Exercise set 4 - Kinematics

Reminders

Simplified notation of sines and cosines

To simplify the notation, we use:

- $\sin(\theta_1) = s_1$
- $\cos(\theta_1) = c_1$
- $1 - \cos(\theta_1) = v_1$
- $\sin(\theta_2) = s_2$
- $\cos(\theta_2) = c_2$
- $1 - \cos(\theta_2) = v_2$
- $\cos(\theta_1 + \theta_2) = c_{1+2}$
- $\sin(\theta_1 + \theta_2) = s_{1+2}$
- $L_1 + L_2 = L_{1+2}$

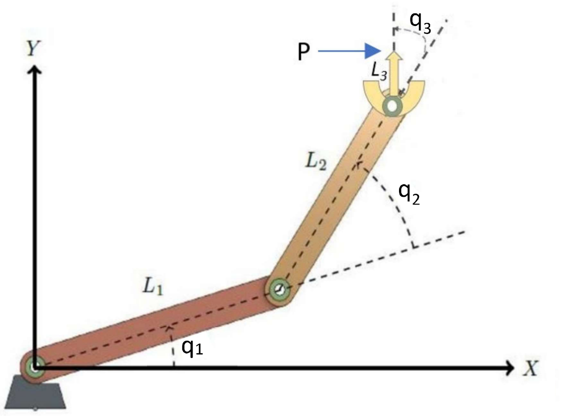
Exercise 1

In this exercise you will work on the geometric model of the SCARA robot. Here we won't consider the rotation of the end effector. The output point will be the point P at the extremity of the second segment L_2 (see figure).

Give the direct geometric model (DGM) that expresses the coordinates (x, y) of point P as a function of the joint coordinates q_1 and q_2.

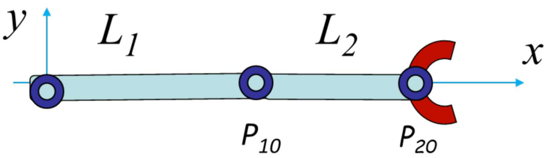
Hint: use the homogeneous matrices of the following transformations:

1. Rotation of q_2 around P_{10} with coordinates $(L_1, 0)$
2. Rotation of q_1 around the origin



Exercise 1 – Solution

To obtain the direct geometric model that expresses the coordinates (x, y) of the end effector P as a function of the joint coordinates q_1 and q_2. To do so, we **first** put the robot in its reference position (figure below) and then respectively develop the homogenous matrices at each joint, starting from the last one.



With $\mathbf{p}_{10} = [L_1, 0]'$

and $\mathbf{p}_{20} = [L_{12}, 0]'$

Figure: The robot in its reference position

Secondly, we develop the homogenous matrices associated with each of the joints, namely:

1. Homogenous matrix corresponding to the rotation q_2 around the point P_{10} with coordinates $(L_1, 0)$
2. Homogenous matrix corresponding to the rotation of q_1 around the origin

Lastly, we express the direct geometric model by multiplying the sequence of the homogenous matrices starting with the last transformation to the first as described in the lecture.

We recall that the homogeneous matrix for a rotation around an arbitrary point **p** is expressed as:

$$H = \begin{bmatrix} R & p - Rp \\ 0 & 1 \end{bmatrix}.$$

By using this relation, we calculate the homogeneous matrices of the transformations described in the hint. The homogenous matrix H_2, corresponding to the rotation with q_2 around the point P_{10}, with the coordinates $(L_1, 0)$ is then:

$$H_2 = \begin{bmatrix} R_2 & p_{10} - R_2 p_{10} \\ 0 & 1 \end{bmatrix}.$$

$$p_{10} - R_2 p_{10} = \begin{pmatrix} L_1 \\ 0 \end{pmatrix} - \begin{pmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{pmatrix} \begin{pmatrix} L_1 \\ 0 \end{pmatrix} = \begin{pmatrix} L_1 - c_2 L_1 \\ -s_2 L_1 \end{pmatrix} = \begin{pmatrix} L_1(1 - c_2)^* \\ -L_1 s_2 \end{pmatrix} = \begin{pmatrix} L_1 v_2 \\ -L_1 s_2 \end{pmatrix}$$

* We recall that 1 - *cos* $(q_2) = v(q_2)$ (definition of the versine function: link)

Thus,

$$\boldsymbol{H_2} = \begin{bmatrix} R_2 & p_{10} - R_2 p_{10} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 & L_1 v_2 \\ s_2 & c_2 & -L_1 s_2 \\ 0 & 0 & 1 \end{bmatrix}$$

The homogenous matrix H_1, corresponding to the rotation q_1 around the origin, is expressed as follows:

$$\boldsymbol{H_1} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The combined homogenous matrix of the sequence of the two rotations, respectively represented by the homogenous matrix H_2 (of angle q_2) then H_1 (of angle q_1), is equal to the following product:

$$\boldsymbol{H} = \boldsymbol{H_1 H_2} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} c_2 & -s_2 & L_1 v_2 \\ s_2 & c_2 & -L_1 s_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{1+2} & -s_{1+2} & L_1(c_1 v_2 + s_1 s_2) \\ s_{1+2} & c_{1+2} & L_1(s_1 v_2 - c_1 s_2) \\ 0 & 0 & 1 \end{bmatrix}$$

To find the coordinates (x, y) of the point P (which is the Tool Center Point), we proceed as follows:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \boldsymbol{H} * \boldsymbol{P_{20}}$$

Thus,

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \boldsymbol{H} * \begin{pmatrix} L_{12} \\ 0 \\ 1 \end{pmatrix}$$

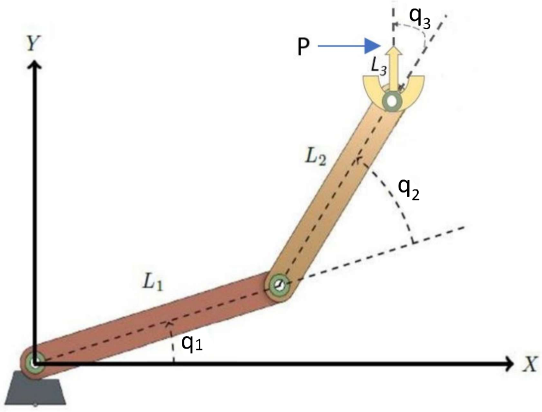
Once we use the calculated homogenous matrix, that gives the following result:

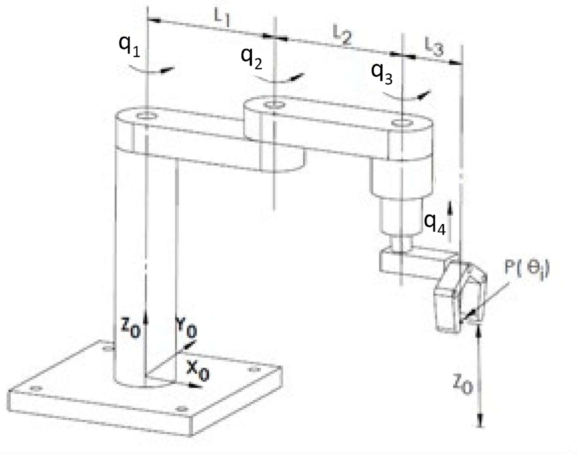
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} L_1 c_1 + L_2 c_{1+2} \\ L_1 s_1 + L_2 s_{1+2} \\ 1 \end{pmatrix}$$

Exercise 2

In this exercise we take the output point as the tip of the end effector, as shown in the figures below. Therefore, here we consider the rotation of the end effector given by q_3. In addition, as illustrated in the right figure below, we consider the possible translation along z given by q_4.

The reference position of the end effector is $P(\theta_i = 0) = \begin{pmatrix} x_0 \\ 0 \\ z_0 \end{pmatrix} = \begin{pmatrix} L_{1+2+3} \\ 0 \\ z_0 \end{pmatrix}$. Give the position P($q_i$) as a function of the variables q_i.





Exercise 2 – Solution

To obtain the direct geometric model that expresses the coordinates (x, y, z) of the end effector P as a function of the joint coordinates q_1, q_2, q_3 and q_4, we **first** put the robot in its reference position (figure below) and then respectively develop the homogenous matrices at each joint, starting from the last coordinate.

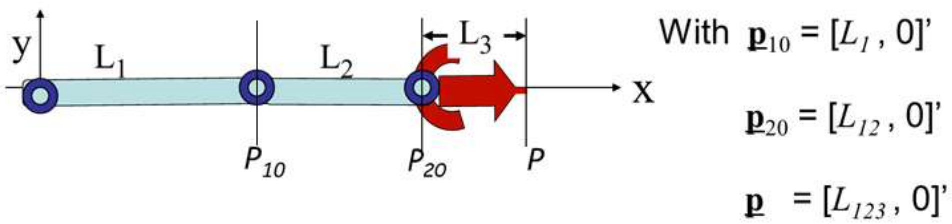


Figure: The robot in its reference position

Secondly, we develop the homogenous matrices associated with each of the joints, namely:

1. Homogenous matrix associated with the translation q_4 in the direction of z axis at the point P_{20}
2. Homogenous matrix corresponding to the rotation q_3 around P_{20} with coordinates (L_{1+2} , *0, 0*)
3. Homogenous matrix corresponding to the rotation of q_2 around P_{10}, with coordinates (L_1, *0, 0*)
4. Homogenous matrix corresponding to the rotation of q_1 around the origin.

Finally, we express the direct geometric model by multiplying the sequence of the homogenous matrices, starting from the last transformation to the first, by the output point P_0 when the robot is at its reference position, here the point is P with coordinates $(L_{1+2+3}, 0, z_0)$ as shown in the figure.

In the same way as before, the homogeneous matrix for a rotation around an arbitrary point **p** is expressed as:

$H = \begin{bmatrix} R & p - Rp \\ 0 & 1 \end{bmatrix}$.

Therefore, we can calculate the homogeneous matrices of the transformations described in points 1, 2, 3 and 4.

The homogenous matrix H_4, associated with the translation q_4, is:

$$\boldsymbol{H_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{[Pure translation]}$$

The homogenous matrix H_3, corresponding to the rotation of q_3 around P_{20} with coordinates $(L_{1+2}, 0, 0)$, is:

$H_3 = \begin{bmatrix} R_{z3} & p_{20} - R_{z3}p_{20} \\ 0 & 1 \end{bmatrix}$.

$$p_{20} - R_{z3}p_{20} = \begin{pmatrix} L_{1+2} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{1+2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{1+2} - c_3 L_{1+2} \\ -s_3 L_{1+2} \\ 0 \end{pmatrix} = \begin{pmatrix} L_{1+2} v_3 \\ -L_{1+2} s_3 \\ 0 \end{pmatrix}$$

We then have:

$$\boldsymbol{H_3} = \begin{pmatrix} c_3 & -s_3 & 0 & L_{1+2} v_3 \\ s_3 & c_3 & 0 & -L_{1+2} s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The homogenous matrix H_2, corresponding to the rotation q_2 around an axis parallel to the z axis and passing through the point P_{10} with the coordinates (L_1, 0, 0), is:

$$\boldsymbol{H_2} = \begin{pmatrix} c_2 & -s_2 & 0 & L_1 v_2 \\ s_2 & c_2 & 0 & -L_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The homogenous matrix H_1, corresponding to the rotation q_1 around the axis z, is:

$$\boldsymbol{H_1} = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The sequence of the 3 rotations and the translation, each represented by its corresponding homogenous matrix, is then expressed by the following product:

$$\boldsymbol{H} = \boldsymbol{H_1 H_2 H_3 H_4}^{(*)}$$

$$\boldsymbol{H} = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & L_1 v_2 \\ s_2 & c_2 & 0 & -L_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 & L_{1+2} v_3 \\ s_3 & c_3 & 0 & -L_{1+2} s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(*) Note that the translation q_4 is carried out first before q_3 in the implementation order as mentioned in the lecture.

Similar to the previous exercise, in order to find the (x,y,z) coordinates of P, one should use the following formula:

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{1} \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & L_1 v_2 \\ s_2 & c_2 & 0 & -L_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 & L_{1+2} v_3 \\ s_3 & c_3 & 0 & -L_{1+2} s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{L_{1+2+3}} \\ \boldsymbol{0} \\ \boldsymbol{z_0} \\ \boldsymbol{1} \end{pmatrix}$$

$$= \begin{pmatrix} c_{1+2} & -s_{1+2} & 0 & L_1(c_1v_2+s_1s_2) \\ s_{1+2} & c_{1+2} & 0 & L_1(s_1v_2-c_1s_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 & L_{1+2}v_3 \\ s_3 & c_3 & 0 & -L_{1+2}s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{1+2+3} \\ 0 \\ z_0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{1+2+3} & -s_{1+2+3} & 0 & L_1(c_1v_2+s_1s_2)+L_{1+2}(c_{1+2}v_3+s_{1+2}s_3) \\ s_{1+2+3} & c_{1+2+3} & 0 & L_1(s_1v_2-c_1s_2)+L_{1+2}(s_{1+2}v_3-c_{1+2}s_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{1+2+3} \\ 0 \\ z_0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{1+2+3} & -s_{1+2+3} & 0 & L_1(c_1v_2+s_1s_2)+L_{1+2}(c_{1+2}v_3+s_{1+2}s_3) \\ s_{1+2+3} & c_{1+2+3} & 0 & L_1(s_1v_2-c_1s_2)+L_{1+2}(s_{1+2}v_3-c_{1+2}s_3) \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{1+2+3} \\ 0 \\ z_0 \\ 1 \end{pmatrix}$$

To simplify, we apply trigonometric transformations such as $c_{1+2+3} = c_{1+2}v_3 - s_{1+2}s_3$.

$$= \begin{pmatrix} L_1c_1 + L_2c_{1+2} + L_3c_{1+2+3} \\ L_1s_1 + L_2s_{1+2} + L_3s_{1+2+3} \\ q_4 + z_0 \\ 1 \end{pmatrix}$$

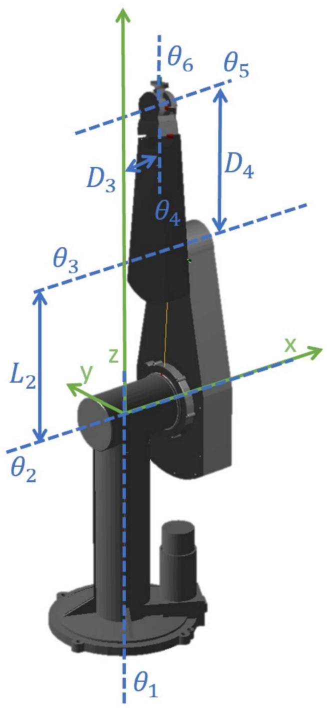
We therefore obtain:

$$\boldsymbol{x} = L_1c_1 + L_2c_{1+2} + L_3c_{1+2+3}$$
$$\boldsymbol{y} = L_1s_1 + L_2s_{1+2} + L_3s_{1+2+3}$$
$$\boldsymbol{z} = q_4 + z_0$$

Like the previous exercise, this one makes the link between DGM, homogeneous transformation matrix and position of the robot/of the robot segments; and illustrates the link between DGM and generalized coordinate system.

Exercise 3

The homogeneous matrices $\mathbf{K_5}$ and $\mathbf{K_6}$ of the DGM of the PUMA robot arm are given in the lecture slides. Give the missing matrices $\mathbf{K_i}$, for i = 1,2,3,4.



Exercise 3 – Solution

The homogeneous matrices of the PUMA DGM (according to the generalized coordinate system of the course) are as follows:

K_4 is the homogeneous matrix corresponding to the rotation θ_4 around an axis parallel to the z axis and passing through the point P_{40}= [D_3, 0, 0].

$$K_4 = \begin{bmatrix} R_{z4} & p_{40} - R_{z4}\, p_{40} \\ 0 & 1 \end{bmatrix}$$

$$p_{40} - R_{z4}\, p_{40} = \begin{pmatrix} D_3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} c_4 & -s_4 & 0 \\ s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} D_3 - D_3\, c_4 \\ -D_3 s_4 \\ 0 \end{pmatrix} = \begin{pmatrix} D_3 v_4 \\ -D_3 s_4 \\ 0 \end{pmatrix}$$

Thus:

$$\boldsymbol{K_4} = \begin{pmatrix} c_4 & -s_4 & 0 & D_3 v_4 \\ s_4 & c_4 & 0 & -D_3 s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

K_3 is the homogeneous matrix corresponding to the rotation θ_3 around an axis parallel to the x axis and passing through the point P_{30}= [0, 0, L_2].

$$K_3 = \begin{bmatrix} R_{x3} & p_{30} - R_{x3}\, p_{30} \\ 0 & 1 \end{bmatrix}$$

$$p_{30} - R_{x3} p_{30} = \begin{pmatrix} 0 \\ 0 \\ L_2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & -s_3 \\ 0 & s_3 & c_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ L_2 \end{pmatrix} = \begin{pmatrix} 0 \\ L_2 s_3 \\ L_2(1 - c_3) \end{pmatrix} = \begin{pmatrix} 0 \\ L_2 s_3 \\ L_2 v_3 \end{pmatrix}$$

Therefore:

$$\boldsymbol{K_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_3 & -s_3 & L_2 s_3 \\ 0 & s_3 & c_3 & L_2 v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

K_2 is the homogeneous matrix corresponding to the rotation θ_2 around the axis x.

$$\boldsymbol{K_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & -s_2 & 0 \\ 0 & s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

K_1 is the homogeneous matrix corresponding to the rotation θ_1 around the axis z.

$$\boldsymbol{K_1} = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$