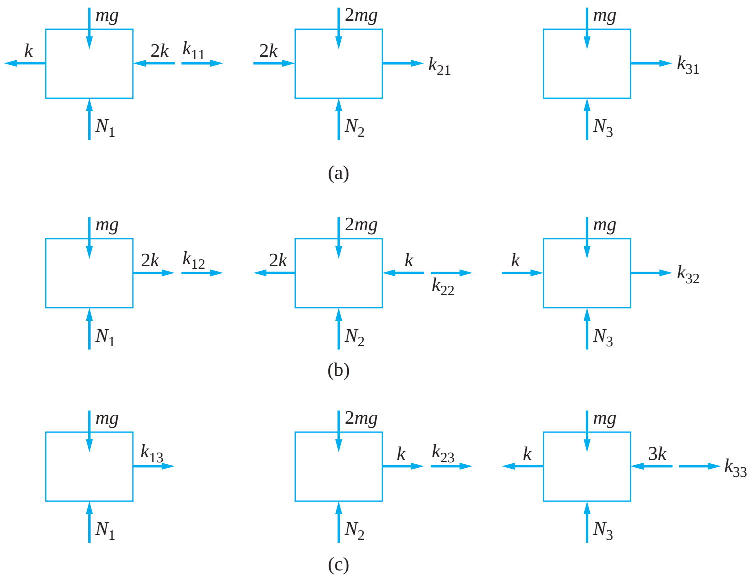


FIGURE 7.11
(a) First column of stiffness matrix is calculated by setting $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$, and determining forces maintaining the position in static equilibrium. (b) Second column of stiffness matrix is calculated by setting $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$, and determining forces maintaining the position in static equilibrium. (c) Third column of stiffness matrix is calculated by setting $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$, and determining forces maintaining the position in static equilibrium.

The second column is obtained by setting $x_2 = 0$, $x_1 = 1$, and $x_3 = 0$. Summing forces on the free-body diagrams yields

Block *a:* $2k + k_{12} = 0 \Rightarrow k_{12} = -2k$

Block *b:* $-2k - k + k_{22} = 0 \Rightarrow k_{22} = 3k$

Block *c:* $k + k_{32} = 0 \Rightarrow k_{32} = -k$

The third column is obtained by setting $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$. Summing forces on the free-body diagrams yields

Block *a:* $\Rightarrow k_{13} = 0$

Block *b:* $k + k_{23} = 0 \Rightarrow k_{23} = -k$

Block *c:* $-k - 3k + k_{33} = 0 \Rightarrow k_{33} = 4k$

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{bmatrix}$$

EXAMPLE 7.12

Use the stiffness influence coefficient method to find the stiffness matrix for the system in Figure 7.12. Use x_A, the downward displacement of block A, x_B, the upward displacement of block B, and θ, the counterclockwise angular rotation of the pulley, as generalized coordinates.

SOLUTION

The first column of the stiffness matrix is obtained by setting $x_A = 1$, $x_B = 0$, and $\theta = 0$, and finding the resulting system of forces and moments to maintain this as an equilibrium position. Note that since θ is an angular coordinate, k_{31} is a moment.

Block *A:* $\sum F = 0 \Rightarrow -k + k_{11} = 0 \Rightarrow k_{11} = k$

Block *B:* $\sum F = 0 \Rightarrow k_{21} = 0$

Pulley: $\sum M_O = 0 \Rightarrow k(r) + k_{31} = 0 \Rightarrow k_{31} = -kr$

The second column is obtained by setting $x_A = 0$, $x_B = 1$, and $\theta = 0$. The equations of equilibrium yield

Block *A:* $\sum F = 0 \Rightarrow k_{12} = 0$

Block *B:* $\sum F = 0 \Rightarrow 3k - k_{22} = 0 \Rightarrow k_{22} = 3k$

Pulley: $\sum M_O = 0 \Rightarrow 3k(2r) + k_{32} = 0 \Rightarrow k_{32} = -6kr$

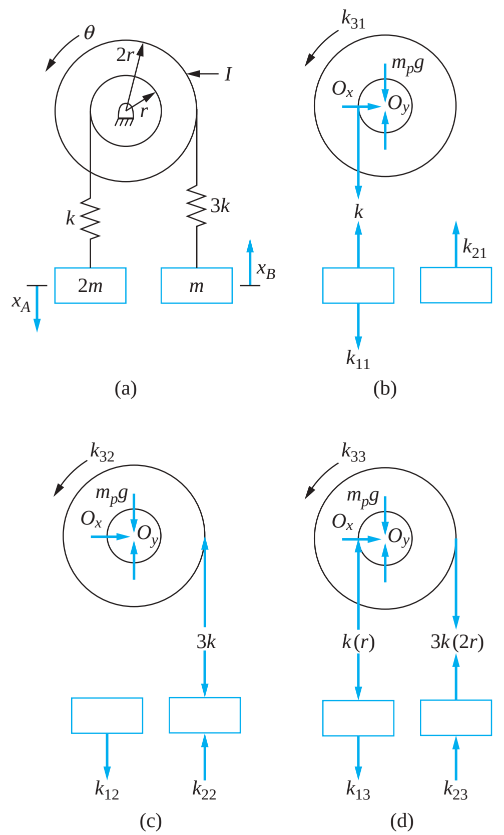


FIGURE 7.12
(a) System of Example 7.12. (b) First column of stiffness matrix is obtained by setting $x_A = 1$, $x_B = 0$, and $\theta = 0$ and calculating forces and moments to maintain the position in static equilibrium. (c) Second column of stiffness matrix is obtained by setting $x_A = 0$, $x_B = 1$, and $\theta = 0$ and calculating forces and moments to maintain the position in static equilibrium. (d) Third column of stiffness matrix is obtained by setting $x_A = 0$, $x_B = 0$, and $\theta = 1$ and calculating forces and moments to maintain the position in static equilibrium.

The third column is obtained by setting $x_A = 0$, $x_B = 0$, and $\theta = 1$. The equations of equilibrium yield

Block *A:* $\sum F = 0 \Rightarrow kr + k_{13} = 0 \Rightarrow k_{13} = -kr$

Block *B:* $\sum F = 0 \Rightarrow 3k(2r) + k_{23} = 0 \Rightarrow k_{23} = -6kr$

Pulley: $\sum M_O = 0 \Rightarrow -k(r)(r) - 3k(2r)(2r) + k_{33} = 0 \Rightarrow k_{33} = 13kr^2$

Thus, the stiffness matrix for this choice of generalized coordinates is

$$\mathbf{K} = \begin{bmatrix} k & 0 & -kr \\ 0 & 3k & -6kr \\ -kr & -6kr & 13kr^2 \end{bmatrix}$$

EXAMPLE 7.13

Use the influence coefficient method to find the stiffness matrix for the system of Figure 7.13 using θ_1, the clockwise angular displacement of bar AB, and θ_2, the counterclockwise angular displacement of bar CD, as generalized coordinates.

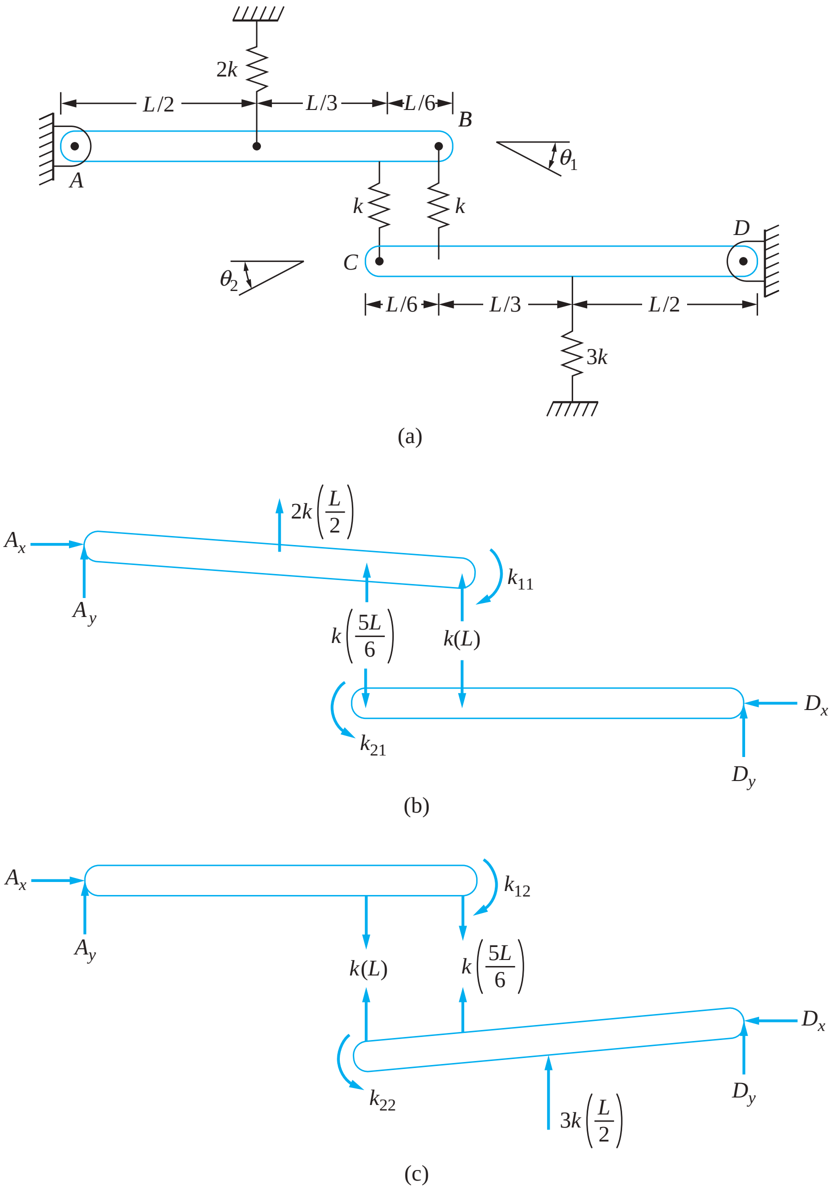


FIGURE 7.13

(a) System of Example 7.13. (b) First column of stiffness matrix is determined by setting $\theta_1 = 1$ and $\theta_2 = 0$, and calculating the applied moments required to maintain this position in equilibrium. (c) Second column of stiffness matrix is determined by setting $\theta_1 = 0$ and $\theta_2 = 1$, and calculating the applied moments required to maintain this position in equilibrium.

SOLUTION

The first column of the stiffness matrix is obtained by setting $\theta_1 = 1$ and $\theta_2 = 0$ and finding the moments that must be applied to the bars to maintain this as an equilibrium position. The small angle assumption is used. Equilibrium equations are applied to the free-body diagrams of Figure 7.13(b).

Taking moments to be positive clockwise about an axis at A and moments to be positive counterclockwise about an axis at D, we have

$$\sum M_A = 0 = -2k\frac{L}{2}\left(\frac{L}{2}\right) - 5k\frac{L}{6}\left(5\frac{L}{6}\right) - kL(L) + k_{11} \Rightarrow k_{11} = \frac{79}{36}kL^2 \quad \text{(a)}$$

$$\sum M_D = 0 = 5k\frac{L}{6}(L) + kL\left(5\frac{L}{6}\right) + k_{21} \Rightarrow k_{21} = -5k\frac{L^2}{3} \quad \text{(b)}$$

The second column is obtained by setting $\theta_1 = 0$ and $\theta_2 = 1$. The equilibrium equations are applied to the free-body diagrams to yield

$$\sum M_A = 0 = kL\left(5\frac{L}{6}\right) + 5k\frac{L}{6}(L) + k_{12} \Rightarrow k_{12} = -5k\frac{L^2}{3} \quad \text{(c)}$$

$$\sum M_D = 0 = -kL(L) - 5k\frac{L}{6}\left(5\frac{L}{6}\right) - 3k\frac{L}{2}\left(\frac{L}{2}\right) + k_{22} \Rightarrow k_{22} = 22k\frac{L^2}{9} \quad \text{(d)}$$

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} \frac{79}{36}kL^2 & -5k\frac{L^2}{3} \\ -5k\frac{L^2}{3} & \frac{22}{9}kL^2 \end{bmatrix} \quad \text{(e)}$$

EXAMPLE 7.14

The transverse vibrations of the cantilever beam Figure 7.14 are to be approximated by modeling the beam as a two degree-of-freedom system. The inertia of the beam is modeled by placing discrete masses at the beam's midspan and end. Calculate the stiffness matrix for this two degree-of-freedom model using the displacements of the midspan and end of the beam as generalized coordinates.

SOLUTION

Calculation of the stiffness matrix requires the evaluation of the deflection of the beam due to a concentrated load at the midspan and a concentrated load at the end of the beam. Perhaps the best way of handling the beam deflection problem is to use the method of superposition as shown in Figure 7.14(b). The elements of the ith column of the stiffness matrix are calculated from

$$y\left(\frac{L}{2}\right) = k_{1i}y_1\left(\frac{L}{2}\right) + k_{2i}y_2\left(\frac{L}{2}\right) \quad \text{(a)}$$

$$y(L) = k_{1i}y_1(L) + k_{2i}y_2(L) \quad \text{(b)}$$

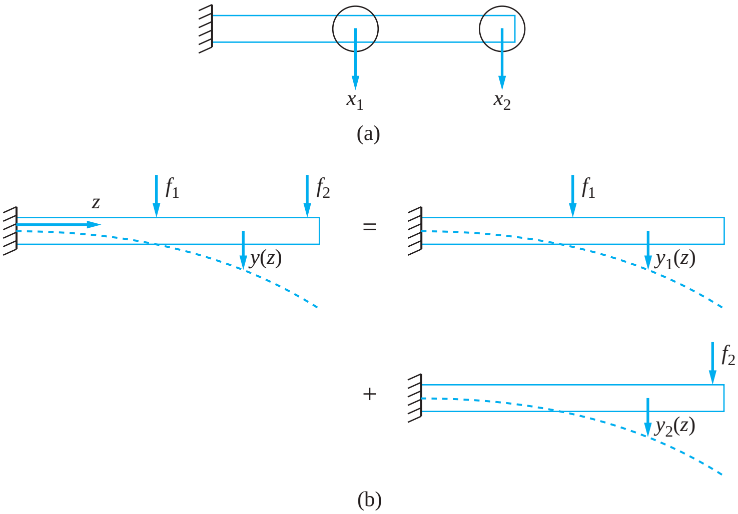


FIGURE 7.14
(a) Two degree-of-freedom model of cantilever beam of Example 7.14. (b) Illustration of the method of superposition used to calculate the stiffness matrix.

where $y(z)$ is the total deflected shape of the beam, $y_1(z)$ is the deflected shape of the beam due to a concentrated unit load at the midspan, and $y_2(z)$ is the deflected shape of the beam due to a concentrated unit load at the end of the beam. From Table D.2, these are evaluated as

$$y_1\left(\frac{L}{2}\right) = \frac{L^3}{24EI} \qquad y_2\left(\frac{L}{2}\right) = \frac{5L^3}{48EI} \tag{c}$$

$$y_1(L) = \frac{5L^3}{48EI} \qquad y_2(L) = \frac{L^3}{3EI} \tag{d}$$

To determine the first column, set $y(L/2) = 1$ and $y(L) = 0$. The equations are solved simultaneously, yielding

$$k_{11} = \frac{768EI}{7L^3} \qquad k_{21} = -\frac{240EI}{7L^3} \tag{e}$$

To determine the second column, set $y(L/2) = 0$ and $y(L) = 1$. The equations are solved simultaneously, yielding

$$k_{12} = -\frac{240EI}{7L^3} \qquad k_{22} = \frac{96EI}{7L^3} \tag{f}$$

7.6 FLEXIBILITY INFLUENCE COEFFICIENTS

Development of the stiffness matrix using stiffness influence coefficients is straight-forward. For mechanical systems, the calculation of stiffness influence coefficients requires the application of the principles of statics and little algebra. However, as shown in Example 7.14, the calculation of a column of stiffness influence coefficients for a structural system modeled with n degrees of freedom requires the solution of n simultaneous equations. This leads to significant computation time for systems with many degrees of freedom. Flexibility influence coefficients provide a convenient alternative. They are easier to calculate than stiffness influence coefficients for structural systems and the knowledge of them is sufficient for solution of the free-vibration problem.

If the stiffness matrix, $\mathbf{K}$, is nonsingular, then its inverse exists. The flexibility matrix, $\mathbf{A}$, is defined by

$$\mathbf{A} = \mathbf{K}^{-1} \tag{7.43}$$

Premultiplying Equation (7.1) by $\mathbf{A}$ gives

$$\mathbf{AM\ddot{x}} + \mathbf{AC\dot{x}} + \mathbf{x} = \mathbf{AF} \tag{7.44}$$

Equation (7.44) shows that knowledge of $\mathbf{A}$ instead of $\mathbf{K}$ is sufficient for solution of a vibration problem.

The elements of $\mathbf{K}$ are determined by using stiffness influence coefficients. Analogously, flexibility influence coefficients can be used to determine $\mathbf{A}$. The flexibility influence coefficient a_{ij} is defined as the displacement of the particle whose displacement is represented by x_i when a unit load is applied to the particle whose displacement is represented by x_j and no other loading is applied to the system. If x_j represents an angular coordinate, then a unit moment is applied.

Suppose an arbitrary set of concentrated loads $\{f_1, f_2, \ldots, f_n\}$ is applied statically to an nDOF system. The load f_i is applied to the particle whose displacement is represented by x_i. Using the definition of flexibility influence coefficients, x_j is calculated from

$$x_j = \sum_{i=1}^{n} a_{ji} f_i \tag{7.45}$$

Equation (7.45) is summarized in matrix form as

$$\mathbf{x} = \mathbf{Af} \tag{7.46}$$

Multiplying Equation (7.46) by $\mathbf{A}^{-1}$ yields

$$\mathbf{f} = \mathbf{A}^{-1}\mathbf{x} = \mathbf{Kx} \tag{7.47}$$

which defines the static relationship between force and displacement. Equation (7.47) shows that the flexibility influence coefficients as defined are the elements of the inverse of the stiffness matrix, called the flexibility matrix.

The procedure for determining the flexibility matrix using influence coefficients is as follows:

1. Apply a unit load at the location whose displacement is defined by x_1. The flexibility influence coefficient in the first column, a_{i1}, is the resulting displacement of the particle whose displacement is x_i.
2. Successively apply concentrated unit loads to particles whose displacements define the remaining generalized coordinates. Calculate column of flexibility influence coefficients using the principles of statics.
3. If x_l is an angular displacement, then a unit moment is applied to calculate a_{jl}, $j = 1, \ldots, n$. The displacements calculated for a_{li}, $i = 1, \ldots, n$, are angular displacements.
4. Since the stiffness matrix is symmetric, the flexibility matrix must also be symmetric. This condition serves as a check on the analysis.

EXAMPLE 7.15

Determine the flexibility matrix for the system in Figure 7.13 of Example 7.13 using flexibility influence coefficients.

SOLUTION

The free-body diagrams of Figure 7.15 show the external forces, in terms of angular displacements, acting on each bar when an arbitrary set of moments is applied. The equations of equilibrium are used to derive equations relating the displacements to the applied forces

$$\text{Bar } AB: \quad \sum M_A = 0 \Rightarrow m_1 = \frac{79kL^2}{36}\theta_1 - \frac{5kL^2}{3}\theta_2 \tag{a}$$

$$\text{Bar } BC: \quad \sum M_D = 0 \Rightarrow m_2 = -\frac{5kL^2}{3}\theta_1 + \frac{22kL^2}{9}\theta_2 \tag{b}$$

The first column of the flexibility matrix is obtained by setting $m_1 = 1$, $m_2 = 0$, $\theta_1 = a_{11}$, $\theta_2 = a_{21}$, and solving the resulting equations simultaneously. The second column is obtained by setting $m_1 = 0$, $m_2 = 1$, $\theta_1 = a_{12}$, $\theta_2 = a_{22}$, and solving the resulting simultaneous equations. The flexibility matrix is

$$\mathbf{A} = \begin{bmatrix} \dfrac{396}{419kL^2} & \dfrac{270}{419kL^2} \\ \dfrac{270}{419kL^2} & \dfrac{711}{838kL^2} \end{bmatrix} \tag{c}$$

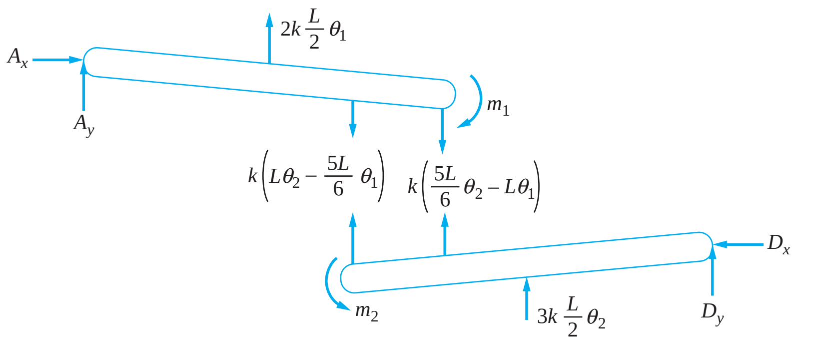


FIGURE 7.15
FBDs of static equilibrium position used to calculate flexibility influence coefficients for system of Example 7.15. For the first column, $m_1 = 1$ and $m_2 = 0$. For the second column, $m_1 = 0$ and $m_2 = 1$.

EXAMPLE 7.16

Two small machines are to be bolted to an overhanging beam as shown in Figure 7.16. The beam is nonuniform; thus prediction of influence coefficients from strength-of-materials concepts is difficult. Instead, the project engineer performs static measurements. After the first machine is installed, the engineer notes that the deflection directly below the machine

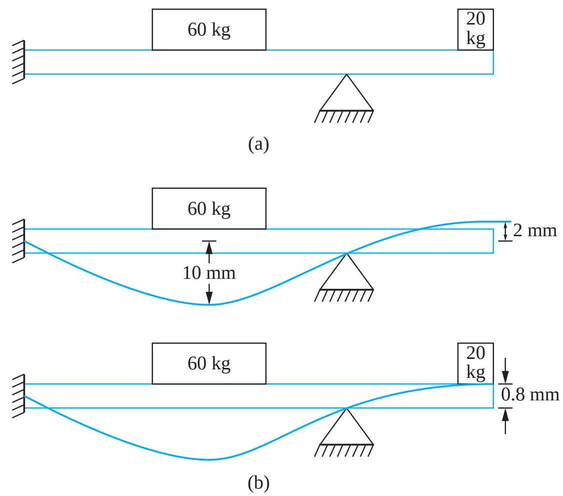


FIGURE 7.16
(a) System of Example 7.16. (b) As each machine is bolted to the beam, static deflection measurements are made.

is 10 mm and the deflection of the end of the beam is 2 mm. After the second machine is also installed, the deflection of the end of the beam is 0.8 mm.

(a) What is the deflection at the location where the first machine is installed after the second machine is installed?
(b) What is the flexibility matrix for this system?

SOLUTION

(a) Assuming a linear system, the principle of superposition yields the following relationships between the static loads, the influence coefficients, and the deflection:

$$x_1 = a_{11}f_1 + a_{12}f_2 \tag{a}$$

$$x_2 = a_{21}f_1 + a_{22}f_2 \tag{b}$$

When only the first machine is installed, $f_1 = (60\text{ kg})(9.81\text{ m/s}^2) = 588.6\text{ N}$, $f_2 = 0$, $x_1 = 0.01\text{ m}$, $x_2 = -0.002\text{ m}$. Substitution into the preceding equations yields $a_{11} = 1.7 \times 10^{-5}\text{ m/N}$, $a_{21} = -3.4 \times 10^{-6}\text{ m/N}$. When the second machine is also installed, $f_1 = 588.6\text{ N}$, $f_2 = (20\text{ kg})(9.81\text{ m/s}^2) = 196.2\text{ N}$, and $x_2 = -0.0008\text{ m}$. Then, since $a_{12} = a_{21}$, the displacement at the location of the first machine when both machines are installed is

$$x_1 = (1.7 \times 10^{-5}\text{ m/N})(588.6\text{ N}) + (-3.4 \times 10^{-6}\text{ m/N})(196.2\text{ N}) = 9.3\text{ mm} \tag{c}$$

(b) The second of the preceding equations yields

$$a_{22} = \frac{x_2 - a_{21}f_1}{f_2} = \frac{[-0.0008\text{ m} - (-3.4 \times 10^{-6}\text{ m/N})(588.6\text{ N})]}{196.2\text{ N}} = 6.1 \times 10^{-6}\text{ m/N} \tag{d}$$

The flexibility matrix is

$$\mathbf{A} = \begin{bmatrix} 1.7 & -0.34 \\ -0.34 & 0.61 \end{bmatrix} 10^{-5}\text{ m/N} \tag{e}$$

EXAMPLE 7.17

Four machines are equally spaced along the length of an 8 m fixed-free beam of elastic modulus 210×10^9 N/m^2 and cross-section moment of inertia 1.6×10^{-5} m^4, as shown in Figure 7.17. Determine the flexibility matrix for a four degree-of-freedom model of the system with the location of the machines as the generalized coordinates.

SOLUTION

The deflection equation for a fixed-free beam taken from Appendix D is

$$w(z;\ a) = \frac{1}{EI}\left[\frac{1}{6}(z-a)^3 u(z-a) - \frac{z^3}{6} + \frac{az^2}{2}\right] \tag{a}$$

The flexibility matrix is calculated sequentially by column in reverse order. Imagine the unit load placed at $a = L = 8$ m. Then

$$a_{41} = w\left(\frac{L}{4};\ L\right) = \frac{1}{EI}\left[-\frac{1}{6}\left(\frac{L}{4}\right)^3 + \frac{1}{2}(L)\left(\frac{L}{4}\right)^2\right] = \frac{11L^3}{384EI}$$
$$= \frac{11(8\text{ m})^3}{384(210\times 10^9\text{ N/m}^2)(1.6\times 10^{-5}\text{ m}^4)} = 4.37\times 10^{-6}\text{ m/N} \tag{b}$$

In a similar manner,

$$a_{42} = w\left(\frac{L}{2};\ L\right) = 1.59\times 10^{-5}\text{ m/N},$$
$$a_{43} = w\left(\frac{3L}{4};\ L\right) = 3.21\times 10^{-5}\text{ m/N},\ a_{44} = w(L;\ L) = 5.08\times 10^{-5}\text{ m/N} \tag{c}$$

Symmetry of the flexibility matrix is used to determine $a_{34} = a_{43}$. Then a unit load is imagined at $a = 3L/4$ and

$$a_{31} = w\left(\frac{L}{4};\ \frac{3L}{4}\right) = 3.17\times 10^{-6}\text{ m/N},\quad a_{32} = w\left(\frac{L}{2};\ \frac{3L}{4}\right) = 1.11\times 10^{-5}\text{ m/N},$$
$$a_{33} = w\left(\frac{3L}{4};\ \frac{3L}{4}\right) = 2.14\times 10^{-5}\text{ m/N} \tag{d}$$

Imagine a unit load placed at $a = L/2$

$$a_{21} = w\left(\frac{L}{4};\ \frac{L}{2}\right) = 1.98\times 10^{-6}\text{ m/N},\quad a_{22} = w\left(\frac{L}{2};\ \frac{L}{2}\right) = 6.35\times 10^{-6}\text{ m/N} \tag{e}$$

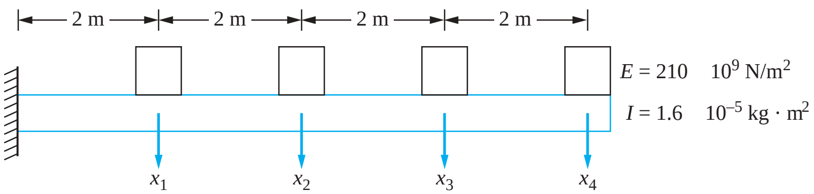


FIGURE 7.17
Four machines along the span of a fixed-free beam used in Example 7.17.

Finally, imagine a unit load placed at $a = L/4$

$$a_{11} = w\left(\frac{L}{4}; \frac{L}{4}\right) = 7.90 \times 10^{-7}\,\text{m/N} \tag{f}$$

The flexibility matrix is

$$\mathbf{A} = 10^{-7}\begin{bmatrix} 7.90 & 19.8 & 31.7 & 43.7 \\ 19.8 & 63.5 & 111.1 & 158.7 \\ 31.7 & 111.1 & 214.3 & 321.4 \\ 43.7 & 158.7 & 321.4 & 507.9 \end{bmatrix}\text{m/N} \tag{g}$$

Systems exist in which the stiffness matrix is singular and hence the flexibility matrix does not exist. These systems are called *semidefinite* or *unconstrained*. It is shown in Chapter 8 that these systems have a lowest natural frequency of zero and a corresponding mode where the system moves as a rigid body.

The system of Figure 7.18(a) has two degrees of freedom and is unconstrained. The stiffness matrix for this system is calculated as

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{7.48}$$

The second row of the stiffness matrix is a multiple of the first row, which implies that the matrix is singular and a flexibility matrix for this system does not exist. Indeed, when the definition of flexibility influence coefficients is applied in an attempt to calculate the flexibility matrix, as shown in Figure 7.18(b), no solution is found. Since the system is unconstrained, when a unit force is applied to either mass, the system cannot remain in equilibrium. Instead, the system will behave as a rigid body with uniform acceleration.

Another example of an unconstrained system is the system of Figure 7.11 in Example 7.10. The stiffness matrix for this example is repeated here

$$\mathbf{K} = \begin{bmatrix} k & 0 & -kr \\ 0 & 3k & -6kr \\ -kr & -6kr & 13kr^2 \end{bmatrix} \tag{7.49}$$

Inspection of this matrix reveals that the first row plus two times the second row is proportional to the third row. Thus, the three rows of the stiffness matrix are dependent, which implies that the stiffness matrix is singular, which, in turn, implies that the flexibility matrix does not exist. If, for example, a unit moment were applied to the pulley, then there are no other external forces which develop a moment about the center of the pulley. Hence, equilibrium cannot be maintained.

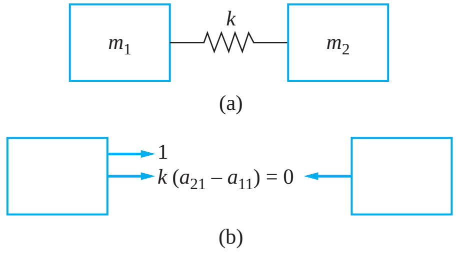


FIGURE 7.18
(a) A two degree-of-freedom unrestrained system. (b) FBDs of a system are used to show that the flexibility matrix does not exist.

A beam pinned at one end with no other support is an example of an unconstrained structural system. Application of a force or moment will lead to rigid body rotation about the pin support. A free-free beam is doubly unconstrained, in that it has two independent rigid-body motions. A free-free beam is unconstrained from transverse motion as well as rigid-body rotation.

Flexibility influence coefficients can be used to calculate the flexibility matrix. Equation (7.44) shows that knowledge of the flexibility matrix instead of knowledge of the stiffness matrix is sufficient to proceed with solution of the system of differential equations governing the vibrations of a MDOF system. The choice of whether to determine the stiffness matrix or the flexibility matrix is usually easy.

For structural systems, calculation of the flexibility matrix is easier than calculation of the stiffness matrix. For these systems, deflection equations from mechanics of solids are used to determine the deflection of a particle due to an applied concentrated load. The deflection equation for the structure is often available in a textbook or handbook (e.g., Appendix D). Thus, calculation of the flexibility matrix is direct, whereas the solution of a system of simultaneous equations is necessary to determine each column of the stiffness matrix. However, calculation of the stiffness matrix is easier than calculation of the flexibility matrix for mechanical systems that comprise rigid bodies connected by flexible elements. For these systems, application of the equations of static equilibrium to appropriate free-body diagrams is sufficient to calculate the stiffness matrix, while calculation of a column of the flexibility matrix also requires the solution of a system of simultaneous equations.

The stiffness matrix must be calculated for unconstrained systems.

7.7 INERTIA INFLUENCE COEFFICIENTS

The mass matrix can be calculated directly from the quadratic form of kinetic energy. It also can be calculated from influence coefficients calculated from an impulse and momentum analysis. Consider a linear system initially at rest in equilibrium. Free vibrations will occur if the system is given either an initial kinetic or potential energy. The stiffness influence coefficients are developed by examining potential energy induced by a static application of a system of forces. Inertia influence coefficients are developed by examining the kinetic energy induced by application of a system of impulses. An instantaneous change in velocity (and hence an instantaneous change in kinetic energy) occurs due to application of an impulse. If a system is dynamically coupled, then an instantaneous change in the velocity associated with one generalized coordinate may cause an instantaneous change in the velocities associated with the other generalized coordinates.

Consider a MDOF system with generalized coordinates $x_1, x_2, \ldots, x_n$. Assume a system of impulses is applied such that I_i is an impulse applied to the particle whose velocity is $\dot{x}_i$. Motion occurs with possibly non-zero velocities in the other generalized coordinates. These velocities are related to the applied impulses by n application of the principle of impulse and momentum. For a linear system, these are

$$I_i = \sum_{i=1}^{n} m_{ij}\,\dot{x}_j \tag{7.50}$$

where m_{ij} are the inertia influence coefficients. Consider in particular a system of applied impulses such that $\dot{x}_k = 1$ and $\dot{x}_j = 0$ for $j \neq k$. Then Equation (7.50) reduces to

$$I_i = m_{ik} \tag{7.51}$$

Thus, the inertia influence coefficient m_{ik} is one component of a system of impulses that is applied to generate an instantaneous velocity $\dot{x}_k = 1$ with $\dot{x}_j = 0$ for $j \neq k$. Specifically, it is the impulse that is applied to the particle whose displacement is represented by x_i. If a system of impulses is applied to a linear system such that the relationship between the applied impulses and the induced velocities is given by Equation (7.50), then the principle of work and energy can be used to show that the kinetic energy developed by the system is the quadratic form of kinetic energy given by Equation (7.22). Thus, the inertia influence coefficients are the elements of the mass matrix.

The following summarizes the calculation of inertia influence coefficients:

1. Assume that a system of impulses, I_i, $i = 1, 2, \ldots, n$ are applied such that $\dot{x}_1 = 1$, $\dot{x}_2 = 0, \dot{x}_2 = 0, \ldots, \dot{x}_n = 0$. Note that I_j is the impulse applied to the particle whose displacement is described by the generalized coordinate x_j. Repeated application of the principle of impulse and momentum allows for the solution of the applied impulse. The inertial influence coefficients are $m_{i1} = I_i$ for $i = 1, 2, \ldots, n$.
2. The procedure in step 1 is repeated with $\dot{x}_k = 1$ and all other velocities equal to zero for $k = 2, 3, \ldots, n$. The inertia influence coefficients are $m_{ik} = I_k$.
3. If x_j represents an angular coordinate, then I_j is an angular impulse and $\dot{x}_j$ is an angular velocity.
4. The mass matrix is symmetric, $m_{ij} = m_{ji}$. This serves as a check on the calculations.

EXAMPLE 7.18

Determine the mass matrix for the system of Figure 7.19(a) using inertia influence coefficients. Use θ and x, as illustrated, as generalized coordinates.

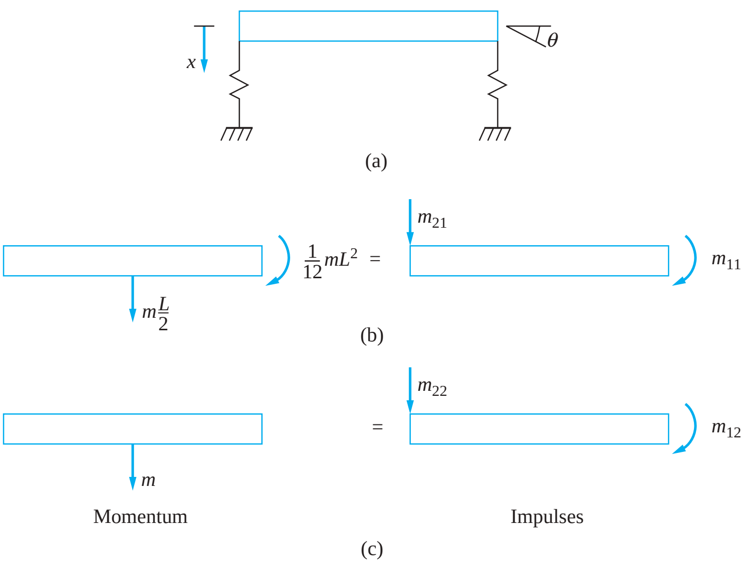


FIGURE 7.19 (a) System of Example 7.18 where θ and x are used as generalized coordinates. (b) Impulse and momentum diagrams of system for set $\dot{\theta} = 1$ and $\dot{x} = 0$. (c) Impulse and momentum diagrams for set $\dot{\theta} = 0$ and $\dot{x} = 1$.

SOLUTION

To determine the first column of the mass matrix, set $\dot{\theta} = 1$ and $\dot{x} = 0$. The angular momentum of the system is equal to $\bar{I}\dot{\theta} = \frac{1}{12}mL^2$. The linear momentum of the system is $m\bar{v}$. If the velocity of the end of the bar is zero but its angular velocity is one, the relative velocity equation is used to determine the velocity of the mass center as $L/2$ directed downward. An angular impulse equal to m_{11} is applied clockwise to the bar, and a linear impulse equal to m_{21} is applied downward at the end of the bar. Impulse and momentum diagrams are shown in Figure 7.19(b). Applying the principle of linear impulse and momentum gives

$$m\frac{L}{2} = m_{21} \tag{a}$$

Applying the principle of angular impulse and angular momentum about the end of the bar to impulse diagram of Figure 7.19(b)

$$m\frac{L}{2}\left(\frac{L}{2}\right) + \frac{1}{12}mL^2 = m_{11} \Rightarrow m_{11} = m\frac{L^2}{3} \tag{b}$$

To determine the second column of the mass matrix, set $\dot{\theta} = 0$ and $\dot{x} = 1$. The angular momentum of the bar is zero, and the linear momentum is simply m. An angular impulse equal to m_{12} is applied clockwise to the bar, and a linear impulse of magnitude m_{22} is applied downward at the end of the bar. Applying the principle of linear impulse and momentum to the impulse diagram of Figure 7.19(c) yields

$$m = m_{22} \tag{c}$$

Of course, the mass matrix is symmetric, so $m_{12} = m_{21} = m\frac{L}{2}$. However, it is best to check the result. Applying the principle of angular impulse and angular momentum to the diagrams of Figure 7.19(c) about an axis at the end of the bar leads to

$$m\frac{L}{2} = m_{12} \tag{d}$$

Thus, the mass matrix for this system is

$$\mathbf{M} = \begin{bmatrix} m\frac{L^2}{3} & m\frac{L}{2} \\ m\frac{L}{2} & m \end{bmatrix} \tag{e}$$

7.8 LUMPED-MASS MODELING OF CONTINUOUS SYSTEMS

Vibrations of continuous systems are governed by partial differential equations. Analytical solutions to partial differential equations are often difficult to obtain. Thus, approximate and numerical methods are often used to approximate the vibration properties and systems response of continuous systems. Some of these, such as the Rayleigh-Ritz method and the finite-element method, are discussed in Chapters 10 and 11. A simpler method of approximation is to replace the distributed inertia of the continuous system by a finite number of lumped inertia elements. A point where a lumped mass is placed is called a *node.* All inertia effects are concentrated at the nodes. The nodes are assumed to be connected by elastic but massless elements. Generalized coordinates are chosen as the displacements of the nodes.

A lumped-mass model of a continuous system is a discrete model of a continuous system. A system with n nodes is modeled as an n degree-of-freedom system. Differential equations of the form of Equation (7.1) or Equation (7.44) are derived to approximate the vibrations of the continuous system. It is necessary to determine the mass matrix, either the stiffness matrix or the flexibility matrix, and the force vector for the discrete approximation.

Unless the system is unconstrained, the flexibility matrix is used in lumped-mass modeling of a continuous system. The flexibility matrix is obtained by using flexibility influence coefficients, as described in Section 7.6. If the system is unconstrained, the stiffness matrix must be determined.

Lumped-mass approximations for modeling a continuous system using one degree of freedom were considered in Chapter 2. Recall that the inertia effects of a linear spring are approximated by placing a particle of mass equal to one-third of the mass of the spring at its end. The one-third approximation determined by calculating the particle mass such that the kinetic energy of the model system is equal to the kinetic energy of the spring, assuming a linear displacement function along the axis of the spring. This model illustrates that it is incorrect to model the inertia effects of the spring by using the full mass of the spring. The kinetic energy of particles near its fixed support is much less than the kinetic energy of the particles near the point of attachment to the system. Kinetic energy considerations could be used to determine the mass matrix for a discrete approximation. However, such a mass matrix, called the *consistent mass matrix*, is difficult to obtain and is not a diagonal matrix. The amount of effort used in determining a consistent mass matrix would be better used in developing a finite-element model for the system.

For simplicity, it is desirable to specify a diagonal mass matrix for a lumped-mass approximation of a continuous system. If a discretization is used where the mass of the system is lumped at nodes, then an obvious approximation to the mass matrix is a diagonal matrix with the nodal masses along the diagonal. In such a situation, the values of the nodal masses affects the accuracy of the system response. Using the one-degree-of-freedom approximation of the inertia effects of a linear spring as a guide, it is clear that using the entire mass of the system in the approximation will lead to errors in the approximation.

When a diagonal matrix is used to model the inertia effects of a continuous system, the mass lumped at each node should represent the mass of an identifiable region of the structure. A good scheme is to define the nodal mass as the mass of a region whose boundaries are halfway between the node and neighboring nodes on its right and left. If the node has no neighbor on one side, but is adjacent to a free end, then all of the mass between the node and the free end is used in calculating the nodal mass. If the particle is adjacent to a support that prevents motion, then only half of the mass between the node and the support is used. The accuracy of this method of approximation is considered in Chapter 8.

Calculation of the force vector may also require additional approximations. As shown in Section 7.3, the force vector is obtained by calculating the generalized forces, which occur when the method of virtual work is used. If a concentrated load is applied at a node, then the generalized force for the node's generalized coordinate is the value of the concentrated load and the generalized forces for all other coordinates are zero. However, if a concentrated load is applied at a location other than a node or the loading is distributed, calculation of the generalized forces requires additional approximations. The dynamic displacement is not available to apply the method of virtual work. In these cases it is suggested that the loading be replaced by a series of concentrated loads, calculated as follows, such that the resulting system is approximately statically equivalent to the applied loading. Static equivalence does not imply dynamic equivalence.

If the applied loading is replaced by a system of concentrated loads, the following method is used. The loading between any two nodes is replaced by a concentrated load at each of the nodes. The two concentrated loads are statically equivalent to the loading between the nodes. The sum of the concentrated loads is the resultant of the loading between the nodes. The moment of the distributed loading about either node is the same as the moment of the two concentrated loads about that point. Thus, the total generalized force applied at a node is approximated by the sum of the contribution from the loading between the node and its neighbor to the left and the contribution from the loading between the node and its neighbor to the right. If the node is adjacent to a free end, the contribution to the loading between the node and the free end is the resultant of the loading. If the particle is adjacent to a support that prevents displacement, only the resultant of the loading between the node and the point halfway between the node and the support is used. In this case, the work done by particles near supports is ignored in modeling the system, just as these particles' kinetic energy is ignored. The concentrated load is not statically equivalent to the actual loading if the particle is adjacent to a free end or a support.

EXAMPLE 7.19

Derive the differential equations whose solution approximates the forced response of the cantilever beam of Figure 7.20. Use four degrees of freedom to discretize the system. The beam is made of a material of elastic modulus E and mass density ρ. It has a cross-sectional area A and moment of inertia I. Neglect damping.

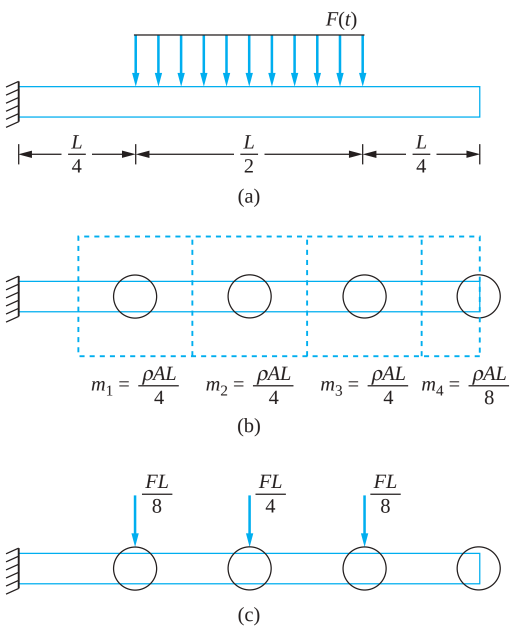


FIGURE 7.20
(a) System of Example 7.19. (b) Calculation of nodal masses. (c) Nodal forces are applied such that the forces are statically equivalent to the distributed loading of Figure 7.20(a).

SOLUTION

The beam is discretized by lumping its mass in four particles as shown in Figure 7.20(b). The nodes are chosen to be equally spaced. The generalized coordinates are the displacements of the nodes. The mass of each particle models the inertia effects of the regions shown in the figure. The loading is replaced by time-dependent concentrated loads at the nodes, as shown in Figure 7.20(c).

The flexibility matrix for this discretized system is determined from flexibility influence coefficients, as described in Section 7.6. The first column is obtained by placing a unit load at the first node and calculating the resulting deflections at each of the nodes. The result is

$$\mathbf{A} = \frac{L^3}{384EI}\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 28 & 40 \\ 8 & 28 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix} \tag{a}$$

The mass matrix is a diagonal matrix with the nodal masses along the diagonal. The force vector is simply the vector of concentrated loads from Figure 7.20(c). Then Equation (7.44) becomes

$$\left(\frac{\rho AL}{4}\right)\left(\frac{L^3}{384EI}\right)\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 28 & 40 \\ 8 & 28 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \left(\frac{L^3}{384EI}\right)\left(\frac{FL}{8}\right)\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 28 & 40 \\ 8 & 28 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \tag{b}$$

which simplifies to

$$\frac{\rho AL^3}{1536\,EI}\begin{bmatrix} 4 & 10 & 16 & 11 \\ 10 & 32 & 56 & 40 \\ 16 & 56 & 108 & 81 \\ 22 & 80 & 162 & 128 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{\rho AL^4 F(t)}{3072\,EI}\begin{bmatrix} 18 \\ 65 \\ 118 \\ 172 \end{bmatrix} \tag{c}$$

7.9 BENCHMARK EXAMPLES

7.9.1 MACHINE ON FLOOR OF AN INDUSTRIAL PLANT

Consider the machine directly bolted to the beam. Four lumped masses, as illustrated in Figure 7.21, are used to represent the motion of the beam, rather than one. The total weight of the beam is 250.8 lbf or a mass of 7.79 slugs. The mass matrix is determined using the methods described in Section 7.8. Each lumped mass has a value of

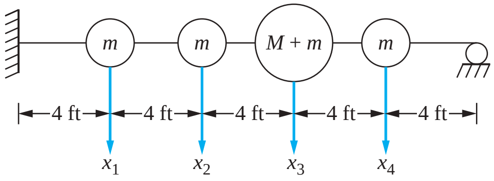


FIGURE 7.21
Four degree-of-freedom model of machine bolted directly to beam.

7.79 slugs/5 = 1.56 slugs. The mass associated with x_3 is the mass of the machine plus the lumped mass:

$$\mathbf{M} = \begin{bmatrix} 1.56 & 0 & 0 & 0 \\ 0 & 1.56 & 0 & 0 \\ 0 & 0 & 32.7 & 0 \\ 0 & 0 & 0 & 1.56 \end{bmatrix} \text{slugs} \tag{a}$$

The flexibility matrix is calculated using Appendix D. For example, calculation of the fourth column of the matrix requires a unit force applied at $a = 16$ ft, and calculation of the deflection at the locations of the generalized coordinates is

$$\begin{aligned} a_{44} = &\frac{1}{\left(30 \times 10^6 \dfrac{\text{lb}}{\text{in}^2}\right)(291 \text{ in}^4)\left(\dfrac{1 \text{ ft}}{12 \text{ in.}}\right)^2}\left\{\frac{1}{2}\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\right. \\ &\times \left[\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right)^2 - 2\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right) - 2\right]\frac{1}{6}(16 \text{ ft})^3 \\ &\left. + \frac{1}{2}(16 \text{ ft})\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\left(2 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\frac{1}{2}(16 \text{ ft})^2\right\} = 7.21 \times 10^{-7} \frac{\text{lb}}{\text{ft}} \end{aligned} \tag{b}$$

$$\begin{aligned} a_{34} = &\frac{1}{\left(30 \times 10^6 \dfrac{\text{lb}}{\text{in}^2}\right)(291 \text{ in}^4)\left(\dfrac{1 \text{ ft}}{12 \text{ in.}}\right)^2}\left\{\frac{1}{2}\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\right. \\ &\times \left[\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right)^2 - 2\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right) - 2\right]\frac{1}{6}(12 \text{ ft})^3 \\ &\left. + \frac{1}{2}(16 \text{ ft})\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\left(2 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\frac{1}{2}(12 \text{ ft})^2\right\} = 8.47 \times 10^{-7} \frac{\text{lb}}{\text{ft}} \end{aligned} \tag{c}$$

$$\begin{aligned} a_{24} = &\frac{1}{\left(30 \times 10^6 \dfrac{\text{lb}}{\text{in}^2}\right)(291 \text{ in}^4)\left(\dfrac{1 \text{ ft}}{12 \text{ in.}}\right)^2}\left\{\frac{1}{2}\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\right. \\ &\times \left[\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right)^2 - 2\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right) - 2\right]\frac{1}{6}(8 \text{ ft})^3 \\ &\left. + \frac{1}{2}(16 \text{ ft})\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\left(2 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\frac{1}{2}(8 \text{ ft})^2\right\} = 5.97 \times 10^{-7} \frac{\text{lb}}{\text{ft}} \end{aligned} \tag{d}$$

$$a_{14} = \frac{1}{\left(30 \times 10^6 \dfrac{\text{lb}}{\text{in}^2}\right)(291 \text{ in}^4)\left(\dfrac{1 \text{ ft}}{12 \text{ in.}}\right)^2}\left\{\frac{1}{2}\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\right.$$
$$\times\left[\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right)^2 - 2\left(\frac{16 \text{ ft}}{20 \text{ ft}}\right) - 2\right]\frac{1}{6}(4 \text{ ft})^3$$
$$\left.+\frac{1}{2}(16 \text{ ft})\left(1 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\left(2 - \frac{16 \text{ ft}}{20 \text{ ft}}\right)\frac{1}{2}(4 \text{ ft})^2\right\} = 2.01 \times 10^{-7} \frac{\text{lb}}{\text{ft}} \quad \textbf{(e)}$$

$$\mathbf{A} = 10^{-7}\begin{bmatrix} 2.14 & 3.67 & 3.43 & 2.01 \\ 3.67 & 9.12 & 9.74 & 5.97 \\ 3.43 & 9.74 & 12.9 & 8.47 \\ 2.01 & 5.97 & 8.47 & 7.21 \end{bmatrix}\frac{\text{ft}}{\text{lb}} \quad \textbf{(f)}$$

The differential equations that model the system are

$$10^{-7}\begin{bmatrix} 2.14 & 3.67 & 3.43 & 2.01 \\ 3.67 & 9.12 & 9.74 & 5.97 \\ 3.43 & 9.74 & 12.9 & 8.47 \\ 2.01 & 5.97 & 8.47 & 7.21 \end{bmatrix}\begin{bmatrix} 1.56 & 0 & 0 & 0 \\ 0 & 1.56 & 0 & 0 \\ 0 & 0 & 32.7 & 0 \\ 0 & 0 & 0 & 1.56 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$
$$= 10^{-7}\begin{bmatrix} 2.14 & 3.67 & 3.43 & 2.01 \\ 3.67 & 9.12 & 9.74 & 5.97 \\ 3.43 & 9.74 & 12.9 & 8.47 \\ 2.01 & 5.97 & 8.47 & 7.21 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ F_0 \sin \omega t \\ 0 \end{bmatrix} \quad \textbf{(g)}$$

or

$$10^{-7}\begin{bmatrix} 3.34 & 5.73 & 112.2 & 3.13 \\ 5.73 & 14.23 & 318.5 & 9.31 \\ 5.31 & 15.19 & 421.8 & 13.21 \\ 3.13 & 9.31 & 277 & 11.25 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 10^{-7}\begin{bmatrix} 3.43 \\ 9.74 \\ 12.9 \\ 8.47 \end{bmatrix} F_0 \sin \omega t \quad \textbf{(h)}$$

Now consider a five degree-of-freedom model including the vibration isolator of stiffness 3.93×10^4 lbf/ft as illustrated in Figure 7.22(a). Let the displacement of the machine be x_5. The first four columns and rows of the flexibility matrix for this model are the same as in Equation (f). The fifth column is calculated by placing a unit load on the machine and no loads anywhere else. However, summing forces on a free-body diagram of the machine Figure 7.22(b) reveal

$$k(a_{55} - a_{35}) = 1 \quad \textbf{(i)}$$

and the force developed in the isolator is unity. Thus, the deflections of the other points on the beam are as if a unit load were applied to the mass whose displacement is x_3. This is the

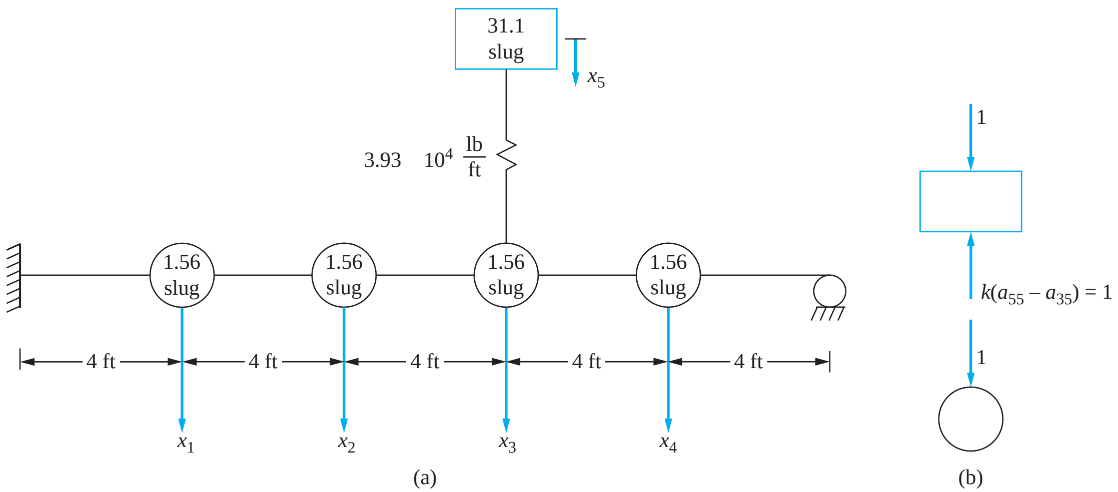


FIGURE 7.22
(a) Five degree-of-freedom model machine on fixed pinned beam. (b) FBD of machine and particle on beam.

displacement as calculated for the third column of the flexibility matrix. Hence, the flexibility matrix for the five degree-of-freedom model is

$$\mathbf{A} = 10^{-7}\begin{bmatrix} 2.14 & 3.67 & 3.43 & 2.01 & 3.43 \\ 3.67 & 9.12 & 9.74 & 5.97 & 9.74 \\ 3.43 & 9.74 & 12.9 & 8.47 & 12.9 \\ 2.01 & 5.97 & 8.47 & 7.21 & 8.47 \\ 3.43 & 9.74 & 12.9 & 8.47 & 255.4 \end{bmatrix}\frac{\text{ft}}{\text{lb}} \tag{j}$$

The mass matrix is

$$\mathbf{M} = \begin{bmatrix} 1.56 & 0 & 0 & 0 & 0 \\ 0 & 1.56 & 0 & 0 & 0 \\ 0 & 0 & 1.56 & 0 & 0 \\ 0 & 0 & 0 & 1.56 & 0 \\ 0 & 0 & 0 & 0 & 31.1 \end{bmatrix}\text{slug} \tag{k}$$

The differential equations modeling the displacement of the system are

$$10^{-7}\begin{bmatrix} 3.34 & 5.73 & 5.35 & 3.14 & 106.7 \\ 5.73 & 14.4 & 15.2 & 9.31 & 302.9 \\ 5.35 & 15.2 & 20.1 & 13.2 & 401.1 \\ 3.14 & 9.31 & 13.2 & 11.2 & 263.4 \\ 5.35 & 15.2 & 20.1 & 13.2 & 794.3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 10^{-7}\begin{bmatrix} 3.43 \\ 9.74 \\ 12.9 \\ 8.47 \\ 255.4 \end{bmatrix} F_0 \sin \omega t \tag{l}$$

7.9.2 SIMPLIFIED SUSPENSION SYSTEM

The distribution of mass about the center of mass is considered to matter such that the vehicle has the four degree-of-freedom model of Figure 7.23. The vehicle is now represented as a nonuniform bar of mass $m_s = 300$ kg. The length of the bar is the length of the vehicle is $l = 3$ m with a mass center 1.3 m from the front axle. The moment of inertia of the vehicle is $l = 225 \text{ kg} \cdot \text{m}^2$. Each axle has a mass $m_a = 25$ kg. The stiffness of each set of tires is $k_t = 100{,}000$ N/m. It is estimated that the damping coefficient of each tire is $10{,}000 \text{ N} \cdot \text{s/m}$. The front wheel has a displacement $y(t)$, and the rear wheel has a displacement $z = y\left(t - \frac{L}{v}\right)$ where v is the constant horizontal speed of the car. The generalized coordinates are x_1 (the displacement of the mass center of the vehicle form the system's equilibrium position), θ (the clockwise angular displacement of the vehicle form the system's equilibrium position), and x_2 (the displacement of the front axle), and x_3 (the displacement of the rear axle), where all are measured from the system's equilibrium position.

Lagrange's equations are employed to derive the governing differential equations. The kinetic energy of the car at an arbitrary instant is

$$T = \frac{1}{2} m_s \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_a \dot{x}_2^2 + \frac{1}{2} m_a \dot{x}_3^2 \tag{a}$$

The potential energy of the car at an arbitrary instant is

$$V = \frac{1}{2} k[x_2 - (x_1 + a\theta)]^2 + \frac{1}{2} k\{x_3 - [x_1 - (L-a)\theta]\}^2 + \frac{1}{2} k_t (y - x_2)^2 + \frac{1}{2} k_t (z - x_3)^2 \tag{b}$$

The system's Lagrangian is

$$\begin{aligned} L = {} & \frac{1}{2} m_s \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}_2^2 + \frac{1}{2} m_a \dot{x}_2^2 + \frac{1}{2} m_a \dot{x}_3^2 \\ & - \left[\frac{1}{2} k[x_2 - (x_1 + a\theta)]^2 + \frac{1}{2} k\{x_3 - [x_1 - (L - a)\theta]\}^2 \right. \\ & \left. + \frac{1}{2} k_t (y - x_2)^2 + \frac{1}{2} k_t (z - x_3)^2 \right] \end{aligned} \tag{c}$$

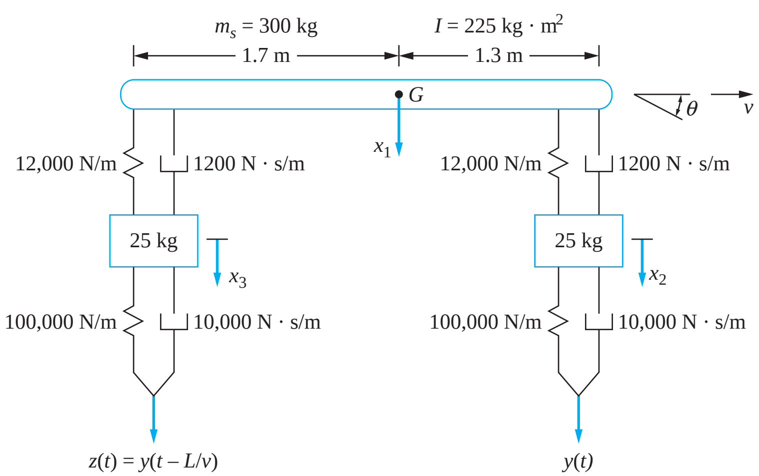


FIGURE 7.23
Four degree-of-freedom model of vehicle suspension system.

Rayleigh's dissipation function is

$$\mathfrak{F} = -\frac{1}{2}c[\dot{x}_2 - (\dot{x}_1 + a\,\dot{\theta})]^2 - \frac{1}{2}c\{\dot{x}_3 - [\dot{x}_1 - (L-a)\dot{\theta}]\}^2$$
$$-\frac{1}{2}c_t(\dot{y} - \dot{x}_2)^2 + \frac{1}{2}c_t(\dot{z} - \dot{x}_3)^2 \quad \text{(d)}$$

Application of Lagrange's equations yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \frac{\partial \mathfrak{F}}{\partial \dot{\theta}} = 0$$

$$I\ddot{\theta} + c\left[a^2 + (L-a)^2\right]\dot{\theta} + cL\dot{x}_1 - ca\dot{x}_2 + c(L-a)\dot{x}_3 + k\left[a^2 + (L-a)^2\right]\theta$$
$$+ kLx_1 - kax_2 + k(L-a)x_3 = 0 \quad \text{(e)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_1} = 0$$

$$m_s\ddot{x}_1 + c(L-2a)\dot{\theta} + 2c\dot{x}_1 - c\dot{x}_2 - c\dot{x}_3 + k(L-2a) + 2kx_1 - kx_2 - kx_1 = 0 \quad \text{(f)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_2} = 0$$

$$m_a\ddot{x}_2 - ca\dot{\theta} - c\dot{x}_1 + (c + c_t)\dot{x}_2 - ka\theta - kx_1 + (k + k_t)x_2 = c_t\dot{y} + k_t y \quad \text{(g)}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} - \frac{\partial \mathfrak{F}}{\partial \dot{x}_3} = 0$$

$$m_a\ddot{x}_3 + c(L-a)\dot{\theta} - c\dot{x}_1 + (c + c_t)\dot{x}_3 + k(L-a)\theta - kx_1 + (k + k_t)x_3 = c_t\dot{z} + k_t z \quad \text{(h)}$$

The equations summarized in matrix form become

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & m_s & 0 & 0 \\ 0 & 0 & m_a & 0 \\ 0 & 0 & 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} c\left[a^2 + (L-a)^2\right] & cL & -ca & c(L-a) \\ cL & 2c & -c & -c \\ -ca & -c & c + c_t & 0 \\ c(L-a) & -c & 0 & c + c_t \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} k\left[a^2 + (L-a)^2\right] & kL & -ka & k(L-a) \\ -k(L-2a) & 2k & -k & -k \\ -ka & -k & k + k_t & 0 \\ k(L-a) & -k & 0 & k + k_t \end{bmatrix}\begin{bmatrix} \theta \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c_t\dot{y} + k_t y \\ c_t\dot{z} + k_t z \end{bmatrix} \quad \text{(i)}$$

Substituting the given values into Equation (i) leads to

$$\begin{bmatrix} 225 & 0 & 0 & 0 \\ 0 & 300 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + 10^3 \begin{bmatrix} 5.5 & -0.48 & -1.56 & 2.04 \\ -0.48 & 2.4 & -1.2 & -1.2 \\ -1.56 & -1.2 & 11.2 & 0 \\ 2.04 & -1.2 & 0 & 1.12 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$+10^4 \begin{bmatrix} 5.5 & -3.60 & -1.56 & 2.04 \\ -1.08 & 2.4 & -1.2 & -1.2 \\ -1.56 & -1.2 & 1.12 & 0 \\ 2.04 & -1.2 & 0 & 1.2 \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \times 10^4 \dot{y} + 1 \times 10^5 y \\ 1 \times 10^4 \dot{z} + 1 \times 10^5 z \end{bmatrix} \quad \text{(j)}$$

7.10 FURTHER EXAMPLES

EXAMPLE 7.20

Refer to the system shown in Figure 7.24(a).

(a) Use Lagrange's equations to derive the differential equations governing the motion of the three degree-of-freedom system shown. Use x_1, x_2, and θ as generalized coordinates. Assume small displacements.

(b) Use stiffness influence coefficients to derive the stiffness matrix.

(c) Use inertia influence coefficients to derive the mass matrix.

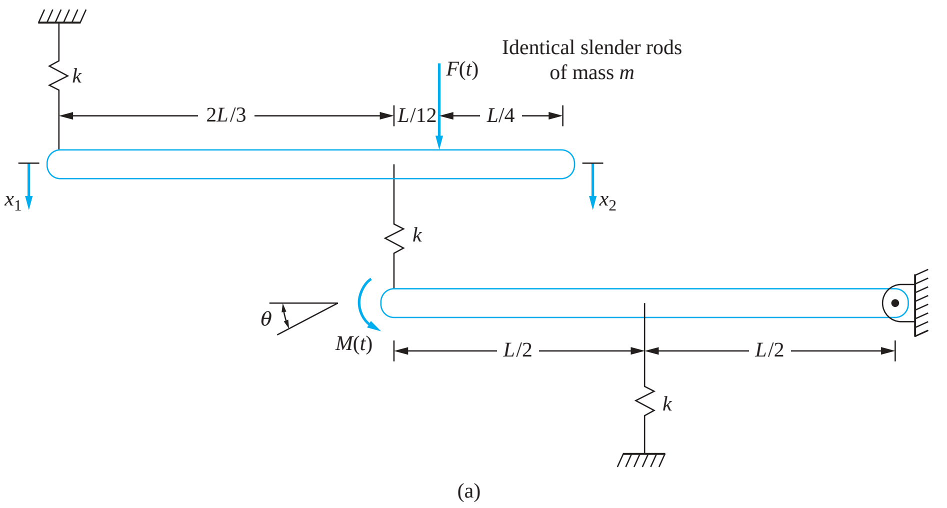


FIGURE 7.24
(a) System of Example 7.20. (b) FBDs for calculation of the first column of stiffness matrix. (c) FBDs for the second column of stiffness matrix. (d) FBDs for the third column of stiffness matrix. (e) Impulse-momentum diagrams to determine the first column of mass matrix. (f) Impulse-momentum diagrams for the second column of mass matrix. (g) Impulse-momentum diagrams for the third column of mass matrix.

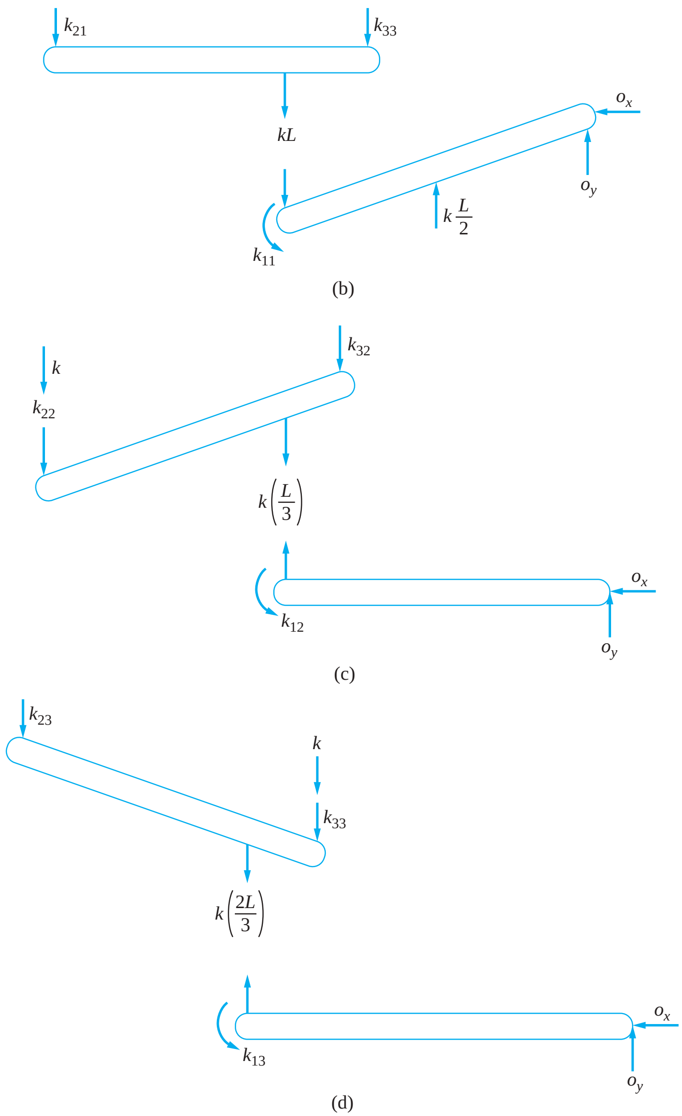


FIGURE 7.24
(Continued)

SOLUTION

(a) The system's kinetic energy at an arbitrary instant is

$$T = \frac{1}{2}m(L\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2 \qquad \text{(a)}$$

The system's potential energy at the same instant is

$$V = \frac{1}{2}k\left(\frac{L}{2}\theta\right)^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(\frac{x_1 + 2x_2}{3} - L\theta\right)^2 \qquad \text{(b)}$$

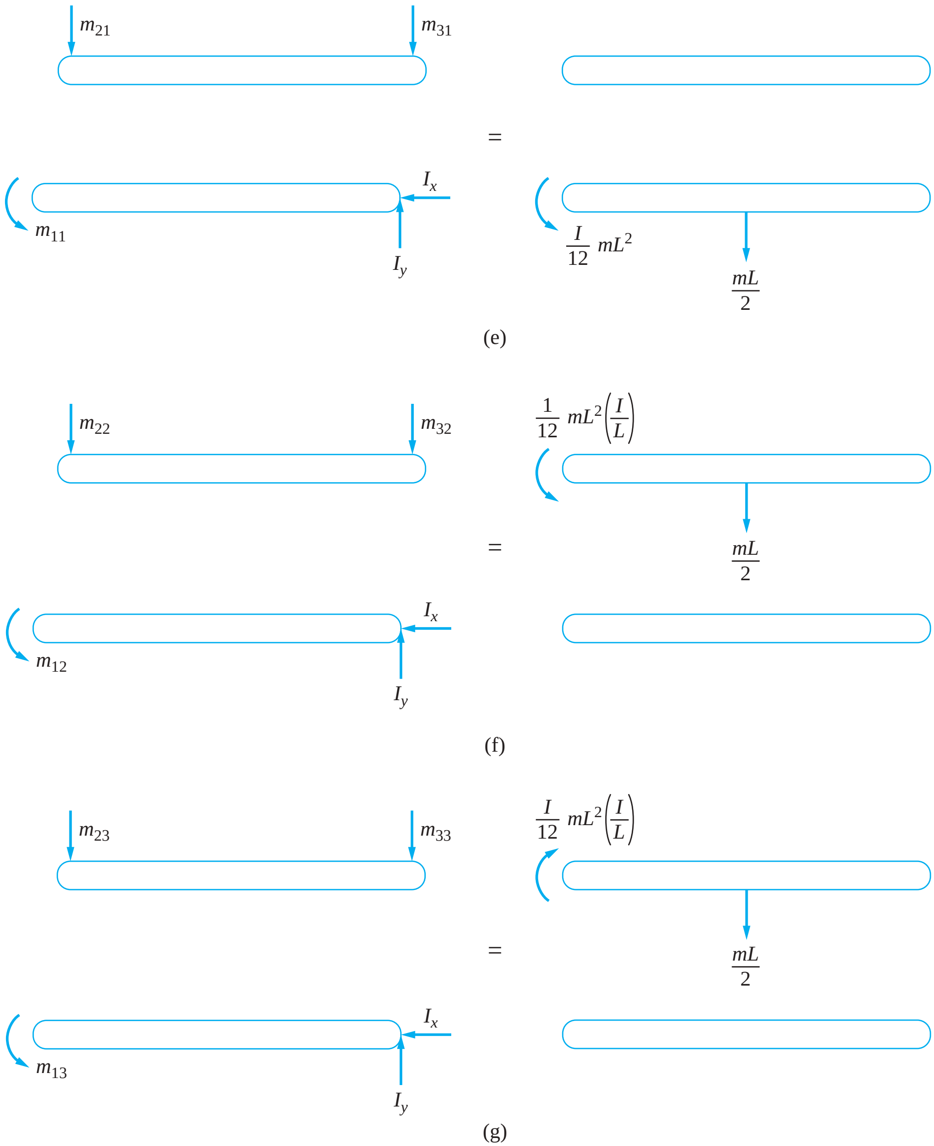


FIGURE 7.24
(Continued)

The Lagrangain becomes

$$L = T - V = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2$$

$$-\left[\frac{1}{2}k\left(\frac{L}{2}\theta\right)^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(\frac{x_1 + 2x_2}{3} - L\theta\right)^2\right] \qquad \text{(c)}$$

The method of virtual work is used to obtain the generalized forces. Assume virtual displacements $\delta\theta$, δx_1, and δx_2. The virtual work done by the external forces is

$$\delta W = M(t)\delta\theta + F(t)\left(\frac{\delta x_1 + 3\delta x_2}{4}\right) \tag{d}$$

Thus, $Q_1 = M(t)$, $Q_2 = \dfrac{F(t)}{4}$, and $Q_3 = \dfrac{3F(t)}{4}$.

Successive application of Lagrange's equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_1 \tag{e}$$

$$\frac{d}{dt}\left[\frac{1}{2}(2)\left(\frac{1}{3}mL^2\right)\dot{\theta}\right] - \left[-\frac{1}{2}(2)k\left(\frac{L}{2}\right)^2\theta - \frac{1}{2}(2)k\left(\frac{x_1 + 2x_2}{3} - L\theta\right)(-L)\right] = M(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = Q_2 \tag{f}$$

$$\frac{d}{dx}\left[\frac{1}{2}(2)m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}(2)\left(\frac{1}{12}mL^2\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)\left(-\frac{1}{L}\right)\right]$$
$$-\left[-\frac{1}{2}(2)kx_1 - \frac{1}{2}(2)k\left(\frac{x_1 + 2x_2}{3} - L\theta\right)\left(\frac{1}{3}\right)\right] = \frac{1}{4}F(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = Q_3 \tag{g}$$

$$\frac{d}{dt}\left[\frac{1}{2}(2)m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}(2)\left(\frac{1}{12}mL^2\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)\left(\frac{1}{L}\right)\right]$$
$$-\left[-\frac{1}{2}(2)k\left(\frac{x_1 + 2x_2}{3} - L\theta\right)\left(\frac{2}{3}\right)\right] = \frac{3}{4}F(t)$$

Cleaning up these equations and writing them in a matrix form gives

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & \frac{1}{3}m & \frac{1}{6}m \\ 0 & \frac{1}{6}m & \frac{1}{3}m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{5}{4}kL^2 & -\frac{1}{3}kL & -\frac{2}{3}kL \\ -\frac{1}{3}kL & \frac{10}{9}k & \frac{2}{9}k \\ -\frac{2}{3}kL & \frac{2}{9}k & \frac{4}{9}k \end{bmatrix}\begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M(t) \\ \frac{1}{4}F(t) \\ \frac{3}{4}F(t) \end{bmatrix} \tag{h}$$

(b) The differential equations are derived assuming the same displacement vector as in part (a). The first column of the stiffness matrix is obtained by setting $\theta = 1$, $x_1 = 0$, and

$x_2 = 0$, as shown in Figure 7.24(b). Summing moments using the FBD of the lower bar, $\sum M_O = 0$ yields

$$k_{11} - (kL)(L) - \left(k\frac{L}{2}\right)\left(\frac{L}{2}\right) = 0 \Rightarrow k_{11} = \frac{5kL^2}{4} \tag{i}$$

Summing moments on the FBD of the upper bar using $\sum M_2 = 0$ yields

$$k_{21}(L) + (kL)\left(\frac{L}{3}\right) = 0 \Rightarrow k_{21} = -\frac{kL}{3} \tag{j}$$

Summing moments on the FBD of the upper bar using $\sum M_1 = 0$ yields

$$k_{31}(L) + (kL)\left(\frac{2L}{3}\right) = 0 \Rightarrow k_{31} = -\frac{2kL}{3} \tag{k}$$

The second column is obtained by setting $\theta = 0,\ x_1 = 1$, and $x_2 = 0$. Summing moments on the upper bar using the FBDs of Figure 7.24(c) yields

$$(k_{22})L - (k)L - \left(\frac{k}{3}\right)\left(\frac{L}{3}\right) = 0 \Rightarrow k_{22} = \frac{10k}{9} \tag{l}$$

and

$$(k_{32})L - \frac{k}{3}\left(\frac{2L}{3}\right) = 0 \Rightarrow k_{32} = \frac{2k}{9} \tag{m}$$

The third column is obtained by setting $\theta = 0$, $x_1 = 0$, and $x_2 = 1$. Summing moments on the upper bar using the FBDs of Figure 7.24(d) yields

$$(k_{33})L - \frac{2k}{3}\left(\frac{2L}{3}\right) = 0 \Rightarrow k_{33} = \frac{4k}{9} \tag{n}$$

The remaining elements of the stiffness matrix are determined using symmetry of the stiffness matrix.

(c) The mass matrix is determined through the use of inertia influence coefficients. The first column is calculated by setting $\dot{\theta} = 1$, $\dot{x}_1 = 0$, and $\dot{x}_2 = 0$. Using the principle of angular impulse and momentum applied to the lower bar about O using impulse momentum diagrams of Figure 7.24(e) leads to

$$m_{11} = \frac{1}{12}mL^2 + \frac{mL}{2}\left(\frac{L}{2}\right) \Rightarrow m_{11} = \frac{mL^2}{3} \tag{o}$$

Applying the principle of impulse and momentum to the upper bar yields

$$m_{21} = m_{31} = 0 \tag{p}$$

The second column of the mass matrix is calculated by setting $\dot{\theta} = 0$, $\dot{x}_1 = 1$, and $\dot{x}_2 = 0$. The induced velocity of the mass center of the upper bar is one-half downward, and the induced angular velocity of the bar is $1/L$ counterclockwise. Using angular momentum about O on the lower bar of the momentum diagrams of Figure 7.24(f) leads to

$$m_{12} = 0 \tag{q}$$

Application of the principle of angular impulse and angular momentum for the upper bar about an axis through the particle whose displacement is x_2 leads to

$$m_{22}(L) = \frac{m}{2}\left(\frac{L}{2}\right) + \frac{1}{12}mL \Rightarrow m_{22} = \frac{m}{3} \tag{r}$$

Application of the principle of angular impulse and angular momentum for the upper bar about an axis through the particle whose displacement is x_1 leads to

$$m_{32}(L) = \frac{m}{2}\left(\frac{L}{2}\right) - \frac{1}{12}mL \Rightarrow m_{32} = \frac{m}{6} \tag{s}$$

The third column of the mass matrix is calculated by setting $\dot{\theta} = 0$, $\dot{x}_1 = 0$, and $\dot{x}_2 = 1$. The induced velocity of the mass center is one-half downward, and the induced angular velocity of the bar is $1/L$ clockwise. Application of the principle of angular impulse and angular momentum for the upper bar about an axis through the particle whose displacement is x_1 using the diagrams of Figure 7.24(g) leads to

$$m_{33}(L) = \frac{m}{2}\left(\frac{L}{2}\right) + \frac{1}{12}mL \Rightarrow m_{33} = \frac{m}{3} \tag{t}$$

The remaining elements of the mass matrix are determined from its symmetry.

EXAMPLE 7.21

The three degree-of-freedom model of a human hand and upper arm when squeezing a handle was first suggested in by Dong, Dong, Wu, and Rakheja. It is illustrated in Figure 7.25. Use Lagrange's equations to derive a mathematical model for the arm.

SOLUTION

The kinetic energy of the system at an arbitrary instant using the generalized coordinates indicated in Figure 7.25(b) is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}m_4\dot{y}^2 + \frac{1}{2}m_5\dot{y}^2 \tag{a}$$

The potential energy at an arbitrary instant is

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3(x_3 - x_2)^2 + \frac{1}{2}k_4(y - x_2)^2 + \frac{1}{2}k_5(y - x_3)^2 \tag{b}$$

The Lagrangian is

$$\begin{aligned} L = {} & \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}m_4\dot{y}^2 + \frac{1}{2}m_5\dot{y}^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2 \\ & - \frac{1}{2}k_3(x_3 - x_2)^2 - \frac{1}{2}k_4(y - x_2)^2 - \frac{1}{2}k_5(y - x_3)^2 \end{aligned} \tag{c}$$

Rayleigh's dissipation function is

$$\mathfrak{F} = -\frac{1}{2}c_1\dot{x}_1^2 - \frac{1}{2}c_2(\dot{x}_2 - x_1)^2 - \frac{1}{2}c_3(\dot{x}_3 - \dot{x}_2)^2 - \frac{1}{2}c_4(\dot{y} - \dot{x}_2)^2 - \frac{1}{2}c_5(\dot{y} - \dot{x}_3)^2 \tag{d}$$

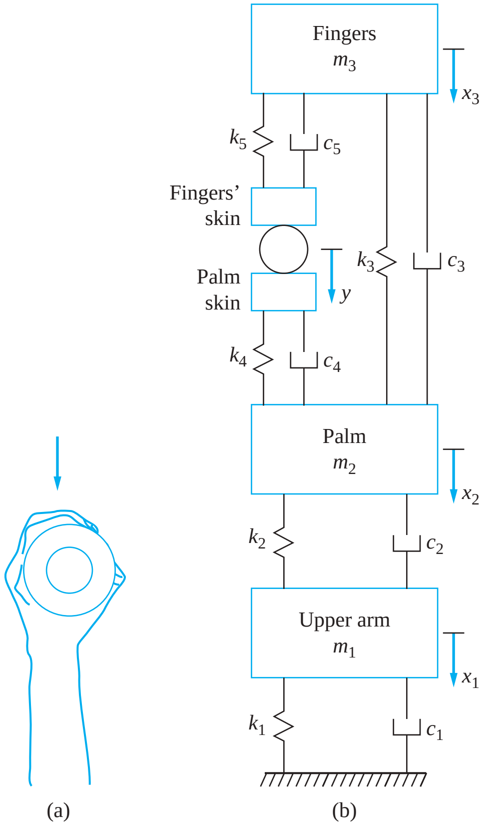


FIGURE 7.25
(a) Hand and upper arm gripping an object. (b) Three degree-of-freedom model of hand and upper arm.

Application of Lagrange's equation for x_1, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \dfrac{\partial \mathfrak{F}}{\partial \dot{x}_1} - \dfrac{\partial L}{\partial x_1} = 0$ yields

$$\frac{d}{dx}(m_1\dot{x}_1) - [-c_1\dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1)(-1)] - [-k_1x_1 - k_2(x_2 - x_1)(-1)] = 0 \tag{e}$$

Application of Lagrange's equation for x_2, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) - \dfrac{\partial \mathfrak{F}}{\partial \dot{x}_2} - \dfrac{\partial L}{\partial x_2} = 0$ yields

$$\begin{aligned}\frac{d}{dx}(m_2\dot{x}_2) &- [-c_2(\dot{x}_2 - \dot{x}_1) - c_3(\dot{x}_3 - \dot{x}_2)(-1) c_4(\dot{y} - \dot{x}_2)(-1)] \\ &- [-k_2(x_2 - x_1) - k_3(x_3 - x_2)(-1) - k_4(y - x_2)(-1)] = 0\end{aligned} \tag{f}$$

Application of Lagrange's equation for x_3, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_3}\right) - \dfrac{\partial \mathfrak{F}}{\partial \dot{x}_3} - \dfrac{\partial L}{\partial x_3} = 0$ yields

$$\begin{aligned}\frac{d}{dx}(m_3\dot{x}_3) &- [-c_3(\dot{x}_3 - x_2) - c_5(\dot{y} - \dot{x}_3)(-1)] - [-k_3(x_3 - x_2) \\ &- k_5(y - x_3)(-1)] = 0\end{aligned} \tag{g}$$

The differential equations are written in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & -0 \\ -c_2 & c_2 + c_3 + c_4 & -c_3 \\ 0 & -c_3 & c_3 + c_5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_4 & -k_3 \\ 0 & -k_3 & k_3 + k_5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ c_4\dot{y} + k_4 y \\ c_5\dot{y} + k_5 y \end{bmatrix} \quad \text{(h)}$$

EXAMPLE 7.22

To study the instability of a missile as it flies, it is modeled as a free-free beam. For ease of modeling, a four degree-of-freedom model is used as shown in Figure 7.26(a). The beam is divided as shown and the masses are lumped as shown. Determine the differential equations for governing the four degree-of-freedom model.

SOLUTION

The flexibility matrix for this unrestrained system does not exist; therefore, we use the stiffness matrix in the modeling. Stiffness influence coefficients are used to develop the stiffness matrix. Consider the deflection of the beam due to concentrated loads applied at $z = 0,\ L/3,\ 2L/3$, and L, as shown in Figure 7.26(b). The deflection of a beam due to this series of concentrated loads is

$$w(z) = \frac{1}{EI}\left[\frac{1}{6}F_1 z^3 + \frac{1}{6}F_2\left(z - \frac{L}{3}\right)^3 u\left(z - \frac{L}{3}\right) + \frac{1}{6}F_3\left(z - \frac{2L}{3}\right)^3 u\left(z - \frac{2L}{3}\right)\right.$$

$$\left. + \frac{1}{6}F_4(z - L)^3 u(z - L) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4\right] \quad \text{(a)}$$

Requiring that $\omega''(0) = 0$ gives $C_2 = 0$. Requiring that $\omega'''(0) = \frac{F_1}{EI}$ leads to $C_1 = 0$. The system is in static equilibrium; thus, $\sum F = 0$, or using the FBD of Figure 7.27(c) yields

$$F_1 + F_2 + F_3 + F_4 = 0 \quad \text{(b)}$$

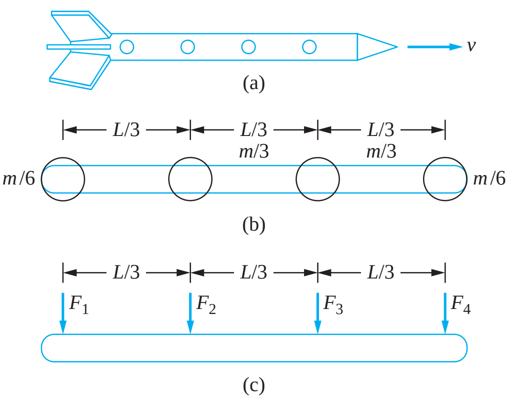


FIGURE 7.26
(a) Missile is modeled as a free-free beam. (b) Four degree-of-freedom model of missile with concentrated masses placed along span of beam. (c) Forces are used to determine the stiffness matrix; since the system is unrestrained, statics must first be used to obtain relations between the forces.

and $\sum M = 0$ about any axis. Choose an axis through $x = L$,

$$F_1 L + F_2 \frac{2L}{3} + F_3 \frac{L}{3} = 0 \tag{c}$$

Solving for F_1 and F_4 from Equations (b) and (c) leads to

$$F_1 = -\frac{2}{3}F_2 - \frac{1}{3}F_3 \tag{d}$$

$$F_4 = -\frac{1}{3}F_2 - \frac{2}{3}F_3 \tag{e}$$

Substituting Equations (d) and (e) into Equation (a) leads to

$$w(z) = \frac{1}{EI}\left[\frac{1}{6}\left(-\frac{2}{3}F_2 - \frac{1}{3}F_3\right)z^3 + \frac{1}{6}F_2\left(z - \frac{L}{3}\right)^3 u\left(z - \frac{L}{3}\right) + \frac{1}{6}F_3\left(z - \frac{2L}{3}\right)^3 u\left(z - \frac{2L}{3}\right)\right.$$
$$\left. + \frac{1}{6}\left(-\frac{5}{3}F_2 - \frac{4}{3}F_3\right)(z - L)^3 u(z - L) + C_3 z + C_4\right] \tag{f}$$

The constants C_3 and C_4 cannot be solved by application of statics or boundary conditions.

The deflections at the points where the forces are applied are

$$x_1 = w(0) = \frac{1}{EI}[C_4] \tag{g}$$

$$x_2 = w\left(\frac{L}{3}\right) = \frac{1}{EI}\left[\frac{1}{6}F_1\left(\frac{L^3}{27}\right) + C_3\left(\frac{L}{3}\right) + C_4\right] \tag{h}$$

$$x_3 = w\left(\frac{2L}{3}\right) = \frac{1}{EI}\left[\frac{1}{6}F_1\left(\frac{8L^3}{27}\right) + \frac{1}{6}F_2\left(\frac{L^3}{27}\right) + C_3\left(\frac{2L}{3}\right) + C_4\right] \tag{i}$$

and

$$x_4 = w(L) = \frac{1}{EI}\left[\frac{1}{6}F_1(L^3) + \frac{1}{6}F_2\left(\frac{8L^3}{27}\right) + \frac{1}{6}F_3\left(\frac{L^3}{27}\right) + C_3(L) + C_4\right] \tag{j}$$

The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$. Substitute Equation (e) into Equations (h) through (j). Solve the resulting equations for F_2, F_3, C_3, and C_4. Substitute into Equations (d) and (e) to find F_1 and F_4. The second column of the stiffness matrix is obtained by setting $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and $x_4 = 0$ and repeating the same procedure. The third column is obtained by setting $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 0$ and repeating the procedure. The fourth column is obtained by setting $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 1$. The stiffness matrix must be symmetric. The result is

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 43.2 & -97.2 & 64.8 & -10.8 \\ -97.2 & 259.2 & -226.8 & 64.8 \\ 64.8 & -226.8 & 259.2 & -97.2 \\ -10.8 & 64.8 & 259.2 & 43.2 \end{bmatrix} \tag{k}$$

The mass matrix is obtained by the methods of Section 7.7. resulting in

$$\mathbf{M} = \frac{m_b}{6}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{l}$$

where m_b is the total mass of the beam.

The differential equations governing the displacements of the lumped masses are

$$\frac{m_b}{6}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 43.2 & -97.2 & 64.8 & -10.8 \\ -97.2 & 259.2 & -226.8 & 64.8 \\ 64.8 & -226.8 & 259.2 & -97.2 \\ -10.8 & 64.8 & 259.2 & 43.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{m}$$

7.11 SUMMARY

7.11.1 IMPORTANT CONCEPTS

- The FBD method can be used to derive the governing differential equations of a MDOF system.
- Lagrange's equations provide an alternative method for deriving differential equation for a MDOF system.
- Lagrange's equations is based upon the calculus of variations. The kinetic energy and the potential energy are calculated at an arbitrary instant in terms of the generalized coordinates.
- The Lagrangian is the difference between kinetic and potential energies written at an arbitrary instant.
- Rayleigh's dissipation function is the power dissipated by viscous damping forces, written at an arbitrary instant.
- The method of virtual work is used to calculate the generalized forces.
- The kinetic energy, the potential energy, and Rayleigh's dissipation function all have quadratic forms for linear systems.
- The mass matrix, stiffness matrix, and damping matrix can be directly calculated from the quadratic forms.
- The mass matrix, damping matrix, and stiffness matrix are all symmetric when Lagrange's equations are used to derive the differential equations.
- When the mass matrix is not a diagonal matrix, the system is said to be dynamically coupled. When the stiffness matrix is not a diagonal matrix, the system is said to be statically coupled.

- The stiffness matrix also may be calculated using stiffness influence coefficients. One column of the stiffness matrix is calculated at a time. If the ith column is being calculated, a unit displacement is assumed for the particle whose displacement is represented by the generalized coordinate x_i with the displacements of the particles whose displacements are represented by x_j for $j = 1, 2, \ldots, n$, but $j \neq i$ set equal to zero. The stiffness influence coefficients are the forces required to maintain this in static equilibrium.
- The flexibility matrix is the inverse of the stiffness matrix. The differential equations can be written using the flexibility matrix.
- The flexibility matrix can be calculated using flexibility influence coefficients. One column of the flexibility matrix is calculated at a time. To calculate the ith column of the flexibility matrix, a unit force is applied at the location described by the generalized coordinate x_i. The flexibility influence coefficients are the displacements at the locations described by the generalized coordinates.
- The flexibility matrix does not exist for unrestrained systems.
- Inertia influence coefficients can be used to calculate the mass matrix. Assume a unit velocity for the ith generalized coordinate $\dot{x}_i = 1$ and all other velocities zero as $\dot{x}_j = 0$ for $j \neq i$. Calculate the system of impulses that would have to be applied to achieve this configuration. These impulses are the ith column of the mass matrix.
- Continuous systems may be modeled as MDOF systems. Flexibility influence coefficients are used to determine the flexibility matrix for a lumped mass model.

7.11.2 IMPORTANT EQUATIONS

Hamilton's Principle

$$\delta \int_{t_1}^{t_2} (T - V + \delta W_{\text{nc}})dt = 0 \tag{7.6}$$

Lagrangian

$$L = T - V \tag{7.7}$$

Lagrange's equations for a conservative system

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial \dot{x}_i} = 0 \qquad i = 1, 2, \ldots, n \tag{7.10}$$

Lagrange's equations for a nonconservative system

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i \qquad i = 1, 2, \ldots, n \tag{7.11}$$

Virtual work by non-conservative forces

$$\delta W_{\text{nc}} = \sum_{i=1}^{n} Q_i \delta x_i \tag{7.12}$$

Rayleigh's dissipation function

$$\mathfrak{F} = -\frac{1}{2}P \tag{7.13}$$

Quadratic forms of potential and kinetic energies

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j \tag{7.21}$$

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{x}_i\dot{x}_j \tag{7.22}$$

Differential equations for a linear system written in matrix form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.31}$$

Quadratic form of Rayleigh's dissipation function

$$\mathfrak{I} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}\dot{x}_i\dot{x}_j \tag{7.32}$$

Flexibility matrix

$$\mathbf{A} = \mathbf{K}^{-1} \tag{7.43}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 7.1 through 7.15, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

7.1 The differential equations for a linear MDOF system can be written in a matrix form.

7.2 Lagrange's equations can be used to derive the differential equations governing the motion only for linear systems.

7.3 Lagrange's equations can be used for conservative systems and nonconservative systems.

7.4 The FBD method, when applied to a MDOF linear system, always leads to symmetric mass, stiffness, and damping matrices.

7.5 Lagrange's equations, when applied to a MDOF linear system, always leads to symmetric mass, stiffness, and damping matrices.

7.6 The quadratic form of the potential energy can be used to determine the stiffness matrix for a linear MDOF system.

7.7 A system is dynamically coupled if the mass matrix for the system is not symmetric.

7.8 The choice of generalized coordinates is irrelevant in deciding whether a system is dynamically coupled.

7.9 The flexibility matrix is the transpose of the stiffness matrix.

7.10 A diagonal stiffness matrix means that $k_{ij} = k_{ji}$ for all $i, j = 1, 2, \ldots, n$.

7.11 Elements of the mass matrix for a MDOF system may have different dimensions.

7.12 The formulation of the stiffness influence coefficient method to determine the stiffness matrix for a linear MDOF system relies on the concept that potential energy is a function of position.

7.13 When flexibility influence coefficients are used to calculate the flexibility matrix for a MDOF system, the flexibility matrix is calculated one column at a time.

7.14 The stiffness matrix for a system always exists but the flexibility matrix does not always exist.

7.15 A system is not statically coupled if its flexibility matrix is a diagonal matrix.

7.16 Lagrange's equations can be used to derive the equations governing the vibrations of three masses along the span of a beam ignoring the inertia of the beam and using three degrees of freedom in the model.

Problems 7.17 through 7.28 require a short answer.

7.17 Write the general matrix form of the differential equations governing the undamped and forced vibrations of a linear *n*DOF system.

7.18 State Lagrange's equations for a conservative system.

7.19 What defines whether a system is dynamically coupled?

7.20 How is Rayleigh's dissipation function used?

7.21 What is a variation?

7.22 How is the method of virtual work applied in the application of Lagrange's equations for a MDOF system?

7.23 What is Maxwell's reciprocity relation and how is it applied?

7.24 Write the differential equations governing a MDOF system in matrix form when the mass matrix, damping matrix, and flexibility matrix are known.

For Problems 7.25 through 7.28, the generalized coordinates for modeling a system have been selected as x_1, x_2, and θ where x_1 and x_2 are linear displacements and θ is an angular coordinate.

7.25 Describe the calculation of the stiffness influence coefficient k_{13}.

7.26 Describe the calculation of the flexibility influence coefficient a_{13}.

7.27 Describe the calculation of the inertia influence coefficient m_{12}.

7.28 Describe the calculation of the inertia influence coefficient m_{31}.

Problems 7.29 through 7.41 require a short calculation.

7.29 What is the kinetic energy of the system of Figure SP7.29 at an arbitrary instant?

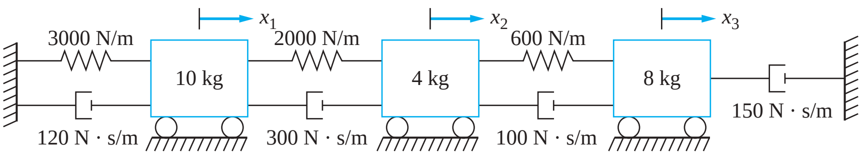


FIGURE SP7.29

7.30 What is the potential energy in the system of Figure SP7.29 at an arbitrary instant?

7.31 What is Rayleigh's dissipation function for the system of Figure SP7.28 at an arbitrary instant?

7.32 What is the result of

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(2\dot{x} - \dot{y})^2\right]$$

7.33 What is virtual work done by the external forces in Figure SP7.33, assuming virtual displacements δx and δy?

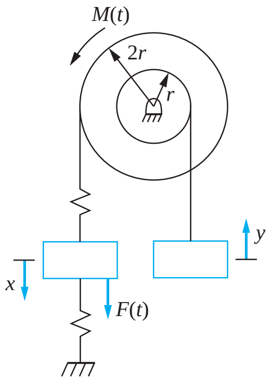


FIGURE SP7.33

7.34 What are the generalized forces for the system of Figure SP7.34 using x and θ as generalized coordinates?

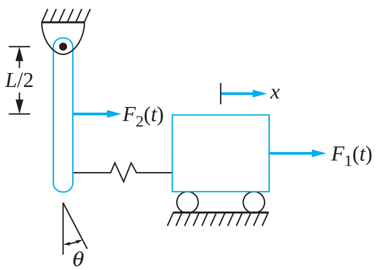


FIGURE SP7.34

7.35 The quadratic form of the potential energy for a three degree-of-freedom system is

$$V = 5x_1^2 + 4x_1x_2 + 2x_1x_3 + 8x_2^2 + 3x_2x_2 + 6x_3^2$$

Determine the stiffness matrix for the system.

7.36 The kinetic energy for a three degree-of-freedom system is

$$T = 3\left(\dot{x}_2 - \frac{1}{2}\dot{x}_1\right)^2 + 12\left(\dot{x}_2 + \frac{1}{3}\dot{x}_1\right)^2 + 4\dot{x}_3^2$$

Determine the mass matrix for the system.

7.37 When a load of 50 N is applied to the 250 kg mass in the system of Figure SP7.37, the displacements of the masses are $x_1 = 3$ mm, $x_2 = 5$ mm, and $x_3 = 2.5$ mm. Determine all possible elements of the system's flexibility matrix.

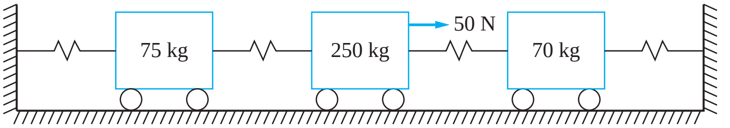


FIGURE SP7.37

7.38 When the block of mass 10 kg is given a displacement of 3 mm in the system of Figure SP7.38 and all other blocks are held in their equilibrium positions, it is found that the forces on the blocks are $F_1 = 0$, $F_2 = 100$ N, and $F_3 = 300$ N. Determine all possible elements of the system's stiffness matrix.

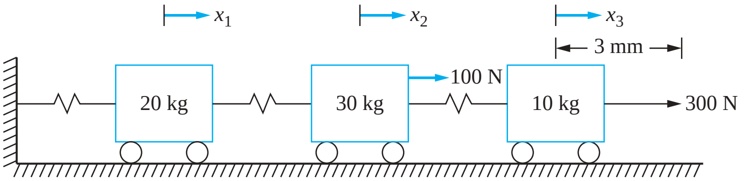


FIGURE SP7.38

7.39 What is the determinant of the stiffness matrix of the system of Figure SP7.39?

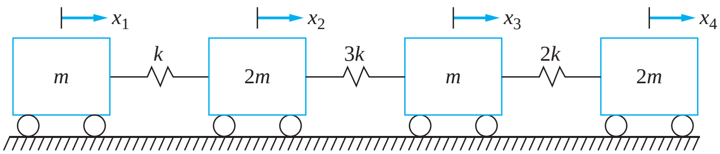


FIGURE SP7.39

7.40 When block A of Figure SP7.40 is given a velocity of 15 m/s and the velocities of blocks B and C remain at rest, an impulse of 3 N · s applied to block A is required. Determine all possible elements of the system's mass matrix.

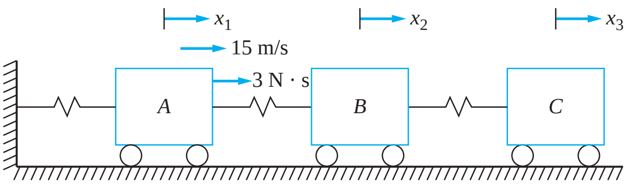


FIGURE SP7.40

7.41 When the right end of the bar of the system of Figure SP7.41 is given a velocity of 3 m/s but the angular velocity of the bar is zero, an impulse of magnitude 6 N · s is required at the right end of the bar and an angular impulse of 10 N · m · s is required. Determine all possible elements of the mass matrix for this two degree-of-freedom system using x, which is the displacement of the right end of the bar, and θ, which is the angular rotation of the mass center of the bar, as generalized coordinates.

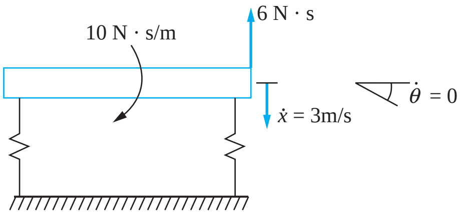


FIGURE SP7.41

7.42 Lagrange's equations are used to derive the differential equations for a three degree-of-freedom system resulting in

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

where x_1 and x_2 are linear displacements and θ is an angular coordinate. Match the term in the equation with its units. Some units may be used more than once, others not at all.

(a)	m_{11}	(i)	N · s/m
(b)	m_{23}	(ii)	N/m
(c)	m_{33}	(iii)	m
(d)	c_{12}	(iv)	kg
(e)	c_{22}	(v)	N · s · m/rad
(f)	c_{33}	(vi)	N · m/rad
(g)	k_{13}	(vii)	rad/s^2
(h)	k_{21}	(viii)	N/rad
(i)	k_{33}	(ix)	N
(j)	F_2	(x)	kg · m^2
(k)	F_3	(xi)	N · m
(l)	x_2	(xii)	N · s/rad
(m)	$\dot{x}_1$	(xiii)	m/s
(n)	$\ddot{x}_3$	(xiv)	N · s^2/m
		(xv)	kg · m

CHAPTER PROBLEMS

7.1–7.7 Use the free-body diagram method to derive the differential equations governing the motion of the systems shown in Figures P7.1 through P7.7 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

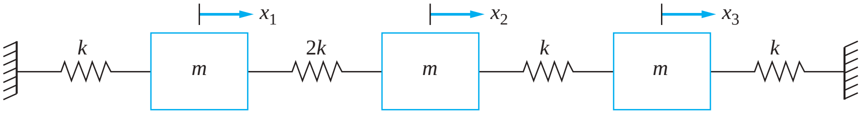


FIGURE P7.1
(Problems 7.1, 7.8, 7.23, 7.30, 7.36, 7.51, 7.66)

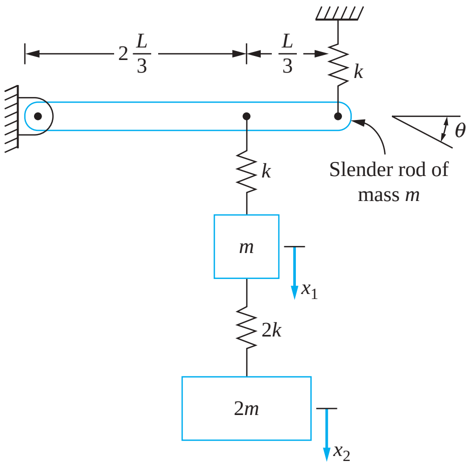


FIGURE P7.2
(Problems 7.2, 7.9, 7.31, 7.37, 7.52, 7.67)

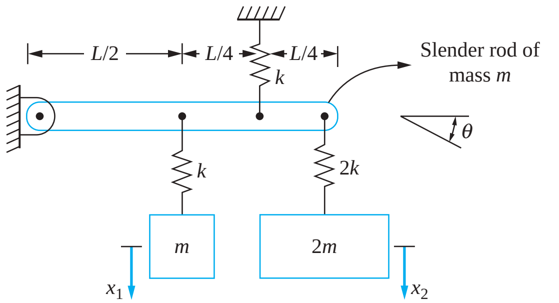


FIGURE P7.3
(Problems 7.3, 7.10, 7.24, 7.38, 7.53, 7.68)

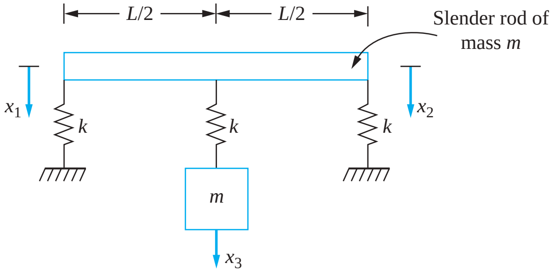


FIGURE P7.4
(Problems 7.4, 7.11, 7.25, 7.39, 7.54, 7.69)

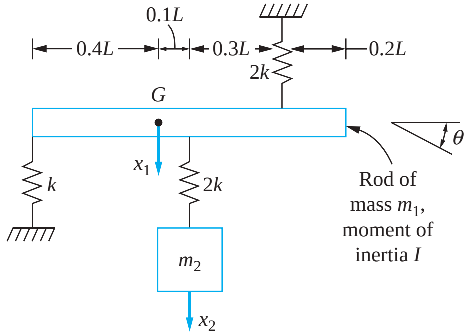


FIGURE P7.5
(Problems 7.5, 7.12, 7.26, 7.40, 7.55, 7.70)

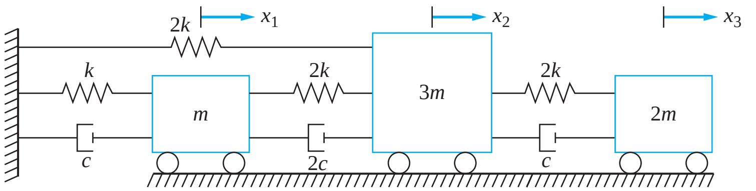


FIGURE P7.6
(Problems 7.6, 7.13, 7.41, 7.56, 7.71)

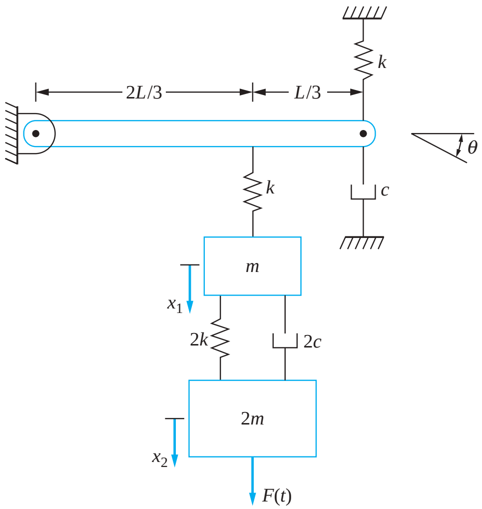
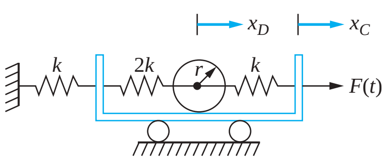


FIGURE P7.7
(Problems 7.7, 7.14, 7.42, 7.57, 7.72)

7.8–7.14 Use Lagrange's equations to derive the differential equations governing the motion of the systems shown in Figures P7.1 through P7.7. Use the indicated generalized coordinates. Make linearizing assumtions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

7.15–7.22 Use Lagrange's equations to drive the differential equations governing the motion of the systems shown in Figures P7.15 through P7.22. Use the indicated generalized coordinates. Make linearizing assumptions, and write the resulting equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.



Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk.

FIGURE P7.15
(Problems 7.15, 7.27, 7.32, 7.43, 7.58, 7.73)

Thin disk of mass m and radius r rolls without slip relative to center of cart. x_D is absolute displacement of mass center of disk.

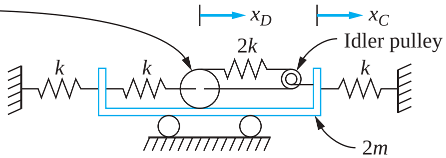


FIGURE P7.16
(Problems 7.16, 7.33, 7.44, 7.59, 7.74)

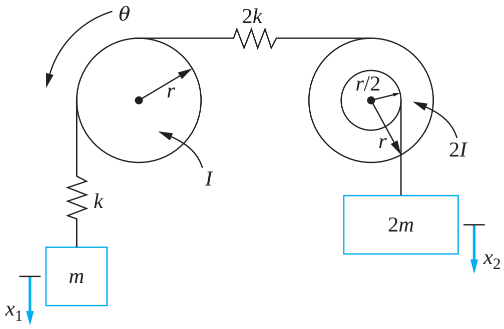


FIGURE P7.17
(Problems 7.17, 7.45, 7.60, 7.75)

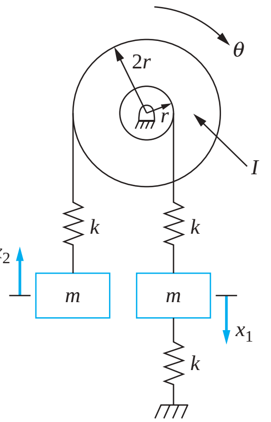


FIGURE P7.18
(Problems 7.18, 7.46, 7.61, 7.76)

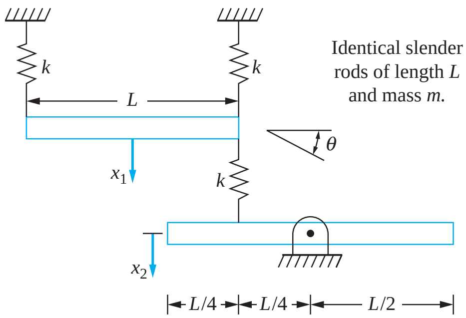


FIGURE P7.19
(Problems 7.19, 7.28, 7.34, 7.47, 7.62, 7.77)

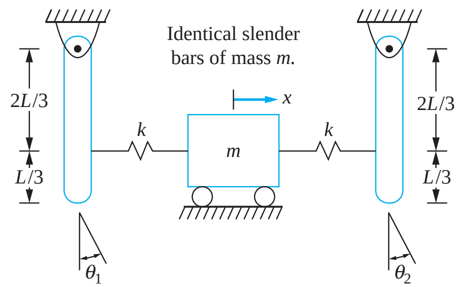


FIGURE P7.20
(Problems 7.20, 7.35, 7.48, 7.63, 7.78)

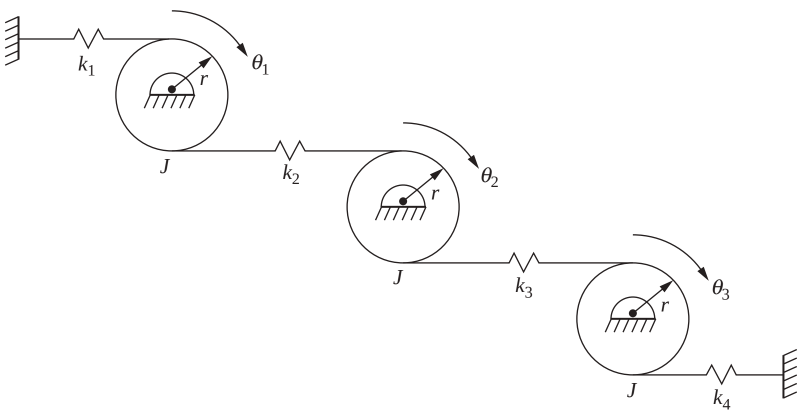


FIGURE P7.21
(Problems 7.21, 7.49, 7.64, 7.79)

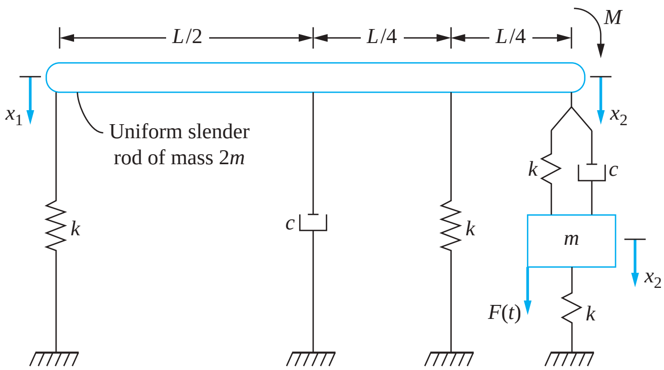


FIGURE P7.22
(Problems 7.22, 7.29, 7.50, 7.65, 7.80)

7.23–7.29 Determine the kinetic energy of the system at an arbitrary instant for the systems of Figures P7.1, P7.3, P7.4, P7.5, P7.15, P7.19, and P7.22. Put the kinetic energy in a quadratic form. Use the quadratic form to determine the mass matrix for the system.

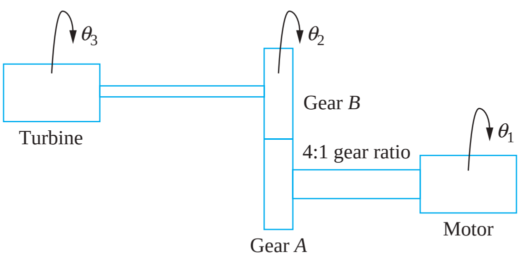
7.30–7.35 Determine the potential energy of the system at an arbitrary instant for the systems of Figures P7.1, P7.2, P7.15, P7.16, P7.19, and P7.20. Put the potential energy in a quadratic form. Use the quadratic form to determine the stiffness matrix for the system.

7.36–7.50 Derive the stiffness matrix for the systems of Figures P7.1, P7.2, P7.3, P7.4, P7.5, P7.6, P7.7, P7.15, P7.16, P7.17, P7.18, P7.19, P7.20, P7.21, and P7.22 using the indicated generalized coordinates and stiffness influence coefficients.

7.51–7.65 Determine the flexibility matrix for the systems of Figures P7.1, P7.2, P7.3, P7.4, P7.5, P7.6, P7.7, P7.15, P7.16, P7.17, P7.18, P7.19, P7.20, P7.21, and P7.22 using the indicated generalized coordinates and flexibility influence coefficients.

7.66–7.80 Determine the mass matrix for the systems of Figures P7.1, P7.2, P7.3, P7.4, P7.5, P7.6, P7.7, P7.15, P7.16, P7.17, P7.18, P7.19, P7.20, P7.21, and P7.22 using the indicated generalized coordinates and inertia influence coefficients.

7.81 Derive the differential equations governing the torsional oscillations of the turbomotor of Figure P7.81. The motor operates at 800 rpm and the turbine shaft turns at 3200 rpm.



Moments of inertia:	Turbine shaft	Motor shaft
Motor 1800 kg · m^2	$G = 80 \quad 10^9$ N/m^2	$G = 80 \quad 10^9$ N/m^2
Turbine 600 kg · m^2	$L = 2.1$ m	$L = 1.4$ m
Gear A 400 kg · m^2	$d = 180$ mm	$d = 305$ mm
Gear B 80 kg · m^2		

FIGURE P7.81

7.82 Derive the differential equations governing the torsional oscillations of the system of Figure P7.82.

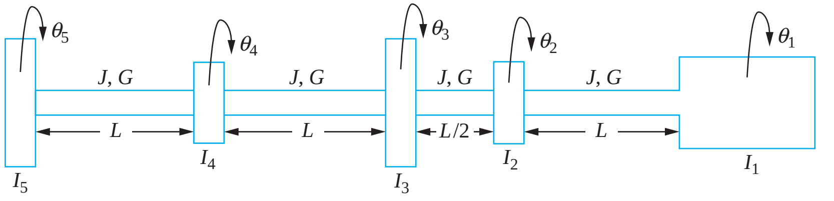


FIGURE P7.82

7.83 A rotor of mass m is mounted on an elastic shaft with journal bearings at both ends. A three degree-of-freedom model of the system is shown in Figure P7.83. Each journal bearing is modeled as a spring in parallel with a viscous damper. Drive the differential equations governing the transverse motion of the system.

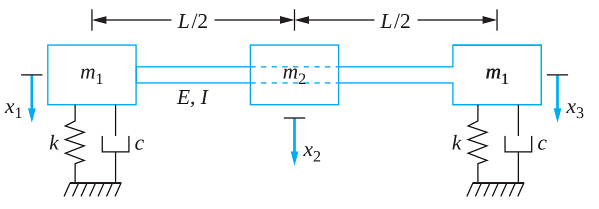


FIGURE P7.83

7.84 A three degree-of-freedom model of a railroad bridge is shown in Figure P7.84. The bridge is composed of three rigid spans. Each span is pinned at its base. Using the angular displacements of the spans as generalized coordinates, derive the differential equations governing the motion of the bridge.

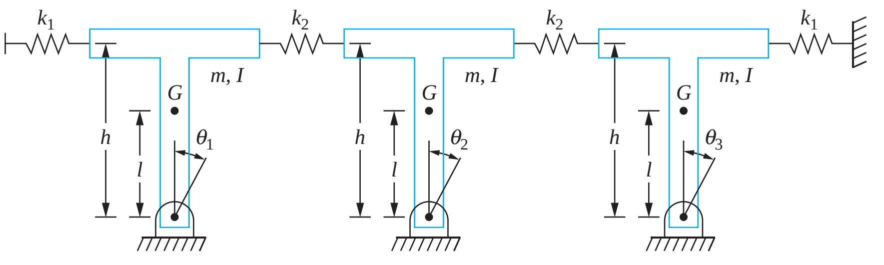


FIGURE P7.84

7.85 A five-degree of model of a railroad bridge is shown in Figure P7.85. The bridge is composed of five rigid spans. The connection between each span and its base is modeled as a torsional spring. Using the angular displacements of the spans as the generalized coordinates, derive the differential equations governing the motion of the bridge.

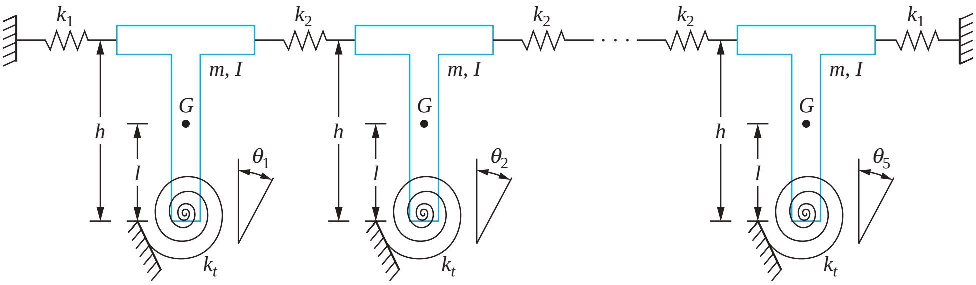


FIGURE P7.85

7.86 A four degree-of-freedom model of an aircraft wing is shown in Figure P7.86. Derive the flexibility matrix for the model.

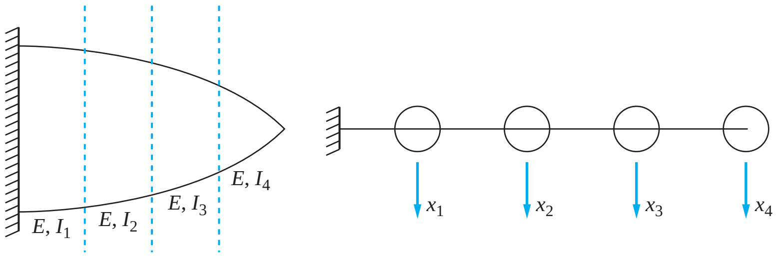


FIGURE P7.86

7.87 Figure P7.87 illustrates a three degree-of-freedom model of an aircraft. A rigid fuselage is attached to two thin flexible wings. An engine is attached to each wing, but the wings themselves are of negligible mass. Derive the differential equations governing the motion of the system.

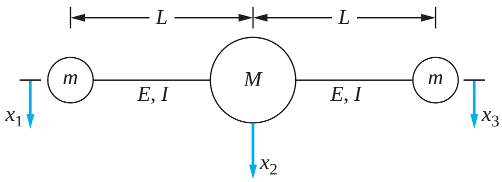


FIGURE P7.87

7.88 An airplane is modeled as two flexible wings attached to a rigid fuselage (Figure P7.88). Use two degrees of freedom to model each wing and derive the differential equations governing the motion of the five degree-of-freedom system.

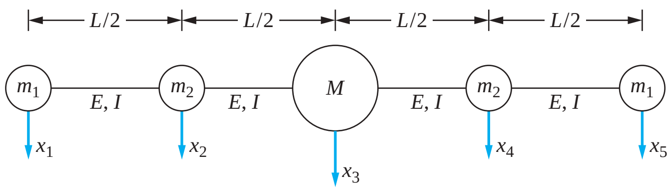


FIGURE P7.88

7.89 A drum of mass m is being hoisted by an overhead crane as illustrated in Figure P7.89. The crane is modeled as a simply supported beam with a winch at its midspan. The cable connecting the crane to the drum is of stiffness k. Derive the differential equations governing the motion of the system using four degrees of freedom to model the system, three degrees of freedom for the beam and one for the displacement of the load.

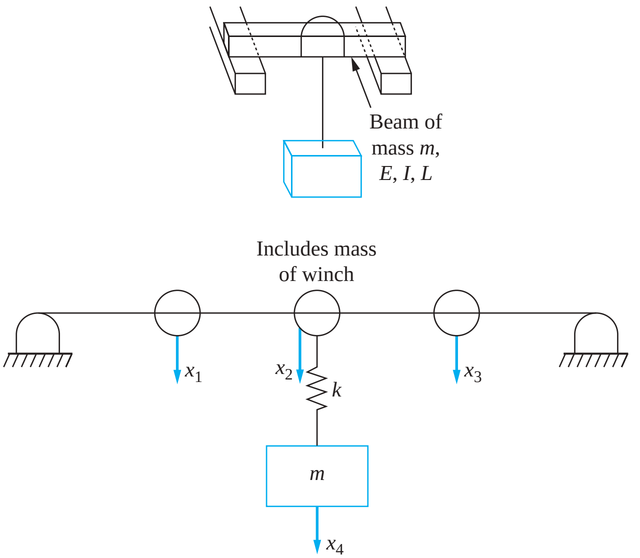


FIGURE P7.89

7.90–7.93 The beams shown in Figures P7.90 through P7.93 are made of an elastic material of elastic modulus 210×10^9 N/m^2 and have a cross-sectional moment of inertia 1.3×10^{-5} m^4. Determine the flexibility matrix when a three degree-of-freedom model is used to analyze the beam's vibrations. Use the displacements of the particles shown as generalized coordinates. Use Table D.2 for deflection calculations.

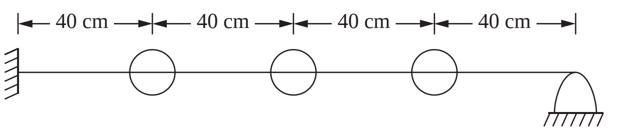


FIGURE P7.90

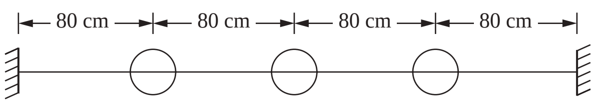


FIGURE P7.91

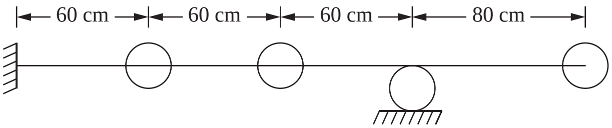


FIGURE P7.92

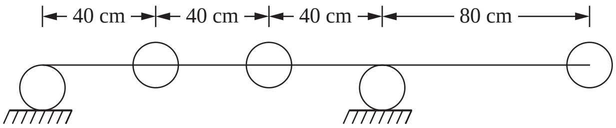


FIGURE P7.93

7.94 Determine the stiffness matrix for the three degree-of-freedom model of the free-free beam of Figure P7.94.

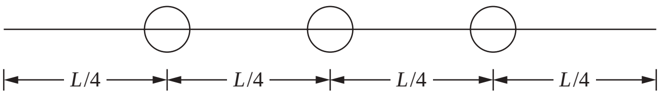


FIGURE P7.94

7.95 Using a two degree-of-freedom model, derive the differential equations governing the forced vibration of the system of Figure P7.95.

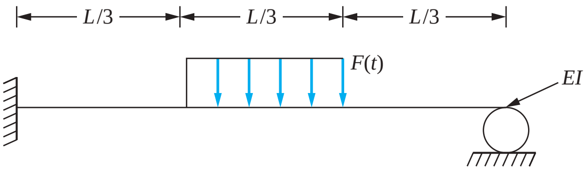


FIGURE P7.95

7.96 Use a two degree-of-freedom model to derive the differential equations governing the motion of the system of Figure P7.96. A thin disk of mass moment of inertia I_D is attached to the end of the fixed-free beam. Use x, the vertical displacement of the disk, and θ, the slope of the end of the beam, as generalized coordinates.

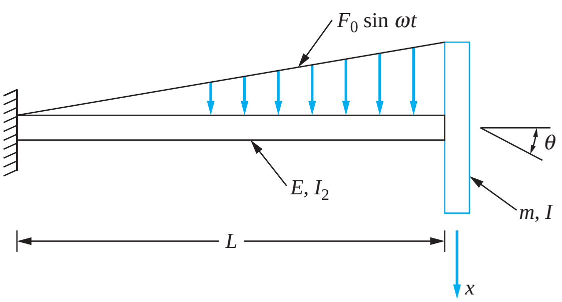


FIGURE P7.96

Chapter 8

FREE VIBRATIONS OF MDOF SYSTEMS

8.1 INTRODUCTION

Free vibrations of an n degree-of-freedom (nDOF) system are governed by a system of n differential equations. If the system is linear, the differential equations can be summarized in matrix form. When the differential equations are derived using Lagrange's equations, the mass, stiffness, and damping matrices are guaranteed to be symmetric. It is assumed that, whatever method is used to derive the differential equations for a linear system, they can be summarized in a matrix form, which for free vibrations is either

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{8.1}$$

or

$$\mathbf{A}\mathbf{M}\ddot{\mathbf{x}} + \mathbf{A}\mathbf{C}\dot{\mathbf{x}} + \mathbf{x} = \mathbf{0} \tag{8.2}$$

The free response of an nDOF system is more complicated than the free response of a one or two degree-of-freedom system. Computation of the response requires matrix algebra. A reader unfamiliar with topics in matrix algebra (such as eigenvalues and eigenvectors) is encouraged to read Appendix C before proceeding.

For an undamped system, the response of a MDOF system is assumed to be synchronous; the particles represented by the generalized coordinates move with the same frequency. This leads to a normal-mode solution in which a mode shape vector provides the relation between the generalized coordinates. The time dependence of the response is expressed by an exponential with a complex exponent equal to $i\omega t$. When the normal mode solution is substituted into the differential equations governing the undamped free response, the natural frequencies are shown to be the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ or the reciprocals of the square roots of the eigenvalues of $\mathbf{AM}$. The mode-shape vectors are the corresponding eigenvectors. An nDOF system has n natural frequencies.

The general free response is a linear combination of all modes in the solution. The constants in the linear combination are determined from the initial conditions, the values of the generalized coordinates at $t = 0$, and their velocities at $t = 0$. There are $2n$ initial conditions required.

Two special cases are considered. When the system is *unrestrained*, it has its lowest natural frequency equal to zero, which corresponds to a rigid-body movement of the system. In *degenerate* systems, two natural frequencies of the system are equal.

If the equations are derived using Lagrange's equations or any method that is derived from Lagrange's equations, the mass matrix and the stiffness matrix are guaranteed to be symmetric. This implies that a kinetic-energy scalar product and a potential-energy scalar product can be defined. This leads to showing that all eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real, all eigenvalues are non-negative, and an orthogonality condition exists for eigenvectors corresponding to distinct natural frequencies of the same system. Also, an expansion theorem is developed for representing a vector by the eigenvectors of a MDOF system.

Any multiple of an eigenvector is also an eigenvector corresponding to the same eigenvalue. The normalized mode-shape vector is defined such that the kinetic-energy scalar product of the vector with itself is one. This has an implication for the potential-energy scalar product of a vector with itself.

Principal coordinates are defined as coordinates which uncouple the differential equations. A method is presented for determination of principal coordinates for a MDOF system.

Rayleigh's quotient provides a method for approximation of the lowest natural frequency of a MDOF system. Numerical methods are presented for determination of the natural frequencies and their mode shapes.

Damping is addressed for MDOF systems. Systems that have *proportional damping* (where the damping matrix is a linear combination of the stiffness matrix and the mass matrix) are uncoupled using the same principal coordinates as the corresponding undamped system. Natural frequencies and modal damping ratios are defined. General viscous damping is considered by rewriting the n second-order differential equations as $2n$ first-order differential equations.

8.2 NORMAL-MODE SOLUTION

The general formulation of the differential equations governing free vibrations of a linear undamped n-degree-of-freedom system is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{8.3}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the symmetric $n \times n$ mass and stiffness matrices, respectively, and $\mathbf{x}$ is the n-dimensional column vector of generalized coordinates.

Free vibrations of a MDOF system are initiated by the presence of an initial potential or kinetic energy. If the system is undamped, there are no dissipative mechanisms and it is expected that the free vibrations described by the solution of Equation (8.3) are periodic. It is assumed that the vibrations are synchronous in that all dependent variables execute motion with the same time-dependent behavior. Thus, when free vibrations at a single frequency are initiated for a particular system, the ratio of any two dependent variables is independent of time. These assumptions lead to hypothesizing the normal-mode solution of Equation (8.3) in the form

$$\mathbf{x}(t) = \mathbf{X}e^{i\omega t} \tag{8.4}$$

where ω is the frequency of vibration and $\mathbf{X}$ is an n-dimensional vector of constants, called a *mode shape.* This hypothesis implies that certain initial conditions lead to a solution of the form of Equation (8.4) for specific values of ω. The values of ω such that Equation (8.4) is a solution of Equation (8.3) are called the *natural frequencies.* Each natural frequency has at least one corresponding mode shape. Since the differential equations represented by Equation (8.3) are linear and homogeneous, their general solution is a linear superposition over all possible modes.

Substitution of Equation (8.4) into Equation (8.3) leads to

$$(-\omega^2\mathbf{MX} + \mathbf{KX})e^{i\omega t} = \mathbf{0} \tag{8.5}$$

Since $e^{i\omega t} \neq 0$, for any real value of t,

$$-\omega^2\mathbf{MX} + \mathbf{KX} = \mathbf{0} \tag{8.6}$$

The mass matrix is nonsingular, and thus $\mathbf{M}^{-1}$ exists. Premultiplying Equation (8.6) by $\mathbf{M}^{-1}$ and rearranging gives

$$(\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I})\mathbf{X} = \mathbf{0} \tag{8.7}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Equation (8.7) is the matrix representation of a system of n simultaneous linear algebraic equations for the n components of the mode shape vector. The system is homogeneous. Application of Cramer's rule gives the solution for the jth component of $\mathbf{X}$, X_j, as

$$X_j = \frac{0}{\det|\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}|} \tag{8.8}$$

Thus the trivial solution ($\mathbf{X = 0}$) is obtained unless

$$\det|\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}| = 0 \tag{8.9}$$

Hence, applying the definitions of Appendix C, ω^2 must be an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$. The square root of a real positive eigenvalue has two possible values, one positive and one negative. While both are used to develop the general solution, the positive square root is identified as a natural frequency. The mode shape is the corresponding eigenvector.

It is shown in Section 7.6 that when the stiffness matrix, $\mathbf{K}$, is nonsingular, its inverse is the flexibility matrix, $\mathbf{A}$. Premultiplying Equation (8.6) by $\mathbf{A}$ leads to

$$(-\omega^2\mathbf{AM} + \mathbf{I})\mathbf{X} = \mathbf{0} \tag{8.10}$$

Dividing by ω^2 gives

$$\left(\mathbf{AM} - \frac{1}{\omega^2}\mathbf{I}\right)\mathbf{X} = \mathbf{0} \tag{8.11}$$

Thus, the natural frequencies are the reciprocals of the positive square roots of the eigenvalues of $\mathbf{AM}$ and the mode shapes are its eigenvectors. The matrix, $\mathbf{AM}$, is often called the *dynamical matrix.*

Natural frequencies of MDOF systems are calculated as either the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ or as the reciprocals of the square roots of the eigenvalues of $\mathbf{AM}$. The mode shapes are the corresponding eigenvectors of either matrix.

8.3 NATURAL FREQUENCIES AND MODE SHAPES

In the previous section, it is shown that the natural frequencies of an nDOF system are the positive square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ or the reciprocals of the positive square roots of the eigenvalues of $\mathbf{AM}$. The mode shape vectors are the corresponding eigenvectors. As shown in Appendix C, the evaluation of Equation (8.9) leads to an nth-order polynomial equation, called the *characteristic equation*, whose roots are the eigenvalues. Since all elements of the mass and stiffness matrices are real, all coefficients in the characteristic equation are real and thus, if complex roots occur, they must occur in complex conjugate pairs. However, it can be shown that, because of the symmetry of $\mathbf{M}$ and $\mathbf{K}$, the characteristic equation has only real roots. Negative roots are possible, but lead to imaginary values of the natural frequency. When the negative square root of a negative eigenvalue is multiplied by i to form the exponent in the normal-mode solution of Equation (8.4), a real positive exponent is developed. This term grows without bound as time increases. Such a system is unstable.

Assume that all eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ corresponding to symmetric mass and stiffness matrices are nonnegative. Then there exist n real natural frequencies that can be ordered by $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$. Each distinct eigenvalue ω_i^2, $i = 1, 2, \ldots, n$ has a corresponding nontrivial eigenvector, $\mathbf{X}_i$, which satisfies

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{X}_i = \omega_i^2\mathbf{X}_i \tag{8.12}$$

This mode shape, $\mathbf{X}_i$, is an n-dimensional column vector of the form

$$\mathbf{X}_i = \begin{bmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{in} \end{bmatrix} \tag{8.13}$$

Since the system of equations represented by Equation (8.12) is homogeneous, the mode shape is not unique. However, if ω_1^2 is not a repeated root of the characteristic equation, then there is only one linearly independent nontrivial solution of Equation (8.12). The eigenvector is unique only to an arbitrary multiplicative constant. Normalization schemes exist such that the constant is chosen so the eigenvector satisfies an externally imposed condition.

If ω_1^2 is a repeated root of the characteristic equation of multiplicity r, there are r linearly independent nontrivial solutions of Equation (8.12). Each of these mode shapes is also unique to a multiplicative constant.

Solution of the eigenvalue-eigenvector problem is an important part of the vibration analysis of MDOF systems. The quadratic formula is used to find the roots of the characteristic equation for a two degree-of-freedom system. The natural frequencies of a three degree-of-freedom system are obtained by finding the roots of a cubic polynomial, which can be done by trial and error or an iterative method. The algebraic complexity of the solution grows exponentially with the number of degrees of freedom. The development of a characteristic equation for an nDOF system requires the evaluation of an $n \times n$ determinant and the natural frequencies are the n roots of the characteristic equation. The determination of each eigenvector requires the solution of n homogeneous simultaneous algebraic equations. Thus, numerical methods which do not require the evaluation of the characteristic equation are used for systems with a large number of degrees of freedom.

EXAMPLE 8.1

Determine the natural frequencies and mode shapes for the system of Figure 8.1. Use θ and x as generalized coordinates.

SOLUTION

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}\left(\frac{1}{12}mL^2\dot{\theta}^2\right) + \frac{1}{2}(2m)\dot{x}^2 \tag{a}$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2}kx^2 \tag{b}$$

Application of Lagrange's equations leads to

$$\begin{bmatrix} \frac{1}{12}mL^2 & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} k\frac{L^2}{4} & -k\frac{L}{2} \\ -k\frac{L}{2} & 2k \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{c}$$

Since the mass matrix is a diagonal matrix, its inverse is also a diagonal matrix with the reciprocals of the diagonal elements of **M** along its diagonal. The matrix $\mathbf{M}^{-1}\mathbf{K}$ is

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{12}{mL^2} & 0 \\ 0 & \frac{1}{2m} \end{bmatrix}\begin{bmatrix} k\frac{L^2}{4} & -k\frac{L}{2} \\ -k\frac{L}{2} & 2k \end{bmatrix} = \begin{bmatrix} \frac{3k}{m} & -\frac{6k}{mL} \\ -\frac{kL}{4m} & \frac{k}{m} \end{bmatrix} = \phi\begin{bmatrix} 3 & -\frac{6}{L} \\ -\frac{L}{4} & 1 \end{bmatrix} \tag{d}$$

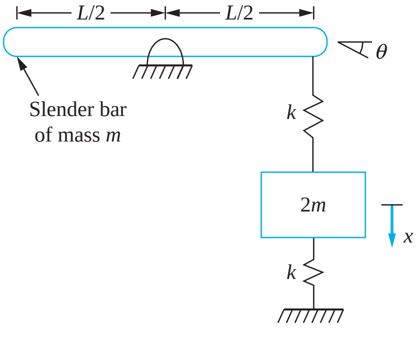


FIGURE 8.1
System of Example 8.1.

where $\phi = \frac{k}{m}$. Calculating the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, we have

$$\det(\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}) = \begin{vmatrix} 3\phi - \lambda & \frac{-6\phi}{L} \\ \frac{-\phi L}{4} & \phi - \lambda \end{vmatrix} = (3\phi - \lambda)(\phi - \lambda) - \left(\frac{-6\phi}{L}\right)\left(\frac{-\phi L}{4}\right)$$

$$= \lambda^2 - 4\phi\lambda + \frac{3}{2}\phi^2 \tag{e}$$

The eigenvalues are obtained by solving

$$\beta^2 - 4\beta + \frac{3}{2} = 0 \tag{f}$$

where $\beta = \lambda/\phi$. The solutions are

$$\beta = \frac{4 \pm \sqrt{(-4)^2 - 4\left(\frac{3}{2}\right)}}{2} = \frac{1}{2}(4 \pm \sqrt{10}) = 0.419,\ 3.58 \tag{g}$$

The natural frequencies are the square roots of the eigenvalues

$$\omega_1 = \sqrt{0.419\frac{k}{m}} = 0.647\sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{3.58\frac{k}{m}} = 1.89\sqrt{\frac{k}{m}} \tag{h}$$

The mode-shape vectors are obtained from

$$\begin{vmatrix} 3\phi - \lambda_i & \frac{-6\phi}{L} \\ \frac{-\phi L}{4} & \phi - \lambda_i \end{vmatrix} \begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{i}$$

for $i = 1,2$. The two equations are linearly dependent when evaluated for the eigenvalues. The first equation gives

$$(3\phi - \lambda_i)X_{i1} - \frac{6\phi}{L}X_{i2} = 0 \tag{j}$$

or

$$X_{i2} = \frac{L(3\phi - \lambda_i)}{6\phi}X_{i1} \tag{k}$$

Recalling that $\lambda_1 = 0.419\phi$,

$$X_{12} = \frac{L(3\phi - 0.419\phi)}{6\phi}X_{11} = 0.430LX_{11} \tag{l}$$

and given that $\lambda_2 = 3.58\phi$,

$$X_{22} = \frac{L(3\phi - 3.58\phi)}{6\phi}X_{21} = -0.0977LX_{21} \tag{m}$$

Arbitrarily taking $X_{i1} = 1$, the mode-shape vectors are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0.430L \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.0977L \end{bmatrix} \tag{n}$$

In the first mode, when x is 1, the value of θ is $0.430L$. The bar and the block are moving in the same direction for the first mode. In the second mode, when x is 1, the value of θ is $-0.977L$, which is a counter-clockwise rotation. The bar and the block move in opposite directions for the second mode. A point of zero displacement must exist in the spring connecting the bar to the block.

EXAMPLE 8.2

Calculate the natural frequencies and the mode shapes for the three degree-of-freedom system of Figure 8.2(a).

SOLUTION

The differential equations for free vibrations using the displacements of the masses from equilibrium as the generalized coordinates are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

Calculating $\mathbf{M}^{-1}\mathbf{K}$ gives

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{2}{m} \end{bmatrix} \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix} = \begin{bmatrix} 3\phi & -2\phi & 0 \\ -2\phi & 3\phi & -\phi \\ 0 & -2\phi & 6\phi \end{bmatrix} \tag{b}$$

where $\phi = k/m$. Application of Equation (8.9) gives

$$\det \begin{bmatrix} 3\phi - \lambda & -2\phi & 0 \\ -2\phi & 3\phi - \lambda & -\phi \\ 0 & -2\phi & 6\phi - \lambda \end{bmatrix} = 0 \tag{c}$$

Expansion of the determinant yields the characteristic equation

$$-\beta^3 + 12\beta^2 - 39\beta + 24 = 0 \tag{d}$$

where $\beta = \lambda/\phi$. A plot of the preceding cubic polynomial is given in Figure 8.2(b). The roots of this equation are

$$\beta = 0.798,\ 4.455,\ 6.747 \tag{e}$$

which leads to the natural frequencies

$$\omega_1 = 0.893\sqrt{\frac{k}{m}} \qquad \omega_2 = 2.110\sqrt{\frac{k}{m}} \qquad \omega_3 = 2.597\sqrt{\frac{k}{m}} \tag{f}$$

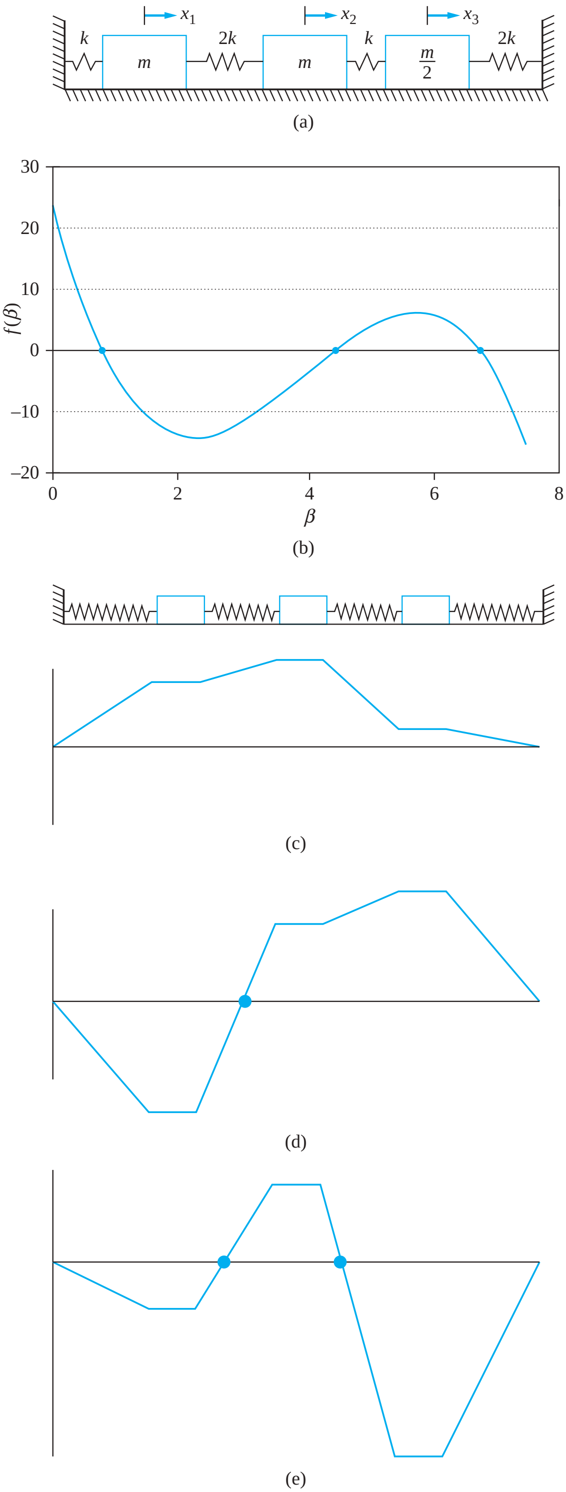


FIGURE 8.2
(a) Three degree-of-freedom system of Example 8.2. (b) Plot of characteristic equation of Example 8.2 where roots occur at values of β where curve intersects horizontal axis. (c) Illustration of mode shape for first mode. (d) Illustration of mode shape for second mode; mode has one node. (e) Illustration of mode shape for third mode; mode has two nodes.

The mode shapes are obtained by finding the nontrivial solutions of

$$\begin{bmatrix} 3\phi - \lambda_i & -2\phi & 0 \\ -2\phi & 3\phi - \lambda_i & -\phi \\ 0 & -2\phi & 6\phi - \lambda_i \end{bmatrix} \begin{bmatrix} X_{i1} \\ X_{i2} \\ X_{i3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{g}$$

The first equation leads to

$$X_{i1} = \frac{2\phi}{3\phi - \lambda_i} X_{i2} \tag{h}$$

while the third equation leads to

$$X_{i3} = \frac{2\phi}{6\phi - \lambda_i} X_{i2} \tag{i}$$

Arbitrarily choosing $\mathbf{X}_{i2} = 1$ leads to the following mode shape vectors:

$$\mathbf{X}_1 = \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \qquad \mathbf{X}_3 = \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix} \tag{j}$$

The graphical representations of the mode shapes in Figure 8.2(c) through (e) are based on the assumption that the displacement in each spring is a linear function of position along the length of the spring. There are no nodes for the first mode. The second mode has a node in the spring between the first and second mass. The third mode has one node in the spring between the first and second mass and one node in the spring between the second and third masses.

EXAMPLE 8.3

An engineer is designing an 18-ft-long steel fixed-pinned beam ($E = 30 \times 10^6$ lb/in^2, $\gamma = 394$ lb/ft^3) for use in an industrial plant. The beam is to support a machine at its midspan. The machine may weigh up to 5 tons and will operate at speeds between 1000 rad/s and 2000 rad/s. The engineer is considering using either a W-shape W16 $\times$ 100 ($I = 712$ in^4, $A = 29.4$ in^2) beam or a W-shape W27 $\times$ 114 beam ($I = 4090$ in^4, $A = 33.5$ in^2) in the design. Use a three degree-of-freedom model of the beam to help decide which shape is the better choice in this design.

SOLUTION

Using a three degree-of-freedom model as shown in Figure 8.3(a), the mass of the beam is lumped at three equally spaced locations along the span of the beam. The mass of each particle is $m_b/4$, where m_b is the total mass of the beam. If β is the mass of the machine, the mass matrix for a three degree-of-freedom model is

$$\mathbf{M} = \begin{bmatrix} \frac{m_b}{4} & 0 & 0 \\ 0 & \frac{m_b}{4} + \beta & 0 \\ 0 & 0 & \frac{m_b}{4} \end{bmatrix}$$

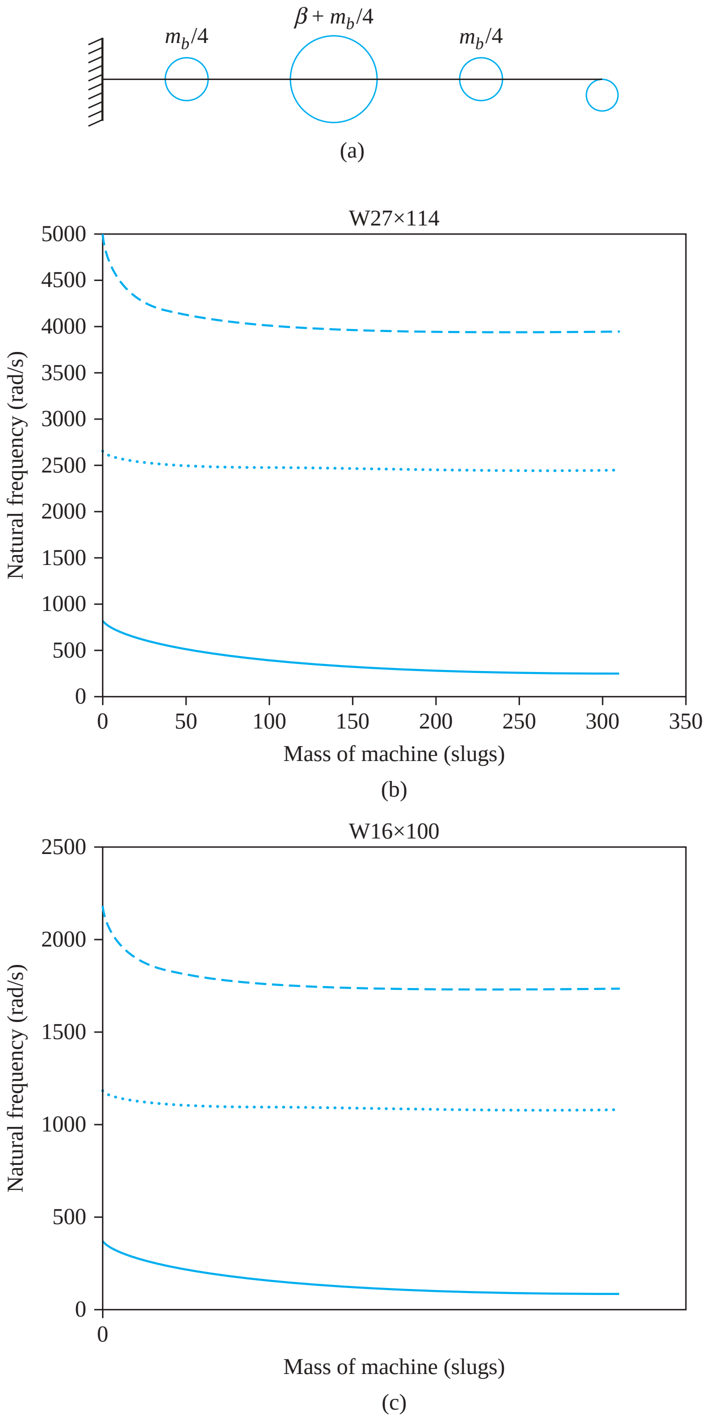


FIGURE 8.3
(a) System of Example 8.3 where inertia of the beam is lumped at three locations along axis of beam. (b) Natural frequencies versus mass of machine for W27×114 beam. (c) Natural frequencies versus mass for W16×100 beam.

The flexibility matrix **A** for the model is determined from Appendix D.

A MATLAB script is written to symbolically determine the eigenvalues of **AM** as a function of the machine mass. The natural frequencies are the reciprocal of the square roots of the eigenvalues. The MATLAB generated plots of the natural frequency approximations as a function of the machine mass for each of the beams under consideration are given in Figures 8.3(b) and (c). These plots show that using the W16×100 shape is not a good choice, as the system's second natural frequency is in this range. The W27×114 shape is a better choice, as the specified operating range of 1000 rad/s to 2000 rad/s is between the system's two lowest natural frequencies for all machines up to 5 tons.

8.4 GENERAL SOLUTION

Equation (8.3) is a homogeneous system of n second-order linear differential equations. The normal-mode assumption, Equation (8.4), leads to the determination of n natural frequencies. If λ is an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$, then both $\omega = +\sqrt{\lambda}$ and $\omega = -\sqrt{\lambda}$ satisfy Equation (8.9) and give rise to the same solution, **X**, of Equation (8.7). The functions $e^{i\omega t}$ and $e^{-i\omega t}$ are linearly independent with each other and linearly independent with other functions of the same form with different values of ω. Thus, the normal-mode solution generates $2n$ linearly independent solutions of Equation (8.3). The most general solution of a linear homogeneous problem is a linear combination of all possible solutions. To this end,

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i(\tilde{C}_{i1} e^{i\omega t} + \tilde{C}_{i2} e^{-i\omega t}) \tag{8.14}$$

Using Euler's identity to replace the complex exponential by trigonometric functions and redefining the arbitrary constants gives

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i(C_{i1} \cos \omega_i t + C_{i2} \sin \omega_i t) \tag{8.15}$$

Trigonometric identities are used to write Equation (8.15) in the alternate form

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i A_i \sin(\omega_i t - \phi_i) \tag{8.16}$$

Initial conditions must be specified for each dependent variable

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} \qquad \dot{\mathbf{x}}(0) = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{bmatrix} \tag{8.17}$$

Application of the $2n$ initial conditions to Equation (8.16) yields $2n$ equations to be solved for the $2n$ integration constants.

$$\mathbf{x}(0) = -\sum_{i=1}^{n} \mathbf{X}_i A_i \sin \phi_i \tag{8.18}$$

and

$$\dot{\mathbf{x}}(0) = \sum_{i=1}^{n} \mathbf{X}_i \omega_i A_i \cos \phi_i \tag{8.19}$$

EXAMPLE 8.4

The block of mass $m/2$ of Figure 8.2(a) is given an initial displacement δ while the other blocks are held in their equilibrium position. The system is then released. What is the response of the system?

SOLUTION

The solution is formed according to Equation (8.16) resulting in

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = A_1 \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \sin\left(0.893\sqrt{\frac{k}{m}}t - \phi_1\right) + A_2 \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \sin\left(2.110\sqrt{\frac{k}{m}}t - \phi_2\right) + A_3 \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix} \sin\left(2.597\sqrt{\frac{k}{m}}t - \phi_3\right) \tag{a}$$

Application of the initial displacements yield

$$\begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} = A_1 \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \sin(-\phi_1) + A_2 \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \sin(-\phi_2) + A_3 \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix} \sin(-\phi_3) \tag{b}$$

Application of initial velocities lead to

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A_1\left(0.893\sqrt{\frac{k}{m}}\right) \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \cos(-\phi_1) + A_2\left(2.110\sqrt{\frac{k}{m}}\right) \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \cos(-\phi_2) + A_3\left(2.597\sqrt{\frac{k}{m}}\right) \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix} \cos(-\phi_3) \tag{c}$$

Equation (c) is satisfied by taking $\cos(-\phi_1) = \cos(-\phi_2) = \cos(-\phi_3) = 0$ or $\phi_1 = \phi_2 = \phi_3 = \pi/2$. Then Equation (b) becomes

$$\begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} = A_1 \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} + A_2 \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} + A_3 \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix} = \begin{bmatrix} 0.908 & -1.375 & -0.534 \\ 1 & 1 & 1 \\ 0.384 & 1.294 & -2.677 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{d}$$

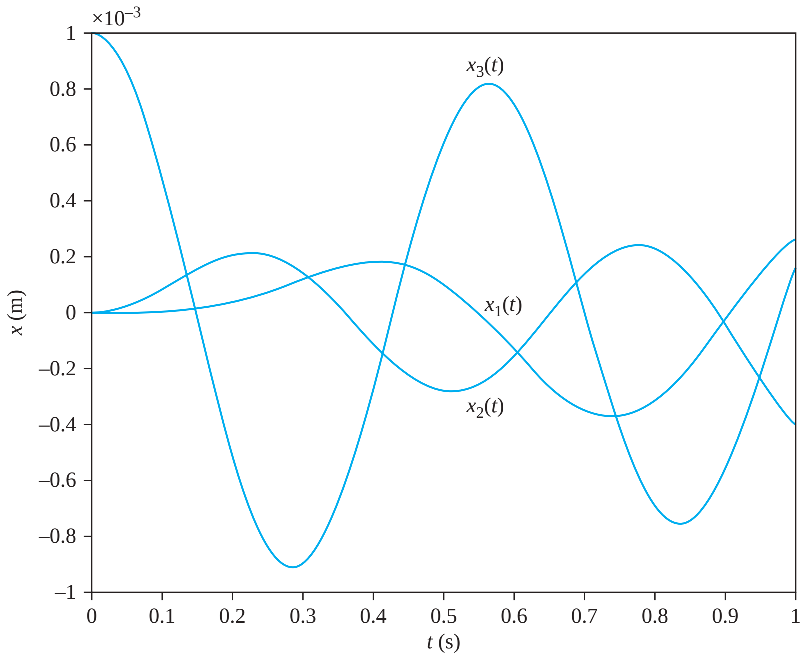


FIGURE 8.4
Solution of Example 8.4.

Equation (d) is solved yielding $A_1 = 0.101\delta$, $A_2 = 0.174\delta$, and, $A_3 = -0.275\delta$. The response of the system is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \delta\left\{\begin{bmatrix} 0.0920 \\ 0.101 \\ 0.0389 \end{bmatrix}\sin\left(0.893\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right) + \begin{bmatrix} -0.239 \\ 0.174 \\ 0.224 \end{bmatrix}\sin\left(2.110\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)\right.$$
$$\left. + \begin{bmatrix} 0.147 \\ -0.275 \\ 0.736 \end{bmatrix}\sin\left(2.597\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)\right\} \tag{e}$$

Equations (e) are plotted in Figure 8.4 for $k = 1000$ N/M, $m = 10$ kg, and $\delta = 1$ mm.

8.5 SPECIAL CASES

8.5.1 DEGENERATE SYSTEMS

Repeated eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{AM}$ occur when the natural frequencies of two distinct modes coincide. It is usually possible to identify the separate modes of vibration. For example, consider the circular cantilever beam of Figure 8.5. The beam has a thin disk attacked at its end. If the disk is vertically displaced and released, the disk undergoes free transverse

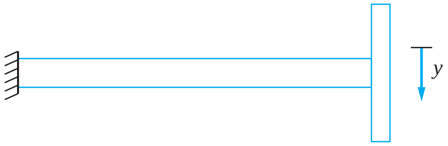


FIGURE 8.5
For certain combinations of parameters, natural frequency of transverse vibration coincides with natural frequency for torsional oscillations.

vibrations. For a SDOF model with inertia effects of the beam ignored, the natural frequency of free transverse vibrations of the disk is

$$\omega_1 = \sqrt{\frac{3EI}{mL^3}} \tag{8.20}$$

where E is the elastic modulus of the beam, I is the cross-sectional moment of inertia of the beam, L is the length of the beam, and m is the mass of the disk. If the disk is twisted and released, it undergoes free torsional oscillations. For a SDOF model, with inertia effects of the beam ignored, the natural frequency of free torsional oscillations is

$$\omega_2 = \sqrt{\frac{JG}{I_D L}} \tag{8.21}$$

where J is the polar moment of inertia of the cross section of the beam, G is the beam's shear modulus, and I_D is the mass moment of inertia of the disk. These two natural frequencies are equal for a steel shaft when the ratio of the length of the beam to the radius of the disk is 1.40. The two modes of vibration are independent but happen to have the same natural frequency.

A system with a repeated natural frequency is called a *degenerate* system. If ω_i is a natural frequency calculated from an eigenvalue of multiplicity m, then only $n - m$ of the linear algebraic equations from which the mode shape is calculated are independent. Thus, m elements of the mode shape can be arbitrarily chosen. The most general mode shape involves m arbitrary constants. Then m linearly independent mode shapes, $\mathbf{X}_i$, $\mathbf{X}_{i+1}$, . . . , $\mathbf{X}_{i+m}$, are specified. The general solution of Equation (8.3) is still given by Equation (8.16), but $\omega_i = \omega_{i+1} = \cdots = \omega_{i+m-1}$.

EXAMPLE 8.5

The two degree-of-freedom system of Figure 8.6 has a natural frequency of $\sqrt{6k/m}$ corresponding to a rotational mode and a natural frequency of $\sqrt{2k/m}$ corresponding to a translational mode. The system is neither statically nor dynamically coupled. A block of mass m is attached to the mass center of the bar through a spring as shown in Figure 8.6(d), adding a degree of freedom and leading to static coupling. The differential equations governing free vibration of this vibration of this three degree-of-freedom system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\dfrac{L^2}{12} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k + k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & k\dfrac{L^2}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

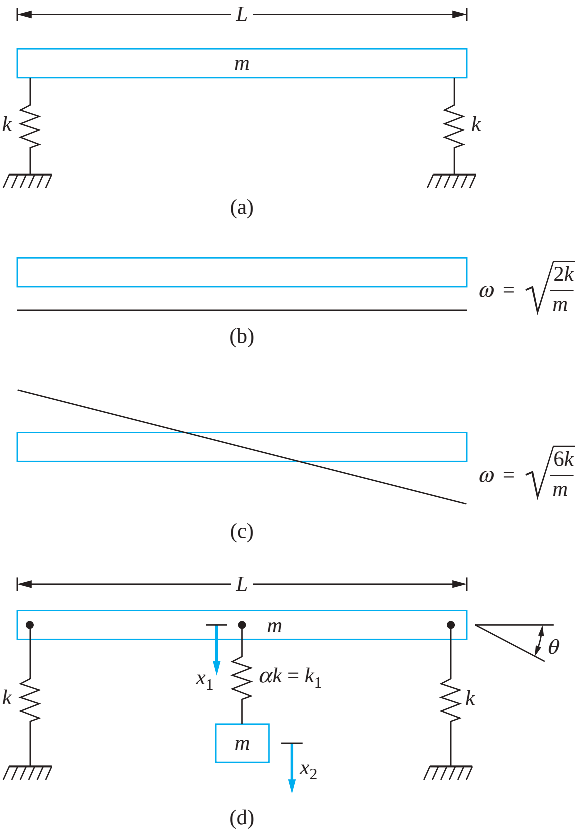


FIGURE 8.6
(a) Original system of Example 8.5. (b) Mode shape for translational mode $\omega = \sqrt{2k/m}$. (c) Mode shape for rotational mode $\omega = \sqrt{6k/m}$. (d) System of Example 8.5 with added mass-spring system. Correct tuning of mass-spring system gives a double root of the characteristic equation resulting in two independent mode shapes for the same natural frequency.

The rotational mode is still uncoupled from the other modes. Find a value of k_1 such that another natural frequency of the system coincides with the natural frequency of the rotational mode. Find the mode shapes corresponding to all modes.

SOLUTION

The determinant leading to the characteristic equation is

$$\det\begin{bmatrix} (2+\alpha)\phi - \lambda & -\alpha\phi & 0 \\ -\alpha\phi & \alpha\phi - \lambda & 0 \\ 0 & 0 & 6\phi - \lambda \end{bmatrix} = 0 \quad \textbf{(b)}$$

where

$$\phi = \frac{k}{m}$$

and

$$\alpha = \frac{k_1}{k}$$

The characteristic equation obtained by row expansion of the determinant, using the third row, is

$$(6 - \beta)[\beta^2 - 2(1 + \alpha)\beta + 2\alpha] = 0 \tag{c}$$

where

$$\beta = \frac{\lambda}{\phi} \tag{d}$$

The roots of the characteristic equation are

$$\beta = 6,\ 1 + \alpha \pm \sqrt{1 + \alpha^2} \tag{e}$$

The $\beta = 6$ root corresponds to the natural frequency of the rotational mode. Requiring one of the other natural frequencies to be equal to the natural frequency of the rotational mode leads to

$$1 + \alpha \pm \sqrt{1 + \alpha^2} = 6 \Rightarrow \alpha = \frac{12}{5} \tag{f}$$

Then the natural frequencies become

$$\omega_1 = \sqrt{\frac{4k}{5m}} \qquad \omega_2 = \omega_3 = \sqrt{6\frac{k}{m}} \tag{g}$$

The mode shape corresponding to the lowest natural frequency is

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1.5 \\ 0 \end{bmatrix} \tag{h}$$

For $\beta = 6$, the mode shapes are determined from

$$\begin{bmatrix} -1.6\phi & -2.4\phi & 0 \\ -2.4\phi & -3.6\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{21} \\ X_{22} \\ X_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{i}$$

The general solution of this system contains two arbitrary constants and can be written as

$$\begin{bmatrix} a \\ -\frac{2}{3}a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{j}$$

Thus, the two linearly independent mode shapes corresponding to $\omega = \sqrt{6k/m}$ are

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix} \qquad \mathbf{X}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{k}$$

Note that the mode corresponding to the lowest natural frequency is a translational mode with extension of the spring. One mode corresponding to $\omega = \sqrt{6k/m}$ is a translational mode with extension in the spring, but with a node in the spring. The second independent mode for $\omega = \sqrt{6k/m}$ is a rigid-body rotation of the bar about its mass center, with no extension in the spring.

8.5.2 UNRESTRAINED SYSTEMS

A second special case occurs when one of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ is zero. The general solution for a system with a zero eigenvalue is

$$x(t) = (C_1 + C_2 t)\mathbf{X}_1 + \sum_{i=2}^{n} A_i \mathbf{X}_i \sin(\omega_i t - \phi_i) \tag{8.22}$$

where C_1, C_2, and A_i are constants determined from application of the initial conditions. The first part of the solution corresponds to a rigid-body motion. The summation term corresponds to oscillatory motion.

A system has a natural frequency of zero only when it is unrestrained. For example, if both masses of the two degree-of-freedom system of Figure 8.7(a) are given the same initial displacement with no initial velocity, they will remain in their displaced positions indefinitely. If the shaft connecting the two flywheels of Figure 8.7(b) is rotating at a constant speed, both flywheels will continue to rotate at this speed.

When motion of an unrestrained system occurs, either linear or angular momentum is conserved for the entire system. Application of the principle of conservation of linear momentum or the principle of conservation of angular momentum provides a relationship between the generalized coordinates of the form

$$\sum_{l=1}^{n} \alpha_l \dot{x}_l = C_1 \tag{8.23}$$

where C_1 is a constant determined from the initial state. Equation (8.23) can be integrated to provide a constraint between the generalized coordinates of the form

$$\sum_{l=1}^{n} \alpha_l x_l = C_1 t + C_2 \tag{8.24}$$

Equation (8.25) could be used to reduce the number of degrees of freedom by one.

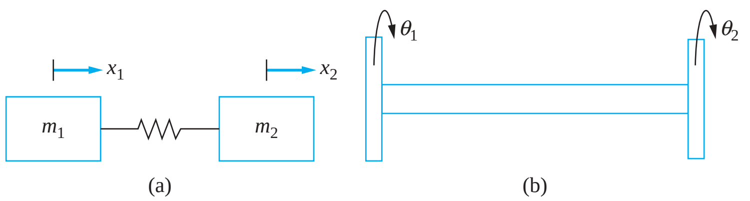


FIGURE 8.7
(a) A two degree-of-freedom unrestrained system. If both blocks are given the same displacement, they will move as a rigid body. If the blocks are given different displacements, free oscillations occur. (b) An unrestrained torsional system.

EXAMPLE 8.6

A railroad car of mass 1500 kg is to be coupled to an assembly of two precoupled identical railroad cars. The couplers are elastic connections of stiffness 4.2×10^7 N/m. The single car is rolled toward the other cars with a velocity of 7 m/s, as shown in Figure 8.8(a). Describe the motion of the three railroad cars after coupling is achieved.

SOLUTION

After coupling, the motion of the three railroad cars is modeled by using three degrees of freedom, as shown in Figure 8.7(b). The differential equations of motion are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

The natural frequencies are determined from

$$\det \begin{bmatrix} \phi - \lambda & -\phi & 0 \\ -\phi & 2\phi - \lambda & -\phi \\ 0 & -\phi & \phi - \lambda \end{bmatrix} = 0 \tag{b}$$

where $\phi = k/m$. The resulting characteristic equation is solved to yield

$$\omega_1 = 0 \qquad \omega_2 = \sqrt{\frac{k}{m}} = 167.3 \text{ rad/s} \qquad \omega_3 = \sqrt{\frac{3k}{m}} = 289.8 \text{ rad/s} \tag{c}$$

The corresponding mode shapes are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{X}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \tag{d}$$

Since the lowest natural frequency is zero, the system is unrestrained. The mode-shape vector for the first mode is that of a rigid-body motion in which all cars move together. In the second mode, the middle car is a node, and the other two cars move in opposite

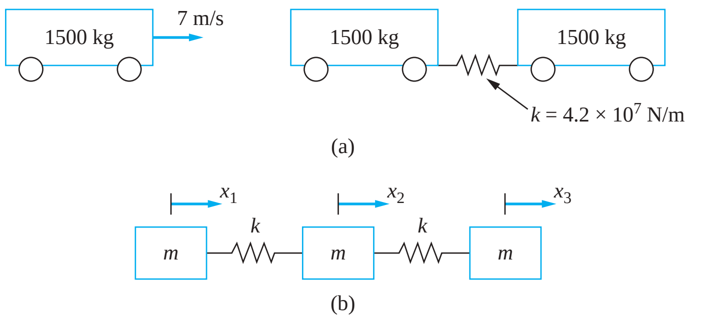


FIGURE 8.8
(a) Shunting of railroad cars. (b) Three degree-of-freedom model once cars are coupled.

directions with the same amplitude. The third mode has two nodes: one in the spring connecting the first car to the middle car and one in the spring connecting the third car to the middle car.

The general solution of the differential equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (C_1 + C_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\sin(167.3t + \phi_1)$$
$$+ C_4\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\sin(289.8t + \phi_2) \quad \text{(e)}$$

Application of the initial conditions leads to

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\sin(-\phi_1) + C_4\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\sin(-\phi_2) \quad \text{(f)}$$

and

$$\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{bmatrix} = \begin{bmatrix} 7\ \text{m/s} \\ 0 \\ 0 \end{bmatrix}$$
$$= C_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3(167.3)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\cos(-\phi_1) + C_4(289.8)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\cos(-\phi_2) \quad \text{(g)}$$

Equations (g) and (h) are satisfied if

$$C_1 = \phi_1 = \phi_2 = 0 \qquad C_2 = 2.32\ \text{m/s} \qquad C_3 = 0.021\ \text{m} \qquad C_4 = 0.004\ \text{m} \quad \text{(h)}$$

The equation expressing conservation of linear momentum of the railroad cars after coupling is achieved is

$$m\dot{x}_1(t) + m\dot{x}_2(t) + m\dot{x}_3(t) = C \quad \text{(i)}$$

EXAMPLE 8.7

Consider the unrestrained three degree-of-freedom system of Example 7.12 and Figure 7.12. Let $mr^2/I = 2$. Calculate the natural frequencies and illustrate the development of the constraint from momentum considerations.

SOLUTION

The differential equations are

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 & -kr \\ 0 & 3k & -6kr \\ -kr & -6kr & 13kr^2 \end{bmatrix}\begin{bmatrix} x_A \\ x_B \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(a)}$$

The characteristic equation is developed from

$$\det \begin{bmatrix} \frac{1}{2}\phi - \lambda & 0 & -\frac{r}{2}\phi \\ 0 & 3\phi - \lambda & -6r\phi \\ -\frac{mr}{I}\phi & -\frac{6mr}{I}\phi & \frac{13mr^2}{I}\phi - \lambda \end{bmatrix} = 0 \qquad \text{(b)}$$

where $\phi = k/m$. The characteristic equation is

$$-\beta^3 + \frac{59}{2}\beta^2 - \frac{39}{2}\beta = 0 \qquad \text{(c)}$$

where $\beta = \lambda/\phi$. The roots of this equation are

$$\beta = 0,\ 0.677,\ 28.82 \qquad \text{(d)}$$

which lead to natural frequencies of

$$\omega_1 = 0 \qquad \omega_2 = 0.823\sqrt{\frac{k}{m}} \qquad \omega_3 = 5.369\sqrt{\frac{k}{m}} \qquad \text{(e)}$$

Application of the principle of conservation of angular momentum about the center of the pulley leads to

$$2mr\dot{x}_A(t) + 2mr\dot{x}_B(t) + I\dot{\theta}(t) = 2mr\dot{x}_A(0) + 2mr\dot{x}_B(0) + I\dot{\theta}(0) \qquad \text{(f)}$$

8.6 ENERGY SCALAR PRODUCTS

A scalar product is an operation performed on two vectors such that the result is a scalar. In order for the operation to be termed a scalar product, it must satisfy certain rules as outlined in Appendix C. When the differential equations governing the motion of a linear nDOF system are formulated by using energy methods, the mass and stiffness matrices are symmetric. Then for a stable restrained system, the following two operations satisfy all requirements to be called scalar products. Let $\mathbf{y}$ and $\mathbf{z}$ be any two n-dimensional vectors; define

$$(\mathbf{y}, \mathbf{z})_K = z^T\mathbf{K}\mathbf{y} \qquad \text{(8.25)}$$

and

$$(\mathbf{y}, \mathbf{z})_M = z^T\mathbf{M}\mathbf{y} \qquad \text{(8.26)}$$

The scalar product defined by Equation (8.25) is called the *potential energy scalar product.* Let $\mathbf{X}_i$ be the mode shape corresponding to a natural frequency ω_i. If the system response includes only this mode, then from Equation (8.16)

$$\mathbf{x}(t) = A_i\mathbf{X}_i \sin(\omega_i t - \phi_i) \qquad \text{(8.27)}$$

From Equation (7.21), the potential energy is calculated as

$$V = \frac{A_i^2}{2} \sin^2(\omega_i t - \phi_i) \sum_{r=1}^{n} \sum_{s=1}^{n} k_{rs} X_{ir} X_{is} = \frac{A_i^2}{2} \sin^2 (\omega_i t - \phi_i)(\mathbf{X}_i, \mathbf{X}_i)_K \tag{8.28}$$

Thus, at a given instant of time, the potential energy scalar product of a mode shape with itself is proportional to the potential energy associated with that mode.

The scalar product defined by Equation (8.26) is called the *kinetic energy scalar product.* It can be shown by using Equations (7.22) and (8.26) that

$$T = \frac{A_i^2}{2} \omega_i^2 \cos^2(\omega_i t - \phi_i)(\mathbf{X}_i, \mathbf{X}_i)_M \tag{8.29}$$

or that for a linear system, the kinetic energy scalar product of a mode shape with itself is proportional to the kinetic energy associated with that mode.

The mass and stiffness matrices for a linear system are guaranteed to be symmetric. In addition, the mass matrix is positive definite. The stiffness matrix for a stable system is positive definite unless it is unrestrained. The stiffness matrix for an unstable system is not positive definite. Thus, from Example C.5 of Appendix C, Equation (8.26) defines a valid scalar product for all nDOF systems and Equation (8.25) defines a valid scalar product for all stable constrained nDOF systems.

The ability to define the potential-energy scalar product and the kinetic-energy scalar product is because **M** and **K** are guaranteed to be symmetric. One property that scalar product defined for real vectors must satisfy is commutivity; that is

$$(\mathbf{y}, \mathbf{z})_\mathbf{K} = (\mathbf{z}, \mathbf{y})_\mathbf{K} \tag{8.30}$$

and

$$(\mathbf{y}, \mathbf{z})_\mathbf{M} = (\mathbf{z}, \mathbf{y})_\mathbf{M} \tag{8.31}$$

Taking the potential-energy scalar product of **y** and **z** using Equation (8.30) implies

$$\mathbf{z}^\mathbf{T}\mathbf{K}\mathbf{y} = \mathbf{y}^\mathbf{T}\mathbf{K}\,\mathbf{z} \tag{8.32}$$

for all n dimensional **y** and **z**, which is true if **K** is symmetric. The commutivity of the kinetic energy scalar product is proved in the same fashion.

Another property of scalar products is that, when a scalar product of a vector is taken with itself, the operation must yield a non-negative quantity and the operation is only zero for the zero vector. This statement, for the potential energy scalar product, is equivalent to

$$\mathbf{y}^\mathbf{T}\mathbf{K}\mathbf{y} \geq 0 \tag{8.33}$$

for all **y** and $\mathbf{y}^\mathbf{T}\mathbf{K}\mathbf{y} = 0$ if and only if $\mathbf{y} = 0$.

Equation (8.33) is also a statement of positive definiteness of the matrix **K**. It can be shown that for all stable systems **K** satisfies the first part of the statement. For restrained systems, **K** satisfies the second part as well. If the system is unrestrained, there exists a $\mathbf{y} \neq 0$ such that $\mathbf{y}^\mathbf{T}\mathbf{K}\mathbf{y} = 0$. This **y** is the mode shape for the rigid-body mode. The kinetic-energy scalar product always satisfies an equivalent statement to Equation (8.33).

For all real n-dimensional vectors **w, y**, and **z** and for all scalars α and β, we have

$$(\alpha\mathbf{w} + \beta\mathbf{y}, \mathbf{z})_\mathbf{K} = \alpha(\mathbf{w}, \mathbf{z})_\mathbf{K} + \beta(\mathbf{y}, \mathbf{z})_\mathbf{K} \tag{8.34}$$

and

$$(\alpha \mathbf{w} + \beta \mathbf{y}, \mathbf{z})_{\mathbf{M}} = \alpha(\mathbf{w}, \mathbf{z})_{\mathbf{M}} + \beta(\mathbf{y}, \mathbf{z})_{\mathbf{M}} \tag{8.35}$$

Equations (8.34) and (8.35) are statements of the linearity of the potential and kinetic energy scalar products.

Two vectors are said to be orthogonal with respect to a scalar product if their scalar product is zero. The n-dimensional vectors $\mathbf{y}$ and $\mathbf{z}$ are orthogonal with respect to the potential-energy scalar product, giving

$$(\mathbf{y}, \mathbf{z})_{\mathbf{K}} = 0 \tag{8.36}$$

The vectors are orthogonal with respect to the kinetic energy scalar product if

$$(\mathbf{y}, \mathbf{z})_{\mathbf{M}} = 0 \tag{8.37}$$

The use of scalar product notation is not essential to analyze and understand free and forced vibrations of MDOF systems. However, writing equations in scalar product notation is usually less confusing than using matrix and vector notation. In addition, since the scalar products have identifiable physical meaning, it may be easier to identify the physical significance of an equation when it is written in scalar product notation. At the very least, the energy scalar products can be thought of as shorthand notation for the products defined by Equations (8.25) and (8.26). For these reasons, the remainder of the discussion in Chapter 8 and the entire discussion in Chapter 7 use scalar product notation. In addition, a scalar product is developed for use with continuous systems in Chapter 10. Many equations are also written using matrix notation for those not comfortable with scalar product notation.

EXAMPLE 8.8

Consider the system of Figure 8.2 and Example 8.2. Define the vectors

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \tag{a}$$

Calculate (a) $(\mathbf{y}, \mathbf{z})_{\mathbf{M}}$, (b) $(\mathbf{y}, \mathbf{z})_{\mathbf{K}}$, and (c) for any three-dimensional vector $\mathbf{x}$ prove Equation (8.33) for this system.

SOLUTION

(a) Using the mass matrix from Example 8.2, we have

$$(\mathbf{y}, \mathbf{z})_M = [2 \;\; -1 \;\; 3]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [2 \;\; -1 \;\; 3]\begin{bmatrix} m \\ 2m \\ 2m \end{bmatrix}$$

$$= 2(m) - 1(2m) + 3(2m) = 6m \tag{b}$$

(b) Using the stiffness matrix from Example 8.2, we have

$$(\mathbf{y}, \mathbf{z})_{\mathbf{K}} = [2 \;\; -1 \;\; 3]\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [2 \;\; -1 \;\; 3]\begin{bmatrix} -k \\ 0 \\ 10k \end{bmatrix}$$

$$= 2(-k) - 1(0) + 3(10k) = 28k \qquad \text{(c)}$$

(c) For an arbitrary $\mathbf{x}$,

$$(\mathbf{x}, \mathbf{x})_{\mathbf{K}} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} 3kx_1 - 2kx_2 \\ -2kx_1 + 3kx_2 - kx_3 \\ -kx_2 + 3kx_3 \end{bmatrix}$$

$$= x_1(3kx_1 - 2kx_2) + x_2(-2kx_1 + 3kx_2 - kx_3) + x_3(-kx_2 + 3kx_3)$$

$$= 3kx_1^2 - 4kx_1x_2 + 3kx_2^2 - 2kx_2x_3 + 3kx_3^2$$

$$= kx_1^2 + 2k(x_2 - x_1)^2 + k(x_3 - x_2)^2 + 2kx_3^2 \qquad \text{(d)}$$

Clearly, Equation (d) is greater than or equal to zero for all choices of $\mathbf{x}$. Additionally, it is obvious that $(\mathbf{x}, \mathbf{x})_{\mathbf{K}} = 0$ if $\mathbf{x} = \mathbf{0}$ and the only $\mathbf{x}$ for which Equation (d) equals zero is $\mathbf{x} = 0$. Equation (d) is twice the potential energy of the system if $\mathbf{x}$ were a mode shape vector.

8.7 PROPERTIES OF NATURAL FREQUENCIES AND MODE SHAPES

Let ω_i and ω_j be distinct natural frequencies of an nDOF system. Let $\mathbf{X}_i$ and $\mathbf{X}_j$ be their respective mode shapes. From Equation (8.6), the equations satisfied by these natural frequencies and mode shapes are

$$\omega_i^2 \mathbf{M}\mathbf{X}_i = \mathbf{K}\mathbf{X}_i \qquad \text{(8.38)}$$

and

$$\omega_j^2 \mathbf{M}\mathbf{X}_j = \mathbf{K}\mathbf{X}_j \qquad \text{(8.39)}$$

Premultiplying Equation (8.38) by $\mathbf{X}_j^T$ gives

$$\omega_i^2 \mathbf{X}_j^T \mathbf{M}\mathbf{X}_i = \mathbf{X}_j^T \mathbf{K}\mathbf{X}_i \qquad \text{(8.40)}$$

or in scalar product notation

$$\omega_i^2 (\mathbf{X}_i, \mathbf{X}_j)_M = (\mathbf{X}_i, \mathbf{X}_j)_K \qquad \text{(8.41)}$$

Premultiplying Equation (8.39) by $\mathbf{X}_i^T$ gives

$$\omega_j^2 (\mathbf{X}_j, \mathbf{X}_i)_M = (\mathbf{X}_j, \mathbf{X}_i)_K \qquad \text{(8.42)}$$

Subtracting Equation (8.42) from Equation (8.41) gives

$$\omega_i^2(\mathbf{X}_i, \mathbf{X}_j)_M - \omega_j^2(\mathbf{X}_j, \mathbf{X}_i)_M = (\mathbf{X}_i, \mathbf{X}_j)_K - (\mathbf{X}_j, \mathbf{X}_i)_K \tag{8.43}$$

On the basis of the commutivity of the scalar products, Equation (8.43) reduces to

$$(\omega_i^2 - \omega_j^2)(\mathbf{X}_i, \mathbf{X}_j)_M = 0 \tag{8.44}$$

Since $\omega_i \neq \omega_j$,

$$(\mathbf{X}_i, \mathbf{X}_j)_M = 0 \tag{8.45}$$

or mode shapes corresponding to distinct natural frequencies are orthogonal with respect to the kinetic energy scalar product. Then from Equation (8.41), these mode shapes are also orthogonal with respect to the potential energy scalar product, or

$$(\mathbf{X}_i, \mathbf{X}_j)_K = 0 \tag{8.46}$$

If a system has a zero natural frequency, then it is strictly improper to define a potential energy scalar product. Property 3 required of scalar products is violated. However, it can be shown that the mode shape for the rigid-body mode for an unrestrained system is orthogonal to all other mode shapes for the system.

If an eigenvalue is not distinct, but has a multiplicity $m > 1$, then there are m linearly independent mode shapes corresponding to that eigenvalue. The preceding analysis shows that each of these mode shapes is orthogonal to mode shapes corresponding to different natural frequencies. Independent mode shapes obtained by solving Equation (8.7) for the same eigenvalue may or may not be mutually orthogonal with respect to the energy scalar products. However, a procedure known as the Gram-Schmidt orthogonalization process can be used to replace these mode shapes with a set of m mutually orthogonal mode shapes. These orthogonalized mode shapes are linearly dependent with the original mode shapes.

EXAMPLE 8.9

Demonstrate orthogonality of the mode shapes with respect to the kinetic energy scalar product for the system of Example 8.2

SOLUTION

The mass matrix, stiffness matrix, and mode shapes are as given in Example 8.2. Orthogonality with respect to the kinetic energy inner product is as follows:

$$\begin{aligned}(\mathbf{X}_2, \mathbf{X}_1)_M &= \mathbf{X}_1^T\mathbf{M}\mathbf{X}_2 \\ &= [0.908 \quad 1 \quad 0.384]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{m}{2} \end{bmatrix}\begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \\ &= [0.908 \quad 1 \quad 0.384]\begin{bmatrix} -1.375m \\ m \\ 0.647m \end{bmatrix}\end{aligned}$$

$$= (0.908)(-1.375m) + (1)(m) + (0.384)(0.647m)$$
$$= -0.000052m \approx 0$$

$$(\mathbf{X}_3, \mathbf{X}_1)_M = \mathbf{X}_1^T \mathbf{M} \mathbf{X}_3$$
$$= [0.908 \;\; 1 \;\; 0.384] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix}$$
$$= [0.908 \;\; 1 \;\; 0.384] \begin{bmatrix} -0.534m \\ m \\ -1.339m \end{bmatrix}$$
$$= (0.908)(-0.534m) + (1)(m) + (0.384)(-1.339m)$$
$$= 0.00095m \approx 0$$

$$(\mathbf{X}_3, \mathbf{X}_2)_M = \mathbf{X}_2^T \mathbf{M} \mathbf{X}_3$$
$$= [-1.375 \;\; 1 \;\; 1.294] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix}$$
$$= [1.375 \;\; 1 \;\; 1.294] \begin{bmatrix} -0.535m \\ m \\ -1.339m \end{bmatrix}$$
$$= (-1.375)(-0.534m) + (1)(m) + (1.294)(-1.339m)$$
$$= -0.00159m \approx 0$$

A version of the preceding argument is used to prove that the eigenvalues are all real. The formal proof of this statement involves the introduction of a scalar product that can be defined to operate on complex vectors and can be evaluated to be a complex number. The properties of a complex scalar product are more general than for a real scalar product. The property of commutivity is generalized to a property where the scalar product is the complex conjugate of its commutative. Assume a complex eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$ or $\mathbf{AM}$ exists and then prove that the eigenvalue must be real due to the symmetry of $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{A}$.

The argument can also be used to show that if $\mathbf{M}$ and $\mathbf{K}$ are positive definite, then the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are all positive. Let $\mathbf{X}_i = \mathbf{X}_j$ in Equation (8.41)

$$\omega_i^2 = \frac{(\mathbf{X}_i, \mathbf{X}_i)_K}{(\mathbf{X}_i, \mathbf{X}_i)_M} \tag{8.47}$$

If $\mathbf{M}$ and $\mathbf{K}$ are positive definite, then both scalar products in the quotient of Equation (8.47) are positive. Hence,

$$\omega_i^2 > 0 \tag{8.48}$$

This, in turn, shows that a system in which both the mass and stiffness matrices are positive definite is stable.

The ratio of Equation (8.47) is called *Rayleigh's quotient.* For a given mode it is the ratio or the potential energy to the kinetic energy.

It is possible to construct n orthogonal, and hence linearly independent, mode shapes for an nDOF system. Thus any n-dimensional vector can be written as a linear combination of these n mode shapes. To this end, if $\mathbf{y}$ is any n-dimensional vector, there exist constants $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{X}_i \tag{8.49}$$

Equation (8.49) is a representation of the *expansion theorem.* Premultiplying Equation (8.49) by $\mathbf{X}_j^T\mathbf{M}$ for some j, $1 \le j \le n$ gives, in scalar product notation

$$(\mathbf{X}_j, \mathbf{y})_M = \left(\mathbf{X}_j, \sum_{i=1}^{n} c_i \mathbf{X}_i\right)_M \tag{8.50}$$

Interchanging the scalar product operation with the summation and using the linearity property of scalar products gives

$$(\mathbf{X}_j, \mathbf{y})_M = \sum_{i=1}^{n} c_i (\mathbf{X}_j, \mathbf{X}_i)_M \tag{8.51}$$

The orthogonality of the mode shapes implies that the only nonzero term in the summation occurs when $i = j$. Then Equation (8.51) reduces to

$$c_j = \frac{(\mathbf{X}_j, \mathbf{y})_M}{(\mathbf{X}_j, \mathbf{X}_j)_M} \tag{8.52}$$

8.8 NORMALIZED MODE SHAPES

A mode shape corresponding to a specific natural frequency of an nDOF system is unique only to a multiplicative constant. The arbitrariness can be alleviated by requiring the mode shape to satisfy the normalization constraint. A mode shape chosen to satisfy the normalization constraint is called a *normalized mode shape.* The normalization constraint, itself, is arbitrary. However, all mode shapes are required to satisfy the same normalization constraint. The constraint should be chosen such that subsequent use of the normalized mode shape is convenient.

It is convenient to normalize mode shapes by requiring that the kinetic energy scalar product of a mode shape with itself is equal to one. That is,

$$(\mathbf{X}_i, \mathbf{X}_i)_M = \mathbf{X}_i^T \mathbf{M} \mathbf{X}_i = 1 \tag{8.53}$$

If the mode shape, $\mathbf{X}_i$, is normalized according to Equation (8.53), then from Rayleigh's quotient, Equation (8.47)

$$\mathbf{X}_i^T \mathbf{K} \mathbf{X}_i = (\mathbf{X}_i, \mathbf{X}_i)_K = \omega_i^2 \tag{8.54}$$

The orthogonality relations, Equations (8.45) and (8.46), the normalization constraint, Equation (8.53), and the subsequent result of the choice of normalization, Equation (8.54), are summarized by

$$(\mathbf{X}_i, \mathbf{X}_j)_M = \delta_{ij} \tag{8.55}$$

and

$$(\mathbf{X}_i, \mathbf{X}_j)_K = \omega_i^2 \delta_{ij} \tag{8.56}$$

where δ_{ij} is the Kronecker delta. From this point, mode shapes will be assumed to be normalized by Equation (8.53).

With the normalization scheme of Equation (8.53), the expansion theorem, Equations (8.49) and (8.52), becomes

$$\mathbf{y} = \sum_{i=1}^{n} (\mathbf{X}_i, \mathbf{y})_M \mathbf{X}_i \tag{8.57}$$

EXAMPLE 8.10

Expand the vector

$$\mathbf{y} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \tag{a}$$

using the normalized mode shapes of Example 8.2.

SOLUTION

The general mode shapes of Example 8.2 are

$$\mathbf{X}_1 = B_1 \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \quad \mathbf{X}_2 = B_2 \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \quad \mathbf{X}_3 = B_3 \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix} \tag{b}$$

where B_1, B_2, and B_3 are arbitrary constants. The normalization of the first mode shape proceeds as follows

$$1 = (\mathbf{X}_1, \mathbf{X}_1)_M = B_1^2 [0.908 \quad 1 \quad 0.384] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \tag{c}$$

which yields $B_1 = 0.726/\sqrt{m}$ and

$$\mathbf{X}_1 = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.659 \\ 0.726 \\ 0.279 \end{bmatrix} \tag{d}$$

The other mode shapes are normalized in the same manner yielding

$$\mathbf{X}_2 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.712 \\ 0.518 \\ 0.670 \end{bmatrix} \qquad \mathbf{X}_3 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.242 \\ 0.453 \\ -1.213 \end{bmatrix} \tag{e}$$

The first coefficient in the expansion is calculated by

$$c_1 = (\mathbf{X}_1, \mathbf{y})_M = \frac{1}{\sqrt{m}}[0.659 \quad 0.726 \quad 0.279]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = 3.284\sqrt{m} \tag{f}$$

The other coefficients are calculated in a similar manner, yielding $c_2 = 0.690\sqrt{m}$, $c_3 = 2.777\sqrt{m}$. Thus,

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = 3.284\begin{bmatrix} 0.659 \\ 0.726 \\ 0.279 \end{bmatrix} + 0.690\begin{bmatrix} -0.712 \\ 0.518 \\ 0.670 \end{bmatrix} + 2.777\begin{bmatrix} -0.242 \\ 0.453 \\ -1.213 \end{bmatrix} \tag{g}$$

8.9 RAYLEIGH'S QUOTIENT

Consider a situation where the free vibrations of a SDOF system are generated such that only one mode is present. The frequency of the mode is ω and its mode shape is **X**. The maximum potential energy associated with this mode of vibration is determined from Equation (8.28) as

$$V_{\max} = \frac{1}{2}(\mathbf{X}, \mathbf{X})_K \tag{8.58}$$

The maximum kinetic energy associated with this mode is determined from Equation (8.29) as

$$T_{\max} = \frac{1}{2}\omega^2(\mathbf{X}, \mathbf{X})_M \tag{8.59}$$

For a conservative system, where a continual process of transfer of kinetic and potential energy occurs without dissipation, the maximum potential energy equals the maximum kinetic energy. Thus, from Equations (8.58) and (8.59)

$$\omega^2(\mathbf{X}, \mathbf{X})_M = (\mathbf{X}, \mathbf{X})_K \tag{8.60}$$

or

$$\omega^2 = \frac{(\mathbf{X}, \mathbf{X})_K}{(\mathbf{X}, \mathbf{X})_M} \tag{8.61}$$

For a general n-dimensional vector $\mathbf{X}$, not necessarily a mode shape, Equation (8.61) is generalized to

$$R(\mathbf{X}) = \frac{(\mathbf{X}, \mathbf{X})_K}{(\mathbf{X}, \mathbf{X})_M} \tag{8.62}$$

The scalar function defined in Equation (8.62) is called *Rayleigh's quotient.* If $\mathbf{X}$ is a mode shape of the linear n degree of freedom whose stiffness and mass matrices are $\mathbf{K}$ and $\mathbf{M}$, respectively, then $R(\mathbf{X})$ takes on the value of the natural frequency associated with that mode. If $\mathbf{X}$ is not a mode shape, then $R(\mathbf{X})$ takes on some other value.

Rayleigh's quotient can be useful in determining an upper bound on the lowest natural frequency. In some cases, it can be used to attain a good approximation to the lowest natural frequency.

From the expansion theorem, an arbitrary vector $\mathbf{X}$ can be written as a linear combination of the normalized mode shapes

$$\mathbf{X} = \sum_{i=1}^{n} c_i \mathbf{X}_i \tag{8.63}$$

Substituting Equation (8.63) in Rayleigh's quotient, using properties of the scalar products and orthonormality of the mode shapes, leads to

$$R(\mathbf{X}) = \frac{\sum_{i=1}^{n} c_i^2 \omega_i^2}{\sum_{i=1}^{n} c_i^2} \tag{8.64}$$

Stationary values of $R(\mathbf{X})$ occur when

$$\frac{\partial R}{\partial c_1} = \frac{\partial R}{\partial c_2} = \cdots = \frac{\partial R}{\partial c_n} = 0 \tag{8.65}$$

The n solutions of Equation (8.65) are summarized by $c_i = \delta_{ij}$ for $j = 1, \ldots, n$. That is, Rayleigh's quotient is stationary only when $\mathbf{X}$ is an eigenvector. It is also possible to show that these stationary values are minimums. Hence ω_1^2 is the minimum value of Rayleigh's quotient.

The preceding result implies that an upper bound and perhaps an approximation for the lowest natural frequency can be obtained by using Rayleigh's quotient. Rayleigh's quotient can be calculated for several trial vectors. The lowest natural frequency can be no greater than the square root of the smallest value obtained. The closer a trial vector is to the actual mode shape, the closer the value of Rayleigh's quotient is to the square of the lowest natural frequency.

EXAMPLE 8.11

Use Rayleigh's quotient to obtain an approximation to the lowest natural frequency of the system of Example 8.2. Use the trial vectors

$$\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

SOLUTION

Calculate Rayleigh's quotient:

$$R(\mathbf{X}) = \frac{\begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}} = 0.823\frac{k}{m} \tag{a}$$

Similar calculations yield

$$R(\mathbf{Y}) = 6.0\frac{k}{m} \qquad R(\mathbf{Z}) = 2.57\frac{k}{m} \tag{b}$$

From the preceding equations, an upper bound on the lowest natural frequency is

$$\omega_1 < 0.907\sqrt{\frac{k}{m}} \tag{c}$$

From Example 8.2, the lowest natural frequency for this system is $0.893\sqrt{k/m}$.

8.10 PRINCIPAL COORDINATES

Let $\omega_1, \omega_2, \ldots, \omega_n$, be the natural frequencies of a linear nDOF system with corresponding normalized mode shapes $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$. The expansion theorem implies that there exists coefficients such that at any time the solution of Equation (8.3) can be expanded in a series of eigenvectors. These coefficients must be continuous functions of time, call them $p_i(t)$, $i = 1, 2, \ldots, n$. The expansion theorem implies

$$\mathbf{x}(t) = \sum_{i=1}^{n} p_i(t)\mathbf{X}_i \tag{8.66}$$

Substitution of Equation (8.66) into Equation (8.3) leads to

$$\mathbf{M}\left(\sum_{i=1}^{n}\ddot{p}_i\mathbf{X}_i\right) + \mathbf{K}\left(\sum_{i=1}^{n}p_i\mathbf{X}_i\right) = 0 \tag{8.67}$$

Taking the standard scalar product of Equation (8.67) with $\mathbf{X}_j$ for an arbitrary j leads to

$$\left(\mathbf{X}_j, \sum_{i=1}^{n}\ddot{p}_i\mathbf{M}\mathbf{X}_i\right) + \left(\mathbf{X}_j, \sum_{i=1}^{n}p_i\mathbf{K}\mathbf{X}_i\right) = 0$$

which, after the properties of scalar products are invoked, becomes

$$\sum_{i=1}^{n}\ddot{p}_i(\mathbf{X}_j, \mathbf{M}\mathbf{X}_i) + \sum_{i=1}^{n}p_i(\mathbf{X}_j, \mathbf{K}\mathbf{X}_i) = 0 \tag{8.68}$$

Using the definitions of the energy scalar products, Equations (8.26) and (8.27), in Equation (8.68) leads to

$$\sum_{i=1}^{n} \ddot{p}_i (\mathbf{X}_j, \mathbf{X}_i)_M + \sum_{i=1}^{n} p_i (\mathbf{X}_j, \mathbf{X}_i)_K = 0 \tag{8.69}$$

Orthogonality and normalization of mode shapes, Equations (8.56) and (8.57), are used in Equation (8.69), leading to

$$\ddot{p}_j + \omega_i^2 p_j = 0 \tag{8.70}$$

Since j was arbitrarily chosen, an equation of the form of Equation (8.70) can be written for each $j = 1, 2, \ldots, n$.

Equation (8.66) can be viewed as a linear transformation between the chosen generalized coordinates, $\mathbf{x}$, and the coordinates $\mathbf{p} = [p_1\, p_2 \, . \; . \; . p_n]^T$, called the *principal coordinates.* The transformation matrix is the matrix whose columns are the normalized mode shapes. This matrix, $\mathbf{P} = [\mathbf{X}_1 \; \mathbf{X}_2 \, . \; . \; . \mathbf{X}_n]$ is called the *modal matrix.* Since the columns of the modal matrix are linearly independent, the modal matrix is nonsingular and the transformations

$$\mathbf{x} = \mathbf{P}\mathbf{p} \qquad \mathbf{p} = \mathbf{P}^{-1}\mathbf{x} \tag{8.71}$$

have a one-to-one correspondence.

The differential equations governing the vibrations of a linear nDOF system are uncoupled when the principal coordinates are used as dependent variables.

EXAMPLE 8.12

(a) Write the differential equations satisfied by the principal coordinates for the system of Example 8.2.

(b) Find the relation between the principal coordinates and the original generalized coordinates and vice versa.

(c) Motion of the system is initiated by moving the third mass a distance δ from equilibrium while holding the other masses in their equilibrium position and then releasing the system from rest. Solve for the response of the principal coordinates.

SOLUTION

(a) Recalling from Example 8.2, the natural frequencies of the system are

$$\omega_1 = 0.893\sqrt{\frac{k}{m}} \qquad \omega_2 = 2.110\sqrt{\frac{k}{m}} \qquad \omega_3 = 2.597\sqrt{\frac{k}{m}} \tag{a}$$

The differential equations governing the principal coordinates are

$$\ddot{p}_1 + \left(0.893\sqrt{\frac{k}{m}}\right)^2 p_1 = 0 \tag{b}$$

$$\ddot{p}_2 + \left(2.110\sqrt{\frac{k}{m}}\right)^2 p_2 = 0 \tag{c}$$

$$\ddot{p}_3 + \left(2.597\sqrt{\frac{k}{m}}\right)^2 p_3 = 0 \tag{c}$$

(b) The normalized eigenvectors are calculated in Example 8.10 as

$$\mathbf{X}_1 = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.659 \\ 0.726 \\ 0.279 \end{bmatrix} \qquad \mathbf{X}_2 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.712 \\ 0.518 \\ 0.670 \end{bmatrix} \qquad \mathbf{X}_1 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.242 \\ 0.453 \\ -1.213 \end{bmatrix} \tag{d}$$

The modal matrix is the matrix whose columns are the normalized eigenvectors

$$\mathbf{P} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.659 & -0.712 & -0.242 \\ 0.726 & 0.518 & 0.453 \\ 0.279 & 0.670 & -1.213 \end{bmatrix} \tag{e}$$

The relation between the two sets of coordinates is given by Equation (8.74)

$$\mathbf{x} = \mathbf{P}\mathbf{p} \Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.659 & -0.712 & -0.242 \\ 0.726 & 0.518 & 0.453 \\ 0.279 & 0.670 & -1.213 \end{bmatrix}\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} \tag{f}$$

The relationship is inverted yielding

$$\mathbf{p} = \mathbf{P}^{-1}\mathbf{x} \Rightarrow \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \sqrt{m}\begin{bmatrix} 0.659 & 0.726 & 0.140 \\ -0.712 & 0.518 & 0.335 \\ -0.242 & 0.453 & -0.607 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{g}$$

(c) The initial conditions for $\boldsymbol{x}$ are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{h}$$

The initial conditions for the principal coordinates are obtained from Equation (g) as

$$\begin{bmatrix} p_1(0) \\ p_2(0 \\ p_3(0) \end{bmatrix} = \sqrt{m}\begin{bmatrix} 0.659 & 0.726 & 0.140 \\ -0.712 & 0.518 & 0.335 \\ -0.242 & 0.453 & -0.607 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} = \sqrt{m}\,\delta\begin{bmatrix} 0.140 \\ 0.335 \\ -0.607 \end{bmatrix} \tag{i}$$

and

$$\begin{bmatrix} \dot{p}_1(0) \\ \dot{p}_2(0) \\ \dot{p}_3(0) \end{bmatrix} = \sqrt{m}\begin{bmatrix} 0.659 & 0.726 & 0.140 \\ -0.712 & 0.518 & 0.335 \\ -0.242 & 0.453 & -0.607 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{j}$$

The general solution for the principal coordinates is

$$p_1(t) = A_1 \sin\left(0.893\sqrt{\frac{k}{m}}t\right) + B_1 \cos\left(0.893\sqrt{\frac{k}{m}}t\right) \tag{k}$$

$$p_2(t) = A_2 \sin\left(2.110\sqrt{\frac{k}{m}}t\right) + B_2 \cos\left(2.110\sqrt{\frac{k}{m}}t\right) \tag{l}$$

$$p_3(t) = A_3 \sin\left(2.597\sqrt{\frac{k}{m}}t\right) + B_3 \cos\left(2.597\sqrt{\frac{k}{m}}t\right) \tag{m}$$

Application of initial conditions, Equations (i) and (j) lead to $B_1 = 0.140\delta\sqrt{m}$, $B_2 = 0.335\delta\sqrt{m}$, $B_3 = -0.607\delta\sqrt{m}$, A1 $= 0$, $A_2 = 0$, and $A_3 = 0$. The original generalized coordinates are obtained using Equation (f) as

$$x_1(t) = 0.0922\delta\cos\left(0.893\sqrt{\frac{k}{m}}t\right) - 0.238\delta\cos\left(2.110\sqrt{\frac{k}{m}}t\right) + 0.147\delta\cos\left(2.597\sqrt{\frac{k}{m}}t\right) \tag{n}$$

$$x_2(t) = 0.102\delta\cos\left(0.893\sqrt{\frac{k}{m}}t\right) + 0.174\delta\cos\left(2.110\sqrt{\frac{k}{m}}t\right) - 0.275\delta\cos\left(2.597\sqrt{\frac{k}{m}}t\right) \tag{o}$$

$$x_3(t) = 0.0389\delta\cos\left(0.893\sqrt{\frac{k}{m}}t\right) + 0.224\delta\cos\left(2.110\sqrt{\frac{k}{m}}t\right) + 0.736\delta\cos\left(2.597\sqrt{\frac{k}{m}}t\right) \tag{p}$$

Equations (n) through (p) are the same as Equation (e) of Example 8.4.

Equation (8.71) shows that the generalized coordinates are linear combinations of the principal coordinates. The generalized coordinates for a linear system are chosen such that the displacement of any particle in the system is a linear combination of the generalized coordinates. Thus, the displacement of any particle in the system is a linear combination of the principal coordinates. This implies that if a particle is a node for the higher mode of a two degree-of-freedom system, then p_1 is proportional to the displacement of that particle. If a particle is a node for the second mode of a three degree-of-freedom system, then a linear combination of the first and third principal coordinates represents the displacement of that point. Nothing can be inferred about the physical interpretation of either principal coordinate.

8.11 DETERMINATION OF NATURAL FREQUENCIES AND MODE SHAPES

The determination of the natural frequencies and mode shapes for a MDOF system requires the solution of a matrix eigenvalue-eigenvector problem. If the system has more than three degrees of freedom, the algebraic and computational burden usually leads one to seek approximate, numerical, or computer solutions. Rayleigh's quotient, presented in Section 8.9, may be used to provide an upper bound to the lowest natural frequency. In the Rayleigh-Ritz method for discrete systems, a linear combination of linearly independent vectors is used in Rayleigh's quotient. The coefficients in the linear combination are chosen to render Rayleigh's quotient stationary.

Most applications require more accurate determination of the natural frequencies and mode shapes than can be provided by Rayleigh's quotient or the Rayleigh-Ritz method. A number of numerical methods lead to accurate numerical determination of natural frequencies and mode shapes. One such is the matrix iteration method. Beginning with a trial mode shape vector $\mathbf{x}_0$, a sequence of vectors $\mathbf{x}_i$ is generated by

$$\mathbf{x}_i = \mathbf{AMx}_{i-1} \tag{8.72}$$

It can be shown that the ratio of two corresponding elements of $\mathbf{x}_i$ and $\mathbf{x}_{i-1}$ approaches ω_1^2 as i gets large and that $\mathbf{x}_i$ approaches the corresponding mode shape vector. Higher natural frequencies and mode shape vectors can be obtained by requiring trial vectors to be orthogonal with respect to the kinetic energy scalar product to all previously obtained mode shape vectors. Matrix iteration has the advantage that natural frequencies and mode shape vectors are determined sequentially and that only the number desired need to be determined.

Jacobi's method is a powerful iterative method that determines all eigenvalues and eigenvectors of a matrix. Jacobi's method uses a series of transformations to convert a symmetric matrix into a diagonal matrix with the eigenvalues along the diagonal. The product of the matrices used in the transformation produces a matrix whose columns are the eigenvectors. The mass and stiffness matrices for a MDOF system are guaranteed to be symmetric, but the matrix $\mathbf{M}^{-1}\mathbf{K}$, whose eigenvalues are the squares of the natural frequencies, is not necessarily symmetric. In this case, it can be shown that there exists a symmetric matrix $\mathbf{D}$ that can be obtained by a method called *Choleski decomposition*, such that the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$ are the same as the eigenvalues and eigenvectors of $\mathbf{D}$.

The above methods are described in other texts on vibrations or numerical analysis texts. These methods are tools that can be used to solve eigenvalue-eigenvector problems and thus, lead to natural frequencies and mode shapes for MDOF systems. However, understanding the mechanics of these methods does not enhance the understanding of vibrations. These methods have been incorporated into the eigenvalue routines used in MATLAB. These MATLAB routines are easy to use.

EXAMPLE 8.13

Study the accuracy of lumped-mass models to approximate the natural frequencies of a simply supported beam. Model the beam using 2, 3, 4, 5, 6, and 7 lumped masses. Compare the natural frequency approximations obtained when each lumped mass is m_b/n, where m_b is the total mass of the beam and n is the number of nodes, to the natural frequencies obtained when the method of Section 7.8 is used to obtain the nodal masses.

SOLUTION

A simply supported beam modeled with n lumped masses is illustrated in Figure 8.9. The nodal masses are of equal value

$$m = \frac{m_b}{\beta} \tag{a}$$

where β is a parameter dependent on the method of discretization. If the sum of the nodal masses equals the total mass of the beam, then $\beta = n$. If each nodal mass represents the mass of a region surrounding the particle, as described and illustrated in Section 7.8, then $\beta = n + 1$.

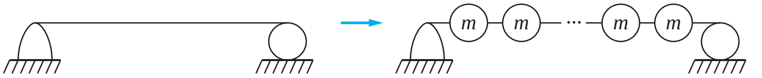


FIGURE 8.9
Lumped model of a simply supported beam by n masses. The generalized coordinates are the transverse displacements of the masses.

The generalized coordinates are the transverse displacements of the lumped masses. The mass matrix is a diagonal matrix with $m_{ii} = m$ as the diagonal element for $i = 1, 2, \ldots, n$.

Flexibility influence coefficients are used to determine the elements of the flexibility matrix. These elements are of the form

$$a_{ij} = \frac{L^3}{EI} q_{ij} \tag{b}$$

where q_{ij} is determined from Appendix D as

$$q_{ij} = \left(\frac{j}{n+1} - 1\right)\left(\frac{i}{n+1}\right)^3 + \frac{1}{6}\left(\frac{j}{n+1}\right)\left(1 - \frac{j}{n+1}\right)\left(2 - \frac{j}{n+1}\right)\left(\frac{i}{n+1}\right) \quad j \geq i \tag{c}$$

Symmetry of the flexibility matrix is used to determine q_{ij} for $j < i$.

The differential equations governing the free vibrations of the approximate system are

$$\phi \mathbf{Q}\ddot{\mathbf{x}} + \mathbf{x} = \mathbf{0} \tag{d}$$

where

$$\phi = \frac{L^3 m_b}{\beta EI} \tag{e}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues of $\phi\mathbf{Q}$, $\omega_i^2 = \frac{1}{\sqrt{\lambda_i}}$. The nondimensional natural frequencies are

$$\omega_i^* = \omega_i \sqrt{\frac{L^3 m_b}{EI}} \tag{f}$$

A MATLAB script is written to determine the non dimensional natural frequencies of the simply supported beam with n discrete masses for $n = 2, 3, \ldots, 7$. The eigenvalues of $\mathbf{Q}$ are summarized in Table 8.1.

TABLE 8.1 Nondimensional frequencies for simply supported beam

	Mode number						
ω	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$n = 2$	5.6922	22.046	—	—	—	—	—
$n = 3$	4.9333	19.596	41.607	—	—	—	—
$n = 4$	4.4133	17.637	39.988	64.202	—	—	—
$n = 5$	4.0290	16.100	36.000	62.356	89.194	—	—
$n = 6$	3.7302	14.913	33.456	58.826	88.776	116.19	—
$n = 7$	3.4894	13.954	31.348	55.427	85.221	117.68	145.52

TABLE 8.2

Dimensional frequencies assuming $\beta = n + 1$

	Mode number						
$\hat{\omega}$	1	2	3	4	5	6	7
Exact	9.8696	39.478	88.826	157.91	246.74	355.31	483.61
$n = 2$	9.8591	38.184	—	—	—	—	—
$n = 3$	9.8666	39.192	83.214	—	—	—	—
$n = 4$	9.8685	39.381	87.179	143.56	—	—	—
$n = 5$	9.8691	39.437	88.182	152.74	218.48	—	—
$n = 6$	9.8693	39.457	88.523	155.64	234.88	307.40	—
$n = 7$	9.8694	39.467	88.664	156.77	241.04	332.85	411.60

$\omega = \dot{\omega}\sqrt{\frac{EI}{\rho A L^4}}$ where ω is the dimensional natural frequency.

TABLE 8.3

Dimensional frequencies assuming $\beta = n$

	Mode number						
$\hat{\omega}$	1	2	3	4	5	6	7
Exact	9.8696	39.478	88.826	157.91	246.74	355.31	483.61
$n = 2$	8.0499	31.177	—	—	—	—	—
$n = 3$	8.5447	33.941	72.065	—	—	—	—
$n = 4$	8.8267	35.223	77.973	128.40	—	—	—
$n = 5$	9.0092	36.000	80.499	139.43	199.44	—	—
$n = 6$	9.1372	36.820	81.956	144.09	217.46	284.60	—
$n = 7$	9.2320	36.918	82.938	146.64	225.47	311.35	295.93

$\omega = \dot{\omega}\sqrt{\frac{EI}{\rho A L^4}}$ where ω is the natural frequency of a simply supported beam.

The natural frequency approximations using $\beta = n + 1$ are summarized in Table 8.2, while the natural frequency approximations for $\beta = n$ are summarized in Table 8.3. When the results are compared to the exact natural frequencies, obtained by the method of Chapter 10, it is clear that using $\beta = n + 1$ leads to a better approximation.

8.12 PROPORTIONAL DAMPING

A MDOF system is said to have *proportional damping* if the viscous damping matrix is a linear combination of the mass matrix and the stiffness matrix,

$$\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M} \tag{8.73}$$

where α and β are constants. The differential equations governing the free vibrations of a linear system with proportional damping are

$$\mathbf{M}\ddot{\mathbf{x}} + (\alpha\mathbf{K} + \beta\mathbf{M})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{8.74}$$

Let $\omega_1 \le \omega_2 \le \cdots \le \omega_n$ be the natural frequencies of an undamped system whose mass matrix is $\mathbf{M}$ and whose stiffness matrix is $\mathbf{K}$. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be the corresponding normalized mode shapes. The expansion theorem implies that $\mathbf{x}(t)$ can be written as a linear combination of the mode shape vectors, as in Equation (8.66). Substituting Equation (8.66) in Equation (8.74) leads to

$$\mathbf{M}\left(\sum_{i=1}^{n} \ddot{p}_i \mathbf{X}_i\right) + (\alpha\mathbf{K} + \beta\mathbf{M})\left(\sum_{i=1}^{n} \dot{p}_i \mathbf{X}_i\right) + \mathbf{K}\left(\sum_{i=1}^{n} p_i \mathbf{X}_i\right) \tag{8.75}$$

Taking the standard scalar product of Equation (8.75) with $\mathbf{X}_j$ for an arbitrary j, and using properties of scalar products and the definitions of energy scalar products, leads to

$$\sum_{i=1}^{n} \ddot{p}_i (\mathbf{X}_j, \mathbf{X}_i)_M + \sum_{i=1}^{n} \dot{p}_i [\alpha(\mathbf{X}_j, \mathbf{X}_i)_K + \beta(\mathbf{X}_j, \mathbf{X}_i)_M] + \sum^{n} p_i (\mathbf{X}_j, \mathbf{X}_i)_K = 0 \tag{8.76}$$

Use of the orthonormality relations, Equations (8.55) and (8.56), in Equation (8.76) leads to

$$\ddot{p}_j + (\alpha\omega_j^2 + \beta)\dot{p}_j + \omega_j^2 p_j = 0 \qquad j = 1, 2, \ldots, n \tag{8.77}$$

The principal coordinates are related to the original generalized coordinates through the linear transformation, Equation (8.71). Thus the same principal coordinates that uncouple the undamped system uncouple the system when proportional damping is added.

Equation (8.77) is analogous to the differential equation governing free vibrations of a SDOF system and by analogy, is rewritten as

$$\ddot{p}_j + 2\zeta_j \omega_j \dot{p}_j + \omega_j^2 p_j = 0 \tag{8.78}$$

where $$\zeta_j = \frac{1}{2}\left(\alpha\omega_j + \frac{\beta}{\omega_j}\right) \tag{8.79}$$

is called the *modal damping ratio.*

The general solution of Equation (8.78) for $\zeta_j < 1$ is

$$p_j(t) = A_j e^{-\zeta_j \omega_j t} \sin\left(\omega_j \sqrt{1 - \zeta_j^2}\, t - \phi_j\right) \tag{8.80}$$

where A_j and ϕ_j are determined from initial conditions. The generalized coordinates are obtained by using Equation (8.71).

Damping in structural systems is mostly hysteretic and hard to quantify. Lacking a better model, proportional damping is often assumed. The modal damping ratios are usually determined experimentally. The equivalent damping ratio for a harmonically excited SDOF system with hysteretic damping is proportional to the natural frequency, and inversely proportional to the excitation frequency. This model fits proportional damping where the damping matrix is proportional to the stiffness matrix. In these cases, the modes with higher frequencies are damped more than modes with lower frequencies. The natural frequencies in stiff structural systems are usually greatly separated. The effect of the higher modes in the free vibration response is often negligible.

EXAMPLE 8.14

The system of Examples 8.2 and 8.12 has damping added, as shown in Figure 8.10. The values of the parameters are $m = 2$ kg, $k = 200$ N/m, and $c = 17$ N · s/m. Motion of the system is initiated by moving the third mass a distance δ from equilibrium while holding the other masses in equilibrium and releasing the system from rest.

(a) Write the differential equations satisfied by the principal coordinates and determine the modal damping ratios.

(b) Find the free response of the system.

SOLUTION

The differential equations of motion are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 51 & -34 & 0 \\ -34 & 51 & -17 \\ 0 & -17 & 51 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} 600 & -400 & 0 \\ -400 & 600 & -200 \\ 0 & -200 & 600 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

The damping matrix is proportional to the stiffness matrix with

$$\alpha = \frac{c}{k} = \frac{17\text{N} \cdot \text{s/m}}{200 \text{ N/m}} = 0.085 \text{ s} \tag{b}$$

The natural frequencies of this system are given by Equation (f) of Example 8.2. They are calculated using the values of the parameters as

$$\omega_1 = 8.93 \text{ rad/s} \qquad \omega_2 = 21.1 \text{ rad/s} \qquad \omega_3 = 25.97 \text{ rad/s} \tag{c}$$

The modal damping ratios are

$$\zeta_1 = \frac{\alpha\omega_1}{2} = \frac{(0.085 \text{ s})(8.93 \text{ rad/s})}{2} = 0.380 \tag{d}$$

$$\zeta_2 = \frac{\alpha\omega_2}{2} = \frac{(0.085 \text{ s})(21.1 \text{ rad/s})}{2} = 0.900 \tag{e}$$

$$\zeta_3 = \frac{\alpha\omega_3}{2} = \frac{(0.085 \text{ s})(25.97 \text{ rad/s})}{2} = 1.10 \tag{f}$$

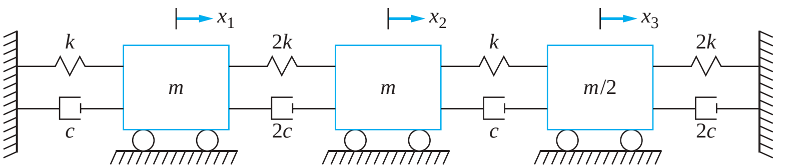


FIGURE 8.10
System of Example 8.14 is the system of Example 8.2, but with viscous damping added.

The first two modes are underdamped; the third is overdamped. The differential equations governing the principal coordinates are

$$\ddot{p}_1 + 6.78\dot{p}_1 + 79.75p_1 = 0 \quad \text{(g)}$$

$$\ddot{p}_2 + 37.84\dot{p}_2 + 445.2p_2 = 0 \quad \text{(h)}$$

$$\ddot{p}_3 + 57.33\dot{p}_3 + 674.4p_3 = 0 \quad \text{(i)}$$

(b) The solutions for the principal coordinates are

$$p_1(t) = A_1 e^{-3.39t} \sin(8.26t - \phi_1) \quad \text{(j)}$$

$$p_2(t) = A_2 e^{-18.92t} \sin(9.33t - \phi_2) \quad \text{(k)}$$

$$p_3(t) = A_3 e^{-19.24t} + A_4 e^{-40.46t} \quad \text{(l)}$$

The initial conditions that the principal coordinates must satisfy are those given in Equations (i) and (j) of Example 8.12. They are applied to Equations (j) through (l) to determine the constants of integration yielding

$$p_1(t) = 0.513\delta e^{-3.39t} \sin(8.26t + 1.81) \quad \text{(m)}$$

$$p_2(t) = 0.148\delta e^{-18.92t} \sin(9.33t + 0.484) \quad \text{(n)}$$

$$p_3(t) = -1.1584\delta e^{-19.24t} + 0.5514\delta e^{-40.46t} \quad \text{(o)}$$

The generalized coordinates are related to the principal coordinates by

$$\mathbf{x} = \mathbf{P}\mathbf{p} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.659 & -0.712 & -0.242 \\ 0.726 & 0.518 & 0.453 \\ 0.279 & 0.670 & -1.213 \end{bmatrix}\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} \quad \text{(p)}$$

which leads to

$$x_1(t) = \delta[0.0997e^{-3.39t} \sin(8.26t + 1.81) - 0.1056e^{-18.92t} \sin(9.33t + 0.484) + 0.2803e^{-19.24t} - 0.1334e^{-40.46t}] \quad \text{(q)}$$

$$x_2(t) = \delta[0.110e^{-3.39t} \sin(8.26t + 1.81) + 0.0678e^{-18.92t} \sin(9.33t + 0.484) - 0.5248e^{-19.24t} + 0.2498e^{-40.46t}] \quad \text{(r)}$$

$$x_3(t) = \delta[0.0422e^{-3.39t} \sin(8.26t + 1.81) + 0.0993e^{-18.92t} \sin(9.33t + 0.484) + 1.405e^{-19.24t} - 0.6688e^{-40.46t}] \quad \text{(s)}$$

8.13 GENERAL VISCOUS DAMPING

The differential equations governing the free vibrations of a MDOF system with viscous damping is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad \text{(8.81)}$$

If the damping matrix is a linear combination of the mass matrix and the stiffness matrix, the system is proportionally damped. In this case, the principal coordinates of the undamped system are used to uncouple the differential equations, Equation (8.76). The differential equation defining each principal coordinate is analogous to the differential equation governing the motion of a linear SDOF system with viscous damping.

If the damping matrix is arbitrary, the principal coordinates of the undamped system do not uncouple Equation (8.81). A more general procedure must be used. Equation (8.81) can be reformulated as $2n$ first-order differential equations by writing

$$\tilde{\mathbf{M}}\dot{\mathbf{y}} + \tilde{\mathbf{K}}\mathbf{y} = \mathbf{0} \tag{8.82}$$

where $$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \qquad \tilde{\mathbf{K}} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} \tag{8.83}$$

A solution to Equation (8.82) is assumed as

$$\mathbf{y} = \Phi e^{-\gamma t} \tag{8.84}$$

Substitution of Equation (8.84) into Equation (8.82) leads to

$$\gamma\tilde{\mathbf{M}}\Phi = \tilde{\mathbf{K}}\Phi \tag{8.85}$$

or $$\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}\Phi = \gamma\Phi \tag{8.86}$$

Thus the values of γ are the eigenvalues of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ and the vectors are the corresponding eigenvectors Φ.

The values of γ occur in complex conjugate pairs. The system is stable only if all eigenvalues have nonnegative real parts. Eigenvectors corresponding to complex conjugate eigenvalues are also complex conjugates of one another. Eigenvectors corresponding to eigenvalues which are not complex conjugates satisfy the orthogonality relation

$$\overline{\Phi}_i^T\tilde{\mathbf{M}}\Phi_j = \mathbf{0} \tag{8.87}$$

EXAMPLE 8.15

Plot the free-vibration response to the system of Figure 8.11 under the initial conditions $x_1(0) = 0$, $x_2(0) = 0.01$ m, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$.

SOLUTION

The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{a}$$

The damping matrix for this system is not a linear combination of the mass matrix and the stiffness matrix. Hence, the principal coordinates of the undamped system cannot be used to uncouple the differential equations. These equations are written in the form of Equation (8.82) where

$$\mathbf{y} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{bmatrix} \quad \tilde{\mathbf{M}} = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & 2m \\ m & 0 & 0 & 0 \\ 0 & 2m & 0 & c \end{bmatrix} \quad \tilde{\mathbf{K}} = \begin{bmatrix} -m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 3k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \tag{b}$$

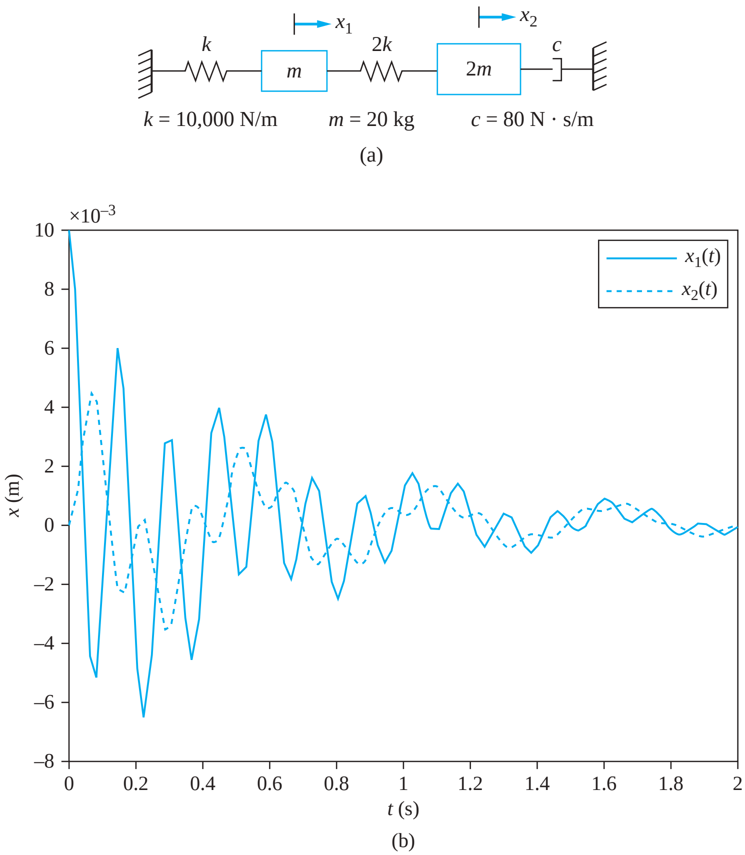


FIGURE 8.11
(a) System of Example 8.15 has a general viscous-damping matrix. (b) Free vibration response of system of Example 8.15.

A solution of Equation (8.82) is assumed in the form of Equation (8.84). The resulting values of γ are the eigenvalues of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$. The eigenvalues obtained by using MATLAB are

$$\gamma_{1,2} = 0.2110 \pm 43.19i \qquad \gamma_{3,4} = 0.7890 \pm 11.50i \tag{c}$$

The corresponding eigenvectors are

$$\mathbf{\Phi}_{1,2} = \begin{bmatrix} -0.924 \mp 0.166\text{i} \\ 0.340 \pm 0.0437i \\ 0.0039 \mp 0.0214i \\ -0.0011 \pm 0.0079i \end{bmatrix} \qquad \mathbf{\Phi}_{3,4} = \begin{bmatrix} 0.4984 \mp 0.3123 \\ 0.6871 \pm 0.4179i \\ 0.0240 \mp 0.0448i \\ 0.0320 \pm 0.0617i \end{bmatrix} \tag{d}$$

The general solution is a linear combination over all modes

$$y = \sum_{j=1}^{4} C_j \mathbf{\Phi}_j e^{-\gamma_i t} \tag{e}$$

where C_j are constants of integration. Application of initial conditions leads to

$$\mathbf{y}_0 = \sum_{j=1}^{4} C_j \Phi_j \tag{f}$$

Since the eigenvalues and eigenvectors are complex conjugate pairs, evaluation of the solution leads to a real response. Evaluation and plotting the response over a period of time leads to Figure 8.11(b).

8.14 BENCHMARK EXAMPLES

8.14.1 MACHINE ON FLOOR OF AN INDUSTRIAL PLANT

The differential equations for free vibration of the machine bolted to the beam illustrated in Figure 7.21 are taken from Equation (h) of Section 7.9.1 with the right-hand side equal to zero as

$$10^{-7}\begin{bmatrix} 3.34 & 5.73 & 112.2 & 3.13 \\ 5.73 & 14.23 & 318.5 & 9.31 \\ 5.31 & 15.19 & 421.8 & 13.21 \\ 3.13 & 9.31 & 277 & 11.25 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

The eigenvalues of **AM** are obtained using MATLAB as

$$\lambda_1 = 5.79 \times 10^{-8} \quad \lambda_2 = 2.19 \times 10^{-7} \quad \lambda_3 = 4.37 \times 10^{-7} \quad \lambda_4 = 4.43 \times 10^{-5} \tag{b}$$

The natural frequencies are reciprocals of the eigenvalues

$$\omega_1 = \frac{1}{\sqrt{\lambda_4}} = 150.2 \text{ rad/s} \qquad \omega_2 = \frac{1}{\sqrt{\lambda_3}} = 1.51 \times 10^3 \text{ rad/s}$$

$$\omega_3 = \frac{1}{\sqrt{\lambda_2}} = 2.14 \times 10^3 \text{ rad/s} \qquad \omega_4 = \frac{1}{\sqrt{\lambda_1}} = 4.16 \times 10^3 \text{ rad/s} \tag{c}$$

The mode shape vectors are

$$\mathbf{X}_1 = \begin{bmatrix} 0.1867 \\ 0.5263 \\ 0.6926 \\ 0.4566 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 0.5533 \\ 0.7226 \\ -0.0203 \\ -0.4113 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} 0.3044 \\ 0.2697 \\ -0.0423 \\ 0.9126 \end{bmatrix} \quad \mathbf{X}_4 = \begin{bmatrix} 0.7698 \\ -0.6335 \\ 0.0155 \\ -0.0764 \end{bmatrix} \tag{d}$$

The mode-shape vectors are illustrated in Figure 8.12.

The differential equations for free vibration of the machine connected to the beam by the isolator of stiffness 3.93×10^4 lbf/ft with the beam modeled with four degrees of freedom, illustrated in Figure 7.22(a), are obtained from Equation (l) of Section 7.9.1 as

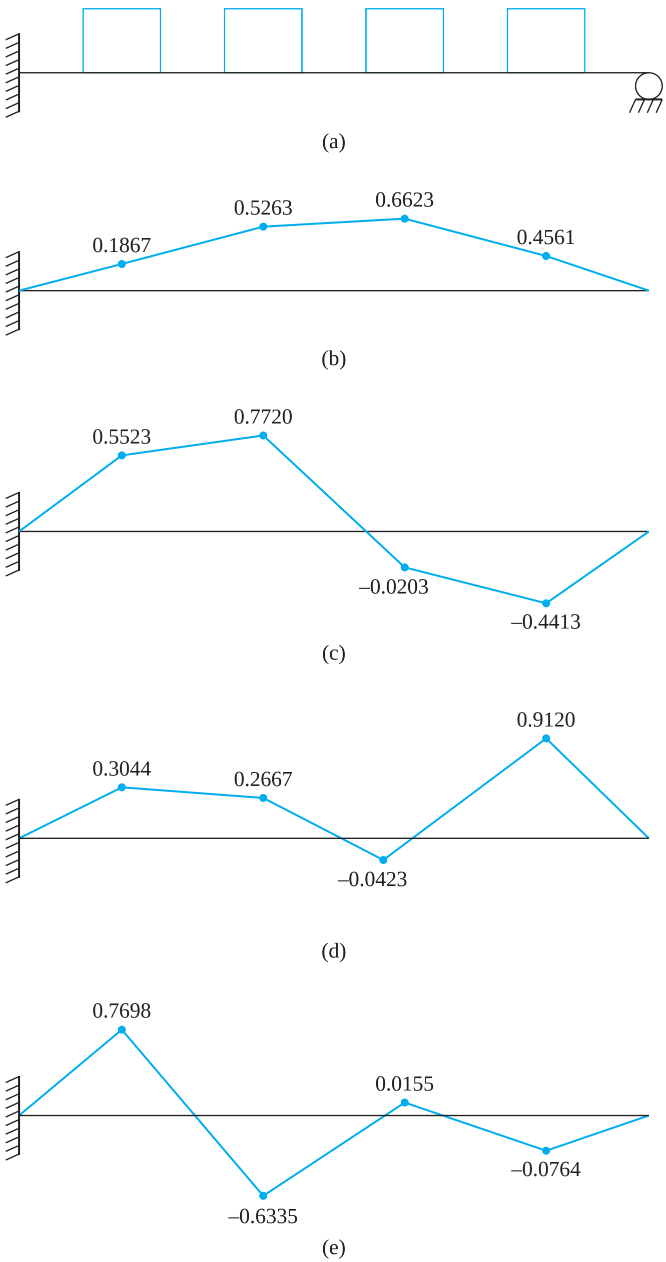


FIGURE 8.12
(a) Four degree-of freedom model of beam on floor of industrial plants. (b) Mode shape for first mode. (c) Mode shape for second mode. (d) Mode shape for third mode. (e) Mode shape for fourth mode.

$$10^{-7}\begin{bmatrix} 3.34 & 5.73 & 5.35 & 3.14 & 106.7 \\ 5.73 & 14.4 & 15.2 & 9.31 & 302.9 \\ 5.35 & 15.2 & 20.1 & 13.2 & 401.1 \\ 3.14 & 9.31 & 13.2 & 12.5 & 263.4 \\ 5.35 & 15.2 & 20.1 & 13.2 & 794.3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(d)}$$

The eigenvalues of **AM** are

$$\lambda_1 = 5.77 \times 10^{-8} \quad \lambda_2 = 1.27 \times 10^{-7} \quad \lambda_3 = 4.11 \times 10^{-7}$$
$$\lambda_4 = 4.10 \times 10^{-6} \quad \lambda_5 = 8.95 \times 10^{-4} \tag{e}$$

The natural frequencies are the reciprocals of the square roots of the eigenvalues

$$\omega_1 = \frac{1}{\sqrt{\lambda_5}} = 35.5 \text{ rad/s} \qquad \omega_2 = \frac{1}{\sqrt{\lambda_4}} = 494.1 \text{ rad/s}$$
$$\omega_3 = \frac{1}{\sqrt{\lambda_3}} = 1.56 \times 10^3 \text{ rad/s} \qquad \omega_4 = \frac{1}{\sqrt{\lambda_2}} = 2.89 \times 10^3 \text{ rad/s}$$
$$\omega_5 = \frac{1}{\sqrt{\lambda_1}} = 4.17 \times 10^3 \text{ rad/s} \tag{f}$$

8.14.2 SIMPLIFIED SUSPENSION SYSTEM

The differential equations governing the free vibrations of the four degree-of-freedom model suspension system illustrated in Figure 7.23 are

$$\begin{bmatrix} 225 & 0 & 0 & 0 \\ 0 & 300 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + 10^{-3} \begin{bmatrix} 5.50 & -0.48 & -1.56 & 2.04 \\ -0.48 & 2.4 & -1.2 & -1.2 \\ -1.56 & -1.2 & 1.12 & 0 \\ 2.04 & -1.2 & 0 & 1.12 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ 10^{-4} \begin{bmatrix} 5.50 & -0.48 & -1.56 & 2.04 \\ -0.48 & 2.4 & -1.2 & -1.2 \\ -1.56 & -1.2 & 1.12 & 0 \\ 2.04 & -1.2 & 0 & 1.12 \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

The system is proportionally damped with the damping matrix proportional to the stiffness matrix with

$$\alpha = \frac{1200 \text{ N} \cdot \text{s/m}}{12{,}000 \text{ N/m}} = 0.1 \text{s} \tag{b}$$

Thus, the methods of Section 8.12 are applicable. The natural frequencies and mode shapes for the undamped system are found by finding the square roots of the eigenvalues of

$$\mathbf{M}^{-1}\mathbf{K} = 10^4 \begin{bmatrix} 4.44 \times 10^{-3} & 0 & 0 & 0 \\ 0 & 3.33 \times 10^{-3} & 0 & 0 \\ 0 & 0 & 4 \times 10^{-2} & 0 \\ 0 & 0 & 0 & 4 \times 10^{-2} \end{bmatrix}$$
$$\times \begin{bmatrix} 5.50 & -0.48 & -1.56 & 2.04 \\ -0.48 & 2.4 & -1.2 & -1.2 \\ -1.56 & -1.2 & 11.2 & 0 \\ 2.04 & -1.2 & 0 & 11.2 \end{bmatrix} = \begin{bmatrix} 244.4 & -21.3 & -69.3 & 90.7 \\ -16 & 80.0 & -40 & -40 \\ -624 & -480 & 4480 & 0 \\ 816 & -480 & 0 & 4480 \end{bmatrix} \tag{c}$$

The eigenvalues and normalized mode shapes are obtained from MATLAB as

$$\lambda_1 = 69.5 \qquad \lambda_2 = 218.7 \qquad \lambda_3 = 4485 \qquad \lambda_4 = 4507 \tag{d}$$

$$\mathbf{X}_1 = \begin{bmatrix} 0.0573 \\ 0.0049 \\ 0.0073 \\ 0.0074 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 0.0064 \\ 0.0134 \\ -0.0090 \\ -0.660 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} 0.0025 \\ -0.1112 \\ -0.1660 \\ 0.0003 \end{bmatrix} \quad \mathbf{X}_4 = \begin{bmatrix} -0.0005 \\ 0.1656 \\ -0.1110 \\ 0.0053 \end{bmatrix} \tag{e}$$

The natural frequencies are the square roots of the eigenvalues

$$\omega_1 = \sqrt{\lambda_1} = 8.33 \text{ rad/s} \quad \omega_2 = \sqrt{\lambda_2} = 14.79 \text{ rad/s} \quad \omega_3 = \sqrt{\lambda_3} = 67.0 \text{ rad/s}$$

$$\omega_4 = \sqrt{\lambda_4} = 67.1 \text{ rad/s} \tag{f}$$

The modal damping ratios are

$$\zeta_1 = \frac{\alpha}{2}\omega_1 = 0.417 \qquad \zeta_2 = \frac{\alpha}{2}\omega_2 = 0.740 \qquad \zeta_3 = \frac{\alpha}{2}\omega_3 = 3.35$$

$$\zeta_4 = \frac{\alpha}{2}\omega_4 = 3.36 \tag{g}$$

The differential equations for the principal coordinates are given by Equation (8.73) that when applied to this problem become

$$\ddot{p}_1 + 6.94\dot{p}_1 + 69.5p_1 = 0 \tag{h}$$

$$\ddot{p}_2 + 21.9\dot{p}_2 + 218.7p_2 = 0 \tag{i}$$

$$\ddot{p}_3 + 448.9\dot{p}_3 + 4485p_3 = 0 \tag{j}$$

$$\ddot{p}_4 + 450.9\dot{p}_4 + 4507p_4 = 0 \tag{k}$$

The solutions of Equations (h) through (k) are

$$p_1(t) = A_1e^{-3.47t}\sin(7.58t - \phi_1) \tag{l}$$

$$p_2(t) = A_2e^{-11.950t}\sin(9.96t - \phi_2) \tag{m}$$

$$p_3(t) = A_3e^{-10.22t} + A_4e^{-438.7t} \tag{n}$$

$$p_4(t) = A_5e^{-10.23t} + A_6e^{-440.7t} \tag{o}$$

The principal coordinates are related to the generalized coordinates by **x=Pp** where **P** is the modal matrix, or the matrix whose columns are the normalized eigenvectors

$$\mathbf{P} = \begin{bmatrix} 0.0573 & 0.0064 & 0.0025 & -0.0005 \\ 0.0049 & 0.0134 & -0.1112 & 0.1656 \\ 0.0073 & -0.0090 & -0.1660 & -0.1110 \\ 0.0074 & -0.660 & 0.0003 & 0.0053 \end{bmatrix} \tag{p}$$

8.15 FURTHER EXAMPLES

EXAMPLE 8.16

Reconsider the three degree-of-freedom model of the hand and upper arm of Example 7.21. Dong et al. report the following data for the parameters in the model for the "grip" condition,

$$m_1 = 5.0516 \text{ kg} \quad m_2 = 1.4295 \text{ kg} \quad m_3 = 0.887 \text{ kg} \quad m_4 = 0.0229 \text{ kg}$$
$$m_5 = 0.0150 \text{ kg}$$
$$k_1 = 149{,}490 \text{ N/m} \quad k_2 = 1726 \text{ N/m} \quad k_3 = 12{,}075 \text{ N/m} \quad k_4 = 29{,}898 \text{ N/m}$$
$$k_5 = 195{,}665 \text{ N/m}$$
$$c_1 = 87.2 \text{ N}\cdot\text{s/m} \quad c_2 = 64.9 \text{ N}\cdot\text{s/m} \quad c_3 = 36.3 \text{ N}\cdot\text{s/m} \quad c_4 = 74.8 \text{ N}\cdot\text{s/m}$$
$$c_5 = 126.0 \text{ N}\cdot\text{s/m}$$

(a) Determine the natural frequencies of free undamped vibration and the normalized mode shapes.

(b) Determine the general form of the solution for the damped response.

SOLUTION

(a) Substituting the given values into Equation (i) of Example 7.21 leads to the following differential equations as

$$\begin{bmatrix} 5.0516 & 0 & 0 \\ 0 & 1.4295 & 0 \\ 0 & 0 & 0.887 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 152.1 & -64.9 & 0 \\ -64.9 & 176.0 & -36.3 \\ 0 & -36.3 & 111.1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+\begin{bmatrix} 151{,}216 & -1726 & 0 \\ -1726 & 43{,}699 & -12{,}075 \\ 0 & -12{,}075 & 207{,}740 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 74.8\dot{y} + 29{,}898y \\ 126\dot{y} + 195{,}695y \end{bmatrix} \tag{a}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. They are calculated as

$$\omega_1 = 171.2 \text{ rad/s} \qquad \omega_2 = 175.0 \text{ rad/s} \qquad \omega_3 = 484.5 \text{ rad/s} \tag{b}$$

The mode-shape vectors are the corresponding eigenvectors. The eigenvectors are normalized such that $\mathbf{X}_i^T\mathbf{M}\mathbf{X}_i = 1$. They are obtained as

$$\mathbf{X}_1 = \begin{bmatrix} 0.3233 \\ 0.5738 \\ 0.0381 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 0.3057 \\ -0.6069 \\ -0.0406 \end{bmatrix} \qquad \mathbf{X}_3 = \begin{bmatrix} 7.3 \times 10^{-4} \\ -0.0439 \\ 1.0603 \end{bmatrix} \tag{c}$$

(b) The damped system is written in the state-space formulation of Equation (8.82) with

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 5.0516 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4295 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.887 \\ 5.0516 & 0 & 0 & 152.1 & -64.9 & 0 \\ 0 & 1.4295 & 0 & -64.9 & 176.0 & -36.3 \\ 0 & 0 & 0.887 & 0 & -36.3 & 111.1 \end{bmatrix} \tag{d}$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} = \begin{bmatrix} -5.0516 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.4295 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.887 & 0 & 0 & 0 \\ 0 & 0 & 0 & 151{,}216 & -1726 & 0 \\ 0 & 0 & 0 & -1726 & 43{,}699 & -12{,}075 \\ 0 & 0 & 0 & 0 & -12{,}075 & 207{,}740 \end{bmatrix} \tag{e}$$

The assumed solution is $\boldsymbol{y} = \Phi e^{-\gamma t}$ where $\boldsymbol{y} = \begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix}$. The values of γ are the eigenvalues of $\tilde{\mathbf{M}}^{-1}\,\tilde{\mathbf{K}}$. They are

$$\gamma_{1,2} = 12.06 \pm 171.7i \quad \gamma_{3,4} = 63.17 \pm 162.3i \quad \gamma_{1,2} = 64.01 \pm 479.1i \tag{f}$$

The corresponding eigenvectors are

$$\Phi_{\mathbf{1,2}} = 10^{-2}\begin{bmatrix} 90.29 \\ 42.89e^{\mp 0.126i} \\ 3.134e^{\pm 0.2085i} \\ 0.5245e^{\mp 1.641i} \\ 0.2489e^{\mp 1.760i} \\ 0.0182e^{\mp 1.360i} \end{bmatrix} \qquad \Phi_{\mathbf{3,4}} = 10^{-2}\begin{bmatrix} 12.76e^{\pm 3.020i} \\ 98.98 \\ 6.255e^{\pm 0.5439i} \\ 0.0731e^{\pm 1.079i} \\ 0.5673e^{\mp 1.941i} \\ 0.0359e^{\mp 1.397i} \end{bmatrix}$$

$$\Phi_{\mathbf{5,6}} = 10^{-2}\begin{bmatrix} 0.2082e^{\pm 2.4639i} \\ 6.841e^{\mp 2.0931} \\ 98.763 \\ 0.0004e^{\pm 0.7603i} \\ 0.0142e^{\pm 2.486i} \\ 0.2064e^{\mp 1.7036i} \end{bmatrix} \tag{g}$$

The general solution is

$$\mathbf{y}(t) = e^{-12.06t}\{C_1\Phi_{\mathbf{1}}e^{i171.7t} + C_2\overline{\Phi}_{\mathbf{1}}e^{-i171.7t}\} + e^{-63.176t}\{C_3\Phi_{\mathbf{3}}e^{i162.3t} + C_4\overline{\Phi}_{\mathbf{3}}e^{-i162.3t}\}$$
$$+ e^{-64.01t}\{C_5\Phi_{\mathbf{5}}e^{i479.1t} + C_2\overline{\Phi}_5e^{-i479.1t}\} \tag{h}$$

Equation (h) can be rewritten as

$$\begin{aligned} \mathbf{y}(t) = {} & e^{-12.06t}\{C_1[\Phi_{\mathbf{1_r}}\cos 171.1t - \Phi_{\mathbf{1i}}\sin 171.1t] \\ & + C_2[\Phi_{\mathbf{1_r}}\sin 171.1t + \Phi_{\mathbf{1i}}\cos 171.1t]\} \\ & + e^{-63.176t}\{C_3[\Phi_{\mathbf{3_r}}\cos 162.3t - \Phi_{\mathbf{3i}}\sin 162.3t] \\ & + C_4[\Phi_{\mathbf{3_r}}\sin 162.3t + \Phi_{\mathbf{3i}}\sin 162.3t]\} \\ & + e^{-64.01t}\{C_5[\Phi_{\mathbf{5_r}}\cos 479.1t - \Phi_{\mathbf{5i}}\sin 479.1t]\} \\ & + C_6[\Phi_{\mathbf{5_r}}\sin 479.1t + \Phi_{\mathbf{5i}}\cos 479.1t]\} \end{aligned} \tag{i}$$

EXAMPLE 8.17

(a) Determine the natural frequencies for the three degree-of-freedom system shown in Figure 8.13.
(b) Calculate and graphically illustrate the normalized mode shape vectors.
(c) Demonstrate mode shape orthogonality.

SOLUTION

The differential equations governing the system may be formulated using Newton's law or Lagrange's equations (the full set or via influence coefficients as)

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

(a) The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$,

$$\det(\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}) = \mathbf{0} \Rightarrow \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{2m} \end{bmatrix}\begin{bmatrix} 2k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\phi - \lambda & -4\phi & 0 \\ -4\phi & 6\phi - \lambda & -2\phi \\ 0 & -\phi & \phi - \lambda \end{bmatrix} \tag{b}$$

where $\phi = \frac{k}{2m}$. Evaluation of the determinant in Equation (b) leads to

$$-\lambda^3 + 11\phi\lambda^2 - 16\phi^2\lambda = 0 \tag{c}$$

The smallest toot of Equation (c) is $\lambda = 0$. The system is unrestrained. The other roots are obtained by solving

$$\lambda^2 - 11\phi\lambda + 16\phi^2 = 0 \tag{d}$$

The solutions of Equation (b) are

$$\lambda = 0,\ 1.725\phi,\ 9.275\phi \tag{e}$$

from which the natural frequencies are obtained as

$$\omega_1 = 0 \qquad \omega_2 = \sqrt{\frac{1.725k}{2m}} = 0.928\sqrt{\frac{k}{m}} \qquad \omega_3 = \sqrt{\frac{9.275k}{2m}} = 2.15\sqrt{\frac{k}{m}} \tag{f}$$

(b) The mode shape vectors are determined from

$$\begin{bmatrix} 4\phi - \lambda & -4\phi & 0 \\ -4\phi & 6\phi - \lambda & -2\phi \\ 0 & -\phi & \phi - \lambda \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{g}$$

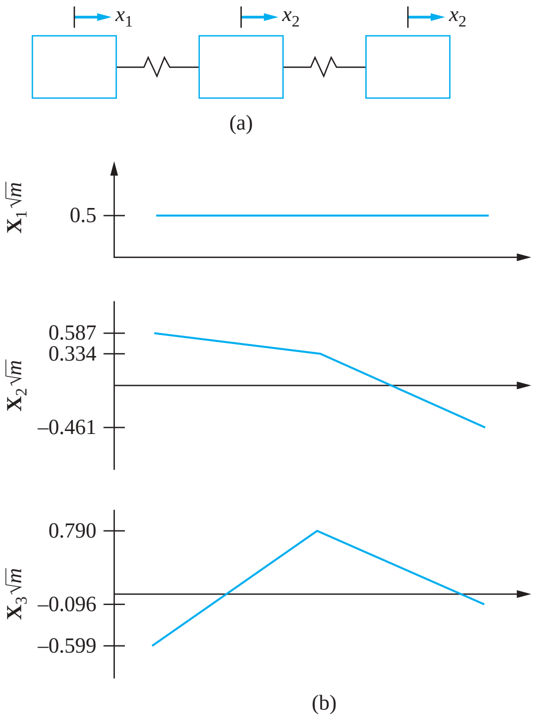


FIGURE 8.13
System of Example 8.17.

The first equation gives

$$X_1 = \frac{4\phi}{4\phi - \lambda} X_2 \tag{h}$$

while the third equation gives

$$X_3 = \frac{\phi}{\phi - \lambda} X_2 \tag{i}$$

When evaluated for the values of λ which are the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ and keeping $X_2 = C$, arbitrary Equations (h) and (i) yield

$$\mathbf{X}_1 = C\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{X}_2 = C\begin{bmatrix} 1.758 \\ 1 \\ -1.379 \end{bmatrix} \qquad \mathbf{X}_3 = C\begin{bmatrix} -0.758 \\ 1 \\ -0.121 \end{bmatrix} \tag{j}$$

The mode shapes are normalized by requiring $(\mathbf{X}_i, \mathbf{X}_i)_\mathbf{M} = \mathbf{X}_i^T\mathbf{M}\mathbf{X}_i = 1$. For example, the normalization of $\mathbf{X}_2$ chooses C such that

$$\mathbf{X}_2^T\mathbf{M}\mathbf{X}_2 = 1 \Rightarrow 1 = C[1.758 \quad 1 \quad -1.3679]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} C\begin{bmatrix} 1.758 \\ 1 \\ -1.379 \end{bmatrix}$$

$$= C^2[1.758 \quad 1 \quad -1.3679]\begin{bmatrix} 1.758m \\ m \\ -2.758m \end{bmatrix}$$

$$= C^2[1.758(1.758m) + 1(m) - 1.479(-2.758m)] = C^2(8.939m) \tag{k}$$

or $C = \frac{0.334}{\sqrt{m}}$. Similar calculations are performed yielding the normalized mode-shape vectors as

$$\mathbf{X}_1 = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \qquad \mathbf{X}_2 = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.587 \\ 0.334 \\ -0.461 \end{bmatrix} \qquad \mathbf{X}_3 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.599 \\ 0.790 \\ -0.096 \end{bmatrix} \tag{l}$$

The normalized mode shapes are illustrated in Figure 8.12(b). The first mode is a rigid-body mode corresponding to the natural frequency of zero. There is a node for the second mode in the spring connecting the second and third masses. Two nodes mark the third mode. One is in the spring connecting the first two masses; the second is in the spring connecting the second and third mass, but not in the same location as the node for the second mode.

(a) Mode-shape orthogonality implies $(\mathbf{X}_i, \mathbf{X}_j)_M = \mathbf{X}_j^{\mathrm{T}}\,\mathbf{M}\mathbf{X}_i = 0$ for $i \neq j$. The demonstration of this relation follows

$$(\mathbf{X}_1, \mathbf{X}_2)_M = \mathbf{X}_2^T\mathbf{M}\,\mathbf{X}_1 = \frac{1}{\sqrt{\mathrm{m}}}[0.587 \quad 0.334 \quad -0.461]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\frac{1}{\sqrt{m}}\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$= \frac{1}{m}[0.587 \quad 0.334 \quad -0.461]\begin{bmatrix} 0.5m \\ 0.5m \\ m \end{bmatrix}$$

$$= 0.587(0.5) + 0.334(0.5) - 0.461(1) = 0 \tag{m}$$

$$(\mathbf{X}_2, \mathbf{X}_3)_M = \mathbf{X}_3^T\mathbf{M}\mathbf{X}_2 = \frac{1}{\sqrt{\mathrm{m}}}[-0.599 \ \ 0.790 \ \ -0.096]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\frac{1}{\sqrt{m}}\begin{bmatrix} 0.587 \\ 0.334 \\ -0.461 \end{bmatrix}$$

$$= \frac{1}{m}[-0.599 \quad 0.790 \quad -0.096]\begin{bmatrix} 0.587m \\ 0.334m \\ 0.922m \end{bmatrix}$$

$$= -0.599(0.587) + 0.790(0.334) - 0.096(-0.922) = 0 \tag{n}$$

$$(\mathbf{X}_3, \mathbf{X}_1)_M = \mathbf{X}_2^T\mathbf{M}\mathbf{X}_1 = \frac{1}{\sqrt{\text{m}}}[0.5 \quad 0.5 \quad -0.5]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\frac{1}{\sqrt{m}}\begin{bmatrix} -0.599 \\ 0.790 \\ -0.096 \end{bmatrix} \quad \text{(o)}$$

$$= \frac{1}{m}[0.5 \quad 0.5 \quad -0.5]\begin{bmatrix} -0.599m \\ 0.790m \\ -0.192m \end{bmatrix}$$

$$= 0.5(-0.599) + 0.5(0.790) + 0.5(-0.192) = 0$$

8.16 SUMMARY

8.16.1 IMPORTANT CONCEPTS

- A nDOF system is governed by n differential equations and has n natural frequencies.
- The natural frequencies of a nDOF system are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$.
- The natural frequencies of a nDOF system are the reciprocals of the square roots of the eigenvalues of **AM**.
- The mode-shape vectors are the corresponding eigenvectors.
- The general solution for the free response is a linear combination of the modes. The constants in the linear combination are determined by application of the initial conditions.
- A degenerate system has repeated natural frequencies.
- An unrestrained system has its lowest natural frequency equal to zero.
- Mode shapes corresponding to distinct frequencies of a MDOF system are mutually orthogonal with respect to the kinetic energy scalar product as well as the potential energy scalar product.
- All eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real.
- If **K** is positive definite, then all eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are positive.
- Any n-dimensional vector can be expanded in a series of mode-shape vectors of a nDOF system.
- Mode-shape vectors are normalized with respect to the kinetic-energy scalar product.
- Principal coordinates are coordinates which uncouple the differential equations.
- The principal coordinates are a linear transformation from the original generalized coordinates.
- The differential equations for a system with proportional damping are uncoupled by the same principal coordinates that uncouple the corresponding undamped system.
- The n second-order equations governing a system with viscous damping are reformulated as $2n$ first-order differential equations for solution.

8.16.2 IMPORTANT EQUATIONS

Normal mode solution

$$\mathbf{x}(t) = \mathbf{X}e^{i\omega t} \tag{8.4}$$

Equations defining mode shapes

$$(\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I})\mathbf{X} = \mathbf{0} \tag{8.7}$$

Natural frequencies from the stiffness matrix

$$\det|\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}| = 0 \tag{8.9}$$

Equations defining mode shapes from flexibility matrix

$$(-\omega^2\mathbf{A}\mathbf{M} + \mathbf{I})\mathbf{X} = \mathbf{0} \tag{8.10}$$

General solution

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i A_i \sin(\omega_i t - \phi_i) \tag{8.16}$$

Potential-energy scalar product

$$(\mathbf{y},\mathbf{z})_K = \mathbf{z}^T\mathbf{K}\mathbf{y} \tag{8.25}$$

Kinetic-energy scalar product

$$(\mathbf{y},\mathbf{z})_M = \mathbf{z}^T\mathbf{M}\mathbf{y} \tag{8.26}$$

Mode-shape orthogonality

$$(\mathbf{X}_i, \mathbf{X}_j)_M = 0 \tag{8.45}$$

$$(\mathbf{X}_i, \mathbf{X}_j)_K = 0 \tag{8.46}$$

Expansion theorem

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{X}_i \tag{8.49}$$

$$c_j = \frac{(\mathbf{X}_j, \mathbf{y})_M}{(\mathbf{X}_j \mathbf{X}_j)_M} \tag{8.52}$$

Normalized mode shapes

$$(\mathbf{X}_i, \mathbf{X}_i)_M = 1 \tag{8.53}$$

$$(\mathbf{X}_i, \mathbf{X}_i)_K = \omega_i^2 \tag{8.54}$$

Rayleigh's quotient

$$R(\mathbf{X}) = \frac{(\mathbf{X},\mathbf{X})_K}{(\mathbf{X},\mathbf{X})_M} \tag{8.62}$$

Principal coordinates

$$\ddot{p}_j + \omega_j^2 p_j = 0 \tag{8.70}$$

$$\mathbf{x} = \mathbf{P}\mathbf{p} \tag{8.71}$$

Proportional damping

$$\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M} \tag{8.73}$$

Principal coordinates for proportional damping

$$\ddot{p}_j + 2\zeta_j\omega_j\dot{p}_j + \omega_j^2 p_j = 0 \tag{8.78}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 8.1 through 8.18, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

8.1 The natural frequencies of a MDOF system are the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$.

8.2 An n degree-of-freedom system has $n + 1$ natural frequencies.

8.3 The mode-shape vector is the solution of $(\mathbf{AM} - \frac{1}{\omega^2}\mathbf{I})\mathbf{X} = 0$.

8.4 A node for a mode is a particle that has zero displacement when the vibrations are solely at that frequency.

8.5 The mode-shape vectors are orthogonal with respect to the standard inner product. That is, $\mathbf{X}_j^T\mathbf{X}_i = 0$.

8.6 The mode-shape vector corresponding to a natural frequency ω for a MDOF system is unique.

8.7 The eigenvectors are normalized by requiring that the kinetic-energy inner product of a mode-shape vector with itself is one.

8.8 The modal matrix is the transpose of the matrix whose columns are the normalized mode-shape vectors.

8.9 Proportional damping occurs when the damping matrix is proportional to the flexibility matrix.

8.10 The natural frequencies of a nDOF system are the roots of a nth-order polynomial.

8.11 $\mathbf{P}^{\mathbf{T}}\mathbf{MP} = \mathbf{I}$ where $\mathbf{P}$ is the modal matrix and $\mathbf{I}$ is the identity matrix.

8.12 If $\mathbf{X}_i$ is a normalized mode shape corresponding to a natural frequency ω_i, then $(\mathbf{X}_i, \mathbf{X}_i)_K = \omega_i^2$.

8.13 The lowest natural frequency when det $\mathbf{K} = 0$ is zero.

8.14 The flexibility matrix does not exist for an unrestrained system.

8.15 Rayleigh's quotient can be applied to obtain a lower-bound on the lowest natural frequency.

8.16 The damping ratio for a proportionally damped system where the proportional damping is proportional to the stiffness matrix is inversely proportional to the natural frequency.

8.17 Matrix iteration is a method used to determine natural frequencies of a MDOF system iteratively.

8.18 If $[1\ 2]^T$ is a mode-shape vector corresponding to a natural frequency of 100 rad/s for a two non-degenerate system, then $[2\ 6]^T$ is also a mode-shape vector corresponding to 100 rad/s.

Problems 8.19 through 8.39 require a short answer.

8.19 What is the normal mode solution?
8.20 What is the dynamical matrix?
8.21 The natural frequencies of an nDOF system are the ________________ of the eigenvalues of **AM**.
8.22 The natural frequencies and mode-shape vectors for a nDOF system have been determined. How is the free response of the system determined?
8.23 What is the name for the mode corresponding to a natural frequency equal to zero?
8.24 How many linearly independent mode-shape vectors correspond to a natural frequency that is a double root of the characteristic equation?
8.25 Define the potential-energy scalar product.
8.26 What does the term "kinetic energy" refer to in the kinetic-energy scalar product?
8.27 How is the property of commutivity of scalar products satisfied for the kinetic-energy scalar product?
8.28 What is meant by mode-shape orthogonality?
8.29 What is a normalized mode-shape vector?
8.30 Define Rayleigh's quotient for an arbitrary n-dimensional vector.
8.31 When is Rayleigh's quotient stationary?
8.32 Why is the modal matrix nonsingular?
8.33 State the expansion theorem.
8.34 What are the principal coordinates for an undamped, linear MDOF system?
8.35 How is matrix iteration used to approximate the lowest natural frequency of a MDOF system?
8.36 What is the modal damping ratio?
8.37 Why can the principal coordinates of an undamped system be used as principal coordinates for a viscously damped system with proportional damping?
8.38 If the lowest natural frequency of a system is zero, what is det $\mathbf{M}^{-1}\mathbf{K}$?
8.39 How many nodes located in the system should be expected for the third mode of a seven degree-of-freedom system?

Problems 8.40 through 8.51 require a short calculation.

8.40 The eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are 20, 50, and 100. What are the eigenvalues of **AM**?
8.41 The eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are 16, 49, 100, and 225. What are the natural frequencies of the system?
8.42 For the system of Figure SP8.42, calculate $(\mathbf{x}, \mathbf{y})_K$ for $\mathbf{x} = [3 \;\; 2 \;\; -1]^T$ and $\mathbf{y} = [1 \;\; -2 \;\; 3]^T$.

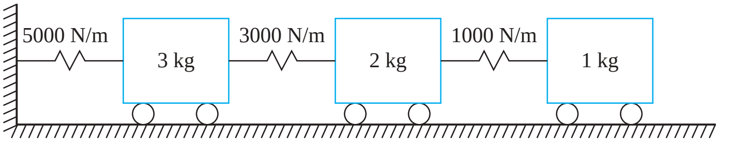


FIGURE SP 8.42

8.43 For the system of Figure SP8.42, calculate Rayleigh's quotient for $\mathbf{x} = [3 \;\; 2 \;\; -1]^T$.

8.44 A mode shape vector of a two degree-of-freedom system is $[1\ 2]^T$. The mass matrix for the system is $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Calculate the second mode-shape vector.

8.45 A mode-shape vector of a two degree-of-freedom system is $[1\ 2]^T$. Is this the mode-shape vector for the first mode, which corresponds to the lowest natural frequency, or the higher mode? Why?

8.46 A mode-shape vector of a two degree-of-freedom system is $[1\ 2]^T$. The mass matrix for the system is $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Normalize the mode-shape vector.

8.47 A normalized mode-shape vector for a two degree-of-freedom system is $[0.1\quad 0.3]^T$. The stiffness matrix for the system is $\boldsymbol{K} = \begin{bmatrix} 200 & -100 \\ -100 & 300 \end{bmatrix}$. Calculate the natural frequency corresponding to this mode.

8.48 Can the vectors $[1\ \ 2\ \ 2.5]^T$ and $[1\ \ 2\ \ -2]^T$ be mode shape vectors of a system with a diagonal mass matrix with all three diagonal elements equal?

8.49 A three degree-of-freedom undamped system has natural frequencies of 10 rad/s, 25 rad/s, and 50 rad/s. What are the differential equations satisfied by the principal coordinates for the system for free vibration?

8.50 A three degree-of-freedom system with viscous damping that is proportional to the stiffness matrix has natural frequencies of 10 rad/s, 25 rad/s, and 50 rad/s. The modal damping ratio for the first mode is 0.1.

(a) What are the modal damping ratios for the higher modes?

(b) Write the differential equations satisfied by the principal coordinates for free vibrations of the system.

8.51 A system has the differential equations

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 50 & -20 & 0 \\ -20 & 100 & -80 \\ 0 & -80 & 120 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Write the system of differential equations as six first-order differential equations.

8.52 Lagrange's equations are used to derive the differential equations for a three degree-of-freedom system resulting in

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

where x_1 and x_2 are linear displacements and θ is an angular coordinate. The damping matrix is such that the system has proportional damping. What are possible units (in SI) for each of the following quantities.

(a) The third natural frequency ω_3

(b) The modal damping ratio ζ_2

(c) The constant of proportionality between the damping matrix and the stiffness matrix α

(d) The third element of the normalized mode-shape vector for the first mode
(e) The second element of the normalized mode-shape vector for the third mode
(f) The principal coordinate p_1
(g) The element of the modal matrix in the first row and second column
(h) The element of the modal matrix in the third row and third column
(i) The constant of proportionality between the mass matrix and the damping matrix

CHAPTER PROBLEMS

8.1–8.7 Calculate the natural frequencies and mode shapes for the system shown in Figures P8.1 through P8.7 by calculating the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. Graphically illustrate the mode shapes. Identify any nodes.

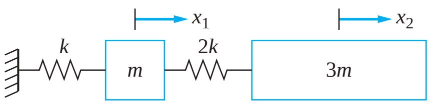


FIGURE P8.1

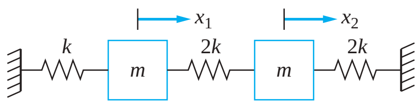


FIGURE P8.2

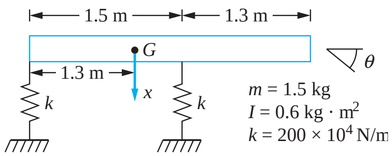


FIGURE P8.3
(Problems 8.3, 8.22.)

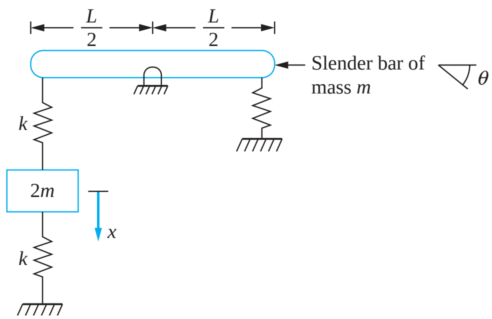


FIGURE P8.4

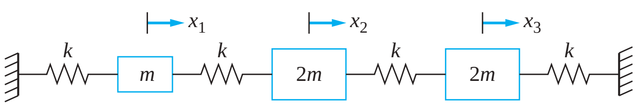


FIGURE P8.5

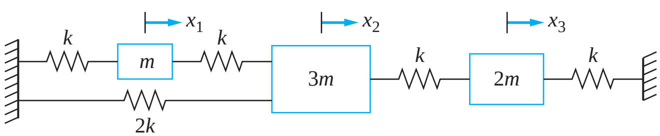


FIGURE P8.6

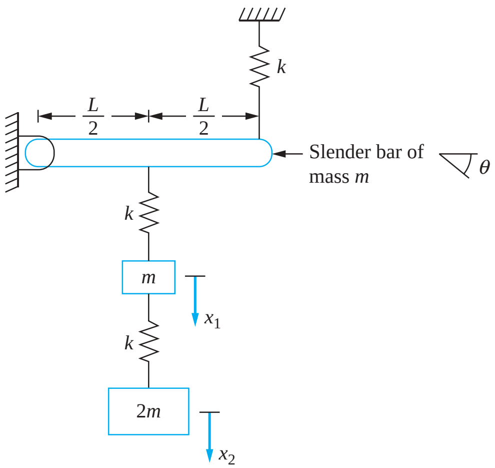


FIGURE P8.7

8.8 Two machines are placed on the massless fixed-pinned beam of Figure P8.8. Determine the natural frequencies for the system.

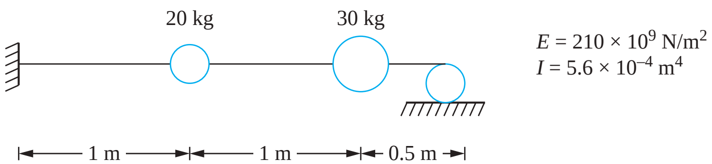


FIGURE P8.8

8.9 Determine the natural frequencies and mode shapes for the system of Figure P7.2 if $k = 3.4 \times 10^5$ N/m, $L = 1.5$ m and $m = 4.6$ kg.

8.10 Determine the natural frequencies of the system of Figure P7.5 if $k = 2500$ N/m, $m_1 = 2.4$ kg, $m_2 = 1.6$ kg, $I = 0.65$ kg · m^2, and $L = 1$ m.

8.11 Determine the natural frequencies and mode shapes for the system of Figure P7.17 if $k = 10{,}000$ N/m, $m = 3$ kg, $I = 0.6$ kg · m^2, and $r = 80$ cm.

8.12 Determine the natural frequencies and mode shapes of the system of Figure P7.19 if $k = 12{,}000$ N/m and each bar is of mass 12 kg and length 4 m.

8.13 A 400 kg machine is placed at the midspan of a 3-m-long, 200-kg simply supported beam. The beam is made of a material of elastic modulus 200×10^9 N/m^2 and has a cross-sectional moment of inertia of 1.4×10^{-5} m^4. Use a three degree-of-freedom model to approximate the system's lowest natural frequency.

8.14 A 500 kg machine is placed at the end of a 3.8-m-long, 190-kg fixed-free beam. The beam is made of a material of elastic modulus 200×10^9 N/m^2 and has a cross-sectional moment of inertia of 1.4×10^{-5} m^4. Use a three degree-of-freedom model to approximate the two lowest natural frequencies of the system.

8.15 Determine the two lowest natural frequencies of the railroad bridge of Chapter Problem 7.84 if $k_1 = 5.5 \times 10^7$ N/m, $k_2 = 1.2 \times 10^7$ N/m, $m = 15{,}000$ kg, $I = 1.6 \times 10^6$ kg · m^2, $l = 6.7$ m, and $h = 8.8$ m.

8.16 Determine the natural frequencies of the system of Chapter Problem 7.89. The beam is of length 5 m, made of a material of elastic modulus 200×10^9 N/m^2, and has a cross-sectional moment of inertia of 1.4×10^{-5} m^4. The total mass of the beam is 320 kg. The mass of the winch is 115 kg. The winch cable is made of a material of elastic modulus 200×10^9 N/m^2 and has a cross-sectional area of 3.4×10^{-2} m^2. The length of the cable is 5.5 m and the mass being lifted is 715 kg.

8.17 Determine the free vibration response of the railroad bridge of Chapter Problem 8.14 if a ground disturbance initially leads to $\theta_1 = 0.8°$ with $\theta_2 = \theta_3 = 0$.

8.18 A robot arm is 60 cm long, made of a material of elastic modulus 200×10^9 N/m^2, and has the cross section of Figure P8.18. The total mass of the arm is 850 g. A tool of mass 1 kg is attached to the end of the arm. Assume one end of the arm is pinned and the other end is free. Use a three degree-of-freedom model to determine the arm's natural frequencies.

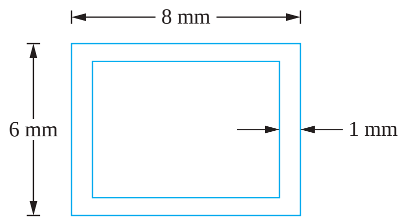


FIGURE P8.18

8.19 A 30,000 kg locomotive is coupled to a fully loaded 20,000 kg boxcar and moving at 6.5 m/s. The assembly is coupled to a stationary and empty 5,000 kg cattle car. The stiffness of each coupling is 5.7×10^5 N/m.
(a) What are the natural frequencies of the three-car assembly?
(b) Mathematically describe the motion of the cattle car after coupling.

8.20 Determine the natural frequencies and mode shapes for the three degree-of-freedom model of an airplane of Chapter Problem 7.87. Assume $m = 3.5$ m.

8.21 Determine the natural frequencies and mode shapes of the torsional system of Problem 7.81.

8.22 Use a four degree-of-freedom model to approximate the two lowest nonzero natural frequencies of a free-free beam.

8.23 A pipe extends from a wall as shown in Figure P8.23. The pipe is supported at A to prevent transverse displacement, but not to prevent rotation. Under what conditions will the pipe's lowest natural frequency of transverse vibrations coincide with its frequency of free torsional vibrations?

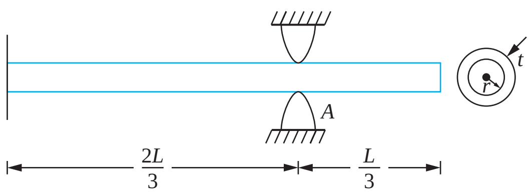


FIGURE P8.23

8.24 Show that Rayleigh's quotient $R(\mathbf{X})$ is stationary if and only if $\mathbf{X}$ is a mode shape vector.

8.25 Use Rayleigh's quotient to determine an upper bound on the lowest natural frequency of the system of Figure P8.7. Use at least four trial vectors.

8.26 An alternative method to derive the uncoupled equations governing the motion of the free vibrations of a nDOF system in terms of principal coordinates is to introduce a linear transformation between the generalized coordinates $\mathbf{x}$ and the principal coordinates $\mathbf{p}$ as $\mathbf{x} = \mathbf{P}\mathbf{p}$, where $\mathbf{P}$ is the modal matrix, the matrix whose columns are the normalized mode shapes. Follow these steps to derive the equations governing the principal coordinates:

(a) Rewrite Equation (8.3) using the principal coordinates as dependent variables by introducing the linear transformation in Equation (8.3).

(b) Premultiply the resulting equation by $\mathbf{P}^T$.

(c) Use the orthonormality of mode shapes to show that $\mathbf{P}^T\mathbf{M}\mathbf{P}$ and $\mathbf{P}^T\mathbf{K}\mathbf{P}$ are diagonal matrices.

(d) Write the uncoupled equations for the principal coordinates.

8.27 Use the method of Chapter Problem 8.26 to derive the uncoupled equations governing the principal coordinates for a system with proportional damping.

8.28 Determine the free vibration response of the system of Figure P8.28 if the system is released from rest after the 3 kg block is displaced 5 mm.

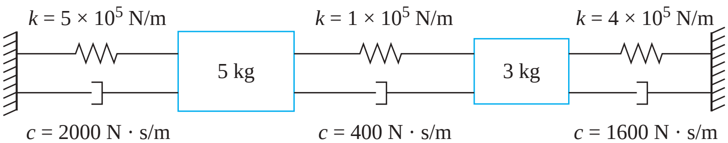


FIGURE P8.28

8.29 If the modal damping ratio for the lowest mode of Chapter Problem 8.13 is 0.03, determine the modal damping ratio for the higher modes and determine the response of the system if the machine is displaced 2 mm and released.

8.30 Determine the free-vibration response of the bar of Figure P8.30 is the mass center is displaced 1 cm from equilibrium while the bar is held horizontal and the system released from this position.

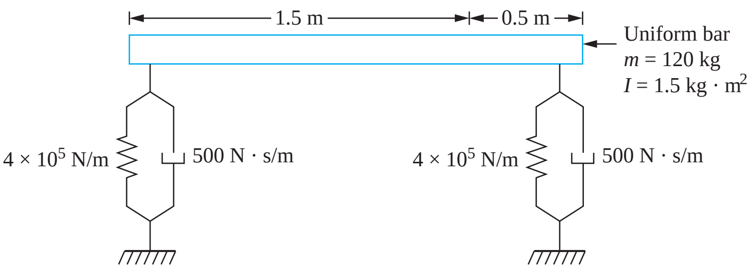


FIGURE P8.30

8.31 Determine the free-vibration response of the system of Figure P8.31.

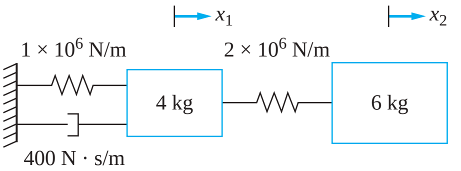


FIGURE P8.31

8.32 Determine the free-vibration response of the system of Figure P8.32.

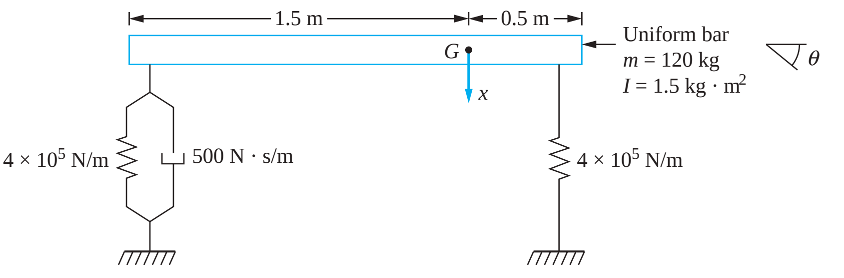


FIGURE P8.32

8.33 Determine the free-vibration response of the system of Chapter Problem 7.87 when $E = 200 \times 10^9$ N/m^2, $I = 1.5 \times 10^{-6}$ m^4, $L = 0.8$ m, $k = 1.5 \times 10^5$ N/m, $c = 250$ N · s/m, $m_1 = 4$ kg, $m_2 = 6.1$ kg.

Chapter 9

FORCED VIBRATIONS OF MDOF SYSTEMS

9.1 INTRODUCTION

The forced response of a linear multiple degree-of-freedom (MDOF) system, as for a linear single degree-of-freedom (SDOF) system, is the sum of a homogeneous solution and a particular solution. The homogeneous solution depends on system properties, while the particular solution is the response due to the particular form of the excitation. The free-vibration response is often ignored for a system whose long-term behavior is important, such as a system subject to a periodic excitation. The free-vibration solution is important for systems in which the short-term behavior is important, such as a system subject to a shock excitation.

Several methods are available to determine the forced response of a MDOF system. The method of undetermined coefficients can be applied to any system subject to a periodic excitation. However, because of algebraic complexity, its usefulness is restricted to systems with only a few degrees of freedom. The Laplace transform method can be applied to determine system properties, but its usefulness is limited because its application requires the solution of a system of simultaneous equations whose coefficients are functions of the transform variable. Both the method of undetermined coefficients and the Laplace transform method can be used to determine the forced response of a system with a general damping matrix.

The method of undetermined coefficients and the Laplace transform method were introduced in Chapter 6 to solve forced-vibration problems involving two degree-of-freedom systems. Their application is the same, except that matrix methods are used in this chapter.

The most useful method for determining the forced-vibration response of a linear MDOF system is modal analysis, which is based on using the principal coordinates to uncouple the differential equations governing the motion of an undamped or proportionally damped system. The uncoupled differential equations are solved by the standard techniques for solution of ordinary differential equations. A more general form of modal analysis involving complex algebra is developed for systems with a general damping matrix.

Often the differential equations cannot be solved in closed form. Modal analysis can still be used to uncouple the differential equations. The differential equations for the principal coordinates can be solved by numerical integration of the convolution integral or direct numerical simulation of the differential equation by a method such as a Runge-Kutta method.

9.2 HARMONIC EXCITATIONS

The response of a MDOF system due to a harmonic excitation is the sum of the homogeneous solution and the particular solution. Even if damping is not included, the homogeneous solution is often ignored. In a real situation, damping is present, causing the homogeneous solution to decay with time. The long-time or steady-state solution is only the particular solution.

The method of undetermined coefficients can be adapted to find the particular solution for a MDOF system subject to a harmonic excitation. The method of undetermined coefficients can be used for damped or undamped systems. Its application for an nDOF system requires the solution of at least one set of n simultaneous equations.

The differential equations governing the motion of an nDOF undamped system subject to a single-frequency excitation with all excitation terms at the same phase are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}\sin\omega t \tag{9.1}$$

where $\mathbf{F}$ is an n-dimensional vector of constants. The method of undetermined coefficients is used and assumes a particular solution of the form

$$\mathbf{x}(t) = \mathbf{U}\sin\omega t \tag{9.2}$$

where $\mathbf{U}$ is an n-dimensional vector of undetermined coefficients. Substituting Equation (9.2) in Equation (9.1) leads to

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{U} = \mathbf{F} \tag{9.3}$$

Equation (9.3) represents a set of n simultaneous algebraic equation to solve for the components of the vector $\mathbf{U}$. A unique solution of Equation (9.3) exists unless

$$|-\omega^2\mathbf{M} - \mathbf{K}| = 0 \tag{9.4}$$

Equation (9.4) is satisfied only when the excitation frequency coincides with one of the system's natural frequencies. When this occurs, the use of Equation (9.2) is inappropriate. The response grows linearly with time, producing a resonance condition.

When a solution of Equation (9.3) exists, it can be written as

$$\mathbf{U} = (-\omega^2\mathbf{M} + \mathbf{K})^{-1}\mathbf{F} \tag{9.5}$$

EXAMPLE 9.1

Determine the forced response of the three degree-of-freedom system shown in Figure 9.1(a)

SOLUTION

The differential equations governing the system of Figure 9.1 are

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 14 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 1500 & -1000 & 0 \\ -1000 & 1700 & -700 \\ 0 & -700 & 700 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \ \sin 10t \end{bmatrix} \tag{a}$$

A steady-state solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \sin 10t \tag{b}$$

which upon substitution into Equation (a) leads to

$$\begin{bmatrix} 500 & -1000 & 0 \\ -1000 & 500 & -700 \\ 0 & -700 & -700 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} \tag{c}$$

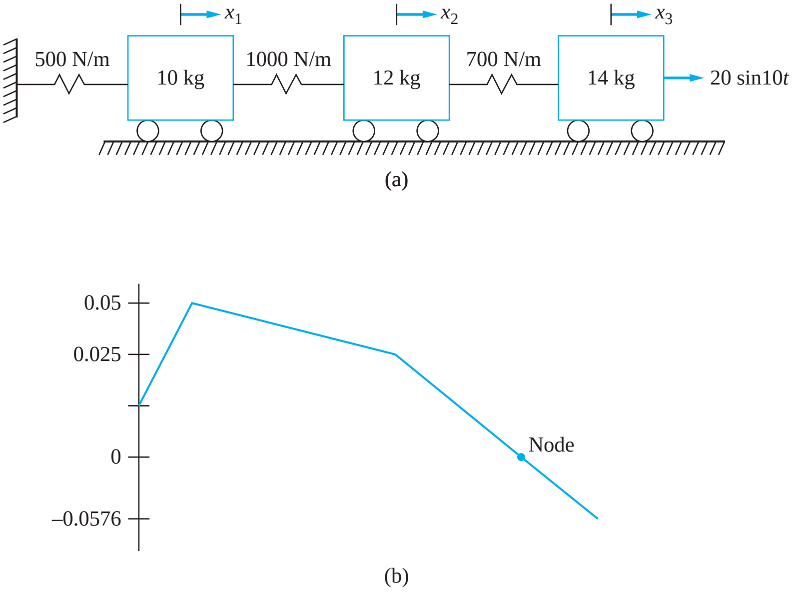


FIGURE 9.1
(a) Three degree-of-freedom system of Example 9.1. (b) Steady-state response of system is determined using the method of undetermined coefficient. The plot is of the steady state amplitudes of the masses versus the position of the mass.

The solution to Equation (c) is

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.025 \\ -0.0536 \end{bmatrix} \tag{d}$$

The vector of solutions is plotted against equilibrium position of the masses as in a mode-shape diagram in Figure 9.1(b). In the steady state, there is a node in the spring between the 12 kg mass and the 14 kg mass. The third mass is out of phase with the excitation.

The differential equations governing the motion of a nDOF system with viscous damping subject to a single-frequency harmonic excitation are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathrm{Im}(\mathbf{F}e^{i\omega t}) \tag{9.6}$$

where $\mathbf{F}$ is an n-dimensional vector of constants. The constants could be complex if each generalized force is not of the same phase and are of the form

$$F_i = f_i e^{i\phi} \tag{9.7}$$

The solution of Equation (9.6) is assumed as

$$\mathbf{x}(t) = \mathrm{Im}(\mathbf{U}e^{i\omega t}) \tag{9.8}$$

where $\mathbf{U}$ is an n-dimensional vector of complex constants. Substitution of Equation (9.8) in Equation (9.6) leads to

$$(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})\mathbf{U} = \mathbf{F} \tag{9.9}$$

The solution of Equation (9.9) is obtained as

$$\mathbf{U} = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}\mathbf{F} \tag{9.10}$$

EXAMPLE 9.2

Determine the steady-state amplitudes of the system of Figure 9.2.

SOLUTION

The differential equations governing the motion of the system shown in Figure 9.2 are

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 50 & 0 & 0 \\ 0 & 100 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} 1500 & -1000 & 0 \\ -1000 & 1700 & -700 \\ 0 & -700 & 700 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10\sin\left(10t + \dfrac{\pi}{4}\right) \\ 0 \\ 20\sin 10t \end{bmatrix} \tag{a}$$

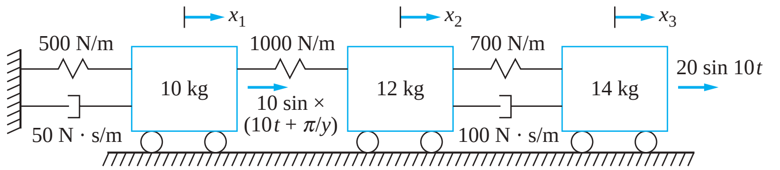


FIGURE 9.2
Three degree-of-freedom system of Example 9.2.

A solution of Equation (a) is assumed to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i10t} \tag{b}$$

Only the imaginary part is used as the solution. Substitution of Equation (b) into Equation (a) leads to

$$\begin{bmatrix} 500 + 500i & -1000 & 0 \\ -1000 & 500 + 1000i & -700 - 1000i \\ 0 & -700 - 1000i & -700 + 1000i \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 10e^{i\frac{\pi}{4}} \\ 0 \\ 20 \end{bmatrix} \tag{c}$$

whose solution is

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^{-3} \begin{bmatrix} 2.43 + 7.54i \\ -9.63 - 3.08i \\ -14.65 - 5.09i \end{bmatrix} \tag{d}$$

The imaginary part of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathrm{Im}\left(10^{-3} \begin{bmatrix} 2.43 + 7.54i \\ -9.63 - 3.08i \\ -14.65 - 5.09i \end{bmatrix} e^{i10t}\right) = 10^{-3} \begin{bmatrix} 2.43 \sin 10t + 7.54 \cos 10t \\ -9.63 \sin 10t - 3.08 \cos 10t \\ -14.65 \sin 10t - 5.09 \cos 10t \end{bmatrix}$$

$$= 10^{-3} \begin{bmatrix} 7.92 \sin(10t - 1.26) \\ 10.1 \sin(10t + 2.93) \\ 15.5 \sin(10t + 2.81) \end{bmatrix} \tag{e}$$

EXAMPLE 9.3

Determine the frequency response of the system of Figure 9.3.

SOLUTION

The differential equations governing the motion of the system shown in Figure 9.3 are

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 50 & 0 & 0 \\ 0 & 100 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 1500 & -1000 & 0 \\ -1000 & 1700 & -700 \\ 0 & -700 & 700 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \sin \omega t \end{bmatrix} \quad \text{(a)}$$

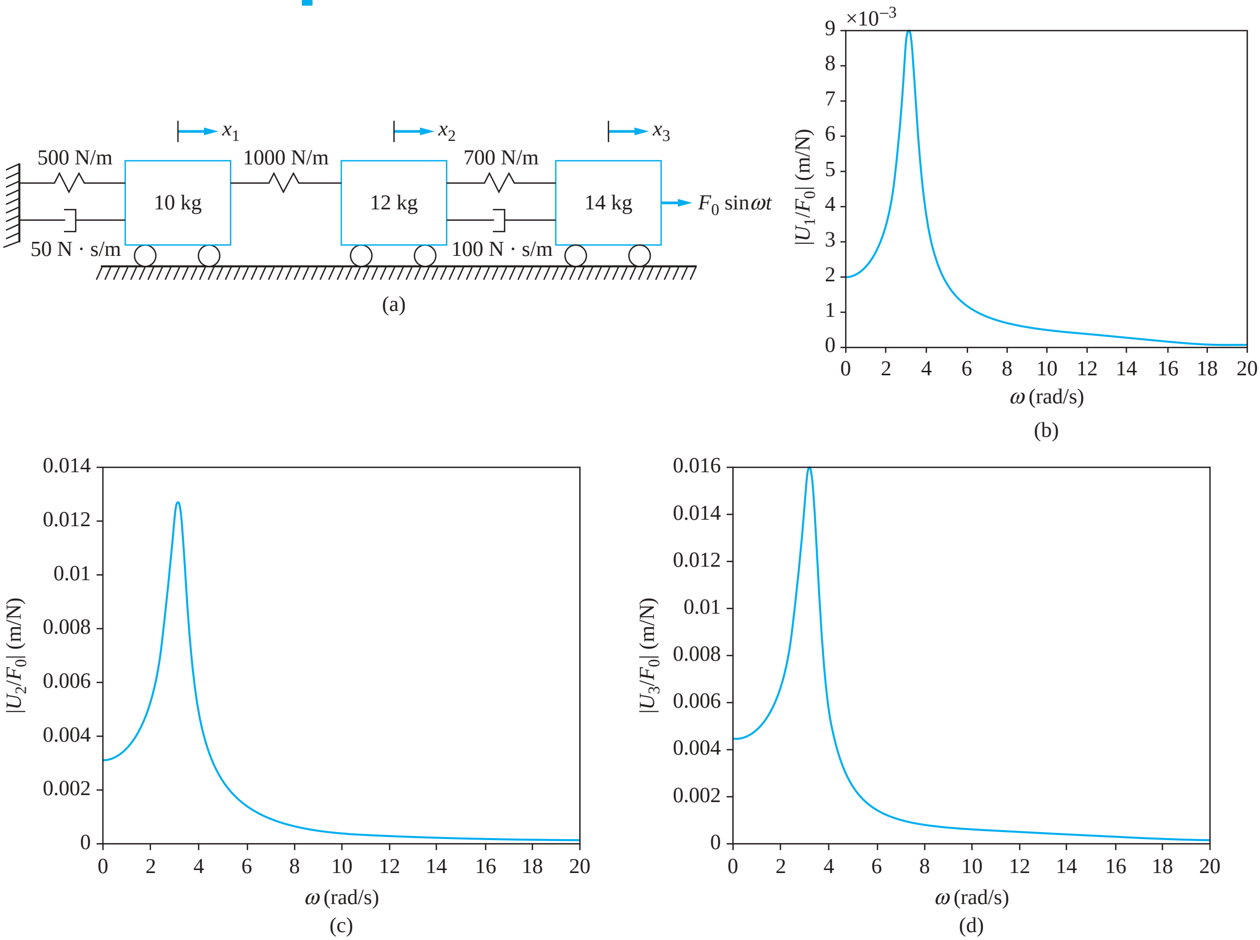


FIGURE 9.3

(a) There degree-of-freedom system of Example 9.3. (b) – (d) Frequency-response curves of Example 9.3.

A solution of Equation (a) is assumed to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i\omega t} \tag{b}$$

Only the imaginary part is used as the solution. Substitution of Equation (b) into Equation (a) leads to

$$\begin{bmatrix} 1500 - 10\omega^2 + 50\omega i & -1000 & 0 \\ -1000 & 1700 - 12\omega^2 + 100i & -700 - 100\omega i \\ 0 & -700 - 100\omega i & 700 - 14\omega^2 + 100i \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ F_0 \end{bmatrix} \tag{c}$$

For a given ω, Equation (c) is solved and the imaginary part of $fe^{i\omega t}$ taken. This leads to the amplitudes of being $|U_0|$. The frequency response curves are given in Figures 9.3(b) through (d).

9.3 LAPLACE TRANSFORM SOLUTIONS

Let $\mathbf{X}(s)$ be the vector of Laplace transforms of the generalized coordinates for an nDOF system. Taking the Laplace transform of the differential equations governing forced vibrations of a linear nDOF system and using linearity of the transform and the property of transform of the first and second derivatives gives

$$(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})\mathbf{X}(s) = \mathbf{F}(s) + (s\mathbf{M} + \mathbf{K})\mathbf{x}(0) + \mathbf{M}\dot{\mathbf{x}}(0) \tag{9.11}$$

where $\mathbf{F}(s)$ is the vector of Laplace transforms of $\mathbf{F}(t)$. If $\mathbf{x}(0) = 0$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$, Equation (9.11) becomes

$$\mathbf{Z}(s)\mathbf{X}(s) = \mathbf{F}(s) \tag{9.12}$$

where

$$\mathbf{Z}(s) = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{9.13}$$

is called the *impedance matrix.* Pre-multiplying Equation (9.13) by $\mathbf{Z}^{-1}(s)$ yields

$$\mathbf{X}(s) = \mathbf{Z}^{-1}(s)\mathbf{F}(s) \tag{9.14}$$

The elements of $\mathbf{Z}^{-1}(s)$ are the transfer functions $G_{k,j}(s)$, which represent the transform of the response of x_k due to a unit impulse applied at the location described by x_j.

The response of the system $\mathbf{x}(t)$ is obtained by inversion of Equation (9.14). If $\mathbf{F}(t)$ is a vector of harmonic forces as $f_j(t) = F_j \sin \omega_j t$, the sinusoidal transfer functions can be used to obtain the response. The solution for the ith component of $\mathbf{X}(s)$ is

$$X_k(s) = \sum_{j=1}^{n} G_{k,j}(s)F_j(s) \tag{9.15}$$

which is inverted as

$$x_k(t) = \sum_{j=1}^{n} |G_{k,j}(i\omega_j)| F_j \sin(\omega_j t + \phi_{k,j}) \tag{9.16}$$

where $i = \sqrt{-1}$ and

$$\phi_{k,j} = \tan^{-1} \frac{\mathrm{Im}[G_{k,j}(i\omega_j)]}{\mathrm{Re}[G_{k,j}(i\omega_j)]} \tag{9.17}$$

EXAMPLE 9.4

Determine the steady-state response of the 10 kg block of Figure 9.4 for the following.

(a) $F_1(t)$ is given in Figure 9.4(a), $F_2(t) = 0$, and $F_3(t) = 0$

(b) $F_1(t) = 20 \sin 10t$, $F_2(t) = 0$, and $F_3(t) = 30 \sin 20t$

SOLUTION

The differential equations governing the motion of the three degree-of-freedom system of Figure 9.4 are

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 50 & 0 & 0 \\ 0 & 100 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 1500 & -1000 & 0 \\ -1000 & 1700 & -700 \\ 0 & -700 & 700 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} \tag{a}$$

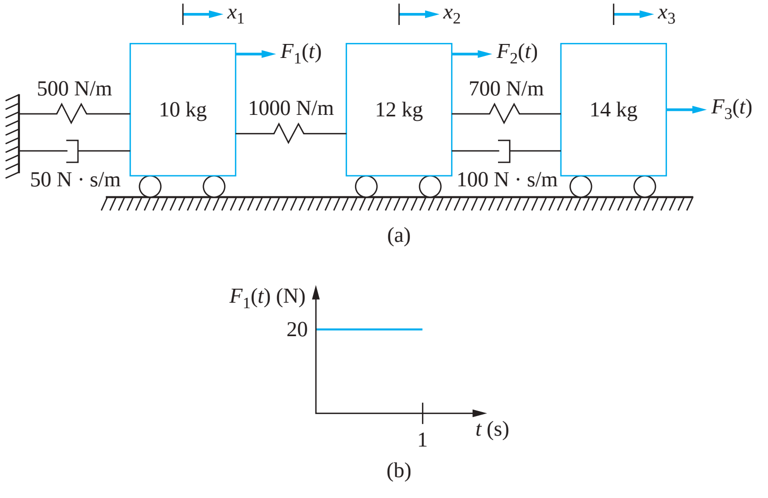


FIGURE 9.4
(a) System of Example 9.4. (b) $F_1(t)$ for part a.

Taking the Laplace transform of Equation (a) and using all initial conditions as zero leads to

$$\begin{bmatrix} 10s^2+50s+1500 & -1000 & 0 \\ -1000 & 12s^2+100s+1700 & -100s-700 \\ 0 & -100s-700 & 14s^2-100s+700 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} F_1(s) \\ F_2(s) \\ F_3(s) \end{bmatrix} \tag{b}$$

The matrix in Equation (b) is $Z(s)$.

(a) Taking the Laplace transform of the excitation leads to

$$\begin{bmatrix} 10s^2+50s+1500 & -1000 & 0 \\ -1000 & 12s^2+100s+1700 & -100s-700 \\ 0 & -100s-700 & 14s^2+100s+700 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s}(1-e^{-0.5s}) \\ 0 \\ F_0/s \end{bmatrix} \tag{c}$$

The inverse of $Z(s)$ is obtained as

$$\mathbf{Z}^{-1}(s) = \frac{1}{D(s)}\left[\begin{matrix} 2.1s^4+32.5s^3+402.5s^2+1250s+8750 \\ 175s^2+1250s+8750 \\ 1250s+8750 \end{matrix}\right. \\ \begin{matrix} 175s^4+1250s+8750 \\ 7s^4+85s^3+1650s^2+9250s+52{,}500 \\ 25s^3+300s^3+4625s+26{,}250 \end{matrix} \\ \left.\begin{matrix} 1250s+8750 \\ 25s^3+300s^3+4625s+26{,}250 \\ 3s^4+40s^3+1000s^2+58{,}755+38{,}750 \end{matrix}\right] \tag{d}$$

where

$$D(s) = 21s^6+430s^5+8800s^4+81{,}375s^3+578{,}750s^2 + 1{,}062{,}500s+4{,}375{,}000 \tag{e}$$

The roots of $D(s)$ are obtained as

$$s = -3.278 \pm 13.95i, \quad -6.550 \pm 7.67i, \quad -0.4097 \pm 3.13i \tag{f}$$

Multiplying $F(s)$ by $Z^{-1}(s)$ and solving for $X_1(s)$ leads to

$$X_1(s) = \frac{(2.1s^4 + 32.5s^3 + 402.5s^2 + 1250s + 8750)(1 - e^{-0.5s})}{s(21s^6 + 430s^5 + 8800s^4 + 81{,}375s^3 + 578{,}750s^2 + 1{,}062{,}500s + 4{,}375{,}000)} \quad \text{(g)}$$

A partial fraction decomposition of Equation (g) leads to

$$X_1(s) = 10^{-4}\left(\frac{20}{s} + \frac{-14s + 16}{s^2 + 0.820s + 9.96} + \frac{-2.93s + 19}{s^2 + 13.71s + 101.73} + \frac{-3.53s + 13}{s^2 + 6.54s + 205.2}\right)(1 - e^{-0.5s}) \quad \text{(h)}$$

Inversion of the transform yields

$$\begin{aligned} x_1(t) = 10^{-4}\{&20 + e^{-0.409t}(-14\cos 3.16t + 4.081\sin 3.16t) \\ &+ e^{-6.56t}(-2.93\cos 10.08t + 2.46\sin 10.08t) \\ &+ e^{-3.28}(-3.53\cos 14.32t + 1.40\sin 14.32t) \\ &- u(t - 0.5)\{20 + e^{-0.409(t-0.5)}[-14\cos 3.16(t - 0.5) \\ &+ 4.081\sin 3.16(t - 0.5]e^{-6.56(t-0.5)}[-2.93\cos 10.08(t - 0.5) \\ &+ 2.36\sin 10.08(t - 0.5)] + e^{-3.28(t-0.5)}[-3.53\cos 14.32(t - 0.5) \\ &+ 1.40\sin 14.32(t - 0.5)]\}\} \end{aligned} \quad \text{(i)}$$

(b) From Equation (9.16) for the given forces

$$x_1(t) = 20|G_{1,1}(10j)|\sin(10t + \phi_{1,1}) + 30|G_{1,3}(20j)|\sin(20t + \phi_{1,3}) \quad \text{(j)}$$

where

$$\begin{aligned} G_{1,1}(10i) &= \frac{2.1(10i)^4 + 32.5(10i)^3 + 402.5(10i)^2 + 1250(10i) + 8750}{21(10i)^6 + 430(10i)^5 + 8800(10i)^4 + 81{,}375(10i)^3 + 578{,}750(10i)^2 + 1{,}062{,}500(10i) + 4{,}375{,}000} \\ &= \frac{-1.05 \times 10^4 - 2.00 \times 10^4 i}{1.26 \times 10^7 - 2.775 \times 10^7 i} = 4.55 \times 10^{-4} - 5.85 \times 10^{-4} i \end{aligned} \quad \text{(k)}$$

and

$$\begin{aligned} G_{1,3}(20i) &= \frac{1250(20i) + 8750}{21(20i)^6 + 430(20i)^5 + 8800(20i)^4 + 81{,}375(20i)^3 + 578{,}750(20i)^2 + 1{,}062{,}500(20i) + 4{,}375{,}000} \\ &= \frac{8.75 \times 10^3 + 2.5 \times 10^4 i}{-1.671 \times 10^8 + 7.463 \times 10^8 i} = 2.94 \times 10^{-5} - 1.83 \times 10^{-5} i \end{aligned} \quad \text{(l)}$$

The steady-state solution is

$$x_1(t) = 20(7.414 \times 10^{-4})\sin(10t - 0.910) + 30(3.463 \times 10^{-5})\sin(20t - 0.557)$$
$$= 0.0148\sin(10t - 0.910) + 0.00106\sin(20t - 0.557) \tag{m}$$

9.4 MODAL ANALYSIS FOR UNDAMPED SYSTEMS AND SYSTEMS WITH PROPORTIONAL DAMPING

The differential equations governing the forced vibrations of an undamped linear nDOF system are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{9.18}$$

The method of *modal analysis* uses the principal coordinates of the system to uncouple the differential equations of Equation (9.18).

Let $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ be the natural frequencies of the system whose equations are given by Equation (9.18). Let $\mathbf{P}$ be the system's modal matrix, the matrix whose columns are the normalized mode shapes, $\mathbf{P} = [\mathbf{X}_1 \ \mathbf{X}_2 \cdots \mathbf{X}_n]$. Using the expansion theorem, as in Section 8.8, the response at any instant of time can be expanded as

$$\mathbf{x}(t) = \sum_{i=1}^{n} p_i(t)\mathbf{X}_i \tag{9.19}$$

where $p_i(t)$ are the system's principal coordinates. Equation (9.19) is equivalent to a linear transformation between the original generalized coordinates and the principal coordinates

$$\mathbf{x} = \mathbf{P}\mathbf{p} \tag{9.20}$$

Substitution of Equation (9.19) in Equation (9.18) leads to

$$\sum_{i=1}^{n} \ddot{p}_i \mathbf{M}\mathbf{X}_i + \sum_{i=1}^{n} p_i \mathbf{K}\mathbf{X}_i = \mathbf{F} \tag{9.21}$$

Taking the standard scalar product of Equation (9.21) with $\mathbf{X}_j$ for an arbitrary j leads to

$$\sum_{i=1}^{n} \ddot{p}_i (\mathbf{X}_j, \mathbf{M}\mathbf{X}_i) + \sum_{i=1}^{n} p_i (\mathbf{X}_j, \mathbf{K}\mathbf{X}_i) = (\mathbf{X}_j, \mathbf{F}) \tag{9.22}$$

On the basis of the definitions of energy scalar products, Equation (9.22) becomes

$$\sum_{i=1}^{n} \ddot{p}_i (\mathbf{X}_j, \mathbf{X}_i)_M + \sum_{i=1}^{n} p_i (\mathbf{X}_j, \mathbf{X}_i)_K = (\mathbf{X}_j, \mathbf{F}) \tag{9.23}$$

Application of mode-shape orthogonality leads to only one nonzero term in each summation, the term corresponding to $i = j$. Since the mode shapes are normalized, Equation (9.23) leads to

$$\ddot{p}_j + \omega_j^2 p_j = g_j(t) \tag{9.24}$$

where

$$g_j(t) = (\mathbf{X}_j, \mathbf{F}) \tag{9.25}$$

An equation of the form of Equation (9.24) can be written for each $j = 1, 2, \ldots, n$. This shows that the principal coordinates that are used to uncouple the differential equations governing free vibrations can also be used to uncouple the differential equations governing forced vibrations. The differential equations of Equation (9.24) can be solved by any useful means. If the initial conditions for p_i are $p_i(0) = 0$ and $\dot{p}_i(0) = 0$, then the convolution integral solution of Equation (9.24) is

$$p_i(t) = \frac{1}{\omega_i}\int_0^t g_i(\tau)\sin[\omega_i(t-\tau)]\,d\tau \tag{9.26}$$

Once the solutions for each p_i have been obtained, Equation (9.19) is used to determine the original generalized coordinates.

The modal analysis procedure to determine the forced response of an undamped linear nDOF system is summarized below.

1. A set of generalized coordinates is chosen. The differential equations governing the motion of the system are derived using Lagrange's equations. The differential equations are written in the matrix form of Equation (9.18).
2. The natural frequencies and normalized mode shapes are obtained. The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ and the mode shapes are the corresponding eigenvectors. The mode shapes are normalized by requiring that the kinetic energy scalar product of a mode shape with itself be equal to one.
3. The elements of the column vector $\mathbf{G}$ are obtained by using Equation (9.25). An alternative method to obtain $\mathbf{G}$ is

$$\mathbf{G} = \mathbf{P}^T\mathbf{F} \tag{9.27}$$

4. Equations of the form of Equation (9.24) are solved to obtain the time-dependent form of the principal coordinates. Equation (9.26) gives the convolution integral solution of Equation (9.24).
5. The time-dependent form of the original generalized coordinates is obtained by using Equation (9.19) or Equation (9.20).

EXAMPLE 9.5

Use modal analysis to determine the time-dependent response of the system of Figure 9.5(a) subject to the excitation of Figure 9.5(b).

SOLUTION

The differential equations governing the motion of the system of Figure 9.5(a) are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix} \tag{a}$$

where from Figure 9.5(b)

$$F(t) = 4000[1 - u(t - 1.2)]\text{ N} \tag{b}$$

where t is in seconds.

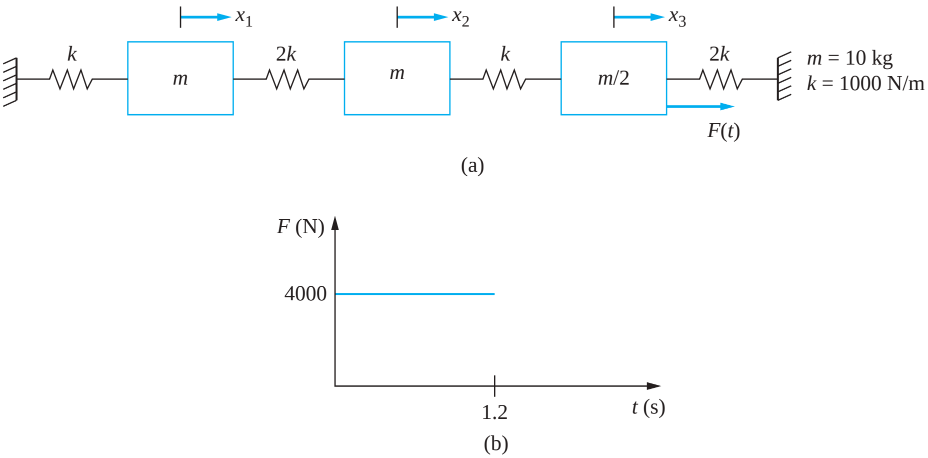


FIGURE 9.5
(a) Three degree-of-freedom system of Example 9.5. (b) Excitation for system of Example 9.5.

The natural frequencies for this system are determined in Example 8.2 and the normalized mode shapes are determined in Example 8.10. Substituting $m = 10$ kg and $k = 1000$ N/m in these results leads to natural frequencies of

$$\omega_1 = 8.936 \text{ rad/s} \qquad \omega_2 = 21.107 \text{ rad/s} \qquad \omega_3 = 25.974 \text{ rad/s} \tag{c}$$

and a modal matrix of

$$\mathbf{P} = \begin{bmatrix} 0.2085 & 0.2252 & 0.0765 \\ 0.2295 & -0.1638 & -0.1432 \\ 0.0882 & -0.2120 & 0.3838 \end{bmatrix} (\text{kg})^{-1/2} \tag{d}$$

The vector $\mathbf{G}(t)$ is then calculated by using Equation (9.27)

$$\mathbf{G}(t) = \mathbf{P}^T\mathbf{F} = \begin{bmatrix} 0.0882 \\ -0.2120 \\ 0.3838 \end{bmatrix} F(t) \tag{e}$$

The differential equations satisfied by the principal coordinates are written by using Equation (9.24)

$$\ddot{p}_1 + 79.852p_1 = 352.8\ [1 - u(t - 1.2)] \tag{f}$$

$$\ddot{p}_2 + 445.5p_2 = -848.0\ [1 - u(t - 1.2)] \tag{g}$$

$$\ddot{p}_3 + 674.6p_3 = 1535.2\ [1 - u(t - 1.2)] \tag{h}$$

The convolution integral is used to solve for p_1 as

$$p_1(t) = \frac{1}{8.936}\int_0^t 352.8\ [1 - u(\tau - 1.2)] \sin 8.936(t - \tau)d\tau$$

$$= 4.418\ \{\cos 8.936t - 1 + u(t - 1.2)[1 - \cos 8.936(t - 1.2)]\} \tag{i}$$

The convolution integral is also used to solve for p_2 and p_3, yielding

$$p_2(t) = -1.903\ \{\cos 21.107t - 1 + u(t-1.2)[1 - \cos 21.107(t-1.2)]\} \tag{j}$$

$$p_3(t) = 2.276\ \{\cos 25.974t - 1 + u(t-1.2)[1 - \cos 25.974(t-1.2)]\} \tag{k}$$

The solution in terms of the original generalized coordinates is obtained by using Equation (9.20)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2085 & 0.2252 & 0.0765 \\ 0.2295 & -0.1638 & -0.1432 \\ 0.0882 & -0.2120 & 0.3838 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} \tag{l}$$

which leads to

$$x_1(t) = 0.921b_1(t) - 0.429b_2(t) + 0.174b_3(t) \tag{m}$$

$$x_2(t) = 1.014b_1(t) + 0.312b_2(t) - 0.326b_3(t) \tag{n}$$

$$x_3(t) = 0.390b_1(t) + 0.403b_2(t) + 0.874b_3(t) \tag{o}$$

where

$$b_1(t) = \cos 8.936t - 1 + u(t-1.2)[1 - \cos 8.936(t-1.2)] \tag{p}$$

$$b_2(t) = \cos 21.107t - 1 + u(t-1.2)[1 - \cos 21.107(t-1.2)] \tag{q}$$

$$b_3(t) = \cos 25.974t - 1 + u(t-1.2)[1 - \cos 25.974(t-1.2)] \tag{r}$$

EXAMPLE 9.6

A machine of mass 150 kg is placed as shown on the simply supported beam of Figure 9.6. The machine has a rotating unbalance of 0.965 kg · m and operates at 1250 rpm. The beam has a total mass of 280 kg, a cross-sectional moment of inertia of 1.2×10^{-4} m^4, a length of 3 m, and an elastic modulus of 210×10^9 N/m^2. Model the beam with three degrees of freedom and use modal analysis to predict the steady-state amplitude of displacement for the point where the machine is attached.

SOLUTION

The beam is modeled as three particles with a mass of 70 kg, as shown in Figure 9.6(b). The mass matrix for this model is

$$\mathbf{M} = \begin{bmatrix} 70 & 0 & 0 \\ 0 & 70 & 0 \\ 0 & 0 & 220 \end{bmatrix} \text{kg} \tag{a}$$

Flexibility influence coefficients are used to determine the flexibility matrix as

$$\mathbf{A} = 10^{-9} \begin{bmatrix} 12.53 & 15.33 & 9.75 \\ 15.33 & 22.29 & 15.33 \\ 9.75 & 15.33 & 12.53 \end{bmatrix} \text{m/N} \tag{b}$$

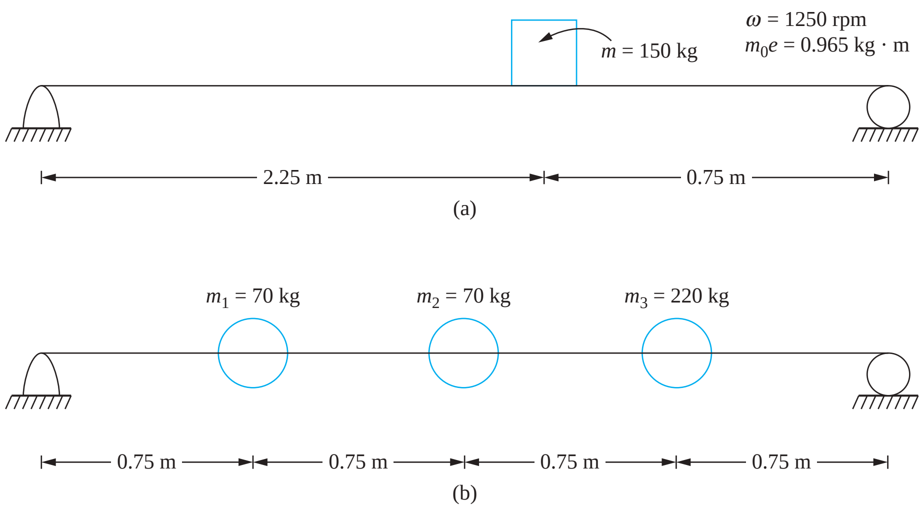


FIGURE 9.6
(a) Machine with rotating unbalance is attached to pinned-pinned beam. (b) Three degree-of-freedom model of beam.

The governing differential equations are

$$\mathbf{AM\ddot{x}} + \mathbf{x} = \mathbf{AF} \tag{c}$$

where

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 0 \\ 16{,}500 \sin 130.9t \end{bmatrix} \text{N} \tag{d}$$

The natural frequencies and normalized mode shapes are determined as the reciprocals of the square roots of the eigenvalues of **AM**. They are

$$\omega_1 = 455.8 \text{ rad/s} \quad \omega_2 = 1.735 \times 10^3 \text{ rad/s} \quad \omega_3 = 4.474 \times 10^3 \text{ rad/s} \tag{e}$$

The normalized eigenvectors comprise the modal matrix **P**, which is

$$\mathbf{P} = \begin{bmatrix} 0.0453 & -0.0851 & -0.0707 \\ 0.0666 & -0.4000 & 0.0908 \\ 0.0498 & 0.0416 & -0.0182 \end{bmatrix} \tag{f}$$

The vector $\mathbf{G}(t)$ is calculated as

$$\mathbf{G}(t) = \mathbf{P}^T\mathbf{F} = \begin{bmatrix} 0.0453 & 0.0666 & 0.0498 \\ -0.0851 & -0.4000 & 0.0416 \\ -0.0707 & 0.0908 & -0.0182 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 16{,}500 \sin 130.9t \end{bmatrix}$$

$$= \begin{bmatrix} 821.8 \\ 687.0 \\ -300.3 \end{bmatrix} \sin 130.9t \text{ N}(\text{kg})^{-1/2} \tag{g}$$

The differential equations for the principal coordinates are written by using Equation (9.24)

$$\ddot{p}_1 + (455.8)^2 p_1 = 821.8 \sin 130.9t \quad \text{(h)}$$

$$\ddot{p}_2 + (1736.5)^2 p_2 = 687.0 \sin 130.9t \quad \text{(i)}$$

$$\ddot{p}_3 + (4474)^2 p_3 = -300.3 \sin 130.9t \quad \text{(j)}$$

The steady-state solution of

$$\ddot{p}_i + \omega_i^2 p_i = F_i \sin \omega t \quad \text{(k)}$$

is

$$p_i(t) = \frac{F_i}{\omega_i^2 - \omega_i^2} \sin \omega t \quad \text{(l)}$$

The steady-state solution for the principal coordinates is

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 10^{-5} \begin{bmatrix} 432.0 \\ 22.93 \\ -1.501 \end{bmatrix} \sin 130.9t \ (\text{kg})^{1/2} \quad \text{(m)}$$

Equation (9.20) is used to determine $x_3(t)$ as

$$x_3(t) = 0.0498 p_1(t) + 0.0416 p_2(t) - 0.0182 p_3(t) = 2.25 \times 10^{-4} \sin 130.9t \text{ m} \quad \text{(n)}$$

Thus, the maximum steady-state displacement of the point on the beam where the machine is placed is 0.225 mm.

The differential equations governing the forced vibrations of a linear system with viscous damping are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \quad \text{(9.28)}$$

If the system is proportionally damped, the damping matrix is a linear combination of the mass matrix and the stiffness matrix.

Modal analysis using the principal coordinates of the undamped system can be used to uncouple the differential equations of a system with proportional damping. Substitution of Equation (9.19) into Equation (9.28) and following a procedure similar to that used for the undamped system leads to the differential equations for the principal coordinates as

$$\ddot{p}_i + 2\zeta_i \omega_i \dot{p}_i + \omega_i^2 p_i = g_i(t) \quad \text{(9.29)}$$

where the modal damping ratio ζ_i is defined in Equation (8.79).

The convolution integral solution of Equation (9.29) for $J_i < 1$ is

$$p_i(t) = \frac{1}{\omega_i \sqrt{1 - \zeta_i^2}} \int_0^t g_i(\tau) e^{-\zeta_i \omega_i (t-\tau)} \sin\left[\omega_i \sqrt{1 - \zeta_i^2}(t - \tau)\right] d\tau \quad \text{(9.30)}$$

The procedure for application of modal analysis to a system with proportional damping is the same as that for an undamped system with the addition of the determination of the modal damping ratios to step 2 and the use of Equation (9.30) as the convolution integral solution.

Damping in structural systems is mostly hysteretic and hard to quantify. Lacking a better model, proportional damping is often assumed. The modal damping ratios are usually determined experimentally. The equivalent damping ratio for a harmonically excited SDOF system with hysteretic damping is proportional to the natural frequency, and inversely proportional to the excitation frequency. This model fits proportional damping where the damping matrix is proportional to the stiffness matrix. In these cases, the higher modes are damped more than the lower modes. The natural frequencies in stiff structural systems are usually greatly separated. The effect of the higher modes in the total response is less than the modes with lower natural frequencies. For these reasons, damping ratios are often specified only for the lower modes.

If proportional damping is assumed, the higher modes are damped more than the lower modes and have a lesser effect on the overall solution. Modes with higher damping ratios die out more quickly when the system is subject to any short-term or shock excitation. If the system is subject to a harmonic excitation, the modes with higher frequencies have lesser effect because their amplitudes are inversely proportional to the square of their frequencies. Thus, fewer modes can be calculated without losing significant accuracy. Hence, in practice, Equation (9.19) is often replaced by

$$\mathbf{x}(t) = \sum_{i=1}^{m} p_i \mathbf{X}_i \tag{9.31}$$

for some $m < n$. Equation (9.31) is often used in situations where the mode shapes are determined experimentally and an experimental modal analysis method is used to determine the response of a system.

EXAMPLE 9.7

The three degree-of-freedom system of Example 9.5 is modified by the addition of dashpots, as shown in Figure 9.7 Determine the forced response of the damped system.

SOLUTION

The damping matrix is

$$\mathbf{C} = \begin{bmatrix} 3c & -2c & 0 \\ -2c & 3c & -c \\ 0 & -c & 3c \end{bmatrix} \tag{a}$$

and is proportional to the stiffness matrix with

$$\alpha = \frac{c}{k} = \frac{40\ \text{N}\cdot\text{s/m}}{1000\ \text{N/m}} = 0.04\ \text{s} \tag{b}$$

Thus, the modal damping ratios are given by

$$\zeta_1 = \frac{\alpha}{2}\omega_1 = 0.178 \quad \zeta_2 = \frac{\alpha}{2}\omega_2 = 0.422 \quad \zeta_3 = \frac{\alpha}{2}\omega_3 = 0.520 \tag{c}$$

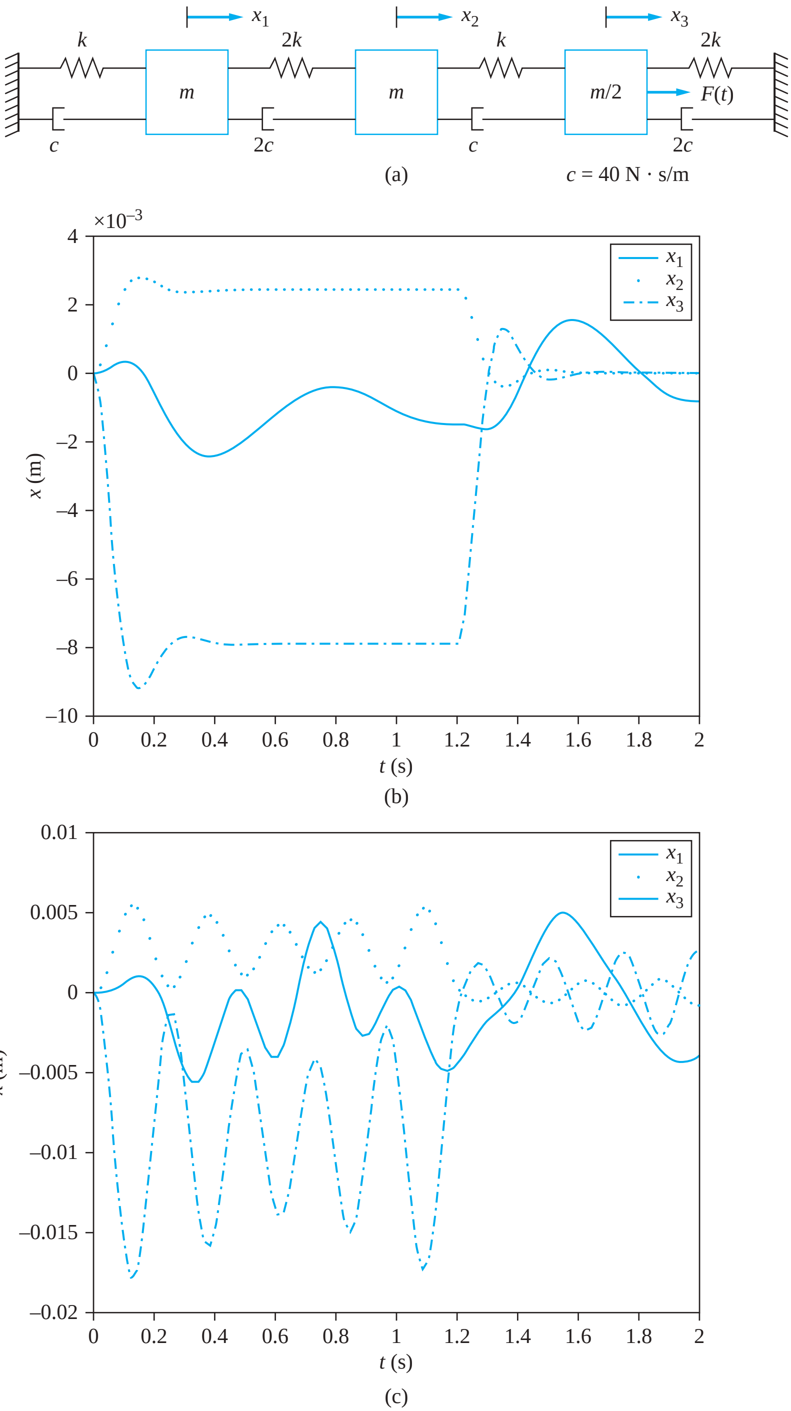


FIGURE 9.7

(a) Three degree-of-freedom system with the damping matrix proportional to the stiffness matrix. (b) System response for $\alpha = 0.04$ s. (c) System response for $\alpha = 0$.

All modes are underdamped. The differential equations governing the principal coordinates are

$$\ddot{p}_1 + 1.60\dot{p}_1 + 79.85p_1 = 0.0882F(t) \tag{d}$$

$$\ddot{p}_2 + 8.91\dot{p}_2 + 445.5p_2 = -0.2120F(t) \tag{e}$$

$$\ddot{p}_3 + 13.49\dot{p}_3 + 674.6p_3 = 0.3838F(t) \tag{f}$$

The solution for the principal coordinates is obtained from the convolution integral. It is noted that

$$\int_0^t [1 - u(\tau - 1.2)]\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\,d\tau$$

$$= -\frac{1-\zeta^2}{\omega_d}\Bigg[1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$- u(t-1.2)\Bigg\{1 - e^{-\zeta\omega_n(t-1.2)}\Bigg[\cos\omega_d(t-1.2)$$

$$+ \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d(t-1.2)\Bigg]\Bigg\}\Bigg] \tag{g}$$

Application of the convolution integral to the first equation leads to

$$p_1(t) = 4.43\,[1 - e^{-1.60t}(\cos 8.79t + 0.181\sin(8.79t)]$$

$$- 4.43u(t-1.2)\{1 - 6.77e^{-1.60t}[\cos(8.79t - 10.55)$$

$$+ 0.181\sin(8.79t - 10.55)]\} \tag{h}$$

The convolution integral solution of Equation (g) is evaluated for the other principal coordinates. The original generalized coordinates are calculated by $\mathbf{x} = \mathbf{P}\mathbf{p}$. The resulting plots for $\alpha = 0.04$ and $\alpha = 0$ are shown in Figure 9.7(b) and (c).

9.5 MODAL ANALYSIS FOR SYSTEMS WITH GENERAL DAMPING

The differential equations governing the forced vibrations of a linear nDOF system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{9.32}$$

can be rewritten as a system of $2n$ linear first-order equations

$$\tilde{\mathbf{M}}\dot{\mathbf{y}} + \tilde{\mathbf{K}}\mathbf{y} = \tilde{\mathbf{F}} \tag{9.33}$$

where $\mathbf{y}$, $\tilde{\mathbf{M}}$, and $\tilde{\mathbf{K}}$ are defined in Equation (8.83) and

$$\tilde{\mathbf{F}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix} \tag{9.34}$$

The homogeneous solution of Equation (9.33) is obtained in Section 8.13. The solution uses eigenvalues and eigenvectors of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$. Eigenvalues occur in complex conjugate pairs. Eigenvectors satisfy the orthogonality relation of Equation (8.84). The eigenvectors can be normalized by requiring

$$\overline{\Phi}_i^T \tilde{\mathbf{M}} \Phi_i = 1 \tag{9.35}$$

The modal matrix $\tilde{\mathbf{P}}$ is the matrix whose columns are the normalized eigenvectors of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$.The principal coordinates are defined by

$$\mathbf{y} = \tilde{\mathbf{P}}\tilde{\mathbf{p}} \tag{9.36}$$

Substituting Equation (9.36) in Equation (9.33) leads to

$$\tilde{\mathbf{M}}\tilde{\mathbf{P}}\dot{\tilde{\mathbf{p}}} + \tilde{\mathbf{K}}\tilde{\mathbf{P}}\tilde{\mathbf{p}} = \tilde{\mathbf{F}} \tag{9.37}$$

Premultiplying Equation (9.37) by $\tilde{\mathbf{P}}^T$ leads to

$$\tilde{\mathbf{P}}^T\tilde{\mathbf{M}}\tilde{\mathbf{P}}\dot{\tilde{\mathbf{p}}} + \tilde{\mathbf{P}}^T\tilde{\mathbf{K}}\tilde{\mathbf{P}}\tilde{\mathbf{p}} = \tilde{\mathbf{P}}^T\tilde{\mathbf{F}} = \tilde{\mathbf{G}} \tag{9.38}$$

Use of mode shape orthonormality in Equation (9.38) results in

$$\dot{\tilde{\mathbf{p}}} + \Lambda\mathbf{p} = \mathbf{G} \tag{9.39}$$

where Λ is a diagonal matrix with the eigenvalues of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ along the diagonal. Thus, the differential equations represented by Equation (9.39) are uncoupled and written as

$$\dot{\tilde{p}}_i + \gamma_i \tilde{p}_i = \tilde{g}_i(t) \qquad i = 1, 2, \ldots, 2n \tag{9.40}$$

The convolution integral solution of Equation (9.40) is

$$\tilde{p}_i = \int_0^t \tilde{g}_i(\tau) e^{-\gamma_i(t-\tau)} d\tau \tag{9.41}$$

Application of modal analysis to systems with general damping is very similar to its application to systems with proportional damping. The procedure is summarized below.

1. The differential equations governing the forced vibrations of the system are derived in terms of a chosen set of generalized coordinates and written in the form of Equation (9.32).
2. The differential equations are reformulated in the form of Equation (9.33), using Equations (8.83) and (9.34).
3. The eigenvalues and eigenvectors of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ are obtained. The eigenvectors are normalized by using Equation (8.87). The modal matrix $\tilde{\mathbf{P}}$ is formed as the matrix whose columns are the normalized mode shapes.
4. The vector $\tilde{\mathbf{G}} = \tilde{\mathbf{P}}^T\mathbf{F}$ is determined.
5. Differential equations of the form of Equation (9.40) are written for each principal coordinate.
6. The differential equations are solved by any convenient method. The convolution integral solution is given by Equation (9.41).
7. The time-dependent behavior of the chosen generalized coordinates is obtained by using Equation (9.36).

EXAMPLE 9.8

Determine the response of the system of Figure 9.8(a) when $F(t) = 50e^{-1.5t}$ N.

SOLUTION

The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{a}$$

The differential equations are written in the form of Equation (9.33) as

$$\begin{bmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & 2m \\ m & 0 & 0 & 0 \\ 0 & 2m & 0 & c \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} + \begin{bmatrix} -m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 3k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix} \tag{b}$$

where $\mathbf{y} = [\dot{x}_1 \quad \dot{x}_2 \quad x_1 \quad x_2]^T$.

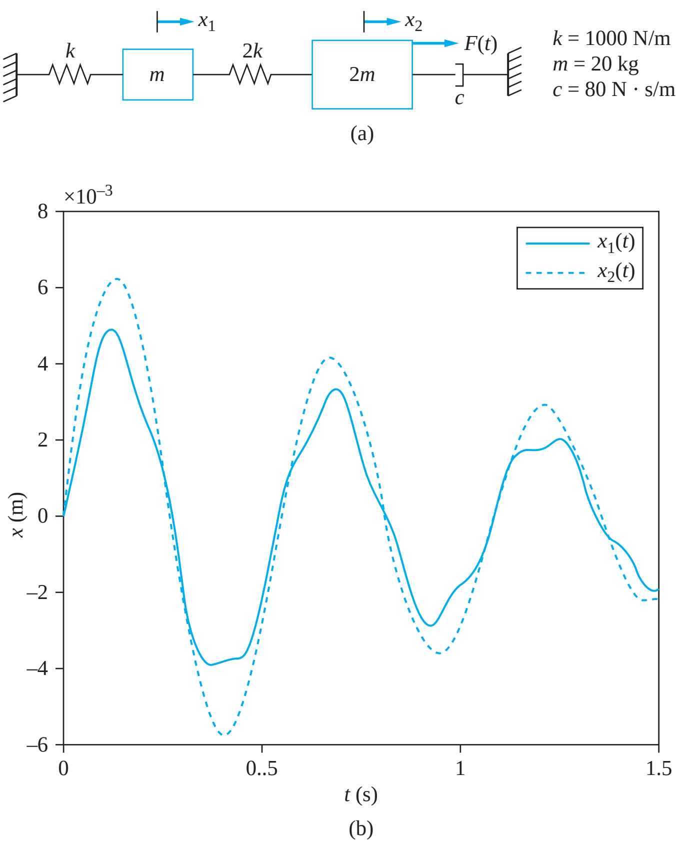


FIGURE 9.8

(a) Two degree-of-freedom system with external excitation and general damping. (b) System response.

A MATLAB program is written to evaluate the forced response for this problem. The free vibration response is calculated first with the eigenvlaues and mode shapes, as in Example 8.16. The modal matrix $\tilde{\mathbf{P}}$ is formed, and the vector $\tilde{\mathbf{G}} = \tilde{\mathbf{P}}^{\mathrm{T}}\mathbf{F}$ is calculated. The differential equation for each principal coordinate is written and solved symbolically by the convolution integral. The response for the original generalized coordinates are obtained from $\mathbf{x} = \tilde{\mathbf{P}}\breve{\mathbf{p}}$. The plot of the output is given in Figure 9.8(b).

9.6 NUMERICAL SOLUTIONS

An exact solution for the forced response of an nDOF linear system is not always possible. The excitation may be such that the convolution integral cannot be evaluated in closed form or the excitation may be known exactly only at discrete values of time. While a closed-form solution is always preferable to a numerical solution, it may be easier to obtain a numerical solution. Even when a closed-form solution is available, it must be evaluated numerically to plot the response.

Numerical difficulties may arise if a direct numerical simulation of Equation (9.18) is used. An nDOF system has n natural frequencies and n natural periods. Hence, there are n time scales implicit in the response. The time step in a numerical simulation must be chosen such that a sufficient number of time steps are taken over each natural period. Thus, the natural periods should be determined before any numerical simulation is attempted.

Since the natural frequencies should be determined before a numerical simulation is attempted, it is suggested that modal analysis be applied before a numerical simulation is attempted. Numerical solutions for the modal equations can be obtained, and Equation (9.20) can be used to obtain the response in terms of the chosen generalized coordinates. This approach has several advantages over direct numerical simulation of Equation (9.18):

1. The natural frequencies and mode shapes are known before the numerical solution begins. This makes it easier to determine an appropriate time step in a numerical approximation.
2. The use of modal analysis provides a choice of numerical solutions. Numerical integration of the convolution integral may be employed or numerical integration of the modal equations based on a method like Runge-Kutta may be used.
3. The numerical solution of n uncoupled equations is simpler and quicker than the numerical solution of n coupled equations.
4. It is not necessary to include all modes in the forced response. If the system is proportionally damped, the higher modes are more highly damped and will contribute less to the overall response. If a large number of degrees of freedom are used in modeling a structural system in order to assure high accuracy for the lowest modes, it is not desirable to include the higher modes in the response, since they provide inaccurate approximations.

9.7 BENCHMARK EXAMPLES

9.7.1 MACHINE ON FLOOR OF INDUSTRIAL PLANT

The differential equations used to model the vibrations of the machine on the floor of the industrial plant using four degrees of freedom to model the vibrations of the floor and another to model the vibrations of the machine and isolator are derived in Section 7.9. Using $F(t) = 20{,}000 \sin 80t$, they are

$$10^{-7}\begin{bmatrix} 3.34 & 5.73 & 5.35 & 3.14 & 106.7 \\ 5.73 & 14.4 & 15.2 & 9.31 & 302.9 \\ 5.35 & 15.2 & 20.1 & 13.2 & 401.1 \\ 3.14 & 9.31 & 13.2 & 12.5 & 263.4 \\ 5.35 & 15.2 & 20.1 & 13.2 & 794.3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 2 \times 10^{-3}\begin{bmatrix} 3.43 \\ 9.74 \\ 12.9 \\ 8.47 \\ 255.1 \end{bmatrix}\sin 80t \tag{a}$$

A steady-state solution is assumed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix}\sin 80t \tag{b}$$

which when substituted into Equation (a) leads to

$$\begin{bmatrix} 0.9979 & -0.0037 & -0.0034 & -0.0020 & -0.683 \\ -0.0037 & 0.9908 & -0.0098 & -0.0060 & -0.1939 \\ -0.0034 & -0.0097 & 0.9871 & -0.0084 & -0.2567 \\ -0.0020 & -0.0060 & -0.0084 & 0.9920 & -0.1686 \\ -0.0034 & -0.0097 & -0.0129 & -0.0084 & -4.0835 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} 0.0069 \\ 0.0195 \\ 0.0258 \\ 0.0169 \\ 0.5708 \end{bmatrix} \tag{c}$$

Simultaneous solution of Equation (c) gives

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} -0.00173 \\ -0.00490 \\ -0.00647 \\ -0.00426 \\ -0.1257 \end{bmatrix} \tag{d}$$

The amplitude of the force transmitted to the beam is

$$k|U_5 - U_3| = (3.93 \times 10^4 \text{ lb/ft})\,|-0.1257 \text{ ft} + 0.00647 \text{ ft}| = 4660 \text{ lb} \tag{e}$$

TABLE 9.1

Model of machine attached to beam with isolator of stiffness 3.93×10^4 lb/ft	$F_T(N)$	Natural frequencies (rad/s)
SDOF model, assumes beam is rigid	5000	35.6
2DOF model, uses equivalent mass and stiffness of beam	4576	34.6, 329
5DOF model, uses flexibility matrix with lumped masses to model beam	4660	35.5, 494.1, 1.56×10^3, 2.83×10^3, 4.186×10^3
Four-element finite-element model of beamresults in a 10DOF system	4610	34.7, 381.7, 1.01×10^3, 2.13×10^3, 3.69×10^3, 6.12×10^3, 9.04×10^3, 1.32×10^4, 1.80×10^4, 2.44×10^4

Since the transmitted force is less than 5,000 lb, the force transmitted through the isolator is still acceptable.

Table 9.1 shows the models of the machine on the floor of the industrial plant with an isolator of stiffness 3.93×10^4 lb/ft. The table includes the natural frequencies of the model as well as the transmitted force between the isolator and the beam. The finite-element model presented in Chapter 11 is included for comparison. The transmitted force predicted using a rigid model for the beam is the largest at 5000 N. The transmitted force in all other models is less. Thus, the SDOF approximation is sufficient for the vibration isolation problem. The lowest natural frequency ranges from 34.6 rad/s for the two DOF model to 35.7 for the SDOF model.

9.7.2 SIMPLIFIED SUSPENSION SYSTEM

The differential equations governing the motion of the vehicle are derived in Section 7.7 as

$$\begin{bmatrix} 225 & 0 & 0 & 0 \\ 0 & 300 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + 10^3 \begin{bmatrix} 5.50 & 3.60 & -1.56 & 2.04 \\ -1.08 & 2.4 & -1.2 & -1.2 \\ -1.56 & -1.2 & 1.12 & 0 \\ 2.04 & -1.2 & 0 & 1.12 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$+ 10^4 \begin{bmatrix} 5.50 & 3.60 & -1.56 & 2.04 \\ -1.08 & 2 & -1.2 & -1.2 \\ -1.56 & -1.2 & 1.12 & 0 \\ 2.04 & -1.2 & 0 & 1.12 \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \times 10^4 \dot{y} + 1 \times 10^5 y \\ 1 \times 10^4 \dot{z} + 1 \times 10^5 z \end{bmatrix} \qquad \text{(a)}$$

The system has proportional damping. Modal analysis is used to solve for the forced response. The natural frequencies, modal damping ratios, and modal matrix are calculated in Section 8.14. The components of the right-hand side vector for the modal equations are calculated as

$$\mathbf{G} = \mathbf{P}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} 0.0169 & 0.0645 & -0.00028 & 0.00450 \\ 0.0560 & -0.00140 & -0.002313 & -0.00046 \\ 0.00709 & 0.00664 & 0.1664 & -0.1105 \\ 0.00262 & -0.0118 & 0.1106 & 0.1661 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1\times 10^4\dot{y} + 1\times 10^5 y \\ 1\times 10^4\dot{z} + 1\times 10^5 z \end{bmatrix}$$

$$= \begin{bmatrix} 97.1\dot{y} + 971y + 97.1\dot{z} + 971z \\ -51.6\dot{y} - 516y - 51.6\dot{z} - 516z \\ 2770\dot{y} + 27{,}700y + 2770\dot{z} + 27{,}700z \\ 556\dot{y} + 5560y + 556\dot{z} + 5560z \end{bmatrix} \tag{b}$$

The vehicle travels over a bump in the road at speed of v, which is given in Section 5.10 as

$$y(t) = 0.02\left[1 - \cos^2\left(\frac{10\pi v}{6}t\right)\right]\left[1 - u\left(t - \frac{0.6}{v}\right)\right] \tag{c}$$

from which

$$\dot{y}(t) = 0.02\left\{-2\left(\frac{10\pi v}{6}\right)\sin\left(\frac{10\pi v}{6}t\right)\cos\left(\frac{10\pi v}{6}t\right)\left[1 - u\left(t - \frac{0.6}{v}\right)\right]\right.$$
$$\left. - \left[1 - \cos^2\left(\frac{10\pi v}{6}t\right)\right]\delta\left(t - \frac{0.6}{v}\right)\right\} \tag{d}$$

The rear wheels traverse the bump at a time $(a + b)/v = 3/v$ later, giving the equation for $z(t)$ as

$$z(t) = 0.02\left\{1 - \cos^2\left[\frac{10\pi v}{6}\left(t - \frac{3}{v}\right)\right]\right\}\left[u\left(t - \frac{3}{v}\right) - u\left(t - \frac{3.6}{v}\right)\right\} \tag{e}$$

from which

$$\dot{z}(t) = 0.02\left[-2\sin\left[\frac{10\pi v}{6}\left(t - \frac{3}{v}\right)\right]\cos\left[\frac{10\pi v}{6}\left(t - \frac{3}{v}\right)\right]\left[u\left(t - \frac{3}{v}\right) - u\left(t - \frac{3.6}{v}\right)\right]\right.$$
$$\left. - \left\{1 - \cos^2\left[\frac{10\pi v}{6}\left(t - \frac{3}{v}\right)\right]\right\}\left[\delta\left(t - \frac{3}{v}\right) - \delta\left(t - \frac{3.6}{v}\right)\right]\right] \tag{f}$$

The differential equations for the modal responses are

$$\ddot{p}_1 + 6.65\dot{p}_1 + 55.2p_1 = 97.1\dot{y} + 971y + 97.1\dot{z} + 971z \tag{g}$$

$$\ddot{p}_2 + 23.18\dot{p}_2 + 193p_2 = -51.6\dot{y} - 516y - 51.6\dot{z} - 516z \tag{h}$$

$$\ddot{p}_3 + 528.5\dot{p}_3 + 4409p_3 = 2770\dot{y} + 27{,}700y + 2770\dot{z} + 27{,}700z \tag{i}$$

$$\ddot{p}_4 + 530.7\dot{p}_4 + 4419p_4 = 556\dot{y} + 5560y + 556\dot{z} + 5560z \tag{j}$$

Convolution integral solutions of Equations (h) through (j) are available for Equations (g) through (j).

$$p_1(t) = \frac{1}{6.64}\int_0^t [97.1\dot{y}(\tau) + 971y(\tau) + 97.1\dot{z}(\tau)$$
$$+ 971z(\tau)]e^{-3.33(t-\tau)}\sin 6.64(t-\tau)d\tau \tag{k}$$

$$p_2(t) = \frac{1}{4.10}\int_0^t [-51.6\dot{y}(\tau) - 516y(\tau)$$
$$- 51.6\dot{z}(\tau) - 516z(\tau)]e^{-11.60(t-\tau)}\sin 4.10(t-\tau)d\tau \tag{l}$$

$$p_3(t) = \frac{1}{512.2}\int_0^t [2770\dot{y}(\tau) + 27{,}700(\tau)$$
$$+ 2770\dot{z}(\tau) + 27{,}700z(\tau)][e^{-8.47(t-\tau)} - e^{-520.7(t-\tau)}]d\tau \tag{m}$$

$$p_4(t) = \frac{1}{513.7}\int_0^t [556\dot{y}(\tau) + 5560y(\tau) + 556\dot{z}(\tau)$$
$$+ 5560z(\tau)][e^{-8.47(t-\tau)} - e^{-522.2(t-\tau)}]d\tau \tag{n}$$

The response of the system in terms of the original generalized coordinates is given by

$$\mathbf{x} = \mathbf{P}\mathbf{p} = \begin{bmatrix} 0.0169 & 0.0645 & -0.00028 & 0.00450 \\ 0.0560 & -0.0140 & -0.00213 & -0.00046 \\ 0.00709 & 0.00664 & 0.1664 & -0.1105 \\ 0.00262 & -0.0118 & 0.1106 & 0.1661 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{o}$$

The third and fourth modes are overdamped and will not have much effect on the response of the system. Thus, only the first two modes are used in the response

$$\begin{bmatrix} \theta \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.0169 & 0.0645 \\ 0.0560 & -0.0140 \\ 0.00709 & 0.00664 \\ 0.00262 & -0.0118 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.0169p_1 + 0.0645p_2 \\ 0.0560p_1 - 0.0140p_2 \\ 0.00709p_1 + 0.00664p_2 \\ 0.00262p_1 - 0.0118p_2 \end{bmatrix} \tag{p}$$

Numerical integration of the convolution integral with piecewise constants is used to determine the time dependence of the principal coordinates. The results are given in Figure 9.9 for $v = 15$ m/s and $v = 60$ m/s.

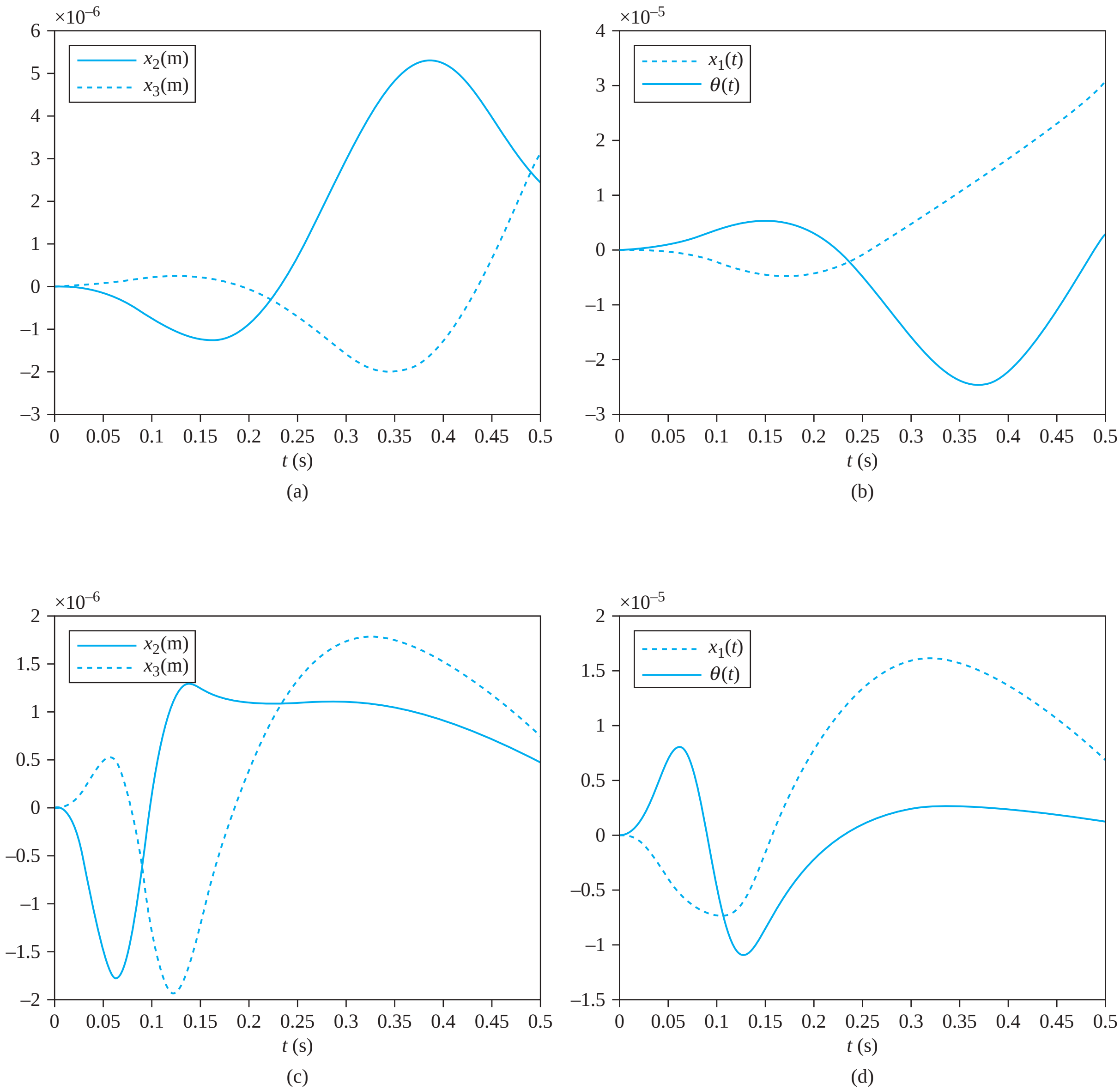


FIGURE 9.9
Numerical integration of convolution integral is used to determine displacement of vehicle traveling over a bump in the road. (a) Displacement of the wheels at $v = 15$ m/s. (b) Displacement of the body of the vehicle and its angular rotation at $v = 15$ m/s. (c) Displacement of the wheels at $v = 60$ m/s. *(d)* Displacement of the body of the vehicle and its angular rotation at $v = 60$ m/s.

9.8 FURTHER EXAMPLES

EXAMPLE 9.9

Reconsider the three degree-of-freedom model of the hand of Example 7.21 and Example 8.16. The mathematical model is repeated as

$$\begin{bmatrix} 5.0516 & 0 & 0 \\ 0 & 1.4295 & 0 \\ 0 & 0 & 0.887 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 152.1 & -64.9 & 0 \\ -64.9 & 176.0 & -36.3 \\ 0 & -36.3 & 111.1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} 151{,}216 & -1726 & 0 \\ -1726 & 43{,}699 & -12{,}075 \\ 0 & -12{,}075 & 207{,}740 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 74.8\dot{y} + 29{,}898y \\ 126\dot{y} + 195{,}695y \end{bmatrix} \tag{a}$$

(a) Determine the steady-state amplitudes when the hand is gripping a power tool that has a vibration of

$$y(t) = 5 \times 10^{-5} \sin 100t \tag{b}$$

(b) Determine the response of the system when the hand is gripping an object that expands according to

$$y(t) = 5 \times 10^{-5}(1 - e^{-50t}) \tag{c}$$

SOLUTION

(a) Substituting for the displacement of the tool into the differential equations leads to

$$\begin{bmatrix} 5.0516 & 0 & 0 \\ 0 & 1.4295 & 0 \\ 0 & 0 & 0.887 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 152.1 & -64.9 & 0 \\ -64.9 & 176.0 & -36.3 \\ 0 & -36.3 & 111.1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} 151{,}216 & -1726 & 0 \\ -1726 & 43{,}699 & -12{,}075 \\ 0 & -12{,}075 & 207{,}740 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.536\sin(100t + 0.254) \\ 9.80\sin(100t + 0.0644) \end{bmatrix} \tag{d}$$

A solution to Equation (c) is assumed to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} e^{i100t} \tag{e}$$

Only the imaginary part of the solution is used for the response. Substitution of Equation (d) into Equation (c) using complex notation for the trigonometric terms leads to

$$10^5\begin{bmatrix} 1.007+0.152i & -0.0173+0.06490i & 0 \\ -0.0173+0.0649i & 0.2877+0.176i & -0.1208-0.0363i \\ 0 & -0.1208-0.0363i & 1.9887+0.111i \end{bmatrix}$$

$$\times\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.536e^{0.254i} \\ 9.80e^{0.0644i} \end{bmatrix} \tag{f}$$

The solution of Equation (f) is

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^{-4}\begin{bmatrix} 0.0270+0.0334i \\ 0.6281-0.1735i \\ 0.5332+0.0029i \end{bmatrix} \tag{g}$$

The steady-state solution for the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathrm{Im}\left(10^{-4}\begin{bmatrix} 0.0270+0.0334i \\ 0.6281-0.1735i \\ 0.5332+0.0029i \end{bmatrix}e^{i100t}\right)$$

$$= 10^{-4}\begin{bmatrix} 0.0270\sin 100t + 0.0334\cos 100t \\ 0.6281\sin 100t - 0.1735\cos 100t \\ 0.5332\sin 100t + 0.0029\cos 100t \end{bmatrix}$$

$$= 10^{-4}\begin{bmatrix} 0.0430\sin(100t+0.892) \\ 0.652\sin(100t-0.270) \\ 0.53332\sin(100t+0.0054) \end{bmatrix} \tag{h}$$

(b) Substituting for the displacement of the object, we have

$$\begin{bmatrix} 5.0516 & 0 & 0 \\ 0 & 1.4295 & 0 \\ 0 & 0 & 0.887 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 152.1 & -64.9 & 0 \\ -64.9 & 176.0 & -36.3 \\ 0 & -36.3 & 111.1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$+\begin{bmatrix} 151{,}216 & -1726 & 0 \\ -1726 & 43{,}699 & -12{,}075 \\ 0 & -12{,}075 & 207{,}740 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.50-1.39e^{-50t} \\ 9.78-9.47e^{-50t} \end{bmatrix} \tag{i}$$

The system has damping, but it is not proportionally damped. Thus, the state–space formulation and a general modal analysis are required. Thus, a six-dimensional vector is defined as $\mathbf{y} = [\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3 \quad x_1 \quad x_2 \quad x_3]^T$. The eigenvalues and eigenvectors of the matrix $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ are calculated in Example 8.16. The force vector is defined as

$$\tilde{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.50 - 1.39e^{-50t} \\ 9.78 - 9.47e^{-50t} \end{bmatrix} \tag{j}$$

The mode shape vectors are normalized according to $\overline{\mathbf{\Phi}}_{\mathbf{i}}^{\mathbf{T}}\tilde{\mathbf{M}}\mathbf{\Phi}_{\mathbf{i}} = 1$. For eigenvalues

$$\gamma_{1,2} = 64.01 \pm 479.1i,\ \gamma_{3,4} = 12.06 \pm 171.7i,\ \gamma_{3,4} = 63.17 \pm 162.3i \tag{k}$$

the modal matrix is

$$\mathbf{P} = \begin{bmatrix} 0.034 - 8.126 \times 10^{-3}i & 0.034 + 8.126 \times 10^{-3}i & -2.609 + 3.047i & -2.609 - 3.047i & -0.514 + 0.877i & -0.514 - 0.877i \\ -0.435 - 1.05i & -0.435 + 1.05i & -1.41 + 1.279i & -1.41 - 1.279i & 3.136 - 7.235i & 3.136 + 7.235i \\ -10.104 + 13.131i & -10.014 - 13.131i & -0.058 + 0.126i & -0.058 - 0.126i & -0.067 - 0.494i & -0.067 + 0.494i \\ 7.458 \times 10^{-6} + 7.114 \times 10^{-7}i & 7.458 \times 10^{-6} - 7.114 \times 10^{-7}i & -0.017 - 0.016i & -0.017 + 0.161i & -3.62 \times 10^{-3} - 4.565 \times 10^{-3}i & -3.62 \times 10^{-3} + 4.565 \times 10^{-3}i \\ 2.272 \times 10^{-3} - 6.039 \times 10^{-4}i & 2.272 \times 10^{-3} - 6.039 \times 10^{-4}i & -6.835 \times 10^{-3} - 8.69 \times 10^{-3}i & -6.835 \times 10^{-3} + 8.69 \times 10^{-3}i & 0.032 + 0.032i & 0.032 - 0.032i \\ -0.024 - 0.024i & -0.024 + 0.024i & -7.085 \times 10^{-4} - 3.85 \times 10^{-4}i & -7.085 \times 10^{-4} + 3.85 \times 10^{-4}i & 2.777 \times 10^{-3} + 6.668 \times 10^{-4}i & 2.777 \times 10^{-3} - 6.668 \times 10^{-4}i \end{bmatrix} \tag{l}$$

The vector of generalized forces is calculated from

$$\tilde{\mathbf{G}} = \tilde{\mathbf{P}}^{\mathbf{T}}\tilde{\mathbf{F}} = \begin{bmatrix} -0.2313 - 0.2438i + (0.2241 + 0.2375i)e^{-50t} \\ -0.2313 + 0.2438i + (0.2241 - 0.2357i)e^{-50t} \\ -0.0172 - 0.0168i + (0.0162 + 0.0157i)e^{-50t} \\ -0.0172 + 0.0168i + (0.0162 - 0.0157i)e^{-50t} \\ 0.0752 + 0.0545i + (-0.0708 - 0.0508i)e^{-50t} \\ 0.0752 - 0.0545 + (-0.0708 + 0.0508i)e^{-50t} \end{bmatrix} \tag{m}$$

The differential equations for the principal coordinates become

$$\dot{p}_1 + (64.01 + 479.1i)p_1 = -0.2313 - 0.2438i + (0.2241 + 0.2375i)e^{-50t} \tag{n}$$

$$\dot{p}_2 + (64.01 - 479.1i)p_2 = -0.2313 + 0.2438i + (0.2241 - 0.2357i)e^{-50t} \tag{o}$$

$$\dot{p}_3 + (63.17 + 162.3i)p_3 = -0.0172 - 0.0168i + (0.0162 + 0.0157i)e^{-50t} \tag{p}$$

$$\dot{p}_4 + (63.17 - 162.3i)p_4 = -0.0172 + 0.0168i + (0.0162 - 0.0157i)e^{-50t} \tag{q}$$

$$\dot{p}_5 + (12.06 + 171.6i)p_5 = 0.0752 + 0.0545i + (-0.0708 - 0.0508i)e^{-50t} \tag{r}$$

$$\dot{p}_6 + (12.06 - 171.6i)p_6 = 0.0752 - 0.0545 + (-0.0708 + 0.0508i)e^{-50t} \tag{s}$$

Equations (n) through (s) are first-order nonhomogenous differential equations. The solution of

$$\dot{p} + \lambda p = A + Be^{-50t} \tag{t}$$

subject to $p(0) = 0$ is

$$p(t) = -\left(\frac{A}{\lambda} + \frac{B}{\lambda - 50}\right)e^{-\lambda t} + \frac{A}{\lambda} - \frac{B}{\lambda - 50}e^{-50t} \tag{u}$$

The solutions to Equations (n) through (s) are

$$\begin{aligned} p_1(t) = 10^{-5}[&(5.435 + 4.535i)e^{-(64.01+479.1i)t} \\ &- 56.33 + 40.75i + (50.89 - 45.28i)e^{-50t}] \end{aligned} \tag{v}$$

$$\begin{aligned} p_2(t) = 10^{-5}[&(5.435 - 4.535i)e^{-(64.01-479.1i)t} \\ &- 56.33 - 40.75i + (50.89 + 45.28i)e^{-50t}] \end{aligned} \tag{w}$$

$$\begin{aligned} p_3(t) = 10^{-5}[&(-15.82 - 3.575i)e^{-(63.17+162.3i)t} \\ &+ 10.42 - 9.13i + (5.407 + 12.70i)e^{-50t}] \end{aligned} \tag{x}$$

$$\begin{aligned} p_4(t) = 10^{-5}[&(-15.82 + 3.575i)e^{-(63.17-162.3i)t} \\ &+ 10.42 + 9.13i + (5.407 - 12.70i)e^{-50t}] \end{aligned} \tag{y}$$

$$\begin{aligned} p_5(t) = 10^{-5}[&(-15.14 - 4.189i)e^{-(12.06+171.6i)t} \\ &+ 34.67 - 41.39i + (-19.52 - 45.57i)e^{-50t}] \end{aligned} \tag{z}$$

$$\begin{aligned} p_6(t) = 10^{-5}[&(-15.14 + 4.189i)e^{-(12.06-171.6i)t} \\ &+ 34.67 + 41.39i + (-19.52 + 45.57i)e^{-50t}] \end{aligned} \tag{aa}$$

The original generalized coordinates and their velocities are obtained by multiplying the modal matrix times the vector of principal coordinates $\mathbf{y} = \tilde{\mathbf{P}}\tilde{\mathbf{p}}$. The values of the original generalized coordinates are $x_1 = y_4$, $x_2 = y_5$, and $x_3 = y_6$.

9.9 CHAPTER SUMMARY

9.9.1 IMPORTANT CONCEPTS

- The method of undetermined coefficients can be used to determine the steady-state response of a system with harmonic input.
- The Laplace transform method leads to a set of algebraic equations in terms of the transform parameter. The elements of the inverse of the impedance matrix are the transfer

functions $G_{i,j}(s)$. The concept of the sinusoidal transfer function can be used to find the steady-state response.

- Modal analysis is a method where principal coordinates are used to uncouple the differential equations and can be applied to systems that are undamped or have proportional damping.
- A modal analysis exists for systems with proportional damping.
- Modal analysis is used to uncouple the differential equations when a numerical integration method is used.
- Numerical methods, such as Runge-Kutta methods or numerical integration of the convolution integral, can be applied to determine the response of a MDOF system.

9.9.2 IMPORTANT EQUATIONS

Steady-state solution of an undamped system using the method of undetermined coefficients

$$\mathbf{U} = (-\omega^2\mathbf{M} + \mathbf{K})^{-1}\mathbf{F} \tag{9.5}$$

Steady-state solution of a damped system using the method of undetermined coefficients

$$\mathbf{U} = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}\mathbf{F} \tag{9.10}$$

Impedance matrix

$$\mathbf{Z}(s) = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{9.13}$$

Solution of equations by Laplace transform method

$$\mathbf{X}(s) = \mathbf{Z}^{-1}(s)\mathbf{F}(s) \tag{9.14}$$

Use of sinusoidal transfer function to determine response of system due to harmonic input

$$x_k(t) = \sum_{j=1}^{n} |G_{k,j}(i\omega_j)| F_j \sin(\omega_j t + \phi_{k,j}) \tag{9.16}$$

Expansion of response in terms of principal coordinates

$$\mathbf{x}(t) = \sum_{i=1}^{n} p_i(t)\mathbf{X}_i \tag{9.19}$$

Differential equations that the principal coordinates satisfy for an undamped system

$$\ddot{p}_j + \omega_j^2 p_j = g_j(t) \tag{9.24}$$

$$g_j(t) = (\mathbf{X}_j, \mathbf{F}) \tag{9.25}$$

Differential equations that the principal coordinates satisfy for a system with proportional damping

$$\ddot{p}_i + 2\zeta_i\omega_i\dot{p}_i + \omega_i^2 p_i = g_i(t) \tag{9.29}$$

Convolution integral solution for principal coordinates

$$p_i(t) = \frac{1}{\omega_i\sqrt{1-\zeta_i^2}}\int_0^t g_i(\tau)e^{-\zeta_i\omega_i(t-\tau)}\sin\left[\omega_i\sqrt{1-\zeta_i^2}(t-\tau)\right]d\tau \tag{9.30}$$

Principal coordinates for system with general damping

$$\dot{\tilde{p}}_i + \gamma_i \tilde{p}_i = \tilde{g}_i(t) \quad (9.40)$$

Convolution integral solution for principal coordinates for system with general damping

$$\tilde{p}_i = \int_0^t \tilde{g}_i(t) e^{-\gamma_i (t-\tau)} d\tau \quad (9.41)$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 9.1 through 9.7, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

9.1 The Laplace transform method cannot be used to determine the response of a system with proportional damping.

9.2 The principal coordinates are used to uncouple the differential equations for forced vibrations.

9.3 For a system with a damping matrix that is proportional to the stiffness matrix, the higher modes are more highly damped and therefore have less of an effect on the forced response.

9.4 The elements of the impedance matrix are the transfer functions $G_{i,j}(s)$.

9.5 The principal coordinates are only used to determine the steady-state response of a system.

9.6 The vector of forces for the right-hand side of the equations defining the principal coordinates is calculated by $\mathbf{G} = \mathbf{P}^T\mathbf{F}$.

9.7 The kth component of G, which is the vector on the right-hand side of the equations defining the generalized coordinate, is calculated by taking the kinetic-energy scalar product of the forced vector with the kth normalized mode shape.

Problems 9.8 and 9.9 require a short answer.

9.8 The determinant of the impedance matrix of an nDOF system is a polynomial of what order?

9.9 The lowest natural frequency of a five degree-of-freedom system is 30 rad/s. Select the differential equation which could be the equation for the principal coordinate.

(a) $\ddot{p}_1 + \dot{p}_1 = g_1(t)$
(b) $\ddot{p}_1 + 30\dot{p}_1 = g_1(t)$
(c) $\ddot{p}_1 + 900\dot{p}_1 = g_1(t)$
(d) $\ddot{p}_1 = g_1(t)$

Problems 9.10 through 9.13 are fill-in-the-blank questions regarding the derivation of modal analysis for an undamped system or a system with proportional damping.

9.10 To derive modal analysis, the ____________ is used to write the general solution as a linear combination of the principal coordinates.
9.11 The ____________ scalar product is taken with both sides of the equation after the linear combination is substituted into the differential equations.
9.12 The equations are ____________ using mode shape ____________ with respect to ____________ and ____________.
9.13 The ____________ integral can be used to solve the resulting nonhomogenous differential equations.

Problems 9.14 through 9.18 are fill-in-the-blank questions regarding the derivation of modal analysis for a system with a general damping matrix.

9.14 For systems with a general damping matrix, the differential equations governing the nDOF system is written as ____________ first-order differential equations.
9.15 The vector $\tilde{\mathbf{F}}$ is defined as the $2n \times 1$ vector ____________.
9.16 The modal matrix $\tilde{\mathbf{P}}$ is defined as the matrix whose columns are normalized by ____________.
9.17 The differential equations governing the principal coordinates of the system are____________.
9.18 The differential equations have a solution, $\tilde{p}_i = \int_0^t \tilde{g}_i(t)e^{-\gamma_i(t-\tau)}d\tau$, called the ____________.
9.19 Give two reasons why modal analysis is convenient to use before solving a system using the Runge-Kutta method.
9.20 Give two reasons why modal analysis should be used before using numerical integration of the convolution integral.

Problems 9.21 through 9.23 require short calculations.

In Problems 9.21 and 9.22, spectral analysis shows that the natural frequencies for a fifth-order system are 20 rad/s, 41 rad/s, 55 rad/s, 93 rad/s, and 114 rad/s. Experimental modal analysis is used to determine that its modal matrix is

$$\mathbf{P} = \begin{bmatrix} 1.3 & 1.0 & 0.7 & 0.5 & 0.1 \\ 1.8 & 1.5 & 1.0 & 0.4 & -0.3 \\ 2.4 & 0.5 & -0.4 & -0.3 & 0.2 \\ 2.9 & -0.2 & -0.7 & 0.5 & -0.5 \\ 2.0 & -0.15 & 0.2 & -0.6 & 0.4 \end{bmatrix}$$

9.21 If the system is undamped and subject to a force vector equal to $\mathbf{F} = [0 \quad 0 \quad \sin 54t \quad 0 \quad 0]^T$, determine the following.

(a) Write the differential equation for the first principal coordinate.
(b) What is the steady-state solution of this differential equation?
(c) Which mode do you expect will have the largest contribution to the response?
(d) What is the relation between the fifth generalized coordinate and the principal coordinates?

9.22 If the system is damped with modal damping ratios of 0.3, 0.615, 0.825, 1.395, and 1.71, and has a forcing vector equal to $\mathbf{F} = [0 \quad 0 \quad \sin 54t \quad 0 \quad 0]^T$, determine the following.

(a) Write the differential equation for p_4.
(b) What is the steady-state solution of this differential equation?
(b) Which modes are overdamped and which are underdamped?
(c) What is the constant(s) of proportionality between the damping matrix and the stiffness and mass matrices?

9.23 The differential equations governing a three degree-of-freedom system are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 5 & -3 & 0 \\ -3 & 7 & -4 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \sin 60t \end{bmatrix}$$

What is the impedance matrix for this system?

CHAPTER PROBLEMS

9.1 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.1. Use the method of undetermined coefficients.

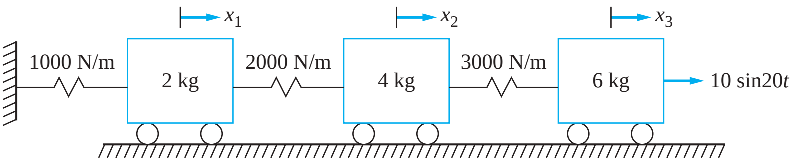


FIGURE P9.1

9.2 Determine the steady-state amplitude for the mass hanging from the end of the bar in the system in Figure P9.2. Use the method of undetermined coefficients.

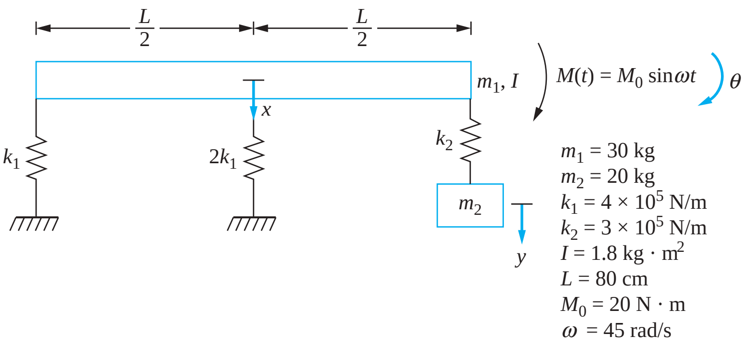


FIGURE P9.2

9.3 Determine the steady-state amplitude of vibration of the mass m_3 of the system in Figure P9.3. Use the method of undetermined coefficients.

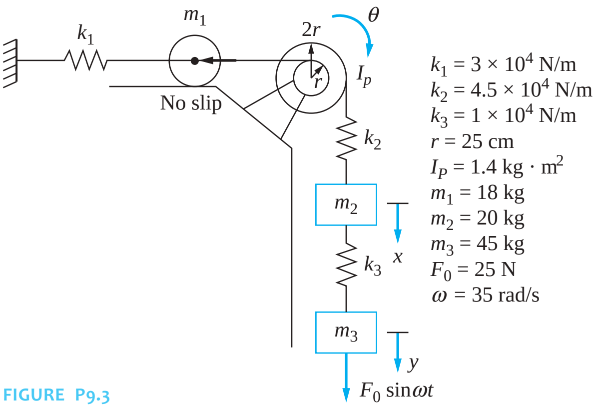


FIGURE P9.3

9.4 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.4. Use the method of undetermined coefficients.

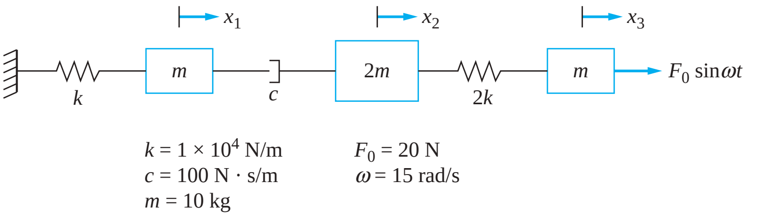


FIGURE P9.4

9.5 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.5. Use the method of undetermined coefficients.

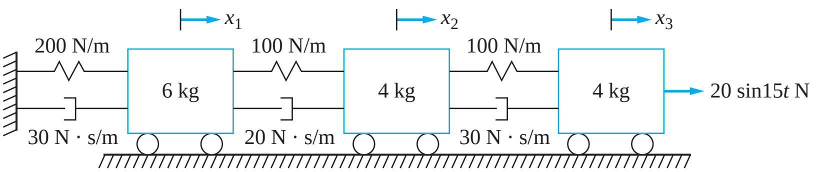


FIGURE P9.5

9.6 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.6. Use the method of undetermined coefficients.

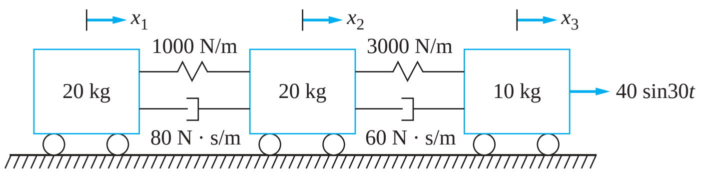


FIGURE P9.6

9.7 Determine the steady-state responses of each of the masses of the system in Figure P9.7. Use the method of undetermined coefficients.

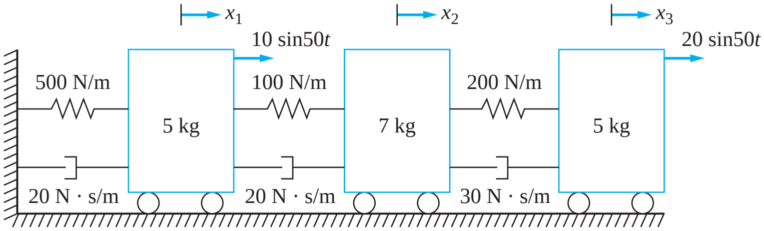


FIGURE P9.7

9.8 Determine the steady-state responses of each of the masses of the system in Figure P9.8. Use the method of undetermined coefficients.

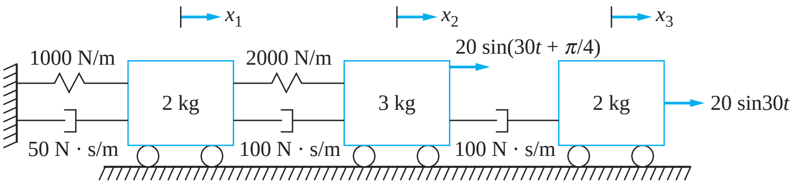


FIGURE P9.8

9.9 Determine the steady-state response of the hanging mass in the system of Figure P9.9. Use the method of undetermined coefficients.

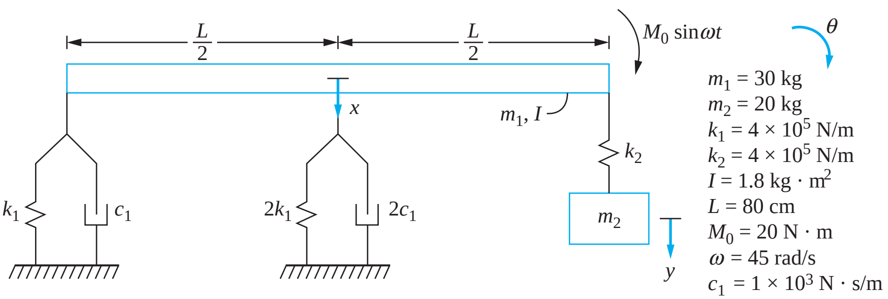


FIGURE P9.9

9.10 Determine the steady-state amplitudes of vibration of each of the masses in the system of Figure P9.1. Use the Laplace transform method.

9.11 Determine the steady-state amplitudes of vibration of the hanging mass in the system of Figure P9.2. Use the Laplace transform method.

9.12 Determine the steady-state amplitude of vibration of the mass m_3 of the system in Figure P9.3. Use the Laplace transform method.

9.13 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.4. Use the Laplace transform method.

9.14 Determine the steady-state amplitudes of vibration of each of the masses of the system in Figure P9.5. Use the Laplace transform method.

9.15 Determine the response of the 2 kg mass of Figure P9.1 if the sinusoidal force is replaced by the triangular pulse of Figure P9.15. Use the Laplace transform method.

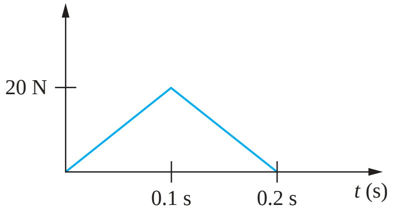


FIGURE P9.15

9.16 Determine the response of the 6 kg mass of Figure P9.1 if the sinusoidal force is replaced by the rectangular pulse of Figure P9.16. Use the Laplace transform method.

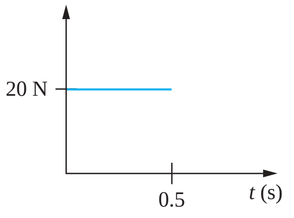


FIGURE P9.16

9.17 Determine the response of the system of Figure P9.2 if the sinusoidal force is replaced by the force of Figure P9.17. Use the Laplace transform method.

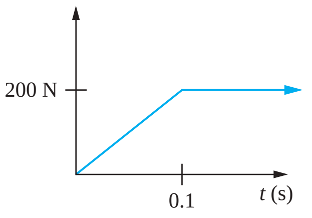


FIGURE P9.17

9.18 Repeat Chapter Problem 9.1 using modal analysis.
9.19 Repeat Chapter Problem 9.2 using modal analysis.
9.20 Repeat Chapter Problem 9.3 using modal analysis.
9.21 Repeat Chapter Problem 9.15 using modal analysis
9.22 Repeat Chapter Problem 9.16 using modal analysis.
9.23 Repeat Chapter Problem 9.7 using modal analysis.
9.24 Repeat Chapter Problem 9.9 using modal analysis.
9.25 Figure P9.25 shows a machine attached to a fixed-pinned beam through an isolator. Design an isolator of damping ratio 0.1 such that the force transmitted

to the beam is 2000 N when the machine is subject to a harmonic excitation with an amplitude of 12,000 N at a frequency of 300 rad/s. Use a three degree-of-freedom lumped-mass model for the beam.

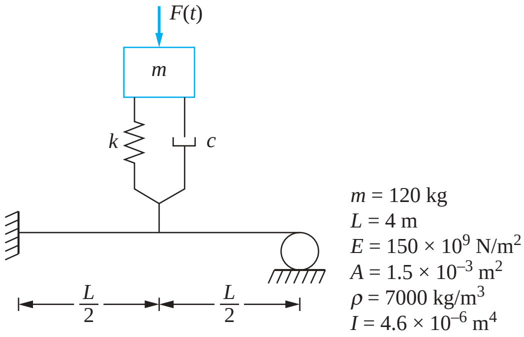


FIGURE P9.25

9.26 Design an isolator with a damping ratio of 0.4 for the system of Figure P9.25 when it is subject to the pulse of Figure P9.26. The maximum force transmitted to the beam should be 500 N.

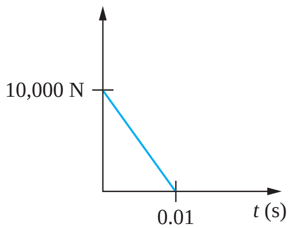


FIGURE P9.25

VIBRATIONS OF CONTINUOUS SYSTEMS

10.1 INTRODUCTION

All solid objects are made of deformable materials. Often a solid is assumed to be rigid. This allows for simpler modeling and leads to information about essential vibrational characteristics. The validity of a rigid-body assumption in modeling the vibrations of a system depends on many factors such as geometry and frequency range. For example, consider a machine mounted on springs and operating in an industrial plant. The floor of the industrial plant is often assumed to be rigid and the vibrations of the machine considered by analyzing a one-degree-of-freedom system. However, if the forces developed in the springs are large, then since the floor is really deformable, vibrations are excited in the floor and perhaps the entire structure. In this case, the vibrations of the machine are coupled to the structural vibrations.

Examples of continuous systems are shown in Figure 10.1. All structural elements such as beams, columns, and plates are continuous systems. This includes the suspended piping system of Figure 10.1(a), simply supported at locations along its length. Vibrations of the pipeline are excited by the fluid flowing through the pipe, the operation of pumps, or structural vibrations. The vibrations are analyzed by considering a continuous beam with simple supports.

All elements of the frame structure of Figure 10.1(b) are continuous structural elements. Often the columns of a frame structure are much more flexible than the girders, and the girders are considered rigid, resulting in the model shown.

The spring of Figure 10.1(c) is a simple continuous system. As one end of the spring is moved relative to the other, a compression wave is generated and travels throughout the

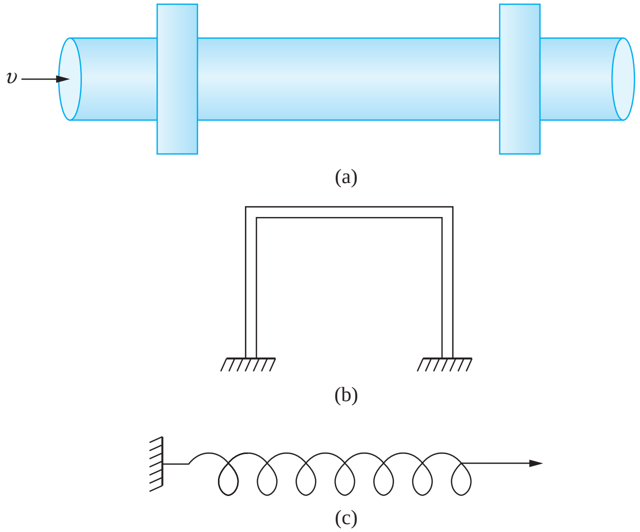


FIGURE 10.1
Examples of continuous systems: (a) simply supported piping system; (b) one-story frame structure; (c) helical coil spring.

spring. If the excitation frequency is near the frequency of the compression waves, a phenomenon called *surge* develops. Surge can be a problem in mechanical systems where one end of a spring is given a harmonic displacement.

The free and forced vibrations of a rigid body attached to a continuous system are approximated by using one degree of freedom in Chapters 2 through 5. The inertia effects of a continuous element are approximated by adding a particle of a calculated equivalent mass at the location of the rigid body. Multiple degree-of-freedom approximations are considered in Chapters 6 through 9.

A variable x, measured along the axis of the bar, is introduced for the analysis of the vibrations in each of the continuous systems in Figure 10.2. The displacement w is measured

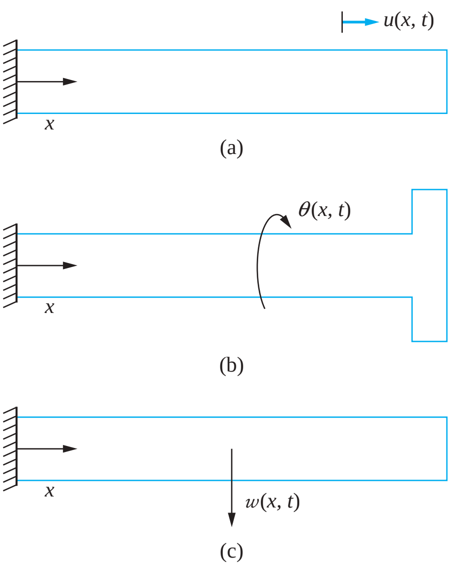


FIGURE 10.2
(a) A coordinate x, measured from the left end of the bar along the axis of the bar, is introduced for the analysis of vibrations of the bar. The displacement of the bar is a function of both x and time t as $u(x, t)$. (b) The angular displacement is a function of x and t as $\theta(x, t)$. (c) The transverse displacement of the beam is a function of x and t as $w(x, t)$, where x is measured along the beam's neutral axis.

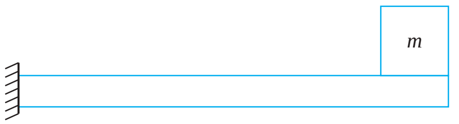


FIGURE 10.3
Discrete approximation works well when m is large compared to the mass of the beam.

as a function of the variable x as well as time as $w(x, t)$. Since w depends upon two independent variables, a governing partial differential equation is derived.

The ordinary differential equations obtained by using a discrete model of the continuous system are easier to solve than the governing partial differential equation. Thus discrete approximations are often used, but have limitations. A continuous system has an infinite, but countable, number of natural frequencies and corresponding mode shapes. A discrete approximation predicts only a finite number of modes. Often a large number of degrees of freedom are needed to attain accurate approximations for higher natural frequencies. Consider, for example, the cantilever beam of Figure 10.3 with a concentrated mass at its end. Figure 10.4 shows the nondimensional lowest natural frequency as a function of β, the ratio of the concentrated mass to the mass of the beam. Figure 10.4 shows natural frequencies calculated using up to six degrees of freedom, as well as a one degree-of-freedom approximation.

The methods used in this chapter are analogous to those used for multiple degree-of-freedom systems. The separation-of-variables method used to determine the natural frequencies is analogous to the normal-mode solution used in Chapter 8. The method used for the analysis of forced vibrations is a direct result of an expansion theorem and is directly analogous to modal analysis. The approximate methods presented are based on energy methods. Indeed, similar notation using energy scalar products can be used. The continuous functions used in the analysis of continuous systems are analogous to the column vectors of generalized coordinates used for discrete systems. Energy scalar products are defined for continuous systems using definite integrals.

A general method for determining the free and forced solutions to continuous vibrations problems is presented in Section 10.2. This method is applied to systems that are

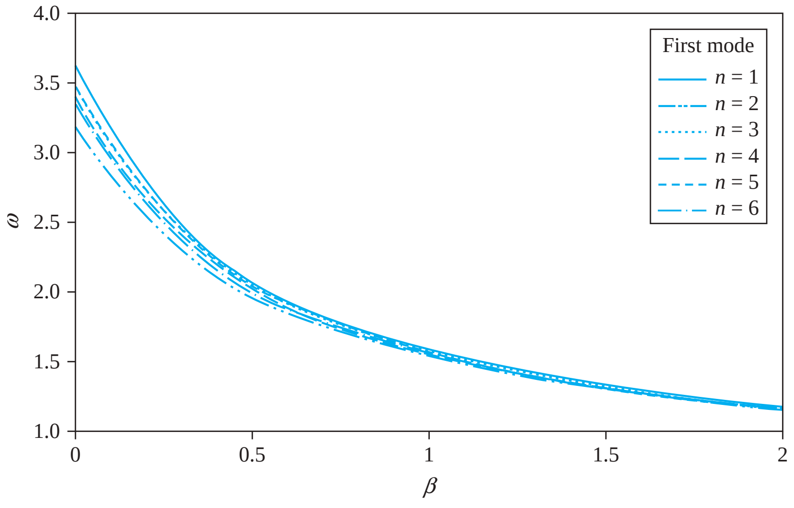


FIGURE 10.4
As the ratio of the concentrated mass to the mass of the beam grows larger, the approximation for the lowest natural frequency using a discrete model with n degrees of freedom improves.

governed by partial differential equations whose highest-order spatial derivative is of second order (second-order systems). Examples of such systems are strings, bars, and shafts. The general method is then applied to systems that are governed by partial differential equations whose highest-order spatial derivative is of fourth order (fourth-order systems). An example of a fourth-order system is a beam. Finally, an energy method is presented as a means of approximating the natural frequencies and mode shapes for second-order and fourth-order systems.

10.2 GENERAL METHOD

This section presents an outline of an exact closed-form method for analyzing vibrations of continuous systems. The method is applied to analyze the torsional oscillations of a circular shaft and the transverse vibrations of a beam in Sections 10.3 and 10.4 respectively. This chapter is intended only as an introduction to vibrations of continuous systems. Thus, it is assumed that the dependent displacement is a function of only one spatial variable and time, all material properties are constant, and all geometries are uniform.

The analysis procedure is broken into three parts: problem formulation, free-vibration analysis, and forced-vibration analysis. The mathematical theory underlying the analysis of vibrations of continuous systems is developed by using an infinite-dimensional vector space, while the mathematical foundation for a MODF is developed by using a finite-dimensional vector space. Many of the concepts developed for finite-dimensional spaces have direct extension to infinite-dimensional spaces.

Part I: Problem Formulation

1. An independent spatial variable is chosen, call it x. This independent spatial variable represents the displacement of a particle from a reference position when the system is in its equilibrium position. A continuous system has an infinite number of degrees of freedom and hence an infinite number of generalized coordinates are required. These are chosen as the displacement of the particles in the system. They can be summarized by a single dependent variable $w(x, t)$.
2. Free-body diagrams (FBDs) of a representative differential element are drawn at an arbitrary instant. The usual assumptions of mechanics of materials are used including plane sections remain plane. Thus, the differential element can be assumed to be undergoing planar motion. Two FBDs are drawn; one showing the external forces acting on the differential element and the second showing the effective forces for that element. The external forces include forces on the surface of the element that are resultants of stress distributions.
3. The appropriate form of Newton's law is applied to the free-body diagrams. Appropriate kinematic conditions and constitutive equations are applied to derive a partial differential equation governing $w(x, t)$.
4. Appropriate boundary conditions, dependent on the end supports of the structural member, are formulated.
5. Appropriate initial conditions are formulated.

6. An optional step is to nondimensionalize the governing equation and boundary conditions by introducing nondimensional forms of the independent and dependent variables. This leads to the formulation of dimensionless parameters which are important in the physical understanding of the results. Assume for the remainder of this discussion that nondimensional variables are introduced and all variables referred to are nondimensional. Also assume that the nondimensional spatial variable x ranges from 0 to 1.

   The governing equations and boundary conditions can also be derived by energy methods. Kinetic and potential energy scalar products directly analogous to those formed for multiple degree-of-freedom systems can be defined.

Part II: Free-Vibration Solution A free-vibration problem is one where $w(x, 0)$ or $\partial w/\partial t(x, 0)$ are nonzero and the partial differential equation and all boundary conditions are homogeneous. The initial potential or kinetic energy drives the vibrations, during which no external forces are applied.

As for MDOF systems, the free-vibration problem is considered to determine the system's natural frequencies and mode shapes. The method presented to solve free vibrations problems for continuous systems is called *separation of variables.* Application of this method requires that the partial differential equation be of an appropriate form, called *separable.* The governing partial differential equations for torsional vibrations of a uniform shaft, longitudinal vibrations of a uniform elastic bar, and transverse vibrations of a uniform beam are all separable.

1. The dependent variable is assumed to be a product of functions of the independent variables,

   $$w(x,\ t) = X(x)T(t) \tag{10.1}$$

   Equation (10.1) is substituted into the governing partial differential equation. If the governing partial differential equation is separable, the resulting equation can be written in the form of $[L_x X(x)]/X(x) = [L_t T(t)]/T(t)$ where L_x and L_t are linear ordinary differential operators. Note that the left-hand side of this equation is a function of x only and the right-hand side is a function of t only. Since x and t are independent, this can only be true if both sides are equal to the same constant, call it $-\lambda$. This argument is called the *separation argument.* Its application leads to ordinary differential equations for $X(x)$ and $T(t)$, both in terms of λ, called the *separation constant.*

2. Equation (10.1) is applied to the boundary conditions to obtain homogeneous boundary conditions for $X(x)$.

3. If the system is undamped, the differential equation for $T(t)$ is

   $$\frac{d^2T}{dt^2} + \lambda T = 0 \tag{10.2}$$

   from which the natural frequencies are deduced to be the square roots of the values of λ. The mode shapes, which are the spatial representation of the solution, are the forms of $X(x)$ corresponding to an appropriate value of λ.

4. The problem for $X(x)$ is

   $$L_x X + \lambda X = 0 \tag{10.3}$$

which is a homogeneous, ordinary differential equation with homogeneous boundary conditions. This is called a *differential eigenvalue problem.* A nontrivial solution is available only for certain values of the separation constant. Standard solution techniques for ordinary differential equations are applied to determine $X(x)$ in terms of arbitrary constants of integration.

5. Application of the boundary conditions leads to a solvability condition of the form $f(\lambda) = 0$. Nontrivial solutions of the eigenvalue problem exist only for values of λ such that $f(\lambda) = 0$. This results in an infinite (but countable) number of solutions $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$. Corresponding to each λ_k, there is an $X_k(x)$, which is unique only to a multiplicative constant.

   If only the natural frequencies and mode shapes are necessary, the solution ends here.

6. An energy scalar product, $(X_i, X_j)_T$, is defined such that $(X_i, X_i)_T$ is proportional to the kinetic energy of the *i*th mode at any instant. It can be shown that for systems governed by the wave equation (torsional vibrations of shafts, longitudinal vibrations of bars) and for uniform beam vibrations, mode shapes for distinct modes are mutually orthogonal with respect to this energy scalar product. For a uniform continuous system (in the absence of discrete masses) the appropriate kinetic energy scalar product is

$$(X_i, X_j)_T = \int_0^1 X_i(x) X_j(x)\, dx \tag{10.4}$$

   If the system has discrete masses, additional terms are added to the integral of Equation (10.4) to account for the kinetic energy of the discrete masses. The mode shapes are normalized by requiring

$$(X_i, X_i)_T = 1 \tag{10.5}$$

7. If the mode shapes are normalized with respect to a scalar product for which they are also mutually orthogonal, then an expansion theorem exists which states that any continuous function, $f(x)$, can be expanded in a series of the mode shapes as

$$f(x) = \sum_{k=1}^{\infty} (f, X_k)_T X_k \tag{10.6}$$

   The expansion converges to $f(x)$ at all x except perhaps at $x = 0$ and $x = 1$. The expansion converges at the boundaries if $f(x)$ satisfies the boundary conditions.

   If a forced-vibration solution is required, the expansion theorem of Equation (10.6) is noted and the solution proceeds to step 1 of the forced response. If a free-vibration solution is required, the solution continues as follows.

8. The general solution is formed by taking a linear combination over all modes

$$w(x, t) = \sum_{k=1}^{\infty} X_k(x)\, T_k(t) \tag{10.7}$$

   Two arbitrary constants for each mode are present from the solution for $T_k(t)$. These constants are determined from application of initial conditions. Often the functions involved in the initial conditions must be expanded by the expansion theorem, Equation (10.6). For example, if $w(x, 0)$ is nonzero and is equal to $f(x)$, then $f(x)$ is expanded by Equation (10.6) and compared to $w(x, 0)$ obtained from Equation (10.7), in terms of arbitrary constants. The linear independence of each $X_k(x)$ is used to determine the constants.

Part III: Forced-Vibration Solution As for discrete systems, there are several methods available to determine the forced response of continuous systems. These include application of the method of undetermined coefficients for harmonic excitations, the Laplace transform method, and modal analysis. Modal analysis is the most powerful and most often used and is described here.

Let $f(x, t)$ represent the nondimensional nonhomogeneous term arising in the partial differential equation as a result of the external forcing. Nonhomogeneous terms can also occur in the boundary conditions.

1. The expansion theorem, Equation (10.6) is used to expand $f(x, t)$ as

$$f(x, t) = \sum_{k=1}^{\infty} C_k(t) X_k(x) \tag{10.8}$$

where

$$C_k(t) = (f(x, t), X_k(x)) \tag{10.9}$$

2. The expansion theorem is also used to expand

$$w(x, t) = \sum_{k=1}^{\infty} p_k(t) X_k(x) \tag{10.10}$$

where the $p_k(t)$ are called the *principal coordinates* for the continuous system. Equations (10.8) and (10.10) are substituted into the governing partial differential equation.

3. The scalar product of the resulting partial differential equation is taken with $X_j(x)$ for an arbitrary j. For a problem whose appropriate scalar product is given by Equation (10.4), this is equivalent to multiplying the equation by $X_j(x)$ and integrating from 0 to 1. Application of the orthogonality condition leads to uncoupled differential equations for the principal coordinates.
4. The uncoupled differential equations are solved to determine each $p_k(t)$.

10.3 SECOND-ORDER SYSTEMS: TORSIONAL OSCILLATIONS OF A CIRCULAR SHAFT

10.3.1 PROBLEM FORMULATION

The circular shaft of Figure 10.5 is made of a material of mass density ρ and shear modulus G and has a length L, cross-sectional area A, and polar moment of inertia J. Let x be the coordinate along the axis of the shaft, measured from its left end. The shaft is subject to a time-dependent torque per unit length, $T(x, t)$. Let $\theta(x, t)$ measure the resulting torsional oscillations where θ is chosen positive clockwise.

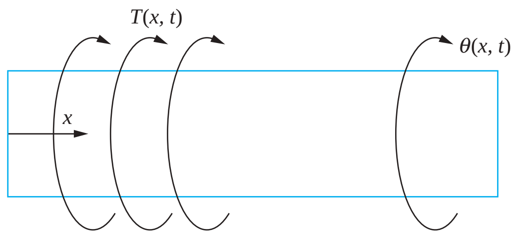


FIGURE 10.5
Circular shaft is subject to torsional loading $T(x, t)$. $\theta(x, t)$ measures angular displacement of the shaft.

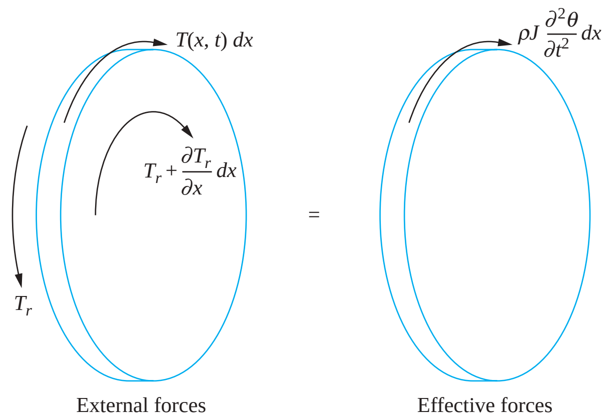


FIGURE 10.6
FBDs of differential element of shaft at an arbitrary instant. $T_r(x, t)$ is the resisting torques in the shaft.

Figure 10.6 shows free-body diagrams of a differential element of the shaft at an arbitrary instant of time. The element is of infinitesimal thickness dx and its left face is a distance x from the left end of the shaft. The assumptions of mechanics of materials imply that sections do not warp thus the problem run be treated assuming the differential element undergos planar motions and the effective-external force method can be used.

The free-body diagram of the external forces shows the time-dependent torque loading as well as the internal resisting torques developed in the cross sections. The internal resisting torques are the resultants of the shear stress distributions. If $T_r(x, t)$ is the resisting torque acting on the left face of the element, then a Taylor series expansion truncated after the linear terms gives

$$T_r(x + dx, t) = T_r(x, t) + \frac{\partial T_r}{\partial X}(x, t)dx \tag{10.11}$$

The directions of the torques shown on the free-body diagram are consistent with the choice of θ positive clockwise.

Since the disk is infinitesimal, the angular acceleration is assumed constant across the thickness. Thus, the free-body diagram of the effective forces simply shows a moment equal to the mass moment of inertia of the disk, which is $\rho J dx$, times its angular acceleration, $\frac{\partial^2\theta}{\partial t^2}$.

Summation of moments about the mass center of the disk

$$\left(\sum M\right)_{\text{ext}} = \left(\sum M\right)_{\text{eff}}$$

gives

$$T(x, t)dx - T_r(x, t) + T_r(x, t) + \frac{\partial T_r}{\partial x}(x, t)dx = \rho J dx \frac{\partial^2\theta}{\partial t^2}(x, t)$$

or

$$T(x, t) + \frac{\partial T_r}{\partial x}(x, t) = \rho J \frac{\partial^2 \theta}{\partial t^2} \tag{10.12}$$

From mechanics of materials,

$$T_r(x, t) = JG\frac{\partial \theta}{\partial x}(x, t) \tag{10.13}$$

which, when substituted in Equation (10.12) for a uniform shaft, leads to

$$T(x, t) + JG\frac{\partial^2 \theta}{\partial x^2} = \rho J \frac{\partial^2 \theta}{\partial t^2} \tag{10.14}$$

The following nondimensional variables are introduced:

$$x^* = \frac{x}{L} \qquad t^* = \sqrt{\frac{G}{\rho}}\frac{t}{L} \tag{10.15}$$

and

$$T^*(x^*, t^*) = \frac{T(x, t)}{T_m} \tag{10.16}$$

where T_m is the maximum value of T. Introduction of Equations (10.15) and (10.16) in Equation (10.14) leads to

$$\left(\frac{L^2 T_m}{JG}\right) T(x, t) + \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2} \tag{10.17}$$

where the * has been dropped from the nondimensional variables.

Boundary conditions are formulated at each end of the shaft. At a free end the shaft is restrained from rotation, thus θ at that end is zero. At a free end there is no torque acting on the free end which implies there is no shear stress distribution at the free end. A linear shear stress-shear strain relation is assumed and the shear strain is given by $\frac{\partial \theta}{\partial x}$. Thus at a free end $\frac{\partial \theta}{\partial x} = 0$. If there is a disk at an end, a moment balance on a FBD of the disk is performed with the resultant of the shear stress distribution from the end of the shaft as the external moment and the inertia of the disk providing the effective moment. Other end conditions such as discrete torsional springs, discrete torsional viscous dampers and applied torques have boundary conditions developed using a moment balance at the end of the shaft. A differential element at the end of the shaft is considered. The sum of the external moments is zero which include the resultant moment form the shear stress distribution. Table 10.1 provides nondimensional boundary conditions for different types of shaft ends.

The problem formulation is completed by specifying appropriate initial conditions of the form

$$\theta(x, 0) = g_1(x) \tag{10.18}$$

and

$$\frac{\partial \theta}{\partial t}(x, 0) = g_2(x) \tag{10.19}$$

Consider the homogeneous form of Equation (10.17),

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2} \tag{10.20}$$

TABLE 10.1 Boundary conditions for torsional oscillations of a circular shaft

End Condition	Boundary Condition	Remarks
Fixed, $x = 0$ or $x = 1$	$\theta = 0$	
Free, $x = 0$ or $x = 1$	$\dfrac{\partial \theta}{\partial x} = 0$	
Torsional spring, $x = 0$	$\dfrac{\partial \theta}{\partial x} = \beta\theta$	$\beta = \dfrac{k_t L}{JG}$
Torsional spring, $x = 1$	$\dfrac{\partial \theta}{\partial x} = -\beta\theta$	$\beta = \dfrac{k_t L}{JG}$
Torsional damper, $x = 0$	$\dfrac{\partial \theta}{\partial x} = \beta\dfrac{\partial \theta}{\partial t}$	$\beta = c_t\sqrt{\dfrac{J}{\rho G}}$
Torsional damper, $x = 1$	$\dfrac{\partial \theta}{\partial x} = -\beta\dfrac{\partial \theta}{\partial t}$	$\beta = c_t\sqrt{\dfrac{J}{\rho G}}$
Attached disk, $x = 0$	$\dfrac{\partial \theta}{\partial x} = \beta\dfrac{\partial^2 \theta}{\partial t^2}$	$\beta = \dfrac{I_D}{\rho J L}$
Attached disk, $x = 1$	$\dfrac{\partial \theta}{\partial x} = -\beta\dfrac{\partial^2 \theta}{\partial t^2}$	$\beta = \dfrac{I_D}{\rho J L}$

Equation (10.20) is a hyperbolic partial differential equation, called the *wave equation*. The wave equation also governs such variables as the axial displacement during the longitudinal motion of a bar, the axial displacement of a particle on a coil spring during a compression wave, and the free vibrations of a taut string. Applications in areas other than vibrations include propagation of surface waves on the interface of two fluids and the velocity potential for supersonic flow in an ideal fluid.

Solutions of the wave equation are rich in physical phenomena. It can be shown that the solutions of the wave equation represent waves propagating through the medium. The speed of propagation is determined from the governing partial differential equation in dimensional form or in the definition of t^*. In general, to arrive at a partial differential equation of the form of Equation (10.20) in which no parameters appear, we have

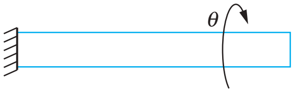
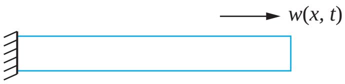
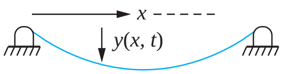
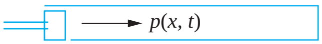
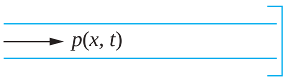
$$t^* = \frac{c}{L}t \tag{10.21}$$

where c is the wave speed. Thus, for torsional oscillations, the wave speed is $\sqrt{G/\rho}$. Table 10.2 gives the wave speed for other situations governed by the wave equation.

10.3.2 FREE-VIBRATION SOLUTIONS

The application of the method discussed in Section 10.2 for calculating the natural frequencies and mode shapes and determining the free response due to non-zero initial conditions for second-order systems is illustrated in the following examples.

TABLE 10.2 Physical problems governed by the wave equation

Problem	Schematic	Nondimensional Wave Equation		Wave Speed
Torsional oscillations of circular cylinder		$\dfrac{\partial^2\theta}{\partial x^2} = \dfrac{\partial^2\theta}{\partial t^2}$	$c = \sqrt{\dfrac{G}{\rho}}$	G = shear modulus ρ = mass density
Longitudinal oscillations of bar		$\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial t^2}$	$c = \sqrt{\dfrac{E}{\rho}}$	E = elastic modulus ρ = mass density
Transverse vibrations of taut spring		$\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$	$c = \sqrt{\dfrac{T}{\mu}}$	T = tension μ = linear density
Pressure waves in an ideal gas		$\dfrac{\partial^2 \rho}{\partial x^2} = \dfrac{\partial^2 \rho}{\partial t^2}$	$c = \sqrt{kRT}$	k = ratio of specific hears R = gas constant T = temperature
Waterhammer waves in rigid pipe		$\dfrac{\partial^2 \rho}{\partial x^2} = \dfrac{\partial^2 \rho}{\partial t^2}$	$c = \sqrt{\dfrac{k}{\rho}}$	k = bulk modulus of fluid ρ = mass density

EXAMPLE 10.1

A moment M is statically applied to the end of a circular shaft, fixed at $x = 0$ and free at $x = 1$, causing the angle of twist to vary linearly over the length of the shaft. Determine the resulting free torsional response when the moment is suddenly removed.

SOLUTION

The free torsional oscillations are governed by Equation (10.20). The boundary condition corresponding to a fixed end at $x = 0$ is

$$\theta(0, t) = 0 \tag{a}$$

and corresponding to a free end at $x = 1$ is

$$\frac{\partial\theta}{\partial x}(1, t) = 0 \tag{b}$$

Static application of the moment M leads to the initial condition

$$\theta(x, 0) = \frac{M}{JG}x = \gamma x \tag{c}$$

Since the shaft is released from rest a second initial condition is

$$\frac{\partial\theta}{\partial t}(x, 0) = 0 \tag{d}$$

A separation-of-variables solution is assumed as

$$\theta(0, t) = X(x)T(t) \tag{e}$$

Substituting Equation (e) into Equation (10.20) and rearranging leads to

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{T(t)}\frac{d^2T}{dt^2} \tag{f}$$

The left-hand side of Equation (f) is a function of x only, while the right-hand side is a function of t only. However, x and t are independent. Thus, Equation (f) is true only if both sides are equal to the same constant, call it $-\lambda$, where λ is called the *separation constant*. Then Equation (f) leads to

$$\frac{d^2T}{dt^2} + \lambda T = 0 \tag{g}$$

and

$$\frac{d^2X}{dx^2} + \lambda X = 0 \tag{h}$$

The solution of Equation (g) is

$$T(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t \tag{i}$$

where A and B are arbitrary constants of integration. From Equation (i) it is obvious that the natural frequencies are the square roots of the separation constant.

The solution of Equation (h) is

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x \tag{j}$$

Application of Equation (a) to Equation (e) yields

$$X(0) = 0 \tag{k}$$

and its subsequent application to Equation (j) gives $C = 0$.

Application of Equation (b) to Equation (e) yields

$$\frac{dX}{dx}(1) = 0 \tag{l}$$

Application of Equation (l) to Equation (j) with $C = 0$ leads to

$$D\sqrt{\lambda}\cos\sqrt{\lambda} = 0 \tag{m}$$

Choosing either $D = 0$ or $\lambda = 0$ leads to the trivial solution. Thus a nontrivial solution is obtained only when

$$\cos\sqrt{\lambda} = 0 \tag{n}$$

or

$$X_k = \left[(2k-1)\frac{\pi}{2}\right]^2 \qquad k = 1, 2, \ldots \tag{o}$$

There are an infinity of solutions of Equation (n), but as evidenced by Equation (o), they are countable. The mode shape corresponding to λ_k is

$$X_k(x) = D_k\sin\left[(2k-1)\frac{\pi}{2}x\right] \tag{p}$$

for any D_k. The mode shapes are orthogonal with respect to the scalar product of Equation (10.4) as follows:

$$(X_k(x), X_j(x))_T = \int_0^1 D_j D_k \sin\left[(2k-1)\frac{\pi}{2}x\right]\sin\left[(2j-1)\frac{\pi}{2}x\right]dx$$
$$= \frac{D_j D_k}{\pi}\left[\frac{1}{j-k}\sin(j-k)\pi - \frac{1}{j+k+1}\sin(j+k+1)\pi\right]$$
$$= 0 \tag{q}$$

The mode shapes are normalized by requiring

$$1 = (X_k, X_k)_T = \int_0^1 D_k^2 \sin^2\left[(2k-1)\frac{\pi}{2}x\right]dx = \frac{D_k^2}{2} \tag{r}$$

which leads to

$$X_k(x) = \sqrt{2}\sin\left[(2k-1)\frac{\pi}{2}x\right] \tag{s}$$

The first three normalized mode shapes are shown in Figure 10.7

The general solution to the free-vibration problem is formed using Equation (10.7)

$$\theta(x,t) = \sum_{k=1}^{\infty}\sqrt{2}\sin\left[(2k-1)\frac{\pi}{2}x\right]\left\{A_k\cos\left[(2k-1)\frac{\pi}{2}t\right] + B_k\sin\left[(2k-1)\frac{\pi}{2}t\right]\right\} \tag{t}$$

Application of the initial condition, Equation (d), yields $B_k = 0$. Application of Equation (c) then gives

$$\gamma x = \sum_{k=1}^{\infty} A_k\sqrt{2}\sin\left[(2k-1)\frac{\pi}{2}x\right] \tag{u}$$

The expansion theorem, Equation (10.6), is used to expand

$$\gamma x = \sum_{k=1}^{\infty}(\gamma x, X_k)_T X_k \tag{v}$$

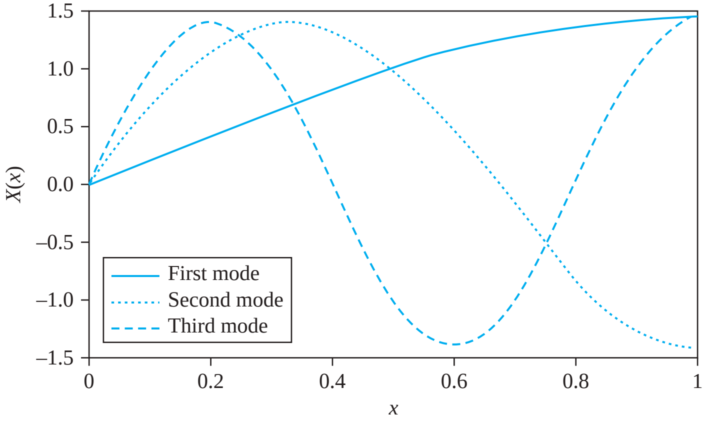


FIGURE 10.7
Normalized mode shapes corresponding to three lowest natural frequencies of a fixed-free shaft.

where

$$(\gamma x, X_k)_T = \int_0^1 \gamma x\sqrt{2}\sin\left[(2k-1)\frac{\pi}{2}x\right]dx$$
$$= \frac{4\gamma\sqrt{2}}{\pi^2(2k-1)^2}\sin(2k-1)\frac{\pi}{2}$$
$$= \frac{4\gamma\sqrt{2}}{\pi^2(2k-1)^2}(-1)^{k+1} \tag{w}$$

Comparison of Equations (u) and (v) yields

$$A_k = (\gamma x, X_k)_T = \frac{4\gamma\sqrt{2}(-1)^{k+1}}{\pi^2(2k-1)^2} \tag{x}$$

Equation (t) becomes

$$\theta(x, t) = \frac{8\gamma}{\pi^2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{(2k-1)^2}\sin\left[(2k-1)\frac{\pi}{2}x\right]\cos\left[(2k-1)\frac{\pi}{2}t\right] \tag{y}$$

The time-dependent angles of twist at four locations along the axis of the shaft, obtained by numerical evaluation of Equation (y), are plotted in Figure 10.8.

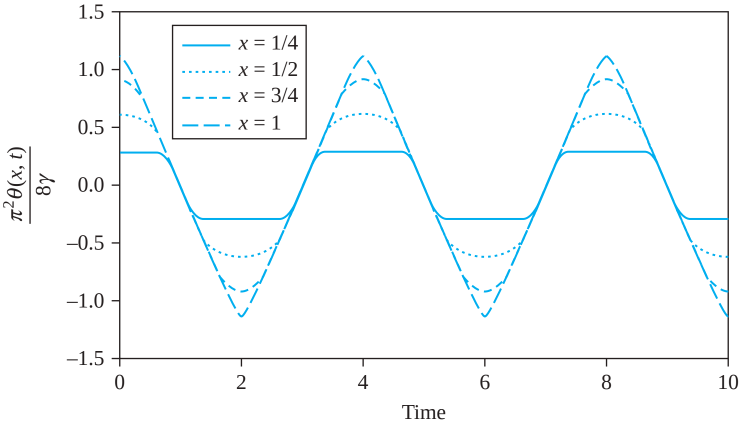


FIGURE 10.8
Time-dependent torsional oscillations of circular fixed-free shaft at different locations on the shaft of Example 10.1.

EXAMPLE 10.2

The circular shaft of Figure 10.9 is fixed at $x = 0$ and has a thin disk of mass moment of inertia I attached at $x = 1$. Determine the natural frequencies for this system, identify the orthogonality condition satisfied by the mode shapes, and determine the normalized mode shapes.

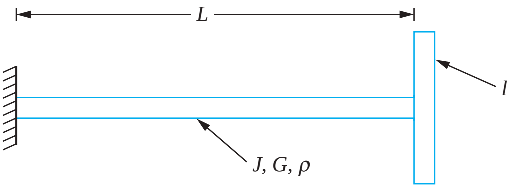


FIGURE 10.9
System of Example 10.2 is a shaft fixed at one end and a disk with a moment of inertia I attached at the other end.

SOLUTION

The partial differential equation governing, this system is Equation (10.20). It is subject to Equation (a) of Example 10.1 and from Table 10.1, giving

$$\frac{\partial \theta}{\partial x}(1, t) = -\beta \frac{\partial^2 \theta}{\partial t^2}(1, t) \tag{a}$$

where

$$\beta = \frac{1}{\rho J L} \tag{b}$$

The solution procedure is similar to that of Example 10.1. Separation of variables is assumed and applied to the partial differential and the boundary conditions leading to the eigenvalue problem

$$\frac{d^2 X}{dx^2} + \lambda X = 0 \tag{c}$$

subject to

$$X(0) = 0 \tag{d}$$

and

$$\frac{dX}{dx}(1) = \beta \lambda X(1) \tag{e}$$

The solution satisfying Equations (c) and (d) is

$$X(x) = D \sin \sqrt{\lambda} x \tag{f}$$

Application of Equation (e) to Equation (f) yields

$$\sqrt{\lambda} \cos \sqrt{\lambda} = \beta \lambda \sin \sqrt{\lambda} \tag{g}$$

or

$$\tan \sqrt{\lambda} = \frac{1}{\beta \sqrt{\lambda}} \tag{h}$$

A graphical solution of the transcendental equation, Equation (h), is shown in Figure 10.10. The values of λ where the curves $\tan \sqrt{\lambda}$ and $1/\beta\sqrt{\lambda}$ intersect are the solutions of Equation (h), and are the values of the separation constant for which nontrivial solutions for $X(x)$ occur. Figure 10.10 shows that there are infinite, but countable, values

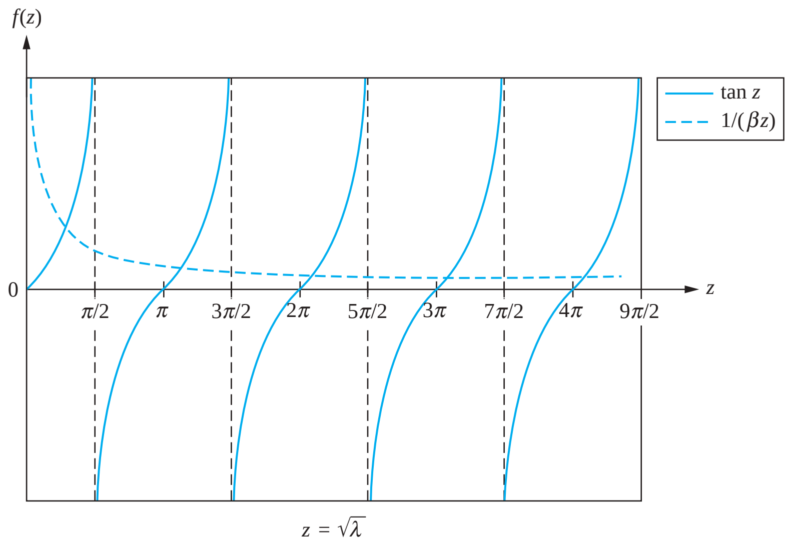


FIGURE 10.10
Graphical solution of transcendental equation $\tan\sqrt{\lambda} = \frac{1}{\beta\sqrt{\lambda}}$ used to determine the natural frequencies of the system of Example 10.2. Values of $\sqrt{\lambda}$ correspond to points of intersection of the two curves.

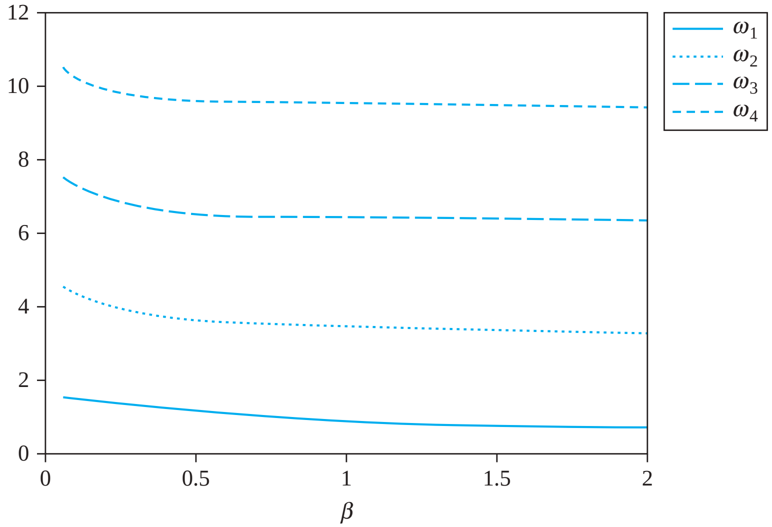


FIGURE 10.11
Nondimensional natural frequencies of Example 10.2 as functions of nondimensional parameter β.

of λ where these curves intersect. Figure 10.10 also shows that for large k, λ_k approaches $[(k-1)\pi]^2$.

The natural frequencies are the square roots of the separation constants. Figure 10.11 shows the first four natural frequencies as a function of β. The first four mode shapes are plotted in Figure 10.12 for $\beta = 0.4$.

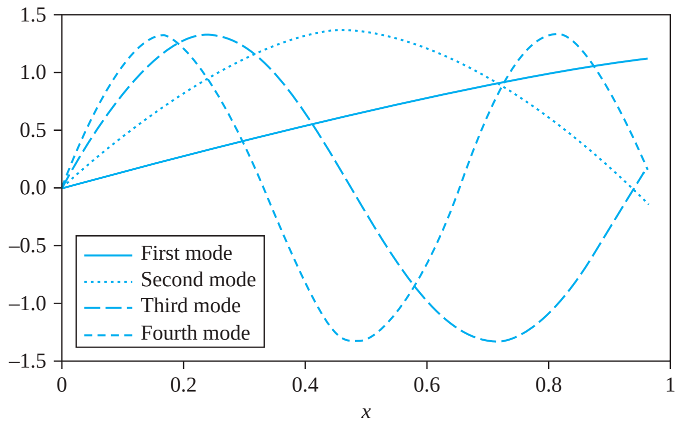


FIGURE 10.12
Mode shapes of Example 10.2 with $\beta = 0.4$.

Let λ_i and λ_j be distinct solutions of Equation (g) with corresponding mode shapes $X_i(x)$ and $X_j(x)$, respectively. The mode shapes satisfy the following problems

$$\frac{d^2X_i}{dx^2} + \lambda_i X_i = 0 \qquad X_i(0) = 0 \qquad \frac{dX_i}{dx}(1) = \beta\lambda_i X_i(1) \tag{i}$$

$$\frac{d^2X_j}{dx^2} + \lambda_j X_j = 0 \qquad X_j(0) = 0 \qquad \frac{dX_j}{dx}(1) = \beta\lambda_j X_j(1) \tag{j}$$

Multiplying Equation (i) by $X_j(x)$ and integrating from 0 to 1 leads to

$$\int_0^1 \frac{d^2X_i}{dx^2}X_j dx + \lambda_i \int_0^1 X_i X_j dx = 0 \tag{k}$$

Using integration by parts on the first integral leads to

$$X_j(1)\frac{dX_i}{dx}(1) - X_j(0)\frac{dX_i}{dx}(0) - \int_0^1 \frac{dX_i}{dx}\frac{dX_j}{dx}dx + \lambda_i \int_0^1 X_i X_j dx = 0 \tag{l}$$

Application of the boundary conditions Equations (d) and (e) in Equation (l) leads to

$$\beta\lambda_i X_i(1)X_j(1) - \int_0^1 \frac{dX_i}{dx}\frac{dX_j}{dx}dx + \lambda_i \int_0^1 X_i X_j dx = 0 \tag{m}$$

Multiplying Equation (j) by $X_i(x)$, integrating from 0 to 1, and performing algebra similar to that leading to Equation (m) gives

$$\beta\lambda_j X_j(1)X_i(1) - \int_0^1 \frac{dX_i}{dx}\frac{dX_j}{dx}dx + \lambda_j \int_0^1 X_i X_j dx = 0 \tag{n}$$

Subtracting Equation (n) from Equation (m) leads to

$$(\lambda_i - \lambda_j)\left(\beta X_i(1)X_j(1) + \int_0^1 X_i X_j dx\right) = 0 \tag{o}$$

Since $\lambda_i \neq \lambda_j$, Equation (o) implies

$$\beta X_i(1)X_j(1) + \int_0^1 X_i X_j dx = 0 \tag{p}$$

If the scalar product of f and g is defined by

$$(f, g) = \int_0^1 f(x)g(x)dx + \beta f(1)g(1) \tag{q}$$

then

$$(X_i, X_j) = 0 \tag{r}$$

Equation (q) defines the energy scalar product with which the mode shapes are mutually orthogonal. Taking (X_k, X_k) gives the nondimensional form of the kinetic energy associated with the mode shape $X_k(x)$. The term $\beta X_k^2(1)$ is the kinetic energy of the attached disk while $\int_0^1 X_k^2(x)dx$ is the kinetic energy of the shaft. Thus the scalar product is a kinetic energy scalar product.

Normalization of the mode shape requires

$$\begin{aligned} 1 = (X_k, X_k) &= \int_0^1 D_k^2 \sin^2 \sqrt{\lambda_k} x dx + D_k^2 \beta \sin^2 \sqrt{\lambda_k} \\ &= D_k^2\left[\int_0^1 \frac{1}{2}(1 - \cos 2\sqrt{\lambda_k} x)dx + \beta \sin^2 \sqrt{\lambda_k}\right] \\ &= D_k^2\left[\frac{1}{2}\left(1 - \frac{1}{2\sqrt{\lambda_k}} \sin 2\sqrt{\lambda_k}\right) + \beta \sin^2 \sqrt{\lambda_k}\right] \end{aligned} \tag{s}$$

Using the trigonometric identity

$$\sin 2\sqrt{\lambda_k} = 2\sin\sqrt{\lambda_k}\cos\sqrt{\lambda_k} \tag{t}$$

and replacing $\cos\sqrt{\lambda_k}$ from Equation (g) leads to

$$D_k = \sqrt{2}(1 + \beta\sin^2\sqrt{\lambda_k})^{-1/2} \tag{u}$$

where λ_k is the kth real solution of Equation (h).

10.3.3 FORCED VIBRATIONS

The application of undetermined coefficients for harmonic excitations is illustrated in the following example. Modal analysis is illustrated with examples in Section 10.4.

EXAMPLE 10.3

The thin disk of Example 10.2 and Figure 10.9 is subject to a harmonic torque,

$$T(t) = T_0 \sin\omega t$$

Determine the steady-state response of the system.

SOLUTION

The torsional oscillations, in terms of nondimensional variables, are governed by Equation (10.20) with

$$\theta(0, t) = 0 \tag{a}$$

and

$$\frac{\partial \theta}{\partial x}(1, t) = -\beta \frac{\partial^2 \theta}{\partial t^2}(1, t) + \frac{T_0 L}{JG} \sin \tilde{\omega} t \tag{b}$$

where

$$\tilde{\omega} = L\sqrt{\frac{\rho}{G}}\omega \tag{c}$$

Since the external excitation is harmonic, the steady-state response is assumed as

$$\theta(x, t) = u(x) \sin \tilde{\omega} t \tag{d}$$

Substituting Equation (d) into Equation (10.20) leads to

$$\frac{d^2 u}{dx^2} \sin \tilde{\omega} t = -\tilde{\omega}^2 u \sin \tilde{\omega} t \tag{e}$$

or

$$\frac{d^2 u}{dx^2} + \tilde{\omega}^2 u = 0 \tag{f}$$

Substituting Equation (d) into the boundary conditions, Equations (a) and (b), leads to

$$u(0) = 0 \tag{g}$$

and

$$\frac{du}{dx}(1) - \beta \tilde{\omega}^2 u(1) = \frac{T_0 L}{JG} \tag{h}$$

The solution of Equation (f) subject to Equations (g) and (h) is

$$u(x) = \frac{T_0 L}{(\tilde{\omega} \cos \tilde{\omega} - \beta \tilde{\omega}^2 \sin \tilde{\omega}) JG} \sin \tilde{\omega} x \tag{i}$$

Note that if $\tilde{v}$ is equal to any of the system's natural frequencies, the denominator vanishes. The assumed form of the solution, Equation (d), must be modified to account for this resonance condition.

The steady-state solution is given by Equation (d), where $u(x)$ is given in Equation (i). The total solution is the steady-state solution plus the homogeneous solution, which is a summation over all free-vibration modes. Initial conditions can then be applied to determine the constants in the linear combination.

10.4 TRANSVERSE BEAM VIBRATIONS

10.4.1 PROBLEM FORMULATION

The uniform beam of Figure 10.13 is made of a material of mass density ρ and elastic modulus E, and has a length L, cross-sectional area A, and centroidal moment of inertia I. Let x be a coordinate along the neutral axis of the beam, measured from its left end. The beam

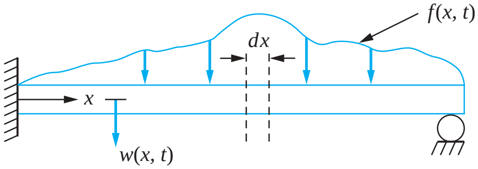
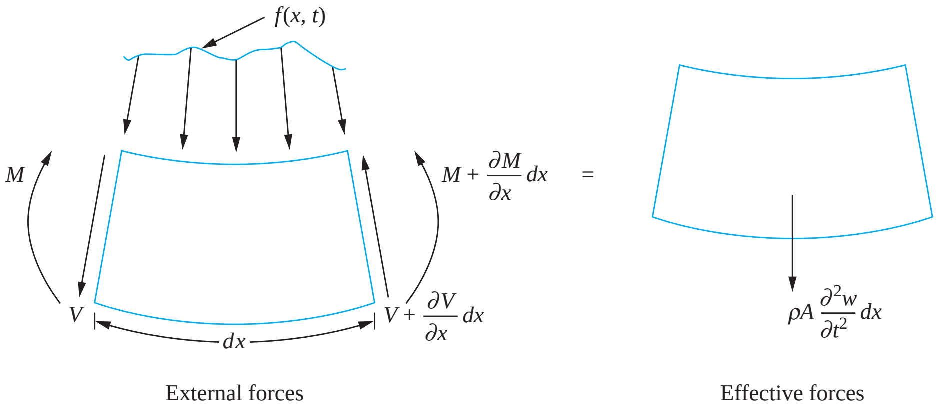


FIGURE 10.13
The beam is undergoing transverse vibrations with $w(x, t)$ being the transverse deflection of the beam measured from its static equilibrium position.

FIGURE 10.14
FBDs of differential beam element at an arbitrary time.



has an external load per unit length, $f(x, t)$. Let $w(x, t)$ be the transverse deflection of the beam, measured from its equilibrium position.

Free-body diagrams of an arbitrary differential element of the beam at an arbitrary instant of time are shown in Figure 10.14. The element is a slice of the beam of thickness dx and its left face is a distance x from the beam's left end. The external forces shown are the external loading, the internal bending moment which is the resultant moment of the normal stress distribution, and the internal shear force, which is the resultant of the shear stress distribution. It is assumed that the resultant of the normal stress distribution is zero. The effective force is the element mass times its acceleration. The element's longitudinal acceleration and angular acceleration are small in comparison to other effects and are thus ignored.

Sum forces in the vertical direction are $(\sum F)_{\text{ext}} = (\sum F)_{\text{eff}}$, so

$$V - \left(V + \frac{\partial V}{\partial x}dx\right) + \int_x^{x+dx} f(\zeta, t)\,d\zeta = \rho A\frac{\partial^2 w}{dt^2}dx \tag{10.22}$$

The mean value theorem implies that there is an $\tilde{x}$, $x < \tilde{x} < x + dx$, such that

$$\int_x^{x+dx} f(\zeta, t)\,d\zeta = f(\tilde{x}, t)\,dx \tag{10.23}$$

Since dx is infinitesimal, $\tilde{x} \approx x$. Equation (10.22) becomes

$$f(x, t) - \frac{\partial V}{\partial x} = \rho A\frac{\partial^2 w}{\partial x^2} \tag{10.24}$$

Sum moments about the neutral axis of the left face of the element are $(\sum M_0)_{\text{ext}} = (\sum M_0)_{\text{eff}}$, so

$$M - \left(M + \frac{\partial M}{\partial x}dx\right) - \left(V + \frac{\partial V}{\partial x}dx\right)dx + \int_x^{x+dx} (\zeta - x)f(\zeta, t)d\zeta = \rho A\frac{\partial^2 w}{\partial x^2}dx\left(\frac{dx}{2}\right) \tag{10.25}$$

Since dx is infinitesimal, terms of order $(dx)^2$ are negligible compared to terms of order dx. When the mean value theorem is used on the integral and since $\zeta - x$ is less than dx over the entire range of integration, it becomes apparent that the term is of order dx^2. Then Equation (10.25) simplifies to

$$V = -\frac{\partial M}{\partial x} \tag{10.26}$$

From mechanics of materials and with the chosen sign conventions,

$$M = -EI\frac{\partial^2 w}{\partial x^2} \tag{10.27}$$

Substitution of Equations (10.26) and (10.27) into Equation (10.24) assuming uniform properties leads to

$$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = f(x, t) \tag{10.28}$$

Equation (10.28) is nondimensionalized by introducing

$$x^* = \frac{x}{L} \qquad t^* = t\sqrt{\frac{EI}{\rho AL^4}} \qquad w^* = \frac{w}{L} \qquad f^* = \frac{f}{f_m} \tag{10.29}$$

where f_m is the maximum value of f. The resulting nondimensional form of Equation (10.28) where the *s have been dropped from nondimensional variables is

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = \frac{f_m L^3}{EI}f(x, t) \tag{10.30}$$

Four boundary conditions, two at $x = 0$ and two at $x = 1$, must be specified. The forms of the boundary conditions depend on the type of end supports. Nondimensional boundary conditions associated with different support conditions are given in Table 10.3.

The nondimensional spatial derivatives of the displacement have the physical meanings:

- $\dfrac{\partial w}{\partial x}$ is the slope of the deflection equation
- $\dfrac{\partial^2 w}{\partial x^2}$ is the internal bending moment
- $\dfrac{\partial^3 w}{\partial x^3}$ is the internal shear force

TABLE 10.3 Boundary conditions for transverse vibrations of beam

End Condition	Boundary Condition *A*	Boundary Condition *B*	Remarks
Free, $x = 0$ or $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = 0$	
Pinned, $x = 0$ or $x = 1$	$w = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	
Fixed, $x = 0$ or $x = 1$	$w = 0$	$\frac{\partial w}{\partial x} = 0$	
Linear spring, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = -\beta w$	$\beta = \frac{kL^3}{EI}$
Linear spring, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = \beta w$	$\beta = \frac{kL^3}{EI}$
Viscous damper, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = -\beta \frac{\partial w}{\partial t}$	$\beta = \frac{cL}{\sqrt{\rho EIA}}$
Viscous damper, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = \beta \frac{\partial w}{\partial t}$	$\beta = \frac{cL}{\sqrt{\rho EIA}}$
Attached mass, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = -\beta \frac{\partial^2 w}{\partial t^2}$	$\beta = \frac{m}{\rho AL}$
Attached mass, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = \beta \frac{\partial^2 w}{\partial t^2}$	$\beta = \frac{m}{\rho AL}$
Attached inertia element, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = -\beta \frac{\partial^3 w}{\partial x \partial t^2}$	$\frac{\partial^3 w}{\partial x^3} = 0$	$\beta = \frac{J}{\rho AL^3}$
Attached inertia element, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = \beta \frac{\partial^3 w}{\partial x \partial t^2}$	$\frac{\partial^3 w}{\partial x^2} = \frac{\partial^3 w}{\partial x \partial t^2}$	$\beta = \frac{J}{\rho AL^3}$

A fixed end is restrained against deflection and slope. A pinned end is restrained against deflection and cannot support an internal moment. There is no normal stress distribution or shear stress distribution at a free end which implies there is no bending moment or shear force. Discrete masses, springs and viscous dampers may be placed at an end of a beam or the end may be subject to an applied force or moment. The appropriate boundary conditions for these situations are developed from application of conservation laws to a FBD of the discrete mass or a differential element at the end of the beam.

A fixed end, say at $x = 0$, is restrained against motion which implies $w(0, t) = 0$ and rotation which implies $\frac{\partial \omega}{\partial x}(0, t) = 0$. A pinned end is restrained against vertical displacement.

The formulation of the mathematical problem is completed by specifying two initial conditions.

Equation (10.30) is the governing nondimensional partial differential equation for forced vibrations of a beam assuming no axial load, longitudinal effects are negligible, rotary inertia and transverse shear are negligible, and other standard assumptions of beam theory from mechanics of materials apply.

10.4.2 FREE VIBRATIONS

When the product solution

$$w(x, t) = X(x)T(t) \tag{10.31}$$

is substituted into Equation (10.30) with $f = 0$, the result is

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = -\frac{1}{X(x)}\frac{d^4X}{dx^4} \tag{10.32}$$

The usual separation argument is used to set both sides of Equation (10.32) equal to the same constant, say $-\lambda$. This leads to

$$\frac{d^2T}{dt^2} + \lambda T = 0 \tag{10.33}$$

and

$$\frac{d^4X}{dt^4} - \lambda X = 0 \tag{10.34}$$

The solution of Equation (10.33) is

$$T(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t \tag{10.35}$$

from which it is obvious that the natural frequencies are the square roots of the separation constants. The general solution of Equation (10.34) is

$$X(x) = C_1\cos\lambda^{1/4}x + C_2\sin\lambda^{1/4}x + C_3\cosh\lambda^{1/4}x + C_4\sinh\lambda^{1/4}x \tag{10.36}$$

The solvability condition is determined by applying the homogeneous boundary conditions to Equation (10.36). Table 10.4 on the next page summarizes the solvability conditions for different types of end supports, provides the first five nondimensional natural frequencies for each entry, their corresponding mode shapes, and specifies the scalar product for which the mode shapes are orthogonal.

Free-free and pinned-free beams are unrestrained and thus their lowest natural frequency is zero, corresponding to a rigid-body mode. The fixed-pinned beam has the same characteristic equations as the pinned-free beam, and $\lambda = 0$ is a solution of this equation. However, $\lambda = 0$ leads to a trivial mode shape for the fixed-pinned beam.

The free beam has a double natural frequency of zero and two rigid-body mode shapes. One corresponds to a translation and one a rotation.

EXAMPLE 10.4

A carbon nanotube is a new engineering material that is made from a graphene sheet rolled to form a tube as shown in Figure 10.15a on page 658. However, the radius of the tube is the radius of a several carbon atoms. The radius of 1 carbon atom is 0.34 nm. If the nanotube is long enough, the continuum assumption may be used and the tube can be modeled as a beam. Determine the five lowest natural frequencies and their corresponding mode shapes of a carbon nanotube with a mean radius of 2 nm and length of 100 nm. The elastic modulus of a carbon nanotube is 1 GPa, and its mass density is 2.3 g/cm^3. Assume the tube is fixed at one end and free at the other.

SOLUTION

The characteristic equation for a fixed-free beam is given in Table 10.4 as

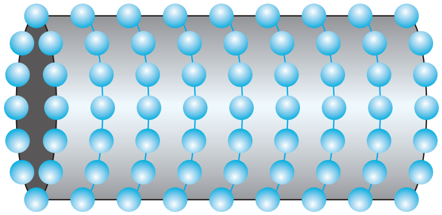
$$\cosh\lambda^{1/4}\cos\lambda^{1/4} = -1 \tag{a}$$

TABLE 10.4 Natural frequencies and mode shapes for beams

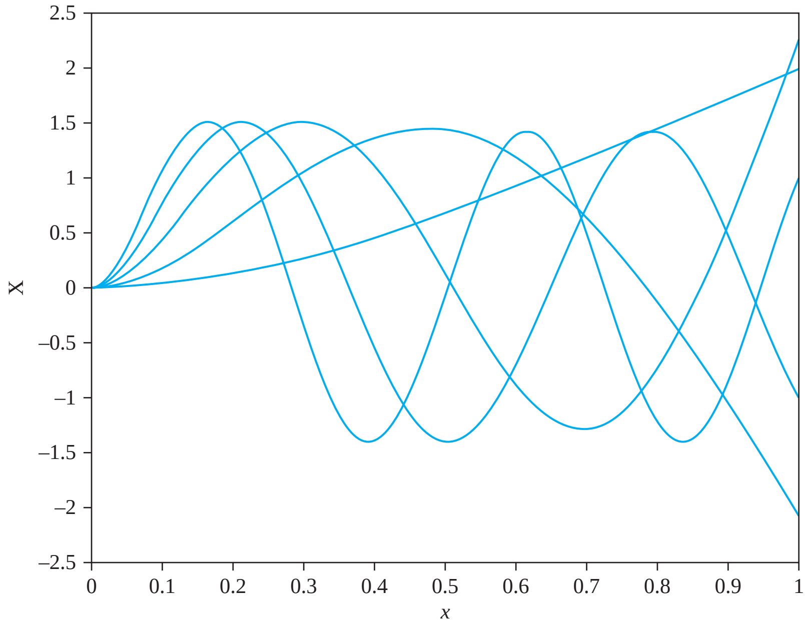
End Conditions $X = 0 \quad X = 1$	Characteristic Equation	Five Lowest Natural Frequencies $\omega_k = \sqrt{\lambda_k}$	Mode Shape	Kinetic Energy Scalar Product $(X_j(x), X_k(x))$
Fixed-fixed	$\cos\lambda^{1/4}\cosh\lambda^{1/4} = 1$	$\omega_1 = 22.37$ $\omega_2 = 61.66$ $\omega_3 = 120.9$ $\omega_4 = 199.9$ $\omega_5 = 298.6$	$C_k\left[\cosh\lambda_k^{1/4}x - \cos\lambda_k^{1/4}x - \alpha_k\left(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \dfrac{\cosh\lambda_k^{1/4} - \cos\lambda_k^{1/4}}{\sinh\lambda_k^{1/4} - \sin\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$
Pinned-pinned	$\sin\lambda^{1/4} = 0$	$\omega_1 = 9.870$ $\omega_2 = 39.48$ $\omega_3 = 88.83$ $\omega_4 = 157.9$ $\omega_5 = 246.7$	$C_k\sin\lambda_k^{1/4}x$	$\int_0^1 X_j(x)X_k(x)dx$
Fixed-free	$\cos\lambda^{1/4}\cosh\lambda^{1/4} = -1$	$\omega_1 = 3.51$ $\omega_2 = 22.03$ $\omega_3 = 61.70$ $\omega_4 = 120.9$ $\omega_5 = 199.9$	$C_k\left[\cosh\lambda_k^{1/4}x - \cos\lambda_k^{1/4}x - \alpha_k\left(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \dfrac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$
Free-free	$\cosh\lambda^{1/4}\cos\lambda^{1/4} = 1$	$\omega_1 = 0$ $\omega_2 = 22.37$ $\omega_3 = 61.66$ $\omega_4 = 120.9$ $\omega_5 = 199.9$	$1, \sqrt{3}x\,(k = 1)$ $C_k\left[\cosh\lambda_k^{1/4}x + \cos\lambda_k^{1/4}x + \alpha_k\left(\sinh\lambda_k^{1/4}x + \sin\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \dfrac{\cosh\lambda_k^{1/4} - \cos\lambda_k^{1/4}}{\sinh\lambda_k^{1/4} - \sin\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$
Fixed-linear spring	$\lambda^{3/4}(\cosh\lambda^{1/4}\cos\lambda^{1/4} + 1)$ $-\beta(\cos\lambda^{1/4}\sin\lambda^{1/4} - \cosh\lambda^{1/4}\sin\lambda^{1/4})$ $= 0$	For $\beta = 0.25$ $\omega_1 = 3.65$ $\omega_2 = 22.08$ $\omega_3 = 61.70$ $\omega_4 = 120.9$ $\omega_5 = 199.9$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x - \alpha_k\left(\sin\lambda_k^{1/4}x - \sinh\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \dfrac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$

Pinned-linear spring	$\cot\lambda^{1/4}\coth\lambda^{1/4} = -\frac{2\beta}{\lambda^{3/4}}$	For $\beta = 0.25$ $\omega_1 = 0.8636$ $\omega_2 = 15.41$ $\omega_3 = 49.47$ $\omega_4 = 104.25$ $\omega_5 = 178.27$	$C_k\left[\sin\lambda_k^{1/4}x + \frac{\sin\lambda_k^{1/4}}{\sinh\lambda_k^{1/4}}\sinh\lambda_k^{1/4}x\right]$	$\int_0^1 X_j(x)X_k(x)dx$
Fixed-attached mass	$\lambda^{1/4}(\cos\lambda^{1/4}\cosh\lambda^{1/4} + 1)$ $+\beta(\cos\lambda^{1/4}\sinh\lambda^{1/4} - \cosh\lambda^{1/4}\sin\lambda^{1/4})$ $= 0$	For $\beta = 0.25$ $\omega_1 = 3.047$ $\omega_2 = 21.54$ $\omega_3 = 61.21$ $\omega_4 = 120.4$ $\omega_5 = 199.4$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x + \alpha_k\left(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \frac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$ $+ \beta X_j(1)X_k(1)$
Pinned-free	$\tan\lambda^{1/4} = \tanh\lambda^{1/4}$	$\omega_1 = 0$ $\omega_2 = 15.42$ $\omega_3 = 49.96$ $\omega_4 = 104.2$ $\omega_5 = 178.3$	$\sqrt{3x}, (k = 1)$ $C_k\left[\sin\lambda_k^{1/4}x + \frac{\sin\lambda_k^{1/4}}{\sinh\lambda_k^{1/4}}\sinh\lambda_k^{1/4}x\right]\ (k > 1)$	$\int_0^1 X_j(x)X_k(x)dx$
Fixed-pinned	$\tan\lambda^{1/4} = \tanh\lambda^{1/4}$	$\omega_1 = 15.42$ $\omega_2 = 49.96$ $\omega_3 = 104.2$ $\omega_4 = 178.3$ $\omega_5 = 272.0$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x - \alpha_k\left(\sin\lambda_k^{1/4}x - \sinh\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \frac{\cos\lambda_k^{1/4} - \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} - \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$
Fixed-attached inertia element	$\cos\lambda^{1/4}\cosh\lambda^{1/4}$ $+\beta(\sin\lambda^{1/4}\cosh\lambda^{1/4} + \cos\lambda^{1/4}\sinh\lambda^{1/4})$ $= -1$	For $\beta = 0.25$ $\omega_1 = 4.425$ $\omega_2 = 27.28$ $\omega_3 = 71.41$ $\omega_4 = 135.4$ $\omega_5 = 219.2$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x + \alpha_k\left(\sin\lambda_k^{1/4}x - \sinh\lambda_k^{1/4}x\right)\right]$ $\alpha_k = \frac{\cos\lambda_k^{1/4} - \cosh\lambda_k^{1/4}}{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)dx$ $+ \beta X_j(1)X_k(1)$

The dimensional natural frequencies are obtained by multiplying the given nondimensional natural frequencies by $\sqrt{EI/\rho AL^4}$; for a given beam β is as defined in Table 10.3.



(a)



(b)

FIGURE 10.15
(a) Carbon nanotube. (b) Five lowest mode shapes.

The nondimensional natural frequencies are the square roots of the solutions of this equation. If ω_k is a nondimensional natural frequency, the corresponding dimensional natural frequency is

$$\tilde{\omega}_k = \omega_k \sqrt{\frac{EI}{\rho A L^4}} \tag{b}$$

where

$$\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(1 \times 10^{12}\frac{\mathrm{N}}{\mathrm{m}^2}\right)\frac{\pi}{4}[(2 \times 10^{-9}\,\mathrm{m})^4 - (1.66 \times 10^{-9}\,\mathrm{m})^4]}{\left(2.3\frac{\mathrm{g}}{\mathrm{cm}^3}\right)\left(\frac{1\,\mathrm{kg}}{1000\,\mathrm{g}}\right)\left(\frac{100\,\mathrm{cm}}{\mathrm{m}}\right)^3 \pi[(2 \times 10^{-9}\,\mathrm{m})^2 - (1.66 \times 10^{-9}\,\mathrm{m})^2](100 \times 10^{-9}\,\mathrm{m})^4}}$$

$$= 2.71 \times 10^9 \frac{1}{\mathrm{s}} \tag{c}$$

The first five nondimensional natural frequencies are given in Table 10.4. They are used to obtain the five lowest dimensional natural frequencies:

$$\tilde{\omega}_1 = 3.51\left(2.71 \times 10^9 \frac{1}{s}\right) = 9.51 \times 10^9 \text{ rad/s} \tag{d}$$

$$\tilde{\omega}_2 = 22.03\left(2.71 \times 10^9 \frac{1}{s}\right) = 5.97 \times 10^{10} \text{ rad/s} \tag{e}$$

$$\tilde{\omega}_3 = 61.70\left(2.71 \times 10^9 \frac{1}{s}\right) = 1.67 \times 10^{11} \text{ rad/s} \tag{f}$$

$$\tilde{\omega}_4 = 120.9\left(2.71 \times 10^9 \frac{1}{s}\right) = 3.28 \times 10^{11} \text{ rad/s} \tag{g}$$

$$\tilde{\omega}_5 = 199.9\left(2.71 \times 10^9 \frac{1}{s}\right) = 5.41 \times 10^{11} \text{ rad/s} \tag{h}$$

The corresponding mode shapes are also given in Table 10.4. For a fixed-free beam they are

$$X_k(x) = C_k[\cosh\lambda^{1/4}x - \cos\lambda^{1/4}x - \alpha_k(\sinh\lambda^{1/4}x - \sin\lambda^{1/4}x)] \tag{i}$$

where

$$\alpha_k = \frac{\cos\lambda^{1/4} + \cosh\lambda^{1/4}}{\sin\lambda^{1/4} + \sinh\lambda^{1/4}} \tag{j}$$

The first five nondimensional mode shapes are

$$X_1(x) = C_1[\cosh 1.87x - \cos 1.87x - 0.73(\sinh 1.87x - \sin 1.87x)] \tag{k}$$

$$X_2(x) = C_2[\cosh 4.69x - \cos 4.69x - 1.02(\sinh 4.69x - \sin 4.69x)] \tag{l}$$

$$X_3(x) = C_3[\cosh 7.86x - \cos 7.86x - 0.999(\sinh 7.86x - \sin 7.86x)] \tag{m}$$

$$X_4(x) = C_4[\cosh 11.0x - \cos 11.0x - \sinh 11.0x + \sin 11.0x] \tag{n}$$

$$X_5(x) = C_5[\cosh 14.14x - \cos 14.14x - \sinh 14.14x + \sin 14.14x] \tag{o}$$

The normalization of the mode shapes is with respect to the kinetic energy scalar product which yields

$$C_i = \frac{1}{\sqrt{\int_0^1 [\cosh\lambda_i x + \cos\lambda_i x - \alpha_i(\sinh\lambda_i x - \sin\lambda_i x)]^2 dx}} \tag{p}$$

Evaluation of the first five constants yields

$$C_1 = 1.003, \; C_2 = 1, \; C_3 = 1, \; C_4 = 1, \; C_5 = 1 \tag{q}$$

The normalized mode shapes are shown in Figure 10.15(b).

EXAMPLE 10.5

Determine the natural frequencies and normalized mode shapes for a simply supported beam.

SOLUTION

The boundary conditions for a simply supported beam are

$$w(0, t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(0, t) = 0 \tag{a}$$

and

$$w(1, t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1, t) = 0 \tag{b}$$

which when applied to Equation (10.36) gives

$$0 = C_1 + C_3 \tag{c}$$

$$0 = -\sqrt{\lambda}C_1 + \sqrt{\lambda}C_3 \tag{d}$$

$$0 = C_1 \cos\lambda^{1/4} + C_2 \sin\lambda^{1/4} + C_3 \cosh\lambda^{1/4} + C_4 \sinh\lambda^{1/4} \tag{e}$$

$$0 = -\sqrt{\lambda}C_1 \cos\lambda^{1/4} - \sqrt{\lambda}C_2 \sin\lambda^{1/4} + \sqrt{\lambda}C_3 \cosh\lambda^{1/4} + \sqrt{\lambda}C_4 \sinh\lambda^{1/4} \tag{f}$$

The first two of these equations imply $C_1 = C_3 = 0$. Then the last two equations become

$$C_2 \sin\lambda^{1/4} + C_4 \sinh\lambda^{1/4} = 0 \tag{g}$$

and

$$-C_2 \sin\lambda^{1/4} + C_4 \sinh\lambda^{1/4} = 0 \tag{h}$$

These equations have a nontrivial solution if and only if

$$\sin\lambda^{1/4} = 0 \tag{i}$$

which is satisfied by

$$\lambda_k = (k\pi)^4 \qquad k = 1, 2, \ldots \tag{j}$$

For these values of λ, $C_4 = 0$ and C_2 remains arbitrary, leading to the mode shape

$$X_k(x) = C_k \sin k\pi x \tag{k}$$

The mode shapes are orthogonal with respect to the scalar product of Equation (10.4), as evidenced by

$$\int_0^1 C_k C_j \sin k\pi x \sin j\pi x\, dx = 0 \qquad k \neq j \tag{l}$$

Normalization with respect to this scalar product yields $C_k = \sqrt{2}$.

EXAMPLE 10.6

Determine the first four natural frequencies for the beam of Figure 10.16

SOLUTION

From Table 10.3, the appropriate boundary conditions are

$$w(0, t) = 0 \qquad \frac{\partial w}{\partial x}(0, t) = 0 \tag{a}$$

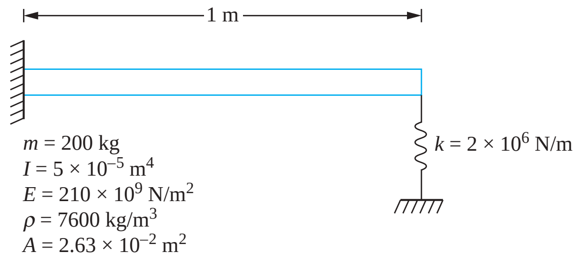


FIGURE 10.16
System of Example 10.6 is a beam that is fixed at one end and attached to a spring at its other end.

and

$$\frac{\partial^2 w}{\partial^2 x}(1,\ t) = 0 \qquad \frac{\partial^3 w}{\partial x^3}(1,\ t) = \beta w(1,\ t) \tag{b}$$

where

$$\beta = \frac{kL^3}{EI} = \frac{(2 \times 10^6\ \text{N/m})(1\ \text{m})^3}{(210 \times 10^9\ \text{N/m}^2)(5 \times 10^{-5}\ \text{m}^4)} = 0.190 \tag{c}$$

Application of the boundary conditions to Equation (10.36) gives

$$0 = C_1 + C_3 \tag{d}$$

$$0 = C_2 + C_4 \tag{e}$$

$$0 = -C_1 \cos\lambda^{1/4} - C_2 \sin\lambda^{1/4} + C_3 \cosh\lambda^{1/4} + C_4 \sinh\lambda^{1/4} \tag{f}$$

$$\begin{aligned} 0 = {} & (\lambda^{3/4}\sin\lambda^{1/4} - \beta\cos\lambda^{1/4})C_1 + (-\lambda^{3/4}\cos\lambda^{1/4} - \beta\sin\lambda^{1/4})C_2 \\ & + (\lambda^{3/4}\sinh\lambda^{1/4} - \beta\cosh\lambda^{1/4})C_3 + (\lambda^{3/4}\cosh\lambda^{1/4} - \beta\sinh\lambda^{1/4})C_4 \end{aligned} \tag{g}$$

The solvability condition is obtained by setting the determinant of the coefficient matrix obtained by writing Equations (d) through (g) in a matrix form to zero yielding

$$\lambda^{3/4}(1 + \cos\lambda^{1/4}\cosh\lambda^{1/4}) = -\beta(\cosh\lambda^{1/4}\sin\lambda^{1/4} - \cos\lambda^{1/4}\sinh\lambda^{1/4}) \tag{h}$$

For $\beta = 0.190$ the first four roots of this equation are

$$\lambda = 13.10, 486.2, 3807.0, 14161.6, \ldots \tag{i}$$

The nondimensional natural frequencies are the square roots of the values of λ that solve the characteristic equation. The dimensional natural frequencies are obtained by noting the relationship between the dimensional time and the nondimensional time and its application to Equation (10.29),

$$\omega = \sqrt{\lambda\frac{EI}{\rho AL^4}} = 229.1\sqrt{\lambda} \tag{j}$$

The first four natural frequencies for this beam are

$$\begin{aligned} \omega_1 &= 829.2\ \text{rad/s} & \omega_2 &= 5.05 \times 10^3\ \text{rad/s} \\ \omega_3 &= 1.41 \times 10^4\ \text{rad/s} & \omega_4 &= 2.13 \times 10^4\ \text{rad/s} \end{aligned} \tag{k}$$

10.4.3 FORCED VIBRATIONS

The modal analysis method, described in Section 10.2, for analyzing the forced vibrations of a continuous system is applied to the following examples.

EXAMPLE 10.7

The simply supported beam of Figure 10.17 is subject to a harmonic excitation over half of its span. Determine the beam's steady-state response.

SOLUTION

The nondimensional force per unit length is

$$f(x, t) = \sin\tilde{\omega}t\left[u\left(x - \frac{1}{4}\right) - u\left(x - \frac{3}{4}\right)\right] \tag{a}$$

where

$$\tilde{\omega} = \omega\sqrt{\frac{\rho AL^4}{EI}} \tag{b}$$

The expansion theorem is used to expand $f(x, t)$ in terms of the normalized mode shapes of the corresponding free-vibration problem, which are determined in Example 10.5 as $X_k(x) = \sqrt{2}\sin k\pi x$. The expansion coefficients are determined using Equation (10.8), with the scalar product defined by Equation (10.4),

$$C_k = \int_0^1 f(x, t)\sqrt{2}\sin(k\pi x)\,dx$$

$$= \sqrt{2}\sin\tilde{\omega}t\int_{1/4}^{3/4}\sin(k\pi x)\,dx$$

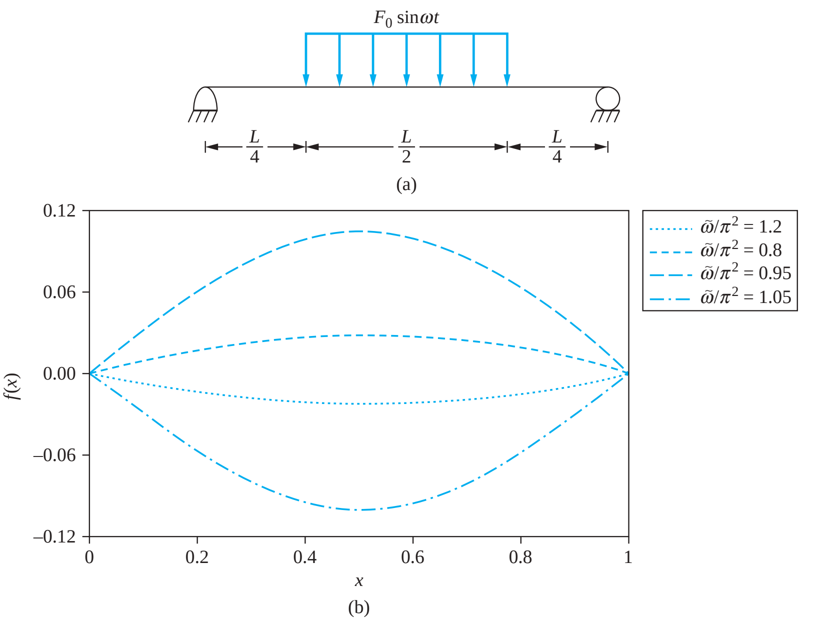


FIGURE 10.17

(a) System of Example 10.7 is a simply supported beam with a harmonic excitation over a portion of its span. (b) Steady-state response for Example 10.7.

$$= \frac{\sqrt{2}}{k\pi}\sin\tilde{\omega}t\left(\cos k\frac{\pi}{4} - \cos k\frac{3\pi}{4}\right)$$

$$= \frac{2}{k\pi}\sin\tilde{\omega}t\begin{cases} 0 & k = 2, 4, 6, \ldots \\ 1 & k = 1, 7, 9, 15, 17, 23, \ldots \\ -1 & k = 3, 5, 11, 13, 19, 21, \ldots \end{cases}$$

$$= \frac{2}{k\pi}a_k\sin\tilde{\omega}t \tag{c}$$

The displacement is expanded as

$$w(x, t) = \sum_{k=1}^{\infty}\sqrt{2}\sin(k\pi x)p_k(t) \tag{d}$$

Substituting for w and f in Equation (10.30) leads to

$$\sum_{k=1}^{\infty}[\ddot{p}_k + (k\pi)^4p_k]\sqrt{2}\sin(k\pi x) = \Lambda\sum_{k=1}^{\infty}C_k(t)\sqrt{2}\sin(k\pi x) \tag{e}$$

where

$$\Lambda = \frac{F_0L^3}{EI} \tag{f}$$

The preceding equation is multiplied by $\sqrt{2}\sin(j\pi x)$ for an arbitrary j and integrated from 0 to 1. This is equivalent to taking the scalar product of both sides of the equation with $X_j(x)$. The orthogonality condition, Equation (10.8), is used such that each sum collapses to a single term, yielding

$$\ddot{p}_j + (j\pi)^4p_j = \Lambda C_j \qquad j = 1, 2, \ldots \tag{g}$$

whose steady-state solution is

$$p_j(t) = \left[\frac{\Lambda}{(j\pi)^4 - \tilde{\omega}^2}\right]\frac{2}{j\pi}a_j\sin\tilde{\omega}t \tag{h}$$

The steady-state response of the beam is

$$\begin{aligned} w(x, t) &= \frac{2\sqrt{2}\Lambda}{\pi}\sin\tilde{\omega}t\left[\frac{1}{\pi^4 - \tilde{\omega}^2}\sin\pi x \right. \\ &\qquad - \frac{1}{3(81\pi^4 - \tilde{\omega}^2)}\sin 3\pi x - \frac{1}{5(625\pi^4 - \tilde{\omega}^2)}\sin 5\pi x \\ &\qquad \left. + \frac{1}{7(150\pi^4 - \tilde{\omega}^2)}\sin 7\pi x + \cdots\right] \\ &= \frac{2\sqrt{2}\Lambda}{\pi}f(x)\sin\tilde{\omega}t \end{aligned} \tag{i}$$

The function $f(x)$ is shown in Figure 10.17(b) for several values of $\tilde{\omega}$. Note that when $\tilde{\omega}$ is close to π^2 the steady-state amplitude is large at the midspan.

EXAMPLE 10.8

A machine of mass 150 kg is attached to the end of the cantilever beam of Figure 10.18. The machine operates at 2000 rpm and has a rotating unbalance of 0.965 kg · m. What is the steady-state amplitude of vibration of the end of the beam?

SOLUTION

The nondimensional formulation of the governing mathematical problem is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \tag{a}$$

subject to

$$w(0, t) = 0 \qquad \frac{\partial w}{\partial x}(0, t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1, t) = 0 \tag{b}$$

$$\frac{\partial^3 w}{\partial x^3}(1, t) = \beta \frac{\partial^2 w}{\partial t^2}(1, t) + \alpha \sin \tilde{\omega} t \tag{c}$$

where

$$\begin{aligned} \tilde{\omega} &= \omega \sqrt{\frac{\rho A L^4}{EI}} \\ &= 209.4 \frac{\text{rad}}{\text{s}} \sqrt{\frac{(280 \text{ kg})(3 \text{ m})^3}{(210 \times 10^9 \text{ N/m}^2)(1.2 \times 10^{-4} \text{ m}^4)}} \\ &= 3.63 \end{aligned} \tag{d}$$

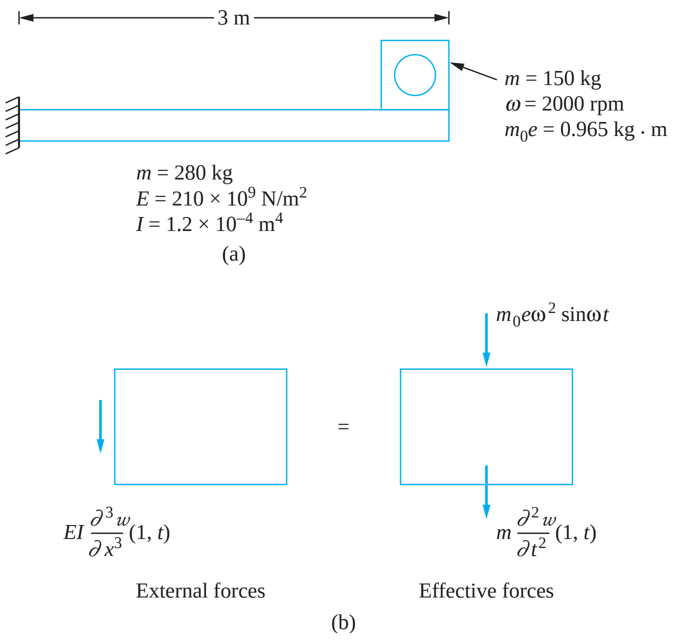


FIGURE 10.18

(a) System of Example 10.8 is a fixed-free beam with a machine with a rotating unbalance at its free end. (b) FBDs of the machine at an arbitrary instant are used to derive a boundary condition.

$$\beta = \frac{m}{\rho AL} = \frac{150 \text{ kg}}{280 \text{ kg}} = 0.536 \tag{e}$$

and

$$\begin{aligned}\alpha &= \frac{m_0 e\omega^2 L^2}{EI} \\ &= \frac{(0.965 \text{ kg} \cdot \text{m})(209.4 \text{ rad/s})^2(3 \text{ m})^2}{(210 \times 10^9 \text{ N/m}^2)(1.2 \times 10^{-4} \text{ m}^4)} \\ &= 0.010\end{aligned} \tag{f}$$

The boundary condition Equation (c) is developed by applying Newton's law to a FBD of the machine as shown in Figure 10.18.(b) The problem is nonhomogeneous due to this boundary condition. From Table 10.4 the characteristic equation for the homogeneous problem of a beam with a concentrated mass at its end is

$$\lambda^{1/4}(1 + \cos\lambda^{1/4}\cosh\lambda^{1/4}) + \beta(\cos\lambda^{1/4}\sinh\lambda^{1/4} - \cosh\lambda^{1/4}\sin\lambda^{1/4}) = 0 \tag{g}$$

The corresponding mode shapes for the homogeneous problem are

$$X_k(x) = C_k\left[\cos\lambda^{1/4}x - \cosh\lambda^{1/4}x + \frac{\cos\lambda^{1/4} + \cosh\lambda^{1/4}}{\sin\lambda^{1/4} + \sinh\lambda^{1/4}}(\sinh\lambda^{1/4}x - \sin\lambda^{1/4}x)\right] \tag{h}$$

where C_k is chosen to normalize the mode shape with respect to the scalar product defined by

$$(X_j(x), X_k(x)) = \int_0^1 X_j(x), X_k(x)dx + \beta X_j(1)X_k(1) \tag{i}$$

The first six nondimensional natural frequencies and normalization constants are given in Table 10.5.

The expansion theorem implies that the solution of the nonhomogeneous problem can be expanded in a series of normalized mode shapes. To this end,

$$w(x, t) = \sum_{k=1}^{\infty} p_k(t)X_k(x) \tag{j}$$

TABLE 10.5 Free-vibration properties for Example 10.8

λ_k	Nondimensional Natural Frequency	Natural Frequency ω_k(rad/s)	C_k
6.71	2.59	149.55	0.715
443.5	21.06	1216.0	0.617
3682.1	60.68	3483.0	0.593
14,371.2	119.88	6922.0	0.584
39,533.3	198.83	11,480.0	0.582
88,513.2	297.51	17,178.0	0.434

Substituting for $w(x, t)$ into the governing partial differential equation, multiplying by $X_j(x)$ for an arbitrary j, and integrating from 0 to 1 leads to

$$\sum_{k=1}^{\infty}(\ddot{p}_k + \lambda p_k)\int_0^1 X_j(x)X_k(X)\,dx = 0 \tag{k}$$

The mutual orthonormality of the mode shapes implies

$$\int_0^1 X_j(x)X_k(x)\,dx = \delta_{jk} - \beta X_j(1)X_k(1) \tag{l}$$

Use of this orthogonality condition leads to

$$\ddot{p}_j + \lambda_j p_j = \sum_{k=1}^{\infty}(\ddot{p}_k + \lambda_k p_k)\beta X_j(1)X_k(1) \tag{m}$$

Substituting for $w(x, t)$ from the expansion theorem in the nonhomogeneous boundary condition leads to

$$\sum_{k=1}^{\infty}\frac{d^3X_k}{dx^3}(1)p_k(t) = \alpha\sin\tilde{\omega}t + \beta\sum_{k=1}^{\infty}X_k(1)\ddot{p}_k(t) \tag{n}$$

The mode shapes satisfy the boundary conditions for the nonhomogeneous problem. Thus,

$$\frac{d^3X}{dx^3}(1) = -\lambda_k\beta X_k(1) \tag{o}$$

which when used in the preceding equation gives

$$\sum_{k=1}^{\infty}(\ddot{p}_k + \lambda_k p_k)X_k(1) = \alpha\sin\tilde{\omega}t \tag{p}$$

and which when substituted into the previously derived differential equations for the principal coordinates uncouples these equations and gives

$$\ddot{p}_j + \lambda_j p_j = \alpha X_j(1)\sin\tilde{\omega}t \qquad j = 1, 2, \ldots \tag{q}$$

The steady-state solution for each of the principal coordinates is now easily obtained and the expansion theorem is used to write the steady-state solution as

$$w(x, t) = \left[\sum_{k=1}^{\infty}\frac{\alpha X_k(1)}{\lambda_k - \tilde{\omega}^2}X_k(x)\right]\sin\tilde{\omega}t \tag{r}$$

The nondimensional steady-state amplitude of the end of the beam is

$$\alpha\sum_{k=1}^{\infty}\frac{X_k^2(1)}{\lambda_k - \tilde{\omega}^2} = 1.67 \times 10^{-4} \tag{s}$$

The dimensional amplitude is obtained using Equation (10.29) as 1.67×10^{-4} (3 m) = 4.0 mm.

10.5 ENERGY METHODS

Consider a differential element of the shaft of Figure 10.5. Assuming elastic behavior throughout, a shear stress distribution is developed across the cross section of the shaft according to

$$\tau = \frac{T_1 r}{J} \tag{10.37}$$

where $T_r(x, t)$ is the resisting torque in the cross section and r is the distance from the center of the shaft to a point in its cross section. The total strain energy in the element is

$$dV = \frac{1}{2G}\left(\int_A \tau^2 dA\right)dx \tag{10.38}$$

Substitution of Equations (10.37) and (10.13) into Equation (10.38) leads to

$$dV = \frac{G}{2}\left(\frac{\partial\theta}{\partial x}\right)^2\left(\int_A r^2 dA\right)dx \tag{10.39}$$

Noting that $J = \int_A r^2 dA$ and integrating over the entire length of the shaft, the total strain energy becomes

$$V = \frac{1}{2}\int_0^L JG\left(\frac{\partial\theta}{\partial x}\right)^2 dx \tag{10.40}$$

The kinetic energy of the differential element is

$$dT = \frac{1}{2}\rho J\left(\frac{\partial\theta}{\partial t}\right)^2 dx \tag{10.41}$$

where ρ is the mass density of the shaft's material. The total kinetic energy of the shaft is

$$T = \frac{1}{2}\int_0^L \rho J\left(\frac{\partial\theta}{\partial t}\right)^2 dx \tag{10.42}$$

For a conservative system, the maximum potential energy is equal to the maximum kinetic energy. Thus, if the free oscillations of the shaft are described by

$$\theta(x, t) = u(x)\sin\omega t \tag{10.43}$$

then

$$\omega^2 = \frac{\displaystyle\int_0^L JG\left(\frac{du}{dx}\right)^2 dx}{\displaystyle\int_0^L \rho J u^2 dx} \tag{10.44}$$

Introducing the nondimensional variables of Equation (10.29) into Equation (10.44) and assuming the shaft is uniform leads to

$$\tilde{\omega}^2 = \frac{\displaystyle\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\displaystyle\int_0^1 u^2 dx} \tag{10.45}$$

where

$$\tilde{\omega} = L\sqrt{\frac{\rho}{G}}\omega \tag{10.46}$$

For any function $w(x)$ which satisfies the boundary conditions specified for the shaft, we define

$$R(w) = \frac{\displaystyle\int_0^L JG\left(\frac{dw}{dx}\right)^2 dx}{\displaystyle\int_0^L \rho J w^2 dx} \tag{10.47}$$

where $R(w)$ is Rayleigh's quotient for this continuous system. If $w(x)$ is a mode shape, then $R(w)$ is equal to the square of the natural frequency of that mode. If $w(x)$ is not a mode shape but satisfies the boundary conditions for the system, then $R(w)$ is a scalar function of w. As for discrete systems, $R(w)$ is a minimum when w is a mode shape. Hence Rayleigh's quotient can be used to approximate the lowest natural frequency for the continuous system.

EXAMPLE 10.9

Use Rayleigh's quotient to approximate the lowest natural frequency of the tapered circular shaft of Figure 10.19.

SOLUTION

The polar moment of inertia varies over the length of the shaft is

$$J(x) = \frac{\pi}{2}(0.2 - 0.05x)^4 \tag{a}$$

A trial function which satisfies the boundary conditions $w(0) = 0$ and $dw/dx(3\text{ m}) = 0$ is

$$w(x) = \sin\frac{\pi}{6}x \tag{b}$$

An upper bound and approximation on the lowest natural frequency is

$$R(w) = \frac{80 \times 10^9 \dfrac{\text{N}}{\text{m}^2}\dfrac{\pi}{2}\displaystyle\int_0^3 (0.2 - 0.05x)^4\left(\frac{\pi}{6}\right)^2 \cos^2\frac{\pi}{6}x\,dx}{7850\dfrac{\text{kg}}{\text{m}^3}\dfrac{\pi}{2}\displaystyle\int_0^3 (0.2 - 0.05x)^4 \sin^2\frac{\pi}{6}x\,dx} \tag{c}$$

$$\omega_1 \le [R(w)]^{1/2} = 3767 \text{ rad/s} \tag{d}$$

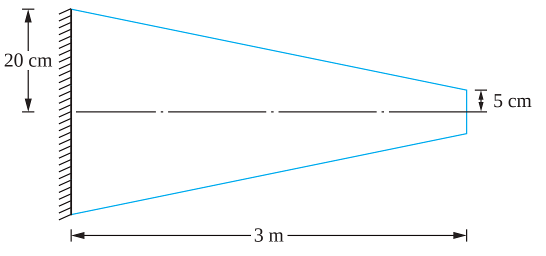


FIGURE 10.19
Tapered circular shaft of Example 10.9.

Rayleigh's quotient can be generalized as the ratio of a potential energy scalar to a kinetic energy scalar product, where the energy products are defined by integrals, perhaps with additional terms to account for discrete masses or springs.

$$R(w) = \frac{(w, w)_V}{(w, w)_T} \tag{10.48}$$

Rayleigh's quotient can be applied to any continuous system. Table 10.6 gives the appropriate form of the scalar products for several continuous systems.

A method based on Rayleigh's quotient, called the *Rayleigh-Ritz method*, can be used to approximate a finite number of the lowest natural frequencies of a continuous system. Let $u_1(x), u_2(x), \ldots, u_n(x)$ be n linearly independent functions, each of which satisfies the boundary conditions for a specific continuous system. An approximation to the free-vibration response of the continuous system is assumed as

$$w(x) = \sum_{i=1}^{n} c_i u_i(x) \tag{10.49}$$

Equation (10.49) is substituted into Rayleigh's quotient which is rewritten as

$$R(w)(w, w)_T = (w, w)_V \tag{10.50}$$

TABLE 10.6 Scalar products for Rayleigh-Ritz method.

Structural Element	Case	$(u, v)_T$	$(u, v)_V$
Torsional shaft	No added disks or springs	$\int_0^L \rho J u(x) v(x)\,dx$	$\int_0^L GJ \frac{du}{dx}\frac{dv}{dx}dx$
	Added disk at $x = \tilde{x}$	$\int_0^L \rho J u(x) v(x)\,dx + I_D u(\tilde{x}) v(\tilde{x})$	$\int_0^L GJ \frac{du}{dx}\frac{dv}{dx}dx$
	Torsional spring at $x = \tilde{x}$	$\int_0^L \rho J u(x) v(x)\,dx$	$\int_0^L GJ \frac{du}{dx}\frac{dv}{dx}dx + k_t u(\tilde{x}) v(\tilde{x})$
Longitudinal bar	No added masses or springs	$\int_0^L \rho A u(x) v(x)\,dx$	$\int_0^L EA \frac{du}{dx}\frac{dv}{dx}dx$
	Added mass at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\,dx + m u(\tilde{x}) v(\tilde{x})$	$\int_0^L EA \frac{du}{dx}\frac{dv}{dx}dx$
	Spring at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\,dx$	$\int_0^L EA \frac{du}{dx}\frac{dv}{dx}dx + k u(\tilde{x}) v(\tilde{x})$
Beam	No added masses, disks, or springs	$\int_0^L \rho A u(x) v(x)\,dx$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx$
	Added mass at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\,dx + m u(\tilde{x}) v(\tilde{x})$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx$
	Added spring at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\,dx$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx + k u(\tilde{x}) v(\tilde{x})$
	Added disk (I_D) at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\,dx + I_D \frac{du(\tilde{x})}{dx}\frac{dv(\tilde{x})}{dx}$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx$

Since $R(w)$ is stationary near a mode shape, the best approximation to the natural frequencies and mode shapes using the functions $u_1(x)$, $u_2(x)$, . . . , $u_n(x)$, is obtained by setting

$$\frac{\partial R}{\partial c_1} = \frac{\partial R}{\partial c_2} = \cdots = \frac{\partial R}{\partial c_n} = 0 \tag{10.51}$$

Differentiating Equation (10.50) with respect to c_k for any $k = 1, 2, \ldots, n$ and using Equation (10.51) gives

$$R(w)\frac{\partial(w, w)_T}{\partial c_k} = \frac{\partial(w, w)_V}{\partial c_k} \tag{10.52}$$

Developing Equation (10.52) for each $k = 1, 2, \ldots, n$ leads to n linear homogeneous equations to solve for $c_1, c_2, \ldots, c_n$ in terms of the parameter $R(w)$. Since the equations are homogeneous, a nontrivial solution is available if and only if the determinant is set equal to zero, yielding an nth-order polynomial equation for $R(w)$. The roots of the polynomial are the squares of the approximations to the lowest natural frequencies. Approximations for the mode shapes can be obtained by returning to the homogeneous equations. The method is illustrated in the following example.

EXAMPLE 10.10

Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of Example 10.1.

SOLUTION

Two polynomials which satisfy the boundary conditions of Example 10.1 are

$$u_1(x) = 2x - x^2 \qquad u_2(x) = 3x - x^3 \tag{a}$$

An approximation to the mode shape is developed as

$$w(x) = c_1(2x - x^2) + c_2(3x - x^3) \tag{b}$$

Calculation of the energy scalar products gives

$$(w, w)_T = \int_0^1 [c_1(2x - x^2) + c_2(3x - x^3)]^2\,dx = \frac{8}{15}c_1^2 + \frac{61}{30}c_1c_2 + \frac{204}{105}c_2^2 \tag{c}$$

$$(w, w)_V = \int_0^1 [c_1(2 - 2x) + c_2(3 - 3x^2)]^2\,dx = \frac{4}{3}c_1^2 + 5c_1c_2 + \frac{24}{5}c_2^2 \tag{d}$$

Application of Equation (10.52) leads to

$$\left(\frac{8}{3} - \frac{16}{15}R\right)c_1 + \left(5 - \frac{61}{30}R\right)c_2 = 0 \tag{e}$$

$$\left(5 - \frac{61}{30}R\right)c_1 + \left(\frac{48}{5} - \frac{136}{35}R\right)c_2 = 0 \tag{f}$$

A nontrivial solution of the preceding equations is obtained if and only if

$$\det\begin{bmatrix} \frac{8}{3} - \frac{16}{15}R & 5 - \frac{61}{30}R \\ 5 - \frac{61}{30}R & \frac{48}{5} - \frac{136}{35}R \end{bmatrix} = 0 \tag{g}$$

Evaluation of the determinant leads to

$$9.24R^2 - 241.0R + 538.0 = 0 \tag{h}$$

whose roots are

$$R = 2.467,\ 23.610 \tag{i}$$

The natural frequency approximations are

$$\omega_1 \approx 1.571 \qquad \omega_2 \approx 4.859 \tag{j}$$

The approximation to the lowest natural frequency is excellent. The approximation to the second natural frequency is also very good, being only 3.3 percent higher than the exact value.

The mode shape approximations are obtained by solving for c_2 in terms of c_1 for each R and then substituting into the expression for $w(x)$ with c_1 remaining arbitrary. This leads to

$$w_1(x) = 7.58x - x^2 - 1.86x^3 \tag{k}$$

$$w_2(x) = 0.4295x - x^2 + 0.5235x^3 \tag{l}$$

The approximate mode shapes plotted in Figure 10.20 have been normalized such that $w_i(1) = 1$. These compare favorably to the first two mode shapes for a fixed-free torsional shaft plotted in Figure 10.7.

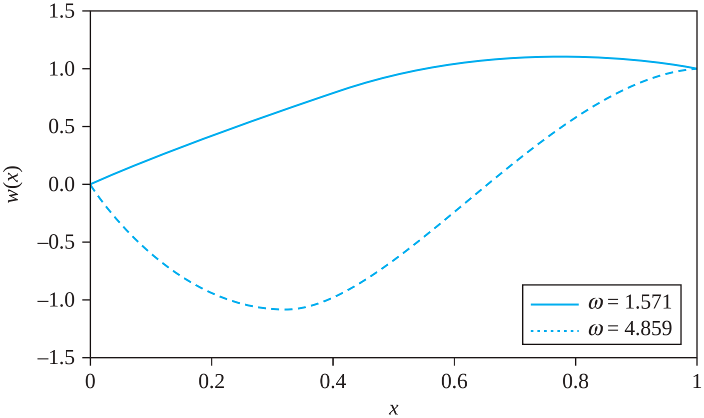


FIGURE 10.20
Rayleigh-Ritz approximations to the mode shapes corresponding to the two lowest natural frequencies of a fixed-free torsional shaft.

10.6 BENCHMARK EXAMPLES

The problem of the machine on the simply supported beam can be formulated using a continuous systems analysis. The problem of where the machine is directly mounted on the beam is formulated such that the governing differential equation is

$$\rho A\frac{\partial^2 w}{\partial t^2} + m\frac{\partial^2 w}{\partial t^2}\delta(x - 12) + EI\frac{\partial^4 w}{\partial x^4} = 0 \tag{a}$$

where $\delta(x - 12)$ is the Dirac delta function introduced in Appendix A. The boundary conditions are those for a fixed-pinned beam, which are

$$w(0, t) = 0 \qquad \frac{\partial w}{\partial x}(0, t) = 0 \qquad w(1, t) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(1, t) = 0 \tag{b}$$

The solutions of Equations (a) and (b) are beyond the scope of this book.

The approach instead is to use the Rayleigh-Ritz method and the scalar products reported in Table 10.6. The energy formulation of Rayleigh's quotient is

$$R(w) = \frac{\displaystyle\int_0^L EI\left(\frac{d^2 w}{dx^2}\right)^2 dx}{\displaystyle\int_0^L \rho A w^2 dx + m[w(12)]^2} \quad \text{or} \quad R(w) = \frac{\displaystyle\int_0^1 \left(\frac{d^2 w}{dx^2}\right)^2 dx}{\displaystyle\int_0^1 w^2 dx + \beta[w(0.6)]^2} \tag{c}$$

(dimensional form) (nondimensional form)

where $\beta = \frac{1000\,\text{lb}}{600\,\text{lb}} = 1.67$. It is desired to approximate the five lowest natural frequencies of the system by

$$w(x) = \sum_{i=1}^{5} c_i u_i(x) \tag{d}$$

The mode shapes for a uniform fixed-free beam reported in Table 10.4 are used as the functions in Equation (d). They are

$$u_1(x) = \cos 3.93x - \cosh 3.93x - 0.998 \sin 3.93x + 0.998 \sinh 3.93x \tag{e}$$

$$u_2(x) = \cos 7.07x - \cosh 7.07x - \sin 7.07x + \sinh 7.073x \tag{f}$$

$$u_3(x) = \cos 10.21x - \cosh 10.21x - \sin 10.21x + \sinh 10.21x \tag{g}$$

$$u_4(x) = \cos 13.35x - \cosh 13.35x - \sin 13.35x + \sinh 13.35x \tag{h}$$

$$u_5(x) = \cos 16.49x - \cosh 16.49x - \sin 16.49x + \sinh 16.49x \tag{i}$$

The mode shaped of Equations (e) through (h) have been normalized. Thus,

$$\int_0^1 u_i(x)u_j(x)\,dx = \delta_{ij} \quad \text{and} \quad \int_0^1 \left(\frac{d^2 u_i}{dx^2}\right)\left(\frac{d^2 u_j}{dx^2}\right)dx = \omega_i^2\delta_{ij} \tag{j}$$

Substituting Equations (e) through (h) into Equation (d) and using Equation (j) leads to

$$\mathbf{K} = \begin{bmatrix} 237.8 & 0 & 0 & 0 & 0 \\ 0 & 2496.0 & 0 & 0 & 0 \\ 0 & 0 & 10857 & 0 & 0 \\ 0 & 0 & 0 & 31790 & 0 \\ 0 & 0 & 0 & 0 & 73984 \end{bmatrix} \tag{k}$$

The mass matrix is determined as

$$\mathbf{M} = \begin{bmatrix} 4.854 & -1.057 & -2.899 & 3.256 & 1.108 \\ -1.057 & 1.298 & 0.807 & -0.857 & -0.309 \\ -2.899 & 0.807 & 3.182 & -2.591 & -0.836 \\ 3.256 & -0.857 & -2.591 & 3.826 & 1.066 \\ 1.108 & -0.309 & -0.836 & 1.066 & 1.320 \end{bmatrix} \quad \text{(l)}$$

The natural frequency approximations are the reciprocals of the square roots of the eigenvalues of **AM**. They are

$$\omega_1 = 7.161 \quad \omega_2 = 48.245 \quad \omega_3 = 84.806 \quad \omega_4 = 166.82 \quad \omega_5 = 275.07 \quad \text{(m)}$$

The dimensional natural frequencies are obtained by multiplying the nondimensional frequencies by

$$\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{\left(30 \times 10^6 \dfrac{\text{lb}}{\text{in}^2}\right)(291 \text{ in}^4)}{\dfrac{\left(30 \dfrac{\text{lb}}{\text{ft}}\right)\left(\dfrac{1 \text{ ft}}{12 \text{ in}}\right)}{386 \dfrac{\text{in}}{\text{s}^2}}(20 \text{ ft})^4\left(\dfrac{12 \text{ in}}{1 \text{ ft}}\right)^4}} = 20.16 \text{ rad/s} \quad \text{(n)}$$

They are

$$\begin{aligned} &\omega_1 = 144.37 \text{ rad/s} \quad \omega_2 = 972.43 \text{ rad/s} \quad \omega_3 = 1.709 \times 10^3 \text{ rad/s} \\ &\omega_4 = 3.363 \times 10^3 \text{ rad/s} \quad \omega_5 = 5.544 \times 10^3 \text{ rad/s} \end{aligned} \quad \text{(o)}$$

The eigenvectors of **AM** are substituted into Equation (d) to provide the mode shape approximations. They are given in Figure 10.21.

Now consider the forced vibrations of the machine when it is subject to a harmonic force $F(t) = 20{,}000 \sin 80t\, \delta(x - 12)$. Nondimensionalizing the force as in Equation (10.29) leads to

$$\frac{f_m L^2}{EI} = \frac{(20{,}000 \text{ lb})(240 \text{ in})^2}{(30 \times 10^6 \text{ lb/in}^2)(291 \text{ in}^4)} = 0.132 \quad \text{(p)}$$

The equations approximating the forced response are Equation (10.57) where the generalized force vector is given by Equation (10.58) and is calculated as

$$f_i(t) = \int_0^1 u_i(x)[0.132 \sin 80t\, \delta(x - 0.6)]\,dx = 0.132 u_i(0.6) \sin 80t \quad \text{(q)}$$

or

$$f(t) = \begin{bmatrix} -1.516 \\ 0.422 \\ 1.143 \\ -1.297 \\ -0.438 \end{bmatrix} 0.132 \sin 80t \quad \text{(r)}$$

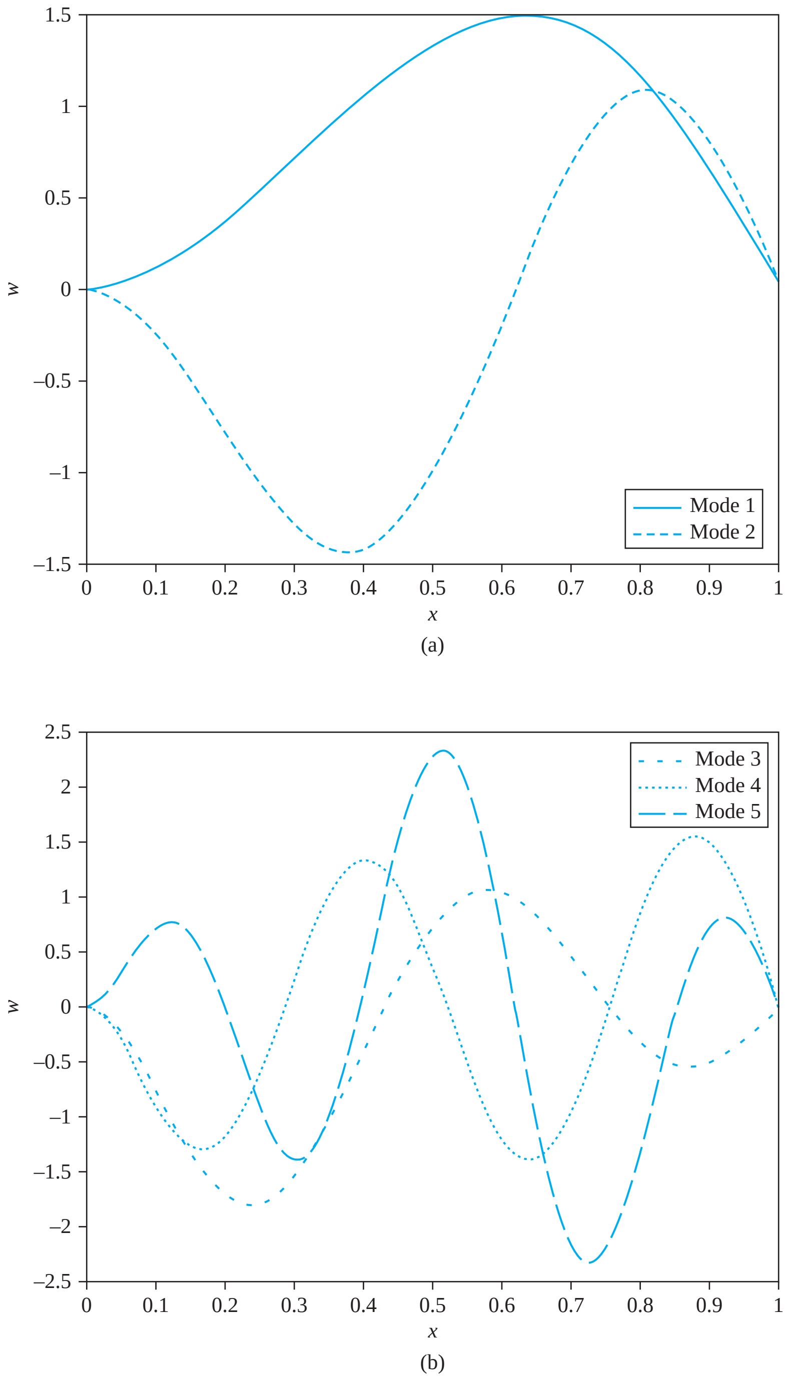


FIGURE 10.21
Mode shape approximations to beam with machine: (a) first and second modes and (b) modes 3, 4, and 5.

The right-hand side of the equation is written in dimensional form, while the left-hand side is written in nondimensional form. The nondimensionalization of time is $t^* = t\sqrt{\dfrac{EI}{\rho A L^4}} = 20.16t$. The nondimensional equations become

$$\begin{bmatrix} 4.854 & -1.057 & -2.899 & 3.256 & 1.108 \\ -1.057 & 1.298 & 0.807 & -0.857 & -0.309 \\ -2.899 & 0.807 & 3.182 & -2.591 & -0.836 \\ 3.256 & -0.857 & -2.591 & 3.826 & 1.066 \\ 1.108 & -0.309 & -0.836 & 1.066 & 1.320 \end{bmatrix} \begin{bmatrix} \ddot{c}_1 \\ \ddot{c}_2 \\ \ddot{c}_3 \\ \ddot{c}_4 \\ \ddot{c}_5 \end{bmatrix}$$

$$+ \begin{bmatrix} 237.8 & 0 & 0 & 0 & 0 \\ 0 & 2496.0 & 0 & 0 & 0 \\ 0 & 0 & 10857 & 0 & 0 \\ 0 & 0 & 0 & 31790 & 0 \\ 0 & 0 & 0 & 0 & 73984 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} -1.516 \\ 0.422 \\ 1.143 \\ -1.297 \\ -0.438 \end{bmatrix} \sin 3.968t \quad \text{(s)}$$

where the * has been dropped on nondimensional variables. It is noted that the frequency of the excitation is near the natural frequency of a uniform fixed-pinned beam, but it is away from the natural frequency of the beam with the machine on it. A steady-state solution to Equation (s) is assumed to be

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \sin 3.968t \quad \text{(t)}$$

When Equation (t) is substituted into Equation (s), we have

$$\begin{bmatrix} 176.7 & 16.64 & 45.64 & -45.67 & -17.44 \\ 16.64 & 2476 & 12.71 & 12.71 & 4.86 \\ 45.64 & -12.71 & 10857 & 34.43 & 13.16 \\ -45.67 & 12.71 & 34.43 & 31790 & -13.19 \\ -17.40 & 4.86 & 13.16 & -13.19 & 73984 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} = 0.132 \begin{bmatrix} -1.516 \\ 0.422 \\ 1.143 \\ -1.297 \\ -0.438 \end{bmatrix} \quad \text{(u)}$$

The solution of Equation (u) is

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} = \begin{bmatrix} -1.142 \times 10^{-3} \\ 3.033 \times 10^{-5} \\ 1.884 \times 10^{-5} \\ -6.439 \times 10^{-6} \\ -1.057 \times 10^{-6} \end{bmatrix} \quad \text{(v)}$$

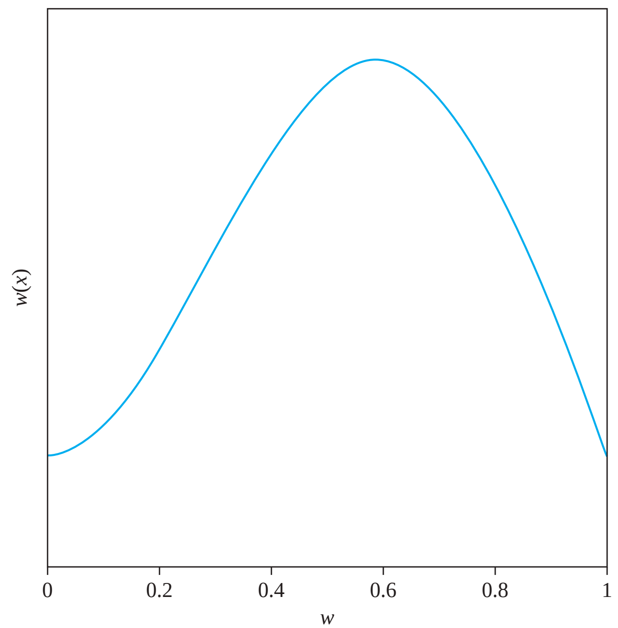


FIGURE 10.22
Plot of the steady-state response mode shape of the beam as a function of position along the beam.

The coefficients are substituted into Equation (d) to approximate the steady-state response of the beam. A plot of the steady-state response at a given time is shown in Figure 10.22. The displacement of the machine is 1.77×10^{-4} ft.

10.7 CHAPTER SUMMARY

10.7.1 IMPORTANT CONCEPTS

- A continuous system is governed by a partial differential equation. The independent variables are a spatial coordinate and time.
- Torsional oscillations of a shaft, longitudinal vibrations of a bar, and transverse vibrations of a string are all governed by the wave equation.
- Transverse vibrations of a beam are governed by a partial differential equation that is of the fourth order in the spatial variable and second order in time.
- The method-of-separation of variables is used to solve the free-vibrations problem.
- A continuous system has an infinite but countable number of natural frequencies ω_k and corresponding mode shapes X_k for $k = 1, 2, 3, \ldots$.
- The natural frequencies and mode shapes are determined by solving a differential eigenvalue problem consisting of a homogeneous differential equation and an appropriate number of homogeneous boundary conditions. The eigenvalue is a parameter in the differential equation for which a non-trivial solution exists only for certain values of the parameter.

- Rayleigh's quotient is the ratio of the potential energy to the kinetic energy if the system has a specified mode shape. It is stationary when the function that it is evaluated for is a mode shape of the system. Rayleigh's quotient has an absolute minimum when the mode shape corresponds to the lowest natural frequency.
- The Rayleigh-Ritz method assumes a solution as a finite, linear combination of n functions which satisfy the boundary conditions of a system. The assumed solution is substituted into Rayleigh's quotient and minimized to approximate the lowest n frequencies and mode shapes of the system.

10.7.2 IMPORTANT EQUATIONS

Product solution for free-vibrations problems

$$w(x, t) = X(x)T(t) \tag{10.1}$$

Separated equations

$$\frac{d^2T}{dt^2} + \lambda T = 0 \tag{10.2}$$

$$L_x X + \lambda X = 0 \tag{10.3}$$

Kinetic-energy scalar product

$$(X_i, X_j)_T = \int_0^1 X_i(x)X_j(x)\,dx \tag{10.4}$$

Normalized mode shapes

$$(X_i, X_j)_T = 1 \tag{10.5}$$

Expansion theorem

$$f(x) = \sum_{k=1}^{\infty}(f, X_k)_T X_k(x) \tag{10.6}$$

General free-vibrations solution

$$w(x) = \sum_{k=1}^{\infty} X_k(x)T_k(t) \tag{10.7}$$

Solution of forced-vibration problem with forcing function $f(x)$

$$f(x) = \sum_{k=1}^{\infty} C_k(t)X_k(x) \tag{10.8}$$

$$C_k(t) = (f(x, t), X_k(x)) \tag{10.9}$$

Wave equation for torsional oscillations of a shaft

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \tag{10.20}$$

Nondimensional partial differential equation governing the transverse forced vibrations of a uniform beam

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = \frac{f_m L^3}{EI} f(x, t) \tag{10.30}$$

Solutions of separated equations when the product solution is assumed for free vibrations of a uniform beam

$$T(t) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \tag{10.35}$$

$$X(x) = C_1\cos\lambda^{1/4}x + C_2\sin\lambda^{1/4}x + C_3\cosh\lambda^{1/4}x + C_4\sinh\lambda^{1/4}x \tag{10.36}$$

Rayleigh's quotient

$$R(w) = \frac{(w, w)_V}{(w, w)_T} \tag{10.48}$$

Rayleigh-Ritz method

$$w(x) = \sum_{i=1}^{\infty} c_i u_i(x) \tag{10.49}$$

Equations to solve for the coefficients in a Rayleigh-Ritz solution

$$R(w)\frac{\partial(w, w)_T}{\partial c_k} = \frac{\partial(w, w)v}{\partial c_k} \tag{10.52}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For all problems, the bar, shaft, string, or beam specified is assumed to be linear, elastic, uniform, and homogenous.

For Problems 10.1. through 10.10, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

10.1 A continuous system is also referred to as a distributed parameter system.

10.2 A continuous system has an infinite number of natural frequencies.

10.3 The longitudinal vibrations of a bar and the transverse vibrations of a beam are both governed by the wave equation.

10.4 A free-free beam is an example of a degenerate system.

10.5 Rayleigh's quotient defined for a system is stationary for any function that satisfies the boundary conditions of that system.

10.6 Four initial conditions are necessary to determine the forced-vibration response of a fixed-free beam.

10.7 The Rayleigh-Ritz method can be used to approximate natural frequencies and forced responses of continuous systems.

10.8 Mode shapes corresponding to distinct natural frequencies of a continuous system are orthogonal with respect to the potential-energy scalar product.

10.9 The mode shape reported in Table 10.4 for a pinned-free beam of $\sqrt{3}x$ is a rigid-body mode.

10.10 The assumption that $M = -EI\frac{\partial^2 w}{\partial x^2}$ is used in the derivation of the differential equation governing the transverse vibrations of a beam.

Problems 10.11 through 10.32 require a short answer.

10.11 What is the method where a product solution is assumed for the free vibrations of a uniform bar called? Is the same method applicable to the free vibrations of a beam?

10.12 What is the order of the highest spatial derivative in the wave equation? What is the order of the highest spatial derivative of the beam equation?

10.13 What is the process of introducing the independent variables t^* and x^* and the dependent variable w^* called?

10.14 How many boundary conditions are required to determine the response of

(a) A beam undergoing transverse vibrations?
(b) A bar undergoing longitudinal vibrations?
(c) A shaft undergoing torsional oscillations?

10.15 What does the boundary condition $\frac{\partial \theta}{\partial x}(L, t)$ mean physically when applied to a torsional shaft.

10.16 What are the boundary conditions for the free vibrations of a longitudinal bar fixed at $x = 0$ and free at $x = L$?

10.17 What are the boundary conditions for the free vibrations of a torsional shaft fixed at $x = 0$ and attached to a thin disk with a mass moment of inertia I at $x = L$?

10.18 What are the boundary conditions for the free vibrations of a torsional shaft free at $x = 0$ and attached to a thin disk with a mass moment of inertia I and a torsional spring of stiffness k_t at $x = L$?

10.19 What are the boundary conditions for the free vibrations of a string fixed at $x = 0$ and attached to a spring of stiffness k at $x = L$?

10.20 What is the relationship between a nondimensional natural frequency and the corresponding dimensional natural frequency for a torsional shaft.

10.21 A bar with a length of L and cross-sectional area A is made of a material with an elastic modulus E and mass density ρ is fixed at $x = 0$ and has a rigid mass m attached at $x = L$. It has a longitudinal mode shape $X_k(x)$ which corresponds to a natural frequency ω_k. What is the normalization condition for this mode?

10.22 A bar with a length of L and cross-sectional area A is made of a material with an elastic modulus E and mass density ρ is fixed at $x = 0$ and is attached to a spring with a stiffness of k at $x = L$. The bar also has a longitudinal mode shape $X_k(x)$ which corresponds to a natural frequency ω_k. What is the normalization condition for this mode?

10.23 The differential equation for the vibrations of a beam is

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = f(x, t)$$

Explain the physical meaning of each term in the equation.

10.24 The characteristic equation for a fixed-free beam is $\cos \lambda^{1/4} \cosh \lambda^{1/4} = -1$. This is an example of a ______________ equation to solve for λ.

10.25 What are the boundary conditions for the free vibrations of a fixed-free beam?

10.26 What are the boundary conditions for the free vibrations of a free-free beam?

10.27 What are the boundary conditions for the free vibrations of a beam that is fixed at $x = 0$ and has a rigid mass m attached at $x = L$?

10.28 The characteristic equation for the fixed-pinned beam is the same as the characteristic equation for the pinned-free beam, yet their lowest natural frequency is different. How is this possible?

10.29 A bar with a length of L and cross-sectional area A is made of a material with an elastic modulus E and mass density ρ and has a normalized longitudinal mode shape $X_k(x)$ which corresponds to a natural frequency ω_k.

(a) What is the potential energy of a system that vibrates with this mode shape?
(b) What is the kinetic energy of a system that vibrates with this mode shape?

10.30 For Short Answer Problem 10.29 what is the value of $R(w)$?

10.31 A beam with a length L cross-sectional area A, and moment of inertia I is made of a material with an elastic modulus E and mass density ρ and has a normalized transverse mode shape $X_k(x)$ which corresponds to a natural frequency ω_k.

(a) What is the potential energy of a system that vibrates with this mode shape?
(b) What is the kinetic energy of a system that vibrates with this mode shape?

10.32 For Short Answer Problem 10.31 what is the value of $R(w)$?

Problems 10.33 through 10.47 require short calculations.

10.33 What is the wave speed for torsional oscillations in a circular shaft made from steel? The shaft is of length 60 cm and has a radius of 3 cm.

10.34 Calculate the wave speed of longitudinal waves in a 3-m long steel bar ($E = 210 \times 10^9\ \mathrm{N/m^2}$, $\rho = 7580\ \mathrm{kg/m^3}$) with a circular cross section of a 20 mm radius.

10.35 Calculate the three lowest natural frequencies of a solid 20-cm radius steel shaft ($G = 80 \times 10^9\ \mathrm{N/m^2}$, $\rho = 7500\ \mathrm{kg/m^3}$) with a length of 1.5 m that is fixed at one end and free at its other end.

10.36 The characteristic equation for a fourth-order continuous system is $\cos \lambda = 0$. What is the lowest natural frequency of the system?

10.37 What are the three lowest positive values of λ that satisfy the equation $\tan \lambda = 6/\lambda$?

10.38 What are the three lowest positive values of λ that satisfy the equation $\tan \lambda = 4\lambda$?

10.39 The nondimensional mode shape of a uniform bar is $\sin 5\pi x$.

(a) Determine the potential energy of this mode.
(b) Determine the kinetic energy of this mode.
(c) What is the nondimensional natural frequency that corresponds to this mode?

10.40 The nondimensional mode shape of a beam is $\sqrt{2}\sin 3\pi x$.

(a) Determine the potential energy of this mode.
(b) Determine the kinetic energy of this mode.
(c) What is the nondimensional natural frequency that corresponds to this mode?

10.41 A circular bar with a length of 80 cm and radius of 3 cm is made of steel which has an elastic modulus 200×10^9 N/m^2 and mass density 7600 kg/m^3. The bar has a mode shape of $X(x) = 33.91 \cos 13.74x$.

(a) Determine the potential energy of this mode.
(b) Determine the kinetic energy of this mode.
(c) What is the natural frequency that corresponds to this mode?

10.42 A carbon nanotube ($E = 1\,GPa$, $\rho = 2.3$ g/cm^3) has a length of 200 nm and radius of 5 nm. Using a fixed-free beam model for the nanotube, calculate its first four natural frequencies.

10.43–10.45 Each of the beams of Figures SP10.43 through SP10.45 is made from a material of $E = 210 \times 10^9$ N/m^2 and $\rho = 7580$ kg/m^3 with $A = 1.2 \times 10^{-2}$ m^2, $I = 4.0 \times 10^{-5}$ m^4, and $L = 1.4$ m. Use Table 10.4 to calculate the beam's three lowest natural frequency of transverse vibrations.

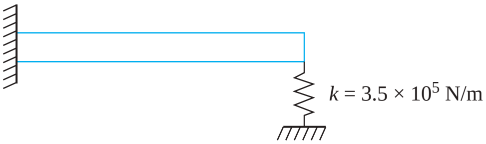


FIGURE SP10.43

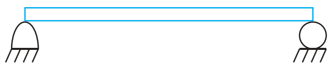


FIGURE SP10.44

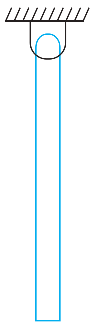


FIGURE SP10.45

10.46 Find all non-trivial solutions to the boundary value problem

$$\frac{d^2X}{dx^2} + \lambda X = 0 \qquad X'(0) = 0 \qquad X'(1) = 0$$

10.47 Find all non-trivial solutions to the boundary value problem

$$\frac{d^4X}{dx^4} - \lambda X = 0 \qquad X(0) = 0 \qquad X''(0) = 0 \qquad X(1) = 0 \qquad X''(1) = 0$$

10.48 Specify the SI units of the given quantity.

(a) Wave speed of longitudinal vibrations in a bar, c
(b) Flexural rigidity of a beam, EI
(c) Natural frequency of sixth mode, ω_6
(d) Nondimensional natural frequency of first mode, ω_1
(e) Rayleigh's quotient, $R(w)$
(f) Inertia term in the beam equation, $\rho A \frac{\partial^2 w}{\partial t^2}$
(g) Kinetic energy of a bar, $T = \int_0^L \rho A (\frac{\partial w}{\partial t})^2 dx$

CHAPTER PROBLEMS

10.1 A 5000 N · m torque is statically applied to the free end of a solid 20-cm radius steel shaft ($G = 80 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3) with a length of 1.5 m that is fixed at one end and free at its other end. The torque is suddenly removed, and torsional oscillations begin. Plot the time-dependent oscillations of the free end of the shaft.

10.2 A 5000 N · m torque is statically applied to a the midspan of a solid 20-cm radius steel shaft ($G = 80 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3)with a length of 1.5 m that is fixed at one end and free at its other end. The torque is suddenly removed, and torsional oscillations begin. Determine an expression for the time-dependent angular displacement of the free end of the shaft.

10.3 A steel shaft ($\rho = 7850$ kg/m^3, $G = 85 \times 10^9$ N/m^2) with a inner radius of 30 mm, outer radius of 50 mm, and length of 1.0 m is fixed at both ends. Determine the three lowest natural frequencies of the shaft.

10.4 A 10,000-N · m torque is applied to the midspan of the shaft of Chapter Problem 10.3 and suddenly removed. Determine the time-dependent angular displacement of the midspan of the shaft.

10.5 A motor of mass moment of inertia 85 kg · m^2 is attached to the end of the shaft of Chapter Problem 10.1. Determine the three lowest natural frequencies of the shaft and motor assembly. Compare the lowest natural frequency to that obtained by making a one-degree-of-freedom model and approximating the inertia effects of the shaft.

10.6 Show the orthogonality of the two lowest mode shapes of the system in Chapter Problem 10.5.

10.7 Operation of the motor attached to the shaft of Chapter Problem 10.5 produces a harmonic torque of amplitude 2000 N · m at a frequency of 110 Hz. Determine the steady-state angular displacement of the end of the shaft.

10.8 A 20-cm-diameter, 2-m-long steel shaft ($\rho = 7600$ kg/m^3, $G = 80 \times 10^9$ N/m^2) has rotors of mass moment of inertia 110 kg · m^2 and 65 kg · m^2 attached to its ends. Determine the three lowest natural frequencies of the shaft. Compare the lowest nonzero natural frequency to that obtained by using a two-degree-of-freedom model, ignoring the inertia of the shaft.

10.9 Determine an expression for the natural frequencies of the shaft of Figure P10.9.

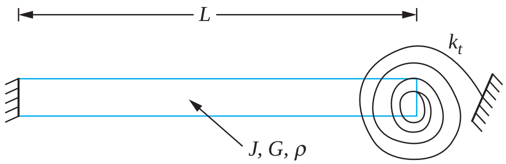


FIGURE P10.9

10.10 An oil well drilling tool is modeled as a bit attached to the end of a long shaft, unrestrained from rotation at its fixed end.

(a) Determine the equation defining the natural frequencies of the drilling tool.
(b) For a particular operation, the shaft ($\rho = 7500 \text{ kg/m}^3$, $G = 80 \times 10^9 \text{ N/m}^2$) is 20 m long with a 20-cm diameter. The tool operates at a speed of 400 rad/s. What are the limits on the moment of inertia of the drill bit such that the two lowest nonzero natural frequencies of the tool are not within 20 percent of the operating speed?

10.11 The shaft of Chapter Problem 10.1 is at rest in equilibrium when the time-dependent moment of Figure P10.11 is applied to the end of the shaft. Determine the time-dependent form of the resulting torsional oscillations.

10.12 The shaft of Chapter Problem 10.1 is at rest in equilibrium when it is subject to the uniform time-dependent torque loading per unit length of Figure P10.12. Determine the time-dependent form of the resulting torsional oscillations.

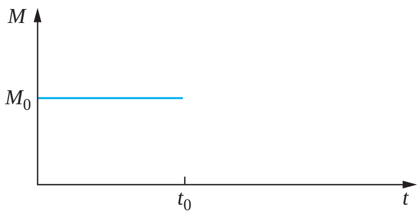


FIGURE P10.11

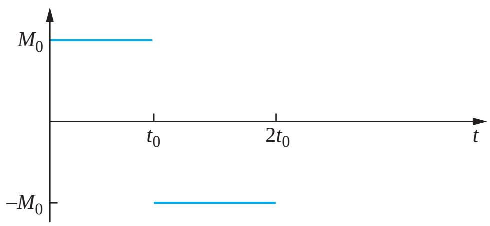


FIGURE P10.12

10.13 The elastic bar of Figure P10.13 is undergoing longitudinal vibrations. Let $u(x, t)$ be the time-dependent displacement of a particle along the centroidal axis of the bar, initially a distance x from the left support.

(a) Draw free-body diagrams showing the external and effective forces acting on a differential element of thickness dx, a distance x from the left end of the bar at an arbitrary instant of time.
(b) Show that the governing partial differential equation is

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

(c) Introduce nondimensional variables to derive a nondimensional partial differential equation.

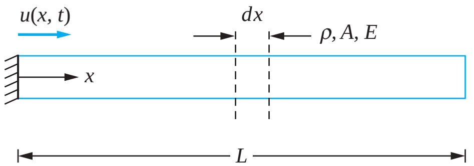


FIGURE P10.13

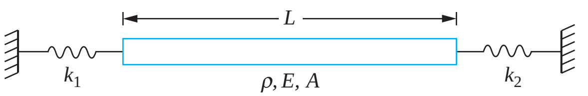
10.14 Using the results of Chapter Problem 10.13, determine the natural frequencies of longitudinal vibrations of a bar fixed at one end and free at the other.

10.15 Show the orthogonality of mode shapes of longitudinal vibration of a bar fixed at one end and free at its other end.

10.16 A large industrial piston operates at 1000 Hz. The piston head has a mass of 20 kg. The shaft is made from steel ($\rho = 7500\ \text{kg/m}^3$, $E = 210 \times 10^9\ \text{N/m}^2$). For what shaft diameters will all natural frequencies be out of the range of 900 to 1100 Hz?

10.17 The free end of the piston of Chapter Problem 10.16 is subject to a force $1000 \sin \omega t$ N, where $\omega = 100$ Hz. If the diameter of the shaft is 8 cm, determine the steady-state response of the piston.

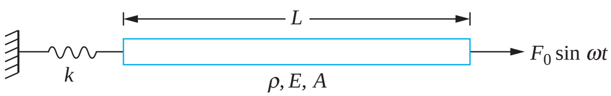
10.18 Determine the five lowest natural frequencies of the system of Figure P10.18.



$\rho = 7500\ \text{kg/m}^3$
$E = 200 \times 10^9\ \text{N/m}^2$
$A = 1.5 \times 10^{-5}\ \text{m}^2$
$L = 3\ \text{m}$
$k_1 = 1 \times 10^6\ \text{N/m}$
$k_2 = 1.5 \times 10^6\ \text{N/m}$

FIGURE P10.18

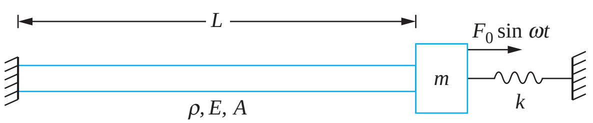
10.19 Determine the steady-state response of the system of Figure P10.19.



$\rho = 7500\ \text{kg/m}^3$
$E = 200 \times 10^9\ \text{N/m}^2$
$A = 4.5 \times 10^{-5}\ \text{m}^2$
$k = 9 \times 10^5\ \text{N/m}$
$F_0 = 800\ \text{N}$
$\omega = 100\ \text{rad/s}$
$L = 3.3\ \text{m}$

FIGURE P10.19

10.20 Determine the steady-state response of the system of Figure P10.20.



$\rho = 7500\ \text{kg/m}^3$
$E = 200 \times 10^9\ \text{N/m}^2$
$A = 4.5 \times 10^{-5}\ \text{m}^2$
$k = 9 \times 10^5\ \text{N/m}$
$m = 2.5\ \text{kg}$
$L = 3.5\ \text{m}$
$F_0 = 600\ \text{N}$
$\omega = 450\ \text{rad/s}$

FIGURE P10.20

10.21 Draw frequency response curves for the response of the disk at the end of the shaft in Example 10.3. Plot the curves for $\beta = 0.5$, $\beta = 2$, and $\beta = 20.0$.

10.22 Determine the steady-state response of a circular shaft subject to a uniform torque per unit length $T_0 \sin \omega t$ applied over its entire length.

10.23 Determine the steady-state response of the system of Figure P10.23.

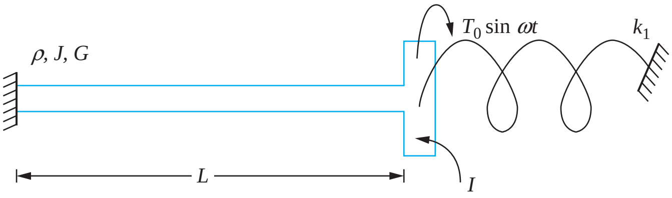


FIGURE P10.23

10.24 Propeller blades totaling 1200 kg with a total mass moment of inertia of 155 kg · m² are attached to a solid circular shaft ($\rho = 5000$ kg/m³, $G = 60 \times 10^9$ N/m², $E = 140 \times 10^9$ N/m²) of radius 40 cm and length 20 m. The other end of the shaft is fixed in an ocean liner. Determine

(a) The lowest natural frequency for torsional oscillations of the propeller.
(b) The lowest natural frequency for longitudinal motion of the propeller.

10.25 A pipe used to convey fluid is cantilevered from a wall. The steel pipe ($\rho = 7500$ kg/m³, $G = 80 \times 10^9$ N/m², $E = 200 \times 10^9$ N/m²) has an inner radius of 20 cm, a thickness of 1 cm, and a length of 4.6 m. For an empty pipe determine

(a) The five lowest natural frequencies for torsional oscillation.
(b) The five lowest natural frequencies for longitudinal vibration.
(c) The five lowest natural frequencies for transverse motion.

10.26 Verify the characteristic equation given in Table 10.4 for a pinned-free beam.

10.27 Verify the characteristic equation given in Table 10.4 for a fixed-fixed beam.

10.28 Verify the orthogonality of the eigenfunctions given in Table 10.4 for a pinned-free beam.

10.29 Verify the orthogonality of the eigenfunctions given in Table 10.4 for a fixed-attached mass beam.

10.30–10.34 Determine the time-dependent displacement for the beam shown in Figures P10.30 through P10.34.

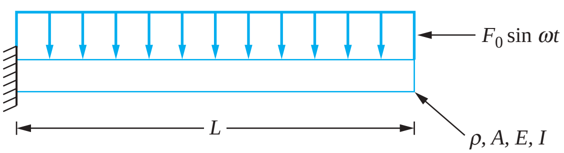


FIGURE P10.30

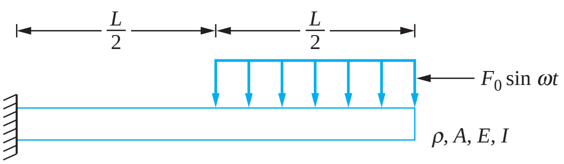


FIGURE P10.31

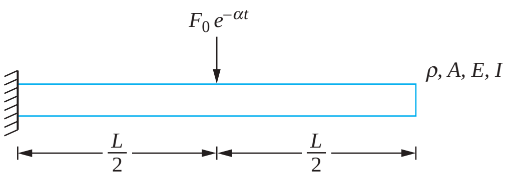


FIGURE P10.32

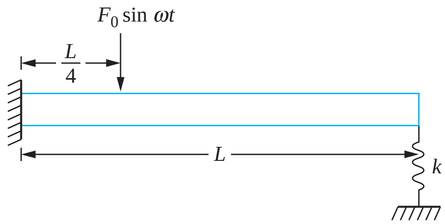


FIGURE P10.33

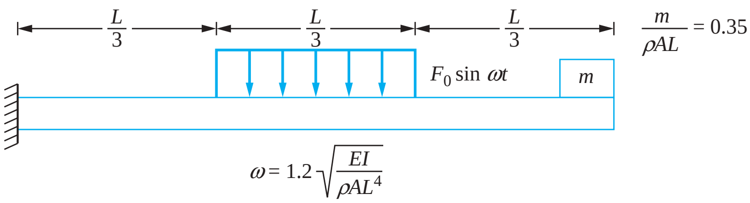


FIGURE P10.34

10.35 A root manipulator is 60 cm long, made of steel ($E = 210 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3) and has the cross section of Figure P10.35. One end of the manipulator is fixed and a 1-kg mass is attached to its opposite end. Determine the three lowest natural frequencies for transverse vibration of the manipulator.

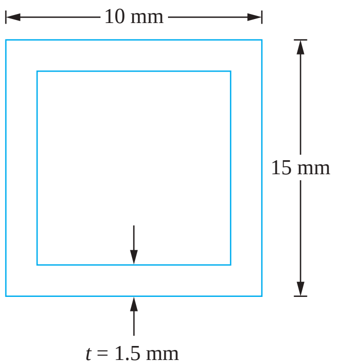


FIGURE P10.35

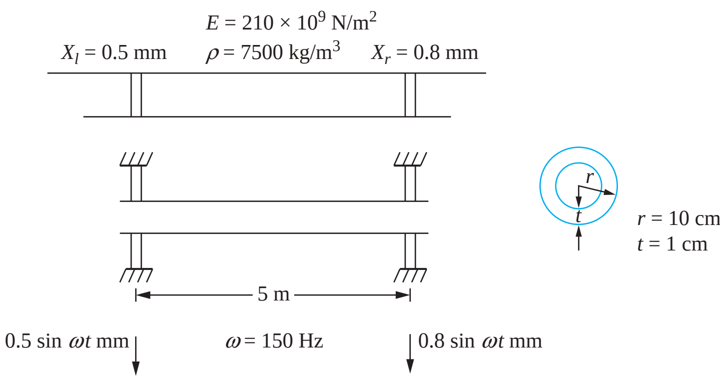


FIGURE P10.36

10.36 The steam pipe of Figure P10.36 is suspended from the ceiling in an industrial plant. A heavy machine with a rotating unbalance is placed on the floor above the machine causing vibrations of the ceiling. If the frequency of the oscillations is 150 Hz and the amplitude of displacement of the pipe's left support is 0.5 mm and the amplitude of displacement of the pipe's right support is 0.8 mm, determine the maximum deflection of the center of the pipe. The pipe is modeld as a simply supported beam of length 5 m and has the cross section shown in Figure P10.36.

10.37 A simplified model of the rocket of Figure P10.37 is a free-free beam.

(a) Calculate the five lowest natural frequencies for longitudinal vibration.
(b) Calculate the five lowest natural frequencies for transverse vibration.

FIGURE P10.37

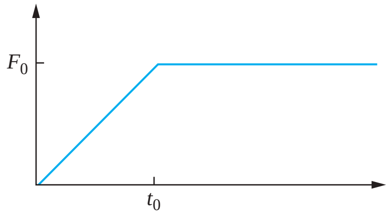


FIGURE P10.38

10.38 Longitudinal vibrations are initiated in the rocket of Figure P10.38 when thrust is developed. Determine the Laplace transform of the transient response U(x, s) when the thrust of Figure P10.38 is developed. Do not invert the transform.

10.39 Determine the response of a cantilever beam when the fixed support is subject to a displacement $f(t) = A \sin \omega t$. Use the Laplace transform method and determine the transform $W(x, s)$. Do not invert.

10.40 The tail rotor blades of a helicopter have a rotating unbalance of magnitude 0.5 kg · m and operate at a speed of 1200 rpm. Modeling the tail section as a cantilever beam of length 3.5 m with $E = 31 \times 10^6$ N · m^2, determine the steady-state response of the tail section.

10.41 Determine the steady-state amplitude of the engine of Figure P10.41.

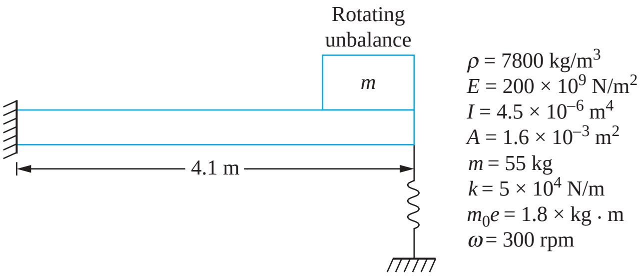


FIGURE P10.41

10.42 Show that the differential equation governing free vibration of a uniform beam subject to a constant axial load, P, is

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial w}{\partial t^2} = 0$$

10.43 Determine the frequency equation for a simply supported beam subject to an axial load.

10.44 Determine the frequency equation for a fixed-pinned beam subject to an axial load.

10.45 A fixed-fixed beam is made of a material with a coefficient of thermal expansion α. After installed, the temperature is decreased by ΔT. Determine the beam's frequency equation.

10.46 Show orthogonality of the mode shapes for a simply supported beam subject to an axial load.

10.47 Use Rayleigh's quotient to approximate the lowest natural frequency of a torsional shaft fixed at both ends.

10.48 Use Rayleigh's quotient to approximate the lowest natural frequency of a torsional shaft with a disk of mass moment of inertia I placed at its midspan. The shaft is fixed at both ends.

10.49 Use Rayleigh's quotient to approximate the lowest natural frequency of a fixed-free beam.

10.50 Use Rayleigh's quotient to approximate the lowest natural frequency of a simply supported beam with a mass m at its midspan. Use $w(x) = \sin(\pi x/L)$ as the trial function.

10.51 Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of a fixed-free beam.

10.52 Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of the system of Figure P10.52.

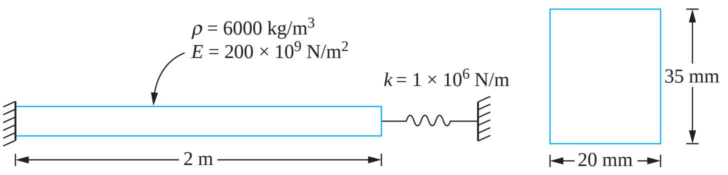


FIGURE P10.52

10.53 Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies for the system of Figure P10.53.

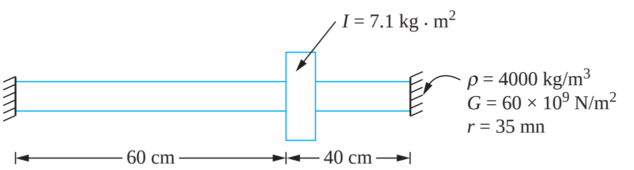


FIGURE P10.53

10.54 Use the Rayleigh-Ritz method to approximate the three lowest natural frequencies of a fixed-pinned beam. Use polynomial of order six or less as trial functions.

10.55 Use the Rayleigh-Ritz method to approximate the three lowest natural frequencies and their corresponding mode shapes of a fixed-free beam. Use polynomials of order six or less as trial functions.

10.56 Use the Rayleigh-Ritz method to approximate the two lowest frequencies of transverse vibration of the system of Figure P10.56.

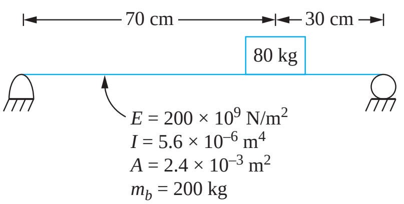


FIGURE P10.56

Chapter 11

FINITE-ELEMENT METHOD

11.1 INTRODUCTION

The finite-element method is a powerful numerical method that is used to provide approximations to solutions of static and dynamic problems for continuous systems. The disciplines in which the finite-element method can be applied include stress analysis, heat transfer, electromagnetics, fluid flow, and vibrations. Application of the finite-element method to a continuous system requires the system be divided into a finite number of discrete elements. Interpolations for the dependent variables are assumed across each element and are chosen to assure appropriate interelement continuity. The interpolating functions are developed in terms of the unknown values of the dependent variables at discrete points, called *nodes*. The nodes for a one-dimensional system are located at element boundaries. A variational principle is applied to derive equations whose solution leads to approximations to the dependent variables at the nodes. The defined interpolations are used to provide approximations to the dependent variables across the system. Lagrange's equations, derived using the application of calculus of variations, is applied for vibrations problems, resulting in a set of differential equations for the dependent variables at the nodes.

The finite-element method for vibration problems could be derived by applying the Rayleigh-Ritz method of Section 10.5 with the interpolating functions, $u_1(x)$, $u_2(x)$, . . . , $u_n(x)$, chosen to be defined piecewise over each element. Consider application of the Rayleigh-Ritz method to approximate the natural frequencies and mode shapes of a bar. The governing equation, the wave equation, has second-order spatial derivatives. Thus the exact solution is at least twice differentiable. However, the energy scalar products used in evaluation of Rayleigh's quotient, given in Table 10.6, only require that approximate solutions be first-order differentiable. Thus functions that are only first-order differentiable are permissible interpolating functions for Rayleigh-Ritz approximations.

Boundary conditions for continuous systems are classified as being of two types. *Geometric boundary conditions* are those that must be satisfied according to geometric constraints. For example, $u(0) = 0$, if $x = 0$ is a fixed support for a bar problem, is a geometric boundary condition. *Natural boundary conditions* are those that must be satisfied as a result of force and moment balances. For example, $du/dx = 0$ at $x = 1$, if $x = 1$ is a free end, is a natural boundary condition. This condition occurs because there is no external force at the free end. Note from Table 10.6 that the energy scalar product definitions include terms arising because of natural boundary conditions. Thus, since the natural boundary conditions are incorporated into the energy scalar products, the chosen interpolating functions for a Rayleigh-Ritz approximation must satisfy only geometric boundary conditions.

The set of *admissible functions* for use as interpolating functions in a Rayleigh-Ritz approximation to solutions of the wave equation consists of those that are first-order differentiable and satisfy all geometric boundary conditions (displacement conditions). By similar arguments, it is determined that the set of admissible functions for use as basis functions in a Rayleigh-Ritz approximation to solutions of the beam equation consists of those that are second-order differentiable and satisfy all geometric boundary conditions (displacement and slope conditions).

The Rayleigh-Ritz can be difficult to apply for vibrations problems. The *assumed modes method*, introduced in the next section, is based on application of Lagrange's equations and leads to the same approximation for the same set of interpolating functions as the Rayleigh-Ritz method. The finite-element method will be developed from the assumed modes method.

11.2 ASSUMED MODES METHOD

Consider the forced vibrations of a longitudinal bar of Figure 11.1. The displacement u is a function of the spatial coordinate x and time t, $u(x, t)$. Let $u_1(x), u_2(x), \ldots, u_n(x)$ be a set of n linearly independent functions that are at least first-order differentiable and satisfy all of the system's geometric boundary conditions. An approximate solution is assumed as

$$u(x, t) = \sum_{i=1}^{n} w_i(t) u_i(x) \tag{11.1}$$

The kinetic energy of the bar, according to the approximation of Equation (11.1), is calculated as

$$\begin{aligned}
T &= \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}m\left[\frac{\partial u}{\partial t}(L)\right]^2 \\
&= \frac{1}{2}\int_0^L \rho A\left(\sum_{i=1}^{n} \dot{w}_i u_i(x)\right)^2 dx + \frac{1}{2}m\left(\sum_{i=1}^{n} \dot{w}_i u_i(L)\right)^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \dot{w}_i \dot{w}_j\left[\int_0^L \rho A u_i(x) u_j(x)\, dx + m u_i(L) u_j(L)\right]
\end{aligned}$$

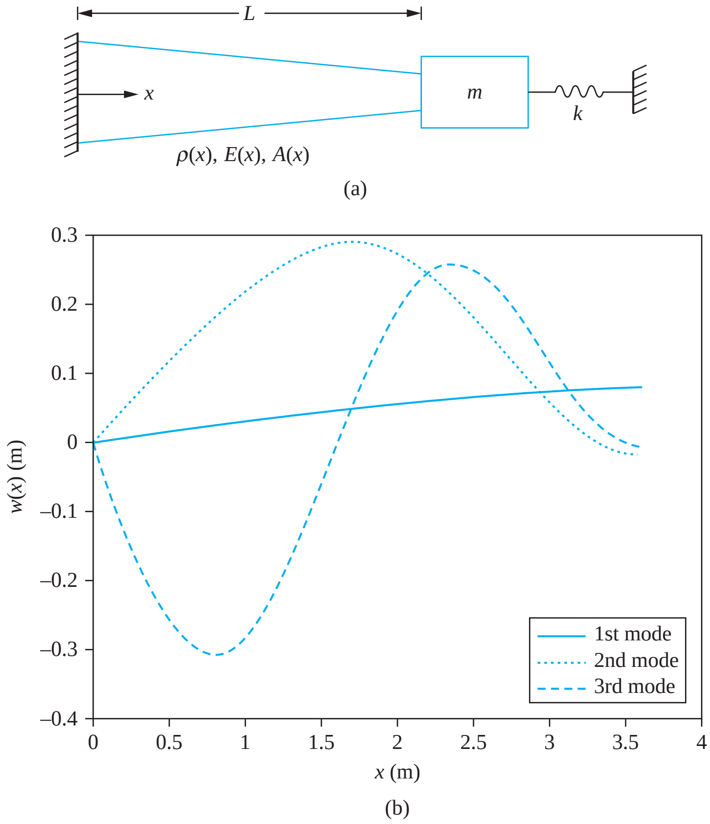


FIGURE 11.1 (a) Forced longitudinal vibrations of a bar are described by a displacement function $u(x, t)$. (b) Assumed mode approximations to mode shapes.

Thus, the kinetic energy has the quadratic form

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{w}_i\dot{w}_j \tag{11.2}$$

where

$$m_{ij} = \int_0^L \rho A u_i(x) u_j(x)\, dx + m u_i(L) u_j(L) \tag{11.3}$$

The potential energy of the system, according to the approximation of Equation (11.1), is

$$\begin{aligned} V &= \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}k[u(L)]^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j\left[\int_0^L EA\frac{du_i}{dx}\frac{du_j}{dx}\, dx + k u_i(L) u_j(L)\right] \end{aligned} \tag{11.4}$$

The potential energy has the quadratic form

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} w_i w_j \tag{11.5}$$

where

$$k_{ij} = \int_0^L EA\frac{du_i}{dx}\frac{du_j}{dx}dx + ku_i(L)u_j(L) \tag{11.6}$$

The virtual work done by the external force $f(x, t)$ due to a virtual displacement $\delta u(x, t)$ is

$$\delta W = \int_0^L f(x, t)\delta u(x, t)\,dx = \sum_{i=1}^{n} \delta w_i \int_0^L f(x, t)u_i(x)\,dx \tag{11.7}$$

The virtual work can be written as

$$\delta W = \sum_{i=1}^{n} Q_i \delta w_i \tag{11.8}$$

where

$$Q_i = \int_0^L f(x, t)u_i(x)\,dx \tag{11.9}$$

The assumed modes method approximates the solution to the forced vibrations of a continuous system with n degrees of freedom. The generalized coordinates for the n-degree-of-freedom model are the coefficient functions $w_1(t), w_2(t), \ldots, w_n(t)$. Quadratic forms of the kinetic and potential energies in terms of the generalized coordinates have been obtained. Use of Lagrange's equations, as applied in Section 7.4 to linear systems with quadratic energy forms, leads to differential equations of the form

$$\mathbf{M}\ddot{\mathbf{w}} + \mathbf{K}\mathbf{w} = \mathbf{F} \tag{11.10}$$

where the elements of the mass matrix **M** are the coefficients of Equation (11.3), the elements of the stiffness matrix **K** are the coefficients of Equation (11.6), and the elements of the force vector **F** are the generalized forces of Equation (11.9). If scalar product notation is used

$$m_{ij} = (u_i, u_j)_T \qquad k_{ij} = (u_i, u_j)_V \qquad Q_i = (f, u_i) \tag{11.11}$$

Approximations to the n lowest natural frequencies are obtained as the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The corresponding mode shape vectors are used in Equation (11.1) to approximate the mode shapes for these frequencies. An approximation to the forced response is obtained by solving Equation (11.10) using the methods of Chapter 9.

EXAMPLE 11.1

Use the assumed modes method to approximate the three lowest natural frequencies and mode shapes for the bar of Figure 11.1(a) with $A(x) = 0.001(1 - 0.002x)$ m^2, $E = 200 \times 10^9$ N/m^2, $\rho = 7600$ kg/m^3, $L = 3.6$ m, $m = 12$ kg, and $k = 4 \times 10^7$ N/m. Use the interpolating functions

$$u_1(x) = \sin\left(\frac{\pi x}{2L}\right) \qquad u_2(x) = \sin\left(\frac{3\pi x}{2L}\right) \qquad u_3(x) = \sin\left(\frac{5\pi x}{2L}\right) \tag{a}$$

which are the first three mode shapes of a uniform fixed-free bar.

SOLUTION

Equations (11.3) and (11.6) are used to determine the elements of the mass and stiffness matrices, respectively, for the assumed modes approximation. For example,

$$m_{12} = \int_0^L \rho[0.001(1 - 0.002x)] \sin\left(\frac{\pi x}{2L}\right) \sin\left(\frac{3\pi x}{2L}\right) dx - m \tag{b}$$

$$k_{12} = \int_0^L E\,[0.001(1 - 0.002x)] \left(\frac{\pi}{2L}\right)\left(\frac{3\pi}{2L}\right) \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{3\pi x}{2L}\right) dx - k \tag{c}$$

A MATLAB script is written using symbolic algebra to determine the mass and stiffness matrices for this assumed modes approximation. The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The eigenvectors are used to develop approximations to the mode shapes. If $\mathbf{X}_1 = [X_{11}\ X_{12}\ X_{13}]^T$ is the eigenvector corresponding to the eigenvalue that gives an approximation to the lowest natural frequency, then the approximation to the corresponding mode shape is

$$w_1(x) = X_{11}u_1(x) + X_{12}u_2(x) + X_{13}u_3(x)$$

The natural frequency approximations are

$$\omega_1 = 1.86 \times 10^3 \text{ rad/s} \qquad \omega_2 = 4.99 \times 10^3 \text{ rad/s} \qquad \omega_1 = 9.72 \times 10^3 \text{ rad/s} \tag{d}$$

The mode shape approximations are illustrated in Figure 11.1(b).

11.3 GENERAL METHOD

Consider again the bar of Figure 11.1(a). The bar is divided into n discrete segments, or elements. For purposes of discussion assume the elements are of equal length $l = L/n$. The discretization of a uniform bar into n elements of equal length l is shown in Figure 11.2(a). The piecewise defined interpolating functions of Figure 11.2(b) are mathematically described as

$$\begin{aligned}
u_0(x) &= \left(-\frac{x}{l} + 1\right)u(x - l) \\
u_j(x) &= \left[\frac{x}{l} + (1 - j)\right][u(x - (j - 1)l) - u(x - jl)] \\
&\quad + \left[-\frac{x}{l} + (1 + j)\right][u(x - jl) - u(x - (j + 1)l)] \qquad 1 \leq j < n \\
u_n(x) &= \left[\frac{x}{l} + (1 - n)\right]u(x - (n - 1)l)
\end{aligned} \tag{11.12}$$

When the functions of Equation (11.12) are used in an assumed modes approximation of the form

$$u(x, t) = \sum_{i=0}^{n} W_i(t)u_i(x) \tag{11.13}$$

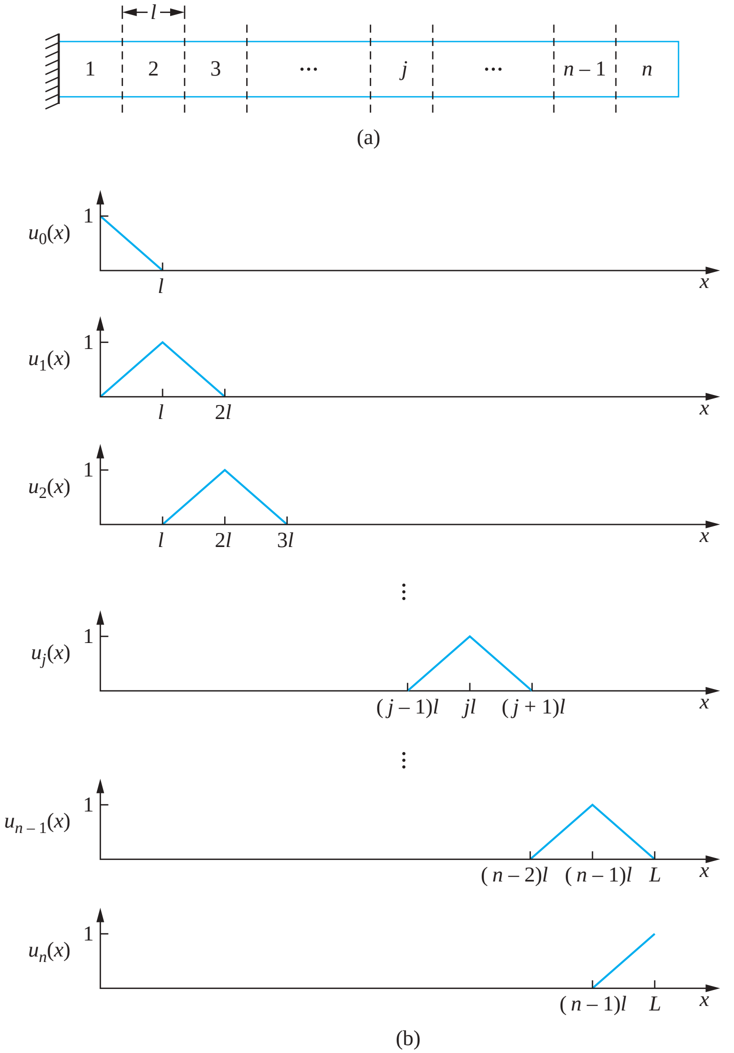


FIGURE 11.2
(a) Discretization of uniform bar into n elements of equal length l. (b) Interpolating functions that can be used in an assumed modes approximation.

then

$$u(jl, t) = W_j \tag{11.14}$$

Thus, the generalized coordinates are the displacements at the element boundaries. The geometric boundary condition $w(0, t) = 0$ can be imposed simply by setting $W_0 = 0$.

The finite-element method is an application of the assumed modes method using piecewise-defined basis functions. The basis function $u_j(x)$ is nonzero only over the jth and $(j + 1)$st elements. The assumed modes method as described in Section 11.2 is applied. The mass matrix is developed from the kinetic energy, the stiffness matrix is developed from the potential energy, and the force vector is developed from the virtual work of the external forces. As a result of the piecewise definition of the interpolating functions, it is noted that $m_{ij} = (u_i, u_j)_T = 0$ unless $i = j - 1, j$, or $j + 1$.

When the interpolating functions of Equation (11.12) are used in the assumed modes method, Equation (11.13) can be rearranged to

$$u(x, t) = \sum_{i=1}^{n} \phi_i(x, t) \tag{11.15}$$

where

$$\begin{aligned}
\phi_1(x, t) &= W_1(t)\frac{x}{l}[u(x) - u(x - l)] \\
\phi_j(x, t) &= \frac{1}{l}[(W_{j+1}(t) - W_j(t))x + (j + 1)lW_j(t) - jlW_{j+1}(t)] \\
&\quad [u(x - jl) - u(x - (j + 1)l)] \quad 2 \le j < n \\
\phi_n(x) &= \left[\frac{1}{l}(W_n - W_{n-1})x + W_n(1 - n) + nW_{n-1}\right][u(x - (n-1)l) - u(x - nl)]
\end{aligned} \tag{11.16}$$

Equations (11.15) and (11.16) are illustrated in Figure 11.3. Equation (11.15) rewrites the assumed modes approximation as a linear combination of functions that are each nonzero only over one element. The functions are in terms of the displacements at the element boundaries. Application of the finite-element method is used to obtain approximations to the displacements of the nodes (the element boundaries). Figure 11.3 illustrates that the finite-element method, as applied to this problem, assumes a linear interpolation between the nodal displacements.

Often a large number of elements are required to obtain accurate results for complex structures. Application of the finite-element method is more convenient when formulated as in Equation (11.15). This allows an approximation function to be defined for each element in terms of the displacements at the element boundaries. The kinetic energy, potential energy, and work done by external forces are calculated for the element in terms of the generalized coordinates representing displacements at element boundaries. For example, the kinetic energy of element j can be written in the quadratic form

$$T_j = \frac{1}{2}\dot{\mathbf{w}}_j^T\mathbf{m}_j\dot{\mathbf{w}}_j \tag{11.17}$$

where $\mathbf{w} = [W_{j-1}\ W_j]^T$ is the element displacement vector, the vector of boundary displacements, written in terms of global generalized coordinates and $\mathbf{m}_j$ is the element mass matrix written in local coordinates. The total kinetic energy of the system is the sum of the element kinetic energies

$$T = \sum_{j=1}^{n} T_j \tag{11.18}$$

and has the quadratic form

$$T_j = \frac{1}{2}\dot{\mathbf{W}}^T\mathbf{M}\dot{\mathbf{W}} \tag{11.19}$$

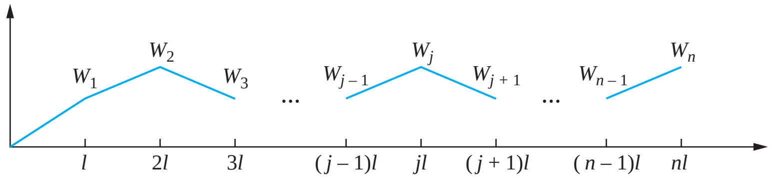


FIGURE 11.3
Linear interpolation between nodes for finite-element model of a bar.

where $\mathbf{W} = [W_1\ W_2\ \ldots\ W_n]^T$ is the global displacement vector, the vector of generalized coordinates, and $\mathbf{M}$ is the global mass matrix.

The above development suggests a computational procedure where the energy methods are used to develop the finite-element model. The system is divided into a finite number of discrete elements. The *global generalized coordinates* are the coordinates representing the degrees of freedom at the nodes. Each element has a specific number of degrees of freedom. The bar element, for example, has two degrees of freedom, the displacements of the ends of the element. A *local coordinate system* is defined for each element in the finite-element model. The kinetic energy, potential energy, and virtual work are determined for each element. The potential energy, for example, is written in quadratic form in terms of an element displacement vector and element stiffness matrix. Model elements for a bar, a torsional system, and a beam are developed in this fashion. The element mass and stiffness matrices are assembled into global mass and stiffness matrices. The differential equations are written in terms of the global generalized coordinates by using the global matrices. The homogeneous solution of the differential equations provides approximations to the natural frequencies and mode shapes. Nonhomogeneous equations are solved to provide approximation to the forced response.

The following sections provide the details of the method. The standard bar element and standard beam element, written in terms of local coordinates, are developed. Methods of assembling the element matrices into global matrices are discussed. Examples of application of the finite-element method to bar, beam, and truss problems are presented.

This chapter provides only an overview of the finite-element method. There is much more to the method that is beyond the scope of the discussion. This includes error analysis, element selection, substructuring, and algorithm development. Many excellent finite-element software packages exist.

11.4 THE BAR ELEMENT

A bar element of length l is illustrated in Figure 11.4. The element has two degrees of freedom represented by w_1, the displacement of the left end of the element, and w_2, the displacement of the right end of the element. Define a local coordinate ξ, $0 \leq \xi \leq l$, along the axis of element. The linear displacement function for the element is

$$u(\xi, t) = \frac{1}{l}(w_2 - w_1)\xi + w_1 \tag{11.20}$$

The kinetic energy of the element, assuming uniform properties, is

$$\begin{aligned} T &= \frac{1}{2}\int_0^l \rho A\left(\frac{\partial u}{\partial t}\right)^2 d\xi \\ &= \frac{1}{2}\int_0^l \rho A\left[\frac{1}{l}(\dot{w}_2 - \dot{w}_1)\xi + \dot{w}_1\right]^2 d\xi \\ &= \frac{1}{2}\frac{\rho A l}{3}(\dot{w}_1^2 + \dot{w}_1\dot{w}_2 + \dot{w}_2^2) \end{aligned} \tag{11.21}$$

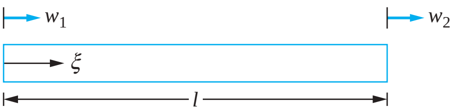


FIGURE 11.4
A bar element of length l has two degrees of freedom. A linear function in terms of local coordinate ξ interpolates the displacement.

Equation (11.21) can be rewritten in the quadratic form

$$T = \frac{1}{2}\dot{\mathbf{w}}^T\mathbf{m}\dot{\mathbf{w}} = \frac{1}{2}\frac{\rho A l}{6}[\dot{w}_1 \quad \dot{w}_2]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} \tag{11.22}$$

Thus the element mass matrix is

$$\mathbf{m} = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{11.23}$$

The potential energy of the element, assuming uniform properties, is

$$\begin{aligned} V &= \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial \xi}\right)^2 d\xi \\ &= \frac{1}{2}\int_0^l EA\left[\frac{1}{l}(w_2 - w_1)\right]^2 d\xi \\ &= \frac{1}{2}\frac{EA}{l}(w_2^2 - 2w_1 w_2 + w_1^2) \end{aligned} \tag{11.24}$$

The potential energy can be written in the quadratic form

$$V = \frac{1}{2}\frac{EA}{l}[w_1 \quad w_2]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{11.25}$$

from which the element stiffness matrix is determined as

$$\mathbf{k} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{11.26}$$

If the element has an external axial load $f(\xi, t)$, then the virtual work done by the load is

$$\begin{aligned} \delta W &= \int_0^l f(\xi, t)\,\delta u(\xi, t)\, d\xi \\ &= \int_0^l f(\xi, t)\left[\frac{1}{l}(\delta w_2 - \delta w_1)\xi + \delta w_1\right] d\xi \\ &= \delta w_1 \int_0^l f(\xi, t)\left(1 - \frac{\xi}{l}\right) d\xi + \delta w_2 \int_0^l f(\xi, t)\frac{1}{l}\, d\xi \end{aligned} \tag{11.27}$$

and the element generalized forces are

$$q_1 = \int_0^l f(\xi, t)\left(1 - \frac{\xi}{l}\right) d\xi \qquad q_2 = \int_0^l f(\xi, t)\frac{1}{l}\, d\xi \tag{11.28}$$

The torsion element of Figure 11.5 is developed in the same manner as the bar element. If w_1 is the angular displacement at the left end of the element and w_2 the angular displacement at the right end of the element, then the finite-element approximation to the

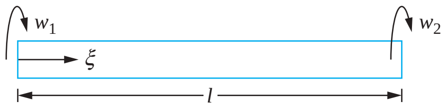


FIGURE 11.5
Uniform torsion element of length l has two degrees of freedom represented by w_1 and w_2, angular displacements at the ends of the element.

angular displacement over the element is given by

$$\theta(\xi, t) = \left(1 - \frac{\xi}{l}\right)w_1 + \frac{1}{l}w_2 \tag{11.29}$$

Application of Equation (11.29) to the kinetic energy

$$T = \frac{1}{2}\int_0^l \rho J\left(\frac{\partial \theta}{\partial t}\right)^2 d\xi \tag{11.30}$$

leads to the element mass matrix

$$\mathbf{m} = \frac{\rho J l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{11.31}$$

Application of Equation (11.29) to the potential energy

$$V = \frac{1}{2}\int_0^l JG\left(\frac{\partial \theta}{\partial \xi}\right)^2 d\xi$$

leads to the element stiffness matrix

$$\mathbf{k} = \frac{JG}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{11.32}$$

EXAMPLE 11.2

Use a one-element, finite-element model to approximate the lowest natural frequencies and mode shapes of a free-free bar.

SOLUTION

The displacements of the ends of the bar are the two generalized coordinates. The differential equations for the model are

$$\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{a}$$

The approximations to the natural frequencies are obtained by assuming a normal-mode solution of Equation (a) as $\mathbf{w} = [1 \quad \chi]^T e^{i\omega t}$, resulting in

$$\begin{bmatrix} -2\omega^2 + \phi & -\phi \\ -\phi & -2\omega^2 + \phi \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b}$$

where $\phi = \frac{6E}{\rho L^2}$. The characteristic equation is obtained by setting the determinant of the coefficient matrix in Equation (b) to zero, giving

$$4\omega^4 - 4\phi\omega^2 = 0 \tag{c}$$

The solutions to Equation (c) are

$$\omega = 0, \sqrt{\frac{6E}{\rho L^2}} \tag{d}$$

The mode shape corresponding to the rigid-body mode is $[1 \quad 1]^T$, while the mode shape corresponding to the second mode is $[1 \quad -1]^T$.

EXAMPLE 11.3

Use a one-element, finite-element model to approximate the lowest natural frequency of a fixed-free bar.

SOLUTION

The differential equations for a one-element, finite-element model are those given in Equation (a) of Example 11.2. Since the bar is fixed at $x = 0$, $w_1 = 0$. When one coordinate is zero, to obtain the appropriate finite-element mode, simply cross-out the row and column associated with the generalized coordinate in the element mass and stiffness matrix. The differential equations, crossing out the first row and first column of the mass and stiffness matrices, reduce to

$$\frac{\rho AL}{6}(2\ddot{w}_2) + \frac{EA}{L}w_2 = 0 \tag{a}$$

or

$$\ddot{w}_2 + \frac{3E}{\rho L^2}w_2 = 0 \tag{b}$$

The approximation to the natural frequency using a one-element approximation is obtained from Equation (b) as

$$\omega = \sqrt{\frac{3E}{\rho L^2}} \tag{c}$$

Note that the one-element, finite-element model of the fixed-free bar leads to the same natural frequency approximation that is obtained by using a SDOF model when an equivalent mass of $\frac{\rho AL}{3}$ is lumped at the end of the bar.

EXAMPLE 11.4

Use a one-element, finite-element model to approximate the lowest natural frequency of torsional oscillations of a fixed-free elastic shaft with a rigid disk of moment of inertia I attached at its free end, as in Figure 11.6.

SOLUTION

The mass matrix and stiffness matrix for a one-element model for the shaft are given by Equations (11.31) and (11.32), respectively. The bar is fixed at $x = 0$; thus, $w_1 = 0$ and the first row and first column of the mass and stiffness matrix are crossed out when developing the model. However, a disk of moment of inertia I is attached at the free end, adding to the kinetic energy such that

$$T = \frac{1}{2}\left(\frac{2\rho JL}{6}\right)\dot{w}_2^2 + \frac{1}{2}I\dot{w}_2^2 \tag{a}$$

Thus, the model of the system is

$$\left(\frac{\rho JL}{3} + I\right)\ddot{w}_2 + \frac{JG}{L}w_2 = 0 \tag{b}$$

or

$$\ddot{w}_2 + \left(\frac{3JG}{\rho JL + 3IL}\right)w_2 = 0 \tag{c}$$

The natural frequency approximation for the system of Figure 11.6 using a one-element, finite-element model is obtained from Equation (c) as

$$\omega = \sqrt{\frac{3JG}{\rho JL + 3IL}} \tag{d}$$

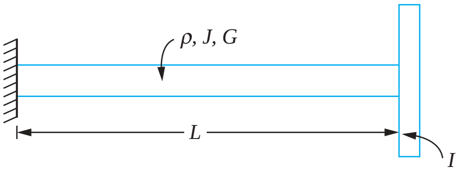


FIGURE 11.6
A one-element, finite-element model is used to approximate the lowest natural frequency of a torsional shaft with an attached disk.

11.5 BEAM ELEMENT

The potential energy scalar product for a beam involves the second spatial derivative of the displacement. Thus a Rayleigh-Ritz or assumed modes approximation must be twice differentiable. When a finite-element model of the beam is developed by the assumed modes method, the requirement that the interpolation be twice differentiable leads to requiring that displacements and slopes (first spatial derivatives) be continuous at element boundaries. In order to enforce this requirement over the entire beam, each beam element has four degrees of freedom represented by the displacements and slopes at the ends of the element. Let w_1 represent the transverse displacement of the left end of the element, w_2 the slope at the left end of the element, w_3 the transverse displacement of the right end of the element, and w_4 the slope at the right end of the element, as illustrated in Figure 11.7. If ξ is the local coordinate over the beam element, the finite element approximation for the displacement across the beam element must satisfy

$$u(0, t) = w_1 \qquad \frac{\partial u}{\partial \xi}(0, t) = w_2 \qquad u(l, t) = w_3 \qquad \frac{\partial u}{\partial \xi}(l, t) = w_4 \tag{11.33}$$

The deflection of a beam element without transverse loading across its span, but with prescribed displacements and slopes at its ends, is

$$u(\xi) = C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4 \tag{11.34}$$

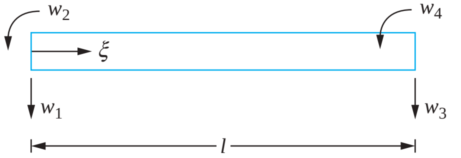


FIGURE 11.7
Beam element has four degrees of freedom, represented by the displacements and slopes of each end.

Using Equation (11.33) in Equation (11.34) to determine the constants leads to

$$C_1 = \frac{1}{l^3}(2w_1 + lw_2 - 2w_3 + lw_4) \tag{11.35}$$

$$C_2 = \frac{1}{l^2}(-3w_1 - 2lw_2 + 3w_3 - lw_4) \tag{11.36}$$

$$C_3 = w_2/l \tag{11.37}$$

$$C_4 = w_1 \tag{11.38}$$

Use of Equations (11.35) through (11.38) in Equation (11.34) and rearranging leads to

$$\begin{aligned} u(\xi, t) = {} & \left(1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3}\right)w_1 + \left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)w_2 \\ & + \left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}\right)w_3 + \left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)w_4 \end{aligned} \tag{11.39}$$

The kinetic energy of the beam element is

$$T = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial u}{\partial t}\right)^2 d\xi \tag{11.40}$$

Use of Equation (11.39) in Equation (11.40) leads to a quadratic form of kinetic energy

$$T = \frac{1}{2}\dot{\mathbf{w}}^T\mathbf{m}\dot{\mathbf{w}} \tag{11.41}$$

where $\dot{\mathbf{w}}^T = [\dot{w}_1 \quad \dot{w}_2 \quad \dot{w}_3 \quad \dot{w}_4]$ and the element (local) mass matrix for a uniform beam element is

$$\mathbf{m} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{11.42}$$

The potential energy of the beam element is

$$V = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 u}{\partial \xi^2}\right)^2 d\xi \tag{11.43}$$

Use of Equation (11.39) in Equation (11.43) leads to the quadratic form of potential energy

$$V = \frac{1}{2}\mathbf{w}^T\mathbf{k}\mathbf{w} \tag{11.44}$$

where the element (local) stiffness matrix for a uniform beam element is

$$\mathbf{k} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{11.45}$$

The method of virtual work is used to obtain the generalized forces as

$$q_1 = \int_0^l f(\xi, t)\left(1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3}\right)d\xi \tag{11.46}$$

$$q_2 = \int_0^l f(\xi, t)\left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)l\,d\xi \tag{11.47}$$

$$q_3 = \int_0^l f(\xi, t)\left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}\right)d\xi \tag{11.48}$$

$$q_4 = \int_0^l f(\xi, t)\left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)l\,d\xi \tag{11.49}$$

EXAMPLE 11.5

Use a one-element, finite-element model to approximate the natural frequencies and mode shapes of a uniform fixed-free beam.

SOLUTION

The differential equations governing the free vibrations of a one-element, finite-element model of a beam are

$$\frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \\ \ddot{w}_4 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{a}$$

The beam is fixed at $x = 0$; thus, $w_1 = 0$ and $w_2 = 0$. Thus, for the one-element, finite-element model, the first and second rows and columns are crossed out, leaving

$$\frac{\rho AL}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}\begin{bmatrix} \ddot{w}_3 \\ \ddot{w}_4 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{bmatrix} w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b}$$

A normal-mode solution is assumed for this two degree-of-freedom system as $\mathbf{w} = [1 \quad \chi]^T e^{i\omega t}$, which when substituted into Equation (b) leads to an eigenvalue problem for ω as

$$-\omega^2\frac{\rho AL}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{c}$$

where the values of ω are the square roots of the eigenvalues of the matrix

$$\mathbf{m}^{-1}\mathbf{k} = \frac{420EI}{\rho AL^4}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}^{-1}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \tag{d}$$

The approximations to the natural frequencies are calculated as

$$\omega_1 = 3.53\sqrt{\frac{EI}{\rho AL^4}} \qquad \omega_1 = 30.8\sqrt{\frac{EI}{\rho AL^4}} \tag{e}$$

The approximations to the mode shapes are represented by the eigenvectors, which are

$$W_1 = \begin{bmatrix} 1 \\ 1.38 \end{bmatrix} \qquad W_2 = \begin{bmatrix} 1 \\ 7.62 \end{bmatrix} \tag{f}$$

The discrete mode shape vectors of Equation (f) are substituted into Equation (11.39) to obtain approximations for the mode shapes. The results are

$$w_1(x) = 3\frac{x^2}{L^2} - 2\frac{x^2}{L^3} + 1.38\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right) \tag{g}$$

and

$$w_2(x) = 3\frac{x^2}{L^2} - 2\frac{x^2}{L^3} + 7.62\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right) \tag{h}$$

Equations (g) and (h) are plotted in Figure 11.8.

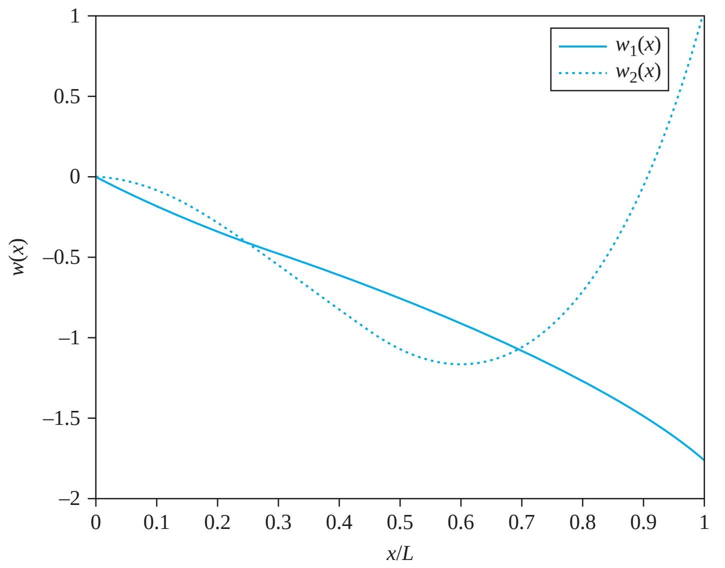


FIGURE 11.8
A one-element, finite-element model of fixed-free beam has two degrees of freedom, which leads to approximations of the first two mode shapes.

EXAMPLE 11.6

Use a one-element, finite-element model to approximate the steady-state response of the pinned-pinned beam of Figure 11.9 with a concentrated load $F(t) = 80 \sin 100t$ at its midspan.

SOLUTION

The differential equations for a one-element, finite-element model of a beam are given in Equation (a) of Example 11.5, except that the right-hand side is $[q_1\ q_2\ q_3\ q_4]^T$. For a pinned-pinned beam, $w_1 = 0$ and $w_3 = 0$. The first and third columns of the stiffness matrix and the mass matrix are crossed out, leading to

$$\frac{\rho AL}{420}\begin{bmatrix} 4L^2 & -3L^2 \\ -3L^2 & 4L^2 \end{bmatrix}\begin{bmatrix} \ddot{w}_2 \\ \ddot{w}_4 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix}\begin{bmatrix} w_2 \\ w_4 \end{bmatrix} = \begin{bmatrix} q_2 \\ q_4 \end{bmatrix} \tag{a}$$

The concentrated load can be represented using the Dirac delta function of Appendix A as

$$F(x, t) = 80 \sin 100t\, \delta\left(x - \frac{L}{2}\right) \tag{b}$$

The generalized forces are obtained by

$$q_2 = \int_0^L 80 \sin 100t\ \delta\left(\xi - \frac{L}{2}\right)\left(\frac{\xi}{L} - 2\frac{\xi^2}{L^2} + \frac{\xi^3}{L^3}\right)d\xi = 10 \sin 100t \tag{c}$$

$$q_4 = \int_0^L 80 \sin 100t\ \delta\left(\xi - \frac{L}{2}\right)\left(-\frac{\xi^2}{L^2} + \frac{\xi^3}{L^3}\right)d\xi = 10 \sin 100t \tag{d}$$

Substituting given values into the differential equations leads to

$$\begin{bmatrix} 0.1737 & -0.1303 \\ -0.1303 & 0.1737 \end{bmatrix}\begin{bmatrix} \ddot{w}_2 \\ \ddot{w}_4 \end{bmatrix} + \begin{bmatrix} 6300 & 3150 \\ 3150 & 6300 \end{bmatrix}\begin{bmatrix} w_2 \\ w_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \end{bmatrix} \sin 100t \tag{e}$$

A steady-state solution is assumed to Equation (e) as

$$\begin{bmatrix} w_2 \\ w_4 \end{bmatrix} = \begin{bmatrix} W_2 \\ W_4 \end{bmatrix} \sin 100t \tag{f}$$

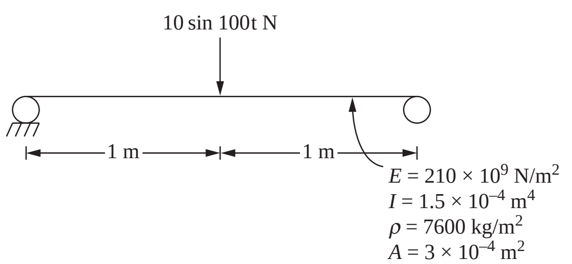


FIGURE 11.9
A one-element, finite-element model is applied to determine the steady-state amplitude of the point of application of a concentrated load.

which when substituted into Equation (e) yields

$$\begin{bmatrix} W_2 \\ W_4 \end{bmatrix} = \begin{bmatrix} 0.09091 \\ -0.09091 \end{bmatrix} \tag{g}$$

The steady-state response is obtained by substituting Equation (g) into Equation (11.39) with $W_1 = W_3 = 0$ yielding

$$W(x) = 0.99091\left(\frac{x}{2} - 2\frac{x^2}{4} + \frac{x^3}{8}\right) - 0.99091\left(-\frac{x^2}{4} + \frac{x^3}{8}\right) \tag{h}$$

11.6 GLOBAL MATRICES

Local mass and stiffness matrices are derived for bar, torsion, and beam elements in Sections 11.4 and 11.5. The accuracy of the finite-element method improves as the number of elements used increases. The use of many elements is necessary in the approximation of complicated systems. Local mass and stiffness matrices are calculated for each element and assembled into global matrices. When many elements are used, an efficient assembly algorithm is necessary.

A bar element has two degrees of freedom. The local generalized coordinates are the displacements of the ends of the elements. An n-element finite-element model of a bar, as illustrated in Figure 11.10, has at most $n + 1$ degrees of freedom. The global generalized coordinates are the displacements of the boundaries between elements and the ends of the bar. Each geometric boundary condition reduces by one the number of global degrees of freedom. For example if the left end of the bar is fixed, then its displacement is zero and the model has n degrees of freedom.

Let $W_1, W_2, \ldots, W_n$ represent the global generalized coordinates. Each local generalized coordinate is one of the global generalized coordinates, unless that element is subject to a geometric boundary condition. The local mass and stiffness matrices can be expanded to include all global generalized coordinates. The total kinetic energy of the system is the sum of the kinetic energies of the elements. Let

$$T_i = \frac{1}{2}\dot{\mathbf{w}}_i^T \mathbf{m}_i \dot{\mathbf{w}}_i \tag{11.50}$$

be the kinetic energy of the ith element. The local mass matrix can be enlarged and the kinetic energy written in terms of the global generalized coordinates as

$$T_i = \frac{1}{2}\dot{\mathbf{W}}^T \tilde{\mathbf{M}}_i \dot{\mathbf{W}} \tag{11.51}$$

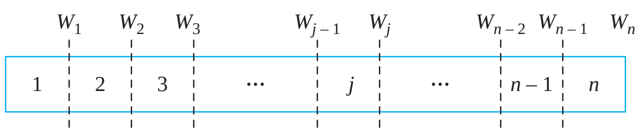


FIGURE 11.10 An n-element model of a fixed-free bar. Global generalized coordinates are the displacements of the nodes, which are located at element boundaries or ends of the bar. The bar is fixed at $x = 0$, $W_0 = 0$.

The total kinetic energy of the system is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2}\sum_{i=1}^{n} \dot{\mathbf{W}}^T \tilde{\mathbf{M}}_i \dot{\mathbf{W}} = \frac{1}{2}\dot{\mathbf{W}}^T\left(\sum_{i=1}^{n}\tilde{\mathbf{M}}_i\right)\dot{\mathbf{W}} \tag{11.52}$$

Thus, the global mass matrix is

$$\mathbf{M} = \sum_{i=1}^{n}\tilde{\mathbf{M}}_i \tag{11.53}$$

The global stiffness matrix and the global force vector can be obtained in an analogous manner.

EXAMPLE 11.7

Derive the global mass matrix for a three-element model of a fixed-free bar.

SOLUTION

The three-element model of a fixed-free bar is shown in Figure 11.11. The three-element model has three degrees of freedom, noting that $u(0) = 0$. The global generalized coordinates are the displacements of the ends of the elements. The assembly of the global mass matrix from the local mass matrices is shown. The global displacement vector is $\mathbf{W} = [W_1 \quad W_2 \quad W_3]^T$. The quadratic form of the kinetic energy is $T = \frac{1}{2}\dot{\mathbf{W}}^T\mathbf{M}\dot{\mathbf{W}}$.

Element 1 Local generalized coordinates:

$$w_1 = 0 \qquad w_2 = W_1 \tag{a}$$

Element mass matrix in terms of local generalized coordinates:

$$\mathbf{m}_1 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{b}$$

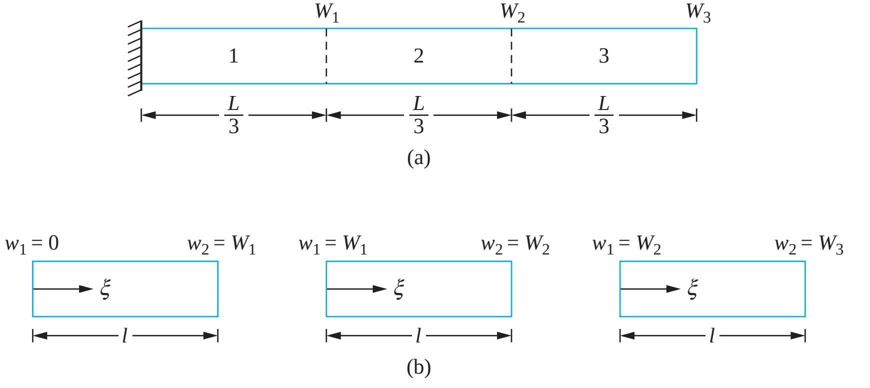


FIGURE 11.11
(a) Three-element model of fixed-free bar has three degrees of freedom. The elements are of equal length $l = \frac{L}{3}$. (b) Local coordinates for each element.

Kinetic energy of element:

$$T = \frac{1}{2}\frac{\rho A l}{6}(2\dot{w}_2^2) = \frac{1}{2}\frac{\rho A l}{6}(2\dot{W}_1^2) = \frac{1}{2}[\dot{W}_1 \quad \dot{W}_2 \quad \dot{W}_3]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{W}_1 \\ \dot{W}_2 \\ \dot{W}_3 \end{bmatrix} \tag{c}$$

Element mass matrix in terms of global generalized coordinates:

$$\tilde{\mathbf{M}}_1 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{d}$$

Element 2 Local generalized coordinates:

$$w_1 = W_1 \qquad w_2 = W_2 \tag{e}$$

Element mass matrix in terms of local generalized coordinates:

$$\mathbf{m}_2 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{f}$$

Element mass matrix in terms of global generalized coordinates:

$$\tilde{\mathbf{M}}_2 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{g}$$

Element 3 Local generalized coordinates:

$$w_1 = W_2 \qquad w_2 = W_3 \tag{h}$$

Element mass matrix in terms of local generalized coordinates:

$$\mathbf{m}_3 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{i}$$

Element mass matrix in terms of global generalized coordinates:

$$\tilde{\mathbf{M}}_3 = \frac{\rho A l}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{j}$$

Thus the global mass matrix is

$$\begin{aligned} \mathbf{M} &= \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 \\ &= \frac{\rho A l}{6}\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) \\ &= \frac{\rho A l}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{aligned} \tag{k}$$

The model of Example 11.7 has only three degrees of freedom, and it is easy to construct $\tilde{\mathbf{M}}$. It is more difficult for systems with a large number of degrees of freedom. For such systems computer analysis will be used to formulate the model and solve the resulting differential equations. Thus it is important to have an efficient algorithm for assembly of the global mass matrices.

Let $\mathbf{S}_i$ be a transformation matrix between the local generalized coordinates for element i and the global generalized coordinates,

$$\mathbf{w}_i = \mathbf{S}_i\mathbf{W} \tag{11.54}$$

The total kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{n}\dot{\mathbf{w}}_i^T\mathbf{m}_i\dot{\mathbf{w}}_i \tag{11.55}$$

Using Equation (11.54) in Equation (11.55) leads to

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{n}(\mathbf{S}_i\dot{\mathbf{W}})^T\mathbf{m}_i(\mathbf{S}_i\dot{\mathbf{W}}) \\ &= \frac{1}{2}\sum_{i=1}^{n}\dot{\mathbf{W}}^T\mathbf{S}_i^T\mathbf{m}_i\mathbf{S}_i\dot{\mathbf{W}} \\ &= \frac{1}{2}\dot{\mathbf{W}}^T\left(\sum_{i=1}^{n}\mathbf{S}_i^T\mathbf{m}_i\mathbf{S}_i\right)\dot{\mathbf{W}} \end{aligned} \tag{11.56}$$

Thus the global mass matrix is

$$\mathbf{M} = \sum_{i=1}^{n}\mathbf{S}_i^T\mathbf{m}_i\mathbf{S}_i \tag{11.57}$$

EXAMPLE 11.8

Illustrate the development of $\tilde{\mathbf{M}}_2$ for the system of Example 11.7 using the transformation matrix.

SOLUTION

The transformation between the local generalized coordinates and the global generalized coordinates for element 2 of Example 11.7 is

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \tag{a}$$

Thus

$$\begin{aligned} \tilde{\mathbf{M}}_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{\rho Al}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \frac{\rho Al}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \frac{\rho Al}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{b}$$

11.7 BENCHMARK EXAMPLE

The response of the machine on the beam is considered using the finite-element method. Five elements are used to model the beam, as shown in Figure 11.12(a), which indicates the global coordinates used. The machine is a discrete mass connected to the beam by an

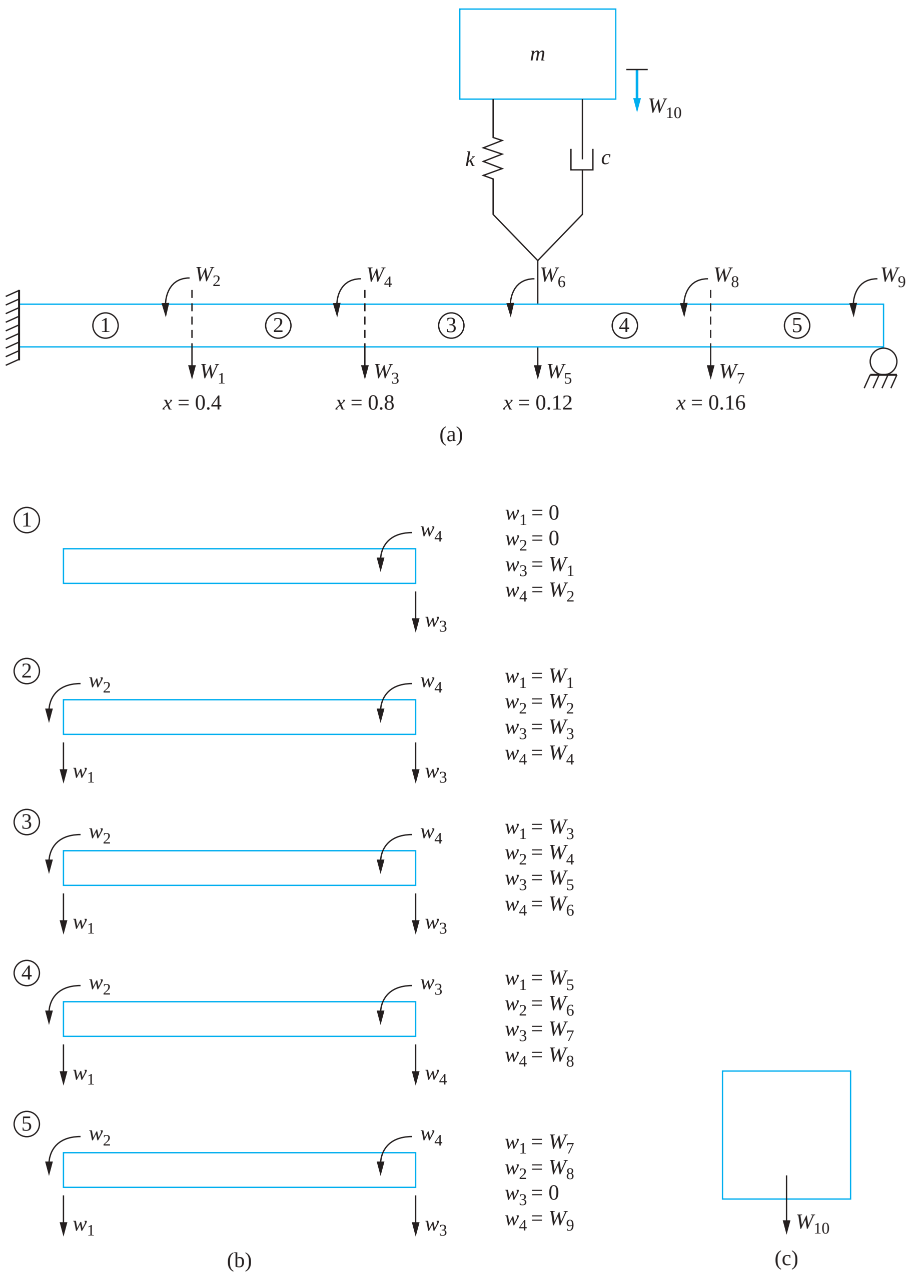


FIGURE 11.12
(a) Five-element, finite-element model of machine attached by isolator to fixed-pinned beam. The model has ten degrees of freedom. (b) Relation between local coordinates and global coordinates for model. (c) The machine adds another degree of freedom to the model.

isolator. The beam is assumed to be undamped. The following describes the construction of the model. There are a total of 10 degrees of freedom in the model. For the element matrices $\ell = \frac{L}{5} = 4\ \text{ft}$. It is noted that

$$\frac{\rho A\ell}{420} = \frac{(30\ \text{lb/ft})(4\ \text{ft})\left(\dfrac{1}{32.2\ \text{ft/s}^2}\right)}{420} = 8.87 \times 10^{-3}\ \text{slugs} \tag{a}$$

and

$$\frac{EL}{\ell^{3}} = \frac{(30 \times 10^6\ \text{psi})(291\ \text{in}^4)\left(\dfrac{1\ \text{ft}}{12\ \text{in}}\right)^2}{(4\ \text{ft})^3} = 9.47 \times 10^5\ \text{lb/ft} \tag{b}$$

The local mass matrix for each element is

$$\mathbf{M} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$
$$= (8.87 \times 10^{-3}\ \text{slugs})\begin{bmatrix} 156 & 88 & 54 & -52 \\ 88 & 64 & 52 & -48 \\ 54 & 52 & 156 & -88 \\ -52 & -48 & -88 & 64 \end{bmatrix} \tag{c}$$

The local stiffness matrix for each element is

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$
$$= (9.47 \times 10^5\ \text{lb/ft})\begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} \tag{d}$$

The relation between the local coordinates for each element and the global coordinates are shown in Figure 11.10(b).

Element 1: $w_1 = 0$, $w_2 = 0$, $w_3 = W_1$ $w_4 = W_2$ The transformation matrix between the local coordinates and the global coordinates is

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{e}$$

Element 2: $w_1 = W_1$ $w_2 = W_2$, $w_3 = W_3$, $w_4 = W_4$ The transformation matrix between the local coordinates and the global coordinates is

$$\mathbf{S}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{f}$$

Element 3: $w_1 = W_3$, $w_2 = W_4$, $w_3 = W_5$, $w_4 = W_6$ The transformation matrix between the local coordinates and the global coordinates is

$$\mathbf{S}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{g}$$

Element 4: $w_1 = W_5$, $w_2 = W_6$, $w_3 = W_7$, $w_4 = W_8$ The transformation matrix between the local coordinates and the global coordinates is

$$\mathbf{S}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{h}$$

Element 5: $w_1 = W_7$, $w_2 = W_8$, $w_3 = 0$, $w_4 = W_9$ The transformation matrix between the local coordinates and the global coordinates is

$$\mathbf{S}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{i}$$

Discrete Mass The displacement of the discrete mass is W_{10}. Its kinetic energy is $T = \frac{1}{2}m\dot{W}_{10}^2$, where $m = 31.06$ slugs. The potential energy of the spring is $V = \frac{1}{2}k(W_{10} - W_5)^2$, $k = 3.93 \times 10^4$ lbf/ft. The 10×10 global matrix for the discrete element is $\tilde{\mathbf{M}}$; the global stiffness matrix is $\tilde{\mathbf{K}}$. Their elements are zero except for

$$\tilde{\mathbf{M}}_{10,10} = 31.06 \text{ slugs} \tag{j}$$

and

$$\tilde{\mathbf{K}}_{5,5} = 3.93 \times 10^4 \text{ lbf/ft} \qquad \tilde{\mathbf{K}}_{5,10} = -3.93 \times 10^4 \text{ lbf/ft}$$
$$\tilde{\mathbf{K}}_{10,5} = -3.93 \times 10^4 \text{ lbf/ft} \qquad \tilde{\mathbf{K}}_{10,10} = 3.93 \times 10^4 \text{ lbf/ft} \tag{k}$$

The global matrices are formed by

$$\mathbf{M} = \sum_{i=1}^{5} \mathbf{S}_i^T \, \mathbf{m} \mathbf{S}_i + \tilde{\mathbf{M}}$$

$$= \begin{bmatrix} 2.768 & 0 & 0.4790 & -0.4612 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.135 & 0.4612 & -0.4258 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4790 & 0.4612 & 2.768 & 0 & 0.4790 & -0.4612 & 0 & 0 & 0 & 0 \\ -0.4612 & -0.4258 & 0 & 1.1354 & 0.4612 & 0.4258 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4790 & 0.4612 & 2.768 & 0 & 0.4790 & -0.4612 & 0 & 0 \\ 0 & 0 & -0.4612 & -0.4258 & 0 & 1.1354 & 0.4612 & -0.4258 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4790 & 0.4612 & 2.7674 & 0 & -0.4612 & 0 \\ 0 & 0 & 0 & 0 & -0.4612 & -0.4258 & 0 & 1.1354 & -0.4258 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.4612 & -0.4258 & 0.5677 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 31.06 & 0 \end{bmatrix} \quad \text{(l)}$$

$$\mathbf{K} = \sum_{i=1}^{5} \mathbf{S}_i^T \, \mathbf{k} \mathbf{S}_i + \tilde{\mathbf{K}}$$

$$= 10^7 \begin{bmatrix} 2.272 & 0 & -1.136 & 2.272 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.12 & -2.272 & 3.030 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.136 & -2.272 & 2.272 & 0 & -1.136 & 2.272 & 0 & 0 & 0 & 0 \\ 2.272 & 3.030 & 0 & 12.12 & -2.272 & 3.030 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.136 & -2.272 & 2.276 & 0 & -1.136 & 2.272 & 0 & -0.00393 \\ 0 & 0 & 2.272 & 3.030 & 0 & 12.12 & -2.272 & 3.030 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.136 & -2.272 & 2.272 & 0 & 2.272 & 0 \\ 0 & 0 & 0 & 0 & 2.272 & 3.030 & 0 & 12.12 & 3.030 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.272 & 3.030 & 6.060 & 0 \\ 0 & 0 & 0 & 0 & -0.00393 & 0 & 0 & 0 & 0.00393 & \end{bmatrix} \quad \text{(m)}$$

The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, which are

$$\begin{gathered} \omega_1 = 34.7 \text{ rad/s} \qquad \omega_2 = 381.7 \text{ rad/s} \qquad \omega_3 = 1.011 \times 10^3 \text{ rad/s} \\ \omega_4 = 2.126 \times 10^3 \text{ rad/s} \qquad \omega_5 = 3.688 \times 10^3 \text{ rad/s} \\ \omega_6 = 6.121 \times 10^3 \text{ rad/s} \qquad \omega_7 = 9.039 \times 10^3 \text{ rad/s} \qquad \omega_8 = 1.322 \times 10^4 \text{ rad/s} \\ \omega_9 = 1.883 \times 10^4 \text{ rad/s} \qquad \omega_{10} = 2.438 \times 10^4 \text{ rad/s} \end{gathered} \quad \text{(n)}$$

The forced response is determined by

$$\mathbf{M}\ddot{\mathbf{W}} + \mathbf{K}\mathbf{W} = \mathbf{F} \quad \text{(o)}$$

where $\mathbf{F}$ is a 10×10 vector with all elements equal to zero except

$$F_{10} = 20{,}000 \sin 80t \text{ lb} \tag{p}$$

A steady-state solution is assumed as $\mathbf{W} = \mathbf{U} \sin 80t$, which leads to

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \end{bmatrix} = \begin{bmatrix} -1.71 \times 10^{-3} \\ -7.3 \times 10^{-4} \\ -4.83 \times 10^{-3} \\ -7.1 \times 10^{-4} \\ -6.38 \times 10^{-3} \\ 5 \times 10^{-5} \\ -4.32 \times 10^{-3} \\ 8.9 \times 10^{-4} \\ 1.18 \times 10^{-3} \\ -1.238 \times 10^{-2} \end{bmatrix} \text{ft} \tag{q}$$

The transmitted force is

$$\begin{aligned} k|U_{10} - U_5| &= (3.93 \times 10^4 \text{ lb/ft})|1.238 \times 10^{-2} \text{ ft} + 6.38 \times 10^{-3} \text{ ft}| \\ &= 4610 \text{ N} \end{aligned} \tag{r}$$

The steady-state approximation is plotted in Figure 11.13.

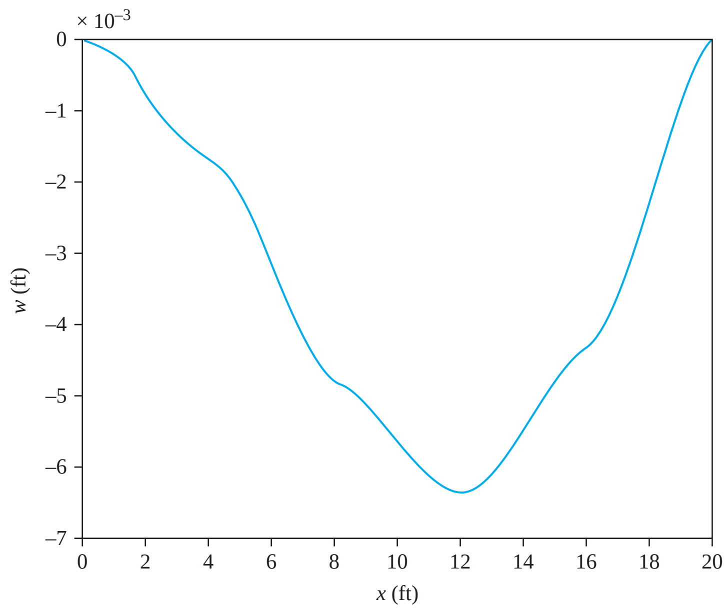


FIGURE 11.13
Steady-state mode shape of beam as predicted by finite-element method.

11.8 FURTHER EXAMPLES

EXAMPLE 11.9

Use a three-element, finite-element model to approximate the lowest natural frequency and its corresponding mode shape for a uniform fixed-free bar.

SOLUTION

The three-element model of a fixed-free bar is illustrated in Figure 11.11. The global mass matrix was derived in Example 11.7. Using the same method, the global stiffness matrix is determined as

$$\mathbf{K} = \frac{EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{a}$$

The differential equations for the bar in the finite-element model are

$$\frac{\rho Al}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \end{bmatrix} + \frac{EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{b}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The mode shape vectors are the corresponding eigenvectors. The lowest natural frequency and mode shape vector are calculated as

$$\omega_1 = 1.584\sqrt{\frac{E}{\rho L^2}} \qquad \mathbf{W} = \begin{bmatrix} 0.577 \\ 1 \\ 1.155 \end{bmatrix} \tag{c}$$

The mode shape vector provides the displacements at the element boundaries. The finite-element approximation to the mode shape is a piecewise linear approximation between the element boundaries.

EXAMPLE 11.10

Use a two-element, finite-element model to approximate the four lowest natural frequencies for the system of Figure 11.14(a). Note that the exact solution for this system was obtained in Example 10.6.

SOLUTION

The two-element, finite-element model for the fixed-free beam illustrating the global generalized coordinates is shown Figure 11.14(b). The beam element of Section 11.5 is used. Note that since the left end of the beam is fixed, the geometric boundary conditions of zero slope and zero displacement must be imposed. The generic element mass and stiffness

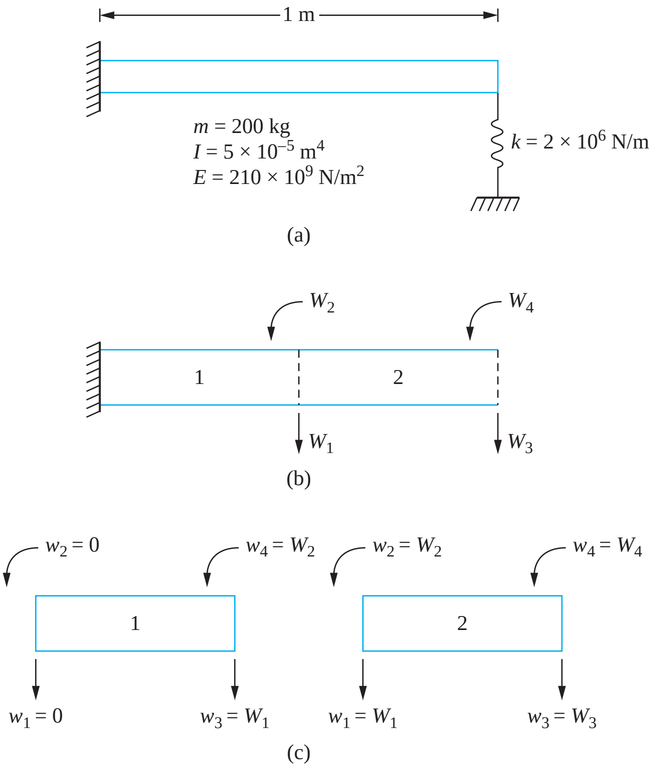


FIGURE 11.14
(a) System of Example 11.10. (b) Two-element, finite-element model of beam illustrating global generalized coordinates. (c) Local generalized coordinates for each element.

matrices for a beam element are

$$\mathbf{m} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{a}$$

$$\mathbf{k} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{b}$$

The potential energy for the discrete spring is incorporated into the local stiffness matrix for element 2. For this model $l = L/2$.

Element 1 Local generalized coordinates:

$$w_1 = 0 \qquad w_2 = 0 \qquad w_3 = W_1 \qquad w_4 = W_2 \tag{c}$$

Element global matrices are

$$\tilde{\mathbf{M}}_1 = \frac{\rho A l}{420}\begin{bmatrix} 156 & -22l & 0 & 0 \\ -22l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_1 = \frac{EI}{l^3}\begin{bmatrix} 12 & -6l & 0 & 0 \\ -6l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{d}$$

Element 2 Local generalized coordinates:

$$w_1 = W_1 \qquad w_2 = W_2 \qquad w_3 = W_3 \qquad w_4 = W_4 \tag{e}$$

The element stiffness matrix for element 2 must be modified to account for the potential energy of the spring, $V = \frac{1}{2}k\omega_3^2$. The stiffness matrix term k_{33} is the only term affected by the discrete spring. The global mass and stiffness matrices for element 2 are

$$\tilde{\mathbf{M}}_2 = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22 & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{f}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 + \dfrac{kl^3}{EI} & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{g}$$

The global mass and stiffness matrices are

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 = \frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{h}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 + \dfrac{kl^3}{EI} & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{i}$$

Substitution of given values leads to

$$\mathbf{M} = \begin{bmatrix} 74.29 & 0 & 12.86 & -1.55 \\ 0 & 0.476 & 1.55 & -0.179 \\ 12.86 & 1.55 & 37.14 & -0.262 \\ -1.55 & -0.179 & -2.62 & 0.238 \end{bmatrix} \tag{j}$$

$$\mathbf{K} = \begin{bmatrix} 2.016 & 0 & -1.008 & 0.252 \\ 0 & 0.108 & -0.252 & 0.042 \\ -1.008 & -0.252 & 1.008 & -0.252 \\ 0.252 & 0.042 & -0.252 & 0.084 \end{bmatrix} 10^9 \tag{k}$$

The natural frequency approximations, the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, are obtained as

$$\begin{aligned} \omega_1 &= 806.0 \text{ rad/s} & \omega_2 &= 5.09 \times 10^3 \text{ rad/s} \\ \omega_3 &= 1.72 \times 10^4 \text{ rad/s} & \omega_4 &= 5.00 \times 10^4 \text{ rad/s} \end{aligned} \tag{l}$$

The exact natural frequencies for this system, obtained in Example 10.6, are

$$\begin{aligned} \omega_1 &= 829 \text{ rad/s} & \omega_2 &= 5.05 \times 10^3 \text{ rad/s} \\ \omega_3 &= 1.41 \times 10^4 \text{ rad/s} & \omega_4 &= 2.73 \times 10^4 \text{ rad/s} \end{aligned} \tag{m}$$

EXAMPLE 11.11

Use a two-element finite-element model for the beam to determine the steady-state response of the system of Figure 11.15(a).

SOLUTION

For a two-element, finite-element model of the beam, the system has five degrees of freedom. The global generalized coordinates are illustrated in Figure 11.15(b). The local mass and stiffness matrices for each element are given by Equations (11.42) and (11.45), respectively.

Element 1

$$w_1 = 0 \qquad w_2 = W_1 \qquad w_3 = W_2 \qquad w_4 = W_3 \tag{a}$$

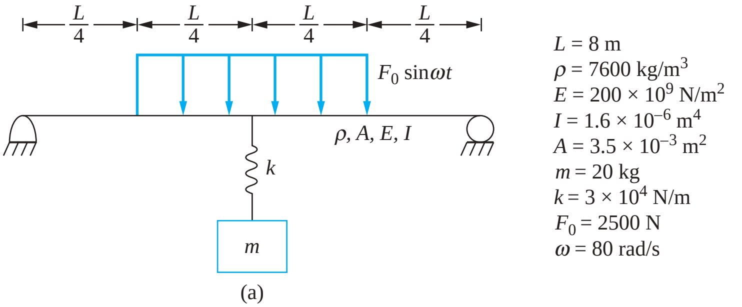
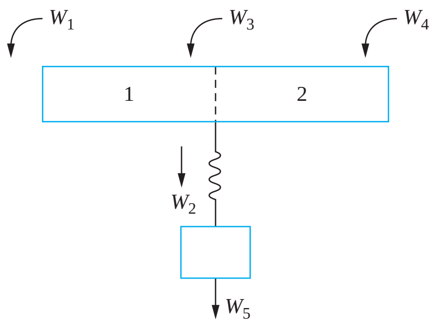
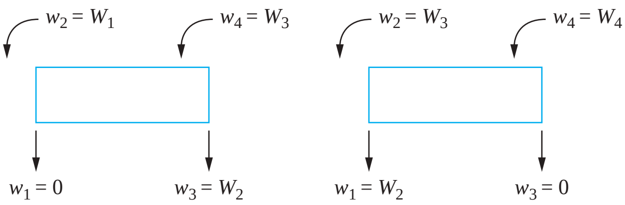


FIGURE 11.15
(a) System of Example 11.11. (b) Two-element model for beam illustrating global generalized coordinates. (c) Relations between local coordinates and global coordinates for each element. (d) Output from MATLAB code. (e) Steady-state mode shape.



(b)



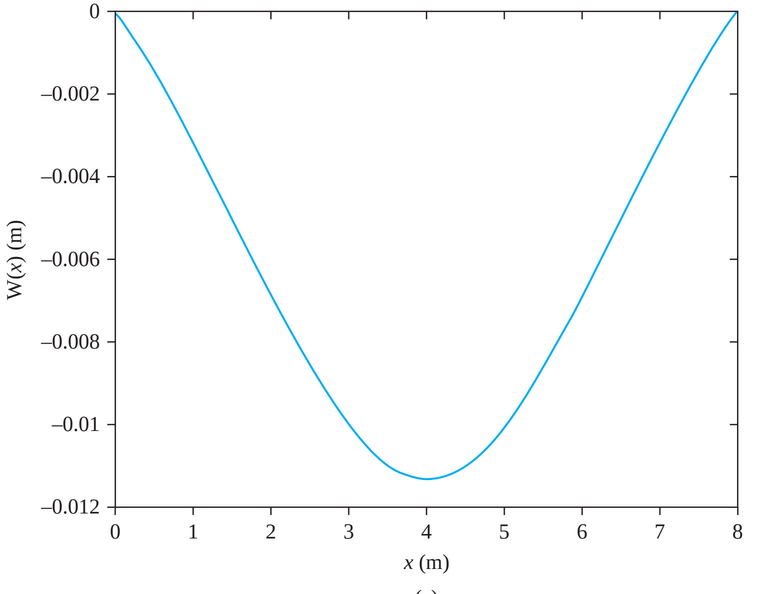
(c)

```
 Global mass matrix
[  16.677,    13.550,   -12.507,          0,        0]
[  13.550,    81.298,         0,    -13.550,        0]
[ -12.507,         0,    33.353,    -12.507,        0]
[       0,   -13.550,   -12.507,     16.677,        0]
[       0,         0,         0,          0,      20.]

 Global stiffness matrix
[   .32000e6,   -.12000e6,    .16000e6,           0,          0]
[  -.12000e6,    .15000e6,           0,    .12000e6,    -30000.]
[   .16000e6,           0,    .64000e6,    .16000e6,          0]
[          0,    .12000e6,    .16000e6,    .32000e6,          0]
[          0,     -30000.,           0,           0,     30000.]

iter =

     2
eigs =

   1.151315789473683e+005
   3.487849420597156e+004
   5.482456140350870e+003
   1.816727650409610e+003
   2.298793581937252e+002
```

FIGURE 11.15
(*Continued*)

```
stopcrit =

   1.245169782336117e-015


Force vector
[   2.60400e2]
[   8.12500e3]
[           0]
[  -2.60400e2]
[           0]


 Natural frequencies in rad/s
[ 339.31, 186.76, 74.044, 42.623, 15.162]

 Steady-state amplitudes in m
[  -0.9758e-2]
[  -0.1133e-1]
[   0.0]
[   0.9758e-3]
[   0.3468e-2]
```

(d)



(e)

FIGURE 11.15
(*Continued*)

Global element matrices are

$$\tilde{\mathbf{M}}_1 = \frac{\rho A l}{420}\begin{bmatrix} 4l^2 & 13l & -3l^2 & 0 & 0 \\ 13l & 156 & -22l & 0 & 0 \\ -3l^2 & -22l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_1 = \frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 & 0 \\ -6l & 12 & -6l & 0 & 0 \\ 2l^2 & -6l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{b}$$

The generalized force vector for element 1 is calculated by using Equations (11.46) through (11.49). Since $w_1 = 0$, q_1 is not calculated.

$$q_2(t) = \int_{l/2}^{l} F_0 \sin\omega t\left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)d\xi = -\frac{1}{48} l\, F_0 \sin\omega t \tag{c}$$

$$q_3(t) = \int_{l/2}^{l} F_0 \sin\omega t\left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}\right)d\xi = \frac{13}{32} l\, F_0 \sin\omega t \tag{d}$$

$$q_4 = \int_{l/2}^{l} F_0 \sin\omega t\left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)d\xi = -\frac{11}{192} l\, F_0 \sin\omega t \tag{e}$$

The global generalized force vector for element 1 is

$$\mathbf{F}_1 = \begin{bmatrix} -\frac{1}{48} \\ \frac{13}{32} \\ -\frac{11}{192} \\ 0 \\ 0 \end{bmatrix} l\, F_0 \sin\omega t \tag{f}$$

Element 2 Local generalized coordinates:

$$w_1 = W_2 \qquad w_2 = W_3 \qquad w_3 = 0 \qquad w_4 = W_4 \tag{g}$$

Global mass and stiffness matrices are

$$\tilde{\mathbf{M}}_2 = \frac{\rho A l}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22l & -13l & 0 \\ 0 & 22l & 4l^2 & -3l^2 & 0 \\ 0 & -13l & -3l^2 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{l^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6l & 6l & 0 \\ 0 & 6l & 4l^2 & 2l^2 & 0 \\ 0 & 6l & 2l^2 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{h}$$

The generalized force vector for element 2 is calculated by using Equations (11.46) through (11.49). Since $w_3 = 0$, q_3 is not calculated.

$$q_1(t) = \int_0^{l/2} F_0 \sin \omega t \left(1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3}\right) d\xi = \frac{13}{32} l F_0 \sin \omega t \tag{i}$$

$$q_2(t) = \int_0^{l/2} F_0 \sin \omega t \left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right) d\xi = \frac{11}{192} l\, F_0 \sin \omega t \tag{j}$$

$$q_4(t) = \int_0^{l/2} F_0 \sin \omega t \left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right) d\xi = -\frac{5}{192} l\, F_0 \sin \omega t \tag{k}$$

The global generalized force vector for element 2 is,

$$\mathbf{F}_2 = \begin{bmatrix} 0 \\ \dfrac{13}{32} \\ \dfrac{35}{192} \\ -\dfrac{5}{192} \\ 0 \end{bmatrix} l\, F_0 \sin \omega t \tag{l}$$

For the discrete spring-mass system.

Potential energy: $\quad V = \dfrac{1}{2} k (W_2 - W_5)^2$ $\qquad$ (m)

Kinetic energy: $\quad T = \dfrac{1}{2} m \dot{W}_5^2$ $\qquad$ (n)

The contributions to the global matrices due to the discrete mass-spring system are

$$\mathbf{M}_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} \qquad \mathbf{K}_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & k \end{bmatrix} \tag{o}$$

Assembling the global mass matrix, global stiffness matrix, and global generalized force vector leads to the following differential equations

$$\frac{\rho A l}{420}\begin{bmatrix} 4l^2 & 13l & -3l^2 & 0 & 0 \\ 13l & 312 & 0 & -13l & 0 \\ -3l^2 & 0 & 8l^2 & -3l^2 & 0 \\ 0 & -13l & -3l^2 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{420m}{\rho A l} \end{bmatrix}\begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \\ \ddot{W}_4 \\ \ddot{W}_5 \end{bmatrix}$$

$$+\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 & 0 \\ -6l & 24+\dfrac{kl^3}{EI} & 0 & 6l & -\dfrac{kl^3}{EI} \\ 2l^2 & 0 & 8l^2 & 2l^2 & 0 \\ 0 & 6l & 2l^2 & 4l^2 & 0 \\ 0 & -\dfrac{kl^3}{EI} & 0 & 0 & \dfrac{kl^3}{EI} \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{48} \\ \dfrac{13}{16} \\ -\dfrac{1}{8} \\ -\dfrac{5}{192} \\ 0 \end{bmatrix} l\, F_0 \sin \omega t \tag{p}$$

The method of undetermined coefficients is used to approximate the steady-state response of the system. The steady-state response is assumed as $\mathbf{W}(t) = \mathbf{S} \sin \omega t$ where $\mathbf{S}$ is the vector of undetermined coefficients. A MATLAB script can be written to determine the natural frequencies and steady-state response. The output from running the script is given in Figure 11.15(d), while the MATLAB-generated plot of the steady-state mode shape is given in Figure 11.15(e). The steady-state amplitude of the discrete mass is $W_5 = 3.3$ mm.

EXAMPLE 11.12

Use the finite-element method to approximate the lowest natural frequency for the truss of Figure 11.16(a). Use one bar element for each truss member.

SOLUTION

The finite-element model of the four-bar truss using one bar element for each member has four degrees of freedom. The global generalized coordinates are illustrated in Figure 11.16(b).

Member 1 The relations between the local generalized coordinates and the global generalized coordinates are $w_1 = 0$, $w_2 = W_1$. The contributions to the global mass and stiffness matrices for element 1 are

$$\tilde{\mathbf{M}}_1 = \frac{\rho A l_1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_1 = \frac{EA}{l_1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{a}$$

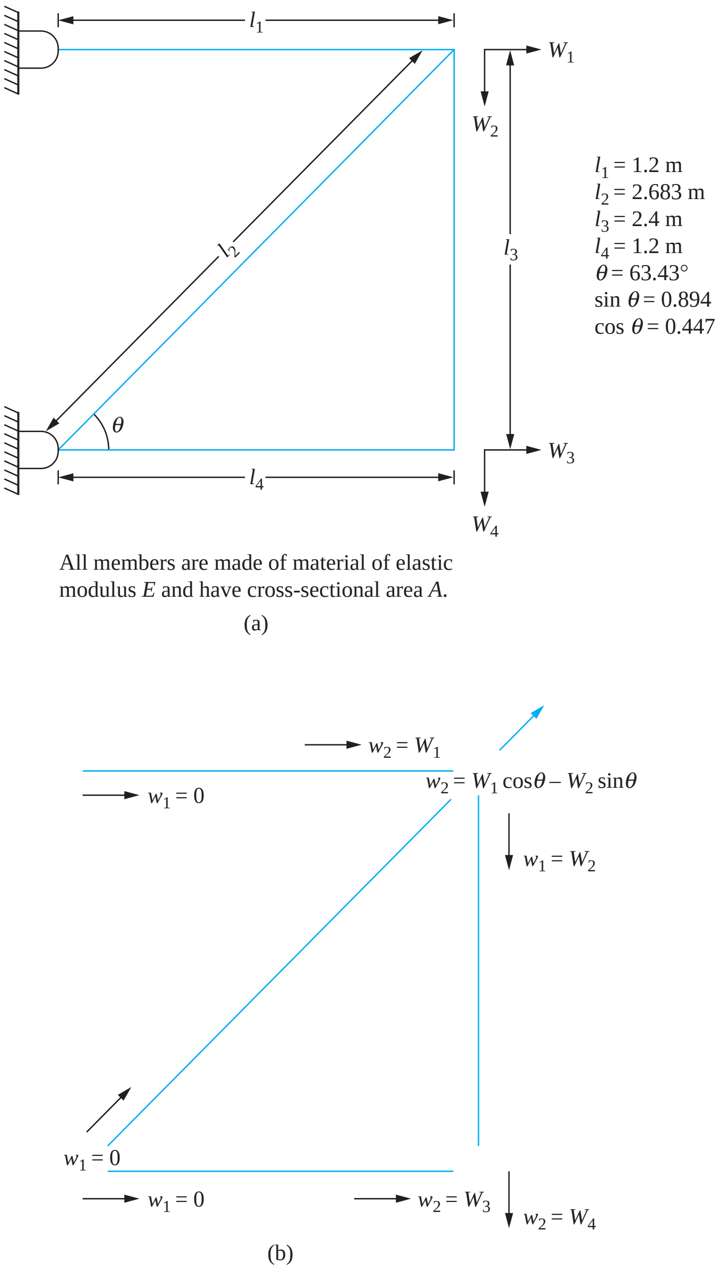


FIGURE 11.16
(a) Four-bar truss of Example 11.12 illustrating global coordinates. (b) Relationships between local coordinates and global coordinates for each truss member. (c) Output from MATLAB code to determine natural frequencies and mode shapes.

```
Global mass matrix
       176.01       -108.78             0             0
      -108.78        460.67             0         121.6
            0             0         121.6             0
            0         121.6             0         243.2

Global stiffness matrix
   7.2634e+009    -1.193e+009             0             0
   -1.193e+009    5.7184e+009             0   -3.3333e+009
             0              0   6.6667e+009             0
             0   -3.3333e+009             0    3.3333e+009
eigs =
    5.482456140350878e+007
    5.482456140350876e+007
    3.162555149229576e+007
    2.146804007227498e+006

Natural frequencies in rad/s
    7404.4     7404.4     5623.7     1465.2
```

(c)

FIGURE 11.16
(Continued)

Member 2 The relations between the local generalized coordinates and the global generalized coordinates for member 2 are $w_1 = 0$, $w_2 = W_1 \cos\theta - W_2 \sin\theta$. The transformation between the nonzero local generalized coordinate and the global generalized coordinates written in matrix from is

$$[w_2] = [\cos\theta \ -\sin\theta \quad 0 \quad 0]\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} \tag{b}$$

The contributions to the global mass and stiffness matrices from element 2 are obtained by using Equation (10.43) with $\mathbf{S}_2 = [\cos\theta \ -\sin\theta \quad 0 \quad 0]$. Note that since $w_1 = 0$, the element mass and stiffness matrices in terms of the local generalized coordinate are

$$\mathbf{m}_2 = \frac{\rho A l_2}{6}[2] \qquad \mathbf{k}_2 = \frac{EA}{l_2}[1] \tag{c}$$

Thus the contribution to the global mass matrix for element 2 is

$$\tilde{\mathbf{M}}_2 = \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \\ 0 \end{bmatrix}\left(\frac{\rho A l_2}{6}\right)[2]\,[\cos\theta \ -\sin\theta \quad 0 \quad 0] \tag{d}$$

$$= \frac{\rho A l_2}{3}\begin{bmatrix} \cos^2\theta & -\cos\theta\ \sin\theta & 0 & 0 \\ -\cos\theta\ \sin\theta & \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{e}$$

The contribution to the global stiffness matrix for element 2 is calculated as

$$\tilde{\mathbf{K}}_2 = \frac{EA}{l_2}\begin{bmatrix} \cos^2\theta & -\cos\theta\ \sin\theta & 0 & 0 \\ -\cos\theta\ \sin\theta & \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{f}$$

Element 3 The relations between the local generalized coordinates and the global generalized coordinates for element 3 are $w_1 = W_2$, $w_2 = W_4$. The contributions to the global mass and stiffness matrices from element 3 are

$$\tilde{\mathbf{M}}_3 = \frac{\rho A l_3}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad \tilde{\mathbf{K}}_3 = \frac{EA}{l_3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{g}$$

Element 4 The relations between the local generalized coordinates and the global generalized coordinates for element 4 are $w_1 = 0$, $w_2 = W_3$. The contributions to the global mass and stiffness matrices from element 4 are

$$\tilde{\mathbf{M}}_4 = \frac{\rho A l_4}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_4 = \frac{EA}{l_4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{h}$$

The global mass matrix is

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 + \tilde{\mathbf{M}}_4 = \frac{\rho A}{6}\begin{bmatrix} 2l_1 + 2l_2\cos^2\theta & -2l_2\cos\theta\sin\theta & 0 & 0 \\ -2l_2\cos\theta\sin\theta & 2l_3 + 2l_2\sin^2\theta & 0 & l_3 \\ 0 & 0 & 2l_4 & 0 \\ 0 & l_3 & 0 & 2l_3 \end{bmatrix} \tag{i}$$

Similar calculations lead to the global stiffness matrix

$$\mathbf{K} = EA\begin{bmatrix} \dfrac{1}{l_1} + \dfrac{(\cos^2\theta)}{l_2} & -\dfrac{(\cos\theta\sin\theta)}{l_2} & 0 & 0 \\ -\dfrac{(\cos\theta\sin\theta)}{l_2} & \dfrac{(\sin^2\theta)}{l_2} + \dfrac{1}{l_3} & 0 & -\dfrac{1}{l_3} \\ 0 & 0 & \dfrac{1}{l_4} & 0 \\ 0 & -\dfrac{1}{l_3} & 0 & \dfrac{1}{l_3} \end{bmatrix} \tag{j}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Output from a MATLAB script to determine the natural frequencies and mode shapes is given in Figure 11.16(c). Note that the results show only three distinct natural frequencies.

11.9 SUMMARY

11.9.1 IMPORTANT CONCEPTS

- Natural boundary conditions are those that are imposed as a result of a force balance, while geometric boundary conditions are those dictated by geometry.
- Admissible functions are functions that satisfy all geometric boundary conditions and have appropriate continuity. For a bar, this implies only that the function is continuous. For a beam, this implies that the function and its first spatial derivative are continuous.
- The assumed-modes method assumes a solution that is a linear combination of admissible functions. The coefficients in the linear combination are unknown functions of time. The linear combination is substituted into Lagrange's equations to derive a set of differential equations for the coefficients.
- The finite-element method uses piecewise defined functions as admissible functions. Only geometric boundary conditions need to be satisfied.
- The finite-element method breaks a complicated structure into element of a finite length. A piecewise defined function is assumed over each element. An elemental mass matrix, stiffness matrix, and force vector are defined.
- The local coordinates (defined for each element) are related to the global coordinates. Global mass and stiffness matrices are defined from local matrices and the transformation between the local coordinate system and the global coordinate system.
- A bar element has two degrees of freedom which are the displacement at each end of the element.
- A beam element has four degree of freedom which are the displacements and slopes at each end of the element.
- The boundary conditions are applied globally.
- The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Approximations to the mode shapes are developed from the eigenvectors.
- The forced-vibration problem can be solved as a forced-vibration problem for a discrete system.

11.9.2 IMPORTANT EQUATIONS

Assumed-modes method

$$u(x, t) = \sum_{i=1}^{n} w_i(t) u_i(x) \tag{11.1}$$

Uniform bar element

$$\mathbf{m} = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{11.23}$$

$$\mathbf{k} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{11.26}$$

$$q_1 = \int_0^\ell f(\xi, t)\left(1 - \frac{\xi}{\ell}\right)d\xi \qquad q_2 = \int_0^\ell f(\xi, t)\frac{1}{\ell}\,d\xi \tag{11.28}$$

Uniform beam element

$$w(x, \xi) = \left(1 - 3\frac{\xi^2}{\ell^2} + 2\frac{\xi^3}{\ell^3}\right)w_1 + \left(\frac{\xi}{\ell} - 2\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right)w_2 + \left(3\frac{\xi^2}{\ell^2} - 2\frac{\xi^3}{\ell^3}\right)w_3 + \left(-\frac{\xi^2}{\ell^2} + \frac{\xi^3}{\ell^3}\right)w_4 \tag{11.39}$$

$$\mathbf{m} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{11.42}$$

$$\mathbf{k} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{11.45}$$

$$q_1 = \int_0^\ell f(\xi, t)\left(1 - 3\frac{\xi^2}{\ell^2} + 2\frac{\xi^3}{\ell^3}\right)d\xi \tag{11.46}$$

$$q_2 = \int_0^l f(\xi, t)\left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)d\xi \tag{11.47}$$

$$q_3 = \int_0^l f(\xi, t)\left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}\right)d\xi \tag{11.48}$$

$$q_4 = \int_0^l f(\xi, t)\left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)d\xi \tag{11.49}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 11.1 through 11.10, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

11.1 A piecewise continuous function that satisfies the boundary conditions is an admissible function for approximation of the natural frequencies of a beam.

11.2 The boundary condition at a free end for a bar is a geometric boundary condition.

11.3 The boundary conditions at a free end for a beam are natural boundary conditions.

11.4 A beam element has four degrees of freedom.

11.5 A finite-element model of a bar with n elements predicts n natural frequencies of the bar.

11.6 Natural frequency approximations using the finite element method are determined as the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ where $\mathbf{M}$ is the global mass matrix and $\mathbf{K}$ is the global stiffness matrix.

11.7 The finite-element method can be used to approximate the displacement of a system subject to initial conditions.

11.8 The global generalized coordinates for a pinned-pinned beam are an accumulation of the local generalized coordinates.

11.9 The stiffness matrix for an interior element of length ℓ for a variable area bar is

$$k = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

11.10 The functions $w_1(x) = x + 1$ and $w_2(x) = x^2 + 1$ can be used as trial functions using the assumed-mode method to predict the lowest natural frequencies of a fixed-free bar.

Problems 11.11 through 11.23 require a short answer.

11.11 What is an admissible function?

11.12 What are natural boundary conditions?

11.13 Give a summary of the assumed-modes method.

11.14 A finite-element model of a bar fixed at $x = 0$ at one end and having a mass m rigidly attached at $x = L$ must satisfy what boundary condition?

11.15 A finite-element model of a torsional shaft that is attached to a spring of torsional stiffness k_{t1} at $x = 0$ and a spring of torsional stiffness k_{t2} at $x = L$ must satisfy what boundary conditons?

11.16 A torsional bar element has two degrees of freedom. What are the generalized coordinates associated with these degrees of freedom?

11.17 What are the local generalized coordinates associated with a beam element.

11.18 How many degrees of freedom are there in a three-element model of a fixed-free bar?

11.19 How many degrees of freedom are there in a two-element model of a fixed-fixed shaft with a rotor at its midspan?

11.20 How many degrees of freedom are there in a two-element model of a fixed-fixed beam?
11.21 How many degrees of freedom are there in a two-element model of a fixed-pinned beam?
11.22 How many degrees of freedom are there in a three-element model of a fixed-free beam?
11.23 How many degrees of freedom are there in a three-element model of beam fixed at one end and attached to a linear spring at its other end?

Problems 11.24 through 11.33 require a short calculation.

11.24 Use a one-element, finite-element model to approximate the lowest natural frequency of a bar (elastic modulus E, density ρ, area A and length L) that is fixed at one end and attached to a discrete spring of stiffness $EA/2L$ at its other end.
11.25 Use a one-element, finite-element model to approximate the lowest torsional natural frequency of a uniform shaft with a length L, polar moment of inertia J, is made from an elastic material of density ρ, and has a shear modulus G that is fixed at one end and has a torsional spring of stiffness k_t at its other end.
11.26 Use a one-element, finite-element model to approximate the lowest torsional natural frequency of a uniform shaft with a length L polar moment of inertia J, is made from an elastic material of density ρ, and has a shear modulus G that is fixed at one end and has a rigid disk with a moment of inertia I attached at its free end.
11.27 Use a one-element, finite-element model to approximate the steady-state amplitude of a uniform bar with a length L, cross-sectional area A, is made from an elastic material of density ρ, and has an elastic modulus G that is fixed at one end and has harmonic force $f(t) = F_0 \sin \omega t$ applied at its free end.
11.28 Develop the element mass matrix for a bar element that is circular in cross section but has a linearly varying radius over the element. The radius is r_1 at $\xi = 0$ and is r_2 at $\xi = \ell$.
11.29 Develop the element stiffness matrix for a bar element that is circular in cross section but has a linearly varying radius over the element. The radius is r_1 at $\xi = 0$ and is r_2 at $\xi = \ell$.
11.30 Develop the element mass matrix for a bar element that is made of a material of varying density. The density varies linearly over the element and is ρ_1 at $\xi = 0$ and ρ_2 at $\xi = \ell$.
11.31 Use a one-element, finite-element model to predict the lowest natural frequency of a beam with a length L, cross-sectional area A, mass moment of inertia I, is made from a material of mass density ρ, and has an elastic modulus E that is fixed at one end and attached to a linear spring of stiffness k at the other end.
11.32 A concentrated load $f(t) = F_0 \sin \omega t$ is acting at the midspan of a simply supported beam with a length L, cross-sectional area A, mass moment of inertia I, is made from a material of mass density ρ, and has an elastic modulus E. Use a one-element, finite-element model to predict the displacement of the midspan of the beam.
11.33 A concentrated load $f(t) = F_0 \sin \omega t$ is applied to the end of a uniform fixed-free beam. Use a one-element, finite element model to predict the steady-state amplitude of displacement of the end of the beam.

CHAPTER PROBLEMS

It may be convenient to use MATLAB to perform natural frequency calculations as well as to solve for forced responses.

11.1 The potential energy scalar product for a uniform bar is defined as

$$(f, g)_v = \int_0^L EAf(x)\frac{d^2g}{dx^2}\,dx$$

Consider the cases where (a) the bar is fixed at $x = 0$ and free at $x = L$ and (b) the bar is fixed at $x = 0$ and attached to a linear spring of stiffness k at $x = L$. Discuss, in each case, the implication of requiring $f(x)$ and $g(x)$ to satisfy only the geometric boundary conditions.

11.2 Use the assumed modes method with trial functions

$$w_1(x) = \sin\left(\pi\frac{x}{L}\right) \qquad w_2(x) = \sin\left(2\pi\frac{x}{L}\right) \qquad w_3(x) = \sin\left(3\pi\frac{x}{L}\right)$$

to approximate the lowest natural frequency and its corresponding mode shape for a uniform fixed-fixed bar of length L.

11.3 Let $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$ be linearly independent polynomials of degree four or less that satisfy the geometric boundary conditions for a bar fixed at $x = 0$ and attached to a spring of stiffness k at $x = L$.

(a) Determine a set of $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$.
(b) Use the assumed modes method with the functions obtained in part (a) as trial functions and $kL^3/EI = 0.5$ to approximate the system's lowest natural frequencies and mode shapes.

11.4 Use the assumed modes method with trial functions

$$w_1(x) = x(x - L) \qquad w_2(x) = x(x - L)^2 \qquad w_3(x) = x(x - L)^3$$

to approximate the two lowest natural frequencies and mode shapes for a simply supported beam.

11.5 Repeat Chapter Problem 11.4 if the beam has a machine of mass $m = 2.0\rho AL$ where ρAL is the total mass of the beam. The machine is placed at the midspan of the beam.

11.6 The mode shapes of a uniform fixed-free bar are of the form

$$\phi_n(x) = \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad n = 1, 2, 3, \ldots$$

Use the assumed modes method with $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ as trial functions to approximate the lowest natural frequency and mode shapes for the tapered bar of Figure P11.6.

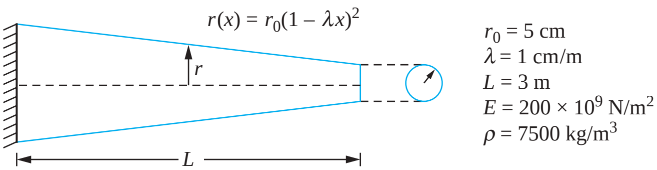


FIGURE P11.6

11.7 Use a one-element, finite-element model to approximate the lowest natural frequency of a uniform bar of mass density ρ, cross-sectional area A, elastic modulus E, and length L that is fixed at one end and has a block of mass m attached at one end.

11.8 Derive the element stiffness and mass matrices for a tapered bar of rectangular cross-section, $A(x) = A_0\,(1 - \mu x)$.

11.9 Use a one-element, finite-element model to approximate the lowest nonzero torsional natural frequency of a uniform shaft of mass density ρ, polar moment of inertia J, shear modulus G, and length L that has a thin disk of mass moment of inertia I_1 attached at one end and a thin disk of mass moment of inertia I_2 attached at the other end.

11.10 Use a one-element, finite-element model to approximate the lowest natural frequencies of a uniform beam of mass density ρ, cross-sectional area A, cross-sectional moment of inertia I, elastic modulus E, and length L that is free at both ends.

11.11 Derive the element m_{34} of the element mass matrix for a beam element.

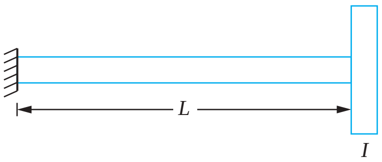
11.12 Derive the element k_{23} of the element stiffness matrix for a beam element.

11.13 Use a two-element, finite-element model to approximate the two lowest natural frequencies and their corresponding mode shapes for the system of Figure P11.13.

$E = 200 \times 10^9$ N/m^2
$A = 1.6 \times 10^{-4}$ m^2
$L = 2.5$ m
$k = 1 \times 10^7$ N/m

FIGURE P11.13

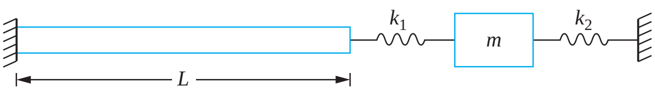
11.14 Use a two-element, finite-element model to approximate the two lowest torsional natural frequencies for the system of Figure P11.14.



$J = 3.2 \times 10^{-5}$ m^4
$G = 80 \times 10^9$ N/m^2
$\rho = 7000$ kg/m^3
$L = 4.2$ m
$I = 1.5$ kg · m^2

FIGURE P11.14

11.15 Use a three-element, finite-element model to approximate the lowest natural frequency and its corresponding mode shape for the system of Figure P11.15.



$E = 200 \times 10^9$ N/m^2
$A = 3.5 \times 10^{-5}$ m^2
$L = 2.5$ m
$\rho = 7000$ kg/m^3

$m = 1.2$ kg
$k_1 = 2 \times 10^6$ N/m
$k_2 = 1.4 \times 10^6$ N/m

FIGURE P11.15

11.16 Use a three-element, finite-element model to approximate the steady-state response of the system of Figure P11.16.

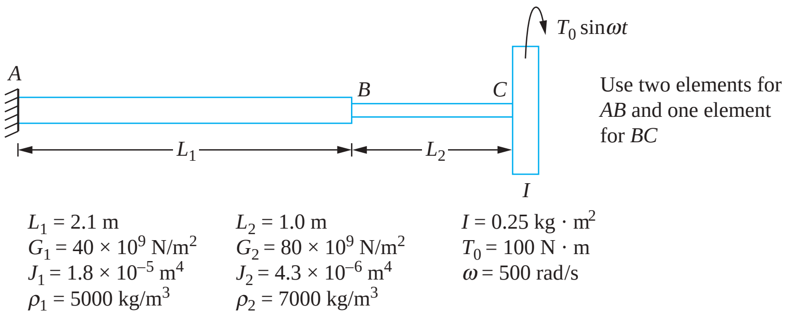


FIGURE P11.16

11.17 Use a three-element, finite-element model to approximate the forced response of the system of Figure P11.15 when the end of the bar is subject to the excitation of Figure P11.17.

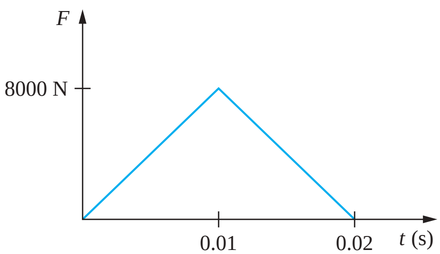


FIGURE P11.17

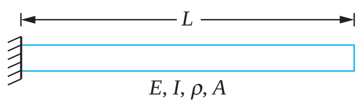


FIGURE P11.18

11.18 Use a two-element, finite-element model to approximate the two lowest natural frequencies of transverse vibration of the beam of Figure P11.18.

11.19 Use a two-element, finite-element model to approximate the lowest natural frequencies of the beam of Figure P11.19.

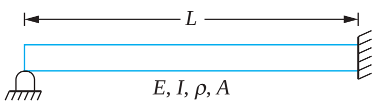


FIGURE P11.19

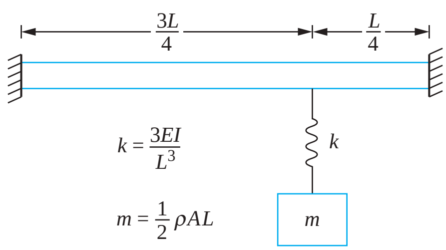


FIGURE P11.20

11.20 Use a two-element, finite-element model to approximate the two lowest natural frequencies of the system of Figure P11.20. Use elements of equal length.

11.21 Use a three-element, finite-element model to approximate the three lowest natural frequencies of the system of Figure P11.21.

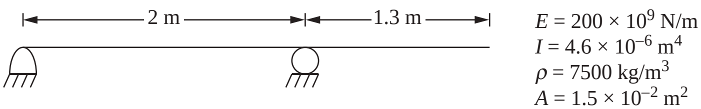


FIGURE P11.21

11.22 Use a two-element, finite-element model to approximate the lowest natural frequency of the system of Figure P11.22.

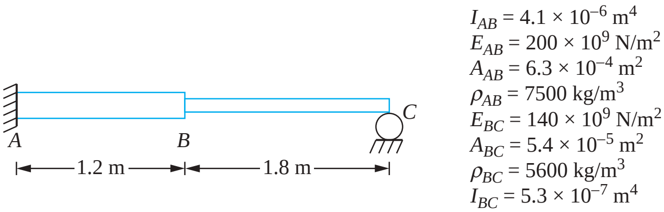


FIGURE P11.22

11.23 Use a two-element, finite-element model for the beam to approximate the two lowest natural frequencies of the system of Figure P11.23.

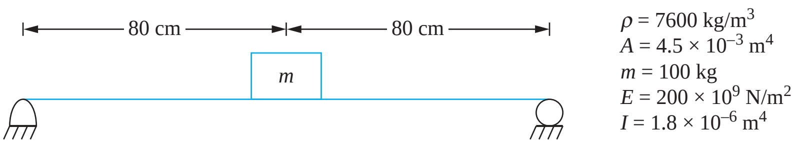


FIGURE P11.23

11.24 Use a two-element, finite-element model to approximate the two lowest natural frequencies of the system of Figure P11.24.

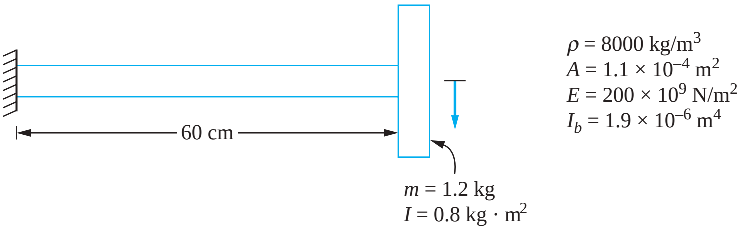


FIGURE P11.24

11.25 Use a three-element, finite-element model to approximate the steady-state amplitude of the machine of the system of Figure P11.25.

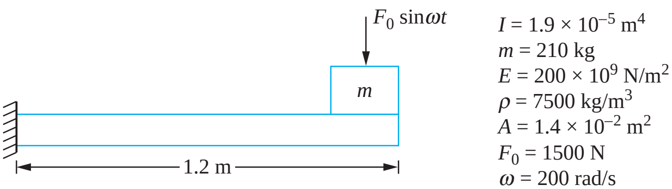


FIGURE P11.25

11.26 Use a three-element, finite-element model to approximate the steady-state amplitude of the machine of the system of Figure P11.26.

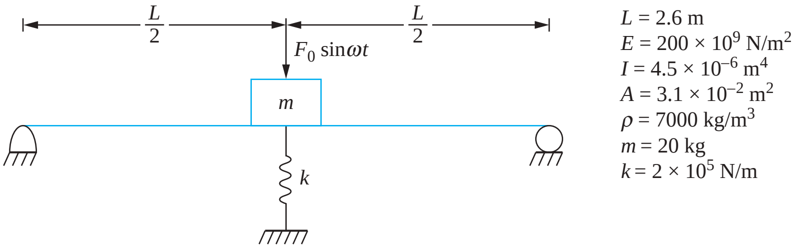


FIGURE P11.26

11.27 The street light has a mass of 25 kg. The wind velocity is 60 m/s, but the force distribution is as shown in Figure P11.27. Use a three-element, finite-element model of the structure to approximate the steady-state amplitude of the light.

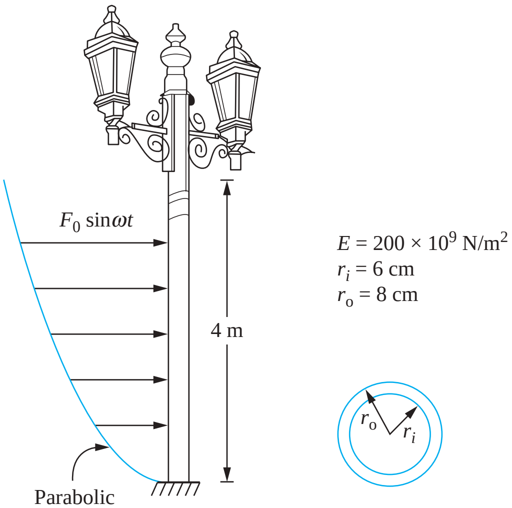


FIGURE P11.27

11.28 Use a three-element, finite-element model to approximate the steady-state response of the system of Figure P11.28.

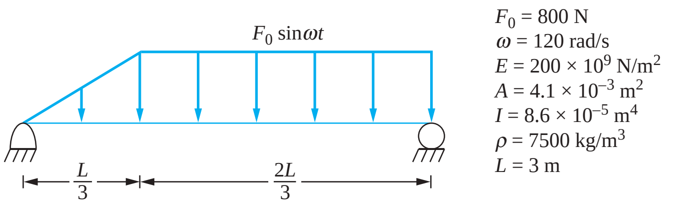


FIGURE P11.28

11.29 A plate and girder bridge is modeled as a simply supported beam, as illustrated in Figure P11.29. A vehicle is traveling across the bridge with the velocity *v.* Use a three-element, finite-element model of the bridge to determine the time-dependent response of the structure as the vehicle is crossing the bridge.

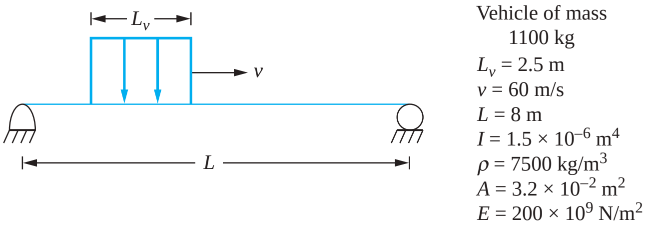


FIGURE P11.29

11.30 A simple model of a one-story frame structure is shown in Figure P11.30(a). Use one beam element to model each of the columns and two bar elements to model the girder. Determine the response of the structure if it is subject to the blast force of Figure P11.30(b).

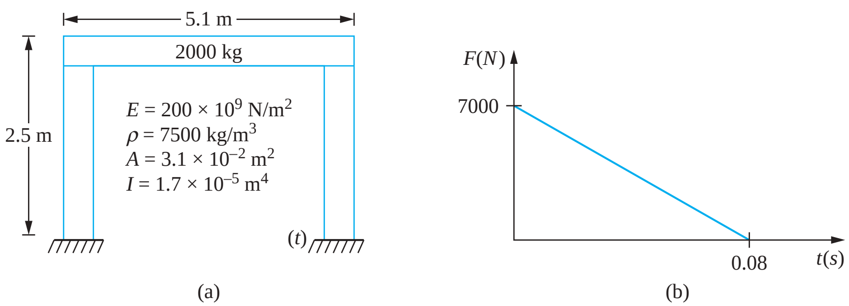


FIGURE P11.30

11.31 Use the finite-element model of Chapter Problem 11.30 to determine the response of the structure if it is subject to the earthquake of Figure P11.31.

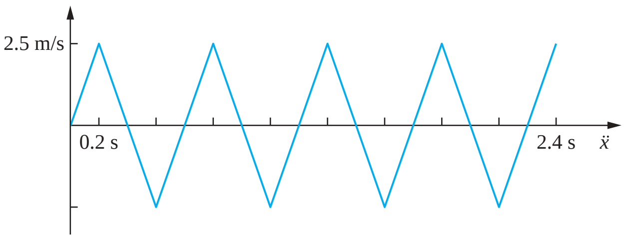


FIGURE P11.31

11.32 Use the finite-element model of Chapter Problem 11.30 to determine the response of the structure if HVAC equipment on the girder produces a lateral harmonic force of magnitude 3000 N at a frequency of 500 rpm.

11.33 Use two bar elements to model each member of the truss of Example 11.12 and approximate the three lowest natural frequencies of the truss.

11.34 Use one bar element to model each member of the truss of Figure P11.34 and approximate its two lowest natural frequencies.

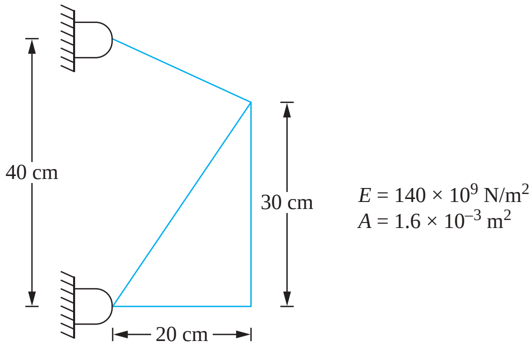


FIGURE P11.34

11.35 Use one bar element to model each member of the truss of Figure P11.35 and approximate its two lowest natural frequencies.

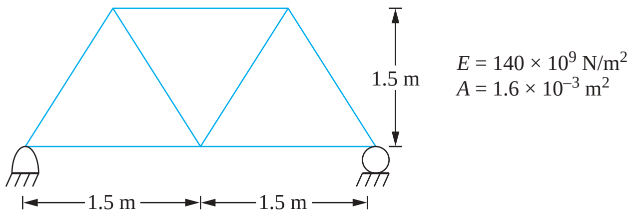


FIGURE P11.35

11.36 A beam is placed on an elastic foundation whose stiffness per unit length is k. Derive the element k_{23} of the local stiffness matrix for a beam element of length l including the stiffness of the elastic foundation.

11.37 A beam is subject to a constant axial load of magnitude P, which is applied along the beam's neutral axis. Derive the element k_{31} of the local stiffness matrix for a beam element of length l, including the effect of transverse displacement due to the axial load.

11.38 A beam is rotating about an axis with an angular velocity ω. Determine the element m_{13} of the local mass matrix for a beam element of length l, including the kinetic energy due to the rotation of the beam. The left end of the element is a distance r from the axis of rotation.

Chapter 12

NONLINEAR VIBRATIONS

12.1 INTRODUCTION

All physical systems are inherently nonlinear. Often assumptions and approximations are made such that the mathematical problem governing the behavior of the system is linear. This is done for an obvious reason; the solution of a linear problem is much easier than the solution of a nonlinear problem. Often, the results obtained using the linear approximation are sufficient for engineering work. Except for the discussions of free and forced oscillations when Coulomb damping is present, this text has thus far considered only linear systems.

Nonlinear systems are much more difficult to analyze than linear systems because the principle of linear superposition is not valid for nonlinear systems. Among the ramifications of the absence of a superposition principle are

- The homogeneous solution of a second-order nonlinear differential equation is not a linear combination of two linearly independent solutions.
- The general solution of a nonlinear differential equation cannot be written as the sum of a homogeneous solution and a particular solution, which is independent of initial conditions. The forced response of a nonlinear system cannot be separated from its free-vibration response.
- The method of superposition cannot be used to add the forced responses due to a combination of excitations. The nonlinearity causes the responses to interact.

- Since the convolution integral is derived by using linear superposition, it does not apply to nonlinear systems. There is no equivalent of the convolution integral for nonlinear systems.
- The Laplace transform cannot be used to derive the solution of nonlinear differential equations.

The focus of this chapter is on the qualitative analysis of nonlinear systems. Quantitative results are presented to show how the nonlinearities act to produce nonlinear phenomena.

12.2 SOURCES OF NONLINEARITY

Let $x_1, x_2, \ldots, x_n$ be the generalized coordinates for a conservative n degree-of-freedom system. The kinetic energy of the system is a function of the generalized coordinates and their derivatives

$$T = T(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \tag{12.1}$$

The potential energy of the system is a function of the generalized coordinates

$$V = V(x_1, x_2, \ldots, x_n) \tag{12.2}$$

If the system is linear, then its kinetic energy is independent of the generalized coordinates and is a quadratic function of their derivatives. A conservative system is nonlinear if either the kinetic or potential energy cannot be written in a quadratic form.

The kinetic energy function contains terms other than quadratic terms when the inertia properties of the system are dependent on the generalized coordinates or other kinematic relationships between the generalized coordinates are nonlinear. Nonlinear terms due to the latter are called *geometric nonlinearities.*

Terms other than quadratic terms appear in the potential energy function because of geometric nonlinearities or nonlinear force-displacement relations in flexible elements. Nonlinear terms due to the latter are called *material nonlinearities.*

EXAMPLE 12.1

Derive the governing differential equation for the simple pendulum of Figure 12.1.

SOLUTION

The kinetic energy function for the pendulum is

$$T = \frac{1}{2}m(l\dot{\theta})^2 \tag{a}$$

With the plane of the support as the datum,

$$V = -mgl\cos\theta \tag{b}$$

The kinetic energy function is quadratic, but the potential energy function is not. The nonquadratic term in the potential energy function is a result of the geometric relationship between the instantaneous position of the particle and the datum.

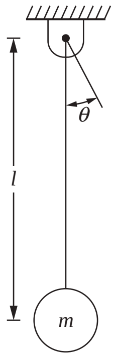


FIGURE 12.1
The differential equation governing oscillations of the simple pendulum of Example 12.1 is nonlinear.

Lagrange's equation, Equation (7.10), is applied with $L = T - V$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{c}$$

giving $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{d}$$

The nonlinear term in the differential equation of Example 12.1 is a transcendental function of the dependent variable. Approximate solutions to such equations are made by replacing the transcendental function by its Taylor series expansion. For the equation of Example 12.1, this leads to

$$\ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right) = 0 \tag{12.3}$$

Approximations can be made by assuming θ is small. A linear approximation is obtained by ignoring all but the linear terms. The simplest nonlinear approximation is obtained by keeping only the largest nonlinear term. Since this term is proportional to the cube of the dependent variable, the nonlinearity is called a *cubic nonlinearity.*

EXAMPLE 12.2

Derive the differential equations governing the motion of the system of Figure 12.2.

SOLUTION

Let x, the change in length of the spring from its length when the system is in equilibrium with a length l, and θ be the generalized coordinates. The system's kinetic energy function is

$$T = \frac{1}{2}m[\dot{x}^2 + (l + x)^2\dot{\theta}^2] \tag{a}$$

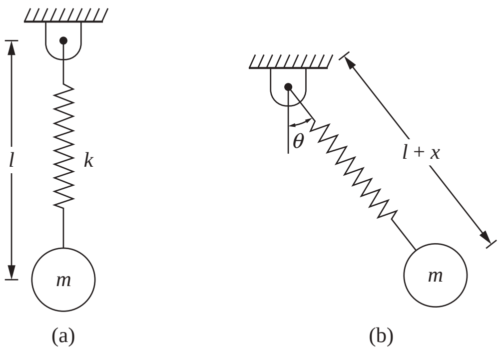


FIGURE 12.2
(a) The "swinging spring" in equlibrium; (b) the oscillations of the swinging spring are described by coupled nonlinear differential equations. The coupling occurs only in the nonlinear terms. The linear approximation calculating the extensional mode is uncoupled from the swinging mode.

Assuming the spring is linear and using the plane of the support as the datum, the system's potential energy function is

$$V = \frac{1}{2}k\left(x + \frac{mg}{k}\right)^2 - mg(l + x)\cos\theta \tag{b}$$

Application of Lagrange's equations leads to

$$m\ddot{x} + kx - m(l + x)\dot{\theta}^2 + mg(1 - \cos\theta) = 0 \tag{c}$$

and $$m(l + x)^2\ddot{\theta} + m(l + x)g\sin\theta + 2m(l + x)\dot{x}\dot{\theta} = 0 \tag{d}$$

If x and θ are assumed small, Taylor series expansions used for the transcendental functions, and only linear terms retained, the differential equations of Example 12.2 becomes

$$m\ddot{x} + kx = 0 \tag{12.4}$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{12.5}$$

Thus, a linear approximation predicts two uncoupled modes: a spring mode with a natural frequency of $\sqrt{k/m}$ and a pendulum mode with a natural frequency of $\sqrt{g/l}$. Coupling occurs only in the nonlinear terms. If only the largest nonlinear terms are retained, the governing differential equations become

$$m\ddot{x} + kx - ml\dot{\theta}^2 + \frac{mg}{2}\theta^2 = 0 \tag{12.6}$$

$$l\ddot{\theta} + g\theta + \frac{g}{l}\theta x + 2\dot{x}\dot{\theta} = 0 \tag{12.7}$$

Since the largest nonlinear terms involve quadratic products of the generalized coordinates and their derivatives, the nonlinearities are termed *quadratic.*

Note that l is not the unstretched length of the spring, but its length when the system is in static equilibrium, $l = l_0 + mg/k$. Hence, the effect of gravity causing a static spring force does not cancel with the static spring force in a nonlinear differential equation. Both must be included in the potential energy formulation.

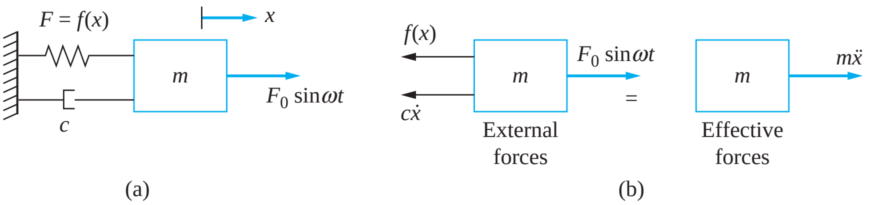


FIGURE 12.3
(a) Model system for an SDOF system with a nonlinear elastic element, viscous damping and harmonic excitation; (b) FBDs used to derive Equation (12.4). Nonlinear terms are due to a material nonlinearity.

A *material nonlinearity* occurs when a flexible component has a nonlinear constitutive equation. The system of Figure 12.3 is used to model most one-degree-of-freedom systems with viscous damping and harmonic excitation. If the spring has a force-displacement relation of the form

$$F = f(x) \tag{12.8}$$

where f is a nonlinear function of x, then the governing differential equation is nonlinear, as

$$m\ddot{x} + c\dot{x} + f(x) = F_0 \sin \omega t \tag{12.9}$$

If the spring is unstretched when it is unloaded, then a Taylor series expansion is used to expand $f(x)$ about $x = 0$. If the spring has the same properties in compression as in tension, only odd powers of x appear in the expansion:

$$m\ddot{x} + c\dot{x} + k_1 x + k_3 x^3 + \cdots = F_0 \sin \omega t \tag{12.10}$$

The values of the coefficients in the Taylor series expansion should decrease as the power increases. The expansion is usually truncated after the cubic term, leading to

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x + \alpha\omega_n^2 x^3 = \frac{F_0}{m} \sin \omega t \tag{12.11}$$

where ω_n is the natural frequency of the corresponding linear system, ζ is the damping ratio for the linear system, and

$$\alpha = \frac{k_3}{k_1} \tag{12.12}$$

A spring for which α is positive is called a *hardening spring*. A spring for which α is negative is called a *softening spring*.

Equation (12.11) is called *Duffing's equation*. Duffing's equation is nondimensionalized by introducing

$$x^* = \frac{x}{\Delta} \qquad t^* = \omega_n t \tag{12.13}$$

where $$\Delta = \frac{mg}{k_1} \tag{12.14}$$

is the static deflection of a linear spring of stiffness k_1. Substituting Equation (12.13) into Equation (12.11), rearranging, and dropping the * from the nondimensional variables leads to

$$\ddot{x} + 2\zeta\dot{x} + x + \epsilon x^3 = \Lambda \sin rt \tag{12.15}$$

where

$$r = \frac{\omega}{\omega_n} \tag{12.16}$$

$$\Lambda = \frac{F_0}{m\omega_n^2\Delta} \tag{12.17}$$

and

$$\epsilon = \alpha\Delta^2 \tag{12.18}$$

It is shown in Chapter 3 that the presence of some forms of damping causes nonlinear terms in the differential equation. If the damping force is a function of the velocity,

$$F_d = g(\dot{x}) \tag{12.19}$$

then for Coulomb damping

$$g(\dot{x}) = \mu mg\frac{\dot{x}}{|\dot{x}|} \tag{12.20}$$

and for aerodynamic drag

$$g(\dot{x}) = c\dot{x}^2 \tag{12.21}$$

The general form of the differential equation for a system subject to a harmonic excitation with nonlinear damping and a nonlinear flexible element is

$$m\ddot{x} + g(\dot{x}) + f(x) = F_0 \sin \omega t \tag{12.22}$$

Nonlinear terms can arise in differential equations because of an external excitation, as in the following example.

EXAMPLE 12.3

The U-tube manometer of Figure 12.4 rotates about an axis other than its centroidal axis with an angular velocity $\omega(t)$. The liquid is incompressible with a mass density ρ, the column has a total length l, and the tube has a cross-sectional area A. If the rotational speed is greater than a critical speed, then all of the fluid is drained from the left leg. Assume the column of liquid

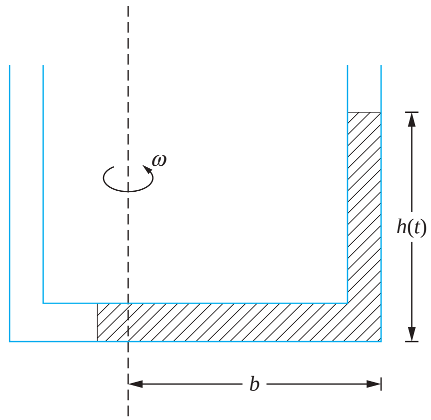


FIGURE 12.4
The oscillations of the column of liquid in a U-tube manometer rotating about a non-centroidal axis. When the angular velocity is large enough to drain fluid from the left leg, the oscillations are governed by a non-linear differential equation.

moves in the manometer as a rigid body and let $h(t)$ represent the instantaneous height of the column in the right leg. The potential energy function for this system is

$$V = \frac{1}{2}\rho g A h^2 \tag{a}$$

The system's kinetic energy function is

$$T = \frac{1}{2}\rho A l\dot{h}^2 + \frac{1}{2}\rho A b^2 h \omega^2 + \int_0^b \rho A r^2 \omega^2 dr + \int_0^{l-b-h} \rho A r^2 \omega^2 dr \tag{b}$$

Neglecting viscous friction, Lagrange's equation is applied to derive

$$l\ddot{h} + gh + \frac{\omega^2}{2}(l - b - h)^2 = \frac{\omega^2 b^2}{2} \tag{c}$$

The differential equation in Example 12.3 has a quadratic nonlinearity which is the result of the externally imposed rotation. If the speed of rotation is time-dependent, the differential equation has variable coefficients and the system is said to *parametrically excited.*

12.3 QUALITATIVE ANALYSIS OF NONLINEAR SYSTEMS

Qualitative analysis of nonlinear systems is of importance since exact analytical solutions are often not available. Qualitative analysis is used to predict general features of the motion including stability and long-time behavior.

The most useful tool for qualitative analysis of a nonlinear system is the state plane, a graphical time history of the relationship between two variables. The state plane for a one degree-of-freedom system is a family of curves showing the history of the relation between velocity and displacement. The curves in the state plane are called *trajectories.* Attractors are points or curves to which the trajectories eventually approach.

EXAMPLE 12.4

Draw the state plane for the unforced Duffing's equation with no damping for a hardening spring.

SOLUTION

Let $v = x$. Then

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{a}$$

Duffing's equation, Equation (12.11), becomes

$$v\frac{dv}{dx} = -x - \epsilon x^3 \tag{b}$$

Integrating both sides with respect to x gives

$$\frac{1}{2}v^2 = C - \frac{1}{2}x^2 - \frac{1}{4}\epsilon x^4 \tag{c}$$

where C is the constant of integration, dependent on initial conditions. The state plane for $\epsilon = \frac{1}{2}$ is shown in Figure 12.5. Different trajectories correspond to different values of C.

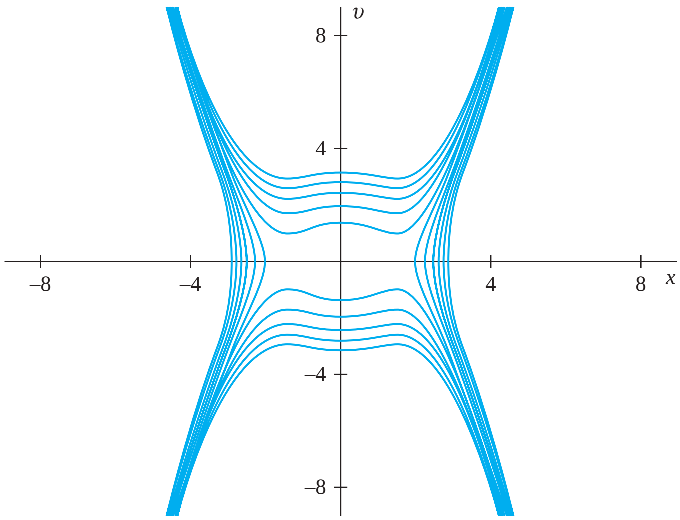


FIGURE 12.5
State plane for unforced and undamped Duffing's equation.

The system of Figure 12.3 is in equilibrium when its velocity is zero and the sum of the spring force and damping force is zero. For a linear system, this occurs only when $v = 0$ and $x = 0$. A nonlinear system may have more than one equilibrium point. An equilibrium point is *stable* if trajectories approach the equilibrium point for large time. An equilibrium point is *unstable* if trajectories diverge from the equilibrium point for large time.

The equilibrium points for a system governed by Equation (12.22) are $v = 0$ and the values of x such that $f = 0$. The stability of an equilibrium point is determined by analyzing the trajectories in the vicinity of the equilibrium point. Let

$$x = x_0 + \Delta x \tag{12.23}$$

be a point in the phase plane in the vicinity of the equilibrium point, x_0. Substituting Equation (12.23) into Equation (12.22) with $F_0 = 0$ leads to

$$\Delta \ddot{x} + g(\Delta \dot{x}) + f(x_0 + \Delta x) = 0 \tag{12.24}$$

Expanding f and g about $x = x_0$ and $\dot{x} = 0$, respectively, and keeping only linear terms gives

$$\Delta \ddot{x} + \frac{dg}{d\dot{x}}(0)\Delta \dot{x} + \frac{df}{dx}(x_0)\Delta x = 0 \tag{12.25}$$

The general solution of Equation (12.25) is

$$\Delta x = Ae^{\beta_1 t} + Be^{\beta_2 t} \tag{12.26}$$

If either β_1 or β_2 have a positive real part, then the equilibrium point is unstable.

If β_1 and β_2 are real and have the same sign, the equilibrium point is called a *node*. If β_1 and β_2 are real and have different signs, the equilibrium point is called a *saddle point*,

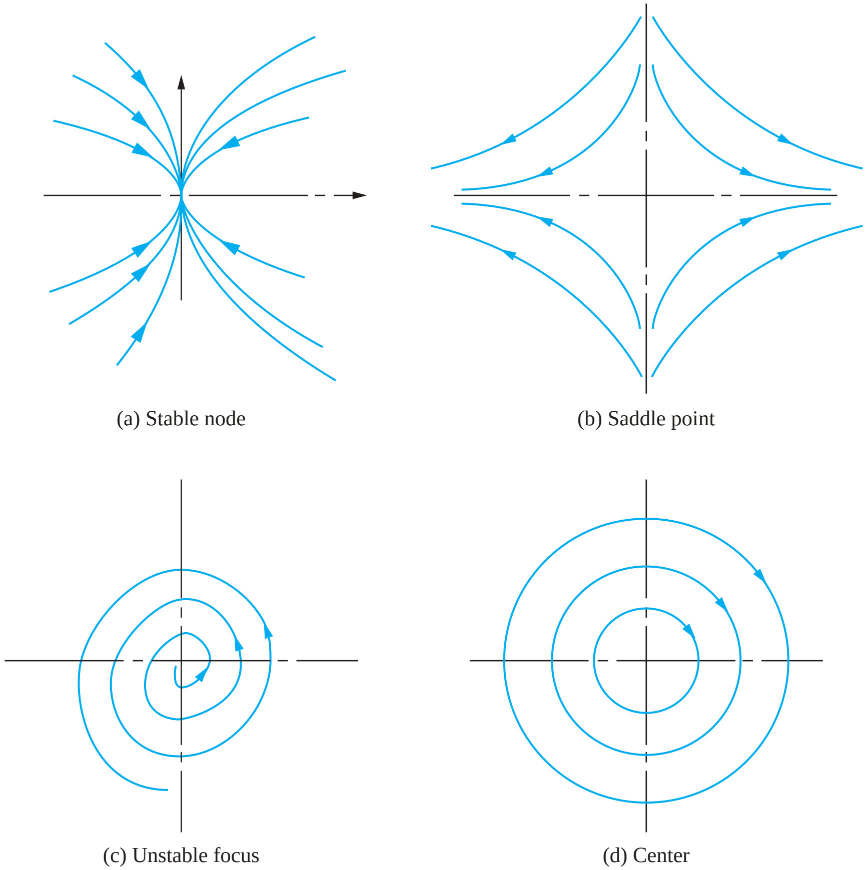


FIGURE 12.6 State planes in the vicinity of equilibrium points: (a) stable node, (b) saddle point, (c) unstable focus, and (d) center.

and is, by definition, unstable. If β_1 and β_2 are complex conjugates, the equilibrium point is called a *focus.* A special case of a focus occurs when β_1 and β_2 are purely imaginary, in which case the equilibrium point is called a *center.* Sketches of state planes in the vicinity of a node, saddle point, focus, and center are given in Figure 12.6.

EXAMPLE 12.5

Determine the equilibrium points and their nature for the damped unforced Duffing's equation.

SOLUTION

The equilibrium points are the values of x such that

$$x + \epsilon x^3 = 0 \tag{a}$$

For a hardening spring, the only equilibrium point for Duffing's equation is $x = 0$. For a softening spring, the system has the additional equilibrium points

$$x_0 = \pm\sqrt{\frac{1}{-\epsilon}} \tag{b}$$

The nature of the equilibrium point corresponding to $x_0 = 0$ is investigated by assuming $x = \Delta x$, which leads to

$$\beta_{1,2} = -\zeta \pm \sqrt{\zeta^2 - 1} \quad \text{(c)}$$

Hence, he equilibrium point $x = 0$ is a stable node if $\zeta \geq 1$, and is a stable focus if $\zeta < 1$.

For a softening spring, the natures of the additional equilibrium points are determined using

$$x = \pm\sqrt{-\frac{1}{\epsilon}} + \Delta x \quad \text{(d)}$$

Substituting into Duffing's equation and linearizing leads to

$$\Delta \ddot{x} + 2\zeta \Delta \dot{x} - 2\Delta x = 0 \quad \text{(e)}$$

and

$$\beta = -\zeta \pm \sqrt{\zeta^2 + 2} \quad \text{(f)}$$

Since the two values of β are real with opposite signs, these equilibrium points are saddle points and thus, by their very nature, unstable.

The phase plane for a system subject to a forced excitation is usually difficult to determine solely by analytical methods. Often, these phase planes must be drawn by graphical methods or numerical results. Figure 12.7 shows several phase planes corresponding to the forced Duffing's equation.

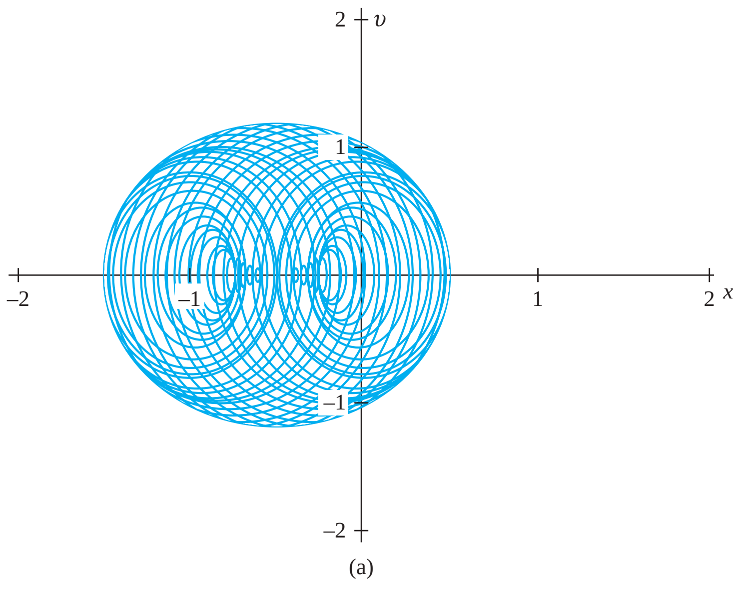


FIGURE 12.7
Examples of state planes for (a) forced, undamped Duffing's equation and (b) forced, damped Duffing's equation.

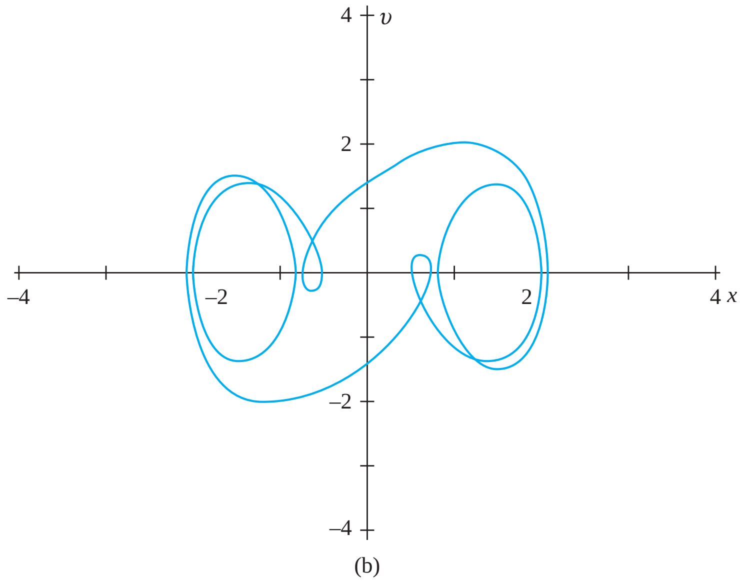


FIGURE 12.7 (*continued*)

12.4 QUANTITATIVE METHODS OF ANALYSIS

Exact solutions to nonlinear vibration problems exist only for a few special free-vibration problems. Exact solutions for nonlinear forced-vibration problems are almost nonexistent. Consider Equation (12.22) with $F_0 = 0$. Let $v = \dot{x}$. Then, using the chain rule for differentiation, as in Example 12.4, Equation (12.22) can be written as

$$v\frac{dv}{dx} + g(v) + f(x) = 0 \tag{12.27}$$

For certain forms of $g(v)$ and $f(x)$, Equation (12.27) can be integrated, yielding $v(x)$, which, in turn, can be integrated, yielding $t(x)$.

Consider an undamped system, $g(v) = 0$. Integrating Equation (12.27) with respect to x and using $x = x_0$ and $v = 0$ when $t = 0$ yields

$$v(x) = \left[2\int_x^{x_0} f(\eta)\,d\eta\right]^{1/2} \tag{12.28}$$

Rearranging and integrating with respect to x gives

$$t = \int_{x_0}^{x} \frac{d\lambda}{\left[2\int_\lambda^{x_0} f(\eta)\,d\eta\right]^{1/2}} \tag{12.29}$$

Since Equation (12.29) gives t as a function of x, it is not useful for computing the time history of motion, but can be used for frequency calculations. For many forms of $f(x)$, closed-form evaluation of the integral does not exist, and numerical integration is used.

Care must be taken when evaluating Equation (12.29) numerically because the integrand is singular for $\lambda = 0$.

Since exact solutions are not often available, numerical solutions are used. Self-starting methods such as Runge-Kutta are convenient for numerical solution of nonlinear equations.

The general form of the equations for a nonlinear n degree-of-freedom system is

$$\begin{aligned}\ddot{x}_1 &= h_1(\mathbf{x}, \dot{\mathbf{x}}, t)\\ \ddot{x}_2 &= h_2(\mathbf{x}, \dot{\mathbf{x}}, t)\\ &\vdots\\ \ddot{x}_n &= h_n(\mathbf{x}, \dot{\mathbf{x}}, t)\end{aligned} \tag{12.30}$$

Let $\mathbf{v} = \dot{\mathbf{x}}$ and $\mathbf{x}$ be independent n-dimensional vectors. Equation (12.30) can be rewritten as two systems of first-order equations

$$\begin{aligned}\frac{dx_1}{dt} &= v_1 &\qquad \frac{dv_1}{dt} &= h_1(\mathbf{x}, \mathbf{v}, t)\\ \frac{dx_2}{dt} &= v_2 &\qquad \frac{dv_2}{dt} &= h_2(\mathbf{x}, \mathbf{v}, t)\\ &\vdots & &\vdots\\ \frac{dx_n}{dt} &= v_n &\qquad \frac{dv_n}{dt} &= h_n(\mathbf{x}, \mathbf{v}, t)\end{aligned}$$

Analytical solutions are preferable to numerical solutions because they can be used to predict trends, analyze the effects of parameters, and develop qualitative results. Thus, approximate analytical methods are often used to approximate the solution of nonlinear problems.

If the magnitude of the nonlinear term is small or the amplitude of motion is small, then a perturbation method can be used to develop an approximate solution. Let ϵ be a small nondimensional parameter, $\epsilon \ll 1$. The small parameter may be a measure of the amplitude or a measure of the nonlinearity. For a one degree-of-freedom system, the generalized coordinate is expanded in a series of powers of ϵ,

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{12.31}$$

Equation (12.31) is substituted into the governing differential equation. Coefficients of like powers of ϵ are collected and set to zero independently. The result is a set of linear differential equations that are successively solved for $x_i(t)$, $i = 1, 2, \ldots$.

The series of Equation (12.31) is convergent. However, it converges slowly and thus a finite number of terms are inadequate to represent the solution for all t. When only a few terms are included, nonperiodic terms appear which cause the solution to be unbounded for large t. The terms which produce these nonuniformities are called *secular terms.* Since it is impossible to include an infinite number of terms in the evaluation, the secular terms must be removed. A variety of perturbation methods have been developed to remove secular terms. These include the method of strained parameters, the method of renormalization, the method of multiple scales, and the method of averaging. The application of these methods to nonlinear oscillation problems is beyond the scope of this book, but an exhaustive treatment is found in Nayfeh and Mook. The method of renormalization is illustrated in Section 12.5.

12.5 FREE VIBRATIONS OF SDOF SYSTEMS

The free vibrations of a conservative system are periodic. If the spring in the system of Figure 12.3 has the same properties in compression as in tension, then each period of motion can be broken into four parts, each of which takes the same amount of time. If the mass is displaced a distance x_0 from equilibrium and released from rest, the period of the resulting motion can be calculated by using Equation (12.29) as four times the time it takes the mass to go from its initial position to $x = 0$,

$$T = \frac{4}{\sqrt{2}}\int_{x_0}^{0}\frac{d\lambda}{\left[\int_{\lambda}^{x_0} f(\eta)\,d\eta\right]^{1/2}} \tag{12.32}$$

Equation (12.32) shows that, in contrast to a linear system, the period and the corresponding natural frequency for a nonlinear system depend on the initial conditions.

EXAMPLE 12.6

A mass, attached to a softening spring with a cubic nonlinearity, is displaced a nondimensional distance x_0 from equilibrium and released from rest. Determine the period of the resulting oscillations as a function of ϵ and x_0.

SOLUTION

In the notation of Section 12.2 and Equations (12.10) through (12.15), the nondimensional force developed in a softening spring is

$$f(x) = x - \epsilon x^3 \qquad \epsilon = \alpha\Delta^2 \tag{a}$$

Thus, the nondimensional period is determined from Equation (12.32)

$$T = \frac{4}{\sqrt{2}}\int_{x_0}^{0}\frac{d\lambda}{\left[\int_{\lambda}^{x_0}(\eta - \epsilon\eta^3)\,d\eta\right]^{1/2}} \tag{b}$$

where x_0 is the nondimensional initial displacement. The dimensional period is the nondimensional period divided by the linear natural frequency. Proceeding with the algebra gives

$$T = \frac{4}{\sqrt{2}}\int_{x_0}^{0}\frac{d\lambda}{\left[\dfrac{x_0^2}{2} - \epsilon\dfrac{x_0^4}{4} - \dfrac{\lambda^2}{2} + \epsilon\dfrac{\lambda^4}{4}\right]^{1/2}} = \frac{4\sqrt{2}}{x_0\sqrt{\epsilon}}\int_{0}^{1}\frac{d\phi}{\sqrt{\dfrac{2}{\epsilon x_0^2} - 1 - \dfrac{2}{\epsilon x_0^2}\phi^2 + \phi^4}}$$

$$= \frac{4\sqrt{2}}{\sqrt{2 - \epsilon x_0^2}}\int_{0}^{1}\frac{d\phi}{(1-\phi^2)(1-k^2\phi^2)} = \frac{4\sqrt{2}}{\sqrt{2 - \epsilon x_0^2}}F\left(k, \frac{\pi}{2}\right) \tag{c}$$

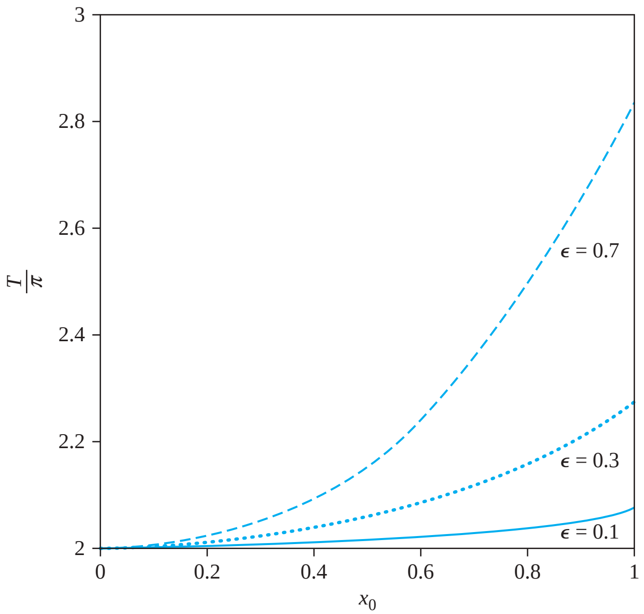


FIGURE 12.8
Period of Duffing's equation as function of displacement x_0 for several values of ε.

where $F(k, \pi/2)$ is the complete elliptic integral of the first kind of argument k, where

$$k = \sqrt{\frac{2 - \epsilon x_0^2}{\epsilon x_0^2}} \tag{d}$$

A table of elliptic integrals, such as in Abramowitz and Stegun, is used to generate Figure 12.8.

When the integral of Equation (12.32) cannot be evaluated in closed form, numerical integration must be used. However, the integrand is singular at $\lambda = x_0$. Let δ be a small nondimensional value. Then for the system of Example 12.6,

$$T = \frac{4\sqrt{2}}{x_0\sqrt{\epsilon}}\left[\int_0^{1-\delta} \frac{d\phi}{\sqrt{\frac{2}{\epsilon x_0^2} - 1 - \frac{2}{\epsilon x_0^2}\phi^2 + \phi^4}} + \int_{1-\delta}^{1} \frac{d\phi}{\sqrt{\frac{2}{\epsilon x_0^2} - 1 - \frac{2}{\epsilon x_0^2}\phi^2 + \phi^4}}\right] \tag{12.33}$$

The first integral is evaluated by numerical integration. The integrand of the second integral is expanded by the binomial theorem, and the resulting expansion is integrated term by term. The expansion is truncated such that desired accuracy is achieved.

Perturbation methods can be applied to approximate the period of a nonlinear system. When the straightforward expansion, Equation (12.31), is substituted into the unforced, undamped Duffing's equation, the results are

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + x_1 + x_0^3) + \epsilon^2(\ddot{x}_2 + x_2 + 3x_0^2 x_1) + \cdots = 0 \tag{12.34}$$

Coefficients of powers of ϵ are set to zero independently, leading to a set of hierarchical equations

$$\ddot{x}_0 + x_0 = 0 \tag{12.35}$$

$$\ddot{x}_1 + x_1 = -x_0^3 \tag{12.36}$$

$$\begin{aligned} \ddot{x}_2 + x_2 &= -3x_0^2 x_1 \\ &\vdots \end{aligned} \tag{12.37}$$

The solution for x_0 is

$$x_0 = A\sin(t + \phi) \tag{12.38}$$

where A and ϕ are determined using initial conditions. Substitution of Equation (12.38) into Equation (12.36) and use of trigonometric identities lead to

$$\ddot{x}_1 + x_1 = -\frac{A^3}{4}[3\sin(t + \phi) - \sin 3(t + \phi)] \tag{12.39}$$

The particular solution of Equation (12.39) is

$$x_1(t) = \frac{A^3}{8} t\cos(t + \phi) - \frac{A^3}{32}\sin 3(t + \phi) \tag{12.40}$$

and the resulting two-term approximation for $x(t)$ is

$$x(t) = A\sin(t + \phi) + \epsilon\left[\frac{3}{8}A^3 t\cos(t + \phi) - \frac{A^3}{32}\sin 3(t + \phi)\right] + \cdots \tag{12.41}$$

Unfortunately, the expansion of Equation (12.41) is not periodic and grows without bound as t gets large. Indeed, when t is as large as $1/\epsilon$, the second term in the expansion is as large as the first term, rendering it invalid.

The problem with the straightforward expansion is that it cannot account for the variation of the period with initial conditions, as mandated by the exact solution. The method of renormalization is used to take this variation into account and render the two-term straightforward expansion uniform. A new time scale is introduced according to

$$t = w(1 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots) \tag{12.42}$$

Equation (12.41) is rewritten with w as the independent variable

$$\begin{aligned} x = {} & A\sin(w + \epsilon\lambda_1 w + \cdots + \phi) \\ & + \epsilon\left[\frac{3}{8}A^3(w + \epsilon\lambda_1 w + \cdots)\cos(w + \epsilon\lambda_1 w + \cdots + \phi)\right. \\ & \left. - \frac{A_3}{32}\sin 3(w + \epsilon\lambda_1 w + \cdots + \phi)\right] + \cdots \end{aligned} \tag{12.43}$$

Taylor series expansions are used to expand the trigonometric functions and coefficients of powers of ϵ are recollected, leading to

$$x = A\sin(w+\phi) + \epsilon\left[A\lambda_1 w\cos(w+\phi) + \frac{3}{8}A^3 w\cos(w+\phi) - \frac{A^3}{32}\sin 3(w+\phi)\right] + \cdots \tag{12.44}$$

The secular term is removed from Equation (12.44) by choosing

$$\lambda_1 = -\frac{3}{8}A^2 \tag{12.45}$$

leading to

$$x = A\sin(w+\phi) - \epsilon\frac{A^3}{32}\sin 3(w+\phi) + \cdots \tag{12.46}$$

where

$$t = w\left(1 - \epsilon\frac{3}{8}A^2 + \cdots\right) \tag{12.47}$$

The binomial expansion is used to invert Equation (12.47)

$$w = t\left(1 + \epsilon\frac{3}{8}A^2 + \cdots\right) \tag{12.48}$$

The amplitude is determined by application of the initial conditions. If $x(0) = \delta$ and $\dot{x}(0) = 0$, then

$$\phi = \frac{\pi}{2} \tag{12.49}$$

$$\delta = A - \epsilon\frac{A^3}{32} \tag{12.50}$$

A natural frequency approximation can be obtained to greater accuracy by calculating higher-order terms in the expansion for x, and choosing the λ_i from Equation (12.48) to eliminate secular terms.

For damped systems, the damping term is often small enough to be ordered with the nonlinearity. To this end, define

$$\zeta = 2\epsilon\mu \tag{12.51}$$

where μ is of order 1. When the straightforward expansion is used in the damped, unforced version of Duffing's equation, the following equations result, defining x_0 and x_1:

$$\ddot{x}_0 + x_0 = 0 \tag{12.52}$$

$$\ddot{x}_1 + x_1 = -x_0^3 - 2\mu\dot{x}_0 \tag{12.53}$$

In order to use the method of renormalization for damped systems, the solutions of Equations (12.52) and (12.53) are written using complex exponentials

$$x_0 = A\cos(t+\phi) = \frac{1}{2}A\left[e^{i(t+\phi)} + e^{-i(t+\phi)}\right] \tag{12.54}$$

When Equation (12.47) is used to remove secular terms from the two-term expansion,

$$\lambda_1 = -\frac{3}{8}A^2 - i\frac{\mu}{2} \tag{12.55}$$

and the resulting two-term uniformly valid expansion is

$$x = Ae^{-\zeta t}\sin\left[(1 + \epsilon\frac{3}{8}A^2)t + \phi\right] \tag{12.56}$$

Thus, when secular terms are removed through x_1, damping has no effect on the natural period. The exponential decay, comparable to that of a linear system, is apparent.

In summary, the natural frequency of a nonlinear system depends on its initial conditions. The straightforward perturbation expansion and the method of renormalization can be used to determine an approximation to the natural frequency when the nonlinearity is small or when the amplitude is small. Small viscous damping has an effect on free vibrations of nonlinear systems similar to that on free vibrations of linear systems, causing an exponential decay of amplitude.

12.6 FORCED VIBRATIONS OF SDOF SYSTEMS WITH CUBIC NONLINEARITIES

Consider the damped Duffing's equation subject to a two-frequency excitation,

$$\ddot{x} + 2\mu\epsilon\dot{x} + x + \epsilon x^3 = F_1\sin r_1 t + F_2\sin r_2 t \qquad r_1 \neq r_2 \tag{12.57}$$

Use of the straightforward expansion, Equation (12.31), produces the following two-term approximation to the solution of Equation (12.57):

$$\begin{aligned}
x = {} & A\sin(t + \phi) + F_1M_1\sin r_1t + F_2M_2\sin r_2t \\
& + \epsilon\Bigg\{-\mu At\sin(t + \phi) - \left(\frac{3}{8}A^3 + \frac{3}{4}AF_1^2M_1^2 + \frac{3}{4}AF_2^2M_2^2\right)t\cos(t + \phi) \\
& - \frac{2\mu F_1M_1r_1}{1 - r_1^2}\cos r_1t - \frac{2\mu F_2M_2r_2}{1 - r_2^2}\cos r_2t \\
& + \frac{3(2A^2F_1M_1 + F_1^3M_1^3 + 2F_1F_2^2M_1M_2^2)}{4(1 - r_1^2)}\sin r_1t \\
& + \frac{3(2A^2F_2M_2 + F_2^3M_2^3 + 2F_1^2F_2M_1^2M_2)}{4(1 - r_2^2)}\sin r_2t \\
& + \frac{A^3}{32}\sin 3(t + \phi) - \frac{3A^2F_1M_1}{4[1 - (2 + r_1)^2]}\sin[(2 + r_1)t + 2\phi] \\
& + \frac{3A^2F_1M_1}{4[1 - (2 - r_1)^2]}\sin[(2 - r_1)t + 2\phi] \\
& - \frac{3A^2F_1^2M_1^2}{4[1 - (1 + 2r_1)^2]}\sin[(1 + 2r_1)t + \phi]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{3AF_1^2M_1^2}{4[1-(1-2r_1)^2]}\sin[(1-2r_1)t+\phi] \\
&- \frac{3AF_2M_2}{4[1-(2+r_2)^2]}\sin[(2+r_2)+2\phi] \\
&+ \frac{3A^2F_2M_2}{4[1-(2-r_2)^2]}\sin[(2-r_2)+2\phi] \\
&- \frac{3AF_2^2M_2^2}{4[1-(1+2r_2)^2]}\sin[(1+2r_2)t+\phi] \\
&+ \frac{3AF_2^2M_2^2}{4[1-(1-2r_2)^2]}\sin[(1-2r_2)t+\phi] \\
&- \frac{3F_1^2F_2M_1^2M_2}{4[1-(2r_1+r_2)^2]}\sin(2r_1+r_2)t \\
&+ \frac{3F_1^2F_2M_1^2M_2}{4[1-(2r_1-r_2)^2]}\sin(2r_1-r_2)t \\
&- \frac{3F_1F_2^2M_1M_2^2}{4[1-(2r_2+r_1)^2]}\sin(2r_2+r_1)t \\
&+ \frac{3F_1F_2^2M_1M_2^2}{4[1-(2r_2-r_1)^2]}\sin(2r_2-r_1)t \\
&- \frac{F_1^3M_1^3}{4(1-9r_1^2)}\sin 3r_1t - \frac{F_2^3M_2^3}{4(1-9r_2^3)}\sin 3r_2t \\
&+ \frac{3AF_1F_2M_1M_2}{2[1-(r_1-r_2+1)^2]}\sin[(r_1-r_2+1)t+\phi] \\
&- \frac{3AF_1F_2M_1M_2}{2[1-(r_1-r_2-1)^2]}\sin[(r_2-r_1-1)t-\phi] \\
&- \frac{3AF-1F-2M_1M_2}{2[1-(r_1+r_2+1)^2]}\sin[(r_1+r_2+1)t+\phi] \\
&+ \left.\frac{3AF_1F_2M_1M_2}{2[1-(r_1+r_2-1)^2]}\sin[(r_1+r_2-1)t-\phi]\right\}\cdots
\end{aligned}
\tag{12.58}
$$

where

$$
M_i = \frac{1}{1-r_i^2} \tag{12.59}
$$

The expansion of Equation (12.58) is nonuniform because of the secular terms arising from the free-vibration solution. Additional nonuniformities occur when the values of r_1 and r_2 are such that the denominators of other terms are very small. Examination of Equation (12.58) suggests that an exhaustive study of the frequency response of a one degree-of-freedom system with a cubic nonlinearity requires the following cases be considered:

1. No resonances.
2. $r_1 = 1$ or $r_2 = 1$, primary resonance.
3. $r_1 = \frac{1}{3}$ or $r_2 = \frac{1}{3}$, superharmonic resonance.
4. $r_1 = 3$ or $r_2 = 3$, subharmonic resonance.
5. $2r_2 + r_1 = 1$, $2r_1 - r_2 = \pm 1$, $2r_2 - r_1 = \pm 1$, $r_1 - r_2 + 1 = -1$, $r_1 - r_2 - 1 = \pm 1$, or $r_1 + r_2 - 1 = 1$, combination resonances.
6. Conditions when two resonances occur simultaneously. For example, when $r_1 = \frac{1}{3}$ and $r_2 = \frac{5}{3}$, both superharmonic and combination resonances occur.

A resonance condition occurs when the free-vibration contribution to the solution does not decay with time. The steady-state solution has a contribution from the free vibrations as well as the forced steady-state response. For a linear system, the free-vibration response is periodic with a frequency equal to the natural frequency, and the forced response due to a harmonic excitation is periodic with a frequency equal to the excitation frequency. For a linear system, only the primary resonance can occur when the excitation frequency is near the natural frequency.

For a system with a cubic nonlinearity, Equation (12.44) shows that the free-vibration response includes a periodic term whose frequency is three times the linear natural frequency. Thus oscillations at this frequency are sustained in the absence of an external excitation. Any additional energy input may lead to growth of the free oscillations and thus produce the subharmonic resonance.

The forced response of a system with a cubic nonlinearity to a harmonic excitation includes a periodic term whose frequency is three times the excitation frequency. Thus, when the excitation frequency is one-third of the natural frequency, this term tends to excite the free vibrations and causes the free-vibration term to be sustained, even in the presence of small damping. This produces the superharmonic resonance.

When a system with a cubic nonlinearity is subject to a multifrequency excitation, the forced response includes periodic terms at frequencies that are combinations of the excitation frequencies. When this combination of frequencies is close to the natural frequency, free oscillations are sustained and a combination resonance exists.

The straightforward expansion is nonuniform for all r_1 and r_2, even when no resonance conditions exist. The method of renormalization can be used to render the two-term expansion uniform, but it can only be used to predict periodic responses, and cannot provide information about the stability of equilibrium points. Possibly the best method for obtaining uniform expansions to approximate the solution of nonlinear forced-vibration problems is the method of multiple scales. The results provided in the following discussion can be obtained using the method of multiple scales. Since its application is beyond the scope of this text, the discussion focuses on qualitative behavior. More detail is available in Nayfeh and Mook.

1. *No resonances.* For most values of r_1 and r_2, no resonance conditions exist. However, the expansion of Equation (12.58) is still nonuniform. When secular terms are removed, the solution is the sum of the free-vibration response and the forced response. The free vibrations decay exponentially, but the frequency of free vibrations depends on the initial conditions and the amplitudes and frequencies of the excitation.

2. *Primary resonance.* A primary resonance occurs when an excitation frequency is near the system's linear natural frequency, corresponding to the nondimensional frequency being near 1. When the amplitude of the excitation is of order 1, the straightforward perturbation expansion predicts an infinite amplitude response, even in the presence of small damping. When the amplitude of the excitation is the same order as the nonlinearity and the damping, secular terms occur in x_1.

The frequency response in the vicinity of the primary resonance is studied by introducing a *detuning parameter*, defined by

$$r_1 = 1 + \epsilon\sigma \tag{12.60}$$

The amplitude and phase of the resulting motion vary with time, but possible steady states can be identified. The following approximate equations can be derived for the steady-state amplitude and the steady-state phase angle:

$$4A^2\left[\mu^2 + \left(\sigma - \frac{3}{8}A^2\right)^2\right] = \hat{F}_1^2 \tag{12.61}$$

$$\phi = -\tan^{-1}\left(\frac{\mu}{\sigma - \frac{3}{8}A^2}\right) \tag{12.62}$$

where $\quad \hat{F}_1 = \dfrac{F_1}{\epsilon}$ $\quad$ **(12.63)**

Equations (12.61) and (12.62) are plotted in Figures 12.9 and 12.10. Note from these figures that there is a frequency range where three possible steady-state amplitudes and phases exist for a single frequency. This leads to an interesting phenomenon, peculiar to nonlinear systems, called the jump phenomenon. Imagine that the amplitude of excitation

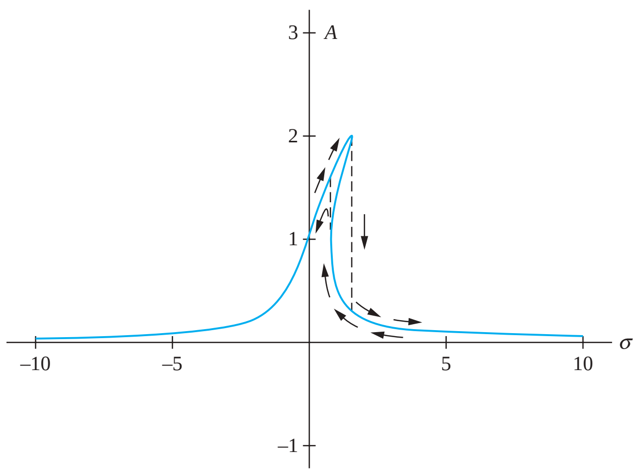


FIGURE 12.9
Frequency response curve for primary resonance of Duffing's equation illustrates the jump phenomenon ($\mu = 0.25$, $\hat{F}_1 = 1$).

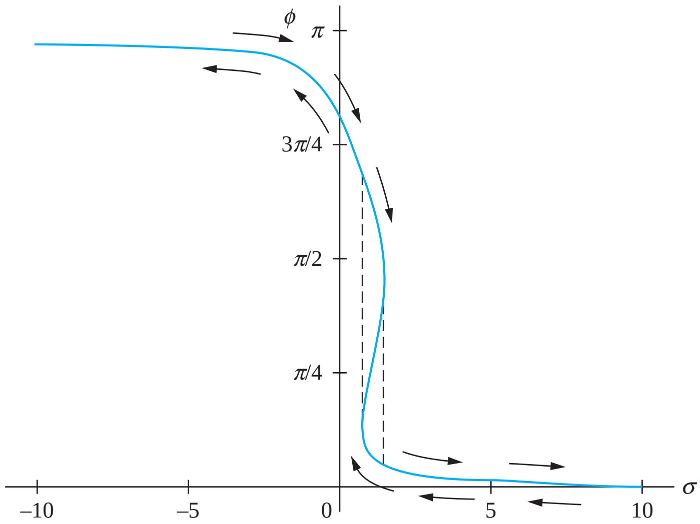


FIGURE 12.10
Phase versus frequency curve for primary resonance of Duffing's equation also shows a jump phenomenon ($\mu = 0.25$, $\hat{F}_1 = 1$).

is fixed, but its frequency is slowly increased, starting slightly below the linear natural frequency. As the frequency is increased the steady-state amplitude follows the upper branch of the frequency response curve, until the point of vertical tangency is reached. When the frequency is increased beyond this critical value, the only possible steady-state amplitude is finitely lower than the amplitude at the critical frequency, and the amplitude will "jump" to this lower value. Now if the frequency is decreased from this value, the steady-state amplitude will follow the lower branch of the frequency response curve, until the point of vertical tangency is reached, when it will "jump" to the upper branch.

A state plane showing the relation between the amplitude and phase can be plotted for Duffing's equation with a primary resonance for parameters where the triple valuedness exists. Two equilibrium points are stable foci corresponding to the points on the upper and lower branches of the frequency response curve. A third equilibrium point is a saddle point corresponding to the intermediate amplitude between the points of vertical tangency. Since this equilibrium point is unstable, it can never be physically attained. The initial conditions dictate which of the two stable foci corresponds to the steady-state solution.

3. *Superharmonic resonance.* When either r_1 or r_2 is near $\frac{1}{3}$, the free-oscillation term does not decay exponentially. The steady-state response then consists of the forced response whose period is three times that of the linear natural period plus the free response, whose frequency is adjusted to three times that of the excitation. Thus the total response is periodic with the period equal to that of the excitation.

Introduction of a detuning parameter when r_1 is near $\frac{1}{3}$, $3r_i = 1 + \epsilon\sigma$, leads to the frequency response equation

$$[\mu^2 + (\sigma - 3F_i^2 - \tfrac{3}{8}A^2)^2]A^2 = F_i^6 \tag{12.64}$$

which is cubic in A^2 and hence has three solutions. For a certain frequency range, three real solutions exist. The triple valuedness of the amplitude leads to a jump phenomenon similar to that for the primary resonance, as shown in Figure 12.11.

4. *Subharmonic resonance.* When an excitation frequency is near three times the linear natural frequency, a subharmonic resonance may occur. The frequency response curve when

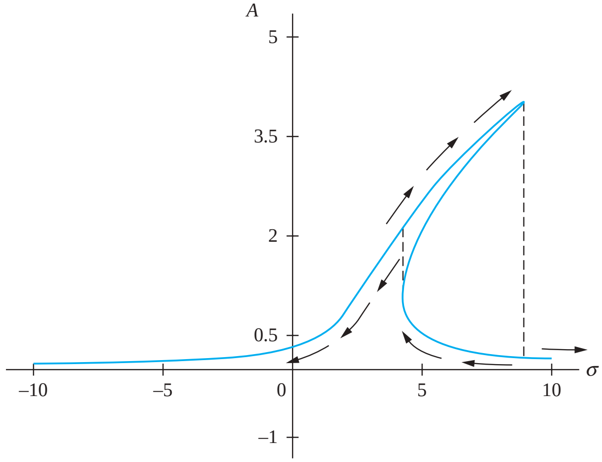


FIGURE 12.11
Frequency response curve in vicinity of superharmonic resonance ($\mu = 0.25$, $\hat{F}_1 = 1$).

r_i is near 3, $r_i = 3 + \epsilon\sigma$ is given by

$$\left[9\mu^2 + \left(\sigma - 9F_i^2 - \frac{9}{8}A^2\right)^2\right]A^2 = \frac{81}{16}F_i^2 A^4 \tag{12.65}$$

Equation (12.65) has the trivial solution, $A = 0$, and two solutions obtained as roots of a quadratic equation in A^2. The quadratic equation yields real solutions for A if and only if the parameters satisfy the following inequality:

$$\frac{\sigma}{\mu} - \sqrt{\left(\frac{\sigma}{\mu}\right)^2 - 63} \le \frac{63F_1^2 M_1^2}{4\mu} \le \frac{\sigma}{\mu} + \sqrt{\left(\frac{\sigma}{\mu}\right)^2 - 63} \tag{12.66}$$

When nontrivial solutions exist, one corresponds to a stable focus and one corresponds to a saddle point. The initial conditions determine whether the steady-state contribution from the free-oscillation term is trivial or approaches the stable focus.

Thus, if Equation (12.66) is satisfied and the initial conditions are appropriate, the free-vibration term will not decay, but will exist with an adjusted frequency of one-third of that of the excitation. The total response is periodic with the period equal to that of the excitation. The frequency response curve is illustrated in Figure 12.12.

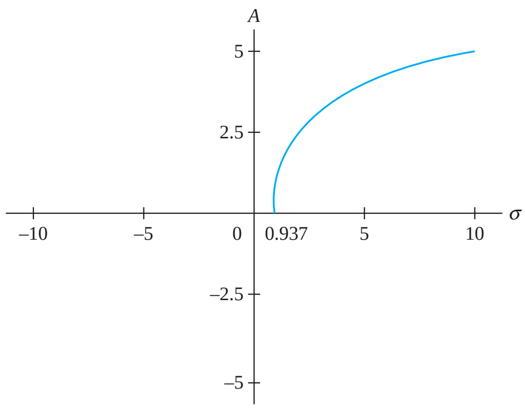


FIGURE 12.12
Frequency response curve for superharmonic resonance with $\mu = 0.25$ and $F_1 = 1$. For $\sigma < 0.937$, only the trivial steady-state response exists.

5. *Combination resonances.* Combination resonances are unique to nonlinear systems and occur because of the nonlinear interaction of the particular solutions from x_0 when x_1 is calculated. When a combination resonance is present, a nontrivial free-vibration solution exists. The nonlinearity tunes the free-vibration response to the appropriate combination of frequencies.

The jump phenomenon does occur when a combination resonance is present.

6. *Simultaneous resonances.* Simultaneous resonances occur when two resonance conditions occur simultaneously. A detuning parameter is introduced for each resonance condition. Analysis of the steady state is much more complicated. For some simultaneous resonances, as many as seven equilibrium points exist in the state plane for the same frequency.

12.7 MDOF SYSTEMS

Nonlinear MDOF systems exhibit behaviors which are not present for linear systems. It is instructive to consider free and forced vibrations of systems with quadratic nonlinearities and systems with cubic nonlinearities. Let $p_1, p_2, \ldots, p_n$ be the principal coordinates for a linearized system with natural frequencies $\omega_1 < \omega_2 < \cdots < \omega_n$, respectively. Principal coordinates that uncouple a linear system do not uncouple the system when nonlinearities are considered. The differential equations for the principal coordinates are coupled through nonlinear terms. For example, the free vibrations of an undamped two degree-of-freedom system with quadratic nonlinearities are governed by

$$\ddot{p}_1 + \omega_1^2 p_1 + \alpha_1 p_1^2 + \alpha_2 p_1 p_2 + \alpha_3 p_2^2 = 0 \tag{12.67}$$

$$\ddot{p}_2 + \omega_2^2 p_2 + \beta_1 p_1^2 + \beta_2 p_1 p_2 + \beta_3 p_2^2 = 0 \tag{12.68}$$

12.7.1 FREE VIBRATIONS

The free-vibration response of a system with quadratic nonlinearities includes periodic terms with frequencies of $\omega_1 + \omega_2$, $\omega_1 - \omega_2$, $2\omega_1$, and $2\omega_1$. If $\omega_2 \approx 2\omega_1$, then the nonlinearity acts as if it excites the system with a harmonic excitation of frequencies ω_1 and ω_2, producing a self-sustaining free oscillation, called an *internal resonance.*

In the absence of the internal resonance, and in the presence of damping, the free oscillations of both modes decay exponentially, and are to first approximation independent. When an internal resonance is present, free oscillations are sustained, even when damping is present and causes coupling between the two modes. Even if only one mode is initially excited, the internal resonance excites the other mode as well. Energy is continually exchanged between the two modes.

An internal resonance occurs in a two degree-of-freedom system with cubic nonlinearities when $\omega_2 \approx 3\omega_1$.

EXAMPLE 12.7

Reconsider the spring pendulum of Example 12.2. The spring has a stiffness 1×10^3 N/m and an unstretched length of 0.5 m. For what values of m will an internal resonance occur?

SOLUTION

Since l is the length of the spring when the system is in equilibrium,

$$l = \left(0.5 + \frac{mg}{k}\right)\text{m} \tag{a}$$

Since the approximate linear system is uncoupled when x and θ are used as generalized coordinates, these are also the principal coordinates and the linear natural frequencies are

$$\omega_1 = \sqrt{\frac{g}{0.5 + \frac{mg}{k}}} \qquad \omega_2 = \sqrt{\frac{k}{m}} \tag{b}$$

Setting $\omega_2 = 2\omega_1$ gives $m = 12.74$ kg.

12.7.2 FORCED VIBRATIONS

The free oscillations are self-sustaining in MDOF systems subject to harmonic excitations when the frequency of excitation is near certain values. A primary resonance occurs if the excitation frequency is near any of the system's natural frequencies. Subharmonic and superharmonic resonances occur as for one degree-of-freedom systems. Other secondary resonances occur when the excitation frequency is near a certain combination of natural frequencies.

For a system with quadratic nonlinearities, these resonances occur when the excitation frequency is near the sum or difference of two natural frequencies. Combination resonances occur for multifrequency excitations. Simultaneous resonance conditions can also exist.

A complete summary of the phenomena present in nonlinear MDOF systems is too extensive. The jump phenomenon occurs for certain types of resonances. Quenching can also occur in certain systems with simultaneous resonances where introduction of the second resonance causes the total response to decrease.

A saturation phenomenon can also occur for systems with quadratic nonlinearities. The amplitude of a specific mode may build up as the amplitude of excitation is increased. When the excitation amplitude reaches a certain value, the mode may become saturated; its amplitude of response remains constant as the excitation amplitude is further increased. The amplitudes of the other modes will continue to grow with the excitation amplitude.

In addition to primary resonances, subharmonic resonances, and superharmonic resonances, combination resonances occur in a two degree-of-freedom system with cubic nonlinearities when one of the following conditions is met:

$$\Omega \approx 2\omega_1 \pm \omega_2 \tag{12.69}$$

$$\Omega \approx 2\omega_2 \pm \omega_1 \tag{12.70}$$

$$\Omega \approx \frac{1}{2}(\omega_2 \pm \omega_1) \tag{12.71}$$

where Ω is the excitation frequency.

12.8 CONTINUOUS SYSTEMS

The nonlinear dimensionless partial differential equation governing transverse vibrations of a uniform beam of length L and radius of gyration r, subject to a transverse load per unit length $F(x, t)$, is

$$\left(\frac{r}{L}\right)^2\left(\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4}\right) = \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx \frac{\partial^2 w}{\partial x^2} + F(x,t) \tag{12.72}$$

The nonlinear term is a result of the midplane stretching and is often ignored.

Let $\omega_1, \omega_2, \ldots$ be the natural frequencies of the linearized system and $\phi_1, \phi_2, \ldots$ be their corresponding normalized mode shapes such that

$$(\phi_i(x), \phi_j(x)) = \delta_{ij} \tag{12.73}$$

for an appropriate scalar product.

The expansion theorem is used to develop an approximation to the solution of Equation (12.72) as

$$w(x, t) = \epsilon \sum_{i=1}^{\infty} p_i(t)\phi_i(x) \tag{12.74}$$

where $\epsilon \ll 1$ is a small dimensionless amplitude. Substituting Equation (12.74) into Equation (12.72), taking the scalar product with respect to $\phi_j(x)$ for an arbitrary j, and using algebra and mode shape orthonormality lead to

$$\ddot{p}_j + \omega_j^2 p_j = \epsilon \left(\frac{L}{r}\right)^2 \left[\frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left(\phi_j, \frac{\partial^2 \phi_k}{\partial x^2}\right) \int_0^1 \frac{\partial \phi_l}{\partial x} \frac{\partial \phi_m}{\partial x} dx \; p_k \, p_l \, p_m \right]$$
$$+ \, (F(x, t), \phi_j(x)) \tag{12.75}$$

The preceding procedure is similar to the modal analysis method of Chapter 10, except that the members of the resulting set of ordinary differential equations are still coupled through the nonlinear terms. The nonlinear terms, due to midplane stretching are cubic nonlinearities. If the excitation is harmonic with a frequency Ω, then from the results of Section. 12.7, the following resonances can occur:

1. Internal resonances occur if $\omega_i \approx 3\omega_j$, or $\omega \approx 2\omega_j + \omega_k$ for any i, j, and k. From Table 10.4, for a fixed-pinned beam, $\omega_2 = 3\omega_1 + 2.30$, and for a fixed-fixed beam $\omega_5 = 2\omega_3 + \omega_2 - 4.86$. Internal resonances occur in each of these beams. It is noted that for a pinned-pinned beam $\omega_3 = 2\omega_2 + \omega_1$. However, the coefficient multiplying $p_2^2 p_1$ in Equation (12.75) is zero for a pinned-pinned beam.
2. Primary resonance occurs if $\Omega \approx \omega_i$ for any i.
3. Superharmonic resonance occurs if $\Omega = \omega_i/3$ for any i.
4. Subharmonic resonance occurs if $\Omega = 3\omega_i$ for any i.
5. Combination resonances occur if $\Omega \approx 2\omega_i \pm \omega_j$, $\Omega \approx \omega_i \pm \omega_j \pm \omega_k$, or $\Omega \approx (\omega_i \pm \omega_i)/2$ for any i, j, and k.

12.9 CHAOS

Recent research in nonlinear phenomena has led to the development of a relatively new branch of physics called chaos. The term *chaos* refers to the seemingly random response of a nonlinear system due to deterministic excitation. Chaos occurs when a periodic excitation leads to a nonperiodic response. It also occurs when slightly different initial conditions lead to divergent responses.

Chaos has been observed and predicted in nonlinear systems in such diverse fields as physics, medicine, economics, and meteorology. Chaos occurs in mechanical systems, electrical systems, and chemical systems. Researchers observed that chicken pox epidemics are periodic while measles epidemics are chaotic. Others have used chaos to model stock market fluctuations. Chaotic fluctuations has been applied to turbulent flows.

Chaotic motion has been observed in many mechanical systems. Chaotic vibrations for systems modeled by Duffing's equation are well documented, as are chaotic motions of a forced pendulum.

Analytical tools have been developed to identify and classify chaotic behavior. These tools can be applied to analytical solutions for vibrating systems as well as experimental observations. Some are described in the following discussion.

1. *State space.* Observation of the state space can indicate whether a system is chaotic. A chaotic motion will have trajectories that do not repeat, when viewed in the phase plane. The trajectories will fill a region of the phase plane without ever repeating. However, viewing of the state plane is by itself insufficient to speculate that a motion is chaotic. An example of a chaotic response from Duffing's equation, Equation (12.15) as viewed in a state plane is shown in Figure 12.13.

2. *Poincaré sections.* A Poincaré section is a graph of the phase plane response taken or sampled only at fixed intervals of time. If the response is periodic and the time interval is equal to the period, then the Poincaré section is only a point, as the same response is obtained on each sampling. If the response is periodic and the time interval is less than the period, but commensurate with the period, the Poincaré section is a finite number of points.

The Poincaré section of a nonlinear system with a quadratic subharmonic resonance, sampled at the period of excitation should have two points. The presence of the subharmonic resonance doubles the period of response. If a system subject to a periodic excitation is sampled at intervals equal to the period of excitation and the Poincaré section is a seemingly random collection of points, the response can be guessed to be chaotic. Poincaré sections for responses of Duffing's equation, Equation (12.15) are given in Figure 12.14. These Poincaré sections illustrate that values of parameters determine whether a response is chaotic.

3. *Fourier transforms.* The Fourier transform of a nonperiodic continuous function is an extension of the Fourier series defined for periodic functions. The Fourier transform is

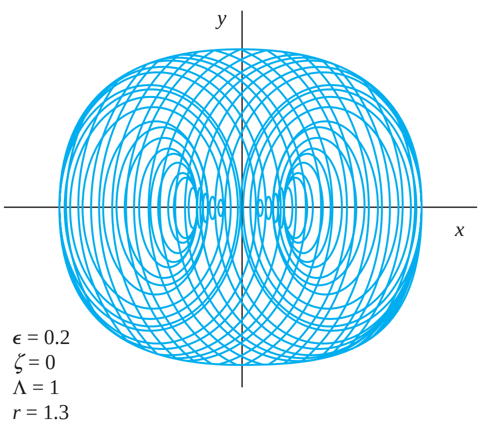


FIGURE 12.13
State plane for an apparently chaotic motion.

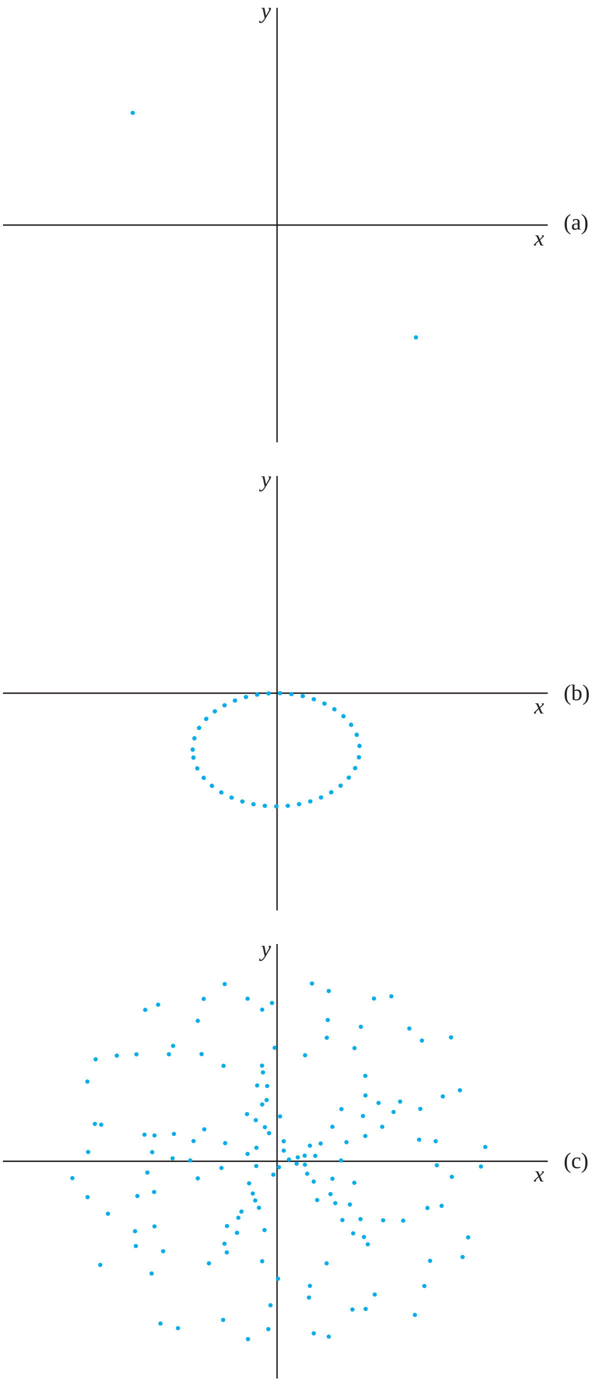


FIGURE 12.14
(a) Poincaré section for periodic motion when sampling interval is equal to half the period. (b) Poincaré section for periodic motion when sampling interval is incommensurate with period. (c) Poincaré section for a chaotic motion.

obtained from the Fourier series by allowing the period to become infinite. The resulting Fourier transform of $f(t)$ is defined as

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{12.76}$$

The transform function, $F(\omega)$, is a function of the transform variable, ω. If the Fourier transform of a periodic function is taken, then $F(\omega) = 0$ unless ω is a multiple of the function's fundamental frequency.

The Fourier transform decomposes a function into its harmonic components. The strength of a component is given by the magnitude of $F(\omega)$. The values of ω which have significant nonzero values of $F(\omega)$ are called the *spectrum* of the function. If the Fourier transform of the response of a nonlinear system due to a periodic response is a continuous spectrum, then the response is chaotic.

For computational purposes the Fourier transform is replaced by the fast Fourier transform. If $f(t)$ is known at k times, $t_1, t_2, \ldots, t_k$, then the discrete fast Fourier transform is given by

$$\bar{F}(j) = \sum_{l=1}^{k} f(t_i)e^{-2\pi i(l-1)(j-1)/k} \tag{12.77}$$

Examples of Fourier transforms are given in Figure 12.15.

4. *Bifurcation diagrams.* Bifurcation diagrams can be used to identify one route to chaos. The steady-state amplitude (or phase) of a nonlinear system as a function of a system parameter is plotted as the parameter is slowly changed. For a nonlinear system the steady-state solution may split at a certain value of the parameter and two possible steady states exist for greater values of the parameter. A bifurcation is said to occur for the value of the parameter where the split occurs. The bifurcation is often the result of the sudden presence of a subharmonic resonance. When this occurs the period of motion doubles. As the parameter is increased, additional bifurcations may occur, where the period again doubles. If the system is chaotic, as the parameter increases, bifurcations and period doubling occur more rapidly. The chaotic response bounces between amplitudes and has no discernible period. The plot of steady-state amplitudes (or phases) as the parameter increases becomes a blur. It is often the case that as the parameter is increased much further, the motion again becomes periodic.

While chaotic motion is characterized by its unpredictable nature, it has some universal features. Feigenbaum showed that, as the number of bifurcations increases, the values of the parameter, call it A, for which the bifurcations occur are given by

$$A_n - A_{n-1} = (4.699\ldots)(A_{n+1} - A_n) \tag{12.78}$$

There are many routes to chaos. The one described here applies to systems undergoing nonlinear oscillations subject to a harmonic excitation and is illustrated by the rotating U-tube manometer of Example 12.3 and Figure 12.16. The manometer is rotated about a vertical axis other than its centroidal axis. The rotational speed of the manometer varies as

$$\omega(t) = A\sin\lambda t \tag{12.79}$$

where A is large enough to cause the fluid to be completely drained from the left leg during an initial transient period. The system is subject to viscous damping from the interaction of the fluid with the wall of the manometer.

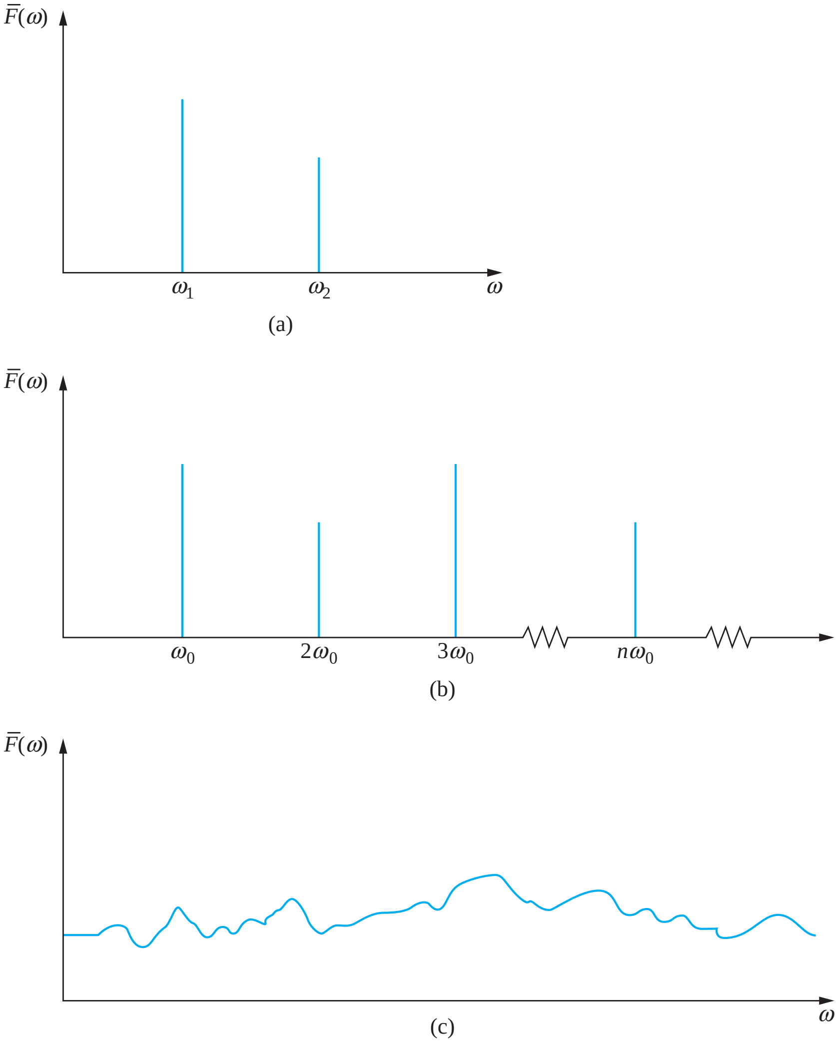


FIGURE 12.15
(a) Fourier transform of a periodic function with two distinct frequencies.
(b) Fourier transform of a periodic function of fundamental frequency ω_o.
(c) Fourier transform of a chaotic response.

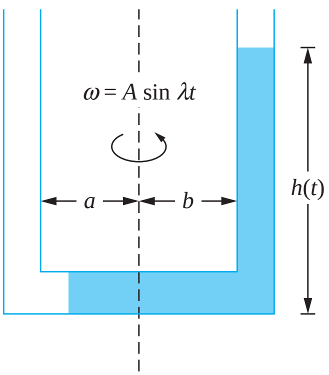
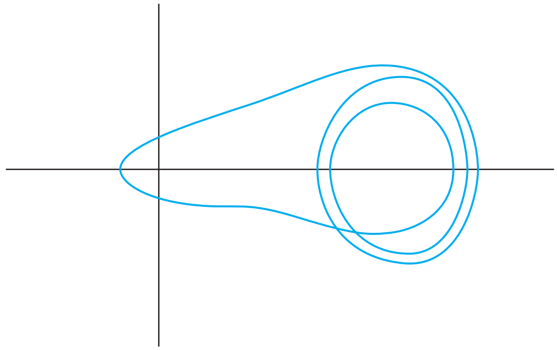
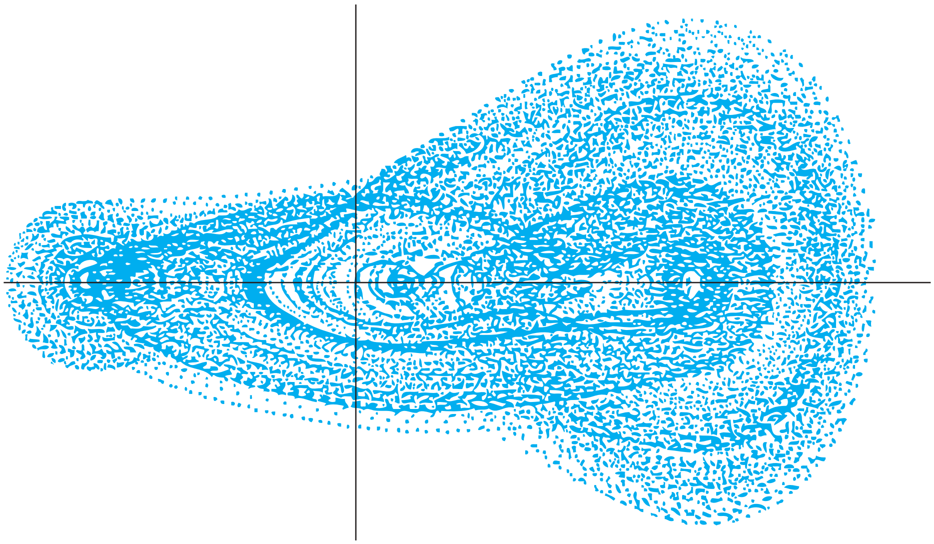


FIGURE 12.16
For certain values of λ and ε the motion of the column of liquid in the U tube manometer can be chaotic when the manometer rotates about a non-centroidal axis.

FIGURE 12.17
A small change in frequency can cause a change from a periodic response to a chaotic response.



(a) $\lambda = 0.27$ $A = 4.5$

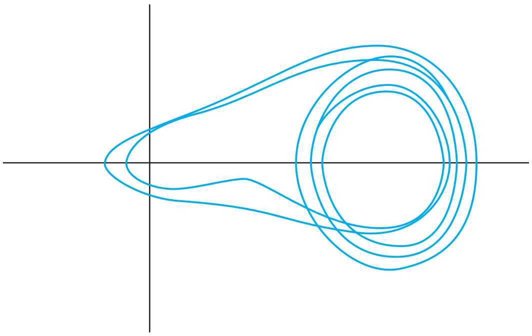


(b) $\lambda = 0.28$ $A = 4.5$

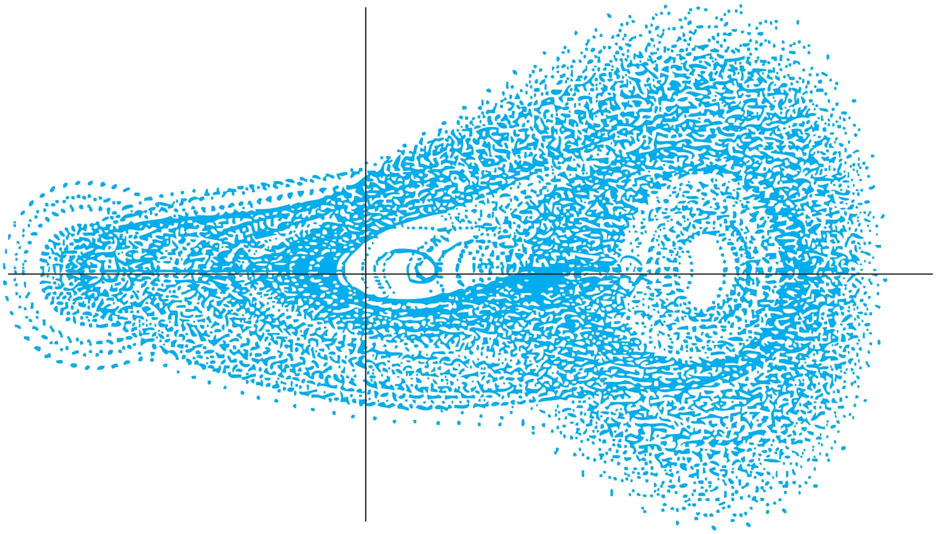
The behavior of a nonlinear system is heavily influenced by the system parameters. This is evidenced by the state planes of Figures 12.17 and 12.18. Figure 12.17 shows the state planes for two slightly different values of the frequency for the same amplitude. A steady state is evident for the motion of Figure 12.17(a), while the motion of Figure 12.17(b) appears chaotic. Chaos is also induced by small amplitude changes for the same frequency as shown in Figure 12.18(a).

A bifurcation diagram for the parameter A is shown in Figure 12.19. The frequency of excitation is fixed as its amplitude varies. For $A < 3.33$, the steady-state motion is periodic. The stationary response is periodic of frequency 2λ and a certain amplitude.

FIGURE 12.18
A small change in the input amplitude can cause a change from a periodic response to a chaotic response.



(a) $\lambda = 0.27$ $A = 4.6$



(b) $\lambda = 0.27$ $A = 4.7$

For $A \approx 3.33$, the parameters change such that a subharmonic resonance becomes present. A bifurcation is said to occur. The presence of the subharmonic resonance means that the steady-state response is the sum of a free-vibration term and a forced-vibration term and that the period of motion is doubled. Two amplitudes are evident in the stationary oscillations.

For $A \approx 3.35$, another bifurcation occurs. A higher-order subharmonic resonance is induced. The response has a period of four times the original period and is made up of four distinct amplitudes.

As A increases, bifurcations occur more rapidly with the period doubling with each bifurcation. Eventually, the response is chaotic. The chaotic response shown in Figure 12.20 bounces between amplitudes and has no discernible period.

For $A \approx 3.36$, the motion ceases to be chaotic and returns to the doubled period. However, bifurcations begin to occur again at $A \approx 3.37$.

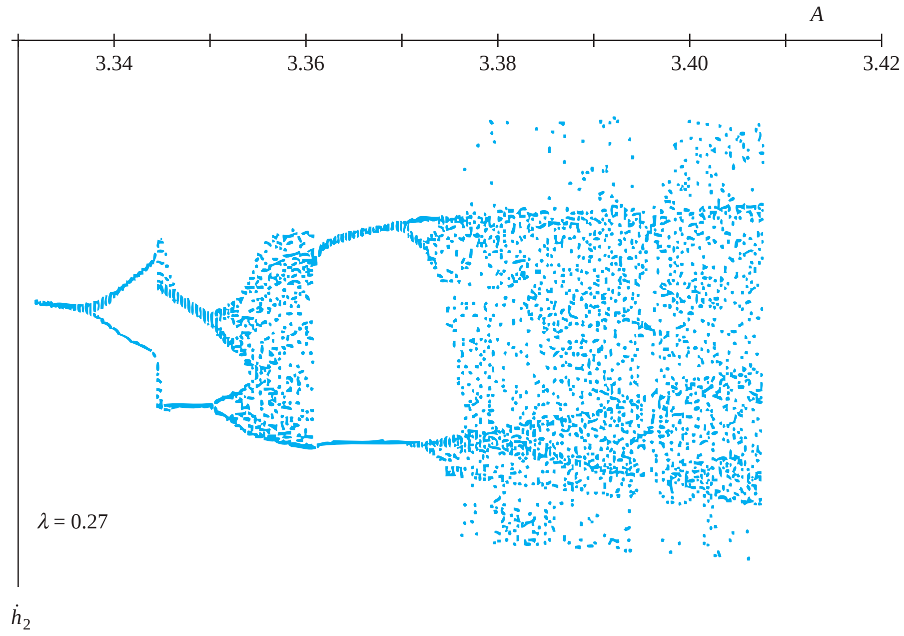


FIGURE 12.19
Bifurcation diagram for rotating manometer. First bifurcation occurs near $A = 3.34$. As A increases chaos develops. Motion is not chaotic for a range of A, then the process to chaos begins again.

The process described previously is called *period doubling through a subharmonic cascade*.

Chaos is the subject of much current research. It is hoped that studying chaos can lead to the better understanding of nonlinear systems like turbulent fluid flows, the flow and pumping of blood through a human heart, weather patterns, and nonlinear vibrations.

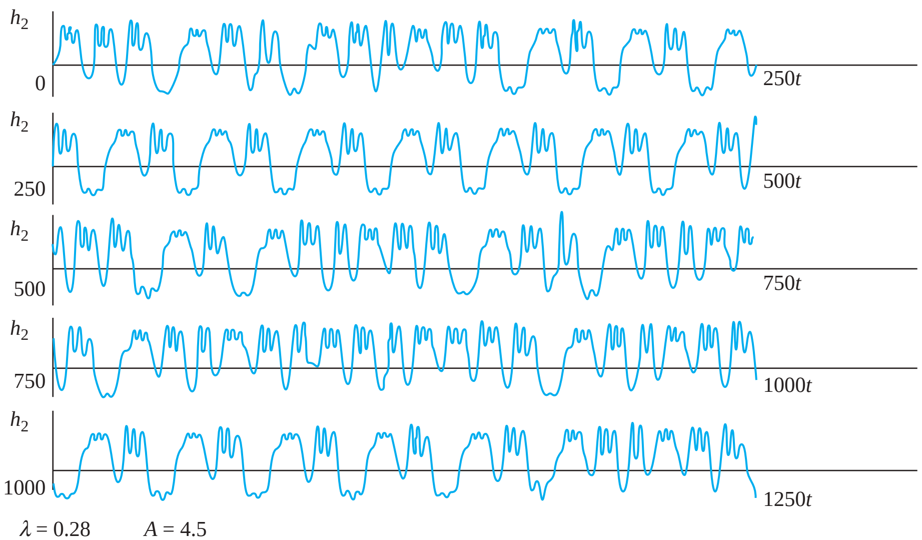


FIGURE 12.20
The time history of motion for these parameters has no discernable period.

12.10 CHAPTER SUMMARY

12.10.1 IMPORTANT CONCEPTS

- Methods of analysis for linear systems are not applicable to nonlinear systems
- A geometric nonlinearity occurs due to the geometry of the system. A material nonlinearity occurs due to nonlinearity in material behavior
- Static spring forces do not cancel with gravity in nonlinear systems
- The state plane is a family of curves showing the history of the relation between displacement and velocity. The curves the state plane are called trajectories.
- An equilibrium point is stable if the trajectories approach the equilibrium point as time gets large. The trajectories are unstable if the trajectories diverge from the equilibrium point.
- An equilibrium point is classified by the eigenvalues of the stability equation β_1 and β_2. If β_1 and β_2 are real and of the same sign, the equilibrium point is called a node. If β_1 and β_2 are real and of opposite signs, the equilibrium point is a saddle point (unstable). If β_1 and β_2 are complex conjugates, the equilibrium point is called a focus. If β_1 and β_2 are purely imaginary, the equilibrium point is called a center.
- Secular terms are terms which produce non-uniformities in perturbation expansions.
- The period for free vibrations of a nonlinear system depends upon initial conditions.
- Small viscous damping leads to linear decay of the free-vibration solution.
- Resonances occur in the forced response of Duffing's equation. Resonances due to a single frequency excitation are classified as primary when $r = 1$, superharmonic when $r = 1/3$, or subharmonic when $r = 3$. Combination resonances and simultaneous resonances occur when the excitation is at two or more frequencies.
- A jump phenomenon occurs when the frequency is in the vicinity of the linear natural frequency, which is characterized by a discrete change in amplitude at critical frequencies. The jump also occurs in the phase.
- A jump response also occurs near the superharmonic resonance, but not near the subharmonic resonance.
- Internal resonances are present in MDOF systems and continuous systems.

12.10.2 IMPORTANT EQUATIONS

Duffing's equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + \alpha\omega_n^2 x^3 = \frac{F_0}{m}\sin\omega t \tag{12.11}$$

Nonlinear differential equation with nonlinear damping and nonlinear flexible element

$$m\ddot{x} + g(\dot{x}) + f(x) = F_0 \sin\omega t \tag{12.22}$$

Stability of an equilibrium point

$$x = x_0 + \Delta x \tag{12.23}$$

$$\Delta x = Ae^{\beta_1 t} + Be^{\beta_2 t} \tag{12.26}$$

General perturbation expansion in terms of a small dimensionless parameter

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{12.31}$$

Period of nonlinear system

$$T = \frac{4}{\sqrt{2}} \int_{x_0}^{0} \frac{d\lambda}{[\int_{\lambda}^{x_0} f(\eta)\, d\eta]^{1/2}} \tag{12.32}$$

Two-term expansion for free vibrations of undamped Duffing's equation

$$x = A \sin(w + \phi) - \varepsilon \frac{A^3}{32} \sin 3(w + \phi) + \cdots \tag{12.46}$$

$$w = t\left(1 + \varepsilon \frac{3}{8} A^2 + \cdots\right) \tag{12.48}$$

Free vibrations of damped Duffing's equation

$$x = Ae^{-\zeta t} \sin\left[\left(1 + \varepsilon \frac{3}{8} A^2\right) t + \phi\right] \tag{12.56}$$

Detuning parameter to allow for study of the frequency response in the neighborhood of resonance

$$r_1 = 1 + \varepsilon\sigma \tag{12.60}$$

Amplitude and phase near primary resonance

$$4A^2\left[\mu^2 + \left(\sigma - \frac{3}{8}A^2\right)^2\right] = \hat{F}_1^2 \tag{12.61}$$

$$\phi = -\tan^{-1}\left(\frac{\mu}{\sigma - \frac{3}{8}A^2}\right) \tag{12.62}$$

Amplitude equation near superharmonic resonance

$$\left[\mu^2 + \left(\sigma - 3F_i^2 - \frac{3}{8}A^2\right)^2\right] A^2 = F_i^6 \tag{12.64}$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 12.1 through 12.14, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

12.1 The convolution integral can be applied to solve nonlinear problems.

12.2 A mass attached to a linear spring sliding on a surface with Coulomb damping is an example of a nonlinear system.

12.3 The swinging spring is an example of a two degree-of-freedom system with a cubic nonlinearity.

12.4 The period of free vibrations of a nonlinear system depends upon initial conditions.

12.5 The free response of a system with a cubic nonlinearity occurs only at the linear natural frequency of the system.

12.6 A focus is always unstable.

12.7 A saddle point is always unstable.

12.8 Secular terms must be removed from the response of a system.

12.9 When a superharmonic resonance occurs, the free oscillation term does not decay exponentially but combines with the forced response.

12.10 A SDOF system with viscous damping subject to a single frequency excitation always has a free response which decays exponentially.

12.11 A SDOF system with a cubic nonlinearity is excited by a harmonic force at a frequency of 100 rad/s. The forced response occurs only at 300 rad/s.

12.12 A MDOF system has a combination resonance when the parameters are such that one of the system's linear natural frequencies is in a certain combination with another of the system's natural frequencies.

12.13 A bifurcation is a split in natural frequencies for one value of a parameter.

12.14 Period doubling is a route to chaos.

Problems 12.15 through 12.38 require a short answer.

12.15 Why can't the Laplace transform method be applied to nonlinear systems?

12.16 A spring has a cubic nonlinearity which is an example of a (geometric, material) ________________ nonlinearity.

12.17 A spring with a cubic nonlinearity equal to $-3x^3$ is an example of a (hardening, softening) ________________ nonlinearity.

12.18 Trajectories near an equilibrium point in the state space are shown in Figure SP12.18. Identify the equilibrium point that is (a) an unstable saddle point, (b) a stable focus, (c) a center, and (d) an unstable node.

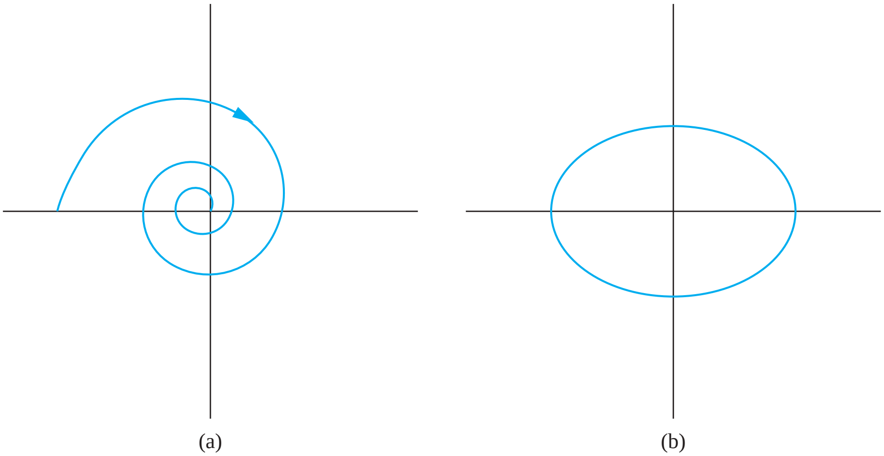


FIGURE SP12.18

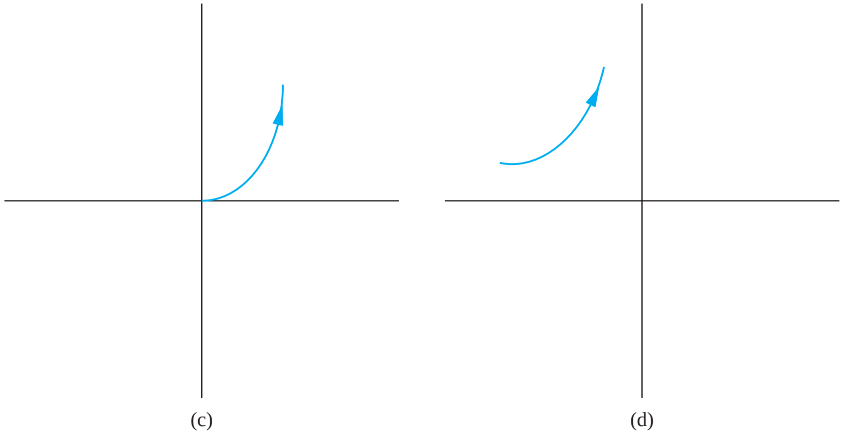


FIGURE SP12.18
(*continued*)

In Problems 12.19 through 12.23, the eigenvalues of the differential equation are β_1 and β_2 when the equation is linearized about an equilibrium point. Determine the type of the equilibrium point and its stability.

12.19 $\beta_1 = 3, \beta_2 = -2$
12.20 $\beta_1 = -3 + 2i, \beta_2 = -3 - 2i$
12.21 $\beta_1 = 2i, \beta_2 = -2i$
12.22 $\beta_1 = -3, \beta_2 = -2$
12.23 $\beta_1 = 3, \beta_2 = 2$
12.24 Explain the use of the detuning parameter.
12.25 The frequency-response curve shown in Figure SP12.25 is for the primary resonance of a SDOF system with a cubic nonlinearity.

(a) Is the curve drawn for a hardening spring or a softening spring?
(b) Explain the significance of points A and B on the diagram.

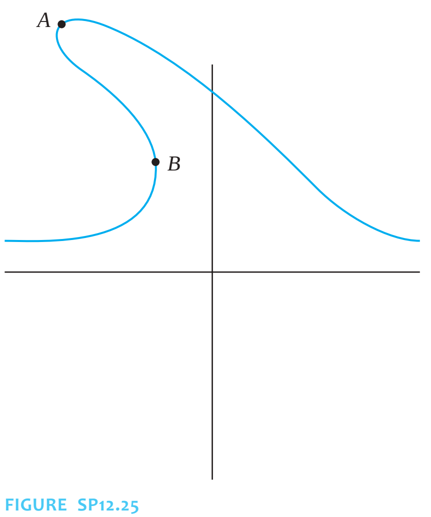


FIGURE SP12.25

12.26 A SDOF system with a cubic nonlinearity has a linear natural frequency of 30 rad/s. At what excitation frequency does the system have

(a) A primary resonance?
(b) A superharmonic resonance?
(c) A subharmonic resonance?

In Problems 12.27 through 12.33, a SDOF system with a cubic nonlinearity has a linear natural frequency of 120 rad/s. The system is forced by harmonic excitations at different frequencies ω_1 and ω_2. What resonances does the system have under the given circumstances.

12.27 ω_1 is near 30 rad/s and ω_2 is near 60 rad/s
12.28 ω_1 is near 90 rad/s and ω_2 is near 60 rad/s
12.29 ω_1 is near 20 rad/s and ω_2 is near 260 rad/s
12.30 ω_1 is near 50 rad/s and ω_2 is near 180 rad/s
12.31 ω_1 is near 40 rad/s and ω_2 is near 200 rad/s
12.32 ω_1 is near 120 rad/s and ω_2 is near 40 rad/s
12.33 ω_1 is near 240 rad/s and ω_2 is near 360 rad/s
12.34 Explain why a superharmonic resonance occurs.
12.35 What is an internal resonance in a MDOF system?
12.36 Describe the Poincaré section corresponding to a periodic function when the sampling interval is one-third of the period.
12.37 What is the signature of the Fourier transform?
12.38 What is Feigenbaum's constant?

Problems 12.39 through 12.53 require short calculations.

12.39 The linearized differential equation around an equilibrium point is

$$\Delta\ddot{x} + 2\Delta\dot{x} + 3\Delta x = 0$$

Classify the equilibrium point and determine its stability.

12.40 The linearized differential equation around an equilibrium point is

$$\Delta\ddot{x} - 2\Delta\dot{x} + 3\Delta x = 0$$

Classify the equilibrium point and determine its stability.

12.41 The linearized differential equation around an equilibrium point is

$$\Delta\ddot{x} + 2\Delta\dot{x} - 3\Delta x = 0$$

Classify the equilibrium point and determine its stability.

12.42 The equation of motion of a simple pendulum of length ℓ is

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

(a) Determine the pendulum's equilibrium points.
(b) Classify the equilibrium points and determine their stability.
(c) Sketch a trajectory in the phase plane corresponding to each equilibrium point.

12.43 The differential equation governing the motion of a nonlinear system is

$$\ddot{x} - 0.5\dot{x} + x - 0.1x^3 = 0$$

(a) Determine the equilibrium points.
(b) Classify the equilibrium points and determine their stability.
(c) Sketch a trajectory in the phase plane corresponding to each equilibrium point.

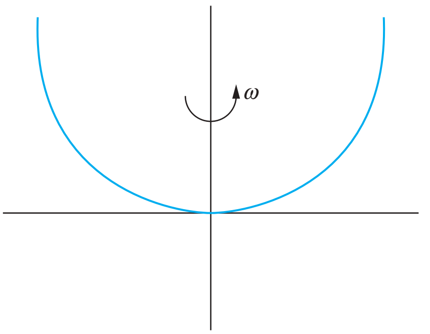
12.44 The differential equation governing the motion of a nonlinear system is

$$\ddot{x} - x + 0.1x^3 = 0$$

(a) Determine the equilibrium points.
(b) Classify the equilibrium points and determine their stability.
(c) Sketch a trajectory in the phase plane corresponding to each equilibrium point.

12.45 The equation of motion for a particle moving on a rotating circular frame (Figure SP12.45) is

$$\ddot{\theta} + \frac{g}{R}\left(\sin\theta - \frac{\omega^2}{g}\cos\theta\right) = 0$$



(b)

FIGURE SP12.45

(a) Determine the equilibrium points.
(b) Classify the equilibrium points and determine their stability for

(i) $\frac{\omega^2}{g} = 0.5$ (ii) $\frac{\omega^2}{g} = 1$ (iii) $\frac{\omega^2}{g} = 1.5$

12.46 Determine the free response to the nondimensional undamped Duffing's equation for $\varepsilon = 0.01$, $x(0) = 1$, and $\dot{x}(0) = 0$.

12.47 Determine the free response to the nondimensional damped Duffing's equation for $\varepsilon = 0.01$, $\zeta = 0.05$, $x(0) = 1$, and $\dot{x}(0) = 0$.

12.48 Determine the steady-state amplitudes for the equation

$$\ddot{x} + 0.05\dot{x} + x + 0.01x^3 = 0.03\sin 1.01t$$

12.49 Determine the steady-state amplitudes for the equation

$$\ddot{x} + 0.05\dot{x} + x + 0.01x^3 = 0.03\sin 0.33t$$

12.50 Determine the steady-state amplitudes for the equation

$$\ddot{x} + 0.05\dot{x} + x + 0.01x^3 = 0.03\sin 3.06t$$

12.51 Suggest any internal resonances for a fixed-free beam.

12.52 Suggest any internal resonances for a beam fixed at one end with a mass of $0.25\rho AL$ attached at its other end.

12.53 Determine the Fourier transform of

$$F(t) = 2\sin 3t + 4\sin 4.5t$$

12.54 What are the dimensions of the following quantities?

(a) Coefficient multiplying x^3 in nonlinear spring stiffness, k_3
(b) The perturbation parameter, ε
(c) A detuning parameter, σ

CHAPTER PROBLEMS

12.1 The free-vibration response of a block hanging from a linear spring is the same as that of the block attached to the same spring, but sliding on a frictionless surface. Is the response the same if the spring has a force-displacement relation given by the following?

(a) $F = k_1x + k_3x^3$
(b) $F = k_1x + k_2x^2$
(c) $F = k_1x,\ x < x_0$
(d) $F = k_2x,\ x > x_0$

12.2 The system of Figure P12.2 is one of the few for which an exact solution is available. Its solution is obtained in a manner analogous to that of free vibrations with Coulomb damping. The block is displaced a distance $x_0 > \delta$ to the right from equilibrium and released. Determine the period of the resulting oscillations.

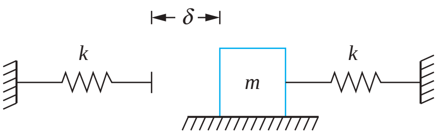


FIGURE P12.2

12.3 The block in Figure P12.3 is not attached to the springs. Determine the period of the resulting oscillations if the block is displaced a distance x_0 to the right from equilibrium and released.

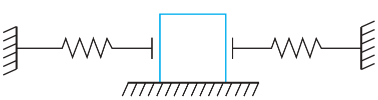


FIGURE P12.3

12.4–12.7 Without making linearizing assumptions, use Lagrange's equations to derive the nonlinear differential equation(s) governing the motion of the systems shown. Use the generalized coordinates indicated in Figures P12.4 through P12.7.

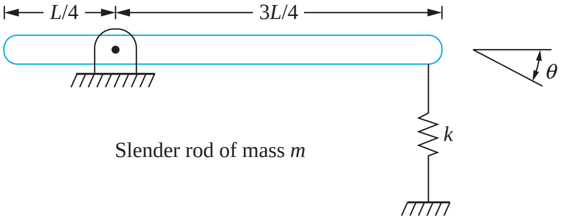


FIGURE P12.4

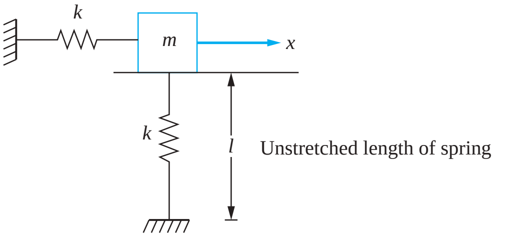


FIGURE P12.5

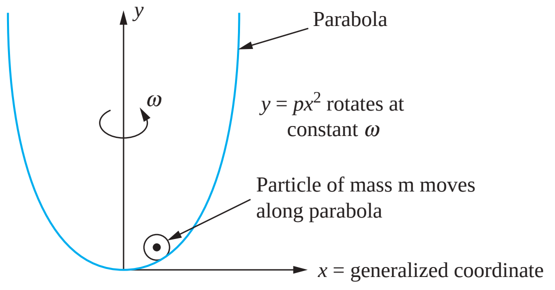


FIGURE P12.6

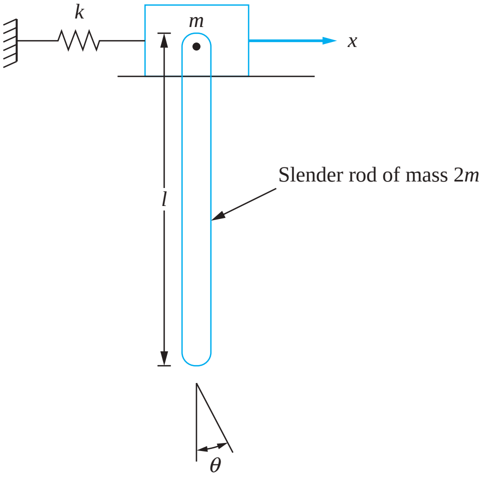


FIGURE P12.7

12.8 A wedge of specific weight γ floats stably on the free surface of a fluid of specific weight γ_w (Figure P12.8). The wedge is given a vertical displacement δ from this equilibrium position.

(a) Derive the differential equation governing the resulting free oscillations of the wedge. Neglect viscous effects and the added mass of the fluid.

(b) What is the equation of the trajectory in the phase plane which describes the resulting motion. Sketch the trajectory.

(c) Assume δ is small and use the method of renormalization to determine a two-term approximation for the frequency-amplitude relationship.

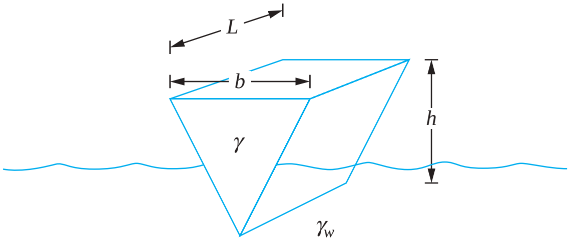


FIGURE P12.8

12.9 Repeat Chapter Problem 12.8 for the inverted cone of Figure P12.9.

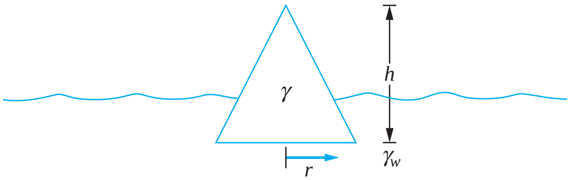


FIGURE P12.9

12.10 Determine the equation defining the state plane for the system of Figure P12.6. Sketch trajectories in the phase plane when the following are given.

(a) $p = 1.5\ \text{m}^{-1}$, $\omega = 5$ rad/s
(b) $p = 1.0\ \text{m}^{-1}$, $\omega = 5$ rad/s
(c) $p = 5.097\ \text{m}^{-1}$, $\omega = 10$ rad/s

12.11 Plot the trajectory in the state plane corresponding to the motion of a mass attached to a linear spring free to slide on a surface with Coulomb damping when the mass is displaced from equilibrium and released from rest.

12.12 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x + \epsilon x^3 = 0$$

12.13 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x - \epsilon x^3 = 0$$

12.14 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} + x + \epsilon x^2 = 0$$

12.15 Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} + x - \epsilon x^2 = 0$$

12.16 The equation of motion for the free oscillations of a pendulum subject to quadratic damping is

$$\ddot{\theta} + 2\zeta\dot{\theta}^2 + \sin\theta = 0$$

(a) Determine an exact equation defining the state plane.
(b) Determine the equilibrium points and their type.

12.17 Determine the period of oscillation of a mass attached to a hardening spring with a cubic nonlinearity.

12.18 Determine an integral expression for the period of oscillation of the system of Figure P12.6.

12.19 Use the method of renormalization to determine a two-term approximation for the frequency-amplitude relation for the system of Figure P12.4. If the bar is rotated 4° from equilibrium and released, what is the period for $L = 4$ m, $k = 1000$ N/m, and $m = 10$ kg?

12.20 A 25-kg mass is attached to a hardening spring with $k_1 = 1000$ N/m and $k_3 = 4{,}000$ N/m^3. The mass is displaced 15 mm from equilibrium and released from rest. What is the period of the ensuing oscillations?

12.21 Suppose the mass of Chapter Problem 12.20 is subject to an impulse which imparts a velocity of 3.1 m/s to the mass when the mass is in equilibrium. What is the period of the ensuing oscillations?

12.22 Suppose the mass of Chapter Problem 12.20 is attached to the same spring when a 50-N force is statically applied and suddenly removed. What is the period of the ensuing oscillations?

12.23 Use the method of renormalization to determine a two-term frequency-amplitude relationship for the particle on the rotating parabola of Figure P12.6, assuming the amplitude is small.

12.24 Use the method of renormalization to determine a two-term frequency-amplitude relationship for a block of mass m attached to a spring with a quadratic nonlinearity. When nondimensionalized the differential equation governing free vibrations of the system is

$$\ddot{x} + \omega^2 x + \epsilon x^2 = 0 \qquad \epsilon \ll 1$$

Problems 12.25 through 12.31 refer to the system of Figure P12.25.

12.25 If $F(t) = F_0 \sin \omega t$, what values of ω will lead to the presence of the following?

(a) A primary resonance
(b) A superharmonic resonance
(c) A subharmonic resonance

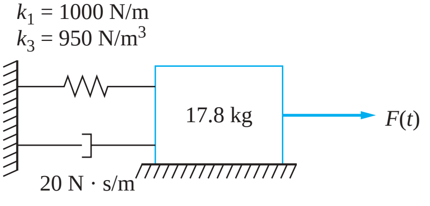


FIGURE P12.25

12.26 When $F(t) = 5 \sin 8t$ N, a primary resonance condition occurs. Determine the amplitude of the forced response.

12.27 When $F(t) = 150 \sin 2.5t$ N a superharmonic resonance condition occurs. Determine the amplitude of the forced response.

12.28 If $F(t) = F_0 \sin \omega t$ N, for what value of ω will a jump in amplitude occur when ω is increased slightly beyond this value when

(a) $F_0 = 5$ N and a primary resonance occurs.
(b) $F_0 = 150$ N and a superharmonic resonance occurs.

12.29 If $F(t) = 25 \sin 22t$ N, will a nontrivial subharmonic response exist?

12.30 If $F(t) = 30 \sin 15t + 25 \sin \omega t$ N, what values of ω lead to a combination resonance?

12.31 If $F(t) = 30 \sin 2.5t + 25 \sin \omega t$ N, what values of ω lead to simultaneous resonances?

Problems 12.32 through 12.35 refer to the systems of Figure P12.32. The spring of stiffness k_2 is a linear spring.

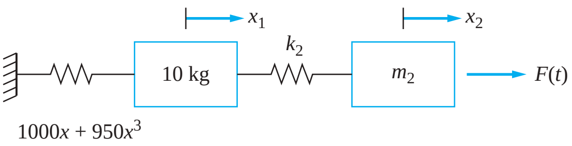


FIGURE P12.32

12.32 If $m_2 = 10$ kg, for what values of k_2 will internal resonances exist?

12.33 For what values of m_2 are internal resonances possible? If an internal resonance is possible in terms of m_2, for what values of k_2 will they exist?

12.34 Consider the system with $m_2 = 10$ kg and $k_2 = 2000$ N/m. The right mass is displaced 10 mm from equilibrium while the left mass is held in place. The system is released from rest from this configuration.

(a) Determine the natural frequencies, mode shapes, and principal coordinates for the linearized system.
(b) Write the nonlinear differential equations governing the system using the principal coordinates of the linearized system as dependent variables.

12.35 If $m_2 = 10$ kg, $k_2 = 1000$ N/m, and $F(t) = 150 \sin \omega t$ N, for what values of ω will the following resonances exist?

(a) Primary resonance
(b) Superharmonic resonance
(c) Subharmonic resonance
(d) Combination resonance

12.36 Consider the system of Figure P12.36.

(a) Derive the nonlinear differential equations governing the motion of the system using the generalized coordinates shown.
(b) Expand trigonometric functions of the generalized coordinates using Taylor series expansions. Rewrite the differential equations keeping only quadratic and cubic nonlinearities.

(c) For what values of l in terms of the other parameters will an internal resonance exist?

(d) In the absence of an internal resonance, for what values of ω will resonance conditions exist?

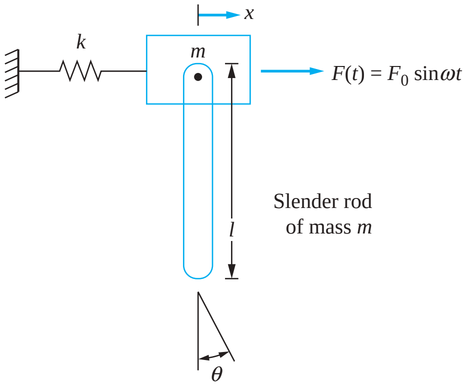


FIGURE P12.36

12.37 Show that the coefficient multiplying $p_2^2 p_1$ for a pinned-pinned beam is zero in Equation (11.50).

12.38 A fixed-free rectangular steel beam ($\rho = 7850$ kg/m^3, $E = 210 \times 10^9$ N/m^2) with a length of 1 m, base of 2 cm, and height of 5 cm is subject to a single-frequency harmonic excitation. List all excitation frequencies that should be avoided to avoid all primary, secondary, and combination resonances involving the three lowest modes.

12.39 If the beam of Chapter Problem 12.38 is fixed-fixed, which of the following excitation frequencies should be avoided and why?

(a) 180 rad/s
(b) 1530 rad/s
(c) 2200 rad/s
(d) 7940 rad/s

Chapter 13

RANDOM VIBRATIONS

13.1 INTRODUCTION

A time-dependent process is deterministic if its properties at a given time t can be predicted in advance. A linear process is deterministic if its input is deterministic. The processes of free and forced vibrations of SDOF, MDOF, and continuous systems (as described previously) are all deterministic; their response can be predicted at any instant of time given a deterministic input, as illustrated in Figure 13.1. A nonlinear system can have a chaotic response due to deterministic input.

Many physical systems do not have a deterministic input, such as those illustrated in Figure 13.2 on page 783. The road contour encountered by the wheels of a vehicle, while described as sinusoidal in previous chapters, is really made up of a series of bumps and depressions that cannot be predicted. Other sources of non-deterministic input to systems are the excitation provided to a building from an earthquake, vortex shedding from a bridge, and the vibration of a floor in an industrial plant. These inputs are said to be random: an input which cannot be predicted at any time. There are many reasons why an input is random. For example, many vehicles of assorted sizes and shapes have travelled over the road; environmental conditions such as temperature, rain, and snow have affected the road over a long period; or imperfections in the road material affect the road contour. An earthquake is random because not enough is known about the origin of the earthquake: the energy released by the earthquake and the propagation of seismic waves are not understood fully. Vortex shedding is a random phenomenon because the wind velocity is uncontrollable and

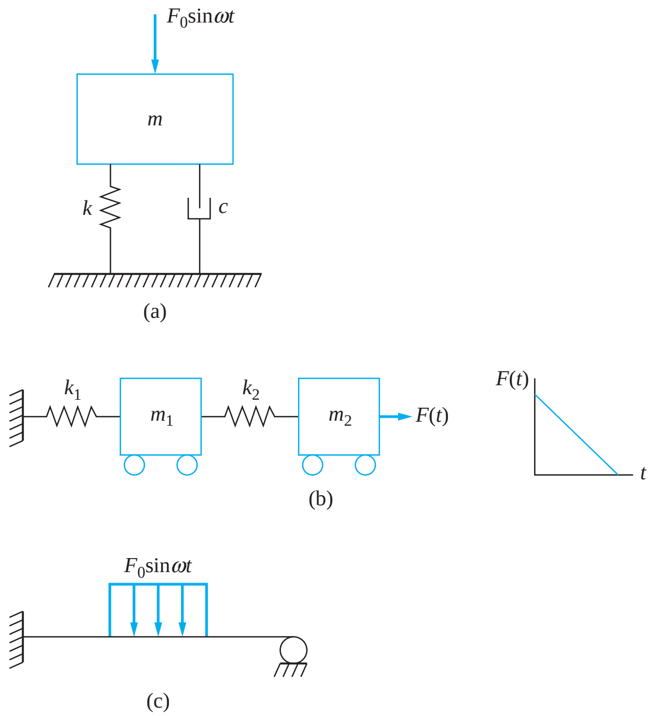


FIGURE 13.1
Examples of deterministic systems. (a) SDOF system subject to harmonic excitation. (b) SDOF system subject to pulse loading. (c) Fixed-pinned beam subject to harmonic loading.

is affected by many factors, including the geometry of the bridge deck. Machines on the floor of an industrial plant are vibrating at different frequencies, and different amplitudes, at different times, providing a random input to anything placed on the floor.

The response of a system due to a random input is also random. Analysis of such systems requires a method of approach which combines the different vibrations with the methods of dealing with random input. Random input can be expressed in terms of statistical quantities and the output can be expressed in terms of its mean square values, which can be translated into probabilities.

Assumptions are made which make the analysis of random vibrations easier. Mathematical functions describing the statistical analysis of a random variable are developed including the mean, standard deviation, and probability distribution. Functions defining the joint probability distribution are developed to include the autocorrelation function. The Fourier transform is employed to derive a transfer function for the system and to relate the autocorrelation function to the power spectral density. The power spectral density is used to describe the random response of a SDOF system.

13.2 BEHAVIOR OF A RANDOM VARIABLE

13.2.1 ENSEMBLE PROCESSES

Consider again the SDOF model of a vehicle suspension system. The following experiment is run. An accelerometer is attached to the wheel of the vehicle and the displacement $y(t)$ of the axle is monitored. The experiment is repeated at the same speed 50 times, but at each

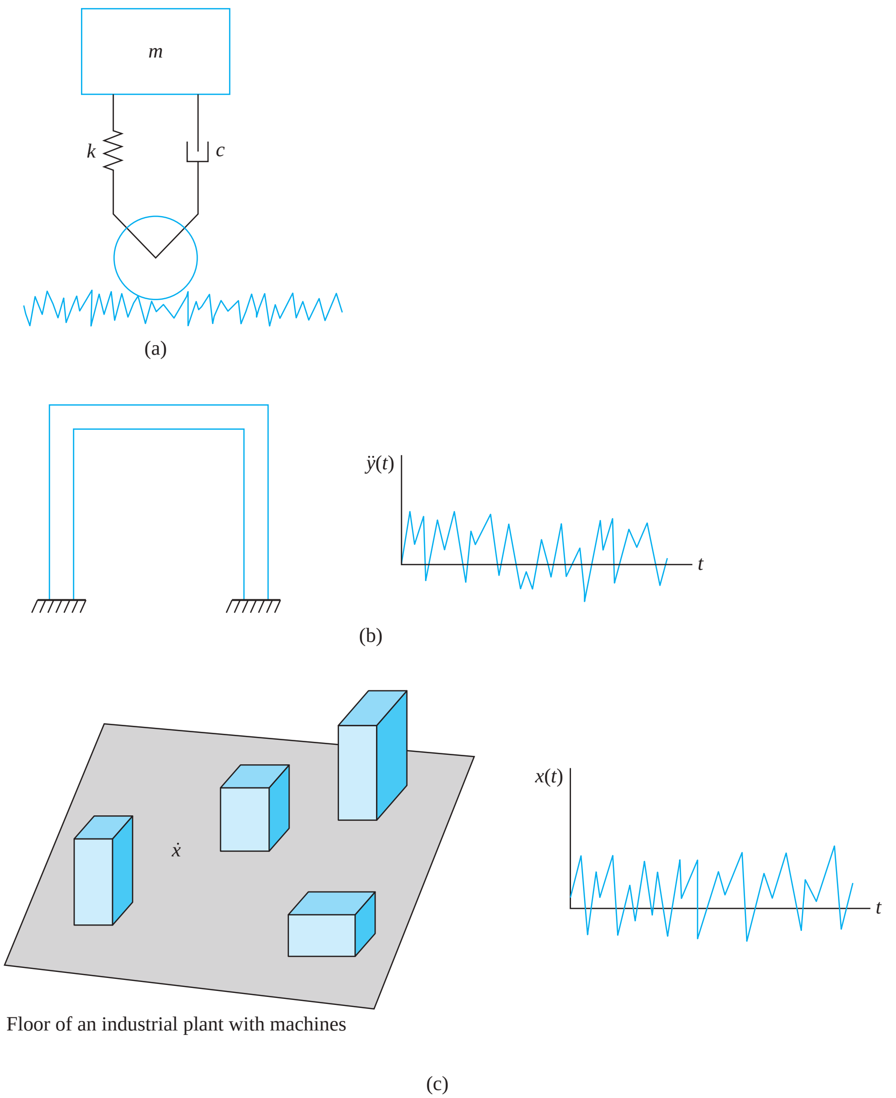


FIGURE 13.2
Systems subject to random input. (a) SDOF model of vehicle suspension system as it traverses a road contour. (b) One-story frame structure subject to an earthquake. (c) Vibrations of the floor in an industrial plant.

time, the displacement measured is different due to variations in wind, temperature, tire conditions, and other factors outside of the control of the experiment. The displacement of the axle is not repeatable and is a random phenomenon. The variable $y(t)$ is a random variable, and each displacement $y_i(t)$ is a sample function of the random variable. All sample functions taken together form a collection or an *ensemble* of sample functions and is expressed as $\{y_i(t)\}$.

13.2.2 STATIONARY PROCESSES

The statistical mean of the ensemble $\{x_i(t)\}$ at a time t_0 is calculated by

$$\bar{x}(t_0) = \frac{1}{n}\sum_{i=1}^{n} x_i(t_0) \tag{13.1}$$

or as n, which is the number of elements in the ensemble, becomes large

$$\mu(t_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i(t_0) \tag{13.2}$$

Statistical definitions of other functions introduced in this chapter can be calculated in the same fashion. For example the statistical definition of the standard deviation is

$$s = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1} \tag{13.3}$$

These statistical averages are functions of time.

The ensemble measurements are a function of time. However, if the measurements have statistical features which are independent of time (such as the mean and standard deviation), the ensemble is said to be *stationary*. The value of μ, as calculated by Equation (13.1), is independent of the time t_0 at which it is calculated. A random stationary process has other implications, which are covered in more detail later. All processes in this chapter are assumed to be stationary.

13.2.3 ERGODIC PROCESSES

The temporal average uses a representative sample function and integrates over time, as

$$\mu_t(i) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_i(t)dt \tag{13.4}$$

or using the notation of improper integrals, as

$$\mu_t(i) = \int_{-\infty}^{\infty} x_i(t)dt \tag{13.5}$$

If the temporal average is the same for all ensemble measurements. That is, μ_t is the same for every i when the random process is *ergodic*. Only stationary ergodic processes are covered in this chapter.

13.3 FUNCTIONS OF A RANDOM VARIABLE

13.3.1 PROBABILITY FUNCTIONS

Consider a random variable y with sample points $y_1, y_2, \ldots, y_n$. The probability that y is less than or equal to a certain value that is less than or equal to a value $\hat{y}$ is the number of values of $y \leq \hat{y}$ divided by the number of sample points n. In the limit, as $n \to \infty$, this defines the probability distribution function $P(y)$. For a random variable that is a function of time, $P(y)$ is defined in the limit as the time span approaches infinity, the total time that the function is less than or equal to y divided by the time span. The probability distribution function has the property that

$$0 \leq P(y) \leq 1 \tag{13.6}$$

subject to $P(-\infty) = 0$ and $P(\infty) = 1$.

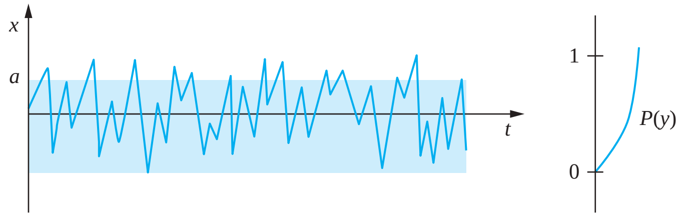


FIGURE 13.3
A random process and its probability distribution.

The time-dependent data for a stationary ergodic process is illustrated in Figure 13.3 along with its probability distribution. $P(a)$ is the probability that the random variable y takes on a value less than or equal to a. The probability that it is greater than a is $1-P(a)$. The probability that the random variable is between a and a value $b > a$ is $P(b) - P(a)$.

This leads to the definition of the probability density function. Taking the limit as Δy goes to zero of the probability that y is between y and $y + \Delta y$ divided by Δy

$$p(y) = \lim_{\Delta y \to 0} \frac{P(y + \Delta y) - P(y)}{\Delta y} \tag{13.7}$$

Equation (13.7) defines the derivative of P with respect to y. Thus,

$$p(y) = \frac{dP}{dy} \tag{13.8}$$

From the fundamental theorem of integral calculus, we have

$$P(y) = \int_{-\infty}^{y} p(\xi)\, d\xi \tag{13.9}$$

where ξ is a dummy variable of integration. It is noted from Equation (13.9) that

$$\int_{-\infty}^{\infty} p(y)\, dy = 1 \tag{13.10}$$

Also,

$$0 \le p(y) \le 1 \tag{13.11}$$

The probability density function, defined by Equation (13.8), illustrated in Figure 13.4.

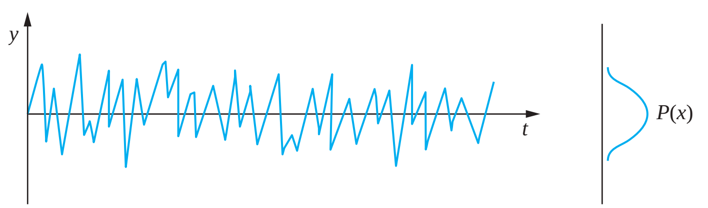


FIGURE 13.4
A random process and its probability density function.

13.3.2 EXPECTED VALUE, MEAN, AND STANDARD DEVIATION

The expected value of a function of a random variable $f(y)$ with a probability density function $p(y)$ is defined as

$$E[f(y)] = \int_{-\infty}^{\infty} f(y)p(y)\,dy \tag{13.12}$$

The mean of a random variable y is the expected value of the random variable

$$\mu = \int_{-\infty}^{\infty} yp(y)\,dy \tag{13.13}$$

The variance of a random variable is the expected valued of $(y-\mu)^2$ and is expressed as

$$\sigma^2 = E[(y-\mu)^2] = \int_{-\infty}^{\infty} (y-\mu)^2 p(y)\,dy \tag{13.14}$$

The standard deviation is the positive square root of the variance and

$$\sigma = \sqrt{\sigma^2} \tag{13.15}$$

13.3.3 MEAN SQUARE VALUE

The mean square value of a function of a random variable y is the expected value of y^2, as

$$\overline{y^2} = E(y^2) = \int_{-\infty}^{\infty} y^2 p(y)\,dy \tag{13.16}$$

The mean square value is related to the variance and mean through

$$\sigma^2 = \int_{-\infty}^{\infty} (y-\mu)^2 p(y)\,dy = \overline{y^2} - 2\mu\int_{-\infty}^{\infty} yp(y)\,dy + \mu^2\int_{-\infty}^{\infty} p(y)dy \tag{13.17}$$

But by definition, the integral of the probability density function is one and the integral of $yp(y)$ is μ. Equation (13.17) becomes

$$\overline{y^2} = \sigma^2 + \mu^2 \tag{13.18}$$

EXAMPLE 13.1

For the process of Figure 13.5, determine (a) the mean, (b) the mean square value, and (c) the standard deviation.

SOLUTION

(a) The mean is calculated two ways. It is equally likely that at any time the value of the function is any value between 0 and A and the function is linear in t. Hence,

$$p(y) = \begin{cases} \dfrac{1}{A} & 0 \le y \le A \\ 0 & y < 0 \text{ or } y > A \end{cases} \tag{a}$$

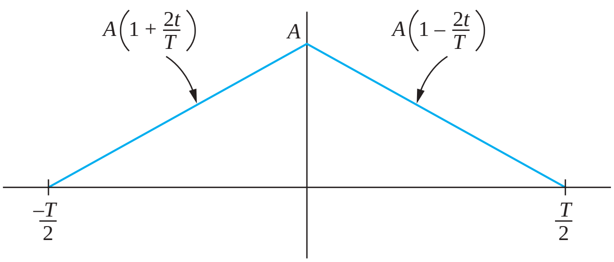


FIGURE 13.5
Process of Example 13.1.

Thus,

$$\mu = \int_{-\infty}^{\infty} yp(y)dy = \int_{0}^{A} x\left(\frac{1}{A}\right)dx = \frac{A}{2} \tag{b}$$

Using the temporal definition of mean, Equation (13.4) becomes

$$\mu = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} A\left(1+\frac{2t}{T}\right)dt + \int_{0}^{\frac{T}{2}} A\left(1-\frac{2t}{T}\right)dt\right] = \frac{A}{2} \tag{c}$$

(b) The mean square value is

$$\overline{y^2} = \int_{-\infty}^{\infty} y^2 p(y)\,dy = \int_{0}^{A} y^2\left(\frac{1}{A}\right)dy = \frac{A^2}{3} \tag{d}$$

(c) The variance is obtained using Equation (13.18) as

$$\sigma^2 = \overline{y^2} - \mu^2 = \frac{A^2}{3} - \frac{A^2}{4} = \frac{A^2}{12} \tag{e}$$

The standard deviation is the square root of the variance, so

$$\sigma = \sqrt{\frac{A^2}{12}} = \frac{A\sqrt{3}}{6} \tag{f}$$

13.3.4 PROBABILITY DISTRIBUTION FOR ARBITRARY FUNCTION OF TIME

Let $y(t)$ be single-valued arbitrary function of time over a period T. Suppose an arbitrary measurement were made. It is equally likely that any time in the interval $0 \le t \le T$ is chosen; thus,

$$p(t) = \frac{1}{T}$$

The following formula is used to find $p(y)$:

$$p(y) = \sum_{i=1}^{n} \frac{p(t)}{\left|\frac{dy}{dt}(t_i)\right|} \tag{13.19}$$

where t_i are all of the values where $y(t_i) = y$.

EXAMPLE 13.2

(a) Find the probability distribution for the rectified sine wave $p(y)$ over the half-period $0 \leq t \leq \frac{T}{2}$.
(b) Use $p(y)$ to calculate the mean.
(c) Use $p(y)$ to determine $\overline{y^2}$.

SOLUTION

(a) The mathematical form of the rectified sine wave is

$$y(t) = A\left|\sin\frac{2\pi}{T}t\right| \tag{a}$$

There are two values of t corresponding to each value of y. The derivatives have equal magnitude, as

$$\left|\frac{dy}{dt}\right| = \frac{2\pi}{T}A\left|\cos\frac{2\pi}{T}t\right| = \frac{2\pi}{T}A\sqrt{1-\left(\frac{y}{A}\right)^2} \tag{b}$$

From Equation (13.19), noting that the interval is $T/2$ and that there are two points corresponding to each y, we have

$$p(y) = 2\left(\frac{2}{T}\right)\left[\frac{\frac{T}{2\pi}}{A\sqrt{1-\left(\frac{y}{A}\right)^2}}\right] = \frac{2}{\pi\sqrt{A^2-y^2}} \tag{c}$$

Thus, since $y < A$, we have

$$p(y) = \begin{cases} \dfrac{2}{\pi\sqrt{A^2-y^2}} & 0 \leq y \leq A \\ 0 & y > A \end{cases} \tag{d}$$

(b) The mean is given by Equation (13.13) as

$$\mu = \int_{-\infty}^{\infty} yp(y)dy = \int_0^A y\left(\frac{2}{\pi\sqrt{A^2-y^2}}\right)dy = \frac{2A}{\pi} \tag{d}$$

The same value of μ is obtained using Equation (13.4).
(c) The mean square value is obtained using Equation (13.16) as

$$\overline{y^2} = \int_{-\infty}^{\infty} y^2p(y)dy = \int_0^A y^2\left(\frac{2}{\pi\sqrt{A^2-y^2}}\right)dy = \frac{A^2}{2} \tag{e}$$

13.3.5 GAUSSIAN PROCESS

The probability density function for a Gaussian distribution of mean μ and standard deviation σ is

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{13.20}$$

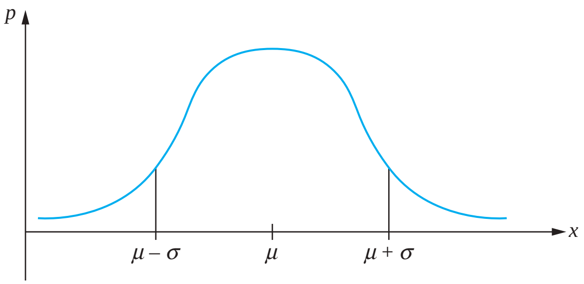


FIGURE 13.6
Gaussian probability density function with a mean of μ and a standard deviation of σ.

The Gaussian distribution is the probability distribution for a normal distribution, such as one defined by the bell-shaped curve, illustrated in Figure 13.6.

$$z = \frac{x - \mu}{\sigma} \tag{13.21}$$

is the standard normal variable. In terms of z, the probability density function is expressed as

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \tag{13.22}$$

Equation (13.22) is the normalized Gaussian probability density function. It has a mean of zero and a standard deviation of one. The normalized Gaussian probability density function is illustrated in Figure 13.7.

The probability distribution is given from the density function by Equation (13.9), as

$$p(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}} d\tau \tag{13.23}$$

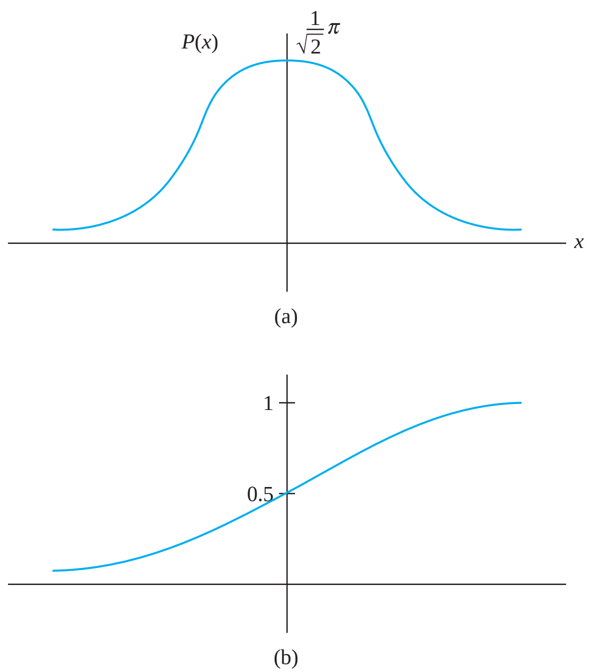


FIGURE 13.7
(a) Normalized Gaussian density function distribution. (b) Gaussian probability distribution.

TABLE 13.1 Probability distribution for a Gaussian process, $P(-z) = 1 - P(z)$

z	P(z)	z	P(z)
0.0	0.5	1.1	0.8643
0.1	0.5398	1.2	0.8849
0.2	0.5793	1.3	0.9032
0.3	0.6179	1.4	0.9192
0.4	0.6554	1.5	0.9382
0.5	0.6915	1.6	0.9452
0.6	0.7257	1.7	0.9554
0.7	0.7580	1.8	0.9641
0.8	0.7881	1.9	0.9713
0.9	0.8159	2.0	0.9773
1.0	0.8413	∞	1

The integral in Equation (13.23) cannot be evaluated analytically, but the values are summarized in a table of normal distributions. An abbreviated one is given in Table 13.1.

EXAMPLE 13.3

The displacement of a machine is a random variable with a mean of 1 mm and a standard deviation of 0.1 mm. What is the probability that at any given time the displacement (a) exceeds 1.05 mm, (b) is less than 0.85 mm, and (c) is between 0.93 mm and 1.01 mm?

SOLUTION

Assume the amplitude of vibration has a Gaussian distribution. The normalized variable is given by Equation (13.20) with $\mu = 1$ mm and $\sigma = 0.1$ mm.

(a) Calculating the value of the normalized variable, we have

$$z = \frac{1.05\,\text{mm} - 1\,\text{mm}}{0.1\,\text{mm}} = 0.5 \tag{a}$$

The probability that $z < 0.5$ is $P(0.5) = 0.6915$

$$\text{Prob}(z > 0.5) = 1 - P(0.5) = 1 - 0.6915 = 0.3085 \tag{b}$$

(b) The normalized variable for $x = 0.85$ mm is

$$z = \frac{0.85\,\text{mm} - 1\,\text{mm}}{0.1\,\text{mm}} = -1.5 \tag{c}$$

The probability that $z < -1.5$ is

$$\text{Prob}(z < -1.5) = P(-1.5) = 1 - P(1.5) = 1 - 0.9832 = 0.0168 \tag{d}$$

(c) The z value corresponding to 0.93 mm is $z = -0.7$ and the z value corresponding to 1.01 mm is $z = 0.1$. The probability that z is between -0.7 and 0.1 is

$$\begin{aligned}\text{Prob}(-0.7 < z < 0.1) &= P(0.1) - P(-0.7)\\ &= 0.5398 - (1 - 0.7580) = 0.2978\end{aligned} \tag{e}$$

13.3.6 RAYLEIGH DISTRIBUTION

The Rayleigh distribution is used for random variables restricted to positive values. The probability density function for the Rayleigh distribution for a positive random variable y is defined by

$$p(y) = \frac{y}{\alpha} e^{-\frac{y^2}{2\alpha}} \tag{13.24}$$

The Rayleigh distribution is illustrated in Figure 13.8 for several values of α.

The probability distribution for the Rayleigh distribution can be obtained by direct integration. Let $u = \frac{y^2}{2\alpha}$. Then

$$du = \frac{y}{\alpha} dy \text{ and } p(y) = \int_0^{\frac{y^2}{2\alpha}} e^{-u} du = 1 - e^{-\frac{y^2}{2\alpha}} \tag{13.25}$$

Representative values of the probability distribution are given in Table 13.2.

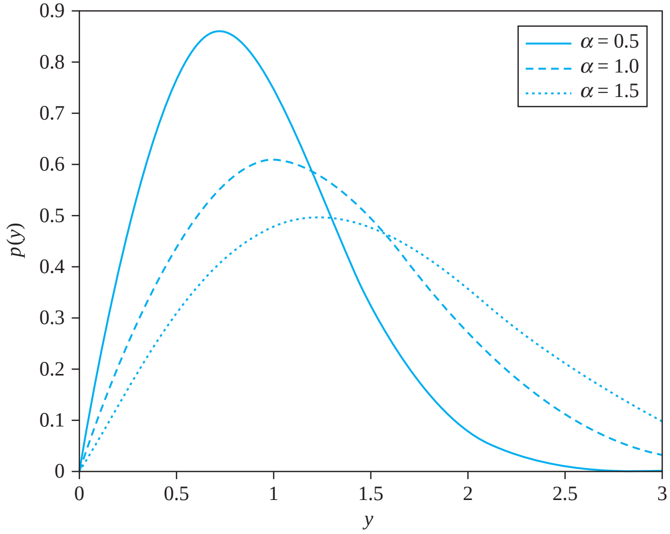


FIGURE 13.8 Rayleigh distribution for several values of α.

TABLE 13.2 Rayleigh probability distribution

$u = \frac{y^2}{2\alpha}$	P(u)	$u = \frac{y^2}{2\alpha}$	P(u)	$u = \frac{y^2}{2\alpha}$	P(u)
0	0	1.0	0.6321	2.0	0.8647
0.1	0.0952	1.1	0.6671	2.1	0.8775
0.2	0.1813	1.2	0.6988	2.2	0.8892
0.3	0.2592	1.3	0.7275	2.3	0.8997
0.4	0.3297	1.4	0.7534	2.4	0.9093
0.5	0.3935	1.5	0.7769	2.5	0.9179
0.6	0.4512	1.6	0.7981	2.6	0.9257
0.7	0.5034	1.7	0.8173	2.7	0.9328
0.8	0.5507	1.8	0.8347	2.8	0.9392
0.9	0.5934	1.9	0.8504	2.9	0.9450

EXAMPLE 13.3

Calculate for the Rayleigh distribution (a) the mean, (b) the mean square value, (c) the standard deviation, and (d) the probability that the random variable is one standard deviation greater than the mean for $\alpha = 4$.

SOLUTION

(a) The mean is calculated using Equation (13.13), knowing that it is only defined for positive values of y, so that

$$\mu = \int_0^\infty y p(y) dy = \int_0^\infty y \left[\frac{y}{\alpha} e^{-\frac{y^2}{2\alpha}}\right] dy = \sqrt{\frac{\pi\alpha}{2}} \tag{a}$$

(b) The mean square value is calculated using Equation (13.16) as

$$\overline{y^2} = \int_0^\infty y^2 p(y) dy = \int_0^\infty y^2 \left[\frac{y}{\alpha} e^{-\frac{y^2}{2\alpha}}\right] dy = 2\alpha \tag{b}$$

(c) The variance is calculated using Equation (13.18) as

$$\sigma^2 = \overline{y^2} - \mu^2 = 2\alpha - \frac{\pi\alpha}{2} = \frac{4-\pi}{2}\alpha = 0.429\alpha \tag{c}$$

The standard deviation of the Rayleigh distribution is

$$\sigma = \sqrt{0.429\alpha} = 0.655\sqrt{\alpha} \tag{e}$$

(d) One standard deviation greater than the mean implies that the variable is greater than

$$\mu + \sigma = \sqrt{\frac{\pi\alpha}{2}} + \sqrt{\left(\frac{4-\pi}{2}\right)\alpha} = 1.910\sqrt{\alpha} = 1.910\sqrt{4} = 3.82 \tag{f}$$

Thus, the probability that y is greater than 3.82 is

$$1 - P(3.82) = 1 - \int_0^{3.82} p(y) dy = 1 - \int_0^{3.82} \frac{y}{4} e^{-\frac{y^2}{8}} dy = 1 - 0.839 = 0.161 \tag{g}$$

13.3.7 CENTRAL LIMIT THEOREM

A random variable can satisfy many probability distributions. Due to the central limit theorem, the Gaussian distribution is the most important. It states that, if a random variable has any distribution, then if an experiment is run n times and the mean and standard deviation of the kth sample are μ_k and σ_k, respectively, then for large n the means of the sample are normally distributed with a mean equal to

$$\mu = \frac{1}{n}\sum_{k=1}^{n} \mu_k \tag{13.26}$$

and a standard deviation equal to $\sigma\sqrt{n}$ where

$$\sigma^2 = \frac{\sum_{k=1}^{n} (\mu_k - \mu)^2}{n-1} \tag{13.27}$$

Stated another way, a random variable is the sum of a large number of random variables, the sum approaches the normal distribution. Thus, if x is a random variable that has a probability distribution $P(x)$, the random variable $X = x_1 + x_2 + \cdots + x_n$ where x_i are different outcomes for x, which is normally distributed. Thus, an ensemble is normally distributed.

13.4 JOINT PROBABILITY DISTRIBUTIONS

13.4.1 TWO RANDOM VARIABLES

An experiment is run repeatedly on a SDOF system subject to a harmonic excitation. The output is expected in the form $x(t) = A\sin(\omega t - \phi)$. However, due to a variety of factors, the experiment does not yield the same amplitude or phase at the same instant of time. Ensembles of each are taken as $\{Ai\}$ and $\{\phi i\}$. Probability density functions can be developed for both A and ϕ, $p(\phi)$ and $p(A)$, respectively. It is known that A and ϕ are related through the frequency ratio of the system and the damping ratio; they are not totally independent. Thus, the joint probability density function is not simply $p(A)p(\phi)$ but is $p(A, \phi)$.

In general, consider two random variables from the same process x and y. The joint probability density function is written as $p(x, y)$. This leads to a joint probability distribution defined by

$$P(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} p(\xi,\tau)\, d\xi\, d\tau \tag{13.28}$$

The joint probability distribution is defined such that $P(x, y)$ is the probability that the first random variable has a value less than x and the second random variable has a value less than y. The probability density functions are defined by

$$p(x) = \int_{-\infty}^{\infty} p(x, y)\, dy \tag{13.29}$$

and

$$p(y) = \int_{-\infty}^{\infty} p(x, y)\, dx \tag{13.30}$$

The means are defined as

$$\mu_x = E(x)\int_{-\infty}^{\infty} xp(x)\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xp(x, y)\, dy\, dx \tag{13.31}$$

and

$$\mu_y = E(y) = \int_{-\infty}^{\infty} yp(y)\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yp(x, y)\, dx\, dy \tag{13.32}$$

The variances are defined by

$$\sigma_x^2 = E[(x - \mu_x)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_x)^2 p(x, y)\, dy\, dx \tag{13.33}$$

and

$$\sigma_y^2 = E[(y - \mu_y)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - \mu_y)^2 p(x, y)\, dx\, dy \tag{13.34}$$

The covariance is

$$\sigma_{xy} = E[(x-\mu_x)(y-\mu_y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)p(x,y)\,dx\,dy \tag{13.35}$$

The coefficient of correlation is defined by

$$r_{xy} = \frac{\sigma_{xy}}{\sigma_x\sigma_y} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)p(x,y)\,dx\,dy}{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)^2 p(x,y)\,dy\,dx\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y-\mu_y)^2 p(x,y)\,dx\,dy\right]^{\frac{1}{2}}} \tag{13.36}$$

13.4.2 AUTOCORRELATION FUNCTION

Let $y(t)$ be a random variable that is a function of time. Consider the variable at a time $t+\tau$. The autocorrelation is the expected value of the product of the function at these two times:

$$R(t,\tau) = E[y(t)y(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t)y(t+\tau)p[y(t),y(t+\tau)]\,dy(t)\,dy(t+\tau) \tag{13.37}$$

For a stationary process, R is independent of t, dependent on τ, and is written as $R(\tau)$. For a stationary ergodic process, the temporal average can be used, as

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)y(t+\tau)\,dt \tag{13.38}$$

If $\tau = 0$,

$$R(0) = E(y^2) \tag{13.39}$$

From Equation (13.18),

$$R(0) = \sigma^2 + \mu^2 \tag{13.40}$$

It can be shown that this is the maximum value of the autocorrelation function.

It is noted that for a stationary process

$$R(-\tau) = E[y(t)y(t-\tau)] = E[y(t)y(t+\tau)] = R(\tau) \tag{13.41}$$

The autocorrelation function is an even function.

EXAMPLE 13.4

Determine the autocorrelation function for the sine wave given by

$$y(t) = A\sin(\omega t - \phi) \tag{a}$$

SOLUTION

The autocorrelation function is given by Equation (13.38) for an egrodic process. The process is periodic of period $T = \frac{2\pi}{\omega}$; thus, the limit of the integral is replaced by its evaluation over one period as

$$R(\tau) = \frac{\omega}{2\pi}\int_0^{\frac{\pi}{\omega}} A\sin(\omega t-\phi)\,\{A\sin[\omega(t+\tau)-\phi]\}\,dt = \frac{A^2}{2}\cos\omega\tau \tag{b}$$

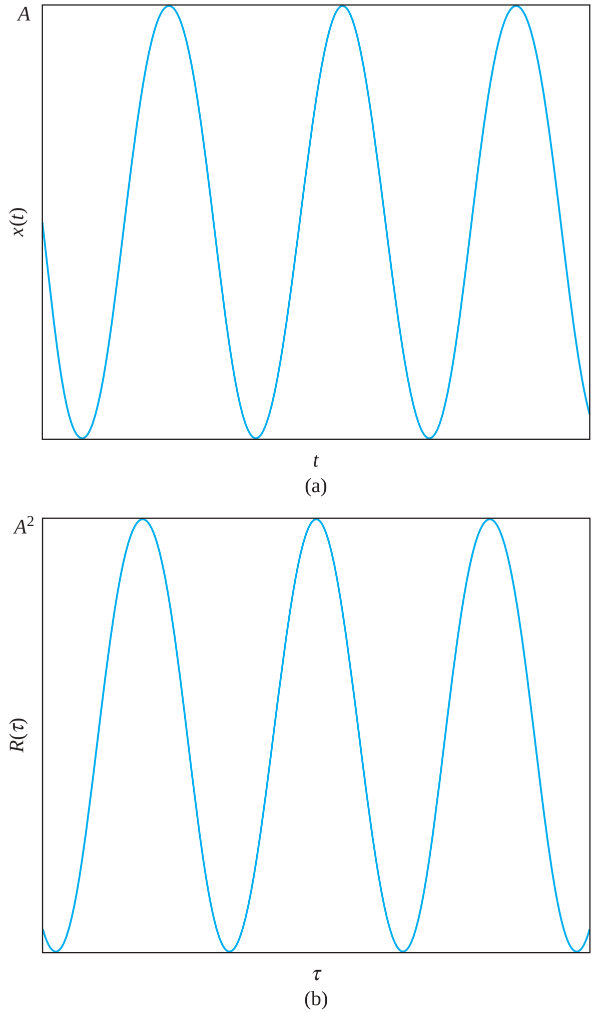


FIGURE 13.9
(a) $y(t) = A\sin(\omega t - \phi)$ for Example 13.4. (b) Autocorrelation function for $y(t)$.

The autocorrelation of a harmonic function is another harmonic function. The function and its autocorrelation are shown in Figure 13.9.

EXAMPLE 13.5

(a) Determine the autocorrelation function for the process shown in Figure 13.10. (b) Verify that $R(0) = E(y^2)$.

SOLUTION

(a) The process is periodic of period T thus the autocorrelation function is

$$R(\tau) = \frac{1}{T}\int_0^T y(t)y(t+\tau)dt = \int_0^{\frac{T}{4}} Ay(t+\tau)dt \tag{a}$$

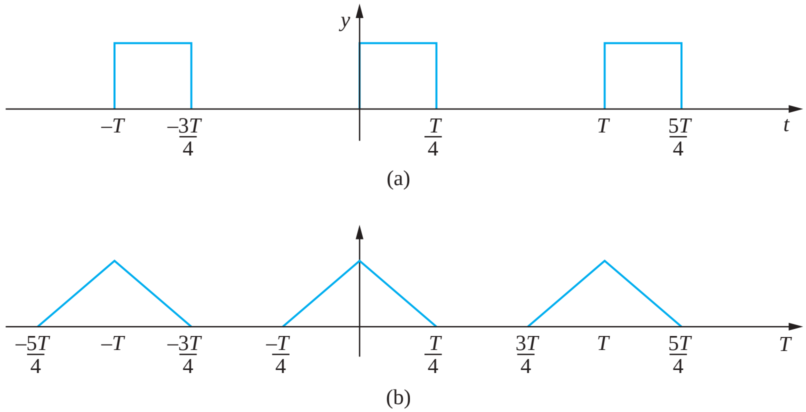


FIGURE 13.10
(a) $y(t)$ for Example 13.5. (b) Autocorrelation function for $y(t)$.

where

$$y(t+\tau) = \begin{cases} A & 0 < \frac{T}{4} - \tau \\ 0 & \frac{T}{4} < \tau < \frac{3T}{4} \\ A & \tau - \frac{3T}{4} < T \end{cases} \tag{b}$$

For $0 < \tau < \frac{T}{4}\tau$

$$R(\tau) = \frac{1}{T}\int_0^{\frac{T}{4}-\tau} A^2 dt = A^2\left(\frac{1}{4} - \frac{\tau}{T}\right) \tag{c}$$

For $\frac{T}{4} < \tau < \frac{3T}{4}$,

$$R(\tau) = 0 \tag{d}$$

For $\frac{3T}{4} < \tau < T$,

$$R(\tau) = \frac{1}{T}\int_{T-\tau}^{\frac{T}{4}} A^2 dt = A^2\left(\frac{\tau}{T} - \frac{3}{4}\right) \tag{e}$$

The autocorrelation is periodic of period T and is illustrated in Figure 13.10.
(b) Using Equation (13.16),

$$E(y^2) = \int_0^T y^2 dt = \int_n^{\frac{T}{4}} A^2 dt = A^2\frac{T}{4} = R(0) \tag{f}$$

13.4.3 CROSS CORRELATIONS

The cross correlation between two stationary ergodic random variables y_1 and y_2 is defined by

$$C(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y_1(t)y_2(t+\tau)dt \tag{13.42}$$

13.5 FOURIER TRANSFORMS

13.5.1 FOURIER SERIES IN COMPLEX FORM

The Fourier series for a periodic function of period T is given by Equation (4.130). Through substitution of the relations between the trigonometric functions and exponentials of complex exponents,

$$\sin\omega t = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}) \tag{13.43}$$

and

$$\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \tag{13.44}$$

the Fourier series representation for a periodic function can be written as

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[\frac{1}{2}(a_k - ib_k)e^{i\omega_k t} + \frac{1}{2}(a_k + ib_k)e^{-i\omega_k t}\right] \tag{13.45}$$

Defining

$$a_k = \frac{1}{2}(a_k - ib_k) \tag{13.46}$$

The Fourier series is written as

$$F(t) = \sum_{k=-\infty}^{\infty}\alpha_k e^{i\omega_k t} \tag{13.47}$$

where $\alpha_{-k} = \alpha_k^*$, which is the complex conjugate of α_k. The above equations can be combined using the definitions of the Fourier coefficients to yield.

$$\alpha_k = \frac{1}{T}\int_0^T F(t)e^{i\omega_k t}dt \tag{13.48}$$

Since $F(t)$ is periodic of period T,

$$\alpha_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(t)e^{i\omega_k t}dt \tag{13.49}$$

The mean square value of $F(t)$ is

$$\overline{F^2} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F^2(t)\,dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\sum_{k=-\infty}^{\infty}\alpha_k e^{i\omega_k t}\right)^2 dt = \sum_{k=-\infty}^{\infty}\alpha_k\alpha_k^* = \sum_{k=-\infty}^{\infty}|\alpha_k|^2 \tag{13.50}$$

Equation (13.50) is called Parseval's identity.

13.5.2 FOURIER TRANSFORM FOR NONPERIODIC FUNCTIONS

A Fourier series can be developed for any function by taking the limit of Equation (13.47) as T approaches infinity, as

$$F(t) = \lim_{T\to\infty}\left\{\sum_{k=-\infty}^{\infty}\left[\frac{1}{T}\int_{\frac{T}{2}}^{\frac{T}{2}} F(t)e^{i\omega_k t}dt\right]e^{i\omega_k t}\right\} \tag{13.51}$$

As $T \to \infty$ the discrete frequencies approach a continuous spectrum $\omega_k \to \omega$, and the integral divided by the period is expressed as a function of the continuous variable ω as $\frac{F(\omega)}{2\pi}$. As T approaches infinity, k becomes a continuous variable, and the infinite sum becomes an integral with a variable of integration of ω. The result of the limiting process is

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \tag{13.52}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt \tag{13.53}$$

Equation (13.53) defines the Fourier transform of a nonperiodic function. Equations (13.52) and (13.53) form a Fourier transform pair.

EXAMPLE 13.6

Determine the Fourier transform of the unit impulse function $\delta(t)$.

SOLUTION

By definition and from Equation (13.53),

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt \tag{a}$$

The unit impulse function is zero everywhere except at $t = 0$, where it is infinite. But it is infinite in such a way (see Appendix A) that

$$\int_{-\infty}^{\infty} \delta(t)F(t)dt = F(0) \tag{b}$$

Thus, the Fourier transform of the unit impulse function is

$$F(\omega) = 1 \tag{c}$$

EXAMPLE 13.7

Determine the Fourier transform of $F_0 \sin \omega_n t$.

SOLUTION

Using Equation (13.43) for $\sin \omega_n t$ and substituting into Equation (13.53) gives the Fourier transform as

$$F(\omega) = \int_{-\infty}^{\infty}\frac{F_0}{2i}(e^{i\omega_n t} - e^{-i\omega_n t})e^{-i\omega t}dt \tag{a}$$

Unlike the Laplace transform, the integration cannot be performed by traditional means. Instead, use Equation (13.52)

$$\frac{F_0}{2i}(e^{i\omega_n t} - e^{-i\omega_n t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \tag{b}$$

Recalling the property of the unit impulse function,

$$\int_{-\infty}^{\infty} F(\omega)\delta(\omega - a)\, d\omega = F(a) \tag{c}$$

leads to the realization that

$$F(\omega) = \frac{F_0}{2i}[\delta(\omega - \omega_n) - \delta(\omega + \omega_n)] \tag{d}$$

The Fourier transform of a periodic function leads to a discrete frequency distribution.

EXAMPLE 13.8

Determine the Fourier transform of the non-periodic function of Figure 13.11.

SOLUTION

The Fourier transform of $F(t)$ is

$$\begin{aligned}
F(\omega) &= F_0\left[\int_{-T}^{-\frac{T}{2}} 2\left(1 + \frac{t}{T}\right)e^{i\omega t}dt + \int_{\frac{T}{2}}^{\frac{T}{2}} e^{i\omega t}dt + \int_{\frac{T}{2}}^{T} 2\left(1 - \frac{t}{T}\right)e^{i\omega t}dt\right] \\
&= F_0\left[\int_{-T}^{-\frac{T}{2}} e^{i\omega t}dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\omega t}dt + 2\int_{\frac{T}{2}}^{T} e^{i\omega t}dt + \frac{2}{T}\int_{-T}^{-\frac{T}{2}} te^{i\omega t}dt + \frac{2}{T}\int_{\frac{T}{2}}^{T} te^{i\omega t}dt\right] \\
&= \frac{F_0}{i\omega}\left[2e^{i\omega t}\Big|_{-T}^{-\frac{T}{2}} + e^{i\omega t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + 2e^{i\omega t}\Big|_{-\frac{T}{2}}^{T}\right. \\
&\quad \left. + \left(te^{i\omega t} - \frac{1}{i\omega}e^{i\omega t}\right)\Big|_{-T}^{-\frac{T}{2}} + \left(te^{i\omega t} - \frac{1}{i\omega}e^{i\omega t}\right)\Big|_{\frac{T}{2}}^{T}\right] \qquad \text{(a)} \\
&= \frac{4F_0}{\omega}\sin\omega T - \frac{2F_0}{\omega}\sin\frac{\omega T}{2} - \frac{iF_0}{\omega}\left[2\sin\frac{\omega T}{2} - \omega T\cos\frac{\omega T}{2}\right. \\
&\quad \left. + 2\,\omega T\cos\omega T - 2\sin\omega T\right]
\end{aligned}$$

The real part of the Fourier transform of $F(t)$ is illustrated in Figure 13.11(b), the imaginary part in Figure 13.11(c).

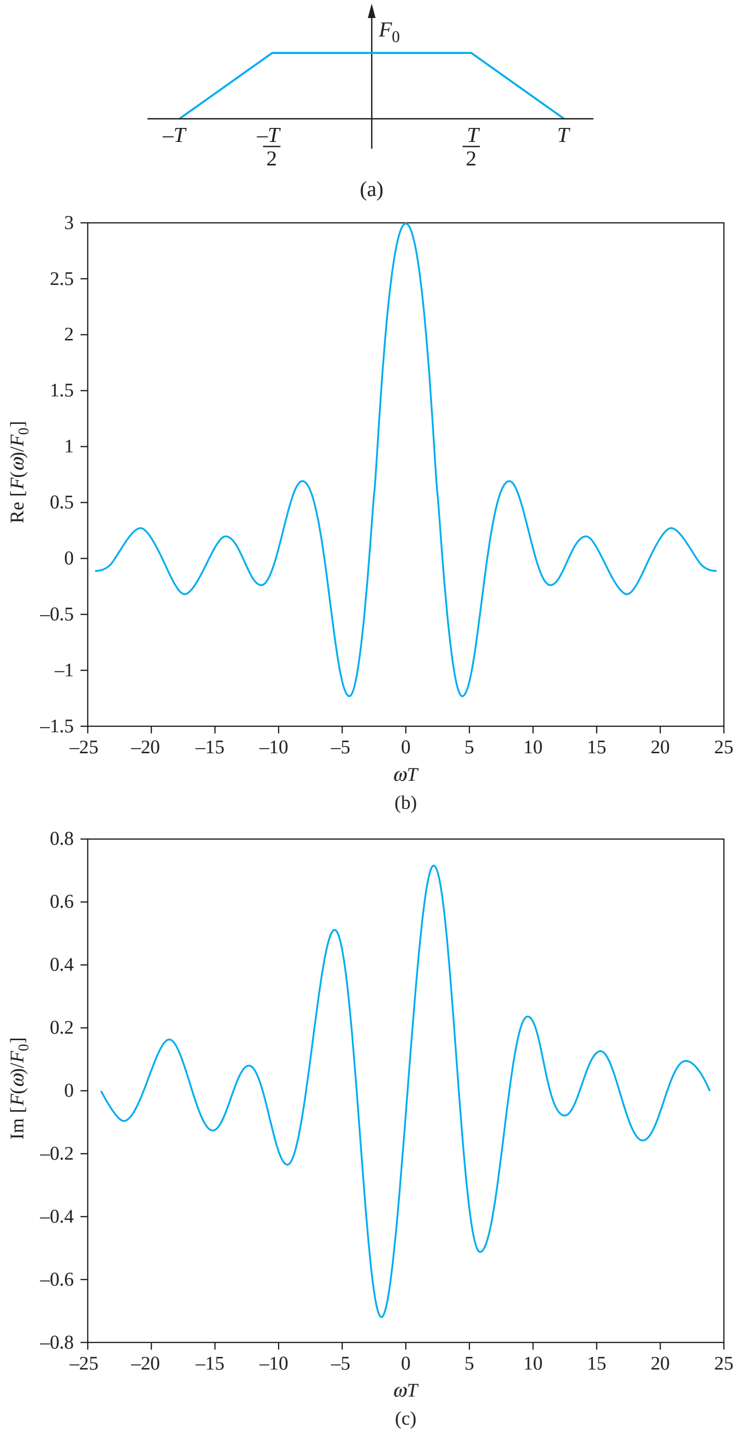


FIGURE 13.11
(a) $F(t)$ for Example 13.8. (b) Real part of $F(\omega)$. (c) Imaginary part of $F(\omega)$.

13.5.3 TRANSFER FUNCTIONS

The Fourier transform has properties similar to that of the Laplace transform. It exists and is unique for functions that can be generated physically. It satisfies a linearity property. Let $\mathcal{T}\{F(t)\}$ represent the Fourier transform of $F(t)$. Let $F(\omega) = \mathcal{T}\{F(t)\}$ and $G(\omega) = \mathcal{T}\{G(t)\}$. Then for any scalars a and b,

$$\mathcal{T}\{aF(t) + bG(t)\} = aF(\omega) + bG(\omega) \tag{13.54}$$

The Fourier transform also satisfies a property of transform of derivatives. Differentiating Equation (13.52) with respect to time leads to

$$\frac{dF}{dt} = \frac{i\omega}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = i\omega F(t) \tag{13.55}$$

Taking the Fourier transform of both sides of this equation using linearity leads to

$$\mathcal{T}\left\{\frac{dF}{dt}\right\} = i\omega\mathcal{T}\{F\} \tag{13.56}$$

The Fourier transform of the second derivative is

$$\mathcal{T}\left\{\frac{d^2F}{dt^2}\right\} = i\omega\mathcal{T}\left\{\frac{dF}{dt}\right\} = -\omega^2\mathcal{T}\{F\} \tag{13.57}$$

EXAMPLE 13.9

Determine the Fourier transform of the solution of

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{a}$$

for an arbitrary $F(t)$.

SOLUTION

Let $F(\omega) = \mathcal{T}\{F(t)\}$ and $\chi(\omega) = \mathcal{T}\{\chi(t)\}$. Taking the Fourier transform of the differential equation leads to

$$\mathcal{T}\{m\ddot{x} + c\dot{x} + kx\} = \mathcal{T}\{F(t)\} \tag{b}$$

Using linearity, Equation (b) becomes

$$m\mathcal{T}\{\ddot{x}\} + c\mathcal{T}\{\dot{x}\} + k\mathcal{T}\{x\} = F(\omega) \tag{c}$$

Applying the property of transform of derivatives, Equation (c) is rewritten as

$$-m\omega^2\chi(\omega) + ic\omega\chi(\omega) + k\chi(\omega) = F(\omega) \tag{d}$$

Solving Equation (d) yields

$$\chi(\omega) = \frac{F(\omega)}{-m\omega^2 + ic\omega + k} \tag{e}$$

Similar to the Laplace transform method, a transfer function can be defined for the Fourier transform as the ratio of the Fourier transform of the output to the Fourier transform of the input,

$$H(\omega) = \frac{\chi(\omega)}{F(\omega)} \tag{13.58}$$

In taking the Fourier transform of a differential equation instead of the Laplace transform, the s from the Laplace transform is replaced by $i\omega$ from the Fourier transform. Thus,

$$H(\omega) = G(i\omega) \tag{13.59}$$

is the same as the sinusoidal transfer function discussed in Section 6.9.

Example 13.6 showed that $\mathcal{T}\{\delta(t)\} = 1]$. Thus, the Fourier transform of the response of a system due to a unit impulse function is $H(\omega)$.

13.5.4 FOURIER TRANSFORM IN TERMS OF f

The frequency parameter ω has units of rad/s and is converted to cycles/s by recognizing that 2π rad $= 1$ cycle. Denoting the frequency in cycles/s by f, we have $\omega = 2\pi f$. Substituting for ω in Equation (13.52) leads to

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(f)e^{i2\pi ft}d(2\pi f) = \int_{-\infty}^{\infty} F(f)e^{i2\pi ft}df \tag{13.60}$$

Substitution into Equation (13.53) gives

$$F(f) = \int_{-\infty}^{\infty} F(t)e^{-i2\pi ft}dt \tag{13.61}$$

Equations (13.60) and (13.61) are the Fourier transform pair in terms of the frequency in cycles/s.

13.5.5 PARSEVAL'S IDENTITY

Applying the definition of the inverse Fourier transform,

$$\int_{-\infty}^{\infty} y^2(t)dt = \int_{-\infty}^{\infty} y(t)\left[\int_{-\infty}^{\infty} Y(f)e^{i2\pi ft}df\right]dt \tag{13.62}$$

Changing the order of integration in Equation (13.62) leads to

$$\int_{-\infty}^{\infty} y^2(t)dt = \int_{-\infty}^{\infty} Y(f)\left[\int_{-\infty}^{\infty} y(t)e^{i2\pi ft}dt\right]df \tag{13.63}$$

The inner integral is the definition of the complex conjugate of the Fourier transform of $y(t)$, leading to

$$\int_{-\infty}^{\infty} y^2(t)dt = \int_{-\infty}^{\infty} Y(f)Y^*(f)df = \int_{-\infty}^{\infty} |Y(f)|^2df \tag{13.64}$$

Equation (13.64) is known as Parseval's identity for nonperiodic functions. Using the circular frequency ω, Parseval's identity is written as

$$\int_{-\infty}^{\infty} y^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2d\omega \tag{13.65}$$

13.6 POWER SPECTRAL DENSITY

The mean square value of a random variable y is calculated according to

$$\overline{y^2} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y^2(t)\,dt \tag{13.66}$$

Using Parseval's identity, Equation (13.65) becomes

$$\overline{y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}\,y(\omega)y^*(\omega)\,d\omega \tag{13.67}$$

Equation (13.67) is written as

$$\overline{y^2} = \int_{-\infty}^{\infty} S(\omega)\,d\omega \tag{13.68}$$

where

$$S(\omega) = \lim_{T\to\infty}\frac{1}{T}\,y(\omega)y^*(\omega) \tag{13.69}$$

is called the *power spectral density*. It represents the energy density associated with a frequency ω.

The power spectral density is written in terms of f as

$$W(f) = 2\pi S(\omega) \tag{13.70}$$

where f can take on positive and negative values. If f is restricted to positive values, in order for

$$E(y^2) = \int_{-0}^{\infty} S(\omega)\,d\omega = \int_{0}^{\infty} W(f)\,df \tag{13.71}$$

the spectral density in terms of f is defined as

$$W(f) = 4\pi S(\omega) \tag{13.72}$$

Consider the Fourier transform of $y(t+\tau)$, defined such that

$$y(t+\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega(t+\tau)}\,d\omega \tag{13.73}$$

Substituting Equation (13.73) into the definition of the autocorrelation function yields

$$\begin{aligned} R(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)y(t+\tau)\,dt \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega(t+\tau)}\,d\omega\right]dt \end{aligned} \tag{13.74}$$

Interchanging the limiting process with the integration, changing the order of the integration, and rearranging leads to

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} y(t)e^{i\omega t}\,d\tau\right]Y(\omega)e^{i\omega\tau}\,d\omega \tag{13.75}$$

Using the definition of the Fourier transform in Equation (13.75) gives

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} Y^*(\omega)\,Y(\omega)e^{i\omega\tau}d\omega \tag{13.76}$$

Using the definition of power spectral density in Equation (13.76) leads to

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega \tag{13.77}$$

Thus, the power spectral density is the Fourier transform of the autocorrelation function over 2π. Using the definition of the transform pair, we have

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega t}dt \tag{13.78}$$

Equations (13.77) and (13.78) are called the *Wiener-Khintchine equations.*

The autocorrelation and the power spectral densities are real functions of ω. In addition, $R(\tau) = R(-\tau)$, so that Equation (13.77) can be written as

$$R(\tau) = 2\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\, d\omega \tag{13.79}$$

A wideband process is one in which a large number of frequencies appear in the time-dependent description of the process, as shown in Figure 13.12(a). The autocorrelation

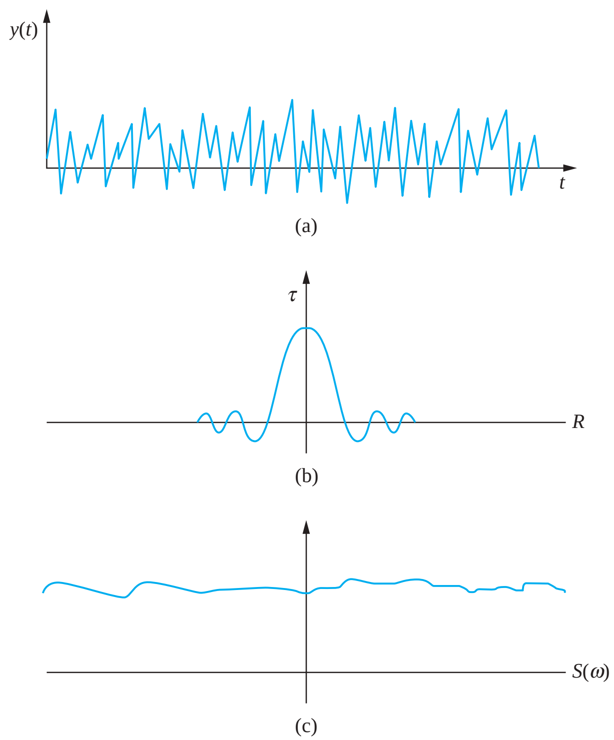


FIGURE 13.12
Wideband process (a) $y(t)$, (b) $R(\tau)$, and (c) $S(\omega)$.

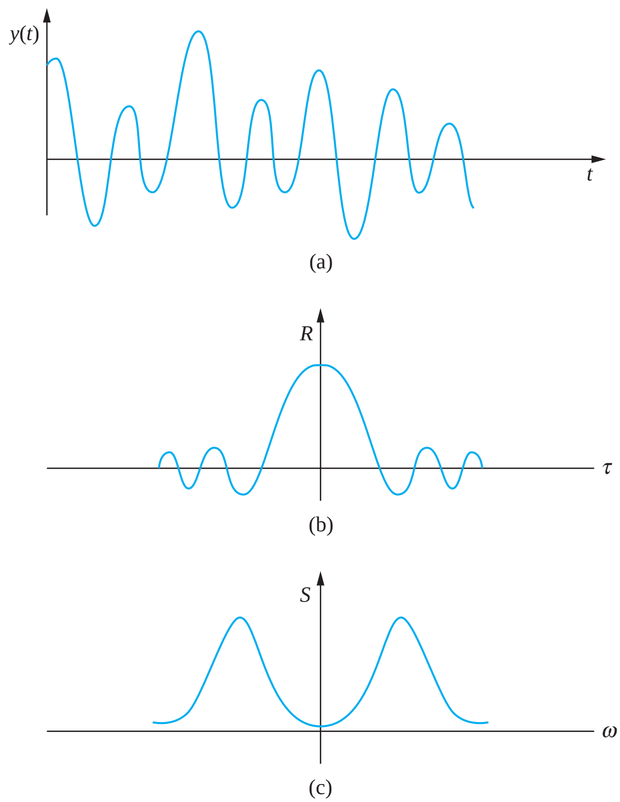


FIGURE 13.13
Narrowband process (a) $y(t)$, (b) $R(\tau)$, (c) $S(\omega)$.

function is large near $\omega = 0$ and decays rapidly, as shown in Figure 13.12(b). The power spectral density has a significant value over a wide range of frequencies, as shown in Figure 13.12(c). Jet engine noise is typically a wideband process.

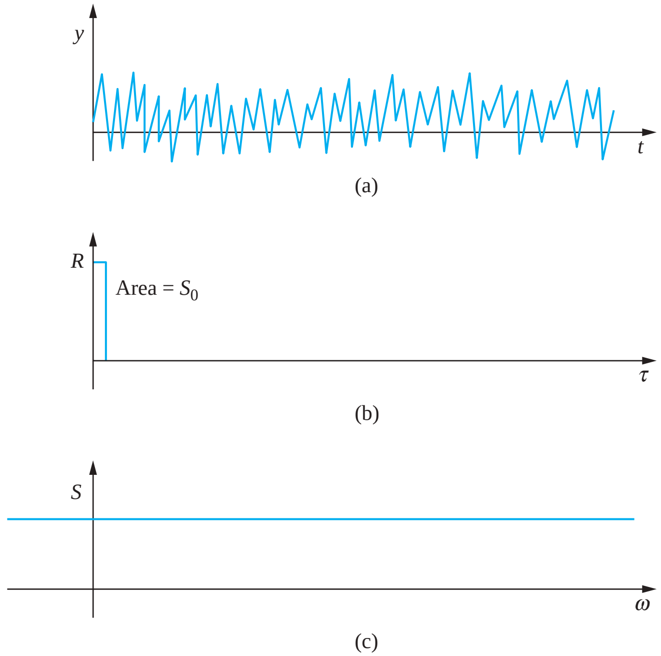
A narrowband process is one in which only a few frequencies with random amplitudes are present in the time-dependent description of the process, as illustrated in Figure 13.13(a). The autocorrelation function appears to be a decaying cosine function, as illustrated in Figure 13.13(b). The power spectral density is large over a narrowband of frequencies, as shown in Figure 13.13(c). Vibration of a floor in an industrial plant is an example of narrowband excitation in which a few frequencies are dominant in the power spectral density.

White noise is a limiting case of a wideband process. Its time dependent description, as illustrated in Figure 13.14(a), is similar to that of a wideband excitation. Its autocorrelation function is proportional to the unit impulse function

$$R(\tau) = S_0\delta(\tau) \tag{13.80}$$

where S_0 is the magnitude of its constant-power spectral density, as shown in Figure 13.14(c), over a theoretically infinite frequency range. White noise is impossible to achieve, as the means square value of the process is infinite but it provides a good approximation for many wide band processes.

FIGURE 13.14
White noise at (a) $y(t)$, (b) $R(\tau)$, and (c) $S(\omega)$.



If the white noise is limited by bandwidth, the power spectral density is as given in Figure 13.14 and is described by

$$S(\omega)\begin{cases} S_0 & \omega_1 < \omega < \omega_2 \\ 0 & \omega < \omega_1 \text{ and } \omega > \omega_2 \end{cases} \tag{13.81}$$

This bandlimited white noise is more realistic, as it has a finite mean square value.

EXAMPLE 13.10

Determine and plot the autocorrelation function for bandlimited white noise, as illustrated in Figure 13.15.

SOLUTION

The autocorrelation function is given by Equation (13.77), which is evaluated for this power spectral density as

$$R(\tau) = 2\int_{\omega_1}^{\omega_2} S_0 \cos \omega\tau \, d\omega = \frac{2F_0}{\tau}(\sin \omega_2\tau - \sin \omega_1\tau) \tag{a}$$

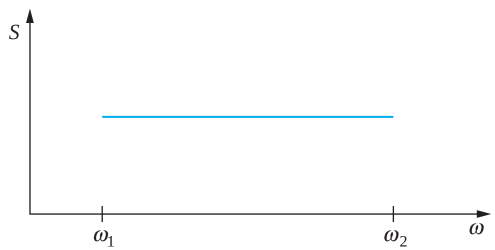


FIGURE 13.15
Bandlimited white noise for Example 13.10.

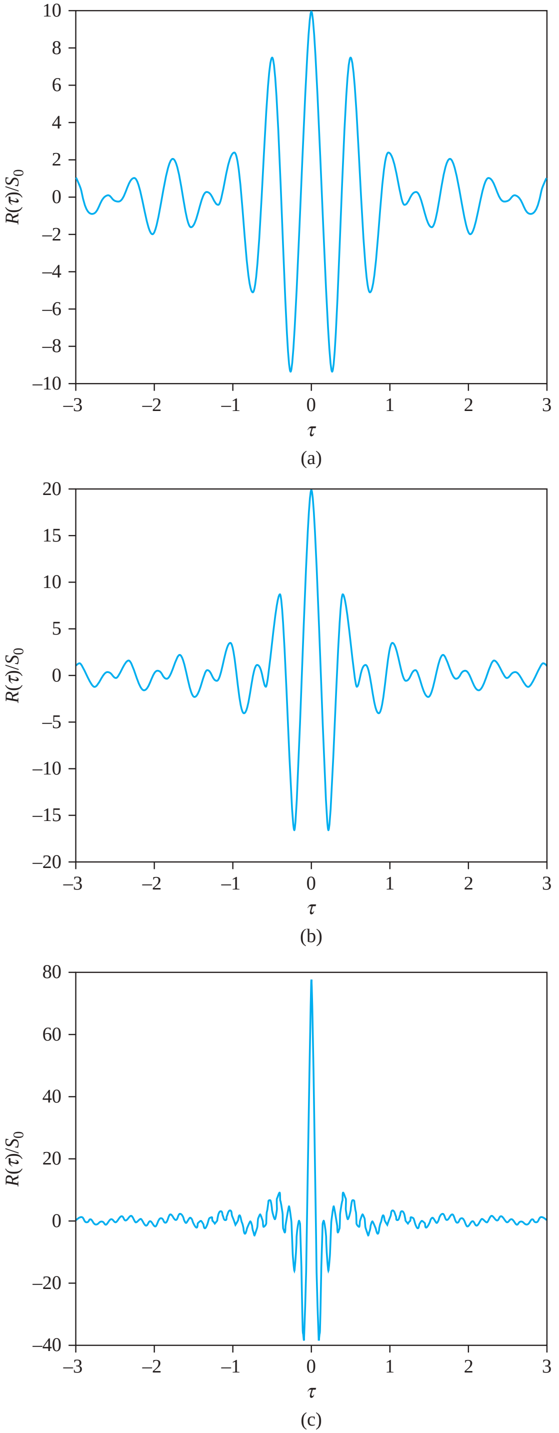


FIGURE 13.16
Autocorrelation function for bandlimited white noise with ω_1 = 10 rad/s at (a) ω_2 = 15 rad/s, (b) ω_2 = 20 rad/s, and (c) ω_2 = 50 rad/s.

The autocorrelation function is plotted in Figure 13.16 for ω_1 = 10 rad/s for different values of ω_2.

13.7 MEAN SQUARE VALUE OF THE RESPONSE

Equation (13.58) implies that the transfer function is the ratio of the Fourier transform of the output to the Fourier transform of the input. Multiplying the transfer function by its complex conjugate yields a real quantity

$$H(\omega)H^*(\omega) = \frac{X(\omega)X^*(\omega)}{F(\omega)F^*(\omega)} \tag{13.82}$$

which is rearranged to yield

$$X(\omega)X^*(\omega) = |H(\omega)|^2 F(\omega)F^*(\omega) \tag{13.83}$$

Let $S_x(\omega)$ represent the spectral density of the output and $S_F(\omega)$ represent the spectral density of the input. The mean square value of the output is calculated as

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(\omega)d\omega = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} X(\omega)X^*(\omega)d\omega \tag{13.84}$$

Substituting Equation (13.83) into Equation (13.84) leads to

$$\overline{x^2} = \int_{-\infty}^{\infty} |H(\omega)|^2 \left[\lim_{T\to\infty} \frac{1}{T} F(\omega)F^*(\omega)\right] d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 S_F(\omega)d\omega \tag{13.85}$$

EXAMPLE 13.11

What is the mean square response of a SDOF system due to (a) white noise, (b) bandlimited white noise if

$$\omega_n = 30 \text{ rad/s}, \; m = 1 \text{ kg}, \; \zeta = 0.1 \; S_0 = 1 \times 10^{-6}\, \text{m}^2 \cdot \text{s/rad} \; \omega_1 = 10 \text{ rad/s}$$

and $\omega_2 = 100$ rad/s

SOLUTION

The results of Example 13.9 are used to determine the transfer function for a SDOF system as

$$H(\omega) = \frac{1}{-m\omega^2 + ic\omega + k} \tag{a}$$

The power spectral density of the output is

$$S_x(\omega) = |H(\omega)|^2 S_F(\omega) = S_0 \left|\frac{1}{-m\omega^2 + ic\omega + k}\right| = \frac{S_0}{(k - m\omega^2)^2 + (c\omega)^2} \tag{b}$$

The mean square response of the output is given by Equation 13.85

$$\overline{x^2} = \int_{-\infty}^{\infty} |H(\omega)|^2 S_F(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{S_0}{(k - m\omega^2)^2 + (c\omega)^2} d\omega \tag{c}$$

The integral is evaluated using Appendix (E) leading to

$$\overline{x^2} = \frac{\pi S_0}{2kc} = \frac{\pi S_0}{2\zeta m^2 \omega_n^3} = \frac{\pi(1 \times 10^{-6}\ \text{m}^2 \cdot \text{s/rad})}{2(0.1)(1\ \text{kg})^2(30\ \text{rad/s})} = 5.81 \times 10^{-10}\ \text{m}^2 \tag{d}$$

(b) The transfer function is written as

$$H(\omega) = \frac{\dfrac{1}{1\ \text{kg}}}{(900 - \omega^2) + i6\omega} = \frac{1}{(10000 - \omega^2) + i6\omega} \tag{e}$$

The square of magnitude of the transfer function is

$$|H(\omega)|^2 = \frac{1}{(900 - \omega^2)^2 + (6\omega)^2} \tag{f}$$

The mean square response of the output is

$$\begin{aligned} \overline{x^2} &= \int_{10}^{100} |H(\omega)|^2 S_F(\omega)\, d\omega = \int_{10}^{100} \frac{1 \times 10^{-6}}{(900 - \omega^2)^2 + (6\omega)^2} d\omega \\ &= \frac{1}{(900)^2} \int_{0.333}^{3.33} \frac{(30)(1 \times 10^{-6})}{\left[1 - \left(\dfrac{\omega}{30}\right)^2\right]^2 + \left[2(0.1)\left(\dfrac{\omega}{30}\right)\right]^2} d\left(\frac{\omega}{30}\right) \end{aligned} \tag{g}$$

The integral is evaluated using Appendix (E) leading to

$$\begin{aligned} \overline{x^2} = (30)(1 \times 10^{-6}) \left[\frac{\tau}{4(0.1)}\right] \Bigg[& \frac{0.1}{2\pi\sqrt{1 - (0.1)^2}} \ln \frac{(3.33)^2 + 2(3.33)\sqrt{1 - (0.1)^2} + 1}{(3.33)^2 - 2(3.33)\sqrt{1 - (0.1)^2} + 1} \\ & + \frac{1}{\pi} \tan^{-1} \frac{3.33 + \sqrt{1 - (0.1)^2}}{0.1} + \frac{1}{\pi} \tan^{-1} \frac{3.33 - \sqrt{1 - (0.1)^2}}{0.1} \\ & - \frac{0.1}{2\pi\sqrt{1 - (0.1)^2}} \ln \frac{(0.333)^2 + 2(0.333)\sqrt{1 - (0.1)^2} + 1}{(0.333)^2 - 2(0.333)\sqrt{1 - (0.1)^2} + 1} \\ & - \frac{1}{\pi} \tan^{-1} \frac{0.333 + \sqrt{1 - (0.1)^2}}{0.1} - \frac{1}{\pi} \tan^{-1} \frac{0.333 - \sqrt{1 - (0.1)^2}}{0.1} \Bigg] \\ = 1.97 \times 10^{-2}\ \text{m}^2 & \end{aligned} \tag{h}$$

EXAMPLE 13.12

A two degree-of-freedom frame structure is subject to wind loading whose power spectral density is measured and given in Table 13.3 on the next page and plotted in Figure 13.17(a) on page 811. The transfer function is also measured and has two peaks, as shown in Figure 13.17.

(a) Determine the mean square value of the response of the system.
(b) Determine the probability that $|x| > 0.02$ m.
(c) Determine the probability that the maximum value of x exceeds 0.02 m.

TABLE 13.3 Measured values of spectral density function and magnitude of transfer function

$\omega\left(\frac{\text{rad}}{\text{s}}\right)$	$S_F(\omega)\left(10^{-5}\frac{\text{m}^2\cdot\text{s}}{\text{rad}}\right)$	$\lvert H(\omega)\rvert\left(\frac{\text{m}}{\text{N}}\right)$
0	0	1
20	0.01	1.01
40	0.03	1.03
60	0.5	1.10
80	1.0	1.4
100	1.6	1.9
120	1.0	2.5
140	0.8	3.1
160	1.3	2.5
180	2.0	1.6
200	3.4	1.1
220	2.0	1.3
240	1.8	2.7
260	1.3	4.0
280	1.0	5.6
300	1.3	3.8
320	1.3	2.1
340	0.9	1.3
360	0.6	0.5
380	0.1	0.1

SOLUTION

The mean square of the response of the system is given by Equation (13.85)

$$\overline{x^2} = \int_{-\infty}^{\infty} |H(\omega)|^2 S_F(\omega)\,d\omega \tag{a}$$

Since the power spectral density and the magnitude of the transfer function are only known at discrete values of omega, a numerical integration procedure must be used. Applying the trapezoidal rule over the range of frequencies $0 \leq \omega \leq 380$ rad/s with the number of intervals equal to 19 and $\Delta\omega = 20$ rad/s leads to

$$\begin{aligned}\overline{x^2} &= \left[S_F(0)|H(0)|^2 + 2\sum_{i=1}^{18} S_F(20i)|H(20i)|^2 + S_F(380)|H(380)|^2\right]\left[\frac{20}{2(19)}\right] \\ &= 1.41 \times 10^{-3}\ \text{m}^2\end{aligned} \tag{b}$$

Assuming the mean is zero, the standard deviation is calculated from Equation (13.18) as

$$\sigma = \sqrt{\overline{x^2}} = 0.0375\ \text{m} \tag{c}$$

(b) The value of x can take on positive and negative values. Thus, using the central limit theorem, it is governed by the Gaussian distribution. The normalized variable is

$$z = \frac{0.02\ \text{m} - 0\ \text{m}}{0.0375\ \text{m}} = 0.5333 \tag{d}$$

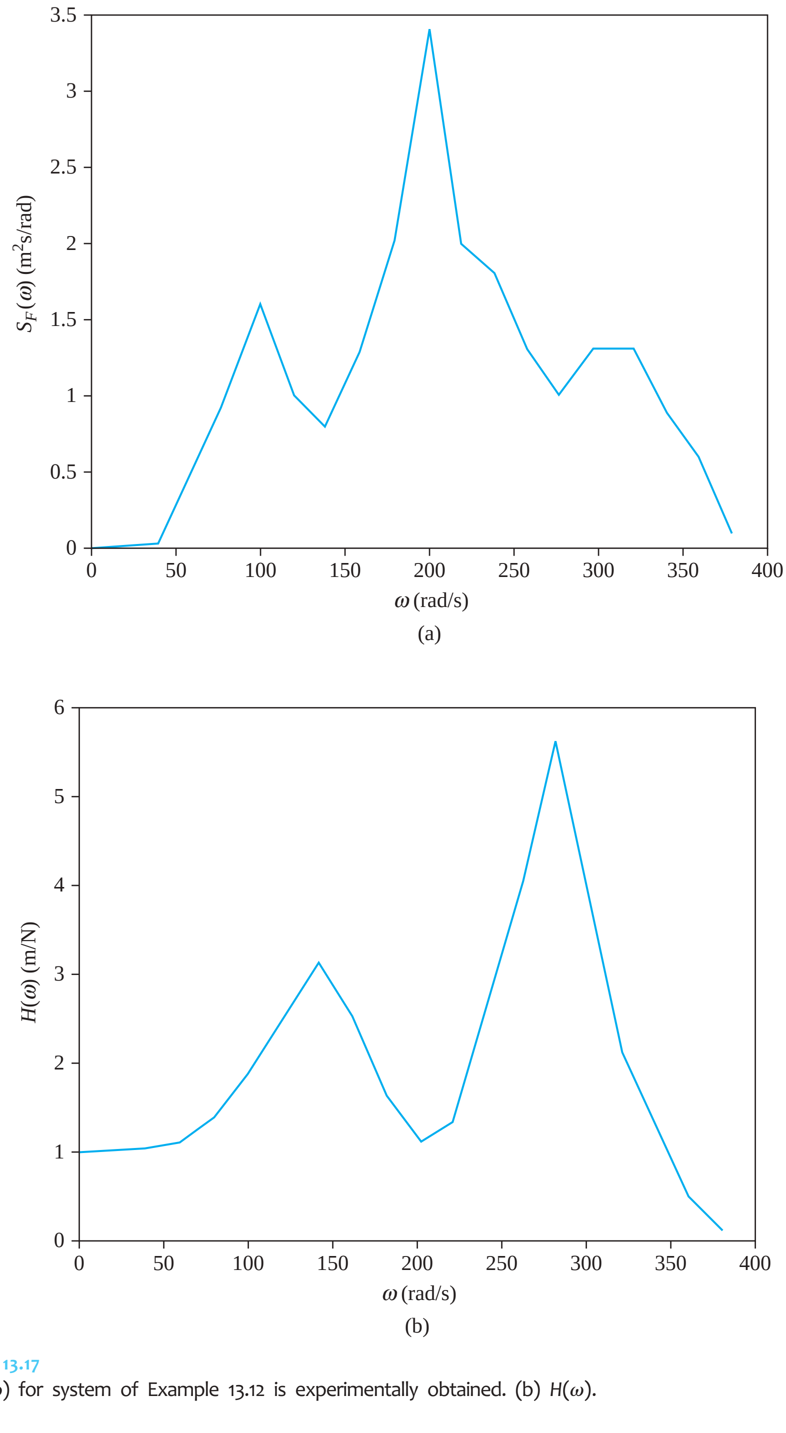


FIGURE 13.17
(a) $S_F(\omega)$ for system of Example 13.12 is experimentally obtained. (b) $H(\omega)$.

The probability that $|x| > 0.02$ is the probability that $z < -0.5333$, which according to Table 13.1 is $1 - 0.701 = 0.299$. The probability that $z > 0.5333$ is 0.299. Thus, the probability that $|z| > 0.533$ is 0.598.

(c) The maximum value of x is only positive. Thus, it likely follows a Rayleigh distribution with $\alpha = 0.0375$. The value of the variable in the Rayleigh distribution is $\frac{(0.02\ \text{m})^2}{2(0.0375\ \text{m})^2} = 0.142$. The probability that the maximum value of x is greater than 0.02 m is

$$e^{-0.142} = 0.867 \tag{e}$$

13.8 BENCHMARK EXAMPLE

Consider the benchmark example of the simplified model of a suspension system subject to a random vibration, as illustrated in Figure 13.18. The differential equation governing the displacement of the body of the vehicle $x(t)$ given the displacement of the wheel $y(t)$ is

$$300\dot{x} + 1200\dot{x} + 12{,}000x = 1200\dot{y} + 12{,}000y \tag{a}$$

The transfer function is obtained by taking the Fourier transform of the differential equation, leading to

$$300(-\omega^2)X(\omega) + 1200i\omega X(\omega) + 12{,}000X(\omega) = 1200i\omega Y(\omega) + 12{,}000Y(\omega) \tag{b}$$

The transfer function is obtained from Equation (b) as

$$H(\omega) = \frac{X(\omega)}{Y(\omega)} = \frac{1200i\omega + 12{,}000}{-300\omega^2 + 1200i\omega + 12{,}000} = \frac{4i\omega + 40}{(40 - \omega^2) + 4i\omega} \tag{c}$$

The transfer function is rewritten as

$$H(\omega) = \frac{4i\omega + 40}{(40 - \omega^2) + 4i\omega}\left[\frac{(40 - \omega^2) - 4i\omega}{(40 - \omega^2) - 4i\omega}\right] = \frac{1600 - 24\omega^2 + 40(2 - \omega)^2 i}{(40 - \omega^2)^2 + (4\omega)^2} \tag{d}$$

The square of the magnitude of the transfer function is

$$|H(\omega)|^2 = \frac{(1600 - 24\omega^2)^2 + 1600(2 - \omega)^4}{(40 - \omega^2)^2 + (4\omega)^2} \tag{e}$$

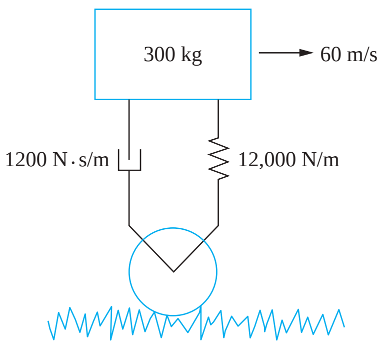


FIGURE 13.18
System of the benchmark example as it travels over a road modeled by white noise.

The vehicle is traveling over a road contour whose power spectral density is that of white noise:

$$S_0 = 2.3 \times 10^{-5} \frac{\text{m}^2}{\text{cycle/m}} \tag{f}$$

If the vehicle is traveling at 60 m/s, we have

$$1 \frac{\text{cycle}}{\text{m}} = 1 \frac{\text{cycle}}{\text{m}} \frac{60\text{m}}{\text{s}} \frac{2\pi\text{rad}}{\text{cycle}} = 120\,\pi\text{rad/s} \tag{g}$$

Thus,

$$S_0 = 2.3 \times 10^{-5} \frac{\text{m}^2}{\text{cycle/m}} \cdot \frac{1\text{ cycle/m}}{120\pi\text{ rad/s}} = 6.10 \times 10^{-8}\,\text{m}^2 \cdot \text{s/rad} \tag{h}$$

The mean square values of the response is

$$\overline{x^2} = \int_{-\infty}^{\infty} \left(6.1 \times 10^{-8} \frac{\text{m}^2 \cdot \text{s}}{\text{rad}}\right) \left\{ \frac{(1600 - 24\omega^2)^2 + 1600(2 - \omega)^4}{(40 - \omega^2)^2 + (4\omega)^2} \right\} d\omega \tag{i}$$

Evaluation of the integral using the formulas of Appendix F leads to

$$\overline{x^2} = (6.1 \times 10^{-8})(404)\pi = 7.74 \times 10^{-5}\,\text{m}^2 \tag{j}$$

Assuming the mean is zero, the standard deviation is

$$\sigma = \sqrt{7.74 \times 10^{-5}\,\text{m}^2} = 8.80\,\text{mm} \tag{k}$$

The random variable x is governed by the Gaussian distribution via the central limit theorem. The maximum absolute value is greater than a if $-a < x$ or $x > a$. The maximum amplitude is a positive random variable and is more likely governed by the Rayleigh distribution. The probabilities of exceeding certain values of the absolute value of x and the maximum amplitude are given in Table 13.4.

TABLE 13.4 Probability that $x > |a|$ and $x_{max} > a$

a (mm)	Normal Variable $\frac{a}{8.80\text{ mm}}$	Probability $\lvert a \rvert > x$	Rayleigh Variable $\frac{a^2}{2(8.80\text{ mm})^2}$	Probability $x_{max} > a$
2	0.227	0.820	0.0258	0.9745
4	0.454	0.645	0.1033	0.9019
6	0.682	0.490	0.2324	0.7926
8	0.909	0.362	0.4132	0.6615
10	1.136	0.250	0.6457	0.5243
12	1.364	0.170	0.9298	0.3947
14	1.591	0.118	1.2655	0.2821
16	1.818	0.068	1.6529	0.1915
18	2.045	0.035	2.0119	0.1234
20	2.272	0.022	2.5826	0.0756

13.9 SUMMARY

13.9.1 IMPORTANT CONCEPTS

- A random variable does not have a defined value, but it is expressed in terms of probabilities.
- A system with a random input has a random response.
- A sample function is one measurement of a random variable $y(t)$.
- An ensemble is a set of sample functions $\{y_i(t)\}$.
- A process is stationary if its defining statistics are independent of time.
- A process is ergoidc if one element of the ensemble is representative of the ensemble. The statistics do not depend on which element is selected.
- The probability distribution function $P(y)$ gives the value of the probability that the random variable is less than y.
- The probability density function is the derivative of the probability distribution function.
- The mean μ of a random variable is the expected value of the random variable.
- The variance σ^2 of a random variable is the expected value of $(y - \mu)^2$.
- The mean square value is the expected value of y^2.
- A Gaussian process defines a bell-shaped curve.
- The central limit theorem implies that the Gaussian process can be used to describe the distribution of the means or random variables which are not Gaussian processes.
- The Rayleigh distribution is used for random variables that only have positive values.
- The autocorrelation function is the expected value of $x(t)x(t + \tau)$. It is a function of τ only for a stationary process.
- The Fourier transform of a nonperiodic function is the application of the Fourier series as the period approaches infinity.
- The transfer function is the ratio of the Fourier transform of the output of a system to the Fourier transform of its input.
- The power spectral density function $S(\omega)$ is the energy density associated with a frequency ω.
- The Wiener-Khintchine equations relate the power spectral density to the Fourier transform of the autocorrelation function.
- A wideband process is one in which the power spectral density has a significant value over a wide range of frequencies.
- A narrowband process is one in which the power spectral density is defined to have a significant value only over a narrow band of frequencies.
- White noise is a wideband process in which the power spectral density is constant over all frequencies.
- The power spectral density of the output is equal to the power spectral density of the input times the square of the transfer function.

- The mean square value of a process is equal to the integral over the entire range of frequencies of the power spectral density.

13.9.2 IMPORTANT EQUATIONS

Range of values of probability distribution function

$$0 \leq P(y) \leq 1 \tag{13.6}$$

Relations between the probability density function and the probability distribution function

$$p(y) = \frac{dP}{dy} \tag{13.8}$$

$$P(y) = \int_{-\infty}^{y} p(\xi)d\xi \tag{13.9}$$

Mean of a random variable

$$\mu = \int_{-\infty}^{\infty} yp(y)dy \tag{13.13}$$

Variance of a random variable

$$\sigma^2 = E[(y-\mu)^2] = \int_{-\infty}^{\infty} (y-\mu)^2 p(y)dy \tag{13.14}$$

Mean square value of a random variable

$$\overline{y^2} = E(y^2) = \int_{-\infty}^{\infty} y^2 p(y)dy \tag{13.16}$$

Relation between the mean, the variance and the mean square value

$$\overline{y^2} = \sigma^2 + \mu^2 \tag{13.18}$$

Probability density function for Gaussian process

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{13.20}$$

Normalized random variable

$$z = \frac{x-\mu}{\sigma} \tag{13.21}$$

Probability distribution function for normalized Gaussian process

$$P(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}} d\tau \tag{13.23}$$

Probability density function for Rayleigh process

$$p(y) = \frac{y}{\alpha} e^{-\frac{y^2}{2\alpha}} \tag{13.24}$$

Autocorrelation function

$$R(t,\tau) = E[y(t)y(t+\tau)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t)y(t+\tau)p[y(t),y(t+\tau)]dy(t)dy(t+\tau) \quad (13.37)$$

Autocorrelation function for an ergodic stationary process

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)y(t+\tau)\,dt \quad (13.38)$$

Relation between autocorrelation function, mean and variance for a stationary process

$$R(0) = \sigma^2 + \mu^2 \quad (13.40)$$

Fourier transform pair

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \quad (13.52)$$

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt \quad (13.53)$$

Transfer function

$$H(\omega) = \frac{\chi(\omega)}{F(\omega)} \quad (13.58)$$

Parseval's identity

$$\int_{-\infty}^{\infty} y^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega \quad (13.65)$$

Power spectral density

$$S(\omega) = \frac{1}{2\pi}\left[\lim_{T\to\infty}\frac{1}{T}Y(\omega)Y^*(\omega)\right] \quad (13.69)$$

Wiener-Khintchine equations

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega \quad (13.77)$$

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega t}dt \quad (13.78)$$

Autocorrelation function for white noise

$$R(\tau) = S_0\delta(\tau) \quad (13.80)$$

Mean square value of output

$$\overline{x^2} = \int_{-\infty}^{\infty} |H(\omega)|^2\left[\lim_{T\to\infty}\frac{1}{T}F(\omega)F^*(\omega)\right]d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 S_F(\omega)d\omega \quad (13.85)$$

PROBLEMS

SHORT ANSWER PROBLEMS

For Problems 13.1 through 13.12, indicate whether the statement presented is true or false. If true, state why. If false, rewrite the statement to make it true.

13.1 The Rayleigh distribution can be applied to random variables with positive values.

13.2 A stationary process is one in which a representative sample of ensemble measurements can be used for the entire process.

13.3 The Weiner-Khintchine equations imply that the autocorrelation function is the Fourier transform of the power spectral density.

13.4 If $P(0) = 0.5$ for a normalized random variable, the probability distribution follows the Gaussian distribution.

13.5 The probability distribution function is the derivative of the probability density function.

13.6 The autocorrelation function is an even function of τ for a stationary process.

13.7 The transfer function is defined as the Fourier transform of the output of a system divided by the Fourier transform of the input to a system is equal to the sinusoidal transfer function for the system.

13.8 If $x(t) = A \sin 5t$, then $p(x) = \frac{1}{A}$ for $|x| < A$.

13.9 The mean of a random function can be calculated by $\mu = \int_{-\infty}^{\infty} xp(x)\,dx$ for a stationary ergodic process.

13.10 The variance is the positive square root of the standard deviation.

13.11 A narrowband process has a power spectral density defined over a narrow band of frequencies.

13.12 For a stationary process $R(0) = 1$.

Problems 13.13 through 13.30 require a short answer.

13.13 What is an ensemble?

13.14 What is a stationary process?

13.15 What is an ergodic process?

13.16 Which is more likely to be a random process, the wind induced vibrations of a bridge or the rotating unbalance of a machine?

13.17 What is the total area under the curve of a probability density function?

13.18 What does the Central Limit theorem imply?

13.19 What is the power spectral density function for ideal white noise?

13.20 What is the autocorrelation function for ideal white noise?

13.21 What is the Fourier transform of $\delta(t)$?

13.22 If the probability density function $p(x)$ is known for a random variable, what is the probability distribution $P(x)$?

13.23 What is $P(0)$ for the normalized Gaussian distribution?

13.24 The probability of the maximum value of the response of a system follows what probability distribution?

13.25 The probability that the absolute value of the response of a system follows what probability distribution?

13.26 A random variable has a probability distribution, $P(x)$. What is $p(\infty)$?

13.27 A random variable has a probability distribution, $P(x)$. What is the probability that $x > b$?

13.28 A random variable has a probability distribution, $P(x)$. What is the probability that $-1 \leq x < 3$?

13.29 If the power spectral density of an input force is $S_F(\omega)$ and the transfer function for the system the force is applied to is $H(\omega)$, what is the power spectral density of the output?

13.30 The spectral density of a random process is $S(\omega)$. How is the mean square value of the process determined?

Problems 13.31 through 13.36 require short calculations.

13.31 For the normalized Gaussian distribution $P(z)$, determine the following.

(a) What is the probability that $z < 1$?
(b) What is the probability that $-2 < z < 1$?
(c) What is the probability that $z > 0.5$?

13.32 A random variable has a Gaussian distribution with $\mu = -1.3$ and $\sigma = 2.8$. Determine the following.

(a) What is the probability that $x < -3.3$?
(b) What is the probability that $x > 3.3$?
(c) What is the probability that $0 < x < 6.3$?

13.33 A random variable has a Rayleigh distribution with $\mu = 3.1$. Determine the following.

(a) What is the probability that $x > 3.1$?
(b) What is the probability that $x < 2.3$?
(c) What is the probability that $2.9 < x < 3.3$?

13.34 The probability density function for the standard Cauchy distribution is

$$p(x) = \frac{1}{\pi(1 + x^2)}$$

(a) What is the probability distribution function for the Cauchy distribution?
(b) What is the mean of the Cauchy distribution?
(c) What is the mean square value of the Cauchy distribution?

13.35 The probability distribution function for the standard Weibull distribution is

$$P(x) = 1 - e^{-x\gamma}$$

What is its probability density function $p(x)$?

13.36 Consider the system shown in Figure SP13.36.

(a) What is the transfer function for the system $H(\omega) = \frac{X(\omega)}{F(\omega)}$?
(b) Determine $|H(\omega)|$.
(c) The system is subject to ideal white noise. What is the power spectral density of the input?
(d) What is the power spectral density of the output?
(e) Determine the mean square response of the system.
(f) If the mean of the response is zero, what is the standard deviation of the response?

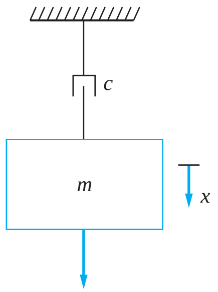


FIGURE SP13.36

13.37 It is desired to approximate the random displacement of a machine due to a random force $F(t)$. What are the SI units of the following.

(a) The power spectral density of the displacement $S_x(\omega)$
(b) The power spectral density of the force $S_F(\omega)$
(c) The Fourier transform of the force $F(\omega)$
(d) The Fourier transform of the displacement $X(\omega)$
(e) The transfer function for the system $H(\omega)$
(f) The mean square value of the displacement $E(x^2)$
(g) The variance of the force σ_F^2
(h) The autocorrelation function for the displacement $R_x(\tau)$
(i) The autocorrelation function for the force $R_F(\tau)$
(j) The probability distribution for the force
(k) The probability density function for the force

CHAPTER PROBLEMS

13.1 Determine the autocorrelation function for $x(t) = A \cos 2t$.
13.2 Determine the autocorrelation function for the rectangular wave shown in Figure P13.2.

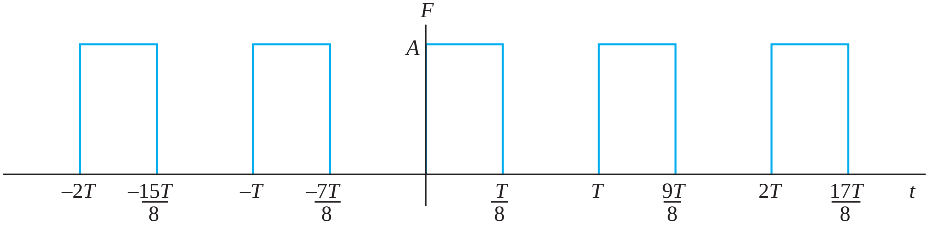


FIGURE P13.2

13.3 Determine the autocorrelation function for the rectangular wave shown in Figure P13.3.

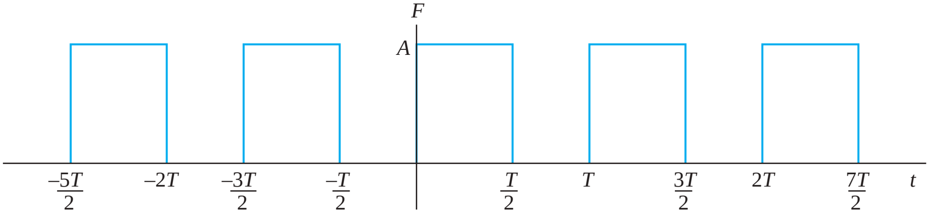


FIGURE P13.3

13.4 Determine the autocorrelation function for the triangular wave shown in Figure P13.4.

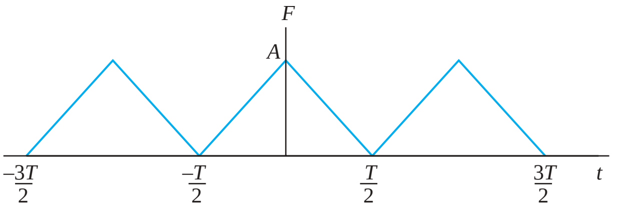


FIGURE P13.4

13.5 Determine the autocorrelation function for the triangular wave shown in Figure P13.5.

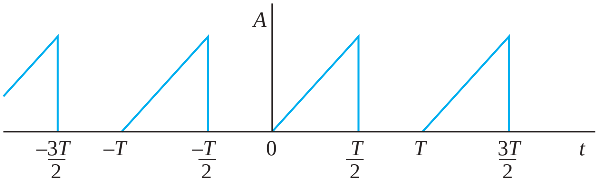


FIGURE P13.5

13.6 Determine the autocorrelation function for the triangular wave shown in Figure P13.6.

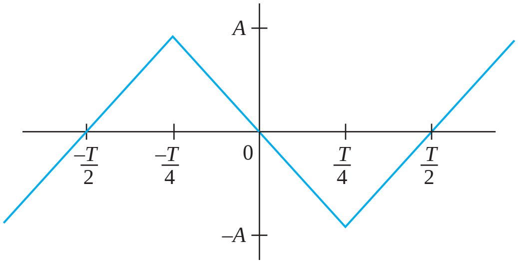


FIGURE P13.6

13.7 A sine wave has the form

$$x(t) = 3 - 2\sin 4t$$

Determine the expected value of x and x^2.

13.8 Assume that t is uniformly distributed.

(a) Determine the probability density function $p(x)$ for the function in Chapter Problem 13.7.
(b) Determine the probability distribution function $P(x)$ for the function in Chapter Problem 13.7.

13.9 Determine the probability density function for the periodic function, one period of which is shown in Figure P13.9

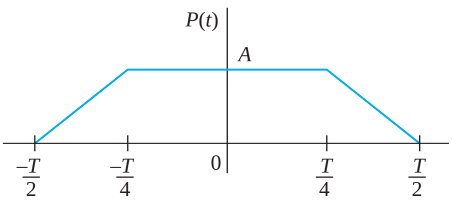


FIGURE P13.9

13.10 Determine the probability density function for the half-period cosine wave of Figure P13.10.

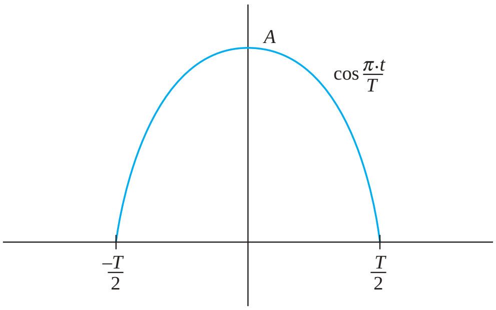


FIGURE P13.10

13.11 Determine the Fourier transform of the rectangular pulse of Figure P13.11.

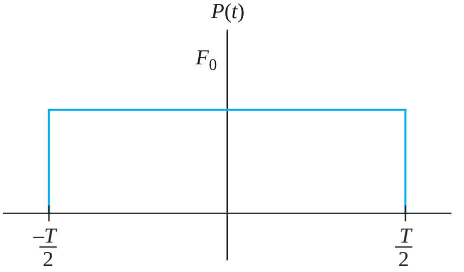


FIGURE P13.11

13.12 Determine the Fourier transform for the triangular pulse of Figure P13.12

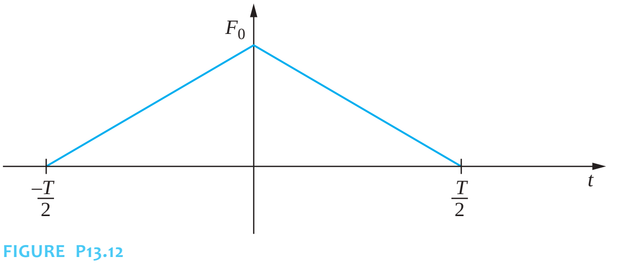


FIGURE P13.12

13.13 Determine the Fourier transform of the half-period cosine wave of Figure P13.10.

13.14 Determine the power spectral density of the wave shown in Figure P13.2.

13.15 Determine the power spectral density of the wave shown in Figure P13.3.

13.16 Determine the power spectral density of the wave shown in Figure P13.4.

13.17 Determine the power spectral density of the wave shown in Figure P13.6.

13.18 A force has band limited white noise with frequency bounds of $\omega_1 = 100$ rad/s and $\omega_2 = 500$ rad/s and magnitude $S_0 = 2 \times 10^2\ \text{N}^2 \cdot \text{s/rad}$ determine the following.

(a) The autocorrelation function for the force
(b) The expected mean square value of the force
(c) Assuming the mean is zero, what is the probability that the magnitude of the force is greater than 1000 N?

13.19 A SDOF system with a mass of 20 kg, $\zeta = 0.1$ and $\omega_n = 100$ rad/s is subject to white noise with $S_0 = 1 \times 10^{-2}\ \text{N}^2 \cdot \text{s/rad}$. What is the power spectral density of the response $S_x(\omega)$?

13.20 A SDOF system with a mass of 30 kg, $\zeta = 0.05$, and $\omega_n = 200$ rad/s is subject to white noise with $S_0 = 1 \times 10^{-2}\ \mathrm{N^2/Hz}$. What is the power spectral density of the response $W_x(f)$?

13.21 A SDOF system with a mass of 20 kg, $\zeta = 0.1$, and $\omega_n = 100$ rad/s is subject to white noise with $S_0 = 1 \times 10^{-2}\ \mathrm{N^2 \cdot s/rad}$.

(a) What is the mean square value of the response of the system?
(b) What is the probability of the response exceeding 5 mm?
(c) What is the probability of the maximum of the response exceeding 5 mm?

13.22 The SDOF system of Figure P13.22 is subject to a white noise with $S_0 = 1 \times 10^{-2}\ \mathrm{m^2/rad \cdot s}$ (the power spectral density of the acceleration of the base). Calculate the mean square value of the acceleration of the 20 kg block.

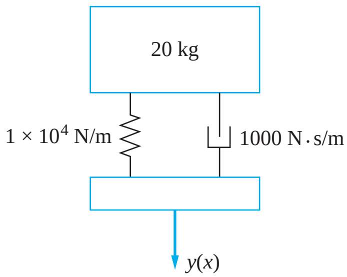


FIGURE P13.22

13.23 The SDOF system of Figure P13.23 is subject to a white noise with $S_0 = 1 \times 10^{-3}\ \mathrm{N^2 \cdot s/rad}$. What is the mean square value of the response of the 300 kg mass.

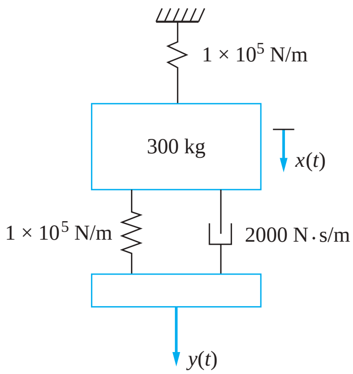


FIGURE P13.23

13.24 Solve Chapter Problem 13.21, assuming the power spectral density is band limited with $\omega_1 = 50$ rad/s and $\omega_2 = 200$ rad/s.

13.25 Solve Chapter Problem 13.21, assuming the force is narrowband with a power spectral density given by $S_F(\omega) = \frac{3 \times 10^{-3}}{2 + 5\omega^2}$.

13.26 Solve Chapter Problem 13.22, assuming the acceleration is band limited with $\omega_1 = 10$ rad/s and $\omega_2 = 30$ rad/s.

13.27 A two SDOF has governing differential equations

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 200 & -100 \\ -100 & 300 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

where $F(t)$ is random with a power spectral density of $S_0 = 5 \times 10^{-2}\ \mathrm{N}^2 \cdot \mathrm{s/rad}$.

(a) Determine the mean square value of x_1.

(b) Determine the mean square value of x_2.

UNIT IMPULSE FUNCTION AND UNIT STEP FUNCTION

Consider the function, $f_\Delta(x;\, a)$, where $f_\Delta(x;\, a)$, as shown in Figure A.1 is defined by

$$f_\Delta(x;\ a) = \begin{cases} 0 & -\infty < x < a - \dfrac{\Delta}{2} \\ \dfrac{1}{\Delta} & a - \dfrac{\Delta}{2} \leq x \leq a + \dfrac{\Delta}{2} \\ 0 & a + \dfrac{\Delta}{2} < x < \infty \end{cases} \tag{A.1}$$

The function has the property

$$\int_{-\infty}^{\infty} f_\Delta(x;\ a)\,dx = 1 \tag{A.2}$$

Taking the limit of $f_\Delta(x;\, a)$ as $\Delta \longrightarrow 0$ yields

$$\lim_{\Delta \to 0} f_\Delta(x;\ a) = \delta(x - a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases} \tag{A.3}$$

From Equation (A.2)

$$\int_{-\infty}^{\infty} \delta(x - a)\,dx = 1 \tag{A.4}$$

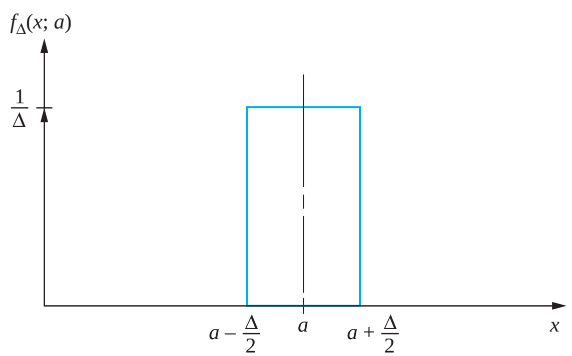


FIGURE A.1
$\delta(x - a) = \lim_{\Delta \to 0} f_\Delta(x;\, a)$.

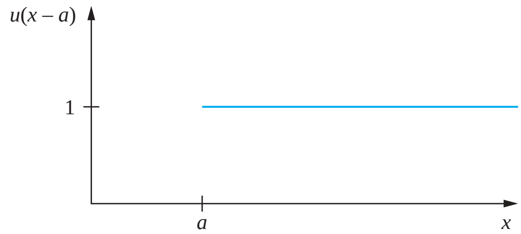


FIGURE A.2
The unit step function $u(x - a)$.

The function defined in Equation (A.3) and whose valuable property is given in Equation (A.4) is called the *unit impulse function*. It has many applications in physics and engineering. It is used to mathematically represent the force that is applied to cause a unit impulse applied at a time $t = a$ in a mechanical system. It is used to represent a unit concentrated load applied at a location $x = a$ to a structure. The unit impulse function, also called the Dirac delta function, is used to represent a unit heat source in a heat transfer problem.

Now define

$$u(x - a) = \int_0^x \delta(x - a)\,dx = \int_0^x \lim_{\Delta \to 0} f_\Delta(x;\, a)\,dx = \begin{cases} 0 & x \le a \\ 1 & x > a \end{cases} \tag{A.5}$$

The function defined in Equation (A.5) is called the *unit step function* and is illustrated in Figure A.2. Differentiating Equation (A.5) gives

$$\frac{du}{dx}(x - a) = \delta(x - a) \tag{A.6}$$

The definitions of the unit impulse function and unit step function can also be used to derive the following integral formulas. For any function $g(t)$,

$$\int_0^t \delta(\tau - a)g(\tau)d\tau = u(t - a)g(a) \tag{A.7}$$

and $$\int_0^t u(\tau - a)g(\tau)d\tau = u(t - a)\int_a^t g(\tau)d\tau \tag{A.8}$$

LAPLACE TRANSFORMS

B.1 DEFINITION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{B.1}$$

If there exist values of α, M, and T such that

$$e^{-\alpha t}|f(t)| < M \qquad \text{for all } t > T \tag{B.2}$$

then $F(s)$ exists for $s > \alpha$. Equation (B.2) is satisfied for all excitations and responses in this text.

The Laplace transform transforms a real-valued function into a function of a complex variable, s. For many functions, the Laplace transform can be obtained by direct integration.

EXAMPLE B.1

Determine the Laplace transform of $f(t) = e^{\alpha t}$.

SOLUTION

$$\mathcal{L}\{e^{\alpha t}\} = \int_0^\infty e^{\alpha t}e^{-st}dt = \frac{1}{\alpha - s}e^{(\alpha - s)t}\Big|_0^\infty = \frac{1}{s-\alpha} \qquad s > \alpha \tag{a}$$

B.2 TABLE OF TRANSFORMS

Equation (B.1) is used to develop a table of transform pairs a table of $f(t)$ versus $F(s)$. Laplace transforms of other functions can be developed using Table B.1 in conjunction with the properties of the transform, provided in Table B.2.

B.3 LINEARITY

The Laplace transform operator is a linear operator. Let $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$, and α and β be any real numbers. Then

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s) \tag{B.3}$$

TABLE B.1

Number	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	t^n	$\frac{n!}{s^{n+1}}$
3	$e^{\alpha t}$	$\frac{1}{s-\alpha}$
4	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
5	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
6	$\delta(t-a)$	e^{-as}
7	$u(t-a)$	$\frac{e^{-as}}{s}$
8	$e^{at}\cos \omega t$	$\frac{s+as}{s^2+2as+\omega^2}$
9	$e^{at}\sin \omega t$	$\frac{\omega}{s^2+2as+\omega^2}$
10	te^{at}	$\frac{1}{(s+a)^2}$
11	$t\cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
12	$t\sin \omega t$	$\frac{2s\omega}{(s^2+\omega^2)^2}$

EXAMPLE B.2

Determine $\mathcal{L}\{\cosh(\omega t)\}$

SOLUTION

Recall that $\cosh(\omega t) = \frac{e^{\omega t} + e^{-\omega t}}{2}$. Then using linearity of the transform

$$\mathcal{L}\{\cosh(\omega t)\} = \frac{1}{2}\mathcal{L}\{e^{\omega t}\} + \frac{1}{2}\mathcal{L}\{e^{-\omega t}\} \tag{a}$$

Using transform pair 3 from Table B.1 with $\alpha = \omega$ and $\alpha = -\omega$ in Equation (a) leads to

$$\mathcal{L}\{\cosh(\omega t)\} = \frac{1}{2}\left(\frac{1}{s-\omega}\right) + \frac{1}{2}\left(\frac{1}{s+\omega}\right) = \frac{1}{2}\left(\frac{s-\omega+s+\omega}{s^2-\omega^2}\right) = \frac{s}{s^2-\omega^2} \tag{b}$$

B.4 TRANSFORM OF DERIVATIVES

The property of the Laplace transform of derivatives along with the linearity of the transform allows easy application of the Laplace transform method to the solution of differential equations. If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{B.4}$$

TABLE B.2 Properties of Laplace Transforms

Name of Property	Statement of Property
Definition	$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$
Linearity of transform	$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
First Shifting Theorem	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
Second Shifting Theorem	$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$
Transform of First Derivative	$\mathcal{L}\left\{\dfrac{df}{dt}\right\} = sF(s) - f(0)$
Transform of Second Derivative	$\mathcal{L}\left\{\dfrac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - \dfrac{df}{dt}(0)$
Convolution	$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$
Inversion Integral	$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$

Note: $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$

EXAMPLE B.3

Use transform pair 5 from Table B.1 and Equation (B.4) to determine $\mathcal{L}\{\sin 2t\}$.

SOLUTION

Noting that

$$\sin 2t = -\frac{1}{2}\frac{d(\cos 2t)}{dt} \tag{a}$$

and applying properties (B.3) and (B.4) with $n = 1$ gives

$$\mathcal{L}\{\sin 2t\} = -\frac{1}{2}(s\mathcal{L}\{\cos 2t\} - 1) \tag{b}$$

Using transform pair 5 from Table B.1,

$$\mathcal{L}\{\sin 2t\} = -\frac{1}{2}\left(\frac{s^2}{s^2+4} - 1\right) = \frac{2}{s^2+4} \tag{c}$$

B.5 FIRST SHIFTING THEOREM

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a) \tag{B.5}$$

EXAMPLE B.4

Use Table B.1 and the first shifting theorem to calculate $\mathcal{L}\{e^{-\zeta\omega_n t}\cos\omega_d t\}$ where $\omega_d = \omega_n\sqrt{(1-\zeta^2)}$.

SOLUTION

Using the first shifting theorem and transform pair 5 from Table B.1,

$$\mathcal{L}\{e^{-\zeta\omega_n t}\cos\omega_d t\} = \left.\frac{s}{s^2+\omega_d^2}\right|_{s\to s+\zeta\omega_n} = \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} = \frac{s+\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{a}$$

B.6 SECOND SHIFTING THEOREM

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \tag{B.6}$$

EXAMPLE B.5

Use Table B.1 and the second shifting theorem to determine the Laplace transform of the function of Figure B.1.

SOLUTION

The function of Figure B.1 is written using unit step functions as

$$\begin{aligned} f(t) &= t[u(t) - u(t-1)] + (2-t)[u(t-1) - u(t-2)] \\ &= tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2) \end{aligned}$$

Use of transform pair 2 from Table B.1 with $n = 1$ and the second shifting theorem give

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - e^{-s}\frac{2}{s} + e^{-2s}\frac{1}{s} = \frac{1}{s}(1 - 2e^{-s} + e^{-2s})$$

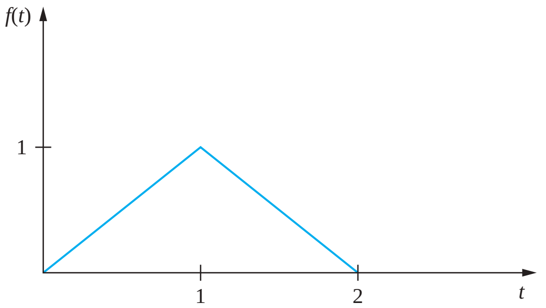


FIGURE B.1

B.7 INVERSION OF TRANSFORM

If $F(s) = \mathcal{L}\{f(t)\}$, then $f(t) = \mathcal{L}^{-1}\{F(s)\}$ where

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}\,ds \tag{B.7}$$

is an integral carried out in the complex s plane. Inverse transforms are often obtained by using Table B.1 in conjunction with transform properties.

EXAMPLE B.6

If

$$e^{-2s}\frac{s+5}{s^2+2s+5} = F(s)$$

find $F(t)$, where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Completing the square of the denominator of $F(s)$ gives

$$F(s) = e^{-2s}\frac{s+5}{(s+1)^2+4} = e^{-2s}\left[\frac{s+1}{(s+1)^2+4} + \frac{4}{(s+1)^2+4}\right] = e^{-2s}G(s)$$

Using linearity, the first shifting theorem, and transform pairs 4 and 5 from Table B.1 leads to

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-t}(\cos 2t + 2\sin 2t)$$

Using the second shifting theorem leads to

$$f(t) = u(t-2)g(t-2) = e^{2-t}[\cos 2(t-2) + 2\sin 2(t-2)]u(t-2)$$

B.8 CONVOLUTION

Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Then

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)G(s) \tag{B.8}$$

where $f(t) * g(t)$ is called the convolution of $f(t)$ and $g(t)$. The property is known as the convolution property. It is usually used to invert transforms.

B.9 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

The properties of linearity of the transform and transform of derivatives are used to solve a linear differential equation with initial conditions.

EXAMPLE B.7

Solve the differential equation

$$\ddot{x} + 16x = f(t) \tag{a}$$

where $f(t)$ is the function of Figure B.1 and

$$x(0) = 0 \qquad \dot{x}(0) = 0 \tag{b}$$

are given initial conditions.

SOLUTION

Taking the Laplace transform of both sides of Equation (a) leads to

$$\mathcal{L}\{\ddot{x} + 16x\} = \mathcal{L}\{f(t)\} \tag{c}$$

Applying the property of linearity of the transform to Equation (c) leads to

$$\mathcal{L}\{\ddot{x}\} + 16\mathcal{L}\{x\} = \mathcal{L}\{f(t)\} \tag{d}$$

Let $X(s) = \mathcal{L}\{x(t)\}$. Using the property of the transform of the second derivative and applying the transform of the function in Figure B.1 leads to

$$s^2X(s) - sx(0) - \dot{x}(0) + 16X(s) = 1 - 2e^{-s} + e^{-2s} \tag{e}$$

Applying the initial conditions, Equation (b) to Equation (e) leads to

$$(s^2 + 16)X(s) = 1 - 2e^{-s} + e^{-2s} \tag{f}$$

The solution for $X(s)$ is

$$X(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s^2 + 16} \tag{g}$$

Noting that $x(t) = \mathcal{L}^{-1}\{X(s)\}$, the solution of the differential equation is obtained by inverting Equation (g).

Using linearity of the transform in the inverse fashion,

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 16}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2 + 16}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2 + 16}\right\} \tag{h}$$

From transform pair 4 of Table B.1, $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 16}\right\} = \frac{1}{4}\sin 4t$. Then using the second shifting theorem in reverse, $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2 + 16}\right\} = \frac{1}{4}\sin 4(t - 1)u(t - 1)$. A similar method is used to invert the final transform, leading to

$$x(t) = \frac{1}{4}\sin 4t - \frac{1}{2}\sin[4(t - 1)]u(t - 1) + \frac{1}{4}\sin[4(t - 2)]u(t - 2) \tag{i}$$

LINEAR ALGEBRA

C.1 DEFINITIONS

1. A *matrix* is a collection of numbers arranged in a specific order in rows and columns. If matrix $\mathbf{A}$ has n rows and m columns, then it is represented by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} \tag{C.1}$$

Throughout this text a single capital letter in boldface is used to represent a matrix. The corresponding lowercase letter with two subscripts is used to refer to a specific element of the matrix. For example, the element a_{ij} resides in the ith row and jth column of $\mathbf{A}$.

A matrix with n rows and m columns is called an $n \times m$ matrix. A square matrix has the same number of rows and columns.

2. A *column vector* is a matrix with only one column. A *row vector* is a matrix with only one row. Usually, a single lowercase letter in boldface is used to represent a column vector or a row vector. The letter with a single subscript refers to a specific element of the vector. A column vector with n rows or a row vector with n columns is said to be an n-dimensional vector. If $\mathbf{x}$ is an n-dimensional column vector, then x_i, $i \leq n$ is the element in the ith row of the vector.

3. A *diagonal matrix* is a square matrix with all off-diagonal elements equal to zero. That is $a_{ij} = 0$ if $i \neq j$.

4. An *identity matrix* is a square diagonal matrix whose diagonal elements are all unity. That is $a_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

5. The *transpose* of the matrix $\mathbf{A}$, denoted by $\mathbf{A}^T$, is the matrix obtained by interchanging the rows and columns of $\mathbf{A}$. If $\mathbf{B} = \mathbf{A}^T$, then $b_{ij} = a_{ji}$. The transpose of a column vector is a row vector and vice versa.

6. A *symmetric matrix* is a square matrix whose transpose is equal to the matrix itself. If $\mathbf{A}$ is an $n \times n$ symmetric matrix, then $a_{ij} = a_{ji}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

C.2 DETERMINANTS

The *determinant* of a square $n \times n$ matrix is a number associated with the matrix that is often of great consequence. It is easiest to define the determinant of a 2×2 matrix and use this definition and properties of determinants to calculate the determinant of larger matrices.

The determinant of the 2×2 matrix $\mathbf{A}$ is

$$\det\{\mathbf{A}\} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{C.2}$$

The *minor* corresponding to the element in the ith row and jth column of an $n \times n$ matrix $\mathbf{A}$, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column from $\mathbf{A}$. The cofactor corresponding to the element in the ith row and jth column of $\mathbf{A}$, denoted by C_{ij}, is

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{C.3}$$

For an i, $i = 1, \ldots, n$, the determinant of $\mathbf{A}$ is obtained by the following row expansion:

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} C_{ij} \tag{C.4}$$

The value of the determinant is the same regardless of the value of i. The determinant can also be calculated by a column expansion according to the formula

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ji} C_{ji} \tag{C.5}$$

Since the minors themselves are determinants, row or columns expansions can be used to express each of the minors in terms of the minors of their corresponding matrix. These expansions continue until the remaining minors are 2×2 determinants.

EXAMPLE C.1

Calculate the determinant of the 4×4 matrix $\mathbf{A}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 1 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

SOLUTION

The determinant is evaluated by a first-row expansion, using Equation (C.4),

$$|\mathbf{A}| = (1)\begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & -2 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 2 & 0 & -2 \end{vmatrix}$$

Expansion by the first row is used to evaluate each of the 3 × 3 determinants, resulting in

$$|\mathbf{A}| = (2)\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - 2\left((1)\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix}\right)$$

The 2 × 2 determinants are evaluated using Equation (C.2), yielding

$$\begin{aligned} |\mathbf{A}| &= (2)[(3)(1) - (1)(-2)] + [(-1)(1) - (1)(0)] \\ &\quad - (2)\{[(-1)(-2) - (3)(0)] - (2)[(2)(-2) - (3)(2)] - [(2)(0) - (-1)(2)]\} \\ &= -31 \end{aligned}$$

The determinant of a matrix is zero if and only if the column vectors that form the matrix are linearly dependent. For example, the determinant of a matrix with a column of zeros is zero. A matrix whose determinant is zero is said to be *singular*. The row vectors of a singular matrix are also linearly dependent.

C.3 MATRIX OPERATIONS

If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then

$$c_{ij} = a_{ij} + b_{ij} \tag{C.6}$$

If the number of columns of **A** equals the number of rows of **B**, then the matrix $\mathbf{C} = \mathbf{AB}$ is defined as a matrix with the number of rows of **A** and the number of columns of **B** and c_{ij} is the sum of the products of the corresponding elements in the ith row of **A** and the jth column of **B**. That is,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{C.7}$$

Matrix multiplication is not commutative, but is associative and distributive. The transpose of the product has the following property. If $\mathbf{C} = \mathbf{AB}$, then

$$\mathbf{C}^T = (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{C.8}$$

EXAMPLE C.2

Calculate **Ax** where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & 6 & 2 \\ 0 & 2 & 3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}$$

SOLUTION

The product of a 4 × 4 matrix and a four-dimensional column vector is a four-dimensional column vector,

$$\mathbf{Ax} = \begin{bmatrix} (1)(1) + (2)(4) + (4)(-1) + (-1)(2) \\ (2)(1) + (3)(4) + (0)(-1) + (4)(2) \\ (1)(1) + (2)(4) + (6)(-1) + (2)(2) \\ (0)(1) + (2)(4) + (3)(-1) + (1)(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 22 \\ 7 \\ 7 \end{bmatrix}$$

C.4 SYSTEMS OF EQUATIONS

Consider the system of n simultaneous equations which are to be solved for the n unknowns $x_1, x_2, \ldots, x_n$,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n \end{aligned} \tag{C.9}$$

Using the definitions of matrix addition and matrix multiplication, the system of Equation (C.9) is written in matrix form as

$$\mathbf{Ax} = \mathbf{y}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{C.10}$$

Cramer's rule can be used to solve for the components of **x**,

$$x_i = \frac{|\mathbf{B}_i|}{|\mathbf{A}|} \tag{C.11}$$

where $\mathbf{B}_i$ is the matrix obtained by replacing the ith column of **A** with **y**. Thus if **A** is singular, a solution of Equation (C.9) exists only for certain forms of **y**. Since its rows are linearly dependent when the matrix is singular, the solution corresponding to special forms of **y** is not unique.

An equation in a system of equations can be replaced, without affecting the solution of the system, by an equation obtained by multiplying the equation by a scalar and adding or subtracting it from another equation. The equations can be so manipulated until one of the equations only has one unknown. This is the basis of the Gauss elimination method.

Matrix formulation of the equations expedites the application of Gauss elimination. The $n \times n$ coefficient matrix is augmented with the right-hand side vector to form an $n \times (n + 1)$ matrix. Each row of the augmented matrix represents one equation. The Gauss elimination procedure is applied by performing manipulations on the rows of the augmented matrix such that coefficients below the diagonal become zero. The elimination procedure results in a coefficient matrix with all zeros below its diagonal. Back substitution is used to determine the solution.

C.5 INVERSE MATRIX

If $\mathbf{A}$ is a nonsingular $n \times n$ matrix, then a matrix $\mathbf{A}^{-1}$, called the *inverse* of $\mathbf{A}$, exists such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{C.12}$$

If $\mathbf{A}^{-1}$ is known, Equation (C.9) can be solved by premultiplying both sides by $\mathbf{A}^{-1}$,

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{C.13}$$

If $\mathbf{y}$ is a column vector with all zeros except $y_i = 1$, then $\mathbf{A}^{-1}\mathbf{y}$ is the ith column of $\mathbf{A}^{-1}$. This provides the basis of an extension of Gauss elimination which is used to determine $\mathbf{A}^{-1}$. The coefficient matrix is augmented by the $n \times n$ identity matrix. The procedure used in Gauss elimination is applied until the identity matrix appears in place of the original matrix. The matrix that augments the identity matrix is $\mathbf{A}^{-1}$.

EXAMPLE C.3

Determine the inverse of

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

SOLUTION

Gauss elimination is applied to the following matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

Gauss elimination is used to develop zeros below the diagonal of the coefficient matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & -4 & 1 & 2 & 0 \\ 0 & 0 & \frac{7}{2} & 1 & 2 & \frac{5}{2} \end{bmatrix}$$

The procedure of Gauss elimination is used to eliminate the zeros above the diagonal of the coefficient matrix. Each row is divided by the value of the element along the diagonal of the matrix that has taken the place of the original coefficient matrix. The result is

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{7} & \frac{3}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} & \frac{6}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{4}{7} & \frac{5}{7} \end{bmatrix}$$

Thus

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{6}{7} & \frac{4}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{5}{7} \end{bmatrix}$$

C.6 EIGENVALUE PROBLEMS

The *eigenvalues* of an $n \times n$ matrix, $\mathbf{A}$, are the values of λ such that the system of equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{C.14}$$

has a nontrivial solution. The nontrivial solution corresponding to an eigenvalue is called an *eigenvector*. Equation (C.14) can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{C.15}$$

From Cramer's rule, Equation (C.11), the solution for x_i is

$$x_i = \frac{0}{|\mathbf{A} - \lambda\mathbf{I}|} \qquad i = 1, \ldots n$$

Thus, for each $i = 1, \ldots, n$, $x_i = 0$, unless

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{C.16}$$

The determinant of Equation (C.16) can be expanded by a row or column expansion. This yields an nth-order polynomial equation of the form

$$\lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots + C_{n-1}\lambda + C_n = 0 \tag{C.17}$$

called the *characteristic equation*. Equation (C.17) has n roots, and $\mathbf{A}$ has n eigenvalues. Since the coefficients in Equation (C.17) are all real, if complex eigenvalues occur, they occur as complex conjugate pairs.

If λ is an eigenvalue of $\mathbf{A}$, then Equation (C.14) has a nontrivial solution, an eigenvector. From Equation (C.l6), the matrix $\mathbf{A} - \lambda \boldsymbol{I}$ is singular. Thus the equations defining the components of the corresponding eigenvector are not all independent and the eigenvector is not unique. The eigenvector is unique only to an arbitrary multiplicative constant.

EXAMPLE C.4

Determine the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

SOLUTION

The eigenvalues of $\mathbf{A}$ are determined by finding the values of λ satisfying Equation (C.16), which for this example become

$$\begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{bmatrix} = 0$$

Expansion of the determinant by its first row gives

$$(2-\lambda)\begin{bmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix} - (-1)\begin{vmatrix} -1 & -2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

When the 2×2 determinants are expanded by using Equation (C.2), the following cubic equation is obtained:

$$-\lambda^3 + 8\lambda^2 - 16\lambda + 7 = 0$$

The eigenvalues are the roots of the cubic equation which are 0.609, 2.227, and 5.164. The eigenvector corresponding to the smallest eigenvalue is obtained by solving

$$\begin{bmatrix} 1.391 & -1 & 0 \\ -1 & 2.391 & -2 \\ 0 & -2 & 2.391 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first equation gives $x_1 = 0.719x_2$. The third equation gives $x_3 = 0.836x_2$. When these relationships are substituted into the second equation, it is identically satisfied. Thus x_2 remains arbitrary and the eigenvector of $\mathbf{A}$ corresponding to $\lambda = 0.609$ is

$$C_1\begin{bmatrix} 0.719 \\ 1 \\ 0.836 \end{bmatrix}$$

where C_1 is an arbitrary constant. The same procedure is followed yielding the eigenvectors corresponding to the second and third eigenvalues. These are

$$C_2\begin{bmatrix} -4.41 \\ 1 \\ 2.59 \end{bmatrix} \qquad C_3\begin{bmatrix} -0.316 \\ 1 \\ -0.924 \end{bmatrix}$$

respectively.

If $\mathbf{A}$ is an $n \times n$ singular matrix, then one of its eigenvalues is zero. If $\mathbf{A}$ is nonsingular, then the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$. The eigenvectors of $\mathbf{A}^{-1}$ are the same as the eigenvectors of $\mathbf{A}$.

C.7 SCALAR PRODUCTS

Let **u**, **v**, and **w** be arbitrary real n-dimensional column vectors. A *scalar product* is an operation among two of these vectors yielding a real value. The scalar product of **u** and **v** is denoted by $(\mathbf{u}, \mathbf{v})$. The scalar product must satisfy four requirements.

1. The scalar product is commutative. That is,

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u}) \tag{C.18}$$

2. For any real α,

$$(\alpha\mathbf{u}, \mathbf{v}) = \alpha(\mathbf{u}, \mathbf{v}) \tag{C.19}$$

3. The scalar product is distributive

$$(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w}) \tag{C.20}$$

4. $(\mathbf{u}, \mathbf{u}) \geq 0$ (C.21)

and $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$ (C.22)

The definition of a scalar product is not unique. The standard scalar product is defined as

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{v} \tag{C.23}$$

Two vectors, **u** and **v**, are said to be *orthogonal* with respect to a scalar product if

$$(\mathbf{u}, \mathbf{v}) = 0 \tag{C.24}$$

A matrix **A** is said to be *positive definite* with respect to a scalar product if

$$(\mathbf{A}\mathbf{u}, \mathbf{u}) \geq 0 \tag{C.25}$$

and

$$(\mathbf{A}\mathbf{u}, \mathbf{u}) = 0 \quad \text{if and only if } \mathbf{u} = \mathbf{0} \tag{C.26}$$

EXAMPLE C.5

Show that if **A** is a positive-definite symmetric matrix, then

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = (\mathbf{A}\mathbf{u}, \mathbf{v}) \tag{C.27}$$

is a valid scalar product where $(\mathbf{u}, \mathbf{v})$ is the standard scalar product defined by Equation (C.23).

SOLUTION

In order for Equation (C.27) to represent a valid scalar product, it is necessary to show that the four properties of Equations (C.18) through (C.22) are true, knowing that they are true for the standard scalar product.

$$
\begin{aligned}
1.\ (\mathbf{u}, \mathbf{v})_A &= (\mathbf{A}\mathbf{u})^T\mathbf{v} && \text{Equations (C.23) and (C.27)}\\
&= \mathbf{u}^T\mathbf{A}^T\mathbf{v} && \text{Equation (C.9)}\\
&= \mathbf{u}^T\mathbf{A}\mathbf{v} && \text{symmetry of } \mathbf{A}\\
&= (\mathbf{u}, \mathbf{A}\mathbf{v}) && \text{Equation (C.23)}\\
&= (\mathbf{A}\mathbf{v}, \mathbf{u}) && \text{Equation (C.18)}\\
&= (\mathbf{v}, \mathbf{u})_A && \text{Equation (C.27)}
\end{aligned}
$$

2. For any real α,

$$\begin{aligned}(\alpha\mathbf{u},\ \mathbf{v})_A &= \alpha(\mathbf{A}\mathbf{u})^T\mathbf{v} \\ &= \alpha(\mathbf{u},\ \mathbf{v})_A\end{aligned}$$

3. $$\begin{aligned}(\mathbf{u}+\mathbf{v},\ \mathbf{w})_A &= [\mathbf{A}(\mathbf{u}+\mathbf{v})]^T\,\mathbf{w} \\ &= [(\mathbf{A}\mathbf{u})^T + (\mathbf{A}\mathbf{v})^T]\,\mathbf{w} \\ &= (\mathbf{A}\mathbf{u})^T\mathbf{v} + (\mathbf{A}\mathbf{v})^T\mathbf{w} \\ &= (\mathbf{u},\ \mathbf{w})_A + (\mathbf{v},\ \mathbf{w})_A\end{aligned}$$

4. The validity of the property 4 for this definition of the scalar product follows directly from the positive definiteness of **A**, Equations (C.25) and (C.26).

The concept of scalar products can be extended to continuous functions. Any operation between two continuous functions that results in a scalar and obeys Equations (C.18) through (C.22) is a valid scalar product. For example, for two functions $f(x)$ and $g(x)$ that are everywhere continuous between $x = 0$ and $x = 1$, a valid scalar product is

$$(f,\ g) = \int_0^1 f(x)g(x)\,dx \tag{C.28}$$

Appendix D

DEFLECTION OF BEAMS SUBJECT TO CONCENTRATED LOADS

Consider a beam of total length L, subject to arbitrary end constraints. Let z be a coordinate along the neutral axis of the beam. The beam has n intermediate simple supports at $z = z_i$, $i = 1, 2, \ldots, n$. It is desired to calculate the deflection of the beam as a function of z due to a concentrated unit load applied at $z = a$. If $y(z)$ is the deflection of the neutral axis of the beam, measured positive downward from the horizontal, then use of the usual assumptions of linear elastic beam theory leads to

$$EI\frac{d^4y}{dz^4} = w(z) \tag{D.1}$$

where $w(z)$ represents the loading, E is the elastic modulus of the beam, and I is the moment of inertia of the cross-sectional area about the neutral axis.

The intermediate supports are replaced by concentrated loads. The analysis requires the deflection to be zero at the intermediate supports.

The mathematical representation for a concentrated load of magnitude P applied at $z = a$ is $P\delta(z - a)$ where $\delta(z)$ is the unit impulse function. Thus the loading function $w(x)$ for the beam of Figure D.1 is written as

$$w(z) = \delta(z - a) + \sum_{i=1}^{n} R_i\delta(z - z_i) \tag{D.2}$$

where R_i, $i = 1, \ldots, n$, are the reactions at the intermediate supports. Equation (D.2) is substituted into Equation (D.1) and the resulting equation is integrated three times, using Equation (A.5), giving

$$EI\frac{d^3y}{dz^3} = u(z - a) + \sum_{i=1}^{n} R_i u(z - z_i) + C_1 \tag{D.3}$$

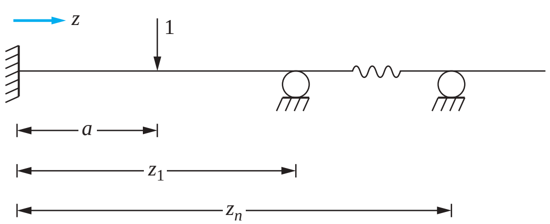


FIGURE D.1
Deflection equation for beam with intermediate supports due to a unit concentrated load is developed by representing the load and the support reactions using the unit impulse function.

TABLE D.1

End Condition	Boundary Condition	Boundary Condition
Free	$EI\dfrac{d^2y}{dx^2} = 0$	$EI\dfrac{d^3y}{dx^3} = 0$
Fixed	$y = 0$	$\dfrac{dy}{dx} = 0$
Pinned	$y = 0$	$EI\dfrac{d^2y}{dx^2} = 0$

$$EI\frac{d^2y}{dz^2} = (z - a)u(z - a) + \sum_{i=1}^{n} R_i(z - z_i)u(z - z_i) + C_1 z + C_2 \tag{D.4}$$

$$EI\frac{dy}{dz} = \frac{1}{2}(z - a)^2u(z - a) + \frac{1}{2}\sum_{i=1}^{n} R_i(z - z_i)^2u(z - z_i) + C_1\frac{z^2}{2} + C_2 z + C_3 \tag{D.5}$$

$$EIy = \frac{1}{6}(z - a)^3u(z - a) + \frac{1}{6}\sum_{i=1}^{n} R_i(z - z_i)^3u(z - z_i) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4 \tag{D.6}$$

where C_1, C_2, C_3, and C_4 are constants of integration which are determined upon application of the appropriate boundary conditions.

The appropriate boundary conditions depend on the type of support at the boundaries. Table D.1 provides the boundary conditions for different types of support. Two boundary conditions are applied at each end of the beam. Thus, $n + 4$ equations are applied to determine the $n + 4$ unknowns, n intermediate support reactions, and four constants of integration.

EXAMPLE D.1

Determine the deflection of a beam fixed at $x = 0$ and pinned at $z = L$ due to a unit concentrated load applied at $z = a$, $0 < a < L$.

SOLUTION

From Table D.1, the appropriate boundary conditions are

$$y(0) = 0 \quad (a) \qquad y(L) = 0 \quad (c)$$

$$\left.\frac{dy}{dz}\right|_{z=0} = 0 \quad (b) \qquad \left.\frac{d^2y}{dz^2}\right|_{z=L} = 0 \quad (d)$$

Application of (a) to Equation (D.6) yields $C_4 = 0$. Application of (b) yields $C_3 = 0$. Application of (c) and (d) yields the following equations:

$$\frac{L^3}{6}C_1 + \frac{L^2}{2}C_2 = -\frac{1}{6}(L - a)^3$$

$$LC_1 + C_2 = -(L - a)$$

respectively. The preceding equations are solved simultaneously, yielding

$$C_1 = \frac{1}{2}\left(1 - \frac{a}{L}\right)\left[\left(\frac{a}{L}\right)^2 - 2\frac{a}{L} - 2\right]$$

$$C_2 = a\left(1 - \frac{a}{L}\right)\left(1 - \frac{a}{2L}\right)$$

Boundary conditions are applied to the beams of Table D.2, resulting in the evaluation of constants and, if applicable, intermediate reactions for each beam. Equation (D.6) is used to calculate the deflection of the beam at any point.

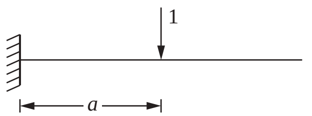
TABLE D.2

The deflection, $y(z)$, of a uniform beam of elastic modulus E and cross-sectional moment of inertia I due to a unit concentrated load applied at $z = a$ is

$$y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-a)^3u(z-a) + \frac{1}{6}\sum_{i=1}^{n} R_i(z-z_i)^3u(z-z_i) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4\right]$$

where R_i is the reaction at an intermediate support located at $z = z_i$. The forms of the constants and the intermediate reactions for common beams are given as follows.

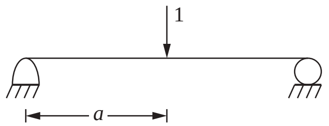
1.



Fixed-free

$$C_1 = -1 \qquad C_3 = 0$$
$$C_2 = a \qquad C_4 = 0$$

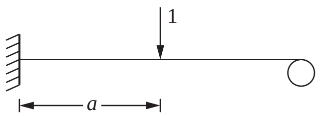
2.



Pinned-pinned

$$C_1 = \frac{a}{L} - 1 \qquad C_3 = \frac{aL}{6}\left(1 - \frac{a}{L}\right)\left(2 - \frac{a}{L}\right)$$
$$C_2 = 0 \qquad C_4 = 0$$

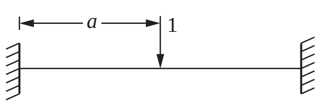
3.



Fixed-pinned

$$C_1 = \frac{1}{2}\left(1 - \frac{a}{L}\right)\left[\left(\frac{a}{L}\right)^2 - 2\frac{a}{L} - 2\right] \qquad C_3 = 0$$
$$C_2 = \frac{1}{2}a\left(1 - \frac{a}{L}\right)\left(2 - \frac{a}{L}\right) \qquad C_4 = 0$$

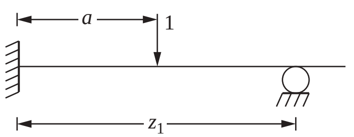
4.



Fixed-fixed

$$C_1 = -\left(1 - \frac{a}{L}\right)^2\left(1 + \frac{2a}{L}\right) \qquad C_3 = 0$$
$$C_2 = a\left(1 - \frac{a}{L}\right)^2 \qquad C_4 = 0$$

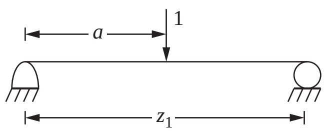
5.



Fixed-free with overhang

$$C_1 = -\frac{3}{2} + \frac{3a}{2z_1} + \frac{1}{2}\left(1 - \frac{a}{z_1}\right)^3 u(z_1 - a) \qquad C_3 = 0$$
$$C_2 = \frac{z_1}{2}\left(1 - \frac{a}{z_1}\right)\left[1 - \left(1 - \frac{a}{z_1}\right)^3 u(z_1 - a)\right] \qquad C_4 = 0$$
$$R_1 = \frac{1}{2} - \frac{3a}{2z_1} - \frac{1}{2}\left(1 - \frac{a}{z_1}\right)^3 u(z_1 - a)$$

6.



Pinned-free with overhang

$$C_1 = \frac{a}{z_1} - 1 \qquad C_3 = -\left(1 - \frac{a}{z_1}\right)\frac{z_1^2}{6}\left[\left(1 - \frac{a}{z_1}\right)^2 u(z_1 - a) - 1\right]$$
$$C_2 = 0 \qquad C_4 = 0$$
$$R_1 = -\frac{a}{z_1}$$

Appendix E

INTEGRALS USED IN RANDOM VIBRATIONS

Lalanne reports the following integral formulas:

$$I_0 = \frac{4\zeta}{\pi}\int \frac{1}{(1-\omega^2)^2 + (2\zeta\omega)^2}\, d\omega$$

$$= \frac{\zeta}{2\pi\sqrt{1-\zeta^2}} \ln \frac{\omega^2 + 2\omega\sqrt{1-\zeta^2} + 1}{\omega^2 - 2\omega\sqrt{1-\zeta^2} + 1}$$

$$+ \frac{1}{\pi}\left[\tan^{-1}\frac{\omega + \sqrt{1-\zeta^2}}{\zeta} + \tan^{-1}\frac{\omega - \sqrt{1-\zeta^2}}{\zeta}\right] \tag{E.1}$$

$$I_1 = \frac{4\zeta}{\pi}\int \frac{\omega}{(1-\omega^2)^2 + (2\zeta\omega)^2}\, d\omega = \frac{1}{\pi\sqrt{1-\zeta^2}} \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1 - 2\zeta^2 - \omega^2} \tag{E.2}$$

and

$$I_2 = \frac{4\zeta}{\pi}\int \frac{\omega^2}{(1-\omega^2)^2 + (2\zeta\omega)^2}\, d\omega$$

$$= \frac{\zeta}{2\pi\sqrt{1-\zeta^2}} \ln \frac{\omega^2 - 2\omega\sqrt{1-\zeta^2} + 1}{\omega^2 + 2\omega\sqrt{1-\zeta^2} + 1}$$

$$+ \frac{1}{\pi}\left[\tan^{-1}\frac{\omega + \sqrt{1-\zeta^2}}{\zeta} + \tan^{-1}\frac{\omega - \sqrt{1-\zeta^2}}{\zeta}\right] \tag{E.3}$$

In general,

$$I_n = \frac{4\zeta}{\pi}\int \frac{\omega^n}{(1-\omega^2)^2 + (2\zeta\omega)^2}\, d\omega \tag{E.4}$$

where

$$I_n = \frac{4\zeta}{\pi}\frac{\omega^{n-3}}{n-3} + 2(1-2\zeta^2)I_{n-2} - I_{n-4} \tag{E.5}$$

Appendix F

VIBES

The software programs, collectively called VIBES, are available at the website www.cengage.com/engineering/kelly. It contains programs written in MATLAB and associated with the calculations involved with vibrations problems and the resulting graphs that can be generated. VIBES also contains all programs that are used in the text in examples, to generate plots, or to simply perform calculations. The following is a brief description of each program that comprises VIBES. The descriptions are arranged in order that they would be useful in the text.

`SPRING.m` Designs a helical coil spring.

`BEAM_STIFFNESS.m` Determines the stiffness of a uniform fixed-free beam as a function of distance from the fixed support.

`BEAM_MASS.m` Determines the equivalent mass of a uniform fixed-free beam as a function of distance from the fixed support.

`MACHINE_A.m` Calculates the natural frequency of a machine mounted on a fixed-pinned beam including inertia effects of the beam.

`DIVER.m` Provides natural-frequency calculations for a diver on a diving board that is modeled as a continuous fixed-pinned beam.

`FREE_VISCOUS.m` Provides the free-vibration response of a system with viscous damping.

`FREE_COULOMB.m` Provides the free response of a system with Coulomb damping.

`SUSPENSION_A.m` Provides the response of a simplified SDOF model of a suspension system when the vehicle encounters a pothole in the road.

`MAGNIF.m` Provides analysis of problems using $M(r, \zeta)$.

`LAMBDA.m` Provides analysis of problems using $\Lambda(r, \zeta)$.

`TRANS.m` Provides analysis of problems using $T(r, \zeta)$.

`SUSPENSION_B.m` Analysis of SDOF model of suspension system as it traverses a sinusoidal road contour.

`ISOL.m` Aids in the design of a vibration isolation system.

`ISOL_FREQ2.m` Aids in the design of a vibration isolation system to protect the foundation over a range of frequencies.

`M_C.m` Provides analysis of problems using $M_C(r, \iota)$.

`FOURIER_A.m` Provides analysis of a machine with a periodic rectangular pulse.

`FOURIER_B.m` Uses symbolic algebra to develop the response of a SDOF system due to a periodic input.

`FOURIER_ISOL.` Aids in the design of vibration isolators to protect a foundation from periodic inputs.

`CONVOL.m` Provides symbolic integration of the convolution integral to solve SDOF problems subject to a general excitation.

`LAPLACE.m` Provides the Laplace transform solution for a SDOF system due to an arbitrary excitation.

`PIECEWISE.m` Provides numerical integration of the convolution integral using piecewise constants to interpolate the excitation.

`RESPONSE_SPECT.m` Uses a MATLAB program `ODE45.m` to numerically integrate the differential equation and develop the response spectrum due to any type of excitation.

`ISOL_EFF.m` Calculates the values of $Q(\zeta)$ and $S(\zeta)$.

`SUSPENSI0N_2DOF.m` Develops a two DOF model for the natural frequencies and mode shapes for a vehicle suspension system.

`FORCED_2.m` Symbolically determines the steady-state response of a two DOF system.

`ABSORB.m` Aids in the design of a undamped vibration absorber.

`FIXED_FREE.m` Develops the flexibility matrix for a six DOF model of a fixed-free beam for equally spaced nodes.

`FIXED_FIXED.m` Calculates the flexibility matrix for an *n*DOF model of a fixed-fixed beam. Nodes may be at any location along axis of the beam.

`FREE_FREE.m` Develops the stiffness matrix for a *n*DOF model of a free-free beam.

`DESIGN_BEAM.m` Provides support for the design of a fixed-pinned beam using a three DOF model.

`NDOF_FREE.m` Determines the natural frequencies and mode shapes for an *n*DOF system given the mass matrix and the stiffness matrix.

`SIMPLY_MASS.m` Calculates the natural frequencies and mode shapes for a *n*DOF model of a simply supported beam with a machine attached by a spring.

`PROPORTIONAL_FREE.m` Calculates the natural frequencies and mode shapes for a four DOF model of a system with proportional damping.

`SUSPENSION_4.m` Calculates the natural frequencies and damping ratios for a four DOF model of a suspension system.

`FORCED_N.m` Determines the steady-state response of an *n*DOF system due to a single frequency harmonic exciation.

`MODAL_3.m` Provides modal analysis on a specific three DOF system given the system input.

`FIXED_PINNED_ISOL.m` Determines the steady-state response of a machine attached to a fixed-pinned beam through an undamped isolator. The machine is subject to a harmonic excitation.

`FIXED_PINNED_ISOLD.m` Determines the steady-state response of a machine attached to a fixed-pinned beam through a damped isolator. The machine is subject to a harmonic excitation.

`FIXED_PINNED_GENERAL.m` Uses modal analysis to develop response of a machine attached to a fixed-pinned beam through a damped isolator. The machine is subject to an arbitrary excitation.

`SUSPESNI0N_6.m` Uses numerical integration of the convolution integral to develop a six DOF model of a vehicle suspension system due to any type of road contour.

`FIXED_FREE_CONT.m` Calculates the natural frequencies, mode shapes, and normalization constants for a continuous system model of a fixed-free beam.

`FIXED_PINNED_CONT.m` Calculates the natural frequencies, mode shapes, and normalization constants for a continuous systems model of a fixed-pinned beam.

`FIXED_SPRING_CONT.m` Calculates the natural frequencies, mode shapes, and normalization constants for a continuous systems model of a beam fixed at one end and attached to a linear spring at its other end.

`PINNED_SPRING_CONT.m` Calculates the natural frequencies, mode shapes, and normalization constants for a continuous systems model of a beam that is pinned at one end and attached to a linear spring at its other end.

`TORSIONAL_CONT.m` Plots the mode shapes for the torsional oscillations of a shaft that is pinned at one end and has a disk attached at its other end.

`FREQ_RESPONSE_CONT` Uses a continuous systems model to develop the frequency response for a beam fixed at one end with a machine attached at its other end. The machine is subject to a frequency squared excitation.

`RAYLEIGH_RITZ.m` Uses a Rayleigh-Ritz method to aid in the design a fixed-pinned beam.

`ASSUMED_FREE.m` Uses the assumed modes method to determine the natural frequencies and mode shapes of a tapered bar with an attached mass and linear spring.

`ASSUMED_FORCED.m` Uses the assumed-mode method to determine the forced response of a tapered bar with an attached mass and linear spring.

`VARIABLE_AREA.m` Develops the local mass and stiffness matrices for a beam element whose cross-sectional properties vary across the span of the element.

`SHAFT_FEM.` Uses the finite-element method to approximate the natural frequencies and mode shapes for torsional oscillation of a free-free shaft with rotors at the end.

`FIXED_PINNED_FREE.m` Uses a finite-element method to calculate the natural frequencies and mode shapes of a fixed-pinned beam with a machine at its midspan.

`FIXED_PINNED_FORCED.m` Uses a finite-element method to calculate the forced response of a fixed-pinned beam with a machine at its midspan.

`SIMPLY-SUPPORTED_FEM` Uses a finite-element method to determine the forced response of a simply supported beam with a discrete mass-spring system attached at its midspan.

`DUFFING.m` Numerically integrates Duffing's equation.

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