A bit about vector spaces, norms and inner products

A *linear vector space* V is a collection of vectors $u, v, w, \ldots$ that obey the following rules:

1. **Vector addition** To every pair of vectors u and v there corresponds a vector $u + v \in V$ that satisfies

   (a) $u + v = v + u$

   (b) (u+v)+w = u +(v+w)

   (c) there exists a unique vector Θ so that $u + \Theta = u$

   (d) to every vector u there exists a vector $-u$ so that $u + (-u) = \Theta$

2. **Scalar multiplication** To every vector u and every real number $\alpha \in \mathbb{R}$ there exists a unique vector αu that obeys:

   (a) $\alpha(\beta u) = (\alpha\beta)u$

   (b) $(\alpha + \beta)u = \alpha u + \beta u$

   (c) $\alpha(u + v) = \alpha u + \beta v$

   (d) $1.u = u.1$

Examples include

The set $\mathcal{P}$ of all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

where $a_0, a_1 \dots$ are real numbers, form a linear vector space with respect to the usual addition and scalar multiplication rules. Also, $\mathcal{P}_n$, the set of polynomials of degree less than or equal to n forms a vector space. However, the space of polynomials of degree exactly equal to n does not form a vector space.

The set

$$S_0 = \left\{ u : u(x) \in C^2[0, L], -\frac{d}{dx}\left(a(x)\frac{du}{dx} \right) + c(x)u = 0, \;\; 0 < x < L \right\},$$

forms a vector space under the usual rules of addition and scalar multiplication.

Vector spaces endowed with a sense of the distance between two ‘points’ is called a *normed vector space*. Norms have been introduced earlier and they meet the following requirements

1. $||u|| \geq 0$ for all u and $||u|| = 0$ only if $u = 0$
2. $||\alpha u|| = \alpha \, ||u||$
3. $||u + v|| \leq ||u|| + ||v||$

The nrom is a operation $||\cdot|| : V \to \mathbb{R}$. It can be used to define the ‘distance’ between two vectors and is called a *natural metric*.

A normed vector space which is *complete* is called a *Banach space*. A complete space is one where every sequence $\{u_j\}$ converges to an element in the set.

Now let us consider some special vector spaces. Consider an open domain $\Omega \in \mathbb{R}^3$ which is a set of points $\boldsymbol{x} = (x_1, x_2, x_3)$. The space of square integrable functions $u(\boldsymbol{x})$ is called the L_2 space and is defined as

$$L_2(\Omega) = \left\{ u(\boldsymbol{x}) : \int_\Omega |u(\boldsymbol{x})|^2 d\boldsymbol{x} \right\},$$

where $\int_\Omega u(\boldsymbol{x}) d\boldsymbol{x}$, and $\int_\Omega |u(\boldsymbol{x})|^2 d\boldsymbol{x}$ exist and are finite.

In general *Lebesgue spaces* are defined for $1 \leq p \leq \infty$,

$$L_p(\Omega) = \left\{ u : ||u||_p < \infty \right\},$$

where,

$$||u||_p = \left[\int_\Omega |u(\boldsymbol{x})|^p d\boldsymbol{x} \right]^{1/p} < \infty.$$

For $p = \infty$ a special case known as the 'sup norm' is defined as

$$||u||_\infty = \sup\{|\mathrm{u}(\boldsymbol{x})| : \boldsymbol{x} \in \Omega\}.$$

Another important normed space is the *Sobolev space* $W^{m,p}(\Omega)$. Let $C^m(\Omega)$ denote the set of all real-valued functions with m continuous derivatives defined in $\Omega \in \mathbb{R}^3$. We define on C^m the norm, called the *Sobolev norm*

$$||u||_{m,p} = \left[\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u(\boldsymbol{x})|^p d\boldsymbol{x} \right]^{1/p},$$

for $1 \leq p \leq \infty$. Here α denotes a set of integers $(\alpha_1, \alpha_2 \ldots \alpha_n)$ and $|\alpha| = \sum \alpha_i$ so that

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

For example, for $m = 1, n = 2$ and $1 \leq p \leq \infty$ we have $\alpha_1, \alpha_2 = 0, 1$ so that

$$||u||_{1,p} = \left\{ \int_{\Omega \in \mathbb{R}^3} \left[|u|^p + \left| \frac{du}{dx} \right|^p + \left| \frac{du}{dy} \right|^p \right] dx dy \right\}^{1/p}.$$

Similar to a norm we define an *inner product* between a pair of vectors. Again, the inner product obeys the following rules

1. $(u, v)_V = (v, u)_V$
2. $(\alpha u, v)_V = \alpha(u, v)_V$
3. $(u_1 + u_2, v)_V = (u_1, v)_V + (u_2, v)_V$
4. $(u, u)_V > 0$ for all $u \neq 0$.

Note that we can associate with every inner product, a norm as

$$||u||_V = \sqrt{(u, u)_V}.$$

Also, two vectors are orthogonal if

$$(u, v)_V = 0.$$

As an example consider the space $W^{m,2}(\Omega)$ (also called $H^m(\Omega)$ and the *Hilbert space of order* m) endowed with a inner product

$$(u,v)_m = \int_\Omega \sum_{|\alpha|\le m} D^\alpha u(\boldsymbol{x}) D^\alpha v(\boldsymbol{x}) d\boldsymbol{x}.$$

For various values of m we have,

$$\begin{aligned}
(u,v)_0 &= \int_\Omega uv dx dy, \\
(u,v)_1 &= \int_\Omega \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx dy
\end{aligned}$$

which induce norms

$$||u||_m = \sqrt{(u,u)_m}.$$

Note that $H^0(\Omega) = L_2(\Omega)$.

Finally, a *linear transformation* of a vector $T : U \to V$ obeys

1. $T(\alpha u) = \alpha T(u)$ for all $u \in U$ and $\alpha \in \mathbb{R}$
2. $T(u_1 + u_2) = T(u_1) + T(u_2)$ for all $u_1, u_2 \in U$.

Transformations that map vectors to real numbers are of special interest and are called *functionals*.

A *linear functional* is a linear transformation $l : V \to \mathbb{R}$. For example,

$$l(u) = \int_a^b f u dx.$$

Similarly, a linear transformation $B : V \times V \to \mathbb{R}$ maps a pair of vectors (u, v) is called a *bilinear form*. For example

$$B(u, v) = \int_a^b \left(\frac{du}{dx}\frac{dv}{dx} + uv \right) dx.$$

An important result on differential equations and functionals

Consider a differential equation written as a linear transformation

$$Au = f,$$

where $A : \mathcal{D} \to H$ and $f \in H$. Here H denotes a Hilbert space.

Examples are

$$A = -\frac{d}{dx}\left(EA\frac{d}{dx}\right) \text{ or } A = \frac{d^2}{dx^2}\left(EI\frac{d^2}{dx^2}\right),$$

in $\Omega = [0, L]$. In these cases, $f \in H^0 = L_2(0, L)$ while $u \in C^2(0, L)$ in the first case and $\in C^4(0, L)$ in the second.

An operator is *self adjoint* or *symmetric* if for all u, v

$$(Au, v)_H = (u, Av)_H.$$

The operator A is *strictly positive* if for all $u \neq 0$

$$(Au, u)_H > 0.$$

Every bilinear form generates a *quadratic form* which is a functional quadratic in its arguments, as

$$B(u, u) = Q(u).$$

Theorem: If A is a strictly positive operator in $\mathcal{D}$, then for $f \in H$,

$$Au = f$$

has at most one solution in $\mathcal{D}$.

Theorem: Let A be a positive operator in $\mathcal{D}$ and $f \in H$. Let $Au = f$ have a solution $u_0 \in \mathcal{D}$. Then the quadratic functional

$$I(u) = \frac{1}{2}(Au, u)_H - (f, u)_H$$

assumes its minimal value in $\mathcal{D}$ for the element u_0. i.e.

$$I(u) \geq I(u_0),$$

except for $u = u_0$ when $I(u) = I(u_0)$.

The above theorem provides an important route to construct weak forms of problems governed by strong forms.

For proofs to the above see, Reddy (2002), *Energy principles and variational methods in applied mechanics*, John Wiley and sons.

Let us consider a differential equation in one variable that governs the transverse deformation $u(x)$ of a cable fixed at both ends and subjected to a transverse load $f(x)$. The tension in the cable is $a(x)$.

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for } 0 < x < L$$

with

$$u(0) = 0, \; u(L) = 0$$

Let us choose $f \in L_2(0, L)$ and $\mathcal{D}$ as the subset of H that contains functions that satisfy the end conditions and are differentiable upto the second order.

The operator

$$A = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)$$

is symmetric as

$$\begin{aligned}
(Au, v)_H &= \int_0^L \left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)\right] v dx \\
&= -a\frac{du}{dx}v\Big|_0^L + \int_0^L \frac{dv}{dx}\left(a\frac{du}{dx}\right) dx \\
&= \int_0^L \left[-\frac{d}{dx}\left(a\frac{dv}{dx}\right)\right] u dx = (u, Av)_H.
\end{aligned}$$

In the above use the fact that as $u, v \in \mathcal{D}$, $u(0) = u(L) = v(0) = v(L) = 0$. Thus the variational principle governing this problem is

$$\Pi(u) = \frac{1}{2}(Au, u)_H - (f, u)_H,$$

i.e.

$$\Pi(u) = \frac{1}{2}\int_)^L a(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^L f u dx.$$

Consider another equation, now in two variables:

$$\nabla^2\phi + c\phi + Q = 0$$

c and Q are functions of position only. The operator is:

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + c, f = -Q$$

The operator is self adjoint as

$$(\mathcal{L}\phi, \psi)_H = \int_V \psi \left\{ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \right\} dV \text{ can be evaluated using}$$

$$\int_V \psi \frac{\partial^2\phi}{\partial x^2} dV = \int_{\partial V} \psi \frac{\partial\phi}{\partial x} n_x dS - \int_V \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} dV$$

Similarly $$\int_V \phi \frac{\partial^2\psi}{\partial x^2} dV = \int_{\partial V} \phi \frac{\partial\psi}{\partial x} n_x dS - \int_V \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} dV$$

Thus,

$$(\mathcal{L}\phi, \psi)_H = (\mathcal{L}\psi, \phi)_H + \int_{\partial V} \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n} \right) dS.$$

The boundary term is zero since at the boundary ∂V, either $\phi = \psi$ is specified or $\phi_{,n} = \psi_{,n}$ is specified. Thus $\mathcal{L}$ is self adjoint.

Thus the variational principle corresponsding to this equation becomes:

$$\Pi = \frac{1}{2}(\mathcal{L}\phi, \phi)_H - (Q, \phi)_H.$$

yielding

$$\int_V \left\{ \frac{1}{2}\phi \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + c\phi \right] + Q\phi \right\} dV$$

or, applying Gauss law,

$$\Pi = \int_V \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{2}c\phi^2 - Q\phi \right] dV + \text{boundary terms}$$

Equations governing bending of a Timoshenko beam with w being the trnsverse deflection and ϕ_x the rotation:

$$\begin{aligned} -\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]+c_f w &= q \\ -\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right)+S\left(\frac{dw}{dx}+\phi_x\right) &= 0 \end{aligned}$$

S shear stiffness, D bending stiffness, w transverse deflection, c_f foundation modulus.

Here,

$$A = \begin{pmatrix} -S\frac{d^2}{dx^2}+c_f & -S\frac{d}{dx} \\ -S\frac{d}{dx} & -D\frac{d^2}{dx^2}+S \end{pmatrix}$$

$$f = \left\{ \begin{array}{c} q \\ 0 \end{array} \right\} \text{ and } \mathrm{u} = \left\{ \begin{array}{c} w \\ \phi_x \end{array} \right\}$$

so that the equations can be represented as

$$Au = f.$$

It is easily shown that under the following boundary conditions:

$$w(0) = \phi_x(0) = 0, \left[S\left(\frac{dw}{dx} + \phi_x\right)\right]_{x=L} = F_0, \left[D\frac{d\phi_x}{dx}\right]_{x=L} = M_0$$

A is self adjoint. Now,

$$\begin{aligned}
(Au, u)_H &= \int_0^L \left[-Sw\frac{d}{dx}\left(\frac{dw}{dx} + \phi_x\right) + S\phi\left(\frac{dw}{dx} + \phi_x\right) + c_f w^2 - D\phi_x \frac{d^2\phi_x}{dx^2}\right] dx \\
&= \int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + c_f w^2\right] dx \\
&\quad - \left[Sw\left(\frac{dw}{dx} + \phi_x\right) + D\phi_x \frac{d\phi_x}{dx}\right]\Bigg|_0^L
\end{aligned}$$

Thus, the variational principle is

$$\begin{aligned}
\Pi(w, \phi_x) &= \frac{1}{2}\int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + c_f w^2\right] dx - \int_0^L wq dx \\
&\quad - (wF_0 + \phi_x M_0)|_L
\end{aligned}$$