Solution 5

1. We start by calculating $\mathbf{Q}_{\mathbf{y}90°}$ and $\mathbf{Q}_{\mathbf{z}90°}$, the quaternions corresponding respectively to $\mathbf{R}_\mathbf{y}(90°)$ and $\mathbf{R}_\mathbf{z}(90°)$:

   For $\mathbf{Q}_{\mathbf{y}90°}$, we have:

   — $\theta_y = 90° \rightarrow \cos(\theta_y/2) = \sin(\theta_y/2) = \frac{\sqrt{2}}{2}$

   — $\boldsymbol{\lambda_y} = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}$ ($\boldsymbol{\lambda}_y$ is the axis of rotation whose norm is $\sin(\theta_y/2)$)

   — $\lambda_{y0} = \cos(\theta_y/2) = \frac{\sqrt{2}}{2}$

   And finally:

$$\mathbf{Q}_{\mathbf{y}90°} = \begin{pmatrix}\lambda_{y0}\\ \boldsymbol{\lambda_y}\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

   For $\mathbf{Q}_{\mathbf{z}90°}$, we have:

   — $\theta_z = 90° \rightarrow \cos(\theta_z/2) = \sin(\theta_z/2) = \frac{\sqrt{2}}{2}$

   — $\boldsymbol{\lambda_z} = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$ ($\boldsymbol{\lambda}_z$ is the axis of rotation whose norm is $\sin(\theta_z/2)$)

   — $\lambda_{z0} = \cos(\theta_z/2) = \frac{\sqrt{2}}{2}$

   And finally:

$$\mathbf{Q}_{\mathbf{z}90°} = \begin{pmatrix}\lambda_{z0}\\ \boldsymbol{\lambda_z}\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

   We notice that the two quaternions are unitary (the opposite would have been surprising).

   We then calculate the two sequences by multiplying the quaternions (product which is of course non-commutative):

   First sequence: $\mathbf{R}_\mathbf{z}(90°) \rightarrow \mathbf{R}_\mathbf{y}(90°)$

$$\begin{aligned}
\mathbf{Q_1} &= \mathbf{Q}_{\mathbf{y}90°}\mathbf{Q}_{\mathbf{z}90°} \\
&= \begin{pmatrix}\lambda_{y0}\lambda_{z0} - \boldsymbol{\lambda_y}\cdot\boldsymbol{\lambda_z}\\ \lambda_{y0}\boldsymbol{\lambda_z} + \lambda_{z0}\boldsymbol{\lambda_y} + \boldsymbol{\lambda_y}\times\boldsymbol{\lambda_z}\end{pmatrix} \\
&= \begin{pmatrix}\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}\\ \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_{1,0}\\ \boldsymbol{\lambda_1}\end{pmatrix}
\end{aligned}$$

Second sequence : $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$

$$\mathbf{Q_2} = \mathbf{Q}_{\mathbf{z}90°}\mathbf{Q}_{\mathbf{y}90°}$$

$$
\begin{aligned}
&= \begin{pmatrix} \lambda_{z0}\lambda_{y0} - \boldsymbol{\lambda_z} \cdot \boldsymbol{\lambda_y} \\ \lambda_{z0}\boldsymbol{\lambda_y} + \lambda_{y0}\boldsymbol{\lambda_z} + \boldsymbol{\lambda_z} \times \boldsymbol{\lambda_y} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} \\ \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_{2,0}\\ \boldsymbol{\lambda_2}\end{pmatrix}
\end{aligned}
$$

The two resulting quaternions are unitary as expected.

2. (a) The calculation of the angles is given below:

First sequence: $\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$

$$\theta_1 = 2\arccos(\lambda_{1,0}) = 2\arccos(1/2) \text{ et } \theta_1 = 2\arcsin(||\boldsymbol{\lambda_1}||) = 2\arcsin\left(\sqrt{3}/2\right)$$

$$\Rightarrow \theta_1 = \frac{2\pi}{3} rad = 120°$$

Second sequence: $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$

$$\theta_2 = 2\arccos(\lambda_{2,0}) = 2\arccos(1/2) \text{ et } \theta_2 = 2\arcsin(||\boldsymbol{\lambda_2}||) = 2\arcsin\left(\sqrt{3}/2\right)$$

$$\Rightarrow \theta_2 = \frac{2\pi}{3} rad = 120°$$

(c) Obtaining the (unitary) axes is as follows:

First sequence: $\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$

$$
\begin{aligned}
\mathbf{k_1} &= \frac{\boldsymbol{\lambda_1}}{\sin(\theta_1/2)} \\
&= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}
\end{aligned}
$$

Second sequence: $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$

$$
\begin{aligned}
\mathbf{k_2} &= \frac{\boldsymbol{\lambda_2}}{\sin(\theta_2/2)} \\
&= \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}
\end{aligned}
$$