INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

Sturm-Liouville Theory and Fourier Series

MSO 203B

September 21, 2016

1. Find the Eigenvalues and Eigenfunctions of the following problem:

$$y'' + \lambda y = 0;\ \ 0 < x < 1$$
$$y(0) = 0$$
$$y(1) - y'(1) = 0$$

2. Find the Eigenpairs for the SLPBVP given by:

$$(x^2 y')' + \lambda y = 0;\ \ 1 < x < 2$$
$$y(1) = y(2) = 0$$

3. Consider the model of wave propogation in a nonhomogeneous string given by

$$u_{tt} = (1+x)^2 u_{xx};\ \ 0 < x < 1;\ \ t > 0$$
$$u(0,t) = 0;\ u(1,t) = 0;\ \ t > 0$$
$$u(x,0) = f(x);\ 0 < x < 1$$
$$u_t(x,0) = g(x);\ 0 < x < 1$$

   Find the solution of the problem in terms of f and g.

4. Find the Fourier Series of the function

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases} \quad \text{and} \ \ f(x+2\pi) = f(x)$$

   Also show that $1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots = \frac{\pi}{4}$.

5. Find the Fourier Series of the function

$$f(x) = \begin{cases} \pi, \ \pi < x < 2\pi \\ x, \ 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$

Also show that $1 + \frac{1}{9} + \frac{1}{25} + \cdots\cdots = \frac{\pi^2}{8}$.

6. Find the Fourier Sine series of $f(x) = x$ on $[0,1]$.

7. Consider the function

$$f(x) = \begin{cases} 1-x, \ 0 < x \le 1 \\ 1, \ 1 < x \le 2 \end{cases}$$

   - Plot its odd and even periodic extensions over (-4,4).
   - Compute its Fourier Cosine Series.

8. The Legendre equation is the second order differential equation given by

$$[(1-x^2)y']' + n(n+1)y = 0 \ \text{ in } \ (-1,1)$$

and has regular singular point at ± 1. We know that P_n is the nth degree Legendre Polynomial which can be expressed by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \ \text{ for } \ n = 0,1,2...$$

which are the solution for the Legendre equation.

   - Prove that $\{P_n\}$ forms an orthogonal system of polynomials on $(-1,1)$.
   - Prove that $||P_n||^2 := \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$
   - Show that for f piecewise smooth in $[-1,1]$ can be represented in terms of P_n. This representation will be called the Fourier-Legendre series.
   - Find the Fourier-Legendre series for the function

$$f(x) = \begin{cases} 1, \ 0 < x < 1 \\ 0, \ -1 < x < 0 \end{cases}$$