ASSIGNMENT IV MSO 202 A

IDENTITY THEOREM, ZEROS OF A HOLOMORPHIC FUNCTION, POLES

Exercises 0.1-0.4 rely on the Identity Theorem: If F is an entire function such that there exists a convergent sequence $\{z_n\}$ such that $F(z_n) = 0$ then $F(z) = 0$ for all $z \in \mathbb{C}$.

Exercise 0.1 : Let f, g be entire functions such that $f(z) = g(z)$ for all $z = x \in (0, 1)$. Show that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

Solution. Apply Identity Theorem to $F(z) = f(z) - g(z)$, $z_n = 1/n$.

Exercise 0.2 : Does there exist an entire function f such that $f(z) = |z|^3$ for all $z = x + iy$, $x \in (-1, 1)$?

Solution. No. Suppose that there exist an entire function f such that $f(z) = |z|^3$ for all $z = x + iy$, $x \in (-1, 1)$. Note that $f(z) = z^3$ for all $z = x + iy$, $x \in (0, 1)$. By Exercise 0.1, $f(z) = z^3$ for all $z \in \mathbb{C}$. However, $f(z) = |z|^3$ for $z = x \in (-1, 0)$, and hence $x^3 = |x|^3$ for $x \in (-1, 0)$, which is not possible.

Exercise 0.3 : Assuming $\sin(2x) = 2\sin(x)\cos(x)$ for all $x \in \mathbb{R}$, show that $\sin(2z) = 2\sin(z)\cos(z)$ for all $z \in \mathbb{C}$.

Remark 0.4 : One can prove similarly that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{0.1}$$

Solution. Apply Exercise 0.1 to $f(z) = \cos(2z)$, $g(z) = 2\sin(z)\cos(z)$.

Exercise 0.5 : Let f be a non-zero entire function. Show that for any $R > 0$, f can have finitely many zeros in the closed disc $\overline{\mathbb{D}_R(0)}$ centred 0 and of radius R.

Solution. Suppose for some $R > 0$, $\overline{\mathbb{D}_R(0)}$ contains infinitely many zeros of f. Let $\{z_n = x + n + iy_n\}$ be a sequence of distinct zeros of f in $\overline{\mathbb{D}_R(0)}$. By Bolazano-Weierstrass Theorem (from MTH101), the bounded sequences $\{x_n\}$ and $\{y_n\}$ have convergent subsequences $\{x_{n_k}\}$

and $\{y_{n_k}\}$ respectively. It follows that $\{z_{n_k}\}$ is convergent. By Identity Theorem, f must be identically 0, which contradicts the assumption that f is non-zero.

Exercise 0.6 : Show that all zeros of $\cos(\frac{\pi}{2}z)$ are at odd integers. Show further that all zeros are simple.

Solution.

(1) Note that $\cos(\frac{\pi}{2}(2k+1)) = 0$ and $\frac{d}{dz}\cos(\frac{\pi}{2}z)|_{z=2k+1} = \frac{\pi}{2}\sin(\frac{\pi}{2}(2k+1)) \neq 0$. This shows that the zeros at odd integers are simple.

(2) Suppose $\cos(\frac{\pi}{2}z_0) = 0$ for some $z_0 = x_0 + iy_0 \in \mathbb{C}$. By (0.1), $\cos(\frac{\pi}{2}z) = \frac{e^{i\frac{\pi}{2}z}+e^{-i\frac{\pi}{2}z}}{2}$. Hence $e^{i\frac{\pi}{2}z_0} = -e^{-i\frac{\pi}{2}z_0}$, that is, $e^{i\pi z_0} = -1$. Taking modulus on both sides, we obtain $e^{-\pi y_0} = 1$. Since e^x is one to one, $y_0 = 0$. Thus $e^{i\pi x_0} = -1$, that is, $\cos(\pi x_0) + i\sin(2\pi x_0) = -1$, and hence x_0 is an odd integer.

Exercise 0.7 : Show that all poles of $\tan(\frac{\pi}{2}z)$ are at odd integers. Show further that all poles are simple.

Solution. Let k be an odd integer. Note that $\sin(\frac{\pi}{2}k) \neq 0$. Also, since $\cos(\frac{\pi}{2}k) = 0$ and $\frac{d}{dz}\cos(\frac{\pi}{2}z)|_{z=k} \neq 0$,

$$\cos\left(\frac{\pi}{2}z\right) = \sum_{n=1}^{\infty} a_n \left(z - \frac{\pi}{2}k\right)^k = \left(z - \frac{\pi}{2}k\right) h(z),$$

where $h(z) \neq 0$ in a neighborhood of $\frac{\pi}{2}k$. It follows that

$$\tan\left(\frac{\pi}{2}z\right) = \sin\left(\frac{\pi}{2}z\right)\frac{1}{h(z)}\frac{1}{z-\frac{\pi}{2}k}. \tag{0.2}$$

Hence, $\frac{1}{\tan(\frac{\pi}{2}z)}$, with 0 redefined at $\frac{\pi}{2}k$, defines a holomorphic function near $\frac{\pi}{2}k$. Its also clear from (0.2) that $z = \frac{\pi}{2}k$ is a pole of order 1.