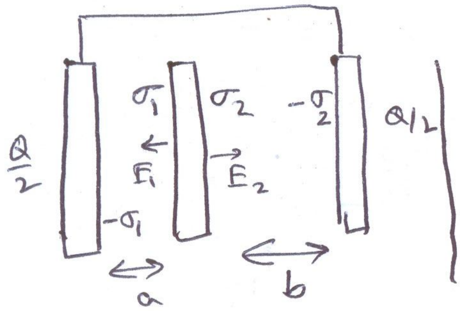
PHY 103

End Sem Exam

Model Solutions

27/04/15

1 (a).



$A(\sigma_1+\sigma_2) = Q$, A = area of each plate

$E_1 = \sigma_1/\epsilon_0$, $E_2 = \sigma_2/\epsilon_0$

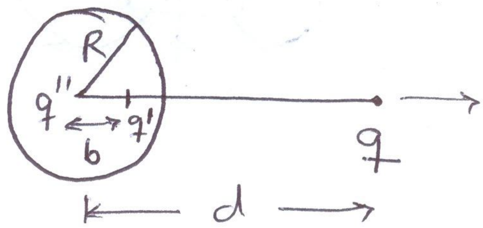
$E_1 a = E_2 b$

Charges from left to right

$\frac{Q}{2}$, $-\frac{b}{a+b}Q$, $\frac{b}{a+b}Q$, $\frac{a}{a+b}Q$, $-\frac{a}{a+b}Q$, $Q/2$.

1. (b) place q'' at the centre such that

$$V_0 = \frac{q''}{4\pi\epsilon_0 R}$$



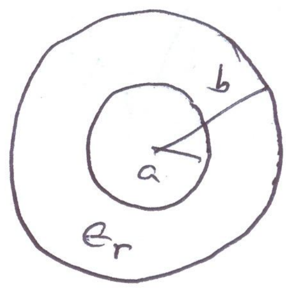
$q' = -\frac{R}{d}q$, $b = R^2/d$

$$\text{Force on the sphere} = \frac{q(-\hat{r})}{4\pi\epsilon_0}\left(\frac{q''}{d^2} + \frac{q'}{(d-b)^2}\right)$$

$$= \left[\frac{qV_0R}{d^2} - \frac{q^2Rd}{4\pi\epsilon_0(d^2-R^2)^2}\right](-\hat{r})$$

1. (c) (i) $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$ $(r>a)$

$= 0$ $(r<a)$



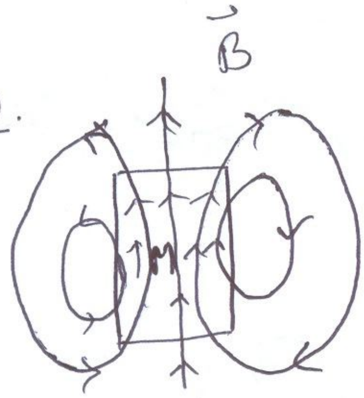
$$\Rightarrow \vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2}\hat{r} & a<r<b \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r>b \end{cases}$$

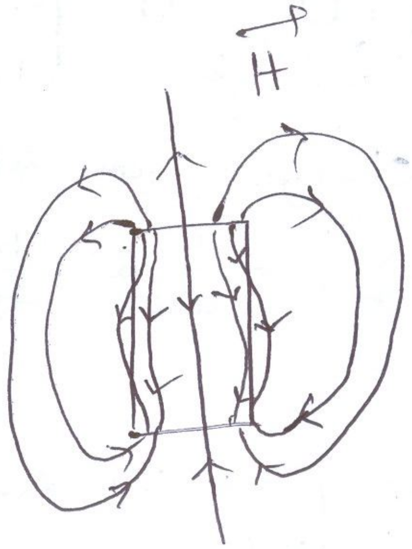
$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\epsilon_r-1}{\epsilon_r}\frac{Q}{4\pi r^2}\hat{r}$, $\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

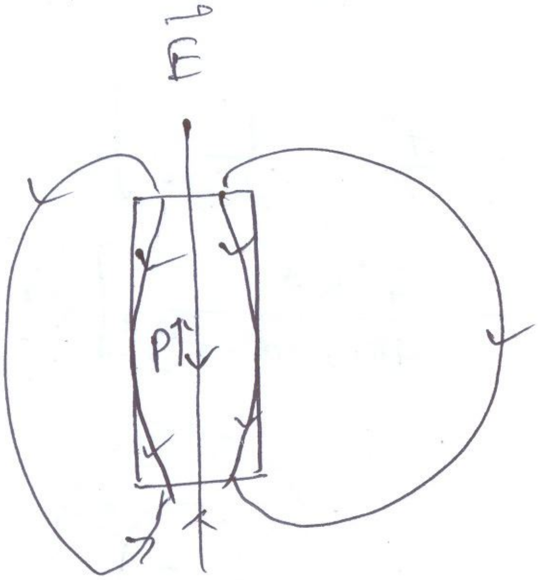
$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} \frac{\epsilon_r-1}{\epsilon_r}\frac{Q}{4\pi b^2} & \text{at } r=b \\ -\frac{\epsilon_r-1}{\epsilon_r}\frac{Q}{4\pi a^2} & \text{at } r=a \end{cases}$$

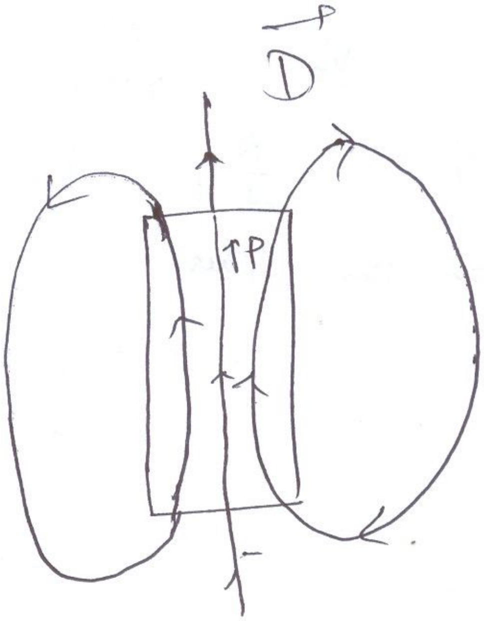
1.c. (ii)
$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$$
$$= \frac{1}{2}\cdot\frac{Q^2}{(4\pi)^2}\left[\frac{1}{\epsilon}\int_{r=a}^{b}\frac{1}{r^2}\,4\pi dr + \frac{1}{\epsilon_0}\int_{r=b}^{\infty}\frac{1}{r^2}\,4\pi dr\right]$$
$$= \frac{Q^2}{8\pi\epsilon_0\epsilon_r}\left[\frac{1}{a} + \frac{\epsilon_r - 1}{b}\right].$$

2. (a).

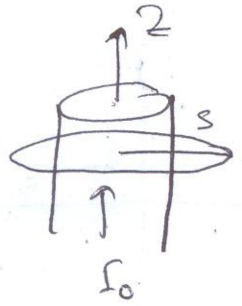








2.(b) (i)



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl.}$$

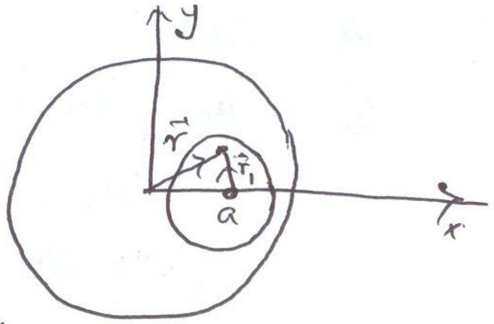
$$\Rightarrow \vec{B} = \begin{cases} \dfrac{\mu_0 I_0}{2\pi s}\hat{\phi} & (s > R) \\[2ex] \dfrac{\mu_0 I_0 s}{2\pi R^2}\hat{\phi} & (s < R). \end{cases}$$

We can choose $\vec{A}$ in $\hat{z}$-direction.

by symmetry, $\vec{A} = A(s)\hat{z}$, $\quad \vec{B} = \vec{\nabla}\times\vec{A} = -\dfrac{\partial A(s)}{\partial s}\hat{\varphi}$

for $s > R$ $\quad \dfrac{\mu_0 I_0}{2\pi s} = -\dfrac{\partial A}{\partial s} \quad \Rightarrow \vec{A} = -\dfrac{\mu_0 I_0}{2\pi}\ln s\,\hat{z}$

for $s < R$ $\quad \dfrac{\mu_0 I_0 s}{2\pi R^2} = -\dfrac{\partial A}{\partial s} \quad \Rightarrow \vec{A} = -\dfrac{\mu_0 I_0}{4\pi R^2}s^2\,\hat{z}$



(ii) $J = \dfrac{I_0}{\pi(R^2-b^2)}$

cylinder with a hole $\equiv$ complete cylinder with $\vec{J}$
+ a cylinder of radius b at $x=a$, with $(-J)$ current density.

For a point inside the hole:

Field due to the bigger cylinder: $\vec{B}_1 = \dfrac{\mu_0 I_0 r}{2\pi(R^2-b^2)}\hat{\varphi}$

field due to the smaller cylinder $\vec{B}_2 = -\dfrac{\mu_0 I_0 r_1}{2\pi(R^2-b^2)}\hat{\varphi}_1$

($\hat{\varphi}_1 \equiv \hat{\varphi}$ with respect to the origin at $x=a$)

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

$$= \frac{\mu_0 I_0}{2\pi(R^2-b^2)}(r\hat{\varphi} - r_1\hat{\varphi}_1)$$

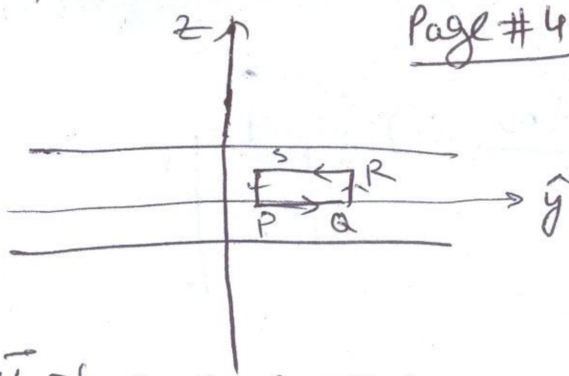
$\hat{z}\times\hat{s} = \hat{\varphi} \Rightarrow \begin{cases}\hat{z}\times\hat{r} = \hat{\varphi} \\ \hat{z}\times\hat{r}_1 = \hat{\varphi}_1\end{cases}$

$$= \frac{\mu_0 I_0}{2\pi(R^2-b^2)}\hat{z}\times(\vec{r}-\vec{r}_1)$$

$$= \frac{\mu_0 I_0}{2\pi(R^2-b^2)}\hat{z}\times(a\hat{x}) = \frac{\mu_0 I_0 a}{2\pi(R^2-b^2)}\hat{y}$$

$|\vec{B}| = \dfrac{\mu_0 I_0 a}{2\pi(R^2-b^2)}$ & in $\hat{y}$ dirn

$\Rightarrow$ uniform field inside the hole.



3.(a) $\vec{J} = J_0 \hat{x}$

by right hand rule, field will be

in $\begin{cases} -\hat{y} & \text{for } z > 0 \\ +\hat{y} & \text{for } z < 0 \end{cases}$

Symmetry $\Rightarrow \begin{cases} H = H(z) \text{ only} \\ H = 0 \text{ at } z = 0 \end{cases} \Rightarrow \vec{H} = \begin{cases} -H(z)\hat{y} & \text{for } z > 0 \\ H(z)\hat{y} & \text{for } z < 0 . \end{cases}$

Amperian loop PQRS $\Rightarrow \oint \vec{H}\cdot d\vec{l} = I_{f\,enc}$.

$\left.\begin{array}{l} H(z) = J_0 z \quad (-a < z < a) \\ \text{if } z > a, \quad H = J_0 a . \end{array}\right\} \Rightarrow \vec{H} = \begin{cases} -J_0 a\hat{y} & \text{for } z > a \\ J_0 a \hat{y} & \text{for } z < -a \\ -J_0 z \hat{y}, & -a < z < a . \end{cases}$

$\vec{M} = \chi_m \vec{H} = -J_0 \chi_m z \hat{y}, \quad -a < z < a,$

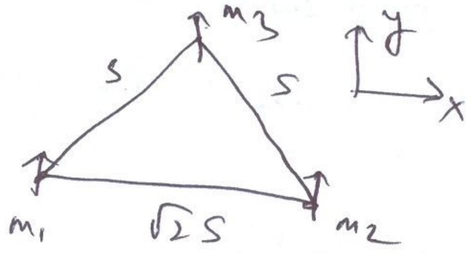
$\vec{B} = \mu_0 (1+\chi_m)\cdot\vec{H} = -\mu_0(1+\chi_m) J_0 z \hat{y} \quad (-a < z < a)$

$\vec{B} = \begin{cases} -\mu_0 J a \hat{y} & z > a \\ \mu_0 J a \hat{y} & z < -a . \end{cases}$

$\vec{K}_b\Big|_{z=a} = \vec{M} \times \hat{n} = -\chi_m J_0 a\, \hat{y} \times \hat{z} = -\chi_m J_0 a \hat{x}$

$\vec{K}_b\Big|_{z=-a} = J_0 a \hat{y} \times (-\hat{z}) = -\chi_m J_0 a \hat{x}$

$\vec{J}_b = \vec{\nabla} \times \vec{M} = \chi_m J_0 \hat{x} \quad (-a < z < a)$



3.(b) (i) field at m_3 due to m_1

$\vec{B}_{31} = \frac{\mu_0 m_0}{4\pi s^3}\left[\frac{3}{2}(\hat{x}+\hat{y}) - \hat{y}\right]$

field at m_3 due to m_2

$\vec{B}_{32} = \frac{\mu_0 m_0}{4\pi s^3}\left[\frac{3}{2}(-\hat{x}+\hat{y}) - \hat{y}\right]$

$\vec{B}_3 = \vec{B}_{31} + \vec{B}_{32} = \frac{\mu_0 m_0}{4\pi s^3}\hat{y}$

Potential energy : $U_3 = -\vec{m}_3 \cdot \vec{B}_3 = \frac{-\mu_0 m_0^2}{4\pi s^3}$

m_3 reversing $\Rightarrow m_0\hat{y} \to -m_0\hat{y}$, work done $= \frac{\mu_0 m_0^2}{4\pi s^3} - \left(-\frac{\mu_0 m_0^2}{4\pi s^3}\right) = \frac{\mu_0 m_0^2}{2\pi s^3}$

ii) $\vec{\tau}_3 = \vec{m}_3 \times \vec{B}_3 = m_0 \hat{y} \times \frac{\mu_0 m_0}{4\pi s^3} \hat{y} = 0$.

iii) $\vec{B}_{12}$ = field at m_1 due to m_2

$$= -\frac{\mu_0 m_0}{4\pi(\sqrt{2}s)^3} \hat{y}$$

$$\vec{B}_{13} = \frac{\mu_0 m_0}{4\pi s^3}\left[\frac{3}{2}(\hat{x}+\hat{y}) - \hat{y}\right] , \quad \vec{B}_1 = \vec{B}_{12} + \vec{B}_{13}$$

$$\vec{\tau}_1 = \vec{m}_1 \times \vec{B}_1 = m_0 \hat{y} \times \frac{\mu_0 m_0}{4\pi s^3}\left(\frac{3}{2}\hat{x}\right) \quad [\because \hat{y} \times \hat{y} = 0]$$

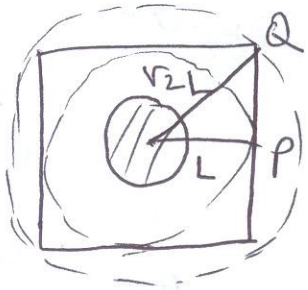
$$= -\frac{3\mu_0 m_0}{8\pi s^3} \hat{z}$$

4.(a) (i) $\mathcal{E} = -\frac{d\phi}{dt} = -\pi R^2 \frac{B_0}{\tau} e^{-t/\tau}$

B increases in time in $\hat{z}$ dirn

$\Rightarrow$ Induced current will be in clockwise dirn (assuming $\hat{z}$ is coming out of the paper)

ii) $\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi}{dt}$



$$\frac{E_P}{E_Q} = \frac{1}{\sqrt{2}}.$$

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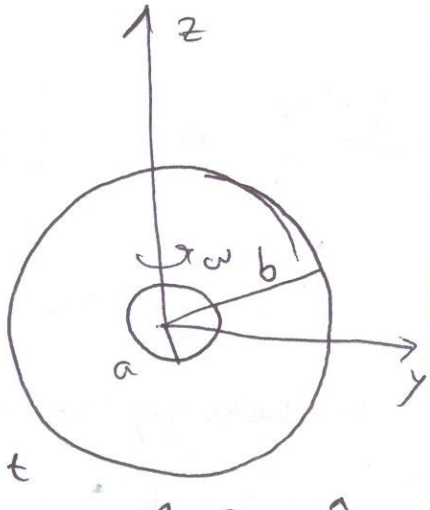
4. (b)



Flux through the small loop

$\Phi = \int \vec{B} \cdot d\vec{a}$, $d\vec{a} = da\,\hat{n}$

$\vec{B}$ = field at the centre due to current in the big loop ($b >> a$)

$\simeq \frac{\mu_0 I}{2b} \hat{x}$,

{ at a time t
$\hat{n} = \cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}$

$$\Phi = \frac{\mu_0 I}{2b} \pi a^2 (\hat{n} \cdot \hat{x}) = \frac{\mu_0 I}{2b} \pi a^2 \cos(\omega t)$$

$$\mathcal{E} = -\frac{d\phi}{dt} = \frac{\mu_0 \pi a^2 I \omega}{2b} \sin \omega t$$

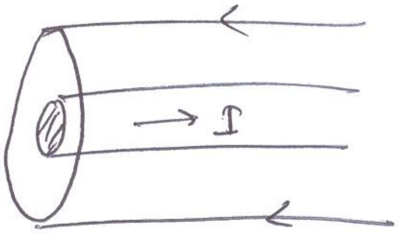
Current in the small loop $I_1 = \dfrac{\mu_0 \pi a^2 I \omega}{2bR} \sin \omega t$

Torque $\vec{\tau} = \vec{m} \times \vec{B} = (I_1 \pi a^2)\, \hat{n} \times \dfrac{\mu_0 I}{2b} \hat{x}$

$$= -\left(\frac{\mu_0 \pi a^2 I}{2b}\right)^2 \frac{\omega}{R} \sin^2 \omega t\, \hat{z}$$

Torque required to keep the loop rotating $\vec{\tau}_{ext} = -\vec{\tau}$.

4.(c) $\vec{B}(\vec{r}) = \begin{cases} \dfrac{\mu_0 I r}{2\pi a^2} & (r \le a) \\ \dfrac{\mu_0 I}{2\pi r} & (a \le r \le b) \\ 0 & (r > b) \end{cases}$



Energy in length l

$$\frac{1}{2} L I^2 = \frac{1}{2\mu_0} \int B^2 d\tau = \frac{1}{2\mu_0}\left[\int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)^2 2\pi r l\, dr + \int_0^a \left(\frac{\mu_0 I r}{2\pi a^2}\right)^2 2\pi r l\, dr\right]$$

$$= \frac{\mu_0 I^2}{4\pi} l \left[\ln(b/a) + \frac{1}{4}\right]$$

$$\Rightarrow \frac{L}{l} = \frac{\mu_0}{2\pi}\left[\ln\left(\frac{b}{a}\right) + \frac{1}{4}\right].$$

5.(a) $\vec{B} = \dfrac{\mu_0 I}{2\pi r} \hat{\phi}$, $\quad \vec{E} = \dfrac{\vec{J}}{\sigma} = \dfrac{I}{\pi a^2 \sigma} \hat{z}$

a = radius of the wire

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = -\frac{I^2}{2\sigma \pi^2 a^3} \hat{s}$$

Total energy coming into the wire in length l

$$= -\oint \vec{S} \cdot d\vec{a} = \frac{I^2}{2\sigma \pi^2 a^3} \cdot 2\pi a l = I^2 \left(\frac{l}{\pi a^2 \sigma}\right) = I^2 R$$

= energy dissipated due to Joule heating.

5.(b) (i) $\vec{E} = \hat{x}\, 10 \sin(\omega t + 6\times 10^5 z)$ V/m.

$$\vec{B} = \frac{\hat{k}\times\vec{E}}{c} \qquad \because \hat{k} = -\hat{z}$$

$$= -3.33\times 10^{-8}\,\hat{y}\,\sin(\omega t + 6\times 10^5 z)\ \text{T}.$$

ii) $$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = -\hat{z}\,\frac{33.3\times 10^{-8}}{4\pi\times 10^{-7}}\sin(\omega t + 6\times 10^5 z)\ \text{W/m}^2$$

$$I = \langle S\rangle = \frac{1}{2}\times\frac{33.3\times 10^{-8}}{4\pi\times 10^{-7}} = \frac{1}{2}\times 33.3\times 10^{-1}\ \text{W/m}^2$$

$$\vec{P} = \frac{\vec{S}}{c^2}$$

iii) $\langle u\rangle = \frac{1}{2}\epsilon_0 E_0^2$

Energy in the cube of 1mm $= \langle u\rangle \times$ volm

$$= \frac{1}{2}\times 8.85\times 10^{-12}\times (10)^2\times 10^{-9}\ \text{N-m}.$$

Volume $= (10^{-3})^3\ \text{m}^3 = 10^{-9}\ \text{m}^3$

Energy enters through the surface at $z = 1$ mm & leaves through the surface at $z = 0$.

5.(c) $$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} = \frac{1}{\sqrt{3}} \Rightarrow \sin\theta_T = \frac{1}{2} \Rightarrow \theta_T = \pi/6$$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \sqrt{3}, \quad \beta = \frac{n_2}{n_1} = \sqrt{3}$$

$\vec{E}$ is in $\hat{x}$-dirn i.e. in xz plane $\Rightarrow$ TM polarization.

$$\frac{E_{0R}}{E_{0I}} = \frac{\alpha-\beta}{\alpha+\beta} = 0 \Rightarrow \text{No reflection}$$

[$\theta_I = \pi/3 \Rightarrow$ Brewster's angle (θ_B)]

6. (a) (i) let $x - ct = u$.

$$\nabla^2 u = \frac{\partial^2 \bar{E}}{\partial x^2} = \frac{\partial^2 \bar{E}}{\partial u^2}$$

$$\frac{\partial^2 \bar{E}}{\partial t^2} = c^2 \frac{\partial^2 \bar{E}}{\partial u^2} \Rightarrow \underbrace{\boxed{\nabla^2 \bar{E} = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}}}_{\text{wave eq}^n}$$

Similarly for $\vec{B}$

(ii) $\bar{\nabla}\cdot\bar{E} = 0 \Rightarrow \frac{\partial f_1}{\partial u} = 0 \Rightarrow f_1 = \text{constant}$

but $\langle f_1 \rangle = 0 \Rightarrow F_1 = 0$

$\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \frac{\partial G_1}{\partial u} = 0$ & $\langle G_1 \rangle = 0 \Rightarrow G_1 = 0$.

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\Rightarrow -\hat{y}\frac{\partial f_3}{\partial u} + \hat{z}\frac{\partial f_2}{\partial u} = \hat{y}\frac{\partial G_2}{\partial u} + \hat{z}\frac{\partial G_3}{\partial u}$$

$\Rightarrow F_3 = -G_2, \quad f_2 = G_3$ [as $\langle f_i \rangle = \langle G_i \rangle = 0$ so no constant term can appear in the solution.

6 (b) $(0,0,0) \to (1,0,1)$

$$\Rightarrow \hat{k} = \frac{\hat{x}+\hat{z}}{\sqrt{2}}, \quad v = \frac{c}{n}, \quad n = \sqrt{\epsilon_r} = 9$$

$$|k| = \frac{\omega}{v} = \frac{6\times10^{14}}{\left(\frac{3\times10^8}{9}\right)} = 18\times10^6.$$

$$\vec{k} = 18\times10^6\left(\frac{\hat{x}+\hat{z}}{\sqrt{2}}\right)$$

$$\vec{E} = 9\,\text{V/m}\,\hat{y}\cos(\vec{k}\cdot\vec{r} - \omega t) = 9\,\text{V/m}\,\hat{y}\cos\left(\frac{18\times10^6}{\sqrt{2}}(x+z) - \omega t\right)$$

$\omega = 6\times10^{14}$ rad/s.

$$\vec{B} = \frac{\hat{k}\times\bar{E}}{v}$$

$$= \frac{81}{c}\left(\frac{\hat{z}-\hat{x}}{\sqrt{2}}\right)\cos\left(|k|\frac{x+z}{\sqrt{2}} - \omega t\right)$$

6.(c) $\sigma = 6\times10^{7}\ (\Omega m)^{-1}$

$\nu = 10^{6}\ Hz \Rightarrow \omega = 2\pi\times10^{6}\ rad/s.$

$\epsilon \simeq \epsilon_0 = 8.85\times10^{-12}\ C^2/Nm^2$

$\frac{\sigma}{\omega\epsilon} >> 1.. \Rightarrow$ Cu - is a good Conductor for the wave.

Skin depth $d = \frac{1}{K_2}$

for good conductor $K_2 = \sqrt{\frac{\omega\mu\sigma}{2}}$

$$\Rightarrow d = \sqrt{\frac{2}{(2\pi\times10^{6})\times6\times10^{7}\times4\pi\times10^{-7}}}\ m$$

$\simeq 0.65$ mm.

27/04/15